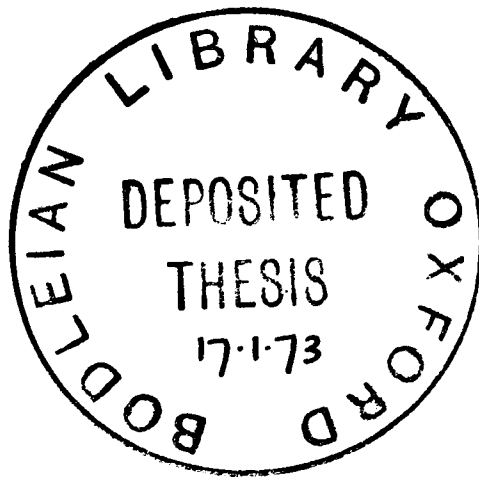


SOME PROBLEMS ON THE RENORMALISATION
OF NON-POLYNOMIAL LAGRANGIANS



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To my parents

and

Maria

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ABSTRACT

A method of analytic renormalisation is developed (in PART I of the thesis) to define the three point time ordered product of massless fields of exponential type as a strictly localisable distribution in the Jaffe Class. The uniqueness property, known for the two point T-product, is verified for the three point T-product for a special choice of finite renormalisation. It is characterised by minimum singularity on the 'light cone' (the Lehmann-Pohlmeyer 'ansatz'); there are no delta function type singularities concentrated on the point $x_1 = x_2 = x_3$.

A model of a massive neutral pseudovector field, W_μ , coupled to a non-conserved fermion current, $j_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi$, is considered (in PART II of the thesis). The generalised Stuckelberg formalism is used to convert the above non-renormalisable coupling into a conventionally renormalisable interaction, $\mathcal{L}_1^{\text{int}} = g : j_\mu A_\mu :$, together with a non-polynomial strictly localisable interaction of the form $\mathcal{L}_2^{\text{int}} = -m \bar{\psi} : (\exp[i\kappa \gamma_5 B] - 1) : \psi$ which can be treated by the methods developed in PART I of this thesis; (A_μ , B) are the Stuckelberg components of the W_μ field, and the B is taken to be a massless pseudoscalar field giving, thus, rise to massless 'superpropagators'. The renormalisation of the model theory is effected with the help of generalised Ward-Takahashi identities by adding suitable gauge invariant counterterms in the original interaction Lagrangian to cancel out the infinities of the theory. Thus the complete theory becomes renormalisable.

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FOREWORD

By 'non-polynomial Lagrangians' we mean strictly localisable Lagrangians that can be expanded out as entire functions, in the free fields, of order less than two. Such Lagrangians belong to the Jaffe class which is characterised by the requirement that the increase of the vacuum expectation values (VEV) of products of interaction Lagrangians, $\langle 0 | \mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n) | 0 \rangle$, in momentum space is bounded by an indicator function $g(t)$ satisfying the condition¹

$$\int_0^\infty \frac{\ln g(t^2)}{1+t^2} dt < \infty$$

Thus, growth such as

$$g(s) \leq \exp[s^\alpha], \quad \alpha < \frac{1}{2}$$

or

$$g(s) \leq \exp \left[s^{1/2} / (\ln s)^2 \right]$$

is allowed.

The thesis consists of two rather disjoint parts. The first part (PART I) - rather mathematical in nature - deals with the exponential self-interaction

$$\mathcal{L}_I(x) = : \left[\exp(g\phi(x)) - 1 - g\phi(x) - g^2/2 \phi(x)^2 \right]:$$

with $\phi(x)$ a scalar massless field, which belongs to the Jaffe class.

The 3-point time ordered product $T \left[\mathcal{L}_I(x_1) \mathcal{L}_I(x_2) \mathcal{L}_I(x_3) \right]$ is defined as a strictly localisable (operator valued) distribution which satisfies the Lehmann-Pohlmeyer ansatz⁵. This part of the work is entirely original, and it has been carried out in collaboration with Dr. P. K. Mitter. We remark that the 3-point function in question has been obtained through a method of analytic renormalisation. The method is quite general, and

can be applied to the case of the n-point time ordered product

$T \left[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n) \right]$ provided a way is found to include finite renormalisations in order to satisfy the 'minimality' criterion of Lehmann, and Pohlmeier.

The second part (PART II) is concerned with the renormalisation of a field theoretic model of a massive neutral vector field coupled to a non-conserved fermion current. This is a non-renormalisable theory, and may serve as a mathematical model suitable for investigating divergence problems associated with more realistic models for Weak Interactions. The model was originally considered by T. D. Lee³¹, and the renormalisation problem, in the context of non-polynomial Lagrangians, was taken up by the authors of Ref. (32,33). Our original contribution lies, firstly, in the possibility of using massless 'superpropagators', and massive fermion propagators, in contrast with the authors of Ref. (32, 33), and secondly, in fixing up the necessary counterterms, in the interaction Lagrangian of the theory, in a gauge invariant manner. This part of the thesis was carried out under the supervision of Dr. J. C. Taylor.

The following notation will be used throughout the thesis:

(a) If x , and y are 4-vectors, we write

$$x \cdot y = \sum_{\mu,\nu=0}^3 x^\mu g_{\mu\nu} y^\nu = x_\mu y_\mu$$

where $g_{\mu\nu} = (1, -1, -1, -1)$

(b) Any n-tuple of variables $(x_1, x_2, \dots, x_n)$ is denoted by underlying: $\underline{x} = (x_1, \dots, x_n)$. The elements of the generalised vector $\underline{x}$ may be 4-vectors, integers, or a set of complex numbers, but the meaning will be clear from the context.

(c) $|S| \equiv \# \{S\}$ = number of elements in a set S .

(d) The space of tempered test functions is denoted by $\mathcal{S}$, and the adjoint space by $\mathcal{S}'$. The Jaffe space of test functions is denoted by $\mathcal{E}$, and the adjoint space by $\mathcal{E}'$, whereas the corresponding spaces in momentum space are denoted by $\mathcal{M}$, and $\mathcal{M}'$.

The organisation of the material of the thesis is as follows. Each PART is divided into a number of Sections. The latter are divided into numbered Subsections, and equations are numbered consecutively in each Section. References are indicated either by the author's name followed by a number or simply by a number. A list of references is found at the end of the thesis.

Finally we would like to remark that the original material of PART I of this thesis has been accepted for publication in the Journal of Mathematical Physics.

PART I

I. INTRODUCTION

The difficulties of "non-renormalisable" field theories, at least at the level of perturbation theory, are well known although, at our present level of understanding, no fundamental objection to (at least a class of) such theories is known¹. At the perturbative level, such theories (e.g., $:\phi^n:$ $n \geq 5$) can indeed be subjected to axiomatic renormalisation^{2,3}, that is, a Lorentz invariant extension procedure of time ordered products of interaction Lagrangians consistent with axiomatic structure (unitarity, causality)⁴; however such procedures are not physically compelling for these theories due to the greater lack of uniqueness of extensions, than is the case of renormalisable theories.

With regards to this problem of uniqueness, Lehmann and Pohlmeyer⁵ made an interesting observation in connection with the two point time ordered product of the interaction Lagrangian:

$$\mathcal{L}_I(x) = : \left[\exp(g\phi(x)) - 1 - g\phi(x) - \frac{g^2}{2} \phi(x)^2 \right] : \quad (1.1)$$

where $\phi(x)$ is a free scalar field in four dimensional space time.

They observed that although $\mathcal{L}_I(x)$ leads to a conventionally non-renormalisable theory, each order of perturbation in $\mathcal{L}_I(x)$ furnishing an infinite set of divergent Feynman graphs, there exists a unique definition of $T \left[\mathcal{L}_I(x_1) \mathcal{L}_I(x_2) \right]$ as a distribution in the sense of Jaffe¹.

The uniqueness was characterised by minimum singularity on the "light cone" corresponding to a certain regularity on the surface $x_1 = x_2$ i.e. absence of $\square^n \delta(x_1 - x_2)$ type singularities. The question naturally arises as to whether higher point time ordered products can be defined with minimum

'light cone' singularity. Since the only ambiguity in the construction of time ordered products, consistent with axiomatic structure, is in the form of distributions concentrated on surfaces of coinciding points³, a construction with minimum singularity, satisfying a regularity condition⁶ on such surfaces, would be automatically unique and would not entail the addition of arbitrary finite counterterms to (1.1). Together with the authors of Ref. (5) we emphasise that, if the question has an affirmative answer (preferably to all orders of perturbation theory), then one might use the regularity condition on the 'light cone vertex' as a boundary condition to fix the dynamics of at least a privileged class of non-renormalisable interaction Lagrangians. This is the motivation for the analysis carried out in this part (PART I) of the thesis. Furthermore the analysis of (1.1) may turn out to be of physical relevance, since non-renormalisable theories of physical interest can be shown to contain such structures⁷.

In this part of the thesis we present a method of analytic renormalisation, motivated by (but different from) that of Speer^{8,9} whereby the divergences of an infinite set of Feynman graphs arising from the three point time ordered product $T \left[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_3) \right]$ is removed, and a minimal extension is secured by a suitable choice of finite renormalisations. The scheme is an extension of the techniques used in an earlier paper of P. K. Mitter⁷, where it was shown, with the restriction that $\phi(x)$ was massless, that the two point time ordered product $T \left[\mathcal{L}_I(x_1) \mathcal{L}_I(x_2) \right]$ could be defined uniquely as a Jaffe distribution with minimum 'light cone' singularity. For simplicity we have considered only the case (1.1). However, the method of renormalisation which we shall develop is readily applicable to the more general case of a strictly localisable¹ interaction Lagrangian of the exponential type:

$$\mathcal{L}_I(x) = \sum_{n=3}^{\infty} \frac{d(n)}{n!} : \phi(x)^n : \quad (1.2)$$

with real coefficients $d(n)$ satisfying

$$\sup_n |d(n)|^{1/n} < \infty \quad (1.3)$$

The material of PART I of the thesis is distributed as follows: In Section II we review the two point T-product $T[\mathcal{L}_I(x_1)\mathcal{L}_I(x_2)]$ using a renormalisation technique different from that of Ref. (5,7). Section III is devoted to the 3-point function. In III.2 we replace the general term, in the formal Wick expansion of $T[\mathcal{L}_I(x_1)\mathcal{L}_I(x_2)\mathcal{L}_I(x_3)]$ in an infinite series of (divergent) Feynman graphs, by an analytically regulated object, a massless generalised Feynman amplitude (GFA) in the sense of Speer^{8,9} which enjoys the following properties: (i) it exists as a well defined tempered distribution and (ii) individual divergences are separately tracked with complex parameters. Once this is done we can define the regulated vacuum expectation value (VEV) of the three point T-product as a series of GFA's converging to a strictly localisable distribution¹. The next step taken in Subsec. III.3 is to generalise the renormalisation procedure of Ref. (7) to the three point T-product. The method is to give an analytic decomposition of the regulated object into a sum of regular and singular parts. Analytic continuation of the regular part to the 'physical' value of the complex parameters gives a renormalisation. The consistency of our procedure is established, through a verification of the unitarity-causality properties of T-products². In Subsec. III.4 it is proved for the three point T-product, that a finite renormalisation may be implemented, so as to secure, precisely as in the case of the two point T-product, a unique extension characterised by a decrease property (in

a subspace of the momentum space) for the real part of the three point function corresponding to minimum singularity on the 'light cone' (in co-ordinate space). We show explicitly the absence of delta function type of singularities concentrated on the surface $x_1 = x_2 = x_3$.

An entirely different approach to this problem has been taken up by K. Pohlmeier¹⁰, and J. G. Taylor¹¹. We wish to emphasise that our motivation in undertaking this analysis has been to extend methods originating in the renormalisation programme of renormalisable theories to interaction Lagrangians of exponential type (i.e. non-polynomial Lagrangians).

Within the frame of Local Field Theory there is a need for a systematic perturbation theory in powers of the Lagrangian (1.1) which is free of ambiguities. However, because of the non-renormalisable nature of (1.1) giving rise to an infinite number of arbitrary parameters due to increasing divergences with increasing order of perturbation, it is far from obvious that such a programme can be carried through successfully. The axioms of renormalisation^{2,3} are too general to restrict the ambiguities of a possible perturbation theory in the interaction Lagrangian (1.1). They must, therefore, be implemented by an extra 'boundary condition', the Lehmann-Pohlmeier ansatz in the form of a regularity condition for the amplitudes on surfaces $x_1 = x_2 = \dots = x_n$. It is worth noting, however, that such a condition is non-trivial. This is because no finite sum of renormalised Feynman graphs can enjoy this property due essentially to the presence of logarithms in renormalised perturbation theory. The first part of the thesis is a report on some progress in the direction of a successful implementation of such a programme.

II. THE TWO POINT FUNCTION AS A DISTRIBUTION

II.1 Preliminaries: We begin by reviewing how the VEV of the two point T-product, $\langle 0 | T [\mathcal{L}_I(x_1) \mathcal{L}_I(x_2)] | 0 \rangle$, may be defined as a generalized function in the space of localisable distributions. In doing so we shall give a partial answer to the following problem proposed by K. Hepp¹².

Problem: Let $\sum_n a_n z^n$ be an entire function such that for $\mathcal{L}_I(x) = \sum a_n : \phi(x)^n :$, and all m , $\langle 0 | \mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_m) | 0 \rangle \in \mathcal{E}'(R^{4m})$ (Jaffe space) where $\phi(x)$ is a free scalar field. Is it possible to define uniquely $\langle 0 | T [\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_m)] | 0 \rangle$ as the limit in the Jaffe Class of the sum of all renormalised graphs for a suitable choice of finite renormalisations? It turns out that in the case of the VEV of the two point, and the three point T-products one can find suitable finite renormalisations so that the Lehmann-Pohlmeyer ansatz⁵ is satisfied thereby ensuring uniqueness of the amplitudes. In this section, however, we shall only deal with the two point function.

Throughout PART I of the thesis we need a form of Wick's theorem applicable to T-products of fields of the case (1.1). This form of the theorem is provided by Hori's formula:

$$T [\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)] = \sum_{i_r=3}^{\infty} \sum_{\substack{\nu_{rs}=0 \\ \nu_{rs}=\nu_{sr}; \nu_{rr}=0}}^{\infty} \left[\prod_{r=1}^n \frac{g^{i_r}}{[i_r - \sum_{s'} \nu_{rs'}]!} \right] \cdot \left[\prod_{1 \leq r < s \leq n} \frac{[g^2 \Delta_F(x_r x_s)]^{\nu_{rs}}}{\nu_{rs}!} \right] : \phi(x_1)^{[i_1 - \sum_{s'} \nu_{rs'}]} \dots \phi(x_n)^{[i_n - \sum_{s'} \nu_{ns'}]} : \quad (2.1)$$

For the 2-point T-product (2.1) implies

$$T [\mathcal{L}_I(x_1) \mathcal{L}_I(x_2)] = \sum_{i_1, i_2=0}^{\infty} \mathcal{T}_2(x_1, x_2) g^{i_1+i_2} : \frac{\phi(x_1)^{i_1}}{i_1!} \frac{\phi(x_2)^{i_2}}{i_2!} :$$

and for connected contributions in the case of a massless scalar field:

$$\mathcal{T}_2(x_1, x_2) = \langle 0 | T [\mathcal{L}_I(x_1) \mathcal{L}_I(x_2)] | 0 \rangle = \sum_{\nu=1}^{\infty} \frac{[g^2]^\nu}{\nu!} \left[(4\pi^2)(-(x_1-x_2)^2 + i0) \right]^{-\nu} \quad (2.2)$$

II.2 Analytic Regularisation: From (2.2) we have

$$\mathcal{T}_2(x_1, x_2) = \sum_{\nu=1}^{\infty} \frac{[g^2]^\nu}{\nu!} \mathcal{T}_2^{(\nu)}(x_1, x_2) \quad (2.3)$$

where

$$\mathcal{T}_2^{(\nu)}(x_1, x_2) = \left[(4\pi^2)(-(x_1-x_2)^2 + i0) \right]^{-\nu} \quad (2.4)$$

Each term of (2.4) may be realised as a connected graph $G(\{V_1, V_2\}, \mathcal{L}(G))$ with a set of vertices $\{V_1, V_2\}$, and a set lines $\mathcal{L}(G)$, with $\# \mathcal{L}(G) = \nu$. In fact, if we make a correspondence with Feynman graphs the line of G is a multiplet of ν lines. The formal nature of (2.3) arises from the lack of definition of (2.4), which involves products of distributions. Therefore, it is necessary to replace $\mathcal{T}_2^{(\nu)}(x_1, x_2)$ by an analytically regulated object, a generalised Feynman amplitude (GFA) in the sense of Speer.

Speer, and Westwater⁹ have shown that in any Feynman amplitude a massless multiplet can be regularised as a whole by means of a single complex parameter. In fact they have proved the following theorem:

Theorem 1. Let $\tilde{F}_G(\underline{\lambda})(\underline{p})$ be the GFA associated with a graph G , and let $\mathcal{X}$ be a massless multiplet in G . Furthermore let G' be the Feynman graph obtained from G by replacing the lines of $\mathcal{X}$ by a single massless line, l_a . Then,

$$\tilde{F}_G(\underline{\lambda})(\underline{p}) = \tilde{F}_{G'}(\underline{\lambda}')(\underline{p})$$

provided

$$\begin{aligned} \lambda_j' &= \lambda_j, \quad j \notin \mathcal{X} \\ \lambda_a' &= \sum_{j \in \mathcal{X}} \lambda_j - [|\mathcal{X}| - 1] \end{aligned}$$

We elaborate briefly on the proof of the theorem in Appendix I.

However, as a direct consequence of the above theorem, and Hori's formula we deduce the following Lemma:

Lemma: The VEV of the n -point T-products of fields of type (1.1) (corresponding to complete graphs) can be analytically regulated by means of $\binom{n}{2}$ complex parameters.

Thus, for the two point function $\mathcal{T}_2(x_1, x_2)$ we need only one complex parameter. In lieu of (2.3), therefore, we consider

$$\mathcal{T}_2(\lambda)(x) = \sum_{\nu=1}^{\infty} \frac{[g^2/(4\pi^2)]^{(1-\lambda)\nu}}{\Gamma[(1-\lambda)\nu+1]} \left[-(x_1 - x_2)^2 + i0 \right]^{(\lambda-1)\nu} \quad (2.5)$$

and the complex parameter λ is restricted to the region:

$$\Lambda = \{ \lambda \mid 0 < \operatorname{Re} \lambda < 1, 0 < \operatorname{Im} \lambda < \infty \}$$

Remark: We have introduced a λ -dependent term $\{\Gamma[(1-\lambda)\nu+1]\}^{-1} [g^2/(4\pi^2)]$ in the place of $\{\Gamma[\nu+1]\}^{-1} [g^2/(4\pi^2)]$. This is equivalent to a choice of a finite renormalisation¹³.

In the next subsection we turn to the renormalisation of (2.5)

II.3 Renormalisation: It is readily seen that (2.5), as a series of meromorphic distributions in $\mathcal{S}'(R^4)$ ¹⁴ converges in the topology of the space $\mathcal{E}'_g(R^4)$ chosen with indicator function of order of growth ρ , $1/3 < \rho < 1/2$, which is consistent with strict localisability¹.

We obtain from (2.5), by continuity of the Fourier operator:

$$\tilde{\mathcal{T}}_2(\lambda)(p) = \lim_{N \rightarrow \infty} \tilde{\mathcal{T}}_2^N(\lambda)(p) \quad (\in \mathcal{M}'_g(R^4)) \quad (2.6)$$

where

$$\tilde{\mathcal{T}}_2^N(\lambda)(p) = \sum_{\nu=1}^N \tilde{\mathcal{T}}_2^\nu(\lambda)(p) \quad (2.7)$$

with $\tilde{\mathcal{T}}_2^\nu(\lambda)(p)$ given by

$$\tilde{\mathcal{J}}_2^\nu(\lambda)(p) = (-i)(4\pi)^2 \frac{[g^2/(4\pi)^2]^{(1-\lambda)\nu}}{\Gamma[(1-\lambda)\nu+1]} e^{-i\pi(1-\lambda)\nu} \frac{\Gamma[(\lambda-1)\nu+2]}{\Gamma[(1-\lambda)\nu]} [p^2 + i0]^{\nu(1-\lambda)-2} \quad (2.8)$$

From the identity

$$\Gamma[2+(\lambda-1)\nu] = \Gamma[\nu\lambda+1] / \left\{ \nu^{\nu-1} \prod_{k=2}^{\nu} [\lambda - (1-k/\nu)] \right\}$$

it is clear that $\lambda = 0$ is a singular point for all terms (excepting the first) of (2.3). The k th term has poles at $\lambda = 1 - k/\nu$, $k=2, \dots, \nu$ (i.e. we have a pole at $\lambda=0$ plus a sequence of neighbouring poles).

Note, however, that the pole nearest to the origin occurs at $\lambda = 1/\nu$ ($\nu \geq 2$).

Hence, as ν increases the neighbouring poles approach $\lambda=0$ so that

$\tilde{\mathcal{J}}_2(\lambda)(p)$ has a non-isolated singularity at $\lambda=0$.

Define $\mathcal{O}_\nu$ to be

$$\mathcal{O}_\nu = \left\{ f(\lambda) \mid f(\lambda) \prod_{k=2}^{\nu} [\lambda - (1-k/\nu)] \text{ is analytic in } |\lambda| < \varepsilon \right\}$$

and

$$\mathcal{O}_N = \bigcup_{\nu=1}^N \mathcal{O}_\nu$$

Then, it is clear that $\tilde{\mathcal{J}}_2^N(\lambda) \in \mathcal{O}_N$. Now, the generalised evaluator of Speer⁸ is defined to be the map:

$$\mathcal{W}_N : \mathcal{O}_N \longrightarrow \mathbb{C}$$

such that (i) $\mathcal{W}_N$ is linear, and (ii) if $f(\lambda) \in \mathcal{O}_N$ is analytic at $\lambda=0$, then $\mathcal{W}_N f(\lambda) = f(0)$. It is easy to see that the following representation satisfies (i), and (ii):

$$\mathcal{W}_N f(\lambda) = \frac{1}{2\pi i} \int_{C_{R_N}} d\lambda \frac{f(\lambda)}{\lambda}, \quad \forall f(\lambda) \in \mathcal{O}_N$$

with C_{R_N} a circle of radius $R_N < 1/N$. Then due to the linearity of

$\mathcal{W}_N$ we obtain

$$\mathcal{W}_N \tilde{\mathcal{J}}_2^N(\lambda)(p) = \sum_{\nu=1}^N \mathcal{W}_N \tilde{\mathcal{J}}_2^\nu(\lambda)(p) \quad (2.9)$$

Because of the restriction on R_N the contour C_{R_N} encloses the simple pole of $\tilde{\mathcal{J}}_2^\nu$ ($\nu=2, \dots, N$) at the origin, and no other singularities.

Thus, defining

$$\tilde{\mathcal{T}}_2^N(p)_R = \omega \tilde{\mathcal{T}}_2^N(\lambda)(p)$$

we obtain from (2.8), and (2.9)

$$\begin{aligned} \tilde{\mathcal{T}}_2^N(p)_R &= \frac{i g^2}{[p^2 + i0]} - \frac{i (4\pi)^2}{[p^2 + i0]^2} \sum_{\nu=2}^N \frac{[g^2]^\nu [p^2 + i0]^\nu}{[(4\pi)^2]^\nu \Gamma[\nu-1] \Gamma[\nu] \Gamma[\nu+1]} \\ &\quad \cdot \left\{ \ln \left[\frac{g^2}{(4\pi)^2} (p^2 + i0) \right] - \psi(\nu-1) - \psi(\nu) - \psi(\nu+1) - i\pi \right\} \quad (2.10) \end{aligned}$$

However, because of the non-isolated singularity of $\tilde{\mathcal{T}}_2(\lambda)$ at $\lambda = 0$

the above procedure does not go through in the limit of $N \rightarrow \infty$.

In order to avoid this difficulty we shall use the concept of weak convergence

in the space $\mathcal{M}'_g$. First we observe that there exists a uniform bound

of (2.10), namely $|\tilde{\mathcal{T}}_2^N(p)_R| \leq C \exp[(\|p\|^2)^{1/3}]$, $\forall N$ where

$\exp[(\|p\|^2)^{1/3}]$ is a locally integrable function which defines a functional on $\mathcal{M}_g(R^4)$. Also $\lim_{N \rightarrow \infty} \tilde{\mathcal{T}}_2^N(p)_R = \tilde{\mathcal{T}}_2(p)_R$

almost everywhere with

$$\begin{aligned} \tilde{\mathcal{T}}_2(p)_R &= \frac{i g^2}{[p^2 + i0]} - \frac{i (4\pi)^2}{[p^2 + i0]^2} \sum_{\nu=2}^{\infty} \frac{[g^2]^\nu [p^2 + i0]^\nu}{[(4\pi)^2]^\nu \Gamma[\nu-1] \Gamma[\nu] \Gamma[\nu+1]} \\ &\quad \cdot \left\{ \ln \left[\frac{g^2}{(4\pi)^2} (p^2 + i0) \right] - \psi(\nu-1) - \psi(\nu) - \psi(\nu+1) - i\pi \right\} \quad (2.11) \end{aligned}$$

Then, the function $\tilde{\mathcal{T}}_2(p)_R$ also defines a continuous linear functional,

$\tilde{\mathcal{T}}_{2,R}$, on the space $\mathcal{M}_g(R^4)$, and $\lim_{N \rightarrow \infty} \tilde{\mathcal{T}}_{2,R}^N = \tilde{\mathcal{T}}_{2,R}$

in the weak topology of $\mathcal{M}'_g(R^4)$. To see this we apply Lebesgue's

dominated convergence theorem to the sequence $\tilde{\mathcal{T}}_2^N(p)_R \tilde{\phi}(p)$,

$\tilde{\phi}(p) \in \mathcal{M}_g(R^4)$ which is majorised by the integrable function

$C \exp[(\|p\|^2)^{1/3}] |\tilde{\phi}(p)|$. Thus we obtain

$$\begin{aligned} \int |\tilde{\mathcal{T}}_2(p)_R \tilde{\phi}(p)| dp &= \lim_{N \rightarrow \infty} \int |\tilde{\mathcal{T}}_2^N(p)_R \tilde{\phi}(p)| dp \\ &\leq C \int \exp[(\|p\|^2)^{1/3}] |\tilde{\phi}(p)| dp \end{aligned}$$

This bound guarantees the absolute convergence of the integral

$$\int \tilde{\mathcal{T}}_2(p)_R \tilde{\phi}(p) dp = (\tilde{\mathcal{T}}_{2,R}, \phi)$$

Finally as the weak limit of continuous functionals, $\tilde{\mathcal{T}}_{2,R}^N$, the functional $\tilde{\mathcal{T}}_{2,R}$ is also continuous.

II.4 Uniqueness: $\tilde{\mathcal{T}}_2(p)_R$, given by (2.11), can be still implemented by a finite renormalisation, which is necessarily a functional of the form $\tilde{\mathcal{X}}(p)$, $\mathcal{X}(s)$ being an entire function of order less than one half. However, we have already chosen special finite renormalisations (See Remark in subsection II.2) and the object of this subsection is to show that such a choice of finite renormalisations leads to a unique definition of the renormalised two point function $\tilde{\mathcal{T}}_2(p)_R$.

Definition:

$$\mathcal{T}_2(x_1, x_2) = \mathcal{F}^{-1} \left[\tilde{\mathcal{T}}_{2,R} \right] (x_1, x_2) \quad (2.12)$$

We claim that the above definition is unique, by virtue of absence of singularities of the form $\square^n \delta(x_1 - x_2)$. To see this we use the following representation for (2.11):

$$\tilde{\mathcal{T}}_2(p)_R = (4\pi)^2 \frac{1}{2} \int_{\Gamma} dz \frac{\cot \pi z [q^2]^z \Gamma[2-z] [-p^2 - i0]^{z-2}}{[(4\pi)^2]^z \Gamma[z+1] \Gamma[z]} \quad (2.13)$$

with the contour Γ shown in fig. (II.4.1) encircling the positive real axis counterclockwise.

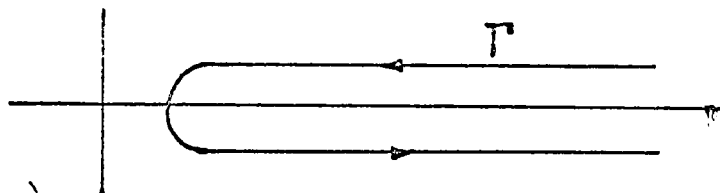


Fig. (II.4.1)

The uniqueness of $\tilde{\mathcal{T}}_2(p)_R$ will be established once the $\text{Re} [i \tilde{\mathcal{T}}_2(p)_R]$ is shown to be unique, because the $\text{Im} [i \tilde{\mathcal{T}}_2(p)_R]$ is fixed by unitarity, and causality³. Now, utilising (2.13) we obtain for the real part of $i \tilde{\mathcal{T}}_2(p)_R$ the following expression

$$\text{Re}[i \tilde{\mathcal{T}}_2(p)_R] = i \int_{\Gamma} dz \frac{[g^2]^z [(4\pi)^2]^{-z} \Gamma[2-z] [-p^2]^{z-2}}{\sin \pi z \Gamma[z+1] \Gamma[z]} \quad (2.14)$$

The contour Γ can be opened up parallel to the imaginary axis without losing convergence. Then (2.14) is rapidly decreasing as $p^2 \rightarrow +\infty$.

In fact the integral can be recognized as a Meijer function G_{03}^{20} with

$g^2/(4\pi)^2 (-p^2)$ as argument, and which decreases exponentially¹⁵, as

$p^2 \rightarrow +\infty$. This implies, by a space smearing argument⁵, that

$\text{Re}[i \mathcal{T}_2(x)]$ is free of delta function type of singularities concentrated at $x=0$. If we add any finite renormalisation, $\tilde{\mathcal{X}}(p)$, to $\tilde{\mathcal{T}}_2(p)_R$ such that $\tilde{\mathcal{T}}_2'(p)_R = \tilde{\mathcal{T}}_2(p)_R + \tilde{\mathcal{X}}(p)$, then by a well known theorem for entire functions of order less than one half the $\text{Re}[i \tilde{\mathcal{T}}_2'(p)_R]$ will not decrease in any direction in momentum space. The F.T. of $\tilde{\mathcal{T}}_2'(p)_R$ will then contain singularities of the form $\square^n \delta(x_1 - x_2)$.

It is therefore, clear that our definition is unique by minimum 'light cone' singularity, the criterion of Ref. (5).

In the subsequent section of PART I we will show how these results may be generalised to the case of the three point function.

III. THE THREE POINT FUNCTION

III.1 In this and the following subsections we turn to the main object of concern of PART I of the thesis, namely the three point time ordered product of (1.1) with minimum 'light cone' singularity. We shall show that it is possible to develop perturbation theory to third order in powers of the interaction Lagrangian (1.1) free of ambiguities which are necessarily present, if one tries to carry out the renormalisation programme in the conventional way of expanding the field (1.1) in powers of the (minor) coupling constant. It is not obvious at all that such a programme can be carried through successfully, but it is highly desirable to know the answer in view of the fact that many models of physical interest in Lagrangian field theory give rise to exponential field interactions of the (1.1)¹⁶. This is really the motivation for the analysis carried out in this part of the thesis.

The inherent ambiguities (in the form of entire functions) of the theory can be traced back to the non-renormalisable nature of the interaction Lagrangian (1.1). Indeed, if we regard renormalisation as a constructive form of the Hahn-Banach theorem¹⁷, the presence of a denumerable set of arbitrary parameters is an indication for the lack of uniqueness of the extension procedure of $\langle 0 | T [\mathcal{L}_I(x_1) \mathcal{L}_I(x_2) \mathcal{L}_I(x_3)] | 0 \rangle$ defined on $\mathcal{G}_0(R^{3 \times 4})$ (the set of test functions in the Jaffe class which vanish together with their derivatives to all orders whenever $x_i = x_j$, $1 \leq i < j \leq 3$) into the whole to Jaffe space $\mathcal{G}(R^{3 \times 4})$. However, the minimal singularity condition of Lehmann, and Pohlmeier can serve as a 'boundary condition' in fixing up possible finite renormalisations to ensure a unique extension. Of course, the physical content of such a criterion is still an open question, but one is tempted to say that

nature singles out the least singular (perturbative) quantum field theory corresponding to a given classical interaction.

In Section II we have seen that the second ordered perturbation expansion in (1.1) is free of any 'ultra violet' divergences leading to finite results for scattering amplitudes (to second order) despite of the non-renormalisable nature of the interaction (1.1). We have indeed demonstrated the high energy damping of the real part of $[i \tilde{\mathcal{T}}_2(p)_R]$, and it is easy to see (by direct computation from (2.13)) that

$$\text{Im} [i \tilde{\mathcal{T}}_2(p)_R] \sim \exp [(p^2)^{1/3}] , p^2 \rightarrow +\infty$$

This latter result is to be expected from phase space considerations⁷ of a system of N massless particles in the limit of $N \rightarrow \infty$. See fig. (III.1.1) below.

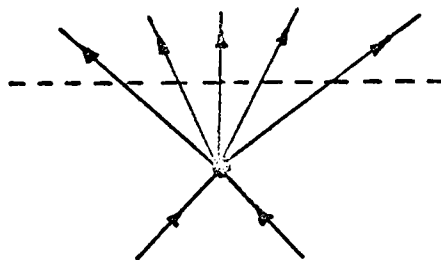


Fig. (III.1.1)

As a consequence cross sections are expected to rise strongly with energy, a situation which implies that perturbation theory can at best be meaningful in the low energy region. Another interesting feature of $\tilde{\mathcal{T}}_2(p)_R$ is the logarithmic dependence in 'g' (due to the presence of double poles in (2.13)) which is closely related to the non-renormalisable nature of the interaction Lagrangian (1.1).

It is interesting to see how these results generalise to the case of the three point function, and this is indeed the subject matter of the remaining subsections of PART I of the thesis. We remark, however, that the three point function, in momentum space, is a function of the three external momenta satisfying a constraint imposed by the overall delta function, and the problem of implementing a unitary and causal amplitude

in order to ensure the correct asymptotic behaviour to satisfy the 'minimality condition' is expected to be far from trivial.

The following subsections deal with the more technical aspects of the problems encountered in the third order perturbation expansion.

III.2 MASSLESS GENERALISED FEYNMAN AMPLITUDES

III.2.1 From (2.1), restricting ourselves to irreducible contributions (these are defined only away from the surface $x_1 = x_2 = x_3$), we obtain

$$\begin{aligned} \mathcal{T}_3(x_1, x_2, x_3) &= \langle 0 | T \left[\mathcal{L}_I(x_1) \mathcal{L}_I(x_2) \mathcal{L}_I(x_3) \right] | 0 \rangle \\ &= \sum_{\underline{\nu}=1}^{\infty} \frac{[g^2]^{\sum_{\ell=1}^3 \nu_{\ell}}}{\prod_{\ell=1,2,3} [\nu_{\ell}!]} \mathcal{T}_3^{(\nu_1, \nu_2, \nu_3)}(x_1, x_2, x_3) \end{aligned} \quad (3.1)$$

where

$$\mathcal{T}_3^{(\nu_1, \nu_2, \nu_3)}(x_1, x_2, x_3) = \prod_{\ell=1}^3 \left[(4\pi^2) \left(-(x_{\ell_i} - x_{\ell_f})^2 + i0 \right) \right] \quad (3.2)$$

Each term of (3.1) may be realised as a connected graph $G(\{V\}, \mathcal{L})$ with a set of vertices $\{V\}$, with $\# \{V\} = 3$ and a set of lines $\mathcal{L}$, with $L = |\mathcal{L}| = 3$. If we make a correspondence with Feynman graphs each line $\ell \in \mathcal{L}$ is a multiplet of ν_{ℓ} massless lines. The formal nature of (3.1 - 3.2) arises from the lack of definition of (3.2) which involves products of distributions. See Appendix I.

$\mathcal{T}_3^{(\nu_1, \nu_2, \nu_3)}$ is a Feynman amplitude with massless lines. Thus

our first step will be to replace it by an analytically regulated object, a generalised Feynman amplitude (GFA) in the sense of Speer^{8,9} which (i) exists as a distribution (ii) reproduces (3.2) formally when regulating parameters are assigned certain values (Subsection III.2.2). We will then analytically continue the massless GFA to a region (Subsection III.2.3)

in which (3.1), with $\mathcal{T}_3^{(\nu_1, \nu_2, \nu_3)}$ replaced by GFA, represents a series of distributions in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$ converging to an element of $\mathcal{E}'_3(\mathbb{R}^{4 \times 3})$ (Subsection III.2.4). At this stage we will have achieved a definition of the regulated version of (3.1) in a form suitable for renormalisation.

III.2.2 In accordance with our programme we will introduce in this subsection an analytically regulated version of the massless Feynman amplitude (3.2), avoiding difficulties with infrared divergences¹⁷. In this subsection ν_1, ν_2, ν_3 in (3.2) are arbitrary positive integers but fixed.

In view of the Lemma of Section II, we expect that $\mathcal{T}_3^{(\nu_1, \nu_2, \nu_3)}(x_1, x_2, x_3)$ can be regularised by means of three complex parameters. Thus, in lieu of the undefined factors in (3.2) we introduce

$$\Delta^{(\nu_\ell)}(\lambda_\ell) = \lim_{\eta \downarrow 0} \frac{1}{[4\pi^2]^{\nu_\ell}} \left[- (x_{\ell_i} - x_{\ell_f})^2 + i\eta \|x_{\ell_i} - x_{\ell_f}\|^2 \right]^{[\lambda_\ell - 1]\nu_\ell} \quad (3.3)$$

which is a meromorphic distribution¹⁴ in $\mathcal{S}'(\mathbb{R}^{4 \times 2})$ (λ_ℓ : complex parameters) and formally reproduces the undefined factors in (3.2) for $\lambda_\ell = 0$. $\| \quad \|$ is the Euclidean norm in $\mathbb{R}^4$. We then introduce a further real (> 0) parameter r :

$$\Delta^{(\nu_\ell)}(\lambda_\ell)(x) = \lim_{\eta \downarrow 0} \lim_{r \downarrow 0} \Delta_{\eta, r}^{(\nu_\ell)}(\lambda_\ell)(x) \quad (3.4)$$

with the R.H.S. of (3.4) defined as the inverse F.T. (in $\mathbb{R}^4$) of

$$\begin{aligned} \Delta_{\eta, r}^{(\nu_\ell)}(\lambda_\ell)(p) &= \frac{i(1-i\eta)(1+\eta^2)^{1/2}}{[4\pi^2]^{\nu_\ell}} e^{\frac{[(\lambda_\ell-1)\nu_\ell]\pi i}{2}} \cdot \frac{2^{2[(\lambda_\ell-1)\nu_\ell+1]}}{\Gamma[(1-\lambda_\ell)\nu_\ell]} \\ &\cdot \int_r^\infty d\alpha_\ell \alpha_\ell^{[(\lambda_\ell-1)\nu_\ell+1]} \exp i\alpha_\ell [p^2 + i\eta \|p^2\|] \end{aligned} \quad (3.5)$$

with $\text{Re } \lambda_\ell < 1$.

$\Delta_{\eta,r}^{(\nu_\ell)}(\lambda_\ell)(x_{\ell_i} - x_{\ell_f})$ is continuous and bounded, the product

$$\mathcal{T}_{\eta,r}^{\underline{\nu}}(\underline{\lambda})(\underline{x}) = \prod_{\ell=1}^3 \Delta_{\eta,r}^{(\nu_\ell)}(\lambda_\ell)(x_{\ell_i} - x_{\ell_f}) \quad (3.6)$$

is well defined and may be evaluated by standard methods⁸ to get the regulated Feynman amplitude in α -space:

$$\tilde{\mathcal{T}}_{\eta,r}^{\underline{\nu}}(\underline{\lambda})(\underline{p}) = [8\pi]^2 \delta\left(\sum_i^3 p_i\right) \prod_{\ell=1}^3 \left[\frac{2^{2[(\lambda_\ell-1)\nu_\ell]} e^{\frac{\pi i}{2}[(\lambda_\ell-1)\nu_\ell]}}{[4\pi^2]^{\nu_\ell} \Gamma[(1-\lambda_\ell)\nu_\ell]} \right] \cdot \frac{1}{[1+\eta^2]^{1/2} [1+i\eta]}$$

$$\cdot \int_r^\infty d\alpha_1 \int_r^\infty d\alpha_2 \int_r^\infty d\alpha_3 \frac{\prod_{\ell=1}^3 [\alpha_\ell^{(\lambda_\ell-1)\nu_\ell+1}]}{[\alpha_1 + \alpha_2 + \alpha_3]^2} \exp \frac{i}{1+\eta^2} \left[\frac{1}{[\alpha_1 + \alpha_2 + \alpha_3]} \right] \cdot \quad (3.7)$$

$$\cdot \left\{ [\alpha_3 \alpha_1 p_1^2 + \alpha_1 \alpha_2 p_2^2 + \alpha_2 \alpha_3 p_3^2] + i\eta [\alpha_3 \alpha_1 \|p_1\|^2 + \alpha_1 \alpha_2 \|p_2\|^2 + \alpha_2 \alpha_3 \|p_3\|^2] \right\}$$

In order to take the $r \rightarrow 0+$ limit on (3.7) in a restricted region of the regulating parameters $\underline{\lambda}$ we introduce⁸ a sectorial decomposition of α -space. We write (3.7) in the form

$$\tilde{\mathcal{T}}_{\eta,r}^{\underline{\nu}}(\underline{\lambda}) = \sum_{\rho} \tilde{\mathcal{T}}_{\eta,r}^{\underline{\nu}}(\underline{\lambda})_{\rho} \quad (3.8)$$

where ρ is a 1-1 map of $\{1,2,3\}$ onto $\mathcal{L}$ and π_{ρ} is the sector in the α -space given by:

$$\pi_p = \{ \underline{\alpha} \mid 0 \leq \alpha_{p(1)} \leq \alpha_{p(2)} \leq \alpha_{p(3)} \leq \infty \} \quad (3.9)$$

and introduce sector coordinates $\alpha_{p(l)} = t_3 \cdots t_l$ such that $0 \leq t_3 \leq \infty$, $0 \leq t_l \leq 1$ ($l=1,2$) with Jacobian $\prod_1^3 t_l^{l-1}$. Then the ' α -integral' in (3.7) in the sector π_p is:

$$(\alpha\text{-integral})_p = \int_r^\infty dt_3 t_3^{\mu_3^p+3} \int_0^1 dt_2 \int_0^1 dt_1 \prod_{l=1}^2 [t_l^{\mu_l^p+2l-1}] \cdot [E(t_1, t_2)]^{-2}.$$

$$\exp \left[i \frac{t_3}{1+\eta^2} \cdot \frac{t_2}{E(t_1, t_2)} \{ P_1^p(t; p) + i\eta P_2^p(t; p) \} \right] \quad (3.10)$$

where (i) $\mu_l^p \equiv \sum_{l'=1}^l [(\lambda_{p(l')} - 1) \nu_{p(l')}]$

(ii) $E(t_1, t_2) \equiv 1 + t_2 + t_1 t_2$

(iii) $P_1^p \equiv p_{p(3)}^2 + t_1 t_2 p_{p(2)}^2 + t_1 p_{p(1)}^2$

$$P_2^p \equiv \|p_{p(3)}\|^2 + t_1 t_2 \|p_{p(2)}\|^2 + t_1 \|p_{p(1)}\|^2 \quad (3.11)$$

and the lower end points of t_1, t_2 integrals are r -dependent, and tend to zero as $r \rightarrow 0^+$. We shall now prove the $r \rightarrow 0^+$ limit. But first we make the following remark:

Remark 3.1

The ' t_2 ' factorisation of the quadratic form in the exponential in (3.10) is characteristic of massless Feynman amplitudes. It is in fact a special case of a more general factorisation for n-point amplitudes (Lemma 2.2.20 of Ref. (9)), and has implications for the structure of singularities in $\underline{\lambda}$ space which correspond to divergences arising from convolution integrals in momentum space. Unfortunately this 't' factorisation has not been taken into account in the analysis for the n-point amplitude of Ref. (11). We remark also that we have introduced imaginary parts through positive definite quadratic forms; as we shall see this enables us to prove easily the existence of the $\eta \downarrow 0$ boundary value in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$ for the massless amplitude. Again the existence of this limit has not been proved in Ref. (11).

