SIAM REVIEW
Vol. 46, No. 4, pp. 717–728

Recovering Holomorphic Functions from Their Real or Imaginary Parts without the Cauchy–Riemann Equations*

William T. Shaw†

Abstract. Students of elementary complex analysis usually begin by seeing the derivation of the Cauchy–Riemann equations. A topic of interest to both the development of the theory and its applications is the reconstruction of a holomorphic function from its real part, or the extraction of the imaginary part from the real part, or vice versa. Usually this takes place by solving the partial differential system embodied by the Cauchy–Riemann equations. Here I show in general how this may be accomplished by purely algebraic means. Several examples are given, for functions with increasing levels of complexity. The development of these ideas within the *Mathematica* software system is also presented. This approach could easily serve as an alternative in the early development of complex variable theory.

Key words. complex analysis, Cauchy–Riemann equations, computer algebra

AMS subject classifications. 30-01, 30-04, 30D05, 30E20

DOI. 10.1137/S0036144503432151

1. Introduction and Examples.

This article presents an alternative approach to that part of complex analysis that deals with relating the real and imaginary parts of complex holomorphic functions to each other and to the underlying holomorphic function from which they are derived. If you have studied or are studying complex analysis, you will have noted the following standard results. Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. If $f(z)$ is *differentiable* at $z = z_0$, then the Cauchy–Riemann equations (C-R) hold at z_0. The C-R are the pair of conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.1}$$

at z_0. Conversely, if the C-R hold and furthermore u, v are both differentiable[1] functions of two real variables at (x_0, y_0), where $z_0 = x_0 + iy_0$, then $f(z)$ is differentiable

*Received by the editors July 25, 2003; accepted for publication (in revised form) August 10, 2004; published electronically October 29, 2004.
http://www.siam.org/journals/sirev/46-4/43215.html

†OCIAM, The Mathematical Institute, 24-29 St. Giles, Oxford OX1 3LB, England (shaww@maths.ox.ac.uk).

[1] By this we do *not* just mean that u, v have first partial derivatives. *Differentiability* of $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the stronger condition that in a neighborhood of (x_0, y_0), we can write $u(x_0 + h, y_0 + k) = u(x_0, y_0) + h\frac{\partial u}{\partial x}(x_0, y_0) + k\frac{\partial u}{\partial y}(x_0, y_0) + R$, $|R|/\sqrt{|h|^2 + |k|^2} \to 0$ as $(h, k) \to (0, 0)$. In fact, it is sufficient that the first partial derivatives of u exist and are continuous in a neighborhood of (x_0, y_0) for u to be differentiable. But we shall not need to worry about such issues here, and the interested reader may wish to consult section 9.10 of [5] for a detailed discussion of real multivariate

at z_0. If such conditions hold for all z in an open subset $U \in \mathbb{C}$, we say that f is *holomorphic* on U. The terms *analytic* or *regular* are often used synonymously with holomorphic. The writer of this article prefers to reserve the term "analytic" to characterize functions having a power series, and to only identify "holomorphic" and "analytic" once one has proved the theorems that show they are equivalent.

Given a function $u(x, y)$, the question often arises as to how to find the corresponding v and hence f. In standard texts this is almost always presented as an exercise in solving the C-R. Note that a preliminary and very sensible step is to check that u (or v if you start with that) satisfies the two-dimensional Laplace equation (this is implied by the C-R once you realize that you can differentiate infinitely often)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ . \tag{1.2}$$

Here is an example often presented in a traditional development. Let $u = x^2 - y^2$. We have, for v, by the C-R,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = 2x \ , \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = -2y \ . \end{aligned} \tag{1.3}$$

We integrate each of these equations for v and obtain

$$\begin{aligned} v &= 2xy + g(x) \ , \\ v &= 2yx + h(y) \ , \end{aligned} \tag{1.4}$$

where g, h are arbitrary functions. We can conclude the matter by arguing that because we have found the consistent contribution $2xy$ to v, equating both the candidate expressions for v requires that both g and h must be constants, and so $v = 2xy + c$ for some *real* constant c, and hence that

$$f(z) = x^2 - y^2 + 2ixy + ic = (x + iy)^2 + ic = z^2 + ic \ , \quad c \in \mathbb{R} \ . \tag{1.5}$$

To summarize, you need to carry out the following steps:

1. differentiate u;
2. integrate the two partial differential equations for v;
3. pick the functions of integration to get consistency;
4. reorganize to identify f.

