

A GERBE FOR THE ELLIPTIC GAMMA FUNCTION

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ABSTRACT. The identities for elliptic gamma functions discovered by A. Varchenko and one of us are generalized to an infinite set of identities for elliptic gamma functions associated to pairs of planes in 3-dimensional space. The language of stacks and gerbes gives a natural framework for a systematic description of these identities and their domain of validity. A triptic curve is the quotient of the complex plane by a subgroup of rank three (it is a stack). Our identities can be summarized by saying that elliptic gamma functions form a meromorphic section of a hermitian holomorphic abelian gerbe over the universal oriented triptic curve.

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1. INTRODUCTION

The elliptic gamma function [24] is a solution of an elliptic version of the Euler functional equation $\Gamma(z+1) = z\Gamma(z)$, in which the rational function z is replaced by a theta function. In the conventions used in [8],

$$\Gamma(z + \sigma, \tau, \sigma) = \theta_0(z, \tau)\Gamma(z, \tau, \sigma),$$

$$\theta_0(z, \tau) = \prod_{j=0}^{\infty} (1 - e^{2\pi i((j+1)\tau - z)})(1 - e^{2\pi i(j\tau + z)}).$$

As in this difference equation three periods $1, \tau, \sigma$ are involved, one should expect that studying modular properties of this function should involve $ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$, just as theta functions have transformation properties under $ISL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Indeed, in [8] three-term functional relations involving Γ at points differing by a $ISL_3(\mathbb{Z})$ -action by fractional linear transformations were discovered. An interpretation of these identities was given in terms of group cohomology.

In this paper we complete the picture and show that the elliptic gamma function is a special case of a family of infinite products defining a *meromorphic section of a holomorphic gerbe* on the complex stack $\mathcal{X}_3 = [X_3/ISL_3(\mathbb{Z})]$, or, equivalently, an equivariant section of a holomorphic equivariant gerbe on X_3 . Here X_3 denotes the total space of the restriction to $\mathbb{CP}^2 - \mathbb{RP}^2$ of the dual tautological bundle $O(1) \rightarrow \mathbb{CP}^2$. This statement, when translated into concrete terms, implies an infinite set of identities between infinite products generalizing the identities found in [8].

Our result is a gerbe version of the fact that the theta function $\theta_0(z, \tau)$ is a holomorphic section of a line bundle on the universal elliptic curve. The analogy is clearer if we extend the definition of $\theta_0(z, \tau)$ from $\text{Im } \tau > 0$ to $\text{Im } \tau \neq 0$ by the rule $\theta_0(z, -\tau) = \theta_0(-z, \tau)^{-1}$. Remarkably, this extended theta function has transformation properties under $ISL_2(\mathbb{Z})$ given by the same formulae, i.e., the multipliers are given by analytic continuation from the domain $\text{Im } \tau > 0$. Passing to homogeneous coordinates, these transformation properties can be rephrased by saying that $\theta_0(w/x_2, x_1/x_2)$ is a meromorphic section of a line bundle on the stack $\mathcal{X}_2 = [X_2/ISL_2(\mathbb{Z})]$, where X_2 is the total space of $O(1) \rightarrow (\mathbb{CP}^1 - \mathbb{RP}^1)$. Equivalently, θ_0 is an equivariant meromorphic section of an equivariant bundle on X_2 .

Equivariant holomorphic (abelian) gerbes can be understood as a generalisation of equivariant line bundles. Here is a naive account of this story, based on local trivializations and cocycles, as in [6] and [18]. A more intrinsic approach is used in Section 5 and the Appendix. Recall that a holomorphic line bundle on a complex manifold X can be described in terms of an open covering $\mathcal{U} = (V_a)_{a \in I}$ of X by holomorphic nowhere vanishing transition functions $\phi_{a,b} \in \mathcal{O}^\times(V_a \cap V_b)$ such that $\phi_{b,a} = \phi_{a,b}^{-1}$ and obeying the cocycle condition

$$\phi_{a,b}(x)\phi_{b,c}(x) = \phi_{a,c}(x), \quad x \in V_a \cap V_b \cap V_c$$

on triple intersections. Now suppose that a group G acts on X by holomorphic diffeomorphisms, and that \mathcal{U} is an invariant open covering, namely that there is a G action on the index set I such that $V_{ga} = gV_a$ for all $g \in G$, $a \in I$. An equivariant structure on a line bundle L is given by a lift of the action, namely the data of linear isomorphisms $\phi(g, x): L_{g^{-1}x} \rightarrow L_x$ between fibres for each $g \in G$ depending holomorphically on x and such that $\phi(gh; x) = \phi(g; x)\phi(h; g^{-1}x)$. In terms of local trivialisations on the charts V_a the lift of the action of $g \in G$ from $V_{g^{-1}a}$ to V_a is given by holomorphic functions $\phi_a(g; \cdot) \in \mathcal{O}^\times(V_a)$ such that

$$\begin{aligned} \phi_{g^{-1}a, g^{-1}b}(g^{-1}x)\phi_b(g; x) &= \phi_a(g; x)\phi_{a,b}(x), & x \in V_a \cap V_b, \\ \phi_a(gh; x) &= \phi_a(g; x)\phi_{g^{-1}a}(h; g^{-1}x), & x \in V_a. \end{aligned}$$

The first equation expresses the fact that the $\phi_a(g; x)$ define a global bundle map and the second is the action property. Let \mathcal{M}^\times denote the sheaf of non-zero meromorphic functions. A (non-zero) meromorphic section of L is then a collection $s_a \in \mathcal{M}^\times(V_a)$, $a \in I$ such that $s_b = s_a\phi_{a,b}$ and an equivariant meromorphic section obeys additionally $s_a(x) = \phi_a(g; x)s_{g^{-1}a}(g^{-1}x)$.

These equations mean that $\phi_{a,b}, \phi_a$ are components of a group-Čech 1-cocycle $\phi \in C_G^{0,1}(\mathcal{U}, \mathcal{O}^\times) \oplus C_G^{1,0}(\mathcal{U}, \mathcal{O}^\times)$ in the double complex $\oplus_{p,q} C_G^{p,q}(\mathcal{U}, \mathcal{O}^\times)$. For any G -equivariant sheaf \mathcal{F} of abelian groups on X , such as $\mathcal{O}^\times, \mathcal{M}^\times$, the Čech complex

$\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of G -modules and thus we have a double complex

$$C_G^{p,q}(\mathcal{U}, \mathcal{F}) = C^p(G, \check{C}^q(\mathcal{U}, \mathcal{F})),$$

with group cohomology differential and Čech differential. The total cohomology of this double complex maps to the equivariant cohomology of the sheaf \mathcal{F} . If D denotes the total differential of this complex, a meromorphic section of a holomorphic G -equivariant line bundle associated to the 1-cocycle ϕ is then given by a 0-chain $s \in C_G^0(\mathcal{U}, \mathcal{M}^\times)$ such that $Ds = \phi \in C_G^1(\mathcal{U}, \mathcal{O}^\times) \subset C_G^1(\mathcal{U}, \mathcal{M}^\times)$.

These notions can be easily extended to the next level: one can define an equivariant holomorphic gerbe (with structure group \mathbb{C}^\times and trivial band) on a complex G -manifold X to be a cocycle $\phi \in C_G^2(\mathcal{U}, \mathcal{O}^\times)$ and an equivariant meromorphic section to be a cochain $s \in C_G^1(\mathcal{U}, \mathcal{M}^\times)$ such that $Ds = \phi$. From a more geometric point of view, gerbes are obtained from line bundles by “replacing functions by line bundles”. Thus a *holomorphic gerbe* is given in terms of the covering \mathcal{U} by a collection $L_{a,b}$ of line bundles on $V_a \cap V_b$, the *transition bundles*, and isomorphisms

$$(1) \quad \phi_{a,b,c}: L_{a,c} \rightarrow L_{a,b} \otimes L_{b,c} \quad \text{on } V_a \cap V_b \cap V_c.$$

The bundles are such that $L_{a,a}$ is the trivial bundle on V_a and $L_{b,a} = L_{a,b}^{-1}$, the dual bundle. The isomorphisms $\phi_{a,b,c}$ are skew-symmetric in a, b, c , when viewed as invertible sections of $L_{a,b} \otimes L_{b,c} \otimes L_{c,a}$ on $V_a \cap V_b \cap V_c$, and are coassociative on fourfold intersections:

$$(\phi_{a,b,c} \otimes \text{id}) \circ \phi_{a,c,d} = (\text{id} \otimes \phi_{b,c,d}) \circ \phi_{a,b,d} \quad \text{on } V_a \cap V_b \cap V_c \cap V_d.$$

Suppose now that G acts on X and \mathcal{U} is a G -invariant open covering. A *G -equivariant holomorphic gerbe* is a holomorphic gerbe on X with the additional data of line bundles $L_a(g) \rightarrow V_a$, $g \in G$, $a \in I$ and isomorphisms

$$(2) \quad \phi_{a,b}(g): (g^{-1})^* L_{g^{-1}a, g^{-1}b} \otimes L_b(g) \rightarrow L_a(g) \otimes L_{a,b} \quad \text{on } V_a \cap V_b$$

$$(3) \quad \phi_a(g, h): L_a(gh) \rightarrow L_a(g) \otimes (g^{-1})^* L_{g^{-1}a}(h) \quad \text{on } V_a,$$

obeying natural coherence conditions. If $L_{a,b}$ is the trivial bundle $V_a \cap V_b \times \mathbb{C}$, then $\phi_{a,b,c}, \phi_{a,b}(g), \phi_a(g, h)$ are invertible holomorphic functions and both associative and coherence condition mean that they form a 2-cocycle in $C_G^2(\mathcal{U}, \mathcal{O}^\times) = C_G^{0,2}(\mathcal{U}, \mathcal{O}^\times) \oplus C_G^{1,1}(\mathcal{U}, \mathcal{O}^\times) \oplus C_G^{2,0}(\mathcal{U}, \mathcal{O}^\times)$.

In the language of transition bundles, a meromorphic section of the gerbe defined by the data (\mathcal{U}, ϕ) is just a collection of meromorphic sections $s_{a,b}$ of $L_{a,b}$ such that $s_{a,b} = s_{b,a}^{-1}$ and $\phi_{a,b,c} \circ s_{a,c} = s_{a,b} \otimes s_{b,c}$. Note that if $s_{a,b}$ are *holomorphic* sections then they form what is called a *trivialisation* of the gerbe. An *equivariant meromorphic section* s is a collection of meromorphic sections $s_{a,b}$ of $L_{a,b}$, forming a meromorphic section, and meromorphic sections $s_a(g)$ of $L_a(g)$ compatible with the maps ϕ . In terms of local trivialisations, these conditions are equivalent to $Ds = \phi$.

Let now $X = X_3$ be the total space of the restriction of the line bundle $\mathcal{O}(1) \rightarrow \mathbb{C}P^2$ to $\mathbb{C}P^2 - \mathbb{R}P^2$. A point of X is an equivalence class of pairs $(w, x) \in \mathbb{C} \times (\mathbb{C}^3 - \mathbb{C} \cdot \mathbb{R}^3)$ modulo $(w, x) \cong (\lambda w, \lambda x)$, $\lambda \in \mathbb{C}^\times$. The group $G = ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ acts on X_3 as follows: $n \in \mathbb{Z}^3$ acts by $(w, x) \mapsto (w - n \cdot x, x)$ and $g \in SL_3(\mathbb{Z})$ acts by $(w, x) \mapsto (x, gx)$. There is an invariant open covering $\mathcal{U} = (V_a)$ indexed by the set of oriented 2-dimensional subspaces of \mathbb{Q}^3 : V_a is the set of $x \in \mathbb{C}^3$ such that $\text{Im}(\alpha \cdot x \beta \cdot x) > 0$ for some (and thus any) oriented basis (α, β) of a .

Theorem 1.1. *The elliptic gamma function $\Gamma(w/x_3, x_1/x_3, x_2/x_3)$ is the component associated to the x - z and y - z planes of a meromorphic section*

$$s = (\Gamma, \Delta) \in C_G^1(\mathcal{U}, \mathcal{M}^\times) = C_G^{0,1}(\mathcal{U}, \mathcal{M}^\times) \oplus C_G^{1,0}(\mathcal{U}, \mathcal{M}^\times)$$

of an equivariant holomorphic gerbe \mathcal{G} on X given by a 2-cocycle $\phi \in C_G^2(\mathcal{U}, \mathcal{O}^\times)$. In explicit terms, we have identities

$$\begin{aligned} \Gamma_{a,b}(y)\Gamma_{b,a}(y) &= 1, & y \in V_a \cap V_b, \\ \phi_{a,b,c}(y)\Gamma_{a,c}(y) &= \Gamma_{a,b}(y)\Gamma_{b,c}(y), & y \in V_a \cap V_b \cap V_c, \\ \phi_{a,b}(g; y)\Gamma_{g^{-1}a, g^{-1}b}(g^{-1}y)\Delta_b(g; y) &= \Delta_a(g; y)\Gamma_{a,b}(y), & y \in V_a \cap V_b, \\ \phi_a(g, h; y)\Delta_a(gh; y) &= \Delta_a(g; y)\Delta_{g^{-1}a}(h; g^{-1}y), & y \in V_a, \end{aligned}$$

for all $a, b, c \in I, g, h \in G$.

The functions Γ, Δ, ϕ are defined in Section 3 and this theorem is proved in Section 4 (see Theorem 4.1). We call *gamma gerbe* the gerbe defined by the cocycle ϕ .

In Section 5 we give a more geometric description of this gerbe. First of all, just like \mathcal{X}_2 is the universal curve over the moduli stack $[(\mathbb{CP}^1 - \mathbb{RP}^1)/SL_2(\mathbb{Z})]$ of elliptic curves, \mathcal{X}_3 is the universal curve over the moduli stack $[(\mathbb{CP}^2 - \mathbb{RP}^2)/SL_3(\mathbb{Z})]$ of *oriented triptic curves*: an oriented triptic curve is a stack of the form $\mathcal{E} = [\mathbb{C}/\iota(\mathbb{Z}^3)]$ for some \mathbb{R} -epimorphism $\iota: \mathbb{R}^3 \rightarrow \mathbb{C}$, equipped with a generator of $H^3(\mathcal{E}, \mathbb{Z}) \simeq \mathbb{Z}$. On this universal curve we introduce geometrically the *pseudodivisor gerbe*; it is given as a sheaf of categories, see Section 5. We then show that the gamma gerbe \mathcal{G}^Γ and the pseudodivisor gerbe \mathcal{G}^{div} are isomorphic by giving a holomorphic trivialisation of $\mathcal{G}^\Gamma \otimes (\mathcal{G}^{div})^*$.

Holomorphic G -equivariant gerbes on X are classified by the second equivariant cohomology group $H_G^2(X, \mathcal{O}^\times)$ [13, 4], the Brauer group $\text{Br}(\mathcal{X})$ of the stack $\mathcal{X} = [X/G]$. The section s defines a class $[s] \in H_G^1(X, \mathcal{M}^\times/\mathcal{O}^\times)$ mapping by the connecting homomorphism to the class $[\phi] \in H_G^2(X, \mathcal{O}^\times)$ of the gerbe. Thus, in analogy with the case of holomorphic line bundles, we can say that s is a *Cartier divisor* for the gerbe \mathcal{G} . The exponential short exact sequence gives then the *Dixmier–Douady class* $c(\mathcal{G})$ of the gerbe, which is the image of $[\phi]$ by the connecting homomorphism $H_G^2(X, \mathcal{O}^\times) \rightarrow H_G^3(X, \mathbb{Z})$. In Section 6 we compute the Brauer group $\text{Br}(\mathcal{E}) = \mathcal{E} \times \mathbb{Z}$ of a triptic curve \mathcal{E} and identify the class of the restriction of the gamma gerbe with $(0, 1)$. For the universal triptic curve, we compute the $H_G^3(X_3, \mathbb{Z})$ modulo torsion and obtain in Section 7 the following result.

Theorem 1.2. *We have*

$$H_G^3(X, \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \oplus \mathbb{Z}$$

and the image of $c(\mathcal{G})$ is a primitive vector in the lattice $\mathbb{Z} \oplus \mathbb{Z}$.

In particular the gerbe is non-trivial. We prove a more precise version of this statement below (Theorem 7.6).

Finally, we show in Section 8 that the gamma gerbe is a *hermitian* holomorphic gerbe. In terms of local trivialisations, this means that the line bundles $L_{a,b}, L_a(g)$ have a hermitian structure and the structure maps (1), (2), (3) are unitary. This result implies that, in spite of the fact that the quotient space is very singular, the gamma gerbe admits an *equivariant connective structure*, i.e., a collection of connections on $L_{a,b}, L_a(g)$ compatible with the structure maps. Indeed, we prove

(Theorem 8.4) that every hermitian equivariant holomorphic gerbe on a complex manifold has a unique equivariant connective structure that is at the same time compatible with the complex and the hermitian structure, generalizing a result of Chatterjee [6]. It is a gerbe analogue of the theorem that holomorphic hermitian line bundles admit a unique compatible connection. In the case of line bundles, the curvature of a connection gives a de Rham representative for the first Chern class and it is a closed form of type (1,1) if the connection comes from a hermitian structure on a holomorphic line bundle. Similarly, hermitian forms on holomorphic gerbes on a complex G -manifold X have an invariant, the $(1,1)$ -curvature in $H_G^1(X, \underline{\Omega}_{cl}^{1,1})$, the first equivariant cohomology group with values in the sheaf of closed (1,1)-forms, see Section 8. An equivariant Čech representative for this class is given by the collection of the curvatures of the connections on $L_{a,b}$, $L_a(g)$. The natural map $H_G^1(X, \underline{\Omega}_{cl}^{1,1}) \rightarrow H_{G,dR}^3(X, \mathbb{C})$ gives then an equivariant de Rham realization of the Dixmier–Douady class $c(\mathcal{G}) \in H_G^3(X, \mathbb{Z})$ (up to a factor $i/2\pi$).

The Appendix contains a description of gerbes on stacks as central extension of groupoids, as well as results on the cohomology on stacks that we use in the paper.

Both the cocycle defining the gamma gerbe and the hermitian structure are constructed using multiple Bernoulli polynomials. The importance of these polynomials in this setting was first recognized by Narukawa [21], who extended the modular relations of [8] to the multiple elliptic gamma functions of Nishizawa [22]. Most of the algebraic identities we prove can be generalized to the multiple gamma function setting, related to $ISL_n(\mathbb{Z})$, and our proofs extend in a straightforward way. However, the geometric setting presented here is for the moment limited to $n \leq 3$.

Note that the identities proven here imply (by taking logarithms) new modular identities for the generalized Eisenstein series considered in [10]. Also, the study of the q -deformed KZB equation on elliptic curve in [11] suggests that there is a non-abelian version of this construction for each representation of a simple Lie algebra.

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2. THETA FUNCTION AND ELLIPTIC GAMMA FUNCTION

2.1. The theta function. For $z \in \mathbb{C}$, $\text{Im } \tau > 0$, let

$$\theta_0(z, \tau) = \prod_{j=0}^{\infty} (1 - e^{2\pi i((j+1)\tau - z)}) (1 - e^{2\pi i(j\tau + z)}).$$

Up to a factor $ie^{-\pi i(z-\tau/6)}\eta(\tau)^{-1}$, where η is the Dedekind eta function, θ_0 is the first (odd) Jacobi theta function. It is holomorphic for $w \in \mathbb{C}$, $\text{Im } \tau > 0$. We extend

θ_0 to a meromorphic function on the domain $\text{Im } \tau \neq 0$ by setting $\theta_0(z, -\tau) = \theta_0(z + \tau, \tau)^{-1}$ (which is also equal to $\theta_0(-z, \tau)^{-1}$).

This function is periodic with period 1 in both arguments. It obeys the functional equation

$$\theta_0(z + \tau, \tau) = -e^{-2\pi iz} \theta_0(z, \tau),$$

and the modular relation

$$\theta_0\left(\frac{w}{x_2}, \frac{x_1}{x_2}\right) \theta_0\left(\frac{w}{x_1}, \frac{x_2}{x_1}\right) = \exp(-\pi i P_2(w, x))$$

where $P_2(w, x) = (w^2 - (x_1 + x_2)w + (x_1^2 + x_2^2 + 3x_1x_2)/6)/x_1x_2$. It follows that θ_0 transforms under $SL_2(\mathbb{Z})$ by

$$\theta_0\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = e^{\pi i Q(g; z, \tau)} \theta_0(z, \tau), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

for some polynomial $Q(g; z, \tau) \in \mathbb{Q}[\tau, \tau^{-1}, z]$ of degree at most 2 in w . An explicit formula for this polynomial is given below.

2.2. The elliptic gamma function. The elliptic gamma function is a meromorphic function $\Gamma(z, \tau, \sigma)$ defined on $\mathbb{C} \times (\mathbb{C} - \mathbb{R}) \times (\mathbb{C} - \mathbb{R})$. For $\text{Im } \tau, \text{Im } \sigma > 0$ it is defined as a convergent infinite product

$$\Gamma(z, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - z)}}{1 - e^{2\pi i(j\tau + k\sigma + z)}},$$

and is extended by the rules

$$(4) \quad \Gamma(z, \tau, \sigma) = \frac{1}{\Gamma(z - \tau, -\tau, \sigma)} = \frac{1}{\Gamma(z - \sigma, \tau, -\sigma)}.$$

For the purpose of this paper, the domain where $\text{Im } \tau < 0$ and $\text{Im } \sigma > 0$ is particularly relevant. There the gamma function is given by the product

$$(5) \quad \Gamma(z, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i(z - (j+1)\tau + k\sigma)}}{1 - e^{2\pi i(-z - j\tau + (k+1)\sigma)}},$$

Clearly Γ is a periodic function of period 1 in all its arguments. Moreover, it obeys an interesting set of identities. First of all, Γ obeys the difference equations

$$\begin{aligned} \Gamma(z + \tau, \tau, \sigma) &= \theta_0(z, \sigma) \Gamma(z, \tau, \sigma) \\ \Gamma(z + \sigma, \tau, \sigma) &= \theta_0(z, \tau) \Gamma(z, \tau, \sigma). \end{aligned}$$

Moreover it obeys the “modular” three-term relations [8]

$$(6) \quad \Gamma(z, \tau, \sigma) = \Gamma(z, \tau, \tau + \sigma) \Gamma(z + \sigma, \tau + \sigma, \sigma).$$

$$(7) \quad \Gamma\left(\frac{w}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right) \Gamma\left(\frac{w}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right) \Gamma\left(\frac{w}{x_2}, \frac{x_3}{x_2}, \frac{x_1}{x_2}\right) = e^{-\pi i P_3(w, x)/3},$$

$$(8) \quad \begin{aligned} P_3(w, x) &= \frac{w^3}{x_1x_2x_3} - 3 \frac{x_1 + x_2 + x_3}{2x_1x_2x_3} w^2 \\ &\quad + \frac{x_1^2 + x_2^2 + x_3^2 + 3x_1x_2 + 3x_2x_3 + 3x_1x_3}{2x_1x_2x_3} w \\ &\quad - \frac{1}{4}(x_1 + x_2 + x_3) \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \right). \end{aligned}$$

2.3. Multiple gamma functions and Bernoulli polynomials. The functions $G_0(z, \tau) = \theta_0(z, \tau)$, $G_1(z, \tau, \sigma) = \Gamma(z, \tau, \sigma)$ are the first two of a sequence introduced by Nishizawa [22] of multiple elliptic gamma functions G_n of $n+2$ variables obeying

$$G_n(z + \tau_i, \tau_0, \dots, \tau_n) = G_{n-1}(z, \tau_0, \dots, \hat{\tau}_i, \dots, \tau_n) G_n(z, \tau_0, \dots, \tau_n).$$

As shown by Narukawa [21], these functions obey functional relations generalizing the modular relations of θ_0, Γ :

$$\prod_{k=1}^r G_{r-2} \left(\frac{w}{x_k}, \frac{x_1}{x_k}, \dots, \frac{\widehat{x_k}}{x_k}, \dots, \frac{x_r}{x_k} \right) = \exp \left(-\frac{2\pi i}{r!} B_{r,r}(w, x) \right),$$

where $B_{r,n}$ are multiple Bernoulli polynomials defined by the generating function

$$e^{wt} \prod_{j=1}^r \frac{t}{e^{x_j t} - 1} = \sum_{n=0}^{\infty} B_{r,n}(w, x_1, \dots, x_r) \frac{t^n}{n!}.$$

In particular, $P_2 = B_{2,2}$, $P_3 = B_{3,3}$. These polynomials obey a number of interesting identities. One of them is the difference relation

$$(9) \quad B_{r,n}(w + x_i, x) - B_{r,n}(w, x) = B_{r-1,n-1}(w, x_1, \dots, \hat{x}_i, \dots, x_r),$$

which can readily be deduced using the generating function.

3. GAMMA FUNCTIONS ASSOCIATED TO WEDGES

Let $\Lambda = \mathbb{Z}^3 \subset V = \mathbb{Q}^3$. We consider V as an oriented vector space with the standard volume form $\det: \wedge^3 V \rightarrow \mathbb{Q}$. The dual lattice $\Lambda^\vee = \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ can be viewed as a lattice in the dual vector space V^* and also comes with an orientation form $\det: \wedge^3 \Lambda^\vee \rightarrow \mathbb{Z}$. To each pair of oriented 2-dimensional subspaces of V^* we associate a function that reduces to the elliptic gamma function for coordinate planes.

3.1. Wedges. An element v of a free \mathbb{Z} -module M is called *primitive* if $\lambda \in \mathbb{Z}$, $v \in \lambda M$ implies $\lambda = \pm 1$ (in particular 0 is not primitive). We denote by $M_{\text{prim}} \subset M$ the set of primitive elements of M . Primitive elements in Λ are in one-to-one correspondence with oriented planes through the origin in V^* . The plane $H(a)$ corresponding to $a \in \Lambda_{\text{prim}}$ has equation $\alpha(a) = 0$ and divides Λ^\vee into two subsets:

$$\Lambda_+^\vee(a) = \{\alpha \in \Lambda^\vee \mid \alpha(a) > 0\}, \quad \Lambda_-^\vee(a) = \{\alpha \in \Lambda^\vee \mid \alpha(a) \leq 0\}.$$

Definition 3.1. A *wedge* is an ordered pair of oriented planes through 0 in V^* or, equivalently, a pair $(a, b) \in \Lambda_{\text{prim}}^2$. A wedge is *in general position* if its planes intersect in a line, i.e., if a, b are linearly independent.

The intersection line of a wedge (a, b) in general position is spanned by the linear form $\det(a, b, \cdot) \in \Lambda^\vee$. This form is not primitive in general, but there exists a unique $\gamma = \gamma_{a,b} \in \Lambda_p^\vee$ and positive integer s such that

$$\det(a, b, x) = s\gamma(x), \quad \forall x \in V.$$

The lattice points in the intersection are then the integer multiples of γ . We call $\gamma_{a,b}$ the *direction vector* of the wedge (a, b) and $s = \text{mod}(a, b)$ its *modulus*. We set $\text{mod}(a, b) = 0$ if a, b are linearly dependent.

Let $\text{Aut}(\Lambda) = \text{SL}_3(\mathbb{Z})$ be the group of linear automorphisms of V preserving the lattice and the volume form. This group acts on Λ preserving Λ_{prim} . The modulus is an invariant of a wedge. There are two orbits with modulus zero and

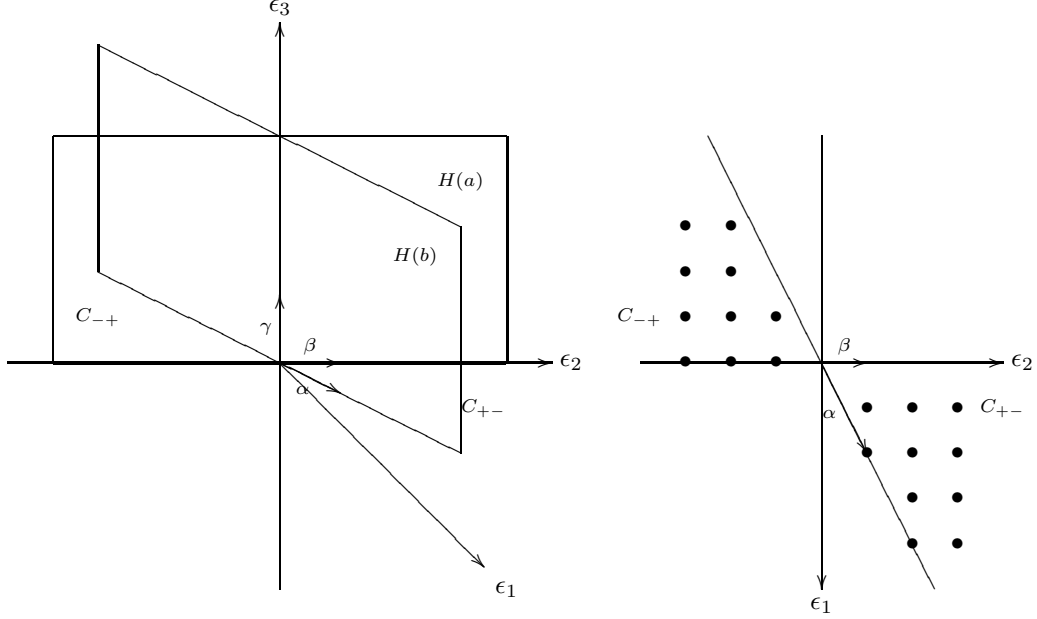


FIGURE 1. These pictures show the situation when $a = e_1$, $b = e_1 - 2e_2$. Then $H(a) = \text{span}\{\epsilon_2, \epsilon_3\}$ and $H(b) = \text{span}\{2\epsilon_1 + \epsilon_2, \epsilon_3\}$, where (ϵ_i) is the basis dual to the standard basis (e_i) of V . Then we take $\alpha = 2\epsilon_1 + \epsilon_2$, $\beta = \epsilon_2$ and $\gamma = \epsilon_3$. The left picture shows the wedge of the ordered pair (a, b) and the right one shows the distribution of δ 's.

the number of orbits with modulus $s > 0$ is given by the Euler ϕ -function $\phi(s)$. Indeed, if e_1, e_2, e_3 is a basis of Λ with $\det(e_1, e_2, e_3) = 1$, then it is easy to see that any wedge (a, b) can be brought by an $SL_3(\mathbb{Z})$ transformation to the normal form $(e_1, \pm e_1)$ or $(e_1, re_1 + se_2)$ with $s = 1, 2, \dots$, $0 \leq r < s$, $\gcd(r, s) = 1$, and that each orbit contains exactly one wedge in normal form.