Theorem 3.2

Let

$$\Lambda_1 = \{ (\lambda_1, \lambda_2, \lambda_3) \mid 1 - \frac{4}{L \cdot \max_{\ell} \{\nu_{\ell}\}} < \operatorname{Re} \lambda_{\ell} < 1, \ 0 < \operatorname{Im} \lambda_{\ell} < a; \ell=1,2,3 \} \quad (3.12)$$

and consider

$$\underline{\lambda} \in \Lambda_1$$

Then $\lim_{\eta \downarrow 0} \lim_{r \downarrow 0} \tilde{\mathcal{T}}_{\eta, r}^{\nu}(\underline{\lambda})$ exists in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$ as a Lorentz invariant distribution. (In Section III.23 we prove analyticity in a region $\Lambda \supset \Lambda_1$ where Λ is ν independent).

Proof:

In order to prove the theorem we study (3.7) in the sector Π_P with the representation (3.10) for the ' α -integral'. Noting the definition
$$\mu_{\ell}^P = \sum_{\ell'=1}^{\ell} [(\lambda_{P(\ell')})^{-1} \nu_{P(\ell')}]$$
, the t_L integral clearly converges as $r \rightarrow 0+$ with the regulating parameters in Λ_1 . We do the t_L integral using:

$$\int_0^{\infty} d\alpha \alpha^{\nu-1} \exp i\alpha\gamma = \Gamma(\nu) \gamma^{-\nu} e^{\frac{\nu\pi i}{2}}, \quad \text{Im}\gamma > 0, \text{Re}\nu > 0$$

to get, in the $r \rightarrow 0+$ limit (which we will justify)

$$(\alpha\text{-integral})_p = e^{\frac{\pi i}{2}[\mu_3^p]} \Gamma[\mu_3^p + 4] \int_0^1 dt_2 t_2^{-[(\lambda_{p(3)}^{-1})\nu_{p(3)}]-1} \\ \cdot \int_0^1 dt_1 t_1^{[(\lambda_{p(1)}^{-1})\nu_{p(1)}]+1} \cdot [E(t_1, t_2)]^{\mu_3^p+2} [P_1^p(t; p) + i\eta P_2^p(t; p)]^{-[\mu_3^p+4]} \quad (3.13)$$

First we note that $E(t_1, t_2)$ is strictly positive. The summability of the (3.13) integrand, with respect to the lower end point, for the t_1 integration follows (as for the t_3 integral) from (3.12), making use of the first inequality for $\text{Re } \lambda_p$. On the other hand the summability, with respect to lower end point of the t_2 integration is assured by the condition $\text{Re } \lambda_p < 1$ in (3.12). Hence, the $r \rightarrow 0+$ limit of $\tilde{J}_{\eta, r}^{\nu}(\lambda)$ exists, since it exists in every sector π_p .

We next prove the existence of the $\eta \downarrow 0$ limit of $\tilde{J}_{\eta, r \downarrow 0}^{\nu}(\lambda)$, in using a method due to Gelfand¹⁸. We smear the integrand of (3.13) (multiplied by $\delta(\sum p_i)$) with a testing function $\tilde{\phi}(p) \in \mathcal{S}(R^{4 \times 3})$ and, observing $\text{Im } \mu_3 > 0$ utilise the following identity¹⁸

$$([P_1 + i\eta P_2]^{-\mu_3-4}, \tilde{\phi}) = [4^k (-[\mu_3+4]+1) \cdots (-[\mu_3+4]+k) \cdot (-[\mu_3+4]+4) \cdots$$

$$\dots (-[\mu_3+4] + 4 + k - 1)]^{-1} \cdot ([P_1 + i\eta P_2]^{k-\mu_3-4}, \mathcal{D}^k \tilde{\phi}) \quad (3.14)$$

where $\mathcal{D}$ is a differential operator of the form

$$\sum_{\substack{0 \leq \mu \leq 3 \\ 1 \leq r, s \leq 2}} a_{rs}^{(\mu)} \frac{\partial}{\partial p_r^\mu} \frac{\partial}{\partial p_s^\mu}$$

$a_{rs}^{(\mu)}$, being the coefficients of the quadratic form $(P_1 + i P_2)$ in p_1, p_2 because of the presence of $\delta(\sum_{i=1}^3 p_i)$.

We choose $k > 4$ and observe $\text{Re } \mu_3 < 0$. The existence of the limit in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$ of (3.13), and hence of $\tilde{\mathcal{T}}_\eta^\mu(\underline{\lambda})$, now follows since the $\eta \downarrow 0$ limit of the (smeared) t-integrand of (3.13) exists and is dominated by a summable function (Lebesgue's theorem). The boundary value is clearly a Lorentz invariant distribution, since P_1 is Lorentz invariant.

Remark 3.3

We shall make one further simplification. We consider in lieu of (3.13), the expression obtained on making the replacement $P_2(\underline{t}; \underline{p}) \rightarrow P_2(\underline{p})$ which is obtained by setting $t_j = 1, j = 1, 2$ in the coefficients of the quadratic form $P_2(\underline{t}; \underline{p})$. Then, repeating the previous chain of arguments, the boundary value $\eta \downarrow 0$ exists as a Lorentz invariant generalized function, and coincides with the boundary value in the preceding theorem. Thus, from now on we shall consider the quadratic form $(P_1(\underline{p}, \underline{t}) + i P_2(\underline{p}))$ in the analytically regulated α -integral (3.13). $P_2(\underline{p}) = \sum_{j=1}^2 \|P_{p(j)}\|^2$

III.2.3 By virtue of Theorem 3.2, and Remark 3.3 the GFA (3.6) with $\eta, r \downarrow 0$ can be written, in the momentum representation, as

$$\tilde{\mathcal{J}}_{\eta \downarrow 0}^{\nu}(\underline{\lambda})(\underline{p}) = \sum_{\underline{p}} \tilde{\mathcal{J}}_{\eta \downarrow 0}^{\nu}(\underline{\lambda})(\underline{p})_{\underline{p}} \quad (3.15)$$

where

$$\begin{aligned} \tilde{\mathcal{J}}_{\eta}^{\nu}(\underline{\lambda})(\underline{p})_{\underline{p}} &= (8\pi)^2 \delta(\sum_j^3 p_j) \exp(\frac{\pi i}{2} \mu_3^{\underline{p}}) \prod_{\ell=1}^3 \left[\frac{2^{2[(\lambda_{p(\ell)}-1)\nu_{p(\ell)}]} e^{\frac{\pi i}{2}[(\lambda_{p(\ell)}-1)\nu_{p(\ell)}]}}{[4\pi^2]^{\nu_{p(\ell)}} \Gamma[(1-\lambda_{p(\ell)})\nu_{p(\ell)}]} \right] \\ &\cdot \Gamma[\mu_3^{\underline{p}}+4] \int_0^1 dt_2 t_2^{-[(\lambda_{p(3)}-1)\nu_{p(3)}]-1} A_{\eta}^{\nu}(\underline{\lambda}; t_2)(\underline{p})_{\underline{p}} \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} A_{\eta}^{\nu}(\underline{\lambda}; t_2)(\underline{p})_{\underline{p}} &= \int_0^1 dt_1 t_1^{[(\lambda_{p(1)}-1)\nu_{p(1)}]+1} [E(t_1, t_2)]^{\mu_3^{\underline{p}}+2} \\ &\cdot [P_1^{\underline{p}}(t_1; \underline{p}) + i\eta P_2^{\underline{p}}(\underline{p})]^{-[\mu_3^{\underline{p}}+4]} \end{aligned} \quad (3.17)$$

with

$$\mu_3^{\underline{p}} \equiv \sum_{\ell'=1}^3 [(\lambda_{p(\ell')}-1)\nu_{p(\ell')}] \quad , \quad E(t_1, t_2) = 1 + t_2 + t_1 t_2$$

$$P_1^{\underline{p}}(t_1; \underline{p}) \equiv p_{p(3)}^2 + t_1 t_2 p_{p(2)}^2 + t_1 p_{p(1)}^2$$

$$P_2^{\underline{p}}(\underline{p}) \equiv \sum_{j=1}^3 \|p_{p(j)}\|^2$$

with the regulating parameters in the region Λ_1 (3.12).

The $\eta \downarrow 0$ boundary value exists as a Lorentz invariant distribution in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$.

In this subsection we will establish an analytic continuation of the GFA (3.15) - (3.17) into the $\underline{\nu}$ -independent region Λ where

$$\Lambda = \{ \lambda \quad -b < \operatorname{Re} \lambda_\ell < 1, \quad 0 < \operatorname{Im} \lambda_\ell < a, \quad \ell = 1, 2, 3 \} \quad (3.18)$$

Returning to (3.16 - 3.17) we observe that when $\operatorname{Re} \lambda_\ell \leq [1 - \frac{4}{L \cdot \operatorname{Max}\{\nu_\ell\}}]$ lower end point singulatities may arise from (3.17) from the factor $t_1^{[(\lambda_{p(1)}-1)\nu_{p(1)}+1]}$ whereas the corresponding t_2 factor in (3.16) will remain integrable. The t_1 end point singularities will be reflected in complicated singularities in the regulating parameters. We first display the singularity structure.

Consider the generalised function:

$$\left[P_1^p(t; p) + i\eta P_2^p(p) \right]^{-[\mu_3^p+4]} \quad (3.19)$$

which appears in (3.17), and where P_2 is a positive definite quadratic form. Such generalised functions have been studied by Gelfand.¹⁸ Now we observe, following Gelfand, that the above generalised function can be analytically continued in the coefficients of the quadratic form. In fact, with $\eta > 0$, it is analytic in t_1 in a complex neighbourhood $K_1 \subset \mathbb{C}$ of the domain of integration $0 \leq t_1 \leq 1$ in (3.17). Furthermore E is analytic in t_1 and strictly positive in $0 \leq t_1 \leq 1$; it is evident that $[E(t_1, t_2)]^{(\mu_3^p+2)}$ too is analytic in a complex neighbourhood K_2 of $0 \leq t_1 \leq 1$. Let K be the common domain of extension.

Then we obtain, by a well known procedure¹⁹

$$A_{\eta}^{\underline{\nu}}(\underline{\lambda}; t_2)(\underline{p})_p = \frac{1}{2i \sin \pi [(\lambda_{p(1)}-1)\nu_{p(1)}+1]} \int_1^{0+} dt_1 (-t_1)^{[(\lambda_{p(1)}-1)\nu_{p(1)}+1]} \cdot [E(t_1, t_2)]^{\mu_3^p+2} [P_1^p(\underline{t}; \underline{p}) + i\eta P_2^p(\underline{p})]^{-[\mu_3^p+4]} \quad (3.20)$$

with $(-t)^y = \exp[ny + i \arg(-t)]$, $|\arg(-t)| \leq \pi$. The t_1 contour begins and ends at 1, in the complex t_1 plane, encircling the origin once counter-clockwise, the contour lying within K .

We remark that, from the identity of (3.20) with (3.17) in the region (3.12), the existence of the $\eta \downarrow 0$ limit in $\mathcal{S}'(R^{4 \times 3})$ of (3.20) follows. (However, in (3.20), the $\eta \downarrow 0$ limit may not be interchanged with the integration.)

Formula (3.20) furnishes us with an explicit analytic continuation of (3.16) from the region Λ_1 (3.12) into the region Λ (3.18) ($\Lambda \supset \Lambda_1$) since the integral in (3.20) is analytic in $\underline{\lambda}$, all the singularities being contained in the factor $\Gamma(\mu_3^p+4) \cdot \{2i \sin \pi [(\lambda_{p(1)}-1)\nu_{p(1)}+1]\}^{-1}$ which is analytic in this region, and the t_2 integral converges. We have introduced the representation (3.20) as it is convenient for future work.

With $\eta > 0$, (3.16) is a distribution in $\mathcal{S}'(R^{4 \times 3})$ with analyticity in Λ . It now remains to prove the existence of $\eta \downarrow 0$ limit in $\mathcal{S}'(R^{4 \times 3})$ of (3.16) in the region Λ . Let us return to (3.16 - 3.17) with regulating parameters in the region Λ_1 . Then by repeated partial integrations⁸ we get:

$$A_{\eta}^{\underline{\nu}}(\underline{\lambda}; t_2)_p = \left[\sum_{k_i=0}^{\nu_i} \left\{ \frac{(-1)^{k_i} \left(\frac{\partial}{\partial t_1} \right)^{k_i} \Big|_{t_1=1}}{\prod_{m_i=0}^{k_i} (\mu_i^p+2+m_i)} \right\} + \frac{(-1)^{\nu_i}}{\prod_{m_i=0}^{\nu_i} (\mu_i^p+2+m_i)} \int_0^1 dt_1 t_1^{\mu_i^p+2+\nu_i} \cdot \left(\frac{\partial}{\partial t_1} \right)^{\nu_i+1} \right] \{ E(t_1, t_2)^{\mu_3^p+2} [P_1^p(\underline{t}; \underline{p}) + i\eta P_2^p(\underline{p})]^{-[\mu_3^p+4]} \} \quad (3.21)$$

(with $\mu_l^p = \sum_i^l [(\lambda_{p(l')} - 1) \nu_{p(l')}]$) which may be expanded out as a sum of terms. This formula with all end point singularities factored out also provides us with an explicit analytic continuation of (3.16) to the region Λ . From the theorem on analytic continuations it follows that, in the region Λ (3.20) and (3.21) (and hence the corresponding representations for (3.16)) are identical. Smear $A_\eta^\nu(\underline{\lambda}; t_2)$ multiplied by $\delta(\sum p)$ with a test function $\tilde{\phi}(\underline{p}) \in \mathcal{S}(\mathbb{R}^{4 \times 3})$ and apply again Gelfand's method (to the representation (3.21)) as in Subsection III.2.2. Note that the application of various derivatives in (3.21) will lower the exponent of the quadratic form (last factor in (3.21) by a finite (ν_1 dependent) amount. On making use again of the identity (3.14) we can raise the power up so that the real part of the exponent is positive. Then

$((P_1 + i\eta P_2)^{-(\mu_3^p + 4)}, \tilde{\phi})$ is continuous in η . The existence of the $\eta \downarrow 0$ limit in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$ of $\delta(\sum p) A_\eta^\nu(\underline{\lambda}; t_2)$ (in Λ), and hence of (3.16) now follows by dominated convergence. (Note that the boundary value is a Lorentz invariant distribution since P_1 is Lorentz invariant).

We have now obtained the definition of our regulated massless Feynman amplitude (3.16) (the regulated form of the formal product (3.2)) as a distribution with regulating parameters in the region of interest (3.18). We state this result as a theorem:

Theorem 3.4

The generalised (massless) Feynman amplitude (corresponding to the formal product (3.2)) is given by the Fourier transform of:

$$\tilde{\mathcal{J}}^\nu(\underline{\lambda})(\underline{p}) = \sum_p \tilde{\mathcal{J}}_{\eta \downarrow 0}^\nu(\underline{\lambda})(\underline{p})_p \quad (3.22)$$

with

$$\begin{aligned} \tilde{\mathcal{J}}_{\eta}^{\nu}(\underline{\lambda})(\underline{p})_p &= (8\pi)^2 \delta(\sum_i p_i) \exp\left[\frac{\pi i}{2} \mu_3^p\right] \prod_{\ell=1}^3 \left[\frac{2^{[\lambda_{p(\ell)}-1]\nu_{p(\ell)}}}{\Gamma[(1-\lambda_{p(\ell)})\nu_{p(\ell)}]} \cdot \frac{e^{\frac{i\pi}{2}[(\lambda_{p(\ell)}-1)\nu_{p(\ell)}]}}{[4\pi^2]^{\nu_{p(\ell)}}} \right] \\ &\quad \cdot \Gamma[\mu_3^p+4] \int_0^1 dt_2 t_2^{-[(\lambda_{p(3)}-1)\nu_{p(3)}]-1} A_{\eta}^{\nu}(\underline{\lambda}; t_2)(\underline{p})_p \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} A_{\eta}^{\nu}(\underline{\lambda}; t_2) &= \{2i \sin \pi [(\lambda_{p(1)}-1)\nu_{p(1)}+1]\}^{-1} \int_1^{0+} dt_1 (-t_1)^{[(\lambda_{p(1)}-1)\nu_{p(1)}+1]} \\ &\quad \cdot [E(t_1, t_2)]^{\mu_3^p+2} [P_1^p(\underline{t}; \underline{p}) + i\eta P_2^p(\underline{p})]^{-[\mu_3^p+4]} \end{aligned} \quad (3.24)$$

where $\mu_3^p = \sum_i^3 (\lambda_{p(i)}-1)\nu_{p(i)}$ and $\lambda_1 \dots \lambda_3$ are restricted to the region Λ . Then (3.22) defines a Lorentz invariant distribution in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$ with analyticity in Λ , and meromorphic elsewhere. For fixed $\nu_1 \dots \nu_3$ (taken arbitrarily) the singularity of $\tilde{\mathcal{J}}_{\eta \downarrow 0}^{\nu}(\underline{\lambda})$ at $\underline{\lambda} = \underline{0}$ is in the form of poles on hyperplanes in $\mathbb{C}^3$:

$$(\lambda_{p(1)}-1)\nu_{p(1)} = -k_1 - 2; \quad \sum_i^3 (\lambda_{p(i)}-1)\nu_{p(i)} = -k_2 - 4$$

with $k_j = 0, 1, 2, \dots$ ($j = 1, 2$). (This is evident from the factors $\Gamma[\mu_3^p+4]$ in (3.23) and $\{2i \sin \pi [(\lambda_{p(1)}-1)\nu_{p(1)}+1]\}^{-1}$ in (3.24). The other poles from this factor in (3.24) are spurious since their residues vanish.)

III.2.4 We are now in a position to turn to our basic object of concern: the analytically regulated version of $\mathcal{J}(\underline{x})$ given in (3.1). Let us define:

$$\tilde{\mathcal{J}}_{\eta}(\underline{\lambda})(\underline{p}) = \sum_{\nu_1, \nu_2, \nu_3 = 1}^{\infty} \frac{[\delta^2]^{\sum_{\ell=1}^3 (1-\lambda_{\ell})\nu_{\ell}}}{\prod_{\ell=1}^3 [\nu_{\ell}!]} \tilde{\mathcal{J}}_{\eta}^{\nu}(\underline{\lambda})(\underline{p}) \quad (3.25)$$

where the GFA $\tilde{\mathcal{J}}_{\eta}^{\nu}$ is given through (3.22 - 3.24) and the $\underline{\lambda}$ regulators are restricted to the region Λ given in (3.19). By Theorem 3.4, each term in (3.25) (together with the $\eta \downarrow 0$ limit) is a distribution in $\mathcal{S}'(\mathbb{R}^{4 \times 3})$. Let us smear each term in (3.25) with a test function $\tilde{\phi}(\underline{p}) \in \mathcal{M}_g(\mathbb{R}^{4 \times 3})$. Returning to (3.23), it is permissible to smear $A_{\eta}^{\nu}((\underline{\lambda}); t_2)$ times $\delta(\sum p)$. In Appendix II we show that for large $\|\nu\|_3 = \sum_{\ell=1}^3 \nu_{\ell}$

$$|(A_{\eta}^{\nu}, \tilde{\phi})| \leq L_{\eta} \tilde{\Psi}_{\eta}^{(\nu)} \|\tilde{\phi}\|_g \|\nu\|_3^{(\frac{1}{\rho} \|\nu\|_3)} \quad (3.26)$$

and that a similar bound holds in the limit $\eta \downarrow 0$. The $\delta(\sum p)$ restriction is understood. Here $\tilde{\Psi}^{(\nu)}$ has growth $(\text{const}) \|\nu\|_3$, $\|\tilde{\phi}\|_g$ is a norm in $\mathcal{M}_g(\mathbb{R}^{4 \times 2})$ and ρ is the order of growth of the indicator function $g(t)^1$. Choosing $\rho > \frac{1}{3}$, as in Section II, and using Stirling's formula, we verify that (3.25) converges in the topology of $\mathcal{M}'_g(\mathbb{R}^{4 \times 3})$, and uniformly in η . Hence $\tilde{\mathcal{J}}_{\eta \downarrow 0}(\underline{\lambda})$, exists in $\mathcal{M}'_g(\mathbb{R}^{4 \times 3})$ and is analytic in Λ , because the convergence is uniform in compact subsets of Λ . We thus have:

Theorem 3.6

$$\mathcal{T}_{\eta \downarrow 0}(\underline{\lambda})(\underline{x}) = \sum_{\nu=1}^{\infty} \frac{[g^2]^{\sum_1^3 (1-\lambda_\ell)\nu_\ell}}{\prod_1^3 [\nu_\ell!]} \mathcal{T}_{\eta \downarrow 0}^{\nu}(\underline{\lambda})(\underline{x}) \quad (3.27)$$

exists as a Lorentz invariant (strictly localisable) generalised function in a space $\mathcal{G}'_g(\mathbb{R}^{4 \times 3})$, with indicator function of order of growth ρ $\frac{1}{3} < \rho < \frac{1}{2}$, analytic in the region Λ .

(3.27) constitutes the analytically regulated version of the formal series (3.1) of divergent Feynman graphs. For later purposes it is convenient to express (3.25), using (3.22 - 3.24), as:

$$\begin{aligned} \tilde{\mathcal{T}}_{\eta}(\underline{\lambda})(\underline{p}) &= (8\pi)^2 \delta\left(\sum_1^3 p_i\right) \sum_{\nu_1, \nu_2, \nu_3=1}^{\infty} \prod_1^3 \left\{ \frac{1}{[(4\pi)^2]^{\nu_\ell} \Gamma[\nu_\ell+1]} \right\} \cdot \\ &\cdot \sum_p \prod_1^3 \left\{ \frac{2^{2\lambda_{p(\ell)}\nu_\ell} e^{[(\lambda_{p(\ell)}-1)\nu_\ell]\pi i}}{[g^2]^{(1-\lambda_{p(\ell)})\nu_\ell}} \right\} \cdot \\ &\cdot \Gamma[\mu_3^p+4] \int_0^1 dt_2 t_2^{-[(\lambda_{p(3)}-1)\nu_3]-1} A_{\eta}^{\nu}(\underline{\lambda}; t_2)(\underline{p})_p \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} A_{\eta}^{\nu}(\underline{\lambda}; t_2) &= \{2i \sin \pi [(\lambda_{p(1)}-1)\nu_1+1]\}^{-1} \int_1^{0+} dt_1 (-t_1)^{[(\lambda_{p(1)}-1)\nu_1+1]} \cdot \\ &\cdot [E(t_1, t_2)]^{\mu_3^p+2} [P_1^p(\underline{t}, \underline{p}) + i\eta P_2^p(\underline{p})]^{-[\mu_3^p+4]} \end{aligned} \quad (3.29)$$

with $\mu_3^p = \sum_1^3 (\lambda_{p(l)} - 1) \nu_l$

In order to derive the above expressions we have exploited the fact that the series coefficients satisfy

$$\frac{[g^2] \sum_1^3 (1 - \lambda_l) \nu_l}{\prod_1^3 \left[[(4\pi)^2]^{\nu_l} \Gamma[\nu_l + 1] \right]} = \frac{[g^2] \sum_1^3 (1 - \lambda_{p(l)}) \nu_{p(l)}}{\prod_1^3 \left[[(4\pi)^2]^{\nu_{p(l)}} \Gamma[\nu_{p(l)} + 1] \right]}$$

for any permutation p of $(1, 2, 3)$. We have then interchanged $\sum_p$ with $\sum_{\{\nu_i\}}$ which is then the same as $\sum_{\{\nu_{p(i)}\}}$. Since the $\{\nu_{p(i)}\}$ are summed over we can replace $\nu_{p(i)} \xrightarrow{\{\nu_{p(i)}\}} \nu_i$ to get the above expression.

III.2.5 In the previous subsections we have given the construction of the analytically regulated Feynman amplitude with massless multiplets for the three point function, since it is with this case that we shall be concerned in this part of the thesis. Our procedure readily generalises for the n -point function on making appropriate use of the $\underline{t}$ -factorisation mentioned in Remark 3.1. However, for the n -point function the parametrisation of the sector co-ordinates, $\{ \alpha_{p(1)}, \dots, \alpha_{p(L)} \}$, where

$$\begin{aligned} \alpha_{p(l)} &= t_L \dots t_l \\ 0 &\leq t_L \leq \infty \\ 0 &\leq t_l \leq 1, \quad l = 1, 2, \dots, (L-1) \end{aligned} \tag{3.30}$$

is too naive, and gives rise to spurious singularities in the $\underline{\lambda}$ -space which are not necessarily true singularities of the Feynman amplitude corresponding to the n -point function. The true singularities of a Feynman amplitude are given by the "singularity family" of Speer⁸. For the simple case, however, of the triangle graph associated with the three point function the parametrisation (3.30) as well as the one suggested by the "singularity

family" of Speer yield identical results.

III.3 RENORMALISATION OF THE THREE POINT FUNCTION

III.3.1 We have already remarked (in III.1) that renormalisation in the abstract sense of BPH (Bogoliubov-Parasiuk-Hepp) may be regarded as an extension procedure of functionals defined only on a subspace (of test functions) into the whole of space of test functions. Indeed

$\mathcal{T}_3(x_1 x_2 x_3)$ is defined only away from the point $x_1 = x_2 = x_3$, and the points $x_i = x_j$ $i, j = 1, 2, 3$, $i \neq j$, because the time ordered product

$T[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_3)]$ is ambiguous at coinciding points. Thus,

$\mathcal{T}_3(x_1 x_2 x_3)$ is defined only up to functionals concentrated on $x_1 = x_2 = x_3$, or $x_i = x_j$, $i \neq j = 1, 2, 3$, that is, $\mathcal{T}_3(x_1 x_2 x_3)$ is defined only on

$\mathcal{E}_0(\mathbb{R}^{3 \times 4})$. The extension, however, of $\mathcal{T}_3(x_1 x_2 x_3)$ as a functional on the whole space of Jaffe test functions $\mathcal{E}(\mathbb{R}^{3 \times 4})$ is non-unique, because by the Hahn-Banach theorem there exist many possible extensions. The axioms of renormalisation, which we shall state below, restrict the possible extensions to the ones which satisfy the general principles of Lorentz invariance, symmetry, causality, and unitarity. We shall, now, turn to the axioms of renormalisation for strictly localisable interactions.

The test function space, $\mathcal{E}_g(\mathbb{R}^4)$, for non-polynomial strictly localisable interactions is the F.T. of the space $\mathcal{M}_g(\mathbb{R}^4)$ of all C^∞ functions f in momentum space the topology being defined by the family of seminorms:

$$\|f\|_{n,m,A} = \sup_{p \in \mathbb{R}^4} g(A\|p\|^2)(1+\|p\|^2)^m |D^n f(p)| < \infty$$

for all integers $n, m \geq 0$, and $A \geq 0$ with

$$g(t^2) = \sum_{r=0}^{\infty} c_{2r} t^{2r} \quad c_{2r} \geq 0, \quad c_0 \geq 0$$

such that

$$\int_0^{\infty} dt \frac{\ln g(t^2)}{1+t^2} < \infty$$

The entire function g is called the indicator function, and characterises the growth of the Wightman functions in momentum space (see foreword).

Definition²: Let $\mathcal{L}_I(x)$ be a strictly localisable interaction (i.e. entire function in the free fields of order less than two). Let $\mathcal{E}$ be the mapping of n -tuples $(\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n))$ into operator-valued distributions $T[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)]$ and $\overline{T}[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)]$ (antichronological product) over $\mathcal{S}_g(\mathbb{R}^{4 \times n})$ which satisfies the following three axioms.

(E1)

$$T[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)] = T[\mathcal{L}_I(x_{p(1)}) \dots \mathcal{L}_I(x_{p(n)})] = \overline{T}[\mathcal{L}_I(x_1)^* \dots \mathcal{L}_I(x_n)^*]^*$$

and

$$T[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)] = U(a, \Lambda) T[\mathcal{L}_I(\Lambda^{-1}(x_1 - a)) \dots \mathcal{L}_I(\Lambda^{-1}(x_n - a))] U(a, \Lambda)^{-1}$$

where $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ is an inhomogeneous Lorentz transformation and

$$p = \begin{pmatrix} 1 & \dots & n \\ p(1) & \dots & p(n) \end{pmatrix}$$

(E2) $\overline{T}[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)]$ give rise to a unitary S matrix,

$SS^\dagger = S^\dagger S = 1$, which implies that

$$0 = \sum_{k=0}^n (-1)^k \sum_{i_1 < \dots < i_k} \overline{T}[\mathcal{L}_I(x_{i_1}) \dots \mathcal{L}_I(x_{i_k})] T[\mathcal{L}_I(x_{i_{k+1}}) \dots \mathcal{L}_I(x_{i_n})]$$

where the second sum extends over all the possible ways of breaking up

the set of points $x_1, \dots, x_n$ into two sets of k and $n-k$ points.

(E3) Causality: Let $\phi(x)$ be a space-time function which controls

the intensity of the interaction at various regions in space-time. Then³

causality implies that

$$\frac{\delta}{\delta\phi(x)} \left[\frac{\delta S(\phi)}{\delta\phi(y)} S^\dagger(\phi) \right] = 0 \quad \forall x \lesssim y$$

where $x \sim y$ denotes that x and y are separated by a space-like interval.

Let

$$R_n(x) = \sum_{k=0}^{n-1} (-1)^k \sum_{1 \leq i_1 < \dots < i_k} \overline{T} \left[\mathcal{L}_I(x_{i_1}) \mathcal{L}_I(x_{i_2}) \dots \mathcal{L}_I(x_{i_k}) \right] \cdot T \left[\mathcal{L}_I(x_{i_{k+1}}) \dots \mathcal{L}_I(x_{i_n}) \right]$$

Then, the above differential form implies that

$$\text{supp } R_n(x) = \{x \mid x_1 - x_k \in \overline{V}_+, \quad 1 \leq k \leq n\}$$

where $\overline{V}_+$ is the closure of $V_+ = \{x \in \mathbb{R}^4 \mid x^0 > |\vec{x}|\}$. Any mapping $\mathcal{E}$ which satisfies the above three axioms is called renormalisation.

In this subsection we shall develop an extension procedure for the three point function, $\mathcal{T}_3(x_1 x_2 x_3)$, which satisfies the axioms (E1,2,3).

III.3.2 Our task is to show how the series of regulated Feynman graphs

(3.27) may be defined at $\underline{\lambda} = \underline{0}$ through a renormalisation. We already

remarked (Theorem 3.4) on the singularity structure of the individual

GFA's $\tilde{\mathcal{T}}_\gamma^\nu(\underline{\lambda})$ for arbitrary (but fixed) $\underline{\nu}$. For an individual

GFA one could renormalise, by procedures suitable for meromorphic functionals,

either through a "generalised evaluation" corresponding to the generalised

evaluator of Speer⁸, or through an "analytic evaluation"²¹ which is equivalent

to an analytic decomposition into regular, and singular parts with a subsequent

analytic continuation of the regular part to $\underline{\lambda} = \underline{0}$ (the 'physical' point).

However, $\tilde{\mathcal{T}}_\gamma(\underline{\lambda})$ being the sum of an infinite series of GFA's is not meromorphic

by the reasoning of Section II. We shall, therefore, generalise the extension procedure of Ref. (7) in order to give an analytic decomposition of $\tilde{J}_\eta(\underline{\lambda})$ into a sum of regular and singular parts.

Returning to (3.28 - 3.29) it is convenient to write the ν_i sum as a z contour integral representation since (as will be clear) the dependent poles on hyperplanes ζ^3 can be analysed as moving z plane poles in the spirit of Ref. (7). We get:

$$\tilde{J}_\eta(\underline{\lambda})(\underline{p}) = (8\pi)^2 \delta(\sum_1^3 p_i) \sum_{\nu_2, \nu_3=1}^{\infty} \prod_{\ell=2}^3 \left\{ \frac{1}{\Gamma[\nu_\ell+1]} \right\} \sum_p \tilde{F}_\eta^{\nu}(\underline{\lambda})(\underline{p})_p \quad (3.31)$$

where

$$\begin{aligned} \tilde{F}_\eta^{\nu}(\underline{\lambda})(\underline{p})_p &= \prod_{\ell=2}^3 \left\{ \frac{2^{\lambda_{p(\ell)}\nu_\ell} e^{(\lambda_{p(\ell)}-1)\nu_\ell \pi i}}{[(4\pi)^2]^{\nu_\ell} \Gamma[(1-\lambda_{p(\ell)})\nu_\ell]} [g^2]^{(1-\lambda_{p(\ell)})\nu_\ell} \right\} \\ &\cdot \frac{1}{2i} \int_{\Gamma_1} dz \frac{2^{\lambda_{p(1)}z} e^{(\lambda_{p(1)}-1)z \pi i}}{[(4\pi)^2]^z \Gamma[(1-\lambda_{p(1)})z] \Gamma[z+1]} [g^2]^{(1-\lambda_{p(1)})z} \cot \pi z. \\ &\cdot \Gamma[\mu_3^p+4] \int_0^1 dt_2 t_2^{-[(\lambda_{p(3)}-1)\nu_3]-1} A_\eta^{\nu}(z, \underline{\lambda}; t_2)(\underline{p})_p \end{aligned} \quad (3.32)$$

with A_η^{ν} given as in (3.29) with the replacement $\nu_i \rightarrow z$ and

$$\mu_3^p \rightarrow \mu_3^p = (\lambda_{p(1)}-1)z + \sum_{\ell=2}^3 (\lambda_{p(\ell)}-1)\nu_\ell$$

The contour Γ_1 is shown in Fig. (III.3.1). From the following analysis it will follow that Γ_1 can be chosen so as to enclose only the fixed poles of $\cot \pi z$ at the positive integer points

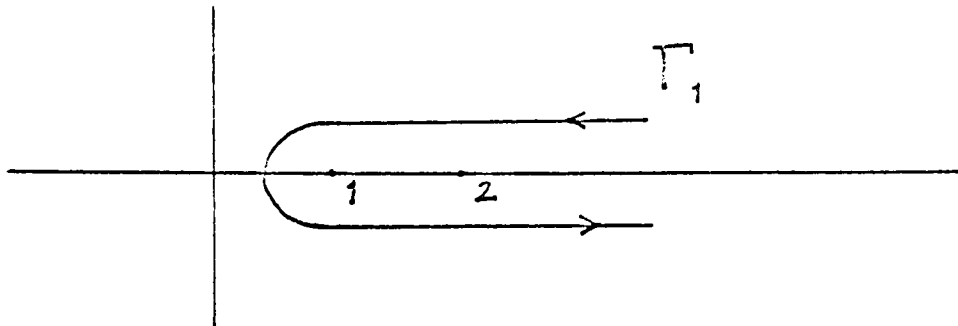


Fig. (III.3.1)

(3.31) when smeared with $\tilde{\phi} \in \mathcal{M}_g(\mathbb{R}^{4 \times 3})$ converges and defines a generalized function in $\mathcal{M}'_g(\mathbb{R}^{4 \times 3})$. Let us now see how the $\underline{\lambda}$ singularities arise.

First, consider a subregion from (3.18) in $\mathbb{C}^3$:

$$\Lambda_s = \Lambda^{(1)} \times \Lambda^{(2)} \times \Lambda^{(3)}$$

$$\Lambda^{(\ell)} = \{ \lambda_\ell \mid \lambda_\ell^{-1} = r_\ell e^{i\Omega_\ell}, \quad \pi/2 < \Omega_\ell < \pi, \quad |r_\ell - 1| < \varepsilon_\ell \} \quad (3.33)$$

is the shaded region in Fig. (III.3.2)

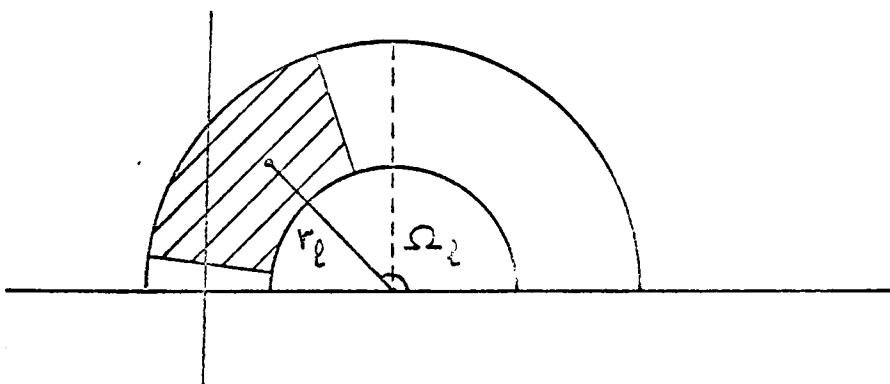


Fig. (III.3.2): The region $\Lambda^{(\ell)}$ (shaded).

Now $\tilde{F}_\gamma^{\underline{\nu}}(\underline{\lambda})$ has possible singularities at the non-analytic points of the integrand of (3.32). Aside from the fixed poles of $\cot \pi z$ these are moving ($\underline{\lambda}$ -dependent) singularities in the z -plane; they lead to singularities of $\tilde{F}$ (as $\underline{\lambda} \rightarrow \underline{0}$) only if they cannot be avoided by Γ_1 contour distortions.

The Γ function in (3.32) and the overall factor (in braces) in the definition (3.29) of $A_\gamma^{\underline{\nu}}$ (in 3.32) lead to the following set of moving z -plane poles:

$$z^{(n_1)}(\underline{\lambda}) = \frac{n_1 + 2}{1 - \lambda_{p(1)}}$$

$$z^{(n_3)}(\underline{\lambda}) = (n_3 + 4 - \sum_{\ell=2}^3 (1 - \lambda_{p(\ell)}) \nu_{\ell}) / (1 - \lambda_{p(1)}) \quad (3.34)$$

with $n_j = 0, 1, 2, \dots (j=1, 3)$, and all permutations ρ . From (3.33) and (3.34) we get (with the convention $\sum = 0$ for $j = 1$, $n_1^0 = 2$, $n_3^0 = 4$)

$$\operatorname{Re} z^{(n_j)}(\underline{\lambda}) = -\frac{1}{r_{p(1)}} \left[(n_j + n_j^0) \cos \Omega_{p(1)} + \sum_{\ell=2}^j r_{p(\ell)} \nu_{\ell} \cos (\Omega_{p(\ell)} - \Omega_{p(1)}) \right] \quad (3.35)$$

$$\operatorname{Im} z^{(n_j)}(\underline{\lambda}) = \frac{1}{r_{p(1)}} \left[(n_j + n_j^0) \sin \Omega_{p(1)} - \sum_{\ell=2}^j r_{p(\ell)} \nu_{\ell} \sin (\Omega_{p(\ell)} - \Omega_{p(1)}) \right]$$

and hence:

$$\operatorname{Im} z^{(n_j)} = -\tan \Omega_{p(1)} \operatorname{Re} z^{(n_j)} - \frac{1}{r_{p(1)} \cos \Omega_{p(1)}} \sum_{\ell=2}^j r_{p(\ell)} \nu_{\ell} \sin \Omega_{p(\ell)} \quad (3.36)$$

From (3.35, 3.36) we see that the $z^{(n_j)}(\underline{\lambda})$ poles with $\operatorname{Re} z^{(n_j)} \geq 0$ lie on a straight line (in the z plane) with a positive intercept and subtending an angle $\Omega'_{p(1)} (= \pi - \Omega_{p(1)})$, $0 < \Omega'_{p(1)} < \pi/2$ on the real axis. Clearly, for $\operatorname{Re} z^{(n_j)} > 0$ we have $\operatorname{Im} z^{(n_j)} > 0$. Thus as shown in Fig. (III.3.3) the Γ_1 contour can be chosen so that all $z^{(n_j)}(\underline{\lambda})$ poles lie exterior to it.

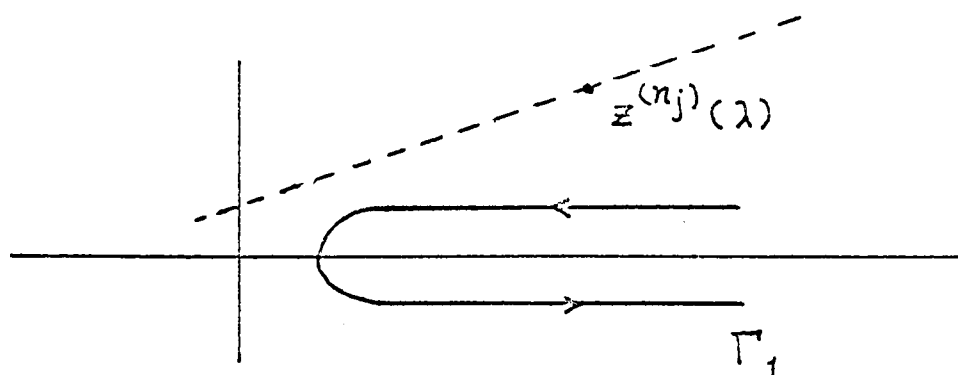


Fig. 4.3 (III.3.3)

In the region Λ_s (4.3), (3.31 - 3.32) is analytic in $\underline{\lambda}$. However,

if we were to continue to $\underline{\lambda} \rightarrow \underline{0}$ the Γ_1 contour would get pinched between the fixed poles at $z = k$ ($k = 1, 2, 3, \dots$) and, from (3.34), those moving poles $z^{(n_j)}(\underline{\lambda})$ which at $\underline{\lambda} = \underline{0}$ satisfy:

$$\begin{aligned} z^{(n_1)} &= n_1 + 2 = k_1 \\ z^{(n_3)} &= n_3 + 4 - \sum_2^3 \nu_2 = k_3 \quad (k_j = 1, 2, 3, \dots) \end{aligned} \quad (3.37)$$

leading to non-analyticity of (3.31 - 3.32) at

In view of the pinch singularities encountered in the continuation to $\underline{\lambda} = \underline{0}$ we generalise the procedure of Ref. (7) to extract an analytic piece out of $\tilde{\mathcal{J}}_\eta(\underline{\lambda})$. To this end we distort the contour $\Gamma_1 \rightarrow \overline{\Gamma}_1$ so as to enclose the $z^{(n_1)}(\underline{\lambda})$ poles and compute the discontinuities using the residue theorem. This leads to the decomposition:

$$\tilde{\mathcal{J}}_\eta(\underline{\lambda}) = \tilde{\mathcal{J}}_\eta(\underline{\lambda})_{R_1} + \Delta_1 \tilde{\mathcal{J}}_\eta(\underline{\lambda}) \quad (3.38)$$

of (3.31) in Λ , (an expression for $\Delta_1 \tilde{\mathcal{J}}_\eta(\underline{\lambda})$ is given in Appendix III). We remark, holding any two λ_j, λ_k fixed in $\Lambda^{(j)}, \Lambda^{(k)}$ respectively that (i) $\tilde{\mathcal{J}}_\eta(\underline{\lambda})_{R_1}$ is analytic in $\lambda_\ell \in \Lambda^{(\ell)}$, and (ii) $\lim_{\lambda_\ell \rightarrow 0} \tilde{\mathcal{J}}_\eta(\underline{\lambda})_{R_1}$ exists uniquely for any sequence $\{\lambda_\ell^i \mid \lambda_\ell^i \in \Lambda^{(\ell)}\}$ since it may be seen that the $z^{(n_j)}(\underline{\lambda})$ do not lead to pinching of $\overline{\Gamma}_1$. Next we consider the second set of poles in (3.34) which will cause a pinch singularity of $\tilde{\mathcal{J}}_\eta(\underline{\lambda})_{R_1}$ when all λ_ℓ ($\ell = 1, 2, 3$). We set $\Omega_\ell = \Omega$ and deduce from (3.35 - 3.36), for $\text{Re } z^{(n_3)} > 0$:

$$0 < \text{Im } z^{(n_3)}(\underline{\lambda}) \leq \frac{1}{r_{p(1)}} (n_3 + 4) \sin \Omega \quad (3.39)$$

(3.39) implies that the imaginary parts of the poles which may pinch

$\overline{\Gamma}_1$ are bounded in a region independent of ν_2, ν_3 . Returning to (3.38) we may repeat the contour distortion procedure $\overline{\Gamma}_1 \rightarrow \overline{\Gamma}$

so as to enclose the $z^{(n_j)}(\underline{\lambda})$ poles (with $\text{Re } z^{(n_j)} \geq 1$), and compute the discontinuity (see Fig (III.3.4 - III.3.5)).