To get a feel for this, or to remind yourself how this works, it is a good idea to try a few examples.

1.1. Examples for You to Try. In more general cases the third step will be less trivial than the example given, and the last step may require quite a bit of algebra and insight. If you have not done an example like this for some time, you should try out the following cases:

$$u(x, y) = x^3 - 3xy^2 \ ,$$

$$u(x, y) = 4xy(y^2 - x^2) \ .$$

differentiability. For further discussion of the interplay between real and complex differentiability, see p. 35 of [3] and Chapter 10 of [6].

These two are reasonably straightforward, but you might find these a little tougher:

$$u(x, y) = \exp\left(\frac{x}{x^2 + y^2}\right) \cos\left(\frac{y}{x^2 + y^2}\right) ,$$

$$u(x, y) = (x^2 + y^2)^{\frac{1}{4}} \cos\left[\frac{1}{2} \tan^{-1}\left(\frac{y}{x}\right)\right] .$$

These examples soon convince you that it would be a good thing if there was an alternative route avoiding the four steps given above.

1.2. The Existence of Another Approach. The solution of the C-R is *not* the *only* route you can take, and this article sets out an alternative and *purely algebraic* approach. No claim for originality is made for the results presented here. Indeed, a simplified form of the basic result (Theorem 2.1) appears, for example, as Exercise 101 of Chapter 3 of the student text by Spiegel [7] and on pages 27–28 of Chapter 2 of Ahlfors [2]. But the general theorem and its applications are conspicuous by their absence from mainstream student texts, and this article sets out not only to remedy this, but also to solicit information on the history of this approach. The author welcomes information on the background to these methods. Ahlfors provides an explanation of the basic result of Theorem 2.1, but appears to restrict attention to rational functions. He also states that "the method can be extended to the general case and that a complete justification can be given." The present author is not aware of where this has been written down, if anywhere, and it is also not clear whether Ahlfors considered the result based on points other than the origin. Another issue raised by Ahlfors is that it is necessary to extend the usual notion of u and v to be complex mappings $\mathbb{C}^2 \longrightarrow \mathbb{C}$, and this is made explicit here. The author also wishes to make it clear that the result of Theorem 2.1 was written down by J. Ockendon in Oxford, having been observed by him to be true based on power series considerations. His challenge to this author[2] was to explain why this result must hold and to find out what happens if $f(0)$ does not exist. So this article is another one of those papers emerging from a math common room challenge!

2. The Theorem Based at the Origin. In this section I present the base case of the theorem that allows a holomorphic function to be reconstructed algebraically from its real part. Throughout the development of the idea we shall use the notion of the reflection of a holomorphic function in the real axis. If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is given by $f(z)$, then its reflection $\hat{f} : \mathbb{C} \longrightarrow \mathbb{C}$ is given by

$$\hat{f}(z) = \overline{f(\overline{z})} . \tag{2.1}$$

Note that if f is holomorphic on $U \subset \mathbb{C}$, then $\hat{f}$ is a holomorphic function on the open set that is the reflection of U in the real axis.

THEOREM 2.1. *Let $f(z)$ be holomorphic in a neighborhood of the origin, with real part $u(x, y)$ and imaginary part $v(x, y)$, where $z = x + iy$. Then*

$$\begin{aligned} f(z) &= 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \overline{f(0)} \\ &= 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) + \overline{f(0)} . \end{aligned} \tag{2.2}$$

[2]This challenge was delivered in late 2002.