3.2. Gamma functions. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ be the complexification of V . We associate to a wedge $(a, b) \in \Lambda_{\text{prim}}^2$ a function $\Gamma_{a,b}$ on an open subset of $\mathbb{C} \times V_{\mathbb{C}}$. If $a = b$ we set

$$\Gamma_{a,a}(w; x) = 1.$$

If a, b are linearly independent, the direction vector $\gamma = \gamma_{a,b}$ spans the intersection line $H(a) \cap H(b) \cap \Lambda^{\vee}$, so the group $\mathbb{Z}\gamma$ acts on

$$C_{\theta\eta}(a, b) = \Lambda_{\theta}^{\vee}(a) \cap \Lambda_{\eta}^{\vee}(b), \quad \theta, \eta \in \{+, -\}$$

by translation. We then set

$$(10) \quad \Gamma_{a,b}(w, x) = \frac{\prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma} (1 - e^{-2\pi i(\delta(x)-w)/\gamma(x)})}{\prod_{\delta \in C_{-+}(a,b)/\mathbb{Z}\gamma} (1 - e^{2\pi i(\delta(x)-w)/\gamma(x)})}.$$

The geometry of this construction is depicted for an example in Fig. 1.

3.3. Domain of definition. Let $a \in \Lambda_{\text{prim}}$. The plane $H(a)$ has a natural orientation: an ordered basis α, β of $H(a)$ is oriented if $\det(\alpha, \beta, \delta) > 0$ whenever $\delta(a) > 0$.

Lemma 3.2. *Let $x \in V_{\mathbb{C}}$. The following are equivalent:*

- (i) $\text{Im}(\alpha(x)\overline{\beta(x)}) > 0$ for some oriented basis α, β of $H(a)$.
- (ii) $\text{Im}(\alpha(x)\overline{\beta(x)}) > 0$ for all oriented bases α, β of $H(a)$.
- (iii) $\beta(x) \neq 0$ and $\text{Im}(\alpha(x)/\beta(x)) > 0$ for some oriented basis of $H(a)$.
- (iv) $\alpha(x) \neq 0$ and $\text{Im}(\beta(x)/\alpha(x)) < 0$ for some oriented basis of $H(a)$.

Proof. The equivalence between (i) and (ii) follows from the fact that the action of $SL_2(\mathbb{Q})$ by fractional linear transformations preserves the upper half-plane. The equivalence with the other statements is clear. \square

Let U_a^+ be the open subset of $x \in V_{\mathbb{C}}$ obeying any of the equivalent conditions of Lemma 3.2.

Proposition 3.3. *Let $a, b \in \Lambda_{\text{prim}}$ be linearly independent. Then the ratio of infinite products (10) converges to a meromorphic function on $U_a^+ \cap U_b^+$. It has simple zeros at $w = \delta(x) + n\gamma(x)$, with $\delta \in C_{+-}(a, b)$, $n \in \mathbb{Z}$, and simple poles at $w = \delta(x) + n\gamma(x)$, with $\delta \in C_{-+}(a, b)$, $n \in \mathbb{Z}$.*

Proof. Let us first consider the denominator. Notice that $C_{-+}(a, b) = (\mathbb{Q}_{\leq 0}\alpha + \mathbb{Q}_{> 0}\beta + \mathbb{Q}\gamma) \cap \Lambda^\vee$, for any $\alpha \in H(b)$, $\beta \in H(a)$ such that $\alpha(a) > 0$, $\beta(b) > 0$. Thus the condition for convergence of the infinite product in the denominator is $\text{Im}(\alpha(x)/\gamma(x)) < 0$ (i.e., $\text{Im}(\gamma(x)/\alpha(x)) > 0$), $\text{Im}(\beta(x)/\gamma(x)) > 0$. Now, by construction, β, γ is an oriented basis of $H(a)$ and γ, α an oriented basis of $H(b)$, so the convergence condition is $x \in U_a^+ \cap U_b^+$. A similar argument applies to the numerator. In this domain we obtain a meromorphic function whose divisor can be read off the zeros of the factors of the infinite products. \square

Remark 3.4. We did not define $\Gamma_{a,b}$ for $a = -b$. There is no need to define it as its domain of definition is empty.

3.4. Expression in terms of ordinary elliptic gamma functions. The gamma function associated to a wedge may be expressed (in many ways) as a finite product of ordinary elliptic gamma functions:

Proposition 3.5. *Let $a, b \in \Lambda_{\text{prim}}$ be linearly independent and let $\gamma \in \Lambda_{\text{prim}}$ be the direction vector of the wedge (a, b) . Let $\alpha, \beta \in \Lambda^\vee$ be such that $\alpha(b) = \beta(a) = 0$ and $\alpha(a) > 0$, $\beta(b) > 0$. Then*

$$\Gamma_{a,b}(w, x) = \prod_{\delta \in F/\mathbb{Z}\gamma} \Gamma\left(\frac{w + \delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right),$$

where F is the set of $\delta \in \Lambda^\vee$ such that

$$0 \leq \delta(a) < \alpha(a), \quad 0 \leq \delta(b) < \beta(b).$$

Proof. Since $C_{+-}(a, b)$ is the subset of Λ defined by the inequalities $\delta(a) > 0$, $\delta(b) \leq 0$, it follows that each $\delta \in C_{+-}(a, b)$ can be uniquely written as $\delta = -\bar{\delta} + (j+1)\alpha - k\beta$, with $\bar{\delta} \in F$ and $j, k \in \mathbb{Z}_{\geq 0}$ (modulo $\mathbb{Z}\gamma$). Similarly if $\delta \in C_{-+}(a, b)$ then $\delta = -\bar{\delta} - j\alpha + (k+1)\beta$ with $\bar{\delta} \in F$ and $j, k \in \mathbb{Z}_{\geq 0}$. \square

Example 3.6. If $a = e_1$, $b = e_2$ are the first two standard basis vectors, $\Gamma_{a,b}$ reduces to the ordinary elliptic gamma function in the domain $\text{Im}(x_2/x_3) > 0$, $\text{Im}(x_3/x_1) > 0$, $w \in \mathbb{C}$:

$$\Gamma_{e_1, e_2}(w, x) = \Gamma\left(\frac{w}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right).$$

In this domain Γ is given by the convergent product (5)

Remark 3.7. One could use Proposition 3.5 to define gamma functions of wedges. The fact that this definition is independent of choices is a consequence of the multiplication formulae [9].

3.5. Inversion and three-term relations. The three-term relations (6), (7) satisfied by the elliptic gamma function are special cases of a collection of three-term relations parametrized by a triple a, b, c of primitive elements of Λ .

Theorem 3.8. *Let $a, b \in \Lambda_{\text{prim}}$. Then, for $x \in U_a^+ \cap U_b^+$ and $w \in \mathbb{C}$ we have the inversion relation*

$$\Gamma_{a,b}(w, x) \Gamma_{b,a}(w, x) = 1.$$

More generally, let $a, b, c \in \Lambda_{\text{prim}}$. Then, for $x \in U_a^+ \cap U_b^+ \cap U_c^+$ and $w \in \mathbb{C}$

$$\Gamma_{a,b}(w, x) \Gamma_{b,c}(w, x) \Gamma_{c,a}(w, x) = \exp\left(-\frac{\pi i}{3} P_{a,b,c}(w, x)\right),$$

for some polynomial $P_{a,b,c}(w, x) \in \mathbb{Q}(x)[w]$ of degree at most 3 in w with rational coefficients.

Before giving the explicit description of the polynomial $P_{a,b,c}$ we need to discuss for which triples the domain of validity of the three-term relation is non-empty.

Lemma 3.9. *Let $a, b, c \in \Lambda_{\text{prim}}$. Then $U_a^+ \cap U_b^+ \cap U_c^+$ is non-empty if and only if a, b, c are on the same side of some plane through the origin*

Proof. Suppose that a, b, c all lie on the same line. Then either $a = b = c$ and $U_a^+ \neq \emptyset$ or the sum of two is zero, in which case the intersection is empty since U_a^+ , U_{-a}^+ are disjoint.

Suppose that a, b, c lie on a plane. Then the corresponding planes $H(a)$, $H(b)$, $H(c)$ intersect in a line $\mathbb{Q}\sigma$ and we may choose $\alpha, \beta, \gamma \in V^*$ so that (α, σ) , (β, σ) , (γ, σ) are oriented bases of these subspaces. Now a, b, c are on the same side of a plane if and only if α, β, γ have the same property. In this case there is a $t \in V$ such that $\alpha(t), \beta(t), \gamma(t) > 0$ and $\sigma(t) = 0$. Then for any $s \in V$ such that $\sigma(s) > 0$, we have $x = s + it \in U_a^+ \cap U_b^+ \cap U_c^+$ so the intersection is non-empty. If α, β, γ are not on the same side of a plane, we may rescale them so that $\alpha + \beta + \gamma = 0$. Then for any x the three complex numbers $\alpha(x)\overline{\sigma(x)}$, $\beta(x)\overline{\sigma(x)}$, $\gamma(x)\overline{\sigma(x)}$ add up to zero, so they cannot all have positive imaginary part and the intersection is empty.

Finally, assume that a, b, c are a basis of V , which we may assume to be oriented, and let α, β, γ be the dual basis of V^* . Both bases are always on the same side of some plane through the origin. Moreover (α, β) , (β, γ) , (γ, α) are oriented bases of $H(c)$, $H(a)$, $H(b)$ respectively. Then for example $x \in V_{\mathbb{C}}$ so that $\alpha(x) = 1, \beta(x) = (-1 - i\sqrt{3})/2, \gamma(x) = (-1 + i\sqrt{3})/2$ belongs to the intersection $U_a^+ \cap U_b^+ \cap U_c^+$. \square

Theorem 3.10. *The polynomial $P_{a,b,c}$ of Theorem 3.8 can be chosen in the following way:*

- (i) If a, b, c are linearly dependent then $P_{a,b,c} = 0$.
- (ii) If a, b, c are linearly independent and $\det(a, b, c) > 0$,

$$\begin{aligned} P_{a,b,c}(w, x) &= \sum_{\delta \in F(a,b,c)} P_3(w + \delta(x), \alpha(x), \beta(x), \gamma(x)) \\ &= \frac{\det(\alpha, \beta, \gamma)}{\alpha(x)\beta(x)\gamma(x)} w^3 + \deg \leq 2 \end{aligned}$$

Here α, β, γ are the direction vectors of the wedges (b, c) , (c, a) and (a, b) , respectively, $P_3 = B_{3,3}$ is the Bernoulli polynomial (8) and $F(a, b, c)$ is the set of $\delta \in \Lambda^\vee$ such that

$$0 \leq \delta(a) < \alpha(a), \quad 0 \leq \delta(b) < \beta(b), \quad 0 \leq \delta(c) < \gamma(c).$$

- (iii) For all permutations $\sigma \in S_3$,

$$P_{a,b,c}(w, x) = \text{sign}(\sigma) P_{\sigma(a), \sigma(b), \sigma(c)}(w, x).$$

Moreover, $P_{a,b,c}(w, x)$ is $SL_3(\mathbb{Z})$ -equivariant:

$$P_{ga,gb,gc}(w, x) = P_{a,b,c}(w, g^{-1}x), \quad g \in SL_3(\mathbb{Z}).$$

Proof of Theorems 3.8 and 3.10 We first prove the inversion relation. The direction vector of (a, b) is the opposite of the direction vector of (b, a) . This implies that $\Gamma_{b,a}$ is obtained from $\Gamma_{a,b}$ by exchanging the numerator and the denominator, proving the claim.

Assume that a, b, c lie on the same plane and let γ be the direction vector of (a, b) . After a cyclic permutation of a, b, c , it may be assumed that γ is also the direction vector of (b, c) and (a, c) . In this situation,

$$C_{+-}(a, c) = C_{+-}(a, b) \sqcup C_{+-}(b, c), \quad C_{-+}(a, c) = C_{-+}(a, b) \sqcup C_{-+}(b, c),$$

implying the identity in the form $\Gamma_{a,b}\Gamma_{b,c} = \Gamma_{a,c}$.

Now let a, b, c be linearly independent and assume that $\det(a, b, c) > 0$. We use the representation of Proposition 3.5. Consider first the factor $\Gamma_{a,b}$. The direction vector α of (b, c) obeys $\alpha(a) > 0$ and $\alpha(b) = 0$ (since α is proportional by a positive integer to $\det(b, c, \cdot)$). Similarly, $\beta(a) = 0$ and $\beta(b) > 0$. Thus we can write $\Gamma_{a,b}$ as product of standard elliptic gamma functions as in Proposition 3.5. Let $F(a, b)$ denote the set F appearing there. Since $\gamma(a) = \gamma(b) = 0$ and $\gamma(c) > 0$, in each class of $F(a, b)/\mathbb{Z}\gamma$ there is a unique representative δ , such that $\delta(c) \in \{0, \dots, \gamma(c) - 1\}$. Therefore we can replace the product in Proposition 3.5 by a product over $F(a, b, c)$. Applying the same procedure to the remaining gamma functions in the product, we obtain a product over $F(a, b, c)$ of triple products of standard elliptic gamma functions. Identity (7) gives then the result.

The coefficient of w^3 in $P_{a,b,c}$ is then the coefficient of P times the cardinality of $F(a, b, c)$. The latter is the volume of the parallelepiped in V^* generated by the vectors α, β, γ .

It is clear that $P_{a,b,c}$, defined by (i), (ii) for $\det(a, b, c) \geq 0$ is invariant under cyclic permutations of a, b, c . Applying the inversion relation to all terms of the three term relation we get $\Gamma_{b,a}\Gamma_{c,b}\Gamma_{a,c} = \exp(i\pi P_{a,b,c})$. Therefore we may extend the definition of P to general triples by setting $P_{c,b,a} = -P_{a,b,c}$. With this definition the three-term relation is obeyed by all triples and (iii) holds.

The equivariance can be easily checked directly from the definition of $P_{a,b,c}$. \square

Remark 3.11. By Proposition 3.5, the identities of Theorems 3.8 and 3.10 can be written as n -term relations for the ordinary elliptic gamma function (5). The inversion relation for $a = e_1, b = e_2$ reads $\Gamma(z, \tau, \sigma)\Gamma(-z, -\sigma, -\tau) = 1$. The three-term relation (7) is the special case of the second identity in Theorem 3.8 where $a = e_1, b = e_2, c = e_3$. If $a = e_1, b = e_1 - e_2, c = e_2$ we obtain the relation

$$\Gamma\left(-\frac{w}{x_3}, -\frac{x_1+x_2}{x_3}, \frac{x_2}{x_3}\right) \Gamma\left(\frac{w}{x_3}, \frac{x_1}{x_3}, \frac{x_1+x_2}{x_3}\right) \Gamma\left(-\frac{w}{x_3}, -\frac{x_1}{x_3}, -\frac{x_2}{x_3}\right) = 1,$$

which reduces to (6) if we apply the inversion relation and (4). For a simple “new” identity, consider the case $a = e_1, b = e_1 + 2e_2, c = e_3$. Then, by using the direction vectors as coordinates, we obtain the four-term relation

$$\begin{aligned} & \Gamma\left(\frac{w}{y_3}, \frac{y_1}{y_3}, \frac{y_2}{y_3}\right) \Gamma\left(\frac{w}{y_3} + \frac{y_1+y_2}{2y_3}, \frac{y_1}{y_3}, \frac{y_2}{y_3}\right) \Gamma\left(\frac{w}{y_1}, \frac{y_1+y_2}{2y_1}, \frac{y_3}{y_1}\right) \Gamma\left(\frac{w}{y_2}, \frac{y_3}{y_2}, \frac{y_1+y_2}{2y_2}\right) \\ &= \exp\left(-\frac{\pi i}{3} (P_3(w, y_1, y_2, y_3) + P_3(w + (y_1+y_2)/2, y_1, y_2, y_3))\right) \end{aligned}$$

3.6. Action of $ISL_3(\mathbb{Z})$. The group $SL_3(\mathbb{Z})$ acts on Λ, Λ^\vee, V and $V_\mathbb{C}$. It acts transitively on Λ_{prim} . If $a \in \Lambda_{prim}$ and $g \in SL_3(\mathbb{Z})$, $U_{ga}^+ = gU_a^+$ and

$$(11) \quad \Gamma_{ga,gb}(w, x) = \Gamma_{a,b}(w, g^{-1}x), \quad x \in U_a^+ \cap U_b^+.$$

The group $ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ acts on $\mathbb{C} \times V_\mathbb{C}$ by

$$(g, \alpha) \cdot (w, x) = (w - \alpha(x), g \cdot x), \quad g \in SL_3(\mathbb{Z}), \quad \alpha \in \Lambda^\vee = \mathbb{Z}^3.$$

The subgroup Λ^\vee maps U_a^+ to itself for all $a \in \Lambda$. Here is a description of the action on gamma functions.

Definition 3.12. A *framing* of Λ_{prim} assigns to each $a \in \Lambda_{prim}$ an oriented basis $f(a) = (\alpha_1, \alpha_2, \alpha_3)$ of the \mathbb{Z} -module Λ^\vee such that $\alpha_1(a) = 1$ and $\alpha_2, \alpha_3 \in H(a)$.

For each framing f we define a family of products of theta functions parametrized by $a \in \Lambda_{prim}$ and $\mu \in \Lambda^\vee$:

$$(12) \quad \Delta_a(\mu; w, x) = \Delta_{a,f}(\mu; w, x) = \prod_{j=0}^{\mu(a)-1} \theta_0\left(\frac{w + j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)}\right).$$

Here the product should be understood as $\prod_{-N \leq j < \mu(a)} / \prod_{-N \leq j < 0}$ for any sufficiently large N so it is defined also for negative $\mu(a)$. Notice moreover that $\Delta_a(\mu; w, x) = 1$, if μ belongs to the plane $H(a)$.

Proposition 3.13. Let $a \in \Lambda_{prim}$, $\mu \in \Lambda^\vee$. The function $(w, x) \rightarrow \Delta_a(\mu; w, x)$ is meromorphic on $\mathbb{C} \times U_a^+$.

Proof. The theta function $\theta(z, \tau)$ is holomorphic for $\text{Im } \tau > 0$. The function Δ_a , being a product of (inverses of) theta functions with $\tau = \alpha_2(x)/\alpha_3(x)$ is meromorphic for $\text{Im}(\alpha_2(x)/\alpha_3(x)) > 0$, i.e., for $x \in U_a$. \square

Proposition 3.14. Let $a, b \in \Lambda_{prim}$ be linearly independent and fix a framing of Λ_{prim} . Then

$$\frac{\Gamma_{a,b}(w + \mu(x), x)}{\Gamma_{a,b}(w, x)} = e^{\pi i P_{a,b}(\mu; x, w)} \frac{\Delta_a(\mu; w, x)}{\Delta_b(\mu; w, x)}$$

for some (framing dependent) polynomial $P_{a,b}(\mu; x, w) \in \mathbb{Q}(x)[w]$ homogeneous of degree 0 in x, w and of degree at most 2 in w .

Note that a change of framing multiplies Δ_a by the exponential of a polynomial of degree at most 2 in w . Indeed a change of basis α_2, α_3 of $H(a)$ results in an $SL_2(\mathbb{Z})$ transformation of the arguments of θ_0 , producing a multiplier given by the exponential of a polynomial of degree ≤ 2 . Also, α_1 is determined up to the addition of an element of $H(a) \cap \Lambda^\vee$. Therefore, under a change of α_1 , the theta functions in Δ_a change by a shift of w by an integer linear combination of α_2 and α_3 . Under such shifts θ_0 changes by the exponential of a polynomial of degree ≤ 1 .

Proof of Proposition 3.14. We consider exemplarily the case $\mu(a) > 0, \mu(b) > 0$; the other cases are treated in a similar way, whenever neither $\mu(a)$ nor $\mu(b)$ vanishes, with due modifications. The cases where at least one of $\mu(a)$ and $\mu(b)$ vanishes are better treated separately, but the main computations are similar to the following ones.

In the ratio of gamma functions infinitely many terms cancel in the ratio of infinite products. The remaining infinitely many terms can be collected as follows. Let γ be the direction vector of (a, b) and let $\alpha, \beta \in \Lambda_{prim}^\vee$ be such that $\alpha(b) = \beta(a) = 0, \alpha(a), \beta(b) > 0$. Then

$$\frac{\Gamma_{a,b}(w + \mu(x), x)}{\Gamma_{a,b}(w, x)} = \prod_{\delta \in F_1} \left(-e^{-2\pi i(w + \delta(x))/\gamma(x)} \right) \frac{\prod_{\delta \in F_2} \theta_0 \left(\frac{w + \delta(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)} \right)}{\prod_{\delta \in F_3} \theta_0 \left(\frac{w + \delta(x)}{-\gamma(x)}, \frac{\alpha(x)}{-\gamma(x)} \right)}.$$

In these products δ runs over the finite subsets of $\Lambda^\vee/\mathbb{Z}\gamma$ defined by the following inequalities:

$$\begin{aligned} F_1: 0 \leq \delta(a) < \mu(a), \quad 0 \leq \delta(b) < \mu(b), \\ F_2: 0 \leq \delta(a) < \mu(a), \quad 0 \leq \delta(b) < \beta(b), \\ F_3: 0 \leq \delta(a) < \alpha(a), \quad 0 \leq \delta(b) < \mu(b). \end{aligned}$$

We need to show that the right-hand side is Δ_a/Δ_b up to multiplication by an exponential functions of a polynomial.

Consider e.g. the product indexed by F_2 : we notice that for each $0 \leq j < \mu(a)$ there is exactly one $\delta \in \Lambda^\vee/\mathbb{Z}\gamma$ such that $\delta(a) = j$ and $0 \leq \delta(b) < \beta(b)$. This can be seen most easily by going to the normal form, where $a = e_1, b = re_1 + se_2, s > 0, 0 \leq r < s, \gcd(r, s) = 1$, see subsection 3.1: if $\epsilon_1, \epsilon_2, \epsilon_3$ is the dual basis of $\Lambda^\vee, \beta = \epsilon_2, \alpha = s\epsilon_1 - r\epsilon_2, \gamma = \epsilon_3$ and the unique δ with these properties is $\delta = j\epsilon_1 + k\epsilon_2 \pmod{\epsilon_3}$ where k is the unique integer such that $0 \leq jr + ks < s$.

If we are given a framing and $(\alpha_1, \alpha_2, \alpha_3)$ is the basis assigned to a , this δ differs from $j\alpha_1$ by a lattice vector in $H(a)$, i.e., an integer linear combination of β and γ . Thus we may replace δ by $j\alpha_1$ in the numerator up to the exponential of a polynomial. A similar reasoning applies to the denominator.

The oriented bases (β, γ) of $H(a)$ and $(\alpha, -\gamma)$ of $H(b)$ can then be replaced by the bases given by the framing up to the exponential of a quadratic polynomial.

This proves the claim. \square

Proposition 3.15. *Fix a framing of Λ_{prim} . Let $\mu, \nu \in \Lambda^\vee, a \in \Lambda_{prim}$. Let $\mu = \sum m_i \alpha_i, \nu = \sum n_i \alpha_i$ be the decomposition of μ, ν in terms of the basis assigned to a by the framing. Then*

$$\Delta_a(\mu + \nu; w, x) = e^{2\pi i P_a(\mu, \nu; w, x)} \Delta_a(\mu; w, x) \Delta_a(\nu; w + \mu(x), x)$$

where

$$P_a(\mu, \nu; w, x) = \frac{n_1 m_2}{\alpha_3(x)} \left[w + \frac{2m_1 + n_1 - 1}{2} \alpha_1(x) + \frac{m_2 - 1}{2} \alpha_2(x) + \frac{1}{2} \alpha_3(x) \right].$$

4. THE GAMMA GERBE

In this section we cast together the identities obtained in the previous sections and describe them geometrically in the gerbe language. After fixing some sign conventions, we begin by reviewing the well-known case of the Jacobi theta function, based on the classical identities discovered by Jacobi and Dedekind.

4.1. Conventions and notations. Our convention for the differential $\check{\delta}$ of the Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$ of a sheaf \mathcal{F} with open covering $\mathcal{U} = (V_a)_{a \in I}$ is

$$\check{\delta} \phi_{a_0, \dots, a_{p+1}} = \phi_{a_1, \dots, a_{p+1}} - \phi_{a_0, a_2, \dots, a_{p+1}} + \dots + (-1)^{p+1} \phi_{a_0, \dots, a_p},$$

if $\phi = \oplus \phi_{a_0, \dots, a_p} \in \check{C}^p(\mathcal{U}, \mathcal{F}) = \oplus_{a_0, \dots, a_p} \mathcal{F}(V_{a_0} \cap \dots \cap V_{a_p})$. The differential δ of the complex $C(G, M) = \oplus_p \text{Maps}(G^p, M)$ of cochains of the group G with values in the left G -module M is

$$\begin{aligned} \delta c(g_1, \dots, g_{p+1}) &= (-1)^{p+1} g_1 \cdot c(g_2, \dots, g_{p+1}) \\ &\quad + \sum_{j=1}^p (-1)^{p+1-j} c(g_1, \dots, g_j g_{j+1}, \dots, g_{p+1}) \\ &\quad + c(g_1, \dots, g_p). \end{aligned}$$

In our case G is a discrete group acting on a complex manifold and M is the Čech complex of an equivariant sheaf \mathcal{F} corresponding to an invariant open covering. Then the double complex $C_G(\mathcal{U}, \mathcal{F}) = \oplus_{p,q} C^p(G, \check{C}^q(\mathcal{U}, \mathcal{F}))$ with total differential $D = \delta + (-1)^p \check{\delta}$ computes the equivariant cohomology $H_G(X, \mathcal{F})$. Many of the sheaves \mathcal{F} we consider, such as \mathcal{O}^\times , \mathcal{M}^\times , are sheaves of multiplicative groups, so that the formulae for the differentials are written multiplicatively.

4.2. The theta line bundle on X_2 . Let X_2 be the total space of the dual tautological line bundle $\mathcal{O}(1) \rightarrow (\mathbb{C}P^1 - \mathbb{R}P^1)$. Thus X_2 is the quotient of $\mathbb{C} \times (\mathbb{C}^2 - \{0\})$ by the action $(w, x) \mapsto (\lambda w, \lambda x)$ of \mathbb{C}^\times . The group $G = ISL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}$ acts on X_2 by $(g, \mu) \cdot (w, x) = (w - \mu_1 x_1 - \mu_2 x_2, g \cdot x)$. As a complex manifold, X_2 is isomorphic to $\mathbb{C} \times (\mathbb{C} - \mathbb{R})$. It is covered by two invariant contractible disjoint open sets $V_\pm = \{(w, x) \in \mathbb{C} \times (\mathbb{C}^2 - \{0\}) \mid \pm \text{Im}(x_1/x_2) > 0\} / \mathbb{C}^\times$. Let

$$\theta_\pm(w, x) = \theta_0 \left(\pm \frac{w}{x_2}, \pm \frac{x_1}{x_2} \right)^{\pm 1}, \quad (w, x) \in V_\pm.$$

Then, for every $h \in G$, $(w, x) \in V_\pm$,

$$\theta_\pm(w, x) = \phi_\pm(g; w, x) \theta_\pm(h^{-1}(w, x)).$$

The group 1-cocycle ϕ_\pm is given by the following formulae: let $N : SL_2(\mathbb{Z}) \rightarrow \mathbb{Z}/12\mathbb{Z}$ be the homomorphism whose value on generators is

$$N \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \equiv 1 \pmod{12}, \quad N \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv -3 \pmod{12}.$$

Then if $g \in SL_2(\mathbb{Z})$

$$\phi_\pm(g; w, x) = \exp \left(-\pi i Q \left(g; \frac{w}{x_2}, \frac{x_1}{x_2} \right) \right),$$

where for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$Q(g; z, \tau) = -\frac{cz^2}{c\tau + d} - \frac{z}{c\tau + d} + \frac{z}{6} \frac{1}{c\tau + d} \frac{a\tau + b}{c\tau + d} - \frac{\tau}{6} + \frac{1}{6} N(g);$$

if $\mu \in \mathbb{Z}^2$,

$$\phi_{\pm}(\mu; w, x) = \exp \left(2\pi i \mu_1 \frac{w}{x_2} + \pi i \mu_1 (\mu_1 - 1) \frac{x_1}{x_2} + \pi i \mu_1 \right).$$

This 1-cocycle extends in a trivial way to a 1-cocycle

$$(1, \phi_{\pm}) \in C_G^{0,1}(\mathcal{U}, \mathcal{O}^{\times}) \oplus C_G^{1,0}(\mathcal{U}, \mathcal{O}^{\times}),$$

in the group-Čech double complex for the covering $\mathcal{U} = \{V_+, V_-\}$ and defines a G -equivariant line bundle on X_2 . The functions θ_{\pm} form then a meromorphic equivariant section in $C_G^{0,0}(\mathcal{U}, \mathcal{M}^{\times})$. Clearly the Čech part of this story is trivial since X_2 is just the disjoint union of two contractible sets, but this will change when we pass to X_3 .

