

On the energy of asymptotically Horowitz-Myers metrics^a

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We present results supporting the Horowitz-Myers conjecture, that the Horowitz-Myers metrics minimise energy amongst metrics on the same spacetime manifold and with the same asymptotic behaviour.

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I. INTRODUCTION

The question of positivity of the total mass in general relativity has played a central role in many investigations in mathematical and theoretical physics, with clear implications for stability, at least at a heuristic level, but also to issues such as the classification of static black hole solutions of Einstein equations [7, 9] or the resolution of the Yamabe problem [21].

The standard setting in this context is that of asymptotically Minkowskian spacetimes, but other asymptotic conditions are also of interest. For instance, Asymptotically Locally Hyperbolic (ALH) metrics have found a prominent place in models of theoretical physics (cf., e.g., [19, 24] and references therein). These models display a fascinating interplay between the positivity of energy and topology: Solutions with spherical conformal infinity are known to have positive mass [11, 14, 25]; some classes of solutions with toroidal infinity are known to have positive mass [25] while some classes of static solutions with toroidal or higher-genus infinity necessarily have negative mass [18, 22]. The fact that a toroidal conformal geometry at infinity can be filled-in by a conformally compact filling in many ways [2] adds further ambiguities to the subject.

The Horowitz-Myers (HM) solutions of vacuum Einstein equations with a negative cosmological constant, also known as AdS solitons, provide an interesting example of ALH strictly static metrics with total mass which is negative when compared with that of a locally maximally symmetric hyperbolic metric with the same toroidal conformal geometry at infinity. The $(n+1)$ -dimensional spacetimes manifolds underlying the HM metrics are $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}^{n-2}$, where \mathbb{T}^{n-2} denotes an $(n-2)$ -dimensional torus, with a conformal boundary at infinity diffeomorphic to $\mathbb{R} \times S^1 \times \mathbb{T}^{n-2} = \mathbb{R} \times \mathbb{T}^{n-1}$. The explicit form of the metric can be found in Appendix A, where we also provide some associated formulae as needed for the remainder of this work.

It has been conjectured in [20, Conjectures 2 and 3] that, in spacetime dimension five and in [17, Section 4] in all dimensions, the negativity of the relative mass is a normalisation artifact, and that all solutions of the general relativistic constraint equations with the same manifold (“bulk”) topology and same conformal geometry at infinity will have energy larger than the corresponding Horowitz-Myers solution. The relevance of the HM conjecture to issues in AdS/CFT has been explained clearly in [17, 20, 23, 26], where it is related to consistency of the whole AdS/CFT program, is shown to imply uniqueness of the AdS soliton [26], and is relevant for a proper understanding of phase transitions [23].

The first aim of this note is to show that, consistently with the above conjectures, the (relative) mass of asymptotically HM solutions of the vacuum Einstein equations is indeed positive for some classes of time-symmetric n -dimensional solutions of the general relativistic constraint equations, namely for $U(1)^{n-1}$ symmetric metrics with an orthogonally-transitive $U(1)^{n-2}$ subgroup, and for a subclass of $U(1)^{n-2}$ symmetric metrics with an orthogonally-transitive $U(1)^{n-2}$ -action.

Next, recall that calculations supporting the conjecture have been given in [17, 20] for linearised perturbations of HM metrics satisfying a transverse-traceless condition. However, the most general linearised perturbations of HM metrics do not belong to this class, as follows from the non-vanishing of the right-hand side of (IV.12) below for general perturbations. The second aim of this note is to provide evidence for positivity of energy for general small perturbations, though it should be recognised that we fail to give a rigorous proof. On the way we fill some gaps in the arguments of [17, 20] for the restricted class of perturbations considered in these references, and promote their linearised-perturbations argument to a result about small perturbations.

II. PRELIMINARIES

Let (M, \bar{g}) be a smooth n -dimensional Riemannian manifold, $n \geq 2$. *Static potentials* are defined as functions V on M satisfying

$$\bar{D}_i \bar{D}_j V = V \left(\bar{R}_{ij} - \frac{\bar{R}}{n-1} \bar{g}_{ij} \right). \quad (\text{II.1})$$

When \bar{g} has constant scalar curvature, an equivalent form is

$$\Delta_{\bar{g}} V + \lambda V = 0, \quad \bar{D}_i \bar{D}_j V = V (\bar{R}_{ij} - \lambda \bar{g}_{ij}), \quad (\text{II.2})$$

for some constant $\lambda \in \mathbb{R}$. Here \bar{R}_{ij} denotes the Ricci tensor of the metric \bar{g} , \bar{D} the Levi-Civita connection of \bar{g} , and $\Delta_{\bar{g}} = \bar{D}^k \bar{D}_k$ is the Laplacian of \bar{g} .

When $\lambda < 0$, rescaling \bar{g} by a constant factor if necessary we can without loss of generality assume that

$$\lambda = -n$$

so that

$$\bar{R} := \bar{g}^{ij} \bar{R}_{ij} = \lambda(n-1) = -n(n-1),$$

and this normalisation will be often chosen. This is equivalent to setting $\ell = 1$ in the HM metric (A.1) below and elsewhere in our equations.

Ignoring an overall dimension-dependent constant, we use the widely accepted definition¹ of the mass m of a Riemannian metric g asymptotic to a metric \bar{g} with a static potential V in an equivalent, flexible form given in [14]:

$$m = \lim_{R \rightarrow \infty} \int_{r=R} [V g^{mj} g^{i\ell} (\bar{D}_m g_{j\ell} - \bar{D}_\ell g_{jm}) + (g^{mj} g^{ki} - g^{ij} g^{km})(g_{jm} - \bar{g}_{jm}) \bar{D}_k V] d\sigma_i. \quad (\text{II.3})$$

The limit as $R \rightarrow \infty$ of the integrand of the mass, whenever it exists, will be referred to as the *mass aspect*.

III. POSITIVITY FOR SELECTED CLASSES OF METRICS

Throughout this section, for convenience we rescale the metric by a constant factor so that $\ell = 1$.

A. Positivity for a class of $U(1)^{n-1}$ invariant metrics

Consider a metric on

$$M = \mathbb{R}^2 \times \mathbb{T}^{n-2} := \mathbb{R}^2 \times \underbrace{S^1 \times \cdots \times S^1}_{n-2 \text{ factors}}$$

which is invariant under rotations of the \mathbb{R}^2 factor as well as rotations of each of the factors S^1 of the torus \mathbb{T}^{n-2} . In coordinates adapted to the symmetry it can be written in the form

$$g = g_{ij} dx^i dx^j, \quad (\text{III.1})$$

where all the g_{ij} 's depend only upon the polar radial coordinate on \mathbb{R}^2 , which will be denoted by r .

A redefinition of the coordinates

$$x^a \mapsto x^a + f^a(r)$$

where $x^1 = r$, $x^a = (\theta, x^A)$, with $a, b = 2, 3, \dots, n$ and $A, B = 3, \dots, n$, with suitably chosen functions f^a allows one to obtain $g_{ra} = 0$, bringing the metric to the form

$$\begin{aligned} g &= e^{2u} dr^2 + e^{2v} d\theta^2 + 2g_{\theta A} d\theta dx^A + g_{AB} dx^A dx^B \\ &\equiv e^{2u} dr^2 + g_{ab} dx^a dx^b, \end{aligned} \quad (\text{III.2})$$

where $u, v, g_{AB}, g_{\theta A}$ are functions of r only. A calculation gives

$$\begin{aligned} R &= -e^{-2u} \left(\hat{W}^2 + 2\partial_r \hat{W} - 2\hat{W} \partial_r u \right) \\ &\quad - \frac{1}{4} e^{-2u} g^{ab} g^{cd} \partial_r g_{ac} \partial_r g_{bd}, \end{aligned} \quad (\text{III.3})$$

where

$$\hat{W} := \frac{1}{2} g^{ab} \partial_r g_{ab}. \quad (\text{III.4})$$

¹ The formula we use for mass assigns the same number to a metric as the formulae due to Abbott-Deser [1], Ashtekar-Magnon [4], de Haro et al. [19], or the definition used in [17], when simultaneously defined. Indeed, the fact that the Abbott-Deser mass coincides with the Hamiltonian mass of [8] follows by inspection of the formulae; the fact that our mass coincides with the Hamiltonian mass has been shown in [15, Appendix B]; the fact that the Abbott-Deser mass coincides with the holographic mass is [16, Equation (5.22)] (the formula there has been written in 3 + 1-dimensions but the calculation and the conclusion are the same in all higher dimensions); the fact that the Ashtekar-Magnon formula coincides with the Hamiltonian one has been shown in [5]; the fact that the mass used in [17] coincides with the Abbott-Deser one follows immediately from the construction of the Abbott-Deser mass.

We will specialise to the case where the orbits of the $U(1)^{n-2}$ -isometry subgroup acting on the torus factor of M are orthogonal to the \mathbb{R}^2 factor; this is sometimes referred to as *orthogonal-transitivity* and is, at least locally, equivalent to the condition that each of the covector fields, say $X_{(A)}^\flat$, $A = 3, \dots, n$, associated with the Killing vectors $X_{(A)}$ generating the $U(1)^{n-2}$ action on the torus factor of M satisfies

$$dX_{(A)}^\flat \wedge X_{(3)}^\flat \wedge \dots \wedge X_{(n)}^\flat = 0. \quad (\text{III.5})$$

In other words, the metric is $U(1)^{n-1}$ -invariant with an orthogonally-transitive $U(1)^{n-2}$ subgroup. In this case there exist coordinates in which the metric takes the form

$$g = e^{2u} dr^2 + e^{2v} d\theta^2 + g_{AB} dx^A dx^B, \quad (\text{III.6})$$

where u, v and the g_{AB} 's still depend only upon r . We then have

$$\begin{aligned} R = & -2e^{-2u} \left(\partial_r^2 v - \partial_r u \partial_r v \right. \\ & + (\partial_r v)^2 + W \partial_r (v - u) + \frac{1}{2} W^2 + \partial_r W \Big) \\ & - \frac{1}{4} e^{-2u} g^{AB} g^{CD} \partial_r g_{AC} \partial_r g_{BD}, \end{aligned} \quad (\text{III.7})$$

with

$$W := \frac{1}{2} g^{AB} \partial_r g_{AB}. \quad (\text{III.8})$$

We wish to show that metrics in this class have positive mass with respect to their asymptotic HM background (with $V = r/\ell$ in the coordinates of (A.1) below), whenever the scalar curvature satisfies

$$R \geq -n(n-1). \quad (\text{III.9})$$

This condition is equivalent to the hypothesis of positivity of energy density for time-symmetric general relativistic initial data sets with negative cosmological constant $\Lambda = -n(n-1)/2$.

Indeed, we claim:

Theorem III.1 *Consider a metric g on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ of the form (III.6) where all the metric functions depend only upon r and which has a well defined total mass m with respect to a Horowitz-Myers metric. If the Ricci scalar R of g satisfies (III.9) then*

$$m \geq 0,$$

vanishing if and only if g coincides with its asymptotic Horowitz-Myers metric.

The reader is referred to [14] for the detailed asymptotic conditions needed for a well-defined mass.