Proof. Given $f(z)$, let $\tilde{u} : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be given by

$$\tilde{u}(w_1, w_2) = \frac{1}{2}\big[f(w_1 + iw_2) + \hat{f}(w_1 - iw_2)\big] \ . \tag{2.3}$$

Note that by setting

$$w_1 = \frac{z}{2} \ , \quad w_2 = \frac{z}{2i} \ , \tag{2.4}$$

where z is a complex number, we obtain

$$\tilde{u}\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\big[f(z) + \hat{f}(0)\big] = \frac{1}{2}\big[f(z) + \overline{f(0)}\big] \ . \tag{2.5}$$

Observe that if $w_1 = x, w_2 = y$, where $x, y \in \mathbb{R}$, then

$$\begin{aligned} \tilde{u}(x, y) &= \frac{1}{2}\big[f(x + iy) + \hat{f}(x - iy)\big] \\ &= \frac{1}{2}\big[f(x + iy) + \overline{f(x + iy)}\big] \\ &= u(x, y) \ , \end{aligned} \tag{2.6}$$

so that $\tilde{u}$ is the natural extension to $\mathbb{C}^2$ of the real function $u(x, y)$. Here, by the *natural extension* of a real function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ we mean the function $u_{ex} : \mathbb{C}^2 \longrightarrow \mathbb{C}$ you get by plugging into u the complex variable w_1 wherever you see an x and the complex variable w_2 wherever you see a y. This u_{ex}, evaluated at $(\frac{z}{2}, \frac{z}{2i})$, is what we meant when we wrote $u(\frac{z}{2}, \frac{z}{2i})$ in (2.2) in the statement of Theorem 2.1.

We showed above that $\tilde{u}$ and u agree when the inputs x, y are real. So $\tilde{u}$ and u have the *same* natural extension to $\mathbb{C}^2$. Moreover, this extension is exactly $\tilde{u} : \mathbb{C}^2 \longrightarrow \mathbb{C}$. So

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = u_{ex}\left(\frac{z}{2}, \frac{z}{2i}\right) = \tilde{u}\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\big[f(z) + \overline{f(0)}\big] \ . \tag{2.7}$$

A slight rearrangement yields the first line of (2.2) as required. The result for the imaginary part follows by defining

$$\tilde{v}(w_1, w_2) = \frac{1}{2i}\big[f(w_1 + iw_2) - \hat{f}(w_1 - iw_2)\big] \tag{2.8}$$

and making similar observations. □

2.1. Comments on the Complexification of the Real and Imaginary Parts. You have seen the introduction of functions $\tilde{u}, \tilde{v}$ that extend u, v from being real mappings to complex mappings. What's going on here? Are there any other senses in which these functions are *natural* extensions of u, v to the complex? First, note that these functions are naturally holomorphic as mappings from $\mathbb{C}^2$ to $\mathbb{C}$, though a formal definition of what this means is not given here.[3] Second, you should also note (and indeed check for yourself) that

$$\frac{\partial^2 \tilde{u}}{\partial w_1^2} + \frac{\partial^2 \tilde{u}}{\partial w_2^2} = 0 = \frac{\partial^2 \tilde{v}}{\partial w_1^2} + \frac{\partial^2 \tilde{v}}{\partial w_2^2} \tag{2.9}$$

[3]In fact, if you take the definition of differentiability $\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by the footnote to the introduction and replace all real objects by complex objects, you get the right notion of differentiability for $\tilde{u} : \mathbb{C}^2 \longrightarrow \mathbb{C}$.

so that the complex functions satisfy the complex Laplace equation in the same way that the real functions satisfy the real Laplace equation. Finally, as you have already seen, the complex functions with real arguments agree with the real ones when those same arguments are used. So these mappings are indeed quite natural.

2.2. Examples. In the following you will see some examples of how these techniques work. You should try the further examples listed at the end for yourself.

