Vectorial problems: sharp Lipschitz bounds and borderline regularity

Cristiana De Filippis

Pembroke College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Hillary 2020

If you find a good move,
look for a better one.

Abstract

This thesis is devoted to the proof of fine regularity properties of solutions to a broad class of variational problems including models from geometry, material science, continuum mechanics and particle physics. Our starting point is the analysis of the behavior of manifold-constrained minima to certain non-homogeneous functionals: under sharp assumptions, we prove that they are regular everywhere, except on a negligible, "singular" set of points, [75]. The presence of the singular set is in general unavoidable. Looking at minima as solutions to the associated Euler-Lagrange system does not help: it presents an additional component generated by the curvature of the manifold having critical growth in the gradient variable. For instance, sphere-valued harmonic maps satisfy in a suitably weak sense

$$-\Delta u = |Du|^2 u.$$

This turns out to be an insurmountable obstruction to regularity, [227]. It is then natural to consider general systems of type

$$-\operatorname{div}\, a(x, Du) = f \tag{0.0.1}$$

and study how the features of f and of the partial map $x \mapsto a(x,z)$ influence the regularity of solutions. In this respect, we are able to cover non-linear tensors with exponential type growth conditions as well as with unbalanced polynomial growth: we prove everywhere Lipschitz regularity for vector-valued solutions to (0.0.1) under optimal assumptions on forcing term and space-depending coefficients, [76]. When the system in (0.0.1) has the Double Phase structure:

$$-\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du\right) = -\operatorname{div}\left(|\mathfrak{F}|^{p-2}\mathfrak{F} + a(x)|\mathfrak{F}|^{q-2}\mathfrak{F}\right)$$
$$0 \le a(\cdot) \in C^{0,\alpha}, \qquad 1 \le \frac{q}{p} \le 1 + \frac{\alpha}{n},$$

we complete the Calderón-Zygmund theory started in [62] by dealing with the delicate borderline case

$$\frac{q}{p} = 1 + \frac{\alpha}{n},$$

which has been left open so far, [78]. Finally, we propose a new approach to the analysis of variational integrals with (p,q)-growth based on convex duality, [73].

Contents

Chapter 1

Introduction

The aim of this thesis it to provide a complete regularity theory for certain non-uniformly elliptic variational problems of geometric or physical nature. This is part of the work done during my PhD studies. Other papers written in this period but that are not covered by this presentation will be listed in the fourth part of the introduction.

1.1 Standard regularity theory: an overview.

We study qualitative features, such as continuity or differentiability, of vector-valued local minimizers $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ to the variational integral

$$W^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega F(x, w, Dw) \ \mathrm{d}x, \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $n \geq 2$ and $N \geq 1$. It is well known that if $F(\cdot)$ is a Carathéodory function convex in the gradient variable and satisfying reasonable coercivity conditions, a minimizer of (1.1.1) exists, [123, Chapter 4]. Moreover, if $F(\cdot)$ is also sufficiently smooth and superlinear at infinity, minima of (1.1.1) can be characterized as weak energy solutions to the Euler-Lagrange system

$$-\operatorname{div}\big(\partial_z F(x, u, Du)\big) + \partial_v F(x, u, Du) = 0 \qquad \text{in} \ \ \Omega,$$

see [46, 48]. It is then natural to wonder how the regularity of the integrand $F(\cdot)$ affects the regularity of minimizers of (1.1.1). Lots of efforts have been made in this direction, starting from [19], which states that any solution $u \in C^3(\mathbb{R}^2)$ of a nonlinear elliptic equation of type

$$F\left(x, y, u, \partial_x u, \partial_y u, \partial_{xy} u, \partial_{xx} u, \partial_{yy} u\right) = 0,$$

with $F(\cdot)$ real analytic in each of its variables, must be real analytic. This can be achieved by means of certain estimates for derivatives of solutions given in form of power series. Through several other works [36–39, 149, 150, 200, 201, 204–207, 220, 229] was established that any sufficiently smooth, say C^1, stationary point of (1.1.1) is analytic provided that the integrand is analytic as well, see also [30, 236]. However, by direct methods we can only prove existence of minimizers in Sobolev spaces, thus the question of raising the regularity of minima from Sobolev to C^0 or even more, is totally rightful. In the scalar setting, this problem was successfully solved by Morrey [202], in two-dimensions and, finally, De Giorgi settled the general case [81], where linear elliptic equations with measurable coefficients of the type

$$\operatorname{div}\big(a(x)Dw\big) = 0 \qquad \text{in} \ \ \Omega \tag{1.1.2}$$

are considered, with $a(\cdot)$ satisfying the growth and ellipticity conditions

$$|a(x)| \le L \quad \text{and} \quad \langle a(x)\xi,\xi\rangle \ge \nu|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $0 < \nu \le L$ are fixed constants. The main outcome is that if $u \in W^{1,2}_{\rm loc}(\Omega)$ is a weak solution of equation (1.1.2), then there exists an exponent $\alpha \equiv \alpha(n, L/\nu) \in (0,1)$ such that $u \in C^{0,\alpha}_{\rm loc}(\Omega)$. The same theorem has been independently obtained by Nash in [213] for parabolic equations, and, later on, those results were proved for solutions of (1.1.2) and its parabolic analog

$$\partial_t w - \operatorname{div}\big(a(x)Dw\big) = 0 \qquad \text{in } \Omega \times [0,T)$$

by Moser via a different technique (Harnack inequality, which implies Hölder continuity), in [209–211]. The point of strength of De Giorgi's method is that $a(\cdot)$ is allowed to be just measurable, otherwise, for continuous coefficients, the same result can be obtained by means of perturbation methods, the so-called Korn's trick, see [119, Section 5.4.2]. The proof of De Giorgi's theorem essentially consists in an innovative iteration procedure which heavily relies on the Caccioppoli inequalities on level sets

$$\int_{B_\varrho \cap \{u(x)\ge\kappa\}} |Du|^2 \, dx \le \frac{c(n,\nu,L)}{(r-\varrho)^2} \int_{B_r \cap \{u(x)\ge\kappa\}} (u-\kappa)_+^2 \, dx \tag{1.1.3}$$

and

$$\int_{B_\varrho \cap \{u(x)\le\kappa\}} |Du|^2 \, dx \le \frac{c(n,\nu,L)}{(r-\varrho)^2} \int_{B_r \cap \{u(x)\le\kappa\}} (u-\kappa)_-^2 \, dx. \tag{1.1.4}$$

A crucial observation to be made here is that, to get (1.1.3)-(1.1.4), the linearity of the equation in (1.1.2) plays no role, therefore, with minor variations, the same techniques have been extended to the non-linear case by [107, 115, 172]. Precisely, in the most general case, there holds that if $u \in W^{1,p}(\Omega)$, $p > 1$, is a local minimizer of the variational integral in (1.1.1), with the integrand $F(\cdot)$ satisfying

$$|z|^p - c_1(|v|^s + 1) \le F(x,v,z) \le c_2|z|^p + c_1(|v|^s + 1), \quad \text{for all } (x,v,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

where c_1, c_2 are positive constants and $p \le s < p^*$, then there exists $\beta_0 \equiv \beta_0(n,p,c_1,c_2,s) \in (0,1)$ such that $u \in C^{0,\beta_0}_{\rm loc}(\Omega)$. It is reasonable to expect higher regularity for minimizers of (1.1.1), as soon as the assumptions on $F(\cdot)$ are strengthened. This is indeed the case. In fact if $(x,v) \mapsto F(x,v,\cdot)$ is sufficiently regular and a suitable control from above and below on $\partial_{zz}F(\cdot,z)$ in terms of powers of $|z|$ is imposed, then it is possible to prove that Du is Hölder continuous. Precisely, if

$$\begin{cases} z \mapsto F(\cdot,z) \in C^1_{\rm loc}(\mathbb{R}^n) \cap C^2_{\rm loc}(\mathbb{R}^n \setminus \{0\}) \\ \nu(\mu^2 + |z|^2)^{\frac{p}{2}} \le F(x,v,z) \le L(1+|z|^2)^{\frac{p}{2}}, \\ \nu(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\xi|^2 \le \partial_{zz}F(x,v,z)\xi\cdot\xi \le L(\mu^2+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \\ |F(x_1,v_1,z) - F(x_2,v_2,z)| \le \omega(|x_1-x_2| + |v_1-v_2|)(1+|z|^2)^{\frac{p}{2}}, \end{cases} \tag{1.1.5}$$

for all $x, x_1, x_2 \in \mathbb{R}^n$, $v, v_1, v_2 \in \mathbb{R}$ and any $z, \xi \in \mathbb{R}^n$, where $\mu \in [0,1]$ and $0 \le \omega(t) \le t^\beta$ for some $\beta \in (0,1]$, then there exists $\beta_0 \equiv \beta_0(n,\nu,L,p,\beta) \in (0,1)$ such that $Du \in C^{0,\beta_0}_{\rm loc}(\Omega,\mathbb{R}^n)$. This result was proved in [117, 172, 180, 242]. It is worth noticing that in the degenerate case, i.e., when the parameter μ appearing in $(1.1.5)_{2,3}$ vanishes, local β_0-Hölder continuity for Du is the best result achievable, in the light of a counterexample due to Ural'tseva [172, 242]. All the theory exposed so far is related to the scalar case $N = 1$. In the vectorial setting $N > 1$, there is no hope of getting analogous results, at least for general integrands. In fact, in [82] is shown that for $N = n \geq 3$ the function

$$u(x) := \frac{x}{|x|^{\alpha}}, \qquad \alpha := \frac{n}{2}\left[1 - \frac{1}{\sqrt{(2n-2)+1}}\right]$$

belongs to $W^{1,2}(B_1, \mathbb{R}^n)$ and locally minimizes the variational integral

$$W^{1,2}(B_1, \mathbb{R}^n) \ni w \mapsto \int_{B_1} F(x, Dw) \ \mathrm{d}x,$$

where

$$(x, z) \in B_1 \times \mathbb{R}^{n \times n} \mapsto F(x, z) := |z|^2 + \left[(n-2)\sum_{i=1}^{n} z_i^i + n \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} z_i^j\right]^2.$$

The key point here is that $x \mapsto F(x, \cdot)$ is discontinuous at the origin, otherwise by perturbation techniques, minimizers are locally Hölder continuous to any exponent $\beta \in (0, 1)$. In [125] a quadratic-type functional

$$W^{1,2}(B_1, \mathbb{R}^n) \ni w \mapsto \int_{B_1} a_{i,j}^{\alpha,\beta}(w) D_i w^{\alpha} D_j w^{\beta} \ \mathrm{d}x \qquad n = N, \tag{1.1.6}$$

with

$$a_{i,j}^{\alpha,\beta}(v) := \delta_{i,j}\delta_{\alpha,\beta} + \left[\delta_{\alpha,i} + \frac{4}{n-2} \cdot \frac{v_\alpha v_i}{1+|v|^2}\right] \cdot \left[\delta_{\beta,j} + \frac{4}{n-2} \cdot \frac{v_\beta v_j}{1+|v|^2}\right]$$

is analyzed. The outcome is that for $n > 2$ large enough, the discontinuous map $u(x) := \frac{x}{|x|}$ is a local minimizer of (1.1.6). This construction works because the map $x \mapsto \tilde{a}_{i,j}^{\alpha,\beta}(x) := a_{i,j}^{\alpha,\beta}(u(x))$ is just measurable. Those counterexamples could lead to the erroneous conclusion that the existence of singular minimizers is due to the way the coefficients depending on (x, v) mix up with the components of the gradient variable z. Anyway, such irregularity is a purely vectorial phenomenon and appears also when the integrand is smooth, uniformly convex and depends only from the gradient variable. The first counterexample in this direction was obtained in [214], where it is shown that, if the ambient space dimension is sufficiently high, the homogeneous degree one map $u(x) := |x|^{-1}(x \otimes x)$ minimizes an autonomous functional with uniformly convex, regular integrand. Such a result was improved later on in [130] to dimension $n = 5$ using the map

$$u(x) := \frac{x \otimes x}{|x|} - \frac{|x|}{n}\mathbb{I}, \tag{1.1.7}$$

which is symmetric and traceless, and so belongs to an $n(n+1)/2 - 1$-dimensional subspace of the set of $n \times n$ matrices. Furthermore, it was proved in [245] that the function in (1.1.7) provides also the example of a non-Lipschitz minimizer in dimensions $n = 3$, $N = 5$ by constructing a null Lagrangian $L(\cdot)$ such that $\partial_z L(Du) = \partial_z F(Du)$ for some smooth, uniformly convex function $F(\cdot)$. The same technique was used in [244] to prove that when $n \geq 3$ continuous minima cannot be expected: precisely when $n = 5$ and $N = 14$ it is found an unbounded minimum of a uniformly convex variational integral (thus proving the sharpness of Campanato's result on the Hölder continuity of minima for $n \leq 4$, cf. [113]), while when $n = 4$, $N = 3$ non-Lipschitz continuous minima coming from the Hopf fibration are constructed. Finally, in [199] is found a singular minimizer of a convex, autonomous functional with bounded second derivative in dimensions $n = 3$, $N = 2$, which are the optimal ones in the light of the results contained in [81, 213]. In this perspective, it is clear that certain structural assumptions must be imposed on the integrand in

(1.1.1) in order to prevent the formation of singularities: for instance the so-called Uhlenbeck structure

$$F(x,v,z)\equiv F(z)\equiv \tilde{F}(|z|) \tag{1.1.8}$$

assures full C^{1,β_0}-regularity, [2, 116, 121, 129, 241]. To determine the regularity for vector-valued minimizers of more general functionals defined by means of integrands not satisfying (1.1.8), in [124, 208] was introduced the theory of partial regularity, which means regularity outside a closed, small singular set. In fact, under assumptions (1.1.5) (obviously recast to fit the vectorial setting), any local minimizer of (1.1.1) is C^{1,β_0}-regular outside a set of zero n-dimensional Lebesgue measure, [115, 116, 120, 153, 154]. The strategy employed in these papers relies on suitable freezing techniques, firstly pioneered in [43] in the case of linear elliptic equations. Partial regularity turns out to be also the correct way to approach the study of regularity for manifold-valued problems, when both, minimizers and competitors take value into a submanifold of $\mathbb{R}^N$. In fact, the prototypical example concerns harmonic maps into spheres, i.e. functions $u\in\mathbb{S}^{N-1}$ a.e. satisfying in the weak sense

$$-\Delta u=|Du|^2u. \tag{1.1.9}$$

The above is a critical system and, as [133, 148, 227] point out, there is no hope of full regularity: in particular, in [227] is built a $\mathbb{S}^2$-valued map satisfying (1.1.9), whose singular set coincides with the whole three-dimensional closed unit ball. Therefore, being merely a weak solution of (1.1.9) is not enough to carry out a decent regularity theory, unless quite restrictive assumptions on the image of solutions are made [95, 140, 141], or the target manifold has a very specific structure [226]. On the other hand, if we look at energy minimizing maps, i.e. local minimizers of the variational integral

$$W^{1,2}(\mathcal{M},\mathcal{N})\ni w\mapsto \int_{B_1}|Dw|^2 \ \mathrm{d}x, \tag{1.1.10}$$

where $\mathcal{M}\subset\mathbb{R}^n$ and $\mathcal{N}\subset\mathbb{R}^N$ are sufficiently regular submanifolds, then the situation greatly improves. In [203] it is observed that in two dimensions, a minimizer of (1.1.10) is Hölder continuous and, by classical results on (unconstrained) harmonic maps, smooth if $\mathcal{M}$ and $\mathcal{N}$ are smooth, while in [95] it was proved that, if $\mathcal{N}$ is compact and has non-positive curvature, any homotopy class of maps from closed $\mathcal{M}$ to $\mathcal{N}$ has a smooth harmonic representative. Moreover, if the image of a minimizer is contained in a convex ball of the target manifold or in a single coordinate chart, then it is possible to recover the majority of the techniques available for the unconstrained case and suitably modify them in order to deal with the complications mostly due to the curvature of the constraint. In this perspective, a complete existence and regularity theory can be found in [115, 118, 141, 142]. A far more general geometric framework is considered in [230] where partial regularity is proved for minimizers of functionals of the type "energy plus lower order terms" defined on maps between two Riemannian manifolds. The strategy followed here heavily relies on the substantial energy structure of the integrand, which turns out to be fundamental for deriving a scaling inequality needed for the construction of smooth maps approximating a minimizer and satisfying the constraint condition. The regularity theory is then obtained by showing that the energy has the right decay to obtain, by Morrey's Lemma, that solutions are locally Hölder continuous whenever their energy is suitably small. Once Hölder continuity is available, it is possible to localize the problem in a single chart and dealing with it as in the unconstrained case [115, 118]. The regularity theory for harmonic maps is completed in [231], where is proved partial regularity up to the boundary and dimension reduction of the singular set of solutions to the Dirichlet problem

$$g+\left(W^{1,2}_0(\Omega,\mathbb{R}^N)\cap W^{1,2}(\Omega,\mathcal{N})\right)\ni w\mapsto \int_{\Omega}|Dw|^2 \ \mathrm{d}x$$

$$g \in W^{1,\infty}(\bar{\Omega}, \mathcal{N}) \quad \text{with} \quad \|Dg\|_{L^\infty(\bar{\Omega})} < \Lambda < 1.$$

In particular, if Λ is sufficiently small, the regularity of the boundary datum propagates around $\partial\Omega$; in other terms, the singular set is strictly contained inside Ω. It is also worth mentioning [179], which treats a general class of functionals with nice blow-ups. The novelty in this approach mainly consists in the construction of comparison maps which involves retractions in Euclidean space. This, in the case of the Dirichlet integral, simplifies the proof in [230]. In [131, 132] is presented a theory analogous to the one contained in [230, 231] for p-harmonic maps, i.e. manifold-valued minimizers of the p-energy

$$W^{1,p}(\mathcal{M}, \mathcal{N}) \ni w \mapsto \int_{\mathcal{M}} |Dw|^p \ \mathrm{d}x$$

as well as partial regularity for constrained minimizers of general functionals of the p-Laplacian type when a certain number of homotopy groups of the target manifold is trivial or at most finite. Such a topological characterization plays a fundamental role also in the question of density of smooth maps in Sobolev spaces defined on manifolds [21, 26, 27, 31, 197], in the extension problem for manifold valued Sobolev maps [20, 22, 26, 197, 198] and in the regularity theory for geometric constrained minima of non-homogeneous variational integrals [52, 67, 69, 75, 90]. In [93] a purely PDE approach for critical systems including the case of manifold-valued p-harmonic maps is proposed: if solutions have sufficiently small energy, then they are $W^{1,p}$-close to an unconstrained p-harmonic map. Then, it is almost straightforward to prove that the p-energy has the right decay, and, as a consequence, solutions are regular outside a negligible set. Analogous results have been obtained in [103] in the autonomous, quadratic quasiconvex case via blow-up techniques and later on extended to holonomic minimizers of general quasiconvex functionals with p-growth [151]: here the question of partial regularity is handled by showing approximate harmonicity for local minimizers. This permits to compare them to solutions of bounded, linear elliptic systems with constant (frozen) coefficients for deriving estimates which show that a suitable excess function has the right decay. As a result, solutions have Hölder continuous gradient away from a relatively closed set of zero n-dimensional Lebesgue measure.
The pathological behavior of weak solution of (1.1.9) lies in the critical growth of the term on the right-hand side. Therefore, when studying systems of the type

$$-\operatorname{div}(\tilde{a}(|Du|)Du) = f \quad \text{in } \Omega, \tag{1.1.11}$$

it is natural to look for optimal conditions to impose on the term on the right-hand side to assure Lipschitz regularity for solutions. This problem has been studied in several papers for vector fields of the p-Laplacian type, i.e. $\tilde{a}(|z|)z \cdot z \sim |z|^p$, see [9, 94, 166, 169, 170] for local regularity results and [56–59] for global ones. The final outcome is that, if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of (1.1.11), then

$$f \in L(n,1)(\Omega, \mathbb{R}^N) \implies Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}). \tag{1.1.12}$$

This amounts to require that

$$\|f\|_{L(n,1)(\Omega)} := \int_0^\infty |\{x \in \Omega\colon f(x) > \lambda\}|^{\frac{1}{n}} \ \mathrm{d}\lambda < \infty. \tag{1.1.13}$$

Such assumption is optimal already in the linear case, when $\tilde{a} \equiv 1$, [55]. For nonlinear problems of p-Laplacian type, the implication in (1.1.12) can be considered as the non-linear extension [169] of [235], which states that any function whose gradient belongs to $L(n,1)$ is continuous and classically differentiable almost everywhere. This result is sharp with respect to immersions because of the strict inclusion

$$L^{n+\sigma} \subset L(n,1) \subset L^n \quad \text{for all } \sigma > 0,$$

see [64], and Morrey's embedding theorem

$$Du \in L^{n+\sigma} \Rightarrow u \in C^{0,\frac{\sigma}{n+\sigma}}.$$

Hence, recalling that Lorentz spaces are interpolation spaces and using classical Calderón-Zygmund theory, it follows that solutions of the Poisson equation have locally continuous gradient under the assumption:

$$\Delta u \in L(n, 1).$$

In particular $\Delta u \in L^n$ does not guarantee continuity, see [55] for issues of optimality. Finally, it is important to stress a remarkable aspect of (1.1.13): it is independent of the operator displayed in (1.1.11); in other terms, no references to p appear there.

1.2 Non-standard growth

Given any variational integral of type

$$W^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega F(Dw) \ \mathrm{d}x \tag{1.2.1}$$

with $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying $z \mapsto F(z) \in C^2(\mathbb{R}^{N\times n} \setminus \{0\}) \cap C^1(\mathbb{R}^{N\times n})$, the quotient

$$\mathcal{R}(z) := \frac{\text{maximum eigenvalue of } \partial_{zz}F(z)}{\text{minimum eigenvalue of } \partial_{zz}F(z)} \tag{1.2.2}$$

defines the ellipticity ratio associated to $F(\cdot)$, a quantity which essentially measures the unbalance in growth of the integrand. A quick check assures that functionals satisfying (1.1.5) have ellipticity ratio bounded from above and below by positive constants depending only from (ν, L, p). Such feature is visible in case of standard p-growth, and, more generally, it is satisfied by the Orlicz functionals examined in [9, 68, 87, 89, 90, 177] under assumptions analogous to (1.1.5), this time with the p-Laplacian structure t^p replaced by the more general (and possibly non-homogeneous) N-function $\varphi(t)$, see [87, 212] for a more detailed discussion. When $\mathcal{R}(z) \sim$ const, the functional is said to be uniformly elliptic. However, there are several model energies available in the literature which do not verify condition $\mathcal{R}(z) \sim$ const: their ellipticity ratio depends on the value of z and may blow up as $|z| \to \infty$. This phenomenon is called non-uniform ellipticity. Consider, for instance, the simplest functional with fast exponential growth:

$$W^{1,p}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega \exp(|Dw|^p) \ \mathrm{d}x, \tag{1.2.3}$$

for which $\mathcal{R}(z) \sim |z|^p$ or, more generally, given any Orlicz function φ whose derivarives do not satisfy the Δ_2-condition, i.e. $\varphi'(t) \not\sim t\varphi''(t)$. In this case, the ellipticity ratio associated to the integral

$$W^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega \varphi(|Dw|) \ \mathrm{d}x$$

is

$$\mathcal{R}(z) \sim \frac{\max\left\{\varphi''(|z|), |z|^{-1}\varphi'(|z|)\right\}}{\min\left\{\varphi''(|z|), |z|^{-1}\varphi'(|z|)\right\}}$$

and no information is available *a priori* on the behavior of $\mathcal{R}(z)$ for large values of z. Another prominent example of non-uniformly elliptic functional is the class of integrals with (p, q)-growth. Precisely, those are functionals of type (1.2.1) whose integrand $F(\cdot)$ verifies

$$\begin{cases} \nu(\mu^2+|z|^2)^{\frac{p}{2}} \le F(z) \le L\left[(\mu^2+|z|^2)^{\frac{p}{2}} + (\mu^2+|z|^2)^{\frac{q}{2}}\right] \\ \nu(\mu^2+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \le \partial_{zz}F(z)\xi\cdot\xi \\ |\partial_{zz}F(z)| \le L\left[(\mu^2+|z|^2)^{\frac{p-2}{2}} + (\mu^2+|z|^2)^{\frac{q-2}{2}}\right] \end{cases} \tag{1.2.4}$$

for all $z,\xi \in \mathbb{R}^{N\times n}$ and some exponents $1<p\le q<\infty$. The systematic study of these variational problems started with Marcellini's seminal works [181,183] and, subsequently, has undergone an intensive development, [1,17,18,28,29,44,45,53,67,70,74,77,97–101,105,112,144,174,184–187]. The quite general structure described by (1.2.4) is aimed at unifying the regularity theory for several models with non-standard growth, such as

$$\mathcal{F}(w,\Omega) := \int_\Omega \left[|Du|^p + \sum_{i=1}^n |\partial_{x_i} u|^q \right] \, \mathrm{d}x \quad \text{with} \quad 1<p\le q<\infty$$

or

$$\mathcal{G}(w,\Omega) := \int_\Omega \sum_{i=1}^k |Du|^{q_i} \, \mathrm{d}x \quad \text{where} \quad 1<p\le q_1 \le \cdots \le q_k < \infty.$$

It is evident from $(1.2.4)_{3,4}$ that for functionals with (p,q)-growth the ellipticity ratio behaves as

$$1 \lesssim \mathcal{R}(z) \lesssim 1+|z|^{q-p}, \tag{1.2.5}$$

therefore it is reasonable to expect that, to get regularity, the exponents p and q cannot be too far apart. Precisely, in the autonomous case,

$$1 \le \frac{q}{p} \le 1+o(n), \tag{1.2.6}$$

where $o(n) \to_{n\to\infty} 0$. This is not only justified in the light of (1.2.5), but (1.2.6) turns out to be a necessary and sufficient condition for regularity, see [114,147,184]. Among these, it is worth recalling the counterexample appearing in [184], where is shown that the unbounded function

$$u(x) := c x_n^{\frac{q}{q-p}} \left(\sum_{i=1}^{n-1} x_i^2 \right)^{-\frac{p}{2(q-p)}},$$

for large values of c is a weak solution of equation

$$\sum_{i=1}^n \partial_{x_i} \left[\left(\sum_{s=1}^{n-1} \partial_{x_s} u \right)^{\frac{p-2}{2}} \partial_{x_i} u \right] + \partial_{x_n} \left(|\partial_{x_n} u|^{q-2} \partial_{x_n} u \right) = 0, \tag{1.2.7}$$

provided that the condition

$$\frac{q}{p} \le \frac{n-1}{n-1-p} \tag{1.2.8}$$

is violated. Recently, in [144] was shown that if p,q satisfy (1.2.8), then minima of (1.2.1) under assumption $(1.2.4)_1$ are locally bounded, so (1.2.8) is sharp, see also [17]. At this point, it is natural to object that equation (1.2.7) is degenerate elliptic in the x_n-direction, i.e., it loses ellipticity when $\partial_{x_n} u$ approaches zero, so this may produce unbounded solutions, rather than the too large distance between p and q. Such issue was fixed in [147], where is shown that a violation of (1.2.8) may result in finding a map

$$u(x) := \sqrt{\frac{n-4}{24}} \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i^2}} - \frac{2}{n-2} \sqrt{\frac{n-4}{24}} \sqrt{\sum_{i=1}^{n-1} x_i^2} \qquad n \ge 6,$$

unbounded on a whole line, which minimizes the regular, non-degenerate variational integral

$$W^{1,2}(\Omega) \ni w \mapsto \int_{\Omega} \left[|Dw|^2 + \frac{1}{2}|Dw|^4 \right] \, dx.$$

Hence, the problem of identifying the optimal relation between p and q which assures that minimizers of (1.2.1) are locally bounded is settled in the scalar case and for vector-valued minima of functionals with radial structure (1.1.8). However, the same cannot be said on the boundedness of the gradient. In [183, 184] was originally proved under assumptions (1.1.8) and (1.2.4) that any $W^{1,p}$-minimizer of (1.2.1) is Lipschitz continuous provided that

$$\frac{q}{p} < 1 + \frac{2}{n}. \tag{1.2.9}$$

The bound in (1.2.9) has a purely interpolative nature and was confirmed by successive papers [16, 99, 100]. However, no counterexamples to gradient boundedness are available, therefore the question of finding the optimal bound for Lipschitz continuity of minima is still open. Condition (1.2.9) was improved for scalar minima in [18] to

$$\frac{q}{p} < 1 + \min\left\{1, \frac{2}{n-1}\right\} \qquad 2 \le p \le q < \infty. \tag{1.2.10}$$

The advance of [18] with respect to [184] stays in a refinement of Moser's iteration technique via optimization on radial cut-off functions, allowing the use of Sobolev inequality on spheres rather than on balls. In this respect, in [73] we propose a new approach to functionals with (p, q)-growth based on convex duality. Precisely, we formulate the growth from above of the second derivative of $F(\cdot)$ in terms of the stress tensor $\partial_z F(\cdot)$ and exploit the strong convexity of both $F(\cdot)$ and of its Fenchel conjugate $F^*(\cdot)$ to gain more informations on the ellipticity of $F(\cdot)$ and, consequently, obtaining higher differentiability of minima under weaker assumptions than those in (1.2.10). Once proven that local minima belong to $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, it is possible to exploit the usual Moser iteration technique appearing in [183] to obtain Lipschitz continuity for minima, provided that $F(\cdot)$ satisfies (1.1.8) in the purely vectorial setting $N > 1$ and the exponents (p, q) verify

$$2 \le p \le q < \frac{np}{n-2} \ \text{ if } \ n \ge 3 \qquad \text{and} \qquad 2 \le p \le q < \infty \ \text{ if } \ n = 2.$$

We also obtain a Morrey-type regularity result in two space dimensions for local minima of non-degenerate functionals in the spirit of [25]. When considering non-autonomous integrals like

$$W^{1,p}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_{\Omega} F(x, Dw) \, dx, \tag{1.2.11}$$

where $F(\cdot)$, in addition to (1.2.4) in the z-variable satisfies also

$$|\partial_z F(x_1, z) - \partial_z F(x_2, z)| \le L|x_1 - x_2|^{\alpha} \left[(\mu^2 + |z|^2)^{\frac{p-1}{2}} + (\mu^2 + |z|^2)^{\frac{q-1}{2}} \right] \tag{1.2.12}$$

for all $x_1, x_2 \in \Omega$ and some $\alpha \in (0, 1]$, the situation becomes much more involved. This time, the functional in (1.2.11) cannot be in general considered a perturbation of the one in (1.2.1) as done in [115], since whether $F(\cdot, Dw)$ is integrable or not, depends on the interaction of Dw with the space-depending coefficient. Moreover, $(1.2.4)_1$ does not give any control on the $W^{1,q}$-norm of Dw. In this framework it is reasonable to expect the occurrence of Lavrentiev phenomenon, i.e.:

$$\inf_{w \in W^{1,p}(\Omega, \mathbb{R}^N)} \int_{\Omega} F(x, Dw) \, dx < \inf_{w \in W^{1,p}(\Omega, \mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)} \int_{\Omega} F(x, Dw) \, dx, \tag{1.2.13}$$

which is by the way excluded in the autonomous setting by convexity (prescribed by $(1.2.4)_2$). Following [32, 101] we can restate (1.2.13) as follows: the Lavrentiev Phenomenon occurs for a map $w \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ when it is not possible to find a sequence of more regular maps $\{w_j\} \subset W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ such that

$$w_j \rightharpoonup w \ \text{ in } \ W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \quad \text{and} \quad \int_B F(x,Dw_j)\ \mathrm{d}x \to \int_B F(x,Dw)\ \mathrm{d}x \tag{1.2.14}$$

for all open subsets $B \Subset \Omega$, see also [70] for an adaptation of this concept to obstacle problems and [77] for the case of non-autonomous functionals with unbalanced growth almost linear from below. Concerning minima of (1.2.11), in [101] is proven that (1.2.14) is equivalent to higher Sobolev regularity. In this respect, it reasonable to foresee that the regularity of the partial map $x \mapsto F(x,\cdot)$ influences in a subtle yet quantifiable way the size of the gap q/p. Precisely, as in the autonomous case, the ratio q/p must be controlled from above by a quantity slightly larger than one, this time depending also on the Hölder exponent appearing in (1.2.12):

$$\frac{q}{p} \le 1 + \frac{\alpha}{n}. \tag{1.2.15}$$

Let us compare the restriction in (1.2.9) with the one displayed above. Clearly, there is a gap between the two bounds: even in the best possible scenario $\alpha = 1$, (1.2.9) and (1.2.15) do not coincide and (1.2.15) is more restrictive than (1.2.9) due to the ineludible presence of an x-component. All those informations are sharply encoded in the Double Phase energy, defined as

$$W^{1,p}(\Omega,\mathbb{R}^N) \ni w \mapsto \mathscr{P}(w,\Omega) := \int_\Omega \left[(\mu^2+|Dw|^2)^{\frac{p}{2}} + a(x)(\mu^2+|Dw|^2)^{\frac{q}{2}}\right]\ \mathrm{d}x \tag{1.2.16}$$
$$1 < p \le q, \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega), \quad \mu \in [0,1].$$

In fact, as soon as (1.2.15) is violated, we can see that there exists a minimizer of (1.2.16) $u \in W^{1,p} \setminus W^{1,q}_{\mathrm{loc}}$ with a one-point singularity which prevents it from being highly Sobolev-regular [101]. Furthermore, elaborating on some constructions of Zhikov [247] in the context of Lavrentiev phenomenon, in [106] is found that if (1.2.15) does not hold, for any $\varepsilon > 0$, there exists a functional as in (1.2.16) having a minimizer $u \in W^{1,p}$ whose singular set is a fractal of Cantor type with almost maximal Hausdorff dimension:

$$n - p \ge \dim_{\mathscr{H}}(\Sigma(u)) > n - p - \varepsilon. \tag{1.2.17}$$

Recalling that the Hausdorff dimension of the set of non-Lebesgue point of $W^{1,p}$-maps is at the most $n-p$, (1.2.17) clearly states that u is nearly as bad as any of its competitors. Also in the non-autonomous case, functionals with (p,q)-growth received lots of attention, see [3, 11, 12, 51, 60–62, 70, 77–79, 96–98, 101, 144, 146, 174, 190] and [194] for a survey. The class of non-autonomous variational integrals with (p,q)-growth includes several models of crucial relevance in materials science and in fluid mechanics, [246–249]. For instance, the $p(x)$-laplacian

$$W^{1,p}(\Omega,\mathbb{R}^N) \ni w \mapsto \mathscr{E}(w,\Omega) := \int_\Omega |Dw|^{p(x)}\ \mathrm{d}x \tag{1.2.18}$$
$$1 < \inf_{x\in\Omega} p(x) \le p(\cdot) \le \sup_{x\in\Omega} p(x) < \infty,$$

studied in [3, 4, 52, 63, 69, 222–225, 237] provides a useful paradigm for the analysis of electrorheological fluids, while the Double Phase energy in (1.2.16), examined in [10, 12, 51, 60–62, 72, 75, 78, 80], describes the behavior of strongly anisotropic materials whose hardening properties, linked to the exponents ruling the growth of the gradient variable, drastically vary with the point. In the regularity perspective, functionals (1.2.18)-(1.2.16) exhibit quite a different behavior. The regularity of $p(x)$-harmonic maps is mainly affected by the modulus of continuity $\omega_p(\cdot)$ of the exponent:

- if $\lim_{\varrho\to 0}\omega_p(\varrho)\log(\varrho^{-1})>0$, then minima are locally β_0-Hölder continuous for some $\beta_0\in(0,1)$;
- if $\lim_{\varrho\to 0}\omega_p(\varrho)\log(\varrho^{-1})=0$, then minima are locally β_0-Hölder continuous for all $\beta_0\in(0,1)$;
- if $\omega_p(\varrho)\sim\varrho^{\alpha}$ for some $\alpha\in(0,1]$, then minima are locally C^{1,β_0}-regular, for some $\beta_0\in(0,1)$,

see [3, 63, 96, 223]. Also the question of density of smooth maps in the Musielak-Orlicz-Sobolev space $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is strongly linked to the modulus of continuity of $p(\cdot)$: in [247, 249] is proven that if $\omega_p(\varrho)=\log(\varrho^{-1})^s$, then when $s\ge 1$ smooth maps are $W^{1,p(\cdot)}$-dense, while for $s\in(0,1)$ density may fail. On the other hand, the regularity of minima of the Double Phase energy is shaped by a very subtle interplay between the Hölder continuity exponent of the modulating coefficient $a(\cdot)$, the exponents (p,q) and further assumptions imposed *a priori* on minima. In fact, let $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of (1.2.16),

- if $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ and (1.2.15) is in force, then $Du\in C^{0,\beta_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$;
- if $u\in W^{1,p}(\Omega,\mathbb{R}^N)\cap L^{\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and $q<p+\alpha$ (large inequality allowed in the scalar case $N=1$), then $Du\in C^{0,\beta_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$;
- if $u\in W^{1,p}(\Omega,\mathbb{R}^N)\cap C^{0,\gamma}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ for some $\gamma\in(0,1]$ and $q<p+\frac{\alpha}{1-\gamma}$, then $Du\in C^{0,\beta_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$,

cf. [12, 60, 61, 75]. The problem of approximation with smooth maps in $W^{1,H(\cdot)}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, the Musielak-Orlicz-Sobolev space built on the Double Phase integrand $H(x,z):=[|z|^p+a(x)|z|^q]$, is again governed by a strict mesh between p,q and α: it turns out that only (1.2.15) assures that regular maps are $W^{1,H(\cdot)}$-dense [101], otherwise, any violation of (1.2.15) results in the possible occurrence of Lavrentiev phenomenon [7, 101, 106]. Models (1.2.16)-(1.2.18) have been recently unified within the concept of Musielak-Orlicz functional

$$W^{1,\varphi(\cdot)}(\Omega,\mathbb{R}^N)\ni w\mapsto\mathcal{F}(w,\Omega):=\int_{\Omega}\varphi(x,|Dw|)\ \mathrm{d}x \tag{1.2.19}$$

introduced in [212] and then intensively developed in a series of papers [5, 13, 75, 135–138, 146] with the scope of carrying on (1.2.19) the majority of the regularity results already available for the usual p-Laplacian integral. Precisely, in [5] is treated the question of density of smooth maps in the Musielak-Orlicz-Sobolev space $W^{1,\varphi(\cdot)}(\Omega,\mathbb{R}^N)$, in [137] Harnack inequalities for quasiminimizers of (1.2.19) are shown and in [146], optimal C^{1,β_0}-regularity is proven for scalar minima of (1.2.19). The strategy here mainly relies on the comparison on balls $B\Subset\Omega$ of the original minimum with the solution of a Dirichlet problems defined by means of variational integrals having as an integrand an autonomous N-function constructed in such a way that it shares the same regularity as $t\mapsto\varphi(\cdot,t)$ and it is equivalent to $\inf_{x\in B}\varphi(x,t)$. Such a procedure allows transferring the regularity from the solutions of the auxiliary boundary value problems [89, 177], to minima of (1.2.19).
The previous discussion enlightens that the matter of regularity for variational integrals with non-standard growth has been deeply investigated and very well understood. However, the question of manifold constrained problems, apart for [90] in the setting of N-functions, remained almost untouched. This topic has been covered by [52, 67, 69, 75] for the first time in the case of non-autonomous functionals with non-standard growth. Precisely, [67] deals with the possible occurrence of Lavrentiev phenomenon in manifold-valued Sobolev spaces defined by means of non-autonomous energies, in [52, 69] are presented sharp dimension estimates on the singular set, interior and up to the boundary partial regularity theory for $p(x)$-harmonic map with values in

manifiold and [75] deals with the matter of partial regularity for sphere-valued minima of general functionals with Double Phase structure of type

$$W^{1,p}(\Omega,\mathbb{S}^{N-1})\ni w\mapsto\int_\Omega b(x,w)\left[|Dw|^p+a(x)|Dw|^q\right]\ \mathrm{d}x.$$

Since the Double Phase integrand features a strong degree of inhomogeneity, we cannot exploit classical monotonicity arguments as in [69,132,230,237] to obtain an accurate dimension analysis of the singular set of minima, therefore we introduce new intrinsic Hausdorff measures aimed at capturing the local geometry of the integrand and then compare them with the natural capacities [13] generated by (1.2.16), and relate the corresponding outcomes to the size of the singular sets. It turns out that singular sets consists of two regions: the one crossing the zero set of the modulating coefficient, which resembles the singular set of p-harmonic maps and its complementary, analogous to the singular set of q-harmonic maps, in perfect accordance with the structure of the Double Phase energy. By now, this is the best description available of the singular set of minima of general functionals with Double Phase. Also non-homogeneous problem (1.1.11) has been studied at length in the literature under uniform ellipticity assumptions, i.e.:

$$\begin{cases} -1<i_a\le\frac{\tilde a'(t)t}{\tilde a(t)}\le s_a<\infty \quad \text{for all}\ \ t>0\\ \tilde a\colon(0,\infty)\to[0,\infty)\ \ \text{is of class}\ \ C^1_{\mathrm{loc}}(0,\infty).\end{cases}\tag{1.2.20}$$

The assumptions listed in (1.2.20) essentially describe operators of the p-Laplacian type or governed by a general n-function. By convexity, solutions of (1.1.11) are local minimizers of the variational integral

$$w\mapsto\int_\Omega\left[\tilde A(|Dw|)-fw\right]\ \mathrm{d}x,\qquad \tilde A(t):=\int_0^t\tilde a(s)s\ \mathrm{d}s.$$

As a direct consequence of $(1.2.20)_1$, the doubling property

$$\tilde A(2t)\le c(i_a,s_a)\tilde A(t)\tag{1.2.21}$$

immediately follows. Moreover, $(1.2.20)_1$ also yields that $\tilde A(\cdot)$ has a growth of power type:

$$t^{i_a+2}\lesssim\tilde A(t)\lesssim t^{s_a+2},$$

therefore the theory contained in [9,56–59,94,166,169,170] does not include functionals with fast growth such as (1.2.3), or

$$W^{1,1}(\Omega,\mathbb{R}^N)\ni w\mapsto\int_\Omega\left[(\exp(\cdots(\exp(\exp(|Dw|^{p_0}))^{p_1})^{p_2})\cdots)^{p_k}-f\cdot w\right]\ \mathrm{d}x\tag{1.2.22}$$

and

$$W^{1,1}(\Omega,\mathbb{R}^N)\ni w\mapsto\int_\Omega\left[\left(\exp(\cdots\exp(\gamma_1(x)\exp(\gamma_0(x)|Dw|^{p_0(x)}))^{p_1(x)})\cdots\right)^{p_k(x)}-f\cdot w\right]\ \mathrm{d}x\tag{1.2.23}$$

with

$$\begin{cases} f\in L(n,1)(\Omega,\mathbb{R}^N) & \text{if}\ \ n\ge3\\ f\in L^2(\mathrm{LogL})^\alpha(\Omega,\mathbb{R}^N)\ \ \text{with}\ \ \alpha>2 & \text{if}\ \ n=2,\end{cases}\tag{1.2.24}$$

for which (1.2.21) fails. Exponential type functionals likely provide the best example to test how far one can go in relaxing uniform ellipticity conditions. From the point of view of regularity theory the functional (1.2.3) for $p=\mathsf{o}(n)$ close to zero has been considered in [185]. This reflects the difficulty in considering exponentials in connection to different growth conditions. The plain functional (1.2.3) with $p = 2$ was first studied in [91, 178]. As for the gradient regularity, paper [188] contains results for functionals featuring an arbitrary large composition of exponentials with constant coefficients as (1.2.22) with $f \equiv 0$. Such type of functional was studied recently in [16], in the non-homogeneous case $f \not\equiv 0$. As it is clear from (1.2.3), the growth and the variability inside the exponential strongly modify the rate of growth. When passing to the case with coefficients, this phenomenon is magnified in a dramatic way. To give an example, let us consider the simplified version of (1.2.23):

$$W^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega \exp(c(x)|Dw|^2)\ \mathrm{d}x.$$

Any minimizer $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ makes this energy finite, nevertheless it may fail to satisfy

$$\int_\Omega \exp(c(x_0)|Du|^2)\ \mathrm{d}x < \infty$$

for a fixed $x_0 \in \Omega$. This basic fact implies that standard perturbation theory does not apply in this case and makes the regularity theory completely non-trivial for fast growth functionals. In other words, dependence on coefficients becomes very sensible in the fast growth case. The difficulties leading to consider small p in (1.2.3), further strengthen considering functionals as

$$W^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega \exp(c(x)|Dw|^{p(x)})\ \mathrm{d}x.$$

Eventually, any composition of exponentials leads to a stronger form of nonuniform ellipticity, magnifying it at every stage of composition, which is the case of (1.2.23). Finding assumptions in order to overcome problems deriving from the simultaneous presence of several exponentials and coefficients is therefore a major challenge here. In turn, such functionals suggest the shape of the most general forms of nonuniform ellipticity to consider when deriving a general theory. These exponential type functionals are useful when dealing for instance with energy approximations of supremum norms. For instance, in [102] are considered functionals of type

$$W^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega \exp(kH(x, Du))\ \mathrm{d}x, \tag{1.2.25}$$

where $H(\cdot)$ is a quadratic, regular Hamiltonian in the context of weak KAM theory, in order to get an approximated variational principle for supremum type functionals. Integral (1.2.25) provides an extremely accurate approximation of supremal functionals when $k \to \infty$. Notice that (1.2.25) belongs to the same class of (1.2.23), but in [102] it can be treated only assuming Lipschitz continuity, and, more importantly, zero boundary conditions. The regularity theory for functionals like (1.2.23) was started in [190] in the non-iterated, homogeneous case and completed in full generality in [76]. As already mentioned, the presence of space-depending exponents and coefficients in a fast exponential growth regime makes the whole theory particularly delicate, since a pointwise control on the ellipticity ratio in terms of (arbitrarily small) powers of the underlying energy turns out to be crucial for reducing non-uniform ellipticity to uniform one in approximating schemes. In fact, in [76] we first need to adjust the definition of ellipticity ratio in such a way that it takes into account also the presence of x-depending coefficients:

$$\mathcal{R}(z) := \sup_{x \in B} \frac{\text{maximum eigenvalue of } \partial_{zz}F(x,z)}{\text{minimum eigenvalue of } \partial_{zz}F(x,z)}, \tag{1.2.26}$$

where $B \Subset \Omega$ is any open ball. We then combine a series of delicate techniques to first, earn enough integrability for the gradient of minima as to treat the part involving the space coefficients as a term of potential and then, via nonlinear iterations, obtain Lipschitz continuity under optimal assumptions on the datum f and on the degree of smoothness of the space-depending coefficients. We get also Lipschitz regularity results for minima of uniformly elliptic variational integrals under limiting regularity assumptions on the coefficients. An interesting phenomenon to be noticed in [76] is the duality (p,q)-growth versus structure: retaining only the growth in the large of the integrand and its derivatives as done in the (p,q) setting [97, 98], permits to prove regularity for a broad class of model energies, but this generality causes a severe loss of informations, which are instead preserved when considering the full structure of the integrand. For instance, let's have a look at non-homogeneous problems involving the Double Phase energy

$$W^{1,1}(\Omega,\mathbb{R}^N) \ni w \mapsto \int_\Omega [|Dw|p + a(x)|Dw|^q - f\cdot w] \, dx, \tag{1.2.27}$$

with f as in (1.2.24) and $a \in W^{1,d}(\Omega)$ for some $d > n$ rather than just Hölder continuous. The (p,q) approach would lead to

$$\frac{p}{q(q-1)} \le \mathcal{R}(z) \le \frac{2qNn}{\min\{p-1,1\}}\left[1 + \frac{q}{p}\|a\|_{L^\infty(B)}|z|^{q-p}\right],$$

so, according to Theorem 12 in Chapter 5, minima are Lipschitz continuous provided that

$$\frac{q}{p} < 1 + \min\left\{\frac{d-n}{nd}, \frac{4(p-1)}{\vartheta p(n-2)}\right\} \quad \text{with} \quad \vartheta := \begin{cases} 1 & \text{if } p \ge 2 \\ 2 & \text{if } 1 < p < 2 \end{cases}.$$

If we instead look at the full Double Phase structure, we immediately see that

$$\mathcal{R}(z) = \frac{2qNn}{\min\{p-1,1\}} \tag{1.2.28}$$

so, applying this time Theorem 13 we obtain Lipschitz continuity under condition

$$\begin{cases} \dfrac{q}{p} \le 1 + \dfrac{1}{n} - \dfrac{1}{d} & \text{if } n \ge 3 \\[2ex] \dfrac{q}{p} \le 1 + \dfrac{1}{2} - \dfrac{1}{d} \ \text{ and } \ q < p^2 & \text{if } n = 2, \end{cases} \tag{1.2.29}$$

which for $n \ge 3$ (or when $p \ge 3/2$ in the two-dimensional case) is precisely the Sobolev counterpart of the bound appearing in [12, 61] when $f \equiv 0$ and, evidently, (1.2.28) is worse than (1.2.29), especially for $p \in (1,2)$. Another prominent subject of interest in regularity theory is the validity of Calderón-Zygmund estimates for solutions of non-autonomous, non-homogeneous systems. In the specific, we shall focus on the Double Phase energy (1.2.16) and consider the equation

$$-\operatorname{div}\left(p|Du|^{p-2}Du + qa(x)|Du|^{q-2}Du\right) = -\operatorname{div}\left(|\mathfrak{F}|^{p-2}\mathfrak{F} + a(x)|\mathfrak{F}|^{q-2}\mathfrak{F}\right), \tag{1.2.30}$$

where $\mathfrak{F}\colon \Omega \to \mathbb{R}^n$ is a given vector field. This naturally occurs, for instance, as the Euler-Lagrange equation of the functional

$$W^{1,p}(\Omega) \ni w \mapsto \mathcal{P}(w,\Omega) - \int_\Omega \left[|\mathfrak{F}|^{p-2}\mathfrak{F} + a(x)|\mathfrak{F}|^{q-2}\mathfrak{F}\right]\cdot Dw \, dx.$$

The main point here is the inference of optimal integrability properties of Du in terms of those of the assigned datum $\mathfrak{F}$. This is quite a classical problem in the linear case $-\Delta u = -\operatorname{div}\mathfrak{F}$ and its solution, together with the analysis of general linear equations dates back to papers [41, 42]. The non-linear Calderón-Zygmund theory for non-homogeneous systems of the p-Laplacian type has been started in [155, 156], see also [40, 86, 128, 161, 195, 196] for important contributions to this line of investigation. Equations modelled on (1.2.30) have been studied in [62], where the sharp implication

$$\left[|\mathfrak{F}|^p + a(\cdot)|\mathfrak{F}|^q\right] \in L^{\gamma}_{\mathrm{loc}}(\Omega) \implies \left[|Du|^p + a(\cdot)|Du|^q\right] \in L^{\gamma}_{\mathrm{loc}}(\Omega) \quad \text{for all} \quad \gamma > 1 \tag{1.2.31}$$

is proved for any solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to (1.2.30), assuming that the strict inequality holds in (1.2.15). In [78] we show the validity of (1.2.31) also in the delicate borderline case

$$\frac{q}{p} = 1 + \frac{\alpha}{n}, \tag{1.2.32}$$

thus completing the Calderón-Zygmund theory for Double Phase problems.

1.3 Organization of the thesis

This thesis is organized as follows.

- In Chapter 2 we display our notation and collect some well-known results which will be recalled several times in the next chapters.
- Chapter 3 is devoted to the proof of partial regularity and capacity estimates of the singular set of sphere-valued minimizers of functionals modelled on (1.2.16), cf. [75].
- The content of Chapter 4 is based on [78], in which we complete the Calderón-Zygmund theory for the non-homogeneous equation (1.2.30) in the limiting case (1.2.32) left open in [62].
- In Chapter 5 we provide sharp Lipschitz regularity for vector-valued solutions to system (0.0.1). We allow very general growth conditions for the non-linear tensor $a(x, z)$: fast exponential growth and unbalanced power growth are both covered, see [76].
- In Chapter 6 we propose a new approach to variational problems with (p, q)-growth based on convex duality, cf. [73].

1.4 Other papers

In this last part of the introduction there is a short summary of the extra research made during my PhD studies, only vaguely related to the theme of the thesis. The main results obtained, together with the reference to the original paper for a complete treatment of the problem are listed below.

Flow of non-uniformly elliptic problems

Joint work with L. Beck (University of Augsburg) and G. Mingione (University of Parma)
***Preprint* (2020)**

We study the gradient flow of the iterated exponential functional

$$\mathcal{E}(w, \Omega) := \int_{\Omega} \left[\exp\left(\exp\left(\cdots \exp(|Dw|^p)\cdots\right)\right) - fw\right] \,\mathrm{d}x, \tag{1.4.1}$$

with

$$p > 1 \quad \text{and} \quad f \in L(n+2, 1), \tag{1.4.2}$$

where $n \geq 2$ is the space dimension. The elliptic counterpart of assumption $(1.4.2)_2$ reads as $f \in L(n,1)$ and it is sharp, see [55,169,235]. In the time-independent setting, variational integrals like (1.4.1) have been treated in [16,76]. In [15], we aim to prove Lipschitz bounds for solutions to the gradient flow of (1.4.1) via a rigorous analysis of the intrinsic parabolic De Giorgi classes associated to (1.4.1) coupled with delicate nonlinear potential theoretic techniques in the spirit of [16,76,160,167]. Our analysis covers also the gradient flow of functionals with (p,q)-growth.

Boundary regularity for manifold constrained $p(x)$-harmonic maps

Joint work with I. Chlebicka (University of Warsaw) and L. Koch (University of Oxford)

***Preprint* (2020) -** `https://arxiv.org/pdf/2001.06243.pdf`

In [52] we complete the partial regularity theory for manifold constrained $p(x)$-harmonic maps started in [68], by proving that solutions to the Dirichlet problem

$$g + \left(W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^N) \cap W^{1,p(\cdot)}(\Omega,\mathcal{M})\right) \ni w \mapsto \min \int_\Omega k(x)|Dw|^{p(x)}\ \mathrm{d}x$$

with

$$g \in W^{1,q}(\bar{\Omega},\mathcal{M}) \quad \text{for some} \quad q > \max\left\{\sup_{x\in\bar{\Omega}} p(x), n\right\},$$

are Hölder continuous around boundary points. In other terms, the singular set of solutions Σ is contained strictly inside Ω, i.e.: $\Sigma \cap \partial\Omega = \emptyset$.

Removable sets in non-uniformly elliptic problems

Joint work with I. Chlebicka (University of Warsaw)

***Ann. Mat. Pura Appl.* (2020) -** `https://doi.org/10.1007/s10231-019-00894-1`

In [51] we provide an upper bound to the maximal size of removable sets for equations of the form

$$-\operatorname{div} A(x,Du) = 0 \quad \text{in} \quad \Omega \setminus E,$$

where the nonlinear tensor $A(\cdot)$ has double-phase structure, in the sense that

$$A(x,z)\cdot z \sim \left[|z|^p + a(x)|z|^q\right].$$

Precisely, the magnitude of removable sets is quantified in terms of the intrinsic Hausdorff measures introduced in [77]. As a byproduct, we develop a regularity theory for solutions to the obstacle problem defined by means of Double Phase operators.

Fully nonlinear elliptic equations with non-homogeneous degeneracy

***Proc. Royal Soc. Edinburgh Math.* (2020) -** `https://doi.org/10.1017/prm.2020.5`

In [72] we prove $C^{1,\gamma}$-local regularity for viscosity solutions to problem

$$\left[|Du|^p + a(x)|Du|^q\right] F(D^2u) = f(x) \quad \text{in} \quad \Omega \quad \text{with} \quad 0 \leq a(\cdot) \in C(\Omega),$$

with inhomogeneous degeneracy term switching between two different powers according to the zero set $\{x \in \Omega\colon a(x) = 0\}$ of the modulating coefficient a. Our result is sharp in the light of the observation made in [152, Example 1]. It is in fact consistent with our case when $a(\cdot) \equiv 0$.

Gradient bounds for solutions to irregular parabolic equations with (p, q)-growth

***Preprint* (2020) -** `https://arxiv.org/pdf/2004.01452.pdf`

In [71] we study the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \text{ div } a(x, t, Du) = 0 & \text{in } \Omega_T \\ u = f & \text{on } \partial_{par}\Omega_T, \end{cases} \tag{1.4.3}$$

where the nonlinear tensor $a\colon \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ features (p, q)-growth, i.e.:

$$\begin{cases} \partial_z a(x,t,z)\xi \cdot \xi \gtrsim (\mu^2 + |z|^2)^{\frac{p-2}{2}} |\xi|^2 \\ |a(x,t,z)| + (\mu^2 + |z|^2)^{\frac{1}{2}} |\partial_z a(x,t,z)| \lesssim \left[(\mu^2 + |z|^2)^{\frac{p-1}{2}} + (\mu^2 + |z|^2)^{\frac{q-1}{2}}\right] \end{cases}$$

and it is only Sobolev-differentiable with respect to the space variable, in the sense that

$$|\partial_x a(x,t,z)| \lesssim \gamma(x,t)|z|^{q-1},$$

where γ possess a suitably high degree of integrability. In this setting, we prove the existence of regular solutions to (1.4.3) satisfying the following L^∞-L^p inequality on parabolic cylinders:

$$\|H(Du)\|_{L^\infty(Q_{\varrho/2})} \le \frac{c}{\varrho^{\beta_1}} \left[1 + \left(\fint_{Q_\varrho} H(Du)^{\frac{p}{2}} \, \mathrm{d}y\right)^{\beta_2}\right],$$

where $H(z) := (\mu^2 + |z|^2)$.

Higher Integrability for Constrained Minimizers of Integral Functionals with (p,q)-Growth in low dimension

***Nonlinear Anal.* (2018) -** `https://doi.org/10.1016/j.na.2017.12.007`

In [67] we study the minimization problem

$$W^{1,p}_{\text{loc}}(\Omega, \mathbb{S}^{N-1}) \ni w \mapsto \mathscr{F}(w, \Omega) := \int_\Omega F(x, Dw) \, \mathrm{d}x,$$

where the integrand $F(\cdot)$ has (p, q) growth and it is α-Hölder continuous in the x-variable, i.e.:

$$\begin{cases} |z|^p \lesssim F(x,z) \lesssim 1 + |z|^q & \text{for some } 1 < p \le q \\ |\partial_z F(x_1, z) - \partial_z F(x_2, z)| \lesssim |x_1 - x_2|^\alpha \left(1 + |z|^{q-1}\right) & \text{with } \alpha \in (0, 1] \\ \left(\mu^2 + |z_1|^2 + |z_2|^2\right)^{\frac{p-2}{2}} |z_1 - z_2|^2 \lesssim \left(\partial_z F(x, z_1) - \partial_z F(x, z_2)\right) \cdot (z_1 - z_2) & \text{for } \mu \in [0, 1]. \end{cases}$$

We prove that if Lipschitz-regular maps are dense in the energy space in the unconstrained case, i.e., for any given map $w \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$ with $\mathscr{F}(w, \Omega) < \infty$, there exists a sequence $\{w_j\} \subset \text{Lip}_{\text{loc}}(\Omega, \mathbb{R}^N)$ so that

$$w_j \to w \text{ in } W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \quad \text{and} \quad F(\cdot, w_j) \to F(\cdot, w) \text{ in } L^1_{\text{loc}}(\Omega)$$

then, given any function $v \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{S}^{N-1})$ we can construct a sequence $\{v_j\} \subset Lip_{\text{loc}}(\Omega, \mathbb{S}^{N-1})$ so that

$$v_j \to v \text{ in } W^{1,p}_{\text{loc}}(\Omega, \mathbb{S}^{N-1}) \quad \text{and} \quad F(\cdot, v_j) \to F(\cdot, v) \text{ in } L^1_{\text{loc}}(\Omega).$$

We also derive higher Sobolev regularity when the ambient space dimension is sufficiently low.

On the regularity of the ω-minima of φ-functionals

Nonlinear Anal. **(2019) -** `https://doi.org/10.1016/j.na.2019.02.017`

In [68] we study ε-regularity and fractional differentiability for the gradient of vector valued ω-minimizers of variational integrals of the type

$$W^{1,\varphi}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \ni w \mapsto \int_\Omega F(x,w,Dw)\ \mathrm{d}x,$$

whose prototypical model example is

$$W^{1,\varphi}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \ni w \mapsto \int_\Omega b(x,w)\varphi(|Dw|)\ \mathrm{d}x,$$

where φ is a Young function satisfying the Δ_2-condition.

Partial regularity for manifold constrained $p(x)$-harmonic maps

Calc. Var. & PDE **(2019) -** `https://doi.org/10.1007/s00526-019-1483-6`

In [69] we prove that manifold-constrained minima of the variable exponent energy

$$W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega,\mathcal{M}) \ni w \mapsto \int_\Omega |Dw|^{p(x)}\ \mathrm{d}x$$

are C^{1,β_0}-regular outside a negligible, "singular" set. We also analyze the structure of the singular set of minima and provide sharp bounds on its maximal Hausdorff measure.

Regularity results for a class of non-autonomous obstacle problems with (p,q)-growth

J. Math. Anal. Appl. **(2019) -** `https://doi.org/10.1016/j.jmaa.2019.123450`

In [72] we analyze the possible occurrence of Lavrentiev phenomenon in obstacle problems defined as

$$\mathcal{K}_{\psi,g}(\Omega) \ni w \mapsto \min \int_\Omega F(x,Dw)\ \mathrm{d}x,$$

where $F(\cdot)$ has (p,q)-growth and Sobolev-regular coefficients and class $\mathcal{K}_{\psi,g}(\Omega)$ is defined as

$$\mathcal{K}_{\psi,g}(\Omega) := \left\{ w \in W^{1,p}(\Omega)\colon w(x) \ge \psi(x) \text{ a.e. in } \Omega \text{ and } \ w|_{\partial\Omega} = g|_{\partial\Omega} \right\}.$$

We then determine sharp conditions which guarantee Lipschitz estimates for solutions.

Uniform ellipticity and p,q-growth

Joint work with F. Leonetti (University of L'Aquila)

Preprint **(2020) -** `https://arxiv.org/pdf/2003.06702.pdf`

In [74], for any two fixed numbers $1 < p \le q$, we give an example of integral functional enjoying uniform ellipticity and (p,q)-growth. The main point here is to emphasize that, for proving regularity of minima of variational integrals with non-standard growth, whenever possible, it is more convenient exploiting the full structure of the integrand rather than retaining only the growth from above and below of its second derivatives: this could lead to a severe loss of information. On this matter, compare the results exposed in [89, 174, 177] with those obtained in [183, 184].

On the regularity of minima of non-autonomous functionals

Joint work with G. Mingione (University of Parma)
***J. Geom. Analysis* (2020) - https://doi.org/10.1007/s12220-019-00225-z**

In [77], we collect a few general results on functionals with (p,q)-growth, i.e.,

$$W^{1,1}_{\rm loc}(\Omega) \ni w \mapsto \int_\Omega F(x, Dw)\ \mathrm{d}x \qquad |z|^p \lesssim F(x,z) \lesssim (1+|z|^2)^{\frac{q}{2}} \tag{1.4.4}$$

that extend those available in the literature in various directions. Precisely we consider conditions where the only possible polynomial bound from below in (1.4.4) allows for the case $p = 1$. In the first result, we relax the lower bound in (1.4.4) to allow nearly linear growth; in this case the model we have in mind is given by a functional of the type

$$w \mapsto \int_\Omega \left[|Dw|\log(1+|Dw|) + a(x)|Dw|^q\right]\ \mathrm{d}x \tag{1.4.5}$$
$$0 \le a(\cdot) \in W^{1,d}(\Omega), \quad d > n.$$

We determine the optimal condition guaranteeing local Lipschitz continuity for minimizers of (1.4.5). We also prove higher Sobolev regularity for *a priori* bounded minimizers of (1.4.4): the boundedness information on minima allows to relax the restriction on the ratio q/p, making it dimension-independent.

Hölder regularity for nonlocal Double Phase equations

Joint work with G. Palatucci (University of Parma)
***J. Differential Equations* (2019) - https://doi.org/10.1016/j.jde.2019.01.017**

In [80] we deal with non-local Double Phase equations; that is, a class of, possible singular and degenerate, integro-differential equations whose leading operator switches between two different fractional elliptic phases according to the zero set of the modulating coefficient $a(\cdot)$, i.e.:

$$\begin{aligned}\mathscr{L}u(x) :=& \mathrm{P.V.}\int_{\mathbb{R}^n} |u(x)-u(x+y)|^{p-2}(u(x)-u(x+y))|y|^{-n-sp}\,\mathrm{d}y\\ &+ \mathrm{P.V.}\int_{\mathbb{R}^n} a(x,y)|u(x)-u(x+y)|^{q-2}(u(x)-u(x+y))|y|^{-n-tq}\,\mathrm{d}y,\end{aligned}$$

with $0 \le a(\cdot) \in C(\Omega)$ and exponents (p,q) satisfying:

$$\begin{cases} p > \frac{1}{1-s} \ \text{ if } \ p < 2, \quad q > \frac{1}{1-t},\\ 1 \le \frac{q}{p} \le \frac{s}{t}, \quad \frac{q}{p} < 1 + \frac{p-1}{p}.\end{cases}$$

Under these assumptions, we prove that bounded viscosity solution of the equation $\mathscr{L}u(x) = f(x)$, where $f \in C(\Omega)$, are local γ-Hölder continuous to some small $\gamma \in (0,1)$.

Regularity for multi-phase variational problems

Joint work with J. Oh (Kyungpook National University)
***J. Differential Equations* (2019) - https://doi.org/10.1016/j.jde.2019.02.015**

In [79], optimal C^{1,β_0}-local regularity is proven for minimizers of Multi Phase variational problems, i.e.,

$$W^{1,p}(\Omega) \ni w \mapsto \int_\Omega \left[|Dw|^p + \sum_{j=1}^k a_j(x)|Du|^{p_j}\right]\ \mathrm{d}x$$
$$0 \le a_j(\cdot) \in C^{0,\alpha_j}(\Omega) \ \text{ and } \ 1 < \frac{p_j}{p} \le 1 + \frac{\alpha_j}{n} \ \text{ for all } \ j \in \{1,\cdots,k\}.$$

The sharpness of this result is assured by the counterexamples contained in [7, 101, 106].

Acknowledgments

The work for this thesis has been supported by the Engineering and Physical Sciences Research Council (EPSRC): CDT Grant Ref. EP/L015811/1.

I would like to thank Professor J. Kristensen for his guidance and constant support and Professors P. Baroni, G. Mingione and G. Palatucci for several enlightening discussions and fruitful collaborations. I am also indebted to Professors I. Fonseca and A. Mondino for their availability at my defense and for their useful comments that have eventually led to a significant improvement of the presentation of this thesis.

Chapter 2

Preliminaries

In this chapter we collect some useful preliminaries such as a complete description of our notation and several well-known results of algebraic or analytical nature which will be helpful in the forthcoming chapters.