PROOF: It turns out that the proof is most transparent for metrics of the form

$$g = e^{2u} dr^2 + e^{2v} d\theta^2 + e^{2w} ((dx^3)^2 + \dots + (dx^n)^2), \quad (\text{III.10})$$

where u, v and w are functions of r . We will therefore first carry-out the proof in this case. For the metric (III.10) we have

$$\begin{aligned} R = & 2e^{-2u} \left[-v'' - (n-2)w'' + u'v' + (n-2)u'w' \right. \\ & \left. - (n-2)v'w' - (v')^2 - \frac{(n-1)(n-2)}{2} (w')^2 \right]. \end{aligned} \quad (\text{III.11})$$

Suppose that g asymptotes to a Horowitz-Myers metric (A.1) with parameter r_0 . We will denote this asymptotic background as g_{HM, r_0} , the associated function u as in (III.10) by u_{HM, r_0} , etc. We have

$$\begin{aligned} u_{\text{HM}, r_0} &= -\ln r - \frac{1}{2} \ln \left(1 - \frac{r_0^n}{r^n} \right), \\ v_{\text{HM}, r_0} &= \ln r + \frac{1}{2} \ln \left(1 - \frac{r_0^n}{r^n} \right), \\ w_{\text{HM}, r_0} &= \ln r. \end{aligned}$$

It is readily seen that $R(g_{\text{HM},r_0}) = -n(n-1)$. In order to obtain a smooth metric at $r = r_0$, θ needs to have period $4\pi/nr_0$.

Using e.g. the perturbation arguments in [10, 13], in order to prove positivity it suffices to assume that the components of the metric, when expressed in terms of orthonormal (ON) frame of the asymptotic background, behave as r^{-n} plus $o(r^{-n})$ terms, and that this behaviour is preserved under differentiation. We can therefore, without loss of generality, assume the asymptotic expansions, for large r ,

$$\begin{aligned} u &= u_{\text{HM},r_0} + \hat{u} = u_{\text{HM},r_0} + u_n r^{-n} + o(r^{-n}) , \\ v &= v_{\text{HM},r_0} + \hat{v} = v_{\text{HM},r_0} + v_n r^{-n} + o(r^{-n}) , \\ w &= w_{\text{HM},r_0} + \hat{w} = w_{\text{HM},r_0} + w_n r^{-n} + o(r^{-n}) , \end{aligned} \quad (\text{III.12})$$

where u_n , v_n and w_n are constants. In order to determine the mass aspect, rather than calculating the integrand of (II.3) one can proceed as follows:

We have

$$\begin{aligned} R &= 2e^{-2u_{\text{HM},r_0}}(1 - 2\hat{u} + O(r^{-2n})) \times \\ &\quad \times \left[-\frac{n(n-1)}{2}e^{2u_{\text{HM},r_0}} - (\hat{v}'' + (n-2)\hat{w}'') + [u'_{\text{HM},r_0}(\hat{v}' + (n-2)\hat{w}') + \hat{u}'(v'_{\text{HM},r_0} + (n-2)w'_{\text{HM},r_0})] \right. \\ &\quad \left. - (n-2)(v'_{\text{HM},r_0}\hat{w}' + \hat{v}'w'_{\text{HM},r_0}) - 2v'_{\text{HM},r_0}\hat{v}' - (n-1)(n-2)w'_{\text{HM},r_0}\hat{w}' + O(r^{-2(n+1)}) \right] \\ &= -n(n-1) + 2n(n-1)\hat{u} + 2r^2 \left[-(\hat{v}'' + (n-2)\hat{w}'') + \frac{n-1}{r}\hat{u}' - \frac{n+1}{r}\hat{v}' - \frac{(n+1)(n-2)}{r}\hat{w}' \right] + O(r^{-2n}) \\ &= -n(n-1) + 2r^{1-n}\frac{d}{dr} \left[(n-1)r^n\hat{u} - r^{n+1}\hat{v}' - (n-2)r^{n+1}\hat{w}' \right] + O(r^{-2n}) . \end{aligned} \quad (\text{III.13})$$

It follows from this last equation and from the way that the mass is calculated in [14] that the mass aspect function of g relative to g_{HM} is the leading term of the expression inside the square bracket. Hence, up to a positive multiplicative constant,

$$\Theta(g) = \frac{2(n-1)}{n}u_n + 2v_n + 2(n-2)w_n . \quad (\text{III.14})$$

The next step of our analysis consists of redefining the coordinate r to a new coordinate \tilde{r} so that the function \hat{u} in the new coordinate system vanishes. In other words, the function u in the new coordinate system will be equal $u_{\text{HM},\tilde{r}_0}$ for some \tilde{r}_0 :

Lemma III.2 *There exists a smooth increasing function $r \mapsto \tilde{r}(r)$ such that*

$$\tilde{r}(r) = r - \frac{r_0^n - \tilde{r}_0^n + 2u_n}{2nr^{n-1}} + o(r^{1-n}) \text{ as } r \rightarrow \infty , \quad (\text{III.15})$$

$$g_{\tilde{r}\tilde{r}} = e^{2u_{\text{HM},\tilde{r}_0}} \text{ with } \tilde{r}_0 = \tilde{r}(r_0) > 0 . \quad (\text{III.16})$$

PROOF: Define

$$F(r) = \int_1^r e^{u_{\text{HM},1}(\xi)} d\xi .$$

Then there exists a constant F_n depending upon the space dimension n such that

$$F(r) = \ln r + F_n - \frac{1}{2nr^n} + o(r^{-n}) \text{ as } r \rightarrow \infty .$$

Let $\tilde{r}_0 > 0$ be such that

$$\lim_{r \rightarrow \infty} \left[\int_{r_0}^r e^{u(\xi)} d\xi - \ln r \right] = F_n - \ln \tilde{r}_0 .$$

The desired function \tilde{r} is then defined by

$$F\left(\frac{\tilde{r}(r)}{\tilde{r}_0}\right) = \int_{r_0}^r e^{u(\xi)} d\xi . \quad (\text{III.17})$$

We proceed to check (III.15)-(III.16). Indeed, we have by the definition of \tilde{r}_0 that

$$\int_{r_0}^r e^{u(\xi)} d\xi = \ln r + (F_n - \ln \tilde{r}_0) - \frac{r_0^n + 2u_n}{2nr^n} + o(r^{-n}) \text{ as } r \rightarrow \infty.$$

Using (III.17) and

$$F\left(\frac{\tilde{r}(r)}{\tilde{r}_0}\right) = \ln \tilde{r} + (F_n - \ln \tilde{r}_0) - \frac{\tilde{r}_0^n}{2n\tilde{r}^n} + o(\tilde{r}^{-n}) \text{ as } \tilde{r} \rightarrow \infty$$

yields (III.15). On the other hand, from (III.17), we have

$$\int_{r_0}^r e^{u(\xi)} d\xi = \int_1^{\frac{\tilde{r}(r)}{\tilde{r}_0}} e^{u_{\text{HM},1}(\xi)} d\xi = \int_{\tilde{r}_0}^{\tilde{r}(r)} e^{u_{\text{HM},\tilde{r}_0}(\xi)} d\xi.$$

Differentiating in r yields

$$e^{u(r)} dr = e^{u_{\text{HM},\tilde{r}_0}(\tilde{r})} d\tilde{r},$$

which gives (III.16). □

Using the variable \tilde{r} given in Lemma III.2, we rewrite the metric (III.10) as

$$g = e^{2\tilde{u}} d\tilde{r}^2 + e^{2\tilde{v}} d\theta^2 + e^{2\tilde{w}} ((dx^3)^2 + \dots + (dx^n)^2), \quad (\text{III.18})$$

where \tilde{u} , \tilde{v} and \tilde{w} are functions of $\tilde{r} \in [\tilde{r}_0, \infty)$, keeping in mind that θ is an angular variable with period $\frac{4\pi}{nr_0}$.

Write

$$\begin{aligned} \tilde{v} &= v_{\text{HM},\tilde{r}_0} + \widehat{\tilde{v}} = v_{\text{HM},\tilde{r}_0} + \tilde{v}_n \tilde{r}^{-n} + o(\tilde{r}^{-n}), \\ \tilde{w} &= w_{\text{HM},\tilde{r}_0} + \widehat{\tilde{w}} = w_{\text{HM},\tilde{r}_0} + \tilde{w}_n \tilde{r}^{-n} + o(\tilde{r}^{-n}), \end{aligned}$$

where \tilde{v}_n and \tilde{w}_n are constants, and note that the function $\widehat{\tilde{u}} := \tilde{u} - u_{\text{HM},\tilde{r}_0}$ is identically zero by construction. It is readily seen that

$$\begin{aligned} \tilde{v}_n &= v_n + \frac{1}{2}(\tilde{r}_0^n - r_0^n) + \frac{r_0^n - \tilde{r}_0^n + 2u_n}{2n}, \\ \tilde{w}_n &= w_n + \frac{r_0^n - \tilde{r}_0^n + 2u_n}{2n}. \end{aligned}$$

Hence, by (III.14),

$$\Theta(g) = -\frac{1}{n}(\tilde{r}_0^n - r_0^n) + 2\tilde{v}_n + 2(n-2)\tilde{w}_n. \quad (\text{III.19})$$

This implies

$$\begin{aligned} \Theta(g) &= -\frac{1}{n}(\tilde{r}_0^n - r_0^n) + \lim_{\tilde{r} \rightarrow \infty} 2\tilde{r}^n(\widehat{\tilde{v}} + (n-2)\widehat{\tilde{w}}) \\ &= -\frac{1}{n}(\tilde{r}_0^n - r_0^n) - \lim_{\tilde{r} \rightarrow \infty} \frac{2}{n}\tilde{r}^{n+1}(\widehat{\tilde{v}} + (n-2)\widehat{\tilde{w}})' \\ &= -\frac{1}{n}(\tilde{r}_0^n - r_0^n) + \frac{2}{n} \int_{\tilde{r}_0}^{\infty} [(-\widehat{\tilde{v}} - (n-2)\widehat{\tilde{w}})' \eta \tilde{r}^{n+1}]' d\tilde{r}, \end{aligned} \quad (\text{III.20})$$

where $'$ now stands for $\frac{d}{d\tilde{r}}$ and η is any function of \tilde{r} which vanishes at $\tilde{r} = \tilde{r}_0$ and $\eta \rightarrow 1$ as $\tilde{r} \rightarrow \infty$.