2.2.1. Example 1. Here is a very simple case. We take $u(x, y) = x^2 - y^2$. You already know what this will turn out to be! Note that $u(0, 0) = 0$, so that $f(0) = i\beta$ for some real β. The theorem says that

$$
\begin{aligned}
f(z) &= 2\left(\left(\frac{z}{2}\right)^2 - \left(\frac{z}{2i}\right)^2\right) - \overline{f(0)} \\
&= 2\left(\frac{z^2}{4} + \frac{z^2}{4}\right) - \overline{(i\beta)} \\
&= z^2 + i\beta \ .
\end{aligned} \tag{2.10}
$$

2.2.2. Example 2. Let's try another polynomial: $u(x, y) = 4xy(y^2 - x^2)$. As in the previous example, we can assert that $f(0) = i\beta$ for some real β. The theorem says that

$$
\begin{aligned}
f(z) &= 2 \times 4 \times \frac{z}{2}\frac{z}{2i}\left(\left(\frac{z}{2i}\right)^2 - \left(\frac{z}{2}\right)^2\right) - \overline{f(0)} \\
&= 8\frac{z^4}{16i}(-2) + i\beta \\
&= i(z^4 + \beta) \ .
\end{aligned} \tag{2.11}
$$

2.2.3. Example 3. Now consider the case $u(x, y) = \exp(x)\cos(y)$. Clearly $u(0, 0) = 1$ so we can assert that $f(0) = 1 + i\beta$ with β real. In what follows it helps to remember that for any complex number p,

$$
\cos p = \frac{1}{2}(\exp(ip) + \exp(-ip)) \ . \tag{2.12}
$$

Then

$$
\begin{aligned}
f(z) &= 2\exp\left(\frac{z}{2}\right)\cos\left(\frac{z}{2i}\right) - \overline{(1 + i\beta)} \\
&= \exp\left(\frac{z}{2}\right)\left[\exp\left(i\frac{z}{2i}\right) + \exp\left(-i\frac{z}{2i}\right)\right] - 1 + i\beta \\
&= \exp\left(\frac{z}{2}\right)\left[\exp\left(\frac{z}{2}\right) + \exp\left(-\frac{z}{2}\right)\right] - 1 + i\beta \\
&= \exp(z) + i\beta \ .
\end{aligned} \tag{2.13}
$$

2.2.4. Examples for You to Try. Try out the following—you will find it easier if you make good use of the expression of trigonometric and hyperbolic functions in terms of exponentials:

1. $u(x, y) = y^3 - 3x^2y$.
2. $u(x, y) = \exp(x)\sin(y)$.
3. $v(x, y) = \cosh(x)\cos(y)$.

3. The Theorem Based on a General Point. The result presented above is all very fine, but what are you to do when $f(0)$ does not exist? A related question would be what should you do when confronted with a function like

$$u(x,y) = \frac{x}{x^2+y^2} \ , \tag{3.1}$$

where the denominator would vanish if you made the replacements $x \to \frac{z}{2}, y \to \frac{z}{2i}$? This u is just the real part of $1/z$, so clearly you want to have a more useful and more general theorem to cope with such a basic case.

THEOREM 3.1. *Let $f(z)$ be holomorphic in a neighborhood of the point a. As before, f has real part $u(x,y)$ and imaginary part $v(x,y)$, where $z = x+iy$. Then*

$$\begin{aligned} f(z) &= 2u\left(\frac{z+\overline{a}}{2}, \frac{z-\overline{a}}{2i}\right) - \overline{f(a)} \\ &= 2iv\left(\frac{z+\overline{a}}{2}, \frac{z-\overline{a}}{2i}\right) + \overline{f(a)} \ . \end{aligned} \tag{3.2}$$

Proof. The details are given this time for the imaginary part. Given $f(z)$, let $\tilde{v} : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be given, as before, by

$$\tilde{v}(w_1, w_2) = \frac{1}{2i}\left[f(w_1 + iw_2) - \hat{f}(w_1 - iw_2)\right] \ . \tag{3.3}$$

Observe that if $w_1 = x, w_2 = y$, where $x, y \in \mathbb{R}$,

$$\begin{aligned} \tilde{v}(x,y) &= \frac{1}{2i}\left[f(x+iy) - \hat{f}(x-iy)\right] \\ &= \frac{1}{2i}\left[f(x+iy) - \overline{f(x+iy)}\right] \\ &= v(x,y) \ . \end{aligned} \tag{3.4}$$