2.1 Notation

We denote by $\Omega \subset \mathbb{R}^n$ an open domain; additional restrictions can be considered. Since our estimates will be local, we shall always assume, without loss of generality, that Ω is also bounded. We denote by c a general constant larger than one. Different occurrences from line to line will be still denoted by c. Special occurrences will be denoted by $c_1, c_2, \tilde{c}$ or likewise. Important dependencies on parameters will be as usual emphasized by putting them in parentheses. We shall denote $\mathbb{N}$ as the set of positive integers and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As usual, we denote by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball with center x_0 and radius $r > 0$; when it is clear from the context, we omit denoting the center, i.e., $B_r \equiv B_r(x_0)$. When not otherwise stated, different balls in the same context will share the same center. We shall also denote $B_1 = B_1(0)$ if not differently specified and set $\omega_n := |B_1|$. Finally, with B being a given ball with radius r and γ being a positive number, we denote by γB the concentric ball with radius γr and by $B/\gamma \equiv (1/\gamma)B$. In denoting several function spaces like $L^p(\Omega), W^{1,p}(\Omega)$, we shall denote the vector valued version by $L^p(\Omega, \mathbb{R}^k), W^{1,p}(\Omega, \mathbb{R}^k)$ in the case the maps considered take values in $\mathbb{R}^k$, $k \in \mathbb{N}$. We shall often abbreviate $L^p(\Omega, \mathbb{R}^k) \equiv L^p(\Omega), W^{1,p}(\Omega, \mathbb{R}^k) \equiv W^{1,p}(\Omega)$. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable subset with bounded positive measure $0 < |\mathcal{B}| < \infty$, and with $g \colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote the integral average of g over $\mathcal{B}$ by

$$(g)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} g(x)\ \mathrm{d}x := \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} g(x)\ \mathrm{d}x\ .$$

We next introduce some quantities which are strictly related to the vectorial framework. We denote $\{e^\alpha\}_{\alpha=1}^N$ and $\{e_i\}_{j=1}^n$ standard bases for $\mathbb{R}^N$ and $\mathbb{R}^n$, respectively; we shall always assume $n \geq 2$ and $N \geq 1$. The general second-order tensor of size (N, n) is defined as $\zeta = \zeta_i^\alpha e^\alpha \otimes e_i$ is identified with an element of $\mathbb{R}^{N \times n}$ (here we use the standard convention on the sum of repeated indices). The Frobenius product of second-order tensors z and ξ is defined as $z \cdot \xi = z_i^\alpha \xi_i^\alpha$; it follows that $\xi \cdot \xi = |\xi|^2$ and in the rest of the paper we shall use the classical Frobenius norm for matrices and tensors. We shall sometimes use the symbol "$\cdot$" also to denote the scalar product in $\mathbb{R}^n$. The gradient of a map $u = u^\alpha e^\alpha$ is thus defined as $Du = \partial_{x_i} u^\alpha e^\alpha \otimes e_i$, and the divergence of a tensor $\zeta = \zeta_i^\alpha e^\alpha \otimes e_i$ as $\operatorname{div}\zeta = \partial_{x_i} \zeta_i^\alpha e^\alpha$. When dealing with the integrands of the type $F \colon \mathbb{R}^{N \times n} \to [0, \infty)$ of the type considered in Chapter 1, we interpret the second differential of $\partial^2 F(z)$ as a fourth-order tensor defined as $\partial^2 F(z) = \partial_{z_j^\beta} \partial_{z_i^\alpha} F(z)(e^\alpha \otimes e_i) \otimes (e^\beta \otimes e_j)$, whenever $z \in \mathbb{R}^{N \times n}$.

2.2 Function spaces and weak differentiability

Let us proceed to present certain function spaces which will appear in the next chapters and their most relevant properties. For the sake of exposition, we shall report only definitions, some basic features and significant embeddings, and refer to [83, 85, 119, 123, 216, 239] for a fully detailed discussion. We start with a few elementary facts on fractional Sobolev spaces.

Definition 1 *Let $\alpha \in (0,1)$, $p \in [1,\infty)$, $k \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be an open subset with $n \geq 2$ (we allow for the case $\Omega = \mathbb{R}^n$). The fractional Sobolev space $W^{\alpha,p}(\Omega,\mathbb{R}^k)$ is defined prescribing that $w\colon \Omega \to \mathbb{R}^k$ belongs to $W^{\alpha,p}(\Omega,\mathbb{R}^k)$ iff the following Gagliardo type norm is finite:*

$$\begin{aligned}\|w\|_{W^{\alpha,p}(\Omega)} &:= \|w\|_{L^p(\Omega)} + \left(\int_\Omega\int_\Omega \frac{|w(x)-w(y)|^p}{|x-y|^{n+\alpha p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\\ &=: \|w\|_{L^p(\Omega)} + [w]_{\alpha,p;\Omega}\,.\end{aligned}$$

Accordingly, in the case $\alpha = [\alpha] + \{\alpha\} \in \mathbb{N} + (0,1) > 1$, we say that $w \in W^{\alpha,p}(\Omega,\mathbb{R}^k)$ iff the following quantity is finite

$$\|w\|_{W^{\alpha,p}(\Omega)} := \|w\|_{W^{[\alpha],p}(\Omega)} + [D^{[\alpha]}w]_{\{\alpha\},p;\Omega}\,.$$

The local variant $W^{\alpha,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$ is defined by requiring that $w \in W^{\alpha,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$ iff $w \in W^{\alpha,p}(\tilde{\Omega},\mathbb{R}^k)$ for every open subset $\tilde{\Omega} \Subset \Omega$.

Now we present an embedding theorem for fractional Sobolev spaces.

Lemma 2.2.1 *Let $f \in W^{\alpha,p}(\Omega,\mathbb{R}^k)$, with $p \geq 1$, $\alpha \in (0,1]$ such that $\alpha p < n$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

- *$\alpha p < n \Rightarrow f \in L^{\frac{np}{n-\alpha p}}(\Omega,\mathbb{R}^k)$ with $\|f\|_{L^{\frac{np}{n-\alpha p}}(\Omega)} \leq c\|f\|_{W^{\alpha,p}(\Omega)}$;*
- *$\alpha p = n \Rightarrow f \in L^t(\Omega,\mathbb{R}^k)$ for all $t \in [p,\infty)$, with $\|f\|_{L^t(\Omega)} \leq c\|f\|_{W^{\alpha,p}(\Omega)}$;*
- *$\alpha p > n \Rightarrow f \in C^{0,\frac{\alpha p-n}{p}}(\Omega,\mathbb{R}^k)$ with $\|f\|_{0,\frac{\alpha p-n}{p};\Omega} \leq c\|f\|_{W^{\alpha,p}(\Omega)}$;*

with c depending at the most from $(n,\alpha,p,t,[\partial\Omega]_{0,1},\mathrm{diam}(\Omega))$.

For a map $w\colon \Omega \to \mathbb{R}^k$ and a vector $h \in \mathbb{R}^n$, we denote by $\tau_h\colon L^1(\Omega,\mathbb{R}^k) \to L^1(\Omega_{|h|},\mathbb{R}^k)$ the standard finite difference operator pointwise defined as $\tau_h w(x) := w(x+h) - w(x)$, whenever $\Omega_{|h|} := \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > |h|\}$ is not empty.

Definition 2 *Let $\alpha \in (0,1)$, $p \in [1,\infty)$, $k \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be an open subset with $n \geq 2$. The Nikol'skii space $N^{\alpha,p}(\Omega,\mathbb{R}^k)$ is defined prescribing that $w \in N^{\alpha,p}(\Omega,\mathbb{R}^k)$ iff*

$$\|w\|_{N^{\alpha,p}(\Omega,\mathbb{R}^k)} := \|w\|_{L^p(\Omega,\mathbb{R}^k)} + \left(\sup_{|h|\neq 0}\int_{\Omega_{|h|}} \frac{|w(x+h)-w(x)|^p}{|h|^{\alpha p}}\,\mathrm{d}x\right)^{1/p}.$$

The local variant $N^{\alpha,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$ is defined by requiring that $f \in N^{\alpha,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$ iff $w \in N^{\alpha,p}(\tilde{\Omega},\mathbb{R}^k)$ for every open subset $\tilde{\Omega} \Subset \Omega$.

It is well-known that $W^{1,p}(\Omega,\mathbb{R}^k) \subsetneqq N^{\alpha,q}(\Omega,\mathbb{R}^k)$. A localized version of such embedding is contained in the next lemma.

Lemma 2.2.2 *Let $B_\varrho \Subset B_r \Subset \Omega$ be concentric balls with $r \le 1$, $1 < p < q < \infty$ and $w \in W^{1,p}(B_r, \mathbb{R}^k)$ with $k \ge 1$. Then there exists a positive constant $c \equiv c(n,k,p,q)$ such that*

$$\|w\|_{N^{\alpha,q}(B_\varrho)} \le c\left[\|Dw\|_{L^p(B_r)} + \frac{1}{r-\varrho}\|w\|_{L^p(B_r)}\right],$$

provided that

$$\alpha = 1 - n(p^{-1} - q^{-1}). \tag{2.2.1}$$

The quantity in (2.2.1) is always less or equal than one provided that $p \le q$. Moreover, when $p < n$ we see that α is positive provided that $q < p^*$, while, for $p \ge n$ any value of p and q assure that $\alpha > 0$.

Let us compare now the two spaces described by Definitions 1-2. We have that $W^{\alpha,p}(\Omega,\mathbb{R}^k) \subsetneqq N^{\alpha,p}(\Omega,\mathbb{R}^k) \subsetneqq W^{\beta,p}(\Omega,\mathbb{R}^k)$, for every $\beta < \alpha$, hold for sufficiently regular domains. A local, quantified version is in the next lemma, see for instance [6].

Lemma 2.2.3 *Let $B_r \subset \mathbb{R}^n$ be a ball with $r \le 1$, $w \in L^p(B_r,\mathbb{R}^k)$, $p > 1$ and assume that, for $\alpha \in (0,1]$, $S \ge 1$ and concentric balls $B_\varrho \Subset B_r$, there holds*

$$\|\tau_h w\|_{L^p(B_\varrho,\mathbb{R}^k)} \le S|h|^\alpha \quad \text{for every } h \in \mathbb{R}^n \text{ with } 0 < |h| \le \tfrac{r-\varrho}{K}, \text{ where } K \ge 1\,.$$

Then $w \in W^{\beta,p}(B_\varrho,\mathbb{R}^k)$ whenever $\beta \in (0,\alpha)$ and

$$\|w\|_{W^{\beta,p}(B_\varrho,\mathbb{R}^k)} \le \frac{c}{(\alpha-\beta)^{1/p}}\left(\frac{r-\varrho}{K}\right)^{\alpha-\beta} S + c\left(\frac{K}{r-\varrho}\right)^{n/p+\beta}\|w\|_{L^p(B_r,\mathbb{R}^k)}\,,$$

holds, where $c \equiv c(n,p)$.

An important fact about translation operators is their continuity in Lebesgue spaces.

Lemma 2.2.4 *Let $\tilde\Omega \Subset \Omega$ be any open set, $h \in \mathbb{R}^n$ be any vector with $|h| \in \left(0, \frac{1}{100}\operatorname{dist}(\tilde\Omega,\partial\Omega)\right)$ and $w \in L^p_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$ with $p \in [1,\infty)$ and $k \in \mathbb{N}$. Then*

$$\|w(\,\cdot + h) - w\|_{L^p(\tilde\Omega)} \to 0 \qquad \text{as} \quad |h| \to 0.$$

Furthermore, whenever $\Omega' \Subset \tilde\Omega$ is another open subset and $|h| \in \left(0, \frac{1}{100}\operatorname{dist}(\Omega',\partial\tilde\Omega)\right)$, there exists an absolute constant $c > 0$ such that

$$\int_{\Omega'}\left[|w(x)|^2 + |w(x+h)|^2\right]^{\frac{p}{2}}\,\mathrm{d}x \le c\int_{\tilde\Omega}|w|^p\,\mathrm{d}x.$$

Via the finite difference operator we can also define the difference quotient of a map $w\colon \Omega \to \mathbb{R}^k$ as

$$\Delta_h w(x) := \frac{w(x+h) - w(x)}{|h|} = |h|^{-1}\tau_h w(x) \quad \text{for} \quad x \in \Omega_{|h|}.$$

It is also useful to recall a basic property of difference quotient.

Lemma 2.2.5 *Let $w \in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$, $k \ge 1$, be any function. There holds that:*

- *If $p > 1$ and $\Omega' \Subset \Omega$ is any open set so that*

$$\sup_{|h|>0}|h|^{-p}\int_{\Omega'}|\tau_h w|^p\ \mathrm{d}x < \infty,$$

  then $Dw \in L^p(\Omega',\mathbb{R}^{k\times n})$ and $\tau_h w \to Dw$ strongly in L^p_{loc};

- *If in addition $w \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$, $\Omega'' \Subset \Omega$ is another open subset and $|h| < \frac{1}{100}\operatorname{dist}(\Omega'',\partial\Omega')$, then*

$$\int_{\Omega''}|\tau_h w|^p\ \mathrm{d}x \le c(n,s)|h|^p\int_{\Omega'}|Dw|^p\ \mathrm{d}x.$$

2.3 Nonlinear potentials

Let us describe now an important nonlinear theoretic potential quantity that will play a crucial role in Chapter 5, with related function spaces. Our main reference here is [16, Section 2]. The modified nonlinear Riesz potential of a map $g \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^k)$ is defined as

$$\mathbf{P}_1^g(x_0, r) := \int_0^r \left(\varrho^2 \fint_{B_\varrho(x_0)} |g|^2 \, \mathrm{d}x \right)^{1/2} \frac{d\varrho}{\varrho}, \tag{2.3.1}$$

for $x_0 \in \mathbb{R}^n$ and $r > 0$. In the nonlinear setting, the potential $\mathbf{P}_1^g(\cdot, r)$ replaces the truncated Riesz potential defined by

$$\mathbf{I}_1^g(x_0, r) := \int_0^r \frac{|g|(B_\varrho(x_0))}{\varrho^{n-1}} \frac{\mathrm{d}\varrho}{\varrho} = \omega_n \int_0^r \fint_{B_\varrho(x_0)} |g| \; \mathrm{d}x \, \mathrm{d}\varrho,$$

provided that the argument function g is at least locally L^1-regular. This can be easily see via Hölder inequality:

$$\mathbf{I}_1^g(x_0, r) \le \omega_n \mathbf{P}_1^g(x_0, r).$$

The potential in (2.3.1) turns out to be the right one to use when dealing with a large class of problems, from degenerate elliptic or parabolic PDE to fully nonlinear equations or systems of differential forms, [8, 65, 165, 232]. The potential $\mathbf{P}_1^g$ defines an operator which is well behaved in various function spaces. We shall focus on the Lorentz space $L(n, 1)$, defined by means of (1.1.13), when $n \ge 3$, and on the Orlicz space $L^2(\mathrm{LogL})^\alpha$, $\alpha \ge 0$, which can be characterized as

$$w \in L^2(\mathrm{LogL})^\alpha(\Omega, \mathbb{R}^k) \iff \int_\Omega |w|^2 \log(1 + |w|)^\alpha \; \mathrm{d}x < \infty. \tag{2.3.2}$$

The connection between such spaces and the potential $\mathbf{P}_1^g$ is given by the fact that, as an operator, $\mathbf{P}_1^g$ is bounded in $L(n, 1)(\Omega, \mathbb{R}^k)$ when $n \ge 3$, and in $L^2(\mathrm{LogL})^\alpha(\Omega, \mathbb{R}^k)$ for $n = 2$ and $\alpha > 2$. In fact, whenever $B_r(x_0) \subset \mathbb{R}^n$ we have that

$$\begin{cases} \mathbf{P}_1^g(x_0, r) \le c(n) \|g\|_{L(n,1)(B_r(x_0))} & \text{if } \ n \ge 3 \\ \mathbf{P}_1^g(x_0, r) \le c(\alpha) \|g\|_{L^2(\mathrm{LogL})^\alpha(B_r(x_0))} & \text{if } \ n = 2 \text{ and } \alpha > 2, \end{cases}$$

cf. [165, Lemma 2.3 and Lemma 2.4] and [16, Section 2]. Therefore, given concentric balls $B_r \subset B_{r+\varrho} \subset \mathbb{R}^n$, with $\varrho \in (0, 1)$, and $g \in L^2(B_{r+\varrho}; \mathbb{R}^k)$, there holds that:

$$\begin{cases} \left\| \mathbf{P}_1^g(\cdot, \varrho) \right\|_{L^\infty(B_r)} \le c(n) \|g\|_{L(n,1)(B_{r+\varrho})}, & n \ge 3 \\ \left\| \mathbf{P}_1^g(\cdot, \varrho) \right\|_{L^\infty(B_r)} \le c(\alpha) \|g\|_{L^2(\log L)^\alpha(B_{r+\varrho})}, & n = 2 \, , \end{cases} \tag{2.3.3}$$

where in the last of the two inequalities $c(\alpha) \to \infty$ when $\alpha \searrow 2$. The potential in (2.3.1) plays also a crucial role in a nonlinear iteration lemma originally proved in [160] and based on the iteration scheme contained in [81]. We present it in a version which is a quite straightforward variation of the one in [16, Section 3].

Lemma 2.3.1 *Let $B_{r_0}(x_0) \subset \mathbb{R}^n$, $n \ge 2$, $\delta \in \big(0, 1/2\big)$ and $v \in W^{1,2}(B_{r_0}(x_0))$ be non-negative and $f_1, f_2 \in L^2(\mathbb{R}^n)$; assume that there exist positive constants $\tilde{c}, M_1, M_2, M_3 \ge 1$ and a number $\kappa_0 \ge 0$ such that for all $\kappa \ge \kappa_0$ and every ball $B_r(x_0) \subset B_{r_0}(x_0)$ the inequality*

$$\fint_{B_{r/2}(x_0)} |D(v - \kappa)_+|^2 \, \mathrm{d}x \le \frac{\tilde{c} M_1^2}{r^2} \fint_{B_r(x_0)} (v - \kappa)_+^2 \, \mathrm{d}x$$

$$+ \tilde{c} M_2^2 \fint_{B_r(x_0)} |f_1|^2 \, \mathrm{d}x + \tilde{c} M_3^2 \fint_{B_r(x_0)} |f_2|^2 \, \mathrm{d}x \tag{2.3.4}$$

holds. If x_0 is a Lebesgue point for v, then

$$v(x_0) \le \kappa_0 + c M_1^{1+\max\left\{\delta, \frac{n-2}{2}\right\}} \left(\fint_{B_{r_0}(x_0)} (v-\kappa_0)_+^2 \, \mathrm{d}x \right)^{1/2}$$
$$+ c M_1^{\max\left\{\delta, \frac{n-2}{2}\right\}} \left[M_2 \mathbf{P}_1^{f_1}(x_0, 2r_0) + M_3 \mathbf{P}_1^{f_2}(x_0, 2r_0) \right] \tag{2.3.5}$$

holds with $c \equiv c(n, \tilde{c}, \delta)$.

2.4 Tools for p-Laplacian type problems

When dealing with p-Laplacian type problems, we shall often use the auxiliary vector fields $V_{\mu,p} \colon \mathbb{R}^{k \times n} \to \mathbb{R}^{k \times n}$, defined by

$$V_{\mu,p}(z) := (\mu^2 + |z|^2)^{(p-2)/4} z, \qquad p \in (1, \infty) \;\text{ and }\; \mu \in [0,1] \tag{2.4.1}$$

whenever $z \in \mathbb{R}^{k \times n}$. If $\mu = 0$ we shall simply write $V_{\mu,p} \equiv V_p$. A useful related inequality is contained in the following

$$|V_{\mu,p}(z_1) - V_{\mu,p}(z_2)| \approx (\mu^2 + |z_1|^2 + |z_2|^2)^{(p-2)/4} |z_1 - z_2|, \tag{2.4.2}$$

where the equivalence holds up to constants depending only on n, k, p. An important property which is usually related to such field is recorded in the following lemma.

Lemma 2.4.1 *Let $p > -1$, $\mu \in [0,1]$ and $z_1, z_2 \in \mathbb{R}^k$ be so that $\mu + |z_1| + |z_2| > 0$. Then*

$$\int_0^1 \left[\mu^2 + |z_1 + \lambda(z_2 - z_1)|^2 \right]^{\frac{p}{2}} \, \mathrm{d}\lambda \sim (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p}{2}},$$

with constants implicit in "$\sim$" depending only from n, k, p.

For more details on this matter we refer to [2, 129]. Finally, the "simple, but fundamental" iteration lemma of [115, Section 1].

Lemma 2.4.2 *Let $\mathcal{Z} \colon [\varrho_0, \varrho_1] \to \mathbb{R}$ be a nonnegative and bounded function, and let $\theta \in (0,1)$ and $A, B \ge 0$, $\gamma_1, \gamma_2 \ge 0$ be numbers. Assume that*

$$\mathcal{Z}(t) \le \theta \mathcal{Z}(s) + \frac{A}{(s-t)^{\gamma_1}} + \frac{B}{(s-t)^{\gamma_2}}$$

holds for $\varrho_0 \le t < s \le \varrho_1$. Then the following inequality holds with $c \equiv c(\theta, \gamma_1, \gamma_2)$:

$$\mathcal{Z}(\varrho_0) \le \frac{cA}{(\varrho_1 - \varrho_0)^{\gamma_1}} + \frac{cB}{(\varrho_1 - \varrho_0)^{\gamma_2}} \, .$$

Chapter 3

Manifold constrained non-uniformly elliptic problems

Joint work with G. Mingione (University of Parma)
***J. Geom. Analysis* (2020) -** https://doi.org/10.1007/s12220-019-00275-3

We want to treat, from the regularity theory viewpoint, a special but yet significant class of non-uniformly variational problems characterized by the fact that minimizers and competitors take their values into the sphere. At the same time, we want to introduce a few intrinsic methods and viewpoints that should be useful in order to prove regularity theorems for more general classes of non-uniformly elliptic equations and functionals with geometric constraints. For this reason we shall combine and present both old techniques from different perspectives and new ones. Specifically, we shall consider a class of variational integrals of the type

$$W^{1,1}(\Omega,\mathbb{R}^N) \ni w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega F(x,w,Dw) \ \mathrm{d}x \tag{3.0.1}$$

where the main model is provided by the so-called Double Phase functional

$$\begin{cases} W^{1,1}(\Omega,\mathbb{R}^N) \ni w \mapsto \mathcal{P}(w,\Omega) := \int_\Omega \left[|Dw|^p + a(x)|Dw|^q\right] \ \mathrm{d}x \\ 1 < p < q < N \\ 0 \le a(\cdot) \in C^{0,\alpha}(\Omega). \end{cases} \tag{3.0.2}$$

Here, $\Omega \subset \mathbb{R}^n$ denotes (unless otherwise specified) a bounded open domain, $n \ge 2$, and, again unless otherwise stated, we consider $N > 1$ and $a(\cdot)$ satisfies the condition in $(3.0.2)_3$ for some $\alpha \in (0,1]$. When $a(\cdot) \equiv 0$, the integral in (3.0.2) reduces to the familiar p-Dirichlet integral

$$w \mapsto \int_\Omega |Dw|^p \ \mathrm{d}x, \tag{3.0.3}$$

whose Euler-Lagrange equation is given by the p-Laplacian system $-\operatorname{div}\left(|Du|^{p-2}Du\right) = 0$. The regularity theory for minimizers of the functional in (3.0.3) has been treated at length starting from the seminal papers of Uraltseva [242] and Uhlenbeck [241], in the scalar and vectorial case, respectively. For results concerning more general functionals as in (3.0.1) with p-growth, that is, modelled on the one in (3.0.2) and therefore satisfying

$$|z|^p \lesssim F(x,w,z) \lesssim |z|^p + 1 \ , \tag{3.0.4}$$

see for instance [168, 170, 180] and related references. Under suitable assumptions, the final outcome is that minima are locally of class $C^{1,\beta}$, for some $\beta \in (0,1)$, on a subset of full n-dimensional Lebesgue measure. The regularity theory in the case when both minimizers and competitors take values into a manifold $\mathcal{M} \subset \mathbb{R}^N$ poses additional difficulties. In particular, the case $\mathcal{M} = \mathbb{S}^{N-1}$ is the $(N-1)$-dimensional sphere in $\mathbb{R}^N$ has been treated extensively. The theory started with the fundamental papers of Eells & Sampson [95] and Schoen & Uhlenbeck [230,231], analyzing harmonic maps, i.e., constrained minimizers of the functional in (3.0.3) for $p = 2$. The extension of such basic results to the case $p \neq 2$ has been done in the by now classical papers of Fuchs [110, 111], Hardt & Lin [132] and Luckhaus [179]. Moreover, several results have been extended to more general functionals with p-growth, that is, functionals as in (3.0.1) with $F(\cdot)$ satisfying (3.0.4); see for instance [131]. On the other hand, we notice that energies of the type in (3.0.2) do not satisfy conditions as in (3.0.4), but rather, the more general and flexible ones

$$|z|^p \lesssim F(x, w, z) \lesssim |z|^q + 1 \,, \quad 1 < p < q \,. \tag{3.0.5}$$

These are known in the literature as (p, q)-growth conditions or non-standard growth conditions. They have pioneered by Uraltseva & Urdaletova [243] and Zhikov [246–248] in the context of Homogenization (see also the recent paper [92]). In the setting of the Calculus of Variations they have been systematically studied by Marcellini [183, 184]. We refer to [194] for a reasonable survey on the subject. Growth conditions of the type in (3.0.5) often occur when considering variational models for physical phenomena. For instance, in the setting of Homogenization, a model as the Double Phase functional can be used to describe a composite of two materials with hardening exponents p and q respectively, whose geometry is dictated by the zero set $\{a(x) = 0\}$ of the coefficient $a(\cdot)$. Obviously, both in the case $a(\cdot) \equiv 0$ and in the one when $\inf a(\cdot) > 0$, we have a functional with standard polynomial growth of the type in (3.0.4) (with p replaced by q in the second case). In the remaining one, the nature of ellipticity of the functional $\mathcal{P}$ switches between the p and q rates accordingly to the value of $a(\cdot)$. For this reason models as those in (3.0.2) are particularly useful to describe strongly anisotropic media. We refer to the papers [12, 50, 61, 79, 80, 219, 248] for more results, different directions and related topics. Another, softer instance of functional with non-standard growth used to describe anisotropic models [4, 246–248] is the variable exponent one

$$w \mapsto \int_\Omega |Dw|^{p(x)} \, dx \,, \tag{3.0.6}$$

that has attracted a lot of attention in the last years [221]; a match between the two cases has been recently proposed in [69, 88, 101, 217, 221, 223, 224]. The growth conditions in (3.0.5) are typically linked to the non-uniform ellipticity of the Euler-Lagrange equations associated to the functionals in question. In case of (3.0.2), such equation is

$$-\operatorname{div} A(x, Du) = 0 \qquad \text{with} \quad A(x, z) = |z|^{p-2}z + (q/p)a(x)|z|^{q-2}z \,, \tag{3.0.7}$$

and therefore, the ellipticity ratio $\mathcal{R}(z, B)$ on any ball $B \Subset \Omega$ touching the transition set $\{a(x) = 0\}$, which is defined by

$$\mathcal{R}(z, B) := \sup_{x \in B} \frac{\text{highest eigenvalue of } \partial_z A(x, z)}{\text{lowest eigenvalue of } \partial_z A(x, z)} \approx 1 + \|a\|_{L^\infty(B)} |z|^{q-p} \,,$$

becomes unbounded as $|z| \to \infty$. This means that the equation in (3.0.7) is non-uniformly elliptic. More specifically, the asymptotics of the ratio $\mathcal{R}(z, B)$ exhibit a delicate interplay between the size of $|z|^{q-p}$ and the one of the coefficient $a(\cdot)$ which is crucially close to the zero set $\{a(x) = 0\}$. As a matter of fact, the rate $a(\cdot)$ approaches to zero rebalances the rate of potential blow-up. This is displayed in the sharp condition $q \leq p + \alpha$, which in [12, 60, 62] is found to be necessary and sufficient for unconstrained, *bounded*, scalar minimizers of the model functional (3.0.2) to be regular; otherwise discontinuous minimizers of $\mathcal{P}$ may exist, [101, 106]. Let us remark that both the model functional in (3.0.2) and the one in (3.0.6) fall in the realm of non-autonomous functionals defined in Musielak-Orlicz spaces. These are functionals of the type

$$w \mapsto \int_\Omega \Phi(x, |Dw|) \; dx \,, \tag{3.0.8}$$

where, $\Phi\colon \Omega\times[0,\infty)\to[0,\infty)$ is a Carathéodory function such that for each choice of $x\in\Omega$, the partial map $t\mapsto\Phi(x,t)$ is a Young function and thereby generates an Orlicz space that changes with x. Such functionals are naturally defined on Musielak-Orlicz-Sobolev spaces/classes $W^{1,\Phi}$, i.e.,

$$W^{1,\Phi}(\Omega,\mathbb{R}^N)=\left\{\ f\in W^{1,1}(\Omega,\mathbb{R}^N)\ :\ \Phi(\cdot,|f|)+\Phi(\cdot,|Df|)\in L^1(\Omega)\ \right\} \tag{3.0.9}$$

For such spaces we refer to [135]. A main problem here is the one of finding general conditions ensuring the regularity of minimizers. This appears as a non-trivial and challenging issue. The main idea is that the regularity of minima of (3.0.8) is governed by a delicate interplay between the regularity of the function $x\mapsto\Phi(x,\cdot)$ and the growth conditions of $t\mapsto\Phi(\cdot,t)$. For instance, in the Double Phase case, relations as $q\le p+\alpha$ or $q/p\le 1+\alpha/n$ define sharp conditions for regularity [12, 33, 60, 61]. In the variable exponent case the log-modulus of continuity of $p(\cdot)$ is another instance of such conditions. See [11] for a picture concerning the similarities between (3.0.2) and (3.0.6) as particular cases of the one in (3.0.8). More general conditions unifying those for (3.0.2) and (3.0.6) have been formulated in [138] and lead to a full De Giorgi-Nash-Moser theory. A general approach to the gradient regularity has been devised and suggested in [12]. The regularity problem in the case of constrained minimizers for functionals with non-standard growth conditions has recently received some attention, see the higher integrability result recently obtained in [67] and the singular set estimates proved for the variable exponent case in [69]. The main difficulties essentially rely in the lack of a certain number of properties, that are typically linked to uniform ellipticity and that are essential in order to treat constrained minimizers. We undertake this issue in the model case of Double Phase energies, i.e., functionals of the type in (3.0.1) controlled by the one in (3.0.2) in the sense of (3.0.12) below. Moreover, we consider the case when the manifold is the $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$, that already incorporates several of the new difficulties. Our aim here is also to propose an intrinsic approach which departs from the usual estimates, and it is designed for treating the quantity $\Phi(\cdot,|Dw|)$ which in this case is $|Dw|^p+a(\cdot)|Dw|^q$, as a sort of replacement of $|Dw|^p$. We shall therefore formulate and use a certain number of tools (harmonic approximation lemmas, a priori estimates and so on) in terms of the quantity $\Phi(\cdot,|Dw|)$. Accordingly to this viewpoint, in order to characterize the singular sets, we shall use an intrinsic Hausdorff type measure aimed at catching the local geometry of the integrand $\Phi(\cdot,t)$. Such measures give back the standard Hausdorff measure in the case $\Phi(\cdot,t)=t^p$ as well as other examples of measures available in the literature. We then compare these measures to the natural capacities generated by functionals of the type in (3.0.8) and relate the corresponding outcomes to the size of the singular sets, that are indeed found to have zero capacity.

3.0.1 Partial regularity

It is convenient to introduce some notation (see also Section 3.1 below). We shall denote

$$H(x,z):=|z|^p+a(x)|z|^q\,,\qquad \text{for all}\ \ z\in\mathbb{R}^{N\times n}\,,\quad x\in\Omega \tag{3.0.10}$$

where $a(\cdot)$ is as in (3.0.2) and recall that in the following it will always be $n\ge 2$ and $N>1$ (in fact, sometimes we shall consider also the case $z\in\mathbb{R}^{(N-1)\times n}$; see Section 3.1 and Step 4 from the proof of Theorem 1 below). Moreover, with $B\Subset\Omega$ being a ball, we introduce the auxiliary Young functions

$$\begin{cases} H_B^-(z):=|z|^p+a_i(B)|z|^q, & \text{where}\ \ a_i(B):=\inf_{x\in B}a(x);\\ H_B^+(z):=|z|^p+a_s(B)|z|^q, & \text{where}\ \ a_s(B):=\sup_{x\in B}a(x)\,. \end{cases} \tag{3.0.11}$$

With abuse of notation, we shall keep on denoting $H(x,t)=t^p+a(x)t^q$ for $t\ge 0$ (and the like for the functions in (3.0.11)), that is when in (3.0.10) z is a non-negative number. From now on, with $B \subset \mathbb{R}^n$ being a ball, we shall denote by $r(B)$ its radius. Following [12], we then consider variational integrals of the type in (3.0.1), where $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand, such that $z \mapsto F(\cdot, z)$ is $C^1(\mathbb{R}^{N\times n}) \cap C^2(\mathbb{R}^{N\times n} \setminus \{0\})$ and satisfies the following assumptions:

$$
\begin{cases}
\nu H(x,z) \le F(x,v,z) \le LH(x,z) \\
|\partial_z F(x,v,z)||z| + |\partial_{zz} F(x,v,z)||z|^2 \le LH(x,z) \\
\nu(|z|^{p-2} + a(x)|z|^{q-2})|\xi|^2 \le \partial_{zz} F(x,v,z)\xi \cdot \xi \\
|\partial_z F(x_1,v,z) - \partial_z F(x_2,v,z)||z| \le L\left[\omega(|x_1-x_2|)[H(x_1,z)+H(x_2,z)] + |a(x_1)-a(x_2)||z|^q\right] \\
|F(x,v_1,z) - F(x,v_2,z)| \le L\omega(|v_1-v_2|)H(x,z)\ .
\end{cases}
\tag{3.0.12}
$$

These are assumed to hold whenever $x, x_1, x_2 \in \Omega$, $v, v_1, v_2 \in \mathbb{R}^N$, $z, z_1, z_2 \in \mathbb{R}^{N\times n} \setminus \{0\}$, $\xi \in \mathbb{R}^{N\times n}$, where $0 < \nu \le 1 \le L$ and $\vartheta \in (0,1)$ are fixed constants and, for every non-negative number t,

$$
\omega(t) := \min\left\{t^{\beta}, 1\right\}\ , \qquad \beta \in (0,1] \tag{3.0.13}
$$

is defined as the standard concave β-Hölder modulus of continuity. We finally consider the necessary structure assumption to deal with the vectorial case, that is, we assume that for every choice of $(x,v) \in \Omega \times \mathbb{R}^N$ there exists a function $\tilde{F}_{x,v}(\cdot) \equiv \tilde{F}(x,v,\cdot)\colon [0,\infty) \to [0,\infty)$ of class $C^1[0,\infty) \cap C^2(0,\infty)$, such that

$$
\begin{cases}
F(x,v,z) = \tilde{F}(x,v,|z|) \text{ holds for every } z \in \mathbb{R}^{N\times n} \text{ with } t \mapsto \tilde{F}(x,y,t) \text{ non-decreasing,} \\
\left|\tilde{F}''(x,v,t+s) - \tilde{F}''(x,v,t)\right| \le \dfrac{LH(x,t)}{t^2}\left(\dfrac{|s|}{t}\right)^{\beta_1}\ .
\end{cases}
\tag{3.0.14}
$$

The inequality in the last line is assumed to hold whenever $s, t \in \mathbb{R}$ are such that $t > 0$ and $2|s| < t$, and with a fixed constant $\beta_1 \in (0,1]$ (which is independent of the considered (x,v)). As for the exponents p, q, we assume

$$
p < q < p + \alpha\,, \qquad q < N. \tag{3.0.15}
$$

We remark that the inequality $q < p + \alpha$ in the last display, apart from the missing equality case, is a sharp condition for regularity, as shown in [101,106]. The second inequality $q < N$ relates the growth conditions of the problem and the topological properties of the target manifold, which is in this case $\mathbb{S}^{N-1}$. This is necessary in order to use certain projection operators (see Lemma 3.2.1 below). Assumptions of this kind are considered by Hardt & Lin [131, 132] in the convex case and by Hopper [151] in the quasiconvex one. The related definition of local minimizer we are going to consider is the following:

Definition 3 *A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{S}^{N-1})$ is a local minimizer of the functional $\mathcal{F}$ defined in (3.0.1) under assumptions $(3.0.12)_1$, if and only if $H(\cdot, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and the minimality condition $\mathcal{F}(u, \mathrm{supp}\,(u-v)) \le \mathcal{F}(v, \mathrm{supp}\,(u-v))$ is satisfied whenever $v \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{S}^{N-1})$ is such that $\mathrm{supp}\,(u-v) \subset \Omega$.*

By definition a local minimizer belongs to $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{S}^{N-1})$; in the rest of the chapter, we shall appeal such local minimizers sometimes as constrained local minimizers to emphasize that the presence of the constraint $|u| = 1$. We notice when $a(\cdot) \equiv 0$ assumptions (3.0.12) reduce to the standard ones considered for functionals with p-growth when considering partial regularity problems (see for instance [162, 163, 191, 194] and related references). In particular, assumptions (3.0.12)-(3.0.14) are devised to cover functionals of the type

$$w \mapsto \int_{\Omega} \left[F_1(x,w,Dw) + a(x)F_2(x,w,Dw)\right] \, \mathrm{d}x \,,$$

where $F_1(\cdot)$ and $F_2(\cdot)$ have p- and q-growth, respectively, accordingly to the standard assumptions described for instance in [162]. Another functional covered by our set of assumptions is

$$W^{1,1}(\Omega) \ni w \mapsto \int_{\Omega} b(x,w)H(x,Dw) \, \mathrm{d}x \,,$$

where $0 < \nu_1 \le b(x,v) \le L_1$, for some constants ν_1, L_1. Here, $b(\cdot)$ is a Hölder-continuous function. For later convenience, we shall denote

$$\texttt{data} \equiv \texttt{data}\left(n, N, \nu, L, p, q, \alpha, [a]_{0,\alpha}\right) \tag{3.0.16}$$

as the set of basic parameters intervening in the problem. Our first main result is the following:

Theorem 1 *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{S}^{N-1})$ be a local minimizer of the functional $\mathcal{F}$ in (3.0.1) under the assumptions (3.0.12)-(3.0.15). There exists $\delta_g \equiv \delta_g(\texttt{data}) > 0$ such that*

$$H(\cdot, Du) \in L^{1+\delta_g}_{\mathrm{loc}}(\Omega). \tag{3.0.17}$$

Moreover, there exist $\beta_0 \equiv \beta_0(\texttt{data}, \beta, \beta_1) > 0$ and an open subset $\Omega_u \subset \Omega$, called the regular set, such that

$$Du \in C^{0,\beta_0}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^{N\times n}) \qquad \text{and} \qquad |\Omega \setminus \Omega_u| = 0 \,. \tag{3.0.18}$$

In the case $p(1+\delta_g) > n$ we have $\Omega = \Omega_u$. When $p(1+\delta_g) \le n$, there exists a number $\varepsilon \equiv \varepsilon(\texttt{data}, \beta)$, such that a point $x_0 \in \Omega$ belongs to Ω_u iff

$$\left[H^-_{B_r(x_0)}\left(\frac{\varepsilon}{r}\right)\right]^{-1} \fint_{B_r(x_0)} H(x,Du) \, \mathrm{d}x < 1 \tag{3.0.19}$$

holds for some ball $B_r(x_0) \Subset \Omega$ with $r \le 1$. Finally, as for the so-called singular set $\Sigma_u := \Omega \setminus \Omega_u$, it follows that

$$\Sigma_u = \left\{ x_0 \in \Omega : \ \limsup_{\varrho \to 0} \left[H^-_{B_\varrho(x_0)}\left(\frac{1}{\varrho}\right)\right]^{-1} \fint_{B_\varrho(x_0)} H(x,Du) \, \mathrm{d}x > 0 \right\}. \tag{3.0.20}$$

The ε-regularity condition (3.0.19) differs from the usual ones given in the case of functionals with p-growth as it gives an intrinsic quantified version of the amount of energy needed for regularity; see also [90] for the case of autonomous functionals. The shape of (3.0.19) suggests an intrinsic path to estimate the size of the so-called singular set $\Omega \setminus \Omega_u$. Indeed, this can be done via a general definition of certain Hausdorff type measures that can be useful in general contexts too; for this we refer to the next section. It is worth remarking that the results of Theorem 1 continue to hold in the case of unconstrained *bounded* minimizers and it is new in the vectorial case (it extends the scalar one in [12]). In the unconstrained case the condition $q < N$ in (3.0.5) can be dropped (see Remark 3.2.1 below).

Remark 3.0.1 It is still possible to get a partial regularity result by weakening the assumptions on the function $\tilde{F}_{x,v}(t) \equiv F(x,v,t)$ considered in (3.0.14). Specifically, we can drop $(3.0.14)_2$, thereby replacing (3.0.18) with the weaker outcome

$$u \in C^{0,\beta_2}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^{N\times n}) \qquad \text{and} \qquad |\Omega \setminus \Omega_u| = 0 \,, \tag{3.0.21}$$

for every $\beta_2 < 1$.

3.0.2 Weighted Hausdorff measures, intrinsic capacities and singular sets

Here we shall be slightly more general than what is needed in the present setting, as we wish to settle down a general approach valid also for other contexts. We shall produce a family of Hausdorff type measures that are naturally linked to general functionals of the type in (3.0.8). In the following we consider a Carathéodory function $\Phi\colon \Omega \times [0,\infty) \to [0,\infty)$, i.e., such that $x \mapsto \Phi(x,t)$ is measurable for every $t \geq 0$ and $t \mapsto \Phi(x,t)$ is continuous and non-decreasing for almost every $x \in \Omega$. Here, $\Omega \subset \mathbb{R}^n$ denotes an open subset. Moreover, we assume that $\Phi(x,0) = 0$ and that $\lim_{t\to\infty} \Phi(x,t) = \infty$ for every $x \in \Omega$. We also assume that

$$\Phi(x,t) \lesssim m(x)t^n \ , \qquad \text{for all } t \geq 1 \ , \quad \text{a.e. } x \in \Omega \ , \quad \text{where } 0 \leq m(\cdot) \in L^1_{\rm loc}(\Omega) \tag{3.0.22}$$

and that

$$\text{there exists } \beta_3 \in (0,1) \text{ such that } \Phi(x,\beta_3) \leq 1 \text{ and } \Phi\left(x, 1/\beta_3\right) \geq 1 \text{ for every } x \in \Omega \ , \tag{3.0.23}$$

$$\frac{\Phi(x,s)}{s} \lesssim \frac{\Phi(x,t)}{t} \qquad \text{whenever} \quad 0 < s \leq t \ , \quad \text{for all} \ \ x \in \Omega \ . \tag{3.0.24}$$

These assumptions, also considered in [13], are trivially verified by all the relevant model examples motivating us; see Remark 3.0.2 below. To proceed, for any n-dimensional open ball $B \subset \Omega$ (there is no difference in the following in taking closed balls in this respect) of radius $r(B) \in (0,\infty)$, we define

$$h_\Phi(B) = \int_B \Phi\left(x, 1/\mathrm{r}(B)\right) \ \mathrm{d}x \tag{3.0.25}$$

Notice that this function is always finite and that this is guaranteed by (3.0.22). It results:

$$h_\Phi(B) \lesssim \fint_B m(x) \ \mathrm{d}x.$$

We then use the standard Carathéodory's construction to obtain an outer measure. For this, let $E \subset \Omega$ be any subset. We define the weighted κ-approximating Hausdorff measure of E, $\mathscr{H}_{\Phi,\kappa}(E)$ with $0 < \kappa \leq 1$, by

$$\mathscr{H}_{\Phi,\kappa}(E) = \inf_{C_E^\kappa} \sum_j h_\Phi(B_j) \ , \tag{3.0.26}$$

$$C_E^\kappa = \left\{ \ \{B_j\}_{j\in\mathbb{N}} \text{ is a countable collection of balls} B_j \subset \Omega \text{ covering } E, \text{ such that } \mathrm{r}(B_j) \leq \kappa \ \right\} .$$

As $0 < \kappa_1 < \kappa_2 < \infty$ implies $C_E^{\kappa_1} \subset C_E^{\kappa_2}$, we have that $\mathscr{H}_{\Phi,\kappa_1}(E) \geq \mathscr{H}_{\Phi,\kappa_2}(E)$ and there exists the limit

$$\mathscr{H}_\Phi(E) := \lim_{\kappa\to 0} \mathscr{H}_{\Phi,\kappa}(E) = \sup_{\kappa>0} \mathscr{H}_{\Phi,\kappa}(E) \ . \tag{3.0.27}$$

When considering functionals of the type in (3.0.8), it is convenient to localize the x-dependence and locally compare the starting integrand $\Phi(\cdot)$ with similar maps that are independent of x. This means that, with a ball $B \subset \Omega$ being fixed, we consider the functions $t \mapsto \operatorname{ess\,inf}_{x\in B} \Phi(x,t)$ and $t \mapsto \operatorname{ess\,sup}_{x\in B} \Phi(x,t)$, and define

$$h^+_\Phi(B) = |B| \operatorname*{ess\,sup}_{x\in B} \Phi\left(x, 1/r(B)\right) \quad \text{and} \quad h^-_\Phi(B) = |B| \operatorname*{ess\,inf}_{x\in B} \Phi\left(x, 1/r(B)\right) \tag{3.0.28}$$

so that $h^-_\Phi(B) \leq h_\Phi(B) \leq h^+_\Phi(B)$. Accordingly, keeping (3.0.26)-(3.0.27), we finally set

$$\mathscr{H}^\pm_{\Phi,\kappa}(E) = \inf_{C_E^\kappa} \sum_j h^\pm_\Phi(B_j) \quad \text{and} \quad \mathscr{H}^\pm_\Phi(E) = \lim_{\kappa\to 0} \mathscr{H}^\pm_{\Phi,\kappa}(E). \tag{3.0.29}$$

The above definitions obviously imply that $\mathscr{H}^-_{\Phi,\kappa}(E) \le \mathscr{H}_{\Phi,\kappa}(E) \le \mathscr{H}^+_{\Phi,\kappa}(E)$ holds for every $\kappa \in (0,1]$ and therefore, upon letting $\kappa \to 0$, it follows that

$$\mathscr{H}^-_{\Phi}(E) \le \mathscr{H}_{\Phi}(E) \le \mathscr{H}^+_{\Phi}(E). \tag{3.0.30}$$

Remark 3.0.2 Definition (3.0.27) is aimed at catching and unifying several instances of similar objects. Furthermore, let us notice that

- In the case $\Phi(x,t) \equiv t^p$ for $p \le n$, then $\mathscr{H}_\Phi$ is equivalent (up to constants) to the usual $(n-p)$-dimensional spherical Hausdorff measure.
- In the case $\Phi(x,t) \equiv t^{p(x)}$ for $p(x) \le n$ being a continuous function defined on an open subset Ω, then $\mathscr{H}_\Phi$ falls in the class of the variable exponent Hausdorff measures studied in [215, 240].
- In the case $\Phi(x,t) \equiv w(x)t^p$ for $p \le n$ and $w(\cdot)$ being a non-negative and measurable function, $\mathscr{H}_\Phi$ is equivalent to the weighted Hausdorff measures introduced in [215, 240], with particular emphasis on the situations when $w(\cdot)$ is a Muckenhoupt weight.
- The case we are mostly interested in is when $\Phi(x,t) = [H(x,t)]^{1+\delta} \equiv [t^p + a(x)t^q]^{1+\delta}$ for some $\delta \ge 0$, with $H(\cdot)$ as in (3.0.10) and under the condition that $q(1+\delta) \le n$. In this case we shall use the notation $\mathscr{H}_\Phi \equiv \mathscr{H}_{H^{1+\delta}}$ *and* $\mathscr{H}^\pm_\Phi \equiv \mathscr{H}^\pm_{H^{1+\delta}}$. These measures reveal to be an essential tool to prove sharp theorems of removable singularities in Double Phase problems, as shown in [51].

Remark 3.0.3 By standard arguments, i.e., those of the type needed in the case of the usual Hausdorff measures, the set function $\mathscr{H}_\Phi$ turns out to be a Borel-regular measure (here we adopt the standard terminology from [104]). We notice that the definition of $\mathscr{H}_{\Phi,\kappa}$ is invariant when using open or closed balls in (3.0.25). As for the set functions $\mathscr{H}^\pm_\Phi$ in (3.0.29), these turn out to be Borel regular measures too. We mention an alternative way to describe measures as $\mathscr{H}^\pm_\Phi$. This occurs upon replacing (3.0.28) by

$$h^+_\Phi(B) = |B| \sup_{x\in B} \Phi\left(x, 1/r(B)\right) \quad \textit{and} \qquad h^-_\Phi(B) = |B| \inf_{x\in B} \Phi\left(x, 1/r(B)\right). \tag{3.0.31}$$

In this case the corresponding set functions $\mathscr{H}^\pm_\Phi$ are again Borel measures and are Borel regular too if $\Phi(\cdot)$ is continuous. Alternatively, one can use in the definition (3.0.31) closed balls instead of open ones, thereby always getting automatically a Borel regular measure.

In order to place the above measures in the setting of regularity of minimizers and to connect the three measures appearing in (3.0.30), we next consider the following assumption:

$$\operatorname*{ess\,sup}_{x\in B} \Phi\left(x, \beta_4 t\right) \le c_d \operatorname*{ess\,inf}_{x\in B} \Phi\left(x, t\right) \tag{3.0.32}$$

to hold whenever $1 \le t \le 1/r(B)$ for all balls $B \subset \Omega$ with $r(B) \le 1$ and for some constants $\beta_4 \in (0,1], c_d \ge 1$. This assumption is known to be crucial to prove the local Hölder continuity of *bounded* minimizers of functionals of the type in (3.0.8) and in certain Harmonic Analysis questions related to Musielak-Orlicz spaces; see [138, 145]. In this respect, assumption (3.0.32) is sharp by the examples in [101, 106]. When applied to the choice $\Phi(x,t) = t^p + a(x)t^q$ and $a(\cdot) \in C^{0,\alpha}(\Omega)$, (3.0.32) amounts to require that $q \le p + \alpha$ as first considered in [60]; see Proposition 3.0.1 and again (3.5.1) below. An immediate consequence of (3.0.32) is the following fact, whose proof is reported in Section 3.5.

Proposition 3.0.1 *Assume that* (3.0.32) *holds. Then, for any subset* $E \subset \Omega$ *it follows that*

$$\mathscr{H}^+_\Phi(E) \le \frac{c_d}{\beta_4^n}\mathscr{H}^-_\Phi(E). \tag{3.0.33}$$

As a consequence, if $a(\cdot) \in C^{0,\alpha}(\Omega)$, $q \le p+\alpha$ *and* $\delta \ge 0$, *then there exists a constant* $c \equiv c([a]_{0,\alpha},\delta) \ge 1$ *such that the following inequality holds for every subset* $E \subset \mathbb{R}^n$*:*

$$\mathscr{H}^-_{H^{1+\delta}}(E) \le \mathscr{H}_{H^{1+\delta}}(E) \le \mathscr{H}^+_{H^{1+\delta}}(E) \le c\mathscr{H}^-_{H^{1+\delta}}(E)\ . \tag{3.0.34}$$

Following [13], we now introduce a notion of (relative) capacity generated by the function $\Phi(\cdot)$. For a compact subset $K \subset \mathbb{R}^n$, we denote

$$Cap^*_\Phi(K) \equiv Cap^*_\Phi(K,\Omega) := \inf_{f\in\mathcal{R}(K)}\int_\Omega \Phi(x,|Df|)\ \mathrm{d}x \tag{3.0.35}$$

where

$$\mathcal{R}(K) := \left\{\ f \in W^{1,\Phi}(\Omega)\cap C_0(\Omega) \colon\ f \ge 1 \text{ in } K,\ f \ge 0\ \right\}\ .$$

As usual, for open subsets $U \subset \Omega$ we set

$$Cap_\Phi(U) := \sup_{K\subset U,\ K \text{ is compact}} Cap^*_\Phi(K)$$

and then, for general sets $E \subset \Omega$ we finally define

$$Cap_\Phi(E) := \inf_{E\subset\tilde U\subset\Omega,\ \tilde U \text{ is open}} Cap_\Phi(\tilde U)\ .$$

It turns out that, under the present assumptions on $\Phi(\cdot)$, we have $Cap^*_\Phi(K) = Cap_\Phi(K)$, whenever $K \subset \Omega$ is a compact subset and therefore the symbol Cap^*_Φ will not be used anymore, see [13, Proposition 6.3]. Anisotropic capacities of this kind have been studied at length in the literature. Classical reference in this respect are [54, 108, 139, 192, 215]. Here we refer to the recent paper [13], where such capacities have been studied in detail under the assumptions in (3.0.23)-(3.0.24) considered here. These ensure that Cap_Φ enjoys the standard properties of Sobolev capacities; in particular Cap_Φ is a Choquet capacity in the sense that

$$Cap_\Phi(E) = \sup\left\{Cap_\Phi(K) :\ K \subset E \text{ and } K \text{ is compact}\right\} \tag{3.0.36}$$

holds for every Borel set $E \subset \Omega$, [13, Remark 3.6]. Needless to say, in the case $\Phi(x,t) \equiv t^p$, Cap_Φ coincides with the usual relative $W^{1,p}$-capacity. In the following we shall denote $Cap_H = Cap_\Phi$ when $\Phi(x,t) = t^p + a(x)t^q$. Exactly as in the case of the $W^{1,p}$-capacity, we can prove a relation between capacity and Haudorff measures. For this, we need some more assumptions. Specifically, we assume that there exist $1 < p < q < \infty$ such that

$$\frac{\Phi(x,s)}{s^p} \le c_g\frac{\Phi(x,t)}{t^p} \quad \text{and} \quad \frac{\Phi(x,t)}{t^q} \le c_g\frac{\Phi(x,s)}{s^q} \qquad \text{whenever } 0 < s \le t\ , \tag{3.0.37}$$

for some $c_g \ge 1$. We then have

Theorem 2 *Assume that* (3.0.32) *and* (3.0.37) *are in force. Let* $E \subset \mathbb{R}^n$ *be such that* $\mathscr{H}_\Phi(E) < \infty$, *then* $Cap_\Phi(E) = 0$.

It is now possible to improve the estimates of the Hausdorff measure of the singular set $\Sigma_u := \Omega \setminus \Omega_u$ from Theorem (1). This is in the following:

Theorem 3 *Let $u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{S}^{N-1})$ be a local minimizer of the functional $\mathcal{F}$ in (3.0.1) under the assumptions (3.0.12)-(3.0.15), and let $\Omega_u \subset \Omega$ be its regular set in the sense of Theorem 1. Assume that $q(1+\delta_g) \leq n$, where δ_g is the number appearing in (3.0.17). Then*

$$\mathcal{H}_{H^{1+\delta_g}}(\Omega \setminus \Omega_u) = 0 \quad \textit{and therefore} \quad Cap_{H^{1+\delta_g}}(\Omega \setminus \Omega_u) = 0 \,. \tag{3.0.38}$$

In particular, we have

$$\mathcal{H}^{n-p-p\delta_g}(\Sigma_u) = 0 \,, \tag{3.0.39}$$

and

$$\mathcal{H}^{n-q-q\delta_g}\big(\Sigma_u \cap \{a(x) > 0\}\big) = 0 \,. \tag{3.0.40}$$

3.0.3 An overview

As mentioned at the beginning of the Introduction, our aim is not only to prove regularity results for constrained local minimizers of Double Phase functionals, but also to expose intrinsic techniques bound to cover general functionals of the type in (3.0.8). In this sense, we further develop the ideas introduced in [12] to get general regularity methods for non-autonomous functionals and also simplifies some of the arguments presented there. Moreover, the techniques considered here provide new results also in the unconstrained case. For instance, a partial regularity theory which is analogous to the classical one for standard p-functionals can be derived in the Double Phase case too (see Remark 3.2.1). The chapter is structured as follows. In Section 3.1 we fix some notation. In Section 3.2 we establish some basic energy and higher integrability inequalities adapting the path developed in [60, 61] to the manifold constrained case. This is based on a projection argument exposed in Lemma 3.2.1. Moreover, we derive the precise form of the Euler-Lagrange equation of functionals of the type in (3.0.1), under assumptions (3.0.12)-(3.0.14). We finally readapt a Morrey type decay estimate originally proved in [60] (see Theorem 4). In Section 3.3 we develop an intrinsic harmonic type approximation result (compactness lemma), which is Lemma 3.3.2. The main novelty is that the energy bounds and the approximation are given directly in the intrinsic terms of a Musielak-Orlicz energy, rather that a more typical Orlicz one, as usually done in the literature [90, 93]. The lemma is quantitative, in the sense that it reveals a power type dependence of the constants. It therefore extends a similar result previously obtained in [12], which was there considered in a more classical Orlicz setting. It is interesting to note that the proof of Lemma 3.3.2 involves the use of an a priori smallness assumption (see (3.3.5) below) which is exactly the one which is needed to prove partial regularity in the subsequent Section 4. The conceptual advantage of using such an approach becomes clear in Section 4, where partial regularity and Theorem 1 are proved. The intrinsic approach adopted in Lemma 3.3.2 allows avoiding to readapt the elaborate arguments of [12, 60, 61] as at this point we can directly use the intrinsic Morrey decay estimate of Theorem 4 as a natural reference estimate. This incorporates the regularity information on the solutions indeed developed in [12, 60, 61]. The final outcome is a treatment which is close to the classical one proposed in [234] in the case of harmonic maps. Finally, in Section 3.5 we develop the arguments concerning the Hausdorff type measures presented in Section 3.0.2.

3.1 Some useful remarks

When dealing with p-Laplacian type problems, we shall often use the auxiliary vector fields $V_p, V_q \colon \mathbb{R}^{k\times n} \to \mathbb{R}^{k\times n}$, $k \geq 1$ defined in (2.4.1). In these terms, we notice that

$$H(x,z) = |V_p(z)|^2 + a(x)|V_q(z)|^2 \,, \qquad \text{for all } x \in \Omega,\ z \in \mathbb{R}^{k\times n} \,. \tag{3.1.1}$$

For the maps in (2.4.1) and also $H(\cdot)$, we shall typically choose $k \in \{N-1, N\}$, where $N > 1$ is the number considered in Theorem 1. Actually, we shall almost always consider the case it is $z \in \mathbb{R}^{N\times n}$. When projecting on the sphere (in Step 4 of the proof of Theorem 1, for example), we shall also consider $z \in \mathbb{R}^{(N-1)\times n}$. Usually, we shall not specify the case occurring, as it will be clear from the context. As a consequence of $(3.0.12)_3$, it can be proved that

$$|V_p(z_1) - V_p(z_2)|^2 + a(x)|V_q(z_1) - V_q(z_2)|^2 \leq c\left[\partial_z F(x, v, z_1) - \partial_z F(x, v, z_2)\right] \cdot (z_1 - z_2) \quad (3.1.2)$$

holds whenever $z_1, z_2 \in \mathbb{R}^{N\times n}$, $x \in \Omega$, $v \in \mathbb{R}^N$ and with $c \equiv c(n, N, \nu, p, q)$. For this see for instance [12, 162]. We similarly have, again from $(3.0.12)_3$

$$\begin{aligned} &|V_p(z_2) - V_p(z_1)|^2 + a(x)|V_q(z_2) - V_q(z_1)|^2 + \partial_z F(x, v, z_1) \cdot (z_2 - z_1) \\ &\quad \leq c\left[F(x, v, z_2) - F(x, v, z_1)\right] , \end{aligned} \quad (3.1.3)$$

whenever $z_1, z_2 \in \mathbb{R}^{N\times n}$, again for $c \equiv c(n, N, \nu, p, q)$. We next recall some basic terminology about Musielak-Orlicz-Sobolev spaces. The space $W^{1,H}(\Omega, \mathbb{R}^N)$ is defined as in (3.0.9) with the choice $\Phi(\cdot) \equiv H(\cdot)$, with the local variants being defined in the obvious way and $W^{1,H}_0(\Omega) = W^{1,H}(\Omega) \cap W^{1,p}_0(\Omega)$. In the same way, we set

$$W^{1,H}(\Omega, \mathbb{S}^{N-1}) := \left\{ w \in W^{1,H}(\Omega, \mathbb{R}^N) : \ |w| = 1 \text{ holds a.e.} \right\} ,$$

with the local variants defined in a similar fashion. Finally, with $u \in W^{1,H}(\Omega, \mathbb{S}^{N-1})$ we denote the Dirichlet class

$$W^{1,H}_u(\Omega, \mathbb{S}^{N-1}) := \left\{ w \in W^{1,H}(\Omega, \mathbb{S}^{N-1}) : \ u - w \in W^{1,1}_0(\Omega, \mathbb{R}^N)\right\} .$$

We similarly define the Dirichlet class of unconstrained maps $W^{1,H}_u(\Omega, \mathbb{R}^N)$. Moreover, with $w \in W^{1,H}(\tilde\Omega, \mathbb{R}^N)$ and $\tilde\Omega$ being a domain that allows for a trace operator (for instance, this happens when $\partial\tilde\Omega$ is Lipschitz), we denote by $\mathtt{tr}(w, \partial\tilde\Omega)$ the trace of w on $\partial\tilde\Omega$.

3.2 Basic material

3.2.1 Caccioppoli's and higher integrability inequalities

Following the path established in [60, 61], in this section we gather a few technical inequalities for minimizers of functionals with Double Phase. The main difference is that now the setting is the one of constrained variational problems. Therefore, in several cases, we shall confine ourselves to give the necessary modifications to the proofs proposed in [60, 61]. We start with the following lemma; this provides an extension result in the spirit of [131].

Lemma 3.2.1 *Let $\tilde\Omega \subset \Omega$ be a bounded, Lipschitz domain in $\mathbb{R}^n$ and $v \in W^{1,H}(\tilde\Omega, \mathbb{R}^N)$ be such that $v(\partial\tilde\Omega) \subset \mathbb{S}^{N-1}$. Then there exists $c \equiv c(n, N, p, q)$ and $\tilde v \in W^{1,H}(\tilde\Omega, \mathbb{S}^{N-1})$ satisfying*

$$\int_{\tilde\Omega} H(x, D\tilde v)\ \mathrm{d}x \leq c\int_{\tilde\Omega} H(x, Dv)\ \mathrm{d}x \quad \textit{and} \quad v - \tilde v \in W^{1,1}_0(\tilde\Omega, \mathbb{R}^N) . \quad (3.2.1)$$

Proof. For $a \in B_{1/2}(0) \subset \mathbb{R}^N$ and v as in the statement of the lemma, define the map

$$v^a(x) := \frac{v(x) - a}{|v(x) - a|} , \qquad x \in \tilde\Omega .$$

Clearly, it is

$$|Dv^a(x)| \leq \frac{2|Dv(x)|}{|v(x) - a|} ,$$

so that we can estimate

$$\begin{aligned}\int_{B_{1/2}(0)} H(x, Dv^a)\ \mathrm{d}a &\leq c\int_{B_{1/2}(0)} H\left(x, \frac{Dv}{|v-a|}\right)\ \mathrm{d}a\\ &= c\left(|Dv|^p \int_{B_{1/2}(0)} \frac{\mathrm{d}a}{|v-a|^p} + a(x)|Dv|^q \int_{B_{1/2}(0)} \frac{\mathrm{d}a}{|v-a|^q}\right)\\ &\leq cH(x, Dv)\ .\end{aligned}$$

Here $c \equiv c(N, p, q)$ and we have used the assumption $(3.0.15)_2$; this makes the integrals in the above line finite. Integrating over $\tilde{\Omega}$, using Fubini's theorem and the content of the last display, we obtain

$$\fint_{B_{1/2}(0)} \int_{\tilde{\Omega}} H(x, Dv^a)\ \mathrm{d}x\,\mathrm{d}a \leq c\int_{\tilde{\Omega}} H(x, Dv)\ \mathrm{d}x\ .$$

By Chebyshev inequality, this yields the existence of $a_0 \in B_{1/2}(0)$ such that

$$\int_{\tilde{\Omega}} H(x, Dv^{a_0})\ \mathrm{d}x \leq c\int_{\tilde{\Omega}} H(x, Dv)\ \mathrm{d}x\ , \tag{3.2.2}$$

with $c \equiv c(n, N, p, q)$. Let us consider the projector

$$\Pi_a(y) := \frac{y-a}{|y-a|}, \quad \text{for} \quad y \in \mathbb{S}^{N-1} \text{ and } a \in B_{1/2}(0)\ .$$

Such a projector is a bilipschitz map $\mathbb{S}^{N-1}$ into itself, and it is such that

$$\left[\nabla(\Pi_a^{-1})\right]_{0,1} \leq c \equiv c(N), \tag{3.2.3}$$

an estimate which is independent of $a \in B_{1/2}(0)$. Since $v^a(x) \in \mathbb{S}^{N-1}$ for a.e. $x \in \Omega$ and all $a \in B_{1/2}(0)$, we may define $\tilde{v} := \Pi_{a_0}^{-1} \circ v^{a_0}$ which has the requested features. In fact, since $v(\partial\tilde{\Omega}) \subset \mathbb{S}^{N-1}$, we have

$$\mathtt{tr}(\tilde{v}, \partial\tilde{\Omega}) = \Pi_{a_0}^{-1}\left(\mathtt{tr}(v^{a_0}, \partial\tilde{\Omega})\right) = \Pi_{a_0}^{-1}\left(\Pi_{a_0}\left(\mathtt{tr}(v, \partial\tilde{\Omega})\right)\right) = \mathtt{tr}(v, \partial\tilde{\Omega})$$

and, by (3.2.3) and (3.2.2),

$$\int_{\tilde{\Omega}} H(x, D\tilde{v})\ \mathrm{d}x \leq c\int_{\tilde{\Omega}} H(x, Dv^{a_0})\ \mathrm{d}x \leq c\int_{\tilde{\Omega}} H(x, Dv)\ \mathrm{d}x\ ,$$

where $c \equiv c(n, N, p, q)$, so that (3.2.1) is proved in view of the last two displays. □

Remark 3.2.1 The condition $q < N$ in (3.0.5) enters only in the proof of the above lemma and therefore can be dropped when adapting the proofs given here to the unconstrained case.

Lemma 3.2.1 allows to derive in the new constrained setting a number of preliminary tools that have been already obtained and used in the unconstrained one [60, 61]. We shortly report them, with some additional modification and informations.

Lemma 3.2.2 (Caccioppoli's Inequality) *Let $u \in W^{1,H}(\Omega, \mathbb{S}^{N-1})$ be a constrained local minimizer of the functional $\mathcal{F}$ in (3.0.1) under (only) assumptions $(3.0.12)_1$ and $q \leq p + \alpha$. Then there exists $c \equiv c(n, N, \nu, L, p, q) > 0$ such that for any choice of concentric balls $B_r \subset B_R \Subset \Omega$ there holds*

$$\int_{B_r} H(x, Du)\ \mathrm{d}x \leq c\int_{B_R} H\left(x, \frac{u-(u)_{B_R}}{R-r}\right)\ \mathrm{d}x \tag{3.2.4}$$

and, if $R \leq 1$, it also holds that

$$\int_{B_{R/2}} H(x, Du) \ \mathrm{d}x \leq c \int_{B_R} H^-_{B_R}\left(\frac{u-(u)_{B_R}}{R}\right) \ \mathrm{d}x \ , \tag{3.2.5}$$

for $c \equiv c(n, N, \nu, L, p, q, [a]_{0,\alpha}) > 0$. Moreover, if

$$\inf_{B_R} a(x) \leq 4[a]_{0,\alpha} R^{\alpha} \tag{3.2.6}$$

holds and again it is $R \leq 1$, then (3.2.5) reduces to

$$\int_{B_{R/2}} H(x, Du) \ \mathrm{d}x \leq c \int_{B_R} \left|\frac{u-(u)_{B_R}}{R}\right|^p \ \mathrm{d}x \ , \tag{3.2.7}$$

with $c \equiv c(n, N, \nu, L, p, q, [a]_{0,\alpha})$. Finally, these facts still hold for an unconstrained local minimizer $u \in (W^{1,H} \cap L^{\infty})(\Omega, \mathbb{R}^N)$, with all the constants depending in addition on $\|u\|_{L^\infty}$.