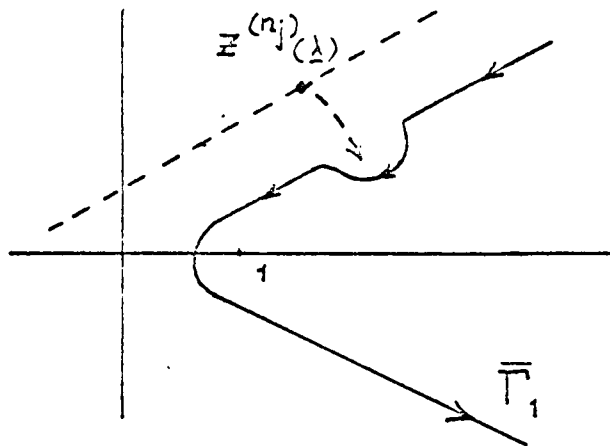


Fig. (III.3.4)

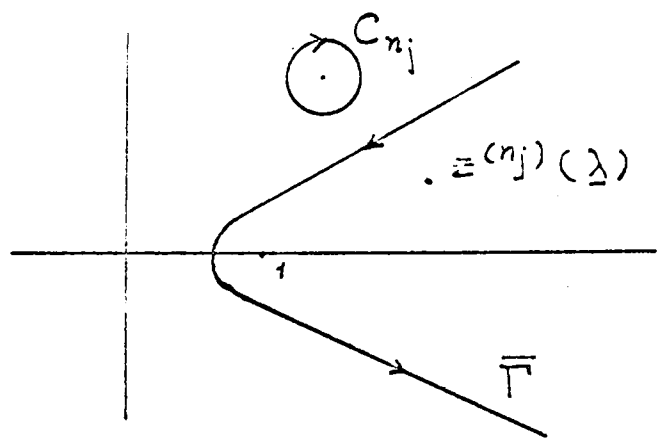


Fig. (III.3.5)

We then get the analytic decomposition in Λ :

$$\tilde{\mathcal{J}}_{\eta}(\underline{\lambda}) = \tilde{\mathcal{J}}_{\eta}(\underline{\lambda})_R + \Delta_1 \tilde{\mathcal{J}}_{\eta}(\underline{\lambda}) + \Delta_3 \tilde{\mathcal{J}}_{\eta}(\underline{\lambda}) \quad (3.40)$$

Now $\Delta_{1,3} \tilde{\mathcal{J}}_{\eta}(\underline{\lambda})$ (which are the residues of the moving poles) have a complicated singularity at $\underline{\lambda} = 0$. However, as shown in Appendix III, $\mathcal{F}^{-1} [\Delta_3 \tilde{\mathcal{J}}_{\eta_0}(\underline{\lambda})](\underline{x})$ is a distribution in $\mathcal{C}'_g(\mathbb{R}^{4 \times 3})$ with support concentrated on $x_1 = x_2 = x_3$. Furthermore $\mathcal{F}^{-1} \Delta \tilde{\mathcal{J}}_{\eta_0}(\underline{\lambda})$ is a sum of distributions with supports concentrated on surfaces $x_i = x_j$. On the other hand, $\tilde{\mathcal{J}}_{\eta}(\underline{\lambda})_R$ can be analytically continued to $\underline{\lambda} = 0$. This continuation may be done by setting all the λ 's equal to some and continuing in the single variable to zero. (The analyticity follows from the fact that in the continuation no further $\bar{\Gamma}$ contour distortion is necessary, the moving $z^{(n_j)}(\underline{\lambda})$ poles fusing with the fixed poles at the positive integers). It now follows that $\mathcal{F}^{-1} [\tilde{\mathcal{J}}_{\eta_0}(\underline{\lambda})](\underline{x})$ is unambiguously defined as a continuous linear functional only on the subspace $\mathcal{C}_0(\mathbb{R}^{4 \times 3})$ of test functions from $\mathcal{C}(\mathbb{R}^{4 \times 3})$ which vanish, together with all derivatives, when any two spacetime points coincide. This was to be expected.

We now obtain a definition of $\mathcal{T}(\underline{\lambda})(\underline{x})$ at $\underline{\lambda} = \underline{0}$ through the following extension to $\mathcal{E}'_g(\mathbb{R}^{4 \times 3})$:

$$\mathcal{T}_3(\underline{x}) = \mathcal{F}^{-1} \left[\tilde{\mathcal{T}}_{\eta \downarrow 0} \right](\underline{x}) \quad (3.41)$$

where:

$$\begin{aligned} \tilde{\mathcal{T}}_{\eta}(\underline{p}) &= \lim_{\underline{\lambda} \rightarrow \underline{0}} \tilde{\mathcal{T}}_{\eta}(\underline{\lambda})(\underline{p})_R \\ &= \delta\left(\sum_{i=1}^3 p_i\right) \sum_{\nu_2, \nu_3=1}^{\infty} \prod_{\ell=2}^3 \left[\frac{(-1)^{\nu_{\ell}} [g^2]^{\nu_{\ell}}}{[(4\pi)^2]^{\nu_{\ell}} \Gamma(\nu_{\ell}) \Gamma(\nu_{\ell}+1)} \right] \cdot \\ &\quad \cdot \frac{1}{2i} \int dz \frac{e^{-i\pi z} \cot \pi z [g^2]^z}{[(4\pi)^2]^z \Gamma(z) \Gamma(z+1)} \cdot \frac{\Gamma[4-z-\sum_{\ell=2}^3 \nu_{\ell}]}{2i \sin \pi(-z+1)} \int_0^1 dt_2 t_2^{\nu_3-1} \cdot \\ &\quad \cdot \int_1^{0+} dt_1 (-t_1)^{-z+1} [E(t_1, t_2)]^{2-z-\sum_{\ell=2}^3 \nu_{\ell}} \sum_P [P_1^P + i\eta P_2^P]^{z+\sum_{\ell=2}^3 \nu_{\ell}-4} \end{aligned} \quad (3.42)$$

From the considerations of Subsec. III.2.4 it follows that $\tilde{\mathcal{T}}_{\eta \downarrow 0}$ is a Lorentz invariant distribution in $\mathcal{M}'(\mathbb{R}^{4 \times 3})$. The extension (3.41 - 3.42) is certainly not unique. We can clearly add to the R.H.S. of (3.41) any translation invariant distribution in $\mathcal{E}'_g(\mathbb{R}^{4 \times 3})$, $\mathcal{K}(x_1, x_2, x_3)$ with support concentrated on $x_1 = x_2 = x_3$. In fact we shall take advantage of this later to obtain a minimal and unique extension. But first we prove that the extension (3.41 - 3.42) constitutes a renormalisation.

III.3.3 We have already shown that (3.41 - 3.42) constitute a Lorentz invariant extension to a distribution in $\mathcal{E}'_g$. Furthermore the extension of $\overline{\mathcal{T}}_3(\underline{x})$ is the complex conjugate of the extension $\mathcal{T}_3(\underline{x})$. This

follows easily on utilising the fact that in Section III.2 the $\overline{\mathcal{T}}^\nu(\underline{\lambda})$ can be obtained from $\mathcal{T}^\nu(\underline{\lambda})$ by complex conjugation for real values of the regulating parameters, and then by analytic continuation in general. Hence, (E1) is satisfied. It remains, now, to prove the basic unitarity - causality relations for the functions. We introduce the following notation:

$$(a) \quad \mathcal{T} [x_1 \dots x_n] \equiv \mathcal{T} [\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)]$$

$$(b) \quad \frac{:\phi(x_1)^{i_1} \dots \phi(x_n)^{i_n}:}{i_1! \dots i_n!} \equiv \frac{:\Phi(\underline{x})^I:}{I!}$$

$$(c) \quad \| I \| \equiv \sum_{r=1}^n i_r$$

$$(d) \quad (x_{j_1} \dots x_{j_m}) \equiv (j_1 \dots j_m)$$

Starting from

$$\mathcal{T} [x_1 \dots x_n] = \sum_I \mathcal{T}_n(x_1 \dots x_n) \delta^{\|I\|} \frac{:\Phi(\underline{x})^I:}{I!}$$

the unitarity relation for the T-products (see III.3.1) implies

$$0 = \sum_I \left[\mathcal{T}_n(\underline{x}) + (-1)^n \overline{\mathcal{T}}_n(\underline{x}) \right] : \frac{\Phi(\underline{x})^I}{I!} : \\ + \sum_{k=1}^{n-1} (-1)^k \sum_{j_1 < \dots < j_k} \sum_{\overline{J}_k, \overline{\overline{J}}_k} \overline{\mathcal{T}}_k(j_1 \dots j_k) \mathcal{T}_{n-k}(j_{k+1} \dots j_n) : \frac{\Phi(j_1 \dots j_k)^{\overline{J}_k}}{\overline{J}_k!} :: \frac{\Phi(j_{k+1} \dots j_n)^{\overline{\overline{J}}_k}}{\overline{\overline{J}}_k!} : \quad (3.43)$$

with

$$\mathcal{J}_k = \{ i_{j_1}, \dots, i_{j_k} \} \quad ; \quad \overline{\mathcal{J}}_k = \{ i_{j_{k+1}}, \dots, i_{j_n} \}$$

$$\mathcal{J}_k \cup \overline{\mathcal{J}}_k = I \quad , \quad \forall \quad k = 1, \dots, n-1$$

Now, from Wick's theorem we obtain

$$\begin{aligned}
& \frac{\Phi(j_1 \dots j_k)^{J_k}}{J_k!} : : \frac{\Phi(j_{k+1} \dots j_n)^{\bar{J}_k}}{\bar{J}_k!} : \\
&= \sum_{\underline{\nu}=0} \left\{ \prod_{\substack{r,s \\ 1 \leq r \leq k \\ k+1 \leq s \leq n}} \frac{[g^2]^{\nu_{jrjs}} \Delta_+^{\nu_{jrjs}}(j_r j_s)}{\nu_{jrjs}!} \right\} \left\{ \prod_{1 \leq r \leq k} \frac{1}{[i_{jr} - \sum_{s'} \nu_{jrjs'}]!} \right\} \cdot \\
&\cdot \left\{ \prod_{\substack{s \\ k+1 \leq s \leq n}} \frac{1}{[i_{js} - \sum_{r'} \nu_{jr'js}]!} \right\} : \phi(j_1)^{i_{j_1} - \sum_{s'} \nu_{j_1 s'}} \dots \phi(j_n)^{i_{j_n} - \sum_{r'} \nu_{j_r j_n}} :
\end{aligned}$$

Hence, with the convention that negative factorials are infinite, we deduce that

$$\begin{aligned}
& \sum_{J_k, \bar{J}_k} \bar{J}_k(j_1 \dots j_k) J_{n-k}(j_{k+1} \dots j_n) : \frac{\Phi(j_1 \dots j_k)^{J_k}}{J_k!} : : \frac{\Phi(j_{k+1} \dots j_n)^{\bar{J}_k}}{\bar{J}_k!} : \\
&= \sum_{J_k, \bar{J}_k} : \frac{\Phi(j_1 \dots j_n)^{J_k} \bar{\Phi}(j_{k+1} \dots j_n)^{\bar{J}_k}}{J_k! \bar{J}_k!} : \\
&\cdot \bar{J}_k(j_1 \dots j_k) \sum_{\underline{\nu}} \left\{ \prod_{\substack{r,s \\ 1 \leq r \leq k \\ k+1 \leq s \leq n}} \frac{[g^2]^{\nu_{jrjs}} \Delta_+^{\nu_{jrjs}}(j_r j_s)}{\nu_{jrjs}!} \right\} J_{n-k}(j_{k+1} \dots j_n)
\end{aligned}$$

Substituting into the unitarity equation (3.41), using the independence of Wick polynomials in the free fields, and noting that $I! = J_k! J_{n-k}!$

$\forall k = 1, 2, \dots, n-1$ we finally obtain

$$0 = \mathcal{T}_n(x) + (-1)^n \overline{\mathcal{T}}_n(x) + \sum_{k=1}^{n-1} (-1)^k \sum_{j_1 < \dots < j_k} \overline{\mathcal{T}}_k(j_1 \dots j_k) \cdot$$

$$\cdot \sum_{\underline{\nu}} \left\{ \prod_{\substack{r,s \\ 1 \leq r \leq k \\ k+1 \leq s \leq n}} \frac{[g^2]^{\nu_{jrjs}} \Delta_+^{\nu_{jrjs}}(jrjs)}{\nu_{jrjs}!} \right\} \mathcal{T}_{n-k}(j_{k+1} \dots j_n)$$

Specialising to the three point function, and using a slightly different notation the unitarity condition reads:

Unitarity

$$0 = \sum_{U \subset G} (-1)^{|U|} \overline{\mathcal{T}}_{G(U)} \sum_{\substack{\nu_2=1 \\ \ell \in M(U,U')}} \left\{ \prod_{\ell \in M} \frac{[g^2]^{\nu_\ell} \Delta_+^{\nu_\ell}(V_{\ell_1}, V_{\ell_2})}{\nu_\ell!} \right\} \mathcal{T}_{G(U')} \quad (3.44)$$

where $U = \{V_{j_1}, \dots, V_{j_k}\}$ is a generalised vertex of $G(\{V_1 \dots V_3\}, \mathcal{L})$,

$U' = C(U)$, $\mathcal{T}_{G(U)} = \mathcal{T}(x_{j_1} \dots x_{j_k})$, and $M(U, U')$ is the set of lines connecting U and U' . (For notational simplicity we have suppressed the space-time dependence). Similarly we may deduce the causality condition:

Causality

$$\text{supp } R_3(x_1 x_2 x_3) \subset \{x_1 - x_j \in \overline{V}_+, j=2,3\} \quad (3.45)$$

where

$$R_3 \equiv \sum_{U \subset G} (-1)^{|U|-1} \overline{\mathcal{T}}_{G(U)} \sum_{\substack{\nu_2=1 \\ \ell \in M(U,U')}} \left\{ \prod_{\ell \in M} \frac{[g^2]^{\nu_\ell} \Delta_+^{\nu_\ell}(V_{\ell_1}, V_{\ell_2})}{\nu_\ell!} \right\} \mathcal{T}_{G(U')} \quad (3.46)$$

where $U_1 = U$ such that $V_1 \in U$.

In order to show that the $\tilde{J}_3(\underline{x})$ functions of (3.41 - 3.42) satisfy the above conditions we first obtain an analytically regulated version²² of these relations.

First we note that the work of Subsection III.2.2 could have been done using Pauli-Villars (P.V.) regulators $\{M_j\}^3$ instead of the regulator "r". Then the analytically and P.V. regulated propagator replacing (3.5) would be proportional to

$$\lim_{\eta \downarrow 0} \int_0^\infty d\alpha_2 \alpha_2^{[(\lambda_2-1)\nu_2]+1} I(\alpha_2, \underline{M}) \exp i\alpha_2 [p^2 + i\eta \|p\|^2] \quad (3.47)$$

where, for sufficiently many regulator masses, $I(\alpha_2, \underline{M})$ can have a zero of any desirable order at $\alpha_2 = 0$. Then the identities:

$$\begin{aligned} \Delta^{(\nu), \underline{M}}(\lambda; x) &= \theta(x^0) \Delta_+^{(\nu), \underline{M}}(\lambda; x) + \theta(-x^0) \Delta_+^{(\nu), \underline{M}}(\lambda; -x) \\ \Delta^{(\nu), \underline{M}}(\lambda; x) &= \theta(-x^0) \Delta_+^{(\nu), \underline{M}}(\lambda; x) + \theta(x^0) \Delta_+^{(\nu), \underline{M}}(\lambda; -x) \end{aligned} \quad (3.48)$$

hold in the sense of continuous functions, and the graphwise unitarity relation:

$$\begin{aligned} 0 &= \sum_{U \in \mathcal{G}} (-1)^{|U|} \left\{ \prod_{l \in \mathcal{L}[G(U)]} \bar{\Delta}^{(\nu_l), \underline{M}}(\lambda_l) \right\} \cdot \\ &\cdot \left\{ \prod_{l \in M(U, U')} \Delta_+^{(\nu_l), \underline{M}}(\lambda_l) \right\} \cdot \left\{ \prod_{l \in \mathcal{L}[G(U')] } \Delta^{(\nu_l), \underline{M}}(\lambda_l) \right\} \end{aligned} \quad (3.49)$$

holds as an algebraic identity between continuous functions²². Here

$\mathcal{L}[G(U)]$ is the set of lines connecting vertices of U . Recall from Section III.2.1 that each "line" is a multiplet of ν_l lines of the

Feynman graph. The first and third factors of (3.49) can be evaluated in momentum space, as in III.2.2 and from Theorem 3.2 it follows that the limit $M_j \rightarrow \infty$ exists, if we restrict $\underline{\lambda}$ to the region Λ_1 . We then get (with the P.V. regulators removed) the generalized unitarity relation for the massless GF. $\tilde{\mathcal{J}}_{\underline{g}}^{\underline{\nu}}(\underline{\lambda})(\underline{x})$ in the region Λ_1 . Now we can carry through the analytic continuation procedure of Subsec. III.2.3 leading to Theorem 3.4 (note that $\Delta_+^{(\nu_i)}(\lambda_i)$ is entire analytic) and obtain the unitarity relation for the massless GFA's:

$$0 = \sum_{U \subset G} (-1)^{|U|} \tilde{\mathcal{J}}_{G(U)}^{\underline{\nu}}(\underline{\lambda}) \left\{ \prod_{\ell \in M(U, U')} \Delta_+^{(\nu_\ell)}(\lambda_\ell) \right\} \tilde{\mathcal{J}}_{G(U')}^{\underline{\nu}}(\underline{\lambda}) \quad (3.50)$$

with the λ 's (which are distinct for each factor) restricted to the region Λ (3.18). We multiply (3.50) by

$$[g^2]^{\sum_i (1-\lambda_i)\nu_i} \prod_{\ell \in \mathcal{L}(G)} \left(\frac{1}{\Gamma[\nu_\ell+1]} \right) \quad (3.51)$$

and redistribute it amongst the various factors in (3.50). We then get:

$$0 = \sum_{U \subset G} (-1)^{|U|} \left\{ \left(\prod_{\ell \in \mathcal{L}[G(U)]} \frac{[g^2]^{(1-\lambda_\ell)\nu_\ell}}{\Gamma[\nu_\ell+1]} \right) \tilde{\mathcal{J}}_{G(U)}^{\underline{\nu}}(\underline{\lambda}) \right\} \cdot \left\{ \prod_{\ell \in M(U, U')} \left[\frac{(g^2)^{(1-\lambda_\ell)\nu_\ell} \Delta_+^{(\nu_\ell)}(\lambda_\ell)}{\Gamma[\nu_\ell+1]} \right] \right\} \cdot \left\{ \left(\prod_{\ell \in \mathcal{L}[G(U')]} \frac{[g^2]^{(1-\lambda_\ell)\nu_\ell}}{\Gamma[\nu_\ell+1]} \right) \tilde{\mathcal{J}}_{G(U')}^{\underline{\nu}}(\underline{\lambda}) \right\} \quad (3.52)$$

Finally, summing over all the ν 's we get noting (3.27) the generalized unitarity relation:

$$0 = \mathcal{T}_3(\underline{\lambda}) - \overline{\mathcal{T}}_3(\underline{\lambda}) + \sum_{\substack{U \subseteq G \\ U \neq G, U \neq \emptyset}} (-1)^{|U|} \overline{\mathcal{T}}_{G(U)}(\underline{\lambda}) .$$

$$+ \sum_{\substack{\nu_\ell=1 \\ \ell \in M}}^{\infty} \left[\prod_{\ell \in M(U, U')} \frac{(g^2)^{(1-\lambda_\ell)\nu_\ell} \Delta_+^{\nu_\ell}(\lambda_\ell)}{\Gamma[\nu_\ell+1]} \right] \mathcal{T}_{G(U')}(\underline{\lambda}) \quad (3.53)$$

with the λ 's in the region $\Lambda_{\overline{\mathcal{T}}_3}$. A generalised causality relation for the analytically regulated $\overline{\mathcal{T}}_3$ functions can be similarly derived.

We can apply to (3.53) the analytic decomposition method of Subsec. III.5.2 in the subregion Λ_S of (3.53). Considering the $\Xi^{(n_i)}(\underline{\lambda})$ poles of (3.34) we get the decomposition (3.56). Hence, (3.53) can be written as:

$$0 = \left[\mathcal{T}_3(\underline{\lambda})_{R_1} - \overline{\mathcal{T}}_3(\underline{\lambda})_{R_1} + \sum_{\substack{U \subseteq G \\ U \neq G, U \neq \emptyset}} (-1)^{|U|} \overline{\mathcal{T}}_{G(U)}(\underline{\lambda})_R \right. \\ \left. + \sum_{\substack{\nu_\ell=1 \\ \ell \in M}}^{\infty} \left\{ \prod_{\ell \in M(U, U')} \frac{(g^2)^{(1-\lambda_\ell)\nu_\ell} \Delta_+^{\nu_\ell}(\lambda_\ell)}{\Gamma[\nu_\ell+1]} \right\} \cdot \mathcal{T}_{G(U')}(\underline{\lambda})_R \right] + \Delta_1 U(\underline{\lambda}) \quad (3.54)$$

where:

$$\Delta_1 U(\underline{\lambda})(x) = \sum_P \sum_{n=2}^{\infty} a_{n,n}^P(\lambda_{P(n)}) \sum_{r=0}^{n-2} f_{r,n}^P(\lambda_{P(2)}, \lambda_{P(3)}) \Xi^n \delta(x_{P(1)}, -x_{P(n)}) \quad (3.55)$$

as follows from Appendix III (A.5.4). The identifications can be easily made. It suffices to note that $a_{n,n}^P(\lambda_{P(n)})$ is proportional to $\cot\left(\frac{\pi n}{1-\lambda_{P(n)}}\right)$ times analytic factors with no zeroes for $\lambda_{P(n)} = 0$

On the other hand $\lim_{\lambda_{p(2)} \rightarrow 0} f_{r,n}^p(\lambda_{p(2)}, \lambda_{p(3)})$ exists for fixed $\lambda_{p(3)}$ and similarly for the other variable. Now we show $\Delta_1 U(\underline{\lambda})(\underline{x}) = 0$. Fixing our attention on $\lambda_{p(1)}$, for some p , we recall (from the remark after (3.38)) that the limit $\lambda_{p(1)} \rightarrow 0$ (for fixed $\lambda_{p(2)}, \lambda_{p(3)}$) exists for the square bracketed in (3.54). Hence, under the same conditions, $\lim_{\lambda_{p(1)} \rightarrow 0} \Delta_1 U(\underline{\lambda})(\underline{x})$ exists. On the other hand, (4.24) can be written as:

$$\Delta_1 U(\underline{\lambda})(\underline{x}) = \sum_p \sum_{r=0}^{\infty} b_r^p(\underline{\lambda}) \square^r \delta(x_{p(1)i} - x_{p(1)f}) \quad (3.56)$$

where

$$b_r^p(\underline{\lambda}) = \sum_{n=r}^{\infty} a_{n+2}^p(\lambda_{p(1)}) f_{r,n+2}^p(\lambda_{p(2)}, \lambda_{p(3)})$$

and we must have $\lim_{\lambda_{p(1)} \rightarrow 0} b_r^p(\underline{\lambda})$ exists for each r . But

$$b_r^p(\underline{\lambda}) - b_{r+1}^p(\underline{\lambda}) = a_{r+2}^p(\lambda_{p(1)}) f_{r,r+2}^p$$

where $a_{r+2}^p(\lambda_{p(1)})$ has a pole at $\lambda_{p(1)} = 0$. Hence $f_{r,r+2} = 0$.

By a trivial induction we get for each r , $f_{r,r+2+k} = 0$, $k = 0, 1, 2, \dots$

so that $b_r^p(\underline{\lambda}) = 0$. Repeating this argument for all p , $\Delta_1 U(\underline{\lambda}) = 0$ as claimed²³.

Thus we are left with (3.54) with only the square bracket term on the R.H.S. Now we consider the $Z^{(n_3)}(\underline{\lambda})$ poles of (3.34) which are singularities of $\mathcal{T}_3$ and $\overline{\mathcal{T}}_3$ only. Applying the decomposition procedure leading to (3.40), we find that $\Delta_3 \mathcal{T}(\underline{\lambda})(\underline{x}) = \Delta_3 \overline{\mathcal{T}}(\underline{\lambda})(\underline{x})$. This is because, as noted earlier, $\overline{\mathcal{T}}^y(\underline{\lambda})$ is obtained by complex conjugation with λ real and then by analytic continuation, and we have that $\Delta_3 \mathcal{T}(\underline{\lambda})(\underline{x})$ is concentrated on $x_1 = x_2 = x_3$ by Appendix III. Hence we get:

$$0 = \mathcal{T}_3(\underline{\lambda})_R - \overline{\mathcal{T}}_3(\underline{\lambda})_R + \sum_{\substack{U \subset G \\ U \neq G, U \neq \emptyset}} (-1)^{|U|} \overline{\mathcal{T}}_{G(U)}(\underline{\lambda})_R.$$

$$\sum_{\substack{\nu_\ell=1 \\ \ell \in M(U, U')}}^{\infty} \left\{ \prod_{\ell \in M} \frac{[g^2]^{(1-\lambda_\ell)\nu_\ell} \Delta_+^{\nu_\ell}(\lambda_\ell)}{\Gamma(\nu_\ell+1)} \right\} \cdot \mathcal{T}_{G(U')}(\underline{\lambda})_R \quad (3.57)$$

Setting all the λ 's equal to a single λ , each term admits analytic continuation to $\lambda=0$. Performing the continuation, we derive the unitarity relation (3.44).

It seems worthwhile to note that our procedure is analogous to what would have happened, if we applied an analytic evaluation (Ref. (21)), instead of a generalised evaluation (Ref. (8)) as done in Ref. (2), to the generalised unitarity relation for a GFA. We also remark that an additive interpretation of our renormalisation is possible; for the three point massless GFA the set of $\underline{\lambda}$ -singularities as obtained from the sectorial representation (3.30) is actually minimal (see III.2.5) and from the work of Appendix III the subtraction terms are 'vertex' parts.

The proof of the causality relation (3.45) follows on exactly the same lines. Thus, (E2,3) are satisfied, and the extension (3.41 - 3.42) is a renormalisation.

III.3.4 The extension procedure of III.3.2 generalises to the case of the n -point function²⁰ (with the parametrisation of the sector co-ordinates given by (3.30)) the discontinuities being, now, functionals concentrated on surfaces of the form $x_{i_1} = x_{i_2} = \dots = x_{i_k}$ ($1 < k < n$), and $x_1 = x_2 = \dots = x_n$. However, for the higher point functions ($n > 3$) such an extension does not necessarily correspond to a renormalisation. The reason is the inherent

lack of symmetry in extracting the discontinuities. This asymmetry comes about because of the way of analysing the λ -dependent singularities (now in $\mathbb{C}^L$, $L = |\mathcal{L}(G)|$) as moving singularities in the Z_1 -plane. This is a serious drawback, because symmetry plays a vital role (in the cancellation of singularities in the analytically regulated unitary equation, see (3.54)) for the higher point functions ($n > 3$), where one has to use the representation of sector co-ordinates induced by the "singularity family" of Speer. The use of the Picard-Lefschetz theorem in extracting the discontinuities could be a possible way out of this difficulty, but we have not investigated this possibility with sufficient detail to draw definite conclusions.

III.4 THE UNIQUE EXTENSION WITH MINIMUM LIGHT CONE SINGULARITY

In this subsection we will show how a finite renormalisation may be implemented so as to ensure for $\mathcal{T}^{(3)}(\underline{x})$ an unique extension characterised by minimum 'light cone' singularity (the Lehmann-Pohlmeyer 'ansatz' for the 3-point function). First we give the following definition.

Definition: A finite renormalisation (in the context of non-polynomial interaction Lagrangians in the Jaffe Class) is a map which, for each generalised vertex $U = \{V'_1 \dots V'_m\}$ of G , gives a distribution $\tilde{\mathcal{X}}(U) \in \mathcal{D}'(\mathbb{R}^{4 \times m})$ of the form

$$\tilde{\mathcal{X}}(U) = \begin{cases} 1, & \text{if } m=1 \\ \delta\left(\sum_i p'_i\right) \tilde{\mathcal{X}}(\underline{p}'), & \text{if } G(V'_1 \dots V'_m) \text{ is IPI} \\ 0, & \text{otherwise.} \end{cases}$$

where $\tilde{\mathcal{X}}(\underline{p}')$ is an entire function in the momenta $\{\underline{p}'\}$ of order less than one half. In co-ordinate space

$$\mathcal{X}(\underline{x}') = f(\square_{x'_j}) \delta(x'_{i_1} - x'_{i_2}) \dots \delta(x'_{i_{m-1}} - x'_{i_m})$$

where f is an entire function in the differential operators $\square_{x'_j}$.

III.4.1 First we recall that $\mathcal{T}^{(2)}(\underline{x})$ has been completely fixed by virtue of unitarity, causality and minimum light cone singularity. Then by virtue of the unitarity causality restrictions (3.44 - 3.45), which our extension for $\mathcal{T}^{(3)}(\underline{x})$ has been proved to satisfy we have only the freedom to add to (3.41) a real, translation invariant distribution $\mathcal{X}(\underline{x})$ from $\mathcal{C}'_g(\mathbb{R}^{4 \times 3})$ concentrated on $x_1 = x_2 = x_3$. In this subsection we will implement a specific finite renormalisation. To this end we consider, in lieu of (3.41 - 3.42), the following:

Definition 3.7

$$\mathcal{Y}^{(3)}(\underline{x}) = \mathcal{F}^{-1} \left[\lim \tilde{\mathcal{Y}}_{\eta}^{(3)} \right](\underline{x}) \quad (3.58)$$

where

$$\begin{aligned} \tilde{\mathcal{Y}}_{\eta}^{(3)}(\underline{p}) &= (8\pi)^2 \delta\left(\sum_1^3 p_i\right) \left[\left\{ \sum_{\nu_2, \nu_3=1}^{\infty} \prod_2^3 \left(\frac{(-1)^{\nu_2} [g^2]^{\nu_2}}{[(4\pi)^2]^{\nu_2} \Gamma(\nu_2) \Gamma(\nu_2+1)} \right) \right. \right. \\ &\cdot \frac{1}{2i} \int_{\bar{\Gamma}_0} dz \frac{[g^2/(4\pi)^2]^z \cot \pi z (1 + \sin^2 \pi z)}{\Gamma(z) \Gamma(z+1)} e^{i\pi z} \frac{\Gamma[4-z-\sum_2^3 \nu_2]}{2i \sin \pi(-z+1)} \int_0^1 dt_2 t_2^{\nu_3-1} \\ &\cdot \int_1^{0+} dt_1 (-t_1)^{-z+1} [E(t_1, t_2)]^{2-z-\sum_2^3 \nu_2} \sum_p \left[P_1^p(\underline{t}; \underline{p}) + i\eta P_2^p(\underline{p}) \right]^{z+\sum_2^3 \nu_2-4} \left. \right\} + \tilde{\chi}(\underline{p}) \right] \quad (3.59) \end{aligned}$$

where the contour $\bar{\Gamma}_0$ is shown in Fig. (III.4.1)

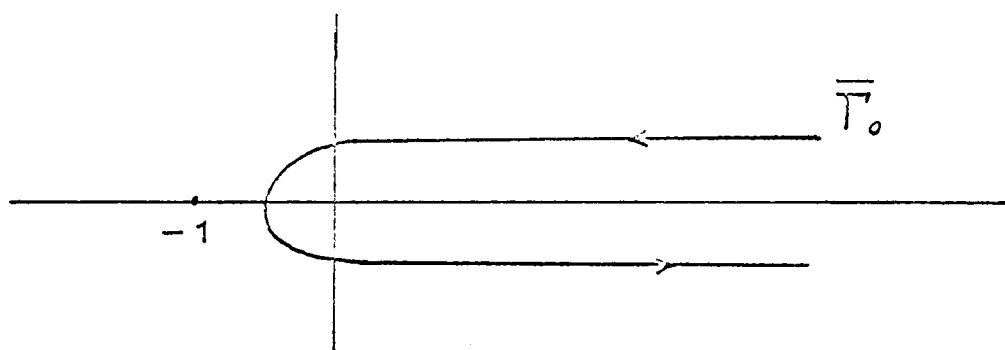


Fig. (III.4.1)

and $\tilde{\chi}(\underline{p})$ is defined by²⁴:

$$\begin{aligned} \tilde{\chi}(\underline{p}) &\equiv 2 \left[\frac{g^2}{(4\pi)^2} \right]^2 \sum_p \int_0^1 dt_2 \int_0^1 dt_1 (1 + t_2 + t_2 t_1)^{-2} (t_1 t_2)^{1/2} \frac{1}{2i} \int_C d\tau e^{t_1 \xi_p \tau} \\ &\cdot I_1 \left[2 \left(\frac{1}{\tau} \right)^{1/2} \right] I_1 \left[2 \left(\frac{t_2}{\tau} \right)^{1/2} \right] \left(\frac{1}{\tau} \right)^{1/2} K_1 \left[2 \left(\frac{1}{t_1 \tau} \right)^{1/2} \right] \quad (3.60) \end{aligned}$$

with $|\arg \tau| \leq \pi$, $\tau^{-1/2} = \exp \left[-\frac{1}{2} \ln |\tau| - \frac{i}{2} \arg \tau \right]$,

$$\zeta_p \equiv \frac{g^2}{(4\pi)^2} \left[\frac{P_{p(3)}^2 + t_1 t_2 P_{p(2)}^2 + t_1 P_{p(1)}^2}{t_1 (1 + t_1 t_2 + t_2)} \right] \quad (3.61)$$

with standard Bessel functions¹⁵ in (3.60).

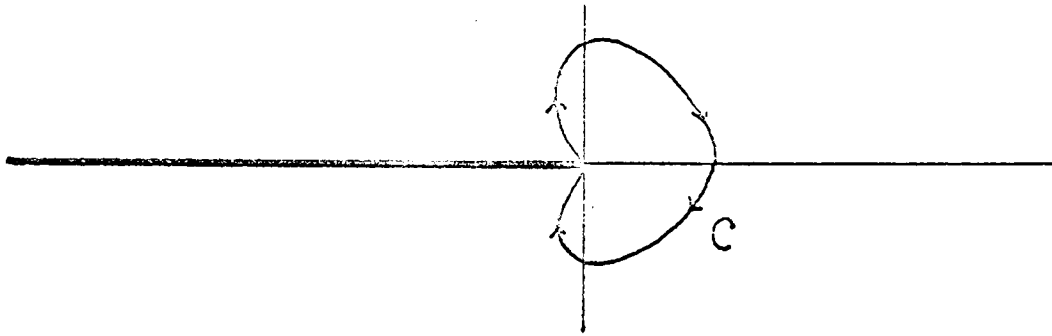


Fig. (III.4.2)

The contour C is shown in Fig. (III.4.2). The contour is constrained, in the neighbourhood of the origin, to be a cardioid; otherwise the τ integral, with the τ integrand analytic in the cut τ -plane with the cut running along the real axis from 0 to $(-\infty)$, is contour independent.

With $\tau = R \exp i\theta$, we choose

$$R = \frac{a}{2} (1 + \cos \theta) \quad (3.62)$$

for the contour C in the neighbourhood of the origin, with any real $a > 0$.

Then the real part of the arguments of the I_1 functions in (3.60) remains bounded in the neighbourhood of the origin. Furthermore since $|\arg \tau| \leq \pi$ it follows, from asymptotic estimates, that for $0 \leq t_1 \leq 1$, $\tau^{-1/2} K_1[2(t_1 \tau)^{1/2}]$ remains bounded for $\tau \in C$. Thus, the τ integral converges uniformly for $0 \leq t_{1,2} \leq 1$ and ζ_p in any compact set in $\mathbb{C}^1$. Since the

τ -integrand in (3.60) is analytic in the cut τ -plane and (3.60) converges for all contour distortions with $|\arg \tau| < \pi$ avoiding the neighbourhood of the origin, by Cauchy's theorem (3.60) is contour independent.

Hence (3.60) defines $\tilde{\chi}(\underline{p})$ unambiguously.

The definition (3.58 - 3.59) differs from the corresponding expressions (3.41 - 3.42) through the addition of real, translation invariant quasi-local distributions concentrated on $x_1 = x_2 = x_3$ which is an allowed finite renormalisation (see the beginning of Subsec. III.4). To verify this we first consider the term in braces in (3.59). This differs from the corresponding expression (3.42) in that (i) the contour $\overline{\Gamma}_0$ passes between zero and minus one, in contrast to $\overline{\Gamma}$ which passes between one and zero (ii) there is the presence of an extra contribution from the $\sin^2 \pi z$ term²⁵ in the integrand. The latter contribution is due to simple z plane poles at the positive integers; it is easily evaluated and leads to an entire function of the (p_j^2) of order of growth $1/3$. Furthermore the difference in the contribution from the $\overline{\Gamma}_0$ and $\overline{\Gamma}$ contour integrals is again due to a simple pole at $z = 0$; the contribution is again an entire function of the (p_j^2) of order of growth $1/3$.

It remains to characterise the contribution of $\tilde{\chi}(\underline{p})$ in (3.59).

To this end we study:

$$\mathcal{J}(\zeta_p, \underline{z}) = \frac{1}{2\pi i} \int_C d\tau e^{t_1 \zeta_p \tau} I_1[2(\frac{1}{\tau})^{1/2}] I_1[2(\frac{t_2}{\tau})^{1/2}] (\frac{1}{t_1 \tau})^{1/2} K_1[2(t_1 \tau)^{-1/2}] \quad (3.63)$$

$|\arg \tau| \leq \pi$, which is the τ -contour integral in (3.60). The integrand is analytic in ζ_p ; since we have a convergent integral with compact domain of integration, $\mathcal{J}(\zeta_p, \underline{z})$ is an entire function of ζ_p . We now estimate the order of growth in ζ_p .

For convenience let the entire contour C be the cardioid (3.62). Then, for $\tau \in C$, we have $\operatorname{Re}(\tau^{-1/2}) = a^{-1/2}$. Also let $|\zeta_p| \leq r$.

Then we have:

$$\sup_{|\zeta_p| = r} |\mathcal{J}(\zeta_p)| \leq C_0 a e^{ra} \sup_{\tau \in C} \left\{ \left| I_1(2(\frac{1}{\tau})^{1/2}) I_1(2(\frac{t_2}{\tau})^{1/2}) (\frac{1}{\tau})^{1/2} K_1(2(\frac{1}{t_1 \tau})^{1/2}) \right| \right\}$$

where C_0 is a constant independent of a, r . Let $a = f(r)$ (which will be determined) so that $a \rightarrow 0$, as $r \rightarrow \infty$, and hence $\operatorname{Re}(\tau^{-1/2}) \rightarrow -\infty$. Then, for large r ,

$$e^{ra} \sup_{\tau \in C} \left| \left\{ \right\} \right| \sim e^{ra} e^{2a^{-1/2}}$$

on using asymptotic estimates of Bessel functions¹⁵. We determine $a = f(r)$, by minimising $g(a) = ra + 2a^{-1/2}$. $g'(a) = 0$ implies $a = r^{-2/3}$ and $g(a) = 3r^{1/3}$.

Hence

$$\lim_{r \rightarrow \infty} \exp(-3r^{1/3}) \sup_{\xi_p = r} \left| \mathcal{J}(\xi_p) \right| = 0 \quad (3.64)$$

Hence $\mathcal{J}_p(\xi_p)$ is an entire function of ξ_p of order not greater than $1/3$.

Recalling the definition (3.61) of ξ_p it follows, because of the compact region of integration in t_1, t_2 in (3.63), that $\tilde{\mathcal{K}}(\underline{p})$ is also an entire function of the (p_j^2) of order not greater than $1/3$.

We conclude that the difference between $\tilde{\mathcal{J}}^{(3)}(\underline{x})$, as defined in (3.58), and the corresponding object (3.41), which was obtained as a renormalisation, is a real, translation invariant distribution in

$\mathcal{C}'_g(\mathbb{R}^{4 \times 3})$ concentrated on $x_1 = x_2 = x_3$, which is an allowed finite renormalisation.

III.4.2 We shall now prove that the definition (3.58) of $\tilde{\mathcal{J}}^{(3)}(\underline{p})$ is unique and is characterised by minimum light cone singularity. We need only study $\operatorname{Re} \tilde{\mathcal{J}}^{(3)}(\underline{p})$, since $\operatorname{Im} \tilde{\mathcal{J}}^{(3)}(\underline{p})$ is fixed, by unitarity, being in the form of a real quasilocal distribution in $\mathcal{C}'_g(\mathbb{R}^{4 \times 3})$ concentrated on $x_1 = x_2 = x_3$ by virtue of causality⁵. Returning to (3.59) we write

$$\tilde{\mathcal{J}}^{(3)}(\underline{p}) = (3\pi)^{-3} \delta\left(\frac{\underline{p}}{3}; i\right) \tilde{\mathcal{J}}^{(3)}(\underline{p}) \quad (3.65)$$

and study $\tilde{T}_\eta(\underline{p})$ with the $\{p_j\}$ restricted to the region

$$\Omega = \{ \underline{p} \mid p_j^2 > 0, j=1,2,3 \} \quad (3.66)$$

In compact subsets of this region $\lim_{\eta \downarrow 0} \tilde{T}_\eta(\underline{p})$ exists in the ordinary sense.

This follows on recalling the definition in (3.19) of $P_1(\underline{t}, \underline{p})$ and on

deforming the t_1 contour in (3.59) for each $\underline{p}$ term as in Fig. (III.4.3)

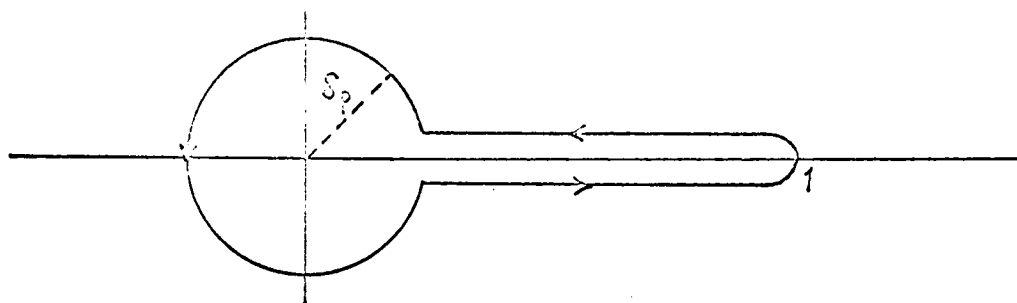


Fig. (III.4.3): The deformed t_1 contour.

with δ_ρ , the radius of the circular part satisfying

$$\delta_\rho < \left[\frac{P_{\rho(3)}^2}{P_{\rho(1)}^2 + P_{\rho(2)}^2} \right] \quad (3.67)$$

with the momenta restricted in any compact subset of Ω , (3.67) implying

that $\text{Re } P_1(\underline{t}; \underline{p}) > 0$. Hence, in taking the real part of $\lim_{\eta \downarrow 0} \tilde{T}_\eta(\underline{p})$ we have¹⁴

$$\frac{1}{2} \{ [P_1^p(\underline{t}; \underline{p}) + i0]^\alpha + [P_1^p(\underline{t}; \underline{p}) - i0]^\alpha \} = [P_1^p(\underline{t}; \underline{p})]^\alpha$$

where $\alpha = z + \frac{3}{2}\nu_2 - 4$. Similarly $e^{i\pi z} \longrightarrow \cos \pi z$ (in taking the real part) which combines with $\cot \pi z (1 + \sin^2 \pi z)$ to give

$\frac{1}{\sin \pi z} (1 - \sin^4 \pi z)$. Hence, on observing that terms involving $\sin^3 \pi z$

give no contribution since they are analytic, we obtain in any compact subset of Ω ,

$$\begin{aligned}
\operatorname{Re}(\lim_{\eta \downarrow 0} \tilde{T}_\eta(\underline{p})) &= \left\{ \sum_{\nu_2, \nu_3=1}^{\infty} \prod_2^3 \left(\frac{(-1)^{\nu_2} [g^2]^{\nu_2}}{[(4\pi)^2]^{\nu_2} \Gamma(\nu_2) \Gamma(\nu_2+1)} \right) \frac{1}{2i} \int_{\bar{\Gamma}_0} dz \cdot \right. \\
&\cdot \frac{[g^2/(4\pi)^2]^z}{\sin \pi z \Gamma(z) \Gamma(z+1)} \frac{\Gamma(4-z-\sum_2^3 \nu_2)}{2i \sin \pi(-z+1)} \int_0^1 dt_2 t_2^{\nu_3-1} \int_1^{0+} dt_1 (-t_1)^{-z+1} \cdot \\
&\cdot [E(t_1, t_2)]^{2-z-\sum_2^3 \nu_2} \sum_p [P_1^p(\underline{t}; \underline{p})]^{z+\sum_2^3 \nu_2-4} \left. \right\} + \tilde{\chi}(\underline{p}) \quad (3.68)
\end{aligned}$$

We can now deform the contour $\bar{\Gamma}_0 \rightarrow L$ (Fig.(III.4.4)), the semicircular contributions vanishing at infinity, and the resulting integral converging along the imaginary axis. Furthermore, after the deformation $\bar{\Gamma}_0 \rightarrow L$, the t_1 Eulerian integral recovers its standard form, the circular contribution (Fig. (III.4.3)) shrinking to zero as $\delta_p \rightarrow 0$.