We merely need to modify our previous proof by picking w_1, w_2 so that

$$\begin{aligned} z &= w_1 + iw_2 \ , \\ \overline{a} &= w_1 - iw_2 \ , \end{aligned} \tag{3.5}$$

with solution

$$\begin{aligned} w_1 &= \frac{z+\overline{a}}{2} \ , \\ w_2 &= \frac{z-\overline{a}}{2i} \ . \end{aligned} \tag{3.6}$$

Substitution then supplies the result

$$2i\tilde{v}\left(\frac{z+\overline{a}}{2}, \frac{z-\overline{a}}{2i}\right) = f(z) - \overline{f(a)} \tag{3.7}$$

as required. The result for u follows by obvious similar manipulations. □

Note that the proof and indeed the application of this theorem involve the replacement of x and y by the complex variables in (3.6). In what follows we shall refer repeatedly to the *replacement rules* for a general base point. The replacement rules are just the substitutions

$$x \longrightarrow \frac{z+\overline{a}}{2} \ , \quad y \longrightarrow \frac{z-\overline{a}}{2i} \ . \tag{3.8}$$

3.1. Examples with a General Base Point. We are now in a position to explore the extraction of the underlying holomorphic function for much more general situations. Some examples are given here for you to follow; then some examples for you to work through are provided. Many of the examples involve a function u containing the combination $x^2 + y^2$ somewhere in its formula. It is helpful to note that under the replacement rules defined by Theorem 3.1,

$$x^2 + y^2 \longrightarrow \left(\frac{z+\bar{a}}{2}\right)^2 + \left(\frac{z-\bar{a}}{2i}\right)^2 = z\bar{a}\ . \tag{3.9}$$

In the following examples we shall see how the use of a general base point and Theorem 3.1 allows the extension of Theorem 2.1 to treat functions with poles, essential singularities, or branch points at the origin.

3.1.1. Example of a Function with a Simple Pole at the Origin. Consider the function

$$u(x,y) = \frac{x}{x^2+y^2}\ . \tag{3.10}$$

Using the replacement rules, we can assert that

$$f(z) = 2\frac{z+\bar{a}}{2}\frac{1}{z\bar{a}} - \overline{f(a)} = \frac{1}{z} + \frac{1}{\bar{a}} - \overline{f(a)}\ . \tag{3.11}$$

Inspection of this relation at $z = a$ implies that the real part of $f(a)$ is the real part of $\frac{1}{\bar{a}}$, so we deduce that

$$f(z) = \frac{1}{z} + i\beta\ , \quad \beta \in \mathbb{R}\ . \tag{3.12}$$

3.1.2. Example of a Function with an Essential Singularity at the Origin. We recall the first of the functions given as a difficult one in subsection 1.1, given by

$$u(x,y) = \exp\left(\frac{x}{x^2+y^2}\right)\cos\left(\frac{y}{x^2+y^2}\right)\ . \tag{3.13}$$

By the replacement rules, the corresponding holomorphic function is given by

$$\begin{aligned} f(z) &= 2\exp\left(\frac{z+\bar{a}}{2z\bar{a}}\right)\cos\left(\frac{z-\bar{a}}{2iz\bar{a}}\right) - \overline{f(a)} \\ &= 2\exp\left(\frac{1}{2\bar{a}} + \frac{1}{2z}\right)\cos\left(\frac{1}{2i\bar{a}} - \frac{1}{2iz}\right) - \overline{f(a)}\ . \end{aligned} \tag{3.14}$$

Using the expression for the cosine function in terms of exponential functions, this simplifies to

$$\begin{aligned} f(z) &= \exp\left(\frac{1}{z}\right) + \exp\left(\frac{1}{\bar{a}}\right) - \overline{f(a)} \\ &= \exp\left(\frac{1}{z}\right) + i\beta\ , \quad \beta \in \mathbb{R}\ . \end{aligned} \tag{3.15}$$

3.1.3. Example of a Function with a Branch Point at the Origin. Consider the function

$$u(x,y) = \log(\sqrt{x^2+y^2}) = \frac{1}{2}\log(x^2+y^2) \ . \tag{3.16}$$

This is now very straightforward. Using the replacement rules, this becomes

$$\begin{aligned} f(z) &= 2\frac{1}{2}\log(z\overline{a}) - \overline{f(a)} = \log(z) + \log(\overline{a}) - \overline{f(a)} \\ &= \log(z) + i\beta \ , \quad \beta \in \mathbb{R} \ . \end{aligned} \tag{3.17}$$