Proof. The proof is a modification of the one originally given in [60, Theorem 1.1, (1.8)]; we furthermore specialize to the case of constrained minimizers, the unconstrained one being totally analogous. In the following all the balls will be concentric to the ones mentioned in the statement of the lemma. With $r \leq t < s \leq R$, we determine a cut-off function $\eta \in C^\infty_c(B_s)$ such that $\chi_{B_t} \leq \eta \leq \chi_{B_s}$ and $|D\eta| \leq 4/(s-t)$. Consider now the function $w(x) = u(x) - \eta(u-(u)_{B_R})$. Since η is smooth and $u \in W^{1,H}(B_s, \mathbb{S}^{N-1})$, then obviously $w \in W^{1,H}(B_s, \mathbb{R}^N)$ and $u - w \in W^{1,1}_0(B_s, \mathbb{R}^N)$. Lemma 3.2.1 yields the existence of $\tilde{w} \in W^{1,H}(B_s, \mathbb{S}^{N-1})$ such that (3.2.1) holds with $\tilde{\Omega} \equiv B_s$, where $c \equiv c(n, N, p, q)$. The minimality of u, $(3.0.12)_1$ and (3.2.1) (with $\Omega \equiv B_s$) yield

$$\begin{aligned}
\nu \int_{B_s} H(x, Du) \ \mathrm{d}x &\leq \int_{B_s} F(x, u, Du) \ \mathrm{d}x \leq \int_{B_s} F(x, \tilde{w}, D\tilde{w}) \ \mathrm{d}x \\
&\leq L \int_{B_s} H(x, D\tilde{w}) \ \mathrm{d}x \leq c \int_{B_s} H(x, Dw) \ \mathrm{d}x \\
&\leq c \int_{B_s \setminus B_t} \left[H(x, (1-\eta)Du) + H(x, (u-(u)_{B_R}) \otimes D\eta)\right] \ \mathrm{d}x \\
&\leq c \int_{B_s \setminus B_t} H(x, Du) \ \mathrm{d}x + c \int_{B_s \setminus B_t} H\left(x, \frac{u-(u)_{B_R}}{s-t}\right) \ \mathrm{d}x
\end{aligned} \tag{3.2.8}$$

with $c \equiv c(N, \nu, L, p, q)$. The proof of (3.2.4) can be now concluded by filling the hole and iteration, as in [61, Theorem 1.1, (1.8)], see also [60]. As for the proof of (3.2.5) we simply estimate (as it is $q < p + \alpha$ and $R \leq 1$), for $x \in B_R$

$$\begin{aligned}
H\left(x, \frac{u-(u)_{B_R}}{R}\right) &\leq H^-_{B_R}\left(\frac{u-(u)_{B_R}}{R}\right) + \sup_{B_R} \left[a(x) - a_{\mathrm{i}}(B_R)\right] \left|\frac{u-(u)_{B_R}}{R}\right|^q \\
&\leq H^-_{B_R}\left(\frac{u-(u)_{B_R}}{R}\right) + 2^{\alpha+q-p}[a]_{0,\alpha} R^{\alpha+p-q} \left|\frac{u-(u)_{B_R}}{R}\right|^p \\
&\leq c H^-_{B_R}\left(\frac{u-(u)_{B_R}}{R}\right) \ ,
\end{aligned}$$

for $c \equiv c(p, q, \alpha)$, and (3.2.5) follows from (3.2.4) with $r = R/2$. Finally, for (3.2.7), we similarly observe that (still $x \in B_R$)

$$a_{\mathrm{i}}(B_R) \left|\frac{u-(u)_{B_R}}{R}\right|^q \stackrel{(3.2.6)}{\leq} 8[a]_{0,\alpha} R^{\alpha+p-q} \left|\frac{u-(u)_{B_R}}{R}\right|^p \leq c \left|\frac{u-(u)_{B_R}}{R}\right|^p \ ,$$

so that (3.2.7) follows from (3.2.5) and the proof is complete. □

We proceed with

Lemma 3.2.3 (Intrinsic Sobolev-Poincaré inequality) *Let $v \in (W^{1,H} \cap L^{\infty})(\Omega, \mathbb{R}^N)$, $N \geq 1$, and $B_r \Subset \Omega$ be a ball with radius $r \leq 1$, and assume that $q \leq p + \alpha$. Then the following inequality holds*

$$\fint_{B_r} H\left(x, \frac{v-(v)_{B_r}}{r}\right) \,\mathrm{d}x \leq c \left(\fint_{B_r} [H(x, Dv)]^d \,\mathrm{d}x\right)^{1/d}, \tag{3.2.9}$$

where $c \equiv c(n, N, p, q, [a]_{0,\alpha}, \|v\|_{L^{\infty}(B_r)}) \geq 1$ and $d \equiv d(n, p, q) < 1$. In (3.2.9) we can replace $v - (v)_{B_r}$ by v in case we also have that $\mathtt{tr}(v, \partial B_r) \equiv 0$. Finally, we can still replace $v - (v)_{B_r}$ by v, provided $v \equiv 0$ on $A \subset B_r$ and $|A|/|B_r| > \gamma > 0$; in this last case the constant c depends also on γ.

Proof. The proof is implicit in the one of [60, Theorem 1.2], with minor modifications that are left to the reader. See also [218]. □

Lemma 3.2.4 (Inner higher integrability) *Let $u \in W^{1,H}(\Omega, \mathbb{S}^{N-1})$ be a constrained local minimizer of the functional $\mathcal{F}$ in (3.0.1) under (only) assumptions $(3.0.12)_1$ and (3.0.15). There exists a positive integrability exponent $\delta_g \equiv \delta_g(\mathtt{data})$, such that the following reverse inequality holds for every $B_{2R} \subset \Omega$ such that $R \leq 1$:*

$$\left(\fint_{B_R} [H(x, Du)]^{1+\delta_g} \,\mathrm{d}x\right)^{1/(1+\delta_g)} \leq c \fint_{B_{2R}} H(x, Du) \,\mathrm{d}x \tag{3.2.10}$$

where $c \equiv c(\mathtt{data})$.

Proof. Also in this case, the proof follows the one for [60, Theorem 1.2], which in turn only uses the assumed bound $q < p + \alpha$ and the validity of (3.2.4). □

Lemma 3.2.5 (Higher integrability up to the boundary) *Let $u \in W^{1,H}(\Omega, \mathbb{S}^{N-1})$ be such that $H(\cdot, Du) \in L^{1+\delta}_{\mathrm{loc}}(\Omega)$, for some $\delta > 0$, and, for $B_R \Subset \Omega$, $R \leq 1$, let $v \in W^{1,H}_u(B_R, \mathbb{S}^{N-1})$ be a solution of*

$$v \mapsto \min_{w \in W^{1,H}_u(B_R, \mathbb{S}^{N-1})} \int_{B_R} F(x, w, Dw) \,\mathrm{d}x\,,$$

where the Carathéodory integrand $F(\cdot)$ satisfies (only) $(3.0.12)_1$ and (3.0.15). Then there exists a positive exponent $\sigma_g \in (0, \delta)$ and a constant $c \geq 1$, both depending on $n, N, \nu, L, p, q, \alpha, [a]_{0,\alpha}$, such that

$$\left(\fint_{B_R} [H(x, Dv)]^{1+\sigma_g} \,\mathrm{d}x\right)^{1/(1+\sigma_g)} \leq c \left(\fint_{B_R} H(x, Du)]^{1+\sigma_g} \,\mathrm{d}x\right)^{1/(1+\sigma_g)}.$$

Moreover, in the above display, σ_g can be replaced by any smaller and positive number.

Proof. With $x_0 \in B_R$, let us fix a ball $B_r(x_0) \subset \mathbb{R}^n$ such that it is $|B_r(x_0) \setminus B_R| > |B_r(x_0)|/10$. Let us fix $r/2 < t < s < r$ and take a cut-off function $\eta \in C^1_c(B_s(x_0))$ such that $\chi_{B_t(x_0)} \leq \eta \leq \chi_{B_s(x_0)}$ and $|D\eta| \leq 4/(s-t)$. The function $v - \eta(v-u)$ coincides with v in ∂B_R (in the sense of traces) and therefore we can apply Lemma 3.2.1. This provides us with a map $w \in W^{1,H}_v(B_s(x_0) \cap B_R, \mathbb{S}^{N-1})$ such that

$$\int_{B_s(x_0) \cap B_R} H(x, Dw) \,\mathrm{d}x \leq c(n, N, \nu, L, p, q) \int_{B_s(x_0) \cap B_R} H(x, D(v - \eta(v-u))) \,\mathrm{d}x\,.$$

The minimality of v and $(3.0.12)_1$, together with the above inequality, yield

$$\begin{aligned}\int_{B_s(x_0)\cap B_R} H(x,Dv)\ \mathrm{d}x &\le \frac{L}{\nu}\int_{B_s(x_0)\cap B_R} H(x,Dw)\ \mathrm{d}x \\ &\le c\int_{B_s(x_0)\setminus B_t(x_0)\cap B_R} H(x,Dv)\ \mathrm{d}x + c\int_{B_s(x_0)\cap B_R} H(x,Du)\ \mathrm{d}x \\ &\le +c\int_{B_s(x_0)\cap B_R} H\left(x,\frac{v-u}{s-t}\right)\ \mathrm{d}x\ ,\end{aligned}$$

with $c \equiv c(n,N,\nu,L,p,q)$. By filling the hole and iterating as for instance done in [61, Proof of Theorem 1.8], we arrive at

$$\int_{B_{r/2}(x_0)\cap B_R} H(x,Dv)\ \mathrm{d}x \le c\int_{B_r(x_0)\cap B_R} H\left(x,\frac{v-u}{r}\right)\ \mathrm{d}x + c\int_{B_r(x_0)\cap B_R} H(x,Du)\ \mathrm{d}x\ ,$$

for $c \equiv c(n,N,\nu,L,p,q)$. From this point on, we can follow the proof of [78, Lemma 5] but using the method of [60, Theorem 1.2], see also [69, Lemma 10]. □

Remark 3.2.2 The assertion of Lemma 3.2.4 continues to hold in the case of unconstrained local minimizers $u \in W^{1,H}(\Omega,\mathbb{R}^N)$, $N > 1$, such that $u \in L^\infty(\Omega)$; in this case c and δ_g also depend on $\|u\|_{L^\infty}$; for this see the original proof in [60] and the extensions made in [218]. Moreover, Lemma 3.2.5 still holds when $u, v \in L^\infty(\Omega,\mathbb{R}^N)$ and, also in this case, c and σ_g again depend on $\|u\|_{L^\infty(B_R)}$ and $\|v\|_{L^\infty(B_R)}$. The proof follows again the one proposed in [78, Lemma 5] and [60, Theorem 1.2]. For later convenience we discuss a case when assumptions in (3.0.12) are relaxed. Instead of $(3.0.12)_1$, we consider

$$\tilde{\nu}(M)H(x,z) \le F(x,v,z) \le \tilde{L}(M)H(x,z) \tag{3.2.11}$$

to be satisfied as in $(3.0.12)_1$, whenever $|v| \le M$, where $0 < \nu(M) \le 1 \le L(M)$ are, respectively, non-increasing and non-decreasing functions of $M \ge 3N$. Both Lemma 3.2.4 and of Lemma 3.2.5 hold assuming (3.2.11) instead of $(3.0.12)_1$, with exponents δ_g, σ_g depending again on $\|u\|_{L^\infty(B_R)}$ and $\|v\|_{L^\infty(B_R)}$. This can be easily seen (for instance in the proof of Lemma 3.2.4) by observing that $\|\tilde{w}\|_{L^\infty(B_R)} \le 3\|u\|_{L^\infty(B_R)}$ and therefore (3.2.11) can be used with M depending only on $\|u\|_{L^\infty(B_R)}$ in (3.2.8). In the same way, the content of Lemma 3.2.2 still holds under assumptions (3.2.11).

Remark 3.2.3 The content of Lemma 3.2.5 applies in particular to the case when the function $a(x) \equiv a_0 \ge 0$ is constant and $H(x,z) \equiv H_0(z) := |z|^p + a_0|z|^q$. In this case assumption $(3.0.15)_1$ is not necessary and the statement continues to hold whenever $p \le q$ are arbitrary.

3.2.2 On the Euler-Lagrange equation under non-standard growth conditions

Let us consider a ball $B_r \Subset \Omega$ and $v \in W^{1,H}(B_r,\mathbb{S}^{N-1})$ being a solution of the frozen Dirichlet problem

$$v \mapsto \min_{w\in W^{1,H}_u(B_r,\mathbb{S}^{N-1})} \int_{B_r} g(x,Dw)\ \mathrm{d}x\ , \tag{3.2.12}$$

where, with $\bar{u} \in \mathbb{R}^N$ being fixed, we have set

$$g(x,z) := F(x,\bar{u},z) = \tilde{F}(x,\bar{u},|z|)\ , \tag{3.2.13}$$

for $x \in \Omega$ and $z \in \mathbb{R}^{N\times n}$ and of course $F(\cdot)$ is the Carathéodory integrand considered in (3.0.1); this time we assume that $F(\cdot)$ satisfies only $(3.0.12)_{1,2}$. By definition, $g(\cdot)$ matches $(3.0.12)_1$-$(3.0.12)_2$. Because of the non-standard growth conditions considered here, we cannot derive the Euler-Lagrange equation for (3.2.12) in the usual way, adopting variations defined through smooth φ and then concluding via a density argument. We shall rather use a direct argument, eventually leading to establish that

$$\int_{B_r} \tilde{F}'(x,\bar{u},|Dv|)\frac{Dv}{|Dv|}\cdot D\varphi - \tilde{F}'(x,\bar{u},|Dv|)|Dv|(v\cdot\varphi)\ \mathrm{d}x$$
$$= \int_{B_r} \partial_z F(x,\bar{u},|Dv|)\cdot D\varphi - \tilde{F}'(x,\bar{u},|Dv|)|Dv|(v\cdot\varphi)\ \mathrm{d}x = 0 \tag{3.2.14}$$

holds whenever $\varphi \in (W^{1,H}_0 \cap L^\infty)(B_r,\mathbb{R}^N)$; needless to say, the symbol $\tilde{F}'$ denotes the derivative of $\tilde{F}$ with respect to the last variable. For the sake of completeness we report all the details. To proceed, for $s \in (0,1)$, define the variation $v_s := \Pi(v+s\varphi)$, where $\Pi(y) = y/|y|$ for $y \in \mathbb{R}^N \setminus \{0\}$. Clearly, for s sufficiently small, $v_s \in W^{1,H}_u(B_r,\mathbb{S}^{N-1})$. The minimality of v and the very definition of v_s tell us that

$$0 \le \int_{B_r} \frac{g(x,Dv_s)-g(x,Dv)}{s}\ \mathrm{d}x$$
$$= \frac{1}{s}\int_{B_r}\left(\int_0^1 \partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)\ \mathrm{d}\lambda\right)\cdot(Dv_s-Dv)\ \mathrm{d}x\ . \tag{3.2.15}$$

We aim to pass to the limit with $s \to 0$ in (3.2.15) via the dominated convergence theorem. A direct computation and the fact that $\nabla\Pi(v)Dv = Dv$ show that

$$\begin{aligned}|Dv_s - Dv| &= |\nabla\Pi(v+s\varphi)(Dv+sD\varphi)-\nabla\Pi(v)Dv| \\ &\le |\nabla\Pi(v+s\varphi)-\nabla\Pi(v)||Dv| + |s\nabla\Pi(v+s\varphi)D\varphi| \le cs\left(|\varphi||Dv|+|D\varphi|\right)\ ,\end{aligned} \tag{3.2.16}$$

where $c \equiv c(\|\nabla\Pi\|_\infty, \|\nabla^2\Pi\|_\infty) \equiv c(N)$. Plugging (3.2.16) into the last term in the right-hand side of (3.2.15) we obtain

$$\frac{1}{s}\left|\left(\int_0^1 \partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)\ \mathrm{d}\lambda\right)\cdot(Dv_s-Dv)\right|$$
$$\le c(|\varphi||Dv|+|D\varphi|)\int_0^1 |\partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)|\ \mathrm{d}\lambda \le c|(\mathrm{I})| + c|(\mathrm{II})|,$$

with $c \equiv c(N)$. From $(3.0.12)_2$, Young's inequality and (3.2.16), we estimate

$$\begin{aligned}|(\mathrm{I})| &\le c\|\varphi\|_{L^\infty(B_r)}\int_0^1 \frac{H(x,\lambda Dv_s+(1-\lambda)Dv)}{|\lambda Dv_s+(1-\lambda)Dv|}|Dv|\ \mathrm{d}\lambda \\ &\le c\|\varphi\|_{L^\infty(B_r)}\int_0^1 [H(x,\lambda Dv_s)+H(x,(1-\lambda)Dv)+H(x,Dv)]\ \mathrm{d}\lambda \\ &\le cH(x,D\varphi)+cH(x,Dv)\ ,\end{aligned}$$

where $c \equiv c(N,L,p,q,\|\varphi\|_{L^\infty(B_r)})$. In a similar way we also have

$$\begin{aligned}|(\mathrm{II})| &\le c\int_0^1 \frac{H(x,\lambda Dv_s+(1-\lambda)Dv)}{|\lambda Dv_s+(1-\lambda)Dv|}|D\varphi|\ \mathrm{d}\lambda \\ &\le c\int_0^1 [H(x,\lambda Dv_s)+H(x,(1-\lambda)Dv)+H(x,D\varphi)]\ \mathrm{d}\lambda\end{aligned}$$

$$\leq \quad cH(x,D\varphi)+cH(x,Dv)\ ,$$

with $c \equiv c(N,L,p,q)$. Merging the content of the last three displays yields

$$\frac{1}{s}\left|\left(\int_0^1 \partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)\ \mathrm{d}\lambda\right)\cdot(Dv_s-Dv)\right| \leq cH(x,Dv)+cH(x,D\varphi)\ , \quad (3.2.17)$$

again for $c \equiv c(N,L,p,q,\|\varphi\|_{L^\infty(B_r)})$. Finally, by the regularity of $z \mapsto g(\cdot,z)$, (3.2.16) and $(3.0.12)_2$ we notice that

$$\begin{cases} |\partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)|\ \mathrm{d}\lambda \leq c\left(\dfrac{H(x,Dv)}{|Dv|}+\dfrac{H(x,D\varphi)}{|D\varphi|}\right) \\ \\ \lim\limits_{s\to 0} \partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)=\partial_z g(x,Dv)\ , \end{cases}$$

thus, by the dominated convergence theorem,

$$\lim_{s\to 0}\int_0^1 \partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)\ \mathrm{d}\lambda = \partial_z g(x,Dv)\ . \quad (3.2.18)$$

Using (3.2.16) and (3.2.18) we can compute

$$\begin{aligned} \lim_{s\to 0}\frac{1}{s}\left(\int_0^1 \partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)\ \mathrm{d}\lambda\right)\cdot(Dv_s-Dv) &= \partial_z g(x,Dv)\cdot\nabla\Pi(v)D\varphi \\ +\lim_{s\to 0}\left(\int_0^1 \partial_z g(x,\lambda Dv_s+(1-\lambda)Dv)\ \mathrm{d}\lambda\right)&\cdot\frac{\nabla\Pi(v+s\varphi)-\nabla\Pi(v)}{s}Dv \\ =\partial_z g(x,Dv)\cdot\Big(\nabla\Pi(v)D\varphi+\nabla^2\Pi(v)\varphi Dv\Big)\ . \end{aligned} \quad (3.2.19)$$

Now, (3.2.15), (3.2.17), (3.2.19) and the dominated convergence theorem render

$$0 \leq \int_{B_r}\partial_z g(x,Dv)\cdot\Big(\nabla\Pi(v)D\varphi+\nabla^2\Pi(v)\varphi Dv\Big)\ \mathrm{d}x\ .$$

The same argument with $s \in (-1,0)$ finally yields

$$\int_{B_r}\partial_z g(x,Dv)\cdot\Big(\nabla\Pi(v)D\varphi+\nabla^2\Pi(v)\varphi Dv\Big)\ \mathrm{d}x = 0\ .$$

Taking into account the symmetry of the Jacobian of the projector, we can conclude that

$$\begin{aligned} 0 &= \int_{B_r}\tilde{F}'(x,\bar{u},|Dv|)\frac{Dv}{|Dv|}\cdot\Big(\nabla\Pi(v)D\varphi+\nabla^2\Pi(v)\varphi Dv\Big)\ \mathrm{d}x \\ &= \int_{B_r}\tilde{F}'(x,\bar{u},|Dv|)\frac{Dv}{|Dv|}\cdot D\varphi-\frac{\tilde{F}'(x,\bar{u},|Dv|)}{|Dv|}A_v(Dv,Dv)\varphi\ \mathrm{d}x \\ &= \int_{B_r}\tilde{F}'(x,\bar{u},|Dv|)\frac{Dv}{|Dv|}\cdot D\varphi-\tilde{F}'(x,\bar{u},|Dv|)|Dv|(v\cdot\varphi)\ \mathrm{d}x\ , \end{aligned}$$

where in the last line we have used the explicit expression of the second fundamental form $A_v(\cdot,\cdot)$ of $\mathbb{S}^{N-1}$; see also [234, Section 2.2]. We have therefore proved the validity of (3.2.14).

3.2.3 A Morrey type decay estimate

In this section we briefly revisit some scalar regularity results reported in [12,60], adapting them to the vectorial case. We consider unconstrained local minimizers of functionals of the type

$$w \mapsto \int_{B_r} g(x, Dw)\ \mathrm{d}x \tag{3.2.20}$$

under the structure condition (3.2.13). We then have the following:

Theorem 4 ([12, 60]) *Let $h \in (W^{1,H} \cap L^\infty)(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional in (3.2.20) under assumptions $(3.0.12)_{1,2,3,4}$, $(3.0.14)_1$, (3.2.13) and $(3.0.15)_1$. Then for every $\sigma \in (0, n]$ there exists a constant $c \equiv c(\boldsymbol{data}, \|u\|_{L^\infty(\Omega)}, \sigma)$ such that*

$$\int_{B_t} H(x, Dh)\, dx \le c \left(\frac{t}{s}\right)^{n-\sigma} \int_{B_s} H(x, Dh)\ \mathrm{d}x \tag{3.2.21}$$

holds whenever $B_t \subset B_s \subset \Omega$ are concentric balls such that $t \le 1$.

Proof. The proof can be obtained tracking the ones given for [60, Proposition 3.4] and [12, Theorem 2], according to the remarks made in the proof of [60, Theorem 5.2], that describes the modifications to make with respect to the scalar case. Reference [12] is actually more suitable as the regularity results are proved for general functionals, without assuming the splitting structure considered in [60]. The remarks given in [60, Section 5,2] will be also useful here. As an outcome of the proofs of [60, Proposition 3.4] and [12, Theorem 2], estimate (3.2.21) follows provided $B_t \subset B_s \subset \Omega_0 \Subset \Omega$ are concentric balls and with an additional dependence of the constant c on $\mathrm{dist}(\Omega_0, \Omega)$, but under the full bound $q \le p+\alpha$. As remarked in [12,60], it is possible to reach the borderline case $q \le p+\alpha$ in the scalar case by using the preliminary local Hölder continuity of h for some exponent $\gamma \in (0, 1)$ (see [60, Proposition 3.1]); the same happens in [12, Theorem 6]. This comes along with an *a priori* estimate of the type $[h]_{0,\gamma;\Omega_0} < c$, where, amongst the other things, c depends also on $\mathrm{dist}(\Omega_0, \Omega)$. This is exactly the point where the dependence on $\mathrm{dist}(\Omega_0, \Omega)$ comes from in the final statement of (3.2.21) from [60, Proposition3.4] and [12, Theorem 2]. On the other hand, as already remarked in the proof of [60, Theorem 5.2], when considering the bound $q < p + \alpha$ we can avoid using that $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ and in this way, taking into account the proofs in [12], we arrive at (3.2.21) with the dependence of the constant c as described in the statement of Theorem 4. Notice that, in order to prove (3.2.21), in [12, 60] it is also necessary to replace the a priori Lipschitz estimate for minima of frozen functionals in [12, (132)] with an analogous one for the vectorial case. This is discussed in Remark 3.2.4 below. Notice that here we are not assuming that the function $\tilde{F}(\cdot)$ satisfies $(3.0.14)_2$. □

Remark 3.2.4 Let us consider a local minimizer $v \in W^{1,H_0}(B_r, \mathbb{R}^{N\times n})$ of the functional

$$w \mapsto \int_{B_r} g(x_0, Dw)\ \mathrm{d}x \qquad x_0 \in \Omega\ , \tag{3.2.22}$$

where $H_0(z) \equiv |z|^p + a(x_0)|z|^q$. The following estimate holds:

$$\sup_{B_{r/2}} H_0(Dw) \le c \fint_{B_r} H_0(Dw)\ \mathrm{d}x, \tag{3.2.23}$$

where $c \equiv c(n, N, \nu, L, p, q)$. This estimate plays a crucial role in the proofs given in [12,60], and these are concerned with the scalar case. To get that this result holds in our vectorial case too it is sufficient to prove that

$$\tilde{F}''(x, \bar{u}, t)t \approx \tilde{F}'(x, \bar{u}, t) \qquad \text{for every } t > 0 \tag{3.2.24}$$

(for implied constants depending only on n, N, ν, L, p, q) and then appeal for instance to [89, Lemma 5.8]. Indeed, following [68, Lemma 3.4], we see that

$$\partial_{zz} g(x_0, z) = \tilde{F}''(x_0, \bar{u}, |z|) \frac{z \otimes z}{|z|^2} + \tilde{F}'(x_0, \bar{u}, |z|) \left[\frac{\mathbb{I}_{N\times n}}{|z|} - \frac{z \otimes z}{|z|^3} \right] , \tag{3.2.25}$$

holds for every $z \in \mathbb{R}^{N\times n}$ such that $|z| \neq 0$; here it is $\mathbb{I}_{N\times n} = \delta_{ij}\delta_{\alpha\beta}$. Testing the above inequality for $\xi \perp z$ and for $\xi = z$ and using $(3.0.12)_{2,3}$ yields

$$\frac{\nu H(x_0, t)}{t} \leq \tilde{F}'(x_0, \bar{u}, t) \leq \frac{LH(x_0, t)}{t} \quad \text{and} \quad \frac{\nu H(x_0, t)}{t^2} \leq \tilde{F}''(x_0, \bar{u}, t) \leq \frac{LH(x_0, t)}{t^2} , \tag{3.2.26}$$

respectively, for every $t > 0$, so that (3.2.24) follows. Notice that here we are only assuming that $\tilde{F}(\cdot)$ satisfies only $(3.0.14)_1$.

Remark 3.2.5 This is a side remark of later use. Assuming that the function $\tilde{F}(\cdot)$ satisfies $(3.0.14)_{1,2}$, as in [68, Lemma 3.4], by using (3.2.26) we get that $(3.0.14)_2$ can be reformulated as

$$\left| \tilde{F}''(x, v, t+s) - \tilde{F}''(x, v, t) \right| \leq c(n, N, \nu, L, p, q) \tilde{F}''(x, v, t) \left(\frac{|s|}{t} \right)^{\beta_1} . \tag{3.2.27}$$

3.3 Harmonic type approximation

In this section we revisit the arguments of [12, 90], to give two kinds of harmonic type approximation lemmas. The most peculiar one is the first, which is given in terms of a generalized Young functions (specifically, $H(\cdot)$), rather than a usual Young function. Therefore all the arguments used there will be of intrinsic type. This perfectly combines with the type of intrinsic estimates already proved in [12, 60, 61], as we shall see in the next section when showing regularity theorems. Accordingly to the notation already established in (3.0.11), with $B_\varrho \Subset \Omega$ being a ball, we shall denote

$$H^-_{B_\varrho}(t) = t^p + a_{\mathrm{i}}(B_\varrho) t^q \qquad \text{and} \qquad H^+_{B_\varrho}(t) = t^p + a_{\mathrm{s}}(B_\varrho) t^q . \tag{3.3.1}$$

We shall again denote, with abuse of notation, $H^-_{B_\varrho}(z) \equiv H^-_{B_\varrho}(|z|)$ and so forth, also in the case $z \in \mathbb{R}^{N\times n}$.

Remark 3.3.1 We collect some features of the functions in (3.3.1). We first notice thatn $H^\pm_{B_\varrho}(\cdot)$ is a Young function in the sense of [89, Section 2], and satisfies the Δ_2-condition. Since $t \mapsto H^\pm_{B_\varrho}(t)$ is strictly increasing and strictly convex, its inverse $(H^\pm_{B_\varrho})^{-1}$ is strictly increasing and strictly concave and $(H^\pm_{B_\varrho})^{-1}(0) = 0$, thus $(H^\pm_{B_\varrho})^{-1}$ is subadditive. Therefore, for all $\lambda \geq 0$, the subadditivity and the monotonicity of $(H^\pm_{B_\varrho})^{-1}(t)$ yield

$$(H^\pm_{B_\varrho})^{-1}(\lambda t) \leq (\lambda + 1)(H^\pm_{B_\varrho})^{-1}(t) \tag{3.3.2}$$

In particular, if $\lambda \geq 1$, $(H^\pm_{B_\varrho})^{-1}(\lambda t) \leq 2\lambda (H^\pm_{B_\varrho})^{-1}(t)$. Next, notice that if $B_\varrho = B_\varrho(x_0) \Subset \Omega$, then the function $(x_0, \varrho, t) \mapsto H^\pm_{B_\varrho(x_0)}(t)$ is continuous on $\Omega \times [0, \infty) \times [0, \infty)$. This easily follows from the Hölder continuity of $a(\cdot)$. Finally, for x_0 fixed, if $\varrho_1 \leq \varrho_2$, then $H^-_{B_{\varrho_1}(x_0)}(t) \geq H^-_{B_{\varrho_2}(x_0)}(t)$, $(H^+_{B_{\varrho_1}(x_0)}(t) \leq H^+_{B_{\varrho_2}(x_0)}(t))$ holds uniformly in $t \geq 0$, and, as a consequence, $(H^-_{B_{\varrho_1}(x_0)})^{-1}(t) \leq (H^-_{B_{\varrho_2}(x_0)})^{-1}(t)$, (resp. $(H^+_{B_{\varrho_1}(x_0)})^{-1}(t) \geq (H^+_{B_{\varrho_2}(x_0)})^{-1}(t)$), holds too for all $t \geq 0$.

We start with a classical lemma (see [10] for a description and references), which is concerned with some properties of Maximal operators with respect to the so called gradient truncation. We recall that the Hardy-Littlewood maximal operator is defined as follows

$$M(f)(x) := \sup_{B_\varrho(x)\subset\mathbb{R}^n} \fint_{B_\varrho(x)} |f(y)| \ \mathrm{d}y \ , \quad x \in \mathbb{R}^n \ ,$$

whenever $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 3.3.1 *Let $B_\varrho \subset \mathbb{R}^n$ be a ball and $w \in W^{1,1}_0(B_\varrho,\mathbb{R}^N)$ (trivially extended by zero outside B_ϱ). Then for any $\lambda > 0$ there exists $w_\lambda \in W^{1,\infty}_0(B_\varrho,\mathbb{R}^N)$ such that*

$$\|Dw_\lambda\|_{L^\infty(B_\varrho,\mathbb{R}^{N\times n})} \le c\lambda \ , \tag{3.3.3}$$

for some positive constant $c \equiv c(n,N)$. Moreover, it holds that

$$B_\varrho \cap \{w \neq w_\lambda\} \subset \big(B_\varrho \cap \{M(|Dw|) > \lambda\}\big) \cup \ \textit{negligible set} \ . \tag{3.3.4}$$

We have a first quantitative harmonic approximation type lemma.

Lemma 3.3.2 (Intrinsic and quantitative $g(x,\cdot)$-harmonic approximation) *Let $B_r \subset \mathbb{R}^n$ be a ball with radius $r \le 1$ and such that $B_{2r} \Subset \Omega$, $\varepsilon \in (0,1)$ and $v \in (W^{1,H} \cap L^\infty)(B_r,\mathbb{R}^N)$, $N \ge 1$, be a function satisfying*

$$\fint_{B_r} H(x,Dv) \ \mathrm{d}x \le c_1 H^-_{B_r}\left(\frac{\varepsilon}{r}\right) \ , \tag{3.3.5}$$

$$\left(\fint_{B_{r/2}} [H(x,Dv)]^{1+\delta} \ \mathrm{d}x\right)^{1/(1+\delta)} \le \tilde{c}_1 \fint_{B_r} H(x,Dv) \ \mathrm{d}x \ , \tag{3.3.6}$$

and

$$\left| \fint_{B_{r/2}} \partial_z g(x,Dv)\cdot D\varphi \ \mathrm{d}x \ \right| \le c_2\varepsilon^t \fint_{B_r} \left[H(x,Dv) + H\left(x, \|D\varphi\|_{L^\infty(B_{r/2})}\right)\right] \ \mathrm{d}x \ , \tag{3.3.7}$$

for some $t \in (0,1]$, $\delta \in (0,1)$ and all $\varphi \in C^\infty_c(B_{r/2},\mathbb{R}^N)$, where $g\colon \Omega\times\mathbb{R}^{N\times n} \to [0,\infty)$ is of the type in (3.2.13) under assumptions $(3.0.12)_{1,2,3,4}$, and where $c_1, \tilde{c}_1$ and c_2 are fixed constants larger than one. Then there exists $h \in W^{1,H}_v(B_{r/2},\mathbb{R}^N)$ such that

$$\int_{B_{r/2}} \partial_z g(x,Dh)\cdot D\varphi \ \mathrm{d}x = 0 \quad \textit{for all} \ \ \varphi \in W^{1,H}_0(B_{r/2},\mathbb{R}^N) \ , \tag{3.3.8}$$

$$\|h\|_{L^\infty(B_{r/2})} \le \sqrt{N}\|v\|_{L^\infty(B_{r/2})} \tag{3.3.9}$$

and

$$\fint_{B_{r/2}} \Big(|V_p(Dv)-V_p(Dh)|^2 + a(x)|V_q(Dv)-V_q(Dh)|^2\Big) \ \mathrm{d}x \le c\varepsilon^m \fint_{B_r} H(x,Dv) \ \mathrm{d}x \ , \tag{3.3.10}$$

with $c \equiv c(\texttt{data}_0)$ and $m = m\left(\texttt{data}, \|v\|_{L^\infty(B_r)}, t, \delta\right)$ (see (3.3.12) below for the meaning of $\texttt{data}_0$). Finally, the function $h \in W^{1,H}_v(B_{r/2},\mathbb{R}^N)$ is the unique solution of the Dirichlet problem

$$h \mapsto \min_{w\in W^{1,H}_v(B_{r/2},\mathbb{R}^N)} \int_{B_{r/2}} g(x,Dw) \ \mathrm{d}x \ . \tag{3.3.11}$$

Remark 3.3.2 The assumptions considered in Lemma 3.3.2 are tailored to the situations where the Lemma will be applied. In the typical applications, v is a minimizer of a constrained problem as considered in Theorem 1. This means that the condition $v \in (W^{1,H} \cap L^\infty)$ is automatically satisfied. For the same reason, assumption (3.3.6) is satisfied by Lemma 3.2.4. Finally, the smallness condition (3.3.5) typically occurs when proving partial regularity theorems (see next section). We also wish to point out that the proof we are going to give here allows for further generalizations to cases where instead of the function $H(\cdot)$ one considers more general instances, as for example those in Section 3.0.2.

Proof. In the following we shall abbreviate, as in (3.0.16), as follows

$$\boldsymbol{data}_0 \equiv \boldsymbol{data}_0 \left(n, N, \nu, L, p, q, \alpha, [a]_{0,\alpha}, \|v\|_{L^\infty(B_r)}, c_1, \tilde{c}_1, c_2\right). \tag{3.3.12}$$

By a standard approximation argument we notice that, if (3.3.7) holds for every $\varphi \in C_c^\infty(B_{r/2}, \mathbb{R}^N)$, then it also holds for every $\varphi \in W_0^{1,\infty}(B_{r/2}, \mathbb{R}^N)$. Now, let $h \in W_v^{1,H}(B_{r/2}, \mathbb{R}^N)$ be the unique solution to the Dirichlet problem (3.3.11). This can be obtained as follows. First, notice that solutions are always unique as a consequence of the strict convexity of $z \mapsto g(\cdot, z)$. Then, existence of $h \in W_v^{1,p}(B_{r/2}, \mathbb{R}^N)$ results by Direct Methods of the Calculus of Variations. By minimality and $(3.0.12)_1$, there holds

$$\fint_{B_{r/2}} H(x, Dh) \, dx \leq \frac{L}{\nu} \fint_{B_{r/2}} H(x, Dv) \, dx , \tag{3.3.13}$$

so h satisfies (3.3.8). Moreover, thanks to the assumptions in (3.2.13) and $(3.0.14)_1$, we can apply the maximum principle in [175, Theorem 2.3] and this yields (3.3.9). In particular, we conclude with $h \in W_v^{1,H}(B_{r/2}, \mathbb{R}^N)$. Thanks to Remark 3.2.2 and (3.3.6), we can also apply Lemma 3.2.5 (with v replaced by h and u replaced by v in the present situation); this, together with (3.3.13), yield that

$$\begin{aligned}\left(\fint_{B_{r/2}} [H(x, Dh)]^{1+\sigma_g} \, dx\right)^{\frac{1}{1+\sigma_g}} &\leq c \left(\fint_{B_{r/2}} [H(x, Dv)]^{1+\sigma_g} \, dx\right)^{\frac{1}{1+\sigma_g}} \\ &\leq c \left(\fint_{B_{r/2}} [H(x, Dv)]^{1+\delta} \, dx\right)^{\frac{1}{1+\delta}} \overset{(3.3.6)}{\leq} c \fint_{B_r} H(x, Dv) \, dx \end{aligned} \tag{3.3.14}$$

for positive constants $c \equiv c(\boldsymbol{data}, c_1)$ and $\sigma_g \equiv \sigma_g\left(\boldsymbol{data}, \|v\|_{L^\infty(B_r)}\right)$, with $c \geq 1$ and $\sigma_g \in (0, \delta)$. This peculiar dependence of the constants is also a consequence of (3.3.9) (see again Remark 3.2.2). In the application of Lemma 3.2.5 we are indeed getting rid of the dependence on $\|h\|_{L^\infty}$ by means of (3.3.9). Now, notice that there is no loss of generality in assuming that $\fint_{B_r} H(x, Dv) \, dx > 0$, otherwise $v \equiv$ const on B_r and the thesis trivially holds for $h \equiv$ const. From Remark 3.3.1, we have that $t \mapsto H_{B_r}^-(t)$ is a bijection, so there is a unique $\lambda > 0$ such that

$$H_{B_r}^-(\lambda) = \mathfrak{M} \fint_{B_r} H(x, Dv) \, dx \tag{3.3.15}$$

holds for some $\mathfrak{M} \geq 1$ whose size will be fixed later. Set $w = v - h \in W_0^{1,H}(B_{r/2}, \mathbb{R}^N)$ and consider $w_\lambda \in W_0^{1,\infty}(B_{r/2}, \mathbb{R}^N)$ given by Lemma 3.3.1, which satisfies (3.3.3) and (3.3.4). We deduce that

$$\frac{|B_{r/2} \cap \{w \neq w_\lambda\}|}{|B_{r/2}|} \overset{(3.3.4)}{\leq} \frac{|B_{r/2} \cap \{M(|Dw|) > \lambda\}|}{|B_{r/2}|}$$

$$\overset{\text{Chebyshev}}{\leq} \frac{c}{[H^-_{B_r}(\lambda)]^{1+\sigma_g}} \fint_{B_{r/2}} [H^-_{B_r}(M(|Dw|))]^{1+\sigma_g} \, dx$$
$$\overset{\text{maximal}}{\leq} \frac{c}{[H^-_{B_r}(\lambda)]^{1+\sigma_g}} \fint_{B_{r/2}} [H^-_{B_r}(Dw)]^{1+\sigma_g} \, dx$$
$$\leq \frac{c}{[H^-_{B_r}(\lambda)]^{1+\sigma_g}} \fint_{B_{r/2}} [H^-_{B_r}(Dh)]^{1+\sigma_g} + [H^-_{B_r}(Dv)]^{1+\sigma_g} \, dx$$
$$\overset{(3.3.14)}{\leq} \frac{c}{[H^-_{B_r}(\lambda)]^{1+\sigma_g}} \left(\fint_{B_r} H(x, Dv) \, dx \right)^{1+\sigma_g}$$
$$\overset{(3.3.15)}{\leq} \frac{c}{\mathfrak{M}^{1+\sigma_g}} \, , \tag{3.3.16}$$

where $c \equiv c(\texttt{data}_0)$. Now we test the weak formulation of (3.3.11) against w_λ to get

$$\begin{aligned} \mathcal{I}_1 :=& \fint_{B_{r/2}} \big(\partial_z g(x, Dv) - \partial_z g(x, Dh)\big) \cdot Dw_\lambda \chi_{\{w=w_\lambda\}} \, dx \\ =& \fint_{B_{r/2}} \partial_z g(x, Dv) \cdot Dw_\lambda \, dx - \fint_{B_{r/2}} \big(\partial_z g(x, Dv) - \partial_z g(x, Dh)\big) \cdot Dw_\lambda \chi_{\{w \neq w_\lambda\}} \, dx \\ =:& \mathcal{I}_2 + \mathcal{I}_3 \, . \end{aligned} \tag{3.3.17}$$

Upon setting (recall the definition in (2.4.1))

$$\mathcal{V}^2 := |V_p(Dv) - V_p(Dh)|^2 + a(x)|V_q(Dv) - V_q(Dh)|^2, \tag{3.3.18}$$

the strict monotonicity (3.1.3) implies there exists a constant $c \equiv c(n, N, \nu, p, q)$ such that

$$\mathcal{I}_1 \geq \frac{1}{c} \fint_{B_{r/2}} \mathcal{V}^2 \chi_{\{w=w_\lambda\}} \, dx.$$

Let us consider term $\mathcal{I}_2$; for this we start observing

$$\begin{aligned} \lambda &\overset{(3.3.15)}{=} \left(H^-_{B_r}\right)^{-1} \left(\mathfrak{M} \fint_{B_r} H(x, Dv) \, dx \right) \\ &\overset{(3.3.2),(3.3.5)}{\leq} 2c_1 \mathfrak{M} \left(H^-_{B_r}\right)^{-1} \left(H^-_{B_r}\left(\frac{\varepsilon}{r}\right)\right) \leq \frac{2c_1 \varepsilon \mathfrak{M}}{r} \, . \end{aligned}$$

From this last inequality and $(3.0.15)_1$ we can estimate

$$\begin{aligned} H^+_{B_r}(\lambda) &= H^-_{B_r}(\lambda) + \left[a_{\rm s}(B_r) - a_{\rm i}(B_r)\right] \lambda^q \\ &\leq H^-_{B_r}(\lambda) + c r^{\alpha-(q-p)} (\varepsilon \mathfrak{M})^{q-p} \lambda^p \leq c \left[1 + (\varepsilon \mathfrak{M})^{q-p}\right] H^-_{B_r}(\lambda) \, , \end{aligned} \tag{3.3.19}$$

with $c \equiv c(p, q, \alpha, [a]_{0,\alpha}, c_1)$. Now we have

$$\begin{aligned} \fint_{B_r} H\left(x, \|Dw_\lambda\|_{L^\infty(B_{r/2})}\right) dx &\overset{(3.3.3)}{\leq} c \fint_{B_r} H(x, \lambda) \, dx \\ &\leq c H^+_{B_r}(\lambda) \overset{(3.3.19)}{\leq} c \left[1 + (\varepsilon \mathfrak{M})^{q-p}\right] H^-_{B_r}(\lambda), \end{aligned} \tag{3.3.20}$$

where $c \equiv c(n, N, p, q, \alpha, [a]_{0,\alpha}, c_1)$, so that

$$|\mathcal{I}_2| \overset{(3.3.7)}{\leq} c_2 \varepsilon^t \fint_{B_r} \left[H(x, Dv) + H\left(x, \|Dw_\lambda\|_{L^\infty(B_{r/2})}\right)\right] dx$$

$$
\overset{(3.3.20)}{\leq} c\varepsilon^t \fint_{B_r} H(x,Dv)\ \mathrm{d}x + c\varepsilon^t \left[1+(\varepsilon\mathfrak{M})^{q-p}\right] H^-_{B_r}(\lambda)
$$
$$
\overset{(3.3.15)}{\leq} c\varepsilon^t \mathfrak{M}\left[1+(\varepsilon\mathfrak{M})^{q-p}\right] \fint_{B_r} H(x,Dv)\ \mathrm{d}x\ ,
$$

for $c \equiv c(n,N,p,q,\alpha,[a]_{0,\alpha},c_1,c_2)$. Finally, for $\mathcal{T}_3$, we fix $\kappa \in (0,1)$ to be chosen later on and estimate as follows:

$$
\begin{aligned}
|\mathcal{T}_3| &\overset{(3.3.17)}{\leq} \fint_{B_{r/2}} \big(|\partial_z g(x,Dh)| + |\partial_z g(x,Dv)|\big)\,|Dw_\lambda|\chi_{\{w\neq w_\lambda\}}\ \mathrm{d}x \\
&\overset{(3.0.12)}{\leq} c\fint_{B_{r/2}} \left(\frac{H(x,Dh)}{|Dh|} + \frac{H(x,Dv)}{|Dv|}\right) |Dw_\lambda|\chi_{\{w\neq w_\lambda\}}\ \mathrm{d}x \\
&\overset{\text{Young}}{\leq} \kappa \fint_{B_{r/2}} \big[H(x,Dh)+H(x,Dv)\big]\ \mathrm{d}x + \frac{c}{\kappa^{q-1}} \fint_{B_{r/2}} H(x,Dw_\lambda)\chi_{\{w\neq w_\lambda\}}\ \mathrm{d}x \\
&\overset{(3.3.13)}{\leq} c\kappa \fint_{B_{r/2}} H(x,Dv)\ \mathrm{d}x + \frac{c}{\kappa^{q-1}} \fint_{B_{r/2}} H(x,Dw_\lambda)\chi_{\{w\neq w_\lambda\}}\ \mathrm{d}x \\
&\overset{(3.3.3)}{\leq} c\kappa \fint_{B_r} H(x,Dv)\ \mathrm{d}x + \frac{c}{\kappa^{q-1}} \frac{|B_{r/2}\cap\{w\neq w_\lambda\}|}{|B_{r/2}|} H^+_{B_r}(\lambda) \\
&\overset{(3.3.16)}{\leq} c\kappa \fint_{B_r} H(x,Dv)\ \mathrm{d}x + \frac{c}{\kappa^{q-1}\mathfrak{M}^{1+\sigma_g}} H^+_{B_r}(\lambda) \\
&\overset{(3.3.19)}{\leq} c\kappa \fint_{B_r} H(x,Dv)\ \mathrm{d}x + \frac{c}{\kappa^{q-1}\mathfrak{M}^{1+\sigma_g}} \left[1+(\varepsilon\mathfrak{M})^{q-p}\right] H^-_{B_r}(\lambda) \\
&\overset{(3.3.15)}{\leq} c\left\{\kappa + \frac{1}{\kappa^{q-1}\mathfrak{M}^{\sigma_g}}\left[1+(\varepsilon\mathfrak{M})^{q-p}\right]\right\} \fint_{B_r} H(x,Dv)\ \mathrm{d}x\ ,
\end{aligned}
$$

with $c \equiv c(\texttt{data}_0)$. Collecting the estimates found for $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ to (3.3.17), we get

$$
\begin{aligned}
&\fint_{B_{r/2}} \mathcal{V}^2 \chi_{\{w=w_\lambda\}}\ \mathrm{d}x \\
&\quad \leq c\left\{\kappa + \varepsilon^t\mathfrak{M} + \varepsilon^{t+q-p}\mathfrak{M}^{q-p+1} + \frac{1}{\kappa^{q-1}\mathfrak{M}^{\sigma_g}} + \frac{(\varepsilon\mathfrak{M})^{q-p}}{\kappa^{q-1}\mathfrak{M}^{\sigma_g}}\right\} \fint_{B_r} H(x,Dv)\ \mathrm{d}x \\
&\quad =: cS(\kappa,\varepsilon,\mathfrak{M}) \fint_{B_r} H(x,Dv)\ \mathrm{d}x,
\end{aligned}
\tag{3.3.21}
$$

again with $c \equiv c(\texttt{data}_0)$. Now let $\theta \in (0,1)$ be a number to be fixed in some lines. From Hölder's inequality, (3.3.16) and (3.3.13) we obtain

$$
\begin{aligned}
\left(\fint_{B_{r/2}} \mathcal{V}^{2\theta}\chi_{\{w\neq w_\lambda\}}\ \mathrm{d}x\right)^{1/\theta} &\leq \left(\frac{|B_{r/2}\cap\{w\neq w_\lambda\}|}{|B_{r/2}|}\right)^{\frac{1-\theta}{\theta}} \fint_{B_{r/2}} \mathcal{V}^2\ \mathrm{d}x \\
&\leq \frac{c}{\mathfrak{M}^{(1+\sigma_g)\frac{1-\theta}{\theta}}} \fint_{B_r} H(x,Dv)\ \mathrm{d}x
\end{aligned}
$$

for $c \equiv c(\texttt{data}_0)$ and, again by Hölder's inequality and (3.3.21),

$$
\left(\fint_{B_{r/2}} \mathcal{V}^{2\theta}\chi_{\{w=w_\lambda\}}\ \mathrm{d}x\right)^{1/\theta} \leq cS(\kappa,\varepsilon,\mathfrak{M}) \fint_{B_r} H(x,Dv)\ \mathrm{d}x\ ,
$$

with $c \equiv c(\texttt{data}_0)$. Merging the content of the last two displays now gives

$$\left(\fint_{B_{r/2}} \mathcal{V}^{2\theta}\ \mathrm{d}x\right)^{1/\theta} \le c\left\{S(\kappa,\varepsilon,\mathfrak{M}) + \frac{1}{\mathfrak{M}^{(1+\sigma_g)\frac{1-\theta}{\theta}}}\right\}\fint_{B_r} H(x,Dv)\ \mathrm{d}x. \tag{3.3.22}$$

In (3.3.22), ε is fixed in the statement of the theorem, while $\kappa \in (0,1)$ and $M \ge 1$ are still free parameters to be chosen arbitrarily. We take

$$\mathfrak{M} = \frac{1}{\varepsilon^{\frac{t}{2}}} > 1 \qquad \text{and} \qquad \kappa = \varepsilon^{\frac{\sigma_g t}{4(q-1)}} \in (0,1)$$

and set

$$\bar{m} := \frac{t\sigma_g}{4}\min\left\{1, \frac{1}{q-1}\right\},$$

so that, recalling the expression of $S(\kappa,\varepsilon,\mathfrak{M})$ in (3.3.21), we find

$$S(\kappa,\varepsilon,\mathfrak{M}) + \frac{1}{\mathfrak{M}^{(1+\sigma_g)\frac{1-\theta}{\theta}}} \le 5\varepsilon^{\bar{m}} + \varepsilon^{\frac{t(1+\sigma_g)(1-\theta)}{2\theta}} \le 6\varepsilon^{\tilde{m}(\theta)}$$

where

$$\tilde{m}(\theta) := \min\left\{\bar{m}, \frac{t(1+\sigma_g)(1-\theta)}{2\theta}\right\} \equiv \tilde{m}(\sigma_g,t,q,\theta) \equiv \tilde{m}(\texttt{data}, \|v\|_{L^\infty(B_r)}, t, \delta, \theta), \tag{3.3.23}$$

and therefore (3.3.22) reads as

$$\left(\fint_{B_{r/2}} \mathcal{V}^{2\theta}\ \mathrm{d}x\right)^{\frac{1}{2\theta}} \le c\varepsilon^{\frac{\tilde{m}(\theta)}{2}}\left(\fint_{B_r} H(x,Dv)\ \mathrm{d}x\right)^{\frac{1}{2}} \tag{3.3.24}$$

with $c \equiv c(\texttt{data}_0)$. The final dependence on the various constants of m in (3.3.23) has been obtained recalling that $\sigma_g \equiv \sigma_g\left(\texttt{data}, \|v\|_{L^\infty(B_r)}\right)$; notice also that the dependence upon the initial higher integrability exponent δ appearing in (3.3.6) comes from the restriction $\sigma_g < \delta$. Next, notice that from the very definition of $\mathcal{V}$ in (3.3.18), and using (3.3.14), we readily infer

$$\left(\fint_{B_{r/2}} \mathcal{V}^{2(1+\sigma_g)}\ \mathrm{d}x\right)^{\frac{1}{2(1+\sigma_g)}} \le c\left(\fint_{B_r} H(x,Dv)\ \mathrm{d}x\right)^{\frac{1}{2}}, \tag{3.3.25}$$

again for $c \equiv c(\texttt{data}_0)$. Next, we choose

$$\theta := \frac{1+\sigma_g}{(1+2\sigma_g)} \equiv \theta\left(\texttt{data}, \|v\|_{L^\infty(B_r)}, \delta\right) \tag{3.3.26}$$

and apply Hölder's inequality with conjugate exponents $\frac{2(1+\sigma_g)}{1+2\sigma_g}$ and $2(1+\sigma_g)$, to get

$$\begin{aligned}\fint_{B_{r/2}} \mathcal{V}^2\ \mathrm{d}x &\le \left(\fint_{B_{r/2}} \mathcal{V}^{\frac{2(1+\sigma_g)}{1+2\sigma_g}}\ \mathrm{d}x\right)^{\frac{1+2\sigma_g}{2(1+\sigma_g)}}\left(\fint_{B_{r/2}} \mathcal{V}^{2(1+\sigma_g)}\ \mathrm{d}x\right)^{\frac{1}{2(1+\sigma_g)}} \\ &\overset{(3.3.24),(3.3.25)}{\le} c\varepsilon^{\frac{\tilde{m}(\theta)}{2}}\fint_{B_r} H(x,Dv)\ \mathrm{d}x,\end{aligned}$$

with where $c \equiv c(\texttt{data}_0)$. This concludes the proof of (3.3.10), and of Lemma 3.3.2, by fixing $m := \tilde{m}(\theta)/2$ and the dependence $m = m\left(\texttt{data}, \|v\|_{L^\infty(B_r)}, t, \delta\right)$ claimed in the statement follows by looking at (3.3.23) and (3.3.26). □

We next report another harmonic type approximation lemma of the type already considered in [12]. On the contrary to Lemma 3.3.2, this one involves a classical Young function $H_0(\cdot)$, i.e., no dependence on x is considered

$$H_0(t) = t^p + a_0 t^q \,, \qquad a_0 \geq 0 \,. \tag{3.3.27}$$

This time we shall consider a $C^1(\mathbb{R}^{N\times n})\cap C^2(\mathbb{R}^{N\times n}\setminus\{0\})$-regular integrand $g_0\colon \mathbb{R}^{N\times n}\to[0,\infty)$ such that

$$g_0(z) = \tilde{g}_0(|z|) \text{ holds for every } z\in\mathbb{R}^{N\times n} \text{ with } t\mapsto \tilde{g}_0(t) \text{ non-decreasing} \tag{3.3.28}$$

where $\tilde{g}_0\colon [0,\infty)\to[0,\infty)$ of class $C^1[0,\infty)\cap C^2(0,\infty)$. We shall consider the following set of assumptions:

$$\begin{cases} \nu H_0(z) \leq g_0(z) \leq L H_0(z) \\ |\partial_z g_0(z)||z| + |\partial_{zz} g_0(z)||z|^2 \leq L H_0(z) \\ \nu(|z|^{p-2} + a_0|z|^{q-2})|\xi|^2 \leq \langle \partial_{zz} g_0(z)\xi,\xi\rangle \,, \end{cases} \tag{3.3.29}$$

considered with the same notation as in (3.0.12), for suitable numbers $0<\nu\leq 1\leq L<+\infty$ (not necessarily the same as appearing in (3.0.12)). We then have the following approximation lemma, which is a different version of [12, Lemma 1]:

Lemma 3.3.3 (Quantitative g_0-harmonic approximation) *Let $B_r\subset\mathbb{R}^n$ be a ball with radius $r\leq 1$, $\varepsilon\in(0,1]$ and $v\in W^{1,H_0}(B_r,\mathbb{R}^N)$, $N\geq 1$, be a function satisfying*

$$\left(\fint_{B_{r/2}} [H_0(Dv)]^{1+\delta}\,\mathrm{d}x\right)^{1/(1+\delta)} \leq \tilde{c}_1 \fint_{B_r} H_0(Dv)\,\mathrm{d}x \,, \tag{3.3.30}$$

for some $\delta\in(0,1)$ and

$$\left|\fint_{B_{r/2}} \partial_z g_0(Dv)\cdot D\varphi\;\mathrm{d}x\right| \leq c_2\varepsilon^t \fint_{B_r}\left[H_0(Dv) + H_0\left(\|D\varphi\|_{L^\infty(B_{r/2})}\right)\right]\;\mathrm{d}x \,, \tag{3.3.31}$$

where $t\in(0,1]$ and all $\varphi\in C^\infty_c(B_{r/2},\mathbb{R}^N)$, where $\tilde{c}_1$ and c_2 are absolute constants and under assumptions (3.3.28)-(3.3.29). Then there exists $h_0\in W^{1,H_0}_v(B_{r/2},\mathbb{R}^N)$ such that

$$\int_{B_{r/2}} \partial_z g_0(Dh_0)\cdot D\varphi\;\mathrm{d}x = 0 \quad \textit{for all} \quad \varphi\in W^{1,H_0}_0(B_{r/2},\mathbb{R}^N)\,, \tag{3.3.32}$$

$$\|h_0\|_{L^\infty(B_{r/2})} \leq \sqrt{N}\|v\|_{L^\infty(B_{r/2})} \tag{3.3.33}$$

and

$$\fint_{B_{r/2}}\left(|V_p(Dv)-V_p(Dh_0)|^2 + a_0|V_q(Dv)-V_q(Dh_0)|^2\right)\;\mathrm{d}x \leq c\varepsilon^m \fint_{B_r} H_0(Dv)\;\mathrm{d}x\,,$$

with $c\equiv c(n,N,\nu,L,p,q,\tilde{c}_1,c_2)$ and $m=m(n,N,\nu,L,p,q,t,\delta)$. Finally, the function $h_0\in W^{1,H_0}_v(B_{r/2},\mathbb{R}^N)$ is the unique solution of the Dirichlet problem

$$h\mapsto \min_{w\in W^{1,H_0}_v(B_{r/2},\mathbb{R}^N)} \int_{B_{r/2}} g_0(Dh_0)\;\mathrm{d}x \,. \tag{3.3.34}$$

Proof. The proof is rather close to that of Lemma 3.3.3. For this reason we only give a sketch of it. Again, in (3.3.31) we can consider $\varphi \in W^{1,\infty}_0(B_{r/2},\mathbb{R}^N)$. This time we define $h_0 \in W^{1,H_0}(B_{r/2},\mathbb{R}^N)$ as the unique solution of the Dirichlet problem (3.3.34), so that (3.3.32)-(3.3.33) hold. Moreover, $(3.3.29)_1$ and minimality yield

$$\fint_{B_{r/2}} H_0(Dh_0)\,\mathrm{d}x \le \frac{L}{\nu}\fint_{B_{r/2}} H_0(Dv)\,\mathrm{d}x\,. \tag{3.3.35}$$

By Lemma 3.2.5 (with constant coefficients, see Remark 3.2.3), we get, as for (3.3.14) and using (3.3.35), that

$$\left(\fint_{B_{r/2}} [H_0(Dh_0)]^{1+\sigma_g}\,\mathrm{d}x\right)^{\frac{1}{1+\sigma_g}} \le c\left(\fint_{B_{r/2}} [H_0(Dv)]^{1+\sigma_g}\,\mathrm{d}x\right)^{\frac{1}{1+\sigma_g}} \overset{(3.3.30)}{\le} c\fint_{B_r} H_0(Dv)\,\mathrm{d}x\,, \tag{3.3.36}$$

holds for $\sigma_g \equiv \sigma_g(n,N,\nu,L,p,q) \in (0,\delta)$ and with $c \equiv c(n,N,\nu,L,p,q,\tilde{c}_1)$. Proceeding as for the proof of Lemma 3.3.2, we find $\lambda > 0$ such that

$$H_0(\lambda) = \mathfrak{M}\fint_{B_r} H_0(Dv)\,\mathrm{d}x, \tag{3.3.37}$$

for some $\mathfrak{M} \ge 1$ to be specified later on. Set $w = v - h_0 \in W^{1,H_0}_0(B_{r/2},\mathbb{R}^N)$ and consider w_λ given by Lemma 3.3.1 matching (3.3.3)-(3.3.4). As for the proof of (3.3.16), but using (3.3.36) and (3.3.37), we deduce that

$$\frac{|B_{r/2}\cap\{w\neq w_\lambda\}|}{|B_{r/2}|} \le \frac{c}{[H_0(\lambda)]^{1+\sigma_g}}\left(\fint_{B_r} H_0(Dv)\,\mathrm{d}x\right)^{1+\sigma_g} \overset{(3.3.37)}{\le} \frac{c}{\mathfrak{M}^{1+\sigma_g}}\,, \tag{3.3.38}$$

with $c \equiv c(n,N,\nu,L,p,q,\tilde{c}_1)$. Now we test (3.3.32) against w_λ and set

$$\begin{aligned}
\mathcal{T}_1 &:= \fint_{B_{r/2}} \big(\partial_z g_0(Dv) - \partial_z g_0(Dh_0)\big)\cdot Dw_\lambda \chi_{\{w=w_\lambda\}}\,\mathrm{d}x \\
&= \fint_{B_{r/2}} \partial_z g_0(Dv)\cdot Dw_\lambda\,\mathrm{d}x - \fint_{B_{r/2}} \big(\partial_z g_0(Dv) - \partial_z g_0(Dh_0)\big)\cdot Dw_\lambda \chi_{\{w\neq w_\lambda\}}\,\mathrm{d}x =: \mathcal{T}_2 + \mathcal{T}_3\,.
\end{aligned}$$

This time, as in (3.3.18), we set $\mathcal{V}_0^2 := |V_p(Dv) - V_p(Dh_0)|^2 + a_0|V_q(Dv) - V_q(Dh_0)|^2$. By monotonicity of $\partial_z g_0(\cdot)$ (which is similar to (3.1.2) for $a(x) \equiv a_0$), there is $c \equiv c(n,N,\nu,p,q)$ such that

$$\mathcal{T}_1 \ge \frac{1}{c}\fint_{B_{r/2}} \mathcal{V}_0^2 \chi_{\{w=w_\lambda\}}\,\mathrm{d}x. \tag{3.3.39}$$

As for $\mathcal{T}_2$, from (3.3.31), (3.3.3) and (3.3.37), we obtain

$$|\mathcal{T}_2| \le c\varepsilon^t\mathfrak{M}\fint_{B_r} H_0(Dv)\,\mathrm{d}x\,, \tag{3.3.40}$$

where $c \equiv c(n,N,L,p,q,c_1)$. Finally, for $\mathcal{T}_3$, we fix $\kappa \in (0,1)$ to be chosen. Then, by using $(3.3.29)_2$, Young's inequality, (3.3.35), (3.3.38) and (3.3.3), and proceeding as in the proof of the analogous term $\mathcal{T}_3$ from Lemma 3.3.2, we have

$$|\mathcal{T}_3| \le c\fint_{B_{r/2}} \left(\frac{H_0(Dh_0)}{|Dh_0|} + \frac{H_0(Dv)}{|Dv|}\right)|Dw_\lambda|\chi_{\{w\neq w_\lambda\}}\,\mathrm{d}x$$

$$
\begin{aligned}
&\le c\kappa \fint_{B_{r/2}} H_0(Dv)\ \mathrm{d}x + \frac{c}{\kappa^{q-1}} \fint_{B_{r/2}} H_0(Dw_\lambda)\chi_{\{w\neq w_\lambda\}}\ \mathrm{d}x \\
&\le c\left(\kappa + \frac{1}{\kappa^{q-1}\mathfrak{M}^{\sigma_g}}\right) \fint_{B_r} H_0(Dv)\ \mathrm{d}x\ , \qquad (3.3.41)
\end{aligned}
$$

with $c \equiv c(n, N, L, p, q, \tilde{c}_1)$. From estimates (3.3.39)-(3.3.41) we get

$$
\fint_{B_{r/2}} \mathcal{V}_0^2(x)\chi_{\{w=w_\lambda\}}\ \mathrm{d}x \le c\left\{\kappa + \varepsilon^t\mathfrak{M} + \frac{1}{\kappa^{q-1}\mathfrak{M}^{\sigma_g}}\right\} \fint_{B_r} H_0(Dv)\ \mathrm{d}x\ ,
$$

with $c \equiv c(n, N, \nu, L, p, q, \tilde{c}_1, c_2)$. Starting from the last inequality, the rest of the proof goes exactly as the one for Lemma 3.3.2, after (3.3.21). □

Finally, an elementary Young type inequality.

Lemma 3.3.4 *Let $H_0(\cdot)$ be the function defined in (3.3.27) . Then, whenever $\kappa \in (0,1)$ it holds that*

$$
st \le \kappa H_0(t) + \kappa^{-1/(p-1)} H_0^*(t)\,, \qquad \textit{for all}\ \ s, t \ge 0\ . \qquad (3.3.42)
$$

where

$$
H_0^*(t) := \sup_{s>0} \left(st - H_0(s)\right)\ , \qquad \textit{for all}\ \ t \ge 0
$$

denotes the convex conjugate function to $H_0(\cdot)$.

Proof. Notice that, for $A \ge 1$, as it is $q \ge p$, we have

$$
\begin{aligned}
H_0^*(At) &= \sup_{s>0} \left(sAt - H_0(s)\right) = A^{\frac{p}{p-1}} \sup_{s>0} \left(sA^{-\frac{1}{p-1}}t - A^{-\frac{p}{p-1}}H_0(s)\right) \\
&\le A^{\frac{p}{p-1}} \sup_{s>0} \left(sA^{-\frac{1}{p-1}}t - H_0(sA^{-\frac{1}{p-1}})\right) = A^{\frac{p}{p-1}} H_0^*(t)\ .
\end{aligned}
$$

Therefore, we find, for $\kappa \in (0,1)$

$$
st \le H_0\left(\kappa^{1/p}t\right) + H_0^*\left(t/\kappa^{1/p}\right) \le \kappa H_0(t) + \kappa^{-1/(p-1)} H_0^*(t)
$$

that is, (3.3.42). □

3.4 Proof of Theorem 1

In the following, $u \in W^{1,H}_{\mathrm{loc}}(\Omega, \mathbb{S}^{N-1})$ is as in the statement of Theorem 1. We start recalling Lemma 3.2.4, according to which there exists $\delta_g \equiv \delta_g(\boldsymbol{data}) > 0$ such that $H(\cdot, Du) \in L^{1+\delta_g}_{\mathrm{loc}}(\Omega)$ holds, i.e., (3.0.17) is proved. For the proof of Theorem 1, we first treat the case when $p(1+\delta_g) \le n$, and then we describe how to get the result in the remaining one $p(1+\delta_g) > n$. The proof now goes in six steps. The first three are devoted to the proof of the partial Hölder continuity of a constrained local minimizer of (3.0.1); in particular, in the third step we describe the regular and the singular sets. In the fourth step we exploit this continuity to move to a single chart. Step five is devoted to show partial Hölder continuity for the gradient in the regular set. In the final step we briefly mention how to treat the case $p(1+\delta_g) > n$.