4.3. The gamma gerbe on X_3 . Let $X = X_3$ be the total space of the line bundle $O(1) \rightarrow (\mathbb{CP}^2 - \mathbb{RP}^2)$. Geometrically we think of $O(1)$ as the dual bundle to the tautological line bundle and of \mathbb{CP}^2 as the projectivization of $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, where $\Lambda = \mathbb{Z}^3$ is a free abelian group of rank 3 equipped with a volume form $\det : \wedge^3 \Lambda \rightarrow \mathbb{Z}$. The group $\text{Aut}(\Lambda) \cong SL_3(\mathbb{Z})$ of linear transformations of $\Lambda_{\mathbb{C}}$ mapping Λ to itself and preserving the volume form, acts naturally on X . The dual lattice $\Lambda^{\vee} = \text{Hom}(\Lambda, \mathbb{Z}) \cong \mathbb{Z}^3 \subset V_{\mathbb{C}}^*$ acts on $O(1)$ fiberwise by translation and we get an action of $\text{Aut}(\Lambda) \ltimes \Lambda^{\vee} \cong ISL_3(\mathbb{Z})$. More explicitly, this group acts linearly on $\mathbb{C} \times V_{\mathbb{C}}$ via $(g, \mu)(w, x) = (w - \alpha(x), gx)$, and this action induces an action on $X = (\mathbb{C} \times (V_{\mathbb{C}} - \mathbb{C} \cdot V_{\mathbb{R}}))/\mathbb{C}^{\times}$.

The complex manifold X has a natural $ISL_3(\mathbb{Z})$ -invariant open covering $\mathcal{U} = (V_a)_{a \in \Lambda_{\text{prim}}}$ labeled by the set of primitive vectors Λ_{prim} in Λ : for $a \in \Lambda_{\text{prim}}$, $V_a = \{(w, x) \mid x \in U_a^+\}/\mathbb{C}^{\times}$, see subsection 3.3.¹

Recall that a *framing* of Λ_{prim} is a map $f : \Lambda_{\text{prim}} \rightarrow \{\text{oriented bases of } \Lambda^{\vee}\}$ such the basis $f(a) = (\alpha_1, \alpha_2, \alpha_3)$ obeys $\alpha_1(a) = 1$ and $\alpha_2(a) = \alpha_3(a) = 0$.

Let us fix a framing f . Then we have the gamma functions $\Gamma_{a,b} \in \mathcal{M}^{\times}(V_a \cap V_b)$, associated to wedges $(a, b) \in \Lambda_{\text{prim}}^2$, and the products of theta functions $\Delta_a = \Delta_{a,f} : \mathbb{Z}^3 \rightarrow \mathcal{M}^{\times}(V_a)$, see Section 3, Eqns. (10), (12). Extend Δ_a to a map $ISL_3(\mathbb{Z}) \rightarrow \mathcal{M}^{\times}(V_a)$ by setting $\Delta_a((g, \mu); w, x) = \Delta_a(\mu \circ g^{-1}; w, x)$ for $g \in SL_3(\mathbb{Z})$, $\mu \in \Lambda^{\vee} = \mathbb{Z}^3$.

Theorem 4.1. *Let $G = ISL_3(\mathbb{Z})$ and f be a framing of Λ_{prim} . View $\Gamma_{a,b}$, Δ_a as the components of a cochain $s = (\Gamma, \Delta) \in C_G^1(\mathcal{U}, \mathcal{M}^{\times})$. Then the functions forming the cocycle $\phi = Ds$ are holomorphic and nowhere vanishing. Thus $\phi \in C_G^2(\mathcal{U}, \mathcal{O}^{\times})$ defines a holomorphic gerbe \mathcal{G} on X_3 and s is a meromorphic section of this gerbe.*

This theorem is a rephrasing Theorem 1.1: the identity $Ds = \phi$, written explicitly, is the set of equations appearing there.

¹One can show that this covering is good, i.e., all non-empty multiple intersections are contractible. Moreover the sets V_a are domains of holomorphy. It follows that the double complex $C(G, \check{C}(\mathcal{U}, \mathcal{F}))$ can be used to compute the equivariant cohomology with values in equivariant constant or analytic coherent sheaves.

Proof. The first thing to check is that $s = (\Gamma, \Delta) \in C^1(\mathcal{U}, \mathcal{M}^\times)$, namely that these functions are meromorphic on their domain of definition. This is the content of Propositions 3.3, 3.13.

Now define ϕ by $\phi = Ds$, namely by the identities of Theorem 1.1. *A priori*, ϕ is a 2-cocycle with values in the sheaf \mathcal{M}^\times of invertible meromorphic functions and we have to show that it takes value in the invertible *holomorphic* functions. It is sufficient to show that it takes values in the exponentials of rational functions, since a meromorphic function of this form is automatically holomorphic and nowhere vanishing. The first identity in Theorem 1.1 is treated in Theorems 3.8, 3.10, implying that $\phi_{a,b,c}$ is indeed holomorphic and invertible on $V_a \cap V_b \cap V_c$. Proposition 3.14 implies that $\phi_{a,b}(g; y)$ is the exponential of a rational function for g in the translation subgroup of G . The second identity of Theorem 1.1 for general g can be obtained from Proposition 3.14 using the equivariance of Γ under $SL_3(\mathbb{Z})$, Equation 11. Again the conclusion is that $\phi_{a,b}$ is the exponential of a rational function. As for the third identity of Theorem 1.1, it appears in Proposition 3.15 for μ, ν in the translation subgroup. Since $\Delta_{a,f}(g; w, x)$ for general g is defined in terms of its restriction to the translation subgroup, the identity of Proposition 3.15 gives an identity for general group elements. However to reduce it to the third identity of Theorem 1.1 we need to relate $\Delta_{g^{-1}a,f}(\mu \circ g; w, g^{-1}x)$ to $\Delta_{a,f}(\mu; w, x)$ for $g \in SL_3(\mathbb{Z})$. The relation involves the natural action of $SL_3(\mathbb{Z})$ on the set of framings:

$$\Delta_{g^{-1}a,f}(\mu \circ g; w, g^{-1}x) = \Delta_{a,gf}(\mu; w, x), \quad g \in SL_3(\mathbb{Z}), \quad \mu \in \Lambda^\vee.$$

As discussed after Proposition 3.14, a change of framing amounts to an $ISL_2(\mathbb{Z})$ -transformation of the theta functions and so we have

$$(13) \quad \Delta_{a,gf}(\mu; w, x) = \psi_{gf,f}(\mu; w, x) \Delta_{a,f}(\mu; w, x), \quad (w, x) \in V_a,$$

for some exponential of rational function ψ , which, by the same argument as above, is an invertible holomorphic function on V_a which enters in the third identity of Theorem 1.1 for general group elements. \square

Remark 4.2. Equation (13) in the proof shows that a change of framing from f to f' amounts to replacing Δ , by $\psi_{f',f} \Delta$ where $\psi_{f',f} \in C^{1,0}(\mathcal{U}, \mathcal{O}^\times)$ and thus ϕ by a cocycle differing from ϕ by the exact cocycle $D(1, \psi_{f',f})$.

By construction, the components of the cocycle are exponentials of rational functions. Their restriction of the components of the cocycle ϕ to the subgroup $\Lambda^\vee = \mathbb{Z}^3$ is given by the explicit formulae calculated in Section 3: for $a, b, c \in \Lambda_{prim}$, $\mu, \nu \in \Lambda^\vee \subset G$,

$$\begin{aligned} \phi_{a,b,c}(w, x) &= e^{-\frac{2\pi i}{3!} P_{a,b,c}(w, x)}, & \phi_{a,b}(\mu; w, x) &= e^{-\frac{2\pi i}{2!} P_{a,b}(\mu; w, x)}, \\ \phi_a(\mu, \nu; w, x) &= e^{-2\pi i P_a(\mu, \nu; w, x)}, \end{aligned}$$

where $P_{a,b,c}(w, x), P_{a,b}(\mu; w, x), P_a(\mu, \nu; w, x) \in \mathbb{Q}(x)[w]$ are the homogeneous rational functions defined in Theorem 3.10, Proposition 3.14 and Proposition 3.15, respectively.

Definition 4.3. The *gamma gerbe* is the equivariant holomorphic gerbe \mathcal{G} on X_3 defined by the cocycle ϕ .

4.4. The gamma gerbe as a groupoid central extension. Now we also state our main result in the groupoid language explained in the Appendix.

We take the invariant covering V_a of $X = X_3$ with $a \in \Lambda_{prim}$. The stack $\mathcal{X} := [X/ISL_3(\mathbb{Z})]$ is presented by the action groupoid $ISL_3(\mathbb{Z}) \times X \rightrightarrows X$. Let $U_0 = \sqcup V_a$ and $U_1 = U_0 \times_X \times (ISL_3(\mathbb{Z}) \times X) \times_X U_0$. Then $U_1 \rightrightarrows U_0$ is a Lie groupoid that also presents \mathcal{X} . Notice that U_1 breaks into pieces: $U_1 = \sqcup_{g,a,g^{-1}b} W_{g,a,g^{-1}b}$ with $W_{g,a,g^{-1}b} = \{(g,y), y \in V_a, g^{-1}y \in V_{g^{-1}b}\}$. Then the gamma gerbe \mathcal{G} is a \mathbb{C}^\times -gerbe on the stack \mathcal{X} . Moreover, the elliptic gamma functions provide a meromorphic section of that gerbe.

Theorem 4.4. *The gamma gerbe \mathcal{G} corresponds to a central extension of groupoids,*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{C}^\times \times U_0 & \longrightarrow & R & \xleftarrow{\Gamma_{a,b}\Delta_b^{-1}} & U_1 \longrightarrow 1 \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ & & U_0 & \xrightarrow{id} & U_0 & \xrightarrow{id} & U_0 \end{array}$$

where R is $L_{a,b}^\times \otimes L_b^\times(g)^{-1}$ on $W_{g,a,g^{-1}b}$, and \square^\times denotes the associated \mathbb{C}^\times -bundle of a line bundle. The maps $\Gamma_{a,b}\Delta_b^{-1}$ form a meromorphic groupoid homomorphism from $U_1 \rightrightarrows U_0$ to $R \rightrightarrows U_0$.

Proof. As shown in Proposition A.5, R is indeed a central extension of $U_1 \rightrightarrows U_0$ and R presents a \mathbb{C}^\times -gerbe \mathcal{G} corresponding to the cohomology class in $H^2(\mathcal{X}, \mathcal{O}^\times)$ represented by $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$. To see that $\Gamma_{a,b}\Delta_b^{-1}$'s provide a meromorphic section of \mathcal{G} in the above sense, we only have to notice

$$\begin{aligned} & (g, y, \Gamma_{a,b}(y)\Delta_b^{-1}(g; y)) \cdot (h, g^{-1}y, \Gamma_{g^{-1}b, g^{-1}c}(g^{-1}y)\Delta_{g^{-1}c}^{-1}(h; g^{-1}y)) \\ &= (gh, y, \Gamma_{a,b}(y)\Delta_b^{-1}(g; y)\Gamma_{g^{-1}b, g^{-1}c}(g^{-1}y)\Delta_{g^{-1}c}^{-1}(h; g^{-1}y)) \\ & \quad \phi_{a,b,c}^{-1}(y)\phi_{b,c}(g; y)\phi_c(g, h; y)) \\ &= (gh, y, \frac{\Gamma_{a,b}(y)\Gamma_{b,c}(y)}{\Gamma_{a,c}(y)} \frac{\Gamma_{g^{-1}b, g^{-1}c}(g^{-1}y)\Delta_c(g, y)}{\Gamma_{b,c}(y)\Delta_b(g; y)} \frac{\Delta_c(gh; y)}{\Delta_c(g; y)\Delta_{g^{-1}c}(h; g^{-1}y)}) \\ & \quad \Gamma_{a,c}(y)\Delta_c^{-1}(gh; y)\phi_{a,b,c}^{-1}(y)\phi_{b,c}(g; y)\phi_c(g, h; y)) \\ &= (gh, y, \Gamma_{a,c}(y)\Delta_c^{-1}(gh; y)). \end{aligned}$$

with the multiplication \cdot on R given in (55) and the identities given in Theorem 1.1. \square

Let $Y = \mathbb{CP}^2 - \mathbb{RP}^2$. The fibres of the projection $[X/ISL_3(\mathbb{Z})] \rightarrow [Y/SL_3(\mathbb{Z})]$ have the form $[\mathbb{C}/\iota(\mathbb{Z}^3)]$ for some linear map $\iota: \mathbb{Z}^3 \rightarrow \mathbb{C}$ of rank 2 over \mathbb{R} . When we restrict the gerbe to a fibre $\mathcal{E} = [\mathbb{C}/\iota(\mathbb{Z}^3)]$, there is a nice groupoid presentation of the gerbe. We may assume that $\iota(n_1, n_2, n_3) = n_1\tau + n_2\sigma + n_3$, with $\tau, \sigma \in \mathbb{C}$ and $\text{Im } \sigma \neq 0$. We regard \mathcal{E} as the stack $[E_\sigma/\mathbb{Z}]$, where $E_\sigma = \mathbb{C}/(\mathbb{Z} + \sigma\mathbb{Z})$ and \mathbb{Z} acts by shifts by $[\tau]$, the class of τ in E_σ . Denote by $L([z])$ a line bundle on E_σ with Chern class 1 corresponding to $[z] \in E_\sigma$, i.e., to the element $([z], 1)$ of the Picard group $H^1(E_\sigma, \mathcal{O}^\times) = E_\sigma \times \mathbb{Z}$. The theta function $\theta_0(z, \sigma)$ is a section of $L([0])$, in the sense that it provides an equivariant section of the trivial line bundle $\mathbb{C} \times \mathbb{C}$ on \mathbb{C} with an $\mathbb{Z} + \sigma\mathbb{Z}$ action. In fact $L([0])$ is the restriction of the theta bundle on a

fibre E_σ . Let

$$\begin{aligned} L^{(n)} &:= L([0])^* \otimes L([- \tau])^* \otimes \dots \otimes L([- (n-1)\tau])^*, \quad n \geq 0, \\ L^{(n)} &:= L([n\tau]) \otimes \dots \otimes L([- \tau]), \quad n < 0, \end{aligned}$$

and note that $L^{(n)} \otimes L^{(m)} = L^{(n+m)}$. Then $\prod_{j=0}^{n-1} \theta_0^{-1}(z + j\tau, \sigma)$ is a section of $L^{(n)}$ (where the product for $n-1 < 0$ is understood as $\prod_{j=-N}^{n-1} / \prod_{j=-N}^0$). The disjoint union $\sqcup_n (L^{(n)})^\times$ of the associated \mathbb{C}^\times -bundles is a \mathbb{C}^\times -bundle over $E_\sigma \times \mathbb{Z}$. Moreover, it is a groupoid over E_σ with source and target maps the composition of the projection $\sqcup_n L^{(n)} \rightarrow E_\sigma \times \mathbb{Z}$ and the source and target maps of $E_\sigma \times \mathbb{Z} \rightrightarrows E_\sigma$. The multiplication is given by the tensor product of line bundles

$$L^{(n)} \times_{E_\sigma} L^{(m)} \rightarrow L^{(n)} \otimes L^{(m)} = L^{(n+m)}$$

since $(\cdot(-n))^* L([k\tau])^* = L([(k+n)\tau])^*$. Here we only demonstrate this for $n, m \geq 0$. The other cases are similar. Then $\sqcup_n L^{(n)} \rightrightarrows E_\sigma$ is a \mathbb{C}^\times -central extension of $E_\sigma \times \mathbb{Z} \rightrightarrows E_\sigma$ and the section $\prod_{j=0}^{n-1} \theta_0^{-1}(z + j\tau, \sigma)$ is a groupoid morphism.

Proposition 4.5. *The gamma gerbe \mathcal{G} restricted to $[\mathbb{C}/\iota(\mathbb{Z}^3)]$ is presented by $\sqcup_n L^{(n)} \rightrightarrows E_\sigma$ and fits in the central extension of groupoids,*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{C}^\times \times E_\sigma & \longrightarrow & \sqcup_n (L^{(n)})^\times & \xrightleftharpoons{\prod \theta_0^{-1}} & E_\sigma \times \mathbb{Z} \longrightarrow 1 \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ & & E_\sigma & \xrightarrow{id} & E_\sigma & \xrightarrow{id} & E_\sigma, \end{array}$$

where \square^\times denotes the associated \mathbb{C}^\times -bundle of a line bundle.

Proof. A fibre at a point $[(w, x_1, x_2, x_3)] \in \mathcal{X}_3$ is $[\mathbb{C}/\iota(\mathbb{Z}^3)]$. By Theorem 4.4, the restriction of the gamma gerbe on this fibre can be presented by $R|_{\mathbb{C}} \rightrightarrows \mathbb{C}$, which is a central extension of $\mathbb{C} \times \mathbb{Z}^3 \rightrightarrows \mathbb{C}$. Suppose that $[(w, x_1, x_2, x_3)] \in V_{e_1}$, then we can choose \mathbb{C} to sit inside V_{e_1} . Denote $[(w, x_1, x_2, x_3)] = [(z, \tau, \sigma, 1)]$. From the same theorem, $\Delta_{e_1}^{-1}$ provides a meromorphic section of this central extension. To show the result, we only have to show that $R|_{\mathbb{C}} \rightrightarrows \mathbb{C}$ is Morita equivalent to $\sqcup_n (L^{(n)})^\times \rightrightarrows E_\sigma$. Since there is a surjective submersion $\mathbb{C} \rightarrow E_\sigma$ by $z \mapsto [z]$, we only have to show that the pull-back groupoid $\sqcup_n (L^{(n)})^\times \times_{E_\sigma \times E_\sigma} \mathbb{C} \times \mathbb{C}$ is the same as $R|_{\mathbb{C}} \rightrightarrows \mathbb{C}$. But what is obvious is that the pull-back groupoid of $E_\sigma \times \mathbb{Z} \times_{E_\sigma \times E_\sigma} \mathbb{C} \times \mathbb{C}$ is the action groupoid $\mathbb{C} \times \mathbb{Z}^2 \times \mathbb{Z} \rightrightarrows \mathbb{C}$, and $\mathbb{C} \times \mathbb{Z}^2 \times \mathbb{Z}$ maps to $E_\sigma \times \mathbb{Z}$ by $(z, n_1, n_2, n_3) \mapsto ([z + n_3 + n_2\sigma], n_1\tau)$. The pull-back groupoid is again a \mathbb{C}^\times -bundle on $\mathbb{C} \times \mathbb{Z}^2 \times \mathbb{Z}$, as the following diagram of pull-back groupoids shows:

$$\begin{array}{ccccc} \sqcup_n (L^{(n)})^\times & \xleftarrow{\prod \theta_0^{-1}} & \sqcup_n (L^{(n)})^\times \times_{E_\sigma \times E_\sigma} \mathbb{C} \times \mathbb{C} & \xleftarrow{\Delta_{e_1}^{-1}} & \mathbb{C} \times \mathbb{Z}^2 \times \mathbb{Z} \\ & \searrow & \searrow & \searrow & \searrow \\ & E_\sigma \times \mathbb{Z} & \xleftarrow{id} & \mathbb{C} \times \mathbb{Z}^2 \times \mathbb{Z} & \\ & \Downarrow & & \Downarrow & \\ & E_\sigma & \xleftarrow{id} & \mathbb{C} & \end{array}$$

Therefore the pull-back groupoid is made up by disjoint union of \mathbb{Z}^2 equivariant \mathbb{C}^\times -bundles on $(\mathbb{C} \times \mathbb{Z}^2) \times \mathbb{Z}$ (hence they are all trivial). So the pull-back groupoid is the same as $R|_{\mathbb{C}}$ as a manifold. By (12), the section $\prod_{j=0}^{n-1} \theta_0^{-1}$ pull backs to $\Delta_{e_1}^{-1}$. Since \mathbb{C}^\times is one-dimensional, the multiplication on the \mathbb{C}^\times -central extension is determined by the multiplication of one section. Since $\prod_{j=0}^{n-1} \theta_0^{-1}$ is a groupoid morphism, $\Delta_{e_1}^{-1}$ is a groupoid homomorphism for the pull-back groupoid, but it is also a groupoid morphism for $R|_{\mathbb{C}}$. Therefore, the pull-back groupoid is $R|_{\mathbb{C}}$. \square

Remark 4.6. This direct fibrewise construction of the gamma gerbe via the theta bundle provides a possibility to build higher gamma (n-) gerbes by the similar procedure.

5. CONSTRUCTING GERBES VIA DIVISORS

The purpose of this section is to give another, more geometrical construction of the gamma gerbe.

First of all, instead of talking about $ISL_3(\mathbb{Z})$ -equivariant gerbes on $X = X_3$, we shall talk about gerbes on the quotient stack $\mathcal{X} = \mathcal{X}_3 := [X/ISL_3(\mathbb{Z})]$. A holomorphic gerbe on \mathcal{X} will then be a gadget (see [5] for a detailed discussion) that assigns to each étale map $U \rightarrow \mathcal{X}$ a category \mathcal{G}_U which, for U small enough, is equivalent to the category of holomorphic line bundles on U . As an example, the trivial gerbe is given by

$$U \mapsto \{\text{holomorphic line bundles on } U\}.$$

If $\mathcal{L}_1, \mathcal{L}_2$ are two objects of \mathcal{G}_U , then $\text{Hom}(\mathcal{L}_1, \mathcal{L}_2)$ can be identified with the set of sections of a line bundle over U . We shall denote this line bundle by $\mathcal{H}om(\mathcal{L}_1, \mathcal{L}_2)$.

We begin by an intrinsic description of \mathcal{X} .

Definition 5.1. A *triptic curve* \mathcal{E} is a pointed holomorphic stack of the form $[\mathbb{C}/\iota(\mathbb{Z}^3)]$, where $\iota : \mathbb{Z}^3 \rightarrow \mathbb{C}$ is a map of (real) rank 2.

An *orientation* of a triptic curve \mathcal{E} is given by the choice of a generator of $H^3(\mathcal{E}, \mathbb{Z}) \simeq \mathbb{Z}$.

Proposition 5.2. *The stack $\text{Tr} := [(\mathbb{CP}^2 - \mathbb{RP}^2)/SL_3(\mathbb{Z})]$ is the moduli space of oriented triptic curves.*

The stack $\mathcal{X} = [X/ISL_3(\mathbb{Z})]$ is the total space of the universal family of triptic curves over Tr .

Proof. The map $[X/\mathbb{Z}^3] \rightarrow \mathbb{CP}^2 - \mathbb{RP}^2$ is an $SL_3(\mathbb{Z})$ -equivariant family of triptic curves. Each fibre has a canonical orientation given by the generator of $H^3(\mathbb{Z}^3, \mathbb{Z}) = \mathbb{Z}$, and these orientations are compatible with the group action. It follows that

$$(14) \quad [[X/\mathbb{Z}^3]/SL_3(\mathbb{Z})] = [X/ISL_3(\mathbb{Z})] \rightarrow [\mathbb{CP}^2 - \mathbb{RP}^2/SL_3(\mathbb{Z})]$$

is a family of oriented triptic curves.

To see that (14) is universal, consider an arbitrary family $\tilde{\mathcal{E}} \rightarrow S$ of oriented triptic curves. We need to show that, locally, $\tilde{\mathcal{E}}$ is the pullback of $[X/\mathbb{Z}^3]$ via some map $S \rightarrow \mathbb{CP}^2 - \mathbb{RP}^2$, and that this map is unique up to an element of $SL_3(\mathbb{Z})$.

Indeed, $\tilde{\mathcal{E}}$ can be written locally as $(S \times \mathbb{C})/\mathbb{Z}^3$, where the action is given by $\mu(w, s) = (w - \iota_s(\mu), s)$ for some map

$$\iota \mapsto \iota_s : S \rightarrow \text{hom}(\mathbb{Z}^3, \mathbb{C}) = \mathbb{C}^3.$$

Modding out by rescaling we get a map to $\mathbb{C}P^2$, and the condition on the rank of ι_s tells us that it doesn't hit $\mathbb{R}P^2$. That map is then uniquely defined up to a change of basis of \mathbb{Z}^3 . \square

Before constructing the gamma gerbe on \mathcal{X} , it is instructive to construct its restriction to the various triptic curves $\mathcal{E} \subset \mathcal{X}$.

5.1. Gerbes on triptic curves. Let \mathcal{E} be a triptic curve, let $\tilde{\mathcal{E}} \simeq \mathbb{C}$ denote the universal cover of \mathcal{E} , and let $\Lambda^\vee := \tilde{\mathcal{E}} \times_{\mathcal{E}} \{0\}$. Note that $\Lambda^\vee \simeq \mathbb{Z}^3$ comes with a canonical map ι to $\tilde{\mathcal{E}}$ and that we have $[\tilde{\mathcal{E}}/\iota(\Lambda^\vee)] = \mathcal{E}$. The action of Λ^\vee on $\tilde{\mathcal{E}}$ can also be identified with the action of $\pi_1(\mathcal{E})$ by deck transformations and thus we get a canonical isomorphism $\Lambda^\vee = \pi_1(\mathcal{E})$.

Let $W := \Lambda^\vee \otimes \mathbb{R}$ and define $K := \ker(\iota_{\mathbb{R}} : W \rightarrow \tilde{\mathcal{E}})$. Observe also that since $\text{rk}(\iota) = 2$, we always have $K \simeq \mathbb{R}$. We shall also let $T := W/\Lambda^\vee$ and $p : T \rightarrow \mathcal{E}$ be the projection. All this can then be assembled in the following short exact sequences (this being of course an abuse of language for what concerns the top and bottom rows):

$$(15) \quad \begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & 0 & \longrightarrow & \Lambda^\vee & \xrightarrow{\iota} & \tilde{\mathcal{E}} & \longrightarrow \mathcal{E} \longrightarrow 0 \\ & & & \downarrow & & & \\ 0 & \longrightarrow & K & \longrightarrow & W & \xrightarrow{\iota_{\mathbb{R}}} & \tilde{\mathcal{E}} \longrightarrow 0 \\ & & & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & T & \xrightarrow{p} & \mathcal{E} \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Lemma 5.3. *An orientation of \mathcal{E} in the sense of Definition 5.1 is equivalent to an orientation of K .*

Proof. Since $\tilde{\mathcal{E}}$ is contractible, the cohomology of \mathcal{E} can be identified with that of its fundamental group. An orientation of \mathcal{E} is therefore the same thing as a choice of generator of $\Lambda^3(\pi_1 \mathcal{E}) \subset \Lambda^3(W)$, which is the same thing as an orientation of W .

Since $\tilde{\mathcal{E}}$ is a complex manifold, it comes with a canonical orientation. We conclude by the middle row of (15) that an orientation of W is equivalent to an orientation of K . \square

From now on, \mathcal{E} shall be an oriented triptic curve. By virtue of Lemma 5.3, we shall then identify the ends of K with $\{\infty, -\infty\}$.

Given an étale map $f : U \rightarrow \mathcal{E}$, we let $T_U := T \times_{\mathcal{E}} U$, $Z_U := \{0\} \times_{\mathcal{E}} U$, and $p_U : T_U \rightarrow U$ be the projection.

$$(16) \quad \begin{array}{ccccc} \{0\} & \hookrightarrow & T & \xrightarrow{p} & \mathcal{E} \\ \uparrow & & \uparrow & & \uparrow f \\ Z_U & \hookrightarrow & T_U & \xrightarrow{p_U} & U \end{array}$$

Lemma 5.4. *The space T_U is a K -principal bundle over U and $Z_U \rightarrow T_U$ is the inclusion of a discrete subset.*

Proof. The first claim follows from the bottom row of (15). The second one holds because $\{0\} \hookrightarrow T$ is discrete and $T_U \rightarrow T$ is étale. \square

We will say that a sequence of points in Z_U *tends to* $+\infty$ (respectively tends to $-\infty$) if its image in U is relatively compact, and if it tends to $+\infty$ (resp. $-\infty$) in the K -coordinate of T_U . Let also “ $<$ ” denote the corresponding total order on K .

Definition 5.5. Let $f : U \rightarrow \mathcal{E}$ be an étale map. A *pseudodivisor* on U is a function $D : Z_U \rightarrow \mathbb{Z}$ such that for every sequence $y_n \in Z_U$ that tends to $+\infty$ we have $\lim D(y_n) = 1$ and for every sequence $z_n \in Z_U$ that tends to $-\infty$ we have $\lim D(z_n) = 0$.

We should note that pseudodivisors are not an empty notion. Indeed, K -principal bundles over manifolds are always trivial. So given an étale open $f : U \rightarrow \mathcal{X}$, we may pick an isomorphism $\varphi : T_U \xrightarrow{\sim} U \times K$. Pick $k_0 \in K$ and let $\text{pr}_2 : U \times K \rightarrow K$ denote the projection. The function $D : Z_U \rightarrow \mathbb{Z}$ given by

$$D(y) = \begin{cases} 1 & \text{if } \text{pr}_2(\varphi(y)) \geq k_0 \\ 0 & \text{if } \text{pr}_2(\varphi(y)) < k_0 \end{cases}$$

is then an example of a pseudodivisor on U . Note also that pseudodivisors form a sheaf on \mathcal{E} .

Lemma 5.6. *If D_1, D_2 are pseudodivisors, then $(p_U)_*(D_1 - D_2)$ is a divisor on U , namely it takes finite values and has discrete support.*

Proof. Let $U' \subset U$ be a subset with compact closure. We need to show that the set of $y \in Z_{U'}$ such that $D_1(y) - D_2(y) \neq 0$ is finite.

Since $T_U \rightarrow U$ is trivial, we may identify it with $U \times K$. We shall thus write $y = (z, t)$, $z \in U$, $t \in K$, for a point $y \in Z_U$. By definition of pseudodivisor, one may pick elements $a, b \in K$ such that $D_i(z, t) = 0$ for all $(z, t) \in Z_{U'}$, $t < a$, and such that $D_i(z, t) = 1$ for all $(z, t) \in Z_{U'}$, $t > b$. It follows that $D_1(z, t) - D_2(z, t) = 0$ for all $(z, t) \in Z_{U'}$, $t \notin [a, b]$.

Since $U' \times [a, b]$ is relatively compact in T_U and since $Z_{U'}$ is discrete, there is only a finite number of points $(z, t) \in Z_{U'}$ such that $t \in [a, b]$. In particular, there is only a finite number of points such that $D_1(y) - D_2(y) \neq 0$. \square

Given two pseudodivisors, we shall henceforth abuse language and identify $D_1 - D_2$ with its pushforward $(p_U)_*(D_1 - D_2)$.