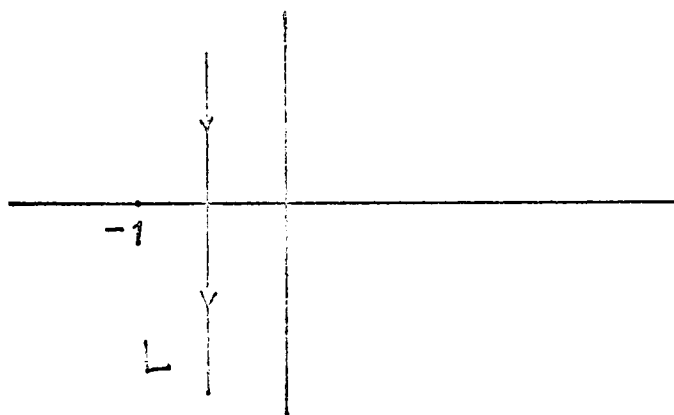


Fig. (III.4.4.): The contour L.

Thus:

$$\begin{aligned}
\operatorname{Re}(\lim_{\eta \downarrow 0} \tilde{T}_\eta(\underline{p})) &= \left\{ \sum_{\nu_2, \nu_3=1}^{\infty} \prod_2^3 \left(\frac{(-1)^{\nu_2} [g^2]^{\nu_2}}{[(4\pi)^2]^{\nu_2} \Gamma(\nu_2) \Gamma(\nu_2+1)} \right) \frac{1}{2i} \int_L dz \frac{[g^2/(4\pi)^2]^z}{\sin \pi z \Gamma(z) \Gamma(z+1)} \cdot \Gamma[4-\sum_2^3 \nu_2-z] \cdot \right. \\
&\cdot \int_0^1 dt_2 t_2^{\nu_3-1} \int_0^1 dt_1 t_1^{-z+1} [E(t_1, t_2)]^{2-z-\sum_2^3 \nu_2} \sum_p [P_1^p(\underline{t}; \underline{p})]^{z+\sum_2^3 \nu_2-4} \left. \right\} + \tilde{\chi}(\underline{p}) \quad (3.69)
\end{aligned}$$

In Appendix IV we show that the double series within braces in (3.69) can be explicitly summed and prove that the L.H.S. of (3.69) is given by:

$$\begin{aligned} \text{Re}(\tilde{T}(\underline{p})) &= 2 \left[\frac{g^2}{(4\pi)^2} \right]^2 \sum_{\underline{p}} \int_0^1 dt_2 \int_0^1 dt_1 (1+t_2+t_2 t_1)^{-2} (t_1 t_2)^{1/2} \int_0^\infty d\tau e^{-t_1 \zeta_p \tau} \\ &\quad \cdot J_1 \left[2 \left(\frac{1}{\tau} \right)^{1/2} \right] J_1 \left[2 \left(\frac{t_2}{\tau} \right)^{1/2} \right] \left(\frac{1}{\tau} \right)^{1/2} J_1 \left[2 \left(\frac{1}{t_1 \tau} \right)^{1/2} \right] \end{aligned} \quad (3.70)$$

with ζ_p as defined in (3.61).

On changing variables $\tau \rightarrow \frac{1}{t_1} \tau$ we get:

$$\begin{aligned} \text{Re}(\tilde{T}(\underline{p})) &= 2 \left[\frac{g^2}{(4\pi)^2} \right]^2 \sum_{\underline{p}} \int_0^1 dt_2 \int_0^1 dt_1 (1+t_2+t_2 t_1)^{-2} t_2^{1/2} \int_0^\infty d\tau e^{-\zeta_p \tau} \\ &\quad J_1 \left[2 \left(\frac{t_1}{\tau} \right)^{1/2} \right] J_1 \left[2 \left(\frac{t_1 t_2}{\tau} \right)^{1/2} \right] \left(\frac{1}{\tau} \right)^{1/2} J_1 \left[2 \left(\frac{1}{\tau} \right)^{1/2} \right] \end{aligned} \quad (3.71)$$

We now consider the behaviour of $\text{Re}(\tilde{T}(\underline{p}))$ when the $p_j^2 \rightarrow +\infty$ specifically for two cases:

- (i) $p_j^2 \rightarrow +\infty$, $j = 1, 2, 3$
- (ii) any two $p_j^2 \rightarrow +\infty$, and the other is bounded.

In either case it follows from the definition (3.61) that $\zeta_p \rightarrow +\infty$, for all permutations $\underline{p}$, uniformly in t_1, t_2 with $0 \leq t_{1,2} \leq 1$. Now the τ integral in (3.71) decreases faster than any inverse power of ζ_p as $\zeta_p \rightarrow +\infty$. When $\zeta_p \rightarrow +\infty$, the dominant contribution is expected in the region $\tau \sim 0$; but in this region rapid oscillation sets in, independent of t_1, t_2 , because of the last Bessel function in (3.71). An asymptotic estimate²⁶ shows that the actual decrease is $\exp[-\varepsilon \zeta_p^{1/3}]$ for some $\varepsilon > 0$ independent of the t_j . Since the region of $t_{1,2}$ integrations is compact, and since the integrand in (3.71) is strongly

decreasing in the cases (i), (ii) above for all t_i , $0 \leq t_i \leq 1$,
it follows that $\text{Re } \tilde{T}(p)$ is a strongly decreasing function in both of the
above cases (i), (ii). We are now in a position to prove the following:

Theorem 3.8

The extension $\mathcal{T}^{(3)}(\underline{x})$, as given in Definition 3.7, is unique
and has minimum light cone singularity.

The proof of uniqueness follows immediately from the previously
established strong decrease in the case (i) $p_j^2 \rightarrow +\infty$, $j = 1, 2, 3$.

For, as has been mentioned earlier, the only arbitrariness is in $\text{Re } \mathcal{T}_3(\underline{x})$
to which we can add a real, translation invariant distribution from

$\mathcal{C}'_g(\mathbb{R}^{4 \times 3})$ concentrated on $x_1 = x_2 = x_3$. In momentum space, after factoring
the overall δ -function, this corresponds to the addition to $\text{Re } \tilde{\mathcal{T}}_3(p)$
of an entire function of (p_j^2) of order $< \frac{1}{2}$ which cannot decrease to zero
in any direction and would destroy the asymptotic property in the region (i).

From the (stronger) property of decrease in the region (ii)²⁷ follows
the absence in $\text{Re } \mathcal{T}_3(\underline{x})$ of distributions concentrated on $x_1 = x_2 = x_3$.

To see this, define:

$$\text{Re } \tilde{\mathcal{T}}_3(p) \equiv \delta^{(4)}(\sum_i p_i) \tilde{T}(p_1^2, p_2^2, p_3^2) \quad (3.72)$$

so that:

$$\text{Re } \mathcal{T}_3(x_1, x_2, x_3) = T(\xi, \eta) \quad (3.73)$$

where

$$T(\xi, \eta) = \int_{\mathbb{R}^4} dp_2 \int_{\mathbb{R}^4} dp_3 e^{-i(p_2 \xi + p_3 \eta)} \tilde{T}((p_2 + p_3)^2, p_2^2, p_3^2)$$

and $\xi = x_1 - x_2$, $\eta = x_1 - x_3$.

We consider

$$t(\xi_0, \eta_0) = (T(\xi, \eta), \phi(\vec{\xi}, \vec{\eta})) \quad (3.74)$$

as a function of ξ_0 and η_0 . Here $\phi(\vec{\xi}, \vec{\eta}) \in \mathcal{C}_g(\mathbb{R}^{3 \times 2})$ such that $\tilde{\phi}$, which is in $\mathcal{M}_g(\mathbb{R}^6)$, is of compact support. Then we have the identity:

$$t(\xi_0, \eta_0) = \int_{R_1} d p_2^0 \int_{R_1} d p_3^0 e^{-i(p_2^0 \xi_0 + p_3^0 \eta_0)} \tilde{t}(p_2^0, p_3^0) \quad (3.75)$$

where:

$$\tilde{t}(p_2^0, p_3^0) = \int_{\mathbb{R}^3} d \vec{p}_2 \int_{\mathbb{R}^3} d \vec{p}_3 \tilde{T}((p_2 + p_3)^2, p_2^2, p_3^2) \tilde{\phi}(\vec{p}_2, \vec{p}_3) \quad (3.76)$$

We now examine (3.76) for two cases:

(i) $|p_2^0|$ bounded, $|p_3^0| \rightarrow +\infty$ (and similarly the other way around).

Since $\tilde{\phi}(\vec{p}_2, \vec{p}_3)$ is of compact support, in the contributing region in (3.76) we will have p_2^2 bounded, $p_3^2 \rightarrow +\infty$. Also $(p_2 + p_3)^2 = (p_2^0 + p_3^0)^2 - (\vec{p}_2 + \vec{p}_3)^2$ must $\rightarrow +\infty$. Because of the known strong decrease property of $\tilde{T}$, when any two variables $\rightarrow +\infty$, and the compact region of integration in (3.76) it follows that $\tilde{t}(p_2^0, p_3^0)$ is strongly decreasing when $|p_3^0| \rightarrow +\infty$, with $|p_2^0|$ bounded and vice versa.

(ii) Both $|p_2^0|, |p_3^0| \rightarrow +\infty$. Again since $\tilde{\phi}$ has compact support, in the contributing region in (3.76), both $p_2^2, p_3^2 \rightarrow +\infty$. On the other hand $(p_2 + p_3)^2 \geq -(\vec{p}_2 + \vec{p}_3)^2$ and is bounded below. Once more from the strong decrease property of $\tilde{T}$, it follows that $\tilde{t}(p_2^0, p_3^0)$ is strongly decreasing when $|p_2^0|, |p_3^0|$ both $\rightarrow +\infty$.

Returning to (3.75) it is now clear that $t(\xi_0, \eta_0)$ is C^∞ in both

ξ , η at the origin which rules out the presence of any distribution of the form $P(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}) \delta^{(4)}(\xi) \delta^4(\eta)$ as claimed. Thus the renormalised three point function (3.58 - 3.59) is unique by virtue of minimum light cone singularity. Q.E.D.

Remark 3.9

The minimal extension is non-analytic in the coupling constant 'g' (a situation familiar from the case of the 2-point function). This follows from the presence of higher order z plane poles within the $\overline{\Gamma}_0$ contour of (3.59).

III.4.3 Relation to other renormalisation schemes: By extending the regularisation procedure of M. K. Volkov⁴² the author of Ref. (10) was able to define a minimally singular three point function of the exponential interaction under consideration. His definition for the 3-point function runs as follows:

$$\begin{aligned} \tilde{T}(p_1, p_2, p_3) &= \frac{\pi^2 \lambda^4}{4} \sum_0^\infty \frac{1}{m_3! (m_3+1)!} \sum_0^\infty \frac{1}{m_2! (m_2+1)!} \\ &\cdot \frac{1}{2\pi i} \int_C ds_1 \frac{\Gamma(-1-s_1)}{\Gamma(3+s_1)} \cos \pi s_1 [1 + \sin^2 \pi s_1] \Gamma(-m_3 - m_2 - s_1). \quad (3.77) \\ &\cdot \left[\frac{\lambda}{4} \right]^{m_3+m_2+s_1} \sum_{\sigma \in \mathcal{P}(3)} \delta\left(\sum_1^3 p_n\right) T_0^{\sigma-}(\{p\}; s_1, m_2-1, m_3-1) + \tilde{\mathcal{R}}^- \end{aligned}$$

where $\lambda = g^2/4\pi^2$, the contour C starts and ends at $+\infty$ encircling the points $-2, -1, 0, +1, \dots$ once counterclockwise, and $\tilde{\mathcal{R}}^-$ is given by

$$\begin{aligned} \tilde{\mathcal{R}}^- &= \frac{\pi^2 \lambda^4}{4} \sum_0^\infty \left(\frac{\lambda}{4}\right)^\ell \frac{1}{\ell!} \frac{1}{(2\pi i)^2} \int_{S_2-i\infty}^{S_2+i\infty} ds_2 \int_{S_1-i\infty}^{S_1+i\infty} ds_1 \prod_1^2 \left[\frac{\Gamma(-1-s_n)}{\Gamma(3+s_n)} \right] \\ &\cdot \frac{\Gamma(s_2+s_1+1-\ell)}{\Gamma(\ell+1-s_1-s_2)} \sum_{\sigma \in \mathcal{P}(3)} \delta\left(\sum_1^3 p_n\right) T_0^{\sigma-}(\{p\}; s_1, s_2, \ell-2-s_1-s_2) \end{aligned} \quad (3.78)$$

$$S_2 < -1, \quad S_1 < -2, \quad -7/2 < S_2 + S_1 < -3$$

where

$$T_0^{G-}(\{p\}; z_1, z_2, z_3) = \frac{(-\pi)}{\sin \pi z_1} \int_0^1 dt_2 t_2^{z_3+1} \frac{1}{2\pi i} \int_1^{0+} dt_1 (-t_1)^{-z_1-1} \cdot$$

$$E(t_1, t_2)^{-2} \left[\frac{- (p_i^2 + t_1 p_j^2 + t_1 t_2 (p_i + p_j)^2)}{E(t_1, t_2)} - i0 \right]^{z_3+z_2+z_1+2} \quad (3.79)$$

Now, $\tilde{\mathcal{F}}(p_1 p_2 p_3)$ is to be compared with our definition given by (3.59').

The first term in (3.59) agrees with the first term in (3.77). The 'difference' lies in the choice of finite renormalisation terms i.e.

$\tilde{\chi}(\underline{p})$ in our equation (3.59), and $\tilde{\mathcal{G}}^-$ in Pohlmeyer's definition.

Although these finite renormalisations appear in different mathematical forms minimum singularity implies their equality. However, a more direct check of the equality is very desirable (due to the importance of the role of uniqueness in non-renormalisable non-polynomial interactions).

IV CONCLUSION

In PART I of the thesis we have taken the point of view that (axiomatic) renormalisation is basic in local field theory (divergences arise in naive manipulations in field theory due to locality). We showed how the two- and the three-point time ordered products in a conventionally non-renormalisable, but strictly local, field theory can be uniquely defined by first effecting a renormalisation, and then imposing, consistently, a 'light cone' boundary condition to fix the allowed finite renormalisations. Thus, the special feature of our particular choices of the two- and three-point functions is the absence of point singularities, that is, the absence in $\text{Re } \mathcal{T}_3(\underline{x})$ (or $\text{Re } \mathcal{T}_2(\underline{x})$) of distributions concentrated on $x_1 = x_2 = x_3$ (or $x_1 = x_2$). It is due to remarkable properties of infinite sets of renormalised Feynman graphs that this situation obtains for the two- and the three-point T-products; it is characteristic of the 'disappearance' of the effects of logarithms, in the kinematic invariants, in the sums. However, the triangle graph (associated with the three-point function) is, topologically, a rather simple object in contrast with the graph associated with the n-point T-product. It is, far from obvious, therefore, that the special features of the two- and three-point functions, characterised by the absence of point singularities, are shared by the n-point function.

It is interesting, however, that our extension procedure generalises (see Appendix II) to the case of the n-point function. It encourages us to believe that the renormalisation on the n-point function can be obtained along similar lines (provided a way can be found of extracting the discontinuities in a symmetric way). The real problem lies in the appropriate choice of finite renormalisations so as to secure for $\mathcal{T}_n(\underline{x})$ a unique extension characterised by minimum 'light cone' singularity.

A successful implementation of such a programme would lead to a control, at a perturbation theoretic level, of at least a privileged class of non-renormalisable interactions in accordance with the general principles of local field theory.

We, finally, remark that, despite of the 'light cone' regularity enjoyed by the two- and three-point functions, the exponential self-coupling interaction is a very singular theory. In perturbation theory cross-sections are expected to rise sharply with energy. Thus, the theory is at best meaningful in the low energy region. If an exact solution exists, it must lead to reasonable cross-sections at high energies⁴⁰. It is, therefore, not unreasonable to believe that, in some way, it is the least singular amplitudes which sum up correctly.

PART II

I. INTRODUCTION

The difficulties associated with the renormalisability of a massive neutral pseudovector field coupled to a non-conserved fermion current are well known. Such a coupling, described by an interaction Lagrangian of the form

$$\mathcal{L}_I = g \bar{\psi} \Gamma \psi W \quad ; \quad \partial(\bar{\psi} \Gamma \psi) \neq 0 \quad (1.1)$$

gives rise to a non-renormalisable theory in the conventional sense.

To see this it is sufficient to consider $\omega(\mathcal{L}_I) = d(\mathcal{L}_I) - 4$. In view of the form of the commutator function

$$[W_\mu(x), W_\nu(y)] = \frac{1}{i} \left[g_{\mu\nu} + \frac{1}{M^2} \frac{\partial^2}{\partial x^\mu \partial y^\nu} \right] D(x-y) \quad (1.2)$$

it is clear that $d(W_\mu) = 2$, and, consequently³⁹, $\omega(\mathcal{L}_I) > 0$ which implies increasing divergences with increasing order of perturbation expansion in the coupling constant. However, as noted in Refs. (7, 32, 33) such a power counting argument could be misleading. Indeed by applying a generalised Stuckelberg formalism³⁴ (in which the masses of the vector, and scalar meson fields are not constrained to be equal) the interaction can be transformed, by means of canonical transformation³⁵, to a renormalisable interaction which is polynomial in the free fields, together with a non-polynomial interaction in the Jaffe Class. The latter is a manifestation of the apparent non- renormalisable nature of the original interaction, and can be treated by non-linear techniques giving rise to a finite number of counterterms.

The material of PART II of the thesis is distributed as follows: In Section II we set up the model of a massive neutral pseudovector field

interacting with a non-conserved fermion current. Subsection II.1 deals with generalised Stuckelberg formalism whereby the pseudovector field is described by means of a doublet of fields with different mass. The resulting derivative coupling is converted, in Subsection II.2, to a non-derivative, non-polynomial interaction by means of a canonical (non-linear) transformation in the Spinor fields. The next step, taken up in Subsection III.3, is to derive a rather complete set of Ward-Takahashi (henceforth denoted by W-T) identities which are the mathematical manifestation of a certain gauge invariance of the Model Lagrangian of Subsec. II.2. Section III deals with the perturbation theory of quantised fields. Some W-T identities are verified in perturbation theory. The renormalisation of the Model theory is taken up in Section IV, where (in conjunction with Appendix V) it is shown that only a finite number of counterterms are needed to render the S-matrix finite.

This part (PART II) of the thesis, however, is rather incomplete in the following respects. Firstly, the uniqueness of the theory has not been established. In a non-renormalisable theory (like the case under consideration) it is, certainly, not sufficient to show the finiteness of the amplitudes, one has further to eliminate possible ambiguities (in the form of entire functions with arbitrary coefficients) arising from finite renormalisations. This is indeed not a trivial problem especially when one is dealing with the general perturbation theoretic term. Secondly, the prescription, consistent with (axiomatic) renormalisation, and the minimality 'ansatz', for obtaining finite parts for the n-point function has not been given in the text. Despite of these defects we present the analysis of PART II to demonstrate the possibility of using non-linear techniques, and yet preserving the gauge invariance of the theory.

II. A MODEL OF A MASSIVE NEUTRAL VECTOR FIELD INTERACTING WITH A NON-CONSERVED FERMION CURRENT

II.1 CLASSICAL FIELDS

We consider the parity conserving interaction of a neutral pseudovector field of mass M with a non-conserved current

$$\mathcal{L}_I = g \int_{\mu} W_{\mu} \quad (2.1)$$

where

$$j_{\mu} = \bar{\psi} \gamma_{\mu} \gamma_5 \psi$$

It is convenient to use the generalised Stuckelberg formalism of Ref. (34) to deal with the pseudovector field. In this formalism a set of field variables, A_{μ} , and B , being subject to a subsidiary condition, are used to describe the pseudovector field W_{μ} . In contrast with the usual Stuckelberg formalism the fields A_{μ} , and B are not constrained to have equal masses. Thus, setting

$$W_{\mu} = A_{\mu} + \frac{1}{M} \partial_{\mu} B$$

the total Lagrangian is taken to be

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} [\partial_{\mu} W_{\nu} - \partial_{\nu} W_{\mu}]^2 + \frac{1}{2} M^2 W_{\mu}^2 - \frac{1}{2} \alpha Z^{-1} [\partial_{\mu} A_{\mu} - \frac{m' \hat{M}}{M} B]^2 \\ & + \left[\frac{i}{2} (\bar{\psi} \gamma_{\mu} \partial_{\mu} \psi - (\partial_{\mu} \bar{\psi}) \gamma_{\mu} \psi) - m \bar{\psi} \psi \right] + g \int_{\mu} W_{\mu} \end{aligned} \quad (2.2)$$

Remark 2.1 We have chosen a convenient form for the gauge term of the Lagrangian (2.2). In Section IV we shall see that Z^{-1} is the wave function renormalisation of the A_{μ} field, and $\hat{M}$ stands for the renormalised mass of the vector field. Thus, in terms of the unrenormalised quantities we do have mixing terms ($\partial \cdot AB$) (but these are cancelled out by renormalisation).

In terms of the A_μ , and B fields we obtain

$$\mathcal{L} = -\frac{1}{4} [\partial_\mu A_\nu - \partial_\nu A_\mu]^2 + \frac{1}{2} M^2 [A_\mu + \frac{1}{M} \partial_\mu B]^2 - \frac{1}{2} \beta [\partial \cdot A - \kappa B]^2 \\ + \left[\frac{i}{2} (\bar{\Psi} \gamma_\mu \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \gamma_\mu \Psi) - m \bar{\Psi} \Psi \right] + g \bar{\Psi} \gamma_\mu \gamma_5 \Psi [A_\mu + \frac{1}{M} \partial_\mu B] \quad (2.3)$$

where $\beta = \alpha Z^{-1}$, and $\kappa = m \hat{M} / M$

Now, from Euler's equations of motion for the A_μ , and B fields it follows that

$$(\square + \kappa M)(\partial \cdot A - \kappa B) = 0 \quad (2.4)$$

which implies that all radiation processes involving the unphysical spin-0 part of the A_μ field are compensated by the corresponding processes involving the B field³¹. Turning this argument around it is clear that the rule for a unitary S-matrix is to confine ourselves to reactions which do not involve in the initial, and the final states any spin-0 part of the A_μ field, nor any B field. Indeed the Fock space generated by A_μ and B (in the quantized theory) is larger than the space of physical states. The physical subspace consists of those state vectors which are annihilated by the positive frequency part of the operator $(\partial \cdot A - \kappa B)$.

Before we investigate the gauge symmetry of the total Lagrangian, it is worth looking at the propagator matrix for the (A_μ, B) fields. To this end we first consider the 'mass' matrix, $\tilde{M}$, obtained from the free part of the Lagrangian (2.3). In momentum space $\tilde{M}$ is given by

$$\tilde{M} = \begin{bmatrix} (-k^2 + M^2) g_{\mu\nu} - (\beta - 1) k_\mu k_\nu & i(\beta\kappa - M) k_\mu \\ -i(\beta\kappa - M) k_\nu & (k^2 - \beta\kappa^2) \end{bmatrix} \quad (2.5)$$

Remark 2.2 It is clear that $\tilde{M}$ is a 2×2 matrix, however, the matrix elements

are characterised by Lorentz indices i.e. $\tilde{M} = \begin{bmatrix} A_{11,\mu\nu} & A_{12,\mu} \\ A_{21,\nu} & A_{22} \end{bmatrix}$ where, for instance, $A_{11,\mu\nu}$ is a Lorentz covariant tensor of rank 2. The inverse of M is a 2×2 matrix, $(\tilde{M}^{\nu\lambda})^{-1}$, with matrix elements which are contravariant Lorentz tensors, such that

$$\tilde{M}_{\mu\nu} (\tilde{M}^{\nu\lambda})^{-1} = \begin{bmatrix} g_{\mu}^{\lambda} & 0 \\ 0 & 1 \end{bmatrix}$$

Now, the complete propagator matrix for the (A_{μ}, B) field is given by the inverse of $\tilde{M}$. Hence

$$\tilde{\Delta}^{\nu\lambda} = \begin{bmatrix} \left\{ (g^{\nu\lambda} - \frac{k^{\nu}k^{\lambda}}{k^2}) [k^2 - M^2]^{-1} + \frac{k^{\nu}k^{\lambda}}{k^2} \cdot \frac{(k^2 - \beta\kappa^2)}{\beta[k^2 - \kappa M]^2} \right\} & \frac{ik^{\nu}(\beta\kappa - M)}{\beta[k^2 - \kappa M]^2} \\ \frac{ik^{\lambda}(\beta\kappa - M)}{\beta[k^2 - \kappa M]^2} & \frac{(\beta k^2 - M^2)}{\beta[k^2 - \kappa M]^2} \end{bmatrix}$$

Due to the presence of the term $-\frac{1}{2} \beta \kappa^2 B^2$ in the free part of the Lagrangian (2.3) it is clear that the square of the mass of the B field is given by $\beta \kappa^2$. Utilising the freedom of the generalised Stuckelberg formalism in which the mass of the vector field A_{μ} is not constrained to be equal to the mass of the scalar field B it is convenient to consider the limit $\kappa \rightarrow 0$. In this limit, with $0 < \beta < \infty$, the propagator matrix becomes

$$\tilde{\Delta}^{\nu\lambda} \rightarrow \begin{bmatrix} \left\{ (g^{\nu\lambda} - \frac{k^{\nu}k^{\lambda}}{k^2}) [k^2 - M^2]^{-1} + \frac{k^{\nu}k^{\lambda}}{k^2} [\beta k^2]^{-1} \right\} & -\frac{ik^{\nu} \cdot M}{k^2 \beta k^2} \\ -\frac{ik^{\lambda} \cdot M}{k^2 \beta k^2} & \frac{(\beta k^2 - M^2)}{k^2} [\beta k^2]^{-1} \end{bmatrix}$$

It is, now, possible to read directly the free field propagators in momentum space. Thus,

(a)

$$\overline{A^\mu A^\nu} \longrightarrow i \left(g^{\mu\nu} - k^\mu k^\nu / k^2 \right) [k^2 - M^2 + i0]^{-1} + i \left(k^\mu k^\nu / k^2 \right) [\beta k^2 + i0]^{-1}$$

(b) $\overline{A^\mu B} \longrightarrow (+1) \left(M k^\mu / k^2 \right) [\beta k^2 + i0]^{-1}$

(c) $\overline{B B} \longrightarrow (i) \left\{ (\beta k^2 - M^2) / k^2 \right\} [\beta k^2 + i0]^{-1}$

It is gratifying to know that these results are identical (for $\beta = 1$) with the results for the free field propagators obtained by the canonical quantization procedure of Ref. (34).

II.2 UNITARY AND GAUGE TRANSFORMATIONS

The non-renormalisable character of the original Lagrangian (2.3) is due to the presence of the derivative coupling between the B, and the fermion fields. This is clear from dimensional arguments: Recall that $d(\partial B) = 1 + d(B)$ which implies that, for $\mathcal{L}_I = g \overline{\psi} \gamma_\mu \gamma_5 \psi \partial^\mu B$, $\omega(\mathcal{L}_I) > 0$. Hence, from the analysis of Ref. (39) one encounters increasing divergences with increasing order of perturbation in g. It is convenient, however, to convert the derivative interaction into a non-derivative coupling. Following Dyson³⁵ we make the unitary transformation

$$\psi(x) = \exp \left[i g / M \gamma_5 B(x) \right] \tilde{\psi}(x) ; \quad \overline{\psi}(x) = \overline{\tilde{\psi}}(x) \exp \left[i g / M \gamma_5 B(x) \right] \quad (2.7)$$

In the new basis the Lagrangian becomes

$$\mathcal{L}(A_\mu, B, \psi, \bar{\psi}) = -\frac{1}{4} [\partial_\mu A_\nu - \partial_\nu A_\mu]^2 + \frac{1}{2} M^2 [A_\mu + \frac{1}{M} \partial_\mu B]^2 - \frac{1}{2} \beta [\partial \cdot A - \kappa B]^2$$

$$+ [i(\bar{\psi} \gamma_\mu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma_\mu \psi)] + g \bar{\psi} \gamma_\mu \gamma_5 \psi A_\mu - m \bar{\psi} \exp[i \frac{2g}{M} \gamma_5 B] \psi \quad (2.8)$$

The two Lagrangians $\mathcal{L}(A_\mu, B, \psi, \bar{\psi})$, and $\mathcal{L}(A_\mu, B, \psi, \bar{\psi})$ being connected by a canonical transformation will give rise to the same S-matrix^{31,36}.

The non-renormalisable derivative coupling of (2.3) has, now, been converted to the exponential type of interaction of (2.8). The latter belongs to the Jaffe Class of non-polynomial interactions. Expanding the exponential as a power series in g we recover couplings of the form

$g^n \bar{\psi} \gamma_5^n \psi B^n$ ($n \geq 2$) which are manifestly non-renormalisable reflecting, thus, the non-renormalisable character of the original derivative coupling. Such an expansion must be avoided (in order to eliminate the presence of an infinite number of counterterms in the Lagrangian of the model theory). Instead one has to make a perturbation expansion in the Lagrangian rather than in the coupling constant 'g'.

We, now, turn to the local gauge transformations which leave (2.8) invariant. Consider the set of local transformations:

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x)$$

$$B(x) \rightarrow B(x) - M \Lambda(x) \quad (2.9)$$

$$\psi(x) \rightarrow \exp[i g \gamma_5 \Lambda(x)] \psi(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) \exp[i g \gamma_5 \Lambda(x)]$$

with the gauge function, Λ , satisfying the equation

$$(\square + \kappa M) \Lambda(x) = 0 \quad (2.10)$$

Then, it is easy to verify that $\mathcal{L}(A_\mu, B, \psi, \bar{\psi})$ remains invariant under (2.9 - 2.10). At this stage we remark that the above gauge symmetry

will give rise to a set of Ward-Takahashi identities which play an important role in the renormalisation of the above model theory.

II.3 GENERALISED WARD-TAKAHASHI IDENTITIES

II.3.1 In this Subsection we shall derive a rather complete set of generalised Ward-Takahashi identities which are the mathematical manifestation of the gauge symmetry of Subsection II.2. To this end we add to the Lagrangian of the theory interaction terms of the form

$$\beta \int [\partial \cdot A - \kappa B] + \int_{\mu} [A_{\mu} + \frac{1}{M} \partial_{\mu} B] + \bar{\eta} \psi + \bar{\psi} \eta$$

where $(J(x), j_{\mu}(x), \eta(x), \bar{\eta}(x))$ are arbitrary C-number functions of space and time (in fact $(\eta(x), \bar{\eta}(x))$ are classical spinor functions) which act as sources. We emphasize that the introduction of external sources is simply a mathematical device and does not presuppose the presence of external charges. We remark, however, that the connected Green's functions of the theory depend now on the above set of external sources through the interaction Lagrangian density, and the S-matrix. We shall make use of this fact later on. Now, denoting collectively the above set of sources by $\underline{J}$ we obtain the following expression for the new Lagrangian, $\mathcal{L}'$:

$$\mathcal{L}'(A_{\mu}, B, \psi, \underline{J}) = \mathcal{L}(A_{\mu}, B, \psi) + \beta \int [\partial \cdot A - \kappa B] + \int_{\mu} [A_{\mu} + \frac{1}{M} \partial_{\mu} B] + \bar{\eta} \psi + \bar{\psi} \eta \quad (2.11)$$

Hence, we deduce that the S-matrix is a functional, through $\mathcal{L}'$, of the external sources.

All the Ward-Takahashi identities can be studied through the response of the generating functional (for connected Green's functions), $\mathbb{Z}$, to the gauge transformations of the Lagrangian under consideration³⁷.

This functional is defined as follows: Let S_0 be the vacuum expectation value of the S-matrix with the external sources set equal to zero in the

interaction Lagrangian. Then,

$$\mathbb{Z} = \langle 0 | S[\underline{J}] | 0 \rangle / S_0 \quad (2.12)$$

Now, supposing the interaction Lagrangian contains terms of the form

$\sum_j \underline{J}_j(x) \mathcal{B}_j(x)$, where $\underline{J}(x)$ is a set of classical sources. Then, from Bogoliubov³, we know that

$$\begin{aligned} \left. \frac{\delta^n \mathbb{Z}}{\delta \underline{J}_1(x_1) \dots \delta \underline{J}_n(x_n)} \right|_{\underline{J}=0} &= (i)^n \langle 0 | T[\mathcal{B}_1(x_1) \dots \mathcal{B}_n(x_n) S] | 0 \rangle_c / S_0 \\ &\equiv (i)^n \langle 0 | T[\bar{\mathcal{B}}_1(x_1) \dots \bar{\mathcal{B}}_n(x_n)] | 0 \rangle_c \end{aligned} \quad (2.13)$$

where $\bar{\mathcal{B}}_i$ is the Heisenberg field corresponding to the free field $\mathcal{B}_i$

(We have used the notation $\langle 0 | T[\dots] | 0 \rangle_c$ for connected Green's functions).

In the following Subsection we shall apply the above techniques to the model Lagrangian (2.11) in connection with the local gauge transformations (2.9) (without imposing the condition (2.10)).

II.3.2 We start with the functional relation (2.13) which yields, for the case of the S-matrix corresponding to the Lagrangian (2.11), the following relation

$$\left. \frac{\delta^3 \mathbb{Z}}{\delta J(x_3) \delta \eta(x_2) \delta \bar{\eta}(x_1)} \right|_{\underline{J}=0} = (i)^3 \langle 0 | T[\psi(x_1) \bar{\psi}(x_2) \beta(\partial \cdot A(x_3) - \kappa B(x_3))] | 0 \rangle \quad (2.14)$$

We, now, perform infinitesimal (local) gauge transformations without imposing the condition, $(\square_x + \kappa M) \wedge(x) = 0$, on the gauge function. Due to the invariance of the Lagrangian, $\mathcal{L}$, the gauge transformations will change only the source, and the gauge terms. Thus,

$$\delta \mathcal{L}' = -\beta [\partial \cdot A - \kappa B] [\square + \kappa M] \Lambda + \beta J [\partial \cdot A - \kappa B] \\ + \beta J [\square + \kappa M] \Lambda + i g (\bar{\eta} \gamma_5 \psi + \bar{\psi} \gamma_5 \eta) + O(\Lambda^2)$$

Constraining $\Lambda(x)$ to satisfy $[\square_x + \kappa M] \Lambda(x) = J(x)$, and taking $J(x)$ to be infinitesimal we can neglect second order terms in J , and Λ .

Hence, we obtain

$$\delta \mathcal{L}' = i g \Lambda [\bar{\eta} \gamma_5 \psi + \bar{\psi} \gamma_5 \eta]$$

which implies the following functional relation

$$\mathbb{Z}[J, j_\mu, \eta] = \mathbb{Z}[0, j_\mu, (1 + i g \Lambda \gamma_5) \eta] \quad (2.15)$$

with $\Lambda(x)$ satisfying

$$(\square_x + \kappa M) \Lambda(x) = J(x) \quad (2.16)$$

In view of (2.15 - 2.16) the L.H.S. of (2.14) becomes

$$\left. \frac{\delta^3 \mathbb{Z}[J, j_\mu, \eta]}{\delta J \delta \eta \delta \bar{\eta}} \right|_{J=0} = (\square + \kappa M)^{-1} \left. \frac{\delta^3 \mathbb{Z}[0, j_\mu, (1 + i g \Lambda \gamma_5) \eta]}{\delta \Lambda \delta \eta \delta \bar{\eta}} \right|_{\Lambda=j_\mu=\eta=0} \quad (2.17)$$

Now, utilising the fact that $\left. \frac{\delta S}{\delta \Lambda} \right|_{\eta=\bar{\eta}=0} = 0$ we deduce from (2.14),

and (2.17) the following W-T identity:

$$[\square_{x_3} + \kappa M] \langle 0 | T [\psi(x_1) \bar{\psi}(x_2) \beta (\partial \cdot A(x_3) - \kappa B(x_3))] | 0 \rangle_c \quad (2.18)$$

$$= g \left[\delta(x_1 - x_3) \gamma_5 \langle 0 | T [\psi(x_1) \bar{\psi}(x_2)] | 0 \rangle_c + \delta(x_2 - x_3) \langle 0 | T [\psi(x_1) \psi(x_2)] | 0 \rangle_c \gamma_5 \right]$$

(2.18) is analogous to the W-T identity in QED, and relates the vertex functions to the self-energy parts of the 'dressed' fermion propagator.

It is of interest, also, to consider the following functional derivatives:

$$(i) \quad \frac{\delta^2 \mathbb{Z}}{\delta J(x_1) \delta J(x_2)} \Big|_{\underline{J} = \underline{0}} \quad (ii) \quad \frac{\delta^2 \mathbb{Z}}{\delta J(x_1) \delta j_\mu(x_2)} \Big|_{\underline{J} = \underline{0}}$$

In view of $\frac{\delta S}{\delta \Lambda} \Big|_{\eta = \bar{\eta} = 0} = 0$, and $\frac{\delta^2 S}{\delta \Lambda^2} \Big|_{\eta = \bar{\eta} = 0} = 0$ both (i), and (ii)

are identically zero. Consequently, we deduce the following set of W-T identities:

$$\left\{ \prod_{r=1}^2 (\square_{x_r} + \kappa M) \right\} \langle 0 | T \left[\left\{ \prod_{r=1}^2 \beta (\partial \cdot A(x_r) - \kappa B(x_r)) \right\} \right] | 0 \rangle_c = 0 \quad (2.19)$$

and

$$(\square_{x_1} + \kappa M) \langle 0 | T \left[(A_\mu(x_2) + 1/M \partial_\mu B(x_2)) \beta (\partial \cdot A(x_1) - \kappa B(x_1)) \right] | 0 \rangle_c = 0 \quad (2.20)$$

(2.19), and (2.20) are the W-T identities connecting the meson (A_μ , B) self energies. Finally we observe that, as a result of

$$0 = \frac{\delta^n \mathbb{Z}}{\delta J(x_n) \dots \delta J(x_1)} \Big|_{\underline{J} = \underline{0}}$$

(2.19) generalises to:

$$\left\{ \prod_{r=1}^n (\square_{x_r} + \kappa M) \right\} \langle 0 | T \left[\left\{ \prod_{r=1}^n \beta (\partial \cdot A(x_r) - \kappa B(x_r)) \right\} \right] | 0 \rangle_c = 0 \quad (2.21)$$

(For the validity of (2.21), for the special case $n=4$, see Appendix V).

II.3.3 It turns out that the W-T relations (2.18 - 2.21) derived in Subsection II.3.2 are somewhat simpler in momentum space. In order to carry out the F.T. of (2.18 - 2.20) we introduce the following compact

notation:

(i) We denote by $\vec{\Phi}$, and $\vec{\Psi}$ the irreducible vertex vector-function,

$\begin{bmatrix} \Gamma_\nu^5 \\ \Gamma^5 \end{bmatrix}$, for the complete fermion pseudovector-meson coupling,
and the vector $\begin{bmatrix} i\beta k_\mu \\ -\beta\kappa \end{bmatrix}$ respectively.

(ii) $\tilde{P}$ stands for the complete meson self-energy matrix,

$$\begin{bmatrix} P_{\mu\nu} & P_\nu \\ P_\mu & P \end{bmatrix}$$

Now, the F.T. of (2.18) yields

$$\begin{aligned} & (-k_3^2 + \kappa M) i S_F'(k_1) \vec{\Phi}(k_3) \tilde{\Delta} \vec{\Psi}(k_3) i S_F'(-k_2) \\ &= i g [\gamma_5 S_F'(-k_2) + S_F'(k_1) \gamma_5] \end{aligned}$$

with $\tilde{\Delta}$ given by (2.6) and $i S_F'(p) = \int d^4x e^{ip \cdot x} \langle 0 | T [\psi(x) \bar{\psi}(0)] | 0 \rangle$

In the limit $\kappa \rightarrow 0$ the above result reduces to the simple form

$$\begin{aligned} & S_F'(k_1) \{ i(k_1+k_2)_\nu \Gamma_\nu^5(k_1, k_2) + M \Gamma^5(k_1, k_2) \} S_F'(-k_2) \\ &= i g \{ \gamma_5 S_F'(-k_2) + S_F'(k_1) \gamma_5 \} \end{aligned}$$

On multiplying on the left by $S_F'(k_1)^{-1}$, and on the right by $S_F'(-k_2)^{-1}$, substituting p' for k_1 , and p for $-k_2$ we obtain the familiar form of the W-T identity (analogous to the W-T identity in QED) for a pseudovector theory:

$$(p'-p)_\nu \Gamma_\nu^5(p, p') - i M \Gamma^5(p, p') = g \left[S_F'(p')^{-1} \gamma_5 + \gamma_5 S_F'(p)^{-1} \right] \quad (2.22)$$

This is depicted in Fig. 13*.

We, now, turn to (2.19 - 2.20). The F.T. of (2.19) yields

$$(-k^2 + \kappa M)^2 \left[\vec{\Psi}^\top(k) \tilde{\Delta}^\top \tilde{P} \tilde{\Delta} \vec{\Psi}(k) \right] = 0$$

In the limit $\kappa \rightarrow 0$ the above equation reduces to :

$$-k_\mu k_\nu P_{\mu\nu} + 2iM k_\mu P_\mu + M^2 P = 0 \quad (2.23)$$

Similarly, from the F.T. of (2.20) we obtain

$$(-k^2 + \kappa M) \left[\vec{\chi}^\top(k) \tilde{P} \tilde{\Delta} \vec{\Psi}(k) \right] = 0$$

with $\vec{\chi} = \begin{bmatrix} 1 \\ ik_\nu/M \end{bmatrix}$, which reduces, in the limit $\kappa \rightarrow 0$,

to:

$$-iM k_\mu P_{\mu\nu} + k_\nu k_\mu P_\mu - M^2 P_\nu - iM k_\nu P = 0 \quad (2.24)$$

However, from Lorentz covariance, the matrix elements of the meson self-energy matrix, $\tilde{P}$, acquire the following form

$$\begin{aligned} P_{\mu\nu}(k) &= A(k^2) k_\mu k_\nu + B(k^2) g_{\mu\nu} \\ P_\mu(k) &= C(k^2) k_\mu \\ P(k) &= D(k^2) \end{aligned} \quad (2.25)$$

Solving (2.23 - 2.24) simultaneously and utilising (2.25) we obtain the following set of W-T identities:

* All figures are shown in a separate section following the final section of PART II. The fermion fields, ψ , and $\bar{\psi}$ are represented by solid lines. The pseudovector field, A_μ , is represented by a wiggly line; it carries a Lorentz vector index μ . The pseudovector, B , field is represented by a dotted line.

$$\begin{bmatrix} P_{\mu\nu}(k) & P_{\mu}(k) \\ P_{\nu}(k) & P(k) \end{bmatrix} \begin{bmatrix} i k_{\nu} \\ M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.26)$$

This set of W-T identities is identical with that of Ref. (33) and is depicted in Fig. 14.

In dealing with the renormalisation of the model Lagrangian theory under consideration we shall make extensive use of the W-T identities derived in this section.