Step 1: Freezing.
Let $B_r = B_r(x_0)$ be any ball such that $B_{2r} \Subset \Omega$ and $r \le 1/2$; more in general, every ball B

considered in the rest of the proof will have radius $r(B) \leq 1/2$. We assume that the smallness condition

$$\fint_{B_{2r}(x_0)} H(x, Du)\ \mathrm{d}x < H^-_{B_{2r}(x_0)}\left(\frac{\varepsilon}{2r}\right) , \tag{3.4.1}$$

holds for some $\varepsilon \in (0,1)$ which is going to be chosen in due course of the proof. Let $v \in W^{1,H}_u(B_r, \mathbb{S}^{N-1})$ be a solution to the frozen Dirichlet problem

$$v \mapsto \min_{w \in W^{1,H}_u(B_r, \mathbb{S}^{N-1})} \int_{B_r} F\left(x, (u)_{B_r}, Dw\right)\ \mathrm{d}x .$$

This functional satisfies the same growth assumptions (in particular $(3.0.12)_1$) of the original one minimized by u and therefore Lemma 3.2.4 applies, giving

$$\left(\fint_{B_{r/2}} [H(x, Dv)]^{1+\delta_g}\ \mathrm{d}x\right)^{1/(1+\delta_g)} \leq \tilde{c}_1 \fint_{B_r} H(x, Dv)\ \mathrm{d}x , \tag{3.4.2}$$

where the exponent $\delta_g \equiv \delta_g(\boldsymbol{data}) > 0$ is the same one appearing in (3.2.10) and $\tilde{c}_1 \equiv \tilde{c}_1(\boldsymbol{data})$. Taking into account the content of Section 3.2.2, and in particular (3.2.14), v solves the Euler-Lagrange equation

$$\fint_{B_r} \partial_z F\left(x, (u)_{B_r}, Dv\right) \cdot D\varphi\ \mathrm{d}x = \fint_{B_r} \tilde{F}'\left(x, (u)_{B_r}, Dv\right) |Dv| (v \cdot \varphi)\ \mathrm{d}x , \tag{3.4.3}$$

which is valid for any $\varphi \in (W^{1,H}_0 \cap L^\infty)(B_r, \mathbb{R}^N)$. Moreover, (3.1.3) becomes

$$\begin{aligned} &|V_p(z_2) - V_p(z_1)|^2 + a(x)|V_q(z_2) - V_q(z_1)|^2 + \partial_z F\left(x, (u)_{B_r}, z_1\right) \cdot (z_2 - z_1) \\ &\quad \leq c\left[F\left(x, (u)_{B_r}, z_2\right) - F\left(x, (u)_{B_r}, z_1\right)\right] \end{aligned} \tag{3.4.4}$$

which holds for any choice of $z_1, z_2 \in \mathbb{R}^{N\times n}$ and $x \in \Omega$, for a constant $c \equiv c(n, N, \nu, p, q)$, see for instance [12, (90)]. The map $w = u - v \in (W^{1,H}_0 \cap L^\infty)(B_r, \mathbb{R}^N)$ is an admissible test function in (3.4.3), therefore we have

$$\begin{aligned} &\fint_{B_r} \left(|V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2\right)\ \mathrm{d}x \\ &\overset{(3.4.4)}{\leq} c\fint_{B_r} \left[F\left(x, (u)_{B_r}, Du\right) - F\left(x, (u)_{B_r}, Dv\right)\right]\ \mathrm{d}x \\ &\quad - \fint_{B_r} \partial_z F\left(x, (u)_{B_r}, Dv\right) \cdot (Du - Dv)\ \mathrm{d}x \\ &\overset{(3.4.3)}{=} c\fint_{B_r} \left[F\left(x, (u)_{B_r}, Du\right) - F\left(x, (u)_{B_r}, Dv\right)\right]\ \mathrm{d}x \\ &\quad - \fint_{B_r} \tilde{F}'\left(x, (u)_{B_r}, Dv\right) |Dv| v \cdot (u - v)\ \mathrm{d}x \\ &\quad = c\fint_{B_r} \left[F\left(x, (u)_{B_r}, Du\right) - F(x, u, Du)\right]\ \mathrm{d}x \\ &\quad + c\fint_{B_r} \left[F(x, u, Du) - F(x, v, Dv)\right]\ \mathrm{d}x \\ &\qquad + c\fint_{B_r} \left[F(x, v, Dv) - F\left(x, (v)_{B_r}, Dv\right)\right]\ \mathrm{d}x \\ &\quad + c\fint_{B_r} \left[F\left(x, (v)_{B_r}, Dv\right) - F\left(x, (u)_{B_r}, Dv\right)\right]\ \mathrm{d}x \end{aligned}$$

$$- \fint_{B_r} \tilde{F}'\left(x, (u)_{B_r}, Dv\right) |Dv| v \cdot (u - v) \, \mathrm{d}x$$
$$=: \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)} \,, \tag{3.4.5}$$

where $c \equiv c(n, N, \nu, p, q)$. Before starting working on terms (I)-(V) in (3.4.5), let us estimate some quantities which will be recurrent in the forthcoming computations. First, notice that the minimality of v and $(3.0.12)_1$ yield

$$\nu \fint_{B_r} H(x, Dv) \, \mathrm{d}x \le \fint_{B_r} F\left(x, (u)_{B_r}, Dv\right) \, \mathrm{d}x$$
$$\le \fint_{B_r} F\left(x, (u)_{B_r}, Du\right) \, \mathrm{d}x \le L \fint_{B_r} H(x, Du) \, \mathrm{d}x \,. \tag{3.4.6}$$

Lemma 3.2.5 gives

$$\left(\fint_{B_r} [H(x, Dv)]^{1+\sigma_g} \, \mathrm{d}x \right)^{1/(1+\sigma_g)} \le c \left(\fint_{B_r} [H(x, Du)]^{1+\sigma_g} \, \mathrm{d}x \right)^{1/(1+\sigma_g)}$$
$$\overset{(3.3.6)}{\le} c \fint_{B_{2r}} H(x, Du) \, \mathrm{d}x \,, \tag{3.4.7}$$

where $0 < \sigma_g \equiv \sigma_g(\texttt{data}) < \delta_g$. We are next going to use the function $H^-_{B_{2r}}(t) := t^p + a_{\mathrm{i}}(B_{2r})t^q$. By Jensen's inequality (recall that $\omega(\cdot)$ in (3.0.13) is a concave function, while $H^-_{B_{2r}}(\cdot)$ is convex), Remark 3.3.1 and the smallness condition (3.4.1), we get

$$\begin{aligned}
\fint_{B_r} \omega\left(|u - (u)_{B_r}|\right) \, \mathrm{d}x \quad & \le \quad \omega\left(r \fint_{B_r} \left| \frac{u - (u)_{B_r}}{r} \right| \, \mathrm{d}x \right) \\
& = \quad \omega\left[r \left(H^-_{B_{2r}} \right)^{-1} \circ \left(H^-_{B_{2r}} \right) \left(\fint_{B_r} \left| \frac{u - (u)_{B_r}}{r} \right| \, \mathrm{d}x \right) \right] \\
& \le \quad \omega\left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_r} H^-_{B_{2r}} \left(\frac{u - (u)_{B_r}}{r} \right) \, \mathrm{d}x \right) \right] \\
& \le \quad \omega\left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_r} H\left(x, \frac{u - (u)_{B_r}}{r} \right) \, \mathrm{d}x \right) \right] \\
& \overset{(3.2.9)}{\le} \quad c\omega\left[r \left(H^-_{B_{2r}} \right)^{-1} \left(c \fint_{B_{2r}} H(x, Du) \, \mathrm{d}x \right) \right] \\
& \overset{(3.3.2),(3.4.1)}{\le} \quad c\omega\left[r \left(H^-_{B_{2r}} \right)^{-1} \circ H^-_{B_{2r}} \left(\frac{\varepsilon}{2r} \right) \right] \le c\varepsilon^{\beta} \,,
\end{aligned} \tag{3.4.8}$$

with $c \equiv c(\texttt{data}, \beta)$. Similarly, we have

$$\begin{aligned}
\fint_{B_r} \omega\left(|v - (v)_{B_r}|\right) \, \mathrm{d}x \quad & \le \quad c\omega\left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_r} H(x, Dv) \, \mathrm{d}x \right) \right] \\
& \overset{(3.4.6)}{\le} \quad c\omega\left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_{2r}} H(x, Du) \, \mathrm{d}x \right) \right] \\
& \overset{(3.4.8)}{\le} \quad c\varepsilon^{\beta} \,,
\end{aligned} \tag{3.4.9}$$

with $c \equiv c(\mathtt{data}, \beta)$. In a totally similar way, in particular again using Lemma 3.2.3 and repeatedly the content of Remark 3.3.1, we get

$$\begin{aligned}
\omega\left(|(u)_{B_r} - (v)_{B_r}|\right) &\leq \omega\left(\fint_{B_r} |u-v| \, dx\right) \\
&\leq \omega\left[r\left(H^-_{B_{2r}}\right)^{-1} \circ \left(H^-_{B_{2r}}\right)\left(\fint_{B_r}\left|\frac{u-v}{r}\right| \, dx\right)\right] \\
&\leq c\omega\left[r\left(H^-_{B_{2r}}\right)^{-1}\left(\fint_{B_r} H(x, Du-Dv) \, dx\right)\right] \\
&\leq c\omega\left[r\left(H^-_{B_{2r}}\right)^{-1}\left(\fint_{B_r} [H(x,Du)+H(x,Dv)] \, dx\right)\right] \\
&\overset{(3.4.6)}{\leq} c\omega\left[r\left(H^-_{B_{2r}}\right)^{-1}\left(\fint_{B_{2r}} H(x,Du) \, dx\right)\right] \overset{(3.4.8)}{\leq} c\varepsilon^{\beta} \qquad (3.4.10)
\end{aligned}$$

again with $c \equiv c(\mathtt{data}, \beta)$. We can now start estimating the terms (I)-(V) in (3.4.5); we have

$$\begin{aligned}
|(\mathrm{I})| &\leq c\fint_{B_r} \omega\left(|u-(u)_{B_r}|\right) H(x,Du) \, dx \\
&\leq c\left(\fint_{B_r} \omega\left(|u-(u)_{B_r}|\right)^{\frac{1+\delta_g}{\delta_g}} \, dx\right)^{\frac{\delta_g}{1+\delta_g}}\left(\fint_{B_r} [H(x,Du)]^{1+\delta_g} \, dx\right)^{\frac{1}{1+\delta_g}} \\
&\overset{(3.2.10)}{\leq} c\left(\fint_{B_r} \omega\left(|u-(u)_{B_r}|\right) \, dx\right)^{\frac{\delta_g}{1+\delta_g}}\left(\fint_{B_{2r}} H(x,Du) \, dx\right) \\
&\overset{(3.4.8)}{\leq} c(\mathtt{data}, \beta)\varepsilon^{\frac{\beta\delta_g}{1+\delta_g}}\fint_{B_{2r}} H(x,Du) \, dx\,. \qquad (3.4.11)
\end{aligned}$$

By minimality we see that

$$\fint_{B_r} F(x,u,Du) \, dx \leq \fint_{B_r} F(x,v,Dv) \, dx \Longrightarrow (\mathrm{II}) \leq 0\,. \qquad (3.4.12)$$

As for (III), we have

$$\begin{aligned}
|(\mathrm{III})| &\leq c\fint_{B_r} \omega\left(|v-(v)_{B_r}|\right) H(x,Dv) \, dx \\
&\leq c\left(\fint_{B_r} \omega\left(|v-(v)_{B_r}|\right)^{\frac{1+\sigma_g}{\sigma_g}} \, dx\right)^{\frac{\sigma_g}{1+\sigma_g}}\left(\fint_{B_r} [H(x,Dv)]^{1+\sigma_g} \, dx\right)^{\frac{1}{1+\sigma_g}} \\
&\overset{(3.4.7)}{\leq} c\left(\fint_{B_r} \omega\left(|v-(v)_{B_r}|\right) \, dx\right)^{\frac{\sigma_g}{1+\sigma_g}}\fint_{B_{2r}} H(x,Du) \, dx \\
&\overset{(3.4.9)}{\leq} c(\mathtt{data}, \beta)\varepsilon^{\frac{\beta\sigma_g}{1+\sigma_g}}\fint_{B_{2r}} H(x,Du) \, dx\,. \qquad (3.4.13)
\end{aligned}$$

The estimation of (IV) is analogous to that of (III), the only difference being that in this case we must use (3.4.10); we end up with

$$|(\mathrm{IV})| \leq c(\mathtt{data}, \beta)\varepsilon^{\frac{\beta\sigma_g}{1+\sigma_g}}\fint_{B_{2r}} H(x,Du) \, dx\,. \qquad (3.4.14)$$

Finally we look at term (V). Proceeding as for the previous terms, and in particular using the smallness condition (3.4.1) as done in the last line of display (3.4.8), we have

$$\begin{aligned}
|(\mathrm{V})| &\leq c \fint_{B_r} H(x,Dv)|u-v| \,\mathrm{d}x \\
&\leq c \left(\fint_{B_r} |u-v|^{\frac{1+\sigma_g}{\sigma_g}} \,\mathrm{d}x \right)^{\frac{\sigma_g}{1+\sigma_g}} \left(\fint_{B_r} [H(x,Dv)]^{1+\sigma_g} \,\mathrm{d}x \right)^{\frac{1}{1+\sigma_g}} \\
&\leq c \left(\fint_{B_r} |u-v| \,\mathrm{d}x \right)^{\frac{\sigma_g}{1+\sigma_g}} \left(\fint_{B_r} [H(x,Dv)]^{1+\sigma_g} \,\mathrm{d}x \right)^{\frac{1}{1+\sigma_g}} \\
&\leq c \left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_r} H^-_{B_{2r}} \left(\frac{u-v}{r} \right) \,\mathrm{d}x \right) \right]^{\frac{\sigma_g}{1+\sigma_g}} \fint_{B_{2r}} H(x,Du) \,\mathrm{d}x \\
&\leq c \left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_r} H\left(x, \frac{u-v}{r} \right) \,\mathrm{d}x \right) \right]^{\frac{\sigma_g}{1+\sigma_g}} \fint_{B_{2r}} H(x,Du) \,\mathrm{d}x \\
&\leq c \left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_r} H(x,Du-Dv) \,\mathrm{d}x \right) \right]^{\frac{\sigma_g}{1+\sigma_g}} \fint_{B_{2r}} H(x,Du) \,\mathrm{d}x \\
&\leq c \left[r \left(H^-_{B_{2r}} \right)^{-1} \left(\fint_{B_{2r}} H(x,Du) \,\mathrm{d}x \right) \right]^{\frac{\sigma_g}{1+\sigma_g}} \fint_{B_{2r}} H(x,Du) \,\mathrm{d}x \\
&\leq c(\boldsymbol{data},\beta)\varepsilon^{\frac{\sigma_g}{1+\sigma_g}} \fint_{B_{2r}} H(x,Du) \,\mathrm{d}x \,. \qquad (3.4.15)
\end{aligned}$$

Connecting estimates (3.4.11)-(3.4.15) to (3.4.5), and recalling that $\varepsilon < 1$, $\beta \leq 1$ and $\sigma_g < \delta_g$, we conclude with

$$\fint_{B_r} \left(|V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 \right) \,\mathrm{d}x \leq c\varepsilon^{\frac{\beta\sigma_g}{1+\sigma_g}} \fint_{B_{2r}} H(x,Du) \,\mathrm{d}x \,, \qquad (3.4.16)$$

holds for $c \equiv c(\boldsymbol{data},\beta)$.

Step 2: $\partial_z F(\cdot,(u)_{B_r},\cdot)$-harmonic approximation.
We aim to show that v matches the assumptions of Lemma 3.3.2, with the choice $g(x,z) \equiv \partial_z F(x,(u)_{B_r},z)$; obviously, $(3.0.12)_{1,2,3,4}$ are satisfied, as well as $(3.0.14)_1$, by the very definition of $g(\cdot)$. As for (3.3.5), we have

$$\begin{aligned}
E(v;B_r) &:= \fint_{B_r} H(x,Dv) \,\mathrm{d}x \overset{(3.4.6)}{\leq} \frac{L}{\nu} \fint_{B_r} H(x,Du) \,\mathrm{d}x \\
&=: \frac{L}{\nu} E(u;B_r) \overset{(3.4.1)}{<} 2^n \frac{L}{\nu} H^-_{B_{2r}} \left(\frac{\varepsilon}{2r} \right) < c_1 H^-_{B_r} \left(\frac{\varepsilon}{r} \right) \,, \qquad (3.4.17)
\end{aligned}$$

which is in fact (3.3.5) with $c_1 := 2^n L/\nu$. On the other hand, the validity of (3.3.6) is stated in (3.4.2). To verify (3.3.7), we look at the Euler-Lagrange equation (3.2.14) solved by v on $B_{r/2}$. By $(3.0.12)_1$, for any $\varphi \in W^{1,\infty}_0(B_{r/2},\mathbb{R}^N)$ we have

$$\left| \fint_{B_{r/2}} \partial_z F\left(x,(u)_{B_r},Dv\right) \cdot D\varphi \,\mathrm{d}x \right| = \left| \fint_{B_{r/2}} \tilde{F}'\left(x,(u)_{B_r},Dv\right) |Dv|(v \cdot \varphi) \,\mathrm{d}x \right|$$

$$
\begin{aligned}
&\leq c\fint_{B_{r/2}} H(x,Dv)|\varphi| \, \mathrm{d}x \\
&\leq cr\|D\varphi\|_{L^\infty(B_{r/2})}\fint_{B_{r/2}} H(x,Dv) \, \mathrm{d}x \\
&\leq cr\|D\varphi\|_{L^\infty(B_{r/2})}E(v;B_r) \qquad (3.4.18)
\end{aligned}
$$

with $c \equiv c(n,L,p,q)$. The last term in display (3.4.18) can be estimated via (3.3.42)

$$
cr\|D\varphi\|_{L^\infty(B_{r/2})}E(v;B_r) \leq \delta_1 H^-_{B_r}\left(\|D\varphi\|_{L^\infty(B_{r/2})}\right) + \frac{c}{\delta_1^{1/(p-1)}}\left(H^-_{B_r}\right)^*\left(rE(v;B_r)\right) \ , \qquad (3.4.19)
$$

with $c \equiv c(n,L,p,q)$ and $\delta_1 \in (0,1)$, where $\left(H^-_{B_r}\right)^*$ denotes the convex conjugate of $H^-_{B_r}$. Since

$$
\left(\left(H^-_{B_r}\right)^*\right)' = \left(\left(H^-_{B_r}\right)'\right)^{-1} \ , \qquad (3.4.20)
$$

then, for

$$
\varepsilon < \frac{p}{c_1} = \frac{p\nu}{2^n L} \qquad (3.4.21)
$$

(recall that ε in (3.4.1) is chosen in due course of the proof via various size restrictions), we find

$$
\begin{aligned}
\left(\left(H^-_{B_r}\right)^*\right)'\left(rE(v;B_r)\right) &\overset{(3.4.17)}{\leq} \left(\left(H^-_{B_r}\right)^*\right)'\left(c_1 r H^-_{B_r}\left(\frac{\varepsilon}{r}\right)\right) \\
&\leq \left(\left(H^-_{B_r}\right)^*\right)'\left(\frac{c_1}{p}\varepsilon\left(H^-_{B_r}\right)'\left(\frac{\varepsilon}{r}\right)\right) \\
&\overset{(3.4.21)}{\leq} \left(\left(H^-_{B_r}\right)^*\right)'\left(\left(H^-_{B_r}\right)'\left(\frac{\varepsilon}{r}\right)\right) \\
&\overset{(3.4.20)}{=} \left(\left(H^-_{B_r}\right)'\right)^{-1}\left(\left(H^-_{B_r}\right)'\left(\frac{\varepsilon}{r}\right)\right) = \frac{\varepsilon}{r} \ ,
\end{aligned}
$$

so we can conclude with

$$
\left(H^-_{B_r}\right)^*\left(rE(v;B_r)\right) \leq rE(v;B_r)\left(\left(H^-_{B_r}\right)^*\right)'\left(rE(v;B_r)\right) \leq \varepsilon E(v;B_r) \ .
$$

In this way (3.4.19) becomes

$$
cr\|D\varphi\|_{L^\infty(B_{r/2})}E(v;B_r) \leq \delta_1 H^-_{B_r}\left(\|D\varphi\|_{L^\infty(B_{r/2})}\right) + \frac{c\varepsilon}{\delta_1^{1/(p-1)}}\fint_{B_r} H(x,Dv) \, \mathrm{d}x \ .
$$

Now select $\delta_1 = \varepsilon^{(p-1)/2}$, and define $2t := \min\{p-1, 1\}$. The last inequality used in (3.4.18) gives

$$
\left|\fint_{B_{r/2}} \partial_z F\left(x,(u)_{B_r},Dv\right)\cdot D\varphi \, \mathrm{d}x\right| \leq c_2\varepsilon^t \fint_{B_r}\left[H(x,Dv) + H\left(x, \|D\varphi\|_{L^\infty(B_{r/2})}\right)\right] \, \mathrm{d}x \ ,
$$

for some $c_2 = c(n,L,p,q)$ which is in fact (3.3.7). So Lemma 3.3.2 applies and yields a $\partial_z F(\cdot,(u)_{B_r},\cdot)$-harmonic map $h \in W^{1,H}_v(B_{r/2},\mathbb{R}^N)$, specifically, a solution to

$$
h \mapsto \min_{w\in W^{1,H}_v(B_{r/2},\mathbb{R}^N)} \int_{B_{r/2}} F\left(x,(u)_{B_r},Dw\right) \, \mathrm{d}x \qquad (3.4.22)
$$

such that (3.3.10) holds; this, together with (3.4.6), allows to get

$$\fint_{B_{r/2}} \left(|V_p(Dv) - V_p(Dh)|^2 + a(x)|V_q(Dv) - V_q(Dh)|^2\right) \, \mathrm{d}x \leq c\varepsilon^m \fint_{B_r} H(x, Du) \, \mathrm{d}x \,, \quad (3.4.23)$$

where $c \equiv c(\textbf{\textit{data}}, \beta)$ and $m = m(\textbf{\textit{data}})$. Moreover, there holds that

$$\|h\|_{L^\infty(B_{r/2})} \leq \sqrt{N} \,. \quad (3.4.24)$$

By virtue of (3.4.22) and of the previous inequality, we are then able to apply Theorem 4. For every $\sigma \in (0, n]$, estimate (3.2.21) reads as

$$\fint_{B_t} H(x, Dh) \, \mathrm{d}x \leq c \left(\frac{t}{s}\right)^{-\sigma} \fint_{B_s} H(x, Dh) \, \mathrm{d}x \,, \quad (3.4.25)$$

that holds whenever $B_t \subset B_s \subset B_{r/2}$ are concentric balls, and where $c \equiv c(\textbf{\textit{data}}, \beta, \sigma)$, again by virtue of (3.4.24); in the following we take $\sigma < 1/4$. With $\tau \in (0, 1/2)$, recalling (3.1.1) and using (3.4.16) and (3.4.23), we can then estimate

$$\begin{aligned}
\fint_{B_{2\tau r}} H(x, Du) \, \mathrm{d}x \leq{}& c \fint_{B_{2\tau r}} \left(|V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2\right) \, \mathrm{d}x \\
&+ c \fint_{B_{2\tau r}} \left(|V_p(Dv) - V_p(Dh)|^2 + a(x)|V_q(Dv) - V_q(Dh)|^2\right) \, \mathrm{d}x \\
&+ c \fint_{B_{2\tau r}} H(x, Dh) \, \mathrm{d}x \\
\leq{}& c\tau^{-n} \fint_{B_r} \left(|V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2\right) \, \mathrm{d}x \\
&+ c\tau^{-n} \fint_{B_{r/2}} \left(|V_p(Dv) - V_p(Dh)|^2 + a(x)|V_q(Dv) - V_q(Dh)|^2\right) \, \mathrm{d}x \\
&+ c \fint_{B_{2\tau r}} H(x, Dh) \, \mathrm{d}x \\
\leq{}& c \left(\tau^{-n} \varepsilon^{\frac{\beta\sigma_g}{1+\sigma_g}} + \tau^{-n}\varepsilon^m + \tau^{-\sigma}\right) \fint_{B_{2r}} H(x, Du) \, \mathrm{d}x \,,
\end{aligned}$$

where $c \equiv c(\textbf{\textit{data}}, \beta, \sigma)$. Recalling the notation adopted in (3.4.17), the conclusion of the last display reads as

$$\begin{aligned}
E(u; B_{2\tau r}) \quad &\leq \quad c(\textbf{\textit{data}}, \beta, \sigma) \left(\tau^{-n} \varepsilon^{\frac{\beta\sigma_g}{1+\sigma_g}} + \tau^{-n}\varepsilon^m + \tau^{-\sigma}\right) E(u; B_{2r}) \\
&= \quad c(\textbf{\textit{data}}, \beta, \sigma)\tau^{\sigma} \left(\tau^{-n-\sigma} \varepsilon^{\frac{\beta\sigma_g}{1+\sigma_g}} + \tau^{-n-\sigma}\varepsilon^m + \tau^{-2\sigma}\right) E(u; B_{2r}) \,. \quad (3.4.26)
\end{aligned}$$

We can now determine $\tau \equiv \tau(\textbf{\textit{data}}, \beta, \sigma) \in (0, 1)$ such that

$$c(\textbf{\textit{data}}, \sigma)\tau^{\sigma} \leq \frac{1}{6}. \quad (3.4.27)$$

It is now time to choose the number ε coming from (3.4.1). Recalling (3.4.21), we now further reduce ε to have

$$\varepsilon < \min\left\{\frac{p\nu}{2^n L}, \tau^{\frac{(n-\sigma)(1+\sigma_g)}{\beta\sigma_g}}, \tau^{\frac{n-\sigma}{m}}\right\} \quad (3.4.28)$$

and notice that this fixes the dependence $\varepsilon \equiv \varepsilon(\texttt{data}, \beta, \sigma)$. By using (3.4.27) and (3.4.28) in (3.4.26), this last inequality reads as

$$E(u; B_{2\tau r}) \leq \tau^{-2\sigma} E(u; B_{2r}) \tag{3.4.29}$$

that is, recalling the definition in (3.4.17)

$$\int_{B_{2\tau r}} H(x, Du)\ \mathrm{d}x \leq \tau^{n-2\sigma} \int_{B_{2r}} H(x, Du)\ \mathrm{d}x\ . \tag{3.4.30}$$

Next, we observe that

$$\begin{aligned} E(u; B_{2\tau r}) &\overset{(3.4.29)}{\leq} \tau^{-2\sigma} E(u; B_{2r}) \overset{(3.4.1)}{<} \tau^{1-2\sigma}\tau^{-1} H^-_{B_{2r}}\left(\frac{\varepsilon}{2r}\right) \\ &\leq \tau^{-1} H^-_{B_{2\tau r}}\left(\frac{\varepsilon}{2r}\right) \leq H^-_{B_{2\tau r}}\left(\frac{\varepsilon}{2\tau r}\right)\ , \end{aligned}$$

and we conclude with

$$E(u; B_{2\tau r}) < H^-_{B_{2\tau r}}\left(\frac{\varepsilon}{2\tau r}\right)\ .$$

We have therefore proved that, for the choice of $\tau \equiv \tau(\texttt{data}, \beta, \sigma)$ and $\varepsilon \equiv \varepsilon(\texttt{data}, \beta, \sigma)$ made in (3.4.27) and (3.4.28), respectively, if the smallness condition (3.4.1) is satisfied on the ball B_{2r} it is also satisfied on the ball $B_{2\tau r}$. We can therefore repeat the whole argument developed after (3.4.1) starting from the ball $B_{2\tau r}$ instead of B_{2r}, thereby arriving at the analog of (3.4.29), that is $E(u; B_{2\tau^2 r}) \leq \tau^{-2\sigma} E(u; B_{2\tau r})$. This argument can obviously be iterated on the family of shrinking balls $\{B_{\tau^j r}\}$, thereby concluding that, for every $j \in \mathbb{N}$, it holds that

$$E(u; B_{2\tau^j r}) < H^-_{B_{2\tau^j r}}\left(\frac{\varepsilon}{2\tau^j r}\right)$$

and

$$\int_{B_{2\tau^j r}} H(x, Du)\ \mathrm{d}x < \tau^{(n-2\sigma)j} \int_{B_{2r}} H(x, Du)\ \mathrm{d}x\ .$$

In turn, a standard interpolation argument leads to conclude that

$$\int_{B_t(x_0)} H(x, Du)\ \mathrm{d}x \leq c\left(\frac{t}{r}\right)^{n-2\sigma} \int_{B_{2r}(x_0)} H(x, Du)\ \mathrm{d}x\ , \qquad \forall\, t \leq 2r\ , \tag{3.4.31}$$

where $c \equiv c(\texttt{data}, \beta, \sigma)$. Notice that the above inequality has been derived for $4\sigma < 1$ but it is then easily seen to hold whenever $\sigma \in (0, 1)$. Going back to (3.4.1), we observe that the two functions

$$x_0 \mapsto \fint_{B_{2r}(x_0)} H(x, Du)\ \mathrm{d}x \qquad \text{and} \qquad x_0 \mapsto H^-_{B_{2r}(x_0)}\left(\frac{\varepsilon}{2r}\right)$$

are continuous. This is a consequence of the absolute continuity of the integral for the former, and of Remark 3.3.1 for the latter. We conclude that, with σ being fixed, if (3.4.1) is satisfied at a point $x_0 \in \Omega$, then there exists ball $B_{4r_{x_0}}(x_0)$ such that

$$y \in B_{4r_{x_0}}(x_0) \Longrightarrow \fint_{B_{2r}(y)} H(x, Du)\ \mathrm{d}x < H^-_{B_{2r}(y)}\left(\frac{\varepsilon}{2r}\right)\ . \tag{3.4.32}$$

We then conclude that (3.4.31) holds (with y replacing x_0), and with the same constant $c \equiv c(\texttt{data}, \beta, \sigma)$, whenever $y \in B_{4r_{x_0}}(x_0)$. By a standard characterization of Hölder continuity it then follows that $u \in C^{0,\gamma}\left(B_{r_{x_0}}(x_0)\right)$ with $\gamma = 1 - 2\sigma/p$ (see Remark 3.4.1 below). As we can choose $\sigma \in (0, 1/4)$ arbitrarily, we have finally proved the following (we can switch from $2r$ to r now):

Proposition 3.4.1 *Let $u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{S}^{N-1})$ be a constrained local minimizer of the functional $\mathcal{F}$ in (3.0.1), under the assumptions (3.0.12)-(3.0.15). Assume that $p(1+\delta_g) \le n$. Then, for every positive exponent $\gamma < 1$, there exists another positive number $\varepsilon_\gamma \equiv \varepsilon_\gamma(\boldsymbol{data}, \beta, \gamma)$ such that if $B_r(x_0) \Subset \Omega$, $r \le 1$, and*

$$\left[H^-_{B_r(x_0)}\left(\frac{\varepsilon_\gamma}{r}\right)\right]^{-1} \fint_{B_r(x_0)} H(x, Du)\ \mathrm{d}x < 1\,, \tag{3.4.33}$$

then u is of class $C^{0,\gamma}$ in a neighbourhood of x_0.

Remark 3.4.1 Let us make the last argument somehow more quantitative. With $\gamma \in (0,1)$ being fixed, let $\tilde r(x_0)$ be the largest radius, such that the smallness condition (3.4.33) is satisfied with $r \equiv \tilde r(x_0)$ (together with $B_{2\tilde r(x_0)} \Subset \Omega$). Then we have that

$$\begin{aligned}\int_{B_t(x_0)} H(x, Du)\ \mathrm{d}x &\le c\left(\frac{t}{\tilde r(x_0)}\right)^{n-p+p\gamma} \int_{B_{2\tilde r(x_0)}(x_0)} H(x, Du)\ \mathrm{d}x \\ &\overset{(3.2.7)}{\le} c t^{n-p+p\gamma}\left[[\tilde r(x_0)]^{p-p\gamma} \fint_{B_{4\tilde r(x_0)}(x_0)} \left(\frac{|u|^p}{[\tilde r(x_0)]^p} + \|a\|_{L^\infty}\frac{|u|^q}{[\tilde r(x_0)]^q}\right)\ \mathrm{d}x\right].\end{aligned}$$

By using Poincarè's inequality, we get

$$\fint_{B_t(x_0)} |u-(u)_{B_t}|^p\ \mathrm{d}x \le \frac{c(\boldsymbol{data}, \|a\|_{L^\infty}, \beta, \gamma)}{[\tilde r(x_0)]^{q-p+p\gamma}} t^{p\gamma}\,.$$

This estimate remains stable whenever x_0 is replaced by $y \in B_{4r_{x_0}}(x_0)$ as in (3.4.32) and therefore, by a standard integral characterization of Hölder continuity, for all $\gamma \in (0,1)$ it follows that

$$[u]_{0,\gamma;B_{r_{x_0}}(x_0)} \le \frac{c(\boldsymbol{data}, \|a\|_{L^\infty}, \beta, \gamma)}{[\tilde r(x_0)]^{q/p-1+\gamma}}\,. \tag{3.4.34}$$

Step 3: Dimension of the singular set; the first estimate and proof of (3.0.20)*.*
Following a standard terminology, we denote by

$$\Omega_u := \left\{x_0 \in \Omega :\ \text{there is } B_{r_{x_0}}(x_0) \subset \Omega \text{ so that } u \in C^{0,\gamma}\left(B_{r_{x_0}}(x_0)\right) \text{ for some } \gamma \in (0,1)\right\}.$$

This set is open by definition and we denote $\Sigma_u := \Omega \setminus \Omega_u$, the so-called singular set. Indeed, we shall later on prove that Du is locally Hölder continuous on Ω_u and this justifies the terminology used here with respect with the one in the statement of Theorem 1. Let us now prove (3.0.20); call Σ the set in the right-hand side of (3.0.20). The inclusion $\Sigma_u \subset \Sigma$ is obvious in view of Proposition 3.4.1. On the other hand, take, by contradiction $x_0 \in \Sigma \setminus \Sigma_u$. Then there exists $\gamma \in (0,1)$ such that u is of class $C^{0,\gamma}\left(B_{r_{x_0}}(x_0)\right)$ for some ball $B_{r_{x_0}}(x_0)$. We then look at the Caccioppoli type inequality in (3.2.5); this implies that, for $\varrho \le r_{x_0}/2$

$$\fint_{B_\varrho(x_0)} H(x, Du)\ \mathrm{d}x \le c \fint_{B_{2\varrho}(x_0)} H^-_{B_{2\varrho}(x_0)}\left(\frac{u-(u)_{B_{2\varrho}(x_0)}}{2\varrho}\right)\ \mathrm{d}x \le c H^-_{B_\varrho(x_0)}\left(\varrho^{\gamma-1}\right)$$

and

$$\left[H^-_{B_\varrho(x_0)}\left(\frac{1}{\varrho}\right)\right]^{-1} \fint_{B_\varrho(x_0)} H(x, Du)\ \mathrm{d}x \le c\varrho^{\gamma p}\,, \tag{3.4.35}$$

for $c \equiv c(\texttt{data}, \beta, [u]_{0,\gamma})$ (here we have used that $H^-_{B_{2\varrho}} \le H^-_{B_\varrho}$). Letting $\varrho \to 0$ in the above display implies $x_0 \notin \Sigma$, a contradiction. This proves that $\Sigma \subset \Sigma_u$ and therefore (3.0.20). Next, observe that by definition we have $H^-_{B_\varrho(x_0)}(1/\varrho) \ge \varrho^{-p}$. Combining this with Lemma 3.2.4 (which is used to assert the integrability of $[H(x, Du)]^{1+\delta_g}$) we conclude with

$$\Sigma_u \subset \left\{ x_0 \in \Omega : \limsup_{\varrho \to 0} \varrho^{p(1+\delta_g)-n} \int_{B_\varrho(x_0)} [H(x, Du)]^{1+\delta_g} \, dx > 0 \right\} . \tag{3.4.36}$$

Giusti's Lemma ([123, Proposition 2.7]) then implies

$$\dim_{\mathcal{H}}(\Sigma_u) \le n - p - p\delta_g \Longrightarrow \mathcal{H}^{n-p}(\Sigma_u) = 0 \, . \tag{3.4.37}$$

Recall we are treating the case when $p(1 + \delta_g) \le n$. In particular, we have $|\Sigma_u| = 0$. Notice also that, once proved that Du is locally Hölder continuous in Ω_u, we shall have proved the validity of (3.0.39). In a totally similar way, assume also that $q(1 + \delta_g) \le n$; we observe that if $x_0 \in \Sigma_u$ is such that $a(x_0) > 0$, then $H^-_{B_\varrho(x_0)}(1/\varrho) \ge a_{\rm i}(B_\varrho(x_0))\varrho^{-q} > 0$ for ϱ sufficiently small, and we have

$$\Sigma_u \cap \{a(x) > 0\} \subset \left\{ x_0 \in \Omega : \limsup_{\varrho \to 0} \varrho^{q(1+\delta_g)-n} \int_{B_\varrho(x_0)} [H(x, Du)]^{1+\delta_g} \, \mathrm{d}x > 0 \right\} .$$

Therefore, again by Giusti's Lemma, we also have

$$\dim_{\mathcal{H}}(\Sigma_u) \le n - q - q\delta_g \Longrightarrow \mathcal{H}^{n-q}(\Sigma_u \cap \{a(x) > 0\}) = 0 \, . \tag{3.4.38}$$

Finally, we observe that in fact we have

$$\Omega_u = \left\{ x_0 \in \Omega : \text{ there is } B_{r_{x_0}}(x_0) \subset \Omega \text{ so that } u \in C^{0,\gamma}\left(B_{r_{x_0}}(x_0)\right) \text{ for every } \gamma \in (0,1) \right\} . \tag{3.4.39}$$

Indeed, call $\tilde{\Omega}_u$ the set in the right-hand side of the previous display; $\tilde{\Omega}_u \subset \Omega_u$, again by Proposition 3.4.1. On the other hand, the us take $x_0 \in \Omega_u$; it follows that there exists $B_{r_{x_0}}(x_0) \subset \Omega$ with $u \in C^{0,\tilde{\gamma}}\left(B_{r_{x_0}}(x_0)\right)$ for some $\tilde{\gamma} < 1$; then, fix $\gamma < 1$ and determine the corresponding $\varepsilon_\gamma \equiv \varepsilon_\gamma(\texttt{data}, \beta, \gamma)$ according to Proposition 3.4.1. We can take ϱ small enough in (3.4.35) (this time with γ replaced by $\tilde{\gamma}$) in such a way that the smallness condition in (3.4.33) is satisfied. This implies that u is γ-Hölder continuous in a neighbourhood of x_0. As $\gamma \in (0, 1)$ has been chosen arbitrarily, we deduce that $x_0 \in \tilde{\Omega}_u$ and therefore $\Omega_u = \tilde{\Omega}_u$, that is, (3.4.39) is completely proved. By (3.4.39) and matching the content of Remark 3.4.1 (in particular, see (3.4.34)) with a standard covering argument, we conclude with

Proposition 3.4.2 *Let $u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{S}^{N-1})$ be a constrained local minimizer of the functional $\mathcal{F}$ in (3.0.1), under the assumptions (3.0.12)-(3.0.15). Assume that $p(1 + \delta_g) \le n$. For every open subset $\Omega_0 \Subset \Omega_u$ and every $\gamma \in (0, 1)$, there exist constants $c, \tilde{c} \equiv c, \tilde{c}(\texttt{data}, \|a\|_{L^\infty}, \beta, \gamma, \Omega_0)$ such that*

$$\fint_{B_\varrho} H(x, Du) \, \mathrm{d}x \le c\varrho^{(\gamma-1)q} \qquad \textit{and} \qquad [u]_{0,\gamma;\Omega_0} \le \tilde{c} \, , \tag{3.4.40}$$

with the first one that holds whenever $B_{2\varrho} \subset \Omega_0$ with $2\varrho \le 1$.

In (3.4.40) we notice that the second inequality actually implies the first one via (3.2.5). As for the rest of the proof, as mentioned above, we only need to show that Du is locally Hölder continuous in Ω_u.

Step 4: Passage to coordinates.
After the proof of the local partial Hölder continuity of u, we can now pass to coordinates using stereographic projections. The procedure is standard in the case of functionals with p-growth but, since we are dealing with non-standard growth conditions, we need to check extra regularity conditions and therefore we shall repeat it in some detail. Having (3.4.40) in mind, we fix a certain initial γ, say $\gamma = 1/2$. We then consider bounded open subsets $\tilde{\Omega} \Subset \Omega_0 \Subset \Omega_u$ and cover $\tilde{\Omega}$ with finitely many balls $B \subset \Omega_0$ with sufficiently small radius (size and number here only depend on n, N, $[u]_{0,1/2;\Omega_0}$ and $\mathrm{diam}(\Omega_0)$) such that $u(B)$ lies in single coordinate neighbourhood of $\mathbb{S}^{N-1}$. More precisely, if $B \equiv B_{r(B)}(x_0)$ is one of such balls, up to rotations we can assume that $u(x_0) = (-1, 0, \cdots, 0)$ and that $u^1(x) \leq -1/2$ for every $x \in B$. Given this, with no loss of generality we can reduce to the case in which we are working on an open subset $\tilde{\Omega} \Subset \Omega$ such that $u^1(x) \leq -1/2$ for every $x \in \tilde{\Omega}$. This is the setting we shall use in the rest of the proof and our next goal is now to prove that Du is locally β_0-Hölder continuous in $\tilde{\Omega}$, with β_0 depending only on $(\texttt{data}, \beta, \beta_1)$, where β_1 is the exponent appearing in $(3.0.14)_2$. The full statement of Theorem 1 then follows again via a standard covering argument. To proceed, denoting by $P(\cdot)$ the usual stereographic projection $P\colon \mathbb{S}^{N-1} \setminus \{(1, \dots, 0)\} \to \mathbb{R}^{N-1}$ and by $S := P^{-1}\colon \mathbb{R}^{N-1} \to \mathbb{S}^{N-1} \setminus \{(1, \dots, 0)\}$ its inverse, i.e.,

$$S(y) = \left(\frac{|y|^2 - 1}{|y|^2 + 1}, \frac{2y}{|y|^2 + 1} \right) , \qquad S^{-1}(v) = \left(\frac{v^i}{1 - v^1} \right)_{i=2}^{N} , \tag{3.4.41}$$

we then define $\tilde{u} := S^{-1}(u)$. We note that

$$\|\nabla S\|_{L^\infty} \leq c(N), \quad \|\nabla^2 S\|_{L^\infty} \leq c(N) \quad \text{and} \quad |\nabla(S^{-1})(u(x))| \leq c(N) , \tag{3.4.42}$$

the last inequality being valid for all $x \in \tilde{\Omega}$ (as $u^1(x) \leq -1/2$ whenever $x \in \tilde{\Omega}$). Recalling (3.4.41), again that $u^1(x) \leq -1/2$ whenever $x \in \tilde{\Omega}$, and that $\tilde{u} = S^{-1}(u)$, we get

$$\|\tilde{u}\|_{L^\infty(\tilde{\Omega})} \leq \frac{2}{3} \leq 1. \tag{3.4.43}$$

Again (3.4.42) implies that if $\tilde{w} \in W^{1,H}(\tilde{\Omega}, \mathbb{R}^{N-1})$, then $S(\tilde{w}) \in W^{1,H}(\tilde{\Omega}, \mathbb{S}^{N-1})$. Therefore, by the minimality of u it follows that the map $\tilde{u} \in W^{1,H}(\tilde{\Omega}, \mathbb{R}^{N-1})$ is a local minimizer of the functional

$$W^{1,H}(\tilde{\Omega}, \mathbb{R}^{N-1}) \ni w \mapsto \int_{\tilde{\Omega}} G(x, w, Dw)\, \mathrm{d}x , \tag{3.4.44}$$

where the integrand $G(\cdot)$ is defined by

$$G(x, y, z) := F\left(x, S(y), \nabla S(y) z\right) \qquad \text{for } x \in \tilde{\Omega},\ y \in \mathbb{R}^{N-1},\ z \in \mathbb{R}^{(N-1)\times n} .$$

As it is

$$|\nabla S(y) z| = \frac{2}{(1 + |y|^2)} |z| \tag{3.4.45}$$

(as it follows from an elementary but lengthy computation), recalling $(3.0.14)_1$ we conclude with

$$G(x, y, z) = \tilde{G}(x, y, |z|) \equiv \tilde{G}_{x,y}(|z|) := \tilde{F}\left(x, S(y), 2(1 + |y|^2)^{-1}|z|\right) . \tag{3.4.46}$$

By using the starting assumptions (3.0.12), it is now not difficult to show that for every $M \geq 3N$ there exist new constants $0 < \tilde{\nu} \equiv \tilde{\nu}(\texttt{data}, M) \leq 1 \leq \tilde{L} \equiv \tilde{L}(\texttt{data})$, such that

$$\begin{cases} \tilde{\nu} H(x,z) \leq G(x,y,z) \leq \tilde{L} H(x,z) \\ |\partial_z G(x,y,z)||z| + |\partial_{zz} G(x,y,z)||z|^2 \leq \tilde{L} H(x,z) \\ \tilde{\nu}(|z|^{p-2} + a(x)|z|^{q-2})|\xi|^2 \leq \partial_{zz} G(x,y,z)\xi \cdot \xi \\ |\partial_z G(x_1,y,z) - \partial_z G(x_2,y,z)||z| \leq \tilde{L}\left[\omega(|x_1-x_2|)[H(x_1,z)+H(x_2,z)] + |a(x_1)-a(x_2)||z|^q\right] \\ |G(x,y_1,z) - G(x,y_2,z)| \leq \tilde{L}\omega(|y_1-y_2|)H(x,z) \ , \end{cases} \tag{3.4.47}$$

hold whenever $x, x_1, x_2 \in \Omega$, $z \in \mathbb{R}^{(N-1)\times n} \setminus \{0\}$, $\xi \in \mathbb{R}^{(N-1)\times n}$ and $y, y_1, y_2 \in \mathbb{R}^{N-1}$ are such that $|y| + |y_1| + |y_2| \leq M$. In the lines above, $\tilde{\nu}$ is a non-increasing function of M. All the inequalities in (3.4.47) are consequences of the definition in (3.4.46) and of (3.4.45) and we leave the details of the verification to the reader. We just spend a few words on the verification of $(3.4.47)_3$. By using (3.2.26), from the explicit representation in (3.4.46) we get

$$\partial_{zz} G(x,y,z) = 4\frac{\tilde{F}''(x,S(y),|\tilde{z}|)}{(1+|y|^2)^2}\frac{\tilde{z}\otimes\tilde{z}}{|\tilde{z}|^2} + 4\frac{\tilde{F}'(x,S(y),|\tilde{z}|)}{(1+|y|^2)^2}\left[\frac{\mathbb{I}_{N\times n}}{|\tilde{z}|} - \frac{\tilde{z}\otimes\tilde{z}}{|\tilde{z}|^3}\right], \tag{3.4.48}$$

where we have denoted $\tilde{z} := 2(1+|y|^2)^{-1}z$. Taking $\xi \in \mathbb{R}^{(N-1)\times n}$ and adding one more null component to both $\tilde{z}$ and ξ (thereby making then $\mathbb{R}^{N\times n}$ matrices), and using (3.2.25) and $(3.0.12)_3$, yields

$$\partial_{zz} G(x,y,z)\xi\cdot\xi \geq \frac{\nu}{2^q}\left(\frac{|z|^{p-2}}{(1+|y|^2)^p} + a(x)\frac{|z|^{q-2}}{(1+|y|^2)^q}\right)|\xi|^2 \geq \frac{\nu(|z|^{p-2}+a(x)|z|^{q-2})}{(2+4M^2)^q}|\xi|^2 \ ,$$

that is, $(3.4.47)_3$. Finally, fix $x \in \tilde{\Omega}$ and $y \in \mathbb{R}^{N-1}$. By $(3.0.14)_1$ and (3.4.46) it follows that

$$t \mapsto \tilde{G}_{x,y}(\cdot,t) \text{ is non-decreasing} \tag{3.4.49}$$

and this means that the structure assumption $(3.0.14)_1$ is verified also by $G(\cdot)$. As for the analog of $(3.0.14)_2$, observe that (3.4.46) implies that $\tilde{G}''_{x,y}(t) = [2(1+|y|^2)^{-1}]^2\tilde{F}''_{x,S(y)}(2(1+|y|^2)^{-1}t)$. Then, with $|s| < t/2$ and $t > 0$, we define $\tilde{s} := 2(1+|y|^2)^{-1}s$ and $\tilde{t} := 2(1+|y|^2)^{-1}t$; taking into account (3.2.27) we find

$$\begin{aligned} |\tilde{G}''_{x,y}(t+s) - \tilde{G}''_{x,y}(t)| &= \frac{4}{(1+|y|^2)^2}|\tilde{F}''_{x,S(y)}(\tilde{t}+\tilde{s}) - \tilde{F}''_{x,S(y)}(\tilde{t})| \\ &\leq \frac{c}{(1+|y|^2)^2}\tilde{F}''_{x,S(y)}(\tilde{t})\left(\frac{|\tilde{s}|}{\tilde{t}}\right)^{\beta_1} = c\tilde{G}''_{x,y}(t)\left(\frac{|s|}{t}\right)^{\beta_1} \ , \end{aligned} \tag{3.4.50}$$

where it is again $c \equiv c(n,N,\nu,L,p,q)$. We moreover remark that, by the growth conditions in (3.4.47), exactly as done for (3.2.24) in Remark 3.2.4 we infer that

$$\tilde{G}''_{x,y}(t)t \approx \tilde{G}'_{x,y}(t) \qquad \text{holds for every } t > 0 \tag{3.4.51}$$

and here the implied constants depend on n, N, ν, L, p, q, M and they are independent of (x,y).

Step 5: Partial Hölder continuity of the gradient.
First of all, let us observe that for reasons that will be clear in a few lines, and in view of (3.4.43), in the following, when considering (3.4.47), we shall permanently use the choice $M = 10N$. Recall from *Step 2* that $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_u, \mathbb{S}^{N-1})$ for every $\gamma \in (0,1)$, and so, by (3.4.42) and (3.4.40), we have that $\tilde{u} \in C^{0,\gamma}(\tilde{\Omega}, \mathbb{R}^{N-1})$ for every $\gamma \in (0,1)$, with

$$[\tilde{u}]_{0,\gamma;\tilde{\Omega}} \leq c(N)[u]_{0,\gamma;\tilde{\Omega}} \leq c(\texttt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega}) \ . \tag{3.4.52}$$

For any ball $B_{4r} \Subset \tilde{\Omega}$ with $r \le 1/8$, as $\tilde{u}$ minimizes the functional in (3.4.44) and (3.4.43) holds, also taking Remark 3.2.2 into account, Lemma 3.2.2 provides

$$\fint_{B_{2r}} H(x, D\tilde{u}) \, dx \le c \fint_{B_{4r}} H^-_{B_{4r}}\left(\frac{\tilde{u}-(\tilde{u})_{B_{4r}}}{4r}\right) \, dx \overset{(3.4.52)}{\le} cH^-_{B_r}\left(r^{\gamma-1}\right) \tag{3.4.53}$$

holds with $c \equiv c(\texttt{data}, \beta, \gamma, \tilde{\Omega})$, for every $\gamma \in (0,1)$. In particular, it follows that

$$\fint_{B_{2r}} H(x, D\tilde{u}) \, dx \le cr^{(\gamma-1)q} \, , \tag{3.4.54}$$

for all $\gamma \in (0,1)$, where $c \equiv c(\texttt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. We start fixing $\gamma \ge 1/2$; we shall further increase the value of γ in due course of the proof (and the constants involved will increase accordingly). Fix $B_r \equiv B_r(x_0)$ such that $B_{4r} \Subset \tilde{\Omega}$, $r \le 1/64$, and let $\tilde{v} \in W^{1,H}(B_r, \mathbb{R}^{N-1})$ be a solution to the frozen Dirichlet problem

$$\tilde{v} \mapsto \min_{w \in W^{1,H}_{\tilde{u}}(B_r, \mathbb{R}^{N-1})} \int_{B_r} G\left(x, (\tilde{u})_{B_r}, Dw\right) \, dx \, . \tag{3.4.55}$$

By (3.4.43) the integrand $G(\cdot, (\tilde{u})_{B_r}, \cdot)$ satisfies assumptions (3.4.47) with $\tilde{\nu}$ and $\tilde{L}$ only depending on $\texttt{data}$ as we have fixed $M = 10N$. In particular, there exist positive numbers $\tilde{\nu}, \tilde{L}$ such that

$$\tilde{\nu}(\texttt{data}) H(x, z) \le G\left(x, (\tilde{u})_{B_r}, z\right) \le \tilde{L}(\texttt{data}) H(x, z) \tag{3.4.56}$$

holds whenever $x \in \Omega$ and $z \in \mathbb{R}^{(N-1)\times n}$. By (3.4.49) and the maximum principle [175, Theorem 2.3] we then have

$$\|\tilde{v}\|_{L^\infty(B_r)} \le \sqrt{N} \|\tilde{u}\|_{L^\infty(B_r)} \overset{(3.4.43)}{\le} \frac{2\sqrt{N}}{3} \, . \tag{3.4.57}$$

The validity of the Euler-Lagrange equation for (3.4.55) can be checked as done in [60] (see also Section 3.2.2 and apply the same arguments exposed there without using projections, that is, when no constraints are involved). Specifically, $\tilde{v}$ solves

$$\int_{B_r} \partial_z G\left(x, (\tilde{u})_{B_r}, D\tilde{v}\right) \cdot D\varphi \, dx = 0 \, , \tag{3.4.58}$$

for all $\varphi \in W^{1,H}_0(B_r, \mathbb{R}^{N-1})$.

Remark 3.4.2 From now on, we adopt the following convention. Also taking the content of Remarks 3.2.2-3.2.3 and (3.4.56)-(3.4.57) into account, the results of Lemma 3.2.4 and 3.2.5 apply to $\tilde{u}, \tilde{v}$ and lead to new higher integrability exponents δ_g and σ_g. The values of δ_g and σ_g are different from those used in the previous steps for u, but essentially equivalent to them. Indeed, they still depend on the same set of parameters, that is $\texttt{data}$. Therefore, with some abuse of notation, we shall keep on denoting by $\sigma_g < \delta_g$ the higher integrability exponents provided by the application of Lemmas 3.2.4-3.2.5 in the present setting (notice that what is denotes by N in Lemmas 3.2.4-3.2.5 is actually $N-1$ here; this can be made rigorous eventually taking the smallest amongst all the exponents considered when the values of ν and L attain their minimum and maximum, respectively). All in all, the following inequalities hold as in (3.4.6) and (3.4.7):

$$\begin{cases} \displaystyle\fint_{B_r} H(x, D\tilde{v}) \, dx \le c \fint_{B_r} H(x, D\tilde{u}) \, dx \\ \displaystyle\left(\fint_{B_{r/2}} [H(x, D\tilde{v})]^{1+\delta_g} \, dx\right)^{1/(1+\delta_g)} \le \tilde{c}_1 \fint_{B_r} H(x, D\tilde{v}) \, dx \end{cases} \tag{3.4.59}$$

and

$$\left(\fint_{B_r}[H(x,D\tilde v)]^{1+\sigma_g}\,\mathrm{d}x\right)^{1/(1+\sigma_g)} \le c\left(\fint_{B_r}[H(x,D\tilde u)]^{1+\sigma_g}\,\mathrm{d}x\right)^{1/(1+\sigma_g)} \le c\fint_{B_{2r}} H(x,D\tilde u)\,\mathrm{d}x\,, \tag{3.4.60}$$

for $c,\tilde c_1 \equiv c,\tilde c_1(\mathtt{data})$. This last dependence on the constants is a consequence of (3.4.56)-(3.4.57). As $\varphi \equiv \tilde u - \tilde v$ is a legal choice in (3.4.58), using (3.4.4) as in (3.4.5), we end up with

$$\begin{aligned}
&c\fint_{B_r}\left(|V_p(D\tilde u)-V_p(D\tilde v)|^2 + a(x)|V_q(D\tilde u)-V_q(D\tilde v)|^2\right)\,\mathrm{d}x\\
&\le \fint_{B_r}\left[G\left(x,(\tilde u)_{B_r},D\tilde u\right)-G\left(x,(\tilde u)_{B_r},D\tilde v\right)\right]\,\mathrm{d}x\\
&= \fint_{B_r}\left[G\left(x,(\tilde u)_{B_r},D\tilde u\right)-G(x,\tilde u,D\tilde u)\right]\,\mathrm{d}x + \fint_{B_r}\left[G(x,\tilde u,D\tilde u)-G(x,\tilde v,D\tilde v)\right]\,\mathrm{d}x\\
&\quad+\fint_{B_r}\left[G\left(x,\tilde v,D\tilde v\right)-G\left(x,(\tilde v)_{B_r},D\tilde v\right)\right]\,\mathrm{d}x\\
&\quad+\fint_{B_r}\left[G\left(x,(\tilde v)_{B_r},D\tilde v\right)-G\left(x,(\tilde u)_{B_r},D\tilde v\right)\right]\,\mathrm{d}x =: \sum_{j=1}^{4}(\mathrm{I})_j\,,
\end{aligned}\tag{3.4.61}$$

with $c \equiv c(\mathtt{data})$. Before taking care of terms $(\mathrm{I})_1$-$(\mathrm{I})_4$, we derive a few preliminary inequalities. The first one is an obvious consequence of (3.4.52) and is

$$\sup_{x\in B_r}\omega\left(|\tilde u(x)-(\tilde u)_{B_r}|\right) \le cr^{\beta\gamma} \le cr^{\beta/2}\,, \tag{3.4.62}$$

for $c \equiv c(\mathtt{data},\|a\|_{L^\infty},\beta,\gamma,\tilde\Omega)$ (here recall that $\gamma \ge 1/2$ and $r \le 1$). Proceeding as for (3.4.9), and recalling (3.4.59), we instead have

$$\begin{aligned}
\fint_{B_r}\omega\left(|\tilde v-(\tilde v)_{B_r}|\right)\,\mathrm{d}x &\le c\omega\left[r\left(H^-_{B_r}\right)^{-1}\left(\fint_{B_r}H(x,D\tilde v)\,\mathrm{d}x\right)\right]\\
&\le c\omega\left[r\left(H^-_{B_r}\right)^{-1}\left(\fint_{B_r}H(x,D\tilde u)\,\mathrm{d}x\right)\right]\\
&\overset{(3.4.53)}{\le} c\omega\left[r\left(H^-_{B_r}\right)^{-1}\left(cH^-_{B_r}\left(r^{\gamma-1}\right)\right)\right] \le cr^{\beta\gamma} \le cr^{\beta/2},
\end{aligned}\tag{3.4.63}$$

with $c \equiv c(\mathtt{data},\|a\|_{L^\infty},\beta,\gamma,\tilde\Omega)$. In a totally similar fashion, as done in (3.4.10), we have

$$\omega\left(|(\tilde u)_{B_r}-(\tilde v)_{B_r}|\right) \le cr^{\beta\gamma} \le cr^{\beta/2}\,, \tag{3.4.64}$$

where $c \equiv c(\mathtt{data},\|a\|_{L^\infty},\beta,\gamma,\tilde\Omega)$. We are now ready to estimate terms $(\mathrm{I})_1$-$(\mathrm{I})_4$. We have

$$(\mathrm{I})_1 \le \fint_{B_r}\omega\left(|\tilde u-(\tilde u)_{B_r}|\right)H(x,D\tilde u)\,\mathrm{d}x \overset{(3.4.62)}{\le} cr^{\beta/2}\fint_{B_{2r}}H(x,D\tilde u)\,\mathrm{d}x\,,$$

with $c \equiv c(\mathtt{data},\|a\|_{L^\infty},\beta,\gamma,\tilde\Omega)$. The minimality of $\tilde u$ gives $(\mathrm{I})_2 \le 0$. As for term $(\mathrm{I})_3$, using (3.4.60) and (3.4.63), we get

$$(\mathrm{I})_3 \le c\left(\fint_{B_r}\omega\left(|\tilde v-(\tilde v)_{B_r}|\right)\,\mathrm{d}x\right)^{\frac{\sigma_g}{1+\sigma_g}}\left(\fint_{B_r}[H(x,D\tilde v)]^{1+\sigma_g}\,\mathrm{d}x\right)^{\frac{1}{1+\sigma_g}}$$

$$\overset{(3.4.60)}{\leq} c\left(\fint_{B_r} \omega\left(|\tilde v-(\tilde v)_{B_r}|\right)\,\mathrm{d}x\right)^{\frac{\sigma_g}{1+\sigma_g}} \fint_{B_{2r}} H(x,D\tilde u)\,\mathrm{d}x$$
$$\overset{(3.4.63)}{\leq} cr^{\frac{\beta\sigma_g}{2(1+\sigma_g)}} \fint_{B_{2r}} H(x,D\tilde u)\,\mathrm{d}x\,,$$

for $c \equiv c(\texttt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde\Omega)$ and, in a totally similar fashion, arguing as for (3.4.15), but using this time (3.4.64), we find

$$(\mathrm{I})_4 \overset{(3.4.62)}{\leq} cr^{\frac{\beta\sigma_g}{2(1+\sigma_g)}} \fint_{B_{2r}} H(x,D\tilde u)\,\mathrm{d}x\,,$$

where, again it is $c \equiv c(\texttt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde\Omega)$. Collecting the estimates found above for the terms $(\mathrm{I})_1$-$(\mathrm{I})_4$ to (3.4.61) yields

$$\fint_{B_r} \left(|V_p(D\tilde u)-V_p(D\tilde v)|^2 + a(x)|V_q(D\tilde u)-V_q(D\tilde v)|^2\right)\,\mathrm{d}x \leq cr^{\frac{\beta\sigma_g}{2(1+\sigma_g)}} \fint_{B_{2r}} H(x,D\tilde u)\,\mathrm{d}x,$$

where $c \equiv c(\texttt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde\Omega)$. In particular, there holds

$$\fint_{B_r} \left(|V_p(D\tilde u)-V_p(D\tilde v)|^2 + a_{\mathrm{i}}(B_r)|V_q(D\tilde u)-V_q(D\tilde v)|^2\right)\,dx \leq cr^{\frac{\beta\sigma_g}{2(1+\sigma_g)}} \fint_{B_{2r}} H(x,D\tilde u)\,dx. \tag{3.4.65}$$

We now select $x_m \in \bar B_r$ such that

$$a(x_m) = a_{\mathrm{i}}(B_r) \tag{3.4.66}$$

and define (keep in mind the notation in (3.4.46))

$$G_m(z) := G\left(x_m, (\tilde u)_{B_r}, z\right) \equiv \tilde G_{x_m,(\tilde u)_{B_r}}(|z|) \qquad \text{for every } z \in \mathbb{R}^{(N-1)\times n}\,. \tag{3.4.67}$$

The newly defined integrand $G_m(\cdot)$ is of the type $g_0(\cdot)$ considered in Lemma 3.3.3 and satisfies assumptions (3.3.29) with the choice $H_0(\cdot) \equiv H^-_{B_r}(\cdot)$ and for suitable constants ν, L depending on $\textbf{\textit{data}}$ (see the discussion in Step 4 and, in particular, (3.4.47) and (3.4.56)). We now proceed applying Lemma 3.3.3 to $\tilde v$; notice that (3.3.30) is automatically satisfied by the second inequality in (3.4.59) and therefore verify (3.3.31). According to the terminology adopted in [12, 60, 61] the p-phase occurs if

$$a_{\mathrm{i}}(B_r) \leq 4[a]_{0,\alpha} r^{\alpha-s}, \quad s := \alpha + (\gamma-1)(q-p) \overset{(3.0.15)_1}{>} 0, \tag{3.4.68}$$

while the (p,q)-phase is defined by the complementary condition

$$a_{\mathrm{i}}(B_r) > 4[a]_{0,\alpha} r^{\alpha-s}. \tag{3.4.69}$$

It is then easy to see that

$$\begin{cases} a_{\mathrm{s}}(B_r) \leq 6[a]_{0,\alpha} r^{\alpha-s} & \text{holds in the } p\text{-phase} \\ a_{\mathrm{s}}(B_r) \leq \frac{3}{2} a_{\mathrm{i}}(B_r) & \text{holds in the } (p,q)\text{-phase}\,. \end{cases} \tag{3.4.70}$$

Notice that the two phases described above depend on the number $\gamma \geq 1/2$, which is going to be chosen later as an absolute function of $\textbf{\textit{data}}$. We start estimating, for every $\varphi \in C^\infty_c(B_{r/2}, \mathbb{R}^N)$

$$\left|\fint_{B_{r/2}} \partial_z G_m(D\tilde v)\cdot D\varphi\,\mathrm{d}x\right| \overset{(3.4.58)}{=} \left|\fint_{B_{r/2}} \left[\partial_z G_m(D\tilde v) - \partial_z G\left(x, (\tilde u)_{B_r}, D\tilde v\right)\right]\cdot D\varphi\,\mathrm{d}x\right| \tag{3.4.71}$$

$$
\begin{aligned}
&\overset{(3.4.47)_4}{\leq} c\|D\varphi\|_{L^\infty(B_{r/2})} \fint_{B_{r/2}} \omega(|x_m - x|) \left[\frac{H(x_m, D\tilde{v})}{|D\tilde{v}|} + \frac{H(x, D\tilde{v})}{|D\tilde{v}|}\right] \, \mathrm{d}x \\
&+ c\|D\varphi\|_{L^\infty(B_{r/2})} \fint_{B_{r/2}} \left[a(x) - a(x_m)\right] |D\tilde{v}|^{q-1} \, \mathrm{d}x \\
&\overset{(3.0.13)}{\leq} cr^\beta \|D\varphi\|_{L^\infty(B_{r/2})} \fint_{B_{r/2}} |D\tilde{v}|^{p-1} \, \mathrm{d}x \\
&+ cr^\beta \|D\varphi\|_{L^\infty(B_{r/2})} \fint_{B_{r/2}} \left(a(x) + a_i(B_r)\right) |D\tilde{v}|^{q-1} \, \mathrm{d}x \\
&+ c[a]_{0,\alpha} r^\alpha \|D\varphi\|_{L^\infty(B_{r/2})} \fint_{B_{r/2}} |D\tilde{v}|^{q-1} \, \mathrm{d}x \\
&=: (\mathrm{II})_1 + (\mathrm{II})_2 + (\mathrm{II})_3 \,,
\end{aligned} \tag{3.4.72}
$$

and proceed estimating the terms appearing in the last three lines. In any case we have, Young's inequality gives

$$
(\mathrm{II})_1 \leq c(\mathtt{data}) r^\beta \fint_{B_r} \left(|D\tilde{v}|^p + \|D\varphi\|^p_{L^\infty(B_{r/2})}\right) \, \mathrm{d}x. \tag{3.4.73}
$$