We are now in position to define the pseudodivisor gerbe on \mathcal{E} . It will assign to each étale map $f : U \rightarrow \mathcal{E}$ a category \mathcal{G}_U , and to each pair of objects $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{G}_U$ a line bundle $\mathcal{H}om(\mathcal{L}_1, \mathcal{L}_2)$ over U .

Definition 5.7. An object of \mathcal{G}_U is a pair (L, D) consisting of a line bundle L on U and a pseudodivisor D on U . A morphism $(L_1, D_1) \rightarrow (L_2, D_2)$ is a section of the twisted line bundle $(\mathcal{H}om(L_1, L_2))(D_2 - D_1)$, where $\mathcal{H}om(L_1, L_2) = L_1^* \otimes L_2$ denotes the usual internal hom of line bundles. In other words

$$\mathcal{H}om_{\mathcal{G}_U}((L_1, D_1), (L_2, D_2)) := (\mathcal{H}om(L_1, L_2))(D_2 - D_1).$$

Note that this definition makes sense since by Lemma 5.6, $D_2 - D_1$ is a divisor on U .

We now globalize the above definition to the whole of \mathcal{X} .

5.2. The global construction. Since all the constructions displayed in (15) are canonical, they carry over to the universal triptic curve $\mathcal{X} \rightarrow \mathcal{T}r$. The only difference is that now all the spaces will come with given maps to $\mathcal{T}r$. We shall call them $\underline{\Delta}^\vee$, \underline{W} , \underline{T} , and \underline{K} to indicate that fact. We also have $\underline{\mathcal{E}} = \mathcal{X} = [X/ISL_3(\mathbb{Z})]$ and $\underline{\hat{\mathcal{E}}} = [X/SL_3(\mathbb{Z})]$. For example, the last row of (15) becomes the following short exact sequence of abelian group objects over $\mathcal{T}r$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{K} & \longrightarrow & \underline{T} & \xrightarrow{p} & \mathcal{X} \longrightarrow 0 \\ & & & \searrow & \downarrow & \swarrow & \\ & & & & \mathcal{T}r & & \end{array}$$

And since \mathcal{X}_3 is a family of oriented triptic curves, we get by Lemma 5.3 a canonical orientation of the line bundle $\underline{K} \rightarrow \mathcal{T}r$.

Given an étale map $f : U \rightarrow \mathcal{X}$, we now let $T_U := \underline{T} \times_{\mathcal{X}} U$, $Z_U := \mathcal{T}r \times_{\mathcal{X}} U$, and let $p_U : T_U \rightarrow U$ denote the projection.

$$(17) \quad \begin{array}{ccccc} \mathcal{T}r & \xrightarrow{\text{zero section}} & \underline{T} & \xrightarrow{p} & \mathcal{X} \\ \uparrow & & \uparrow & & \uparrow f \\ Z_U & \xrightarrow{\quad} & T_U & \xrightarrow{p_U} & U \end{array}$$

Once again, T_U is a \underline{K} -principal bundle over U . The only difference with (16) is that $Z_U \hookrightarrow T_U$ is not discrete any more. Instead, it is a union of submanifolds of (real) codimension 3.

The analogues of Definition 5.5 and Lemma 5.6 are now straightforward.

Definition 5.8. Let $f : U \rightarrow \mathcal{X}$ be an étale map. A pseudodivisor on U is a continuous function $D : Z_U \rightarrow \mathbb{Z}$ such that for every point $\{q\} \rightarrow \mathcal{T}r$ with corresponding fibre $\mathcal{E} = \{q\} \times_{\mathcal{T}r} \mathcal{X}$, the restriction of D to $\{q\} \times_{\mathcal{T}r} Z_U$ is a pseudodivisor in the sense of Definition 5.5.

Lemma 5.9. *If D_1, D_2 are pseudodivisors, then $(p_U)_*(D_1 - D_2)$ is a divisor on U .*

Proof. The zero section $\mathcal{T}r \rightarrow \mathcal{X}$ is locally like the inclusion of a complex hypersurface. Therefore the same holds for $p_U : Z_U \rightarrow U$. So we just need to show that $(p_U)_*(D_1 - D_2)$ is supported on locally finitely many pieces.

Let $U' \subset U$ be a subset with compact closure. By the same argument as in Lemma 5.6, the support of $(D_1 - D_2)|_{Z_{U'}}$ is relatively compact in T_U . The push-forward $(p_U)_*(D_1 - D_2)$ is therefore a divisor. \square

The pseudodivisor gerbe \mathcal{G} on \mathcal{X} (soon to be identified with the gamma gerbe) can now be defined in a way much similar to Definition 5.7. Once again, we identify $D_1 - D_2$ with $(p_U)_*(D_1 - D_2)$.

Definition 5.10. To an étale map $U \rightarrow \mathcal{X}$, the pseudodivisor gerbe \mathcal{G} assigns the category \mathcal{G}_U given as follows: An object of \mathcal{G}_U is a pair (L, D) consisting of a line bundle L over U and a pseudodivisor D on U . The morphisms are given by

$$\text{Hom}_{\mathcal{G}_U}((L_1, D_1), (L_2, D_2)) := (\text{Hom}(L_1, L_2))(D_2 - D_1).$$

5.3. Identifying the two gerbes. Let \mathcal{G}^Γ , \mathcal{G}^{div} denote the gamma gerbe and the pseudodivisor gerbe respectively, as given in Definitions 4.3 and 5.10. We shall show that $\mathcal{G}^\Gamma \simeq \mathcal{G}^{div}$ by producing a nowhere vanishing holomorphic section of $\mathcal{G}^\Gamma \otimes (\mathcal{G}^{div})^*$.

To describe $(\mathcal{G}^{div})^*$, we introduce the notion of a *dual pseudodivisor*. It is obtained replacing the condition “for every sequence $y_n \in Z_U$ that tends to $+\infty$ we have $\lim D(y_n) = 1$ ” by “ $\dots = -1$ ”. The dual gerbe $(\mathcal{G}^{div})^*$ is then given by taking Definition 5.10 and replacing pseudodivisors by dual pseudodivisors.

The holomorphic section of $\mathcal{G}^\Gamma \otimes (\mathcal{G}^{div})^*$ will be defined with respect to the same invariant covering $\{V_a\}$ that was used in section 4.3. It consists of the following data:

- For each $a \in \Lambda_{prim}$, a pair (L_a, D_a) , where L_a is a line bundle over V_a , and $D_a : Z_{V_a} \rightarrow \mathbb{Z}$ is a dual pseudodivisor.
- For each $a, b \in \Lambda_{prim}$, a nowhere vanishing holomorphic section $s_{a,b}$ of the line bundle $(\mathcal{H}om(L_b, L_a))(D_a - D_b)$.
- For each $a \in \Lambda_{prim}$, $g \in G$ a non-vanishing holomorphic section $s_{a;g}$ of the line bundle $(\mathcal{H}om(L_a, g_* L_{g^{-1}a}))(g_* D_{g^{-1}a} - D_a)$.

satisfying the relations (we set $\phi_{a,b;g}(y) = \phi_{a,b}(g; y)$ etc.)

$$(18) \quad \begin{aligned} \phi_{a,b;c} s_{a,c} &= s_{a,b} \circ s_{b,c}, \\ \phi_{a,b;g} g_*(s_{g^{-1}a, g^{-1}b}) \circ s_{b;g} &= s_{a;g} \circ s_{a,b}, \\ \phi_{a;g,h} s_{a;gh} &= s_{a;g} \circ g_*(s_{g^{-1}a;h}). \end{aligned}$$

The above relations are probably best understood by saying that the following diagrams commute up to multiplication by the functions drawn in their middle.

$$\begin{array}{ccc} & L_b & \\ s_{a,b} \swarrow & & \searrow s_{b,c} \\ L_a & \xleftarrow{s_{a,c}} & L_c \end{array} \quad \begin{array}{ccc} & g_* L_{g^{-1}a} & \xleftarrow{s_{a;g}} L_a \\ g_*(s_{g^{-1}a, g^{-1}b}) \uparrow & & \uparrow \phi_{a,b;g} \\ g_* L_{g^{-1}b} & \xleftarrow{s_{b;g}} & L_b \end{array} \quad \begin{array}{ccc} & g_* L_{g^{-1}a} & \\ g_*(s_{g^{-1}a;h}) \swarrow & & \searrow s_{a;g} \\ (gh)_* L_{(gh)^{-1}a} & \xleftarrow{s_{a;gh}} & L_a \end{array}$$

We now proceed to construct these pieces of data. First of all, we let $L_a := V_a \times \mathbb{C}$ be the trivial line bundle. To write down D_a , we need some more explicit descriptions of $T_a := T_{V_a}$, $Z_a := Z_{V_a}$, and $p := p_{V_a}$.

The space $\mathbb{CP}^2 - \mathbb{RP}^2$ is the moduli space of triptic curves \mathcal{E} equipped with a trivialisation of their fundamental groups $\Lambda^\vee = \pi_1(\mathcal{E})$. So when restricted to $\mathbb{CP}^2 - \mathbb{RP}^2$, the bundles $\underline{\Lambda}^\vee$, \underline{W} , \underline{T} defined in (15) have canonical trivialisations

$$\begin{aligned} \underline{\Lambda}^\vee \times_{Tr} (\mathbb{CP}^2 - \mathbb{RP}^2) &= \mathbb{Z}^3 \times (\mathbb{CP}^2 - \mathbb{RP}^2), \\ \underline{W} \times_{Tr} (\mathbb{CP}^2 - \mathbb{RP}^2) &= \mathbb{R}^3 \times (\mathbb{CP}^2 - \mathbb{RP}^2), \\ \underline{T} \times_{Tr} (\mathbb{CP}^2 - \mathbb{RP}^2) &= (\mathbb{R}^3 / \mathbb{Z}^3) \times (\mathbb{CP}^2 - \mathbb{RP}^2). \end{aligned}$$

Recall that $V_a = (\mathbb{C} \times U_a^+)/\mathbb{C}^\times$ is an open subset of X_3 . We can then compute T_a and Z_a by means of the following pullback squares

$$\begin{array}{ccccccc}
 & & \text{zero section} & & & & \\
 & & \xrightarrow{\quad} & & \xrightarrow{\quad} & & \\
 Tr & \xrightarrow{\quad} & T & \xrightarrow{\quad} & \mathcal{X}_3 & \xrightarrow{\quad} & Tr \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 (CP^2 - \mathbb{R}P^2) & \xrightarrow{\quad} & (\mathbb{R}^3/\mathbb{Z}^3) \times (CP^2 - \mathbb{R}P^2) & \xrightarrow{\quad} & X_3/\mathbb{Z}^3 & \xrightarrow{\quad} & (CP^2 - \mathbb{R}P^2) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \mathbb{Z}^3 \times (CP^2 - \mathbb{R}P^2) & \xrightarrow{\quad} & \mathbb{R}^3 \times (CP^2 - \mathbb{R}P^2) & \xrightarrow{\quad} & X_3 & & \\
 \uparrow & & \uparrow & & \uparrow & & \\
 Z_a = \mathbb{Z}^3 \times (U_a^+/\mathbb{C}^\times) & \xrightarrow{\quad} & T_a = \mathbb{R}^3 \times (U_a^+/\mathbb{C}^\times) & \xrightarrow{p} & V_a & &
 \end{array}$$

The projection p can then be written explicitly as $p : (\mu, [x]) \mapsto [(\mu(x), x)]$. Given the above description of Z_a , we can now define D_a :

$$D_a : Z_a \rightarrow \mathbb{Z} := \begin{cases} -1 & \text{on } \Lambda_+^\vee(a) \times (U_a^+/\mathbb{C}^\times) \\ 0 & \text{on } \Lambda_-^\vee(a) \times (U_a^+/\mathbb{C}^\times) \end{cases}$$

The definition of U_a^+ is made up so that all sequences $y_n \in \Lambda^\vee$ tending to $+\infty$ eventually land in $\Lambda_+^\vee(a)$, and that all sequences tending to $-\infty$ eventually land in $\Lambda_-^\vee(a)$. It follows that D_a is indeed a dual pseudodivisor.

Since the L_a are trivial, $s_{a,b}$ and $s_{a,g}$ are just functions. We let $s_{a,b} := \Gamma_{a,b}$ and $s_{a,g} := \Delta_a(g; \cdot)$. The equations (18) are then identical to those in Theorem 1.1.

So we just need to show that when viewed as section of $\mathcal{O}(D_a - D_b)$ and $\mathcal{O}(g_*D_{g^{-1}a} - D_a)$ respectively, $s_{a,b}$ and $s_{a,g}$ are non-vanishing and holomorphic. In other words, we must show that the divisors of $s_{a,b}$ and $s_{a,g}$ are given by $-p_*(D_a - D_b)$ and $-p_*(g_*D_{g^{-1}a} - D_a)$. For this we rewrite (10) formally as

$$(19) \quad \Gamma_{a,b}(w, x) = \frac{\prod_{\delta \in \Lambda_+^\vee(a)/\mathbb{Z}\gamma} (1 - e^{-2\pi i(\delta(x) - w)/\gamma(x)})}{\prod_{\delta \in \Lambda_+^\vee(b)/\mathbb{Z}\gamma} (1 - e^{2\pi i(\delta(x) - w)/\gamma(x)})}.$$

Neither the numerator nor the denominator converge, but (19) is good enough to compute the zeroes and poles of $\Gamma_{a,b}$. The numerator has zeroes at $p(\Lambda_+^\vee(a) \times (U_a^+/\mathbb{C}^\times))$ and the denominator zeroes at $p(\Lambda_+^\vee(b) \times (U_a^+/\mathbb{C}^\times))$. It follows that the divisor of $s_{a,b} = \Gamma_{a,b}$ is $-p_*(D_a - D_b)$, which is what we wanted to show. Similarly, we can rewrite (12) formally as

$$(20) \quad \Delta_a(g; w, x) = \frac{\prod_{j=-\infty}^{\mu(a)-1} \theta_0\left(\frac{w+j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)}\right)}{\prod_{j=-\infty}^{-1} \theta_0\left(\frac{w+j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)}\right)}$$

where μ is the \mathbb{Z}^3 component of $g \in ISL_3(\mathbb{Z})$. The numerator has zeroes at $p((\mu + \Lambda_+^\vee(a)) \times (U_a^+/\mathbb{C}^\times))$ and the denominator has zeroes at $p(\Lambda_+^\vee(a) \times (U_a^+/\mathbb{C}^\times))$. It follows that the divisor of $s_{a,g} = \Delta_a(g; \cdot)$ is $-p_*(\mu_*D_a - D_a) = -p_*(g_*D_{g^{-1}a} - D_a)$, as desired. Thus we have proven:

Theorem 5.11. *The two gerbes over \mathcal{X} described in Definitions 4.3 and 5.10 are isomorphic.*

6. THE BRAUER GROUP OF TRIPTIC CURVES

In this section, we shall compute the Brauer group $Br(\mathcal{E}) := H^2(\mathcal{E}, \mathcal{O}^\times)$ of a generic triptic curve \mathcal{E} . We shall then identify the class of the gamma gerbe inside this Brauer group.

6.1. The cohomology of \mathcal{E} with coefficients in \mathcal{O}_{an}^\times . Let x_1, x_2, x_3 be three complex numbers, and let $\iota : \mathbb{Z}^3 \rightarrow \mathbb{C}$ denote the linear map sending e_i to x_i . We assume for the moment that the x_i are \mathbb{Q} -linearly independent and that they span \mathbb{C} as an \mathbb{R} -vector space. So in particular ι is injective. Let \mathcal{E} denote the triptic curve² $[\mathbb{C}/\iota(\mathbb{Z}^3)]$.

Let $\mathcal{O}_{an}^\times = (\mathcal{O}_{an}^\times, \cdot)$ the sheaf of invertible analytic functions. The goal of this section is to compute the cohomology of \mathcal{E} with coefficients in \mathcal{O}_{an}^\times .

In order to avoid redundant arguments, we put ourselves in a slightly more general situation. So let $r \geq 2$ be an integer, let $x_j, j = 1 \dots r$ be \mathbb{Q} -linearly independent complex numbers, let ι be the map $\mathbb{Z}^r \rightarrow \mathbb{C} : e_j \mapsto \tau_j$, and let $\mathcal{E} = [\mathbb{C}/\iota(\mathbb{Z}^r)]$. We also assume that x_1 and x_2 span a lattice in \mathbb{C} , and call E the elliptic curve $\mathbb{C}/\mathbb{Z}\{x_1, x_2\}$.

Let $\mathcal{O}_{an} = (\mathcal{O}_{an}, +)$ denote the sheaf of analytic functions, and consider the exponential sequence

$$(21) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{an} \longrightarrow \mathcal{O}_{an}^\times \longrightarrow 0.$$

We will compute $H^*(\mathcal{E}, \mathcal{O}_{an}^\times)$ by first calculating $H^*(\mathcal{E}, \mathbb{Z})$ and $H^*(\mathcal{E}, \mathcal{O}_{an})$, and then using the long exact sequence in cohomology.

We first compute $H^*(\mathcal{E}, \mathbb{Z})$. Since \mathbb{Z} is a constant sheaf, that's just ordinary integral cohomology applied to the étale homotopy type of \mathcal{E} [1, Corollary 9.3]. Let N denote the Čech nerve associated to the cover $\mathbb{C} \rightarrow \mathcal{E}$. It is the simplicial manifold given by all the n -tuple fibered products $\mathbb{C} \times_{\mathcal{E}} \dots \times_{\mathcal{E}} \mathbb{C} = (\mathbb{Z}^r)^{n-1} \times \mathbb{C}$.

$$(22) \quad N = \left(\dots \rightrightarrows (\mathbb{Z}^r)^2 \times \mathbb{C} \rightrightarrows \mathbb{Z}^r \times \mathbb{C} \rightrightarrows \mathbb{C} \right)$$

Since each degree of (22) is a disjoint union of contractible components, the cover $\mathbb{C} \rightarrow \mathcal{E}$ is a good cover. We can therefore compute the étale homotopy type of \mathcal{E} by applying π_0 degree-wise to N . The resulting simplicial set is then the standard simplicial model for $B\mathbb{Z}^r$. We deduce that

$$H^*(\mathcal{E}, \mathbb{Z}) = H^*(B\mathbb{Z}^r, \mathbb{Z}) = \Lambda^*(\mathbb{Z}^r).$$

Morally, this argument can be summarized as follows. The sheaf \mathcal{E} is a free quotient of a contractible space by the group \mathbb{Z}^r , it is therefore a model for $B\mathbb{Z}^r$.

Let \mathcal{O}_{pol} denote the locally constant sheaf of polynomial functions on \mathbb{C} , and let the same symbol also denote its pullback to \mathcal{E} . We will use \mathcal{O}_{pol} as a first approximation to \mathcal{O}_{an} . Let $\mathcal{O}_{\leq n}$ denote the subsheaf of \mathcal{O}_{pol} of functions of degree at most n . Clearly, \mathcal{O}_{pol} is the colimit of the sheaves $\mathcal{O}_{\leq n}$.

Lemma 6.1. *The group $H^i(\mathcal{E}, \mathcal{O}_{pol})$ is the colimit over n of $H^i(\mathcal{E}, \mathcal{O}_{\leq n})$.*

Proof. The sheaves $\mathcal{O}_{\leq n}$ and \mathcal{O}_{pol} are locally constant and their stalks are given respectively by $\mathbb{C}\{1, \dots, z^n\}$ and $\mathbb{C}[z]$. It follows that

$$H^i(\mathcal{E}, \mathcal{O}_{pol}) = H^i(B\mathbb{Z}^r, \mathbb{C}[z]), \quad H^i(\mathcal{E}, \mathcal{O}_{\leq n}) = H^i(B\mathbb{Z}^r, \mathbb{C}\{1, \dots, z^n\}),$$

²Note that since ι is injective, the stack $[\mathbb{C}/\iota(\mathbb{Z}^3)]$ is actually a sheaf.

where the right hand side denotes cohomology with local coefficients. The result follows since $B\mathbb{Z}^r$ admits a model with finitely many cells. \square

Remark 6.2. If $\{\mathcal{F}_n\}$ is an arbitrary directed system of sheaves on \mathcal{E} , then it is also true that $H^i(\mathcal{E}, \text{colim } \mathcal{F}_n) = \text{colim } H^i(\mathcal{E}, \mathcal{F}_n)$. This holds because \mathcal{E} can be resolved by a simplicial manifold which is compact in each degree.

We now compute $H^*(\mathcal{E}, \mathcal{O}_{pol})$ using the spectral sequence associated to the filtration by the $\mathcal{O}_{\leq n}$. This spectral sequence converges by virtue of Lemma 6.1. The associated graded sheaves $\mathcal{O}_{\leq n}/\mathcal{O}_{\leq n-1}$ being isomorphic to the constant sheaf \mathbb{C} , we have

$$H^*(\mathcal{E}, \mathcal{O}_{\leq n}/\mathcal{O}_{\leq n-1}) = H^*(\mathcal{E}, \mathbb{C}) = H^*(B\mathbb{Z}^r, \mathbb{C}) = \Lambda^*(\mathbb{C}^r).$$

So our spectral sequence looks like

$$(23) \quad E_1^{i,n} = \Lambda^i(\mathbb{C}^r) \Rightarrow H^i(\mathcal{E}, \mathcal{O}_{pol}),$$

with differentials d_k of bidegree $(1, -k)$.

Let α_j , $j = 1 \dots r$ denote the generators of $E_1^{1,0} = \mathbb{C}^r$. We want to compute the image under d_1 of the standard generator $z \in E_1^{0,1} = \mathbb{C}$. For this, we need to find an $(\mathcal{O}_{\leq 1})$ -valued Čech 0-cochain c lifting the $(\mathcal{O}_{\leq 1}/\mathcal{O}_{\leq 0})$ -valued 0-cocycle z , and take its coboundary. We define c with respect to the cover $\mathbb{C} \rightarrow \mathbb{C}/\iota(\mathbb{Z}^r) = \mathcal{E}$. It is then the usual coordinate z on \mathbb{C} . Its coboundary δc is then a function on $\mathbb{C} \times \mathbb{Z}^r$ which can easily be computed to be $\sum a_i x_i$ on the component $\mathbb{C} \times (a_1, \dots, a_r)$. This $(\mathcal{O}_{\leq 0})$ -valued 1-cocycle represents $\sum x_i \alpha_i$. We have therefore computed

$$(24) \quad d_1(z) = \sum x_i \alpha_i.$$

All the other d_1 differentials follow from (24) by multiplicativity.

More precisely, letting $x = (x_1, \dots, x_r) \in \mathbb{C}^r$, we can identify $d_1 : E_1^{i,n} \rightarrow E_1^{i+1,n-1}$ with $-\wedge nx : \Lambda^i(\mathbb{C}^r) \rightarrow \Lambda^{i+1}(\mathbb{C}^r)$. The sequence

$$\Lambda^{i-1}(\mathbb{C}^r) \xrightarrow{\wedge x} \Lambda^i(\mathbb{C}^r) \xrightarrow{\wedge x} \Lambda^{i+1}(\mathbb{C}^r)$$

being exact, we deduce that $E_2^{i,n}$ is zero for $n \geq 0$ and $\Lambda^i(\mathbb{C}^r)/(\Lambda^{i-1}(\mathbb{C}^r) \wedge x)$ for $n = 0$. The latter can also be identified with $\Lambda^i(\mathbb{C}^r/x\mathbb{C})$. We have then proven

Proposition 6.3. $H^*(\mathcal{E}, \mathcal{O}_{pol}) = \Lambda^*(\mathbb{C}^r/x\mathbb{C})$. \square

For $r = 3$, this is what the spectral sequence (23) looks like in coordinates:

$$(25) \quad \begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ & \mathbb{C} & & \mathbb{C}^3 & & \mathbb{C}^3 & & \mathbb{C} \\ & \searrow 2 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & & \searrow 2 \cdot \begin{pmatrix} x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{pmatrix} & & \searrow 2 \cdot (x_3 - x_2 \ x_1) & & \\ & \mathbb{C} & & \mathbb{C}^3 & & \mathbb{C}^3 & & \mathbb{C} \\ & \searrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & & \searrow \begin{pmatrix} x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{pmatrix} & & \searrow (x_3 - x_2 \ x_1) & & \\ & \mathbb{C} & & \mathbb{C}^3 & & \mathbb{C}^3 & & \mathbb{C} \end{array}$$

For $r = 2$, $\mathcal{X} = E$ is an elliptic curve. We have then computed that $H^0(E, \mathcal{O}_{pol}) = H^1(E, \mathcal{O}_{pol}) = \mathbb{C}$, and that the other cohomology groups vanish. By Riemann-Roch and Serre's GAGA principle, we also have $H^0(E, \mathcal{O}_{an}) = H^1(E, \mathcal{O}_{an}) = \mathbb{C}$,

and $H^i(E, \mathcal{O}_{an}) = 0$ for $i \geq 2$. This shows that $H^i(E, \mathcal{O}_{pol})$ and $H^i(E, \mathcal{O}_{an})$ are abstractly isomorphic.

Lemma 6.4. *The natural map $H^i(E, \mathcal{O}_{pol}) \rightarrow H^i(E, \mathcal{O}_{an})$ is an isomorphism.*

Proof. Clearly, the only nontrivial case is $i = 1$. Since $H^1(E, \mathbb{C}) \rightarrow H^1(E, \mathcal{O}_{pol})$ is surjective and $\dim H^1(E, \mathcal{O}_{pol}) = \dim H^1(E, \mathcal{O}_{an})$, it is enough to show that $H^1(E, \mathbb{C}) \rightarrow H^1(E, \mathcal{O}_{an})$ is surjective. Consider the short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{an} \xrightarrow{d} \Omega_{an}^1 \longrightarrow 0$$

and the corresponding long exact sequence in cohomology

$$H^1(E, \mathbb{C}) \rightarrow H^1(E, \mathcal{O}_{an}) \rightarrow H^1(E, \Omega_{an}^1) \rightarrow H^2(E, \mathbb{C}) \rightarrow H^2(E, \mathcal{O}_{an})$$

The map $H^1(E, \Omega_{an}^1) \rightarrow H^2(E, \mathbb{C})$ is surjective because $H^2(E, \mathcal{O}_{an}) = 0$. But it is also the inclusion of the middle summand in the Hodge decomposition $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. So it is an isomorphism. It follows that $H^1(E, \mathbb{C}) \rightarrow H^1(E, \mathcal{O}_{an})$ is surjective. \square

As a corollary, we have the following result.

Lemma 6.5. *The cohomology groups $H^i(\mathcal{E}, \mathcal{O}_{an}/\mathcal{O}_{pol})$ are zero for all $i \geq 0$ and all $r \geq 2$.*

Proof. For $r = 2$, this follows from Lemma 6.4 and from the long exact sequence in cohomology. For $r > 2$, let us identify the cohomology of \mathcal{E} with the (\mathbb{Z}^{r-2}) -equivariant cohomology of E . The latter can be computed as the cohomology of the simplicial manifold

$$(26) \quad N = \left(\cdots \rightrightarrows (\mathbb{Z}^{r-2})^2 \times E \rightrightarrows \mathbb{Z}^{r-2} \times E \rightrightarrows E \right).$$

We then have a spectral sequence [12, Prop. 3.2.]

$$E_1^{i,j} = H^i(N_j, \mathcal{O}_{an}/\mathcal{O}_{pol}) \Rightarrow H^{i+j}(N, \mathcal{O}_{an}/\mathcal{O}_{pol}) = H^{i+j}(\mathcal{X}, \mathcal{O}_{an}/\mathcal{O}_{pol})$$

Since each N_j is a disjoint union of E 's, we know from the $r = 2$ case that $H^i(N_j, \mathcal{O}_{an}/\mathcal{O}_{pol}) = 0$. The spectral sequence vanishes identically thus proving the result. \square

Corollary 6.6. *The natural map $H^i(\mathcal{E}, \mathcal{O}_{pol}) \rightarrow H^i(\mathcal{E}, \mathcal{O}_{an})$ is an isomorphism.*

We can now state the main theorem of this section.

Theorem 6.7. *The cohomology of \mathcal{E} with coefficients in \mathcal{O}_{an}^\times reads*

$$(27) \quad H^i(\mathcal{E}, \mathcal{O}_{an}^\times) = \Lambda^i(\mathbb{C}^r/x\mathbb{C})/\Lambda^i(\mathbb{Z}^r) \quad \text{for } 0 \leq i \leq r-2$$

$$(28) \quad H^{r-1}(\mathcal{E}, \mathcal{O}_{an}^\times) = \mathcal{E} \times \mathbb{Z}$$

$$(29) \quad H^i(\mathcal{E}, \mathcal{O}_{an}^\times) = 0 \quad \text{for } i \geq r,$$

where x denotes the vector $(x_1, \dots, x_r) \in \mathbb{C}^r$.

The map $\mathcal{E} \rightarrow \mathcal{E} : w \mapsto w + z$ induces the identity on $H^i(\mathcal{E}, \mathcal{O}_{an}^\times)$ for $i \leq r-2$, and induces $(w, n) \mapsto (w + nz, n)$ on $H^{r-1}(\mathcal{E}, \mathcal{O}_{an}^\times) = \mathcal{E} \times \mathbb{Z}$.

Proof. The cohomology $H^*(\mathcal{E}, \mathbb{Z}) = \Lambda^*(\mathbb{Z}^r)$ lands in the first row of the spectral sequence (25) and hits the standard basis of $\Lambda^*(\mathbb{C}^r)$. The map $H^i(\mathcal{E}, \mathbb{Z}) \rightarrow H^i(\mathcal{E}, \mathcal{O}_{an})$ is therefore the composite

$$(30) \quad H^i(\mathcal{E}, \mathbb{Z}) = \Lambda^i(\mathbb{Z}^r) \longrightarrow \Lambda^i(\mathbb{C}^r) \longrightarrow \Lambda^i(\mathbb{C}^r/x\mathbb{C}) = H^i(\mathcal{E}, \mathcal{O}_{an}).$$

By Lemma 6.9, (30) is injective for $i \leq r-1$. The long exact sequence corresponding to (21) therefore splits into short exact sequences

$$(31) \quad 0 \longrightarrow H^i(\mathcal{E}, \mathbb{Z}) \longrightarrow H^i(\mathcal{E}, \mathcal{O}_{an}) \longrightarrow H^i(\mathcal{E}, \mathcal{O}_{an}^\times) \longrightarrow 0$$

for $i \leq r-2$, and a four term exact sequence

$$(32) \quad 0 \rightarrow H^{r-1}(\mathcal{E}, \mathbb{Z}) \rightarrow H^{r-1}(\mathcal{E}, \mathcal{O}_{an}) \rightarrow H^{r-1}(\mathcal{E}, \mathcal{O}_{an}^\times) \rightarrow H^r(\mathcal{E}, \mathbb{Z}) \rightarrow 0.$$

Equation (27) follows from (31) and our computation of $H^i(\mathcal{E}, \mathbb{Z})$ and $H^i(\mathcal{E}, \mathcal{O}_{an})$.