3.2. Examples for You to Try. Here are some further examples of functions with poles, essential singularities, or branch points at the origin. In the examples involving the inverse tangent function, the algebra will be easier if you use the logarithmic form of the arctangent function $\tan^{-1}$, given by

$$p = \tan(q) \Longleftrightarrow q = \frac{1}{2i}\log\left(\frac{1+ip}{1-ip}\right) \ .$$

3.2.1. Example 1.

$$u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2} \ .$$

3.2.2. Example 2.

$$u(x,y) = \frac{x^3-3xy^2}{(x^2+y^2)^3} \ .$$

3.2.3. Example 3.

$$u(x,y) = \exp\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)\cos\left(\frac{2xy}{(x^2+y^2)^2}\right) \ .$$

3.2.4. Example 4.

$$u(x,y) = (x^2+y^2)^{\frac{1}{4}}\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)\right] \ .$$

3.2.5. Example 5.

$$u(x,y) = (x^2+y^2)^{\frac{1}{2n}}\cos\left[\frac{1}{n}\tan^{-1}\left(\frac{y}{x}\right)\right] \ , \quad n \in \mathbb{Z} \ .$$

4. The Modulus-Argument Variation. Suppose that instead of decomposing $f(z)$ into its real and imaginary parts you write it in modulus-argument form:

$$f = R(x,y)\exp(i\Theta(x,y)) \ . \tag{4.1}$$

Note that while extracting Θ from f is subject to the usual ambiguity of an integer multiple of 2π, if instead Θ is prescribed as a suitably smooth function, there is no difficulty with the reverse process. You can apply Theorem 3.1 to

$$g = \log f = \log R + i\Theta \ , \tag{4.2}$$

and it is a simple matter to deduce that

$$f(z) = \overline{f(a)}\exp\left[2i\Theta\left(\frac{z+\overline{a}}{2},\frac{z-\overline{a}}{2i}\right)\right] = \frac{1}{\overline{f(a)}}R^2\left(\frac{z+\overline{a}}{2},\frac{z-\overline{a}}{2i}\right) . \tag{4.3}$$

In addition to working through the derivation of these identities, you should also check through the following examples for yourself.

4.0.1. An Example Where the Modulus Is Specified. Show that

$$R(x,y) = (x^2+y^2)^{\frac{3}{2}} \Rightarrow f(z) = z^3\exp(i\alpha) , \quad \alpha \in \mathbb{R} .$$

4.0.2. An Example Where the Argument Is Specified. Show that

$$\Theta(x,y) = y \Rightarrow f(z) = \alpha\exp(z) , \quad \alpha \in \mathbb{R} .$$

You might like to try out some other examples.

5. A Novel Use of Poisson's Formula for a Half-plane. *A full appreciation of this section and working out the examples requires some understanding of contour integration.*

So far we have assumed that somehow we can start with an explicit "worked out" formula for u, v, R, or Θ. This is not in fact necessary, and we can make use of other representations. An important case is given by Poisson's formula for a half-plane. If u is a solution of Laplace's equation (1.2) in the set $\{(x,y)|x,y \in \mathbb{R}, y>0\}$ such that $u(x,y) \to 0$ as $\sqrt{x^2+y^2} \to \infty$, and $u(x,0)$ is specified, then Poisson's formula states that

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\ u(s,0)}{(x-s)^2+y^2}ds . \tag{5.1}$$

If you have not met this formula before, you should note that if you want to check that this formula recovers the value of u when $y=0$, you must take the limit $y \to 0$ *after* working out the integral. This formula can be established in at least three different ways:

1. by the use of Green's function methods within potential theory [4];
2. by the use of transform techniques (see, e.g., [1]);
3. as an *application* of the Cauchy integral formula to a half-plane (see, e.g., [1, 6, 7]).