III PERTURBATION THEORY OF QUANTIZED FIELDS

III.1 In this Section we shall develop a perturbation theory for the model Lagrangian (2.8). Separating out the fermion mass term the interaction Lagrangian to be used is given by

$$\mathcal{L}_I = g \bar{\psi} \gamma_\mu \gamma_5 \psi A_\mu - m \bar{\psi} \left(\exp \left[\frac{2ig}{M} \gamma_5 B \right] - 1 \right) \psi$$

The quantized interaction Lagrangian of perturbation theory is defined as the normal-ordered classical expression given above. Hence,

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$$

with

$$\mathcal{L}_1 = g : \bar{\psi} \gamma_\mu \gamma_5 \psi A_\mu : \quad ; \quad \mathcal{L}_2 = -m \bar{\psi} : \left(\exp \left[\frac{2ig}{M} \gamma_5 B \right] - 1 \right) : \psi \quad (3.1)$$

As noted in Ref. (7,33) this Wick ordering eliminates the most divergent parts of the theory arising from the tadpole diagrams of Fig.1.

We seek to develop a systematic perturbation expansion in powers of the interaction Lagrangian, (3.1). The S-matrix of the theory has a formal functional expansion of the form³

$$S = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int T \left[\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n) \right] dx_1 \dots dx_n \quad (3.2)$$

where the T-ordered products of the interaction Lagrangian can be obtained using Wick's theorem. As an example we shall discuss the case of

$T \left[\mathcal{L}_2(x_1) \dots \mathcal{L}_2(x_n) \right]$. To this end we introduce the following notations:

$$\Lambda_{\pm} = \frac{1 \pm \gamma_5}{2} \quad , \quad \kappa = \left(\frac{2g}{M} \right)^2$$

and the fields

$$\Theta^{\pm} = : \left(\exp \left[\pm i \kappa^{1/2} B \right] - 1 \right) :$$

Then, $\mathcal{L}_2(x)$ can be expressed in the form⁷,

$$\mathcal{L}_2 = (-m) [\bar{\psi} \Lambda_+ \psi \Theta^+ + \bar{\psi} \Lambda_- \psi \Theta^-] \quad (3.3)$$

Hence,

$$\begin{aligned} & T[\mathcal{L}_2(x_1) \cdots \mathcal{L}_2(x_n)] \\ &= (-m)^n \left\{ T[\cdots \bar{\psi}(x_i) \Lambda_{\pm} \psi(x_i) \cdots \bar{\psi}(x_j) \Lambda_{\mp} \psi(x_j) \cdots] \cdot \right. \\ & \quad \left. \cdot T[\cdots \Theta^{\pm}(x_i) \cdots \Theta^{\mp}(x_j) \cdots] \right\} \end{aligned} \quad (3.4)$$

with $T[\cdots \Theta^{\pm}(x_i) \cdots \Theta^{\mp}(x_j) \cdots]$ given by (2.1) (PART I) i.e.

$$\begin{aligned} T[\cdots \Theta^{\pm}(x_i) \cdots \Theta^{\mp}(x_j) \cdots] &= \sum_{\underline{i}=1}^{\infty} \sum_{\substack{\{\nu_{rs}\}=0 \\ \nu_{rs}=\nu_{sr} \\ \nu_{rr}=0}}^{\infty} \left\{ \prod_{r=1}^n \frac{(\pm i\kappa^{1/2})^{i_r}}{[i_r - \sum_{s'} \nu_{rs'}]!} \right\} \cdot \\ & \cdot \left\{ \prod_{1 \leq r, s \leq n} \frac{\Delta^{\nu_{rs}}(x_r, x_s)}{\nu_{rs}!} \right\} : \prod_{r=1}^n B(x_r)^{i_r - \sum_{s'} \nu_{rs'}} : \end{aligned} \quad (3.5)$$

Formulae (3.4 - 3.5) are, indeed, the ones to be used for calculations involving 'supergraphs', however, in order to check the algebraic structure of Ward identities, for instance, the following formula is more convenient:

$$\begin{aligned} T[\mathcal{L}_2(x_1) \cdots \mathcal{L}_2(x_n)] &= \sum_{\underline{i}=1}^{\infty} \sum_{\substack{\{\nu_{rs}\}=0 \\ \nu_{rs}=\nu_{sr} \\ \nu_{rr}=0}}^{\infty} \left\{ \prod_{1 \leq r < s \leq n} \frac{\Delta^{\nu_{rs}}(x_r, x_s)}{\nu_{rs}!} \right\} \cdot \\ & : \prod_{r=1}^n \frac{B(x_r)^{i_r - \sum_{s'} \nu_{rs'}}}{[i_r - \sum_{s'} \nu_{rs'}]!} : T[\bar{\psi}(x_1) \gamma_s^{i_1} \psi(x_1) \cdots \bar{\psi}(x_n) \gamma_s^{i_n} \psi(x_n)] \end{aligned} \quad (3.6)$$

Similar expressions can be obtained for the T-ordered products of 'mixed' interactions. We shall not give the corresponding formulae here, instead we turn to the W-T identities of Subsection II.3.

III.2 THE W-T IDENTITIES IN PERTURBATION THEORY

It will be instructive to see how the W-T identities of Subsection II.3 are satisfied in perturbation theory especially in the case of a theory giving rise to graphs containing 'super-vertices', and 'superpropagators' (due to the exponential character of the interaction Lagrangian $\mathcal{L}_2$). In this Subsection we shall verify the W-T identity (2.22).

Let Λ_μ^5 , and Λ^5 be the corrections (due to higher order processes) to the elements of the vector $\vec{\Phi}$ (see II.3.3) defined by

$$\Gamma_\mu^5 = g \gamma_\mu \gamma_5 + \Lambda_\mu^5$$

$$\Gamma^5 = (-m) 2ig \not{M} \gamma_5 + \Lambda^5$$

Then, using $S_F(p)^{-1} = \not{p} - m - \Sigma(p)$, we may rewrite (2.22) as:

$$(p-p')_\mu \Lambda_\mu^5(p,p') - iM \Lambda^5(p,p') = -g [\Sigma(p) \gamma_5 + \gamma_5 \Sigma(p')] \quad (3.7)$$

To a given order of perturbation theory all the diagrams contributing to can be divided into two classes³⁸:

(i) diagrams in which the axial vector vertex, $g \gamma_\mu \gamma_5$, is attached to a fermion line with p' and p the in - and out - momentum respectively.

The general diagram is illustrated in Fig. 2.

(ii) diagrams in which the axial vector vertex, $g \gamma_\mu \gamma_5$, is attached to an internal closed fermion loop. This case is illustrated in Fig. 3.

Firstly, we examine the contributions to (3.7) due to diagrams belonging to class (i). Two such diagrams are shown in Fig. 4.(I & II). The contribution from Fig. 4 (I) has the form

$$A_\mu^5(I) = \sum_{\underline{p}} \int d\underline{p} \dots \frac{1}{\not{p} + \not{p}_{k-1} - m} \Gamma^{(k)} \frac{1}{\not{p} + \not{p}_k - m} \gamma_\mu \gamma_5 \frac{1}{\not{p}' + \not{p}_k - m} \dots$$

where

$$\Gamma^{(k)} = \begin{cases} \gamma_\mu^{(k)} \gamma_5 & , \text{ for an axial vector coupling vertex} \\ \gamma_5^{(k)} & , \text{ for a 'super-vertex' .} \end{cases}$$

Multiplying $A_\mu^5(I)$ by $(p-p')_\mu$ and utilising the identity

$$(\not{p}-\not{p}')\gamma_5 = (\not{p}+\not{p}_{k-1}-m)\gamma_5 + \gamma_5(\not{p}'+\not{p}_k-m) + 2m\gamma_5$$

we obtain

$$\begin{aligned} (p-p)_\mu A_\mu^5(I) &= (\dots) \frac{1}{\not{p}+\not{p}_{k-1}-m} \Gamma^{(k)} \frac{1}{\not{p}+\not{p}_k-m} (2m\gamma_5) \frac{1}{\not{p}'+\not{p}_k-m} (\dots) \\ &+ (\dots) \frac{1}{\not{p}+\not{p}_{k-1}-m} \Gamma^{(k)} \frac{1}{\not{p}+\not{p}_k-m} \gamma_5 (\dots) + (\dots) \frac{1}{\not{p}+\not{p}_{k-1}-m} \Gamma^{(k)} \gamma_5 \frac{1}{\not{p}'+\not{p}_k-m} (\dots) \end{aligned}$$

Similarly, the contribution from Fig. 4 (II) yields

$$\begin{aligned} (p-p')_\mu A_\mu^5(II) &= (\dots) \frac{1}{\not{p}+\not{p}_{k-1}-m} (2m\gamma_5) \frac{1}{\not{p}'+\not{p}_{k-1}-m} \Gamma^{(k)} \frac{1}{\not{p}'+\not{p}_k-m} (\dots) \\ &+ (\dots) \frac{1}{\not{p}+\not{p}_{k-1}-m} \gamma_5 \Gamma^{(k)} \frac{1}{\not{p}'+\not{p}_k-m} (\dots) + (\dots) \gamma_5 \frac{1}{\not{p}'+\not{p}_{k-1}-m} \Gamma^{(k)} \frac{1}{\not{p}'+\not{p}_k-m} (\dots) \end{aligned}$$

Thus,

$$(p-p')_\mu [A_\mu^5(I) + A_\mu^5(II)] = \dots + (\dots) \frac{1}{\not{p}+\not{p}_{k-1}-m} \left[\Gamma^{(k)} \gamma_5 + \gamma_5 \Gamma^{(k)} \right] \frac{1}{\not{p}'+\not{p}_k-m} (\dots)$$

There are, now, two cases to be considered

(a) $\Gamma^{(k)} = \gamma_\mu^{(k)} \gamma_5$. Then, because of the anticommutator

$\{\gamma_\mu, \gamma_5\} = 0$, $\Gamma^{(k)} \gamma_5 + \gamma_5 \Gamma^{(k)} = 0$. Hence, summing over k we get vanishing contributions except when $k = 1$, and $k = n$.

Therefore,

$$\sum_k (p-p')_\mu \left[A_\mu^5(I) + A_\mu^5(II) \right] = \dots - \left\{ (\dots) \Gamma^{(2n-1)} \frac{1}{\not{p} + \not{p}_{2n-1} - m} \Gamma^{(2n)} \right\} \gamma_5$$

$$- \gamma_5 \left\{ \Gamma^{(1)} \frac{1}{\not{p}' + \not{p}_1 - m} \Gamma^{(2)} \frac{1}{\not{p}' + \not{p}_2 - m} (\dots) \right\} + \sum_{k=1}^{2n} (\dots) \frac{1}{\not{p} + \not{p}_{k-1} - m} \Gamma^{(k)} \frac{1}{\not{p} + \not{p}_k - m} (2m\gamma_5) \frac{1}{\not{p}' + \not{p}_k - m} (\dots)$$

where the first two terms of the above equation correspond to the fermion self-energy parts $-\Sigma(p)\gamma_5$, and $-\gamma_5\Sigma(p')$, whereas the last term is a contribution to $\Lambda^5(p,p')$.

(b) $\Gamma^{(k)} = \gamma_5 \gamma_k$. In this case we obtain

$$(p-p')_\mu \left[A_\mu^5(I) + A_\mu^5(II) \right] = \dots + (\dots) \frac{1}{\not{p} + \not{p}_{k-1} - m} (2\gamma_5) \gamma_k \frac{1}{\not{p}' + \not{p}_k - m} (\dots)$$

$$+ (\dots) \frac{1}{\not{p} + \not{p}_{k-1} - m} (2m\gamma_5) \frac{1}{\not{p}' + \not{p}_{k-1} - m} \Gamma^{(k)} \frac{1}{\not{p}' + \not{p}_k - m} (\dots)$$

The first term is precisely the contribution from the diagram of Fig. 5.

Moreover, for the special case of $k=1$ we obtain

$$\Gamma^{(1)} \gamma_5 \frac{1}{\not{p}' + \not{p}_1 - m} \Gamma^{(2)} (\dots) = \gamma_5 \gamma_1 \frac{1}{\not{p}' + \not{p}_1 - m} \Gamma^{(2)} (\dots)$$

$$= 2\gamma_5 \Gamma^{(1)} \frac{1}{\not{p}' + \not{p}_1 - m} \Gamma^{(2)} (\dots) - \gamma_5 \Gamma^{(1)} \frac{1}{\not{p}' + \not{p}_1 - m} \Gamma^{(2)} (\dots)$$

(this is the crucial step in the proof) where the first term is the contribution to $\Lambda^5(p,p')$ from the diagram of Fig. 6, whereas the second term is the corresponding contribution to $-\gamma_5\Sigma(p')$.

A similar argument holds in the case of $k=2n$, whereby we get the contribution from the diagram of Fig. 7, together with a piece corresponding to

$-\Sigma(p)\gamma_5$. Hence, we may conclude, by summing over k , that

$\sum_k (p-p')_\mu \left[A_\mu^5(I) + A_\mu^5(II) \right]$ is equal to the corresponding contributions to $\Lambda(p,p')$ together with the self-energy pieces corresponding to $-\sum(p)\gamma_5$, and $-\gamma_5 \sum(p')$.

Therefore (due to (a) and (b), we obtain (after utilising the structure of the pseudo-scalar coupling)

$$(p-p')_\mu \Lambda_\mu^{5,(i)}(p,p') = iM \Lambda^{5,(i)}(p,p') - g \left[\sum(p)\gamma_5 + \gamma_5 \sum(p') \right] \quad (3.8)$$

Secondly, we turn to diagrams belonging to class (ii). A typical contribution to $\Lambda_\mu^5(p,p')$ has the form

$$\int d_4 q \text{Tr} \left\{ \sum_{k=1}^m \prod_{j=1}^{k-1} \left[\Gamma^{(j)} \frac{1}{\not{q} + \not{p}_j - m} \right] \Gamma^{(k)} \frac{1}{\not{q} + \not{p}_k - m} \gamma_\mu \gamma_5 \frac{1}{\not{q} + \not{p}_k + \not{p}' - \not{p} - m} \right. \\ \left. \prod_{j=k+1}^m \left[\Gamma^{(j)} \frac{1}{\not{q} + \not{p}_j + \not{p}' - \not{p} - m} \right] \right\} (\dots)$$

Multiplying by $(p-p')_\mu$ we obtain

$$\int d_4 q \text{Tr} \left\{ \sum_{k=1}^m \prod_{j=1}^{k-1} \left[\Gamma^{(j)} \frac{1}{\not{q} + \not{p}_j - m} \right] \Gamma^{(k)} \left[\frac{1}{\not{q} + \not{p}_k - m} (2m\gamma_5) \frac{1}{\not{q} + \not{p}_k + \not{p}' - \not{p} - m} \right. \right. \\ \left. \left. + \frac{1}{\not{q} + \not{p}_k - m} \gamma_5 + \gamma_5 \frac{1}{\not{q} + \not{p}_k + \not{p}' - \not{p} - m} \right] \prod_{j=k+1}^m \left[\Gamma^{(j)} \frac{1}{\not{q} + \not{p}_j + \not{p}' - \not{p} - m} \right] \right\} (\dots)$$

Now, if $\Gamma^{(k)} = \gamma^{(k)} \gamma_5$ we get cancellations on summing over k (due to the anticommutator $\{\gamma_\mu, \gamma_5\} = 0$) except for the special cases when $k=1$, and $k=2m$. For these case we obtain, using a well - known theorem on traces,

$$\int d_4 q \text{ Tr } \left\{ -\gamma_5 \prod_{j=1}^m \left(\Gamma^{(j)} \frac{1}{\not{q} + \not{p}_j + \not{p}' - \not{p} - m} \right) + \gamma_5 \prod_{j=1}^m \left(\Gamma^{(j)} \frac{1}{\not{q} + \not{p}_j - m} \right) \right\} (\dots)$$

A change of integration variable in the second term ($q \longrightarrow q + p' - p$)

shows that the two terms cancel out. Thus, we are left only with the

contributions to $\Lambda^5(p, p')$ arising from the diagrams of Fig. 8, and Fig. 9.

Therefore,

$$(p - p')_\mu \Lambda_\mu^{5, (ii)}(p, p') = i M \Lambda^{5, (ii)}(p, p') \quad (3.9)$$

The W-T identity (3.7) is obtained by adding (3.8), and (3.9). Q.E.D.

Other W-T identities can be verified along similar lines.

Having establish the (algebraic) validity of the W-T identities we shall turn, in the following section, to the renormalisation of the Model Lagrangian theory with an interaction Lagrangian given by (3.1).

IV RENORMALISATION OF THE MODEL THEORY

IV.1 It is an inherent feature of any Lagrangian field theory that the procedure of the subtraction of divergences, arising from T-ordered products of the interaction Lagrangian, may be implemented by means of counterterms added to the original Lagrangian^{2,3}. A renormalisable theory is characterised by the requirement that the number of counterterms induced by the renormalisation procedure is finite. A more stringent definition of renormalisability, however, requires the possibility of absorbing such a finite number of counterterms into a redefinition of the parameters of the original Lagrangian. In this way all the 'infinities' of the theory may be absorbed by the parameters of the new Lagrangian.

Starting with the Lagrangian of the form

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} [\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu]^2 + \frac{1}{2} \hat{M}^2 [\hat{A}_\mu + \frac{1}{\hat{M}} \partial_\mu \hat{B}]^2 - \frac{1}{2} \alpha [\partial \cdot A - m' \hat{B}]^2 \\ & + \left\{ \frac{i}{2} [\bar{\hat{\psi}} \gamma_\mu \partial_\mu \hat{\psi} - (\partial_\mu \bar{\hat{\psi}}) \gamma_\mu \hat{\psi}] - m \bar{\hat{\psi}} \hat{\psi} \right\} + \hat{g} : \bar{\hat{\psi}} \gamma_\mu \gamma_5 \hat{\psi} \hat{A}_\mu : \quad (4.1) \\ & - m \bar{\hat{\psi}} : (\exp [i 2 \hat{g} / \hat{M} \gamma_5 \hat{B}] - 1) : \hat{\psi} \end{aligned}$$

we consider the limit as $m' \rightarrow 0$, and $\alpha \rightarrow \infty$ in such a way that $\alpha m' = M$.

Then, there are no 'mixing terms' ($A \cdot \partial B$) in (4.1), and B is now a massless scalar field. Consequently (i) the propagator matrix given by (2.5) is now diagonal, and (ii) the exponential interaction gives rise only to massless 'superpropagators'.

Remark 4.1: A massive 'superpropagator' is, mathematically, a rather complicated object, and the problem of uniqueness of the higher point functions (n-point functions with $n \geq 3$) involving massive superpropagators has not been solved yet. The existing literature deals only with higher point functions involving massless 'superpropagators'.

We recall that the interaction Lagrangian consists of two parts

$$\mathcal{L}_I(x) = \mathcal{L}_1(x) + \mathcal{L}_2(x)$$

where

$$\mathcal{L}_1 = \hat{g} : \bar{\hat{\psi}} \gamma_\mu \gamma_5 \hat{\psi} \hat{A}_\mu :$$

$$\mathcal{L}_2 = (-m) \left(\bar{\hat{\psi}} \wedge_+ \hat{\psi} \hat{\Theta}^+ + \bar{\hat{\psi}} \wedge_- \hat{\psi} \hat{\Theta}^- \right)$$

Now, let $C[\mathcal{L}^{\text{int}}(x_1) \dots \mathcal{L}^{\text{int}}(x_n)]$ be the 'discontinuity' between the totally advanced, and totally retarded operators of the product of n interaction Lagrangians $\mathcal{L}^{\text{int}}(x_1) \dots \mathcal{L}^{\text{int}}(x_n)$. Then Epstein, and Glaser³⁹ have shown that

$$\omega(\langle 0 | C[\mathcal{L}^{\text{int}}(x_1) \dots \mathcal{L}^{\text{int}}(x_n)] | 0 \rangle) = \sum_{i=1}^n \omega(\mathcal{L}^{\text{int}}) + 4$$

where the L.H.S. stands for the degree of the tempered distribution

$\langle 0 | C[\mathcal{L}^{\text{int}}(x_1) \dots \mathcal{L}^{\text{int}}(x_n)] | 0 \rangle$ which is a measure of its increase at infinity in momentum space, and

$$\omega(\mathcal{L}^{\text{int}}) = d(\mathcal{L}^{\text{int}}) - 4$$

$d(\mathcal{L}^{\text{int}})$ being the dimension of the interaction Lagrangian, $\mathcal{L}^{\text{int}}$.

For the case $\mathcal{L}^{\text{int}} \equiv \mathcal{L}_1$ the form of the propagator $\overline{\hat{A}_\mu} \hat{A}_\nu$ (derived in II.1) implies that $d(\mathcal{L}_1) = 4$. Hence, $\omega(\langle 0 | C[\mathcal{L}_1(x_1) \dots \mathcal{L}_n(x_n)] | 0 \rangle)$ remains bounded with increasing order of perturbation theory. The interaction Lagrangian $\mathcal{L}_1(x)$ is, therefore, renormalisable.

We shall, now, classify the divergent graphs of the $\bar{\hat{\psi}} \hat{\psi} \hat{A}$ theory (i.e. the graphs arising from the VEV of T-ordered products of $\mathcal{L}_1(x)$).

To this end we introduce a convenient form for the superficial degree of divergence of a given Feynman graph G (see Appendix I). For a renormalisable polynomial (in the free fields interaction this form is given by³

$$\omega(G) = 4 - \frac{1}{2} \sum_{\ell_{\text{ext}}} (r_\ell + 2)$$

(4.2)

where r_l is the degree of the polynomial in the numerator of the causal propagator corresponding to l^{th} external line of G . A given IPI (see Appendix I) graph is divergent provided $\omega(G) \geq 0$. Thus, utilising the above formula, we deduce that there exist five types of divergent diagrams (neglecting vacuum diagrams) in the $\bar{\psi}\psi\hat{A}$ theory shown in Fig. 10. The two types (d), and (e) require no counterterms, because they give rise to finite contributions to S-matrix. Hence, we are left with three types of divergent diagrams corresponding to the self-energies of the $\hat{A}_\mu$, and $\hat{\psi}$ fields, and the $\bar{\psi}\psi\hat{A}$ vertex function.

We consider, now, the contribution to the S-matrix arising from 'mixed' interactions. In this case we are interested in T-ordered products of interaction Lagrangians of the form $T \left[\dots \mathcal{L}_1(x_1) \dots \mathcal{L}_2(x_j) \dots \right]$. The power counting theorem of Epstein, and Glaser is rather misleading in this case due to an inherent high energy damping of massless 'superpropagators'³². However, it is easy to see that divergences may arise from diagrams shown in Fig. 11 together with their associated diagrams shown in Fig. 12. Before we proceed any further, we remark that the tadpole diagram of Fig. 11 (d), and the associated diagram of Fig. 12 (d) arise because the interactions $\bar{\psi} \wedge \pm \hat{\psi} \hat{\Theta}^\pm$ are not completely Wick ordered. It turns out, however, that the presence of such tadpole-like diagrams is essential in order to satisfy the W-T identities of Subsec. II.3. The latter are depicted, in a somewhat sketchy way, in Figs. 13, 14, 15.

Counterterms: Having classified the divergent diagrams of the theory we turn, at present, to the structure of the necessary counterterms cancel out the various divergences. Denoting collectively the counterterms by $\delta\mathcal{L}$ we maintain that the effect of renormalisation is to change (4.1) into

$$\begin{aligned} \mathcal{L} - \delta\mathcal{L} = & -\frac{1}{4} [\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu]^2 + \frac{1}{2} \hat{M}^2 [\hat{A}_\mu + \frac{1}{\hat{M}} \partial_\mu \hat{B}]^2 - \frac{1}{2} \alpha [\partial \cdot \hat{A} - m \cdot \hat{B}]^2 \\ & + \left\{ \frac{i}{2} [\bar{\hat{\psi}} \gamma_\mu \partial_\mu \hat{\psi} - (\partial_\mu \bar{\hat{\psi}}) \gamma_\mu \hat{\psi}] - m \bar{\hat{\psi}} \hat{\psi} \right\} + \hat{g} : \bar{\hat{\psi}} \gamma_\mu \gamma_5 \hat{\psi} \hat{A}_\mu : \\ & - m \bar{\hat{\psi}} : (\exp[i 2 \hat{g}/\hat{M} \gamma_5 \hat{B}] - 1) : \hat{\psi} - \frac{1}{4} (Z-1) [\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu]^2 \quad (4.3) \end{aligned}$$

$$\begin{aligned} & + \frac{1}{2} Z_3 [\hat{A}_\mu + \frac{1}{\hat{M}} \partial_\mu \hat{B}]^2 + (Z-1) \left\{ \frac{i}{2} [\bar{\hat{\psi}} \gamma_\mu \partial_\mu \hat{\psi} - (\partial_\mu \bar{\hat{\psi}}) \gamma_\mu \hat{\psi}] \right\} \\ & + [Z_1 \delta m - (Z_1 - 1)m] \bar{\hat{\psi}} : \exp[2i \hat{g}/\hat{M} \gamma_5 \hat{B}] : \hat{\psi} + \hat{g} (Z_2 - 1) : \bar{\hat{\psi}} \gamma_\mu \gamma_5 \hat{\psi} \hat{A}_\mu : \end{aligned}$$

To see this we shall study in some detail the role of individual counterterms which appear in (4.3).

We start with the meson self-energy diagrams. Although such diagrams, see Fig. 10 (b), and Fig. 11 (b), have (at worst) a superficial degree of divergence equal to two, they are at most logarithmically divergent due to the pseudovector character of the $:\hat{\int}_\mu^5 \hat{A}_\mu:$ coupling. Thus, in view of the set of W-T identities (2.26) (see also Fig. 14) it is easy to see (using the regularisation procedure of Ref. (41)) that the quadratic divergences arising from diagrams of the type shown in Fig. 11 (c,d) cancel out among themselves, and the remaining logarithmic divergences (due to the totality of diagrams of the types shown in Fig. 10 (b), Fig. 11 (b, c, d)) can be cancelled out by means of the gauge invariant counterterm

$$\frac{1}{4} (Z-1) [\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu]^2 - \frac{1}{2} Z_3 [\hat{A}_\mu + \frac{1}{\hat{M}} \partial_\mu \hat{B}]^2 \quad (4.4)$$

Furthermore the counterterm

$$- \hat{g} (Z_2 - 1) : \hat{\psi} \gamma_\mu \gamma_5 \hat{\psi} \hat{A}_\mu : \quad (4.5)$$

will cancel out the logarithmic (at worst) divergences associated with diagrams of the type shown in Fig. 10 (a). Note, however, that (4.5) by itself is not gauge-invariant.

We, now, proceed to examine the fermion self-energy diagrams. Such diagrams shown in Fig. 10 (c) have a superficial degree of divergence equal to one. Hence they are linearly divergent (at worst), and a counterterm of the form

$$-(Z_1-1)\left\{\frac{i}{2}\left[\bar{\psi}\gamma_\mu\partial_\mu\psi-(\partial_\mu\bar{\psi})\gamma_\mu\psi\right]-m\bar{\psi}\psi\right\}-Z_1\delta m\bar{\psi}\psi \quad (4.6)$$

is required to cancel out the corresponding divergences. Such a counterterm is not present in (4.3). Instead we have introduced the following counterterm

$$\begin{aligned} &-(Z_1-1)\left\{\frac{i}{2}\left[\bar{\psi}\gamma_\mu\partial_\mu\psi-(\partial_\mu\bar{\psi})\gamma_\mu\psi\right]\right\} \\ &\quad -\left[Z_1\delta m-(Z_1-1)m\right]\bar{\psi}:(\exp[2i\hat{\partial}/\hat{M}\gamma_5\hat{B}]):\psi \quad (4.7) \end{aligned}$$

(For an excellent discussion in connection with the presence of the exponential counterterm see Ref. (33)). Hopefully (4.7) not only will cancel out the divergences arising from 'pure' fermion self-energy diagrams, but also divergences arising from pseudoscalar vertex diagrams (see Fig. 11 (a, e)) as well as the associated diagrams shown in Fig. 12 (a, e). However, such a statement is far from obvious and needs some clarification.

To this end we first observe that diagrams of the type shown in Fig. 12 (a) with an odd number of B fields emerging from the same vertex have the same singularity structure as the corresponding diagrams with an even number of B fields emerging from the same vertex. Now, let Z_5 be the renormalisation constant of the pseudoscalar vertex function. Due to the presence of the counterterm (4.5) the corresponding renormalisation constant for the pseudovector vertex function is Z_2 . It will be instructive to find out, what relations exist among the renormalisation constants Z_1 , Z_2 , and Z_5 implied by the W-T identity (2.22). We start with the identity itself:

$$(p-p')_\mu\Gamma_\mu^5(p,p')-i\hat{M}\Gamma^5(p,p')=\hat{g}\left[S_F'(p)^{-1}\gamma_5+\gamma_5S_F'(p')^{-1}\right]$$

Factoring out the coupling constant $\hat{g}$ from Γ_μ^5 , and $(-m)2i\hat{g}/\hat{M}$ from Γ_5 the above W-T identity reduces (with some abuse of notation) to

$$(p-p')_\mu \Gamma_\mu^5(p,p') - 2m \Gamma^5(p,p') = [S_F'(p)\bar{\gamma}_5 + \gamma_5 S_F'(p')^{-1}] \quad (4.8)$$

We, now, present the following argument due to Adler³⁸. The argument is carried out in two steps after introducing a cutoff Λ into the Feynman amplitudes. Because of the cutoff the renormalisation constants are cutoff-dependent but finite (they diverge, however, as $\Lambda \rightarrow \infty$).

(a) Utilising the relations

$$\Gamma_\mu^5 = Z_2^{-1} \tilde{\Gamma}_\mu^5 ; \quad \Gamma^5 = Z_5^{-1} \tilde{\Gamma}^5 ; \quad S_F' = Z_1 \tilde{S}_F'$$

(with the tilde quantities being cutoff independent) and setting $p = p'$ in (4.8) we obtain

$$-2m Z_1 \Gamma^5(p,p') = [\tilde{S}_F'(p)\bar{\gamma}_5 + \gamma_5 \tilde{S}_F'(p')^{-1}]$$

Since the R.H.S. is finite (i.e. cutoff independent) we see that

$2m Z_1 \Gamma^5$ must also be finite. Now, bearing in mind, that

$Z_5 \Gamma^5$ is finite, by definition, we may conclude that, in the

limit $\Lambda \rightarrow \infty$,

$$\frac{2m Z_1}{Z_5} = \alpha \quad (4.9)$$

where α is finite.

(b) Multiplying (4.8) by Z_2 we obtain

$$(p-p')_\mu \tilde{\Gamma}_\mu^5(p,p') = \frac{Z_2}{Z_1} \left[\left(\frac{2m Z_1}{Z_5} \right) \tilde{\Gamma}^5(p,p') + (\tilde{S}_F'(p)\bar{\gamma}_5 + \gamma_5 \tilde{S}_F'(p')^{-1}) \right] \quad (4.10)$$

Now, the L.H.S. is finite, and the term in the square bracket on the

R.H.S. of (4.10) is also finite due to (4.9). Thus, (4.10) implies that (in the limit $\Lambda \longrightarrow \infty$)

$$\frac{Z_2}{Z_1} = \beta \quad (4.11)$$

where β is finite.

From (4.9), and (4.11) we conclude that, up to arbitrary finite factors, the axial-vector, and the pseudovector vertex renormalisations are just Z_1 , and $2mZ_1$ respectively.

Remark 4.2: We may define Z_1 by setting $\beta = 1$ (4.11) i.e. $Z_1 = Z_2$.

This, however, will have certain implications concerning the experimental determination of the coupling constant g_{phys} . The latter may be determined by observing the decay of the physical spin-1 field (W_μ field) into a fermion pair. Because we have chosen arbitrarily $Z_1 = Z_2$ (and Z_1 is fixed according to the usual procedure) the axial-vector vertex corrections may not vanish, when all particles are on the mass-shell, and consequently the experimentally determined constant will not correspond to g .

We may, now, see (taking into account (4.9), and (4.11)) that the counterterm (4.7) has the correct structure to deal with divergences arising from fermion self-energy diagrams together with divergences associated with pseudoscalar vertex diagrams shown in Fig. 10 (a, e). Furthermore we believe that the above counterterm will deal successfully with divergences arising from the associated diagrams (with more than one B-emission lines), but we have not carried out any detailed investigations.

Let us, now, turn to divergent diagrams of the type shown in Fig. 12 (b, c). We have not introduced any counterterm to cancel out the corresponding divergences for the following reason. Firstly, we observe that in a given physical process (spinor-spinor scattering for instance) both types will contribute, and secondly, there is a mutual cancellation of

divergences arising from the two types of diagrams as one may readily verify by using the regularisation procedure of Ref. (41) in connection with the W-T identity depicted in Fig. 15. This state of affairs is demonstrated in Appendix V. In the same appendix we have also checked explicitly the cancellation of divergences to order $g^8 \log g^2$ for the set of diagrams shown in Fig. 16. Similar considerations hold for the type of diagrams shown in Fig. 12 (c, d, f, j); also for the set shown in Fig. 12 (h, i) (see Appendix V). Cancellation of divergences are also expected to occur among the diagrams of the type shown in Fig. 17.

IV.2 To see the physical meaning of the renormalisation procedure we define

$$A_\mu = Z^{1/2} \hat{A}_\mu \quad ; \quad B = M/\hat{M} Z^{1/2} \hat{B} \quad ; \quad \psi = Z_1^{1/2} \hat{\psi}$$

$$g = Z^{-1/2} Z_1^{-1} Z_2 \hat{g} \quad ; \quad M^2 = (\hat{M}^2 + Z_3) Z^{-1}$$

Then, $\mathcal{L} - \delta\mathcal{L}$, given by (4.3), may be rewritten as

$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{4} [\partial_\mu A_\nu - \partial_\nu A_\mu]^2 + \frac{1}{2} M^2 [A_\mu + \frac{1}{M} \partial_\mu B]^2 - \frac{1}{2} \beta [\partial \cdot A - \kappa B]^2 \\ & + \left\{ \frac{i}{2} [\bar{\psi} \gamma_\mu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma_\mu \psi] - m \bar{\psi} \psi \right\} + g : \bar{\psi} \gamma_\mu \gamma_5 \psi A_\mu : \\ & - m \bar{\psi} : (\exp [i 2 \hat{g} Z^{-1/2} / M \gamma_5 B] - 1) : \psi \\ & + \delta m \bar{\psi} : (\exp [i 2 \hat{g} Z^{-1/2} / M \gamma_5 B]) : \psi \end{aligned}$$

Now, setting $Z_1 = Z_2$ (see also Remark 4.2), and $m_0 = m - \delta m$, $\mathcal{L}_0$ acquires the simple form

$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{4} [\partial_\mu A_\nu - \partial_\nu A_\mu]^2 + \frac{1}{2} M^2 [A_\mu + \frac{1}{M} \partial_\mu B]^2 - \frac{1}{2} \beta [\partial \cdot A - \kappa B]^2 \\ & + \left\{ \frac{i}{2} [\bar{\psi} \gamma_\mu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma_\mu \psi] - m_0 \bar{\psi} \psi \right\} + g : \bar{\psi} \gamma_\mu \gamma_5 \psi A_\mu : \\ & - m_0 \bar{\psi} : (\exp [i 2 g / M \gamma_5 B] - 1) : \psi \end{aligned}$$

This is, however, the same as our starting Lagrangian (4.1), except that the various coefficients have been changed. The fields A_μ , B , and ψ are the unrenormalised fields. The quantities M , m_0 are the bare masses, and g stands for the bare coupling constant. Thus, if we choose the bare masses, and coupling constant in an appropriate cutoff-dependent fashion, and rescale the fields in an appropriate cutoff-dependent way (recall that Z , Z_1 , Z_2 , Z_3 , and δm are cutoff dependent, and diverge as the cutoff parameters tend to infinity), all the divergences (hopefully) disappear in perturbation theory, and the Lagrangian (4.1) gives rise to a renormalisable theory (despite of the apparent non-renormalisable nature of the interaction).

CONCLUSION

In PART II of the thesis the conventionally non-renormalisable coupling of a massive neutral pseudovector to a non-conserved fermion current has been analysed using the generalised Stuckelberg formalism. We emphasize at the outset, however, that this analysis has not been exhaustive. By converting the above coupling into a polynomial interaction of the renormalisable type, and a non-polynomial interaction of the exponential type (in a massless field) we are able to fix up a finite number of gauge invariant counterterms to cancel out the infinities of the model theory. Such a state of affairs is materialised only because of an inherent damping for large momenta provided by the exponential interaction, when the latter is treated by non-linear techniques developed in PART I of the thesis. Our analysis is rather unsatisfactory, however, due to a lack of a proof concerning the uniqueness of the theory. This requires an extensive work on the asymptotic behaviour of Feynman amplitudes. We hope to return to this problem elsewhere in the near future.