Recall formula (III.11), which in the current coordinate system translates to

$$R = 2e^{-2\tilde{u}} \left[-\tilde{v}'' - (n-2)\tilde{w}'' + \tilde{u}'\tilde{v}' + (n-2)\tilde{u}'\tilde{w}' - (n-2)\tilde{v}'\tilde{w}' - (\tilde{v}')^2 - \frac{(n-1)(n-2)}{2}(\tilde{w}')^2 \right]. \quad (\text{III.21})$$

Using that $g_{\text{HM}, \tilde{r}_0}$ has curvature $-n(n-1)$, we thus have

$$\begin{aligned} R + n(n-1) &= 2e^{-2u_{\text{HM}, \tilde{r}_0}} \left[-\widehat{v}'' - (n-2)\widehat{w}'' + (u'_{\text{HM}, \tilde{r}_0} - 2v'_{\text{HM}, \tilde{r}_0} - (n-2)w'_{\text{HM}, \tilde{r}_0})\widehat{v}' \right. \\ &\quad \left. + (n-2)(u'_{\text{HM}, \tilde{r}_0} - v'_{\text{HM}, \tilde{r}_0} - (n-1)w'_{\text{HM}, \tilde{r}_0})\widehat{w}' - (n-2)\widehat{v}'\widehat{w}' - (\widehat{v}')^2 - \frac{(n-1)(n-2)}{2}(\widehat{w}')^2 \right]. \end{aligned} \quad (\text{III.22})$$

Using $\eta = \tilde{r}^{-(n+1)}e^{2v_{\text{HM}, \tilde{r}_0} + (n-1)w_{\text{HM}, \tilde{r}_0} + \widehat{v}}$ in (III.20) and noting that $u_{\text{HM}, \tilde{r}_0} = -v_{\text{HM}, \tilde{r}_0}$, we arrive at

$$\begin{aligned} \Theta(g) &= -\frac{1}{n}(\tilde{r}_0^n - r_0^n) + \frac{2}{n} \int_{\tilde{r}_0}^{\infty} \left[(-\widehat{v} - (n-2)\widehat{w})' e^{2v_{\text{HM}, \tilde{r}_0} + (n-1)w_{\text{HM}, \tilde{r}_0} + \widehat{v}} \right]' d\tilde{r} \\ &= -\frac{1}{n}(\tilde{r}_0^n - r_0^n) + \frac{2}{n} \int_{\tilde{r}_0}^{\infty} e^{2v_{\text{HM}, \tilde{r}_0} + (n-1)w_{\text{HM}, \tilde{r}_0} + \widehat{v}} \left[-\widehat{v}'' - (n-2)\widehat{w}'' \right. \\ &\quad \left. - (\widehat{v}' + (n-2)\widehat{w}') (2v'_{\text{HM}, \tilde{r}_0} + (n-1)w'_{\text{HM}, \tilde{r}_0} + \widehat{v}') \right] d\tilde{r} \\ &= -\frac{1}{n}(\tilde{r}_0^n - r_0^n) + \frac{2}{n} \int_{\tilde{r}_0}^{\infty} e^{2v_{\text{HM}, \tilde{r}_0} + (n-1)w_{\text{HM}, \tilde{r}_0} + \widehat{v}} \left\{ \frac{1}{2} e^{2u_{\text{HM}, \tilde{r}_0}} (R + n(n-1)) \right. \\ &\quad \left. + (v'_{\text{HM}, \tilde{r}_0} - w'_{\text{HM}, \tilde{r}_0})\widehat{v}' + \frac{(n-1)(n-2)}{2}(\widehat{w}')^2 \right\} d\tilde{r}. \end{aligned} \quad (\text{III.23})$$

The term containing \widehat{v}' can be computed as follows:

$$\frac{2}{n} \int_{\tilde{r}_0}^{\infty} e^{2v_{\text{HM}, \tilde{r}_0} + (n-1)w_{\text{HM}, \tilde{r}_0} + \widehat{v}} (v'_{\text{HM}, \tilde{r}_0} - w'_{\text{HM}, \tilde{r}_0})\widehat{v}' d\tilde{r} \quad (\text{III.24})$$

$$= \tilde{r}_0^n \int_{\tilde{r}_0}^{\infty} e^{\widehat{v}} \widehat{v}' d\tilde{r} = \tilde{r}_0^n e^{\widehat{v}} \Big|_{\tilde{r}=\tilde{r}_0}^{\tilde{r}=\infty} = \tilde{r}_0^n (1 - e^{\widehat{v}(\tilde{r}_0)}). \quad (\text{III.25})$$

As g is regular at $\tilde{r} = \tilde{r}_0$ and θ has period $\frac{4\pi}{nr_0}$, we have that

$$e^{\widehat{v}(\tilde{r}_0)} = \frac{r_0}{\tilde{r}_0}. \quad (\text{III.26})$$

Altogether we obtain

$$\begin{aligned} \Theta(g) &= -\frac{1}{n}(\tilde{r}_0^n - r_0^n) + \tilde{r}_0^{n-1}(\tilde{r}_0 - r_0) \\ &\quad + \frac{2}{n} \int_{\tilde{r}_0}^{\infty} e^{2v_{\text{HM}, \tilde{r}_0} + (n-1)w_{\text{HM}, \tilde{r}_0} + \widehat{v}} \\ &\quad \left\{ \frac{1}{2} e^{2u_{\text{HM}, \tilde{r}_0}} (R + n(n-1)) \right. \\ &\quad \left. + \frac{(n-1)(n-2)}{2}(\widehat{w}')^2 \right\} d\tilde{r}. \end{aligned} \quad (\text{III.27})$$

The quantity $-\frac{1}{n}(\tilde{r}_0^n - r_0^n) + \tilde{r}_0^{n-1}(\tilde{r}_0 - r_0)$ is non-negative due to the convexity of the function $t \mapsto t^n$, which establishes that $\Theta(g)$ is positive or vanishes.

The case $m = 0$ implies $\Theta(g) = 0$, and (III.27) gives

$$\widehat{w}' \equiv 0 \equiv R + n(n-1), \quad r_0 = \tilde{r}_0.$$

We see that $\widehat{w} \equiv 0 = \widehat{w}_n = \widehat{v}_n$ as well, and (III.22) gives

$$0 = -\widehat{v}'' + (u'_{\text{HM}, \tilde{r}_0} - 2v'_{\text{HM}, \tilde{r}_0} - (n-2)w'_{\text{HM}, \tilde{r}_0} - \widehat{v}')\widehat{v}', \quad (\text{III.28})$$

while from (III.26) we obtain

$$e^{\widehat{v}(\tilde{r}_0)} = 1. \quad (\text{III.29})$$

The maximum principle shows that $\widehat{v} \equiv 0$, and we have proved:

Proposition III.3 *If the metric g in (III.10) satisfies $R \geq -n(n-1)$ then g has non-negative mass, vanishing if and only if g coincides with the corresponding Horowitz-Myers metric.* \square

We now pass to general $U(1)^{n-1}$ -orthogonally-transitive-invariant metrics (III.6). For this let us write

$$\partial_r g_{AB} = \frac{2W}{n-2} g_{AB} + H_{AB}, \quad (\text{III.30})$$

with W as in (III.8), thus $g^{AB} H_{AB} = 0$. This allows us to rewrite the last term appearing in the formula (III.7) for the Ricci scalar of g as

$$-\frac{1}{4} e^{-2u} g^{AB} g^{CD} \partial_r g_{AC} \partial_r g_{BD} = -\frac{1}{4} e^{-2u} \left(g^{AB} g^{CD} H_{AC} H_{BD} + \frac{4W^2}{n-2} \right) \leq -\frac{e^{-2u} W^2}{n-2}. \quad (\text{III.31})$$

Inserting this into (III.7) one obtains

$$R \leq -2e^{-2u} \left(\partial_r^2 v - \partial_r u \partial_r v + (\partial_r v)^2 + W \partial_r (v-u) + \frac{1}{2} W^2 + \partial_r W \right) - \frac{e^{-2u} W^2}{n-2}. \quad (\text{III.32})$$

Defining

$$w' := \frac{W}{n-2}, \quad (\text{III.33})$$

the inequality (III.32) can be rewritten as

$$\begin{aligned} R \leq 2e^{-2u} \left[-v'' - (n-2)w'' + u'v' + (n-2)u'w' \right. \\ \left. - (n-2)v'w' - (v')^2 - \frac{(n-2)(n-1)}{2} (w')^2 \right]. \end{aligned} \quad (\text{III.34})$$

This coincides with (III.21) except that the equality there is changed to an inequality consistent with what we need to prove. With the definition (III.33) the formula (III.14) (derived as the mass aspect of the metric (III.10)) provides also the correct formula for the metric (III.6). The argument of the proof of Proposition III.3 leads again to (III.23) and (III.27) with the equalities there replaced by \geq , which establishes that $m \geq 0$.

If $m = 0$ all the inequalities arising in the argument have to be equalities, in particular (III.31) with \leq replaced by an equality implies that $\partial_r g_{AB}$ is pure trace, and Proposition III.3 applies. The proof of Theorem III.1 is complete. \square

B. Positivity for a class of orthogonally-transitive- $U(1)^{n-2}$ -invariant metrics

It turns out that the arguments given so far partially generalise to metrics which are invariant under an orthogonally-transitive action of $U(1)^{n-2}$ by isometries of the torus factor of M . Such metrics can be written in the form

$$g = e^{2u} dr^2 + e^{2v} d\theta^2 + g_{AB} dx^A dx^B, \quad (\text{III.35})$$

where u, v, g_{AB} are functions of (r, θ) . One finds

$$\begin{aligned} R = -2e^{-2u} \left(\partial_r^2 v - \partial_r u \partial_r v + (\partial_r v)^2 + \partial_r (v-u) W^r + \frac{1}{2} (W^r)^2 + \partial_r W^r \right) - \frac{1}{4} e^{-2u} g^{AB} g^{CD} \partial_r g_{AC} \partial_r g_{BD} \\ - 2e^{-2v} \left(\partial_\theta^2 u - \partial_\theta u \partial_\theta v + (\partial_\theta u)^2 - \partial_\theta (v-u) W^\theta + \frac{1}{2} (W^\theta)^2 + \partial_\theta W^\theta \right) - \frac{1}{4} e^{-2v} g^{AB} g^{CD} \partial_\theta g_{AC} \partial_\theta g_{BD}, \end{aligned} \quad (\text{III.36})$$

with

$$W^r = \frac{1}{2} g^{AB} \partial_r g_{AB}, \quad W^\theta = \frac{1}{2} g^{AB} \partial_\theta g_{AB}. \quad (\text{III.37})$$

A useful device in the $U(1)^{n-1}$ -symmetric case was the introduction of a new radial coordinate \tilde{r} so that $g_{\tilde{r}\tilde{r}}$ takes a canonical form. This does not seem to go through in the general case above while preserving a form of the metric which is convenient for the remaining arguments. On the other hand, the proof generalises if we assume at the outset that

$$u \equiv u_{\text{HM}}. \quad (\text{III.38})$$

We then have:

Theorem III.4 *Consider a metric g on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ of the form (III.35) where the metric functions depend only upon (r, θ) and which has a well defined total mass m with respect to a Horowitz-Myers background metric. If (III.38) holds and if the Ricci scalar R of g satisfies $R \geq -n(n-1)$ then*

$$m \geq 0,$$

vanishing if and only if g coincides with its asymptotic Horowitz-Myers metric.