The holomorphic extraction Theorem 3.1 allows a different view to be taken of the Poisson formula. If we take the Poisson formula as given to us from real potential theory, we can show that it *implies* the Cauchy integral formula for a half-plane in a very direct way. If we apply Theorem 3.1 to (5.1), we deduce, after some elementary simplification, that

$$f(z) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{(z-\overline{a})u(s,0)ds}{(s-z)(s-\overline{a})} - \overline{f(a)} . \tag{5.2}$$

This can be simplified further by expanding the integrand into partial fractions. You should check that this means you can write

$$f(z) = h(z) - h(\overline{a}) - \overline{f(a)} , \tag{5.3}$$

where

$$h(z) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u(s,0)ds}{(s-z)} \ . \tag{5.4}$$

For our purposes, rather than analyzing this representation further, it is more useful to take the route of letting the base point a tend to infinity, and simply choose the imaginary constant in f to be such that $f \to 0$ at infinity. Then we can see immediately that

$$f(z) = h(z) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u(s,0)ds}{(s-z)} \ . \tag{5.5}$$

This is of course a stronger result than the Cauchy integral formula applied to the upper half-plane, since only the real part is used along the straight part of the boundary. This result can be used as is, to find the $f(z)$ that is holomorphic in the upper half-plane with a given real part along the x-axis, or to provide a quick derivation of the Hilbert transform. The following two examples, which require some expertise in contour integration, should make this clear.

5.0.1. An Example of Calculating f. Let

$$u(s,0) = \frac{1}{s^2+b^2} \ , \quad b \in \mathbb{R} \ , \quad b > 0 \ .$$

Show that the function of z holomorphic in the upper half-plane with this real part on $y = 0$ is

$$f(z) = \frac{i}{b(z+ib)} \ .$$

5.0.2. The Hilbert Transform. By letting $z = t + i\epsilon$, where $\epsilon \to 0_+$, deduce the *Hilbert transform*—that the imaginary part of f along the $y = 0$ axis is given in terms of the real part by

$$v(t,0) = -\frac{1}{\pi} PV \int_{-\infty}^{\infty}\frac{u(s,0)ds}{s-t} \ .$$

Here the *principal value* (PV) of the integral is interpreted as truncating the integral symmetrically about the singular point:

$$PV\int_{-\infty}^{\infty} = \lim_{\epsilon\to 0_+}\int_{-\infty}^{t-\epsilon} + \int_{t+\epsilon}^{\infty} \ .$$

5.1. Some Interesting Further Topics to Explore. In the Poisson half-plane formula discussed here, attention is restricted to where u is specified along the entire real axis, and attention is further restricted to where u and f are to be found in the upper half-plane only. If you look at the lower half-plane at the same time, matters become even more interesting. You might like to explore the use of the Poisson formula for the lower half-plane, and then the more general set of problems known as *Riemann–Hilbert* problems. A brief introduction to this is given in Chapter 5, particularly section 5.9.2, of [4]. If instead you allow, for example, u to be specified on *part* of the real line (e.g., $x > 0$) and you specify, e.g., $\partial u/\partial y$ or v along a complementary region (e.g., $x < 0$), you are taken into an area of complex variable theory known as the *Wiener–Hopf method*. This is a powerful and practical method for solving *mixed* boundary value problems. An introduction to this is given in section 5.9.4 of [4]. An excellent and extensive discussion of all these matters is given in Chapter 7 of the text by Ablowitz and Fokas [1].

6. Development of Automatic Symbolic Computation Routines. Given the purely algebraic nature of the extraction procedure, it is a straightforward matter to persuade a computer algebra system to carry out the calculation for you. In this section I give some examples using *Mathematica* [8]. If you have this particular system, you might like to explore the use of these routines (and e-mail the author if you find examples where they break down!). If you have another system, you may find it useful to transfer these examples to your system. Either way, you should note the use of the *Mathematica* function `FullSimplify` in dealing with more difficult examples (it is not needed on simple cases, but it is on some examples, such as the square root case).