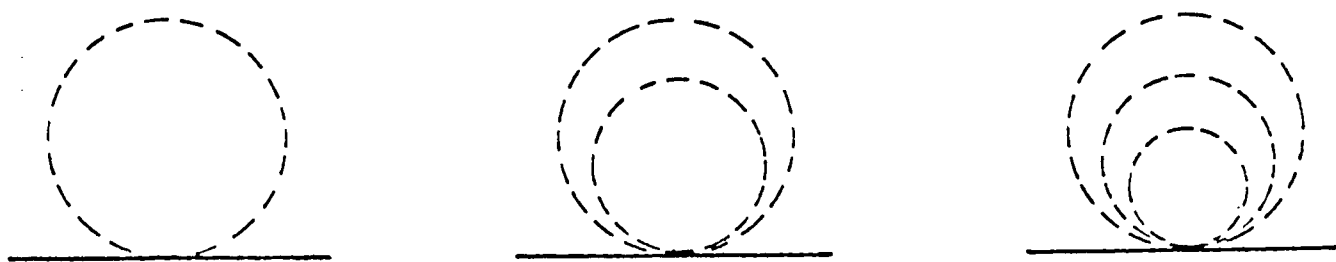


Fig. 1

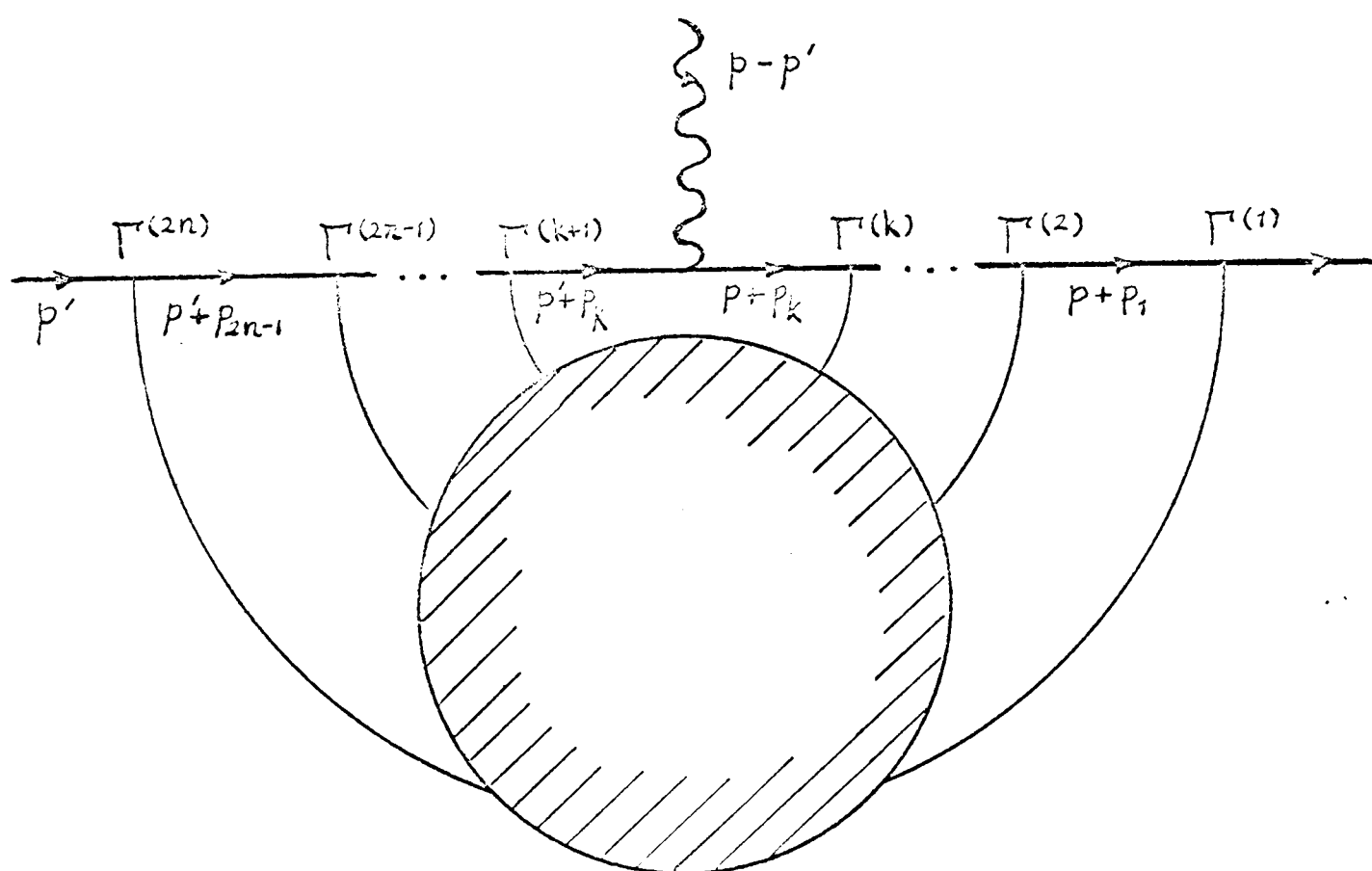


Fig. 2

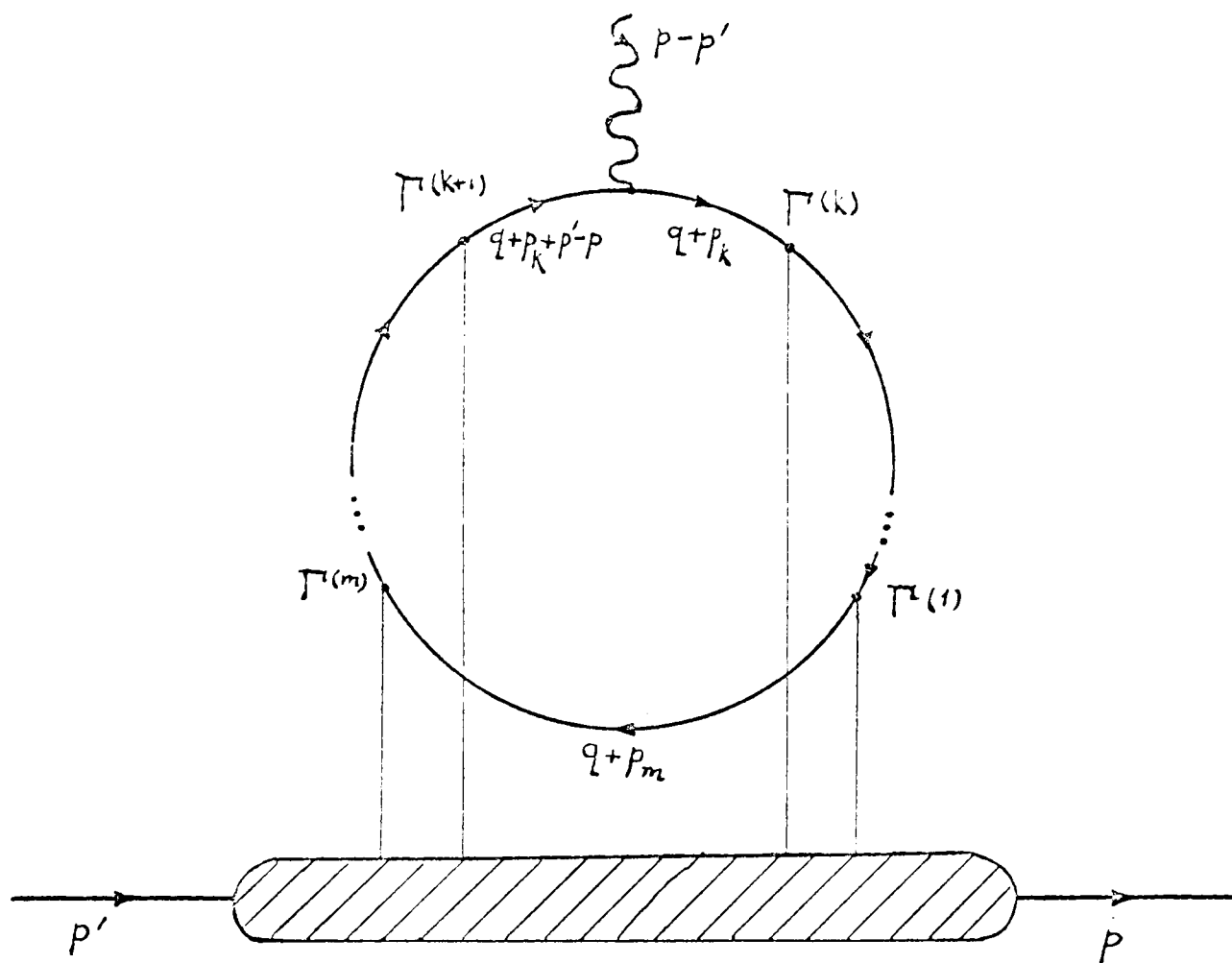


Fig. 3

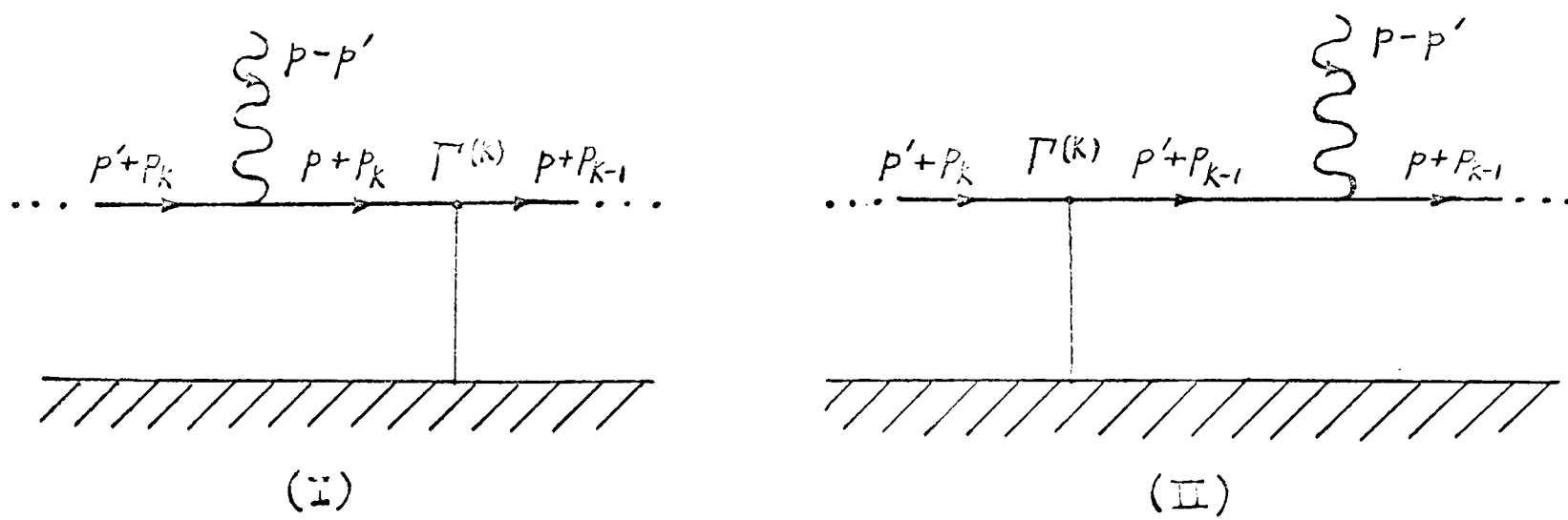


Fig. 4

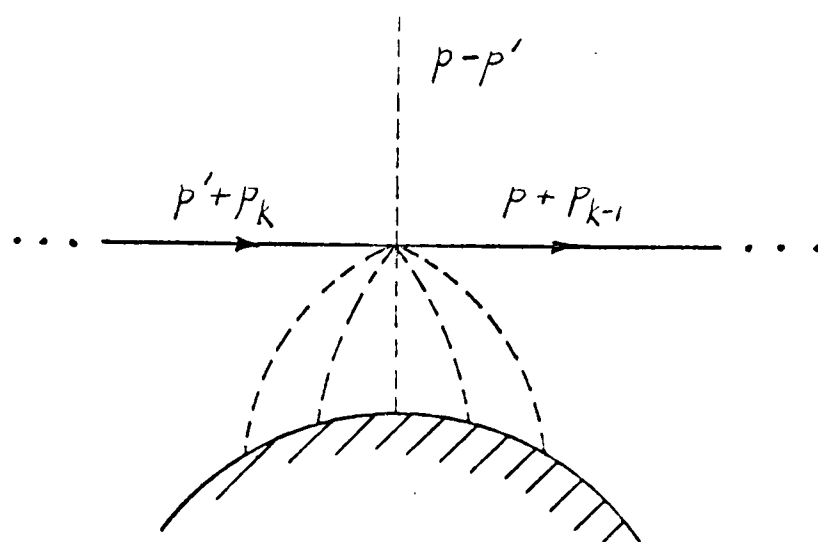


Fig. 5

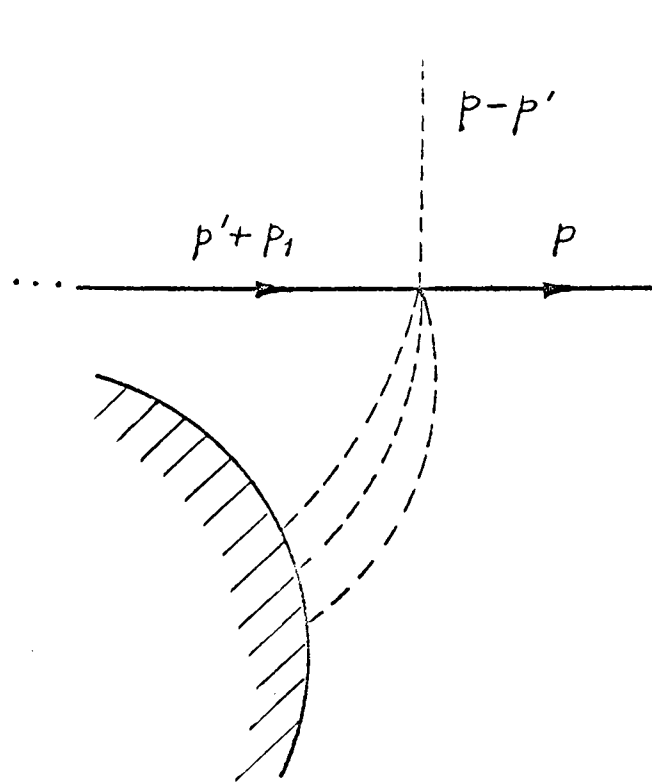


Fig. 6

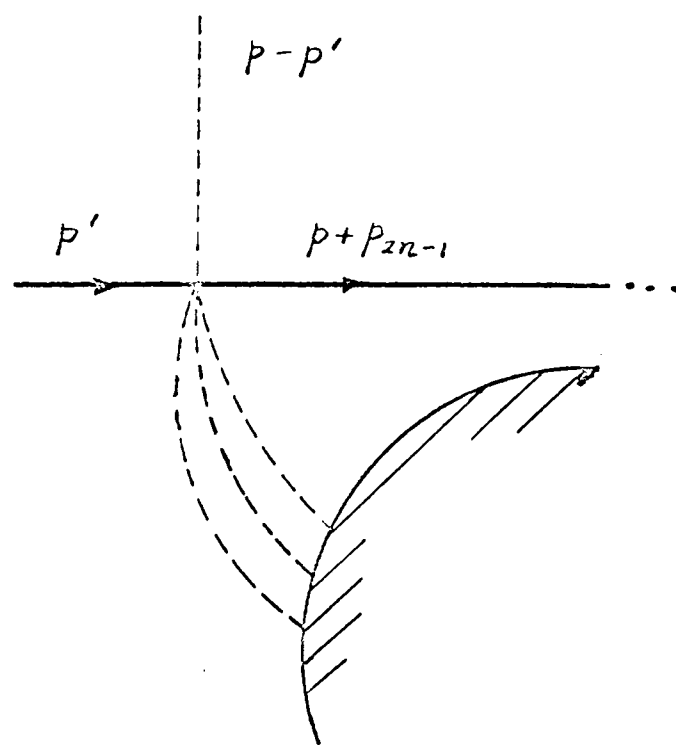


Fig. 7

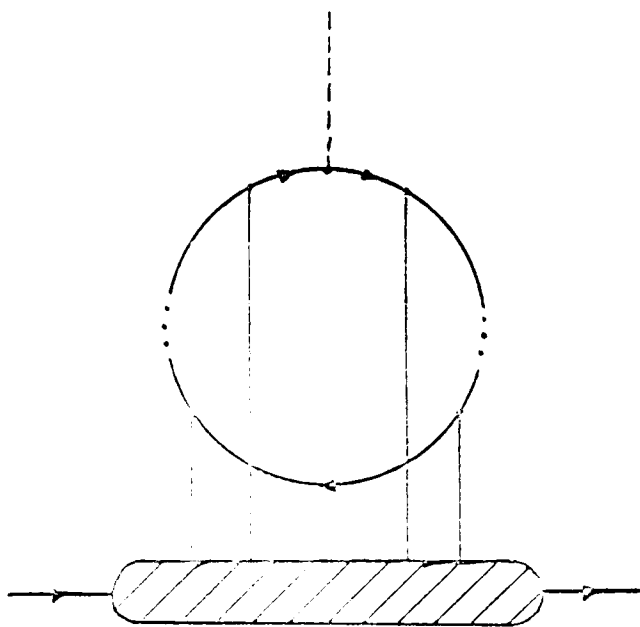


Fig. 8

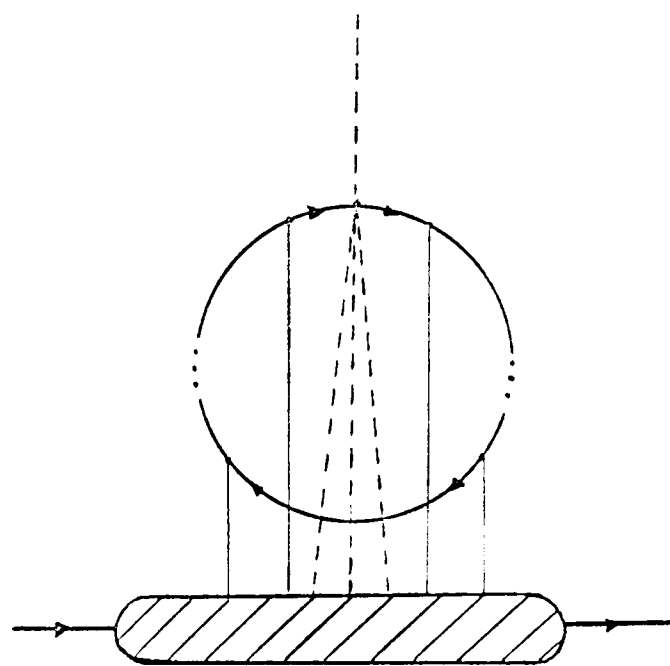


Fig. 9

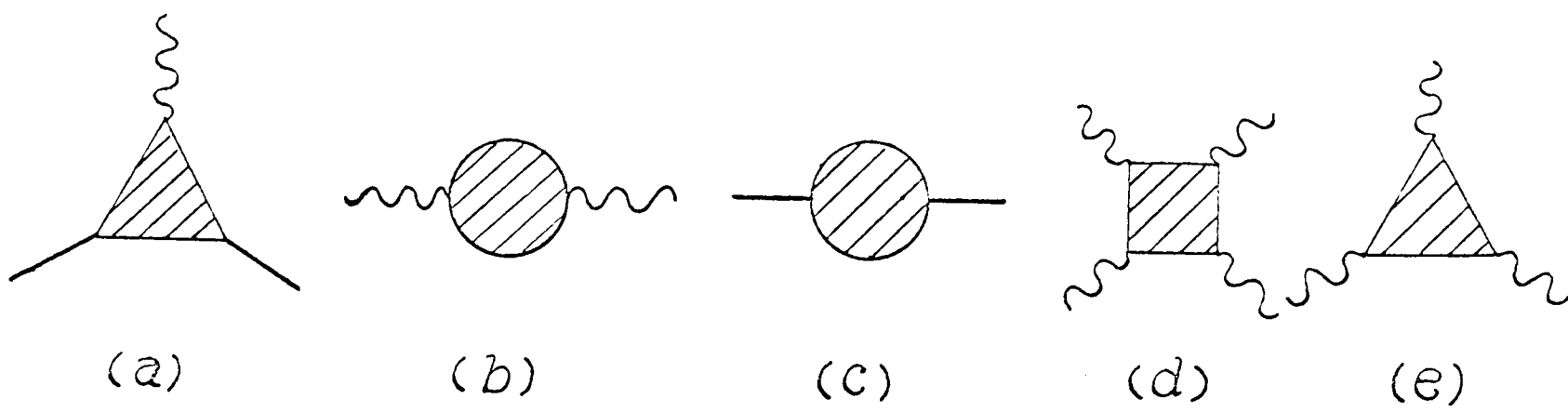


Fig.10

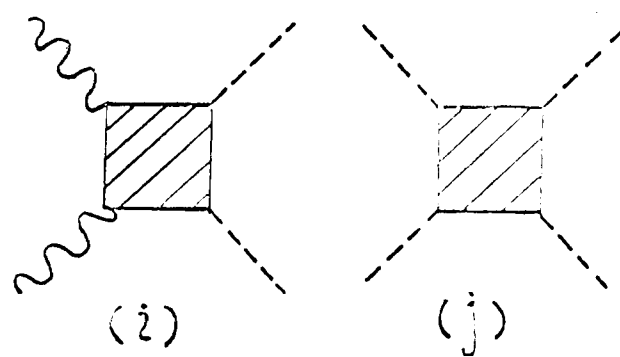
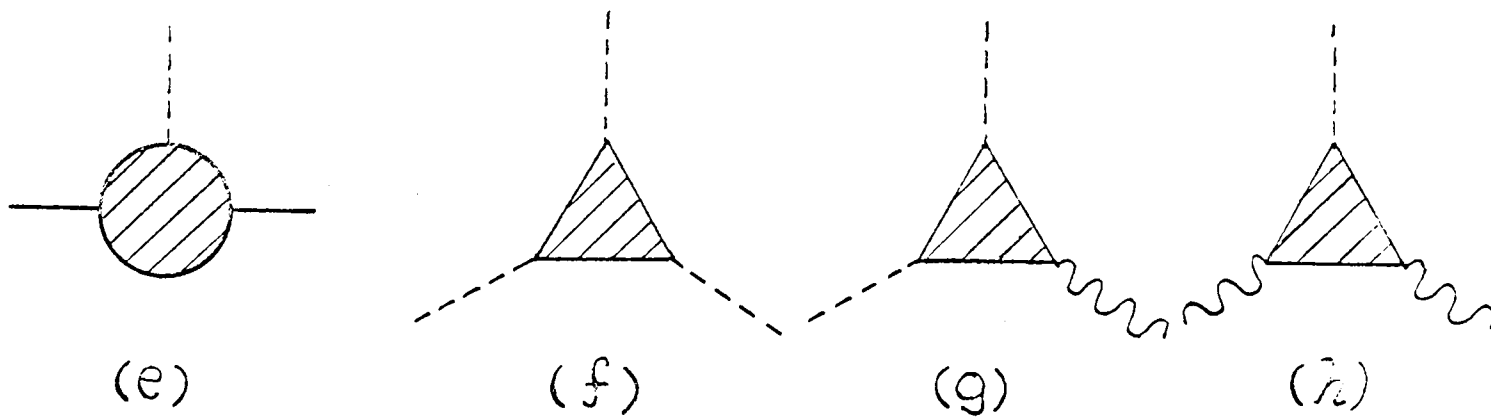
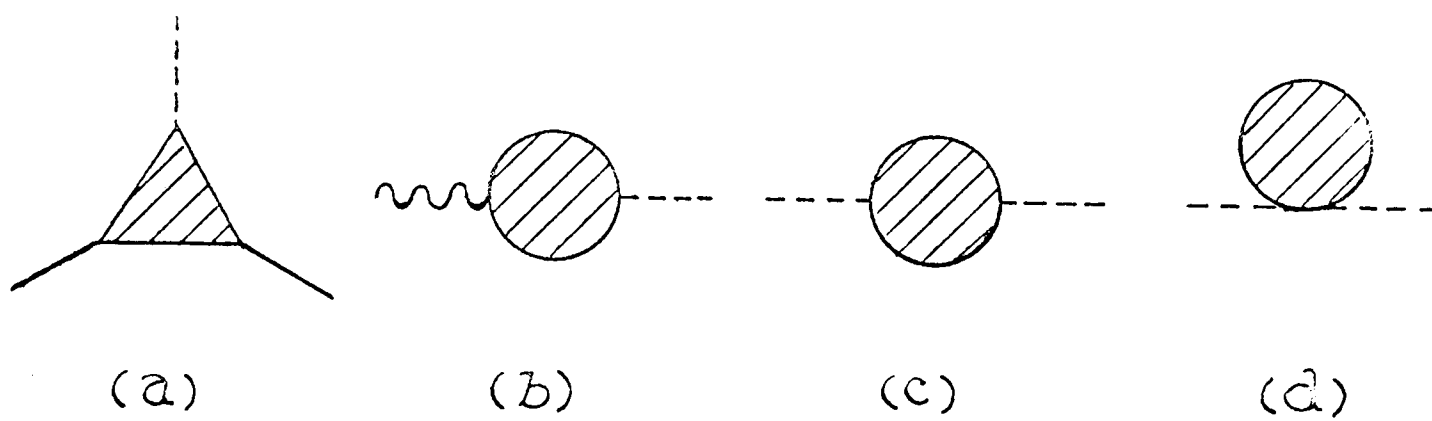


Fig.11

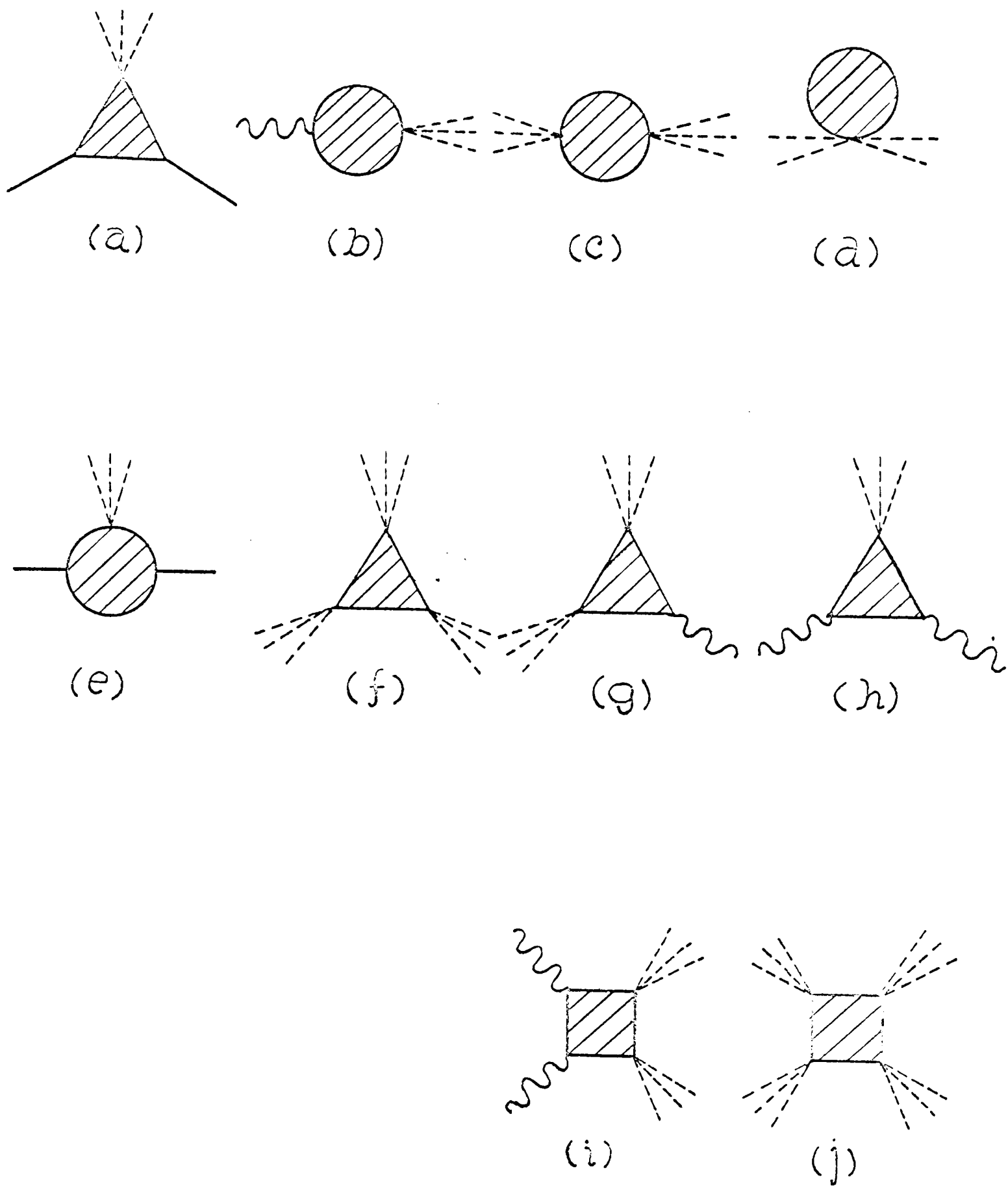


Fig.12

$$\begin{aligned}
& k_\mu \left[\text{triangle diagram with wavy line } \mu \right] - iM \left[\text{triangle diagram with dashed line} + \text{circle diagram with dashed line} \right] \\
&= (-9) \left[\gamma_5 - \text{circle diagram} + \text{circle diagram} - \gamma_5 \right]
\end{aligned}$$

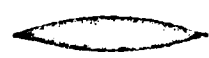
Fig. 13

$$\begin{aligned}
& k_\mu \left[\text{circle diagram with wavy lines } \mu, \nu \right] - iM \left[\text{circle diagram with wavy line } \nu \text{ and dashed line} \right] = 0 \\
& k_\mu \left[\text{circle diagram with wavy line } \mu \text{ and dashed line} \right] - iM \left[\text{circle diagram with dashed lines} + \text{circle diagram with dashed line} \right] = 0
\end{aligned}$$

Fig. 14

$$\begin{aligned}
& \left\{ k_\mu k_\nu k_\rho k_\sigma \left[\begin{array}{c} \mu \text{ wavy} \quad \nu \text{ wavy} \\ \text{shaded square} \\ \sigma \text{ wavy} \quad \rho \text{ wavy} \end{array} \right] - iM k_\mu k_\nu k_\rho \left[\begin{array}{c} \mu \text{ wavy} \quad \nu \text{ wavy} \\ \text{shaded square} \\ \rho \text{ wavy} \quad \text{dashed line} \end{array} \right] \right. \\
& - M^2 k_\mu k_\nu \left[\begin{array}{c} \mu \text{ wavy} \quad \text{dashed line} \\ \text{shaded square} \\ \nu \text{ wavy} \quad \text{dashed line} \end{array} \quad \begin{array}{c} \text{dashed line} \quad \text{dashed line} \\ \text{shaded triangle} \\ \mu \text{ wavy} \quad \nu \text{ wavy} \end{array} \right] \\
& + iM^3 k_\mu \left[\begin{array}{c} \mu \text{ wavy} \quad \text{dashed line} \\ \text{shaded square} \\ \text{dashed line} \quad \text{dashed line} \end{array} \quad \begin{array}{c} \text{dashed line} \quad \text{dashed line} \\ \text{shaded triangle} \\ \mu \text{ wavy} \quad \text{dashed line} \end{array} \quad \begin{array}{c} \mu \text{ wavy} \\ \text{shaded circle} \\ \text{dashed lines} \end{array} \right] \\
& + M^4 \left[\begin{array}{c} \text{dashed line} \quad \text{dashed line} \\ \text{shaded square} \\ \text{dashed line} \quad \text{dashed line} \end{array} \quad \begin{array}{c} \text{dashed line} \quad \text{dashed line} \\ \text{shaded triangle} \\ \text{dashed line} \quad \text{dashed line} \end{array} \quad \begin{array}{c} \text{dashed line} \\ \text{shaded circle} \\ \text{dashed lines} \end{array} \right] \\
& \left. + \left[\begin{array}{c} \text{dashed lines} \\ \text{shaded circle} \\ \text{dashed lines} \end{array} \quad \begin{array}{c} \text{dashed lines} \\ \text{shaded circle} \\ \text{dashed lines} \end{array} \quad \begin{array}{c} \text{dashed line} \\ \text{shaded circle} \\ \text{dashed lines} \end{array} \right] \right\} = 0
\end{aligned}$$

Fig.15

 = superpropagator

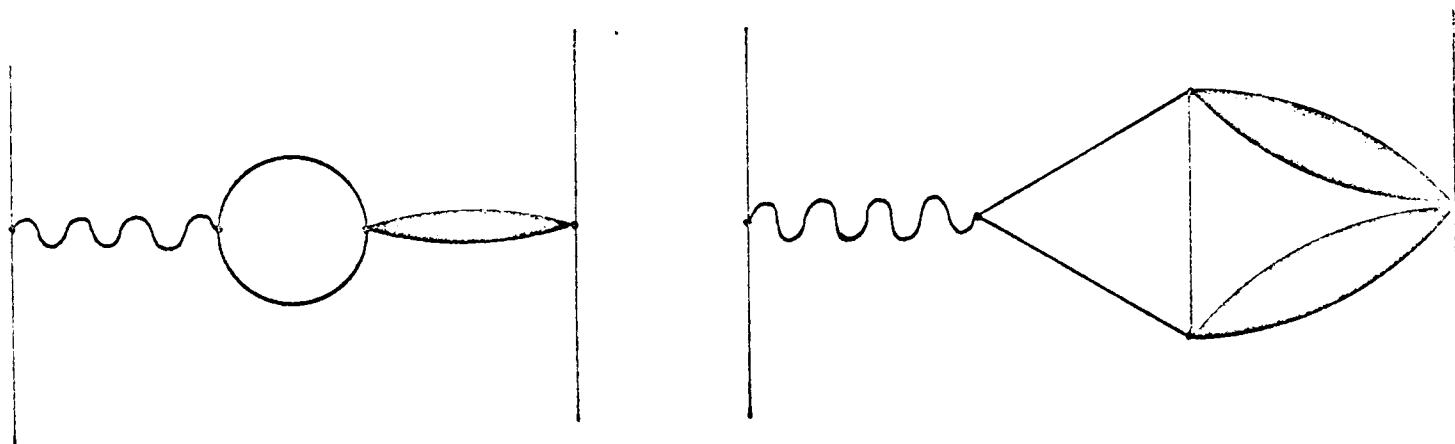


Fig.16

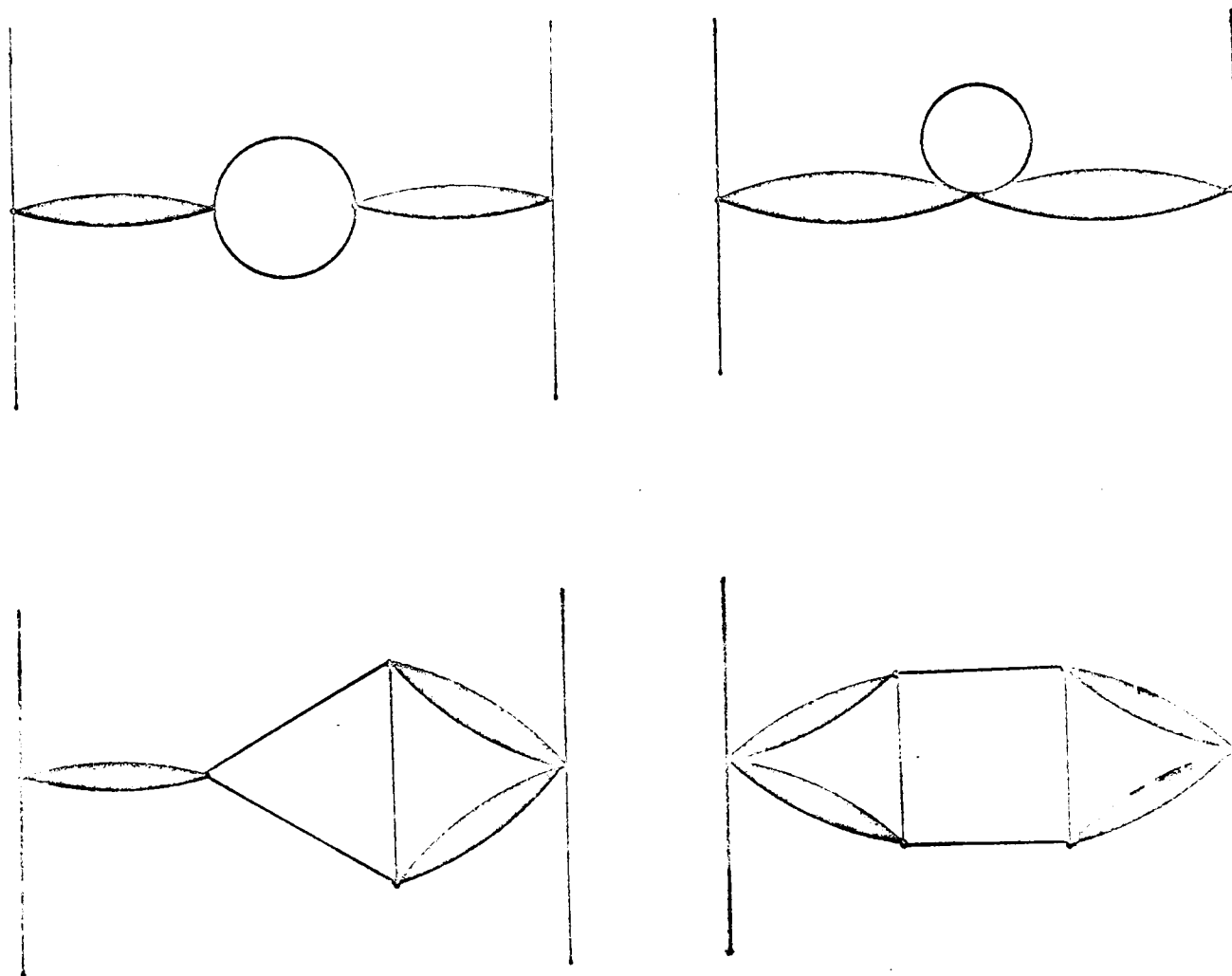


Fig.17

APPENDIX I

The purpose of this appendix is twofold. Firstly, we give the necessary terminology associated with Feynman graphs, and secondly we sketch the proof of Theorem 1 of Section II (PART I) which justifies our regularisation procedure for graphs with 'superpropagators'.

(A) In this part of the appendix we shall give the basic concepts of graph theory associated with Feynman graphs^{8,9}.

Definition 1: A Feynman graph, $G(M, \mathcal{L})$, is a set of vertices $M = \{V_1, \dots, V_n\}$, and a set of lines, $\mathcal{L}$, together with a mapping $i; \mathcal{L} \rightarrow M \times M$. This mapping assigns to each line $l \in \mathcal{L}$ an initial V_{l_i} , and a final V_{l_f} vertex; collectively V_{l_i} and V_{l_f} are called the end points of l . A multiplet with respect to any two vertices, V_i and V_j , of the graph G is the set of lines in G which have V_i and V_j as their end points. Finally, a generalised vertex of G is any subset $U = \{V'_1 \dots V'_m\} \subseteq M$.

Definition 2: An r-tree, in a connected graph $G(M, \mathcal{L})$, is a subgraph $H(M, T_r)$, where $T_r \subset \mathcal{L}$, such that $N(H(M, T_r)) = 0$, and $C(H(M, T_r)) = r$, where N is the number of independent closed loops, and C the number of components of $H(M, T_r)$ respectively. Thus, a tree, T , is a 1-tree which contains $n(G)-1$ lines ($n = \# \{M\}$), and each vertex of G is in T .

It is also clear that a 2-tree, T_2 , is a tree minus one line.

Definition 3: A graph $G(M, \mathcal{L})$ is one particle irreducible (IPI), if it is connected, and any subgraph obtained from G by removing a single line is also connected.

The superficial degree of divergence of Feynman graph $G(M, \mathcal{L})$ is given by

$$\omega(G) = 2L(G) + \sum_{l \in \mathcal{L}} r_l - 4[n(G) - 1]$$

where $L(G) = |\mathcal{L}|$, $n(G) = |M|$, and r_l is the degree of the polynomial

(in momentum) of the numerator of the Feynman propagator corresponding to the line $\ell \in \mathcal{L}$. $\omega(G)$ is a measure of the degree of divergence of the convolution integrals (in momentum space) of the Feynman amplitude associated with G .

(B) We, now, turn to Theorem 1 of Section II (PART I) (i.e. Theorem 2.3.1 of Ref. (9)):

In co-ordinate space the Feynman amplitude corresponding to a given graph $G(M, \mathcal{L})$ is given by the following product of causal propagators

$$F_G(\underline{x}) = \prod_{\ell \in \mathcal{L}} \Delta^{(\ell)}(x_{\ell_f} - x_{\ell_i})$$

(For definiteness we shall take $r_\ell = 0 \quad \forall \ell \in \mathcal{L}$)

$F_G(\underline{x})$ is not well defined being a product of distributions which are very singular on surfaces where two or more x_j coincide. This difficulty shows up also in the F.T. of $F_G(\underline{x})$. Indeed the product of distributions becomes a convolution, with convolution integrals which diverge for large momenta. This being the case one may defined a generalised Feynman amplitude (GFA)^{8,9} given by

$$F_G(\underline{\lambda})(\underline{x}) = \prod_{\ell \in \mathcal{L}} \Delta^{(\ell)}(\lambda_\ell)(x_{\ell_f} - x_{\ell_i})$$

where $\Delta^{(\ell)}(\lambda_\ell)(x_{\ell_f} - x_{\ell_i})$ is given by the inverse F.T. of

$$\tilde{\Delta}^{(\ell)}(\lambda_\ell)(p) = \begin{cases} \frac{e^{i\pi/2(\lambda_\ell+1)}}{(2\pi)^4} [p^2 - m^2 + i0]^{-\lambda_\ell-1} & \text{(massive line)} \\ \frac{e^{i\pi/2(\lambda_\ell+1)}}{(2\pi)^4} [p^2 + i0]^{-\lambda_\ell-1} & \text{(massless line)} \end{cases}$$

with λ_ℓ being a complex number. We observe that $F_G(\underline{\lambda})(\underline{x})$ formally reproduces $F_G(\underline{x})$ for $\lambda_1 = \lambda_2 = \dots = \lambda_L = 0$, and the original divergence difficulties appear now as a singularity of the GFA at this point.

Now, in momentum space the GFA associated with the graph $G(M, \mathcal{L})$ having L lines, n vertices, and $K=L-n+1$ independent loops is given by

$$\tilde{F}_G(\lambda)(p) = f_G(\lambda) \int_{\gamma_0} \prod_{\ell \in \mathcal{L}} d\beta_\ell \beta_\ell^{\lambda_\ell} \frac{[d_G(\beta)]^{\lambda_0}}{[D_G(\beta, p)]^\mu} \quad (A.1.1)$$

where

$$\mu = L - 2N + \sum_{\ell \in \mathcal{L}} \lambda_\ell \quad ; \quad \lambda_0 = \mu - 2$$

and $\beta_\ell, \forall \ell \in \mathcal{L}$ are homogeneous co-ordinates in the $(L-1)$ -Dimensional projective space ($\alpha_\ell = t \beta_\ell, \forall \ell \in \mathcal{L}$ are the usual Feynman parameters).

The integration contour is the simplex $\gamma_0 = \{ \beta \mid \beta_\ell \geq 0 \forall \ell \}$ whereas

$$f_G(\lambda) = \frac{\Gamma(\mu)}{\prod_{\ell \in \mathcal{L}} \{ \exp(-i\pi\sigma_\ell) \Gamma(\sigma_\ell) \}}, \quad \sigma_\ell = -\lambda_\ell + 1, \lambda_\ell \in \mathbb{C}$$

The functions $d_G(\beta)$, and $D_G(\beta, p)$ are given explicitly below:

$$(i) \quad d_G(\beta) = \sum_{T^G} \left(\prod_{\ell \notin T^G} \beta_\ell \right) \quad (A.1.2)$$

where the summation is over all the trees of the graph G ;

$$(ii) \quad D_G(\beta, p) = \sum_{r,s} p_r A_G(i^r_s) p_s - d_G(\beta) \left[\sum_{\ell \in \mathcal{L}^M} \beta_\ell m_\ell^2 - i\varepsilon \right] \quad (A.1.3)$$

with $\mathcal{L}^M$ being the set of massive lines of G , and

$$A(i^r_s) = \sum'_{T_2^G} \left(\prod_{\ell \notin T_2^G} \beta_\ell \right) \quad (A.1.4)$$

where the summation is over all the 2-trees of G such that the vertices $\{V_i\}$, and $\{V_r, V_s\}$ lie on different components of T_2^G .

Consider a massless multiplet $\mathcal{X}$ such that $|\mathcal{X}| = 2$. Then, from Speer, and Westwater, we know that there exists a region $\Lambda_\pi \subset \mathbb{C}^L$ such that for $\underline{\lambda} \in \Lambda_\pi$ (A.1.1) is absolutely convergent when integrated over the region

$$0 \leq \beta_b, \beta_c, \quad b, c \in \mathcal{X}$$

$$0 \leq \beta_{\pi(1)} \leq \dots \leq \beta_{\pi(L-2)}$$

where $\pi \in \mathcal{P}(L-2)$, the group of permutations of $(L-2)$ objects. Then,

$$\begin{aligned} \tilde{F}_G^\pi(\underline{\lambda})(\underline{p}) &= f_G(\underline{\lambda}) \int_0^\infty d\beta_b d\beta_c \int_0^1 d\beta_{\pi(L-3)} \dots \int_0^{\beta_{\pi(2)}} d\beta_{\pi(1)} \prod_{l \in \mathcal{L}} \beta_l^{\lambda_l} \\ &\quad \frac{[d_G(\underline{\beta})]^{\lambda_0}}{[D_G(\underline{\beta}, \underline{p})]^r} \end{aligned} \quad (\text{A.1.5})$$

and

$$\tilde{F}_G(\underline{\lambda})(\underline{p}) = \sum_{\pi \in \mathcal{P}(L-2)} \tilde{F}_G^\pi(\underline{\lambda})(\underline{p}) \quad (\text{A.1.6})$$

Let $G'(M, \mathcal{L}')$ be the Feynman graph obtained from G by replacing the lines of $\mathcal{X}$ by a single massless line, ℓ_a . Then

$$d_G(\underline{\beta}) = d_{G'}(\underline{\beta}') (\beta_b + \beta_c) \quad (\text{A.1.7})$$

$$D_G(\underline{\beta}, \underline{p}) = D_{G'}(\underline{\beta}', \underline{p}) (\beta_b + \beta_c)$$

These functional relations are simply stated but not proved in Ref. (9).

For completeness we shall give the proof here.

Proof: There are no trees in G which contain both ℓ_b , and ℓ_c , $b, c \in \mathcal{X}$.

Hence, all trees in G can be divided into three classes: (a) trees that contain neither ℓ_b , nor ℓ_c , (b) trees that contain ℓ_b , and not ℓ_c , and

finally (c) trees that contain l_c , and not l_b . Similarly all the trees in G' can be divided into two classes: (a) trees that do not contain l_a , and (b) trees that contain l_a . Then from (A.1.2) we obtain

$$\begin{aligned}
 d_G(\beta) &= \sum_{r=a,b,c} \sum_{T_{(r)}^G} \prod_{l \notin T_{(r)}^G} \beta_l \\
 &= (\beta_b + \beta_c) \sum_{T_{(a)}^G} \frac{\beta_b \beta_c}{\beta_b + \beta_c} \prod_{\substack{l \notin T_{(a)}^{G'} \\ l \neq l_a}} \beta'_l + (\beta_b + \beta_c) \sum_{T_{(b)}^G} \prod_{l \notin T_{(b)}^{G'}} \beta'_l \\
 &= (\beta_b + \beta_c) d_{G'}(\beta') \Big|_{\beta_a = \frac{\beta_b \beta_c}{\beta_b + \beta_c}}
 \end{aligned}$$

This is the first result. In order to prove the second result, it is necessary, and sufficient, from (A.1.3), to show that

$$A_G \begin{pmatrix} i & r \\ i & s \end{pmatrix} = (\beta_b + \beta_c) A_{G'} \begin{pmatrix} i & r \\ i & s \end{pmatrix}$$

To this end we observe that there are no 2-trees in G that contain both l_b , and l_c . Moreover the set of 2-trees in summation $\sum'_{T_2^G}$ in (A.1.4) can be divided again into the following three classes: (a) 2-trees that contain neither l_b , nor l_c , (b) 2-trees that contain l_b , and not l_c , and (c) 2-trees that contain l_c , and not l_b . Thus, in general we obtain

$$\sum'_{T_2^G} \prod_{l \notin T_2^G} \beta_l = \sum_{r=a,b,c} \sum'_{T_{2,(r)}^G} \prod_{l \notin T_{2,(r)}^G} \beta_l$$

However, any one of the summation $\sum'_{T_{2,(r)}^G}$ $r=a,b,c$ may be over an empty set, since in the subset of the 2-trees, $\sum'_{T_{2,(r)}^G}$ (with some abuse of notation), there may be no 2-trees from any one of the three possible classes. We have, therefore, to examine individual cases:

(i) If all the three classes are present, then analysis similar to the case for the $d_G(\frac{\beta}{\gamma})$ function shows that

$$A_G\left(\begin{smallmatrix} i & r \\ j & s \end{smallmatrix}\right) = (\beta_b + \beta_c) A_{G'}\left(\begin{smallmatrix} i & r \\ j & s \end{smallmatrix}\right)$$

(ii) If there are only two classes present, then these classes must necessarily be (b), and (c). Hence

$$\sum'_{T_2^G} \prod_{\ell \notin T_2^G} \beta_\ell = (\beta_b + \beta_c) \sum'_{T_{2,(b)}^{G'}} \prod_{\ell \notin T_{2,(b)}^{G'}} \beta_\ell = (\beta_b + \beta_c) A_{G'}\left(\begin{smallmatrix} i & r \\ j & s \end{smallmatrix}\right)$$

(iii) The last possibility is to have only 2-trees from class (a) present. Then

$$\begin{aligned} \sum'_{T_2^G} \prod_{\ell \notin T_2^G} \beta_\ell &= \beta_b \beta_c \sum'_{T_{2,(a)}^G} \prod_{\substack{\ell \notin T_{2,(a)}^G \\ \ell \notin X}} \beta_\ell \\ &= (\beta_b + \beta_c) \sum'_{T_{2,(a)}^G} \frac{\beta_b \beta_c}{\beta_b + \beta_c} \prod_{\substack{\ell \notin T_{2,(a)}^G \\ \ell \notin X}} \beta_\ell \\ &= (\beta_b + \beta_c) A_{G'}\left(\begin{smallmatrix} i & r \\ j & s \end{smallmatrix}\right) \Big|_{\beta_a = \frac{\beta_b \beta_c}{\beta_b + \beta_c}} \end{aligned}$$

This completes the proof.