PROOF: As before, the proof is most transparent for metrics of the form

$$g = e^{2u} dr^2 + e^{2v} d\theta^2 + e^{2w} ((dx^3)^2 + \dots + (dx^n)^2), \quad (\text{III.39})$$

where we allow now u , v and w to depend both upon r and θ . Then

$$\begin{aligned} R = 2e^{-2u} & \left[-v_{,rr} - (n-2)w_{,rr} + u_{,r}v_{,r} + (n-2)u_{,r}w_{,r} - (n-2)v_{,r}w_{,r} - (v_{,r})^2 - \frac{(n-1)(n-2)}{2}(w_{,r})^2 \right] \\ & + 2e^{-2v} \left[-u_{,\theta\theta} - (n-2)w_{,\theta\theta} + u_{,\theta}v_{,\theta} + (n-2)v_{,\theta}w_{,\theta} - (n-2)u_{,\theta}w_{,\theta} - (u_{,\theta})^2 - \frac{(n-1)(n-2)}{2}(w_{,\theta})^2 \right]. \end{aligned} \quad (\text{III.40})$$

If we assume (III.38) and write

$$v = v_{\text{HM}} + \hat{v} \text{ and } w = w_{\text{HM}} + \hat{w}$$

then, using (III.22),

$$\begin{aligned} R + n(n-1) = 2e^{-2u_{\text{HM}}} & \left[-\hat{v}_{,rr} - (n-2)\hat{w}_{,rr} + (u'_{\text{HM}} - 2v'_{\text{HM}} - (n-2)w'_{\text{HM}})\hat{v}_{,r} \right. \\ & + (n-2)(u'_{\text{HM}} - v'_{\text{HM}} - (n-1)w'_{\text{HM}})\hat{w}_{,r} \\ & \left. - (n-2)\hat{v}_{,r}\hat{w}_{,r} - (\hat{v}_{,r})^2 - \frac{(n-1)(n-2)}{2}(\hat{w}_{,r})^2 \right] \\ & + 2(n-2)e^{-2v_{\text{HM}}-2\hat{v}} \left[-\hat{w}_{,\theta\theta} + \hat{v}_{,\theta}\hat{w}_{,\theta} - \frac{n-1}{2}(\hat{w}_{,\theta})^2 \right]. \end{aligned}$$

Already-mentioned perturbation arguments allow us to assume that g is asymptotic to g_{HM} in the sense of (III.12), where the expansion coefficients are now allowed to depend upon θ . We have

$$\Theta(g) = \lim_{r \rightarrow \infty} 2r^n (\hat{v} + (n-2)\hat{w}) = - \lim_{r \rightarrow \infty} \frac{2}{n} r^{n+1} \partial_r (\hat{v} + (n-2)\hat{w}) = \frac{2}{n} \int_{r_0}^{\infty} \partial_r [\partial_r (-\hat{v} - (n-2)\hat{w}) \eta r^{n+1}] dr,$$

where η is any function which vanishes at $r = r_0$ and $\eta \rightarrow 1$ as $r \rightarrow \infty$. In the sequel we take $\eta = r^{-(n+1)} e^{2v_{\text{HM}} + (n-1)w_{\text{HM}} + \hat{v}}$.

Using $u_{\text{HM}} = -v_{\text{HM}}$ we find

$$\begin{aligned}
\Theta(g) &= \frac{2}{n} \int_{r_0}^{\infty} \partial_r [\partial_r (-\hat{v} - (n-2)\hat{w}) e^{2v_{\text{HM}} + (n-1)w_{\text{HM}} + \hat{v}}] dr \\
&= \frac{2}{n} \int_{r_0}^{\infty} e^{2v_{\text{HM}} + (n-1)w_{\text{HM}} + \hat{v}} \left[-\hat{v}_{,rr} - (n-2)\hat{w}_{,rr} \right. \\
&\quad \left. - (\hat{v}_{,r} + (n-2)\hat{w}_{,r}) \times \right. \\
&\quad \left. \times (2v'_{\text{HM}} + (n-1)w'_{\text{HM}} + \hat{v}_{,r}) \right] dr \\
&= \frac{2}{n} \int_{r_0}^{\infty} e^{2v_{\text{HM}} + (n-1)w_{\text{HM}} + \hat{v}} \left\{ \frac{1}{2} e^{2u_{\text{HM}}} (R + n(n-1)) \right. \\
&\quad \left. + (v'_{\text{HM}} - w'_{\text{HM}})\hat{v}_{,r} + \frac{(n-1)(n-2)}{2} (\hat{w}_{,r})^2 \right. \\
&\quad \left. - (n-2)e^{-4v_{\text{HM}} - 2\hat{v}} \times \right. \\
&\quad \left. \times \left[-\hat{w}_{,\theta\theta} + \hat{v}_{,\theta}\hat{w}_{,\theta} - \frac{n-1}{2} (\hat{w}_{,\theta})^2 \right] \right\} dr.
\end{aligned}$$

Regularity of the metric at the core geodesic $r = r_0$ requires $\hat{v} = 0$ there. As in (III.25) we have now

$$\begin{aligned}
&\frac{2}{n} \int_{r_0}^{\infty} e^{2v_{\text{HM}} + (n-1)w_{\text{HM}} + \hat{v}} (v'_{\text{HM}} - w'_{\text{HM}})\hat{v}_{,r} dr \\
&= r_0^n \int_{r_0}^{\infty} e^{\hat{v}} \hat{v}_{,r} dr = r_0^n e^{\hat{v}} \Big|_{r=r_0}^{r=\infty} = 0.
\end{aligned}$$

The term containing $-\hat{w}_{,\theta\theta} + \hat{v}_{,\theta}\hat{w}_{,\theta}$ cancels away after integrating over θ :

$$\int e^{-\hat{v}} (-\hat{w}_{,\theta\theta} + \hat{v}_{,\theta}\hat{w}_{,\theta}) d\theta = \int \partial_{\theta} (-e^{-\hat{v}} \hat{w}_{,\theta}) d\theta = 0. \quad (\text{III.41})$$

So, under the hypotheses of Theorem III.4, we obtain non-negativity of mass for metrics satisfying moreover (III.39). The vanishing of the mass implies $\hat{w} \equiv 0$ and one concludes by an argument similar to that in the proof of Proposition III.3.

We consider now the general case. Formula (III.36) with $\hat{u} \equiv 0$ implies

$$\begin{aligned}
R &\leq -2e^{-2u} \left(\partial_r^2 v - \partial_r u \partial_r v + (\partial_r v)^2 + \partial_r (v-u) W^r + \frac{(n-1)}{2(n-2)} (W^r)^2 + \partial_r W^r \right) \\
&\quad - 2e^{-2v} \left(\partial_{\theta}^2 u - \partial_{\theta} u \partial_{\theta} v + (\partial_{\theta} u)^2 - \partial_{\theta} (v-u) W^{\theta} + \frac{(n-1)}{2(n-2)} (W^{\theta})^2 + \partial_{\theta} W^{\theta} \right), \quad (\text{III.42})
\end{aligned}$$

where we have used (III.31). Introducing

$$w_{,r} = \frac{W^r}{n-2} \quad (\text{III.43})$$

allows us to rewrite (III.42) as

$$\begin{aligned}
R &\leq 2e^{-2u} \left[-v_{,rr} - (n-2)w_{,rr} + u_{,r}v_{,r} + (n-2)u_{,r}w_{,r} - (n-2)v_{,r}w_{,r} - (v_{,r})^2 - \frac{(n-1)(n-2)}{2} (w_{,r})^2 \right] \\
&\quad - 2e^{-2v} \left(\partial_{\theta}^2 u - \partial_{\theta} u \partial_{\theta} v + (\partial_{\theta} u)^2 - \partial_{\theta} (v-u) W^{\theta} + \frac{(n-1)}{2(n-2)} (W^{\theta})^2 + \partial_{\theta} W^{\theta} \right). \quad (\text{III.44})
\end{aligned}$$

This differs from (III.40) by the replacement of an equality sign by \leq , and replacement of $w_{,\theta}$ by $W^{\theta}/(n-1)$. The rest of the proof requires only trivial changes, for example (III.41) is replaced by

$$\int e^{-\hat{v}} (-W_{,\theta} + \hat{v}_{,\theta} W^{\theta}) d\theta = \int \partial_{\theta} (-e^{-\hat{v}} W^{\theta}) d\theta = 0. \quad (\text{III.45})$$

The details are left to the reader. \square

IV. PERTURBATIONS OF STATIC RIEMANNIAN METRICS

In the remainder of this work we wish to address the question of positivity of the relative mass for small perturbations of the Horowitz-Myers metrics. We start with some general considerations.

Given a metric g asymptotic to a background metric \bar{g} , we define

$$h_{ij} := g_{ij} - \bar{g}_{ij}, \quad (IV.1)$$

$$\psi^j := \bar{D}_i g^{ij} \iff g^{ij} \bar{D}_i h_{j\ell} = -g_{\ell j} \psi^j, \quad (IV.2)$$

$$\phi := g^{ij} h_{ij} \implies \bar{\phi} := \bar{g}^{ij} h_{ij} = \phi + O(|h|_{\bar{g}}^2). \quad (IV.3)$$

We will denote by \check{h} , respectively by \hat{h} , the g -trace-free, respectively the \bar{g} -trace-free, part of h :

$$\check{h} := h - \frac{1}{n} \phi g, \quad \hat{h} := h - \frac{1}{n} \bar{\phi} \bar{g}. \quad (IV.4)$$

The most relevant fields for our purposes here are ϕ and \hat{h} , and we emphasise that \check{h} and \hat{h} differ by terms quadratic in ε if $h = O(\varepsilon)$ and if ε is small; similarly for ϕ and $\bar{\phi}$. We also use the notation

$$g^{ij} = \bar{g}^{ij} - h^{ij} + \chi^{ij}, \quad (IV.5)$$

where

$$h^{ij} = \bar{g}^{ik} \bar{g}^{j\ell} h_{k\ell} \quad \text{and} \quad \chi^{ij} := \bar{g}^{ik} \bar{g}^{j\ell} \bar{g}^{mn} h_{km} h_{n\ell} + O(|h|_{\bar{g}}^3) = O(|h|_{\bar{g}}^2). \quad (IV.6)$$

In order to address the question of gauge-freedom, it would be convenient to apply a diffeomorphism to g so that

$$\check{\psi}^i := \psi^i + \frac{1}{2} g^{ik} \bar{D}_k \phi \quad (IV.7)$$

vanishes. The equation $\check{\psi}^i = 0$ will be referred-to as the *harmonicity condition*, which is motivated by the fact that it reduces to the harmonic-coordinates condition in the case of a flat background. It is likely that the harmonicity condition can be achieved in whole generality for perturbations of a Horowitz-Myers background, but this is irrelevant for the current work as our analysis is inconclusive anyway.

In [6] the following formula was derived, which holds for any asymptotically hyperbolic background (M, \bar{g}) with a static potential V , under the usual conditions for existence of the hyperbolic mass m :

$$\begin{aligned} m = \int_M & \left[(R - \bar{R})V + \left(\frac{n+2}{8n} |\bar{D}\phi|_{\bar{g}}^2 + \frac{1}{4} |\bar{D}\hat{h}|_{\bar{g}}^2 \right. \right. \\ & - \frac{1}{2} \hat{h}^{i\ell} \hat{h}^{jm} \bar{R}_{\ell mij} - \frac{n+2}{2n} \phi \hat{h}^{ij} \bar{R}_{ij} - \frac{n^2-4}{8n^2} \lambda \phi^2 \\ & - \frac{1}{2} (|\check{\psi}|_{\bar{g}}^2 - \check{\psi}^i \bar{D}_i \phi) \Big] V + \left(h^k{}_i \check{\psi}^i + \frac{1}{2} \phi \check{\psi}^k \right) \bar{D}_k V \\ & + \left(O(|h|_{\bar{g}}^3) + O(|h|_{\bar{g}} |\bar{D}h|_{\bar{g}}^2) \right) V \\ & + O(|h|_{\bar{g}}^2 |\bar{D}h|_{\bar{g}}) |\bar{D}V|_{\bar{g}} \Big] d\mu_{\bar{g}}. \end{aligned} \quad (IV.8)$$

In this equation all indices are raised and lowered using the background metric \bar{g} .