In order to bound the remaining two terms we distinguish between the p-phase (3.4.68) and the (p,q)-phase (3.4.69). We start noticing that in the p-phase we have

$$
\fint_{B_{r/2}} H(x, D\tilde{v}) \, \mathrm{d}x \overset{(3.4.59)}{\leq} c \fint_{B_r} H(x, D\tilde{u}) \, \mathrm{d}x \overset{(3.4.53)}{\leq} cH^-_{B_r}(r^{\gamma-1}) \overset{(3.4.68)}{\leq} cr^{(\gamma-1)p} \,, \tag{3.4.74}
$$

for $c \equiv c(\mathtt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. As $q - 1 < p$, using Hölder's inequality we have

$$
\begin{aligned}
(\mathrm{II})_2 &\overset{(3.4.70)_1}{\leq} cr^{\beta+\alpha-s} \|D\varphi\|_{L^\infty(B_{r/2})} \left(\fint_{B_{r/2}} |D\tilde{v}|^p \, \mathrm{d}x\right)^{\frac{q-p}{p}} \left(\fint_{B_{r/2}} |D\tilde{v}|^p \, \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\leq cr^{\beta+\alpha-s} \|D\varphi\|_{L^\infty(B_{r/2})} \left(\fint_{B_{r/2}} H(x, D\tilde{v}) \, \mathrm{d}x\right)^{\frac{q-p}{p}} \left(\fint_{B_{r/2}} |D\tilde{v}|^p \, \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\overset{(3.4.74)}{\leq} cr^{\beta+\alpha-s+(\gamma-1)(q-p)} \|D\varphi\|_{L^\infty(B_{r/2})} \left(\fint_{B_{r/2}} |D\tilde{v}|^p \, \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\overset{(3.4.68)}{\leq} cr^\beta \fint_{B_r} \left(|D\tilde{v}|^p + \|D\varphi\|^p_{L^\infty(B_{r/2})}\right) \, \mathrm{d}x \,,
\end{aligned}
$$

for $c \equiv c(\mathtt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. Finally, we similarly have

$$
\begin{aligned}
(\mathrm{II})_3 &\leq cr^s r^{\alpha-s} \|D\varphi\|_{L^\infty(B_{r/2})} \left(\fint_{B_{r/2}} |D\tilde{v}|^p \, \mathrm{d}x\right)^{\frac{q-p}{p}} \left(\fint_{B_{r/2}} |D\tilde{v}|^p \, \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\overset{(3.4.68),(3.4.74)}{\leq} cr^s \fint_{B_r} \left(|D\tilde{v}|^p + \|D\varphi\|^p_{L^\infty(B_{r/2})}\right) \, \mathrm{d}x \,,
\end{aligned}
$$

with $c \equiv c(\mathtt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. We now consider the occurrence of the (p,q)-phase (3.4.69). We have, again by Hölder's inequality

$$
(\mathrm{II})_2 \overset{(3.4.70)_2}{\leq} cr^\beta \|D\varphi\|_{L^\infty(B_{r/2})} \left[a_{\mathrm{i}}(B_r)\right]^{\frac{1}{q}} \fint_{B_r} \left[a_{\mathrm{i}}(B_r)\right]^{\frac{q-1}{q}} |D\tilde{v}|^{q-1} \, \mathrm{d}x
$$

$$\leq \quad cr^{\beta}\|D\varphi\|_{L^{\infty}(B_{r/2})}\left[a_{\rm i}(B_r)\right]^{\frac{1}{q}}\left(\fint_{B_r} a_{\rm i}(B_r)|D\tilde{v}|^q \ {\rm d}x\right)^{\frac{q-1}{q}}$$

$$\leq \quad cr^{\beta}\fint_{B_r}\left(a_{\rm i}(B_r)|D\tilde{v}|^q + a_{\rm i}(B_r)\|D\varphi\|^q_{L^{\infty}(B_{r/2})}\right) \ {\rm d}x$$

and

$$
\begin{aligned}
({\rm II})_3 \quad & = \quad c[a]_{0,\alpha}r^{\alpha-s}r^{s}\|D\varphi\|_{L^{\infty}(B_{r/2})}\fint_{B_{r/2}}|D\tilde{v}|^{q-1} \ {\rm d}x \\
& \overset{(3.4.69)}{\leq} \quad cr^{s}\|D\varphi\|_{L^{\infty}(B_{r/2})}\left[a_{\rm i}(B_r)\right]^{\frac{1}{q}}\fint_{B_r}\left[a_{\rm i}(B_r)\right]^{\frac{q-1}{q}}|D\tilde{v}|^{q-1} \ {\rm d}x \\
& \leq \quad cr^{s}\fint_{B_r}\left(a_{\rm i}(B_r)|D\tilde{v}|^q + a_{\rm i}(B_r)\|D\varphi\|^q_{L^{\infty}(B_{r/2})}\right) \ {\rm d}x \ ,
\end{aligned}
$$

where $c \equiv c(\texttt{data}, \beta)$. Collecting the estimates founds for the terms $({\rm II})_1, ({\rm II})_2, ({\rm II})_3$ to (3.4.72) and recalling (3.4.68), in any case we conclude with

$$\left|\fint_{B_{r/2}}\partial_z G_m(D\tilde{v})\cdot D\varphi \ {\rm d}x\right| \leq cr^{\min\{\beta,\alpha-(q-p)/2\}}\fint_{B_r}\left[H^-_{B_r}(D\tilde{v}) + H^-_{B_r}\left(\|D\varphi\|_{L^{\infty}(B_{r/2})}\right)\right] \ {\rm d}x \ ,$$

with $c \equiv c(\texttt{data}, \|a\|_{L^{\infty}}, \beta, \gamma, \tilde{\Omega})$. Here we have used $\gamma \geq 1/2$, so that it is $s \geq \alpha-(q-p)/2 > 0$. Lemma 3.3.3 (notice that the number N used there is actually $N-1$ in this context; recall again that here it is $N > 1$) yields the existence of $\tilde{h} \in W^{1,H^-_{B_r}}_{\tilde{v}}(B_{r/2}, \mathbb{R}^{N-1})$ satisfying

$$\fint_{B_{r/2}}\partial_z G_m(D\tilde{h})\cdot D\varphi \ {\rm d}x = 0 \quad \text{for every } \varphi \in W^{1,H^-_{B_r}}_0(B_{r/2}, \mathbb{R}^{N-1}) \ ,$$

and, such that

$$\fint_{B_{r/2}}\left(|V_p(D\tilde{v})-V_p(D\tilde{h})|^2 + a_{\rm i}(B_r)|V_q(D\tilde{v})-V_q(D\tilde{h})|^2\right) \ {\rm d}x \leq cr^m\fint_{B_r}H^-_{B_r}(D\tilde{v}) \ {\rm d}x \quad (3.4.75)$$

where $c \equiv c(\texttt{data}, \|a\|_{L^{\infty}}, \beta, \gamma, \tilde{\Omega})$ and $m = m(n, N, \nu, L, p, q, \alpha)$. Needless to say, $\tilde{h}$ solves

$$\tilde{h} \mapsto \min_{w \in W^{1,H^-_{B_r}}_{\tilde{v}}(B_{r/2},\mathbb{R}^{N-1})}\fint_{B_{r/2}}G_m(Dw) \ {\rm d}x \quad (3.4.76)$$

so that, recalling again (3.4.66), we find

$$\fint_{B_{r/2}}H^-_{B_r}(D\tilde{h}) \ {\rm d}x \leq c\fint_{B_r}H^-_{B_r}(D\tilde{v}) \ {\rm d}x \leq \fint_{B_{r/2}}H(x, D\tilde{v}) \ {\rm d}x \overset{(3.4.59)}{\leq} c\fint_{B_r}H(x, D\tilde{u}) \ {\rm d}x \ . \quad (3.4.77)$$

Hence, from (3.4.65), (3.4.75), and the last inequality in the above display, we obtain

$$\fint_{B_{r/2}}\left(|V_p(D\tilde{u})-V_p(D\tilde{h})|^2 + a_{\rm i}(B_r)|V_q(D\tilde{u})-V_q(D\tilde{h})|^2\right) \ dx \leq cr^{\kappa}\fint_{B_{2r}}H(x, D\tilde{u}) \ {\rm d}x, \quad (3.4.78)$$

for

$$\kappa := \min\left\{m, \frac{\beta\sigma_g}{2(1+\sigma_g)}\right\} < 1$$

and $c \equiv c(\mathtt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. After some standard manipulations on (3.4.78), see e.g. [12, Section 10] and [61, pp. 483], we have

$$\fint_{B_{r/2}} |D\tilde{u} - D\tilde{h}|^p \, dx \le \fint_{B_{r/2}} H^-_{B_r}\left(D\tilde{u} - D\tilde{h}\right) \, dx \le c r^{\kappa/2} \fint_{B_{2r}} H(x, D\tilde{u}) \, dx, \tag{3.4.79}$$

for $c \equiv c(\mathtt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. Now we make a further restriction on the size of γ imposing that

$$\gamma \ge 1 - \frac{\kappa}{4q} \tag{3.4.80}$$

(which is still larger than 1/2), and apply (3.4.53); we use the resulting inequality in (3.4.79) to obtain

$$\fint_{B_{r/2}} |D\tilde{u} - D\tilde{h}|^p \, dx \le c r^{\kappa/4} \, , \tag{3.4.81}$$

where $c \equiv c(\mathtt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. By using the content of Remark 3.4.3 below we have that

$$\fint_{B_\varrho} H^-_{B_r}\left(D\tilde{h} - (D\tilde{h})_{B_\varrho}\right) \, dx \le c \left(\frac{\varrho}{r}\right)^\mu \fint_{B_r} H(x, D\tilde{u}) \, dx \, , \tag{3.4.82}$$

holds for concentric balls $B_\varrho \subset B_{r/2}$; here we take $\varrho \le r/8$. Here it is $c \equiv c(n, N, \nu, L, p, q)$ and $\mu \equiv \mu(n, N, \nu, L, p, q, \beta_1)$. We estimate

$$\begin{aligned}
\fint_{B_\varrho} |D\tilde{u} - (D\tilde{u})_{B_\varrho}|^p \, dx \quad & \le \quad c \fint_{B_\varrho} |D\tilde{u} - (D\tilde{h})_{B_\varrho}|^p \, dx \\
& \overset{(3.4.81)}{\le} \quad c\left\{\left(\frac{r}{\varrho}\right)^n r^{\kappa/4} + \fint_{B_\varrho} H^-_{B_r}\left(D\tilde{h} - (D\tilde{h})_{B_\varrho}\right) \, dx\right\} \\
& \overset{(3.4.82)}{\le} \quad c\left\{\left(\frac{r}{\varrho}\right)^n r^{\kappa/4} + \left(\frac{\varrho}{r}\right)^\mu \fint_{B_r} H(x, D\tilde{u}) \, dx\right\} \\
& \overset{(3.4.53)}{\le} \quad c\left\{\left(\frac{r}{\varrho}\right)^n r^{\kappa/4} + \left(\frac{\varrho}{r}\right)^\mu r^{(\gamma-1)q}\right\} \, ,
\end{aligned} \tag{3.4.83}$$

where $c \equiv c(\mathtt{data}, \|a\|_{L^\infty}, \beta, \gamma, \tilde{\Omega})$. In (3.4.83) we pick

$$\varrho = \frac{r^{1+a}}{8} \quad \text{with} \quad a := \frac{(1-\gamma)q + \kappa/4}{\mu + n} \quad \text{and} \quad \gamma = 1 - \frac{\kappa\mu}{8nq} \, ,$$

(γ also meets the condition in (3.4.80)) so that (3.4.83) yields

$$\fint_{B_\varrho} |D\tilde{u} - (D\tilde{u})_{B_\varrho}|^p \, dx \le c \varrho^{\frac{\kappa/4 - na}{1+a}} \le c\varrho^{\beta_0 p} \, , \qquad \beta_0 := \frac{\kappa\mu}{16p(n+\mu)} \, . \tag{3.4.84}$$

The classical Hölder continuity characterization of Campanato and Meyers, a standard covering argument, and the fact that $\tilde{\Omega} \Subset \Omega_u$ is arbitrary, allow to conclude that $D\tilde{u} \in C^{0,\beta_0}_{\rm loc}\left(\Omega_u, \mathbb{R}^{(N-1)\times n}\right)$, with β_0 as in (3.4.84) and therefore depends on $(\mathtt{data}, \beta, \beta_1)$. Finally, using $(3.4.42)_{1,2}$ we get $Du = D(S(\tilde{u})) \in C^{0,\beta_0}_{\rm loc}\left(\Omega_u, \mathbb{R}^{N\times n}\right)$ and the proof of the partial local Hölder continuity of the gradient as stated in (3.0.18) is complete in the case it is $p(1+\delta_g) \le n$.

Step 6: The case $p(1+\delta_g) > n$.
In this case the singular set is empty $\Sigma_u = \Omega_u = \Omega$ as the right-hand side in (3.4.36) is empty. Therefore we see from *Step 3* that $u \in C^{0,\gamma}_{\rm loc}(\Omega, \mathbb{S}^{N-1})$ for every $\gamma < 1$ and the rest of the proof, i.e., the local Hölder continuity of Du follows as for the case when $p(1+\delta_g) \le n$.

Remark 3.4.3 We briefly explain how to get estimate (3.4.82) from the results of [89]. Recalling the notation in (3.4.67), by (3.4.51) we can argue exactly as in Remark 3.2.4 to get that

$$\sup_{B_{r/4}} H^-_{B_r}\left(D\tilde{h}\right) \le c \fint_{B_{r/2}} H^-_{B_r}\left(D\tilde{h}\right)\, \mathrm{d}x \tag{3.4.85}$$

holds for a constant $c \equiv c(n, N, \nu, L, p, q)$. Moreover, by (3.4.50) we are able to satisfy [90, Assumption 2.2], where we can take $\varphi(\cdot) \equiv \tilde{G}_{x_m,(\tilde{u})_{B_r}}(\cdot)$ and therefore (also taking into account the definitions in (2.4.1) and in [90, (1.3)]) we can apply [90, Theorem 6.4] that in the present setting gives

$$\begin{aligned}&\fint_{B_\varrho} \left(|V_p(D\tilde{h}) - (V_p(D\tilde{h}))_{B_\varrho}|^2 + a_{\rm i}(B_r)|V_q(D\tilde{h}) - (V_q(D\tilde{h}))_{B_\varrho}|^2\right)\, \mathrm{d}x \\ &\quad \le c\left(\frac{\varrho}{r}\right)^{2\mu} \fint_{B_{r/2}} \left(|V_p(D\tilde{h}) - (V_p(D\tilde{h}))_{B_{r/2}}|^2 + a_{\rm i}(B_r)|V_q(D\tilde{h}) - (V_q(D\tilde{h}))_{B_{r/2}}|^2\right)\, \mathrm{d}x\end{aligned} \tag{3.4.86}$$

whenever $B_\varrho \subset B_{r/2}$ is concentric to $B_{r/2}$, where $c \ge 1$ and $\mu \in (0, 1/2)$ are both depending on n, N, p, q, ν, L and β_1. Following the method explained in [11, Theorem 3.1], see also [68, Proposition 3.3], and combining (3.4.85)-(3.4.86) with (2.4.2), finally yields

$$\fint_{B_\varrho} H^-_{B_r}\left(D\tilde{h} - (D\tilde{h})_{B_\varrho}\right)\, \mathrm{d}x \le c\left(\frac{\varrho}{r}\right)^{\mu} \fint_{B_{r/2}} H^-_{B_r}(D\tilde{h})\, \mathrm{d}x\,,$$

from which (3.4.82) obviously follows.

3.5 Weighted Hausdorff measures and singular sets

3.5.1 Proof of Proposition 3.0.1

Observe that, for a ball B such that $\beta_4^{-1}B \subset \Omega$ and $r(B) \le 1$ we then have

$$\begin{aligned}|\beta_4^{-1}B| \operatorname*{ess\,sup}_{x\in\beta_4^{-1}B} \Phi(x, 1/r(\beta_4^{-1}B)) &= \frac{|B|}{\beta_4^n} \operatorname*{ess\,sup}_{x\in\beta_4^{-1}B} \Phi(x, \beta_4/r(B)) \\ &\stackrel{(3.0.32)}{\le} \frac{c_d}{\beta_4^n}|B| \operatorname*{ess\,inf}_{x\in\beta_4^{-1}B} \Phi(x, 1/r(B)) \le \frac{c_d}{\beta_4^n}|B| \operatorname*{ess\,inf}_{x\in B} \Phi(x, 1/r(B))\,.\end{aligned}$$

By taking balls B such that $r(B) \le \kappa$ and $\kappa \le 1$, we find $\mathscr{H}^+_{\Phi,\kappa/\beta_4} \le (c_d/\beta_4^n)\mathscr{H}^-_{\Phi,\kappa}$, from which (3.0.33) follows by letting $\kappa \to 0$. As for the proof of (3.0.34), we observe that in this case it is $\Phi(x,t) = [t^p + a(x)t^q]^{1+\sigma} \approx t^{p(1+\sigma)} + [a(x)]^{1+\sigma}t^{q(1+\sigma)}$, with constants implicit in "$\approx$" depending on σ. Then, for $B \subset \Omega$ and $1 \le t \le 1/r(B)$, we have

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in B} \Phi(x,t) &\le \operatorname*{ess\,inf}_{x\in B} \Phi(x,t) + \left\{[a_{\rm s}(B)]^{1+\sigma} - [a_{\rm i}(B)]^{1+\sigma}\right\} t^{q(1+\sigma)} \\ &\le \operatorname*{ess\,inf}_{x\in B} \Phi(x,t) + c[r(B)]^{\alpha(1+\sigma)}t^{q(1+\sigma)} \\ &\le \operatorname*{ess\,inf}_{x\in B} \Phi(x,t) + c[r(B)]^{(\alpha-q+p)(1+\sigma)}t^{p(1+\sigma)} \\ &\le c \operatorname*{ess\,inf}_{x\in B} \Phi(x,t)\end{aligned} \tag{3.5.1}$$

where $c \equiv c([a]_{0,\alpha}, \sigma)$. Therefore (3.0.32) is satisfied for $\beta_4 = 1$ and assertion (3.0.34) follows by (3.0.33).

3.5.2 Proof of Theorem 2

The proof is a suitable modification of the one which is valid for the standard $W^{1,p}$-capacity. Thanks to the Choquet property (3.0.36), we can reduce to the case when E is a compact subset. Therefore, recalling that $Cap^*_\Phi(K) = Cap_\Phi(K)$ whenever $K \subset \Omega$ is a compact subset, we can then compute $Cap_\Phi(E)$ via (3.0.35). We now claim that there exists a positive constant c, essentially depending on E, such that if V is a bounded open set such that $E \Subset V \subset \Omega$, then there exists an open set W and a function $f \in \mathcal{R}(E)$ with the following features:

$$\begin{cases} E \subset W \subset \{x \in \Omega \colon f(x) = 1\} \ , \quad \operatorname{supp} f \subset V \ , f \in C_0(\Omega); \\ \displaystyle\int_\Omega \Phi(x, |Df|) \ \mathrm{d}x < c \ . \end{cases} \tag{3.5.2}$$

Let $V \subset \Omega$ be an open set as above and fix $\kappa = \frac{1}{4} \min \{ \operatorname{dist}(E, \mathbb{R}^n \setminus V), 1 \}$. Since $\mathcal{H}_\Phi(E) < \infty$ and E is compact, there exists a positive integer $m = m(E)$ and a finite collection of open balls $\{B_{r_j}(x_j)\}_{j \le m} \in \mathcal{C}_E^{\kappa/2}$ such that $\{x_j\}_{j\le m} \subset E$, $B_{2r_j}(x_j) \Subset \Omega$ for all $j \in \{1, \cdots, m\}$ and

$$B_{r_j}(x_j) \cap E \neq \emptyset \text{ for all } j \in \{1, \cdots, m\} \quad \text{and} \quad \sum_{j=1}^m \int_{B_{r_j}(x_j)} \Phi\left(x, 1/r_j\right) \ \mathrm{d}x \le 2 \left[\mathcal{H}_\Phi(E) + 1\right] \ . \tag{3.5.3}$$

We introduce $W := \bigcup_{j=1}^m B_{r_j}(x_j)$ and the maps f_j, defined on the whole $\mathbb{R}^n$, such that $f_j(x) := 1$ if $|x - x_j| \le r_j$, $f_j(x) := 2 - |x - x_j|/r_j$ if $r_j < |x - x_j| \le 2r_j$ and $f_j(x) := 0$ if $2r_j < |x - x_j|$. With $\beta_4 \in (0, 1)$ being the constant appearing in (3.0.32), we have

$$\begin{aligned} \int_\Omega \Phi(x, |Df_j|) \ \mathrm{d}x \quad &\le \quad \int_{B_{2r_j}(x_j)} \Phi\left(x, 1/r_j\right) \ \mathrm{d}x \\ &\overset{(3.0.37)}{\le} \quad \frac{c_g}{\beta_4^q} \int_{B_{2r_j}(x_j)} \Phi\left(x, \beta_4/r_j\right) \ \mathrm{d}x \\ &\le \quad \frac{2^n c_g}{\beta_4^q} |B_{r_j}(x_j)| \operatorname*{ess\,sup}_{x \in B_{2r_j}(x_j)} \Phi\left(x, \beta_4/r_j\right) \\ &\overset{(3.0.32)}{\le} \quad \frac{2^n c_g c_d}{\beta_4^q} |B_{r_j}(x_j)| \operatorname*{ess\,inf}_{x \in B_{r_j}(x_j)} \Phi\left(x, 1/r_j\right) \\ &\le \quad \frac{2^n c_g c_d}{\beta_4^q} \int_{B_{r_j}(x_j)} \Phi\left(x, 1/r_j\right) \ \mathrm{d}x \ . \end{aligned}$$

In particular, it follows that $f_j \in W^{1,\Phi}(\Omega)$. We then set

$$f := \max_{j \in \{1, \cdots, m\}} f_j \ ,$$

which is continuous, as every f_j is. Moreover, if $x \in W$, then $x \in B_{r_j}(x_j)$ for some $j \in \{1, \cdots, m\}$ and, as a consequence, $f_j(x) = 1$; thus $f(x) = 1$, so $f(W) \equiv \{1\}$ and using the content of the last display, the lattice property of Sobolev functions, see [139, Theorem 1.20] and (3.5.3), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x, |Df|) \ \mathrm{d}x \quad &\le \quad \sum_{j=1}^m \int_\Omega \Phi(x, |Df_j|) \ \mathrm{d}x \\ &\le \quad c \sum_{j=1}^m \int_{B_{r_j}(x_j)} \Phi\left(x, 1/r_j\right) \ \mathrm{d}x \overset{(3.5.3)}{\le} c \left[\mathcal{H}_\Phi(E) + 1\right] \ , \end{aligned}$$

with $c = c(n, \tilde{c}_g, c_d, \beta_4, q)$, hence $f \in W^{1,\Phi}(\Omega) \cap C_0(\Omega)$. Finally, observe that if $x \in \mathbb{R}^n \setminus V$, then $|x - x_j| \geq \operatorname{dist}(E, \mathbb{R}^n \setminus V) \geq 4\kappa$, so $f(x) = 0$ and $\operatorname{supp} f \subset V$. This completes the proof of (3.5.2). Using the above construction inductively, for any $k \in \mathbb{N}$ we find a collection of open sets $\{V_k\}_k$, with $V_0 = \emptyset$ and a sequence of functions $\{\tilde{f}_k\}_k$ such that

$$
\begin{cases} E \subset V_{k+1} \subset V_k \ , \quad \overline{V}_{k+1} \subset \{x \in \Omega \colon \ \tilde{f}_k(x) = 1\} \ , \quad \operatorname{supp} \tilde{f}_k \subset V_k \\ \displaystyle\int_\Omega \Phi(x, |D\tilde{f}_k|) \, \mathrm{d}x \leq c_* \ , \end{cases} \tag{3.5.4}
$$

with c_* being independent of $k \in \mathbb{N}$. For $j \in \mathbb{N}$, define

$$
S_j = \sum_{k=1}^{j} \frac{1}{k} \qquad \text{and} \qquad g_j = \frac{1}{S_j} \sum_{k=1}^{j} \frac{\tilde{f}_k}{k} \ .
$$

From the above discussion, $\tilde{f}_j$ belongs to $\mathcal{R}(E)$ for every j, so, by construction, $g_j \in \mathcal{R}(E)$, and, given that $\operatorname{supp} |D\tilde{f}_k| \subset V_k \setminus \overline{V}_{k+1}$, we find

$$
\begin{aligned}
Cap_\Phi(E) &\leq \int_{\mathbb{R}^n} \Phi(x, |Dg_j|) \ \mathrm{d}x \\
&= \sum_{k=1}^{j} \int_{V_k \setminus V_{k+1}} \Phi(x, S_j^{-1} k^{-1} |D\tilde{f}_k|) \ \mathrm{d}x \overset{(3.0.37),(3.5.4)}{\leq} \frac{c_* \tilde{c}_g}{S_j^p} \sum_{k=1}^{j} \frac{1}{k^p} \to 0
\end{aligned}
$$

as $j \to \infty$, because $p > 1$. The proof is complete.

3.5.3 Proof of Theorem 3

The assertions (3.0.39)-(3.0.40) have already been proved in *Step 3* from the proof of Theorem 1; see (3.4.37)-(3.4.38). We therefore proceed with the proof of (3.0.38). We abbreviate $\Sigma_u = \Sigma_u^p \cup \Sigma_u^q$ where $\Sigma_u^p := \Sigma_u \cap \{x_0 \in \Omega \colon a(x_0) = 0\}$ and $\Sigma_u^q := \Sigma_u \cap \{x_0 \in \Omega \colon a(x_0) > 0\}$. It is therefore sufficient to show

$$
\mathcal{H}_{H^{1+\delta_g}}(\Sigma_u^p) = 0 \quad \text{and} \quad \mathcal{H}_{H^{1+\delta_g}}(\Sigma_u^q) = 0. \tag{3.5.5}
$$

The implication concerning the capacity in (3.0.38) will then be a consequence of Theorem 2. To prove (3.5.5), we use (3.0.34) from Proposition 3.0.1, that gives, in particular, that $\mathcal{H}_{H^{1+\delta_g}} \lesssim \mathcal{H}^-_{H^{1+\delta_g}}$. On the other hand, by the very definition of Σ_u^p, we have that

$$
\mathcal{H}^-_{H^{1+\delta_g}}(\Sigma_u^p) \lesssim \mathcal{H}^{n-p-p\delta_g}(\Sigma_u^p) \ . \tag{3.5.6}
$$

Indeed, taking a covering from $\mathcal{C}^\kappa_{\Sigma_u^p}$ for any $\kappa \in (0,1)$ as in (3.0.26), we see that every ball B of the covering (that for obvious reasons can be assumed to touch Σ_u^p) is such that $a_{\mathrm{i}}(B) = 0$. Therefore $\mathcal{H}^-_{H^{1+\delta_g}}(\Sigma_u^p)$ is equivalent to the $(n-p-p\delta_g)$-dimensional spherical Hausdorff measure of Σ_u^p and (3.5.6) is proved. We conclude that the first equality in (3.5.5) follows from the already proved fact that $\mathcal{H}^{n-p-p\delta_g}(\Sigma_u^p) = 0$. We now prove the second equality in (3.5.5). For this we show that $\mathcal{H}_{H^{1+\delta_g}}(\Sigma_{u,m}^q) = 0$ for every integer m, where $\Sigma_{u,m}^q := \Sigma_u^q \cap \{a(x) > 1/m\}$. The key observation is that there exists a positive number $\tilde{\kappa} \equiv \tilde{\kappa}(\alpha, [a]_{0,\alpha}, m) \in (0,1)$ such that, if $r(B) < \tilde{\kappa}$ and B touches $\Sigma_{u,m}^q$, then $2ma_{\mathrm{i}}(B) > 1$. Therefore, we take a covering $\{B_j\} \in \mathcal{C}^\kappa_{\Sigma_{u,m}^q}$ (again we assume that each of the balls from the covering is touching $\Sigma_{u,m}^q$) with $\kappa < \tilde{\kappa}$, and estimate as follows

$$
\sum_{j \in \mathbb{N}} [r(B_j)]^n \left\{ [r(B_j)]^{-p} + a_{\mathrm{i}}(B_j) [r(B_j)]^{-q} \right\}^{1+\delta_g}
$$

$$\leq cm^{1+\delta_g} \sum_{j\in\mathbb{N}} [r(B_j)]^n \left\{ a_{\rm i}(B_j)[r(B_j)]^{-p} + a_{\rm i}(B_j)[r(B_j)]^{-q} \right\}^{1+\delta_g}$$
$$\leq cm^{1+\delta_g} \|a\|_{L^\infty}^{1+\delta_g} \sum_{j\in\mathbb{N}} [r(B_j)]^{n-q(1+\delta_g)} \ .$$

The above relation implies then that

$$\mathscr{H}^-_{H^{1+\delta_g}}(\Sigma^q_{u,m}) \lesssim \mathscr{H}^{n-q-q\delta_g}(\Sigma^q_{u,m}) \leq \mathscr{H}^{n-q-q\delta_g}(\Sigma^q_u) \Longrightarrow \mathscr{H}^-_{H^{1+\delta_g}}(\Sigma^q_{u,m}) = 0$$

by (3.0.40) for every positive integer m. By again appealing to (3.0.34) we deduce that $\mathscr{H}_{H^{1+\delta_g}}(\Sigma^q_{u,m}) = 0$ for every positive integer m and the proof is complete.

Chapter 4

A borderline case of Calderón-Zygmund estimates for non-uniformly elliptic problems

Joint work with G. Mingione (University of Parma)
***St. Petersburg Math. J.* (2019) - http://mi.mathnet.ru/eng/aa1653**

We aim to complete the Calderón-Zygmund type theory obtained in [62], achieving a delicate borderline case that has been left open there. Let us briefly summarize the situation. In [10, 12, 60, 61, 75] the authors have provided a basic regularity theory for minimizers of double phase functionals of the type

$$\mathscr{P}(w,\Omega) := \int_\Omega \left[|Dw|^p + a(x)|Dw|^q\right] \ \mathrm{d}x \tag{4.0.1}$$

where

$$1 < p < q\,, \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)\,, \qquad \alpha \in (0,1]. \tag{4.0.2}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded open subset of $\mathbb{R}^n$ and $n \geq 2$, we refer to Section 4.2 below for more notation. The functional $\mathscr{P}(\cdot)$ is characterized by the fact that it changes the rate of ellipticity according to the positivity of the coefficient $a(x)$: when $a(x) > 0$ the integrand has q-polynomial behaviour with respect to the gradient, otherwise it shows p-polynomial one. Needless to say, when $a(x) \equiv 0$ the functional $\mathscr{P}(\cdot)$ reduces to the standard p-Laplacian functional (we refer to the papers [168, 170] for a recent update of regularity theory in the standard p-case and to [241, 242] for the beginnings). This kind of functional has been introduced by Zhikov in a series of remarkable papers [246–249]. He was motivated by speculations on theoretical aspects of the Calculus of Variations such as the Lavrentiev phenomenon, and by some problems arising in the homogenization of composite and strongly anisotropic materials. We refer the reader to the introductory sections of [60, 61], and especially of [11], for a comprehensive discussion on the subject and its role in the setting of non-uniformly elliptic problems and modern regularity theory. Recently, in [62], is investigated the validity of Calderón-Zygmund estimates for solutions to non-homogenous equations that are naturally connected to the Euler-Lagrange equations of the functional in (4.0.1). The model equation is in this case given by

$$-\operatorname{div}\left[p|Du|^{p-2}Du + qa(x)|Du|^{q-2}Du\right] = -\operatorname{div}\left[|\mathfrak{F}|^{p-2}\mathfrak{F} + a(x)|\mathfrak{F}|^{q-2}\mathfrak{F}\right], \tag{4.0.3}$$

which is in fact the Euler-Lagrange equation of the functional

$$W^{1,p}(\Omega) \ni w \to \mathscr{P}(w,\Omega) - \int_\Omega \left[|\mathfrak{F}|^{p-2}\mathfrak{F} + a(x)|\mathfrak{F}|^{q-2}\mathfrak{F}\right]\cdot Dw \ \mathrm{d}x, \tag{4.0.4}$$

where $\mathfrak{F}\colon \Omega \to \mathbb{R}^n$ is a given vector field. For this situation one of the main results of [62] claims the validity of the sharp implication

$$|\mathfrak{F}|^p + a(\cdot)|\mathfrak{F}|^q \in L^\gamma_{\rm loc}(\Omega) \Longrightarrow |Du|^p + a(\cdot)|Du|^q \in L^\gamma_{\rm loc}(\Omega) \qquad \text{for all} \ \ \gamma > 1\,, \tag{4.0.5}$$

under the main assumption

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{4.0.6}$$

As in fact shown in [101], (4.0.5) fails to hold in the case $q/p > 1 + \alpha/n$. Here, we show that (4.0.5) still holds in the limiting case $q/p = 1 + \alpha/n$, thereby replacing (4.0.6) by

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n}. \tag{4.0.7}$$

Borderline cases are always delicate, and, indeed, we shall exploit a few subtle facts that have been overlooked in [62]. We again refer to the Introduction of [62] for a description of the problems concerning the type of results in (4.0.5) and their place in what is nowadays called Nonlinear Calderón-Zygmund theory. We just confine ourselves to remark that, in view of the example in [101], the condition in (4.0.7) is necessary to obtain our main result, that is Theorem 5 below. We also remark that, in the standard case $a(\cdot) \equiv 0$ or $p = q$, the results in (4.0.5) gives back the classical Calderón-Zygmund estimate for p-Laplacian type operators. Our result here actually holds for a class of equations that are more general of the one in (4.0.3) - and in fact this has already been dealt with in [62]. We shall consider systems of the type

$$-\operatorname{div} A(x, Du) = -\operatorname{div} G(x, \mathfrak{F}) \qquad \text{in} \ \ \Omega \subset \mathbb{R}^n\,. \tag{4.0.8}$$

The assumptions on the continuous vector field $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are now as follows:

$$\begin{cases} A \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n) \ \text{ and } \ z \mapsto A(\cdot, z) \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n) \\ |A(x,z)| + |\partial_z A(x,z)||z| \leq L\left(|z|^{p-1} + a(x)|z|^{q-1}\right) \\ \nu\left(|z|^{p-2} + a(x)|z|^{q-2}\right)|\xi|^2 \leq \partial_z A(x,z)\xi \cdot \xi \\ |A(x_1,z) - A(x_2,z)| \leq L|a(x_1) - a(x_2)||z|^{q-1} \end{cases}, \tag{4.0.9}$$

for all $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$, $x, x_1, x_2 \in \Omega$, where $0 < \nu \leq L < \infty$ are fixed ellipticity constants. The coefficient $a(\cdot)$ and the numbers p, q, α have already been specified in (4.0.2). As for the right-hand side, the Carathéodory regular vector field $G\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is instead assumed to verity the following natural growth conditions:

$$|G(x,z)| \leq L\left(|z|^{p-1} + a(x)|z|^{q-1}\right)\,. \tag{4.0.10}$$

Assumptions (4.0.9)-(4.0.10) perfectly cover the case displayed in (4.0.3) and are in fact modelled on them. Finally, before going on, let us mention that we shall use the compact notation

$$\texttt{data} \equiv \texttt{data}\left(n, p, q, \alpha, \nu, L, \|a\|_{L^\infty}, [a]_\alpha, \|H(\cdot, Du)\|_{L^1(\Omega)}\right), \tag{4.0.11}$$

including several data of the problem considered. We shall denote

$$H(x,z) := \left[|z|^p + a(x)|z|^q\right] \tag{4.0.12}$$

whenever $x \in \Omega$ and $z \in \mathbb{R}^n$; we shall, with some abuse of notation, keep on denoting as in (4.0.12) also in the case $z \in \mathbb{R}$. Our main result is the following:

Theorem 5 *Let $u \in W^{1,1}(\Omega)$ be a distributional solution to* (4.0.8)*, such that $H(\cdot, Du), H(\cdot, \mathfrak{F}) \in L^1(\Omega)$ and under the assumptions* (4.0.2), (4.0.7), (4.0.9), (4.0.10)*. Then* (4.0.5) *holds. Moreover, fix open subsets $\Omega_0 \Subset \tilde{\Omega}_0 \Subset \Omega$ with $\mathrm{dist}(\Omega_0, \partial\tilde{\Omega}_0) \approx \mathrm{dist}(\tilde{\Omega}_0, \partial\Omega) \approx \mathrm{dist}(\Omega_0, \partial\Omega)$; for every $\gamma > 1$, there exist a radius $r > 0$ and a constant $c \geq 1$, both depending on* **data**, $\mathrm{dist}(\Omega_0, \partial\Omega), \gamma$ *and $\|\mathfrak{F}\|_{L^\gamma(\tilde{\Omega}_0)}$, such that the inequality*

$$\left(\fint_{B_{\varrho/2}} [H(x, Du)]^\gamma \, \mathrm{d}x\right)^{1/\gamma} \leq c \fint_{B_\varrho} H(x, Du) \, \mathrm{d}x + c \left(\fint_{B_\varrho} [H(x, \mathfrak{F})]^\gamma \, \mathrm{d}x\right)^{1/\gamma} \tag{4.0.13}$$

holds for every ball $B_\varrho \subset \Omega_0$ such that $\varrho \leq r$.

Starting from the procedure developed in [62], we are able to achieve the limiting case in (4.0.7) by using an improved approach to the fractional estimates originally developed in [61]. These are estimates aimed at proving that the gradient of solutions to homogeneous equations as

$$-\operatorname{div} A(x, Dw) = 0 \,, \tag{4.0.14}$$

under assumptions as (4.0.7) and (4.0.9) belong to suitable fractional Sobolev spaces. See Theorem 8 below. Let us remark that fractional differentiability properties of solutions to various types of potentially degenerate elliptic equations are a powerful tool in regularity theory (see for instance [193] where these are employed to estimate singular sets of solutions), and have been recently the object of investigation in different settings for both local and nonlocal operators including rough data too [6,168,171]. The improvement in the arguments of [61] then comes from a further application of a preliminary higher integrability result for solutions to (4.0.14), that uses certain classical self-improving properties of reverse Hölder inequalities. Once this improvement is reached we can revisit the proof given in [62] to get the statement of Theorem 5. Let us finally observe that the type of problems considered in this work are related to so called functionals with (p, q)-growth conditions. These have been extensively treated in the literature over the last years starting by the papers of Marcellini [183,184]. Eventually, several contributions have been given in this direction, see for instance [67,69,79,134,138,145,176,177,217,218,223,238] amongst the most closely related to the setting we are considering, that is the one of non-autonomous functionals with non-standard growth conditions.

4.1 The vectorial case

We now state the analogous of Theorem 5 in the vectorial framework. We have to restrict to minima of (4.0.4), since, as stressed in Chapter 1, vector-valued solutions to general equations or variational problems with no additional structure such as in (4.0.3)-(4.0.4), exhibit singularities already in the standard case $a(\cdot) \equiv 0$.

Theorem 6 *Let $u \in W^{1,1}(\Omega, \mathbb{R}^N)$, $N \geq 1$, be a local minimizer of* (4.0.4) *with $H(\cdot, Du), H(\cdot, \mathfrak{F}) \in L^1(\Omega)$ and* (4.0.2) *and* (4.0.7) *being in force. Then* (4.0.5) *holds together with estimate* (4.0.13) *as described in Theorem 5.*

4.2 Basic material

As usually done in the framework of p-Laplacian type problems, we shall use the auxiliary vector fields $V_p, V_q \colon \mathbb{R}^n \to \mathbb{R}^n$ defined in (2.4.1). As a consequence of $(4.0.9)_3$, it holds that

$$\begin{cases} |V_p(z_1) - V_p(z_2)|^2 + a(x)|V_q(z_1) - V_q(z_2)|^2 \leq c\big(A(x, z_1) - A(x, z_2)\big) \cdot (z_1 - z_2) \\ H(x, z) \leq cA(x, z) \cdot z \end{cases} \tag{4.2.1}$$

whenever $z, z_1, z_2 \in \mathbb{R}^n$, $x \in \Omega$, and with $c \equiv c(n, \nu, p, q)$. Let us recall some basic terminology about generalized Orlicz-Sobolev spaces, i.e., Sobolev spaces defined by the fact that the distributional derivatives lie in a suitable Orlicz-Musielak space. We are mainly interested in spaces related to the Young type function defined in (4.0.12) and its constant coefficients variants (see for instance (4.2.4) below). These are defined by

$$W^{1,H}(\Omega) := \left\{ u \in W^{1,1}(\Omega) : \ H(\cdot, Du) \in L^1(\Omega) \right\}, \tag{4.2.2}$$

with the local variant being defined in the obvious way and $W^{1,H}_0(\Omega) := W^{1,H}(\Omega) \cap W^{1,p}_0(\Omega)$. For more details we refer to [11] and related references. Next to equations as in (4.0.8) and (4.0.14), we shall consider boundary value problems involving operators with constant coefficients of the type

$$\begin{cases} -\operatorname{div} A_0(Dv) = 0 \ \text{ in } B \\ v \in w + W^{1,H_0}_0(B) , \end{cases} \tag{4.2.3}$$

where $B \subset \mathbb{R}^n$ is a ball, $w \in W^{1,H_0}(B)$ and

$$H_0(z) := \left[|z|^p + a_0 |z|^q \right] , \tag{4.2.4}$$

for some $a_0 \geq 0$. We consider the following assumptions on $A_0(\cdot)$, that are parallel to those in (4.0.9) (and actually coincide with (4.0.9) when $a(\cdot) \equiv a_0$) and indeed follow a notation similar to the one in (4.0.9):

$$\begin{cases} A_0 \in C(\mathbb{R}^n, \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n) \\ |A_0(z)| + |\partial_z A_0(z)||z| \leq L(|z|^{p-1} + a_0|z|^{q-1}) \\ \nu(|z|^{p-2} + a_0|z|^{q-2})|\xi|^2 \leq \langle \partial_z A_0(z)\xi \cdot \xi. \end{cases} \tag{4.2.5}$$

Solvability of (4.2.3) follows using monotonicity methods as described in [62]. We then have the following result obtained in [62, Section 5]. We only mention that in the next statement no restriction occurs on the ratio q/p; the proof is exactly the same as the one presented in [62].

Theorem 7 *Under assumptions (4.2.5) with $1 < p < q$, let $v \in W^{1,H_0}(B)$ be the unique distributional solution to (4.2.3) with $w \in W^{1,H_0}(B)$. Then*

$$H_0(Dw) \in L^\gamma(B) \Rightarrow H_0(Dv) \in L^\gamma(B) \qquad \textit{holds for every } \gamma > 1 .$$

Moreover, for every $\gamma > 1$, there exists a constant $c \equiv c(n, p, q, \nu, L, \gamma)$, which is a non decreasing function of $|B|$ and, in particular, it is independent of a_0, such that the following inequality holds:

$$\fint_B [H_0(Dv)]^\gamma \, \mathrm{d}x \leq c \fint_B [H_0(Dw)]^\gamma \, \mathrm{d}x . \tag{4.2.6}$$

The following approximation property extends the ones already exploited in [61, 101].

Lemma 4.2.1 *Under assumptions (4.0.2) and (4.0.7), if $v \in W^{1,H}(\Omega)$ is such that $H(\cdot, Dv) \in L^{1+\delta}_{\mathrm{loc}}(\Omega)$ for some $\delta \geq 0$, then there exists a sequence of smooth functions $\{v_k\}_{k \in \mathbb{N}} \subset W^{1,\infty}_{\mathrm{loc}}(\Omega)$ such that, for all $B \Subset \Omega$ with radius r, $r \leq 1$, there holds that $v_k \to v$ strongly in $W^{1,p(1+\delta)}_{\mathrm{loc}}(\Omega)$ and*

$$\int_B [H(x, Dv_k)]^{1+\delta} \, \mathrm{d}x \to \int_B [H(x, Dv)]^{1+\delta} \, \mathrm{d}x .$$

Proof. Let us define $H_\delta(x,z) := \left[|z|^p + a(x)|z|^q\right]^{1+\delta} \approx \left[|z|^{p(1+\delta)} + [a(x)]^{1+\delta}|z|^{q(1+\delta)}\right]$. It is sufficient to show that

$$v_k \to v \text{ strongly in } W^{1,p(1+\delta)}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \int_B H_\delta(x, Dv_k)\ \mathrm{d}x \to \int_B H_\delta(x, Dv)\ \mathrm{d}x\ .$$

This is now a consequence of the arguments in [61, 101] since the function $x \mapsto [a(x)]^{1+\delta}$ is still $C^{0,\alpha}$-regular and the newly defined integrand $H_\delta(\cdot)$ satisfies the conditions detailed in [61, Section 4], with p and q replaced by $p(1+\delta)$ and $q(1+\delta)$, respectively. □

We conclude with a lemma has been proved in [62, Proposition 3.1] assuming (4.0.6), but for its proof the bound in (4.0.7) is actually sufficient.

Lemma 4.2.2 *Under assumptions* (4.0.2) *and* (4.0.7)*, let* $B \Subset \Omega$ *be a ball and let* $S\colon B \to \mathbb{R}^n$ *be a measurable vector field such that* $H(\cdot, S) \in L^1(B)$ *and which is a distributional solution to the equation* $-\operatorname{div} T(x,S) = 0$ *in* B*. Here we assume that the vector field* $T\colon B \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfies the growth conditions*

$$|T(x,z)| \lesssim \left[|z|^{p-1} + a(x)|z|^{q-1}\right]$$

for every $x \in B$ *and* $z \in \mathbb{R}^n$*. Then every* $\varphi \in W^{1,1}_0(B)$ *such that* $H(\cdot, D\varphi) \in L^1(B)$ *satisfies*

$$\int_B T(x,S) \cdot D\varphi\ \mathrm{d}x = 0.$$

4.3 Higher integrability estimates

In this section we fix a ball B, with radius $r > 0$, such that $B \Subset \Omega$ and $r \le 1$, and we provide a few existence results and regularity estimates for solutions $w \in W^{1,H}(B)$ to Dirichlet boundary value problems of the type

$$\begin{cases} -\operatorname{div} A(x, Dw) = 0 & \text{in } B \\ w \in w_0 + W^{1,H}_0(B)\,, \end{cases} \tag{4.3.1}$$

where $w_0 \in W^{1,H}(B)$ is given boundary datum such that

$$H(\cdot, Dw_0) \in L^{1+\delta}(B) \qquad \text{for some } \delta > 0 \tag{4.3.2}$$

and

$$\|H(\cdot, Dw_0)\|_{L^1(B)} \le L_1 \tag{4.3.3}$$

for some finite constant $L_1 \ge 0$. Needless to say, in (4.3.1) and for the rest of the section, the vector $A(\cdot)$ satisfies the assumptions of Theorem 5, that is (4.0.2), (4.0.7) and (4.0.9). We start introducing the perturbed vector fields $A_k\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$A_k(x,z) := A(x,z) + \varepsilon_k |z|^{s-2} z \tag{4.3.4}$$

where $s > q$ and $\{\varepsilon_k\}$ is a sequence of positive numbers such that $\varepsilon_k \to 0$. Moreover, with $B \Subset \Omega$, let us consider a sequence $\{\tilde{w}_k\} \subset W^{1,\infty}(B)$ such that

$$\int_B H(x, D\tilde{w}_k)\ \mathrm{d}x \to \int_B H(x, Dw_0)\ \mathrm{d}x \quad \text{and} \quad \int_B [H(x, D\tilde{w}_k)]^{1+\delta}\ \mathrm{d}x \to \int_B [H(x, Dw_0)]^{1+\delta}\ \mathrm{d}x. \tag{4.3.5}$$

The existence of such a sequence is ensured by Lemma 4.2.1 and by (4.3.2). Beside the functions $\tilde{w}_k$, we consider another sequence $\{w_k\} \subset W^{1,s}(B)$ in such a way that for every $k \in \mathbb{N}$, w_k is the unique solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div} A_k(x, Dw_k) = 0 \quad \text{in } B \\ w_k \in \tilde{w}_k + W_0^{1,s}(B) \ . \end{cases} \tag{4.3.6}$$

We start with a first higher integrability result extending those originally found in [60, 61].

Lemma 4.3.1 *Under assumptions* (4.0.2), (4.0.7) *and* (4.0.9), *assume also that*

$$\sup_{k\in\mathbb{N}} \|H(\cdot, Dw_k)\|_{L^1(B)} \leq \tilde{c} \tag{4.3.7}$$

holds for a positive constant $\tilde{c}$. *Then there exist two positive constants* $\delta_1 > 0$ *and* c, *both depending on* $\tilde{c}, n, p, q, \nu, L, s, [a]_{0,\alpha}$ *and* α, *but otherwise independent of* k, *such that*

$$\left(\fint_{B_\varrho} [H(x, Dw_k) + \varepsilon_k |Dw_k|^s]^{1+\delta_1} \, \mathrm{d}x \right)^{\frac{1}{1+\delta_1}} \leq c \fint_{B_{2\varrho}} \left[H(x, Dw_k) + \varepsilon_k |Dw_k|^s \right] \, \mathrm{d}x \ . \tag{4.3.8}$$

holds for every ball $B_{2\varrho} \subset B$ *and every* $k \in \mathbb{N}$.

Proof. The proof combines a suitable Caccioppoli type estimate with the Sobolev-Poincaré type inequalities obtained in [60, 61, 218]. Let $B_{2\varrho} \Subset B$ be a ball and $\eta \in C_c^1(B_{2\varrho})$ be so that $\chi_{B_\varrho} \leq \eta \leq \chi_{B_{2\varrho}}$ and $|D\eta| \leq 4\varrho^{-1}$. We test the weak formulation of (4.3.6) against $\varphi = \eta^s(w_k - (w_k)_{B_{2\varrho}})$, which is admissible since $w_k \in W^{1,s}(B, \mathbb{R}^N)$, $(s > q)$ for all $k \in \mathbb{N}$. By means of (4.2.1), Young inequality, and recalling that

$$s < \frac{q(s-1)}{q-1} < \frac{p(s-1)}{p-1} \ ,$$

we obtain, for $\varepsilon \in (0,1)$

$$\begin{aligned} &\int_{B_{2\varrho}} \eta^s \left[H(x, Dw_k) + \varepsilon_k |Dw_k|^s \right] \, \mathrm{d}x \\ &\leq c \int_{B_{2\varrho}} \eta^{s-1} \left(|Dw_k|^{p-1} + a(x)|Dw_k|^{q-1} + \varepsilon_k |Dw_k|^{s-1} \right) \left| \frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho} \right| \, \mathrm{d}x \\ &\leq \varepsilon \int_{B_{2\varrho}} \eta^s \left[H(x, Dw_k) + \varepsilon_k |Dw_k|^s \right] \, \mathrm{d}x \\ &\quad + c_\varepsilon \int_{B_{2\varrho}} H\left(x, \frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho} \right) + \varepsilon_k \left| \frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho} \right|^s \, \mathrm{d}x \ . \end{aligned}$$

Choosing ε small enough and reabsorbing terms, we can conclude that

$$\fint_{B_\varrho} \left[H(x, Dw_k) + \varepsilon_k |Dw_k|^s \right] \, \mathrm{d}x \leq c \fint_{B_{2\varrho}} H\left(x, \frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho} \right) + \varepsilon_k \left| \frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho} \right|^s \, \mathrm{d}x \tag{4.3.9}$$

holds for $c \equiv c(n, p, q, \nu, L, s)$. We can then use the intrinsic Sobolev-Poincaré's inequality developed in [61, 217] to get

$$\fint_{B_{2\varrho}} H\left(x, \frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho} \right) \, \mathrm{d}x \leq c \left(\fint_{B_{2\varrho}} [H(x, Dw_k)]^d \, \mathrm{d}x \right)^{1/d} \ , \tag{4.3.10}$$

with $c \equiv c(n,p,q,\alpha,[a]_{0,\alpha},\tilde{c}) \geq 1$ and $d \equiv d(n,p,q,\alpha,[a]_{0,\alpha}) < 1$. Moreover, defining $s_* := \max\left\{1, \frac{ns}{n+s}\right\}$, we obtain

$$\fint_{B_{2\varrho}} \varepsilon_k \left|\frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho}\right|^s \, dx \leq c \left(\fint_{B_{2\varrho}} (\varepsilon_k |Dw_k|^s)^{s_*/s} \, dx\right)^{s/s_*}, \tag{4.3.11}$$

with $c \equiv c(n,s)$. Let $\tilde{d} := \max\{s_*/s, d\} < 1$ and combine (4.3.10) and (4.3.11) with Hölder inequality to obtain

$$\begin{aligned}\fint_{B_{2\varrho}} & \left[H\left(x, \frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho}\right) + \varepsilon_k \left|\frac{w_k - (w_k)_{B_{2\varrho}}}{\varrho}\right|^s\right] \, dx \\ & \leq c \left(\fint_{B_{2\varrho}} [H(x, Dw_k) + \varepsilon_k |Dw_k|^s]^{\tilde{d}} \, dx\right)^{1/\tilde{d}},\end{aligned} \tag{4.3.12}$$

where $c \equiv c(n,p,q,\nu,L,s,[a]_{0,\alpha},\alpha,\tilde{c})$ and $\tilde{d} = \tilde{d}(n,p,q,s)$. From (4.3.9) and (4.3.12) we obtain

$$\fint_{B_\varrho} [H(x, Dw_k) + \varepsilon_k |Dw_k|^s] \, dx \leq c \left(\fint_{B_{2\varrho}} [H(x, Dw_k) + \varepsilon_k |Dw_k|^s]^{\tilde{d}} \, dx\right)^{1/\tilde{d}}, \tag{4.3.13}$$

so we can apply a standard variant of Gehring's lemma to conclude with the statement. □

We then obtain a global version of the above result.

Lemma 4.3.2 *Under the assumptions* (4.0.2), (4.0.7), (4.0.9), (4.3.2), (4.3.3), (4.3.7), *suppose also that*

$$\sup_{k \in \mathbb{N}} \|H(\cdot, D\tilde{w}_k)\|_{L^1(B)} \leq \bar{c} \tag{4.3.14}$$

holds for a positive constant $\bar{c}$. *Then there exist a positive exponent* $\delta_2 \leq \delta_1$, *and constant* c, *both depending only on* $\tilde{c}, \bar{c}, n, p, q, \alpha, \nu, L, [a]_{0,\alpha}$, *such that*

$$\left(\fint_B [H(x, Dw_k) + \varepsilon_k |Dw_k|^s]^{1+\sigma} \, dx\right)^{\frac{1}{1+\sigma}} \leq c \left(\fint_B [H(x, D\tilde{w}_k) + \varepsilon_k |D\tilde{w}_k|^s]^{1+\sigma} \, dx\right)^{\frac{1}{1+\sigma}}$$

holds whenever $\sigma \in [0, \delta_2)$, *for every* $k \in \mathbb{N}$.

Proof. We first test (4.3.6) against $\phi := w_k - \tilde{w}_k$ to obtain by (4.2.1) (arguing for instance as in (see again [62, Theorem 3.1]))

$$\int_B [H(x, Dw_k) + \varepsilon_k |Dw_k|^s] \, dx \leq c \int_B [H(x, D\tilde{w}_k) + \varepsilon_k |D\tilde{w}_k|^s] \, dx, \tag{4.3.15}$$

with $c \equiv c(n,p,q,\nu,L,s)$. Now, for $x_0 \in B$, let $B_{2\varrho}(x_0) \subset \mathbb{R}^n$ be any ball such that $|B_{2\varrho}(x_0) \setminus B| > |B_{2\varrho}(x_0)|/10$ and let $\varphi := \eta^s(w_k - \tilde{w}_k)$, with $\eta \in C^1_c(B_{2\varrho})$ so that $\chi_{B_\varrho} \leq \eta \leq \chi_{B_{2\varrho}}$ and $|D\eta| \leq 4\varrho^{-1}$. Notice that φ is admissible for testing and $\mathrm{supp}\varphi \subset B \cap B_{2\varrho}(x_0)$. From the weak formulation of (4.3.6) and (4.2.1), we obtain

$$\begin{aligned}\int_{B \cap B_{2\varrho}(x_0)} \eta^s [H(x, Dw_k) + \varepsilon_k |Dw_k|^s] \, dx &\leq c \int_{B \cap B_{2\varrho}(x_0)} \eta^{s-1} |A_k(x, Dw_k)| \left|\frac{w_k - \tilde{w}_k}{\varrho}\right| \, dx \\ &+ c \int_{B \cap B_{2\varrho}(x_0)} \eta^s |A_k(x, Dw_k)| |D\tilde{w}_k| \, dx =: \mathrm{(I)} + \mathrm{(II)},\end{aligned} \tag{4.3.16}$$

with $c \equiv c(n,p,q,\nu,L,s)$. Again Young inequality gives, for any $\varepsilon \in (0,1)$

$$|(\mathrm{I})| \le \varepsilon \int_{B\cap B_{2\varrho}(x_0)} \eta^s [H(x,Dw_k)+\varepsilon_k |Dw_k|^s] \,\mathrm{d}x \\ + c_\varepsilon \int_{B\cap B_{2\varrho}(x_0)} H\left(x, \frac{w_k-\tilde{w}_k}{\varrho}\right) + \varepsilon_k \left|\frac{w_k-\tilde{w}_k}{\varrho}\right|^s \,\mathrm{d}x \ ,$$

where $c_\varepsilon \equiv c_\varepsilon(n,p,q,\nu,L,s,\varepsilon)$ and, similarly,

$$|(\mathrm{II})| \le \varepsilon \int_{B\cap B_{2\varrho}(x_0)} \eta^s [H(x,Dw_k)+\varepsilon_k |Dw_k|^s] \,\mathrm{d}x \\ + c_\varepsilon \int_{B\cap B_{2\varrho}(x_0)} H(x,D\tilde{w}_k) + \varepsilon_k |D\tilde{w}_k|^s \,\mathrm{d}x,$$

where $c_\varepsilon \equiv c_\varepsilon(n,p,q,\nu,L,s,\varepsilon)$. Taking ε small enough and inserting the content of the last two displays in (4.3.16), we obtain

$$\int_{B\cap B_{\varrho}(x_0)} \left[H(x,Dw_k)+\varepsilon_k |Dw_k|^s\right] \,\mathrm{d}x \le c\int_{B\cap B_{2\varrho}(x_0)} \left[H\left(x,\frac{w_k-\tilde{w}_k}{\varrho}\right) + \varepsilon_k \left|\frac{w_k-\tilde{w}_k}{\varrho}\right|^s\right] \,\mathrm{d}x \\ + c\int_{B\cap B_{2\varrho}(x_0)} \left[H(x,D\tilde{w}_k)+\varepsilon_k |D\tilde{w}_k|^s\right] \,\mathrm{d}x \ .$$

Applying Sobolev-Poincaré's inequality as follows

$$\fint_{B\cap B_{2\varrho}(x_0)} H\left(x,\frac{w_k-\tilde{w}_k}{\varrho}\right) \,\mathrm{d}x \le c\left(\fint_{B\cap B_{2\varrho}(x_0)} [H(x,Dw_k-D\tilde{w}_k)]^d \,\mathrm{d}x\right)^{1/d} \\ \le c\left(\fint_{B\cap B_{2\varrho}(x_0)} [H(x,Dw_k)]^d \,\mathrm{d}x\right)^{1/d} + c\fint_{B\cap B_{2\varrho}(x_0)} H(x,D\tilde{w}_k) \,\mathrm{d}x,$$

with c, d as in (1.2.3) (this holds with the proof given in [61, 218]) and additionally depending on constant $\bar{c}$ appearing , after standard manipulations as in the proof of Lemma 4.3.1, we get

$$\fint_{B\cap B_{\varrho}(x_0)} \left[H(x,Dw_k)+\varepsilon_k |Dw_k|^s\right] \,\mathrm{d}x \le c\left(\fint_{B\cap B_{2\varrho}(x_0)} [H(x,Dw_k)+\varepsilon_k|Dw_k|^s]^{\tilde{d}} \,\mathrm{d}x\right)^{1/\tilde{d}} \\ + c\fint_{B\cap B_{2\varrho}(x_0)} \left[H(x,D\tilde{w}_k)+\varepsilon_k |D\tilde{w}_k|^s\right] \,\mathrm{d}x$$

with $c \equiv c(\tilde{c},n,p,q,\alpha,\nu,L,s,[a]_{0,\alpha})$. We next consider the situation when it is $B_{2\varrho}(x_0) \Subset B$. In this case we can proceed as for the interior case treated in Lemma 4.3.1 getting (4.3.8), thereby getting (4.3.13). The two cases can be combined via a standard covering argument. More precisely, upon defining

$$V_k(x) := \begin{cases} [H(x,Dw_k)+\varepsilon_k|Dw_k|^s]^{\tilde{d}} & B \\ 0 & \mathbb{R}^n \setminus B \end{cases}$$

and

$$U_k(x) := \begin{cases} H(x,D\tilde{w}_k)+\varepsilon_k|D\tilde{w}_k|^s & B \\ 0 & \mathbb{R}^n \setminus B \end{cases}$$

we easily get

$$\fint_{B_\varrho(x_0)} [V_k(x)]^{1/\tilde{d}} \, \mathrm{d}x \le c \left\{ \left(\fint_{B_{2\varrho}(x_0)} V_k(x) \, \mathrm{d}x \right)^{1/\tilde{d}} + \fint_{B_{2\varrho}(x_0)} U_k(x) \, \mathrm{d}x \right\} ,$$

with $c = c(\tilde{c}, n, p, q, \nu, L, s, [a]_{0,\alpha}, \alpha)$ and $0 < \tilde{d} < 1$. At this point the conclusion follows once again by the usual variant of Gehring's lemma. □

We proceed with a fractional differentiability result, following a strategy that has been initially implemented in [61].

Theorem 8 *Under assumption (4.0.2), (4.0.7), (4.0.9), (4.3.2) and (4.3.3), there exists a unique solution $w \in w_0 + W^{1,H}_0(B)$ to (4.3.1) and it satisfies*

$$Dw \in L^{\frac{np}{n-2\beta}}_{\mathrm{loc}}(B, \mathbb{R}^n) \cap W^{\min\{2\beta/p, \beta\}, p}_{\mathrm{loc}}(B, \mathbb{R}^n) \tag{4.3.17}$$

for every $\beta < \alpha$. In particular, it follows

$$Dw \in L^{2q-p}_{\mathrm{loc}}(B, \mathbb{R}^n) . \tag{4.3.18}$$

The energy estimate

$$\int_B H(x, Dw) \, \mathrm{d}x \le c_1 \int_B H(x, Dw_0) \, \mathrm{d}x \tag{4.3.19}$$

holds for a constant $c_1 \equiv c_1(n, \nu, L, p, q)$, while the global higher integrability estimate

$$\int_B [H(x, Dw)]^{1+\sigma} \, \mathrm{d}x \le c_2 \int_B [H(x, Dw_0)]^{1+\sigma} \, \mathrm{d}x \tag{4.3.20}$$

holds for positive constants $c_2, \sigma \equiv c_2, \sigma(n, p, q, \alpha, \nu, L, [a]_{0,\alpha}, L_1)$ and it is $\sigma < \delta$. Moreover, in (4.3.20) the exponent σ can be replaced by any smaller positive number. Finally, the solution w can be obtained as the limit of solutions $\{w_k\}$ to problems (4.3.6) in the sense that, up to relabelled subsequences, it holds that

$$w_k \rightharpoonup w \quad \text{in } W^{1,p(1+\sigma)}(B) \qquad \text{and} \qquad w_k \to w \quad \text{in } W^{1,\frac{np}{n-2\beta}}_{\mathrm{loc}}(B) \tag{4.3.21}$$

for every $\beta < \alpha$. In particular, we can choose β such that

$$p(1+\sigma) < 2q - p < \frac{np}{n-2\beta} .$$

Proof. Step 1: Approximation. We preliminary recall that $\tilde{w}_k \in W^{1,\infty}(B)$ for every integer k; for a number $\sigma > 0$ which is yet to be defined, the quantity

$$\varepsilon_k := \left(k + \|D\tilde{w}_k\|^{3(2q-p)}_{L^{2q-p}(B)} + \|D\tilde{w}_k\|^{3(1+\sigma)(2q-p)}_{L^{(1+\sigma)(2q-p)}(B)} \right)^{-1} \tag{4.3.22}$$

is well-defined and finite. Moreover, observe that the convergence

$$\lim_{k\to\infty} \varepsilon_k \int_B |D\tilde{w}_k|^{2q-p} \, \mathrm{d}x = 0$$

happens to be uniform with respect to the choice of $\sigma \ge 0$. We take this choice of $\{\varepsilon_k\}$ in (4.3.4), where we also choose $s := 2q - p$. We later specify the actual value of σ. The weak form of (4.3.6) is

$$\int_B A_k(x, Dw_k) \cdot D\varphi \, \mathrm{d}x = 0 \quad \text{for all} \quad \varphi \in W^{1,2q-p}_0(B) \tag{4.3.23}$$

and (4.3.15) becomes

$$\int_B \left[H(x, Dw_k) + \varepsilon_k |Dw_k|^{2q-p}\right] \, \mathrm{d}x \le c \int_B \left[H(x, D\tilde{w}_k) + \varepsilon_k |D\tilde{w}_k|^{2q-p}\right] \, \mathrm{d}x \tag{4.3.24}$$

with $c \equiv c(n, \nu, L, p, q)$, so that, recalling (4.3.22) we also have, for k large enough, we have

$$\int_B H(x, D\tilde{w}_k) \, \mathrm{d}x \overset{(4.3.5)}{\le} 2 \int_B H(x, Dw_0) \, \mathrm{d}x \le 2L_1 := \bar{c}, \tag{4.3.25}$$

and

$$\int_B H(x, Dw_k) \, \mathrm{d}x \overset{(4.3.5)}{\le} 2 \int_B H(x, Dw_0) \, \mathrm{d}x + c \overset{(4.3.3)}{\le} c(L_1 + 1) := \tilde{c}, \tag{4.3.26}$$

again for $c \equiv c(n, \nu, L, p, q)$. This now fixes the choice of the numbers $\tilde{c}$ and $\bar{c}$ appearing in (4.3.7) and (4.3.14), respectively, and therefore this ultimately reflects in the value of the two higher integrability exponents $\delta_2 \le \delta_1$, appearing in Lemmas (4.3.1) and (4.3.2), respectively, that are independent of the number $\sigma > 0$ introduced in (4.3.22). Needless to say, and with no loss of generality, we shall consider always the indexes k large enough for which (4.3.25)-(4.3.26) hold. We fix a positive σ such that

$$\sigma < \frac{\delta_2}{2}, \quad p(1+\sigma) < 2q - p \quad \text{and} \quad \sigma < \frac{2\alpha}{n}, \tag{4.3.27}$$

and this finally fixes the choice in (4.3.22); observe that σ exhibits the following dependence:

$$\sigma \equiv \sigma\left(n, p, q, \alpha, \nu, L, [a]_{0,\alpha}, L_1\right). \tag{4.3.28}$$

Similarly, using this time Lemma 4.3.2, and again (4.3.22) we estimate

$$\begin{aligned}
\int_B |Dw_k|^{p(1+\sigma)} \, \mathrm{d}x &\le \int_B [H(x, Dw_k)]^{1+\sigma} \, \mathrm{d}x \le \int_B [H(x, Dw_k) + \varepsilon_k |Dw_k|^{2q-p}]^{1+\sigma} \, \mathrm{d}x \\
&\le c \int_B [H(x, D\tilde{w}_k) + \varepsilon_k |D\tilde{w}_k|^{2q-p}]^{1+\sigma} \, \mathrm{d}x \le c \int_B [H(x, Dw_0)]^{1+\sigma} \, \mathrm{d}x + c\,,
\end{aligned} \tag{4.3.29}$$

for $c \equiv c(n, p, q, \alpha, \nu, L, [a]_{0,\alpha}, L_1)$, for k large enough. We can therefore assume that, up to passing to not relabelled subsequences, $w_k \rightharpoonup w$ in $W^{1,p(1+\sigma)}(B)$ for some $w \in w_0 + W_0^{1,p(1+\sigma)}(B)$. By lower semicontinuity in (4.3.24) and (4.3.29), and again recalling (4.3.22), we find:

$$\int_B H(x, Dw) \, \mathrm{d}x \le c_1 \int_B H(x, Dw_0) \, \mathrm{d}x \tag{4.3.30}$$

and

$$\int_B [H(x, Dw)]^{1+\sigma} \, \mathrm{d}x \le c_2 \int_B [H(x, Dw_0)]^{1+\sigma} \, \mathrm{d}x, \tag{4.3.31}$$

with $c_1 \equiv c_1(n, \nu, L, p, q)$ and $c_2 \equiv c_2(n, p, q, \alpha, \nu, L, [a]_{0,\alpha}, L_1)$.