Now, due to factorisation (A.1.7) equation (A.1.5) becomes

$$\tilde{F}_G^\pi(\underline{\lambda})(\underline{p}) = f_G(\underline{\lambda}) \int_0^\infty d\beta_b d\beta_c \beta_b^{\lambda_b} \beta_c^{\lambda_c} (\beta_b + \beta_c)^{\lambda_0 - \mu}.$$

$$\cdot \int_0^1 d\beta_{\pi(L-3)} \cdots \int_0^{\beta_{\pi(2)}} d\beta_{\pi(1)} \prod_{\ell \in \mathcal{L}-\mathcal{X}} \beta_\ell^{\lambda_\ell} \frac{[d_{G'}(\underline{\beta})]^{\lambda_0}}{[D_{G'}(\underline{\beta}, \underline{p})]^\mu}$$

Introducing new variables $\beta_a = \frac{\beta_b \beta_c}{\beta_b + \beta_c}$, $\tau = \frac{\beta_b}{\beta_b + \beta_c}$

(with Jacobian $J = \beta_a \tau^{-2} (1-\tau)^{-2}$) and carrying out the integration over the τ -variable we obtain

$$\tilde{F}_G^\pi(\underline{\lambda})(\underline{p}) = f'_G(\underline{\lambda}) \int_0^\infty d\beta_a \beta_a^{\lambda_b + \lambda_c - 1} \int_0^1 d\beta_{\pi(L-3)} \cdots \int_0^{\beta_{\pi(2)}} d\beta_{\pi(1)} \cdot \prod_{\ell \in \mathcal{L}-\mathcal{X}} \beta_\ell^{\lambda_\ell} \frac{[d_{G'}(\underline{\beta}')]^{\lambda_0}}{[D_{G'}(\underline{\beta}', \underline{p})]^\mu}$$

with

$$f'_G(\underline{\lambda}) = \frac{\Gamma(\mu)}{\left\{ \prod_{\ell \in \mathcal{L}-\mathcal{X}} \exp(-i\pi\sigma_\ell) \Gamma(\sigma_\ell) \right\} \exp(-i\pi(\sigma_b + \sigma_c)) \Gamma(\sigma_b + \sigma_c)}$$

Thus, if we associate with λ_a the regulating parameter

$$\lambda_a = \sum_{\ell \in \mathcal{X}} \lambda_\ell - (|\mathcal{X}| - 1) = \lambda_b + \lambda_c - 1$$

and with the rest of the lines in G' we associate the regulating parameters

$$\lambda'_\ell = \lambda_\ell \quad , \quad \forall \ell \in \mathcal{L} - \mathcal{X}$$

then we obtain

$$\tilde{F}_G^\pi(\underline{\lambda}) = \tilde{F}_{G'}^\pi(\underline{\lambda}')$$

This result together with (A.1.6) imply that

$$\tilde{F}_G(\underline{\lambda}) = \tilde{F}_{G'}(\underline{\lambda}')$$

By repeated application the same is true for a multiplet $\mathcal{X}$ with any multiplicity.

Hence, the theorem.

Now, using the method for the analytic continuation of the GFA, $\tilde{F}(\underline{\lambda})(p)$ to a neighbourhood of the physical point $\underline{\lambda} = \underline{0}$ of Speer⁸, it is clear, from the above theorem, that a massless multiplet will always give rise to a $\underline{\lambda}$ -space singularity in the sum of the regulating parameters associated with the lines of the multiplet. If we set

$$\lambda_1 = \lambda_2 = \dots = \lambda_\nu = \lambda_\ell$$

then the new regulating parameter for the l th multiplet is given by

$$\lambda_a = \nu \lambda_\ell - \nu + 1$$

In terms of the regulating parameter λ_ℓ the contribution from the l th multiplet, shown in fig. (A.1.1),

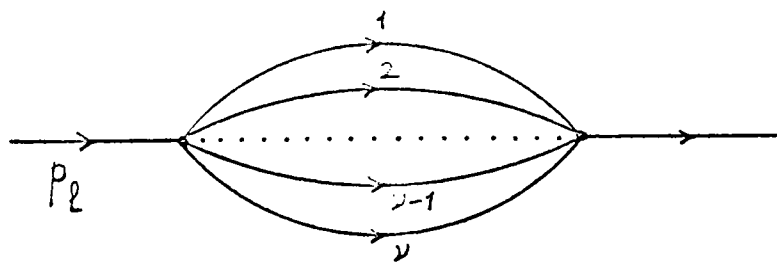


Fig. (A.1.1)

becomes

$$[p_\ell^2 + i0]^{-\lambda_a - 1} \longrightarrow [p_\ell^2 + i0]^{\nu(1-\lambda_\ell) - 2}$$

A direct comparison with formula (3.8) (section II) shows the validity of our regularisation procedure.

APPENDIX II

In this appendix we verify some assertions of Subsection III.2.4.

I. To begin with take any $\eta > 0$, and consider the representation (3.24) for A_{η}^{ν} . Suppressing the dependence on λ and t_2 , we have:

$$A_{\eta}^{\nu} = \int_C dt_1 G^{\nu}(t_1) [P_1(\underline{p}, t_1) + i\eta P_2(\underline{p})]^{-(\mu_3+4)}$$

where the $\delta(\Sigma p)$ restriction is understood. $G^{\nu}(\underline{t})$ is analytic on C with growth (in ν) bounded by $c\|\nu\|_3$, P_2 is positive definite, and it is permissible to write:

$$(A_{\eta}^{\nu}, \tilde{\phi}) = \int_C dt_1 G^{\nu}(t_1) ([P_1(t_1) + i\eta P_2]^{-(\mu_3+4)}, \tilde{\phi})$$

with $\tilde{\phi} \in \mathcal{M}_g(R^{4 \times 2})$. We have

$$\begin{aligned} |(A_{\eta}^{\nu}, \tilde{\phi})| &\leq L_{\eta} \sup_{t_1 \in C} |G^{\nu}(t_1)| \sup_{t_1 \in C} |([P_1(t_1) + i\eta P_2]^{-(\mu_3+4)}, \tilde{\phi})| \\ &\leq L_{\eta} C_1^{\|\nu\|_3} \int_{R^{4 \times 2}} d\underline{p} |\tilde{\phi}(\underline{p})| \sup_{t_1 \in C} |[P_1(\underline{p}, t_1) + i\eta P_2(\underline{p})]^{a\|\nu\|} | \end{aligned}$$

with $0 < \operatorname{Re} a < 1$, and for large $\|\nu\|_3$ can ignore -4 , which can always be compensated by integration by parts (III.2.3). Thus,

$$|(A_{\eta}^{\nu}, \tilde{\phi})| \leq L_{\eta} C_2^{\|\nu\|_3} \|\tilde{\phi}\|_g \int_{R^{4 \times 2}} d\underline{p} g(\|\underline{p}\|^2)^{-1} \left(\sup_{t_1 \in C} |P_1(\underline{p}, t_1)| + \eta P_2(\underline{p}) \right)^{\|\nu\|}$$

where $g(\|\underline{p}\|^2)$ is the indicator function¹, and $\|\tilde{\phi}\|_g$ is a norm in $\mathcal{M}_g(R^{4 \times 2})$ (See III.3.1). Introducing hyperspherical coordinates in $R^{4 \times 2}$ we get

$$|(A_{\eta}^{\nu}, \tilde{\phi})| \leq L_{\eta} \|\tilde{\phi}\|_g \tilde{\psi}_{\eta}^{\nu} \int_0^{\infty} dR R^7 [g(R^2)]^{-1} (R^2)^{\|\nu\|_3}$$

$$\text{with } \tilde{\psi}_{\eta}^{\nu} = C_2^{\|\nu\|_3} \int_{\text{compact region}} d\Omega [\hat{P}_1(\Omega) + \hat{P}_2(\Omega)]^{\|\nu\|_3}$$

whence, utilising the decrease property of $[g(R^2)]^{-1}$, we get

$$|(A_{\eta}^{\nu}, \tilde{\phi})| \leq L_{\eta} \tilde{\psi}_{\eta}^{\nu} \|\nu\|^{\left(\frac{1}{p}\|\nu\|_3\right)} \|\tilde{\phi}\|_g \quad (\text{A.2.1})$$

where p is the order of growth of $g(R^2)$. Obviously $\tilde{\psi}_{\eta}^{\nu} \sim (\text{const})^{\|\nu\|_3}$ for large $\|\nu\|_3$.

II. We now show that the bound continues to hold in the limit $\eta \rightarrow 0$.

As in Subsection III.2.3 we return to the equivalent representation (3.21).

Then:

$$A_{\eta}^{\nu} = \left[\sum_{k_i=0}^{\nu_1} \left\{ \frac{(-1)^{k_1} \left(\frac{\partial}{\partial t_1}\right)^{k_1} \Big|_{t_1=1}}{\prod_{m_1=0}^{k_1} (\mu_1+2+m_1)} \right\} + \frac{(-1)^{\nu_1}}{\prod_{m_1=0}^{\nu_1} (\mu_1+2+m_1)} \int_0^1 dt_1 t_1^{\mu_1+2+\nu_1} \left(\frac{\partial}{\partial t_1}\right)^{\nu_1+1} \right] \\ \cdot \left\{ E(\underline{t})^{\mu_3+2} [P_1(\underline{p}, \underline{t}) + i\eta P_2(\underline{p})]^{-(\mu_3+4)} \right\}$$

which consists of ν_1+1 terms. Because of the action of derivatives each term is a sum of at most ν_1+1 terms of the form (assume large $\|\nu\|_3$)

$$T_{\eta}^{\nu} \sim (\text{const})^{\nu_1} F_{\omega}(p^2, \underline{t}) [P_1(\underline{p}, \underline{t}) + i\eta P_2(\underline{p})]^{a\|\nu\|_3 - \omega - 4}$$

where $P_W(p^2)$ is a polynomial in the $p_i \cdot p_j$ of degree $W \leq \nu_1 + 1$, and $0 < \operatorname{Re} a < 1$. In smearing A_η^ν with a test function $\tilde{\phi} \in \mathcal{S}(\mathbb{R}^{4 \times 2})$, we have to consider (with large $\|\nu\|_3$),

$$([P_1 + i\eta P_2]^{a\|\nu\|_3 - W - 4}, F_W \tilde{\phi}) \sim \left[\frac{\Gamma(a\|\nu\|_3)}{\Gamma(a\|\nu\|_3 - W)} \right]^{-2} ([P_1 + i\eta P_2]^{a\|\nu\|_3}, \mathcal{O}^{W+4}(F_W \tilde{\phi}))$$

on utilising (3.14). The right hand side is continuous in η . One readily obtains:

$$\begin{aligned} |(T_\eta^\nu, \tilde{\phi})| &\leq C \|\tilde{\phi}\|_g \tilde{\psi}_\eta^\nu \|\nu\|_3^{\frac{1}{p}(\|\nu\|_3 + pW)} \|\nu\|_3^{-2W} \\ &\leq C \|\tilde{\phi}\|_g \tilde{\psi}_\eta^\nu \|\nu\|_3^{\frac{1}{p}\|\nu\|_3} \end{aligned}$$

for large $\|\nu\|_3$ where $\tilde{\psi}_\eta^\nu$ is continuous in η and so we obtain easily the bound:

$$|(A_\eta^\nu, \tilde{\phi})| \leq C' \|\tilde{\phi}\|_g \tilde{\psi}_{2,\eta}^\nu \|\nu\|_3^{\frac{1}{p}\|\nu\|_3} \quad (\text{A.2.2})$$

for large $\|\nu\|_3$ with $\tilde{\psi}_{2,\eta}^\nu \sim (\text{const}) \|\nu\|_3$ and continuous in η in the neighbourhood of zero. This suffices to establish the uniformity of convergence stated after (3.26).

APPENDIX III

In this appendix we verify the support properties of the singular (in $\underline{\lambda}$) pieces $\Delta_j \tilde{\mathcal{J}}_\eta(\underline{\lambda})$ of (3.40) stated after that equation.

First consider $\Delta_3 \tilde{\mathcal{J}}_\eta(\underline{\lambda})$. We have, by definition

$$\Delta_3 \tilde{\mathcal{J}}_\eta(\underline{\lambda}) = (8\pi)^2 \delta\left(\sum_1^3 p_i\right) \sum_{\nu_2, \nu_3=1}^{\infty} \frac{1}{\Gamma(\nu_2+1) \Gamma(\nu_3+1)} 2\pi i \sum_p \sum_{n_3 \geq \text{Max}\{0, \nu_2+\nu_3-4\}}^{\infty}$$

$$\cdot \text{Res}_{z=z^{(n_3)}} \{ z \text{ integrand of (3.32)} \} \quad (\text{A.3.1})$$

where $z^{(n_3)}$ is given in (3.34). Hence:

$$\Delta_3 \tilde{\mathcal{J}}_{\eta \downarrow 0}(\underline{\lambda}) = (8\pi)^2 \delta\left(\sum_1^3 p_i\right) \sum_{\nu_2, \nu_3=1}^{\infty} \frac{1}{\Gamma(\nu_2+1) \Gamma(\nu_3+1)} \sum_p \prod_{\ell=2}^3 \left\{ \frac{2^{\lambda_{p(\ell)} \nu_\ell} e^{(\lambda_{p(\ell)}-1) \nu_\ell \pi i}}{\Gamma((1-\lambda_{p(\ell)}) \nu_\ell)} \right\}$$

$$\cdot \sum_{n_3 \geq \text{Max}\{0, \nu_2+\nu_3-4\}} \frac{\pi(-1)^{n_3} \cot \pi z^{(n_3)} 2^{2\lambda_{p(1)} z^{(n_3)}} e^{(\lambda_{p(1)}-1) z^{(n_3)} \pi i}}{\Gamma(n_3+1) \Gamma(z^{(n_3)}) \Gamma(z^{(n_3)}+1) [(4\pi)^2]^{z^{(n_3)}} (g^2)^{n_3}}$$

$$\cdot \int_0^1 dt_2 t_2^{-[(\lambda_{p(3)}-1) \nu_3]-1} \{ 2i \sin \pi [(\lambda_{p(1)}-1) \nu_1+1] \}^{-1} \int_1^{0+} dt_1 (-t_1)^{(\lambda_{p(1)}-1) z^{(n_3)}}$$

$$\cdot [E(t_1 t_2)]^{-(n_3+2)} [P_1^p(p, \underline{t})]^{n_3} \quad (\text{A.3.2})$$

Omitting the $\delta(\sum_1^3 p_i)$, (A.3.2) represents an entire function of all the p_j^2 of order $< \frac{1}{2}$ (for the definition of order of an entire function

of several variables see A. Jaffe, Ann. of Physics 32, 127 (1965)).

Hence $\mathcal{F}^{-1} [\Delta_3 \tilde{\mathcal{J}}_{\eta \downarrow 0}(\underline{\lambda})](\underline{x})$ is a distribution in $\mathcal{C}'_g(\mathbb{R}^{4 \times 3})$ concentrated

on $x_1 = x_2 = x_3$.

Next we consider $\Delta_1 \tilde{\mathcal{J}}_\eta(\underline{\lambda})$. According to its definition:

$$\Delta_1 \tilde{\mathcal{J}}_\eta(\underline{\lambda}) = (8\pi)^2 \delta\left(\sum_1^3 p_i\right) \sum_{\nu_2, \nu_3=1}^{\infty} \frac{1}{\Gamma(\nu_2+1)\Gamma(\nu_3+1)} 2\pi i \sum_p \sum_{n_1=0}^{\infty} \text{Res.}_{z=z^{(n_1)}} \cdot \{z \text{ integrand of (3.32)}\} \quad (\text{A.3.3})$$

where $z^{(n_1)}$ is defined in (3.34). Hence:

$$\Delta_1 \tilde{\mathcal{J}}_{\eta \downarrow 0}(\underline{\lambda}) = (8\pi)^2 \delta\left(\sum_1^3 p_i\right) \sum_p \sum_{n_1=2}^{\infty} a_{n_1}^p(\lambda_{p(1)}) \mathcal{P}_{n_1}^p(\lambda_{p(2)}, \lambda_{p(3)})(\underline{p}) \quad (\text{A.3.4})$$

where

$$(i) \quad a_{n_1}^p(\lambda_{p(1)}) = \frac{2^{\frac{\lambda_{p(1)} n_1}{1-\lambda_{p(1)}}} (-1)^{n_1}}{[(4\pi)^2]^{\frac{n_1}{1-\lambda_{p(1)}}} \Gamma(n_1) \Gamma\left(\frac{n_1}{1-\lambda_{p(1)}} + 1\right)} \cot \pi \frac{n_1}{1-\lambda_{p(1)}}$$

$$(ii) \quad \mathcal{P}_{n_1}^p(\lambda_{p(2)}, \lambda_{p(3)})(\underline{p}) = \sum_{\nu_2, \nu_3=1}^{\infty} \prod_{\ell=2}^3 \left\{ \frac{2^{\lambda_{p(\ell)} \nu_\ell} e^{(\lambda_{p(\ell)}-1)\nu_\ell \pi i} [g^2]^{(1-\lambda_{p(\ell)})\nu_\ell}}{\Gamma(\nu_\ell+1) \Gamma((1-\lambda_{p(\ell)})\nu_\ell)} \right\} \cdot [g^2]^{n_1} \Gamma\left(4-n_1-\sum_2^3 (1-\lambda_{p(\ell)})\nu_\ell\right) \int_0^1 dt_2 t_2^{-[(\lambda_{p(3)}-1)\nu_3]-1} \frac{1}{2i} \lim_{\eta \rightarrow 0+} \int_1^{0+} dt_1 t_1^{-n_1+1}$$

$$\cdot [E(t_1, t_2)]^{2-n_1-\sum_2^3 (1-\lambda_{p(\ell)})\nu_\ell} [P_1^p(\underline{p}, \underline{t}) + i\eta P_2^p(\underline{p})]^{\sum_2^3 (1-\lambda_{p(\ell)})\nu_\ell + n_1 - 4}$$

Now the t_1 contour integral can be evaluated since we have a pole of order

(n_1-1) , and all other factors are analytic; its evaluation gives:

$$(iii) \quad \frac{2\pi i}{(n_1-2)!} \sum_{r=0}^{n_1-2} \binom{n_1-2}{r} \left\{ \left(\frac{\partial}{\partial t_1} \right)^{n_1-2-r} [E(t_1, t_2)]^{2-n_1-\sum \frac{3}{2}(1-\lambda_{p(e)})\nu_e} \right. \\ \left. \cdot \left\{ \left(\frac{\partial}{\partial t_1} \right)^r [P_1^p(p, t) + i\eta P_2^p(p)] \right\}_{\eta \downarrow 0}^{\left[\frac{3}{2}(1-\lambda_{p(e)})\nu_e + n_1 - 4 \right]} \right\}$$

evaluated at $t_1 = 0$. Recalling that $P_1(p, t) = p^2_{p(3)} + t_1 t_2 p^2_{p(2)} + t_1 p^2_{p(1)}$,

the last factor in braces in (iii) gives a polynomial of degree 'r' in $p^2_{p(2)}, p^2_{p(1)}$. Hence $\mathcal{F}^{-1}(\delta(\sum p_i) \times \text{last } \{ \}_{\eta \downarrow 0})$ is concentrated on $x_{(1)} = x_{(2)}$. Putting (i), (ii) and (iii) in (A.3.4), we have that

$\mathcal{F}^{-1}(\Delta, \mathcal{T}_{\eta \downarrow 0}(\underline{\lambda}))(\underline{x})$ consists of a sum of terms, each one a convergent (in $\mathcal{C}'_g$) series of distributions concentrated on $x_{p(1)} = x_{p(2)}$, etc.

The above analysis generalises to the case of the n-point function corresponding to a graph $G(\{V_1 \dots V_n\}, \mathcal{L})$. If we confine ourselves to the sector $0 \leq \alpha_1 \leq \alpha_2 \dots \leq \alpha_L$; $L = |\mathcal{L}|$, and introduce sector co-ordinates $\alpha_l = t_L \dots t_1$, $l \in \mathcal{L}$, then in every given term of the corresponding expression for (ii) (see above) we pick up a sub-integral of the form

$$\frac{1}{2i} \lim_{\eta \downarrow 0} \int_1^{0+} dt_2 (t_2)^{-n_1-1} [E(t_1, t_2, \dots, t_L)]^{[b_L-2]} \{P_1(p, t') + i\eta P_2(p)\}^{-b_L}$$

where

$$b_L = \mu_L(\Xi^{(n_2)}) + 2(L-N) \quad , \quad \Xi^{(n_2)}_i(\lambda) = \frac{n_2 + 2(\ell - N_2) + \sum_{\ell'=2}^{\ell} (\lambda_{\ell'} - 1)\nu_{\ell'}}{1 - \lambda_1} \quad .$$

with $l = 1, 2, \dots, (l_p - 1)$, where l_p is a positive integer determined by the topology of G , and N_1 is the number of loops in the subgraph $G^l_p \subset G$ consisting of all lines l' with $l' \leq l$, and $n_1 = 0, 1, 2, \dots$. The t_1 integral can be evaluated since we have a pole of order $(n_1 + 1)$ (in general). Hence by the residue theorem we obtain

$$\frac{2\pi i}{n_\ell!} \sum_{r=0}^{n_\ell} \binom{n_\ell}{r} \left\{ \left(\frac{\partial}{\partial t_\ell} \right) [E(\underline{t}')]^{b_L-2} \right\} \left\{ \left(\frac{\partial}{\partial t_\ell} \right) [P_1(\underline{p}, \underline{t}') + i\eta P_2(\underline{p})]_{\eta \rightarrow 0}^{-b_L} \right\}$$

evaluated at $t_\ell = 0$. For a given r the last factor in the above expression

consists of a sum of terms of the form $P_{r;k}[\{p_i \cdot p_j\}, (v_i, v_j) \in G_\ell] Q_\eta^{(k)}[\underline{t}'; \{p_i \cdot p_j\}, (v_i, v_j) \in G-G_\ell] \Big|_{t_\ell=0}$ with $0 \leq k \leq r$, where:

(a) $P_{r;k}$ is a polynomial of degree ' r ' in the $(p_i \cdot p_j)$, with $\{v_i, v_j\} \in G_\ell$.

(b) the function $Q^{(k)}$ depends on the $(p_i \cdot p_j)$ with $\{v_i, v_j\} \subset G-G_\ell$.

We shall now justify (a), and (b). First we note that $P_1(\underline{p}, \underline{t}')$ is defined in terms of the quadratic form (see Appendix I)

$$\sum_{r,s \neq i} p_r \cdot A_E \begin{pmatrix} i & r \\ & s \end{pmatrix} \cdot p_s$$

taking into account any possible t -factorisation. Similarly $P_2(\underline{p}) \equiv P_2(\underline{p}, \underline{t}'=1)$

is defined in terms of the above quadratic form using the Euclidean metric.

Both (a), and (b) follow from the definitions of $P_1(\underline{p}, \underline{t}')$, and $P_2(\underline{p})$.

Now, from the definition $\alpha_\ell = t_L \dots t_\ell$, $\ell \in \mathcal{L}$ when $t_\ell \rightarrow 0$,

$\{\alpha_1, \dots, \alpha_\ell\} \rightarrow 0$, and the corresponding subgraph G_ℓ contracts

to a point. Associated are the polynomials $P_{r;k}$. On the other hand,

in computing $Q^{(k)}$, which depends on the external momenta associated with

the vertices of $G-G_\ell$, the restriction E (in the definition of the quadratic form) (due to $\delta(\sum_i^n p_i)$) can be imposed with respect to any momentum p_i .

Hence, we can choose $p_i = -\sum_{j \neq i} p_j$ with $v_i \in G_\ell$. From this, statement (b) follows.

Now, it follows from (a), and (b) that

$$\mathcal{F}^{-1} \left[\delta(\sum p) \cdot \left\{ \left(\frac{\partial}{\partial t_\ell} \right)^r [P_1 + i\eta P_2]^{-b_L} \right\} \Big|_{t_\ell=0} \right]$$

is concentrated on $x_{i_1} = x_{i_2} = \dots = x_{i_k}$ ($1 < k < n$). From this

it is easy to deduce that $\mathcal{F}^{-1} \left[\Delta_\ell \tilde{\mathcal{J}}_{\eta \downarrow 0}^{(n)}(\lambda) \right] (\underline{x})$ consists of a sum of terms, each one a convergent (in $\mathcal{E}'_g$) series of distributions concentrated on $x_{i_1} = x_{i_2} = \dots = x_{i_k}$ ($1 < k < n$). Finally, we remark that it is even more straightforward to show that

$\mathcal{F}^{-1} \left[\Delta_L \tilde{\mathcal{J}}_{\eta \downarrow 0}^{(n)}(\lambda) \right] (\underline{x})$ is a distribution (in $\mathcal{E}'_g$) with support the point $x_1 = x_2 = \dots = x_n$.

APPENDIX IV

In this appendix²⁸ we prove (3.70) starting from (3.69). Defining $\lim_{\eta \downarrow 0} T_\eta(\underline{p}) \equiv T(\underline{p})$, we write

$$\text{Re } \tilde{T}(\underline{p}) = \tilde{F}(\underline{p}) + \chi(\underline{p}) \quad (\text{A.4.1})$$

where $\tilde{F}(\underline{p})$ represents the term in braces in (3.69). We have

$$\begin{aligned} \tilde{F}(\underline{p}) = & (-\pi) \sum_{\nu_2, \nu_3=1}^{\infty} \prod_{\ell=2}^3 \left\{ \frac{[g^2/(4\pi)^2]^{\nu_\ell}}{\Gamma(\nu_\ell)\Gamma(\nu_\ell+1)} \right\} \frac{1}{2i} \int_L dz \frac{[\frac{g^2}{(4\pi)^2}]^z}{\sin^2 \pi z \Gamma(z)\Gamma(z+1)\Gamma(z+\sum_{\ell=2}^3 \nu_\ell - 3)} \\ & \cdot \int_0^1 dt_2 t_2^{\nu_3-1} \int_0^1 dt_1 t_1^{-z+1} (1+t_2+t_2 t_1)^{[2-z-\sum_{\ell=2}^3 \nu_\ell]} \sum_p [p_{p(3)}^2 + t_1 t_2 p_{p(2)}^2 + t_1 p_{p(1)}^2]^{(z+\sum_{\ell=2}^3 \nu_\ell - 4)} \end{aligned} \quad (\text{A.4.2})$$

with the contour L as given in Fig. (III.4.3). Interchanging the z integration with the t_1, t_2 integrations, which is valid due to uniform convergence, introducing the variable ζ_p defined in (3.61), and recalling that $\sum_p$ is a finite sum, we have:

$$\begin{aligned} \tilde{F}(\underline{p}) = & (-\pi) \left[\frac{g^2}{(4\pi)^2} \right]^2 \sum_p \sum_{\nu_2, \nu_3=1}^{\infty} \prod_{\ell=2}^3 \left\{ \frac{1}{\Gamma(\nu_\ell)\Gamma(\nu_\ell+1)} \right\} \int_0^1 dt_2 t_2^{\nu_3-1} \\ & \cdot \int_0^1 dt_1 (1+t_2+t_1 t_2)^{-z} t_1 (t_1 \zeta_p)^{\nu_2+\nu_3-4} \frac{1}{2i} \int_L dz \frac{(\zeta_p)^z}{\sin^2 \pi z \Gamma(z)\Gamma(z+1)\Gamma(z+\sum_{\ell=2}^3 \nu_\ell - 3)} \end{aligned} \quad (\text{A.4.3})$$

(The dependence of ζ_p on t_1, t_2 and $\{p_j^2\}$ should be kept in mind).

It is also permissible, due to uniform convergence, to interchange the

t_1, t_2 integration with the double series summation, so that we have:

$$\tilde{F}(p) = (-\pi) \left[\frac{g^2}{(4\pi)^2} \right]^2 \sum_p \int_0^1 dt_2 \int_0^1 dt_1 (1+t_2+t_1 t_2)^{-2} t_1 (t_1 \xi_p)^{-4} \tilde{G}(t_1, \xi_p) \quad (A.4.4)$$

where

$$(A.4.5)$$

$$\begin{aligned} \tilde{G}(t_1, \xi_p) &\equiv \sum_{\nu_2, \nu_3=1}^{\infty} \prod_{\ell=2}^3 \left\{ \frac{(t_1 \xi_p)^{\nu_\ell}}{\Gamma(\nu_\ell) \Gamma(\nu_\ell+1)} \right\} t_2^{\nu_3-1} \frac{1}{2i} \int_L dz \frac{(\xi_p)^z}{\sin^2 \pi z \Gamma(z) \Gamma(z+1) \Gamma(z+\frac{3}{2} \nu_\ell-3)} \\ &= \frac{1}{2i} \int_L dz \frac{(\xi_p)^z}{\sin^2 \pi z \Gamma(z) \Gamma(z+1)} \left\{ \sum_{\nu_2, \nu_3=1}^{\infty} \prod_{\ell=2}^3 \left(\frac{(t_1 \xi_p)^{\nu_\ell}}{\Gamma(\nu_\ell) \Gamma(\nu_\ell+1)} \right) t_2^{\nu_3-1} \frac{1}{\Gamma(z+\frac{3}{2} \nu_\ell-3)} \right\} \end{aligned} \quad (A.4.6)$$

the last interchange being valid due to uniform convergence, as follows readily on using Stirling's formula. We now introduce in (A.4.6) the integral representation¹⁵

$$\frac{1}{\Gamma(z+\nu_2+\nu_3-3)} = \frac{1}{2\pi i} \int_{-\infty}^{0+} d\tau e^{\tau} \tau^{-(z+\nu_2+\nu_3-3)} \quad (A.4.7)$$

with $|\arg \tau| \leq \pi$, τ^{-z} being defined as $\exp[-z(\log|\tau| + i \arg \tau)]$.

The contour in (A.4.7) begins and ends at $(-\infty)$ clockwise encircling the origin once (Fig. (A.4.1)). It is convenient for the subsequent development to replace (A.4.7) by

$$\frac{1}{\Gamma(z+\nu_2+\nu_3-3)} = \frac{1}{2\pi i} \int_{\gamma} d\tau e^{\tau} \tau^{-(z+\nu_2+\nu_3-3)} \quad (A.4.8)$$

with the deformed contour γ submitting to $\frac{\pi}{2} < |\arg \tau| < \pi$ as shown in Fig. (A.4.2), the deformation being valid due to strong decrease at infinity.

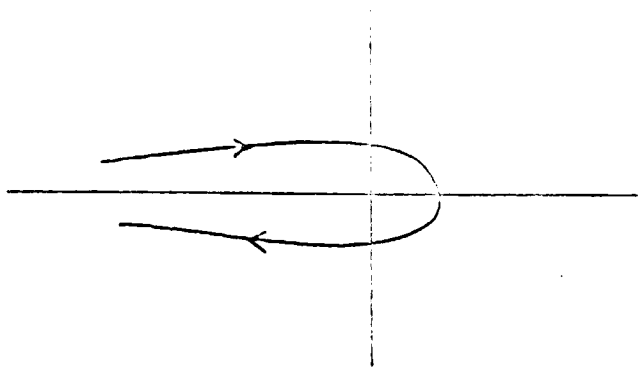


Fig. (A.4.1)

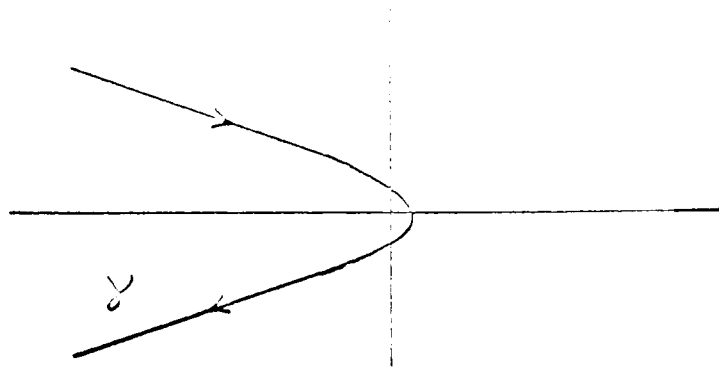


Fig. (A.4.2)

Introducing the representation (A.4.8) in (A.4.6) we interchange, by virtue of uniform convergence, the double series summation with the τ integration , to get:

$$\begin{aligned} \tilde{G}(t, \xi_p) &= \frac{1}{2i} \int_L dz \frac{(\xi_p)^z}{\sin^2 \pi z \Gamma(z) \Gamma(z+1)} \frac{1}{2\pi i} \int_{\gamma} d\tau e^{\tau} \tau^{3-z} \left\{ \sum_{\nu_2=1}^{\infty} \frac{\left(\frac{t_1 \xi_p}{\tau}\right)^{\nu_2}}{\Gamma(\nu_2) \Gamma(\nu_2+1)} \right\} \\ &\quad \cdot \left\{ \frac{1}{t_2} \sum_{\nu_3=1}^{\infty} \frac{\left(\frac{t_1 t_2 \xi_p}{\tau}\right)^{\nu_3}}{\Gamma(\nu_3) \Gamma(\nu_3+1)} \right\} \\ &= t_1^3 t_2^{1/2} \xi_p^3 \frac{1}{2i} \int_L dz \frac{(\xi_p)^z}{\sin^2 \pi z \Gamma(z) \Gamma(z+1)} \frac{1}{2\pi i} \int_{\gamma} d\tau e^{\tau} \tau^{-z} I_1\left(2\left(\frac{t_1 \xi_p}{\tau}\right)^{1/2}\right) I_1\left(2\left(\frac{t_1 t_2 \xi_p}{\tau}\right)^{1/2}\right) \end{aligned} \quad (A.4.9)$$

on making appropriate identifications of the two series in braces with Bessel functions¹⁵.

Finally, on noting the restriction $\frac{\pi}{2} < |\arg \tau| < \pi$ for $\tau \in \gamma$, and that ξ_p is real and positive, the τ and z integrations in (A.4.9)

may be interchanged due to uniform convergence. In fact the convergence of the τ integral is independent of z ($z \in L$) and the convergence of the z integral is uniform in τ because $|\arg \tau| < \pi$, $\tau \in \gamma$. Performing the interchange and recognising²⁹ that

$$-\frac{1}{2\pi i} \int_L dz \Gamma(1-z) \Gamma(-z) \left(\frac{\xi_p}{\tau}\right)^z = 2 \left(\frac{\xi_p}{\tau}\right)^{1/2} K_1 \left(2 \left(\frac{\xi_p}{\tau}\right)^{1/2}\right)$$

where K_1 is the modified Bessel function, we get, on making the change of variable $\tau \rightarrow t, \xi_p \tau$,

$$\begin{aligned} \tilde{G}(t, \xi_p) &= \frac{1}{\pi} (t, \xi_p)^4 t_2^{1/2} \frac{1}{2\pi i} \int_{\gamma} d\tau e^{t_1 \xi_p \tau} I_1 \left(2 \left(\frac{1}{\tau}\right)^{1/2}\right) I_1 \left(2 \left(\frac{t_2}{\tau}\right)^{1/2}\right) \\ &\quad \cdot 2 \left(\frac{1}{t_1 \tau}\right)^{1/2} K_1 \left(2 \left(\frac{1}{t_1 \tau}\right)^{1/2}\right) \end{aligned} \quad (\text{A.4.10})$$

Next, we deform the contour γ of Fig. (A.4.2) to that of Fig. (A.4.3).

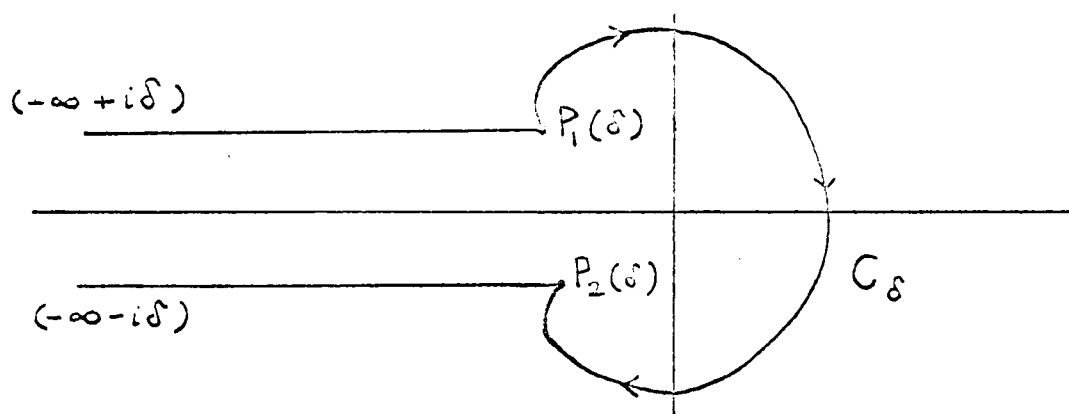


Fig. (A.4.3)

The points $P_1(\delta)$, $P_2(\delta)$ are the intersections of the two straight portions, parallel to the real axis, with the curve (3.62). We choose, for convenience, the portion C_δ of the contour, starting at $P_1(\delta)$ and ending at $P_2(\delta)$, to lie on (3.62). We express (A.4.10) as the sum of two parts:

$$\tilde{G}(t, \xi_p) = \tilde{G}_\delta^{(1)} + \tilde{G}_\delta^{(2)} \quad (\text{A.4.11})$$

where $\tilde{G}_\delta^{(1)}$ receives its contribution from the portions $(-\infty + i\delta, P_1(\delta))$ and $(P_2(\delta), -\infty - i\delta)$ and $G_\delta^{(2)}$ receives its contribution from the portion C_δ of the contour.

We shall now take the limit $\delta \rightarrow 0$. In the limit, $P_{1,2}(\delta) \rightarrow 0$ and we recover $C_\delta \rightarrow C$, the contour of (3.62). Then

$$\lim_{\delta \rightarrow 0} \tilde{G}_\delta^{(2)} = \frac{2}{\pi} (t_1 \xi_p)^4 t_2^{1/2} \frac{1}{2\pi i} \int_C d\tau e^{t_1 \xi_p \tau} I_1\left(2\left(\frac{1}{\tau}\right)^{1/2}\right) I_1\left(2\left(\frac{t_2}{\tau}\right)^{1/2}\right) \cdot \left(\frac{1}{t_1 \tau}\right)^{1/2} K_1\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right) \quad (\text{A.4.12})$$

The existence and contour independence (avoiding distortion in the neighbourhood of the origin) follows from the arguments after (3.61). The limit as $\delta \rightarrow 0$ of $\tilde{G}_\delta^{(1)}$ also exists. To calculate it we need to compute the discontinuity of the integrand of (A.4.10). Now, the product of the two I_1 functions in (A.4.10) is analytic in the punctured (at the origin) τ plane, whereas for the non-analytic part we use the identity³⁰

$$\frac{2 K_1\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right)}{2\left(\frac{1}{t_1 \tau}\right)^{1/2}} = K_0\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right) + K_2\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right)$$

together with¹⁵

$$K_{2n}\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right) \text{ non-analytic piece} = -I_{2n}\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right) \log \tau^{-1/2}$$

for the piece contributing to the discontinuity to get:

$$\text{disc} \left\{ \frac{2 K_1\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right)}{2\left(\frac{1}{t_1 \tau}\right)^{1/2}} \right\} = -2\pi i \frac{2 I_1\left(2\left(\frac{1}{t_1 \tau}\right)^{1/2}\right)}{2\left(\frac{1}{t_1 \tau}\right)^{1/2}}$$

on using the similar identity for the I_n functions. Thus we get:

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \tilde{G}_\delta^{(1)} &= \frac{1}{\pi} (t_1 \xi_p)^4 t_2^{1/2} \int_0^{-\infty} d\tau e^{t_1 \xi_p \tau} I_1 \left(2 \left(\frac{1}{\tau} \right)^{1/2} \right) I_1 \left(2 \left(\frac{t_2}{\tau} \right)^{1/2} \right) \\
&\quad \cdot 2 \left(\frac{1}{t_1 \tau} \right)^{1/2} I_1 \left(2 \left(\frac{1}{t_1 \tau} \right)^{1/2} \right) \\
&= -\frac{2}{\pi} (t_1 \xi_p)^4 t_2^{1/2} \int_0^{\infty} d\tau e^{-t_1 \xi_p \tau} J_1 \left(2 \left(\frac{1}{\tau} \right)^{1/2} \right) J_1 \left(2 \left(\frac{t_2}{\tau} \right)^{1/2} \right) \\
&\quad \cdot \left(\frac{1}{t_1 \tau} \right) J_1 \left(2 \left(\frac{1}{t_1 \tau} \right)^{1/2} \right) \quad (A.4.13)
\end{aligned}$$

which also exists. Hence from (A.4.11 - A.4.13) we get:

$$\begin{aligned}
\tilde{G}(t, \xi_p) &= -\frac{2}{\pi} (t_1 \xi_p)^4 t_2^{1/2} \int_0^{\infty} d\tau e^{-t_1 \xi_p \tau} J_1 \left(2 \left(\frac{1}{\tau} \right)^{1/2} \right) J_1 \left(2 \left(\frac{t_2}{\tau} \right)^{1/2} \right) \\
&\quad \cdot J_1 \left(2 \left(\frac{1}{t_1 \tau} \right)^{1/2} \right) \left(\frac{1}{t_1 \tau} \right)^{1/2} + \frac{2}{\pi} (t_1 \xi_p)^4 t_2^{1/2} \frac{1}{2\pi i} \int_C d\tau e^{t_1 \xi_p \tau} I_1 \left(2 \left(\frac{1}{\tau} \right)^{1/2} \right) \\
&\quad \cdot I_1 \left(2 \left(\frac{t_2}{\tau} \right)^{1/2} \right) \left(\frac{1}{t_1 \tau} \right) K_1 \left(2 \left(\frac{1}{t_1 \tau} \right)^{1/2} \right) \quad (A.4.14)
\end{aligned}$$

Combining (A.4.14) with (A.4.4), and noting the definition (3.60)

of $\tilde{\chi}(\underline{p})$ we get:

$$\begin{aligned}
\tilde{F}(\underline{p}) &= 2 \left[\frac{g^2}{(4\pi)^2} \right]^2 \sum_{\underline{p}} \int_0^1 dt_2 \int_0^1 dt_1 (1+t_2+t_2 t_1)^{-2} (t_1 t_2)^{1/2} \int_0^{\infty} d\tau e^{-t_1 \xi_p \tau} \\
&\quad \cdot J_1 \left(2 \left(\frac{1}{\tau} \right)^{1/2} \right) J_1 \left(2 \left(\frac{t_2}{\tau} \right)^{1/2} \right) \left(\frac{1}{\tau} \right)^{1/2} J_1 \left(2 \left(\frac{1}{t_1 \tau} \right)^{1/2} \right) - \tilde{\chi}(\underline{p}) \quad (A.4.15)
\end{aligned}$$

Combining (A.4.15) with (A.4.1) we get the result (3.70) as was claimed. Q.E.D.

APPENDIX V

In this appendix we examine, firstly, the validity, to fourth order, $O(g^4)$, in the minor coupling constant, of the W-T identity depicted in Fig. 15, and secondly, possible cancellations (motivated by the validity of this W-T identity) of divergences arising from a certain class of 'super-graphs' (see Figs. (16, 17)). Such a cancellation mechanism is very desirable, because otherwise such divergences will lead to non-gauge invariant counterterms. We shall make use of the gauge-invariant regularisation of Ref. (41) which involves an analytic continuation in the dimension of space-time. Such a regularisation procedure is, indeed, gauge invariant, because the W-T identities being algebraic relations which do not involve the dimensionality of space-time explicitly are clearly satisfied by the regularised Feynman amplitudes (once it is shown that they are formally satisfied).

I. Let $\mathcal{T}(\text{diagram})$ be the amplitude due to the totality of diagrams with four external B-lines. Then, to order $O(g^4)$, we obtain

$$\mathcal{T}(\text{diagram}) \Big|_{g^4} = \begin{array}{c} \text{(a)} + \text{(b)} + \text{(c)} + \text{(d)} \\ \text{(e)} + \text{(f,g)} + \text{(h)} + \text{(i,j)} \end{array} \quad (\text{A.5.1})$$

Our task is to demonstrate a mutual cancellation of divergences among the graphs of the above set. To this end we first regularise the Feynman amplitudes corresponding to individual graphs of the set (A.5.1), and then show explicitly how the mutual cancellations occur.