We will also need another formula from [6]:

$$R = \bar{R}_{ij} g^{ij} + \bar{D}_k [g^{ij} g^{k\ell} (\bar{D}_i h_{j\ell} - \bar{D}_\ell h_{ji})] + Q, \quad (IV.9)$$

where

$$\begin{aligned} Q := \frac{1}{4} g^{ij} g^{kp} g^{\ell q} & \left(2 \bar{D}_p h_{j\ell} \bar{D}_q h_{ki} \right. \\ & \left. - \bar{D}_\ell h_{kp} \bar{D}_q h_{ij} - \bar{D}_i h_{pq} \bar{D}_j h_{k\ell} \right). \end{aligned} \quad (IV.10)$$

In the notation of (IV.2)-(IV.7), the identity (IV.9) becomes

$$-\frac{1}{2}\overline{D}_k(g^{kl}D_l\phi) = R - \overline{R} + \overline{R}_{ij}h^{ij} - \underbrace{\overline{D}_k(g^{k\ell}h_{ji}\overline{D}_\ell g^{ij} - \check{\psi}^k)}_{\text{gauge and higher order terms}} \underbrace{-Q}_{O(|\overline{D}h|^2)}. \quad (\text{IV.11})$$

It follows that a metric perturbation will satisfy the linearised time-symmetric scalar constraint equation if and only if

$$-\frac{1}{2}\overline{D}_k(g^{kl}D_l\phi) - \frac{\overline{R}}{n}\phi = \overline{R}_{ij}\hat{h}^{ij}. \quad (\text{IV.12})$$

If \overline{g} is a space-form the term linear in \hat{h}_{ij} at the right-hand side vanishes, which typically implies that ϕ itself is higher order (compare the discussion before Equation (2.14) in [12]). However, this is not true for general \overline{g} and \hat{h} , in particular one cannot assume that $\phi = 0$ for general perturbations of the Horowitz-Myers metrics in harmonic gauge.

Let us consider a one-parameter family of perturbations h of the metric of the form

$$h_{ij} = \epsilon \overset{(1)}{h}_{ij} + \epsilon^2 \overset{(2)}{h}_{ij} + O(\epsilon^3), \quad (\text{IV.13})$$

where $\overset{(1)}{h}$ and $\overset{(2)}{h}$ are independent of ϵ . We assume that this expansion is preserved by differentiation. Subsequently, the mass will have an expansion

$$m = \epsilon \overset{(1)}{m} + \epsilon^2 \overset{(2)}{m} + O(\epsilon^3), \quad (\text{IV.14})$$

We will use similar notation for expansions of \hat{h} and ϕ :

$$\hat{h}_{ij} = \epsilon \overset{(1)}{\hat{h}}_{ij} + \epsilon^2 \overset{(2)}{\hat{h}}_{ij} + O(\epsilon^3), \quad \phi = \epsilon \overset{(1)}{\phi} + \epsilon^2 \overset{(2)}{\phi} + O(\epsilon^3). \quad (\text{IV.15})$$

Suppose, first, that h satisfies the constraint equation up to terms of order ϵ^2 ; equivalently, that $\overset{(1)}{h}$ satisfies the linearised constraint equation. Dividing (IV.8) by ϵ and passing to the limit $\epsilon = 0$ one obtains the well-known result, that static metrics are local extrema of mass on the set of solutions of the constraint equations:

$$\overset{(1)}{m} = 0. \quad (\text{IV.16})$$

Suppose, next, that h satisfies the constraint equation up to terms of order ϵ^3 and that the gauge condition $\check{\psi}^k = 0$ holds. Dividing (IV.8) now by ϵ^2 and passing to the limit $\epsilon = 0$ one obtains

$$\begin{aligned} \overset{(2)}{m} = \int_M & \left[\left(\frac{n+2}{8n} |\overline{D} \overset{(1)}{\phi}|_{\overline{g}}^2 + \frac{1}{4} |\overline{D} \overset{(1)}{\hat{h}}|_{\overline{g}}^2 \right. \right. \\ & \left. \frac{1}{2} \overset{(1)}{\hat{h}}^{i\ell} \overset{(1)}{\hat{h}}^{jm} \overline{R}_{\ell mij} - \frac{n+2}{2n} \overset{(1)}{\phi} \overset{(1)}{\hat{h}}^{ij} \overline{R}_{ij} \right. \\ & \left. \left. + \frac{n^2-4}{8n\ell^2} (\overset{(1)}{\phi})^2 \right] V d\mu_{\overline{g}}. \end{aligned} \quad (\text{IV.17})$$

We note that the knowledge of the perturbed metric to first order in ϵ suffices to obtain a formula for the mass which is accurate to second order in ϵ .

To simplify notation, we will from now on interchangeably write $(\overset{(1)}{\phi}, \overset{(1)}{\hat{h}})$ and (ϕ, \hat{h}) , the smallness parameter ϵ being implicitly understood whenever required.

V. PERTURBATIONS OF HOROWITZ-MYERS METRICS

If \bar{g} is the space-part of the Horowitz-Myers metric, the curvature-dependent terms in (IV.17) read (see Appendix A)

$$\bar{R}_{ij}\hat{h}^{ij} = \frac{n}{2\ell^2} \left(\frac{r_0}{r}\right)^n \left(\hat{h}^{\hat{1}\hat{1}} + \hat{h}^{\hat{2}\hat{2}}\right), \quad (\text{V.1})$$

$$\begin{aligned} \bar{R}_{ijkl}\hat{h}^{ik}\hat{h}^{jl} = & f'' \left[(\hat{h}^{\hat{1}\hat{2}})^2 + \frac{1}{4} (\hat{h}^{\hat{1}\hat{1}} - \hat{h}^{\hat{2}\hat{2}})^2 \right] \\ & + \left[\frac{(3-n)f}{(n-2)r^2} + \frac{f'}{r} - \frac{f''}{4} \right] (\hat{h}^{\hat{1}\hat{1}} + \hat{h}^{\hat{2}\hat{2}})^2 \\ & + \frac{f}{r^2} |\hat{h}|_{\bar{g}}^2, \end{aligned} \quad (\text{V.2})$$

where

$$f(r) := \frac{r^2}{\ell^2} \left[1 - \left(\frac{r_0}{r}\right)^n \right], \quad (\text{V.3})$$

$$\hat{\hat{h}}_{\hat{A}\hat{B}} := \hat{h}_{\hat{A}\hat{B}} - \frac{1}{n-2} \bar{g}^{\hat{C}\hat{D}} \hat{h}_{\hat{C}\hat{D}} \bar{g}_{\hat{A}\hat{B}}, \quad (\text{V.4})$$

with $\hat{\hat{h}}_{\hat{a}\hat{i}} \equiv 0 \equiv \hat{\hat{h}}_{\hat{a}\hat{b}}$, where from now on $a, b \in \{1, 2\}$, and where hatted indices denote frame components with respect to the \bar{g} -orthonormal frame (A.8) below.

The question arises, whether the quadratic form (IV.17) in the fields $(\overset{(1)}{\phi}, \overset{(1)}{\hat{h}}_{ij})$ is positive definite. An affirmative answer would establish the Horowitz-Myers conjecture for small perturbations of Horowitz-Myers metrics. We show in Appendix B that this is not the case: there exist fields so that the right-hand side of (IV.17) is negative.

However, the examples there satisfy neither the linearised constraint equations nor the harmonicity conditions, as would have been needed to invalidate the conjecture. And we have neither been able to find fields satisfying all necessary requirements, nor to prove that no such fields with negative $\overset{(2)}{m}$ exist. The analysis that we present below suggests strongly, but fails to prove, that if such fields existed, then there would also exist negative mass configurations depending only upon r . Since we have just proved that any metric in the relevant class depending only upon r has positive mass, we are led to suspect that the Horowitz-Myers conjecture is also correct for small perturbations of the Horowitz-Myers metrics.

An obvious approach to analyse the sign of the right-hand side of (IV.17) is to estimate $\overset{(2)}{m}$ from below by discarding all positive terms which do not contain radial derivatives of the fields. Thus, from the term $\frac{1}{4} |\bar{D}\hat{h}|_{\bar{g}}^2$ there we will only keep the following, using (A.15) below,

$$\begin{aligned} \frac{1}{4} |\bar{D}\hat{h}|_{\bar{g}}^2 & \geq \frac{f}{4} (|\partial_r \hat{h}_{\hat{1}\hat{1}}|^2 + 2|\partial_r \hat{h}_{\hat{1}\hat{2}}|^2 + |\partial_r \hat{h}_{\hat{2}\hat{2}}|^2 + |\partial_r \hat{\hat{h}}|_{\bar{g}}^2) \\ & = \frac{f}{4} \left(\frac{1}{2} |\partial_r (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}})|^2 + 2|\partial_r \hat{h}_{\hat{1}\hat{2}}|^2 + \frac{1}{2} |\partial_r (\hat{h}_{\hat{1}\hat{1}} - \hat{h}_{\hat{2}\hat{2}})|^2 + |\partial_r \hat{\hat{h}}|_{\bar{g}}^2 \right). \end{aligned} \quad (\text{V.5})$$

This gives, after ignoring further irrelevant-looking obviously-positive terms,

$$\begin{aligned} \overset{(2)}{m} & \geq \int_M \left[\left(\frac{n+2}{8n} f |\partial_r \phi|^2 + \frac{f}{4} \left(\frac{1}{2} |\partial_r (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}})|^2 + 2|\partial_r \hat{h}_{\hat{1}\hat{2}}|^2 + \frac{1}{2} |\partial_r (\hat{h}_{\hat{1}\hat{1}} - \hat{h}_{\hat{2}\hat{2}})|^2 + |\partial_r \hat{\hat{h}}|_{\bar{g}}^2 \right) \right. \right. \\ & \quad \left. \left. - \frac{f''}{2} \left[(\hat{h}_{\hat{1}\hat{2}})^2 + \frac{1}{4} (\hat{h}_{\hat{1}\hat{1}} - \hat{h}_{\hat{2}\hat{2}})^2 \right] - \frac{1}{2} \left[\frac{(3-n)f}{(n-2)r^2} + \frac{f'}{r} - \frac{f''}{4} \right] (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}})^2 \right. \right. \\ & \quad \left. \left. - \frac{n+2}{4\ell^2} \left(\frac{r_0}{r}\right)^n (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}}) \phi + \frac{n^2-4}{8n\ell^2} \phi^2 - \frac{f}{2r^2} |\hat{h}|_{\bar{g}}^2 \right] V d\mu_{\bar{g}}, \end{aligned} \quad (\text{V.6})$$

with equality attained on those perturbations with $h_{rA} = 0 = h_{\theta A}$ which depend only upon r .