To get (28), we note that $H^r(\mathcal{E}, \mathbb{Z}) = \mathbb{Z}$. The rightmost arrow of (32) therefore splits, and we get

$$H^{r-1}(\mathcal{E}, \mathcal{O}_{an}^\times) = \left(\Lambda^{r-1}(\mathbb{C}^r/x\mathbb{C}) / \Lambda^{r-1}(\mathbb{Z}^r) \right) \times \mathbb{Z}.$$

To identify $\Lambda^{r-1}(\mathbb{C}^r/x\mathbb{C}) / \Lambda^{r-1}(\mathbb{Z}^r)$ with \mathcal{E} , consider the isomorphism

$$(33) \quad \begin{aligned} & \Lambda^{r-1}(\mathbb{C}^r/x\mathbb{C}) && \rightarrow \mathbb{C} \\ \bar{e}_j &:= (-1)^j e_1 \wedge \dots \wedge e_{j-1} \wedge e_{j+1} \wedge \dots \wedge e_r && \mapsto x_j \end{aligned}$$

It is an isomorphism because it is not zero and $\Lambda^{r-1}(\mathbb{C}^r/x\mathbb{C})$ is one dimensional. It is well defined because the relation $x_k \bar{e}_j = x_j \bar{e}_k$ holds in $\Lambda^{r-1}(\mathbb{C}^r/x\mathbb{C})$:

$$x_k \bar{e}_j - x_j \bar{e}_k = x \wedge (e_1 \wedge \dots \wedge e_{j-1} \wedge e_{j+1} \wedge \dots \wedge e_{k-1} \wedge e_{k+1} \wedge \dots \wedge e_r) = 0.$$

We now observe that

$$\Lambda^{r-1}(\mathbb{C}^r/x\mathbb{C}) / \Lambda^{r-1}(\mathbb{Z}^r) = \Lambda^{r-1}(\mathbb{C}^r/x\mathbb{C}) / \mathbb{Z}\{\bar{e}_j\} = \mathbb{C} / \mathbb{Z}\{s_j\} = \mathcal{E},$$

thus proving (28). At last, (29) holds because $H^i(\mathcal{E}, \mathcal{O}_{an}) = H^{i+1}(\mathcal{E}, \mathbb{Z}) = 0$ for $i \geq r$.

We now examine the map $z_* : H^*(\mathcal{E}, \mathcal{O}_{an}^\times) \rightarrow H^*(\mathcal{E}, \mathcal{O}_{an}^\times)$ induced by $w \mapsto w + z$. The latter being homotopic to the identity, it induces the identity on $H^*(\mathcal{E}, \mathbb{C})$. As a consequence, z_* is the identity on the spectral sequence (23), and thus also on the short exact sequence (31). This shows that z_* is the identity on $H^{\leq r-2}(\mathcal{E}, \mathcal{O}_{an}^\times)$.

For similar reasons, the induced map on $H^{r-1}(\mathcal{E}, \mathcal{O}_{an}^\times) = \mathcal{E} \times \mathbb{Z}$ is the identity on the subgroup \mathcal{E} and on the quotient \mathbb{Z} . To further determine z_* , we consider the quotient $\mathcal{E}' := [\mathcal{E}/z\mathbb{Z}]$. Its cohomology is related to that of \mathcal{E} by means of the following long exact sequence:

$$\dots \longrightarrow H^i(\mathcal{E}, \mathcal{O}_{an}^\times) \xrightarrow{1-z_*} H^i(\mathcal{E}, \mathcal{O}_{an}^\times) \longrightarrow H^{i+1}(\mathcal{E}', \mathcal{O}_{an}^\times) \longrightarrow \dots$$

Letting $i = r-1$, we get

$$\dots \longrightarrow \mathcal{E} \times \mathbb{Z} \xrightarrow{1-z_*} \mathcal{E} \times \mathbb{Z} \longrightarrow \mathcal{E}' \times \mathbb{Z} \longrightarrow 0.$$

It follows that (up to a choice of sign), the map $1 - z_*$ is given by $(w, n) \mapsto (nz, 0)$. \square

As a special case of Theorem 6.7, we have:

Corollary 6.8. *The cohomology of a generic triptic curve $\mathcal{E} = \mathbb{C}/\iota(\mathbb{Z}^3)$ is given by*

$$\begin{aligned} H^0(\mathcal{E}, \mathcal{O}_{an}^\times) &= \mathbb{C}^\times & H^1(\mathcal{E}, \mathcal{O}_{an}^\times) &= (\mathbb{C}^\times)^3 / (e^{\lambda x_1}, e^{\lambda x_2}, e^{\lambda x_3}), \quad \lambda \in \mathbb{C} \\ H^2(\mathcal{E}, \mathcal{O}_{an}^\times) &= \mathcal{E} \times \mathbb{Z} & H^i(\mathcal{E}, \mathcal{O}_{an}^\times) &= 0 \quad \text{for } i \geq 3. \end{aligned}$$

Lemma 6.9. *The map $\Lambda^i(\mathbb{Z}^r) \rightarrow \Lambda^i(\mathbb{C}^r/x\mathbb{C})$ is injective for all $i \leq r-1$.*

Proof. Let $b_1 := 1$, $b_j := -x_j/x_1$, $j = 2 \dots r$. We then have

$$e_1 = \sum_{j=2}^r b_j e_j$$

in $\mathbb{C}^r/\tau\mathbb{C}$. The x_j were assumed \mathbb{Q} -linearly independent, so we may complete the set of b 's to a \mathbb{Q} -basis $\{b_\alpha\}$ of \mathbb{C} . Since $\{e_j\}_{2 \leq j \leq r}$ is a \mathbb{C} -basis of $\mathbb{C}^r/x\mathbb{C}$, the set $\{e_J\}$, $J \subset \{2, \dots, r\}$, $|J| = i$ is a \mathbb{C} -basis of $\Lambda^i(\mathbb{C}^r/x\mathbb{C})$. It follows that

$$(34) \quad \{b_\alpha e_J\}_{J \subset \{2, \dots, r\}, |J|=i}$$

is a \mathbb{Q} -basis of $\Lambda^i(\mathbb{C}^r/x\mathbb{C})$. We now expand the \mathbb{Z} -basis $\{e_I\}$ of $\Lambda^i(\mathbb{Z}^r)$ in terms of (34):

$$\begin{aligned} e_I &= b_1 e_I & \text{if } 1 \notin I \\ e_I &= e_1 \wedge e_J = \sum_{j=2}^r b_j e_j \wedge e_J = \sum_{\substack{j=2, \\ j \notin J}}^r b_j e_{J \cup \{j\}}, & \text{if } I = \{1\} \cup J. \end{aligned}$$

Each element of (34) appears in the expansion of at most one e_I . Unless one of the e_I maps to zero (which happens only if $i = r$), this means that they are \mathbb{Q} -linearly independent in $\Lambda^i(\mathbb{C}^r/x\mathbb{C})$. The map $\Lambda^i(\mathbb{Z}^r) \rightarrow \Lambda^i(\mathbb{C}^r/x\mathbb{C})$ is therefore injective. \square

6.2. The Dixmier–Douady class. We now proceed to identify the class of the gamma gerbe inside the Brauer group $Br(\mathcal{E}) = H^2(\mathcal{E}, \mathcal{O}^\times) = \mathcal{E} \times \mathbb{Z}$. We first determine its Dixmier–Douady class.

Proposition 6.10. *Let \mathcal{G} be the restriction of the gamma gerbe to a generic triptic curve \mathcal{E} . Then its Dixmier–Douady class*

$$c(\mathcal{G}) \in H^3(\mathcal{E}, \mathbb{Z}) = \mathbb{Z}$$

is a generator.

Proof. Pick an open V_a covering \mathcal{E} . By Theorem 4.1, a cocycle representing $c(\mathcal{G})$ is given by

$$\psi = \frac{1}{2\pi i} \delta \log \phi_a \in C^3(\mathbb{Z}^3, \check{C}^0(V_a, \mathbb{Z})).$$

This can be computed from Proposition 3.15: in the notation used there, let $\lambda = \sum_i \ell_i \alpha_i$, $\mu = \sum_i m_i \alpha_i$, $\nu = \sum_i n_i \alpha_i$. Then ψ is the cocycle

$$\begin{aligned} \psi(\lambda, \mu, \nu) &= \delta P_a(\lambda, \mu, \nu) \\ &= -P_a(\mu, \nu; w + \lambda(x), x) + P_a(\lambda + \mu, \nu; w, x) \\ &\quad - P_a(\lambda, \mu + \nu; w, x) + P_a(\lambda, \mu; w, x) \\ &= -m_1 n_2 \ell_3, \end{aligned}$$

representing a generator of $H^3(\mathbb{Z}^3, \mathbb{Z}) = \mathbb{Z}$. \square

It is now tempting to try to determine the \mathcal{E} coordinate of $c(\mathcal{G})$ in $Br(\mathcal{E}) = \mathcal{E} \times \mathbb{Z}$. However, this question doesn't make any sense. Indeed, we don't have yet a canonical isomorphism between $Br(\mathcal{E})$ and $\mathcal{X} \times \mathbb{Z}$, we just have the short exact sequence

$$(35) \quad 0 \longrightarrow \mathcal{E} \longrightarrow Br(\mathcal{E}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

coming from (32). So instead, we shall use Proposition 6.10 to construct a splitting of (35) by sending the generator of \mathbb{Z} to $c(\mathcal{G}) \in Br(\mathcal{E})$. This has the effect of fixing the isomorphism $Br(\mathcal{E}) = \mathcal{E} \times \mathbb{Z}$.

Now that we have fixed this isomorphism, we can construct gerbes representing each element of $Br(\mathcal{E})$.

Proposition 6.11. *The class $(z, n) \in \mathcal{E} \times \mathbb{Z} = Br(\mathcal{E})$ is represented by the gerbe*

$$\mathcal{G}^{\otimes n-1} \otimes z_*(\mathcal{G}),$$

where \mathcal{G} is the gamma gerbe, and $z_*(\mathcal{G})$ is its translate by z .

Proof. Let $c \in \mathcal{E} \times \mathbb{Z}$ denote the class of \mathcal{G} . By Proposition 6.10, the \mathbb{Z} component of c is 1, and by the above convention, the \mathcal{E} component of c is 0. By Theorem 6.7, the \mathbb{Z} component of $z_*(c)$ is then also 1, and its \mathcal{E} component is z . \square

7. THE CHARACTERISTIC CLASS OF THE GAMMA GERBE

Let as above $G = ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ act on X , the total space of the dual tautological bundle $O(1)$ on $Y = \mathbb{C}P^2 - \mathbb{R}P^2$. The Dixmier–Douady class $c(\mathcal{G})$ of the gamma gerbe \mathcal{G} is a class in the equivariant cohomology $H_G^3(X, \mathbb{Z})$. In this section we show that $H_G^3(X, \mathbb{Z})/\text{torsion} = \mathbb{Z} \oplus \mathbb{Z}$ and that the image of $c(\mathcal{G})$ is a primitive vector in this group. In particular, this implies that the gerbe is topologically non-trivial. We first compute the restriction of $c(\mathcal{G})$ to a generic fibre. We then compute $H_G^3(X, \mathbb{Z})$ relating it to group cohomology via a spectral sequence.

7.1. The restriction to a generic fibre. The bundle projection $p: X \rightarrow \mathbb{C}P^2 - \mathbb{R}P^2$ is $ISL_3(\mathbb{Z})$ equivariant (with trivial action of \mathbb{Z}^3 on the base). The isotropy group of the generic fibre $F = p^{-1}(x) \cong \mathbb{C}$ is \mathbb{Z}^3 . The action of $n \in \mathbb{Z}^3$ is $z \rightarrow z + \sum n_i x_i$ and here generic means that the x_i are linearly independent over \mathbb{Q} . Correspondingly, the restriction to F induces a homomorphism $r: H_G^3(X, \mathbb{Z}) \rightarrow H_{\mathbb{Z}^3}^3(F, \mathbb{Z})$. It is the composition of the natural homomorphisms (restriction to a subgroup and pull-back by the inclusion map)

$$H_G^3(X, \mathbb{Z}) \rightarrow H_{\mathbb{Z}^3}^3(X, \mathbb{Z}) \rightarrow H_{\mathbb{Z}^3}^3(F, \mathbb{Z}).$$

Since F is contractible, $H_{\mathbb{Z}^3}^\bullet(F, \mathbb{Z}) = H^\bullet(\mathbb{Z}^3, \mathbb{Z}) = \wedge(\mathbb{Z}^3)$. Then Prop. 6.10 gives:

Proposition 7.1. *The restriction to the generic fibre*

$$r: H_G^3(X, \mathbb{Z}) \rightarrow H_{\mathbb{Z}^3}^3(F, \mathbb{Z}) = \mathbb{Z},$$

sends the Dixmier–Douady class $c(\mathcal{G})$ to a generator. In particular $c(\mathcal{G})$ is non-trivial.

7.2. The topology of X .

Proposition 7.2. *The manifold X retracts to the 2-sphere S^2 , embedded as the rational curve $x_1^2 + x_2^2 + x_3^2 = 0$, $w = 0$. The retraction $r: X \rightarrow S^2$ is $ISL_3(\mathbb{Z})$ -equivariant for the following action on S^2 : \mathbb{Z}^3 acts trivially; if S^2 is viewed as the space of row vectors of norm 1 in \mathbb{R}^3 then $g \in SL_3(\mathbb{Z})$ acts on $v \in S^2$ by $g \cdot v = vg^{-1}/|vg^{-1}|$.*

Proof. Clearly X retracts to its zero section Y . Since the subgroup \mathbb{Z}^3 acts on the fibres of $X \rightarrow Y$, the retraction is $ISL_3(\mathbb{Z})$ -equivariant for the trivial action of \mathbb{Z}^3 on the zero section. It is thus sufficient to show that we have an embedding $j: S^2 \rightarrow Y$ and an $SL_3(\mathbb{Z})$ -equivariant deformation retraction $r: Y \rightarrow S^2$.

The rational curve $x_1^2 + x_2^2 + x_3^2 = 0$ in \mathbb{CP}^2 does not have any real points so it is contained in Y . Thus we have an embedding $j: S^2 \cong \mathbb{CP}^1 \rightarrow Y$, induced by the \mathbb{C}^\times -equivariant map $\mathbb{C}^2 - \{0\} \rightarrow \mathbb{C}^3 - \{0\}$

$$j(z_1, z_2) = (i(z_1^2 - z_2^2), z_1^2 + z_2^2, 2iz_1z_2).$$

To construct the retraction $r: Y \rightarrow S^2$, notice that to each point $[x] \in \mathbb{CP}^2 - \mathbb{RP}^2$, $x \in \mathbb{C}^3$ there corresponds an ordered pair $(\operatorname{Re} x, \operatorname{Im} x)$ of linearly independent vectors in \mathbb{R}^3 , and all such pairs arise in this fashion. Moreover the oriented plane defined by $[x]$ is independent of the choice of representative x . So we have a map r from Y to the manifold of oriented planes through the origin in \mathbb{R}^3 . This manifold is identified with S^2 via the normal vector map: in terms of the cross product,

$$r([x]) = \frac{\operatorname{Re} x \times \operatorname{Im} x}{|\operatorname{Re} x \times \operatorname{Im} x|}.$$

The induced action of SL_3 on S^2 is obtained from the fact that the cross product is an isomorphism of $SL_3(\mathbb{Z})$ -modules $\wedge^2(\mathbb{R}^3) \rightarrow (\mathbb{R}^3)^*$.

A simple calculation shows that $r \circ j = \operatorname{Id}$ if we identify S^2 with \mathbb{CP}^1 via the south pole stereographic projection $(x, y, z) \rightarrow [x + iy, 1 + z]$. The map $r: Y \rightarrow S^2$ is a fibre bundle with contractible fibres: the fibre over a is the space of oriented bases of the plane normal to a modulo rotations and rescaling. Thus $r \circ j$ is homotopic to the identity. \square

7.3. The cohomology of $ISL_3(\mathbb{Z})$.

Proposition 7.3. *The first few integral cohomology groups of $ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ are*

$$H^j(ISL_3(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0, \\ 0, & j = 1, 2, \end{cases}$$

and there is an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H^3(ISL_3(\mathbb{Z}), \mathbb{Z})/\text{torsion} \xrightarrow{\operatorname{res}} H^3(\mathbb{Z}^3, \mathbb{Z}) \cong \mathbb{Z},$$

where res is induced by the restriction to the subgroup $\mathbb{Z}^3 \subset ISL_3(\mathbb{Z})$ and the group \mathbb{Z} on the left is $H^2(SL_3(\mathbb{Z}), \mathbb{Z}^3)/\text{torsion}$. (We will also show below that res is surjective).

Proof. The Lyndon–Hochschild–Serre spectral sequence of the exact sequence of groups

$$0 \rightarrow \mathbb{Z}^3 \rightarrow ISL_3(\mathbb{Z}) \rightarrow SL_3(\mathbb{Z}) \rightarrow 1,$$

converging to $H^\bullet(ISL_3(\mathbb{Z}), \mathbb{Z})$, has E_2 term

$$E_2^{p,q} = H^p(SL_3(\mathbb{Z}), H^q(\mathbb{Z}^3, \mathbb{Z})).$$

Now $H^q(\mathbb{Z}^3, \mathbb{Z}) = \wedge^q(\mathbb{Z}^3)$ with the natural action of $SL_3(\mathbb{Z})$ on column vectors.

The relevant part of E_2 is ($n\mathbb{Z}_p$ denotes the direct sum of n copies of $\mathbb{Z}/p\mathbb{Z}$)

$$\begin{array}{cccccc}
 q \geq 4 & 0 & 0 & 0 & 0 & \\
 q = 3 & \mathbb{Z} & & & & \\
 q = 2 & 0 & 0 & \mathbb{Z} \oplus \mathbb{Z}_{2^r} & & \\
 q = 1 & 0 & 0 & \mathbb{Z} \oplus \mathbb{Z}_{2^r} & & \\
 q = 0 & \mathbb{Z} & 0 & 0 & 2\mathbb{Z}_2 & 2\mathbb{Z}_3 \oplus 2\mathbb{Z}_4 \\
 & p=0 & p=1 & p=2 & p=3 & p=4
 \end{array}$$

for some integer r . In the lowest row we have the groups $E_2^{p,0} = H^p(SL_3(\mathbb{Z}), \mathbb{Z})$ computed by Soulé (see [26], Theorem 4). The $SL_3(\mathbb{Z})$ -modules \mathbb{Z}^3 and $\wedge^2(\mathbb{Z}^3) \cong (\mathbb{Z}^3)^*$ are related by an automorphism of $SL_3(\mathbb{Z})$ (inversion composed with transposition). Therefore $E_2^{p,1} \cong E_2^{p,2}$. As for $E_2^{p,1}$, we have $H^0(SL_3(\mathbb{Z}), \mathbb{Z}^3) = 0$ (there are no invariant column vectors), $H^1(SL_3(\mathbb{Z}), \mathbb{Z}^3) = 0$ (Sah, [25], Theorem III.6) and $H^2(SL_3(\mathbb{Z}), \mathbb{Z}^3) = \mathbb{Z} \oplus \mathbb{Z}_{2^r}$ for some r (Hewitt, [17]). Finally $\wedge^3(\mathbb{Z}^3) \cong \mathbb{Z}$ so $E_2^{p,3} = E_2^{p,0}$.

From this description of the E_2 term we obtain the groups $H^j(ISL_3(\mathbb{Z}), \mathbb{Z})$ for $j = 0, 1, 2$. For $j = 3$, higher differentials should be considered. The group $E_\infty^{2,1}$ is the kernel of $d_2: E_2^{2,1} \rightarrow E_2^{4,0}$. Since $E_2^{4,0}$ is torsion, this kernel is infinite cyclic plus torsion. Since $E_2^{3,0}$ is torsion, we have an embedding $\mathbb{Z} = H^2(SL_3(\mathbb{Z}), \mathbb{Z})/\text{torsion} \hookrightarrow H^3(ISL_3(\mathbb{Z}), \mathbb{Z})/\text{torsion}$. The cokernel of this map is $E_\infty^{0,3} \subset E_2^{0,3} = H^3(\mathbb{Z}^3, \mathbb{Z}) \cong \mathbb{Z}$, which is vulnerable to d_2, d_3, d_4 . The edge homomorphism $H^3(ISL_3(\mathbb{Z}), \mathbb{Z}) \rightarrow E_2^{0,3} = H^3(\mathbb{Z}^3, \mathbb{Z})$ is induced by the restriction map. \square

7.4. The equivariant cohomology of X . There is a spectral sequence converging to the equivariant cohomology of X with values in the constant sheaf \mathbb{Z} , alias the integral cohomology of the stack $[X/ISL_3(\mathbb{Z})]$. Its E_2 term is $E_2^{p,q} = H^p(G, H^q(X, \mathbb{Z}))$. Since X is connected, $H^0(X, \mathbb{Z}) = \mathbb{Z}$ (with trivial action of G) and we have a natural edge homomorphism

$$E_2^{p,0} = H^p(G, \mathbb{Z}) \rightarrow H_G^p(X, \mathbb{Z}).$$

Proposition 7.4. *Let $G = ISL_3(\mathbb{Z})$. Then $H_G^0(X, \mathbb{Z}) = H_G^2(X, \mathbb{Z}) = \mathbb{Z}$ and $H_G^1(X, \mathbb{Z}) = 0$. The edge homomorphism $H^3(G, \mathbb{Z}) \rightarrow H_G^3(X, \mathbb{Z})$ is surjective with finite kernel. In particular,*

$$H_G^3(X, \mathbb{Z})/\text{torsion} \cong H^3(G, \mathbb{Z})/\text{torsion}.$$

Proof. Since X is a deformation retract of S^2 , we have $H^0(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = \mathbb{Z}$ and $H^q(X, \mathbb{Z}) = 0$ for $q \neq 0, 2$. The action of $ISL_3(\mathbb{Z})$ on these groups is trivial since it is the restriction of an action of $ISL_3(\mathbb{R})$ and any homomorphism $G_{\mathbb{R}} \rightarrow \text{Aut}(\mathbb{Z}) = \{\pm \text{Id}\}$ from a connected Lie group $G_{\mathbb{R}}$ is necessarily trivial. Thus $E_2^{p,0} = E_2^{p,2} = H^p(G, \mathbb{Z})$ and all other $E_2^{p,q}$ are trivial. By Proposition 7.3, $H^1(G, \mathbb{Z}) = H^2(G, \mathbb{Z}) = 0$, so the E_2 term is

$$\begin{array}{cccccc}
 \vdots & & & & \vdots & \\
 0 & 0 & 0 & 0 & 0 & \dots \\
 \mathbb{Z} & 0 & 0 & H^3(G, \mathbb{Z}) & H^4(G, \mathbb{Z}) & \dots \\
 0 & 0 & 0 & 0 & 0 & \dots \\
 \mathbb{Z} & 0 & 0 & H^3(G, \mathbb{Z}) & H^4(G, \mathbb{Z}) & \dots
 \end{array}$$

This implies the claim for H_G^0 , H_G^1 and gives an exact sequence

$$0 \rightarrow H_G^2(X, \mathbb{Z}) \rightarrow \mathbb{Z} \xrightarrow{d_3} H^3(G, \mathbb{Z}) \rightarrow H_G^3(X, \mathbb{Z}) \rightarrow 0.$$

To complete the proof it is thus sufficient to show that the image of $H_G^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{Z}) = \mathbb{Z}$ is non-zero. We can use de Rham cohomology to prove this. A 2-form representing the generator of $H^2(X, \mathbb{Z})$ can be constructed with the Fubini–Study Kähler form ω_2 on \mathbb{CP}^2 . On \mathbb{CP}^n ω_n is represented by the \mathbb{C}^\times -basic 2-form on $\mathbb{C}^{n+1} - \{0\}$

$$\omega_n = d d^c \log |x|^2, \quad d^c = \frac{1}{4\pi i}(\partial - \bar{\partial}).$$

Lemma 7.5. *Let $j : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ be the parametrization of the rational curve $x_1^2 + x_2^2 + x_3^2 = 0$ induced by the map $\mathbb{C}^2 \rightarrow \mathbb{C}^3$*

$$(z_1, z_2) \rightarrow (i(z_1^2 - z_2^2), z_1^2 + z_2^2, 2iz_1z_2).$$

Then

$$j^* \omega_2 = 2\omega_1,$$

Proof. This follows from the identity $|j(z)|^2 = 2|z|^4$. □

Let us also denote by $\omega_2 \in \Omega^2(X)$ the pull-back of $\omega_2 \in \Omega^2(\mathbb{CP}^2)$ by the composition $X \rightarrow Y \hookrightarrow \mathbb{CP}^2$. Since ω_1 is normalized so that $\int_{\mathbb{CP}^1} \omega_1 = 1$, it follows that the class of $\frac{1}{2}\omega_2 \in \Omega^2(X)$ is a generator of $\mathbb{Z} = H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$. Consider the double complex $C^{p,q} = C^p(G, \Omega^q(X))$ computing the equivariant cohomology with real coefficients. Denote the two differentials by d , the de Rham differential, and by δ , the differential of group cohomology. We must show that $\omega_2 \in C^{0,2}$ extends to an equivariant cocycle in $C^{0,2} \oplus C^{1,1} \oplus C^{2,0}$. We have $d\omega_2 = 0$ and

$$\delta\omega_2(g) = \omega_2 - (g^{-1})^* \omega_2 = d\psi(g),$$

where $\psi \in C^{1,1}$ is

$$\psi(g) = d^c \log \frac{|x|^2}{|g^{-1}x|^2}$$

Note that the argument of the logarithm is \mathbb{C}^\times -invariant, so $\psi(g)$ is a well-defined form on X . Finally

$$\delta\psi(g, h) = d^c \left(\log \frac{|x|^2}{|g^{-1}x|^2} - \log \frac{|x|^2}{|(gh)^{-1}x|^2} + \log \frac{|g^{-1}x|^2}{|h^{-1}g^{-1}x|^2} \right) = 0.$$

It follows that $(\omega_2, \psi, 0) \in C^{0,2} \oplus C^{1,1} \oplus C^{2,0}$ represents a class in $H_G^2(X, \mathbb{R})$ mapping to the non-trivial class $[\omega_2] \in H^2(X, \mathbb{R})$. □

Theorem 7.6. *The exact sequence of Proposition 7.3 is also exact on the right and thus we have an exact sequence*

$$0 \rightarrow \mathbb{Z} \rightarrow H_G^3(X, \mathbb{Z})/\text{torsion} \rightarrow H^3(\mathbb{Z}^3, \mathbb{Z}) \cong \mathbb{Z} \rightarrow 0$$

The image of the Dixmier–Douady class $c(\mathcal{G}) \in H_G^3(X, \mathbb{Z})$ of the gamma gerbe is a generator of $H^3(\mathbb{Z}^3, \mathbb{Z})$.

Proof. This follows from the commutativity of the diagram of restriction maps and edge homomorphisms

$$\begin{array}{ccccc} H^3(G, \mathbb{Z}) & \longrightarrow & H^3(\mathbb{Z}^3, \mathbb{Z}) & & \\ \downarrow & & \downarrow & \searrow & \\ H_G^3(X, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}^3}^3(X, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}^3}^3(F, \mathbb{Z}) \cong \mathbb{Z}. \end{array}$$

The composition in the bottom row sends $c(\mathcal{G})$ to a generator of \mathbb{Z} by Proposition 7.3 and is thus surjective. The left vertical arrow is surjective by Proposition 7.4 and the map $H^3(\mathbb{Z}^3, \mathbb{Z}) \rightarrow H_{\mathbb{Z}^3}^3(F, \mathbb{Z})$ is an isomorphism. It follows that the restriction map $H^3(G, \mathbb{Z}) \rightarrow H^3(\mathbb{Z}^3, \mathbb{Z})$ is also surjective. \square

8. HERMITIAN STRUCTURE AND CONNECTIVE STRUCTURE

In this section we introduce the notion of a hermitian structure on a gerbe and its canonical connective structure. Then we construct a hermitian structure on the gamma gerbe and compute its $(1,1)$ -curvature in $H^1(\mathcal{X}, \underline{\Omega}_{cl}^{1,1})$.

8.1. Hermitian structures on line bundles and gerbes. Recall that a hermitian line bundle is a holomorphic line bundle with a hermitian metric on each fibre depending smoothly on the base. A hermitian line bundle has a canonical connection, the unique connection preserving the hermitian metric and compatible with the holomorphic structure. The curvature F of the canonical connection is a closed $(1,1)$ -form, and $\frac{i}{2\pi}F$ is a de Rham representative of the first Chern class of the bundle. Suppose that a discrete group G acts on a complex manifold X . Then a G -equivariant hermitian line bundle on X is a hermitian line bundle $L \rightarrow X$ with a lift of the action to L preserving the hermitian metric. In this case the canonical connection is G -equivariant, the curvature is a G -invariant $(1,1)$ -form F and $\frac{i}{2\pi}F$ is an equivariant de Rham representative of the first Chern class. In terms of local trivializations, suppose that a holomorphic line bundle is given by transition functions $\phi_{a,b} \in \mathcal{O}^\times(V_a \cap V_b)$ with respect to an open covering $\mathcal{U} = (V_a)_{a \in I}$. Then a hermitian metric is given by positive smooth functions h_a on V_a , the norms squared of the trivializing sections, such that $h_b/h_a = |\phi_{a,b}|^{-2}$. The canonical connection on V_a is $d + \theta_a$ where $\theta_a = \partial \log h_a$ and the curvature is the global $(1,1)$ -form whose restriction to V_a is $F = d\theta_a = \bar{\partial} \partial \log h_a$. In the equivariant case, with an invariant open covering \mathcal{U} , the lift of the action to the line bundle is given by functions ϕ_a on $G \times V_a$, as described in the introduction and an equivariant hermitian structure obeys additionally $h_a(x)/h_{g^{-1}a}(g^{-1}x) = |\phi_a(g; x)|^{-2}$ on $G \times V_a$. The curvature F is then G -invariant and $\frac{i}{2\pi}F$ represents the image in the equivariant de Rham cohomology of first Chern class.