Step 2: Fractional Sobolev embedding and interpolation. Here we modify the arguments of [61, Section 5]. Let us take a ball $B_{2\varrho} \Subset B$ (not necessarily concentric to B); the computations made in [61, Section 5, p. 470], which hold when assuming (4.0.7) too, give that, for $0 < \varrho \le t < s \le 2\varrho$ (and after scaling back in the proof given in [61])

$$\|V_p(Dw_k)\|^2_{L^{\frac{2n}{n-2\beta}}(B_t)} + [V_p(Dw_k)]^2_{W^{\beta,2}(B_t)}$$

$$\leq \frac{c}{(s-t)^{2\beta}}\|Dw_k\|^p_{L^p(B_s)} + \frac{c}{(s-t)^{2\beta}}\left(\|a\|^2_{L^\infty(B_s)} + \varrho^{2\alpha}[a]^2_{0,\alpha;B_s} + \varepsilon_k\right)\|Dw_k\|^{2q-p}_{L^{2q-p}(B_s)}\,, \tag{4.3.32}$$

for every $\beta < \alpha$ and $k \in \mathbb{N}$, with $c \equiv c(n,p,q,\nu,L,\alpha,\beta)$. In the above display we are using the standard notation for the Gagliardo seminorm

$$[V_p(Dw_k)]^2_{W^{\beta,2}(B_t)} := \int_{B_t}\int_{B_t} \frac{|V_p(Dw_k(x)) - V_p(Dw_k(y))|^2}{|x-y|^{n+2\beta}}\, dx\, dy\,.$$

We refer to [85] for basic properties about fractional Sobolev spaces, and to [61, 101] for the specific ones that are relevant here. We now aim at estimating the second term in the right-hand side of (4.3.32). In particular there holds:

$$\|Dw_k\|^p_{L^{\frac{np}{n-2\beta}}(B_t)} \leq \frac{c\|Dw_k\|^p_{L^p(B_s)}}{(s-t)^{2\beta}} + \frac{cT_k\|Dw_k\|^{2q-p}_{L^{2q-p}(B_s)}}{(s-t)^{2\beta}} \tag{4.3.33}$$

for $\beta \in (0,\alpha)$, where

$$T_k := \|a\|^2_{L^\infty(B_\varrho)} + \varrho^{2\alpha}[a]^2_{0,\alpha;B_\varrho} + \varepsilon_k \tag{4.3.34}$$

and $c \equiv c(n,p,q,\nu,L,\alpha,\beta)$. Notice that this last constant blows-up when $\beta \to \alpha$. Next, we start taking $\beta < \alpha$ such that

$$2q - p < \frac{np}{n-2\beta} \tag{4.3.35}$$

which is implied, by virtue of (4.0.7), by

$$\frac{\alpha}{1+2\alpha/n} < \beta < \alpha$$

which is in particular satisfied by choosing

$$\frac{\alpha}{1+\sigma} < \beta < \alpha \tag{4.3.36}$$

in view of the last inequality in (4.3.27). Keeping (4.3.27) and (4.3.35) in mind, we now look for $\theta \in (0,1)$ such that

$$\frac{1}{2q-p} = \frac{1-\theta}{p(1+\sigma)} + \frac{(n-2\beta)\theta}{np} \tag{4.3.37}$$

and this gives

$$\theta = \frac{n(2q-2p-p\sigma)}{[2\beta-(n-2\beta)\sigma](2q-p)}\,.$$

We notice that, keeping also (4.0.7) in mind,

$$\theta(2q-p) < p \Leftrightarrow \frac{n(2q-2p-p\sigma)}{2\beta-(n-2\beta)\sigma} < p \Leftrightarrow 2n(q-p) < 2\beta p(1+\sigma) \Leftarrow \alpha < \beta(1+\sigma)\,, \tag{4.3.38}$$

and the last inequality is true by virtue of the choice made in (4.3.36). By the choices made above, and in particular by (4.3.37), we use the interpolation inequality

$$\|Dw_k\|_{L^{2q-p}(B_s)} \leq \|Dw_k\|^{(1-\theta)}_{L^{p(1+\sigma)}(B_s)}\|Dw_k\|^{\theta}_{L^{\frac{np}{n-2\beta}}(B_s)}\,.$$

In (4.3.38), we saw that $(2q-p)\theta < p$, so we may apply Young inequality with conjugate exponents

$$t_1 = \frac{p}{\theta(2q-p)} \quad \text{and} \quad t_2 = \frac{p}{p-(2q-p)\theta} \, ,$$

to obtain, for $\varepsilon \in (0,1)$

$$\begin{aligned} \frac{T_k}{(s-t)^{2\beta}} \|Dw_k\|_{L^{2q-p}(B_s)}^{2q-p} &\le \frac{T_k}{(s-t)^{2\beta}} \|Dw_k\|_{L^{p(1+\sigma)}(B_s)}^{(1-\theta)(2q-p)} \|Dw_k\|_{L^{\frac{np}{n-2\beta}}(B_s)}^{\theta(2q-p)} \\ &\le \varepsilon \|Dw_k\|_{L^{\frac{np}{n-2\beta}}(B_s)}^{p} + c_\varepsilon \left[\frac{T_k}{(s-t)^{2\beta}} \|Dw_k\|_{L^{p(1+\sigma)}(B_s)}^{(1-\theta)(2q-p)} \right]^{\frac{p}{p-(2q-p)\theta}} . \end{aligned} \tag{4.3.39}$$

Merging (4.3.39) with (4.3.33) and choosing

$$\varepsilon \equiv \varepsilon(n,p,q,\nu,L,\alpha,\beta,\sigma) \equiv \varepsilon \left(n,p,q,\alpha,\nu,L,[a]_{0,\alpha},\beta,L_1\right) \tag{4.3.40}$$

small enough, we conclude that

$$\begin{aligned} \|Dw_k\|_{L^{\frac{np}{n-2\beta}}(B_t)}^{p} \le \frac{1}{2} \|Dw_k\|_{L^{\frac{np}{n-2\beta}}(B_s)}^{p} + \frac{c}{(s-t)^{2\beta}} \|Dw_k\|_{L^p(B_s)}^{p} \\ + c \left[\frac{T_k}{(s-t)^{2\beta}} \|Dw_k\|_{L^{p(1+\sigma)}(B_s)}^{(1-\theta)(2q-p)} \right]^{\frac{p}{p-(2q-p)\theta}} , \end{aligned}$$

where c exhibits the same dependence on the constants appearing in (4.3.40) as an effect we also determining the value of c_ε in (4.3.39). From Lemma 2.4.2, we have

$$\|Dw_k\|_{L^{\frac{np}{n-2\beta}}(B_\varrho)}^{p} \le \frac{c}{\varrho^{2\beta}} \|Dw_k\|_{L^p(B_{2\varrho})}^{p} + \left[\frac{T_k}{\varrho^{2\beta}} \|Dw_k\|_{L^{p(1+\sigma)}(B_{2\varrho})}^{(1-\theta)(2q-p)} \right]^{\frac{p}{p-(2q-p)\theta}} , \tag{4.3.41}$$

again with c depending as in (4.3.40) and for every $k \in \mathbb{N}$. By (4.3.35) and (4.3.41), Hölder inequality, (4.3.26) and (4.3.29), and recalling that $\varrho \le 1$, we can conclude that

$$\begin{aligned} &\|Dw_k\|_{L^{2q-p}(B_\varrho)} + \|Dw_k\|_{L^{\frac{np}{n-2\beta}}(B_\varrho)} \\ &\quad \le \frac{c}{\varrho^{2\beta/p}} \|Dw_k\|_{L^p(B_{2\varrho})} + c \left[\frac{T_k}{\varrho^{2\beta}} \|Dw_k\|_{L^{p(1+\sigma)}(B_{2\varrho})}^{(1-\theta)(2q-p)} \right]^{\frac{1}{p-(2q-p)\theta}} \\ &\quad \le \frac{c}{\varrho^{2\beta/p}} \left[\|H(\cdot,Dw_0)\|_{L^1(B)}^{1/p} + 1 \right] \\ &\qquad + c \left[\frac{T_k}{\varrho^{2\beta}} \left(\|H(\cdot,Dw_0)\|_{L^{1+\sigma}(B)}^{1/p} + 1 \right)^{(1-\theta)(2q-p)} \right]^{\frac{1}{p-(2q-p)\theta}} \end{aligned}$$

holds with $c \equiv c(n,p,q,\alpha,\nu,L,[a]_{0,\alpha},L_1)$ and for sufficiently large k. All in all, we deduce that

$$\|Dw_k\|_{L^{2q-p}(B_\varrho)} + \|Dw_k\|_{L^{\frac{np}{n-2\beta}}(B_\varrho)} \le c\varrho^{-\lambda} \, , \tag{4.3.42}$$

holds for $c, \lambda \equiv c, \lambda(n,p,q,\nu,L,\|a\|_\infty,[a]_{0,\alpha},\alpha,\beta,L_1)$, which is independent of k. This last estimate used together with (4.3.32) and a standard covering argument (keep again the proof of [62, Theorem 3.1] in mind), allows to get the new bound

$$\|Dw_k\|_{W^{\min\{2\beta/p,\beta\},p}(U)} + \|V_p(Dw_k)\|_{W^{\beta,2}(U)} \le c \tag{4.3.43}$$

for any open subset $U \Subset B$ and any $\beta < \alpha$, with c depending as in (4.3.42), and additionally on $\operatorname{dist}(U,\partial B)$. By (4.3.29) and (4.3.43) we can use the standard compact embedding theorems of fractional Sobolev spaces and again a standard diagonal argument allows to conclude that, up to (not relabelled) subsequence, it holds that $Dw_k \to Dw$ strongly in $L^{2q-p}_{\mathrm{loc}}(B)$ and a.e. (thus completely proving (4.3.21)). This allows to let $k \to \infty$ in (4.3.23), obtaining that w is a distributional solution to (4.3.1). Using lower semicontinuity in (4.3.43) then yields (4.3.17)-(4.3.18), while (4.3.19)-(4.3.20) are a consequence of (4.3.30). Finally, let $\tilde{w}$ be another distributional solution to (4.3.1) such that $\tilde{w} \in W^{1,H}(B)$. By Lemma 4.2.2 it follows that we can use $\varphi = w - \tilde{w}$ as test function in the weak formulation

$$\int_B (A(x, Dw) - A(x, D\tilde{w})) \cdot (Dw - D\tilde{w}) \, \mathrm{d}x = 0 \, .$$

At this point, the strict monotonicity (4.2.1) of the vector field $A(\cdot)$ gives that $\tilde{w} = w$. □

4.4 Another higher integrability estimate

This is in the following:

Theorem 9 *Let $u \in W^{1,H}(\Omega)$ be a solution to the equation (4.0.8), under the assumptions (4.0.7) and (4.0.9)-(4.0.10). Assume, moreover, that $H(\cdot, \mathfrak{F}) \in L^{\gamma}_{\mathrm{loc}}(\Omega)$, for some $\gamma > 1$. There exists a positive higher integrability exponent $\delta < \gamma - 1$, depending only on* **data***, such that $H(\cdot, Du) \in L^{1+\delta}_{\mathrm{loc}}(\Omega)$; moreover, the reverse Hölder type inequality*

$$\left(\fint_{B_\varrho} [H(x, Du)]^{1+\delta} \, \mathrm{d}x \right)^{\frac{1}{1+\delta}} \leq c \fint_{B_{2\varrho}} H(x, Du) \, \mathrm{d}x + c \left(\fint_{B_{2\varrho}} [H(x, \mathfrak{F})]^{1+\delta} \, \mathrm{d}x \right)^{\frac{1}{1+\delta}} \tag{4.4.1}$$

holds for every ball $B_{2\varrho} \subset \Omega$, where the constant c depends again only on **data***. In particular, in the case $\mathfrak{F} \equiv 0$, for every open subset $\Omega_0 \Subset \Omega$ there exists a constant $c \equiv c(\mathtt{data}, \operatorname{dist}(\Omega_0, \partial\Omega)) \geq 1$ such that*

$$\|H(x, Du)\|_{L^{1+\delta}(\Omega_0)} \leq c \, .$$

Moreover, in (4.4.1) the exponent δ can be replaced by any smaller number.

Proof. The proof is similar to the one of Lemma 4.3.1 and to the one offered in [61, Theorem 1.1], and we shall report only a brief sketch. Let $B_{2\varrho} \Subset \Omega$. We test the weak formulation of (4.0.8) against $\varphi := \eta^q(u - (u)_{B_{2\varrho}})$, where $\eta \in C^1_c(B_{2\varrho})$, $\chi_{B_\varrho} \leq \eta \leq \chi_{B_{2\varrho}}$ and $|D\eta| \leq 4/\varrho$. Notice that φ is admissible by Lemma 4.2.2. Using (4.0.9)-(4.0.10), the fact that $q < \frac{p(q-1)}{p-1}$, and Young inequality we get

$$\int_{B_{2\varrho}} H(x, Du)\eta^q \, \mathrm{d}x \leq c \int_{B_{2\varrho}} H\left(x, \frac{u - (u)_{B_{2\varrho}}}{\varrho}\right) \, \mathrm{d}x + c \int_{B_{2\varrho}} H(x, \mathfrak{F}) \, \mathrm{d}x; \tag{4.4.2}$$

with $c \equiv c(n, \nu, L, p, q)$. Now we apply Sobolev-Poincaré's inequality to the first term on the right-hand side of (4.4.2) to have

$$\fint_{B_\varrho} H(x, Du) \, \mathrm{d}x \leq c \left(\fint_{B_{2\varrho}} [H(x, Du)]^d \, \mathrm{d}x \right)^{1/d} + c \fint_{B_{2\varrho}} H(x, \mathfrak{F}) \, \mathrm{d}x,$$

with $c \equiv c(\mathtt{data})$ and $d \equiv d(n, p, q) \in (0, 1)$. Now, since $H(\cdot, \mathfrak{F}) \in L^{\gamma}_{\mathrm{loc}}(\Omega)$ and $\gamma > 1$, by a variant of Gehring's Lemma we can conclude with (4.4.1) for a number δ as described in the statement of the theorem, and the proof is complete. □

4.5 Proof of Theorem 5: A conditional reverse Hölder inequality

In this section we start the proof of Theorem 5. Our aim is to prove the reverse type inequality in Theorem 10 below. This extends a similar fact obtained in [61, Theorem 5.1] under the assumption (4.0.6); we now replace this by (4.0.7). The result we are going to develop here is in fact a technical tool in the forthcoming proof of Theorem 5 contained in the next section. We start considering the original solution u from Theorem 5. By Theorem 9 and a standard covering argument, we know that for every choice of open subset $\Omega_0 \Subset \tilde{\Omega}_0 \Subset \Omega$ (with $\operatorname{dist}(\Omega_0, \partial\tilde{\Omega}_0) \approx \operatorname{dist}(\tilde{\Omega}_0, \partial\Omega) \approx \operatorname{dist}(\Omega_0, \partial\Omega)$) as in the statement of Theorem 5, there exists a constant $c \geq 1$ such that

$$\|H(\cdot, Du)\|_{L^{1+\delta}(\Omega_0)} \leq c\left(\texttt{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(\cdot, \mathfrak{F})\|_{L^{\gamma}(\tilde{\Omega}_0)}\right) \tag{4.5.1}$$

holds for some exponent δ depending only on $\texttt{data}$ and which is such that $\delta < \gamma - 1$. We then use the setting of Section 4.3 with $w_0 \equiv u$, by considering problems of the type

$$\begin{cases} -\operatorname{div} A(x, Dw) = 0 & \text{in } B_{4\varrho} \\ w \in u + W_0^{1,H}(B_{4\varrho}), \end{cases} \tag{4.5.2}$$

where $B_{4\varrho}$ is a ball such that $B_{8\varrho} \subset \Omega_0$ and $8\varrho \leq 1$, with the number δ coming from (4.5.1) as the one fixed in (4.3.2); the constant L_1 appearing in is obviously fixed by $L_1 := \|H(\cdot, Du)\|_{L^1(B_{4\varrho})}$. Moreover, by (4.3.19), we have

$$\|H(\cdot, Dw)\|_{L^1(B_{4\varrho})} \leq c(n, p, q, \nu, L)\|H(\cdot, Du)\|_{L^1(B_{4\varrho})} . \tag{4.5.3}$$

In view of this last inequality and by Theorem 9, this time applied with $\mathfrak{F} \equiv 0$, we obtain that $Dw \in L^{1+\delta_0}_{\mathrm{loc}}(4B, \mathbb{R}^n)$ for some $\delta_0 \equiv \delta_0(\texttt{data}) > 0$. Moreover, recalling that in Theorem 8 we can take $\sigma \in (0, \delta]$ as small as we like, and in particular $\sigma \leq \delta_0$, we have the following reverse Hölder type inequality:

$$\left(\fint_{B_{2\varrho}} [H(x, Dw)]^{1+\sigma} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma}} \leq c \fint_{B_{4\varrho}} H(x, Dw) \, \mathrm{d}x , \tag{4.5.4}$$

for a constant $c \equiv c(\texttt{data})$ and for a final number σ depending only on $\texttt{data}$. The main result of this section looks like this.

Theorem 10 *Let $w \in W^{1,H}(B_{4\varrho})$ be a solution to (4.5.2) under the assumptions (4.0.7), (4.0.9). Assume that*

$$\sup_{x \in B_\varrho} a(x) \leq K[a]_{0,\alpha}\varrho^{\alpha} \tag{4.5.5}$$

holds for some $K \geq 1$. Then, for any $\bar{q} < np/(n - 2\alpha)$ ($= \infty$ when $\alpha = 1$ and $n = 2$) there exists a positive constant $c \equiv c(\texttt{data}, \bar{q}, K)$, such that the following reverse Hölder type inequality holds:

$$\left(\fint_{B_\varrho} |Dw|^{\bar{q}} \, \mathrm{d}x\right)^{1/\bar{q}} \leq c\left(\fint_{B_{4\varrho}} H(x, Dw) \, \mathrm{d}x\right)^{1/p} . \tag{4.5.6}$$

Moreover, let $1 < \gamma_1 < \gamma_2 \leq 4$, the inequality

$$\left(\fint_{B_{\gamma_1 \varrho}} |Dw|^{2q-p} \, \mathrm{d}x\right)^{\frac{1}{2q-p}} \leq c\left(\fint_{B_{\gamma_2 \varrho}} H(x, Dw) \, \mathrm{d}x\right)^{1/p} \tag{4.5.7}$$

holds for a constant c additionally depending on γ_1, γ_2.

Proof. We shall revisit the arguments in the proof of Theorem 8, and we keep the notation introduced there; in particular, we shall retain (4.3.27) and (4.3.36) concerning β and σ (and of course σ obeys the smallness conditions enumerated before the statement). Given the reference ball $B_{4\varrho}$ mentioned in the statement of the theorem, all the remaining balls will be concentric unless otherwise stated. We go back to the proof of Theorem 8, *Step 2*, where we consider this time concentric balls $B_\varrho \subset B_{4\varrho} \equiv B$, and the approximate solutions $\{w_k\}$ defined in (4.3.6) with the choice $w_0 = u$; ultimately, this means we are considering problems (4.5.2). Theorem 8 implies, in particular, that $Dw_k \to Dw$ strongly in $L^{p(1+\sigma)}(B_{2\varrho},\mathbb{R}^n)$; letting $k\to\infty$ in (4.3.41) and recalling the definition (4.3.34), we conclude with

$$
\begin{aligned}
\|Dw\|_{L^{\frac{np}{n-2\beta}}(B_\varrho)} &\le \frac{c}{\varrho^{2\beta/p}}\|Dw\|_{L^p(B_{2\varrho})}\\
&\quad + c\left[\frac{1}{\varrho^{2\beta}}\left(\|a\|^2_{L^\infty(B_{2\varrho})} + c\varrho^{2\alpha}[a]^2_{0,\alpha;B_{2\varrho}}\right)\|Dw\|^{(1-\theta)(2q-p)}_{L^{p(1+\sigma)}(B_{2\varrho})}\right]^{\frac{1}{p-(2q-p)\theta}}, \qquad (4.5.8)
\end{aligned}
$$

with $c \equiv c(\textit{\textbf{data}},\beta)$. Notice now that condition (4.5.5) is stable when the radius increases for nested balls. In fact, (4.5.5) and the α-Hölder continuity of $a(\cdot)$ imply

$$
\sup_{x\in B_{M\varrho}} a(x) \le (K+3M)\varrho^\alpha [a]_{0,\alpha}\,, \qquad \text{for all} \quad M\in(1,4)\,, \qquad (4.5.9)
$$

so, with (4.5.9), (4.5.8) becomes (recall $K\ge 1$)

$$
\|Dw\|^p_{L^{\frac{np}{n-2\beta}}(B_\varrho)} \le \frac{c}{\varrho^{2\beta}}\|Dw\|^p_{L^p(B_{2\varrho})} + c\left(\varrho^{2(\alpha-\beta)}[a]^2_{0,\alpha;B_{2\varrho}}K^2\|Dw\|^{(1-\theta)(2q-p)}_{L^{p(1+\sigma)}(B_{2\varrho})}\right)^{\frac{p}{p-(2q-p)\theta}}. \qquad (4.5.10)
$$

Define

$$
b_1 := \frac{p}{p-(2q-p)\theta} \quad \text{and} \quad b_2 := \frac{(1-\theta)(2q-p)}{(1+\sigma)[p-(2q-p)\theta]}\,.
$$

In these terms, after averaging and making a few elementary manipulations, (4.5.10) reads as

$$
\begin{aligned}
\left(\fint_{B_\varrho}|Dw|^{\frac{np}{n-2\beta}}\,\mathrm{d}x\right)^{\frac{n-2\beta}{np}} &\le c\left(\fint_{B_{2\varrho}}|Dw|^p\,\mathrm{d}x\right)^{1/p}\\
&\quad + c\left([a]^2_{0,\alpha;B_{2\varrho}}K^2\right)^{b_1/p}\varrho^{\frac{2(\alpha-\beta)b_1+2\beta+n(b_2-1)}{p}}\left(\fint_{B_{2\varrho}}|Dw|^{p(1+\sigma)}\,\mathrm{d}x\right)^{b_2/p}.
\end{aligned}
$$

Let us re-write the last term in the above inequality as follows:

$$
\begin{aligned}
\left(\fint_{B_{2\varrho}}|Dw|^{p(1+\sigma)}\,\mathrm{d}x\right)^{b_2/p} &= \left(\fint_{B_{2\varrho}}|Dw|^{p(1+\sigma)}\,\mathrm{d}x\right)^{\frac{b_2(1+\sigma)-1}{p(1+\sigma)}}\left(\fint_{B_{2\varrho}}|Dw|^{p(1+\sigma)}\,\mathrm{d}x\right)^{\frac{1}{p(1+\sigma)}}\\
&= \omega_n^{\frac{1-b_2(1+\sigma)}{p(1+\sigma)}}\varrho^{-n\frac{b_2(1+\sigma)-1}{p(1+\sigma)}}\|Dw\|^{\frac{b_2(1+\sigma)-1}{p(1+\sigma)}}_{L^{p(1+\sigma)}(B_{2\varrho})}\left(\fint_{B_{2\varrho}}|Dw|^{p(1+\sigma)}\,\mathrm{d}x\right)^{\frac{1}{p(1+\sigma)}}.
\end{aligned}
$$

Now notice that

$$
\begin{aligned}
\frac{1}{p}\left[2(\alpha-\beta)b_1+2\beta+n(b_2-1)-nb_2+\frac{n}{(1+\sigma)}\right] &\ge \frac{1}{p}\left(2\beta-\frac{n\sigma}{1+\sigma}\right)\\
&\overset{(4.3.36)}{\ge} \frac{2\alpha-n\sigma}{p(1+\sigma)} \overset{(4.3.27)}{>} 0,
\end{aligned}
$$

so, merging the content of the previous three displays and using Hölder inequality, we conclude with

$$
\begin{aligned}
&\left(\fint_{B_\varrho} |Dw|^{\frac{np}{n-2\beta}}\,\mathrm{d}x\right)^{\frac{n-2\beta}{np}}\\
&\quad\le c\left\{1+\left([a]^2_{0,\alpha;B_{2\varrho}}K^2\right)^{b_1/p}\|Dw\|^{\frac{b_2(1+\sigma)-1}{p(1+\sigma)}}_{L^{p(1+\sigma)}(B_{2\varrho})}\right\}\left(\fint_{B_{2\varrho}}|Dw|^{p(1+\sigma)}\,\mathrm{d}x\right)^{\frac{1}{p(1+\sigma)}}\\
&\quad\overset{(4.5.3)}{\le} c\left(\fint_{B_{2\varrho}}|Dw|^{p(1+\sigma)}\,\mathrm{d}x\right)^{\frac{1}{p(1+\sigma)}}\overset{(4.5.4)}{\le} c\left(\fint_{B_{4\varrho}}H(x,Dw)\,\mathrm{d}x\right)^{\frac{1}{p}}, \qquad (4.5.11)
\end{aligned}
$$

with $c \equiv c(\boldsymbol{data},\beta,K)$. We have therefore proved (4.5.6) for the values of $\bar q$ such that $np/(n-\beta p)\le \bar q$, where β is such that $\alpha/(1+\sigma)<\beta<\alpha$, and with σ coming from Theorem 8 as specified at the beginning of the proof; the same obviously follows using Hölder inequality. In particular, (4.5.6) results from a suitable choice of β. As for the (4.5.7), this is a consequence of a variant of a standard covering argument starting from the validity of (4.5.6); let us briefly recall it. We can cover B_ϱ by a finite number $k\equiv k(n,\gamma)$ of balls $\{B_i\}_{i\le k}$ touching $B_{\gamma_1\varrho}$ and with radius $\tilde\varrho=(\gamma_2-\gamma_1)\varrho/100$. Obviously, it is $2B_i\Subset B_{\gamma_2\varrho}$ and these balls are not necessarily concentric to the starting ball B_ϱ. Notice that, for every $i\le k$, it is

$$
\sup_{x\in B_i} a(x)\le \sup_{x\in B_{\gamma\varrho}} a(x)\le (K+3\gamma)[a]_{0,\alpha}\varrho^\alpha\le\left[\frac{100(K+3\gamma_2)}{\gamma_2-\gamma_1}\right][a]_{0,\alpha}\tilde\varrho^\alpha\,.
$$

We can therefore apply (4.5.6) to each of the balls B_i, thereby getting

$$
\left(\fint_{B_i}|Dw|^{\bar q}\,\mathrm{d}x\right)^{1/\bar q}\le c\left(\fint_{2B_i}H(x,Dw)\,\mathrm{d}x\right)^{1/p},
$$

for a constant c now depending also on γ. By summing up the above inequalities with respect to i and again increasing the involved constant in a way that depends only on n,γ, we finally arrive at (4.5.7) and the proof is complete. □

4.6 Proof of Theorem 5: Exit time arguments and conclusion

The general scheme of the proof of Theorem 5 is the same one of [62, Theorem 1.1]. We ask the reader to have [62] at hand since we shall essentially indicate the relevant modifications and we shall follow exactly the same steps as described there, reporting them with the same titles.

Step 1: Exit time and covering of the level set. We recall that with open subset $\Omega_0\Subset\tilde\Omega_0\Subset\Omega$ as in the statement of the theorem, we have that (4.5.1) holds. Then we consider $B_R\subset\Omega_0$, where $R\le r$ (and r is the small radius appearing in the statement of Theorem 5 and to be determined at the end of the proof). We proceed with the exit time and covering argument as in the proof of [62, Theorem 1.1]. In particular, this yields the family of balls $\{B_i\}\equiv\{B_{\varrho_i}(x_i)\}\equiv\{5\tilde B_i\}$ as indicated in [62, (4.9)-(4.11)]. All the balls in question are contained in B_R. Before going on, similarly to (4.0.11), we set

$$
\boldsymbol{data}_0\equiv\boldsymbol{data}_0\left(n,p,q,\alpha,\nu,L,\|a\|_{L^\infty},[a]_\alpha,\|H(\cdot,Du)\|_{L^1(\Omega)},\operatorname{dist}(\Omega_0,\partial\Omega),\|H(\cdot,\mathfrak{F})\|_{L^\gamma(\tilde\Omega_0)},\gamma\right).
$$

Step 2: A first comparison function. We recover the setting of Sections (4.3) and (4.5). By (4.5.1) we use the setting of Section 4.3 with this choice of δ in (4.3.2); in this way we are also ready to use the setting of Section 4.5 and Theorem 10. We are therefore able to apply Theorem 8 that, in turn, allows to define $w_i \in u + W_0^{1,H}(4B_i)$ as the solutions to the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x, Dw_i) = 0 \ \text{ in } 4B_i \\ w_i \in u + W_0^{1,H}(4B_i) \ . \end{cases} \tag{4.6.1}$$

where the balls B_i are from Step 1. Again, by Theorem 8 with $w_0 \equiv u$ as explained in the previous section, we have

$$w_i \in W^{1,2q-p}_{\text{loc}}(4B_i), \quad Dw_i \in L^{\frac{np}{n-2\beta}}_{\text{loc}}(4B_i, \mathbb{R}^n) \ \text{ for all } \ \beta < \alpha \tag{4.6.2}$$

and the estimates

$$\begin{cases} \displaystyle \fint_{4B_i} H(x, Dw_i) \ \mathrm{d}x \le c_1 \fint_{4B_i} H(x, Du) \ \mathrm{d}x \\ \displaystyle \int_{4B_i} [H(x, Dw_i)]^{1+\sigma} \ \mathrm{d}x \le c_2 \int_{4B_i} [H(x, Du)]^{1+\sigma} \ \mathrm{d}x \end{cases} \tag{4.6.3}$$

hold for positive constants $c_1 \equiv c_1(n, \nu, L, p, q)$, $c_2, \sigma \equiv c_2, \sigma(\texttt{data})$. As in [62, (4.17)], we gain that for every $\varepsilon \in (0, 1)$ there exists a constant c_ε, depending also by n, ν, L, p, q, such that

$$\begin{aligned} &\fint_{4B_i} \left[|V_p(Du) - V_p(Dw_i)|^2 + a(x)|V_q(Du) - V_q(Dw_i)|^2 \right] \ \mathrm{d}x \\ &\qquad \le \varepsilon \fint_{4B_i} H(x, Du) \ \mathrm{d}x + c_\varepsilon \fint_{4B_i} H(x, \mathfrak{F}) \ \mathrm{d}x. \end{aligned} \tag{4.6.4}$$

Step 3: A second comparison function. As in [62], we consider a point $x_{i,m} \in \overline{2B_i}$ such that

$$a(x_{i,m}) = \sup_{x \in 2B_i} a(x). \tag{4.6.5}$$

By (4.6.2) we have in particular that $w_i \in W^{1,q}(2B_i)$ so that, by setting

$$H_{i,m}(z) := H(x_{i,m}, z), \tag{4.6.6}$$

we have that $w_i \in W^{1,H_{i,m}}(2B_i)$, and, in particular $w_i \in W^{1,2q-p}(2B_i)$. We can therefore use Theorem 8 for the special case of H-function in (4.6.6). This yields $v_i \in W^{1,H}(2B_i) \cap W^{1,H_{i,m}}(2B_i)$ as the unique solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x_{i,m}, Dv_i) = 0 \ \text{ in } 2B_i \\ v_i \in w_i + W_0^{1,H_{i,m}}(2B_i) \ , \end{cases}$$

for which we get

$$\fint_{2B_i} H(x_{i,m}, Dv_i) \ \mathrm{d}x \le c \fint_{2B_i} H(x_{i,m}, Dw_i) \ \mathrm{d}x \ , \tag{4.6.7}$$

with $c \equiv c(n, \nu, L, p, q)$. Notice that the right-hand side of (4.6.7) is finite, sice $Dw_i \in L^q(2B_i)$ (by (4.6.2)). The weak form of the equations solved by w_i and v_i respectively can be rewritten as

$$\fint_{2B_i} (A(x_{i,m}, Dv_i) - A(x_{i,m}, Dw_i)) \cdot D\varphi \ \mathrm{d}x = \fint_{2B_i} (A(x, Dw_i) - A(x_{i,m}, Dw_i)) \cdot D\varphi \ \mathrm{d}x, \tag{4.6.8}$$

that holds for every choice of smooth test function $\varphi \in C_c^\infty(2B_i)$. As in [62], the function $\varphi := v_i - w_i$ is admissible in (4.6.8) and this gives, via (4.2.1) and $(4.0.9)_4$ (see also [62, (4.26)]),

$$
\begin{aligned}
&\fint_{2B_i} \left[|V_p(Dv_i) - V_p(Dw_i)|^2 + a(x_{i,m})|V_q(Dv_i) - V_q(Dw_i)|^2 \right] \, dx \\
&\qquad \le c \left(\operatorname*{osc}_{2B_i} a \right) \fint_{2B_i} |Dw_i|^{q-1}|Dv_i - Dw_i| \, dx =: (\mathrm{I}) \, ,
\end{aligned} \tag{4.6.9}
$$

where $c \equiv c(n, \nu, L, p, q)$.

Step 4: Two different phases. We are now aiming at estimating the term (I) appearing in the last display. For this, following [61, 62], we distinguish the two phases, that is

$$
\inf_{x \in 2B_i} a(x) > K[a]_{0,\alpha}\varrho_i^\alpha \, , \tag{4.6.10}
$$

which is called, as in [61], the (p, q)-phase, and

$$
\inf_{x \in 2B_i} a(x) \le K[a]_{0,\alpha}\varrho_i^\alpha . \tag{4.6.11}
$$

The number $K \ge 4$ is to be determined towards the proof as a quantity depending only on n, p, q, ν, L, γ.

Step 5: Estimates in the (p, q)-phase. We consider the case (4.6.10) holds. Notice that

$$
\operatorname*{osc}_{2B_i} a \le 4[a]_{0,\alpha}\varrho_i^\alpha \le \frac{4a(x)}{K} \quad \text{for every} \quad x \in 2B_i \, ,
$$

so that, Young inequality yields

$$
(\mathrm{I}) \le \frac{c}{K} \fint_{2B_i} a(x)|Dw_i|^{q-1}|Dv_i - Dw_i| \, dx \le \frac{c}{K} \fint_{2B_i} a(x)(|Dw_i|^q + |Dv_i|^q) \, dx \, , \tag{4.6.12}
$$

with $c \equiv c(n, p, q, \nu, L, [a]_{0,\alpha})$. Observing that

$$
a(x_{i,m}) \le a(x) + 4[a]_{0,\alpha}\varrho_i^\alpha \le a(x) + \frac{4a(x)}{K} \le 2a(x) \, , \tag{4.6.13}
$$

we estimate

$$
\begin{aligned}
\fint_{2B_i} a(x)(|Dw_i|^q + |Dv_i|^q) \, dx &\overset{(4.6.5)}{\le} c \fint_{2B_i} a(x_{i,m}) \left(|Dw_i|^q + |Dv_i|^q \right) \, dx \\
&\le c \fint_{2B_i} H(x_{i,m}, Dw_i) \, dx \\
&+ c \fint_{2B_i} H(x_{i,m}, Dv_i) \, dx \\
&\overset{(4.6.7)}{\le} c \fint_{2B_i} H(x_{i,m}, Dw_i) \, dx \\
&\overset{(4.6.13)}{\le} c \fint_{2B_i} H(x, Dw_i) \, dx \\
&\overset{(4.6.3)}{\le} c \fint_{4B_i} H(x, Du) \, dx \, ,
\end{aligned}
$$

with $c \equiv c(n, p, q, \nu, L, [a]_{0,\alpha})$. Using this with (4.6.12) in (4.6.9) yields

$$
\fint_{2B_i} \left[|V_p(Dv_i) - V_p(Dw_i)|^2 + a(x_{i,m})|V_q(Dv_i) - V_q(Dw_i)|^2 \right] \, dx \le \frac{c}{K} \fint_{4B_i} H(x, Du) \, dx \, , \tag{4.6.14}
$$

for a constant $c \equiv c(n,p,q,\nu,L,[a]_{0,\alpha})$ which is independent of K. We single out the following estimate from the second-last display

$$\fint_{2B_i} H(x_{i,m},Dw_i)\,\mathrm{d}x \le c\fint_{4B_i} H(x,Du)\,\mathrm{d}x\,, \tag{4.6.15}$$

for $c \equiv c(n,p,q,\nu,L)$.

Step 6: Estimates in the p-phase. Here we consider the case in which (4.6.11) is in force. In this case it is

$$a(x_{i,m}) \le 4[a]_{0,\alpha}\varrho_i^\alpha + \inf_{x\in 2B_i} a(x) \le (4+K)[a]_{0,\alpha}\varrho_i^\alpha\,. \tag{4.6.16}$$

Therefore, as described in Step 2, we apply Theorem 10, estimate (4.5.7), that gives

$$\left(\fint_{2B_i} |Dw_i|^q\,\mathrm{d}x\right)^{1/q} \le c\left(\fint_{4B_i} H(x,Dw_i)\,\mathrm{d}x\right)^{1/p}\,, \tag{4.6.17}$$

with $c \equiv c(\texttt{data}_0,K)$. Back to the estimation of the last term in (4.6.9), Young inequality gives

$$\begin{aligned}
\text{(I)} &\le ca(x_{i,m})\fint_{2B_i} (|Dw_i|+|Dv_i|)^{q-1}|Dw_i-Dv_i|\,\mathrm{d}x \\
&\le c\fint_{2B_i} a(x_{i,m})|Dv_i|^q\,\mathrm{d}x + c\fint_{2B_i} a(x_{i,m})|Dw_i|^q\,\mathrm{d}x\,,
\end{aligned} \tag{4.6.18}$$

for $c \equiv c(n,\nu,L,p,q)$. We start estimating the last term in (4.6.18) as follows

$$\begin{aligned}
\fint_{2B_i} a(x_{i,m})|Dw_i|^q\,\mathrm{d}x &\overset{(4.6.16)}{\le} c\varrho_i^\alpha\fint_{2B_i} |Dw_i|^q\,\mathrm{d}x \\
&\overset{(4.6.17)}{\le} c\varrho_i^\alpha\left(\fint_{4B_i} H(x,Dw_i)\,\mathrm{d}x\right)^{\frac{q-p}{p}}\fint_{4B_i} H(x,Dw_i)\,\mathrm{d}x \\
&\overset{\text{Hölder}}{\le} c\varrho_i^\alpha\left(\fint_{4B_i} [H(x,Dw_i)]^{1+\sigma}\,\mathrm{d}x\right)^{\frac{q-p}{p(1+\sigma)}}\fint_{4B_i} H(x,Dw_i)\,\mathrm{d}x \\
&\le c\|H(\cdot,Dw_i)\|_{L^{1+\sigma}(4B_i)}^{\frac{q-p}{p(1+\sigma)}}\varrho_i^{\alpha-\frac{n(q-p)}{p(1+\sigma)}}\fint_{4B_i} H(x,Dw_i)\,\mathrm{d}x \\
&\overset{(4.6.3)}{\le} c\|H(\cdot,Du)\|_{L^{1+\sigma}(4B_i)}^{\frac{q-p}{p(1+\sigma)}}\varrho_i^{\alpha-\frac{n(q-p)}{p(1+\sigma)}}\fint_{4B_i} H(x,Dw_i)\,\mathrm{d}x \\
&\overset{(4.5.1)}{\le} c\varrho_i^{\alpha-\frac{n(q-p)}{p(1+\sigma)}}\fint_{4B_i} H(x,Dw_i)\,\mathrm{d}x \\
&\overset{(4.6.3)}{\le} c\varrho_i^{\alpha-\frac{n(q-p)}{p(1+\sigma)}}\fint_{4B_i} H(x,Du)\,\mathrm{d}x,
\end{aligned}$$

for $c \equiv c(\texttt{data}_0,K)$. Letting

$$\kappa_1 := \alpha - \frac{n(q-p)}{p(1+\sigma)} > \alpha - n\left(\frac{q}{p}-1\right) \overset{(4.0.7)}{\ge} 0 \tag{4.6.19}$$

and we conclude with

$$\varrho_i^\alpha\fint_{2B_i} |Dw_i|^q\,\mathrm{d}x + \fint_{2B_i} a(x_{i,m})|Dw_i|^q\,\mathrm{d}x \le c\varrho_i^{\kappa_1}\fint_{4B_i} H(x,Du)\,\mathrm{d}x. \tag{4.6.20}$$

In order to estimate the remaining term in the last line of (4.6.18), the one featuring v_i, we need to make use of Theorem 7. Precisely, we aim to apply it with the choice $\gamma = q/p$ and then combine the outcome with Theorem 10 for $(2q-p) = q^2/p$. We first show that q^2/p enters in the range of exponents covered by (4.3.17). For this, we notice that

$$\frac{n\alpha(\alpha+2n)}{2(n+\alpha)^2} < \beta < \alpha \Longrightarrow \frac{q^2}{p^2} \stackrel{(4.0.7)}{\leq} \left(1+\frac{\alpha}{n}\right)^2 < \frac{n}{n-2\beta} < \frac{n}{n-2\alpha} \, .$$

The last quantity in the previous display is meant to be ∞ when $\alpha = 1$ and $n = 2$. Therefore we have that $w_i \in W^{1,q^2/p}(2B_i)$ and the reverse inequality holds as a consequence of (4.5.1)

$$\left(\fint_{2B_i} |Dw_i|^{q^2/p} \, \mathrm{d}x\right)^{p/q^2} \leq c \left(\fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x\right)^{1/p} , \tag{4.6.21}$$

where $c \equiv c(\texttt{data}, K)$. Estimate (4.2.6) (applied with $\gamma = q/p$ and $a_0 = a(x_{i,m})$, so that $H_0(z) = H(x_{i,m}, z)$) reads as

$$\fint_{2B_i} [H(x_{i,m}, Dv_i)]^{q/p} \, \mathrm{d}x \leq c \fint_{2B_i} [H(x_{i,m}, Dw_i)]^{q/p} \, \mathrm{d}x \, , \tag{4.6.22}$$

for $c \equiv c(n, \nu, L, p, q)$ and with a finite right-hand side. We then argue as follows:

$$\begin{aligned}
\fint_{2B_i} a(x_{i,m})|Dv_i|^q \, \mathrm{d}x &\stackrel{(4.6.16)}{\leq} c\varrho_i^{\alpha} \fint_{2B_i} |Dv_i|^q \, \mathrm{d}x \leq c\varrho_i^{\alpha} \fint_{2B_i} [H(x_{i,m}, Dv_i)]^{q/p} \, \mathrm{d}x \\
&\stackrel{(4.6.22)}{\leq} c\varrho_i^{\alpha} \fint_{2B_i} [H(x_{i,m}, Dw_i)]^{q/p} \, \mathrm{d}x \\
&\stackrel{(4.6.16)}{\leq} c\varrho_i^{\alpha} \fint_{2B_i} |Dw_i|^q \, \mathrm{d}x + c\varrho_i^{\alpha(1+q/p)} \fint_{2B_i} |Dw_i|^{q^2/p} \, \mathrm{d}x \\
&\stackrel{(4.6.20)}{\leq} c\varrho_i^{\kappa_1} \fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x + c\varrho_i^{\alpha(1+q/p)} \fint_{2B_i} |Dw_i|^{q^2/p} \, \mathrm{d}x,
\end{aligned} \tag{4.6.23}$$

with $c \equiv c(\texttt{data}_0, K)$. Notice that, upon defining

$$\kappa_2 := \alpha\left(1+\frac{q}{p}\right) - \frac{n}{1+\sigma}\left[\left(\frac{q}{p}\right)^2 - 1\right] > \left(1+\frac{q}{p}\right)\left[\alpha - n\left(\frac{q}{p}-1\right)\right] \stackrel{(4.0.7)}{\geq} 0 \, , \tag{4.6.24}$$

we have

$$\begin{aligned}
\varrho_i^{\alpha(1+q/p)} & \fint_{2B_i} |Dw_i|^{q^2/p} \, \mathrm{d}x \\
&\stackrel{(4.6.21)}{\leq} c\varrho_i^{\alpha(1+q/p)} \left(\fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x\right)^{q^2/p^2-1} \fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x \\
&\stackrel{\text{Hölder}}{\leq} c\varrho_i^{\alpha(1+q/p)} \left(\fint_{4B_i} [H(x, Dw_i)]^{1+\sigma} \, \mathrm{d}x\right)^{\frac{q^2/p^2-1}{1+\sigma}} \fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x \\
&\leq c\varrho_i^{\alpha(1+q/p)-n\frac{q^2/p^2-1}{1+\sigma}} \|H(\cdot, Dw_i)\|_{L^{1+\sigma}(4B_i)}^{q^2/p^2-1} \fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x \\
&\stackrel{(4.6.3)}{\leq} c\varrho_i^{\kappa_2} \|H(\cdot, Du)\|_{L^{1+\sigma}(4B_i)}^{q^2/p^2-1} \fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x \\
&\leq c\varrho_i^{\kappa_2} \fint_{4B_i} H(x, Dw_i) \, \mathrm{d}x
\end{aligned}$$

$$\overset{(4.6.3)}{\leq} \quad c\varrho_i^{\kappa_2} \fint_{4B_i} H(x, Du)\ \mathrm{d}x, \tag{4.6.25}$$

where $c \equiv c(\boldsymbol{data}_0, K)$. Collecting estimates (4.6.23), (4.6.25) yields

$$\fint_{2B_i} a(x_{i,m})|Dv_i|^q\ \mathrm{d}x \leq c\varrho_i^{\kappa_2} \fint_{4B_i} H(x, Du)\ \mathrm{d}x\ .$$

Using this last estimate and (4.6.20) in (4.6.18), we again obtain

$$\text{(I)} \leq c\varrho_i^{\kappa_1} \fint_{4B_i} H(x, Du)\ \mathrm{d}x,$$

with $c \equiv c(\boldsymbol{data}_0, K)$. We have also used that $\kappa_1 < \kappa_2$ (compare (4.6.19) and (4.6.24)) and that $\varrho_i \leq 1$. Finally, this last estimate and (4.6.9) lead to

$$\fint_{2B_i} \left[|V_p(Dv_i) - V_p(Dw_i)|^2 + a(x_{i,m})|V_q(Dv_i) - V_q(Dw_i)|^2\right]\ \mathrm{d}x \leq c_*\varrho_i^{\kappa_1} \fint_{4B_i} H(x, Du)\ \mathrm{d}x \tag{4.6.26}$$

for $c_* \equiv c_*(\boldsymbol{data}_0, K)$. We also observe that, by first using (4.6.20) and then (4.6.3) we obtain the following analog of (4.6.15):

$$\fint_{2B_i} H(x_{i,m}, Dw_i)\ \mathrm{d}x \leq c \fint_{4B_i} H(x, Du)\ \mathrm{d}x\ , \tag{4.6.27}$$

where this time it is $c \equiv c(\boldsymbol{data}_0, K)$.

Step 7: Matching the two phases and comparison estimates. Summarizing the content of (4.6.14) and (4.6.26), we have in both cases (4.6.10) and (4.6.11) that the following inequality holds:

$$\begin{aligned}
&\fint_{2B_i} \left[|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x)|V_q(Dw_i) - V_q(Dv_i)|^2\right]\ \mathrm{d}x \\
&\quad \leq \fint_{2B_i} \left[|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,m})|V_q(Dw_i) - V_q(Dv_i)|^2\right]\ \mathrm{d}x \\
&\quad \leq \left(\frac{\bar{c}}{K} + c_*\varrho_i^{\kappa_1}\right) \fint_{4B_i} H(x, Du)\ \mathrm{d}x\ ,
\end{aligned}$$

with $\bar{c} \equiv \bar{c}(n, \nu, L, p, q)$ and $c_* \equiv c_*(\boldsymbol{data}_0, K)$, where $K \geq 4$ is still to be chosen and κ_1 has been defined in (4.6.19). This last estimate and (4.6.4) (recall that the ε in (4.6.4) is arbitrary) then give

$$\begin{aligned}
&\fint_{2B_i} \left[|V_p(Dv_i) - V_p(Du)|^2 + a(x)|V_q(Dv_i) - V_q(Du)|^2\right]\ \mathrm{d}x \\
&\quad \leq \left(2\varepsilon + 2c_*r^{\kappa_1} + \frac{2\bar{c}}{K}\right) \fint_{4B_i} H(x, Du)\ \mathrm{d}x + 2c_\varepsilon \fint_{4B_i} H(x, \mathfrak{F})\ \mathrm{d}x\ ,
\end{aligned} \tag{4.6.28}$$

for all $\varepsilon \in (0, 1)$, where again it is $\bar{c} \equiv \bar{c}(n, \nu, L, p, q)$ and $c_* \equiv c_*(\boldsymbol{data}_0, K)$, where $K \geq 4$, while $c_\varepsilon \equiv c_\varepsilon(n, p, q, \nu, L, \varepsilon)$ and $\kappa_1 > 0$ has been defined in (4.6.19). Here we also used the fact that $\varrho_i \leq r$ (see *Step 1*). We now adopt the notation

$$S(\varepsilon, r, K, M) := 2\varepsilon + 2c_*r^{\kappa_1} + \frac{2\bar{c}}{K} + \frac{2c_\varepsilon}{M}$$

and, using the information contained in [62, $(4.14)_2$] in (4.6.28), we establish that for every $K \geq 4$ the estimate

$$\fint_{2B_i} \left[|V_p(Dv_i) - V_p(Du)|^2 + a(x)|V_q(Dv_i) - V_q(Du)|^2\right] \, dx \leq S(\varepsilon, r, K, M)\lambda \tag{4.6.29}$$

holds for all the balls B_i from the covering displayed in [62, (4.11)]. Notice that at this stage this in an estimate which is phase-independent: no matter of which among (4.6.10) and (4.6.11), we have that occurs (4.6.29) holds in any case. Moreover, we are still free to choose $K \geq 4$.

Step 8: The two phases at a different threshold. The goal here is to show that the estimate

$$\fint_{2B_i} H(x_{i,m}, Dv_i) \, dx = \fint_{2B_i} \left[|Dv_i|^p + a(x_{i,m})|Dv_i|^q\right] \, dx \leq c\lambda, \tag{4.6.30}$$

holds for $c \equiv c(\texttt{data}_0)$. For this we simply consider the two alternatives

$$\inf_{x\in 2B_i} a(x) > 10[a]_{0,\alpha}\varrho_i^\alpha \quad \text{and} \quad \inf_{x\in 2B_i} a(x) \leq 10[a]_{0,\alpha}\varrho_i^\alpha \ , \tag{4.6.31}$$

which are nothing but (4.6.10) and (4.6.11) with $K = 10$ respectively. We start considering the case in which the first inequality in (4.6.31) holds; by using (4.6.7) first and then (4.6.15) we get that

$$\fint_{2B_i} H(x_{i,m}, Dv_i) \, dx \leq c \fint_{4B_i} H(x, Du) \, dx \ , \tag{4.6.32}$$

for $c \equiv c(n, \nu, L, p, q)$. At this stage (4.6.30) follows recalling the setting of [62, $(4.14)_2$]. In case the second inequality in (4.6.31) holds, we similarly use (4.6.7) again and (4.6.27) with $K = 10$, that renders (4.6.32) with a constant $c \equiv c(\texttt{data}_0)$. Therefore, (4.6.30) follows in every case.

Step 9: A priori estimates for Dv_i. The same as in the proof of [62, Theorem 1.1].

Step 10: Estimates involving level sets. The same as in the proof of [62, Theorem 1.1].

Step 11: Integration and conclusion. The same as in the proof of [62, Theorem 1.1] and this concludes the proof of Theorem 5. We only remark that the peculiar dependence on the constants on $\texttt{data}_0$ comes from the estimates in *Steps 2-8*, and finally reflects in (4.0.13).

Chapter 5

Lipschitz bounds and non-autonomous functionals

Joint work with G. Mingione (University of Parma)
***Preprint* (2020)**

We aim to provide a comprehensive treatment of Lipschitz regularity of solutions for a very large class of vector-valued non-autonomous variational problems, involving integral functionals of the type

$$W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \ni w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega \left[F(x,Dw) - f\cdot w\right]\, \mathrm{d}x\ . \tag{5.0.1}$$

These are defined on Sobolev spaces and here $\Omega \subset \mathbb{R}^n$ is an open subset with $n \geq 2$. In the rest of the chapter we shall assume that $F(x,Dw) \equiv \tilde{F}(x,|Dw|)$, which is a natural assumption in the vectorial case, where $\tilde{F} : \Omega \times [0,\infty) \to [0,\infty)$ is a suitably regular function (see Section 5.1.1 below for the precise assumptions). The vector field $f\colon \Omega \mapsto \mathbb{R}^N$ will be at least L^n-integrable

$$f \in L^n_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\ . \tag{5.0.2}$$

The notion of local minimizers used here is quite standard in the literature.

Definition 4 *A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a* local minimizer *of the functional $\mathcal{F}$ in* (5.0.1) *with $f \in L^n_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ if, for every open subset $\tilde{\Omega} \Subset \Omega$, we have $\mathcal{F}(u,\tilde{\Omega}) < \infty$ and if $\mathcal{F}(u;\tilde{\Omega}) \leq \mathcal{F}(w,\tilde{\Omega})$ holds for every competitor $w \in u + W^{1,1}_0(\tilde{\Omega},\mathbb{R}^N)$.*

The analysis of (5.0.1) involves the related Euler-Lagrange system

$$-\operatorname{div}(\tilde{a}(x,|Du|)Du) = f\ , \qquad \tilde{a}(x,|Du|) := \frac{\tilde{F}'(x,|Du|)}{|Du|}\ , \tag{5.0.3}$$

where $\tilde{F}'(\cdot)$ denotes the partial derivative of $\tilde{F}(\cdot)$ with respect to the second variable. Our main focus is on obtaining sharp conditions on the datum f, and on the degree of smoothness of the partial map $x \mapsto F(x,\cdot)$, ensuring the local Lipschitz continuity of minimizers. This problem has recently been the object of intensive investigation in the uniformly elliptic, autonomous case $\tilde{F}(x,t) \equiv \tilde{F}(t)$, so that $\tilde{a}(t) = t^{-1}\tilde{F}'(t)$ satisfies (1.2.20), see for instance the recent papers [56–59] for global estimates and [168–170] for local ones. A special, yet important uniformly elliptic model case is given by the p-Laplacian system with coefficients

$$\begin{cases} -\operatorname{div}(c(x)|Du|^{p-2}Du) = f \\ p > 1 \ \text{ and } \ 0 < \nu \leq c(\cdot) \leq L\ . \end{cases} \tag{5.0.4}$$

In this respect, [169, 170] provide us with the following theorem.

Theorem 11 *Let $u \in W^{1,p}_{\rm loc}(\Omega, \mathbb{R}^N)$ be a weak solution to (5.0.4). If $f \in L_{\rm loc}(n,1)(\Omega, \mathbb{R}^N)$ and $c(\cdot)$ is Dini continuous, then Du is continuous.*

In particular, Du is locally bounded. We recall that $f \in L_{\rm loc}(n,1)(\Omega, \mathbb{R}^N)$ means that (1.1.13) is satisfied on any open subset $\tilde{\Omega} \Subset \Omega$, while, denoting by $\omega(\cdot)$ the modulus of continuity of $c(\cdot)$, its Dini continuity amounts to require that

$$\int_0^\tau \omega(\varrho)\,\frac{d\varrho}{\varrho} < \infty \quad \text{for some} \quad \tau \in (0,1]. \tag{5.0.5}$$

Theorem 11 extends to general scalar equations [168] and to systems depending on forms [232]; it is also extension of a classical result of Uhlenbeck [241]. The terminology is motivated by the fact that, for $c(\cdot) \equiv 1$ and $p = 2$, Theorem 11 is a classical result of Stein [235]. It is optimal both with respect to (1.1.13), see [55], and with respect to condition (5.0.5), cf. [157]. The relevant fact in Theorem 11 is that the conditions on f and $c(\cdot)$ implying local Lipschitz continuity are independent of p, and, for more general equations, are in fact independent of the operator considered under the symbol of divergence. This applies both in the case of standard p-growth conditions [168], and in the one of general autonomous uniformly elliptic operators characterized by (5.0.3)-(1.2.20); see [9]. In the case of nonuniformly elliptic operators, the problem of deriving sharp conditions with respect to data for Lipschitz regularity is considerably more difficult. When $f \neq 0$, it has been attacked only recently in [16], but only for the case of autonomous functionals in the principal part, i.e. when $F(\cdot)$ is independent of x. The outcome is that when $n \geq 3$ condition (1.1.13) is still sufficient to guarantee the local Lipschitz regularity of minima, thereby revealing itself as a sort of universal property. In the case $n = 2$, the alternative (actually stonger) borderline condition $L^2_{\rm loc}(\mathrm{LogL})^\alpha(\Omega, \mathbb{R}^N)$ with $\alpha > 2$, implies Lipschitz continuity. In this paper we deal with the general, fully non-autonomous case in (5.0.1). This is by no means a technical extension as in fact, when passing from the uniformly to the nonuniformly elliptic case, the situation with respect to coefficients drastically changes as they can no longer be treated via perturbation as in [169]. To give a glimpse of the situation, let us considered the Double Phase energy, given by

$$W^{1,p}_{\rm loc}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega \left[|Dw|^p + a(x)|Dw|^q\right]\ dx, \tag{5.0.6}$$

with $1 < p < q$, $0 \leq a(\cdot) \in L^\infty(\Omega)$. As shown in [101,106], already when $f \equiv 0$, local minima fail to be continuous if the ratio q/p is too far from one, depending on the rate of α-Hölder continuity of the modulating coefficient $a(\cdot)$. Specifically, condition

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n}\,, \qquad 0 \leq a(\cdot) \in C^{0,\alpha}_{\rm loc}(\Omega)\,, \qquad \alpha \in (0,1] \tag{5.0.7}$$

is necessary and sufficient to get gradient local continuity; see [12]. Condition in (5.0.7) encodes a typical balancing phenomenon between the decay of the space depending coefficient and the gap of the exponents. These is indeed a subtle interaction between the growth of $F(\cdot)$ with respect to the gradient variable z, and its smoothness with respect to the x, determines whether minima are regular or not. This is in fact the main theme of this chapter. Nonuniform ellipticity is a very classical topic in partial differential equations, and it is often motivated by geometric and physical problems. Several classical papers have been devoted to this subject, see for instance [91, 173, 233, 246, 248]. In the setting of the Calculus of Variations there is a wide literature available, starting from the basic papers of Urdaletova & Ural'tseva [243] and Marcellini [183–187]. More recently, the study of the non-autonomous case has intensified; many papers have been devoted to study specific structures [12, 14, 16, 23, 24, 28, 33–35, 49, 77, 84, 126, 127, 218]. Connections to related function spaces have been studied in [88, 135, 221]. The results obtained in this chapter are very general. In fact, in order to catch several model cases simultaneously, their formulation involves a rather generous list of assumptions; see Section 5.1.6 below. Anyway, when applied to single models, such assumptions reveal to be minimal and they produce sharp results. Moreover, in the autonomous case $F(x,z) \equiv F(z)$, the assumptions considered here essentially coincide with those introduced in [16]. For this reason, and also to ease the reading, in this introductory part we shall present a few main corollaries of the general theorems, in connection to a some relevant instances of nonuniformly elliptic functionals. We shall divide our results in three different general classes, detailed in Sections 5.0.1-5.0.3 below. We just remark that, thanks to (5.0.2) and Sobolev embedding, requiring that $\mathcal{F}(u;\tilde{\Omega}) < \infty$ in Definition 4 is the same than requiring that $F(\cdot, Du) \in L^1(\tilde{\Omega})$.

5.0.1 Nonuniform ellipticity at polynomial rates

To provide a general treatment, we start considering functionals featuring (p,q)-growth conditions, [184, 186]. The idea is to provide general conditions on the partial map $x \mapsto F(x,\cdot)$ matching those suggested by counterexamples [101, 106]. In this respect, we consider an integrand $F(\cdot)$ satisfying (5.1.1) for some $\tilde{F}(\cdot)$ as in (5.1.2), whose growth can be framed as

$$\begin{cases} \nu(|z|^2+\mu^2)^{p/2} \le F(x,z) \le \Lambda(|z|^2+\mu^2)^{q/2} + \Lambda(|z|^2+\mu^2)^{p/2} \\ (|z|^2+\mu^2)|\partial^2 F(x,z)| \le \Lambda(|z|^2+\mu^2)^{q/2} + \Lambda(|z|^2+\mu^2)^{p/2} \\ \nu(|z|^2+\mu^2)^{(p-2)/2}|\xi|^2 \le \langle \partial^2 F(x,z)\xi,\xi\rangle \ , \end{cases} \tag{5.0.8}$$

for every choice of $z, \xi \in \mathbb{R}^{N\times n}$ such that $|z| \neq 0$ and for exponents $1 < p \le q$. Here $0 < \nu \le 1 \le \Lambda$ are fixed ellipticity constants and $\mu \in [0,1]$. We also assume that

$$t \mapsto \tilde{F}'(x,t)(t^2+\mu^2)^{\frac{2-p}{2}}/t \quad \text{is non-decreasing} \tag{5.0.9}$$

for every $x \in \Omega$. As for the crucial dependence on x, we assume that for every $t \ge 0$ it holds that $x \mapsto \tilde{F}'(x,t) \in W^{1,d}(\Omega, \mathbb{R}^{N\times n})$ and that

$$|\partial_{xz} F(x,z)| \le h(x)\left[(|z|^2+\mu^2)^{\frac{q-1}{2}} + (|z|^2+\mu^2)^{\frac{p-1}{2}}\right], \qquad h(\cdot) \in L^d_{\rm loc}(\Omega), \quad d > n, \tag{5.0.10}$$

holds for every $x \in \Omega$ and $z \in \mathbb{R}^{N\times n}$. This is a natural approach that has also been considered elsewhere in the work of Marcellini and coauthors, [97, 98].

Theorem 12 *Let $u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of (5.0.1) under assumptions (5.0.8)-(5.0.10). Assume (5.1.24) and*

$$\frac{q}{p} < 1 + \min\left\{\mathfrak{m}_p, \frac{1}{n} - \frac{1}{d}\right\} \quad \text{with} \ \ \mathfrak{m}_p := \begin{cases} \frac{4(p-1)}{\vartheta p(n-2)} & \text{if} \ \ n \ge 3 \\ 1 - \frac{1}{p} & \text{if} \ \ n = 2, \end{cases} \tag{5.0.11}$$

where $\vartheta = 1$ if $p \ge 2$ and $\vartheta = 2$ otherwise. Then $Du \in L^\infty_{\rm loc}(\Omega, \mathbb{R}^{N\times n})$. Moreover, whenever $\mathcal{B} \Subset \Omega$ is a ball with radius $\mathbf{r}(\mathcal{B}) \le 1$ the following L^∞-estimate holds:

$$\|Du\|^p_{L^\infty(t\mathcal{B})} \le \frac{c}{(1-t)^\beta [\mathbf{r}(\mathcal{B})]^\beta}\left[\|F(\cdot,Du)\|_{L^1(\mathcal{B})} + \|f\|_{X(\mathcal{B})} + 1\right]^\theta, \tag{5.0.12}$$

for all $t \in (0,1)$, $c \equiv c(\texttt{data}_\infty, \|h\|_{L^d(\mathcal{B})})$ (resp. $c \equiv c(\texttt{data}_{\rm two}, \|h\|_{L^d(\mathcal{B})})$ if $n = 2$) and $\beta, \theta \equiv \beta, \theta(n,d,p,q)$.

For the Double Phase functional in (5.0.6), condition (5.0.10) amounts to assume that

$$0 \le a(\cdot) \in W^{1,d}_{\rm loc}(\Omega) \qquad \text{for some} \ \ d > n, \tag{5.0.13}$$

and indeed we have

Theorem 13 *Let $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional in (5.0.6) such that (5.0.13) is in force and (5.1.24) holds together with*

$$\begin{cases} \dfrac{q}{p} \le 1 + \dfrac{1}{n} - \dfrac{1}{d} & \text{if } n \ge 3 \\ \dfrac{q}{p} \le 1 + \dfrac{1}{2} - \dfrac{1}{d} \ \text{ and } \ q < p^2 & \text{if } n = 2. \end{cases} \tag{5.0.14}$$

Then $Du \in L^{\infty}_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$. Furthermore, for all balls $\mathcal{B} \Subset \Omega$ with $\mathbf{r}(\mathcal{B}) \le 1$, the following Lipschitz bound holds:

$$\begin{aligned} &\|Du\|^p_{L^\infty(t\mathcal{B})} + \|H(\cdot,Du)\|_{L^\infty(t\mathcal{B})} \\ &\quad \le \frac{c}{(1-t)^\beta[\mathbf{r}(\mathcal{B})]^\beta}\left[\|H(\cdot,Du)\|_{L^{1+\sigma_g}(\mathcal{B})} + \|f\|_{X(\mathcal{B})} + 1\right]^\theta, \end{aligned} \tag{5.0.15}$$

for all $t \in (0,1)$, where $\beta,\theta \equiv \beta,\theta(n,d,p,q)$, $\sigma_g \equiv \sigma_g(n,N,p,q,\|a\|_{W^{1,d}(\mathcal{B})},\|f\|_{L^n(\mathcal{B})})$ and $c \equiv c(\texttt{data}_\infty,\|\partial_x a\|_{L^d(\mathcal{B})},\|H(\cdot,Du)\|_{L^1(\mathcal{B})},\|f\|_{L^n(\mathcal{B})})$ when $n \ge 3$, or, in two dimensions, $c \equiv c(\texttt{data}_{\rm two},\|\partial_x a\|_{L^d(\mathcal{B})},\|H(\cdot,Du)\|_{L^1(\mathcal{B})},\|f\|_{L^2(\mathcal{B})})$. In (5.0.15), we adopted the shorthand notation $H(x,z) := [|z|^p + a(x)|z|^q]$.

Theorem 13 allows to clarify in which sense assumptions (5.0.10) (i.e. (5.0.13) for the Double Phase energy) and (5.0.14) are sharp. Indeed, notice that by Sobolev-Morrey embedding and (5.0.13), we have that $a \in C^{0,\alpha}_{\rm loc}(\Omega)$ with $\alpha := 1 - n/d$. This last identity makes conditions (5.0.7) and (5.0.13) coincide. Therefore, assumption (5.0.10) is the sharp differentiable version of (5.0.7), which is stronger than (5.0.7), but weaker than assuming that $a(\cdot)$ is Lipschitz, as usually done in several places [186, 194]. Notice that improvement between (5.0.11) and (5.0.14) is due to the special structure considered in Theorem 13. A standard consequence Theorem 12 is about splitting structures as

$$W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N) \ni w \mapsto \int_\Omega \left[c(x)F(Dw) - f\cdot w\right]\,{\rm d}x\,, \qquad 0 < \nu \le c(\cdot) \le L\,. \tag{5.0.16}$$

In this case, with $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ being a local minimizer, assuming that $F(\cdot)$ satisfies (5.1.1) for some $\tilde F(\cdot)$ as in (5.1.2), that it grows as prescribed by (5.0.8)-(5.0.9), and taking $c \in W^{1,d}_{\rm loc}(\Omega)$ with $d > n$ yield that $Du \in L^\infty_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$ provided that (5.1.24)-(5.0.11) hold.