A. Positive contribution from $h_{\hat{1}\hat{2}}$

Let us, first, consider those terms in (V.6) which involve $h_{\hat{1}\hat{2}}$. Using

$$\begin{aligned} f'' &= \ell^{-2} \left(2 - (n-2)(n-1) \frac{r_0^n}{r^n} \right), \quad \sqrt{\det \bar{g}} = \frac{r^{n-2}}{\ell^{n-2}}, \\ V &= \frac{r}{\ell}, \end{aligned} \tag{V.7}$$

we need to analyse the integral

$$\int_{r_0}^{\infty} \left(\frac{f}{2} |\partial_r h_{\hat{1}\hat{2}}|^2 - \frac{f''}{2} |h_{\hat{1}\hat{2}}|^2 \right) \frac{r^{n-1}}{\ell^{n-1}} dr. \tag{V.8}$$

To this end, we use the identity

$$\int_{r_0}^{\infty} 2f \partial_r h_{\hat{1}\hat{2}} h_{\hat{1}\hat{2}} r^{n-2} dr + \int_{r_0}^{\infty} |h_{\hat{1}\hat{2}}|^2 (r \partial_r f + (n-2)f) r^{n-3} dr = 0. \tag{V.9}$$

Hence, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \int_{r_0}^{\infty} (f |\partial_r h_{\hat{1}\hat{2}}|^2 - f'' |h_{\hat{1}\hat{2}}|^2) r^{n-1} dr &= \int_{r_0}^{\infty} (f |\partial_r h_{\hat{1}\hat{2}}|^2 - f'' |h_{\hat{1}\hat{2}}|^2) r^{n-1} dr \\ &+ \alpha \int_{r_0}^{\infty} 2f \partial_r h_{\hat{1}\hat{2}} h_{\hat{1}\hat{2}} r^{n-2} dr + \alpha \int_{r_0}^{\infty} |h_{\hat{1}\hat{2}}|^2 (r \partial_r f + (n-2)f) r^{n-3} dr \\ &= \int_{r_0}^{\infty} \left[f |\partial_r h_{\hat{1}\hat{2}} + \frac{\alpha}{r} h_{\hat{1}\hat{2}}|^2 + W_{\alpha}(f) |h_{\hat{1}\hat{2}}|^2 \right] r^{n-1} dr, \end{aligned} \tag{V.10}$$

where

$$\begin{aligned} W_{\alpha}(f) &= -f'' + \frac{\alpha}{r} \partial_r f + \frac{(n-2)\alpha - \alpha^2}{r^2} f \\ &= \frac{1}{\ell^2} \left(-(\alpha^2 - n\alpha + 2) + (n^2 - 3n + 2 + \alpha^2) \left(\frac{r_0}{r} \right)^n \right). \end{aligned}$$

Choosing $\alpha = \frac{n}{2}$ we have $W_{n/2}(f) \geq (n^2 - 8)/(4\ell^2) > 0$ for $n \geq 3$, and so with this choice the integral in (V.8) is non-negative.

The reader will note that the argument leading to (V.10) establishes the following inequality, for any function ζ and any $\alpha \in \mathbb{R}$,

$$\int_M f (\partial_r \zeta)^2 d\mu_{\bar{g}} \geq \int_M (\alpha (r \partial_r f + (n-2)f) - \alpha^2 f) r^{-2} \zeta^2 d\mu_{\bar{g}}. \tag{V.11}$$

Remark V.1 An identical calculation applies to those terms in (V.6) which involve $\hat{h}_{\hat{1}\hat{1}} - \hat{h}_{\hat{2}\hat{2}}$. However, these terms are coupled with the remaining ones through the harmonicity condition, and their positivity compensates for the negativity of the remaining contributions in the radial case, so one should not discard them when estimating $\overset{(2)}{m}$ from below.

B. Positive contribution from $\hat{h}_{\hat{A}\hat{B}}$

We consider now those terms in (V.6) which explicitly involve $\hat{h}_{\hat{A}\hat{B}}$. These are, up to irrelevant numerical factors and an inessential integration over the remaining variables,

$$\int_{r_0}^{\infty} f \left(|\partial_r \hat{h}|_{\bar{g}}^2 - \frac{2}{r^2} |\hat{h}|_{\bar{g}}^2 \right) \frac{r^{n-1}}{\ell^{n-1}} dr. \tag{V.12}$$

Using the inequality

$$|\partial_r \hat{h}|_{\bar{g}} \leq |\partial_r \hat{h}|_{\bar{g}},$$

the analysis of the sign of (V.12) can be reduced to that of the sign of the integral

$$\int_{r_0}^{\infty} f((\partial_r \zeta)^2 - \frac{2}{r^2} \zeta^2) \frac{r^{n-1}}{\ell^{n-1}} dr, \quad (\text{V.13})$$

for differentiable functions ζ . A calculation as in (V.10) with f'' there replaced by $2f/r^2$ gives, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} & \int_{r_0}^{\infty} (f|\partial_r h_{\hat{1}\hat{2}}|^2 - \frac{2}{r^2} |h_{\hat{1}\hat{2}}|^2) r^{n-1} dr \\ &= \int_{r_0}^{\infty} \left[f|\partial_r h_{\hat{1}\hat{2}} + \frac{\alpha}{r} h_{\hat{1}\hat{2}}|^2 + \tilde{W}_\alpha(f) |h_{\hat{1}\hat{2}}|^2 \right] r^{n-1} dr, \end{aligned} \quad (\text{V.14})$$

where

$$\begin{aligned} \tilde{W}_\alpha(f) &= \frac{\alpha}{r} \partial_r f + \frac{(n-2)\alpha - \alpha^2 - 2}{r^2} f \\ &= \frac{1}{\ell^2} \left(-(\alpha^2 - n\alpha + 2) + (\alpha^2 + 2) \left(\frac{r_0}{r} \right)^n \right). \end{aligned} \quad (\text{V.15})$$

The choice $\alpha = \frac{n}{2}$ leads similarly to $\tilde{W}_{n/2}(f) \geq (n^2 - 8)/(4\ell^2) > 0$ for $n \geq 3$, which shows that the integral in (V.12) is non-negative.

C. The remainder

After discarding those fields which have been shown to give a positive contribution to the mass so far, and ignoring the warning in Remark V.1, one is left to face a lower bound for $\overset{(2)}{m}$ governed by the integral

$$\begin{aligned} I &:= \int_{r_0}^{\infty} \left[\frac{n+2}{8n} f |\partial_r \phi|^2 + \frac{f}{8} |\partial_r (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}})|^2 \right. \\ &\quad - \frac{1}{2} \left[\frac{(3-n)f}{(n-2)r^2} + \frac{f'}{r} - \frac{f''}{4} \right] (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}})^2 \\ &\quad \left. - \frac{n+2}{4\ell^2} \left(\frac{r_0}{r} \right)^n (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}}) \phi + \frac{n^2 - 4}{8n\ell^2} \phi^2 \right] \frac{r^{n-1}}{\ell^{n-1}} dr. \end{aligned} \quad (\text{V.16})$$

As already mentioned, we show in Appendix B that I can take negative values, when viewed as a functional of

$$\xi := h_{\hat{1}\hat{1}} + h_{\hat{2}\hat{2}}$$

after taking $\phi = 0$, in all dimensions $n \geq 3$. However, after enforcing the linearised scalar constraint equation,

$$-\frac{1}{2} \overline{D}_{\hat{k}} \left(\hat{g}^{\hat{k}\hat{l}} \overline{D}_{\hat{l}} \phi \right) + \frac{(n-1)}{\ell^2} \phi = \frac{n}{2\ell^2} \left(\frac{r_0}{r} \right)^n (\hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}}), \quad (\text{V.17})$$

the fields ϕ and ξ are not independent anymore. For example, if $\phi \equiv 0$ we obtain $\xi \equiv 0$, then $I = 0$, and from what has been said so far $\overset{(2)}{m} \geq 0$ for such variations. We recover a result already observed by Horowitz and Myers [20], that $\overset{(2)}{m}$ is positive for transverse-traceless perturbations of the metric satisfying the linearised constraint equations.

On the other hand, if $n = 3$ and if we take $\phi(r) = r^{-5}$, Equation (V.17) gives

$$\xi(r) = \frac{30 - 11r^3}{3r^5}, \quad (\text{V.18})$$

and $I = -79/84$.

When $n = 4$ a negative value of $I \approx -0.575$ is obtained by setting $\phi(r) = \tanh(r)/r^6$, which results in

$$\xi(r) = \frac{3(7 - 2r^4) \tanh(r) + r(4r^4 + (r^4 - 1)r \tanh(r) - 6) \operatorname{sech}^2(r)}{2r^6}. \quad (\text{V.19})$$

We note that the ansatz $\phi(r) = r^{-c}$ leads to a positive I in all dimensions that we tried, namely $4 \leq n \leq 16$, regardless of the choice of the exponent c for which the integral converges.

D. A functional on ϕ

The above does not invalidate the HM conjecture in space-dimensions three and four because imposing harmonicity leads to a non-vanishing field $h_{\hat{1}\hat{1}} - h_{\hat{2}\hat{2}}$, the contribution of which restores positivity of $\hat{m}^{(2)}$ for r -dependent perturbations. Indeed, if we prescribe a function $\phi(r)$, we can then calculate ξ from (V.17). If we consider metric variations in harmonic gauge satisfying

$$\hat{h}_{\hat{1}\hat{2}} = 0 = \hat{h}_{\hat{1}\hat{A}} = \hat{h}_{\hat{2}\hat{A}}, \quad (\text{V.20})$$

with all metric perturbations depending only upon r , then the harmonicity conditions reduce to the equation (D.5) for $\hat{h}_{\hat{1}\hat{1}}$,

$$\sqrt{f}\partial_r(\hat{h}_{\hat{1}\hat{1}} - \frac{n-2}{2n}\phi) = \frac{f'}{2\sqrt{f}}(\hat{h}_{\hat{2}\hat{2}} - \hat{h}_{\hat{1}\hat{1}}) - \frac{\sqrt{f}}{r}(\xi + (n-2)\hat{h}_{\hat{1}\hat{1}}), \quad (\text{V.21})$$

where $\hat{h}_{\hat{2}\hat{2}}$ is viewed as a function of the already-known field ξ and of $\hat{h}_{\hat{1}\hat{1}}$:

$$\hat{h}_{\hat{2}\hat{2}} = \xi - \hat{h}_{\hat{1}\hat{1}}. \quad (\text{V.22})$$

In space-dimension $n = 3$ we then necessarily have

$$\hat{h}_{\hat{3}\hat{3}} = -\xi,$$

and (V.21) becomes

$$r^{-(n-2)}\partial_r(r^{n-2}f\hat{h}_{\hat{1}\hat{1}}) = \left(\frac{f'}{2} - \frac{f}{r}\right)\xi + \frac{n-2}{2n}f\partial_r\phi, \quad (\text{V.23})$$

Integrating from r_0 , regularity at r_0 enforces the solution to be

$$\hat{h}_{\hat{1}\hat{1}} = r^{-(n-2)}f^{-1} \int_{r_0}^r \left(\left(\frac{f'}{2} - \frac{f}{r}\right)\xi + \frac{n-2}{2n}f\partial_r\phi\right) \Big|_{r=s} s^{n-2} ds. \quad (\text{V.24})$$

The function $\hat{h}_{\hat{2}\hat{2}}$ can now be determined using (V.22), and we obtain a linearised metric perturbation satisfying the gauge conditions and the linearised constraint equation.

The right-hand side of (V.6) with $\hat{h}_{AB} = h_{r\theta} = 0$ becomes thus a functional of $\phi^{(1)}$, the positivity of which follows in an indirect way from Proposition III.3. However, positivity when $\phi^{(1)}$ is allowed to depend upon all variables is not clear.