These notions have a straightforward generalization to (abelian) holomorphic gerbes:

Definition 8.1. A *hermitian structure* on a holomorphic gerbe given by line bundles $L_{a,b}$ on $V_a \cap V_b$ is a collection of hermitian structures $\|\cdot\|_{a,b}$ on $L_{a,b}$ such that the maps $\phi_{a,b,c}$, see (1), are unitary isomorphisms. A *hermitian gerbe* is a holomorphic gerbe with a hermitian structure.

Definition 8.2. An *equivariant hermitian structure* on a G -equivariant holomorphic gerbe given by line bundles $L_{a,b}$, $L_a(g)$ with respect to an invariant open covering is the additional data of hermitian structures $\|\cdot\|_{a,g}$ on $L_a(g)$ such that

also (2), (3) are unitary. A holomorphic gerbe with an equivariant hermitian structure will be called *equivariant hermitian gerbe*.

The notion of equivalence of hermitian gerbes is patterned on the corresponding notion for gerbes defined by transition bundles [6, 18]: an equivariant hermitian gerbe defined with respect to an invariant open covering induces by restriction equivariant hermitian gerbes for all equivariant refinements of the covering. Two equivariant hermitian gerbes $\mathcal{G}, \mathcal{G}'$ given by line bundles $(L_{a,b}, L_a(g))$ and $(L'_{a,b}, L'_a(g))$, respectively, are equivalent if, possibly after passing to a common refinement (V_a) , there are hermitian line bundles M_a on V_a with unitary isomorphisms $L'_{a,b} \cong M_a \otimes L_{a,b} \otimes M_b^*$, $L'_a(g) \cong M_a^* \otimes L_a(g) \otimes (g^{-1})^* M_{g^{-1}a}$.

By definition, a G -equivariant hermitian gerbe on a complex manifold X is the same as a hermitian gerbe on the stack $\mathcal{X} = [X/G]$. More generally, in the language of groupoid presentations introduced in the Appendix, a hermitian gerbe on a stack is a hermitian structure on the central extension of a presentation groupoid, compatible with the groupoid multiplication.

Recall that a connective structure on a holomorphic gerbe given in terms of line bundles $L_{a,b}$ is given by smooth complex connections $\nabla_{a,b}$ on $L_{a,b}$ compatible with the structure maps $\phi_{a,b,c}$, see (1), meaning that $\phi_{a,b,c} \circ \nabla_{a,c} = (\nabla_{a,b} \otimes \text{id} + \text{id} \otimes \nabla_{b,c}) \circ \phi_{a,b,c}$, on sections of $L_{a,c}$ restricted to $V_a \cap V_b \cap V_c$. In the equivariant case, we have additional connections $\nabla_{a;g}$ on $L_a(g)$ compatible with the structure maps (2), (3). In the language of groupoid presentations, a connective structure is a connection on the \mathbb{C}^\times -bundle defining the groupoid central extension compatible with the groupoid multiplication.

Definition 8.3. A *compatible connective structure* on an equivariant hermitian gerbe given by hermitian line bundles $L_{a,b}, L_a(g)$ is a connective structure given by connections $\nabla_{a,b}, \nabla_{a;g}$ preserving the hermitian structure and compatible with the holomorphic structure.

Theorem 8.4. *An equivariant hermitian gerbe possesses a unique compatible connective structure.*

Proof. Let the gerbe \mathcal{G} be represented by line bundles $L_{a,b}, L_a(g)$. Since $L_{a,b}$ and $L_a(g)$ are hermitian line bundles, they possess unique canonical compatible connections $\nabla_{a,b}$ and $\nabla_{a;g}$. It remains to check compatibility of the connections with the isomorphism (1)-(3). This follows immediately from the fact that these isomorphism preserve the hermitian structure of $L_{a,b}$ and because tensor products and pull-backs preserve the corresponding canonical connections. \square

Recall that equivalence classes of hermitian line bundles on a complex manifold $\mathcal{X} = X$ or stack $\mathcal{X} = [X/G]$ form a group $\widehat{\text{Pic}}(\mathcal{X})$, an extension of the Picard group $\text{Pic}(\mathcal{X}) = H^1(\mathcal{X}, \mathcal{O}^\times)$ of \mathcal{X} , with respect to tensor product. The curvature $F = F(\nabla_h)$ of the canonical connection ∇_h of a hermitian structure h on a line bundle L gives a group homomorphism

$$\begin{aligned} \widehat{\text{Pic}}(\mathcal{X}) &\rightarrow H^0(\mathcal{X}, \underline{\Omega}_{cl}^{1,1}) \\ (L, h) &\mapsto \frac{i}{2\pi} F \end{aligned}$$

to the group of global closed differential forms of type (1, 1). The normalization is chosen so that the class of the image in de Rham cohomology is a representative of

the image of the first Chern class of L . In fancier terms, we have a commutative diagram

$$\begin{array}{ccc} \widehat{\text{Pic}}(\mathcal{X}) & \longrightarrow & \text{Pic}(\mathcal{X}) \\ \downarrow & & \downarrow \\ H^0(\mathcal{X}, \underline{\Omega}_{cl}^{1,1}) & \longrightarrow & H_{dR}^2(\mathcal{X}, \mathbb{C}). \end{array}$$

The first arrow forgets the hermitian structure. The right vertical arrow sends a line bundle to the image of the first Chern class in de Rham cohomology and the lower arrow is induced by the inclusion of closed 2-forms in the de Rham complex of sheaves $\underline{\Omega}^0 \rightarrow \underline{\Omega}^1 \rightarrow \dots$ of smooth complex-valued differential forms, whose hypercohomology is $H_{dR}^\bullet(\mathcal{X}, \mathbb{C})$.

Similarly, the curvatures of the canonical connections on the hermitian line bundles $L_{a,b}$, $L_a(g)$ form a 1-cocycle $F = (F_{a,b}, F_{a,g}) \in C^{0,1}(G, \underline{\Omega}_{cl}^{1,1}) \oplus C^{1,0}(G, \underline{\Omega}_{cl}^{1,1})$ with values in the sheaf of closed $(1,1)$ -forms. We call the class of F in $H_G^1(X, \underline{\Omega}_{cl}^{1,1}) = H^1(\mathcal{X}, \underline{\Omega}_{cl}^{1,1})$ the $(1,1)$ -curvature of the hermitian gerbe. Hermitian gerbes also form a group under tensor product. The group of equivalence classes of hermitian gerbes $\widehat{\text{Br}}(\mathcal{X})$ on \mathcal{X} is an extension of the Brauer group $\text{Br}(\mathcal{X}) = H^2(\mathcal{X}, \mathcal{O}^\times)$ of equivalence classes of holomorphic gerbes. The $(1,1)$ -curvature of a tensor product is the sum of the $(1,1)$ -curvatures of the factors. Moreover, the 1-cocycles of equivalent hermitian gerbes differ by the differential of a 0-cochain, consisting of the curvatures of the line bundles M_a defining the equivalence. Thus we have a group homomorphism

$$\begin{aligned} \widehat{\text{Br}}(\mathcal{X}) &\rightarrow H^1(\mathcal{X}, \underline{\Omega}_{cl}^{1,1}) \\ (\mathcal{G}, h) &\mapsto \frac{i}{2\pi} F. \end{aligned}$$

The normalization is again chosen so that we have a commutative diagram

$$\begin{array}{ccc} \widehat{\text{Br}}(\mathcal{X}) & \longrightarrow & \text{Br}(\mathcal{X}) \\ \downarrow & & \downarrow \\ H^1(\mathcal{X}, \underline{\Omega}_{cl}^{1,1}) & \longrightarrow & H_{dR}^3(\mathcal{X}, \mathbb{C}), \end{array}$$

with the right vertical arrow mapping a gerbe to the image in de Rham cohomology of its Dixmier–Douady class.³

In explicit terms, suppose that a G -equivariant holomorphic gerbe \mathcal{G} is given on an invariant open covering $\mathcal{U} = (V_a)$ with trivial bundles $L_{a,b}$, $L_a(g)$ as in the case of the gamma gerbe. Then a hermitian structure is given by the collection $h_{a,b}$, $h_{a,g}$ of the norms squared of the trivializing sections. These functions form an equivariant cochain

$$(h^{0,1}, h^{1,0}) \in C_G^1(\mathcal{U}, \mathbb{R}_+) = C^0(G, \check{C}^1(\mathcal{U}, \mathbb{R}_+)) \oplus C^1(G, \check{C}^0(\mathcal{U}, \mathbb{R}_+)),$$

with values in the sheaf of positive smooth functions. The compatibility condition with the structure maps is then $Dh = |\phi|^{-2}$, where $|\phi|^{-2} \in C^2(\mathcal{U}, \mathbb{R}_+)$ is the cocycle $(|\phi_{a,b,c}|^{-2}, |\phi_{a,b}|^{-2}, |\phi_a|^{-2})$. The canonical connective structure is given by the connections $\nabla_{a,b} = d + \partial \log h_{a,b}$, $\nabla_{a,g} = d + \partial \log h_{a,g}$. The $(1,1)$ -curvature is then the class of the cocycle F with

$$F_{a,b} = \bar{\partial} \partial \log h_{a,b}, \quad F_{a,g} = \bar{\partial} \partial \log h_{a,g}.$$

³Note that F is given by imaginary $(1,1)$ -forms, so $\frac{i}{2\pi} F$ is real and we may take the *real* de Rham complex.

Finally the complex de Rham cohomology of \mathcal{X} may be realized as the cohomology of the complex $\Omega_G^n(X) = \oplus_{p+q=n} C^p(G, \Omega^q(X))$ of group cochains with values in smooth complex differential forms. A representative of the image of the Dixmier–Douady class of the gerbe is obtained by unravelling the definitions: a *curving*, see [5], [18], is a collection of $(1, 1)$ -forms B_a on V_a so that $B_b - B_a = F_{a,b}$. A curving always exists and can be constructed by a partition of unity. The *3-curvature* dB_a is then a globally defined 3-form. Also $F_{a;g} - B_a + (g^{-1})^* B_{g^{-1}a}$ is the restriction to V_a of a globally defined $(1, 1)$ -form and

$$\frac{i}{2\pi}(dB_a, F_{a;g} - B_a + (g^{-1})^* B_{g^{-1}a}, 0, 0) \in \Omega_G^3(X),$$

is a cocycle representing the image in $H_{dR}^3(\mathcal{X}, \mathbb{C})$ of the Dixmier–Douady class.

8.2. The case of the theta function. It is a well-known fact that the theta bundle has an equivariant hermitian structure. The norm squared of a local section $s(z, \tau)$ is $h_2(z, \tau)|s(z, \tau)|^2$. With our conventions,

$$h_2(z, \tau) = \exp\left(-2\pi \frac{(\operatorname{Im} z)^2}{\operatorname{Im} \tau} + 2\pi \operatorname{Im}(z - \tau/6)\right).$$

The statement that this function defines an equivariant hermitian structure on the theta bundle is equivalent to the fact that the smooth function on $\mathbb{C} \times H_+$

$$h_2(z, \tau)|\theta_0(z, \tau)|^2$$

is invariant under $ISL_2(\mathbb{Z})$. Moreover h_2 is defined on $\mathbb{C} \times (\mathbb{C} - \mathbb{R})$ and obeys $h_2(-z, -\tau) = h_2(z, \tau)^{-1}$. As a consequence the invariance property extends to the lower half-plane (recall that we set $\theta_0(z, \tau) = \theta_0(-z, -\tau)^{-1}$). In homogeneous coordinates we then have the result:

Proposition 8.5. *Let $p: X_2 \times \mathbb{C} \rightarrow X_2$ be the $ISL_2(\mathbb{Z})$ -equivariant theta line bundle. Then*

$$\|\zeta\|^2 = h_2\left(\frac{w}{x_2}, \frac{x_1}{x_2}\right) |\zeta|^2, \quad \zeta \in p^{-1}(w, x) = \mathbb{C},$$

is an equivariant hermitian structure on the theta bundle. In other words, the function

$$h_2\left(\frac{w}{x_2}, \frac{x_1}{x_2}\right) \left| \theta_0\left(\frac{w}{x_2}, \frac{x_1}{x_2}\right) \right|^2$$

is invariant under the action of $ISL_2(\mathbb{Z})$.

To complete the picture, we write down the explicit formula for the $(1, 1)$ -form $c_1 = \frac{i}{2\pi} \bar{\partial} \partial \log h_2$, which represents the first Chern class of the theta bundle. In homogeneous coordinates $z = w/x_2$, $\tau = x_1/x_2$,

$$c_1 = \frac{i}{2\pi} \bar{\partial} \partial \log h_2 = \frac{i}{2 \operatorname{Im} \tau} \left(dz - \frac{\operatorname{Im} z}{\operatorname{Im} \tau} d\tau \right) \wedge \overline{\left(dz - \frac{\operatorname{Im} z}{\operatorname{Im} \tau} d\tau \right)}.$$

This 2-form is by construction $ISL_2(\mathbb{Z})$ -invariant and restricts to the standard normalized volume form on each fibre $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

8.3. A hermitian structure on the gamma gerbe. As in Subsection 4.3, let $G = ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ act on $X = X_3$ and let \mathcal{G} be the gamma gerbe defined in terms of the covering $(V_a)_{a \in \Lambda_{\text{prim}}}$ by Theorem 4.1. By the preceding discussion, an equivariant hermitian structure on the gamma gerbe is given by a set of positive smooth functions $h_{a,b}$ on $V_a \cap V_b$ and h_a on $G \times V_a$ (such that $h_{a;g}(y) = h_a(g; y)$) obeying the conditions:

$$(36) \quad h_{a,b}(y)h_{b,a}(y) = 1, \quad y \in V_a \cap V_b,$$

$$(37) \quad h_{a,b}(y)h_{b,c}(y)h_{c,a}(y) = |\phi_{a,b,c}(y)|^{-2}, \quad y \in V_a \cap V_b \cap V_c,$$

$$(38) \quad h_{g^{-1}a, g^{-1}b}(g^{-1}y)h_b(g; y) = |\phi_{a,b}(g; y)|^2 h_a(g; y), \quad y \in V_a \cap V_b,$$

$$(39) \quad h_a(g_1 g_2; x) = |\phi_a(g_1, g_2; y)|^2 h_a(g_1; y) h_{g_1^{-1}a}(g_2; g_1^{-1}y), \quad y \in V_a,$$

for all $a, b, c \in I, g, g_1, g_2 \in G$.

Introduce the function of three complex variables

$$h_3(z, \tau, \sigma) = \exp\left(-\frac{2\pi}{3}R_3(\text{Im } z, \text{Im } \tau, \text{Im } \sigma)\right),$$

$$R_3(\zeta, t, s) = \frac{\zeta^3}{ts} - \frac{3}{2}\left(\frac{1}{t} + \frac{1}{s}\right)\zeta^2 + \left(\frac{t}{2s} + \frac{s}{2t} + \frac{3}{2}\right)\zeta - \frac{t+s}{4}.$$

Note that both h_2 and h_3 (and their higher n analogues) can be expressed in terms of multiple Bernoulli polynomials (see Subsection 2.2)

$$(40) \quad h_n(z, \tau_1, \dots, \tau_{n-1}) = \exp(-(4\pi/n!)B_{n-1,n}(\zeta, t_1, \dots, t_{n-1})),$$

where $\zeta = \text{Im } z, t_j = \text{Im } \tau_j$. In particular, $R_3 = B_{2,3}$.

Theorem 8.6. Fix a framing of Λ_{prim} and let $(V_a)_{a \in \Lambda_{\text{prim}}}$ be the open covering of X_3 of Section 4. For $a \neq b \in \Lambda_{\text{prim}}$, let

$$h_{a,b}(w, x) = \prod_{\delta \in F/\mathbb{Z}\gamma} h_3\left(\frac{w + \delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right), \quad (w, x) \in V_a \cap V_b,$$

where α, β, γ and F are as in Proposition 3.5, and set $h_{a,a} = 1$. Let $a \in \Lambda_{\text{prim}}$ and $(\alpha_1, \alpha_2, \alpha_3)$ the basis of Λ^\vee associated to a by the framing and define for $\mu \in \Lambda^\vee = \mathbb{Z}^3$

$$h_a(\mu; w, x) = \prod_{j=0}^{\mu(a)-1} h_2\left(\frac{w + j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)}\right), \quad (w, x) \in V_a.$$

Extend h_a to $ISL_3(\mathbb{Z})$ by setting $h_a((g, \mu); w, x) = h_a(\mu \circ g^{-1}; w, x)$. Then the 1-cochain $(h^{0,1}, h^{1,0})$, specified by $h^{1,0} = (h_a)_{a \in \Lambda_{\text{prim}}}$ and $h^{0,1} = (h_{a,b})_{a,b \in \Lambda_{\text{prim}}}$, with values in the sheaf of positive smooth functions, defines a hermitian structure on the gamma gerbe.

Proof. We first need to show that $h_{a,b}$ is well-defined, independently of the choice of α, β . The general α, β obeying the conditions of Proposition 3.5 are of the form

$$\alpha = n\alpha_0 + m\gamma, \quad \beta = n'\beta_0 + m'\gamma, \quad n, n' \in \mathbb{Z}_{>0}, \quad m, m' \in \mathbb{Z},$$

where $\alpha_0(a), \beta_0(b)$ are minimal (and equal to $\text{mod}(a, b)$). It is clear that $h_{a,b}$ does not change if we add to α or β a multiple of γ . If we replace α_0 by a multiple $n\alpha_0$ then F is replaced by the union of n shifts of F and the claim reduces to:

Lemma 8.7. $h_3(z, \tau, \sigma) = \prod_{j=0}^{n-1} h_3(z + j\tau, n\tau, \sigma)$.

Proof. This is the special case of a family of identities for multiple Bernoulli polynomials following from the trivial identity of generating series

$$\begin{aligned} \sum_{j=0}^{\infty} B_{r,j}(\zeta, t_1, \dots, t_r) \frac{u^j}{j!} &= e^{\zeta u} \prod_{i=1}^r \frac{u}{e^{t_i u} - 1} \\ &= \sum_{j=0}^{n-1} e^{(\zeta + jt_1)u} \frac{u}{e^{nt_1 u} - 1} \prod_{i=2}^r \frac{u}{e^{t_i u} - 1}. \end{aligned}$$

In our case $r = 2$ and comparing the coefficients of u^3 gives

$$R_3(\zeta, t, s) = B_{2,3}(\zeta, t, s) = \sum_{j=0}^{n-1} R_3(\zeta + jt, nt, s),$$

which is equivalent to the claim. \square

The case of a multiple of β_0 is treated in the same way. This shows that $h_{a,b}$ is well defined.

It is immediate to check that the functions $h_{a,b}$, h_a are indeed smooth and positive on their domain of definition and that $h_{a,b}h_{b,a} = 1$. There remains to prove the conditions (37)–(39).

Proof of (37): The first condition $h_{a,b}h_{b,c}h_{c,a} = |\phi_{a,b,c}|^{-2}$ are checked directly as in the proof of Theorem 3.8.

Consider first the case of linearly independent a, b, c . Then we proceed as in the proof of Theorem 3.8 and use the direction vectors α, β, γ of the three pairs in the definition of $h_{a,b}$, $h_{b,c}$ and $h_{c,a}$. The equation then reduces to

$$R_3(\text{Im}(w/x_3), \text{Im}(x_1/x_3), \text{Im}(x_2/x_3)) + \text{cycl.} = \text{Im } P_3(w, x).$$

This identity can be easily proved using the following formula.

Lemma 8.8. *Let $z_1, \dots, z_n, w_1, \dots, w_n$ be complex numbers such that $\text{Im}(w_i \bar{w}_j) \neq 0$, $i \neq j$. Then*

$$\text{Im} \frac{z_1 \cdots z_n}{w_1 \cdots w_n} = \sum_{j=1}^n \frac{\text{Im}(z_1/w_j) \cdots \text{Im}(z_n/w_j)}{\text{Im}(w_1/w_j) \cdots \widehat{\text{Im}(w_j/w_j)} \cdots \text{Im}(w_n/w_j)}.$$

Proof. Apply the Cauchy residue theorem to the meromorphic differential

$$\omega = \frac{1}{2i} \frac{(\zeta z_1 - \bar{z}_1) \cdots (\zeta z_n - \bar{z}_n)}{(\zeta w_1 - \bar{w}_1) \cdots (\zeta w_n - \bar{w}_n)} \frac{d\zeta}{\zeta}.$$

The left-hand side of the claimed identity is $-\text{res}_{\zeta=\infty} \omega - \text{res}_{\zeta=0} \omega$ and the right-hand side is the sum of the residues at the remaining poles $\zeta = \bar{w}_i/w_i$. \square

Consider now the case where the vectors a, b, c span a plane. We can assume that a, b, c are pairwise linearly independent since otherwise the identity reduces to the inversion relation. Again we follow the pattern of proof of Theorem 3.8, for which we need an analogue for $h_{a,b}$ of the infinite product representation for $\Gamma_{a,b}$. This is again provided by “Bernoulli calculus”:

Lemma 8.9. *Let $a, b \in \Lambda_{\text{prim}}$ and γ be the corresponding direction vector. As in Section 3 introduce $C_{+-}(a, b) = \{\delta \in \Lambda^\vee \mid \delta(a) > 0, \delta(b) \leq 0\}$. Then for $(w, x) \in$*

$V_a \cap V_b$ the series

$$S_{a,b}(t) = -t^2 \sum_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma} \exp\left(t \operatorname{Im} \frac{w - \delta(x)}{\gamma(x)}\right),$$

convergent for $\operatorname{Im} t > 0$, is holomorphic around $t = 0$ and

$$h_{a,b}(w, x) = e^{-(2\pi/3)S_{a,b}'''(0)}.$$

Proof. By (40) and the formula for the generating function of multiple Bernoulli polynomials, $h_{a,b}$ is defined as $\exp(-(2\pi/3)\hat{S}_{a,b}'''(0))$ where

$$\begin{aligned} \hat{S}_{a,b}(t) &= \sum_{\delta \in F/\mathbb{Z}\gamma} \exp\left(t \operatorname{Im} \frac{w - \delta(x)}{\gamma(x)}\right) \frac{t^2}{(e^{t \operatorname{Im} \tau} - 1)(e^{t \operatorname{Im} \sigma} - 1)} \\ &= \sum_{\delta \in F/\mathbb{Z}\gamma} \exp\left(t \operatorname{Im} \frac{w - \delta(x)}{\gamma(x)}\right) \frac{t^2 e^{-t \operatorname{Im} \sigma}}{(e^{t \operatorname{Im} \tau} - 1)(1 - e^{-t \operatorname{Im} \sigma})}, \end{aligned}$$

where $\tau = \alpha(x)/\gamma(x)$ and $\sigma = \beta(x)/\gamma(x)$, Note that $\operatorname{Im} \tau < 0$ and $\operatorname{Im} \sigma > 0$ on $V_a \cap V_b$. So for $\operatorname{Re} t > 0$ we may expand the geometric series and obtain, as in the proof of Proposition 3.5, that $S_{a,b}(t) = \hat{S}_{a,b}(t)$. \square

Using this representation, we proceed as in the proof of Theorem 3.8: we may assume that (a, b) , (b, c) , (a, c) have the same direction vector γ and deduce from $eC_{-+}(a, c) = C_{-+}(a, b) \sqcup C_{-+}(b, c)$, that

$$S_{a,c}(t) = S_{a,b}(t) + S_{b,c}(t).$$

Taking the third derivative at $t = 0$ implies that $h_{a,b}h_{b,c} = h_{a,c}$. This completes the proof of (37).

Proof of (38),(39): Since the other components of the 2-cocycle are not so explicit and depend on the framing, it is better to reformulate them in terms of norms of local sections. Let $\|\Gamma_{a,b}\|^2 = h_{a,b}|\Gamma_{a,b}|^2$ be the norm squared of the section $\Gamma_{a,b}$ of $L_{a,b}$ (a more precise notation for this norm would be $\|\cdot\|_{a,b}$). It is a smooth function defined almost everywhere on $V_a \cap V_b$. Similarly let $\|\Delta_a\|^2 = h_a|\Delta_a|^2$. Then the remaining conditions (38), (39) read

$$(41) \quad \|\Gamma_{a,b}(w + \mu(x), x)\|^2 \|\Delta_b(\mu; w, x)\|^2 = \|\Gamma_{a,b}(w, x)\|^2 \|\Delta_a(\mu; w, x)\|^2,$$

$$(42) \quad \|\Delta_a(\mu + \nu; w, x)\|^2 = \|\Delta_a(\mu; w, x)\|^2 \|\Delta_a(\nu; w + \mu(x), x)\|^2,$$

$$(43) \quad \|\Delta_a(\mu \circ g^{-1}; w, x)\|^2 = \|\Delta_{g^{-1}a}(\mu; w, g^{-1}x)\|^2,$$

for all $\mu, \nu \in \Lambda^\vee = \mathbb{Z}^3$, $g \in \operatorname{Aut}(\Lambda) = SL_3(\mathbb{Z})$.

We start with (43). For this it is better to temporarily make the dependence on the framing explicit: let $\Delta_{a,f}$ denote Δ_a calculated with the framing $f: \Lambda_{prim} \rightarrow \{\text{Bases of } \Lambda^\vee\}$. Then $\Delta_{g^{-1}a,f}(\mu \circ g; w, g^{-1}x) = \Delta_{a,gf}(\mu; w, x)$, for the natural action of $SL_3(\mathbb{Z})$ on the set of framings. Thus (43) follows from the following remark.

Lemma 8.10. *The normed squared $\|\Delta_{a,f}(w, x)\|^2$ is independent of the framing f . Moreover, $\|\Delta_{a,f}(w + \mu(x), x)\|^2 = \|\Delta_{a,f}(w, x)\|^2$ if $\mu \in H(a)$.*

Proof. Both h_a and Δ_a are defined using the basis $f(a) = (\alpha_1, \alpha_2, \alpha_3)$ assigned to a by the framing. By definition, α_2, α_3 are an oriented basis of $H(a)$ and $\alpha_1(a) = 1$. This basis can be changed in two ways: α_2, α_3 can be replaced by another oriented

basis of $H(a)$ or one can add to α_1 an integer linear combination of α_2 and α_3 . The normed squared is a product of factors

$$h_2 \left(\frac{w + j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)} \right) \left| \theta_0 \left(\frac{w + j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)} \right) \right|^2,$$

and a change of basis amounts to an $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ -transformation under which, by Proposition 8.5, each factor is invariant. Similarly, a shift of w by a linear combination of $\alpha_1(x), \alpha_2(x)$ does not change the factors of the product, proving the second statement. \square

Let us turn to Equation (42). Clearly it is sufficient to prove it for ν belonging to a basis, which we take to be the one given by the framing. Note that $\Delta_a(\mu + \nu; w, x) = \Delta_a(\mu; w, x)$ if $\nu \in H(a)$ and the same holds for h_a . Also $h_a(0; w, x) = \Delta_a(0; w, x) = 1$. This proves the claim for $\nu \in H(a)$. It remains to check the identity for $\nu = \alpha_1$. In this case inserting the definitions gives

$$\|\Delta_a(\mu + \alpha_1; w, x)\|^2 = \|\Delta_a(\mu; w, x)\|^2 \|\Delta_a(\alpha_1; w + \mu(a)\alpha_1(x), x)\|^2.$$

This implies the claim by Lemma 8.10 since $\mu(a)\alpha_1 = \mu \pmod{H(a)}$.

Finally, let us prove (41). Again, it is sufficient to prove it for μ belonging to a basis of Λ^\vee . Let $\gamma = \gamma_{a,b}$ be the direction vector of (a, b) and let α, β be such that $\alpha(b) = \beta(a) = 0$ and that $\alpha(a), \beta(b)$ are positive and minimal (thus both equal to $\text{mod}(a, b)$). If $\mu = \alpha$ or β , formulae are simple: if $\mu = \alpha$, then $\Delta_b(\mu; w, x) = 1$ and by Proposition 3.5

$$(44) \quad \frac{\Gamma_{a,b}(w + \alpha(x), x)}{\Gamma_{a,b}(w, x)} = \prod_{\delta \in F} \theta_0 \left(\frac{w + \delta(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)} \right).$$

Suppose that the framing assigns to a a basis of the form $(\alpha_1, \beta, \gamma)$. Then the right-hand side differs from $\Delta_a(\alpha; w, x)$ by a shift of the first argument of the theta function by a multiple of $\beta(x)/\gamma(x)$. On the other hand we have, thanks to the identity

$$h_3(z + \tau, \tau, \sigma) = h_2(z, \sigma) h_3(z, \tau, \sigma),$$

following from (9),

$$(45) \quad \frac{h_{a,b}(w + \alpha(x), x)}{h_{a,b}(w, x)} = \prod_{\delta \in F} h_2 \left(\frac{w + \delta(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)} \right).$$

Multiplying (45) with the absolute value squared of (44) gives

$$\frac{\|\Gamma_{a,b}(w + \alpha(x), x)\|^2}{\|\Gamma_{a,b}(w, x)\|^2} = \|\Delta_a(\mu; w, x)\|^2,$$

since by Proposition 8.5 the right-hand side is invariant under shifts of the first argument of θ_0 by $\beta(x)/\gamma(x)$ or by change of framing. Similarly,

$$\frac{\|\Gamma_{a,b}(w + \beta(x), x)\|^2}{\|\Gamma_{a,b}(w, x)\|^2} = \|\Delta_b(\mu; w, x)\|^{-2}.$$

Also, both $\Gamma_{a,b}$ and $h_{a,b}$ are invariant under a shift of w by $\gamma(x)$. This proves (41) if μ is in the sublattice M generated by α, β, γ . Let $\mu_0 \in \Lambda^\vee$ be such that $\mu_0(a) = 1$ and $0 \leq \mu_0(b) < \alpha(a)$. Such a vector is unique modulo $\mathbb{Z}\gamma$. Then the quotient Λ^\vee/M is cyclic of order $s = \text{mod}(a, b)$ generated by the image of μ_0 . All this can be best seen by going to the normal form, where $a = e_1$, $b = re_1 + se_2$, $\alpha = (s, -r, 0)$, $\beta = (0, 1, 0)$, $\gamma = (0, 0, 1)$ and $\mu_0 = (1, 0, 0)$.