5.0.2 Nonuniform ellipticity at fast rates

We now come to consider functionals with growth conditions that are not of polynomial type. A prototype we have in mind is given by

$$\begin{gathered} W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N) \ni w \mapsto \int_\Omega \left[c_1(x)\exp(c_2(x)|Dw|^{p(x)}) - f\cdot w\right]\,{\rm d}x \\ p_m := \inf_{x\in\Omega} p(x) > 1\,, \quad 0 < \nu \le c_1(\cdot), c_2(\cdot) \le L\,. \end{gathered} \tag{5.0.17}$$

Functionals as the one in (5.0.17) are not easy to deal with, as their integrands do not satisfy the so called Δ_2-condition, i.e. $\tilde F(x,2t) \lesssim \tilde F(x,t)$. Looking at the case of polynomial growth in Section 5.0.1, from (5.0.10) and (5.0.11) we see that the required integrability rate of coefficients d increases with the ratio q/p. A naive bet would then be that in the exponential case a more stringent condition on d is required. As a matter of fact, the situations reverses, and any exponent $d > n$ implies local Lipschitz continuity:

Theorem 14 *Let $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional in (5.0.17) such that $c_1, c_2, p \in W^{1,d}_{\rm loc}(\Omega)$ with $d > n$ and f satisfies (5.1.24). Then $Du \in L^\infty_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$. Moreover, for all balls $\mathcal{B} \Subset \Omega$ with $\mathbf{r}(\mathcal{B}) \le 1$ we have*

$$\begin{aligned}\|Du\|^{p_m}_{L^\infty(t\mathcal{B})} &+ \|c_1(\cdot)\exp(c_2(\cdot)|Du|^{p(\cdot)})\|_{L^\infty(t\mathcal{B})}\\ &\le \frac{c}{(1-t)^\beta[\mathbf{r}(\mathcal{B})]^\beta}\left[\|c_1(\cdot)\exp(c_2(\cdot)|Du|^{p(\cdot)})\|_{L^1(\mathcal{B})} + \|f\|_{X(\mathcal{B})} + 1\right]^\theta,\end{aligned} \tag{5.0.18}$$

for all $t \in (0,1)$, where $c \equiv c(\texttt{data}_\infty, \|c_1\|_{L^d(\mathcal{B})}, \|c_2\|_{L^d(\mathcal{B})}, \|p\|_{L^d(\mathcal{B})})$ and $\beta,\theta \equiv \beta,\theta(n,d,\sigma,\hat\sigma,p_m)$. In the two-dimensional case, it is $c \equiv c(\texttt{data}_{\rm two}, \|c_1\|_{L^d(\mathcal{B})}, \|c_2\|_{L^d(\mathcal{B})}, \|p\|_{L^d(\mathcal{B})})$.

The same applies to more general functionals, involving arbitrary compositions of exponentials. Specifically, we fix sequences of exponent functions $\{p_k(\cdot)\}$ and coefficients$\{c_k(\cdot)\}$, all defined on the open subset $\Omega \subset \mathbb{R}^n$, such that

$$1 < p_m \le p_k(\cdot) \le p_M\,, \quad 0 \le \nu \le c_k(\cdot) \le L\,, \quad p_k, c_k \in W^{1,d}_{\rm loc}(\Omega)\,. \tag{5.0.19}$$

We then inductively define, for every $k \in \mathbb{N}$, the functions $\mathbf{e}_k \colon \Omega \times [0,\infty) \to \mathbb{R}$ as

$$\begin{cases} \mathbf{e}_{k+1}(x,t) & \exp\left[c_{k+1}(x)\left(\mathbf{e}_k(x,t)\right)^{p_{k+1}(x)}\right] \\ \mathbf{e}_0(x,t) \quad := & \exp\left(c_0(x)t^{p_0(x)}\right)\,, \end{cases} \tag{5.0.20}$$

and consider the variational integrals

$$W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N) \ni w \mapsto \int_\Omega \left[\mathbf{e}_k(x,|Dw|) - fw\right]\,dx\,. \tag{5.0.21}$$

For vector-valued minima of (5.0.21) we have the following result.

Theorem 15 *Let $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional in (5.0.21) for some $k \in \mathbb{N}$, under assumptions (5.1.24) and (5.0.19) with $d > n$. Then $Du \in L^\infty_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$. Furthermore, for any ball $\mathcal{B} \Subset \Omega$ with $\mathbf{r}(B) \le 1$ there holds that*

$$\begin{aligned}\|Du\|^{p_m}_{L^\infty(t\mathcal{B})} &+ \|\mathbf{e}_k(\cdot,|Du|)\|_{L^\infty(t\mathcal{B})}\\ &\le \frac{c}{(1-t)^\beta[\mathbf{r}(\mathcal{B})]^\beta}\left[\|\mathbf{e}(\cdot,Du)\|_{L^1(\mathcal{B})} + \|f\|_{X(\mathcal{B})} + 1\right]^\theta,\end{aligned} \tag{5.0.22}$$

for all $t \in (0,1)$, with $c \equiv c(\texttt{data}_\infty, \|c_j\|_{L^d(\mathcal{B})}, \|p_j\|_{L^d(\mathcal{B})})$, ($c \equiv c(\texttt{data}_{\rm two}, \|c_j\|_{L^d(\mathcal{B})}, \|p_j\|_{L^d(\mathcal{B})})$ if $n = 2$) for any $j \in \{1,\cdots,k\}$ and $\beta,\theta \equiv \beta,\theta(n,d,\sigma,\hat\sigma,p_m)$.

To explain the improvement with respect to the polynomial case, we recall that the rate of nonuniform ellipticity of the functional in (5.0.1) can be measured by the ellipticity ratio defined in (1.2.26). Recall that, in case $\mathcal{R}(z,B) \to \infty$ as $|z| \to \infty$, the problem is said to be nonuniformly elliptic, and conditions as in (5.0.11) are devised to bound the rate of blow-up of $\mathcal{R}(z,B)$. Now, although $\mathcal{R}(z,B)$ blows-up faster in the exponential case, what really matters to get regularity is the rate with which the renormalized ratio

$$\overline{\mathcal{R}}(z,B) := \sup_{x\in B}\left[\frac{\text{maximum eigenvalue of } \partial^2_z F(x,z)}{\text{minimum eigenvalue of } \partial^2_z F(x,z)}\cdot\left(\frac{1}{F(x,z)}\right)\right]$$

goes to zero as $|z| \to \infty$. Now, notice that $\overline{\mathcal{R}}(z,B) \to 0$ polynomially fast in the case of functionals satisfying (5.0.8). On the other hand, with functionals in (5.0.21), we see that $\overline{\mathcal{R}}(z,B) \to 0$ exponentially fast. Therefore less regularity of $F(\cdot)$ is required with respect to the space variable.

5.0.3 New results in the uniformly elliptic setting

New results also follow in the uniformly elliptic setting. In particular, when considering the classical problem (5.0.4), we get a new criterion on coefficients $c(\cdot)$ ensuring the local Lipschitz continuity of minima (and solutions) that goes beyond the one in (5.0.5). This time the model is

$$W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N) \ni w \mapsto \int_\Omega \left[A(x,|Dw|) - f\cdot w\right] \, dx \qquad (5.0.23)$$
$$A(x,t) := c(x)\int_0^t \tilde{a}(s)s \, ds \quad \text{for } t>0,$$

with (1.2.20) being in force and such that $0<\nu\le c(\cdot)\le L$ and $c\in W^{1,1}_{\rm loc}(\Omega)$. Under such conditions, every solution to the system in (5.0.3) is a local minimizer of the functional in (5.0.23) and the second identity in (5.0.3) is automatically satisfied.

Theorem 16 *Let $u\in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional in (5.0.23) under assumptions (1.2.20). If $|f|,|Dc|\in X_{\rm loc}(\Omega)$ as defined in (5.1.24), then $Du\in L^\infty_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$. Moreover, for all balls $\mathcal{B}\Subset\Omega$ so that $\mathbf{r}(B)\le 1$ there holds that*

$$\|A(|Du|)\|_{L^\infty(t\mathcal{B})} \le \frac{c}{(1-t)^n[\mathbf{r}(\mathcal{B})]^n}\left[\|F(|Du|)\|_{L^1(\mathcal{B})}+1\right] + c\|f\|_{X(\mathcal{B})}^{i_a+2/(i_a+1)}, \qquad (5.0.24)$$

for all $t\in(0,1)$ with $c\equiv c(\mathtt{data}_{\rm uni})$.

In other words, f and Dc this time enjoy the same degree of regularity. Theorem 16 applies to (5.0.4) by taking $\tilde{a}(x,t)\equiv c(x)t^{p-2}$ and it is sufficient to require that $Dc\in L_{\rm loc}(n,1)(\Omega,\mathbb{R}^n)$. Notice that this is a new criterion, which is alternative to the known and classical one in (5.0.5). Indeed, $Dc\in L_{\rm loc}(n,1)(\Omega,\mathbb{R}^n)$ implies that $c(\cdot)$ is continuous by the result in [235], but not necessarily with a modulus of continuity $\omega(\cdot)$ satisfying (5.0.5). Moreover, this criterion works for the general cases as in (5.0.23), to which the methods in [169] do not apply under the only considered structure assumption (1.2.20).

5.0.4 Additional results and remarks

When $n\ge 3$, in Theorems 12-16, we can replace assumption (5.1.24) with the weaker one $f\in L^n(\Omega)$, getting, as a corresponding outcome, that $Du\in L^\gamma_{\rm loc}(\Omega)$ for every $\gamma<\infty$. This result is new in the nonuniformly elliptic case and is in perfect accordance with the nonlinear theory known for the uniformly elliptic one. Indeed, let us for instance consider the system in (5.0.4). From $Dc\in L^n$ it follows that $c(\cdot)\in$ VMO, the space of functions with vanishing mean oscillations [228]. At this point the arbitrary higher integrability of Du, with every exponent, is a consequence of the standard Nonlinear Calderón-Zygmund theory (see for instance [34]). For the higher integrability results under the weaker integrability assumption (5.0.2) see Theorem 19 in Section 5.1.6 below.

We remark that some of the methods here also extend to the general scalar functionals, i.e., when minima and competitors are real valued functions. In this case there is no need to assume the radial structure condition $F(x,Dw)\equiv\tilde{F}(x,|Dw|)$. On the other hand, additional conditions ensuring the absence of the so called Lavrentiev phenomenon are needed to build a suitable approximation argument, for this we refer to the approach explained in [77, 101]. We anyway remark that in the vectorial case it is necessary to assume additional conditions such as the radial structure, otherwise singular minimizers occur under the most favourable conditions, see for instance [194, 199, 244, 245] and related references.

Finally, notice that obstacle problems represent a concrete application of our results. In fact, by now standard procedures allow rearranging problems of type

$$\mathcal{K}_f(\Omega)\ni w\mapsto \min\int_\Omega F(x,Dw)\, dx \quad \text{with} \quad \mathcal{K}_f(\Omega):=\left\{w\in W^{1,1}_{\rm loc}(\Omega)\colon w(x)\ge f(x) \text{ in } \Omega\right\}$$

in form (5.0.1), cf. [109], therefore, with minor variations to our techniques, we can prove results analogous to those exposed in Theorems 12-16 for the obstacle problem defined by means of energies of type (5.0.6), (5.0.8), (5.0.17), (5.0.21) or (5.0.23), provided that the obstacle $f\colon\Omega\to\mathbb{R}$ satisfies

$$f\in W_{\rm loc}(2;n,1)(\Omega)\ \text{ if }\ n\geq 3\quad\text{and}\quad f\in W_{\rm loc}(2;\mathrm{L}^2(\mathrm{LogL})^{\alpha})(\Omega),\ \text{with } \alpha>2\ \text{ if }\ n=2.$$

This means that we can not only extend under sharp assumptions on the regularity of the obstacle [55], the results obtained in [47, 51, 70, 112, 158], but also introduce for the first time in the framework of obstacle problems non-autonomous models with fast exponential growth such as (5.0.17) or (5.0.21), we refer to [16] for the autonomous case.

5.1 Assumptions and general results

In this section we are going to describe a number of conditions aimed at proving our main results, that is Theorems 19-20 in Section 5.1.6 below; in turn, these will imply Theorems 12-16 from the Introduction. The assumptions might appear technical at a first sight and the whole list might look long; indeed it is. On the other hand, when applied to each single model case, such conditions reveal to be minimal and produce sharp results.

5.1.1 Basic structural assumptions, and consequences

We assume that the integrand $F(\cdot)$ has radial structure, i.e.

$$F(x,z)=\tilde F(x,|z|)\quad\text{for all } (x,z)\in\Omega\times\mathbb{R}^{N\times n}\,. \tag{5.1.1}$$

In order to describe the ellipticity and growth assumptions of the integrand $F(\cdot)$, we shall use three continuous functions $g_i\colon\Omega\times(0,\infty)\to[0,\infty)$, for $i\in\{1,2,3\}$, whose precise properties will be described later on. Indeed, $\tilde F\colon\Omega\times[0,\infty)\to[0,\infty)$ is assumed to satisfy

$$\begin{cases} t\mapsto \tilde F(\cdot,t)\in C^1_{\rm loc}[0,\infty)\cap C^2_{\rm loc}(0,\infty)\\ x\mapsto \tilde F'(x,t)\in W^{1,n}_{\rm loc}(\Omega)\ \text{ for every } t>0\\ |\partial_x\tilde F'(x,t)|\leq h(x)g_3(x,t)\quad\text{for all } x\in\Omega \text{ and } t>0\,, \end{cases} \tag{5.1.2}$$

where $0\leq h(\cdot)\in L^n_{\rm loc}(\Omega)$. Notice that here, as in the rest of the paper, we are using the notation $\partial_t\tilde F(x,t)\equiv\tilde F'(x,t)$. Moreover, we assume that there exists $T>0$ such that

$$\begin{cases} z\mapsto F(\cdot,z) & \text{is convex}\\ |\partial_{zz}F(x,z)|\leq g_2(x,|z|) & \text{for all } x\in\Omega \text{ on } \{|z|\geq T\}\\ g_1(x,|z|)|\xi|^2\leq\partial_{zz}F(x,z)\xi\cdot\xi & \text{on } \{|z|\geq T\} \text{ and for all } x\in\Omega \text{ and } \xi\in\mathbb{R}^{N\times n}\,. \end{cases} \tag{5.1.3}$$

Needless to say, that all the maps and functions considered in (5.1.2) and (5.1.3) are Carathéodory regular. Using the notation (see also (5.0.3))

$$\tilde a(x,t):=\frac{\tilde F'(x,t)}{t}\quad\text{for all } (x,t)\in\Omega\times(0,\infty)\,,$$

we shall assume, for fixed numbers $\gamma>1$, $\mu\in[0,1]$ $\nu\in(0,1]$, and for every $x\in\Omega$, that

$$\begin{cases} t\mapsto\dfrac{\tilde a(x,t)}{(t^2+\mu^2)^{\frac{\gamma-2}{2}}}\quad\text{and}\quad t\mapsto\dfrac{g_1(x,t)}{(t^2+\mu^2)^{\frac{\gamma-2}{2}}}\quad\text{are non-decreasing on } (0,\infty)\\ \qquad\qquad\nu(t^2+\mu^2)^{\frac{\gamma-2}{2}}\leq g_1(x,t)\quad\text{for } t\geq T\,. \end{cases} \tag{5.1.4}$$

Remark 5.1.1 In most of the relevant model examples it will be $\tilde{a}(\cdot) \equiv g_1(\cdot)$, this justifies the double assumption in (5.1.4). The hypotheses in (5.1.3) are bound to describe ellipticity of $\partial_{zz}F(x,z)$ outside the ball $\{|z| > T\}$ and this allows to cover functionals loosing their ellipticity properties on a bounded set. This assumption is natural when dealing with nonuniformly elliptic problems. Indeed, in such cases certain inequalities valid when $|z|$ is large reverse when $|z|$ is small. For instance, this is the case for (5.0.6): $|z|^p \lesssim |z|^q$ for $|z|$ large, while the opposite holds when $|z|$ is small. Notice that we could have done the same also with respect to the partial derivative $\partial_{xz}F$ in $(5.1.2)_3$, but we preferred not to follow this path as this would have only added useless technical difficulties.

Let us now draw a few consequences of (5.1.1)-(5.1.4). Notice that $(5.1.2)_3$ implies that $x \mapsto \partial_z F(x,z) \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^{N\times n})$ for all $z \in \mathbb{R}^{N\times n}$ such that $z \neq 0$, with

$$|\partial_{xz}F(x,z)|, |\partial_x \tilde{a}(x,|z|)||z| \leq h(x)g_3(x,|z|) \quad \text{on } \{|z|>0\} \text{ and for all } x \in \Omega\,. \tag{5.1.5}$$

Again from the very definition of $\tilde{a}(\cdot)$, we have

$$\tilde{F}(x,t) = \int_0^t \tilde{a}(x,s)s\,\mathrm{d}s \quad \text{for all } (x,t) \in \Omega \times (0,\infty) \tag{5.1.6}$$

and

$$\partial_{zz}F(x,z) = \partial_z[\tilde{a}(x,|z|)z] = \tilde{a}(x,|z|)\mathbb{I}_{N\times n} + \tilde{a}'(x,|z|)|z|\frac{z\otimes z}{|z|} \quad \text{for } |z| \neq 0 \tag{5.1.7}$$

so that, using (5.1.3) with $\xi = z$ and $\xi \perp z$, we obtain

$$\begin{cases} \tilde{a}(x,|z|) + \tilde{a}'(x,|z|)|z| \geq g_1(x,|z|) \\ \tilde{a}(x,|z|) \geq g_1(x,|z|) \end{cases} \quad \text{for any } |z| \geq T \text{ and all } x \in \Omega\,, \tag{5.1.8}$$

and

$$\begin{cases} \tilde{a}(x,|z|) + \tilde{a}'(x,|z|)|z| \leq g_2(x,|z|) \\ \tilde{a}(x,|z|) \leq g_2(x,|z|) \end{cases} \quad \text{for any } |z| \geq T \text{ and all } x \in \Omega \tag{5.1.9}$$

respectively. In particular, it follows that $g_1(\cdot,t) \leq g_2(\cdot,t)$ for $t \geq T$, and by $(5.1.4)_2$ and $(5.1.8)_2$

$$\tilde{a}(x,t) \geq \nu(t^2+\mu^2)^{\frac{\gamma-2}{2}} \quad \text{holds for all } (x,t) \in \Omega \times [T,\infty)\,. \tag{5.1.10}$$

Now we introduce two functions that will play a crucial role in the rest of the paper, i.e.,

$$G(x,t) := \int_T^{\max\{t,T\}} g_1(x,s)s\,\mathrm{d}s \quad \text{and} \quad \bar{G}(x,t) := G(x,t) + (T^2+1)^{\gamma/2}\,. \tag{5.1.11}$$

Remark 5.1.2 Before going on, we notice that all the estimates considered in this paper essentially depend on the behavior of the derivatives of $F(\cdot)$ with respect to the gradient variable. Therefore, upon replacing $\tilde{F}(x,t)$ with $[\tilde{F}(x,t) - \tilde{F}(x,0)]$ - a replacement that does not change the set of local minimizers - in light of (5.1.2)-(5.1.3), we can always assume that

$$\tilde{F}(x,0) = 0 \qquad \text{holds for every } x \in \Omega\,. \tag{5.1.12}$$

5.1.2 Quantification of nonuniform ellipticity, $n > 2$

We describe the basic properties of the functions $g_i(\cdot)$. These quantify ellipticity of $F(\cdot)$ via (5.1.2); here we consider the case $n > 2$. We assume that $g_1^-(T) := \inf_{x\in\Omega} g_1(x,T) > 0$ and that

$$T \le s < t \Longrightarrow \frac{g_2(x,s)}{g_1(x,s)} \le c_a \frac{g_2(x,t)}{g_1(x,t)} \quad \text{and} \quad g_1(x,s)s \le g_1(x,t)t \tag{5.1.13}$$

for all $x \in \Omega$ and some $c_a \ge 1$. This implies, in particular, that $0 < g_1^-(T)T/t \le g_1(x,t) \le g_2(x,t)$ whenever $(x,t) \in \Omega \times [T,\infty)$. In particular, $g_1(\cdot,t)$ and $g_2(\cdot,t)$ never vanish for $t \ge T$. Next, we consider an integrability d and dual exponents $\sigma, \hat\sigma \ge 0$ such that

$$h(\cdot) \in L^d_{\rm loc}(\Omega)\,, \qquad n < d\,, \qquad \sigma + \hat\sigma < \frac{1}{n} - \frac{1}{d}\,, \tag{5.1.14}$$

with the additional condition that $\hat\sigma = 0$ if $x \mapsto g_1(x,t) \equiv \text{const}$ for all $t \ge T$. In the case it is $\hat\sigma > 0$, we assume that $x \mapsto g_1(x,t) \in W^{1,d}_{\rm loc}(\Omega)$ for all $t > T$, and

$$|\partial_x g_1(x,t)| \le h(x)[\bar G(x,t)]^{\hat\sigma} g_1(x,t) \qquad \text{for } (x,t) \in \Omega \times [T,\infty)\,. \tag{5.1.15}$$

Needless to say, $\partial_x g_1(\cdot)$ is also assumed to be Carathéodory regular on $\Omega \times [T,\infty)$. We come to the structural assumptions codifying nonuniform ellipticity and its interaction with the presence of x. For all $(x,t) \in \Omega \times [T,\infty)$ we assume that

$$\begin{cases} g_3(x,t)\sqrt{t^2+\mu^2} \le c_b[\bar G(x,t)]^{1+\sigma} \\[2mm] \dfrac{[g_3(x,t)]^2}{g_1(x,t)} \le c_b[\bar G(x,t)]^{1+2\sigma} \end{cases} \tag{5.1.16}$$

and introduce another parameter

$$0 \le \sigma' < \frac{4(\gamma-1)}{\gamma\vartheta(n-2)} \qquad \text{for } \vartheta := \begin{cases} 1 & \text{when } \gamma \ge 2 \\ 2 & \text{when } 1 < \gamma < 2, \end{cases} \tag{5.1.17}$$

so that the rebalancing condition, serving to bound the ratio $g_2(\cdot)/g_1(\cdot)$,

$$\frac{g_2(x,t)}{g_1(x,t)} \le c_b \min\left\{\bar G_T(x,t)^{\sigma}, \bar G_T(x,t)^{\sigma'}\right\} \tag{5.1.18}$$

holds for any $(x,t) \in \Omega \times [T,\infty)$. Here, $c_b \ge 1$ is a fixed constant. The use of parameters σ, $\hat\sigma$ and σ' is a novelty here, and allows to treat, with larger flexibility, different type of non-autonomous problems and identify certain borderline behaviours. For more on this aspect, see Sections 5.0.1-5.0.2.

5.1.3 Quantification of nonuniform ellipticity, $n = 2$

When $n = 2$ we essentially keep the assumptions for the case $n > 2$ but with a slightly different bound on σ. Moreover, σ' will not appear, whereas the values of $\sigma, \hat\sigma \ge 0$ will be slightly modified. Specifically, this time we take

$$h(\cdot) \in L^d(\Omega)\,, \qquad 2 < d\,, \qquad \sigma + \hat\sigma < \min\left\{\frac{1}{2} - \frac{1}{d}, 1 - \frac{1}{\gamma}\right\} \tag{5.1.19}$$

and we assume $\hat\sigma = 0$ if $x \mapsto g_1(x,t) \equiv \text{const}$ for all $t \ge T$. Next, instead of (5.1.18), we assume that

$$\frac{g_2(x,t)}{g_1(x,t)} \le c_b \bar G_T(x,t)^{\sigma} \tag{5.1.20}$$

holds for all $(x,t) \in \Omega \times [T,\infty)$.

5.1.4 The uniformly elliptic case

Here we describe the relevant assumptions to treat the uniformly elliptic case, and therefore models as those appearing in Section 5.0.3. In this case we shall retain the structure assumptions in Section 5.1.1 and this time consider

$$\begin{cases} g_i(x,t) \equiv g_i(t), \quad i \in \{1,2,3\} \\ g_1(t) \le g_2(t) \le K g_1(t) \\ g_1(t)\sqrt{t^2+\mu^2} \le g_3(t) \le K g_1(t)\sqrt{t^2+\mu^2} \end{cases} \tag{5.1.21}$$

for all $t \in [T, \infty)$, where $K \ge 1$.

5.1.5 Summary of assumptions and dependence on the constants

Here we summarize the basic structural assumptions on the integrand $F(\cdot)$ that, beside those considered on f, shall imply various degrees of gradient regularity for local minimizers. These are

$$\begin{cases} \mathtt{set}_{\mathrm{m}} := \big\{(5.1.1)\text{-}(5.1.5), (5.1.12)\text{-}(5.1.18)_1\big\} \\ \mathtt{set}_{\infty} := \big\{(5.1.1)\text{-}(5.1.5), (5.1.12)\text{-}(5.1.18)\big\} \\ \mathtt{set}_{\mathrm{two}} := \big\{(5.1.1)\text{-}(5.1.5), (5.1.12)\text{-}(1.1.11), (5.1.19), (5.1.20)\big\} \\ \mathtt{set}_{\mathrm{uni}} := \big\{(5.1.1)\text{-}(5.1.12), (5.1.21)\big\}\ . \end{cases} \tag{5.1.22}$$

Specifically, we denote by $\mathtt{set}_{\mathrm{m}}$ the assumptions used to get results of the type $Du \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ for every $p < \infty$ and $n \ge 2$. The assumptions $\mathtt{set}_{\infty}$ and $\mathtt{set}_{\mathrm{two}}$ are used to prove that $Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ when $n > 2$ and $n = 2$, respectively. Notice that in $\mathtt{set}_{\mathrm{m}}$ we only retain the first condition in (5.1.18), the one involving σ as in (5.1.14); we shall use the whole assumption (5.1.18) only when showing L^∞-estimates for Du. Finally, the set $\mathtt{set}_{\mathrm{uni}}$ is used to get gradient bounds when dealing with uniformly elliptic problems. In order to abbreviate the dependence on the constants, we shall denote

$$\begin{cases} \mathtt{data}_m := (n, N, \nu, \gamma, \tilde{a}_+, T, c_a, c_b, d, \sigma, \hat{\sigma})\ , \\ \mathtt{data}_\infty := \big(n, N, \nu, \gamma, \tilde{a}_+, T, c_a, c_b, d, \sigma, \hat{\sigma}, \sigma'\big) \\ \mathtt{data}_{\mathrm{two}} := (N, \nu, \gamma, \tilde{a}_+, T, c_a, c_b, d, \sigma, \hat{\sigma}, \alpha) \\ \mathtt{data}_{\mathrm{uni}} := (n, N, T, K)\,, \end{cases} \tag{5.1.23}$$

where $\tilde{a}_+ := \sup_{x\in\Omega} \tilde{a}(x, \max\{1,T\}) \max\{1,T\}$. It is important to notice that in the rest of the paper, all the constants depending on the set of parameters in (5.1.23) blow-up when $T \to \infty$, while remaining bounded when $T \to 0$.

5.1.6 General results

Here we report our main results for minima of (5.0.1) in full generality. However, the minimal regularity assumption (5.0.2) on f is not enough for proving Lipschitz regularity, therefore in certain cases we shall strengthen it by requiring that

$$f \in X_{\mathrm{loc}}(\Omega, \mathbb{R}^N) := \begin{cases} L_{\mathrm{loc}}(n,1)(\Omega, \mathbb{R}^N) & \text{if } n \ge 3 \\ L^2_{\mathrm{loc}}(\mathrm{LogL})^\alpha(\Omega, \mathbb{R}^N)\,,\ \alpha > 2 & \text{if } n = 2. \end{cases} \tag{5.1.24}$$

The following two theorems are our main results on gradient boundedness for minima of (5.0.1) in full generality.

Theorem 17 *Let $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of (5.0.1) under assumptions $\texttt{set}_\infty$ and (5.1.24) with $n \geq 3$. Then $Du \in L^\infty_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$. Moreover there exists a positive radius $R_* \equiv R_*(\texttt{data}_\infty, f(\cdot)) \leq 1$ such that if $\mathcal{B} \Subset \Omega$ is a ball such that $\boldsymbol{r}(\mathcal{B}) \leq R_*$, then*

$$
\begin{aligned}
&\|Du\|^\gamma_{L^\infty(t\mathcal{B})} + \|G_T(\cdot,|Du|)\|_{L^\infty(t\mathcal{B})} \\
&\qquad\qquad \leq \frac{c}{(1-t)^\beta[\boldsymbol{r}(\mathcal{B})]^\beta}\left[\|F(\cdot,Du)\|_{L^1(\mathcal{B})} + \|f\|_{L(n,1)(\mathcal{B})} + 1\right]^\theta
\end{aligned}
\tag{5.1.25}
$$

holds for every $t \in (0,1)$, where $c \equiv c(\texttt{data}_\infty, \|h\|_{L^d(\mathcal{B})}) \geq 1$, $\beta,\theta \equiv \beta,\theta(n,d,\sigma,\hat{\sigma}) > 0$.

Theorem 18 *Let $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of (5.0.1) under assumptions $\texttt{set}_{\rm two}$ and (5.1.24) with $n = 2$. Then $Du \in L^\infty_{\rm loc}(\Omega,\mathbb{R}^{N\times 2})$. Moreover, if $\mathcal{B} \Subset \Omega$ is a ball such that $\boldsymbol{r}(\mathcal{B}) \leq 1$, then*

$$
\begin{aligned}
&\|Du\|^\gamma_{L^\infty(t\mathcal{B})} + \|G_T(\cdot,|Du|)\|_{L^\infty(t\mathcal{B})} \\
&\qquad\qquad \leq \frac{c}{(1-t)^\beta[\boldsymbol{r}(\mathcal{B})]^\beta}\left[\|F(\cdot,Du)\|_{L^1(\mathcal{B})} + \|f\|_{L^2(\mathrm{LogL})^\alpha(\mathcal{B})} + 1\right]^\theta ,
\end{aligned}
\tag{5.1.26}
$$

holds for every $t \in (0,1)$, where $c \equiv c(\texttt{data}_{\rm two}, \|h\|_{L^d(\mathcal{B})}) \geq 1$, $\beta,\theta \equiv \beta,\theta,(n,d,\sigma,\hat{\sigma}) > 0$.

In deriving Theorem 17 we need to prove higher integrability bounds for the gradient, that are worth being singled out in the following:

Theorem 19 *Let $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of (5.0.1) and assume that $\texttt{set}_{\rm m}$ and (5.0.2) hold with $n \geq 3$. Then $Du \in L^p_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$ for every $p \in [1,\infty)$. Moreover, for every $p \in [1,\infty)$ there exists a positive radius $R_* \equiv R_*(\texttt{data}_m, f(\cdot), p) \leq 1$ such that if $\mathcal{B} \Subset \Omega$ is a ball such that $\boldsymbol{r}(\mathcal{B}) \leq R_*$, then*

$$
\|G(\cdot,|Du|)\|_{L^p(t\mathcal{B})} \leq \frac{c}{(1-t)^{\beta_p}[\boldsymbol{r}(\mathcal{B})]^{\beta_p}}\left[\|F(\cdot,Du)\|_{L^1(\mathcal{B})} + 1\right]^{\theta_p}
\tag{5.1.27}
$$

holds for every $t \in (0,1)$, where $c \equiv c(\texttt{data}_m, \|h\|_{L^d(\mathcal{B})}, p)$, $\theta_p,\beta_p \equiv \theta_p,\beta_p(n,d,\sigma,\hat{\sigma},p)$.

Finally, we give the main Lipschitz regularity result in the uniformly elliptic case.

Theorem 20 *Let $u \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of (5.0.1) and assume that $\texttt{set}_{\rm uni}$ holds for $n \geq 2$ and that $h, |f| \in X_{\rm loc}(\Omega)$. Then, $u \in W^{1,\infty}_{\rm loc}(\Omega,\mathbb{R}^N)$. Moreover, there exists a positive radius $R_* \equiv R_*(\texttt{data}_{\rm uni}, h(\cdot)) \leq 1$ such that if $\mathcal{B} \Subset \Omega$ is a ball such that $\boldsymbol{r}(\mathcal{B}) \leq R_*$, then*

$$
\|F(|Du|)\|_{L^\infty(t\mathcal{B})} \leq \frac{c}{(1-t)^n[\boldsymbol{r}(\mathcal{B})]^n}\left[\|F(|Du|)\|_{L^1(\mathcal{B})} + 1\right] + \|f\|^{\gamma/(\gamma-1)}_{X(\mathcal{B})}
\tag{5.1.28}
$$

holds with $c \equiv c(\texttt{data}_{uni})$.

5.2 Approximation

Here we implement a truncation scheme aimed at approximating the original integrand $(x,z) \mapsto F(x,z)$ with a family $\{F_\varepsilon(\cdot)\}$ of integrands with standard polynomial growth, and suitably converging to $F(\cdot)$. The new integrands preserve structure properties as (5.1.1)-(5.1.3), with corresponding control functions $g_{i,\varepsilon}(\cdot)$, still satisfying relations as for instance in (5.1.16)-(5.1.18). For this we shall revisit and extend a few arguments used in [16]. In this section we shall permanently assume that (5.1.1)-(5.1.4) and (5.1.13) are in force, with their consequences. Therefore, all in all, we shall permanently work using also (5.1.5)-(5.1.12). Additional assumptions as (1.1.10)-(5.1.18) and (5.1.20) shall also be considered; in such cases all the results will work indifferently with the values of $\sigma, \hat{\sigma}, \sigma'$ considered in (5.1.14), (5.1.17) or (5.1.19); we shall therefore omit to specify which value of such exponents is precisely occurring. In the following we use a parameter ε such that $0 < \varepsilon < \min\{1,T\}/4$.

5.2.1 General setup

We shall use the numbers

$$\mu_\varepsilon := \mu + \varepsilon\,, \qquad T_\varepsilon := T + 1/\varepsilon\,, \tag{5.2.1}$$

and set

$$E(t) := \sqrt{t^2+\mu^2}\,, \qquad E_\varepsilon(t) := \sqrt{t^2+(\mu+\varepsilon)^2}\,, \tag{5.2.2}$$

for $t \geq 0$, where T and μ have been introduced in (5.1.3) and (5.1.4) respectively. We start considering the functions $\tilde{a}_\varepsilon\colon \Omega \times [0,\infty) \to [0,\infty)$ defined as

$$\tilde{a}_\varepsilon(x,t) := \begin{cases} \frac{\tilde{a}(x,\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } t \in [0,\varepsilon) \\ \tilde{a}(x,t) & \text{if } t \in [\varepsilon, T_\varepsilon) \\ \frac{\tilde{a}(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } t \in [T_\varepsilon, \infty)\,, \end{cases} \tag{5.2.3}$$

for every $x \in \Omega$. Then, for $t \geq 0$ we define

$$\begin{cases} F_\varepsilon(x,z) := \tilde{F}_\varepsilon(x,|z|) \quad \text{for} \quad \tilde{F}_\varepsilon(x,t) := \displaystyle\int_0^t \tilde{a}_\varepsilon(x,s)s\,\mathrm{d}s + \varepsilon L_{\gamma,\varepsilon}(t) \\ L_{\gamma,\varepsilon}(t) := \dfrac{1}{\gamma}\left[(t^2+\mu_\varepsilon^2)^{\gamma/2} - \mu_\varepsilon^\gamma\right] = \displaystyle\int_0^t (s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s\,\mathrm{d}s\,, \end{cases} \tag{5.2.4}$$

so that, in view of (5.1.2) and (5.1.5), it follows

$$\begin{cases} t \mapsto \tilde{F}_\varepsilon(x,t) \in C^1_{\text{loc}}[0,\infty) \cap W^{2,\infty}_{\text{loc}}[0,\infty) \cap C^2_{\text{loc}}\big([0,\infty)\setminus\{\varepsilon, T_\varepsilon\}\big) \quad \text{for all} \quad x \in \Omega \\ x \mapsto \tilde{F}'_\varepsilon(x,t) \in W^{1,n}_{\text{loc}}(\Omega) \quad \text{for all} \quad t \in [0,\infty) \\ t \mapsto \tilde{a}_\varepsilon(x,t) \in W^{1,\infty}_{\text{loc}}[0,\infty) \cap C^1_{\text{loc}}\big([0,\infty)\setminus\{\varepsilon,T_\varepsilon\}\big) \\ t \mapsto \tilde{F}_\varepsilon(x,t) \quad \text{is strictly convex for all } x \in \Omega \\ t \mapsto \tilde{F}_\varepsilon(x,t) \quad \text{is non-decreasing for all } x \in \Omega\,. \end{cases} \tag{5.2.5}$$

The above definitions lead to the introduction of the following truncated control functions $g_{i,\varepsilon}\colon \Omega\times[0,\infty) \to (0,\infty)$, $i \in \{1,2,3\}$:

$$g_{1,\varepsilon}(x,t) := \mathfrak{g}_1 \begin{cases} \frac{g_1(x,\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } t \in [0,\varepsilon) \\ g_1(x,t) & \text{if } t \in [\varepsilon,T_\varepsilon) \\ \frac{g_1(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } t \in [T_\varepsilon,\infty)\,, \end{cases} \tag{5.2.6}$$

$$g_{2,\varepsilon}(x,t) := \mathfrak{g}_2 \begin{cases} \left[\frac{g_2(x,\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} + \varepsilon\right](t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } t \in [0,\varepsilon) \\ g_2(x,t) + \varepsilon(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } t \in [\varepsilon,T_\varepsilon) \\ \left[\frac{g_2(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} + \varepsilon\right](t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } t \in [T_\varepsilon,\infty)\,, \end{cases} \tag{5.2.7}$$

$$g_{3,\varepsilon}(x,t) := \mathfrak{g}_3 \begin{cases} \frac{g_3(x,\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-1}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-1}{2}} & \text{if } t \in [0,\varepsilon) \\ g_3(x,t) & \text{if } t \in [\varepsilon,T_\varepsilon) \\ \frac{g_3(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-1}{2}}}(t^2+\mu_\varepsilon^2)^{\frac{\gamma-1}{2}} & \text{if } t \in [T_\varepsilon,\infty)\,, \end{cases} \tag{5.2.8}$$

where the constants $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$ are defined by

$$\mathfrak{g}_1 := \min\{1,\gamma-1\} \leq 1 \leq \mathfrak{g}_2 := 4\left(\sqrt{Nn}+\gamma\right) \quad \text{and} \quad \mathfrak{g}_3 := 2n. \tag{5.2.9}$$

We next introduce also the truncated counterparts of the maps defined in (5.1.11), i.e.

$$G_\varepsilon(x,t) := \int_T^{\max\{t,T\}} g_{1,\varepsilon}(x,s)s\,\mathrm{d}s\ , \quad \bar{G}_\varepsilon(x,t) := G_\varepsilon(x,t) + (T^2+1)^{\gamma/2}\ . \tag{5.2.10}$$

In the following we shall use repeatedly that

$$1 \leq \left(\frac{s^2+\mu_\varepsilon^2}{s^2+\mu^2}\right) \leq 4 \quad \text{provided that } s \geq T\ . \tag{5.2.11}$$

5.2.2 Five technical lemmas

We start with an analysis of the growth and coercivity properties of the integrands $\{F_\varepsilon(\cdot)\}$.

Lemma 5.2.1 *There exist constants $\{\Lambda_\varepsilon\}$ and $\{L_\varepsilon\}$ such that the following properties hold:*

$$\begin{cases} |\partial_{zz}F_\varepsilon(x,z)| \leq g_{2,\varepsilon}(x,|z|) & \text{if } (x,z) \in \Omega \times \left\{|z| \geq T\,, |z| \neq T_\varepsilon\right\} \\ g_{1,\varepsilon}(x,|z|)|\xi|^2 \leq \partial_{zz}F_\varepsilon(x,z)\xi\cdot\xi & \text{if } (x,z) \in \Omega \times \left\{|z| \geq T\,, |z| \neq T_\varepsilon\right\} \\ |\partial_{xz}F_\varepsilon(x,z)| \leq h(x)g_{3,\varepsilon}(x,|z|) & \text{if } (x,z) \in \Omega \times \mathbb{R}^{N\times n} \end{cases} \tag{5.2.12}$$

and

$$\begin{cases} |\partial_{zz}F_\varepsilon(x,z)| \leq \Lambda_\varepsilon(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} & \text{if } (x,z) \in \Omega \times \left\{|z| \neq \varepsilon, T_\varepsilon\right\} \\ \varepsilon\mathfrak{g}_1(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}|\xi|^2 \leq \partial_{zz}F_\varepsilon(x,z)\xi\cdot\xi & \text{if } (x,z) \in \Omega \times \left\{|z| \neq \varepsilon, T_\varepsilon\right\} \\ |\partial_{xz}F_\varepsilon(x,z)| \leq L_\varepsilon h(x)(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-1}{2}} & \text{if } (x,z) \in \Omega \times \mathbb{R}^{N\times n}\ , \end{cases} \tag{5.2.13}$$

for all $\xi \in \mathbb{R}^{N\times n}$. Moreover we have:

$$0 < s \leq t \quad \Longrightarrow \quad \begin{array}{l} \dfrac{\tilde{a}_\varepsilon(x,s)}{(s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \leq c(\gamma)\,\dfrac{\tilde{a}_\varepsilon(x,t)}{(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \\ \frac{g_{1,\varepsilon}(x,s)}{(s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \leq c(\gamma)\,\frac{g_{1,\varepsilon}(x,t)}{(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \end{array} \tag{5.2.14}$$

and for every $(x,z) \in \Omega\times\{|z| \geq T\}$, there holds that:

$$c(\nu,\gamma)(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \leq g_{1,\varepsilon}(x,t), \qquad F(x,z) \geq G(x,|z|), \qquad F_\varepsilon(x,z) \geq G_\varepsilon(x,|z|)\ , \tag{5.2.15}$$

$$\begin{cases} c(\nu,\gamma)F(x,z) \geq (|z|^2+\mu^2)^{\gamma/2} - (T^2+\mu^2)^{\gamma/2} \\ c(\nu,\gamma)F_\varepsilon(x,z) \geq (|z|^2+\mu^2)^{\gamma/2} - (T^2+\mu^2)^{\gamma/2}\ , \end{cases} \tag{5.2.16}$$

$$[E_\varepsilon(t)]^\gamma - [E_\varepsilon(T)]^\gamma \leq c(\nu,\gamma)\int_T^t g_{1,\varepsilon}(x,s)s\,\mathrm{d}s \quad \text{for } (x,t) \in \Omega\times[T,\infty)\ , \tag{5.2.17}$$

where $c(\nu,\gamma) \geq 1$. Finally, for another constant $c \equiv c(\nu,\tilde{a}_+)$, the following holds:

$$F_\varepsilon(x,z) \leq c\left[F(x,z) + T^\gamma + \mu_\varepsilon^\gamma\right] \qquad \text{for all } (x,z) \in \Omega\times\mathbb{R}^{N\times n} \tag{5.2.18}$$

and

$$\frac{\varepsilon}{\gamma}(t^2+\mu_\varepsilon^2)^{\gamma/2} - \frac{\varepsilon\mu_\varepsilon^\gamma}{\gamma} \leq \tilde{F}_\varepsilon(x,t) \leq c_\varepsilon(t^2+\mu_\varepsilon^2)^{\gamma/2} \quad \text{for all } (x,t) \in \Omega\times[0,\infty)\ . \tag{5.2.19}$$

Proof. By (5.2.4)-(5.2.5), we first notice that $\partial_{zz}F_\varepsilon(x,z)$ exists for all $(x,z) \in \Omega \times \{|z| \neq \varepsilon, T_\varepsilon\}$ with

$$\partial_{zz}F_\varepsilon(x,z) = \begin{cases} (|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \left[\frac{\tilde{a}(x,\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} + \varepsilon\right] \mathcal{C}_\varepsilon(z) & \text{if } (x,z) \in \Omega \times \{|z| < \varepsilon\} \\ \partial_{zz}F(x,z) + \varepsilon(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}\mathcal{C}_\varepsilon(z) & \text{if } (x,z) \ \in \Omega \times \{\varepsilon < |z| < T_\varepsilon\} \\ (|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \left[\frac{\tilde{a}(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} + \varepsilon\right] \mathcal{C}_\varepsilon(z) & \text{if } (x,z) \in \Omega \times \{T_\varepsilon < |z|\} \,, \end{cases}$$

where

$$\mathcal{C}_\varepsilon(z) := \mathbb{I}_{N\times n} + (\gamma-2)\frac{z \otimes z}{|z|^2+\mu_\varepsilon^2} \quad \text{for } z \in \mathbb{R}^{N\times n} \,.$$

Moreover, recalling (5.2.3), we have

$$\partial_{xz}F_\varepsilon(x,z) = \begin{cases} \frac{\partial_x\tilde{a}(x,\varepsilon)}{(\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} z & \text{if } (x,z) \in \Omega \times \{|z| < \varepsilon\} \\ \partial_x\tilde{a}(x,|z|)z & \text{if } (x,z) \ \in \Omega \times \{\varepsilon \le |z| < T_\varepsilon\} \\ \frac{\partial_x\tilde{a}(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} z & \text{if } (x,z) \in \Omega \times \{T_\varepsilon \le |z|\} \,. \end{cases}$$

Then (5.2.12)-(5.2.13) directly follow by (5.1.3),(5.1.5),(5.2.3),(5.2.6)-(5.2.8) and with the choice

$$\Lambda_\varepsilon := \mathfrak{g}_2 \left[1 + \sup_{x\in\Omega} \frac{g_2(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} + \sup_{x\in\Omega, |z|\in[\varepsilon,T_\varepsilon]} \frac{|\partial_{zz}F(x,z)|}{(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}\right],$$

$$L_\varepsilon := 2n \sup_{x\in\Omega, t\in[\varepsilon,T_\varepsilon]} \frac{g_3(x,t)}{(t^2+\mu_\varepsilon^2)^{\frac{\gamma-1}{2}}} \,.$$

Notice that we are also using $(5.1.9)_2$ and $(5.1.4)_1$ in order to get upper bounds for $|\partial_{zz}F_\varepsilon|$, in $(5.2.12)_1$ and $(5.2.13)_1$, respectively. We also use $(5.1.8)_2$ to get $(5.2.12)_2$. Conditions (5.2.14) are a direct consequence of $(5.1.4)_1$, (5.2.11) and of the definitions in (5.2.3), (5.2.6)-(5.2.8). The inequality in $(5.2.15)_1$ comes from $(5.1.4)_2$ and (5.2.11). For $(5.2.15)_2$-$(5.2.16)_1$ we notice

$$\begin{aligned} F(x,z) \quad &= \quad \int_T^{|z|} \tilde{a}(x,s)s\,\mathrm{d}s \overset{(5.1.8)_2}{\ge} \int_T^{|z|} g_1(x,s)s\,\mathrm{d}s \overset{(5.1.11)}{=} G(x,|z|) \\ &\overset{(5.1.4)_2}{\ge} \quad \nu \int_T^{|z|} (s^2+\mu^2)^{\frac{\gamma-2}{2}} s\,\mathrm{d}s \overset{(5.2.2)}{=} \frac{\nu}{\gamma}\left\{[E(|z|)]^\gamma - [E(T)]^\gamma\right\} \,. \end{aligned}$$

To show $(5.2.15)_3$-$(5.2.16)_2$ and (5.2.17), we proceed as before when $|z| \in [T, T_\varepsilon)$. Instead, when $T_\varepsilon < |z|$, again using $(5.2.14)_2$, we estimate

$$\begin{aligned} F_\varepsilon(x,z) &\ge \int_T^{T_\varepsilon} \tilde{a}(x,s)s\ \mathrm{d}s + \frac{\tilde{a}\,(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \int_{T_\varepsilon}^{|z|} (s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s\ \mathrm{d}s \\ &\ge \int_T^{T_\varepsilon} g_1(x,s)s\ \mathrm{d}s + \frac{g_1\,(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \int_{T_\varepsilon}^{|z|} (s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s\ \mathrm{d}s \\ &= G_\varepsilon(x,|z|) = \int_T^t g_{1,\varepsilon}(x,s)s\,\mathrm{d}s \\ &\ge \nu\mathfrak{g}_1 \int_T^{T_\varepsilon} (s^2+\mu^2)^{\frac{\gamma-2}{2}} s\ \mathrm{d}s + \nu\mathfrak{g}_1 \frac{(T_\varepsilon^2+\mu^2)^{\frac{\gamma-2}{2}}}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} \int_{T_\varepsilon}^{|z|} (s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} s\ \mathrm{d}s \\ &\ge \frac{1}{c(\nu,\gamma)} \int_T^{|z|} (s^2+\mu^2)^{\frac{\gamma-2}{2}} s\ \mathrm{d}s \ge \frac{1}{c(\nu,\gamma)} \left\{[E(|z|)]^\gamma - [E(T)]^\gamma\right\} \,. \end{aligned}$$

Finally, the proof of (5.2.18) can be obtained as in [16, Lemma 5.2, (5.20)], taking into account that $\tilde{F}(\cdot)$ also depends on x and keeping in mind the definition in $(5.2.4)_1$, while (5.2.19) is direct consequence of the definition in (5.2.4). □

The aim of next lemma is to transfer the various features of functions $g_1(\cdot)$-$g_2(\cdot)$ to their truncated counterparts $g_{1,\varepsilon}(\cdot)$-$g_{2,\varepsilon}(\cdot)$.

Lemma 5.2.2 *For every $0 < \varepsilon < \min\{1, T\}/4$, it holds that*

- $$T \leq s \leq t \quad \Longrightarrow \quad \begin{aligned} &\frac{g_{2,\varepsilon}(x,s)}{g_{1,\varepsilon}(x,s)} \leq c\,\frac{g_{2,\varepsilon}(x,t)}{g_{1,\varepsilon}(x,t)} \qquad \text{for } c \equiv c(n, N, \nu, \gamma, c_a) \\ &g_{1,\varepsilon}(x,s)s \leq g_{1,\varepsilon}(x,t)t \end{aligned} \tag{5.2.20}$$

  for all $x \in \Omega$.

- *If (5.1.18) holds, then for all $(x,t) \in \Omega \times [T, \infty)$,*

  $$\frac{g_{2,\varepsilon}(x,t)}{g_{1,\varepsilon}(x,t)} \leq c \min\left\{\bar{G}_\varepsilon(x,t)^\sigma, \bar{G}_\varepsilon(x,t)^{\sigma'}\right\} \tag{5.2.21}$$

  holds for $t \geq T$ with σ, σ' as in (5.1.14)-(5.1.17) respectively, while, if (5.1.20) is in force we instead have

  $$\frac{g_{2,\varepsilon}(x,t)}{g_{1,\varepsilon}(x,t)} \leq c\bar{G}_\varepsilon(x,t)^\sigma \ , \tag{5.2.22}$$

  again for all $t \geq T$ and this time with σ as in (5.1.19). In (5.2.21)-(5.2.22) it is $c \equiv c(n, N, \nu, \gamma, c_b)$.

- *If (5.1.15) holds, and if $\hat{\sigma}$ satisfies the bounds in (5.1.14) (or (5.1.19) when $n = 2$), then*

  $$|\partial_x g_{1,\varepsilon}(x,t)|t \leq ch(x)\partial_t[\bar{G}_\varepsilon(x,t)]^{1+\hat{\sigma}} \quad \text{for all} \ \ (x,t) \in \Omega \times [T, \infty) \ , \tag{5.2.23}$$

  where $c \equiv c(\nu, \gamma)$.

- *For all $(x,t) \in \Omega \times [T, \infty)$, and with $c(\nu, \gamma) \geq 1$, there holds*

  $$\begin{cases} (t^2 + \mu_\varepsilon^2)^{\gamma/2} \leq c(\nu, \gamma)\bar{G}_\varepsilon(x,t) \\ [g_{1,\varepsilon}(x,t)]^{-1} G_{T,\varepsilon}(x,t) \leq c(\nu, \gamma)[\bar{G}_\varepsilon(x,t)]^{2/\gamma} \ . \end{cases} \tag{5.2.24}$$

- *For all $(x,t) \in \Omega \times [0, \infty)$*

  $$\varepsilon_1 < \varepsilon_2 < \min\{1, T\}/4 \Longrightarrow G_{\varepsilon_2}(x,t) \leq c(\gamma)G_{\varepsilon_1}(x,t) \tag{5.2.25}$$

  holds true.

Proof. The properties in (5.2.20) follow from (1.1.3) and the definitions in (5.2.6)-(5.2.7); see also [16, Lemma 5.3]. Arguing as in this last paper it is also not difficult to see that

$$T \leq s \leq T_\varepsilon \quad \Longrightarrow \quad \frac{g_{2,\varepsilon}(x,s)}{g_{1,\varepsilon}(x,s)} \leq c\,\frac{g_2(x,s)}{g_1(x,s)} \ , \qquad c \equiv c(n, N, \nu, \gamma) \tag{5.2.26}$$

for every $x \in \Omega$. Moreover, from the very definitions in (5.2.6)-(5.2.7) it holds that

$$\frac{g_{2,\varepsilon}(x,s)}{g_{1,\varepsilon}(x,s)} = \frac{g_{2,\varepsilon}(x,T_\varepsilon)}{g_{1,\varepsilon}(x,T_\varepsilon)} \qquad \text{for all } (x,s) \in \Omega \times [T_\varepsilon \ \infty) \ , \tag{5.2.27}$$

and $(5.2.20)_1$ follows by means of $(5.1.13)_1$ and $(5.2.15)_1$. Combining $(5.1.13)_2$ with the fact that being $\gamma > 1$, the map $t \mapsto (t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}t$ is increasing, we obtain $(5.2.20)_2$. The following useful estimate is a consequence of $(5.2.20)_2$:

$$G_{T,\varepsilon}(x,t) = \int_T^t g_{1,\varepsilon}(x,s)s \, \mathrm{d}s \leq g_{1,\varepsilon}(x,t)t \int_T^t \, \mathrm{d}s = g_{1,\varepsilon}(x,t)t(t-T) \leq g_{1,\varepsilon}(x,t)t^2 \, , \tag{5.2.28}$$

that holds whenever $t \geq T$. We are now ready to check the validity of (5.2.21). It is sufficient to show that

$$T \leq t \implies \frac{g_{2,\varepsilon}(x,t)}{g_{1,\varepsilon}(x,t)} \leq c\bar{G}_\varepsilon(x,t)^{\tilde{\sigma}} \qquad \tilde{\sigma} \in \{\sigma, \sigma'\} \tag{5.2.29}$$

with $c \equiv c(n,N,\nu,\gamma,c_b)$ to conclude. The proof of (5.2.22) is exactly the same, with a different value of σ, of course. We have:

$$T \leq t \leq T_\varepsilon \implies \frac{g_{2,\varepsilon}(x,t)}{g_{1,\varepsilon}(x,t)} \overset{(5.2.26)}{\leq} c\frac{g_2(x,t)}{g_1(x,t)} \overset{(5.1.18)}{\leq} c\bar{G}_T(x,t)^{\tilde{\sigma}} \overset{(5.2.6)}{\leq} c\bar{G}_\varepsilon(x,t)^{\tilde{\sigma}}$$

and

$$\begin{aligned} T_\varepsilon \leq t \implies \frac{g_{2,\varepsilon}(x,t)}{g_{1,\varepsilon}(x,t)} &\overset{(5.2.27)}{\leq} \frac{c\mathfrak{g}_2}{\mathfrak{g}_1}\left[\frac{g_2(x,T_\varepsilon)+\varepsilon(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}{g_1(x,T_\varepsilon)}\right] \\ &\overset{(5.1.4)_2}{\leq} c\left[\frac{g_2(x,T_\varepsilon)}{g_1(x,T_\varepsilon)}+\frac{\varepsilon}{\nu}\right] \overset{(5.1.18)}{\leq} c\bar{G}_T(x,T_\varepsilon)^{\tilde{\sigma}} \overset{(5.2.6)}{\leq} c\bar{G}_\varepsilon(x,t)^{\tilde{\sigma}}, \end{aligned}$$

and (5.2.29) is proved with c having the dependencies outlined in (5.2.21). The proof of (5.2.23) is a straightforward consequence of the definition in (5.2.6) and (5.1.15). Let us now take care of $(5.2.24)_1$. For $t \in [T, T_\varepsilon)$, using $(5.1.4)_2$ and (5.2.11), we see that (recall that $\varepsilon \leq T/4$)

$$\bar{G}_\varepsilon(x,t) \geq \frac{\nu}{\gamma}\left[(t^2+\mu^2)^{\gamma/2}-(T^2+\mu^2)^{\gamma/2}\right]+\mathfrak{g} \geq \frac{\nu}{2^\gamma\gamma}(t^2+\mu_\varepsilon^2)^{\gamma/2} \, ,$$

while, when $t \geq T_\varepsilon$ by analogous means, we have

$$\begin{aligned} \bar{G}_\varepsilon(x,t) &= \int_{T_\varepsilon}^t \frac{g_1(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}s\,\mathrm{d}s + \int_T^{T_\varepsilon} g_1(x,s)s\,\mathrm{d}s + \mathfrak{g} \\ &\geq \frac{1}{c(\nu,\gamma)}\int_T^t (s^2+\mu^2)^{\frac{\gamma-2}{2}}s\,\mathrm{d}s + \mathfrak{g} \geq \frac{(t^2+\mu_\varepsilon^2)^{\gamma/2}}{c(\nu,\gamma)} \, . \end{aligned}$$

As for $(5.2.24)_2$, this follows using (5.2.28) with $(5.2.24)_1$. Concerning (5.2.25), we see that it is trivially verified if $t \in [0,T]$ because of the definition in (5.2.10). On the other hand, by (5.2.6), $(5.1.4)_1$ and (5.2.11), for all $(x,t) \in \Omega \times [T,\infty)$ we have

$$\varepsilon_1 < \varepsilon_2 < \min\{1,T\}/4 \implies g_{1,\varepsilon_2}(x,t) \leq c(\gamma)g_{1,\varepsilon_1}(x,t),$$

so (5.2.25) follows as a consequence of the positions in (5.2.10) and the proof of the lemma is complete. □

Let us connect now $g_1(\cdot)$-$g_3(\cdot)$ with $g_{1,\varepsilon}(\cdot)$-$g_{3,\varepsilon}(\cdot)$.

Lemma 5.2.3 *Assume that* (1.1.11) *is in force; then, for every* $0 < \varepsilon < \min\{1, T\}/4$

$$\begin{cases} g_{3,\varepsilon}(x,t)\sqrt{t^2+\mu_\varepsilon^2} \le c[\bar{G}_\varepsilon(x,t)]^{1+\sigma} \\[2mm] \dfrac{[g_{3,\varepsilon}(x,t)]^2}{g_{1,\varepsilon}(x,t)} \le c[\bar{G}_\varepsilon(x,t)]^{1+2\sigma}\,, \end{cases} \tag{5.2.30}$$

hold for all $(x,t) \in \Omega \times [T,\infty)$*, for* $c \equiv c(n,N,\nu,\gamma,c_b)$*.*

Proof. When $t \in [T, T_\varepsilon)$, the proof of (5.2.30) follows directly by the definitions (5.2.6), (5.2.8) and from assumption (1.1.11). Therefore we restrict ourselves to the case $t \ge T_\varepsilon$, set $t_\varepsilon := t/T_\varepsilon$, introduce the quantity:

$$\mathcal{Q}_\varepsilon(t) := \frac{\bar{G}_\varepsilon(x,T_\varepsilon)}{\bar{G}_\varepsilon(x,t)} = \frac{\bar{G}(x,T_\varepsilon)}{\bar{G}_\varepsilon(x,t)} \le 1 \tag{5.2.31}$$

and bound via $(1.1.11)_1$ and (5.2.11):

$$\begin{aligned} g_{3,\varepsilon}(x,t)\sqrt{t^2+\mu_\varepsilon^2} &\le c g_3(x,T_\varepsilon)\sqrt{T_\varepsilon^2+\mu^2}\left(\frac{t^2+\mu_\varepsilon^2}{T_\varepsilon^2+\mu_\varepsilon^2}\right)^{\gamma/2} \\ &\le c[\bar{G}_\varepsilon(x,T_\varepsilon)]^{1+\sigma}t_\varepsilon^\gamma \le c(\gamma)[\mathcal{Q}_\varepsilon(t)]^{1+\sigma}t_\varepsilon^\gamma[\bar{G}_\varepsilon(x,t)]^{1+\sigma}\,. \end{aligned}$$

Observe that, if $t_\varepsilon \le 1000$, then $(5.2.30)_1$ follows using (5.2.31). In the case $t_\varepsilon > 1000$ we instead estimate

$$\begin{aligned} \mathcal{Q}_\varepsilon(t) &\overset{(5.2.10)}{\le} \frac{\bar{G}(x,T_\varepsilon)}{\displaystyle\int_{T_\varepsilon}^t g_{1,\varepsilon}(x,s)s\,\mathrm{d}s + T^\gamma + 1} \\ &\overset{(5.2.6)}{\le} \frac{\bar{G}(x,T_\varepsilon)}{\frac{g_1(x,T_\varepsilon)}{\gamma(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}T_\varepsilon^\gamma\left[\left(t_\varepsilon^2+\left(\frac{\mu_\varepsilon}{T_\varepsilon}^2\right)\right)^{\gamma/2} - \left(1+\left(\frac{\mu_\varepsilon}{T_\varepsilon}^2\right)\right)^{\gamma/2}\right]} \\ &\overset{t_\varepsilon>1000}{\le} \frac{c(\gamma)\bar{G}(x,T_\varepsilon)(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}{g_1(x,T_\varepsilon)T_\varepsilon^\gamma t_\varepsilon^\gamma} \\ &\overset{(5.2.28)}{\le} c(\gamma)\left[\frac{g_1(x,T_\varepsilon)T_\varepsilon^2(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}{g_1(x,T_\varepsilon)T_\varepsilon^\gamma t_\varepsilon^\gamma} + \frac{(T^\gamma+1)(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}{g_1(x,T_\varepsilon)T_\varepsilon^\gamma t_\varepsilon^\gamma}\right] \\ &\overset{(5.1.4)_2}{\le} \frac{c(\gamma)}{t_\varepsilon^\gamma}\left[1+\frac{T^\gamma+1}{\nu T_\varepsilon^\gamma}\right] \le \frac{c(\nu,\gamma)}{t_\varepsilon^\gamma}\,, \end{aligned}$$

therefore

$$g_{3,\varepsilon}(x,t)\sqrt{t^2+\mu_\varepsilon^2} \le ct_\varepsilon^{-\sigma\gamma}[\bar{G}_\varepsilon(x,t)]^{1+\sigma} \le c[\bar{G}_\varepsilon(x,t)]^{1+\sigma},$$

where $c \equiv c(n,N,\nu,\gamma,c_b)$. As for $(5.2.30)_2$, similarly to $(5.2.30)_1$, using $(1.1.11)_2$ we have

$$\frac{[g_{3,\varepsilon}(x,t)]^2}{g_{1,\varepsilon}(x,t)} \le \frac{[g_{3,\varepsilon}(x,T_\varepsilon)]^2}{g_{1,\varepsilon}(x,T_\varepsilon)}\left(\frac{t^2+\mu_\varepsilon^2}{T_\varepsilon^2+\mu_\varepsilon^2}\right)^{\gamma/2} \le c[\mathcal{Q}_\varepsilon(t)]^{1+2\sigma}t_\varepsilon^\gamma[\bar{G}_\varepsilon(x,t)]^{1+2\sigma}$$

and $(5.2.30)_2$ follows arguing as for $(5.2.30)_1$. □

We next present some convergence arguments which will be very helpful for passing to the limit in certain approximating problems defined by means of the $\{F_\varepsilon(\cdot)\}$.

Lemma 5.2.4 *Let $B \Subset \Omega$ be a ball with $\mathrm{rad}(B) \le 1$ and $0 < \varepsilon \le \min\{1, T\}/4$ and let $w \in W^{1,\gamma}(B, \mathbb{R}^N)$. Let σ be as in (5.1.14)-(5.1.19) and assume that, for some $p > 1+\sigma$ there exists a positive, finite constant so that*

$$\liminf_{\varepsilon\to 0} \int_B \bar{G}_\varepsilon(x, |Dw|)^p \, \mathrm{d}x \le \mathcal{C}. \tag{5.2.32}$$

Then

$$\int_B |F(x, Dw) - F_\varepsilon(x, Dw)| \, \mathrm{d}x \to 0 \quad \text{and} \quad \int_B |G_\varepsilon(\cdot, |Dw|) - \mathfrak{g}_1 G_T(\cdot, |Dw|)| \, \mathrm{d}x \to 0, \tag{5.2.33}$$

where $\mathfrak{g}_1$ is as in (5.2.9).