Appendix A: Horowitz-Myers metrics

The Horowitz-Myers “soliton” metric \tilde{g}_{HM} reads

$$\begin{aligned} \tilde{g}_{\text{HM}} = & -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2} \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^n} \\ & + \frac{r^2}{\ell^2} \left[1 - \left(\frac{r_0}{r}\right)^n\right] d\theta^2 + \frac{r^2}{\ell^2} \sum_{A=1}^{n-2} (dx^A)^2, \end{aligned} \quad (\text{A.1})$$

where

$$\ell^2 = -\frac{n(n-1)}{2\Lambda}, \quad (\text{A.2})$$

and where $r_0 > 0$ is a constant, θ is an angle with the period

$$\beta = \frac{4\pi\ell^2}{nr_0},$$

and the x^A 's, $A \in \{3, \dots, n\}$, are local coordinates on an $(n-2)$ -dimensional flat manifold. Setting

$$f(r) := \frac{r^2}{\ell^2} \left[1 - \left(\frac{r_0}{r} \right)^n \right], \quad (\text{A.3})$$

the spatial part, say \bar{g}_{HM} , of (A.1) takes the form

$$\bar{g}_{\text{HM}} = \frac{dr^2}{f(r)} + f(r)d\theta^2 + \ell^{-2}r^2\bar{\delta}, \quad (\text{A.4})$$

where $\bar{\delta} = \delta_{AB}dx^A dx^B$ is flat. We will write g_{HM, r_0} for g_{HM} when r_0 needs to be made explicit.

Dividing the metric by ℓ^2 and rescaling the coordinates (t, θ, x^A) suitably we can always achieve $\ell = 1$. A subsequent rescaling of r leads to $r_0 = 1$. This can be used to reduce the analysis to one where

$$\ell = 1 = r_0, \quad \beta = \frac{4\pi}{n}. \quad (\text{A.5})$$

The nonvanishing, up to index-symmetries, Christoffel symbols read

$$\bar{\Gamma}_{rr}^r = -\frac{f'}{2f}, \quad \bar{\Gamma}_{\theta\theta}^r = -\frac{1}{2}ff', \quad \bar{\Gamma}_{AB}^r = -\ell^{-2}rf\delta_{AB}, \quad \bar{\Gamma}_{r\theta}^\theta = \frac{f'}{2f}, \quad \bar{\Gamma}_{rB}^A = \frac{1}{r}\delta_{AB}. \quad (\text{A.6})$$

The nontrivial components, again up to index symmetries, of the Riemann tensor of the (Riemannian) metric \bar{g}_{HM} are (cf., e.g., [3, Appendix D])

$$\bar{R}_{r\theta r\theta} = -\frac{1}{2}f'', \quad \bar{R}_{rArB} = -\frac{rf'}{2f\ell^2}\delta_{AB}, \quad \bar{R}_{\theta A\theta B} = -\frac{rf'f}{2\ell^2}\delta_{AB}, \quad \bar{R}_{ABCD} = -\ell^{-4}r^2f(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad (\text{A.7})$$

It is convenient to introduce the orthonormal co-frame

$$\bar{\theta}^{\hat{1}} = \frac{dr}{\sqrt{f}}, \quad \bar{\theta}^{\hat{2}} = \sqrt{f}d\theta, \quad \bar{\theta}^{\hat{A}} = \ell^{-1}r dx^A. \quad (\text{A.8})$$

We have the following non-vanishing connection one-forms

$$-\bar{\omega}_{\hat{1}\hat{2}} = \bar{\omega}_{\hat{2}\hat{1}} = \frac{f'}{2\sqrt{f}}\bar{\theta}^{\hat{2}}, \quad -\bar{\omega}_{\hat{1}\hat{A}} = \bar{\omega}_{\hat{A}\hat{1}} = \frac{\sqrt{f}}{r}\bar{\theta}^{\hat{A}}, \quad (\text{A.9})$$

with $\hat{a}, \hat{b} \in \{\hat{1}, \hat{2}\}$, $\hat{A}, \hat{B} \in \{\hat{3}, \dots, \hat{n}\}$, and

$$\bar{R}_{\hat{1}\hat{2}\hat{1}\hat{2}} = -\frac{1}{2}f'', \quad \bar{R}_{\hat{a}\hat{A}\hat{b}\hat{B}} = -\frac{f'}{2r}\bar{g}_{\hat{a}\hat{b}}\bar{g}_{\hat{A}\hat{B}}, \quad \bar{R}_{\hat{A}\hat{B}\hat{C}\hat{D}} = -\frac{f}{r^2}(\bar{g}_{\hat{A}\hat{C}}\bar{g}_{\hat{B}\hat{D}} - \bar{g}_{\hat{A}\hat{D}}\bar{g}_{\hat{B}\hat{C}}). \quad (\text{A.10})$$

The nontrivial frame components of the Ricci tensor read

$$\bar{R}_{\hat{a}\hat{b}} = -\frac{1}{2}\left[f'' + (n-2)\frac{f'}{r}\right]\bar{g}_{\hat{a}\hat{b}}, \quad \bar{R}_{\hat{A}\hat{B}} = -\left[\frac{f'}{r} + (n-3)\frac{f}{r^2}\right]\bar{g}_{\hat{A}\hat{B}}. \quad (\text{A.11})$$

Let us define

$$\hat{\hat{h}}_{\hat{A}\hat{B}} := \hat{h}_{\hat{A}\hat{B}} - \frac{1}{n-2}\bar{g}^{\hat{C}\hat{D}}\hat{h}_{\hat{C}\hat{D}}\bar{g}_{\hat{A}\hat{B}}, \quad (\text{A.12})$$

with $\hat{\hat{h}}_{\hat{a}\hat{a}} := 0$. Using this notation, we have

$$\begin{aligned} \bar{R}_{ij}\hat{h}^{ij} &= -\frac{1}{2}\left[f'' + (n-4)\frac{f'}{r} - 2(n-3)\frac{f}{r^2}\right]\bar{g}_{\hat{a}\hat{b}}\hat{h}^{\hat{a}\hat{b}} \\ &= \frac{n}{2\ell^2}\left(\frac{r_0}{r}\right)^n\left(\hat{h}^{\hat{1}\hat{1}} + \hat{h}^{\hat{2}\hat{2}}\right), \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} \bar{R}_{ijkl}\hat{h}^{ik}\hat{h}^{jl} &= -f''(\hat{h}^{\hat{1}\hat{1}}\hat{h}^{\hat{2}\hat{2}} - (\hat{h}^{\hat{1}\hat{2}})^2) + \left(\frac{f'}{r} - \frac{f}{r^2}\right)(\hat{h}^{\hat{1}\hat{1}} + \hat{h}^{\hat{2}\hat{2}})^2 + \frac{f}{r^2}\bar{g}_{\hat{A}\hat{D}}\bar{g}_{\hat{B}\hat{C}}\hat{h}^{\hat{A}\hat{C}}\hat{h}^{\hat{B}\hat{D}} \\ &= f''\left[(\hat{h}^{\hat{1}\hat{2}})^2 + \frac{1}{4}(\hat{h}^{\hat{1}\hat{1}} - \hat{h}^{\hat{2}\hat{2}})^2\right] + \left[\frac{(3-n)f}{(n-2)r^2} + \frac{f'}{r} - \frac{f''}{4}\right](\hat{h}^{\hat{1}\hat{1}} + \hat{h}^{\hat{2}\hat{2}})^2 + \frac{f}{r^2}|\hat{\hat{h}}|_{\bar{g}}^2. \end{aligned} \quad (\text{A.14})$$

1. Covariant derivatives

We have

$$\bar{D}_1 \hat{h}_{\hat{j}\hat{k}} = \sqrt{f} \partial_r \hat{h}_{\hat{j}\hat{k}}. \quad (\text{A.15})$$

Further,

$$\bar{D}_2 \hat{h}_{\hat{j}\hat{k}} = \frac{1}{\sqrt{f}} \partial_\theta \hat{h}_{\hat{j}\hat{k}} - \bar{\omega}_{\hat{j}\hat{2}}^{\hat{\ell}} \hat{h}_{\hat{\ell}\hat{k}} - \bar{\omega}_{\hat{k}\hat{2}}^{\hat{\ell}} \hat{h}_{\hat{\ell}\hat{j}}. \quad (\text{A.16})$$

The only nonvanishing connection coefficient relevant for this equation is, up to index-symmetries, $\bar{\omega}_{\hat{2}\hat{2}}^{\hat{1}} = -\frac{f'}{2\sqrt{f}}$. This yields

$$\begin{aligned} \bar{D}_2 \hat{h}_{\hat{1}\hat{1}} &= \frac{1}{\sqrt{f}} \left(\partial_\theta \hat{h}_{\hat{1}\hat{1}} - f' \hat{h}_{\hat{1}\hat{2}} \right), \\ \bar{D}_2 \hat{h}_{\hat{1}\hat{2}} &= \frac{1}{\sqrt{f}} \left[\partial_\theta \hat{h}_{\hat{1}\hat{2}} - \frac{f'}{2} \left(\hat{h}_{\hat{2}\hat{2}} - \hat{h}_{\hat{1}\hat{1}} \right) \right], \\ \bar{D}_2 \hat{h}_{\hat{2}\hat{2}} &= \frac{1}{\sqrt{f}} \left(\partial_\theta \hat{h}_{\hat{2}\hat{2}} + f' \hat{h}_{\hat{1}\hat{2}} \right), \\ \bar{D}_2 \hat{h}_{\hat{1}\hat{A}} &= \frac{1}{\sqrt{f}} \left(\partial_\theta \hat{h}_{\hat{1}\hat{A}} - \frac{f'}{2} \hat{h}_{\hat{2}\hat{A}} \right), \\ \bar{D}_2 \hat{h}_{\hat{2}\hat{A}} &= \frac{1}{\sqrt{f}} \left(\partial_\theta \hat{h}_{\hat{2}\hat{A}} + \frac{f'}{2} \hat{h}_{\hat{1}\hat{A}} \right), \\ \bar{D}_2 \hat{h}_{\hat{A}\hat{B}} &= \frac{1}{\sqrt{f}} \partial_\theta \hat{h}_{\hat{A}\hat{B}}. \end{aligned} \quad (\text{A.17})$$

Finally, we have

$$\bar{D}_A \hat{h}_{\hat{j}\hat{k}} = \frac{\ell}{r} \partial_{\hat{A}} \hat{h}_{\hat{j}\hat{k}} - \bar{\omega}_{\hat{j}\hat{A}}^{\hat{\ell}} \hat{h}_{\hat{\ell}\hat{k}} - \bar{\omega}_{\hat{k}\hat{A}}^{\hat{\ell}} \hat{h}_{\hat{\ell}\hat{j}}. \quad (\text{A.18})$$