The extraction functions may also be regarded as the inverse of operations to take the real or imaginary part of the function. These exist in *Mathematica* as the functions `ComplexExpand[Re[expression]]`, `ComplexExpand[Im[expression]]`, where the target functions for `ComplexExpand` have been set so as to "work out the formulae" in as much detail as possible—i.e., it is not allowed to leave functions like `Arg` in the output. You will need to consider this if operating in another computer algebra system. Note that a *Mathematica* notebook, containing the code discussed in this section and several examples, is available on the Internet at http://www.maths.ox.ac.uk/~shaww/SIAMcr.nb. Other applications of *Mathematica* to complex analysis are given in [6]. Now we define the *Mathematica* function that extracts the holomorphic function from its real part:

```
RealToHolo[expr_, anum_, {xsym_, ysym_, zsym_}] := Module[
{abar = Conjugate[anum]}, func= 2*expr /.  {xsym → (zsym+abar)/2, ysym
→ (zsym-abar)/(2*I)};
basecorr = - expr/.{xsym → Re[anum], ysym → Im[anum]};
FullSimplify[func+basecorr+i*β ]]
```

We can test the operation of `RealToHolo` on a list of functions all at once. First we define a list of interesting functions:

```
TestUSet = {x^2 - y^2, 4 x y(y^2 - x^2), Exp[x] Cos[y],
Exp[x/(x^2 + y^2)] Cos[y/(x^2 + y^2)], 1/2 Log[x^2 + y^2],
(x^2 + y^2)^(1/4) Cos[1/2 ArcTan[x, y]]};
```

Then we `Map` our function onto this list using a base point $a = 1$ throughout:

```
Map[RealToHolo[#, 1, x, y, z] &, TestUSet]
```

$$\left\{z^2 + i\beta, i(z^4 + \beta - 1), i\beta + e^z, i\beta + e^{\frac{1}{z}}, i\beta + \log(z), i\beta + \sqrt{z}\right\}$$

The output demonstrates the reliability of the method and its ease of implementation.

6.1. Inversion of the *Mathematica* `ComplexExpand` function. *Mathematica* contains a function `ComplexExpand` that is a good companion to the holomorphic extraction routines. The `ComplexExpand` function offers a neat way of finding the u or v

function given the holomorphic function f. To apply it, we need to get this function to work everything out as much as possible, and not leave functions such as `Arg`, for the argument, within the formula. This is accomplished by an option setting:

```
SetOptions[ComplexExpand, TargetFunctions -> Re, Im];
```

Now one example will suffice:

```
ComplexExpand[Re[Exp[1/(x+I*y)]]]
```

$$e^{\frac{x}{x^2+y^2}} \cos\left(\frac{y}{x^2+y^2}\right)$$

```
RealToHolo[%, 1, x, y, z]
```

$$i\beta + e^{\frac{1}{z}}$$

You should try the other examples in this article to see how `RealToHolo[]` acts as a natural inverse to `ComplexExpand[Re[]]`. Other examples and `ImToHolo` are given in the notebook available over the Internet.

Acknowledgments. The author wishes to thank the referees and editors for their detailed study of this paper and their numerous helpful and very detailed suggestions for its improvement. Thanks are due again to John Ockendon for suggesting this topic.

REFERENCES

[1] M. J. Ablowitz and A. S. Fokas, *Complex Variables, Introduction and Applications*, 2nd ed., Cambridge Texts Appl. Math., Cambridge University Press, Cambridge, UK, 2003.
[2] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1979.
[3] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Grad. Texts in Math. 11, Springer, New York, 1978, 1995.
[4] J. Ockendon, S. Howison, A. Lacey, and A. Movchan, *Applied Partial Differential Equations*, revised ed., Oxford University Press, Oxford, 2003.
[5] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1976.
[6] W. T. Shaw, *Complex Analysis with Mathematica*, Cambridge University Press, Cambridge, UK, forthcoming, 2005.
[7] M. Spiegel, *Theory and Problems of Complex Variables*, Schaum's Outline Series, McGraw-Hill, Singapore, 1981.
[8] S. Wolfram, *The Mathematica Book*, 5th ed., Wolfram Media, Champaign, IL, 2004.