(a)

$$\frac{i}{1!} \tilde{\mathcal{T}}^{(a)}_{(0)} = \frac{(-1)}{[2\pi]^4} \int dk \frac{\text{Tr} \{ k+m \}}{[k^2 - m^2 + i0]} \quad (\text{A.5.2})$$

Using the Pauli metric ($x_\mu = (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, ix_0)$), and $x_\mu x_\mu = \sum_1^4 x_\mu^2 = -x \cdot x$ (Bjorken-Drell metric) the regularised⁴¹ amplitude yields

$$\frac{i}{1!} \text{reg } \tilde{J}_1^{(a)}(z)(0) = \frac{4m}{[2\pi]^4} \frac{i\pi^{z/2}}{[m^2]^{1-z/2}} \Gamma(1-z/2) \quad (\text{A.5.3})$$

The logarithmic and quadratic divergences of (A.5.2) appear now as z -plane singularities at the physical point $z = 4$ (the dimension of space-time), and the point $z = 2$ respectively. Utilising the identity

$$\Gamma[n - z/2] = \frac{\Gamma[n + k - z/2]}{\prod_{r=0}^{k-1} [n - z/2 + r]} \quad (\text{A.5.4})$$

we can easily deduce that

$$\text{Res.} \left[\frac{i}{1!} \text{reg } \tilde{J}_1^{(a)}(z)(0) \right] = \begin{cases} \frac{8im^3}{16\pi^2}, & \text{at } z=4 \\ \frac{(-i)m}{2\pi^3}, & \text{at } z=2 \end{cases} \quad (\text{A.5.5})$$

(b)

$$\frac{i^2}{2!} \tilde{J}_2^{(b)}(p_1) = \frac{1}{2! [2\pi]^4} \int d^4k \frac{\text{Tr} \{ (-\not{k} + m)((\not{k} + \not{p}_1) + m) \}}{[k^2 - m^2 + i0][(k+p_1)^2 - m^2 + i0]}$$

which becomes in the Pauli metric

$$\frac{i^2}{2!} \tilde{J}_2^{(b)}(p_1) = \frac{1}{2! [2\pi]^4} \int d^4k \frac{\text{Tr} \{ (-i\not{k} - m)(i(\not{k} + \not{p}_1) - m) \}}{[k^2 + m^2 - i0][(k+p_1)^2 + m^2 - i0]} \quad (\text{A.5.6})$$

(A.5.6) yields for the regularised amplitude

$$\frac{i^2}{2!} \text{reg} \tilde{J}_2^{(b)}(z)(p_1) = \frac{4}{2! [2\pi]^4} \int_0^1 dx \int d_z K \frac{[-i^2 (k^2 + k \cdot p_1) + m^2]}{[k^2 + 2p_1 \cdot kx + m^2 + x p_1^2]^2} \quad (A.5.7)$$

From the set of formulae given in Appendix A of Ref. (41) (A.5.7) reduces to

$$\frac{4 i \pi^{z/2}}{2! [2\pi]^4 \Gamma(z)} \int_0^1 dx \frac{1}{[m^2 + x p_1^2 - x^2 p_1^2]^{2-z/2}} \cdot \left\{ -i^2 \Gamma(2-z/2) p_1^2 x^2 \right. \\ \left. - i^2 \Gamma(1-z/2) \frac{z}{2} [m^2 + x p_1^2 - x^2 p_1^2] - i^2 \Gamma(2-z/2) p_1 \cdot (-x p_1) + m^2 \Gamma(2-z/2) \right\}$$

Hence,

$$\text{Res.} \left[\frac{i^2}{2!} \text{reg} \tilde{J}_2^{(b)}(z)(p_1) \right] = \begin{cases} \frac{4i}{2! 16\pi^2} [-p_1^2 + 2m^2], & \text{at } z=4 \\ \frac{(-i)}{2! 2\pi^3}, & \text{at } z=2 \end{cases} \quad (A.5.8)$$

(c)

$$\frac{i^2}{2!} \tilde{J}_2^{(c)}(q) = \frac{1}{2! [2\pi]^4} \int d^4 k \frac{\text{Tr} \{ (k+m)((k+q)+m) \}}{[k^2 - m^2 + i0][(k+q)^2 - m^2 + i0]}$$

Following the procedure of the case (b) we finally obtain

$$\text{Res.} \left[\frac{i^2}{2!} \text{reg} \tilde{J}_2^{(c)}(z)(q) \right] = \begin{cases} \frac{4i}{2! 16\pi^2} [-q^2 - 6m^2], & \text{at } z=4 \\ \frac{i}{2! 2\pi^3}, & \text{at } z=2 \end{cases} \quad (A.5.9)$$

The remaining graphs of the set (A.5.1) are logarithmically divergent, the logarithmic singularity being of the form $\alpha m^2 \log \Lambda$ (where Λ is the cutoff parameter). We expect, therefore, to see cancellation of quadratic, and logarithmic (of the form $\beta p_i^2 \log \Lambda$) divergences among the graphs (a) - (d). This is indeed the case:

(i) Taking into account the combinatorial factors arising from Wick's theorem, and Reduction formula we obtain from (A.5.5), (A.5.8), and (A.5.9).

$$\begin{aligned} & \frac{(-m)^2 i^2}{2!} \left[\left\{ \binom{4}{1} \text{Res. reg } \tilde{\mathcal{T}}_2^{(b)}(z)(p_1) + \binom{4}{3} \text{Res. reg } \tilde{\mathcal{T}}_2^{(d)}(z)(p_1) \right. \right. \\ & \quad \left. \left. + \binom{4}{2} \text{Res. reg } \tilde{\mathcal{T}}_2^{(c)}(z)(q) \right\} \Big|_{z=2} \right] + \frac{(-m)i}{1!} \binom{4}{4} \text{Res. reg } \tilde{\mathcal{T}}_1^{(a)}(z)(0) \Big|_{z=2} \\ & = \frac{i(-m)^2}{2! 2\pi^3} \left[\sum_{r=0}^4 (-1)^r \binom{4}{r} \right] = 0 \end{aligned}$$

The vanishing residue at $z = 2$ implies that $\mathcal{T}(\text{diagram})g^4$ is free of quadratic divergences.

(ii) To see the cancellation of momentum-dependent logarithmic divergences we take the average* over the four external momenta. Thus, from (A.5.8), and (A.5.9) we obtain

$$\begin{aligned} & \frac{i^2}{2!} \left[\left\{ \binom{4}{1} + \binom{4}{3} \right\} \frac{1}{4} \text{Res.} \left\{ \sum_{i=1}^4 \text{reg } \tilde{\mathcal{T}}_2^{(b)}(z)(p_i) \right\} \Big|_{z=4} \right. \\ & \quad \left. + \binom{4}{2} \frac{1}{3} \text{Res.} \left\{ \sum_{s,t,u} \text{reg } \tilde{\mathcal{T}}_2^{(c)}(s) \right\} \Big|_{z=4} \right] \\ & = \frac{i}{2! 4\pi^2} \left\{ -\frac{1}{4} \left[\binom{4}{1} + \binom{4}{3} \right] \left(\sum_{i=1}^4 p_i^2 \right) + \frac{1}{3} \binom{4}{2} (s+t+u) \right\} \end{aligned}$$

* I am indebted to Dr. J. C. Taylor for pointing out this possibility.

where s, t, u are the usual Mandelstam variables. Since

$s + t + u = \sum_1^4 p_i^2$, it is clear that the residue vanishes with the

implication that $\mathcal{T}(\text{diagram})$ is free of momentum-dependent logarithmic divergences.

(e) - (g)

In the Pauli metric we have to consider the logarithmically divergent integral

$$\frac{i^3}{3!} \tilde{\mathcal{T}}_3^{(e)}(\underline{p}) = \frac{(-1)}{3! [2\pi]^4} \int d^4 k \frac{\text{Tr} \{ \gamma_5 (i\hat{k} - m) \gamma_5^2 (i(\hat{k} - \hat{p}_1 - \hat{p}_2) - m) \gamma_5 (i(\hat{k} - \hat{p}_1) - m) \}}{[k^2 + m^2 - i0] [(k - p_1 - p_2)^2 + m^2 - i0] [(k - p_1)^2 + m^2 - i0]}$$

Separating out the divergent part we obtain

$$I(\underline{p}) = \frac{m}{3! [2\pi]^4} \int d^4 k \frac{\text{Tr} \{ \hat{k} \hat{k} \}}{[] [] []}$$

Hence,

$$\text{reg } I(z)(\underline{p}) = \frac{m}{3! [2\pi]^4} 2! \int_0^1 dx_1 \int_0^{x_1} dx_2 \int d^4 k \frac{\text{Tr} \{ \hat{k} \hat{k} \}}{[k^2 + 2k \cdot P(\underline{x}; \underline{p}) + Q(\underline{x}; \underline{p})]^3} \quad (\text{A.5.10})$$

where

$$P(\underline{x}; \underline{p}) = -[(x_1 - x_2)p_2 + (1 - x_1)p_1]$$

$$Q(\underline{x}; \underline{p}) = [(p_1 + p_2)^2(x_1 - x_2) + p_1^2(1 - x_1)]$$

From (A.5.10) we easily deduce (utilising the results of Appendix A of Ref. (41)) that

$$\text{Res. reg } I(z)(\underline{p}) \Big|_{z=4} = \frac{(-8)(m) i\pi^2}{3! [2\pi]^4}$$

Taking into account the combinatorics of Wick's theorem and the Reduction formula we obtain

$$\text{Res. reg } \tilde{J}_3^{(e)-(f)}(z)(p) \Big|_{z=4} = \frac{i 96 m^4}{16 \pi^2} \quad (\text{A.5.11})$$

$$(h) - (j)$$

$$\frac{i^4}{4!} \tilde{J}_4^{(h)}(p) = \frac{i^4}{4! [2\pi]^4} \int d^4 k \text{Tr} \{ i^4 \gamma_5 (k+m) \gamma_5 (k+p_2+m) \gamma_5 (k+p_2+p_3+m) \cdot \\ \cdot \gamma_5 (k-p_1+m) \} / \{ [k^2 - m^2 + i0] [(k+p_2)^2 - m^2 + i0] [(k+p_2+p_3)^2 - m^2 + i0] [(k-p_1)^2 - m^2 + i0] \}$$

Separating out the divergent part we obtain (in the Pauli metric) for the regularised integral

$$\text{reg } J(z)(p) = \frac{3!}{4! [2\pi]^4} \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int d^4 k \frac{\text{Tr} \{ k \not{x} k \not{x} \}}{[k^2 + 2k \cdot P'(x;p) + Q'(x;p)]^4}$$

Utilising the following formula

$$\int d^4 k \frac{k_\mu k_\nu k_\rho k_\sigma}{[k^2 + 2k \cdot P + Q]^\alpha} = \frac{i \pi^{\frac{4}{2}}}{[Q^2 - P]^\alpha} \cdot \frac{1}{\Gamma(\alpha)} \cdot \{ \Gamma(\alpha - \frac{4}{2}) k_\mu k_\nu k_\rho k_\sigma + \Gamma(\alpha - \frac{4}{2} - 1) \frac{1}{2} [Q^2 - P] (g_{\mu\nu} k_\rho k_\sigma + g_{\mu\rho} k_\nu k_\sigma + g_{\mu\sigma} k_\nu k_\rho + g_{\nu\sigma} k_\mu k_\rho + g_{\rho\sigma} k_\mu k_\nu) + \Gamma(\alpha - \frac{4}{2} - 2) \frac{1}{4} (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\nu\rho} g_{\mu\sigma}) [Q^2 - P]^2 \}$$

we obtain (taking into consideration the appropriate combinatorial factors)

$$\text{Res. reg } \tilde{\mathcal{T}}_4^{(h)-(j)}(z)(p) \Big|_{z=4} = \frac{(-48) i m^4}{16 \pi^2} \quad (\text{A.5.13})$$

Gathering up the contributions from all the graphs in the set (A.5.1) we obtain for the momentum-independent logarithmic divergences the following residue

$$\frac{i(-m)^4}{16\pi^2} \left\{ \frac{1}{1!} \binom{4}{4} (-8) + \frac{1}{2!} \left[\binom{4}{1} + \binom{4}{3} \right] (8) + \frac{1}{2!} \binom{4}{2} (-24) + 96 - 48 \right\} = 0$$

We may now conclude that $\mathcal{T}(\text{diagram})|_{g^4}$ is free of divergences, because of the vanishing residues of $\text{reg } \mathcal{T}(\text{diagram})|_{g^4}(z)$ at $z = 4$, and $z = 2$.

We, now, turn to diagrams with three B- and one A -external lines.

Let $\mathcal{T}(\text{diagram})_\mu$ be the corresponding amplitude. Then, to $O(g^4)$ we obtain

$$\begin{aligned} \mathcal{T}(\text{diagram})_\mu \Big|_{g^4} = & \left\{ \text{(a)} + \text{(b)} \right\} + \\ & \text{(I)} \\ & + \left\{ \text{(c)} + \text{(d)} + \text{Perms.} \right\} + \left\{ \text{(e)} + \text{Perms.} \right\} \\ & \text{(II)} \qquad \qquad \qquad \text{(III)} \end{aligned} \quad (\text{A.5.14})$$

Graphs belonging to the subset (III) are convergent, whereas the graphs of (I), and (II) exhibit logarithmic divergences. We shall follow the procedure of the previous paragraph to show that (A.5.14) is free of divergences.

(a)

$$\frac{i^2}{2!} \tilde{\mathcal{J}}_2^{(a)}(p)_\mu = \frac{1}{2! [2\pi]^4} \int dk \frac{\text{Tr} \{ \gamma_\mu \gamma_5 (k+m) \gamma_5 (k+p+m) \}}{[k^2 - m^2 + i0] [(k+p)^2 - m^2 + i0]}$$

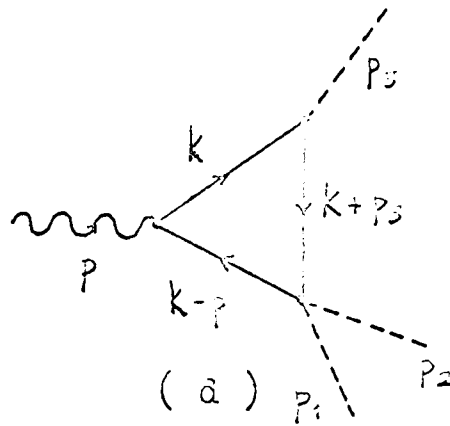
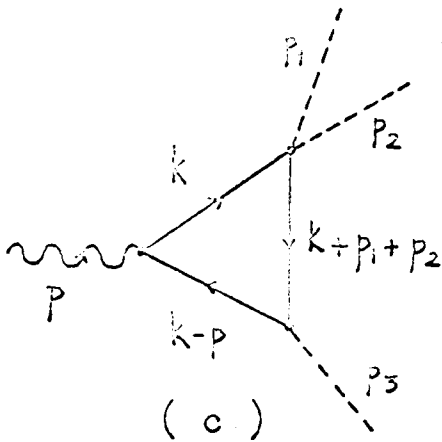
Using the Pauli metric we obtain for the regularised version of $p_\mu \{ \frac{i^2}{2!} \tilde{\mathcal{J}}_2^{(a)}(p)_\mu \}$

$$p_\mu \{ \frac{i^2}{2!} \text{reg} \tilde{\mathcal{J}}_2^{(a)}(z)(p)_\mu \} = \frac{4im p^2}{2! [2\pi]^4} \int_0^1 dx \frac{i\pi^{z/2}}{[m^2 + x p^2 - p^2 x^2]^{2-z/2}} \frac{\Gamma(2-z/2)}{\Gamma(2)}$$

Thus,

$$\text{Res.} \left\{ p_\mu \frac{i^2}{2!} \tilde{\mathcal{J}}_2^{(a)}(z)(p)_\mu \right\} \Big|_{z=4} = \frac{8im p^2}{2! 16\pi^2} \quad (\text{A.5.15})$$

(c)-(d)



$$\frac{i^3}{3!} \tilde{\mathcal{J}}_3^{(c)}(p)_\mu = \frac{i^3}{3! [2\pi]^4} \int dk \frac{\text{Tr} \{ i^3 \gamma_\mu \gamma_5 (k+m) \gamma_5^2 (k+p_1+p_2+m) \gamma_5 (k-p+m) \}}{[k^2 - m^2 + i0] [(k+p_1+p_2)^2 - m^2 + i0] [(k-p)^2 - m^2 + i0]}$$

Separating out the divergent part we obtain

$$\tilde{\mathcal{I}}^{(c)}(p)_\mu = \int dk \frac{\text{Tr} \{ \gamma_\mu \not{k} \not{k} \not{k} + \gamma_\mu \not{k} (p_1 + p_2) \not{k} - \gamma_\mu \not{k} \not{k} \not{p} \}}{[\quad] [\quad] [\quad]}$$

A rather long calculation shows that the residue of the regularised version of $p_\mu I^{(c)}(p)_\mu$ is given by

$$i\pi^2 \left\{ 4p \cdot [(p_1 + p_2) - p] - 4p \cdot (p_1 + p_2) + 8p^2 \right\} \quad (\text{A.5.16})$$

The corresponding contribution from (d) is of the form

$$i\pi^2 \left\{ -4p \cdot (p_3 - p) - 4p \cdot p_3 + 8p^2 \right\} \quad (\text{A.5.17})$$

From (A.5.16), and (A.5.17) it follows that the total contribution to the residue is given by

$$(-8) i\pi^2 p \cdot p_3$$

Averaging over the external momenta p_i , $i = 1, 2, 3$ we finally obtain

$$(-8) i\pi^2 \frac{1}{3} p \cdot \left[\sum_i^3 p_i \right] = \frac{8i\pi^2}{3} p^2$$

(due to the δ -function, $\delta(p + \sum_i^3 p_i)$, constraint). Hence, the residue of the regulated amplitude (Lorentz multiplied by p_μ), in the Pauli metric, corresponding to (c), and (d) is given by

$$\frac{1}{3! [2\pi]^4} \cdot \frac{8i\pi^2}{3} p^2 \quad (\text{A.5.18})$$

On taking into account the combinatorial factors arising from Wick's theorem, and the perturbation expansion of the S-matrix (A.5.15), and (A.5.18) yield for the total contribution to the residue of $p_\mu \text{reg } \mathcal{T} \left| \text{diagram} \right|_{g^4}^{(z)}_\mu$ at $z=4$

$$\text{Res. reg } \mathcal{T} (\quad)_{g^4}^{(z)} \Big|_{z=4} \quad (\text{A.5.19})$$

$$= (2 \times 1) \frac{1}{3!} \left[\frac{(-m)g}{2!} \cdot \frac{8im p^2}{16\pi^2} \right] + (3 \times 2) \frac{1}{2!} \left[\frac{(-m)^2 g}{3!} \cdot \frac{8i p^2}{3 \times 16\pi^2} \right] = 0$$

(A.5.19) implies the absence of divergences in (A.5.14). We would like also to remark that similar considerations hold for the total amplitude (to $O(g^4)$) corresponding to graphs with two B - and two A_μ -external lines. The calculation is rather laborious, and we shall not present it here.

Finally, we examine $\mathcal{T}(\text{diagram})$ to $O(g^4)$. To this order we obtain

$$\mathcal{T}(\text{diagram})_{\mu\nu\rho\sigma} \Big|_{g^4} = \text{(a)} + \text{(b)} + \text{(c)} \quad (\text{A.5.20})$$

Consider

$$\frac{z^4}{4!} p_{1,\mu} p_{2,\nu} p_{3,\rho} p_{4,\sigma} \tilde{\mathcal{T}}_4^{(a)}(z)(p)_{\mu\nu\rho\sigma} \quad (\text{A.5.21})$$

The divergent piece is given by the integral

$$\mathcal{J}^{(a)}(p) = \frac{1}{4! [2\pi]^4} \int dk \frac{\text{Tr} \{ \not{p}_1 \not{k} \not{p}_2 \not{k} \not{p}_3 \not{k} \not{p}_4 \not{k} \}}{[k^2 + m^2 - i0][(k+p_2)^2 + m^2 - i0][(k+p_2+p_3)^2 + m^2 - i0][(k-p_1)^2 + m^2 - i0]}$$

Hence, utilising (A.5.12) we obtain for the residue of the regularised version of (A.5.21)

$$\text{Res.} \left\{ \frac{i^4}{4!} p_{1,\mu} p_{2,\nu} p_{3,\rho} p_{4,\sigma} \text{reg} \tilde{\mathcal{T}}_4^{(a)}(z)(p)_{\mu\nu\rho\sigma} \right\} \Big|_{z=4} \quad (\text{A.5.22})$$

$$= \frac{i}{16\pi^2} \cdot \frac{(-1)}{12 \times 4!} [8(p_1 \cdot p_2)(p_3 \cdot p_4) + 8(p_1 \cdot p_4)(p_2 \cdot p_3) - 16(p_1 \cdot p_3)(p_2 \cdot p_4)]$$

The contributions from (b), and (c) are readily obtained from (A.5.22)

by interchanging $p_1 \leftrightarrow p_2$, and $p_2 \leftrightarrow p_3$ respectively. Hence, the total

contribution to $\text{Res} \left\{ p_{1,\mu} p_{2,\nu} p_{3,\rho} p_{4,\sigma} \text{reg} \mathcal{T}(\text{diagram})_{\mu\nu\rho\sigma} \right\} \Big|_{g^4}(z)(p)$ is given by

$$\begin{aligned}
& \text{Res.} \left\{ p_{1,\mu} p_{2,\nu} p_{3,\rho} p_{4,\sigma} \text{ reg } \mathcal{T} \left(\quad \right)_{g^4(z)(p)}{}_{\mu\nu\rho\sigma} \right\} \Big|_{z=4} \\
&= \frac{i}{16\pi^2} \frac{(-1)}{12 \times 4!} \left[\left\{ 8(p_1 \cdot p_2)(p_3 \cdot p_4) + 8(p_1 \cdot p_4)(p_2 \cdot p_3) - 16(p_1 \cdot p_3)(p_2 \cdot p_4) \right. \right. \\
&\quad \left. \left. + \{ p_1 \leftrightarrow p_2 \} + \{ p_2 \leftrightarrow p_3 \} \right\} \right] = 0
\end{aligned}$$

Hence, (A.5.20) is free of divergences.

II. Motivated by the cancellation of divergences among graphs satisfying the W-T identity of Fig. 15 we conjecture that similar cancellations are also likely to occur among 'supergraphs '. For definiteness we consider a specific example (taken from fermion-fermion scattering) shown in Fig. 16. In this case we expect a mutual cancellation of logarithmic divergences arising from the fermion loops in Fig. 16 (a, b). In the following subsections we shall carry out a calculation to show the plausibility of our conjecture. The actual calculation, however, turns out to be rather long, and we are, therefore, inclined to give only the main steps.

(a) A typical contribution to the amplitude corresponding to the diagram in Fig. 16 (a) comes from $T \left[\mathcal{L}_1(x_1) \mathcal{L}_1(x_2) \mathcal{L}_2(x_3) \mathcal{L}_2(x_4) \right]$ which yields (using the notation of Subsec. III.1 and taking into account the S-matrix perturbation expansion (3.2))

$$\begin{aligned}
& \frac{i^4}{4!} g^2 (-m)^2 T \left[\left\{ \prod_1^2 : \bar{\psi}(x_i) \gamma_\mu \gamma_5 \psi(x_i) A_\mu(x_i) : \right\} \left\{ \prod_3^4 \left[\bar{\psi}(x_j) \Lambda_+ \psi(x_j) \Theta^+(x_j) \right. \right. \right. \\
&\quad \left. \left. \left. + \bar{\psi}(x_j) \Lambda_- \psi(x_j) \Theta^-(x_j) \right] \right\} \right\} \\
&= \frac{i^4}{4!} g^2 (-m)^2 \left\{ : \bar{\psi}(1) \gamma_\mu \gamma_5 \psi(1) \bar{\psi}(4) \Lambda_+ \psi(4) : D_{\mu\nu}^c(1,2) \text{Tr} [\gamma_\nu \gamma_5 \cdot \right. \\
&\quad \left. \cdot S^c(2,3) \Lambda_+ S^c(3,2)] \mathcal{D}^-(3,4) + \text{other contributions} \right\}
\end{aligned}$$

Thus, a typical contribution to the Feynman amplitude corresponding to Fig. 16 (a) is given by

$$\mathcal{T}_4(\underline{x})_\alpha = D_{\alpha\beta}^c(x_1, x_2) \text{Tr} \{ \gamma_\beta \gamma_5 S^c(x_2, x_3) \Lambda_+ S^c(x_3, x_2) \} \mathcal{D}^-(x_3, x_4) \quad (\text{A.5.24})$$

where

$$\mathcal{D}^\pm(x) = \exp \left[\pm \frac{2g}{M} D^c(x) \right] - 1$$

with $D^c(x) = (4\pi^2)^{-1} [-x^2 + i0]^{-1}$

In momentum space (A.5.24) yields

$$\begin{aligned} \tilde{\mathcal{T}}_4(q)_\alpha &= \frac{1}{2} \left[\tilde{D}_{\alpha\beta}^c(q) \tilde{\mathcal{D}}^-(q) \right] \frac{1}{[2\pi]^4} \int dk \text{Tr} \left[\gamma_\beta \gamma_5 i(k+m)(1+\gamma_5) \cdot \right. \\ &\quad \left. \cdot i(k-q+m) \right] \cdot \{ [k^2 - m^2 + i0] [(k-q)^2 - m^2 + i0] \}^{-1} \end{aligned} \quad (\text{A.5.25})$$

The integration over the loop momentum diverges logarithmically. Hence, using the Pauli metric in conjunction with the regularisation procedure of Ref. (41) we obtain for the regularised version of (A.5.25)

$$\begin{aligned} \text{reg } \tilde{\mathcal{T}}_4(z)(q)_\alpha &= \frac{1}{2} \left[\tilde{D}_{\alpha\beta}^c(q) \tilde{\mathcal{D}}^-(q) \right] \cdot \frac{4mq_\beta i\pi^{z/2} \Gamma(2-z/2)}{[2\pi]^4} \\ &\quad \cdot \int_0^1 dx \frac{1}{[m^2 + xq^2 - x^2q^2]^{2-z/2}} \end{aligned}$$

The divergence of (A.5.25) manifests itself as a simple pole in the z -plane at $z = 4$ (the dimension of space-time). The residue is given by

$$\text{Res.}_{z \rightarrow 4} \{ \text{reg } \tilde{\mathcal{T}}_4(z)(q)_\alpha \} = \left(\frac{(-i)m}{(2\pi)^2} \right) \left[\tilde{D}_{\alpha\beta}^c(q) \tilde{\mathcal{D}}^-(q) \right] q_\beta \quad (\text{A.5.26})$$

with (see Sec. II PART I)

$$\begin{aligned} \mathcal{D}^-(q) = & (-i) \left\{ \frac{(-\kappa)}{q^2 - i0} - \left(\frac{4\pi}{q^2} \right)^2 \sum_{n=2}^{\infty} \frac{\left[\frac{\kappa q^2}{(4\pi)^2} \right]^n}{\Gamma(n-1)\Gamma(n)\Gamma(n+1)} \right. \\ & \left. \cdot \left[\frac{1}{2} n \left(\frac{\kappa (p^2 - i0)}{(4\pi)^2} \right) - \psi(n-1) - \psi(n) - \psi(n+1) \right] \right\} \end{aligned} \quad (\text{A.5.27})$$

where $\kappa = (2g/M)^2$. Thus, one could check order by order in g^2 (up to factors in $\ln g^2$) the cancellation of divergences (i.e. the vanishing of the total residue) between the regularised amplitudes corresponding to the diagrams of Fig. 16 (a), and Fig. 16 (b).

(b) Again a typical contribution to the Feynman amplitude corresponding to the diagram in Fig. 16 (b) comes from $T[\mathcal{L}_1(x_1)\mathcal{L}_1(x_2)\mathcal{L}_2(x_3)\mathcal{L}_2(x_4)\mathcal{L}_2(x_5)]$. Thus, we consider

$$\begin{aligned} & \frac{i^5}{5!} g^2 (-m)^3 T \left[\left\{ \prod_1^2 : \bar{\psi}(x_i) \gamma_\mu \gamma_5 \psi(x_i) A_\mu(x_i) : \right\} \left\{ \prod_3^5 [\bar{\psi}(x_j) \Lambda_+ \psi(x_j) \Theta^+(x_j) \right. \right. \\ & \quad \left. \left. + \bar{\psi}(x_j) \Lambda_- \psi(x_j) \Theta^-(x_j)] \right\} \right] \end{aligned} \quad (\text{A.5.28})$$

and look for divergent contributions of the form

$$: \bar{\psi}(x_i) \gamma_\alpha \gamma_5 \psi(x_i) \bar{\psi}(x_j) \Lambda_+ \psi(x_j) : \mathcal{T}_5(x)_\alpha, \quad i=1,2; j=3,4,5$$

Such contributions arise from the following terms in (A.5.28)

$$\begin{aligned} & T \left[\left\{ \prod_1^2 : \bar{\psi}(x_i) \gamma_\alpha \gamma_5 \psi(x_i) A_\alpha(x_i) : \right\} \left[\bar{\psi}(x_3) \Lambda_+ \psi(x_3) \Theta^+(x_3) \bar{\psi}(x_4) \Lambda_- \psi(x_4) \Theta^-(x_4) \right. \right. \\ & \quad \left. \left. + \bar{\psi}(x_5) \Lambda_+ \psi(x_5) \Theta^+(x_5) + \text{cyclic Perms.} \right] \right] \end{aligned} \quad (\text{A.5.29})$$

Let us examine in some detail the first of the terms in (A.5.29) Then,

$$\begin{aligned}
& T \left[\left[\prod_1^2 : \bar{\psi}(x_i) \gamma_\alpha \gamma_5 \psi(x_i) A_\alpha(x_i) : \right] \bar{\psi}(x_3) \Lambda_+ \psi(x_3) \Theta^+(x_3) \bar{\psi}(x_4) \Lambda_- \psi(x_4) \Theta^-(x_4) \bar{\psi}(x_5) \Lambda_+ \psi(x_5) \Theta^+(x_5) \right] \\
& = \left\{ \left[: \bar{\psi}(x_1) \gamma_\alpha \gamma_5 \psi(x_1) \bar{\psi}(x_5) \Lambda_+ \psi(x_5) : \left\{ D_{\alpha\beta}^c(x_1, x_2) \text{Tr} [\gamma_\beta \gamma_5 S^c(x_2, x_3) \Lambda_+ \right. \right. \right. \\
& \quad \left. \left. S^c(x_3, x_4) \Lambda_- S^c(x_4, x_2) \right] + 3 \leftrightarrow 4 \right\} + : \bar{\psi}(x_1) \gamma_\alpha \gamma_5 \bar{\psi}(x_3) \Lambda_+ \psi(x_3) : \right. \\
& \quad \left. \left\{ D_{\alpha\beta}^c(x_1, x_2) \text{Tr} [\gamma_\beta \gamma_5 S^c(x_2, x_4) \Lambda_- S^c(x_4, x_5) \Lambda_+ S^c(x_5, x_2)] + 5 \leftrightarrow 4 \right\} \right. \\
& \quad \left. + [1 \leftrightarrow 2] \right\} \cdot T [\Theta^+(x_3) \Theta^-(x_4) \Theta^+(x_5)]
\end{aligned} \tag{A.5.30}$$

Due to the multiplicity of contributions we are content to examine the contribution to the Feynman amplitude corresponding to the diagram of Fig. 16 (b) due to the first term in (A.5.30). The other contributions could be worked out in a similar way. To this end we start with the formal expression

$$\mathcal{T}_5(\underline{x})_\alpha = \sum_{\nu_1, \nu_2=1}^{\infty} \frac{[-\kappa]^{\nu_1} [\kappa]^{\nu_2}}{\nu_1! \nu_2!} D_{\alpha\beta}^c \mathcal{T}^\nu(\underline{x})_\beta \tag{A.5.32}$$

where

$$\begin{aligned}
\mathcal{T}^\nu(\underline{x})_\beta &= \text{Tr} \{ \gamma_\beta \gamma_5 S^c(x_2, x_3) \Lambda_+ S^c(x_3, x_4) \Lambda_- S^c(x_4, x_2) \} \cdot \\
&\quad \cdot [D^c(x_3, x_4)^{\nu_1} D^c(x_4, x_5)^{\nu_2}] \tag{A.5.33}
\end{aligned}$$

The formal nature of (A.5.32) arises due to the lack of definition of (A.5.33). Thus, we introduce, in momentum space, the following regularised amplitude corresponding to (A.5.33):

$$\text{reg } \tilde{J}^{\underline{\nu}}(\underline{\lambda}, \underline{z})(q)_{\beta} = \prod_{\ell=1}^2 \left\{ \frac{(4\pi)^2 (-i) 2^{2\lambda_{\ell} \nu_{\ell}} \Gamma((\lambda_{\ell}-1)\nu_{\ell}+2)}{[(4\pi)^2]^{\nu_{\ell}} \Gamma((1-\lambda_{\ell})\nu_{\ell})} \right\} \frac{1}{[2\pi]^{4 \times 2}}.$$

$$\cdot \int d_{z_2} k_2 \int d_{z_1} k_1 \text{Tr} \left\{ \gamma_{\beta} \gamma_5 i(k_1+m) \left(\frac{1+\gamma_5}{2} \right) i(k_1+k_2+m) \left(\frac{1-\gamma_5}{2} \right) i(k_1-q+m) \right\} \cdot$$

$$\cdot \frac{[-k_2^2 - i\varepsilon]^{(1-\lambda_1)\nu_1-2} [-(k_2+q)^2 - i\varepsilon]^{(1-\lambda_2)\nu_2-2}}{[k_1^2 - m^2 + i\varepsilon] [(k_1+k_2)^2 - m^2 + i\varepsilon] [(k_1-q)^2 - m^2 + i\varepsilon]}$$

where, for fixed $\underline{\nu}$, $(\lambda_1, \lambda_2) \in \Lambda_1$, $(z_1, z_2) \in Z \subset \mathbb{C}^2$ and

$$\Lambda_1 = \left\{ \underline{\lambda} \mid 1 - \frac{1}{\max\{\nu_{\ell}\}} < \text{Re } \lambda_{\ell} < 1, \quad 0 < \text{Im } \lambda_{\ell} < b, \quad \ell=1,2 \right\}$$

If we separate out the piece in the trace which gives rise to divergences, when the regulating parameters take on their physical values (i.e. $\lambda_{\ell} = 0$, $z_{\ell} = 4$, $\ell = 1,2$), and concentrate (for the sake of argument) only on one of the divergent contributions we obtain (with abuse of notation)

$$\text{reg } \tilde{J}^{\underline{\nu}}(\underline{\lambda}, \underline{z})(q)_{\beta} = \prod_{\ell=1}^2 \left\{ \frac{(4\pi)^2 (-i) 2^{2\lambda_{\ell} \nu_{\ell}} [(4\pi)^2]^{-\nu_{\ell}}}{\Gamma((1-\lambda_{\ell})\nu_{\ell})} \right\} \cdot$$

$$\cdot \frac{i \pi^{z_1/2} \Gamma[4 + (\lambda_1 - \nu_1) + (\lambda_2 - 1)\nu_2]}{4 (2\pi)^{2 \times 4}} \int_0^1 dx \, x^{(\lambda_1-1)\nu_1+1} (1-x)^{(\lambda_2-1)\nu_2+1}$$

$$\cdot \int d_{z_2} k_2 \frac{1}{[k^2 + 2 k_2 \cdot q (1-x) + q^2 (1-x) - i\varepsilon]^{\sum_1^2 (\lambda_{\ell}-1)\nu_{\ell}+4}} \cdot$$

$$\cdot \int_0^1 dy_1 \int_0^{y_1} dy_2 \frac{1}{[Q(q, k_2; \underline{y}) - P(q, k_2; \underline{y})^2]^{3-z_1/2}} \cdot (-P_{\beta}(q, k_2; \underline{y})).$$

$$\cdot \left\{ \Gamma(3-z_1/2) P^2 + \Gamma(2-z_1/2) \frac{(z_1+2)}{2} [Q - P^2] \right\}$$

(A.5.34)

where

$$\begin{aligned} P_\beta(q, k_2; \underline{y}) &= [k_2(y_1 - y_2) - q(1 - y_1)]_\beta \\ Q(q, k_2; \underline{y}) &= m^2 + k_2^2(y_1 - y_2) + q^2(1 - y_1) \end{aligned} \quad (\text{A.5.35})$$

We now adopt the following representation for the integration with respect to x :

$$\begin{aligned} &\int_0^1 dx \, x^{\alpha-1} (1-x)^{\beta-1} f(x) \\ &= \frac{e^{-i\pi(\alpha+\beta)}}{[2i \sin \pi \alpha \cdot 2i \sin \pi \beta]} \int_A^{(1+, 0+, 1-, 0-)} dx \, x^{\alpha-1} (1-x)^{\beta-1} f(x) \end{aligned}$$

where $0 < \text{Re} A < 1$, $\text{Im} A = 0$ and $f(x)$ is any function analytic in x .

The above formula used in connection with (A.5.34) furnishes us with an explicit analytic continuation of (A.5.34) from the region Λ_1 into a $\underline{y}$ -independent region Λ

$$\Lambda = \{ \underline{\lambda} \mid -a < \text{Re} \lambda_\ell < 1, \quad 0 < \text{Im} \lambda_\ell < b, \quad 0 \leq a, \quad \ell = 1, 2 \}$$

(see PART I III.2.3). Then we obtain after summing with respect to

ν_1 , and ν_2

$$\begin{aligned} \text{reg } \tilde{J}_5(\underline{\lambda}, \underline{z})(q)_\alpha &= \frac{i\pi^{z_1/2}}{4(2\pi)^{2 \times 4}} \sum_{\nu_1, \nu_2=1}^{\infty} \frac{[-\kappa]^{\nu_1} [\kappa]^{\nu_2}}{\Gamma(\nu_1+1) \Gamma(\nu_2+1)} D_{\alpha\beta}^c(q) \cdot \\ &\cdot \left\{ \prod_{\ell=1}^2 \frac{(4\pi)^2 (-i) 2^{2\lambda_\ell \nu_\ell}}{[(4\pi)^2]^{\nu_\ell} \Gamma((1-\lambda_\ell)\nu_\ell)} \right\} \frac{\Gamma(4 + \sum_1^2 (\lambda_\ell - 1)\nu_\ell) e^{-i\pi(\sum_1^2 (\lambda_\ell - 1)\nu_\ell + 4)}}{\prod_1^2 [2i \sin \pi((\lambda_\ell - 1)\nu_\ell + 2)]} \cdot \\ &\cdot \int_A^{(1+, 0+, 1-, 0-)} x^{(\lambda_1 - 1)\nu_1 + 1} (1-x)^{(\lambda_2 - 1)\nu_2 + 1} \int_{z_2}^1 d_{z_2} k_2 [k_2^2 + 2k_2 \cdot q(1-x) + q^2(1-x) - i\varepsilon]^{\sum_1^2 (1-\lambda_\ell)\nu_\ell - 4} \cdot \\ &\cdot \int_0^1 dy_1 \int_0^{y_1} dy_2 \frac{1}{[Q - P^2]^{3-z_1/2}} (-P_\beta) \{ \Gamma(3-z_1/2) P^2 + \\ &\Gamma(2-z_1/2) (\frac{z_1+2}{2}) (Q - P^2) \}. \end{aligned} \quad (\text{A.5.36})$$

The original divergence associated with the fermion loop integral shows up now as a simple pole in the z_1 -plane at $z_1 = 4$. The residue is given by

$$\begin{aligned} \text{Res.}_{z_1 \rightarrow 4} &= \int_0^1 dy_1 \int_0^{y_1} dy_2 (-4) (-P_\beta(q, k_2; \underline{y})) \\ &= -\frac{2}{3} (k_2 - q)_\beta \end{aligned}$$

Inserting this in (A.5.36), summing up the ν_1 -series by Watson's method, and carrying out the integral with respect to k_2 we finally obtain

$$\begin{aligned} \text{Res}_{z_1 \rightarrow 4} \left[\text{reg } \tilde{\mathcal{J}}_5(\underline{\lambda}, \underline{z})(q)_\alpha \right] &= B \sum_{\nu_2=1}^{\infty} \frac{\left[\frac{\kappa}{(4\pi)^2} \right]^{\nu_2} 2^{2\lambda_2 \nu_2}}{\Gamma(\nu_2+1) \Gamma((1-\lambda_2)\nu_2)} \cdot \\ &\cdot D_{\alpha\beta}^c(q) \frac{1}{2i} \int_{\Gamma} ds_1 \frac{\left[\frac{\kappa}{(4\pi)^2} \right]^{s_1} 2^{2\lambda_1 s_1}}{\sin \pi s_1 \Gamma(s_1+1) \Gamma((1-\lambda_1)s_1)} \Gamma[4-z_2/2 + (\lambda_1-1)s_1 + (\lambda_2-1)\nu_2] \cdot \\ &\cdot \int_0^1 dx \ x^{(1-\lambda_2)\nu_2 + z_2/2 - 3} (1-x)^{(1-\lambda_1)s_1 + z_2/2 - 3} \\ &\cdot [-q(1-x) - q]_\beta [q^2 - i0]^{(1-\lambda_1)s_1 + (1-\lambda_2)\nu_2 + z_2/2 - 4} \end{aligned}$$

Holding λ_1, λ_2 in Λ we set $z_2 = 4$. Then following the techniques of Subsec. III.3 (PART I) we obtain the following expression for the residue

$$\begin{aligned}
& \text{Res.} \left[\tilde{\mathcal{J}}_5(\underline{\lambda}, \underline{z})_R(q)_\alpha \right] \Bigg|_{\substack{z_1 = z_2 = 4 \\ \underline{\lambda} = \underline{0}}} \\
&= B D_{\alpha\beta}^c(q) \sum_{\nu_2=1}^{\infty} \frac{\left[\frac{K}{(4\pi)^2} \right]^{\nu_2}}{\Gamma(\nu_2+1)\Gamma(\nu_2)} \frac{1}{2i} \int_{\overline{\Gamma}} ds_1 \frac{\left[\frac{K}{(4\pi)^2} \right]^{s_1}}{\sin \pi s_1 \Gamma(s_1+1)\Gamma(s_1)} \quad (\text{A.5.37}) \\
&\cdot \Gamma(2-s_1-\nu_2) (-q)_\beta \left\{ \frac{\Gamma(s_1)\Gamma(\nu_2)}{\Gamma(\nu_2+s_1)} + \frac{\Gamma(s_1+1)\Gamma(\nu_2)}{\Gamma(\nu_2+s_1+1)} \right\} [q^2-i0]^{s_1+\nu_2-2}
\end{aligned}$$

From (A.5.26), (A.5.27), and (A.5.37) one can easily see that the amplitudes corresponding to the diagrams shown in Fig. 16 (a), Fig. 16 (b) have the same singularity structure. This is very promising, and demonstrates the plausibility of our conjecture. We would like to believe that the verification of our conjecture is only a matter of a detailed calculation.

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17. How to do this has been shown in Ref. (9), Lemmata 2.2.20, 2.2.33, and Theorem 2.3.1. See also Appendix I of the thesis.
18. See Ref. (14), Ch. III. Sections 2.3, 2.4, for quadratic forms raised to powers as distributions. Equation (6), p.272, is useful for taking boundary values.
19. The usual procedure for Eulerian integrals. See e.g. Ref. (15), p. 60.
20. M. Daniel, and P. K. Mitter (unpublished).
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22. We follow the method given in Ref. (2), Ch. IV. We emphasize that in the following equations (3.48 - 3.49) the $\eta \rightarrow 0+$ limit has already been taken.
23. It is possible, although laborious, to verify explicitly the cancellation mechanism at least for the first few terms.
24. The choice of $\tilde{\chi}(p)$, which may seem mysterious at this stage, is motivated by the work of Appendix IV, where the real part of the double series in (3.59), in the region (3.66), is actually summed and the step from (3.69) - (3.70) is justified.
25. The choice of $\sin^2 \pi z$ term was motivated by a suggestion of H. Lehmann.
26. To get the asymptotic estimate, we divide the region of τ integration into two parts: $(0, (\frac{1}{5})^{2/3})$, and $((\frac{1}{5})^{2/3}, \infty)$. In the second interval the integral may be bounded, using Poisson's bound for Bessel functions, and we get the asserted decrease. In the first

interval, for ζ sufficiently large, replace $J_1(2(\frac{1}{\zeta})^{1/2})$ by its asymptotic value; also the dominant contribution is for $t_1 \zeta^{2/3}$ bounded, $t_1 t_2 \zeta^{2/3}$ bounded. After changing variables $\tau = \zeta^{-2/3} \xi$, a steepest descent estimate leads to the asserted decrease.

27. The necessity of obtaining a decrease property in region (ii) was pointed out to us by H. Lehmann. A similar argument is given in Ref. (10).
28. The method of Appendix IV was developed in collaboration with D. I. Fivel (University of Maryland).
29. See Ref. (15), page 216, formula (4).
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ERRATA

On page 2 'satisfaying' should read 'satisfying'.

On page 16 'decompoition' should read 'decomposition'.

On page 44 'bracked' should read 'bracket'.