In this case the only relevant nonvanishing components of the connection coefficient are, again up to index-symmetries, $\bar{\omega}_{\hat{1}\hat{B}}^{\hat{A}} = \frac{\sqrt{f}}{r} \delta_{\hat{B}}^{\hat{A}}$ and $\bar{\omega}_{\hat{A}\hat{B}}^{\hat{1}} = -\frac{\sqrt{f}}{r} \bar{g}_{\hat{A}\hat{B}}$ which implies

$$\begin{aligned} \bar{D}_{\hat{A}} \hat{h}_{\hat{1}\hat{1}} &= \frac{\ell}{r} \partial_{\hat{A}} \hat{h}_{\hat{1}\hat{1}} - \frac{2\sqrt{f}}{r} \hat{h}_{\hat{A}\hat{1}}, \\ \bar{D}_{\hat{A}} \hat{h}_{\hat{1}\hat{2}} &= \frac{\ell}{r} \partial_{\hat{A}} \hat{h}_{\hat{1}\hat{2}} - \frac{\sqrt{f}}{r} \hat{h}_{\hat{A}\hat{2}}, \\ \bar{D}_{\hat{A}} \hat{h}_{\hat{2}\hat{2}} &= \frac{\ell}{r} \partial_{\hat{A}} \hat{h}_{\hat{2}\hat{2}}, \\ \bar{D}_{\hat{A}} \hat{h}_{\hat{1}\hat{B}} &= \frac{\ell}{r} \partial_{\hat{A}} \hat{h}_{\hat{1}\hat{B}} - \frac{\sqrt{f}}{r} \left(\hat{h}_{\hat{A}\hat{B}} - \hat{h}_{\hat{1}\hat{1}} \bar{g}_{\hat{A}\hat{B}} \right), \\ \bar{D}_{\hat{A}} \hat{h}_{\hat{2}\hat{B}} &= \frac{\ell}{r} \partial_{\hat{A}} \hat{h}_{\hat{2}\hat{B}}, \\ \bar{D}_{\hat{A}} \hat{h}_{\hat{B}\hat{C}} &= \frac{\ell}{r} \partial_{\hat{A}} \hat{h}_{\hat{B}\hat{C}}. \end{aligned} \quad (\text{A.19})$$

Appendix B: The lower bound is not positive for some variations

Recall that

$$\xi := \hat{h}_{\hat{1}\hat{1}} + \hat{h}_{\hat{2}\hat{2}}.$$

Consider that part of the contribution of ξ to (V.16) which decouples from ϕ :

$$\int_M \left\{ \frac{f}{8} |\partial_r \xi|^2 - \frac{1}{2} \left[\frac{(3-n)f}{(n-2)r^2} + \frac{f'}{r} - \frac{f''}{4} \right] \xi^2 \right\} V d\mu_{\bar{g}}. \quad (\text{B.1})$$

Note that

$$-\frac{1}{2} \left[\frac{(3-n)f}{(n-2)r^2} + \frac{f'}{r} - \frac{f''}{4} \right] = -\frac{2n - (n(n(n+3) - 6) - 12) \frac{r_0^n}{r^{n+2}}}{8\ell^2(n-2)}.$$

We show below that, in any dimension, there exist functions which render this integral negative. Hence one cannot neglect the constraint equations and/or the gauge condition when attempting to prove positivity of the mass for small perturbations of the metric.

If ξ is supported near infinity, the radial part of the integral (B.1) can be written as

$$\int_{r_0}^{\infty} \left\{ \frac{1}{8} |\partial_r \xi|^2 - \frac{n}{4(n-2)r^2} \xi^2 + \text{l.o.t.} \right\} \frac{r^{n+1}}{\ell^{n+1}} dr, \quad (\text{B.2})$$

where l.o.t. denotes terms which can be made arbitrarily small in comparison with the remaining ones for r 's large enough. In view of the sharp constant for Hardy's inequality in $(n+2)$ -dimensions,

$$\int_0^{\infty} |\xi'|^2 r^{n+1} dr \geq \frac{n^2}{4} \int_0^{\infty} |\xi|^2 r^{n-1} dr,$$

the main terms in (B.2) are non-negative for all ξ if and only if

$$\frac{2n}{n-2} \leq \frac{n^2}{4} \iff n \geq 4. \quad (\text{B.3})$$

This implies that when $n = 3$, the integral is negative for open families of functions ξ which are supported sufficiently far away from the origin.

In higher dimensions some more work is needed. In view of (V.9), we have

$$\int_{r_0}^{\infty} f |\partial_r \xi|^2 r^{n-1} dr = \int_{r_0}^{\infty} \left[f |\partial_r \xi + \frac{\alpha}{r} \xi|^2 + \left(\frac{\alpha}{r} \partial_r f + \frac{(n-2)\alpha - \alpha^2}{r^2} f \right) |\xi|^2 \right] r^{n-1} dr,$$

and so the integral in (B.1) becomes

$$\frac{1}{8} \int_{r_0}^{\infty} f |\partial_r \xi + \frac{\alpha}{r} \xi|^2 \frac{r^{n-1}}{\ell^{n-1}} dr + \frac{1}{8} \int_{r_0}^{\infty} \hat{W}_\alpha(f) |\xi|^2 \frac{r^{n-1}}{\ell^{n-1}} dr, \quad (\text{B.4})$$

where

$$\begin{aligned} \hat{W}_\alpha(f) &= \frac{\alpha-4}{r} f' + \frac{(n-2)\alpha - \alpha^2 + \frac{4(n-3)}{n-2}}{r^2} f + f'' \\ &= -\frac{1}{\ell^2} (\alpha^2 - n\alpha + \frac{2n}{n-2}) \left[1 - \left(\frac{r_0}{r} \right)^n \right] - \frac{1}{\ell^2} n(n+1-\alpha) \left(\frac{r_0}{r} \right)^n. \end{aligned}$$

In particular if we choose $\xi = r^{-\alpha}$ with, when $n \geq 4$,

$$\frac{1}{2} \left(n + \sqrt{n^2 - \frac{8n}{n-2}} \right) < \alpha < n+1, \quad (\text{B.5})$$

then $\hat{W}_\alpha(f) < 0$. Therefore, for $\xi = r^{-\alpha}$ with α satisfying (B.5), the integral in (B.4) is negative.

Appendix C: Regularity at $r = r_0$

We normalise the metric and r so that $r_0 = 1 = \ell$.

For completeness we verify that metric perturbations of the form

$$h = h_{\hat{1}\hat{1}}(\theta^{\hat{1}})^2 + h_{\hat{2}\hat{2}}(\theta^{\hat{2}})^2 + h_{\hat{A}\hat{B}}\theta^{\hat{A}}\theta^{\hat{B}}, \quad (\text{C.1})$$

with the $\theta^{\hat{a}}$'s defined in (A.8), where the functions $h_{\hat{i}\hat{j}}$ are smooth functions of r satisfying

$$h_{\hat{1}\hat{1}}|_{r=1} = h_{\hat{2}\hat{2}}|_{r=1}, \quad (\text{C.2})$$

define a smooth tensor field on the Horowitz-Myers manifold $\mathbb{R}^2 \times \mathbb{T}^{n-2}$, where the last factor denotes an $(n-2)$ -dimensional torus. Here \mathbb{R}^2 is parameterised by polar coordinates (ρ, φ) , where $\rho \in [0, \infty)$ and φ is 2π -periodic, defined by the equations

$$(\rho, \varphi) := \left(\frac{2}{\sqrt{n}} \sqrt{r-1}, \frac{n}{2} \theta \right). \quad (\text{C.3})$$

In particular functions which are smooth in r near $r = 1$ are smooth functions of ρ^2 near $\rho = 0$, as necessary for smooth rotation-invariant functions. It follows from the last equation that we have

$$(\theta^{\hat{2}})^2 = f d\theta^2 = (1 + \rho^2 f_1(\rho^2)) \rho^2 d\varphi^2,$$

for some function f_1 which is smooth in its argument near 0. Now,

$$\begin{aligned} h_{\hat{1}\hat{1}}(\theta^{\hat{1}})^2 + h_{\hat{2}\hat{2}}(\theta^{\hat{2}})^2 \\ &= h_{\hat{1}\hat{1}}((\theta^{\hat{1}})^2 + (\theta^{\hat{2}})^2) + (h_{\hat{2}\hat{2}} - h_{\hat{1}\hat{1}})(\theta^{\hat{2}})^2 \\ &= h_{\hat{1}\hat{1}}((\theta^{\hat{1}})^2 + (\theta^{\hat{2}})^2) + f_2(\rho^2) \rho^4 d\varphi^2, \end{aligned} \quad (\text{C.4})$$

for some function f_2 which is smooth in its argument near zero. The tensor field $(\theta^{\hat{1}})^2 + (\theta^{\hat{2}})^2$ is smooth. In Cartesian coordinates we have $\rho^4 d\varphi^2 = (xdy - ydx)^2$ and $\rho^2 = x^2 + y^2$, and smoothness of h readily follows.

Appendix D: Gauge conditions

Written out in detail, the harmonicity conditions,

$$\bar{D}^i h_{ij} = \frac{1}{2} \bar{D}_j \phi \iff \bar{D}^i \hat{h}_{ij} = \frac{n-2}{2n} \bar{D}_j \phi, \quad (\text{D.1})$$

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$$\bar{D}_1 (\hat{h}_{11} - \frac{n-2}{2n} \phi \bar{g}_{11}) = -\bar{D}_2 \hat{h}_{21} - \bar{D}^A \hat{h}_{A1}, \quad (\text{D.2})$$

$$\bar{D}_1 \hat{h}_{12} = \frac{n-2}{2n} \bar{D}_2 \phi - \bar{D}_2 \hat{h}_{22} - \bar{D}^A \hat{h}_{A2}, \quad (\text{D.3})$$

$$\bar{D}_1 \hat{h}_{1B} = \frac{n-2}{2n} \bar{D}_B \phi - \bar{D}_2 \hat{h}_{2B} - \bar{D}^A \hat{h}_{AB}. \quad (\text{D.4})$$

Equivalently, using (A.15)-(A.19),

$$\sqrt{f} \partial_r (\hat{h}_{\hat{1}\hat{1}} - \frac{n-2}{2n} \phi) = -\frac{1}{\sqrt{f}} \partial_\theta \hat{h}_{\hat{1}\hat{2}} - \frac{\ell}{r} \partial_A \hat{h}_{\hat{1}\hat{A}} + \frac{f'}{2\sqrt{f}} (\hat{h}_{\hat{2}\hat{2}} - \hat{h}_{\hat{1}\hat{1}}) - \frac{\sqrt{f}}{r} (\xi + (n-2) \hat{h}_{\hat{1}\hat{1}}), \quad (\text{D.5})$$

$$\sqrt{f} \partial_r \hat{h}_{\hat{1}\hat{2}} = \frac{n-2}{2n} \frac{1}{\sqrt{f}} \partial_\theta \phi - \frac{1}{\sqrt{f}} \partial_\theta \hat{h}_{22} - \frac{\ell}{r} \partial_A \hat{h}_{\hat{2}\hat{A}} - \frac{f'}{\sqrt{f}} \hat{h}_{\hat{1}\hat{2}}, \quad (\text{D.6})$$

$$\sqrt{f} \partial_r \hat{h}_{\hat{1}\hat{B}} = \frac{n-2}{2n} \frac{\ell}{r} \partial_B \phi - \frac{1}{\sqrt{f}} \partial_\theta \hat{h}_{\hat{2}\hat{B}} - \frac{\ell}{r} \partial_A \hat{h}_{\hat{A}\hat{B}} - \frac{f'}{2\sqrt{f}} \hat{h}_{\hat{1}\hat{B}}. \quad (\text{D.7})$$

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