The proof is reduced to checking (41) for $\mu = \mu_0$. Let $s = \text{mod}(a, b) = \alpha(a) = \beta(b)$ and $r = \mu_0(b)$ be the invariants of the wedge (a, b) . In this case the formula in the proof of Proposition 3.14 reduces to

$$(46) \quad \frac{\Gamma_{a,b}(w + \mu_0(x), x)}{\Gamma_{a,b}(w, x)} = \frac{\theta_0\left(\frac{w+\beta(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right)}{\prod_{\delta \in F_3} \theta_0\left(\frac{w+\delta(x)}{-\gamma(x)}, \frac{\alpha(x)}{-\gamma(x)}\right)},$$

where $F_3 \subset \Lambda^\vee / \mathbb{Z}\gamma$ is defined by the inequalities $0 \leq \delta(a) < s, 0 \leq \delta(b) < r$. On the other hand, shifting δ by $-\mu_0$ in the product defining $h_{a,b}$ gives

$$h_{a,b}(w + \mu_0(x), x) = \prod_{\delta \in F'} h_3\left(\frac{w + \delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right).$$

Here F' is the set of $\delta \in \Lambda^\vee / \mathbb{Z}\gamma$ obeying $0 < \delta(a) \leq s, r \leq \delta(b) < s + r$. For each value of $\delta(a)$ in this range there is a unique $\delta \in F'$. In particular $\delta(a) = s$ only for $\delta = \alpha + \beta$. On the corresponding factor we apply the equation relating h_3 to h_2 :

$$\frac{h_3\left(\frac{w+\beta(x)+\alpha(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right)}{h_3\left(\frac{w+\beta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right)} = h_2\left(\frac{w + \beta(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right).$$

This allows us to replace the range F' by the range F'' determined by $0 \leq \delta(a) < s, r \leq \delta(b) < s + r$. Similarly, the factors corresponding to $\delta \in F''$ with $\delta(b) \geq s$ may be related by a shift of w by $\beta(x)$ to factors with $0 \leq \delta(b) < r$. The result is

$$(47) \quad \frac{h_{a,b}(w + \mu_0(x), x)}{h_{a,b}(w, x)} = \frac{h_2\left(\frac{w+\beta(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right)}{\prod_{\delta \in F_3} h_2\left(\frac{w+\delta(x)}{-\gamma(x)}, \frac{\alpha(x)}{-\gamma(x)}\right)},$$

As above, we obtain the claim by comparing (46) with (47). \square

8.4. The (1,1)-curvature. Here is an explicit formula for the (1,1)-curvature $F = \bar{\partial}\partial \log h$. The components $h^{0,1}, h^{1,0}$ are given in terms of the functions h_3, h_2 , respectively. Thus we first compute $\bar{\partial}\partial \log h_3, \bar{\partial}\partial \log h_2$. The latter was already computed in Subsection 8.2. The former is given by the following expression:

$$\begin{aligned} \bar{\partial}\partial \log h_3 = & \frac{2\pi}{3} \left[\left(\frac{6\zeta}{st} - 3 \left(\frac{1}{s} + \frac{1}{t} \right) \right) dz \wedge d\bar{z} - \right. \\ & - \left(-\frac{3\zeta^2}{t^2s} + \frac{3\zeta}{t^2} + \frac{1}{2s} - \frac{s}{2t^2} \right) (dz \wedge d\bar{\tau} - d\bar{z} \wedge d\tau) - \\ & - \left(-\frac{3\zeta^2}{ts^2} + \frac{3\zeta}{s^2} + \frac{1}{2t} - \frac{t}{2s^2} \right) (dz \wedge d\bar{\sigma} - d\bar{z} \wedge d\sigma) - \\ & - \left(\frac{2\zeta^3}{t^3s} - \frac{3\zeta^2}{t^3} + \frac{s\zeta}{t^3} \right) d\tau \wedge d\bar{\tau} - \\ & - \left(\frac{\zeta^3}{t^2s^2} - \left(\frac{1}{2s^2} + \frac{1}{2t^2} \right) \zeta \right) (d\tau \wedge d\bar{\sigma} - d\bar{\tau} \wedge d\sigma) - \\ & \left. - \left(\frac{2\zeta^3}{ts^3} - \frac{3\zeta^2}{s^3} + \frac{t\zeta}{s^3} \right) d\sigma \wedge d\bar{\sigma} \right]. \end{aligned}$$

By Theorem 8.6, the components $F_{a,b}$ of the $(1,1)$ -curvature are then

$$F_{a,b} = \bar{\partial}\partial \log h_{a,b} = \sum_{\delta \in F/\mathbb{Z}\gamma} \Phi_{a,b,\delta}^* (\bar{\partial}\partial \log h_3),$$

where $\Phi_{a,b,\delta}$ is the map from $V_a \cap V_b$ to $\mathbb{C} \times H_+ \times H_+$, defined via $(w, x) \mapsto \left(\frac{w-\delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)} \right)$. As for the other components of F , we have

$$F_{a;g} = \bar{\partial}\partial \log h_{a;g} = \sum_{j=0}^{\mu(a)} \Phi_{a,j}^* \bar{\partial}\partial \log h_2,$$

where the notations are as in Theorem 8.6, an explicit expression for the closed $(1,1)$ -form $\bar{\partial}\partial \log h_2$ can be found in Subsection 8.2, and the map $\Phi_{a,j}$ from V_a to $\mathbb{C} \times H_+$ is defined via $(w, x) \mapsto \left(\frac{w+j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)} \right)$, for any choice of a framing f . We notice that, despite being present in the Definition of $\Phi_{a,j}$, the differential form $\bar{\partial}\partial \log h_{a;g}$ does not depend on the choice of a framing f , since a change of framing corresponds to multiplying $h_{a;g}$ by the absolute value squared of a holomorphic function.

APPENDIX A. GERBES ON STACKS

Gerbes have been extensively studied in algebraic geometry. Gerbes on manifolds have also been studied for example in [5] [18] [20]. They were recently studied over differentiable stacks [3]. In this section, we summarize and prove certain necessary facts used in the main part of the paper about stacks and gerbes on stacks in the differentiable and in the analytic category.

A.1. Definitions. First of all, we recall that a stack \mathcal{X} is a category fibred in groupoids over the category of smooth manifolds \mathcal{C} satisfying two conditions: “isomorphisms form a sheaf” and “effective data descends”. (See detailed treatments for example in [3] [19] [23]). A manifold M is a stack by viewing it as the category of all morphisms from some manifold to it $\{U \rightarrow M\}$. Morphisms between stacks are functors between fibred categories. A stack morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called a *representable submersion* if for any morphism $Z \rightarrow \mathcal{Y}$ where Z is a manifold, the fibre product $\mathcal{X} \times_{\mathcal{Y}} Z$ is again a manifold and $\mathcal{X} \times_{\mathcal{Y}} Z \rightarrow Z$ is a submersion. We call f a *representable surjective submersion* if it is furthermore an epimorphism of fibred categories. We call f *étale* if when $\mathcal{X} \times_{\mathcal{Y}} Z$ is a manifold, the morphism $\mathcal{X} \times_{\mathcal{Y}} Z \rightarrow Z$ is an étale map, namely a local diffeomorphism.

A *differentiable stack* \mathcal{X} is a stack with a representable surjective submersion from a manifold X to it. We call such a manifold X a *chart* of \mathcal{X} . Then $X \times_{\mathcal{X}} X \rightrightarrows X$ is a Lie groupoid whose classifying space is isomorphic to \mathcal{X} as a stack. We call such a Lie groupoid a *groupoid presentation* of \mathcal{X} . Different charts give rise to Morita equivalent Lie groupoids. Morita equivalent Lie groupoids have isomorphic classifying spaces viewed as stacks. Therefore one can go back and forth between differentiable stacks and Lie groupoids. Further \mathcal{X} is an *étale stack* if it admits a chart X such that the map $X \rightarrow \mathcal{X}$ is étale.

Example A.1. Let a group G act on a manifold M . The action groupoid $G \ltimes M \rightrightarrows M$ is defined by $\mathbf{t}(g, x) = x$ and $\mathbf{s}(g, x) = g^{-1}x$ as target and source maps, and $(g, x) \cdot (h, y) = (gh, x)$ when $\mathbf{s}(g, x) = \mathbf{t}(h, y)$. If the action is free and proper, then the quotient M/G is again a manifold whose presenting groupoid $M/G \rightrightarrows M/G$

is Morita equivalent to the action groupoid $G \ltimes M \rightrightarrows M$. If the action is not free and proper, then the quotient might not be a manifold. But one could still understand it as a differentiable stack $[M/G]$ presented by the action groupoid. In this language, both manifolds and orbifolds are differentiable stacks.

Given an abelian Lie group A , an A -central extension of a groupoid $X_1 \rightrightarrows X_0$ is a groupoid $R \rightrightarrows X_0$ fitting in the following short exact sequence of groupoids:

$$(48) \quad \begin{array}{ccccccc} 1 & \longrightarrow & A \times X_0 & \longrightarrow & R & \longrightarrow & X_1 \longrightarrow 1 \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ & & X_0 & \xrightarrow{id} & X_0 & \xrightarrow{id} & X_0 \end{array}$$

in such a way that $R \rightarrow X_1$ is a principal A -bundle over X_1 . It is equivalent to the following three conditions: $R|_{X_0}$ is the trivial A -bundle; $pr_1^* R \otimes m^* R^{-1} \otimes pr_2^* R \cong 1$ via an isomorphism ϕ with $pr_i : X_1 \times_{X_0} X_1 \rightarrow X_1$ the projections and m the multiplication of $X_1 \rightrightarrows X_0$; ϕ satisfies a higher coherence condition. Two central extensions $R \rightrightarrows X_0$ and $R' \rightrightarrows X_0$ are isomorphic if all the involved groupoids in (48) are isomorphic correspondingly and all the expected diagrams commute.

An A -gerbe over \mathcal{X} is a stack presented by an A -central extension of a groupoid presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} . Two gerbes \mathcal{G} and \mathcal{G}' over \mathcal{X} are *isomorphic* to each other if their respective presentations $R \rightrightarrows X_0$ and $R' \rightrightarrows X_0$ are isomorphic as A -central extensions of some groupoid presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} . This goes under the name of “stable isomorphism” in the framework of bundle gerbes of Murray [20].

We can generalize the above to the holomorphic setting too. A complex groupoid is a groupoid in the category of complex manifolds, namely a groupoid $G_1 \rightrightarrows G_0$ with both G_1 and G_0 complex manifolds and structure maps holomorphic maps. Morita equivalence between two complex groupoids is given by a complex manifold which is a Morita bibundle in the usual sense but with holomorphic action from both groupoids. Then an *étale complex stack* \mathcal{X} is presented by Morita equivalent étale complex groupoids. Here étale means that the source map is étale. By definition, an étale complex stack has charts X ’s which are complex manifolds and the projections $X \rightarrow \mathcal{X}$ are étale. A \mathbb{C}^\times - (holomorphic) gerbe \mathcal{G} over an étale complex stack \mathcal{X} then is a stack presented by a \mathbb{C}^\times -central extension of complex groupoids of a groupoid presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} . Isomorphisms of \mathbb{C}^\times -gerbes are similarly defined as for gerbes (note that the isomorphisms used are now in the category of complex groupoids). To relate this to the categorical interpretation of gerbes, one takes the category of all the torsors (principal bundles) of the \mathbb{C}^\times -central extension. We also relate this to the equivariant gerbe described in the introduction in Proposition A.5.

With the viewpoint of groupoid central extension, some definitions become short. A *hermitian structure* of a \mathbb{C}^\times -gerbe \mathcal{G} over \mathcal{X} is a hermitian structure h on the line bundle $L \rightarrow X_1$ associated to $R \rightarrow X_1$ such that the isomorphism ϕ sends the tensored hermitian structures from $pr_1^* L \otimes m^* L^{-1} \otimes pr_2^* L$ to the trivial hermitian structure on $\mathbb{C} \times X_1$. A hermitian structure of \mathcal{G} comes with a *canonical connective structure* which is the canonical connection 1-form θ on R associated to h , and this is exactly the unique connective structure constructed in Theorem 8.4.

A.2. Sheaf and Čech cohomology of stacks. To describe the Dixmier–Douady class of a gerbe over a stack, we need to introduce cohomology of stacks. Here we

first summarize the results in [7] [12], then define the Čech and sheaf cohomology for stacks.

Given a differentiable stack \mathcal{X} presented by a groupoid $X_1 \rightrightarrows X_0$, the nerve of the groupoid is a simplicial manifold X_\bullet given by

$$\dots X_2 \rightrightarrows X_1 \rightrightarrows X_0,$$

where $X_n = X_1 \times_{X_0} X_1 \cdots \times_{X_0} X_1$ with n -copies of X_1 and the face maps $d_k : X_n \rightarrow X_{n-1}$ are given by $d_k(\gamma_1, \dots, \gamma_n) = (\gamma_1, \dots, \gamma_k \gamma_{k+1}, \dots, \gamma_n)$ for $k \in [1, n-1]$, $d_0(\gamma_1, \dots, \gamma_n) = (\gamma_2, \dots, \gamma_n)$ and $d_n(\gamma_1, \dots, \gamma_n) = (\gamma_1, \dots, \gamma_{n-1})$. There are also degeneracy maps in the structure of a simplicial manifold but we will omit them here since they are used only implicitly in our context.

A covering \mathcal{V} of a simplicial manifold X_\bullet is made up by open coverings $\mathcal{V}_n = \{V_{n,\alpha}\}$ for every manifold X_n compatible with the structure maps of the simplicial manifold. More precisely, we have $n+1$ maps ∂_k from the set of indices of \mathcal{V}_n to that of \mathcal{V}_{n-1} satisfying compatibility conditions as that of d_k 's. Moreover if $x \in V_{n,\alpha}$ then $d_k(x) \in V_{n-1,\partial_k(\alpha)}$. To better understand it, we give two examples.

Example A.2. [pull-back coverings] In the case when X_\bullet is the nerve of some groupoid $X_1 \rightrightarrows X_0$, a covering \mathcal{V} of X_\bullet can be induced by a covering of X_0 . Let $\mathcal{V}_0 = \{V_i\}$ be a covering of X_0 . We define the covering \mathcal{V}_n of X_n to be the collection of the following open sets,

$$(49) \quad V_{i_0 i_1 \dots i_n} := \{(\gamma_1, \gamma_2, \dots, \gamma_n) : \mathbf{t}(\gamma_1) \in V_{i_0}, \mathbf{s}(\gamma_1) \in V_{i_1}, \dots, \mathbf{s}(\gamma_n) \in V_{i_n}\}$$



It is not hard to see that \mathcal{V}_n is a covering of X_n and the map between indices $\partial_k(i_0 \dots i_n) = i_0 \dots \hat{i}_k \dots i_n$ makes \mathcal{V} all together a covering of X_\bullet . Moreover, $Y_n := \sqcup_{i_0 \dots i_n} V_{n, i_0 \dots i_n}$ is the nerve of the groupoid $Y_1 \rightrightarrows Y_0$. The structure maps of Y_\bullet are naturally inherited from X_\bullet . For example, $\mathbf{t} : (g, x) \mapsto x$ and $\mathbf{s} : (g, x) \mapsto g^{-1}x$. Moreover $Y_1 \rightrightarrows Y_0$ is Morita equivalent to $X_1 \rightrightarrows X_0$ (see also the proof of Proposition A.4).

Example A.3. [invariant coverings] In the case when X_\bullet is the nerve of some action groupoid $G \ltimes X_0 \rightrightarrows X_0$, there is another covering of X_\bullet induced by an invariant covering \mathcal{V}_0 of X_0 . By invariance, we mean that there is an action of G on the set of indices $\{a\}$, such that $gV_a := \{gx : x \in V_a\}$ is equal to V_{ga} . A covering \mathcal{V} of X_\bullet can be induced by a covering of X_0 . An element in X_n can be written as $(g_1, g_2, \dots, g_n, x)$ representing $((g_1, x), (g_2, g_1^{-1}x), \dots, (g_n, g_{n-1}^{-1} \cdots g_1^{-1}x))$ in $(G \ltimes X_0) \times_{X_0} \cdots \times_{X_0} (G \ltimes X_0)$. Then we define \mathcal{V}_n to be the collection of the following open sets,

$$V_{g_1, \dots, g_n, a} := \{(g_1, \dots, g_n, x) : x \in V_a\}.$$

It is easy to see that \mathcal{V}_n is a covering of X_n and the map

$$\partial_k(g_1, \dots, g_n, a) = \begin{cases} (g_2, \dots, g_n, g_1^{-1}a) & k = 0 \\ (g_1, \dots, g_k g_{k+1}, \dots, g_n, a) & 0 < k < n, \\ (g_1, \dots, g_{n-1}, a) & k = n, \end{cases}$$

altogether makes \mathcal{V} a covering of X_\bullet . We notice that the intersection $V_{g_1, \dots, g_n, a} \cap V_{h_1, \dots, h_n, b}$ is non-empty only if $(g_1, \dots, g_n) = (h_1, \dots, h_n)$ and then the intersection as a set equals to $V_a \cap V_b$. We also notice that this notion of covering works well in the case when G is a discrete group, but not for a general Lie group. Moreover, in the first case, such an invariant covering always exists. For example, we can take a covering of V_a of X , then translate it around and define V_{ga} as gV_a .

Given such a simplicial covering \mathcal{V} of X_\bullet , we can define a double complex

$$(50) \quad \begin{array}{ccccccc} & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \dots \\ & & \vdots & & \vdots & & \vdots & & \\ V_\bullet \cap V_\bullet \cap V_\bullet & C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \dots \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \dots \\ & \vdots & & \vdots & & \vdots & & \\ V_\bullet \cap V_\bullet & C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & \dots \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \dots \\ & \vdots & & \vdots & & \vdots & & \\ V_\bullet & C^{0,0} & \xrightarrow{\delta} & C^{1,0} & \xrightarrow{\delta} & C^{2,0} & \xrightarrow{\delta} & \dots \\ & X_0 & & X_1 & & X_2 & & \end{array}$$

where

$$(51) \quad C^{n,k} := \{\text{global invertible holomorphic functions on the } (k+1)\text{-intersections of } V_{n,\alpha}\}.$$

The columns are Čech cochains on X_n with covering \mathcal{V}_n and Čech differential $\check{\delta}$, and the rows are simplicial cochains, so that the horizontal differential is defined by

$$(52) \quad (\delta f)_{\cap_i V_{n,\alpha_i}} = \prod_{j=0}^n (d_j^*(f_{\cap_i V_{n-1,\partial_j(\alpha_i)}}))^{(-1)^{j+n}}.$$

As usual, the cohomology $H_{\mathcal{V}}^\bullet(X_\bullet, \mathcal{O}^\times)$ of the double complex is defined as the cohomology of the total complex $C^N = \bigoplus_{n+k=N} C^{n,k}$ with the total differential $D = \sum_{n+k=N} (\delta^{n,k} + (-1)^n (\check{\delta}^{n,k})) : C^N \rightarrow C^{N+1}$ and \mathcal{O}^\times is interpreted as a sheaf of additive groups with $+$ the multiplication. In fact, \mathcal{O}^\times can be replaced by any sheaf \mathcal{F} of abelian groups over the stack \mathcal{X} . In fact, \mathcal{F} induces sheaves \mathcal{F}^n on X_n for all n that automatically satisfies the expected compatibility conditions [7] 5.1.6, so that $\mathcal{F}^\bullet = (\mathcal{F}^n)_{n \geq 0}$ is a simplicial sheaf on the simplicial manifold X_\bullet . Then one replaces invertible holomorphic functions by sections of \mathcal{F}^n in the definition of $C^{n,k}$ and everything else remains the same.

The *sheaf cohomology* $H^i(X_\bullet, \mathcal{F}^\bullet)$ is defined as the derived functor of the invariant section functor $\Gamma_{inv}(X_\bullet, \mathcal{F}^\bullet) := \ker(\Gamma(X_0, \mathcal{F}^0) \rightrightarrows \Gamma(X_1, \mathcal{F}^1))$ ([7] 5.2). Here $\Gamma(\cdot)$ denotes the set of global sections. More precisely, we take an injective resolution $I^{\bullet,i}$ of the simplicial sheaf \mathcal{F}^\bullet , then $H^i(X_\bullet, \mathcal{F}^\bullet)$ is the i -th cohomology of the complex

$$0 \rightarrow \Gamma_{inv}(X_\bullet, I^{\bullet,0}) \rightarrow \Gamma_{inv}(X_\bullet, I^{\bullet,1}) \rightarrow \Gamma_{inv}(X_\bullet, I^{\bullet,2}) \rightarrow \dots$$

It can be also calculated via an acyclic resolution $K^{p,i}$ of \mathcal{F}^p , namely a resolution so that $H^{\geq 1}(X_p, K^{p,i}) = 0$. Then $H^*(X_\bullet, \mathcal{F}^\bullet)$ is the total cohomology of the complex $\Gamma(X_p, K^{p,q})$. As shown in [16], the sheaf cohomology is invariant under Morita

equivalence. We define $H^i(\mathcal{X}, \mathcal{F}) := H^i(X_\bullet, \mathcal{F}^\bullet)$ for an étale groupoid presentation $X_1 \rightrightarrows X_0$. For face maps $d_j^n : X_n \rightarrow X_{n-1}$ for $j = 0, \dots, n$ and degeneracy maps $s_i^n : X_n \rightarrow X_{n+1}$ for $i = 0, \dots, n$, we have $d_j^{n+1}s_i^n = s_{i-1}^{n-1}d_j^n$ for $0 \leq i \leq j-2$, $d_{i+1}^{n+1}s_i^n = id = d_i^{n+1}s_i^n$ for $0 \leq i \leq n$, and $d_j^{n+1}s_i^n = s_{i-1}^{n-1}d_j^n$ for $j+1 \leq i \leq n$. Therefore $\delta^n s^{n-1} + s^n \delta^{n+1} = 0$ for $n \geq 1$, where $s^n := \sum_i (-1)^i s_i^n$. So the cohomology groups H^i of the complex $\dots \xrightarrow{\delta} \Gamma(X_n, \mathcal{F}^n) \xrightarrow{\delta} \Gamma(X_{n+1}, \mathcal{F}^{n+1}) \dots$ vanish when $i \leq 1$ for any \mathcal{F}^\bullet . Hence, $H^i(X_\bullet, \mathcal{F}^\bullet)$ can be also calculated as the total cohomology of the double complex $\Gamma(X_p, I^{p,q})$.

Given a covering \mathcal{V} of \mathcal{X} , $C^q(\mathcal{V}_p, \mathcal{F}^p)$ is a sheaf, where the sections of $C^q(\mathcal{V}_p, \mathcal{F}^p)$ on some open set U are sections of \mathcal{F}^p on U restricted on $q+1$ -fold intersections of $V_{p,j}$'s. We call $C^q(\mathcal{V}_p, \mathcal{F}^p)$ the Čech resolution associated to a covering \mathcal{V} of X_\bullet with values in a sheaf \mathcal{F}^\bullet of abelian groups. In fact, we have the following complex of sheaves:

$$0 \rightarrow \mathcal{F}^p \rightarrow C^0(\mathcal{V}_p, \mathcal{F}^p) \rightarrow C^1(\mathcal{V}_p, \mathcal{F}^p) \rightarrow \dots$$

Notice that $H^i(X_p, C^q(\mathcal{V}_p, \mathcal{F}^p)) = H^i(\sqcup_{j=0, \dots, q} V_{p,j}, \mathcal{F}^p)$. The covering \mathcal{V} of X_\bullet is acyclic, that is $H^{\geq 1}(\cap_{j=0, \dots, q} V_{p,j}, \mathcal{F}^p) = 0$, for all p , iff $C^\bullet(\mathcal{V}, \mathcal{F}^\bullet)$ is an acyclic resolution of \mathcal{F}^\bullet , that is, $H^{\geq 1}(X_p, C^q(\mathcal{V}_p, \mathcal{F}^p)) = 0$ for all p . For constant sheaves such as \mathbb{Z} or \mathbb{C} , if on each level n the intersections $\cap_\alpha V_{n,\alpha}$ are contractible, then \mathcal{V} is acyclic. For \mathcal{O} (resp. \mathcal{O}^\times), one needs the intersections to be Stein (resp. Stein and contractible) to guarantee being acyclic.

Using the Čech resolution $C^\bullet(\mathcal{V}_p, \mathcal{F}^p)$ for \mathcal{F}^p , the total cohomology of the double complex $\Gamma(X_p, C^q(\mathcal{V}_p, \mathcal{F}^p))$ is exactly $H_V^i(X_\bullet, \mathcal{F}^\bullet)$. Since every resolution of \mathcal{F}^\bullet maps to an injective resolution of \mathcal{F}^\bullet , we have the induced map $H_V^i(X_\bullet, \mathcal{F}^\bullet) \rightarrow H^i(X_\bullet, \mathcal{F}^\bullet)$. Moreover, this map is an isomorphism, i.e. $H_V^i(X_\bullet, \mathcal{F}^\bullet) \cong H^i(X_\bullet, \mathcal{F}^\bullet)$ when \mathcal{V} is acyclic since $C^q(\mathcal{V}, \mathcal{F}^p)$ is an acyclic resolution of \mathcal{F}^p . The coverings \mathcal{V} 's of X_\bullet form a direct system by defining $\mathcal{V} \prec \mathcal{U}$ (\mathcal{V} is finer than \mathcal{U}) if $\mathcal{V}_n \prec \mathcal{U}_n$ (that is for all $V_{n,\alpha} \in \mathcal{V}_n$ there is a $U_{n,\beta} \in \mathcal{U}_n$ such that $V_{n,\alpha} \subset U_{n,\beta}$) for all n and the maps $\alpha \rightarrow \beta$ are compatible with the facial maps. Then the map $\varinjlim H_V^i(X_\bullet, \mathcal{F}^\bullet) \rightarrow H^i(X_\bullet, \mathcal{F}^\bullet)$ is an isomorphism when an acyclic covering exists⁴. Therefore, we can use Čech cohomology to interpret $H^i(\mathcal{X}, \mathcal{F})$.

A.3. Geometric objects for H^1 and H^2 and Dixmier–Douady class. A holomorphic \mathbb{C}^\times -principal bundle on an étale complex stack \mathcal{X} is defined as an epimorphism of étale complex stacks $\mathcal{L} \rightarrow \mathcal{X}$ such that for any étale chart $X \rightarrow \mathcal{X}$ the pull-back of \mathcal{L} on X , namely $X \times_{\mathcal{X}} \mathcal{L}$, is a holomorphic \mathbb{C}^\times -principal bundle on X . Let $X_1 = X \times_{\mathcal{X}} X$ and $X_0 = X$. The holomorphic \mathbb{C}^\times -principal bundles on \mathcal{X} are in 1-1 correspondence with the X_1 equivariant holomorphic \mathbb{C}^\times -principal bundles on X_0 . See [3] for details. We will prove a similar statement for gerbes in Proposition A.5.

A holomorphic \mathbb{C}^\times -principal bundle on a stack \mathcal{X} is determined by a class in $H^1(\mathcal{X}, \mathcal{O}^\times)$. We choose a groupoid presentation of \mathcal{X} and a covering of the nerve X_\bullet of the groupoid. Suppose we have a cocycle $(\eta, \xi) \in C^{0,1} \oplus C^{1,0}$ in the double

⁴One could drop the condition of existence of an acyclic coverings, see [27], but this condition always holds in the cases we consider in the main part of the paper.

complex. Then $\check{\delta}(\eta) = 1$, $\check{\delta}(\xi) = \delta(\eta)$ and $\delta(\xi) = 1$. So η corresponds to a \mathbb{C}^\times -principal bundle L_0 on X_0 . We define an X_1 action on it by

$$\underbrace{(x, \lambda)}_{V_{0,i}} \cdot \underbrace{\gamma}_{V_{1,\alpha}} = \underbrace{(\mathbf{s}(\gamma), \lambda \xi_\alpha(\gamma))}_{V_{0,\partial_1(\alpha)}},$$

where $\partial_0 \alpha = i$. One can always choose such α for a γ since $d_0(\gamma) = x$. Then $\check{\delta}(\xi) = \delta(\eta)$ ensures that the action is independent of choice of the charts of (x, λ) and of γ . Moreover $\delta(\xi) = 1$ ensures that what we define is indeed an action. Moreover the action obviously commutes with \mathbb{C}^\times multiplication. Therefore L_0 is an X_1 equivariant \mathbb{C}^\times -bundle over X_0 which corresponds to a \mathbb{C}^\times -bundle \mathcal{L} on \mathcal{X} . This construction works for a general covering, in particular for the two coverings mentioned in the examples.

On the other hand, a \mathbb{C}^\times -bundle \mathcal{L} on \mathcal{X} gives an X_1 -equivariant line bundle L_0 on X_0 for any groupoid presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} . We choose a covering on X_0 , where L_0 trivializes. This covering generates a covering \mathcal{V} of the nerve of the groupoid X_\bullet as in Example A.2. Then the transition functions of L_0 give $\eta \in C^{0,1}$ and the X_1 action gives $\xi \in C^{1,0}$ as described above. Then the cohomology class of (η, ξ) in $H^1_{\mathcal{V}}(X_\bullet, \mathcal{O}^\times)$ converges to an element in $H^1(\mathcal{X}, \mathcal{O}^\times)$ and that is the class corresponding to this equivariant \mathbb{C}^\times -bundle.

Recall that a local section of any \mathbb{C}^\times -bundle L over an open set U of the base manifold X is a function θ_0 defined on $\sqcup(V_i \cap U)$ such that $\check{\delta}\theta_0 = \eta$, for a covering $\{V_i\}$ of X . A *local section* of \mathcal{L} is a function θ defined on an open subset of $\sqcup V_{0,i}$ such that $\check{\delta}\theta = \eta$ and $\delta\theta = \xi$, namely $D(\theta) = (\eta, \xi)$.

It is not hard to see that the entire construction applies to holomorphic line bundles over stacks. In fact, holomorphic line bundles are equivalent to holomorphic \mathbb{C}^\times -principal bundles.

Proposition A.4. *A holomorphic \mathbb{C}^\times -gerbe on a stack \mathcal{X} is characterized by a class in $H^2(\mathcal{X}, \mathcal{O}^\times)$.*

Proof. Take a groupoid presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} and a covering \mathcal{V}_\bullet in the form of Example A.2 of the nerve X_\bullet . To be more consistent with other part of the paper and reduce the usage of notations, we assume $X_1 \rightrightarrows X_0$ is an action groupoid, but the whole proof works for general stacks. Let $(\Phi, \Theta, \Psi) \in C^{0,2} \oplus C^{1,1} \oplus C^{2,0}$ be a 2-cocycle representing the class in $H^2(\mathcal{X}, \mathcal{O}^\times)$. We have,

$$(53) \quad \check{\delta}\Phi = 1, \quad \delta\Phi = \check{\delta}\Theta, \quad \delta\Theta \cdot \check{\delta}\Psi = 1, \quad \delta\Psi = 1.$$

Let $Z_1 := \sqcup(V_i \cap V_j)$ and $Y_1 := \sqcup V_{ij}$ and $U_0 := \sqcup V_i$. Then $Y_1 \rightrightarrows U_0$ is a groupoid Morita equivalent to $X_1 \rightrightarrows X_0$ via the Morita bibundle $U_0 \times_{pr, X_0, t} X_1$. In fact, $(Y_1 = U_0 \times_{X_0} X_1 \times_{X_0} U_0) \rightrightarrows U_0$ is the pull-back groupoid by the covering map $U_0 \rightarrow X_0$. Moreover, Φ and Θ are determined by Ψ via the following equations,

$$(54) \quad \Phi_{i,j,k}(y) = \Psi_{ijk}(1, 1; y)^{-1}, \quad \Theta_{ij,i'j'}(g; y) = \Psi_{ijj'}(g, 1; y) \Psi_{ii'j'}(1, g; y)^{-1}.$$

There is a trivial \mathbb{C}^\times -bundle L_Θ on Y_1 given by the cocycle $\Theta_{ij,i'j'}^{-1} \Theta_{ij,i''j''}$ (namely with $\Theta_{ij,i'j'}$ as the trivialisation function) with respect to the covering $V_{ij} \cap V_{i'j'}$ of V_{ij} , where $\Theta_{ij,i'j'}$ is defined on $V_{ij} \cap V_{i'j'}$. Then the third and fourth equation of (53) ensure that $L_\Theta \rightrightarrows U_0$ is a \mathbb{C}^\times -groupoid central extension of $Y_1 \rightrightarrows U_0$ with the multiplication on L_Θ

$$(55) \quad (g, y, \lambda) \cdot (h, g^{-1}y, \mu) = (gh, y, \lambda\mu\Psi_{ijk}(g, h; y)),$$

for global sections (g, y, λ) (appearing as $(g, y, \lambda\Theta_{ij,i'j'})$ on $V_{ij} \cap V_{i'j'}$) over V_{ij} and $(h, g^{-1}y, \mu)$ over V_{jk} . Therefore this central extension gives us the \mathbb{C}^\times -gerbe \mathcal{G} on \mathcal{X} .

It is not surprising by (54) that the first and second conditions in (53) are not used. More precisely, $\check{\delta}\Phi = 1$ implies that Φ gives a \mathbb{C}^\times -gerbe \mathcal{G}_Φ on X_0 . In particular, Φ gives a \mathbb{C}^\times -bundle L_Φ on Z_1 by the 1-cocycle $\Phi_{ikk'}\Phi_{jkk'}^{-1} = \Phi_{ijk'}\Phi_{ijk}^{-1}$ (namely with Φ_{ijk} as the trivialisation function) with respect to the covering $(V_i \cap V_j) \cap V_k$ on each $V_i \cap V_j$. Notice that $i : Z_1 \rightarrow Y_1$ by $y \in V_i \cap V_j$ mapping to $(1; y) \in V_{ij}$ is an embedding. The second equation of (53) tells us that $i^*L_\Theta = L_\Phi$.

Now we show that different cocycles representing the same cohomology class give the same gerbe. If we have another cocycle representing the same cohomology class as (Φ, Θ, Ψ) possibly on a different covering of a different groupoid, then the corresponding $Y'_1 \rightrightarrows U'_0$ however is still Morita equivalent to $Y_1 \rightrightarrows U_0$ via some Morita bibundle which we call U''_0 . The pull-back groupoid $Y'_1 \rightrightarrows U''_0$ via $U''_0 \rightarrow U_0$ viewed as the common refinement of the two coverings is Morita equivalent to both $Y_1 \rightrightarrows U_0$ and $Y' \rightrightarrows U'_0$ via groupoid morphisms. (The conditions on Morita bibundle ensure that the pull-backs via $U''_0 \rightarrow U_0$ and $U'_0 \rightarrow U_0$ give the same groupoid). Notice that the pull-back groupoid of $R \rightrightarrows U_0$ on $Y'' \rightrightarrows U''_0$ is Morita equivalent to $R \rightrightarrows U_0$, and (Φ, Θ, Ψ) converges to the same cohomology class as its pull-back cocycle on the covering generated by U''_0 . We conclude that we only have to show the statement for two classes on the same covering of the same groupoid, namely if (Φ, Θ, Ψ) and (Φ', Θ', Ψ') differ by a coboundary, which says

$$(56) \quad \Phi = \Phi' \cdot \check{\delta}\eta, \quad \Theta = \Theta' \cdot \delta\eta \cdot (\check{\delta}\xi)^{-1}, \quad \Psi = \Psi' \cdot \delta\xi,$$

then their corresponding central extensions are isomorphic. We construct an isomorphism $f : L_\Theta \rightarrow L'_\Theta$ by defining it on the local charts $V_{ij} \cap V_{i'j'} \times \mathbb{C}^\times \rightarrow V_{ij} \cap V_{i'j'} \times \mathbb{C}^\times$, with

$$(g, y, \lambda) \mapsto (g, y, \lambda \cdot \frac{\eta_{ii'}(y)}{\eta_{jj'}(g^{-1}x)\xi_{i'j'}(g; x)}).$$

The second equation in (56) guarantees that they glue to a global map.

In local charts, the trivialisation of L_Θ on $V_{ij} \cap V_{i'j'}$ is given by function $\Theta_{ij,i'j'}$. Using the third condition in (53) and (55), the multiplication of L_Θ written in local trivialisation is

$$(g, y, \lambda) \cdot (h, g^{-1}y, \mu) = (gh, y, \lambda\mu\Psi_{i'j'k'}(g, h; y)),$$

for $(g, y) \in V_{ij} \cap V_{i'j'}$ and $(h, g^{-1}y) \in V_{jk} \cap V_{j'k'}$. From this it is easy to see that f is a groupoid homomorphism. It is not hard to conclude that f is an isomorphism of central extensions.

For the converse, a \mathbb{C}^\times -gerbe \mathcal{G} over the stack \mathcal{X} can be viewed as a \mathbb{C}^\times -central extension of a groupoid $X_1 \rightrightarrows X_0$ presenting \mathcal{X} . Take a covering V_i of X_0 . Then it generates a covering \mathcal{V} of the simplicial manifold X_\bullet as in Example A.2. By using the same notation as before, $Y_1 \rightrightarrows U_0$ is the pull-back groupoid also presenting \mathcal{X} , so the central extension can be pulled back to this new groupoid and we call it $L \rightrightarrows U_0$. Then $d_0^*L \otimes d_1^*L^{-1} \otimes d_2^*L \cong 1$ via a trivialisation function Ψ_{ijk} over $Y_1 \times_{U_0} Y_1$, recalling that $Y_1 \times_{U_0} Y_1 = \sqcup V_{ijk}$ with $V_{ijk} = \{(g, h, y) : y \in V_i, g^{-1}y \in V_j, (gh)^{-1}y \in V_k\}$. By (54), Ψ_{ijk} determines a class $[(\Phi, \Theta, \Psi)]$ in $H_{\mathcal{V}}^2(Y_\bullet, \mathcal{O}^\times)$ and it converges to a class in $H^2(\mathcal{X}, \mathcal{O}^\times)$. Moreover, if we have two isomorphic \mathbb{C}^\times -central extension R and R' of $X_1 \rightrightarrows X_0$ given by isomorphisms $f_{ij,i'j'}$ locally defined

on the piece $V_{ij} \cap V_{i'j'}$, then their corresponding cocycles differ by a coboundary $D(\eta, \xi)$ with $\eta_{ij}(y) = f_{ij,ij}(1, y)$ and $\xi_{ij}(g, y) = f_{ij,ij}(g, y)$. \square

As the proof above shows, $L_\Theta|_{Z_0} \cong L_\Phi$. Therefore the gerbe given by L_Θ can be viewed as giving a G -equivariant structure to the gerbe \mathcal{G}_Φ on X_0 given by L_Φ .

Using the short exact sequence of sheaves,

$$\mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp 2\pi i} \mathcal{O}^\times,$$

we will have a long exact sequence of cohomology groups. In particular, we have $H^2(\mathcal{X}, \mathcal{O}^\times) \rightarrow H^3(\mathcal{X}, \mathbb{Z})$. According to the above proposition, a gerbe \mathcal{G} over \mathcal{X} corresponds to a class in $H^2(\mathcal{X}, \mathcal{O}^\times)$. The image of this class in $H^3(\mathcal{X}, \mathbb{Z})$ is called the *Dixmier–Douady class* of \mathcal{G} .

A.4. Equivariant gerbes and gerbes over stacks. In this subsection, we relate the two sorts of gerbes: the one described in the introduction and the one in this section. Let a group G act on a manifold X . Then we have the action groupoid $G \times X \rightrightarrows X$. The *equivariant cohomology* $H_G^i(X, \mathcal{F})$ is defined as the cohomology of the nerve of the action groupoid (an equivariant sheaf \mathcal{F} on X induces a simplicial sheaf \mathcal{F}^\bullet on the nerve of the action groupoid). Therefore we have $H_G^i(X, \mathcal{F}) = H^i([X/G], \mathcal{F})$, where $[X/G]$ is the differentiable stack presented by $G \times X \rightrightarrows X$. Moreover, an invariant covering \mathcal{V} of X induces a covering (still denoted by \mathcal{V}) of the nerve $G^\bullet \times X$ as in Example A.3. Then as in Section A.2, we have a map $H_{\mathcal{V}}^i(G^\bullet \times X, \mathcal{F}) \rightarrow H^i([X/G], \mathcal{F})$. It is an isomorphism when \mathcal{V} is an acyclic covering, which is true in our cases. Recall that a 2-cocycle $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$ representing a class in $H_{\mathcal{V}}^2(G^\bullet \times X, \mathcal{O}^\times)$ corresponds to an equivariant holomorphic bundle gerbe as described in the introduction. Under $H_{\mathcal{V}}^2(G^\bullet \times X, \mathcal{O}^\times) \rightarrow H^2([X/G], \mathcal{O}^\times)$, the image of $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$ corresponds to a \mathbb{C}^\times -gerbe on $[X/G]$. In the following proposition, we construct this gerbe on $[X/G]$ in a concrete geometrical way from $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$.

Proposition A.5. *If we use an invariant covering as in Example A.3 for an action groupoid $X_1 := G \times X$ over $X_0 := X$, then an equivariant holomorphic bundle gerbe on X defined by a cocycle $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$ corresponds to a holomorphic \mathbb{C}^\times -gerbe on $[X/G]$ given as a groupoid central extension.*

Proof. Take a 2-cocycle $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$ representing a class in $H_{\mathcal{V}}^2(G^\bullet \times X, \mathcal{O}^\times)$. Then $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$ satisfies the conditions

$$(57) \quad \check{\delta}(\phi_{\dots}) = 1, \quad \delta(\phi_{\dots}) = \check{\delta}(\phi_{\dots}), \quad \delta(\phi_{\dots}) = \check{\delta}(\phi_{\dots})^{-1}, \quad \delta(\phi_{\dots}) = 1.$$

Denote by (V_a) the covering of X that generates the invariant covering \mathcal{V} as in Example A.3. Let $U_0 = \sqcup V_a$, $Y_1 := \sqcup V_{g,a}$ and $Z_1 := \sqcup (V_a \cap V_b)$. The groupoid $Y_1 \rightrightarrows U_0$ is not Morita equivalent to $X_1 \rightrightarrows X_0$ any more. But the groupoid “generated” by Y_1 and Z_1 is the pull-back groupoid $U_1 := U_0 \times_{X_0} X_1 \times_{X_0} U_0$, therefore Morita equivalent to $X_1 \rightrightarrows X_0$. More precisely, Y_1 and Z_1 are subgroupoids of U_1 over the same base U_0 . For an element $y \in V_a \cap V_b$ in Z_1 and an element $(g, y) \in V_{g,b}$ in Y_1 , the multiplication is $y \cdot (g, y) = (g, y) \in W_{g,a,g^{-1}b}$, where $W_{g,a,g^{-1}b} = \{(g, y), y \in V_a, g^{-1}y \in V_{g^{-1}b}\}$. As in the case of equivariant gerbes, there is a trivial \mathbb{C}^\times -bundle L_{11} on Y_1 given by the trivialisation function $\phi_{a,a'}(g, -)$ with the covering $V_{g,a} \cap V_{g,a'}$ of $V_{g,a}$, and a trivial \mathbb{C}^\times -bundle L_{20} on Z_1 given by the trivialisation function $\phi_{a,b,c}$ with the covering $(V_a \cap V_b) \cap V_c$ of $V_a \cap V_b$. The corresponding holomorphic line bundles of L_{11} and L_{20} are exactly what we called $L_a(g)$ and L_{ab} before. Notice

that $U_1 = \sqcup_{g,a,g^{-1}b} W_{g,a,g^{-1}b}$. We define a \mathbb{C}^\times -bundle on U_1 piece by piece, namely by $L|_{W_{g,a,g^{-1}b}} := \iota_{a,b}^* L_{20} \otimes \iota_{g,b}^* L_{11}^{-1}$, where $\iota_{a,b} : W_{g,a,g^{-1}b} \rightarrow V_a \cap V_b$, by $(g, y) \mapsto y$ and $\iota_{g,b} : W_{g,a,g^{-1}b} \rightarrow V_{g,b}$, by $(g, y) \mapsto (g, y)$, where the second piece has to be understood as an element of $V_{g,b}$. Then equation (57) ensures that $L \rightrightarrows U_0$ is a \mathbb{C}^\times -central extension of $U_1 \rightrightarrows U_0$. More precisely, L being a \mathbb{C}^\times -central extension is equivalent to that $L|_{U_0} \cong \mathbb{C}^\times \times U_0$,

$$(58) \quad d_0^* L \otimes d_1^* L^{-1} \otimes d_2^* L \cong 1$$

on $U_2 := U_1 \times_{U_0} U_1$, and the isomorphisms satisfy further coherence condition, where $d_i : U_2 \rightarrow U_1$ are the facial maps. Below, we verify these conditions one by one. Since $\phi_{a,a,c}(y) = 1$ and $\phi_{a,a'}(1; y) = 1$ by requirements of the equivariant cocycles, we have $L|_{U_0} = 1$. For (58), we notice that the second (resp. the first and third) condition in (57) says that $\phi_{b,c}(g; y)$ (resp. $\phi_{a,b,c}^{-1}(y)\phi_c(g, h; y)$) gives the first (resp. second) isomorphism “ \cong ” in the following chain of isomorphisms,

$$\begin{aligned} d_0^* L \otimes d_2^* L &= (\iota_{a,b}^* L_{20} \otimes \iota_{g,b}^* L_{11}^{-1}) \otimes (\iota_{g^{-1}b,g^{-1}c}^* L_{20} \otimes \iota_{h,g^{-1}c}^* L_{11}^{-1}) \\ &= \iota_{a,b}^* L_{20} \otimes (\iota_{g,b}^* L_{11}^{-1} \otimes \iota_{g^{-1}b,g^{-1}c}^* L_{20}) \otimes \iota_{h,g^{-1}c}^* L_{11}^{-1} \\ &\cong \iota_{a,b}^* L_{20} \otimes (\iota_{b,c}^* L_{20} \otimes \iota_{g,c}^* L_{11}^{-1}) \otimes \iota_{h,g^{-1}c}^* L_{11}^{-1} \\ &= (\iota_{a,b}^* L_{20} \otimes \iota_{b,c}^* L_{20}) \otimes (\iota_{g,c}^* L_{11}^{-1} \otimes \iota_{h,g^{-1}c}^* L_{11}^{-1}) \\ &\cong \iota_{a,c}^* L_{20} \otimes \iota_{gh,c}^* L_{11}^{-1} = d_1^* L, \end{aligned}$$

where we omit certain pull-backs for simplicity. Therefore we only have to check that the higher coherence, namely

$$\Psi_{a,g^{-1}b,(gh)^{-1}c}(g, h; y) := (\phi_{a,b,c}^{-1}(y)\phi_{b,c}(g; y)\phi_a(g, h; y))$$

satisfies $\delta(\Psi)_{a,g^{-1}b,(gh)^{-1}c,(ghf)^{-1}d}(g, h, f; y) = 1$ on the triple fold $W_{g,a,g^{-1}b} \cap W_{h,g^{-1}b,(gh)^{-1}c} \cap W_{f,(gh)^{-1}c,(ghf)^{-1}d}$. It follows from (57) and the following calculation:

$$\begin{aligned} &\delta(\Psi)_{a,g^{-1}b,(gh)^{-1}c,(ghf)^{-1}d}(g, h, f; y) \\ &= (\delta\phi_{\dots})_{b,c,d}^{-1}(g; y)(\check{\delta}\phi_{\dots})_{c,d}(g, h; y)(\delta\phi_{\dots})_{c,d}^{-1}(g, h; y)(\check{\delta}\phi_{\dots})_{b,c,d}(g; y). \end{aligned}$$

Therefore L is a \mathbb{C}^\times -central extension and gives a \mathbb{C}^\times -gerbe \mathcal{G} on \mathcal{X} .

Finally, we notice that $V_a, W_{g,a,g^{-1}b}$ etc. actually gives a covering as in Example A.2, which is a refinement of the invariant covering. From the above construction, the cocycle corresponds to the constructed gerbe \mathcal{G} is (Φ, Θ, Ψ) with Φ and Θ determined by Ψ as in (54). It is easy to see that this new cocycle restricting on the invariant covering is exactly $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$. Therefore the \mathbb{C}^\times -gerbe \mathcal{G} corresponds to the cohomology class in $H^2(\mathcal{X}, \mathcal{O}^\times)$ represented by $(\phi_{a,b,c}, \phi_{a,b}, \phi_a)$. \square

A.5. Equivariant cohomology and group cohomology. Since we use the definition of equivariant cohomology with action groupoid instead of with a classifying space EG , we supply the proof in this setting of a basic fact of equivariant cohomology we have used.

Theorem A.6. *Let G be a discrete group acting (not necessarily freely) on a manifold X . Then for a sheaf \mathcal{F} on the stack $[X/G]$, namely an equivariant sheaf \mathcal{F} on X , there is a spectral sequence*

$$E_2^{p,q} = H^p(G, H^q(X, \mathcal{F})) \Rightarrow H_G^{p+q}(X, \mathcal{F}) = H^{p+q}([X/G], \mathcal{F}).$$

Proof. An equivariant sheaf \mathcal{F} on X is a sheaf with isomorphisms of sheaves $\phi_g : g_*\mathcal{F} \rightarrow \mathcal{F}$ such that $\phi_g\phi_h = \phi_{gh}$. We take an acyclic equivariant resolution K^q of \mathcal{F} . There always exists one (for example one could use Godement's resolution). Then K^q generate a simplicial sheaf $K^{\bullet,q}$ on $G^\bullet \times X$. We form a spectral sequence associated with the double complex $\Gamma(G^p \times X, K^{p,q})$ of maps $G^p \rightarrow \Gamma(X, K^q)$, namely $E_0^{p,q} := \Gamma(G^p \times X, K^{p,q})$, filtered by $\oplus_{p \geq n} E_0^{p,q}$ (in the direction of p). On one hand, the double complex calculates the sheaf cohomology $H^i([X/G], \mathcal{F})$. On the other hand, there is a G action on $H^i(X, \mathcal{F})$ induced by the G action on X . In fact, $E_1^{p,q} = C^p(G, H^i(X, \mathcal{F}))$ since K^q is acyclic. Hence the first page $(E_1^{p,q}, d_1)$ is exactly the complex $(C^p(G, H^i(X, \mathcal{F})), \delta_G)$ to calculate the group cohomology. Therefore $E_2^{p,q} = H^p(G, H^q(X, \mathcal{F}))$. \square

A.6. Hypercohomology on stacks. Hypercohomology of complexes of sheaves on manifolds and action groupoids is explicitly studied in [5], [14]. Here we extend it to the category of complex stacks and relate it to our construction using invariant coverings. Our treatment is similar to the one in the references.

We first discuss the concept of hypercohomology of a complex of sheaves on a complex stack. Take a stack \mathcal{X} , and a complex of sheaves of abelian groups

$$(59) \quad \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \dots \rightarrow \mathcal{F}_m$$

over \mathcal{X} . Take a presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} and form the nerve X_\bullet . Then we naturally have a complex of simplicial sheaves

$$(60) \quad \mathcal{F}_0^\bullet \rightarrow \mathcal{F}_1^\bullet \rightarrow \dots \rightarrow \mathcal{F}_m^\bullet,$$

in the sense of Section 3.2 in [14]. Following this reference⁵, we take an injective resolution of (60), which looks like a triple complex with one direction a complex of injective sheaves $I_0^{p,q} \rightarrow I_1^{p,q} \rightarrow \dots \rightarrow I_n^{p,q}$ and one direction an injective resolution $I_r^{p,\bullet}$ of \mathcal{F}_r^p . Furthermore, the maps satisfy similar compatibility conditions as in [5] 1.2.6., where the author defines an injective resolution for a complex of sheaves over a manifold. We define the *hypercohomology* $H^i(X_\bullet, \mathcal{F}_0^\bullet \rightarrow \dots \rightarrow \mathcal{F}_m^\bullet)$ as the total cohomology of the double complex

$$(61) \quad C^{q,r} := \Gamma_{inv}(X_\bullet, I_r^{\bullet,q}).$$

In fact it is also the total cohomology of the triple complex

$$(62) \quad C^{p,q,r} := \Gamma(X_p, I_r^{p,q}).$$

This is so as in the case of sheaf cohomology: we filter $C^{p,q,r}$ by $\oplus_{q+r \geq n} C^{p,q,r}$, then we get the E_1 page with a lot of zeros except $C^{q,r}$. We take the spectral sequence with $E_0^{q,r} := C^{q,r} = \Gamma_{inv}(X_\bullet, I_r^{\bullet,q})$ and filtered by $\oplus_{r \geq n} C^{q,r}$. Then the first page $E_1^{q,r} = H^q(X_\bullet, \mathcal{F}_r^\bullet) = H^q(\mathcal{X}, \mathcal{F}_r)$ which is independent of the choice of groupoid presentation of \mathcal{X} . Hence the hypercohomology $H^i(X_\bullet, \mathcal{F}_0^\bullet \rightarrow \dots \rightarrow \mathcal{F}_m^\bullet)$ is independent of the choice of the groupoid presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} . Therefore, we define the *hypercohomology of a stack* \mathcal{X} as $H^i(\mathcal{X}, \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \dots \rightarrow \mathcal{F}_m) := H^i(X_\bullet, \mathcal{F}_0^\bullet \rightarrow \dots \rightarrow \mathcal{F}_m^\bullet)$ for some groupoid presentation $X_1 \rightrightarrows X_0$ of \mathcal{X} .

Take a simplicial open covering \mathcal{V} of X_\bullet —the nerve of the groupoid $X_1 \rightrightarrows X_0$ presenting \mathcal{X} . Then we let

$$(63) \quad \check{C}^{p,q,r} = \Gamma(\cap_{i=0}^q V_{p,\alpha_i}, \mathcal{F}_r^p).$$

⁵Notice that there are enough injective objects in the category of sheaves of abelian groups over stacks, as explained in [15].

Then we have $\delta : \check{C}^{p,q,r} \rightarrow \check{C}^{p+1,q,r}$ defined as in (52) and $\check{\delta} : \check{C}^{p,q,r} \rightarrow \check{C}^{p,q,r+1}$ the Čech differential. They are defined exactly as the δ and $\check{\delta}$ in the double complex (50). Moreover, there is another differential $d : \check{C}^{p,q,r} \rightarrow \check{C}^{p,q,r+1}$, induced by the differential $\mathcal{F}_r \rightarrow \mathcal{F}_{r+1}$ in the sheaf complex. Then $(\check{C}^{p,q,r}, \delta, \check{\delta}, d)$ is a triple complex and the total complex is $\check{C}^N := \oplus_{N=p+q+r} \check{C}^{p,q,r}$ with the total differential $D_3 = \delta + (-1)^q \check{\delta} + (-1)^{q+r} d$. We define the Čech hypercohomology $\check{H}_{\mathcal{V}}^{\bullet}(X_{\bullet}, \mathcal{F}_0^{\bullet} \rightarrow \dots \rightarrow \mathcal{F}_m^{\bullet})$ with respect to the covering \mathcal{V} as the total cohomology of this triple complex.

Then we have the following proposition which generalizes the case of the relation between Čech cohomology and sheaf cohomology.

Proposition A.7. *In the above setup, there is a map*

$$\check{H}_{\mathcal{V}}^n(X_{\bullet}, \mathcal{F}_0^{\bullet} \rightarrow \dots \rightarrow \mathcal{F}_m^{\bullet}) \rightarrow H^n(\mathcal{X}, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m).$$

In particular, when \mathcal{V} is an acyclic covering of \mathcal{X} , $\check{H}_{\mathcal{V}}^n(\mathcal{X}, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m) = H^n(\mathcal{X}, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)$.

Proof. Take the Čech resolution $C^q(\mathcal{V}_p, \mathcal{F}_r^p)$ of a covering \mathcal{V} of X_{\bullet} . Then the triple complex in (63), $\check{C}^{p,q,r}$ is $\Gamma(X_p, C^q(\mathcal{V}_p, \mathcal{F}_r^p))$. Since every resolution maps to an injective resolution, we arrive at a morphism of triple complexes (63) and (62): $\check{C}^{p,q,r} \rightarrow C^{p,q,r}$. Therefore there is a map on the level of cohomology.

Notice that the cohomology of the triple complex $\check{C}^{p,q,r}$ can be also calculated as the cohomology of the double complex $\check{C}^{n,r} := \oplus_{p+q=n} \check{C}^{p,q,r}$ with one direction $D : \check{C}^{n,r} \rightarrow \check{C}^{n+1,r}$ the total differential in (50) with sheaf \mathcal{F}_r and the other direction $d : \mathcal{F}_r \rightarrow \mathcal{F}_{r+1}$. Moreover, the hypercohomology $H^n(\mathcal{X}, \mathcal{F}_r)$ of stack \mathcal{X} can be calculated by the double complex $C^{q,r}$ in (61). The first pages $E_1^{n,r}$ of \check{C}^{\bullet} and C^{\bullet} are $H_{\mathcal{V}}^n(\mathcal{X}, \mathcal{F}_r) = H^n(\mathcal{X}, \mathcal{F}_r)$ respectively when \mathcal{V} is a acyclic covering of \mathcal{X} . Therefore we immediately have $\check{H}_{\mathcal{V}}^n(\mathcal{X}, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m) = H^n(\mathcal{X}, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)$. \square

For the stack $\mathcal{X} = [X/G]$, we have a result relating hypercohomology of stacks and group cohomology as in Section A.5.

Proposition A.8. *The hypercohomology $H^{\bullet}([X/G], \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)$ can be calculated via a spectral sequence with E_2 page*

$$E_2^{p,q} := H^p(G, H^q(X, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)).$$

Proof. We also denote \mathcal{F}_r the equivariant sheaf on X . Then it has an acyclic equivariant resolution K_r^q (for example Godement's construction). Then K_r^q generates the simplicial sheaves $K_r^{p,q}$ on $G^p \times X$. Consider the triple complex $C^{p,q,r} := \Gamma(G^p \times X, K_r^{p,q})$ the maps $G^p \rightarrow \Gamma(X, K_r^q)$. Then if we filter by r , namely, we form the double complex $C^{n,r} := \oplus_{p+q=n} C^{p,q,r}$ and filtered in the direction of r . Then we arrive at the first page $H^n(G^{\bullet} \times X, \mathcal{F}_r^{\bullet})$ which is the same as the first page of (61). Therefore this triple complex has the total cohomology $H^{\bullet}([X/G], \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)$. In particular, the double complex $\Gamma(X, K_r^q)$ has total cohomology $H^i(X, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)$. This tells us that if we filter $C^{p,q,r}$ by p first, we arrive at the first page $E_1^{p,n} = C^p(G, H^n(X, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m))$. Then $(E_1^{p,n}, d_1)$ is exactly the complex $(C^p(G, H^n(X, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)), \delta_G)$. Therefore, $E_2^{p,n} = H^p(G, H^n(X, \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m))$ and it converges to $H^{\bullet}([X/G], \mathcal{F}_0 \rightarrow \dots \rightarrow \mathcal{F}_m)$. \square

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