Proof. Assumption (5.2.32), definitions (5.1.11) and (5.2.6) and Fatou's Lemma yield

$$\begin{aligned}\mathfrak{g}_1^p \int_B G_T(x, |Dw|)^p \, \mathrm{d}x &\le \liminf_{\varepsilon\to 0} \int_{B\cap\{|Dw|<T_\varepsilon\}} (\mathfrak{g}_1 G_T(x, |Dw|))^p \, \mathrm{d}x \\ &\le \liminf_{\varepsilon\to 0} \int_B G_\varepsilon(x, |Dw|)^p \, \mathrm{d}x \le \mathcal{C},\end{aligned}$$

which in turn implies that

$$G_T(\cdot, |Dw|) \in L^p(B). \tag{5.2.34}$$

Notice that, by Markov and Hölder inequalities we have

$$|\{|Dw| > T_\varepsilon\}| \le T_\varepsilon^{-\gamma} \|Dw\|_{L^\gamma(B)}^\gamma \le \varepsilon^\gamma \|Dw\|_{L^\gamma(B)}^\gamma. \tag{5.2.35}$$

From the definition in (5.2.3) we see that $\tilde{a}_\varepsilon(x,t) \equiv \tilde{a}(x,t)$ for all $(x,t) \in B \times [\varepsilon, T_\varepsilon)$, so, using this fact, together with (5.2.4), $(5.2.14)_1$, (5.1.9), (5.1.13), (5.1.18)-(5.1.20) and (5.2.35) we get

$$\begin{aligned}\int_B |F(x, Dw) - F_\varepsilon(x, Dw)| \, \mathrm{d}x &\le \frac{\varepsilon}{\gamma} \int_B \left[(|Dw|^2 + \mu_\varepsilon^2)^{\frac{\gamma}{2}} - \mu_\varepsilon^\gamma \right] \mathrm{d}x \\ &+ 4 \int_B \int_0^\varepsilon \left[\tilde{a}(x,s) - \frac{\tilde{a}(x,\varepsilon)}{(\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} (s^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \right] s \, \mathrm{d}s \, \mathrm{d}x \\ &+ \int_{B\cap\{|Dw|>T_\varepsilon\}} \int_{T_\varepsilon}^{|Dw|} \left[\tilde{a}(x,s) + \frac{\tilde{a}(x,T_\varepsilon)}{(T_\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}}} (s^2 + \mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \right] s \, \mathrm{d}s \, \mathrm{d}x \\ &\le \frac{\varepsilon}{\gamma} \int_B \left[(|Dw|^2 + \mu_\varepsilon^2)^{\frac{\gamma}{2}} - \mu_\varepsilon^\gamma \right] \mathrm{d}x + c(n, \gamma, \tilde{a}^+) \left[(\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma}{2}} - \mu_\varepsilon^\gamma \right] \\ &+ c(\gamma) \int_{B\cap\{|Dw|>T_\varepsilon\}} \left(\int_0^{|Dw|} \tilde{a}(x,s) s \, \mathrm{d}s \right) \mathrm{d}x \\ &\le \frac{\varepsilon}{\gamma} \int_B \left[(|Dw|^2 + \mu_\varepsilon^2)^{\frac{\gamma}{2}} - \mu_\varepsilon^\gamma \right] \mathrm{d}x + c(n, \gamma, \tilde{a}^+) \left[(\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma}{2}} - \mu_\varepsilon^\gamma \right] \\ &+ c(\gamma, \tilde{a}^+) |B \cap |Dw| > T_\varepsilon| + c(\gamma, c_a, c_b) \int_{B\cap\{|Dw|>T_\varepsilon\}} \bar{G}_T(x, |Dw|)^{1+\sigma} \, \mathrm{d}x \\ &\le \frac{\varepsilon}{\gamma} \int_B \left[(|Dw|^2 + \mu_\varepsilon^2)^{\frac{\gamma}{2}} - \mu_\varepsilon^\gamma \right] \mathrm{d}x + c(n, \gamma, \tilde{a}^+) \left[(\varepsilon^2 + \mu_\varepsilon^2)^{\frac{\gamma}{2}} - \mu_\varepsilon^\gamma \right]\end{aligned}$$

$$+ c(\gamma,\tilde{a}^+)\varepsilon^\gamma + c(\gamma,c_a,c_b)\varepsilon^{\frac{\gamma(p-1-\sigma)}{p}}\left(\int_{B\cap\{|Dw|>T_\varepsilon\}}\bar{G}(x,|Dw|)^p\ \mathrm{d}x\right)^{\frac{1+\sigma}{p}} \to 0,$$

because of (5.2.32), (5.2.34) and the dominated convergence theorem. Now, using (5.2.34), (5.1.11), (5.2.6), (5.2.10), $(5.2.14)_2$ and the dominated convergence theorem we get

$$\begin{aligned}
&\int_B |G_\varepsilon(\cdot,|Dw|) - \mathfrak{g}_1 G_T(\cdot,|Dw|)|\ \mathrm{d}x \\
=&\mathfrak{g}_1\int_{B\cap\{|Dw|>T_\varepsilon\}}\left|\int_{T_\varepsilon}^{|Dw|}\left[g_1(x,s)s - \frac{g_1(x,T_\varepsilon)}{(T_\varepsilon^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}}(s^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}s\right]\right|\ \mathrm{d}s\,\mathrm{d}x \\
\leq& c(\gamma)\varepsilon^{\frac{\gamma(p-1)}{p}}\left(\int_{B\cap\{|Dw|>T_\varepsilon\}}G_T(x,|Dw|)^p\ \mathrm{d}x\right)^{\frac{1}{p}} \to 0,
\end{aligned}$$

and (5.2.33) is completely proven. □

Lemma 5.2.5 *Let $B \Subset \Omega$ be a ball with $\mathrm{rad}(B) \leq 1$ and $0 < \varepsilon_1 < \varepsilon_2 \leq \min\{1,T\}/4$ and let $w \in W^{1,\gamma}(B,\mathbb{R}^N)$. If $\bar{G}_T(\cdot,|Dw|) \in L^p(B)$ for some $p > 1+\sigma$, with σ as in (5.1.14)-(5.1.19), then*

$$\int_B \left|F_{\varepsilon_1}(x,Dw) - F_{\varepsilon_2}(x,Dw)\right|\ \mathrm{d}x \leq \mathsf{o}(\varepsilon_2)\left(\int_B [\bar{G}_{\varepsilon_1}(x,|Dw|)]^p\,\mathrm{d}x + 1\right)^{\frac{p+1+\sigma}{p}}, \tag{5.2.36}$$

where $\mathsf{o}(\varepsilon_2)$ denotes a quantity such that $\mathsf{o}(\varepsilon_2) \to 0$ as $\varepsilon_2 \to 0$. Moreover, there holds that

$$\int_B \left|F(x,Dw) - F_{\varepsilon_2}(x,Dw)\right|\ \mathrm{d}x \leq \mathsf{o}(\varepsilon_2)\left(\int_B [\bar{G}_T(x,|Dw|)]^p\,\mathrm{d}x + 1\right)^{\frac{p+1+\sigma}{p}}. \tag{5.2.37}$$

Proof. We denote

$$\bar{F}_\varepsilon(x,t) = \tilde{F}_\varepsilon(x,t) - \varepsilon L_{\gamma,\varepsilon}(t) = \int_0^t \tilde{a}_\varepsilon(x,s)s\ \mathrm{d}s \tag{5.2.38}$$

and in the following we always take $x \in B$. Proceeding as for (5.1.8)-(5.1.9), thanks to (5.2.12) we get

$$\tilde{a}_\varepsilon(x,|z|) \leq g_{2,\varepsilon}(x,|z|)\,, \quad \text{for every } \varepsilon \leq \min\{1,T\}/4\,, \tag{5.2.39}$$

provided $|z| \geq T$. In the case it is $|z| \leq \varepsilon_2$, by $(5.1.4)_1$ we easily have

$$\begin{aligned}
&\left|\bar{F}_{\varepsilon_1}(x,|z|) - \bar{F}_{\varepsilon_2}(x,|z|)\right| \leq \left|\bar{F}_{\varepsilon_1}(x,\varepsilon_2)\right| + \left|\bar{F}_{\varepsilon_2}(x,\varepsilon_2)\right| \\
&\quad\leq c\|\tilde{a}(\cdot,1)\|_{L^\infty(B)}\int_0^{\varepsilon_2}(s^2+\mu_{\varepsilon_1}^2)^{\frac{\gamma-2}{2}}s\ \mathrm{d}s + c\|\tilde{a}(\cdot,1)\|_{L^\infty(B)}\int_0^{\varepsilon_2}(s^2+\mu_{\varepsilon_2}^2)^{\frac{\gamma-2}{2}}s\ \mathrm{d}s \\
&\quad\leq \mathsf{o}(\varepsilon_2)\,.
\end{aligned}$$

Recalling that $\tilde{a}_{\varepsilon_1}(x,t) \equiv \tilde{a}_{\varepsilon_2}(x,t)$ when $\varepsilon_2 \leq |z| \leq T_{\varepsilon_2}$, by also using the information in the last display, we find

$$\left|\bar{F}_{\varepsilon_1}(x,|z|) - \bar{F}_{\varepsilon_2}(x,|z|)\right| \leq \left|\bar{F}_{\varepsilon_1}(x,\varepsilon_2)\right| + \left|\bar{F}_{\varepsilon_2}(x,\varepsilon_2)\right| \leq \mathsf{o}(\varepsilon_2)\,. \tag{5.2.40}$$

Finally, when $|z| > T_{\varepsilon_2}$ we have, using also (5.2.11)

$$\left|\bar{F}_{\varepsilon_1}(x,|z|) - \bar{F}_{\varepsilon_2}(x,|z|)\right| \overset{(5.1.4)_1}{\leq} \left|\bar{F}_{\varepsilon_1}(x,T_{\varepsilon_2}) - \bar{F}_{\varepsilon_2}(x,T_{\varepsilon_2})\right| + c\int_{T_{\varepsilon_2}}^{|z|}\tilde{a}_{\varepsilon_1}(x,s)s\ \mathrm{d}s$$

$$
\begin{aligned}
&\overset{(5.2.40)}{\leq} \quad c\int_{T_{\varepsilon_2}}^{|z|} \tilde{a}_{\varepsilon_1}(x,s)s\, \mathrm{d}s + \mathsf{o}(\varepsilon_2)\\
&\overset{(5.2.39)}{\leq} \quad c\int_{T_{\varepsilon_2}}^{|z|} g_{2,\varepsilon_1}(x,s)s\, \mathrm{d}s + \mathsf{o}(\varepsilon_2)\\
&\overset{(5.2.20)_1}{\leq} \quad c\frac{g_{2,\varepsilon_1}(x,|z|)}{g_{1,\varepsilon_1}(x,|z|)}\int_{T_{\varepsilon_2}}^{|z|} g_{1,\varepsilon_1}(x,s)s\, \mathrm{d}s + \mathsf{o}(\varepsilon_2)\\
&\overset{(5.2.22)}{\leq} \quad c[\bar{G}_{\varepsilon_1}(x,|z|)]^{1+\sigma} + \mathsf{o}(\varepsilon_2)\ .
\end{aligned}
$$

Using the content of the last four displays, and also Hölder's inequality, we get

$$
\begin{aligned}
\int_B \big|F_{\varepsilon_1}(x,|Dw|) - F_{\varepsilon_2}(x,|Dw|)\big|\, \mathrm{d}x &\leq \mathsf{o}(\varepsilon_2)|B| + c\int_{B\cap\{|Dw|>T_{\varepsilon_2}\}} [\bar{G}_{\varepsilon_1}(x,|Dw|)]^{1+\sigma}\, \mathrm{d}x\\
&\leq \mathsf{o}(\varepsilon_2)|B| + |B\cap\{|Dw|>T_{\varepsilon_2}\}|^{\frac{p-1-\sigma}{p}}\left(\int_B [\bar{G}_{\varepsilon_1}(x,|Dw|)]^p\, \mathrm{d}x\right)^{(1+\sigma)/p}
\end{aligned}
$$

for $c \equiv c(\nu,\gamma,p)$. On the other hand, observe that

$$
|B\cap\{|Dw|>T_{\varepsilon_2}\}| \leq T_{\varepsilon_2}^{-\gamma p}\int_B |Dw|^{\gamma p}\, dx \overset{(5.2.24)_1}{\leq} c\varepsilon_2^{\gamma p}\int_B [\bar{G}_{\varepsilon_1}(x,|Dw|)]^p\, dx\ .
$$

Combining the previous two displays yields

$$
\int_B \left|\tilde{F}_{\varepsilon_1}(x,|Dw|) - \tilde{F}_{\varepsilon_2}(x,|Dw|)\right|\, \mathrm{d}x \leq |B|\mathsf{o}(\varepsilon_2) + c\varepsilon_2^{\gamma(p-1-\sigma)}\left(\int_B [\bar{G}_{\varepsilon_1}(x,|Dw|)]^p\, \mathrm{d}x\right)^{\frac{p+1+\sigma}{p}}\ ,
$$

where $c \equiv c(n,N,\nu,\gamma,c_a,c_b)$. Next, notice that by using again $(5.2.24)_1$ and Hölder's inequality, we get

$$
\begin{aligned}
\int_B \big|\varepsilon_1 L_{\gamma,\varepsilon_1}(Dw) - \varepsilon_2 L_{\gamma,\varepsilon_2}(Dw)\big|\, \mathrm{d}x &\leq c\varepsilon_2\int_B (|Dw|^2+1)^{\gamma/2}\, dx\\
&\leq c\varepsilon_2\left(\int_B [\bar{G}_{\varepsilon_1}(x,|Dw|)]^p\, \mathrm{d}x\right)^{1/p}\ .
\end{aligned}
$$

Combining the content of the last two displays and recalling (5.2.4) and (5.2.38) yields (5.2.36). As for (5.2.37), this follows from (5.2.36) and (5.2.33) letting $\varepsilon_1 \to 0$. Indeed, notice that, since $\bar{G}_T(\cdot,|Dw|) \in L^p(B)$ with $p > 1+\sigma$ and σ as in (5.1.14)-(5.1.19), then, by (5.2.25) we deduce that (5.2.32) is verified. Hence, by $(5.2.33)_1$, Fatou's lemma works for the left-hand side; as for the right-hand side, we can exploit $(5.2.33)_2$, (5.2.25) and the dominated convergence theorem to conclude. □

5.3 A priori estimates

This section is the core of the paper as we here develop the basic a priori estimates. These are obtained for local minimizers of more regular functionals approximating the original one in (5.0.1). We shall permanently assume that assumptions $\mathtt{set}_{\mathrm{m}}$ are in force, therefore all properties (5.1.1)-(5.1.16) and (5.1.20) will be available. Eventually, we shall use additional, more restrictive conditions depending on the degree of regularity we are going to be interested in. With $0 < \varepsilon < \min\{1,T\}/4$ and $\mathcal{B} \Subset \Omega$ being a ball such that $\mathtt{r}(\mathcal{B}) \leq 1$, we consider a weak solution $u \in W^{1,\gamma}(B;\mathbb{R}^N)$ to the system

$$
-\operatorname{div} a_\varepsilon(x,Du) = f \qquad \text{in } \mathcal{B}, \tag{5.3.1}
$$

with

$$f \in L^{\infty}(\mathcal{B},\mathbb{R}^N), \qquad |f| \le |\mathfrak{f}| \ \text{ for some } \mathfrak{f} \in L^n(\mathcal{B},\mathbb{R}^N) \tag{5.3.2}$$

and $a_\varepsilon \colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ being defined as

$$\partial_z F_\varepsilon(x,z) = a_\varepsilon(x,z) = \bar{a}_\varepsilon(x,|z|)z\,, \qquad \bar{a}_\varepsilon(x,t) := \tilde{a}_\varepsilon(x,t) + \varepsilon(t^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}}\,, \tag{5.3.3}$$

and where $\tilde{a}_\varepsilon \colon \Omega \times [0,\infty) \to [0,\infty)$ has been introduced in (5.2.3). The growth and ellipticity properties of the vector field $a_\varepsilon(\cdot)$ are expressed in (5.2.12)-(5.2.13) and it follows that u is a local minimizer of the functional

$$W^{1,\gamma}(\mathcal{B},\mathbb{R}^N) \ni w \mapsto \int_{\mathcal{B}} \left[F_\varepsilon(x,Dw) - f\cdot w\right]\,\mathrm{d}x = \int_{\mathcal{B}} \left[\tilde{F}_\varepsilon(x,|Dw|) - f\cdot w\right]\,\mathrm{d}x\,. \tag{5.3.4}$$

Summarizing, $\bar{a}_\varepsilon(\cdot)$ is such that $x \mapsto \bar{a}_\varepsilon(x,t) \in W^{1,d}_{\rm loc}(\Omega)$ for all $t \ge 0$, $t \mapsto \bar{a}_\varepsilon(x,t) \in W^{1,\infty}_{\rm loc}[0,\infty) \cap C^1_{\rm loc}([0,\infty)\setminus\{\varepsilon,T_\varepsilon\})$ i.e., it is locally C^1-regular outside $\{\varepsilon,T_\varepsilon\}$ and it is such that $\bar{a}'_\varepsilon(x,0)=0$ for all $x \in \Omega$. This implies that $x \mapsto a_\varepsilon(x,z) \in W^{1,d}_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$ for all $z \in \mathbb{R}^{N\times n}$ and $z \mapsto a_\varepsilon(x,z) \in W^{1,\infty}_{\rm loc}(\mathbb{R}^{N\times n})$ for all $x \in \Omega$. Finally, the functions $t \mapsto \bar{a}_\varepsilon(\cdot,t)$ and $g_{1,\varepsilon}(\cdot,t)$ are non-decreasing when $\gamma \ge 2$; this is indeed an easy consequence of assumption $(5.1.4)_1$. As for (5.1.8)-(5.1.9), from (5.2.12) and (5.3.3) it follows

$$g_{1,\varepsilon}(x,|z|) \le \bar{a}_\varepsilon(x,|z|) \le g_{2,\varepsilon}(x,|z|) \tag{5.3.5}$$

and

$$\begin{cases} \bar{a}_\varepsilon(x,|z|) + \bar{a}'_\varepsilon(x,|z|)|z| \ge g_{1,\varepsilon}(x,|z|) \\ \bar{a}_\varepsilon(x,|z|) + \bar{a}'_\varepsilon(x,|z|)|z| \le g_{2,\varepsilon}(x,|z|) \end{cases} \tag{5.3.6}$$

for all $(x,z) \in \Omega \times \{|z| \ge T,\ |z| \ne T_\varepsilon\}$. A direct consequence of (5.3.5)-(5.3.6) is

$$|\bar{a}'_\varepsilon(x,|z|)||z| \le g_{2,\varepsilon}(x,|z|) \quad \text{for all} \quad (x,z) \in \Omega \times \{|z| \ge T,\ |z| \ne T_\varepsilon\}\,, \tag{5.3.7}$$

while, similarly to (5.3.5), by (5.2.13) we have that

$$\varepsilon\mathfrak{g}_1(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \le \bar{a}_\varepsilon(x,|z|) \le \Lambda(|z|^2+\mu_\varepsilon^2)^{\frac{\gamma-2}{2}} \tag{5.3.8}$$

holds this time for all $(x,z) \in \Omega \times \mathbb{R}^{N\times n}$ (recall that $z \mapsto \bar{a}_\varepsilon(x,|z|)$ is continuous). From properties (5.2.12)-(5.2.13) we can conclude that Theorem 21 applies to u and this yields

$$Du \in L^{\infty}_{\rm loc}(\mathcal{B},\mathbb{R}^{N\times n}), \quad u \in W^{2,2}_{\rm loc}(\mathcal{B},\mathbb{R}^N), \quad \bar{a}_\varepsilon(\cdot,Du) \in W^{1,2}_{\rm loc}(\mathcal{B},\mathbb{R}^{N\times n})\,. \tag{5.3.9}$$

As for the expression of $Da_\varepsilon(\cdot,Du)$, we need to apply a non-autonomous chain rule [66, Theorem 1.5]. We remark that the crucial point to apply the results from [66] is that the set of non-differentiable points of the partial map $t \mapsto \bar{a}_\varepsilon(x,t)$ is $\{\varepsilon,T_\varepsilon\}$ and is therefore independent of x (here this holds for every x, but a.e. x is still allowed). Specifically, with slight abuse of notation, let us write

$$(\partial_z \bar{a}_\varepsilon(x,z))^{\alpha\beta}_{ij} = \bar{a}_\varepsilon(x,|z|)\delta_{ij}\delta_{\alpha\beta} + \mathbb{1}_{\mathcal{D}}(|z|)\bar{a}'_\varepsilon(x,|z|)|z|\frac{z_i^\alpha z_j^\beta}{|z|^2}\,, \quad \text{for } z \in \mathbb{R}^{N\times n}\,. \tag{5.3.10}$$

and here we are denoting by $\mathbb{1}_{\mathcal{D}}(\cdot)$ the indicator function of the set $\mathcal{D} := \mathbb{R}\setminus\{\varepsilon,T_\varepsilon\}$. Then, recalling that $|Du| \in W^{1,2}_{\rm loc}(\Omega)$ by (5.3.9), we have

$$\begin{aligned} D_s[a_\varepsilon(\cdot,Du)] =& \partial_{x_s} a_\varepsilon(x,Du) + \partial_z a_\varepsilon(x,Du)DD_su \\ =& \partial_{x_s}\bar{a}_\varepsilon(x,|Du|)Du + \bar{a}_\varepsilon(x,|Du|)\mathbb{I}_{N\times n}DD_su \end{aligned}$$

$$+ \mathbb{1}_{\mathcal{D}}(|Du|)\bar{a}'_\varepsilon(x,|Du|)|Du|\frac{Du \otimes Du}{|Du|^2}DD_s u\,. \tag{5.3.11}$$

Notice that, exactly as in the usual autonomous case, the presence of $\mathbb{1}_{\mathcal{D}}(|Du|)$ in (5.3.10)-(5.3.11) accounts for the fact that terms as $\bar{a}'_\varepsilon(x,|Du|)D_s|Du|$ are interpreted as zero at those points where $|Du| \in \{\varepsilon, T_\varepsilon\}$, i.e., where $\bar{a}'_\varepsilon(x,|Du|)$ alone does not make sense; see [66]. These same arguments apply to G_ε; indeed, notice that for every $x \in \Omega$, the function $t \mapsto G_\varepsilon(x,t)$ is differentiable at every point but T and by (1.1.10) satisfies the assumptions of [66, Theorem 1.5]. Therefore it is $G_\varepsilon(\cdot, Du) \in W^{1,2}_{\rm loc}(B,\mathbb{R}^{N\times n})$ and, on $\{|Du|>T\}$, we have

$$\begin{aligned}
D_i G_\varepsilon(x,|Du|) &= g_{1,\varepsilon}(x,|Du|)|Du|D_i|Du| + \int_T^{|Du|} \partial_{x_i} g_{1,\varepsilon}(x,t)t\,{\rm d}t \\
&= g_{1,\varepsilon}(x,|Du|)\sum_{\alpha=1}^N\sum_{s=1}^n D_iD_s u^\alpha D_s u^\alpha + \int_T^{|Du|} \partial_{x_i} g_{1,\varepsilon}(x,t)t\,{\rm d}t
\end{aligned} \tag{5.3.12}$$

for every $i \in \{1,\ldots,n\}$, where we have also used that

$$D_i|Du| = \frac{1}{|Du|}\sum_{\alpha=1}^N\sum_{s=1}^n D_iD_s u^\alpha D_s u^\alpha\,. \tag{5.3.13}$$

Notice also that $D_iG_\varepsilon(x,|Du|) \equiv 0$ a.e. on the complement of $\{|Du|>T\}$. We conclude this part with

Lemma 5.3.1 *For $\lambda \equiv (\lambda_i) \in \mathbb{R}^n$ and $z = (z_i^\alpha) \in \mathbb{R}^{N\times n}$, $1 \le i \le n$ and $1 \le \alpha \le N$*

$$\begin{aligned}
&\bar{a}_\varepsilon(x,|z|)\lambda\cdot\lambda + \mathbb{1}_{\mathcal{D}}(|z|)\bar{a}'_\varepsilon(x,|z|)|z|\sum_{\alpha=1}^N \frac{|\lambda\cdot z^\alpha|^2}{|z|^2} \\
&\quad \ge \min\left\{\bar{a}_\varepsilon(x,|z|), \bar{a}_\varepsilon(x,|z|) + \mathbb{1}_{\mathcal{D}}(|z|)\bar{a}'_\varepsilon(x,|z|)|z|\right\}|\lambda|^2 \ge g_{1,\varepsilon}(x,|z|)|\lambda|^2
\end{aligned} \tag{5.3.14}$$

holds for every $x \in \Omega$.

Proof. Indeed, (5.3.14) is trivial by (5.3.5) if $\bar{a}'_\varepsilon(x,|z|) \ge 0$. Otherwise, we can estimate simply $\bar{a}'_\varepsilon(x,|z|)|\lambda\cdot z^\alpha|^2 \ge \bar{a}'_\varepsilon(x,|z|)|z^\alpha|^2|\lambda|^2$ for every α and then use $(5.3.6)_1$. □

5.3.1 Caccioppoli inequality for powers

The main result here is

Proposition 5.3.1 *Let $u \in W^{1,\gamma}(\mathcal{B},\mathbb{R}^N)$ be a solution to (5.3.1), under assumptions $\texttt{set}_{\rm m}$ for $n \ge 3$. Then, for each $p \in [1,\infty)$, there exists a positive radius $R_* \equiv R_*(\texttt{data}_{m}, \mathfrak{f}(\cdot), p) \le 1$ such that if $\mathbf{r}(\mathcal{B}) \le R_*$ and $B_\varsigma \Subset B_\varrho$ are concentric balls contained in $\mathcal{B}$, then*

$$\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_\varsigma)} \le \frac{c}{(\varrho-\varsigma)^{\beta_p}}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)} + 1\right]^{\theta_p} \tag{5.3.15}$$

holds with $c \equiv c(\texttt{data}_{m}, \|h\|_{L^d(\mathcal{B})}, p) \ge 1$, $\beta_p, \theta_p \equiv \beta_p, \theta_p(n,d,\sigma,\hat{\sigma},p) > 0$.

The proof will take this and the subsequent Sections 5.3.2-5.3.3; in the following, all the balls but the initial one $\mathcal{B}$, will be concentric with those from the statement of Proposition 5.3.1. We notice that all the foregoing computations, except those involving f, still work in the case $n = 2$, the analysis of which will be done in Section 5.3.7 below. To start the proof of Proposition 5.3.1, we observe that properties (5.3.9) legitimate the passage to the differentiated form of system (5.3.1), that is

$$\sum_{s=1}^{n}\int_{\mathcal{B}}\left[\partial_z a_\varepsilon(x,Du)DD_s u\cdot D\varphi+\partial_{x_s}a_\varepsilon(x,Du)\cdot D\varphi+f\cdot D_s\varphi\right]\,\mathrm{d}x=0\,, \tag{5.3.16}$$

which holds for all $\varphi\in W^{1,2}_0(B;\mathbb{R}^N)$. Notice that we are going too use (5.3.10)-(5.3.11) repeatedly. In the following, we shall consider an arbitrary fixed ball $B_{R_*}\Subset B$ such that

$$0<\varrho\le R_*\le 1\quad\text{and}\quad B_{R_*}\Subset\mathcal{B}\,. \tag{5.3.17}$$

The precise size of R_* will be quantified in due course of the proof. Moreover we shall consider concentric balls $B_\varsigma\subset B_{\tau_1}\Subset B_{\tau_2}\subset B_\varrho\subset B_{R_*}$; in particular it is $\tau_1<\tau_2$. By (5.3.9), the integral identity in (5.3.16) can be tested against $\varphi_s:=\eta^2[G_\varepsilon(x,|Du|)]^{\kappa+1}D_s u$ for $s\in\{1,\dots,n\}$, where $\kappa\ge 0$ and $\eta\in C^1_c(B_\varrho)$ satisfies $\mathbb{1}_{B_{\tau_1}}\le\eta\le\mathbb{1}_{B_{\tau_2}}$ and $|D\eta|\lesssim 1/(\tau_2-\tau_1)$. It follows

$$\begin{aligned}D\varphi_s=&\eta^2[G_\varepsilon(x,|Du|)]^{\kappa+1}DD_s u+(\kappa+1)\eta^2[G_\varepsilon(x,|Du|)]^{\kappa}D_s u\otimes DG_\varepsilon(x,|Du|)\\&+2\eta[G_\varepsilon(x,|Du|)]^{\kappa+1}D_s u\otimes D\eta\,.\end{aligned}\tag{5.3.18}$$

By (5.3.18) we have

$$\begin{aligned}&\sum_{s=1}^{n}\int_{\mathcal{B}}\partial_z a_\varepsilon(x,Du)DD_s u\cdot D\varphi_s\,\,\mathrm{d}x\\&\quad=\sum_{s=1}^{n}\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^{\kappa+1}\partial_z a_\varepsilon(x,Du)DD_s u\cdot DD_s u\,\,\mathrm{d}x\\&\qquad+(\kappa+1)\sum_{s=1}^{n}\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^{\kappa}\partial_z a_\varepsilon(x,Du)DD_s u\cdot\big(D_s u\otimes DG_\varepsilon(x,|Du|)\big)\,\,\mathrm{d}x\\&\qquad+2\sum_{s=1}^{n}\int_{\mathcal{B}}\eta[G_\varepsilon(x,|Du|)]^{\kappa+1}\partial_z a_\varepsilon(x,Du)DD_s u\cdot(D_s u\otimes D\eta)\,\,\mathrm{d}x\\&\quad=:(\mathrm{I})_z+(\mathrm{II})_z+(\mathrm{III})_z\,.\end{aligned}\tag{5.3.19}$$

Notice that in the display above and in the following ones until (5.3.26), as $G_\varepsilon(t)\equiv 0$ for $t\le T$, all the integrals above actually extend only on $B\cap\{|Du|>T\}$, therefore we can always use (5.3.12) when computing the derivatives of G_ε. To proceed, we have

$$(\mathrm{I})_z\overset{(5.2.12)_2}{\ge}\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^{\kappa+1}g_{1,\varepsilon}(x,|Du|)|D^2u|^2\,\,\mathrm{d}x=:\mathcal{S}_1\,.$$

We temporarily shorten the notation as follows:

$$\begin{aligned}&\mathcal{H}_\kappa(Du):=(\kappa+1)[G_\varepsilon(x,|Du|)]^{\kappa}\frac{\tilde a_\varepsilon(x,|Du|)}{g_{1,\varepsilon}(x,|Du|)}\,,\\&\mathcal{H}'_\kappa(Du):=(\kappa+1)\mathbb{1}_{\mathcal{D}}(|Du|)[G_\varepsilon(x,|Du|)]^{\kappa}\frac{\tilde a'_\varepsilon(x,|Du|)|Du|}{g_{1,\varepsilon}(x,|Du|)}\,,\\&m:=\frac{d}{d-2}>1\,.\end{aligned}$$

Recalling (5.3.10)-(5.3.13), we then re-write

$$(\mathrm{II})_z=\int_{\mathcal{B}}\eta^2\mathcal{H}_\kappa(Du)DG_\varepsilon(x,|Du|)\cdot DG_\varepsilon(x,|Du|)\,\,\mathrm{d}x$$

$$
\begin{aligned}
&+\sum_{\alpha=1}^{N}\int_{\mathcal{B}}\eta^2\mathcal{H}_\kappa'(Du)\frac{\big(DG_\varepsilon(x,|Du|)\cdot Du^\alpha\big)^2}{|Du|^2}\ \mathrm{d}x\\
&-\int_{\mathcal{B}}\eta^2\mathcal{H}_\kappa(Du)\int_T^{|Du|}\partial_x g_{1,\varepsilon}(x,t)t\,\mathrm{d}t\cdot DG_\varepsilon(x,|Du|)\ \mathrm{d}x\\
&-\sum_{\alpha=1}^{N}\int_{\mathcal{B}}\eta^2\mathcal{H}_\kappa'(Du)\left(\int_T^{|Du|}\partial_x g_{1,\varepsilon}(x,t)t\,\mathrm{d}t\cdot Du^\alpha\right)\frac{(DG_\varepsilon(x,|Du|)\cdot Du^\alpha)}{|Du|^2}\ \mathrm{d}x\\
=:\ &(\mathrm{II})_{z,1}+(\mathrm{II})_{z,2}+(\mathrm{II})_{z,3}+(\mathrm{II})_{z,4}\,. \qquad (5.3.20)
\end{aligned}
$$

We now observe that

$$
(\mathrm{II})_z^1+(\mathrm{II})_z^2\geq(\kappa+1)\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^\kappa|DG_\varepsilon(x,|Du|)|^2\ \mathrm{d}x=:\mathcal{S}_2\,. \qquad (5.3.21)
$$

Indeed, this follows from (5.3.14) with $\lambda\equiv DG_\varepsilon$ and $z\equiv Du$. Then we have, by using (5.2.22)-(5.2.23), (5.3.5) and (5.3.7)

$$
\begin{aligned}
|(\mathrm{II})_{z,3}|+|(\mathrm{II})_{z,4}|&\leq c(\kappa+1)\int_{\mathcal{B}}\eta^2h(x)[G_\varepsilon(x,|Du|)]^\kappa\frac{g_{2,\varepsilon}(x,|Du|)}{g_{1,\varepsilon}(x,|Du|)}\\
&\qquad\cdot\left(\int_T^{|Du|}\partial_t[\bar{G}_\varepsilon(x,t)]^{1+\hat\sigma}\ \mathrm{d}t\right)|DG_\varepsilon(x,|Du|)|\ \mathrm{d}x\\
&\leq c(\kappa+1)\int_{\mathcal{B}}\eta^2h(x)[G_\varepsilon(x,|Du|)]^\kappa[\bar{G}_\varepsilon(x,|Du|)]^{1+\sigma+\hat\sigma}|DG_\varepsilon(x,|Du|)|\ \mathrm{d}x\\
&\leq\bar\varepsilon\mathcal{S}_2+\frac{c(\kappa+1)\|h\|_{L^d(\mathcal{B})}^2}{\bar\varepsilon}\left(\int_{\mathcal{B}}\eta^{2m}[G_\varepsilon(x,|Du|)]^{m\kappa}[\bar{G}_{T,\varepsilon}(x,|Du|)]^{2m(1+\sigma+\hat\sigma)}\ \mathrm{d}x\right)^{1/m}
\end{aligned}
$$

where $c\equiv c(n,N,\nu,\gamma,c_b)$, and arbitrary $\bar\varepsilon\in(0,1)$. Notice that in last two lines we have used Young and Hölder's inequalities. Similarly to (5.3.20), we also have

$$
\begin{aligned}
(\mathrm{III})_z&=2\int_{\mathcal{B}}\frac{\eta G_\varepsilon(x,|Du|)\mathcal{H}_\kappa(Du)}{\kappa+1}DG_\varepsilon(x,|Du|)\cdot D\eta\ \mathrm{d}x\\
&\quad+2\sum_{\alpha=1}^{N}\int_{\mathcal{B}}\frac{\eta G_\varepsilon(x,|Du|)\mathcal{H}_\kappa'(Du)}{(\kappa+1)|Du|^2}(DG_\varepsilon(x,|Du|)\cdot Du^\alpha)(Du^\alpha\cdot D\eta)\ \mathrm{d}x\\
&\quad-2\int_{\mathcal{B}}\frac{\eta G_\varepsilon(x,|Du|)\mathcal{H}_\kappa(Du)}{\kappa+1}\left(\int_T^{|Du|}\partial_x g_{1,\varepsilon}(x,t)t\,\mathrm{d}t\cdot D\eta\right)\ \mathrm{d}x\\
&\quad-2\sum_{\alpha=1}^{N}\int_{\mathcal{B}}\frac{\eta G_\varepsilon(x,|Du|)\mathcal{H}_\kappa'(Du)}{(\kappa+1)|Du|^2}\left(\int_T^{|Du|}\partial_x g_{1,\varepsilon}(x,t)t\,\mathrm{d}t\cdot Du^\alpha\right)(Du^\alpha\cdot D\eta)\ \mathrm{d}x\\
&=:(\mathrm{III})_{z,1}+(\mathrm{III})_{z,2}+(\mathrm{III})_{z,3}+(\mathrm{III})_{z,4}\,. \qquad (5.3.22)
\end{aligned}
$$

Using again (5.2.22), (5.3.5), (5.3.7), Young and Hölder inequalities, we have, for every $\bar\varepsilon\in(0,1)$

$$
\begin{aligned}
|(\mathrm{III})_{z,1}|+|(\mathrm{III})_{z,2}|&\leq c\int_{\mathcal{B}}\eta[G_\varepsilon(x,|Du|)]^{\kappa+1}\frac{g_{2,\varepsilon}(x,|Du|)}{g_{1,\varepsilon}(x,|Du|)}|DG_\varepsilon(x,|Du|)||D\eta|\ \mathrm{d}x\\
&\leq c\int_{\mathcal{B}}\eta[G_\varepsilon(x,|Du|)]^{\kappa+1}[\bar{G}_\varepsilon(x,|Du|)]^{\sigma}|DG_\varepsilon(x,|Du|)||D\eta|\ \mathrm{d}x\\
&\leq\bar\varepsilon\mathcal{S}_2+\frac{c}{\bar\varepsilon(\kappa+1)}\int_{\mathcal{B}}|D\eta|^2[G_\varepsilon(x,|Du|)]^\kappa[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\sigma)}\ \mathrm{d}x\\
&\leq\bar\varepsilon\mathcal{S}_2+\frac{c}{\bar\varepsilon(\kappa+1)}\left(\int_{\mathcal{B}}|D\eta|^{2m}[G_\varepsilon(x,|Du|)]^{m\kappa}[\bar{G}_\varepsilon(x,|Du|)]^{2m(1+\sigma)}\ \mathrm{d}x\right)^{1/m},
\end{aligned}
$$

and, using now also (5.2.23), we get

$$
\begin{aligned}
|(\mathrm{III})_{z,3}|+|(\mathrm{III})_{z,4}| &\le c\int_{\mathcal{B}}\eta h(x)[G_\varepsilon(x,|Du|)]^{\kappa+1}[\bar G_\varepsilon(x,s)]^{1+\sigma+\hat\sigma}|D\eta|\ \mathrm{d}x\\
&\le c\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^{\kappa+2}\ \mathrm{d}x\\
&\quad +c\|h\|^2_{L^d(\mathcal{B})}\left(\int_{\mathcal{B}}|D\eta|^{2m}[G_\varepsilon(x,|Du|)]^{m\kappa}[\bar G_\varepsilon(x,|Du|)]^{2m(1+\sigma+\hat\sigma)}\ \mathrm{d}x\right)^{1/m}
\end{aligned}
$$

with $c\equiv c(n,N,\nu,\gamma,c_b)$. Now we look at the second group of terms stemming from (5.3.16)

$$
\begin{aligned}
\sum_{s=1}^n\int_{\mathcal{B}}\partial_{x_s}a_\varepsilon(x,Du)\cdot D\varphi_s\ \mathrm{d}x &= \sum_{s=1}^n\int_{\mathcal{B}}\eta^2G_\varepsilon(x,|Du|)^{\kappa+1}\partial_{x_s}a_\varepsilon(x,|Du|)\cdot DD_su\ \mathrm{d}x\\
&+(\kappa+1)\sum_{s=1}^n\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^{\kappa}\partial_{x_s}a_\varepsilon(x,Du)\cdot(D_su\otimes DG_\varepsilon(x,|Du|))\ \mathrm{d}x\\
&+2\sum_{s=1}^n\int_{\mathcal{B}}\eta[G_\varepsilon(x,|Du|)]^{\kappa+1}\partial_{x_s}a_\varepsilon(x,|Du|)\cdot(D_su\otimes D\eta)\ \mathrm{d}x =: (\mathrm{I})_x+(\mathrm{II})_x+(\mathrm{III})_x\,.
\end{aligned}
$$

From $(5.2.12)_3$, $(5.2.30)_2$, Hölder and Young inequalities we obtain, for arbitrary $\bar\varepsilon\in(0,1)$

$$
\begin{aligned}
|(\mathrm{I})_x| \le& c\int_{\mathcal{B}}\eta^2h(x)[G_\varepsilon(x,|Du|)]^{\kappa+1}g_{3,\varepsilon}(x,|Du|)|D^2u|\ \mathrm{d}x\\
\le&\bar\varepsilon\mathcal{S}_1+\frac{c}{\bar\varepsilon}\int_{\mathcal{B}}\eta^2[h(x)]^2[G_\varepsilon(x,|Du|)]^{\kappa+1}\frac{[g_{3,\varepsilon}(x,|Du|)]^2}{g_{1,\varepsilon}(x,|Du|)}\ \mathrm{d}x\\
\le&\bar\varepsilon\mathcal{S}_1+\frac{c}{\bar\varepsilon}\|h\|^2_{L^d(\mathcal{B})}\left(\int_{\mathcal{B}}\eta^{2m}[G_\varepsilon(x,|Du|)]^{m\kappa}[\bar G_\varepsilon(x,|Du|)]^{2m(1+\sigma)}\ \mathrm{d}x\right)^{1/m}\ ,
\end{aligned}
$$

where $c\equiv c(n,N,\nu,\gamma,c_b)$. Using this time $(5.2.30)_1$ we get

$$
\begin{aligned}
|(\mathrm{II})_x| \le& c(\kappa+1)\int_{\mathcal{B}}\eta^2h(x)[G_\varepsilon(x,|Du|)]^{\kappa}g_{3,\varepsilon}(x,|Du|)|Du||DG_\varepsilon(x,|Du|)|\ \mathrm{d}x\\
\le&\bar\varepsilon\mathcal{S}_2+\frac{c(\kappa+1)}{\bar\varepsilon}\int_{\mathcal{B}}\eta^2[h(x)]^2[G_\varepsilon(x,|Du|)]^{\kappa}[g_{3,\varepsilon}(x,|Du|)]^2(|Du|^2+\mu_\varepsilon^2)\ \mathrm{d}x\\
\le&\bar\varepsilon\mathcal{S}_2+\frac{c(\kappa+1)}{\bar\varepsilon}\|h\|^2_{L^d(\mathcal{B})}\left(\int_{\mathcal{B}}\eta^{2m}[G_\varepsilon(x,|Du|)]^{m\kappa}[\bar G_\varepsilon(x,|Du|)]^{2m(1+\sigma)}\ \mathrm{d}x\right)^{1/m}
\end{aligned}
$$

with $c\equiv c(n,N,\nu,\gamma,c_b)$. Again, $(5.2.30)_1$, Hölder and Young inequalities render that

$$
\begin{aligned}
|(\mathrm{III})_x| \le& c\int_{\mathcal{B}}\eta h(x)[G_\varepsilon(x,|Du|)]^{\kappa+1}g_{3,\varepsilon}(x,|Du|)|Du||D\eta|\ \mathrm{d}x\\
\le& c\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^{\kappa+2}\ \mathrm{d}x\\
&+c\int_{\mathcal{B}}|D\eta|^2[h(x)]^2[G_\varepsilon(x,|Du|)]^{\kappa}[g_{3,\varepsilon}(x,|Du|)]^2(|Du|^2+\mu_\varepsilon^2)\ \mathrm{d}x\\
\le& c\int_{\mathcal{B}}\eta^2[G_\varepsilon(x,|Du|)]^{\kappa+2}\ \mathrm{d}x\\
&+c\|h\|^2_{L^d(\mathcal{B})}\left(\int_{\mathcal{B}}|D\eta|^{2m}[G_\varepsilon(x,|Du|)]^{m\kappa}[\bar G_\varepsilon(x,|Du|)]^{2m(1+\sigma)}\ \mathrm{d}x\right)^{1/m}\ ,
\end{aligned}
\tag{5.3.23}
$$

where $c \equiv c(n, N, \nu, \gamma, \mathfrak{g}, c_b)$. Finally, we examine the contributions to (5.3.16) coming from the terms featuring f:

$$
\begin{aligned}
\sum_{s=1}^{n} \int_{\mathcal{B}} f \cdot D_s \varphi_s \, \mathrm{d}x \quad &= \quad \sum_{s=1}^{n} \int_{\mathcal{B}} \eta^2 [G_\varepsilon(x, |Du|)]^{\kappa+1} f \cdot D_s D_s u \ \mathrm{d}x \\
&\quad +(\kappa+1) \sum_{s=1}^{n} \int_{\mathcal{B}} \eta^2 [G_\varepsilon(x, |Du|)]^{\kappa} D_s G_\varepsilon(x, |Du|)(f \cdot D_s u) \ \mathrm{d}x \\
&\quad +2 \sum_{s=1}^{n} \int_{\mathcal{B}} \eta [G_\varepsilon(x, |Du|)]^{\kappa+1} D_s \eta (f \cdot D_s u) \ \mathrm{d}x \\
&=: \quad (\mathrm{I})_f + (\mathrm{II})_f + (\mathrm{III})_f \, .
\end{aligned}
$$

Using $(5.2.24)_2$, Hölder and Young inequalities we get

$$
\begin{aligned}
|(\mathrm{I})_f| \leq & c \int_{\mathcal{B}} \eta^2 [G_\varepsilon(x, |Du|)]^{\kappa+1} |f| |D^2 u| \ \mathrm{d}x \\
\leq & \bar{\varepsilon} \mathcal{S}_1 + \frac{c}{\bar{\varepsilon}} \int_{\mathcal{B}} \eta^2 [G_\varepsilon(x, |Du|)]^{\kappa} |f|^2 [g_{1,\varepsilon}(x, |Du|)]^{-1} [G_\varepsilon(x, |Du|)] \ \mathrm{d}x \\
\leq & \bar{\varepsilon} \mathcal{S}_1 + \frac{c}{\bar{\varepsilon}} \int_{\mathcal{B}} \eta^2 |f|^2 [G_\varepsilon(x, |Du|)]^{\kappa} [\bar{G}_\varepsilon(x, |Du|)]^{2/\gamma} \ \mathrm{d}x \\
\leq & \bar{\varepsilon} \mathcal{S}_1 + \frac{c}{\bar{\varepsilon}} \|f\|^2_{L^n(\mathcal{B})} \left(\int_{\mathcal{B}} \eta^{2^*} [G_\varepsilon(x, |Du|)]^{2^* \kappa/2} [\bar{G}_\varepsilon(x, |Du|)]^{2^*} \ \mathrm{d}x \right)^{2/2^*} ,
\end{aligned}
$$

with $c \equiv c(n, N, \nu, \gamma)$ and where, according to the standard notation, it is

$$
2^* := \begin{cases} \frac{2n}{n-2} & \text{if } \ n > 2 \\ \text{any number larger than } \ 2 & \text{if } \ n = 2 \, . \end{cases} \tag{5.3.24}
$$

Here recall that $n \geq 3$. Moreover, by $(5.2.24)_1$, Hölder and Young inequalities we get

$$
\begin{aligned}
|(\mathrm{II})_f| \leq & c(\kappa+1) \int_{\mathcal{B}} \eta^2 [G_\varepsilon(x, |Du|)]^{\kappa} |f| |DG_\varepsilon(x, |Du|)| |Du| \ \mathrm{d}x \\
\leq & \bar{\varepsilon} \mathcal{S}_2 + \frac{c(\kappa+1)}{\bar{\varepsilon}} \int_{\mathcal{B}} \eta^2 |f|^2 [G_\varepsilon(x, |Du|)]^{\kappa} [\bar{G}_\varepsilon(x, |Du|)]^{2/\gamma} \ \mathrm{d}x \\
\leq & \bar{\varepsilon} \mathcal{S}_2 + \frac{c(\kappa+1) \|f\|^2_{L^n(\mathcal{B})}}{\bar{\varepsilon}} \left(\int_{\mathcal{B}} \eta^{2^*} [G_\varepsilon(x, |Du|)]^{2^* \kappa/2} [\bar{G}_\varepsilon(x, |Du|)]^{2^*} \ \mathrm{d}x \right)^{2/2^*} ,
\end{aligned}
$$

for $c \equiv c(n, N, \nu, \gamma)$. Finally, from $(5.2.24)_1$, Hölder and Young inequalities we deduce that

$$
\begin{aligned}
|(\mathrm{III})_f| \leq & c \int_{\mathcal{B}} \eta [G_\varepsilon(x, |Du|)]^{\kappa+1} |f| |D\eta| |Du| \ \mathrm{d}x \\
\leq & c \int_{\mathcal{B}} |D\eta|^2 [G_\varepsilon(x, |Du|)]^{\kappa+2} \ \mathrm{d}x + c \int_{\mathcal{B}} \eta^2 |f|^2 [G_\varepsilon(x, |Du|)]^{\kappa} [\bar{G}_\varepsilon(x, |Du|)]^{2/\gamma} \ \mathrm{d}x \\
\leq & c \int_{\mathcal{B}} |D\eta|^2 [G_\varepsilon(x, |Du|)]^{\kappa} [\bar{G}_\varepsilon(x, |Du|)]^2 \ \mathrm{d}x \\
& + c \|f\|^2_{L^n(\mathcal{B})} \left(\int_{\mathcal{B}} \eta^{2^*} [G_\varepsilon(x, |Du|)]^{2^* \kappa/2} [\bar{G}_\varepsilon(x, |Du|)]^{2^*} \ \mathrm{d}x \right)^{2/2^*} ,
\end{aligned} \tag{5.3.25}
$$

where $c \equiv c(n, N, \nu, \gamma)$. In the previous three displays, we also used that, by $(5.2.10)_2$ it is $\bar{G}_\varepsilon(\cdot) \geq 1$ for all $(x, t) \in \Omega \times [0, \infty)$, thus $[\bar{G}_\varepsilon(\cdot)]^{1/\gamma} \leq \bar{G}_\varepsilon(\cdot)$. Merging the content of all the displays, from (5.3.19) to (5.3.25), with (5.3.16), choosing $\bar{\varepsilon} > 0$ small enough (in order to reabsorb terms in the usual way), we get, after a few standard manipulations, and again using that $\bar{G}_\varepsilon(\cdot) \geq 1$

$$\begin{aligned}\mathcal{S}_1 + \mathcal{S}_2 &\leq c\int_{\mathcal{B}} \left(\eta^2 + |D\eta|^2\right) [G_\varepsilon(x,|Du|)]^{\kappa}[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\sigma)} \ \mathrm{d}x \\ &+c(\kappa+1)\|h\|^2_{L^d(\mathcal{B})}\left(\int_{\mathcal{B}} \left(\eta^{2m} + |D\eta|^{2m}\right) [G_\varepsilon(x,|Du|)]^{m\kappa}[\bar{G}_\varepsilon(x,|Du|)]^{2m(1+\sigma+\hat{\sigma})} \ \mathrm{d}x\right)^{\frac{1}{m}} \\ &+c(\kappa+1)\|f\|^2_{L^n(\mathcal{B})}\left(\int_{\mathcal{B}} \eta^{2^*}[G_\varepsilon(x,|Du|)]^{2^*\kappa/2}[\bar{G}_\varepsilon(x,|Du|)]^{2^*} \ \mathrm{d}x\right)^{2/2^*}, \end{aligned} \tag{5.3.26}$$

with $c \equiv c(n, N, \nu, \gamma, c_b)$. Now we notice that, by $(5.2.10)_2$ it follows

$$G_\varepsilon(x,t) \leq \bar{G}_\varepsilon(x,t) \leq \mathfrak{g}\left[G_\varepsilon(x,t)+1\right], \tag{5.3.27}$$

for all $(x,t) \in \Omega \times [0,\infty)$, therefore, recalling (5.3.21), estimate (5.3.26) can be rearranged as

$$\begin{aligned}&(\kappa+1)\int_{\mathcal{B}} \eta^2[G_\varepsilon(x,|Du|)]^{\kappa}|DG_\varepsilon(x,|Du|)|^2 \ \mathrm{d}x \\ &\leq c(\kappa+1)\left(\|h\|^2_{L^d(\mathcal{B})}+1\right)\left(\int_{\mathcal{B}} \left(\eta^{2m} + |D\eta|^{2m}\right)\left[[G_\varepsilon(x,|Du|)]^{m(\kappa+2+2\sigma+2\hat{\sigma})}+1\right] \ \mathrm{d}x\right)^{\frac{1}{m}} \\ &\quad +c_{\mathrm{ab}}(\kappa+1)\|f\|^2_{L^n(\mathcal{B})}\left(\int_{\mathcal{B}} \eta^{2^*}\left[[G_\varepsilon(x,|Du|)]^{\frac{2^*}{2}(\kappa+2)}+1\right] \ \mathrm{d}x\right)^{2/2^*},\end{aligned}$$

where $c, c_{\mathrm{ab}} \equiv c, c_{\mathrm{ab}}(n, N, \nu, \gamma, c_b)$ and every number $\kappa \geq 0$. Combining Sobolev-Poincaré inequality and (5.3.28) we obtain

$$\begin{aligned}&\left(\int_{\mathcal{B}} \eta^{2^*}\left[[G_\varepsilon(x,|Du|)]^{\frac{2^*}{2}(\kappa+2)}+1\right] \ \mathrm{d}x\right)^{\frac{2}{2^*}} \leq \left(\int_{\mathcal{B}} \eta^{2^*}\left[[G_\varepsilon(x,|Du|)]^{\frac{\kappa}{2}+1}+1\right]^{2*} \ \mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\leq c\int_{\mathcal{B}} |D(\eta([G_\varepsilon(x,|Du|)]^{\frac{\kappa}{2}+1}+1))|^2 \,\mathrm{d}x \\ &\leq c(\kappa+1)^2\left(\|h\|^2_{L^d(\mathcal{B})}+1\right)\left(\int_{\mathcal{B}} \left(\eta^{2m} + |D\eta|^{2m}\right)\left[[G_\varepsilon(x,|Du|)]^{m(\kappa+2+2\sigma+2\hat{\sigma})}+1\right] \ \mathrm{d}x\right)^{\frac{1}{m}} \\ &\quad +c_{\mathrm{ab}}(\kappa+1)^2\|f\|^2_{L^n(\mathcal{B})}\left(\int_{\mathcal{B}} \eta^{2^*}\left[[G_\varepsilon(x,|Du|)]^{\frac{2^*}{2}(\kappa+2)}+1\right] \ \mathrm{d}x\right)^{2/2^*},\end{aligned} \tag{5.3.28}$$

with $c, c_{\mathrm{ab}} \equiv c, c_{\mathrm{ab}}(n, N, \nu, \gamma, \mathfrak{g}, c_b)$. We now determine the radius R_* such that $\mathtt{r}(\mathcal{B}) \leq R_*$ mentioned in the statement of Proposition 5.3.1. Specifically, we fix $\bar{\kappa} \geq 0$ and, using the absolute continuity of the integral, determine $R_* \equiv R_*(\mathtt{data}_m, \mathfrak{f}(\cdot), \bar{\kappa}) \in (0,1)$ such that

$$c_{\mathrm{ab}}(\bar{\kappa}+1)^2\|f\|^2_{L^n(B_{\tau_2})} \overset{(5.3.1)}{\leq} c_{\mathrm{ab}}(\bar{\kappa}+1)^2\|\mathfrak{f}\|_{L^n(B_{R_*})} \leq \frac{1}{2}. \tag{5.3.29}$$

With (5.3.29) being now in force, for all $\kappa \leq \bar{\kappa}$ estimate (5.3.28) becomes

$$\begin{aligned}&\left(\int_{\mathcal{B}} \eta^{2^*}\left[[G_\varepsilon(x,|Du|)]^{\frac{2^*}{2}(\kappa+2)}+1\right] \ \mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\leq c(\kappa+1)^2\left(\|h\|^2_{L^d(\mathcal{B})}+1\right)\left(\int_{\mathcal{B}} \left(\eta^{2m} + |D\eta|^{2m}\right)\left[[G_\varepsilon(x,|Du|)]^{m(\kappa+2+2\sigma+2\hat{\sigma})}+1\right] \ \mathrm{d}x\right)^{\frac{1}{m}}\end{aligned} \tag{5.3.30}$$

for $c \equiv c(n, N, \nu, \gamma, \mathfrak{g}, c_b)$. For the following we set

$$1 \overset{n<d}{<} \chi := \frac{2^*}{2m} = \frac{n}{n-2}\frac{d-2}{d}. \tag{5.3.31}$$

5.3.2 Moser's iteration in finite steps

With $1 \leq i \in \mathbb{N}$, we inductively define the exponents

$$\kappa_1 := 0, \quad \kappa_{i+1} := \chi(\kappa_i + 2) - 2(1 + \sigma + \hat{\sigma}), \quad s_i := m(\kappa_i + 2 + 2\sigma + 2\hat{\sigma}) \, . \tag{5.3.32}$$

Notice that (5.1.14) yields

$$\sigma + \hat{\sigma} < \chi - 1 \tag{5.3.33}$$

and this implies that $\{\kappa_i\}$ and $\{s_i\}$ are increasing sequences; moreover, it holds that

$$\kappa_{i+1} = 2\left[\chi^i - (\sigma + \hat{\sigma}) \sum_{j=0}^{i-1} \chi^j - 1\right] \quad \text{and} \quad s_{i+1} = 2m\left[\chi^i - (\sigma + \hat{\sigma}) \sum_{j=1}^{i-1} \chi^j\right] \, . \tag{5.3.34}$$

Notice that this implies

$$s_{i+1} = \frac{2^*}{2}(\kappa_i + 2) \quad \text{and therefore} \quad k_{i+1} \leq s_{i+1} \leq 2m\chi^i \, , \quad k_{i+1} \leq 2\chi^i \, . \tag{5.3.35}$$

Again (5.3.33) implies

$$s_{i+1} = 2m\chi^i \left[1 - \frac{(\sigma + \hat{\sigma})(1 - \chi^{1-i})}{\chi - 1}\right] > 2m\chi^i \left(1 - \frac{\sigma + \hat{\sigma}}{\chi - 1}\right) \Longrightarrow \lim_{i \to \infty} s_{i+1} = \infty \, , \tag{5.3.36}$$

so that from the first relation in (5.3.35) it also follows that $\kappa_i \to \infty$. For $0 < \varsigma \leq \tau_1 < \tau_2 \leq \varrho$, we consider a sequence $\{B_{\varrho_i}\}$ of shrinking balls where $\varrho_i := \tau_1 + (\tau_2 - \tau_1)2^{-i+1}$. Notice that $\{\varrho_i\}$ is a decreasing sequence with $\varrho_1 = \tau_2$ and $\varrho_i \to \tau_1$, therefore it is $\bigcap_{i \in \mathbb{N}} B_{\varrho_i} = \bar{B}_{\tau_1}$ and $B_{\varrho_1} = B_{\tau_2}$. Accordingly, we fix corresponding cut-off functions $\{\eta_i\} \subset C^1_c(B)$ with

$$\mathbb{1}_{B_{\varrho_{i+1}}} \leq \eta_i \leq \mathbb{1}_{B_{\varrho_i}} \quad \text{and} \quad |D\eta_i| \lesssim \frac{1}{\varrho_i - \varrho_{i+1}} \sim \frac{2^i}{\tau_2 - \tau_1} \, .$$

Choosing $\eta \equiv \eta_i$ in (5.3.30) and making elementary manipulations, we get that

$$\left(\int_{B_{\varrho_{i+1}}} \left[[G_\varepsilon(x, |Du|)]^{s_{i+1}} + 1\right] \, dx\right)^{1/\chi}$$
$$\leq \frac{c^{2m}\left(\|h\|_{L^d(\mathcal{B})} + 1\right)^{2m} 2^{2mi}(\kappa_i + 1)^{2m}}{(\tau_2 - \tau_1)^{2m}} \int_{B_{\varrho_i}} \left[[G_\varepsilon(x, |Du|)]^{s_i} + 1\right] \, dx \, , \tag{5.3.37}$$

hods whenever $\kappa_i \leq \bar{\kappa}$, with $c \equiv c(n, N, \nu, \gamma, \mathfrak{g}, c_b)$. Finally, we set $c_h := (c\|h\|_{L^d(\mathcal{B})} + c)^{2^*}$ and

$$\mathcal{G}_i := \left(\int_{B_{\varrho_i}} \left[[G_\varepsilon(x, |Du|)]^{s_i} + 1\right] \, dx\right)^{1/s_i} \, ,$$

so that (5.3.37) (recall that $2* = 2m\chi$ by (5.3.31)) becomes

$$\mathcal{G}_{i+1} \leq \left[\frac{c_h 2^{2^* i}(\kappa_i + 1)^{2^*}}{(\tau_2 - \tau_1)^{2^*}}\right]^{\frac{1}{s_{i+1}}} \mathcal{G}_i^{\frac{\chi s_i}{s_{i+1}}} \, .$$

Iterating the above inequality yields that

$$\mathcal{G}_{i+1} \leq \prod_{j=0}^{i-1} \left[\frac{c_h 2^{2^*(i-j)}(\kappa_{i-j} + 1)^{2^*}}{(\tau_2 - \tau_1)^{2^*}}\right]^{\frac{\chi^j}{s_{i+1}}} \mathcal{G}_1^{\frac{\chi^i s_1}{s_{i+1}}} \qquad \text{holds provided } \kappa_i \leq \bar{\kappa} \, . \tag{5.3.38}$$

Now, from (5.3.36) we deduce that

$$\frac{\chi^{i+1}}{s_{i+1}} \le \frac{\chi(\chi-1)}{2m(\chi-1-\sigma-\hat{\sigma})} \,. \tag{5.3.39}$$

The function $t \mapsto t/\chi^t$ is decreasing on $[1/\log\chi, \infty)$ and using this fact one sees that

$$\sum_{j=1}^{\infty} \frac{j}{\chi^j} \lesssim \frac{1}{(\log\chi)^2} \le \frac{c}{(\chi-1)^2} \,. \tag{5.3.40}$$

We then write

$$\prod_{j=0}^{i-1}\left[\frac{c_h 2^{2^*(i-j)}(\kappa_{i-j}+1)^{2^*}}{(\tau_2-\tau_1)^{2^*}}\right]^{\frac{\chi^j}{s_{i+1}}}$$
$$= \exp\left\{\log\left(\frac{c_h}{(\tau_2-\tau_1)^{2^*}}\right)\frac{1}{s_{i+1}}\sum_{j=0}^{i-1}\chi^j + \frac{2^*\log 2}{s_{i+1}}\sum_{j=0}^{i-1}(i-j)\chi^j + \frac{2^*}{s_{i+1}}\sum_{j=0}^{i-1}\chi^j\log(\kappa_{i-j}+1)\right\} \,,$$

and notice that, for every integer $i \ge 1$, we have

$$\frac{1}{s_{i+1}}\sum_{j=0}^{i-1}\chi^j \overset{(5.3.39)}{\le} c \,, \qquad \frac{1}{s_{i+1}}\sum_{j=0}^{i-1}(i-j)\chi^j \le c\sum_{j=0}^{i-1}\frac{i-j}{\chi^{i-j}} \overset{(5.3.40)}{\le} c$$

and

$$\frac{2^*\log 2}{s_{i+1}}\sum_{j=0}^{i-1}\chi^j\log(\kappa_{i-j}+1) \overset{(5.3.35),(5.3.39)}{\le} c\sum_{j=0}^{i-1}\frac{i-j}{\chi^{i-j}} \overset{(5.3.40)}{\le} c \,, \tag{5.3.41}$$

where $c \equiv c(n,d,\sigma,\tilde{\sigma})$ in all cases. Using the content of the last three displays yields

$$\prod_{j=0}^{i-1}\left[\frac{c_h 2^{2^*(i-j)}(\kappa_{i-j}+1)^{2^*}}{(\tau_2-\tau_1)^{2^*}}\right]^{\frac{\chi^j}{s_{i+1}}} \le \frac{c}{(\tau_2-\tau_1)^{\beta}} < \infty \,,$$

where c depends on $n, d, \sigma, \tilde{\sigma}$ and $\|h\|_{L^d(\mathcal{B})}$, and it is $\beta \equiv \beta(n,d,\sigma,\hat{\sigma})$. Notice that such constants blow-up when $\chi \to 1+\sigma+\hat{\sigma}$; in particular, this happens when $d \to n$. Using (5.3.41) in (5.3.38), and keeping (5.3.39) in mind, yields

$$\begin{aligned}\|G_\varepsilon(x,|Du|)\|_{L^{s_{i+1}}(B_{\tau_1})} &\le \mathcal{G}_{i+1} \\ &\le \frac{c\,\mathcal{G}_1^{\frac{\chi^i s_1}{s_{i+1}}}}{(\tau_2-\tau_1)^{\beta}} \le \frac{c}{(\tau_2-\tau_1)^{\beta}}\left[1+\|G_\varepsilon(\cdot,|Du|)\|_{L^{s_1}(B_{\tau_2})}^{\frac{\chi^i s_1}{s_{i+1}}}\right] \,,\end{aligned} \tag{5.3.42}$$

with $c \equiv c(\mathtt{data}_m, \|h\|_{L^d(\mathcal{B})})$ and $\beta \equiv \beta(n,d,\sigma,\hat{\sigma})$, for every $i \in \mathbb{N}$ such that $\kappa_i \le \bar{\kappa}$.

5.3.3 Sobolev regularity

For every index $i \in \mathbb{N}$, we consider the interpolation inequality

$$\|G_\varepsilon(\cdot,|Du|)\|_{L^{s_1}(B_{\tau_2})} \le \|G_\varepsilon(\cdot,|Du|)\|_{L^{s_{i+1}}(B_{\tau_2})}^{\lambda_{i+1}}\|G_\varepsilon(\cdot,|Du|)\|_{L^1(B_{\tau_2})}^{1-\lambda_{i+1}}, \tag{5.3.43}$$

with λ_{i+1} being defined by

$$\frac{1}{s_1} = 1-\lambda_{i+1}+\frac{\lambda_{i+1}}{s_{i+1}} \Rightarrow \lambda_{i+1} = \frac{s_{i+1}(s_1-1)}{s_1(s_{i+1}-1)} \,. \tag{5.3.44}$$

Let us show there exist $\bar{\vartheta} \equiv \bar{\vartheta}(n,d,\sigma,\hat{\sigma}) < 1$ and $i_1 \in \mathbb{N}$ such that

$$i > i_1 \Longrightarrow \frac{\lambda_{i+1}\chi^i s_1}{s_{i+1}} \leq \bar{\vartheta} < 1 \,. \tag{5.3.45}$$

For this, it is sufficient to observe that

$$\lim_{i\to\infty} \frac{\lambda_{i+1}\chi^i s_1}{s_{i+1}} = \lim_{i\to\infty} \frac{\chi^i(s_1-1)}{s_{i+1}-1} = \tilde{l} := \frac{1+\sigma+\hat{\sigma}-\frac{1}{2m}}{1-\frac{\sigma+\hat{\sigma}}{\chi-1}} \overset{(5.1.14)}{<} 1 \,. \tag{5.3.46}$$

Note that the last inequality is actually equivalent to (5.1.14). Now, fix a number $p > 1$, and determine another index $k > i_1$ such that $s_{k+1} \geq p$; accordingly, we consider the number κ_k related to s_k via (5.3.35). We now choose the number $\bar{\kappa} \equiv \bar{\kappa}(p)$ in (5.3.29) as $\bar{\kappa} := \kappa_k$, and accordingly we determine $R_* \equiv R_*(\texttt{data}_m, \mathfrak{f}(\cdot), p)$ via (5.3.29). It follows that (5.3.42) holds in the case $i \equiv k$ and therefore we can plug (5.3.43) in it, thereby obtaining

$$\begin{aligned}
&\|G_\varepsilon(\cdot,|Du|)\|_{L^{s_{k+1}}(B_{\tau_1})} \\
&\leq \frac{c}{(\tau_2-\tau_1)^\beta}\left[\|G_\varepsilon(\cdot,|Du|)\|_{L^{s_{k+1}}(B_{\tau_2})}^{\frac{\lambda_{k+1}\chi^k s_1}{s_{k+1}}}\|G_\varepsilon(\cdot,|Du|)\|_{L^1(B_{\tau_2})}^{\frac{(1-\lambda_{k+1})\chi^k s_1}{s_{k+1}}}+1\right] .
\end{aligned} \tag{5.3.47}$$

On the other hand, as $k > i_1$, then (5.3.45) holds with $i \equiv k$; we can therefore apply Young inequality with conjugate exponents $\left(\frac{s_{k+1}}{\lambda_{k+1}\chi^k s_1}, \frac{s_{k+1}}{s_{k+1}-\lambda_{k+1}\chi^k s_1}\right)$ in (5.3.47); this yields

$$\begin{aligned}
\|G_\varepsilon(\cdot,|Du|)\|_{L^{s_{k+1}}(B_{\tau_1})} \;\leq\; & \frac{1}{2}\|G_\varepsilon(\cdot,|Du|)\|_{L^{s_{k+1}}(B_{\tau_2})} \\
& + \frac{c}{(\tau_2-\tau_1)^{\beta_k}}\left[\|G_\varepsilon(\cdot,|Du|)\|_{L^1(B_\varrho)}^{\theta_k}+1\right],
\end{aligned}$$

where

$$\beta_k := \frac{s_{k+1}\beta}{s_{k+1}-\lambda_{k+1}\chi^k s_1} \qquad \text{and} \qquad \theta_k := \frac{(1-\lambda_{k+1})\chi^{k+1}s_1}{s_{k+1}-\lambda_{k+1}\chi^k s_1} \,. \tag{5.3.48}$$

Lemma 2.4.2 with the choice $\mathcal{Z}(s) \equiv \|G_\varepsilon(\cdot,|Du|)\|_{L^{s_{k+1}}(B_s)}$, finally renders that

$$\|G_\varepsilon(\cdot,|Du|)\|_{L^{s_{k+1}}(B_\varsigma)} \leq \frac{c}{(\varrho-\varsigma)^{\beta_k}}\left[\|G_\varepsilon(\cdot,|Du|)\|_{L^1(B_\varrho)}^{\theta_k}+1\right] , \tag{5.3.49}$$

with $c \equiv c(\texttt{data}_m, \|h\|_{L^d(\mathcal{B})})$. All in all, recalling $(5.2.15)_2$, (5.3.27) and that $p \leq s_{k+1}$, completes the proof of Proposition 5.3.1 with $\beta_p := \beta_{\kappa(p)}, \theta_p := \theta_{\kappa(p)}$. We remark that in (5.3.15) the exponents θ_p, β_p can be replaced by exponents $\tilde{\beta}, \tilde{\theta} \equiv \tilde{\beta}, \tilde{\theta}(n,d,\sigma,\hat{\sigma})$ that are independent of p. For this, observe that

$$\beta_k, \beta_p \to \frac{\beta}{1-\tilde{l}} =: \tilde{\beta} \,, \quad \lambda_k \to 1-\frac{1}{s_1} \,, \quad \theta_k, \theta_p \to \frac{\chi\tilde{l}}{(s_1-1)(1-\tilde{l})} := \tilde{\theta} \,. \tag{5.3.50}$$

The only dependence on p in (5.3.15) comes through the threshold radius R_*; it is $R_*(p) \to 0$ as $p \to \infty$ unless $f \equiv 0$.

5.3.4 A Lipschitz bound in the homogeneous case $f \equiv 0$.

The above reasoning, eventually culminating in Proposition 5.3.1, immediately leads to Lipschitz estimates when $f \equiv 0$. The result, when combined with the approximation of Section 5.4 below, extends those in [97, 186, 189, 190] to the case of non-autonomous functionals with superlinear growth as in (5.0.1). For this, the key observation is that it is not necessary to consider balls with small radii R_* as in (5.3.29), as the last term in (5.3.28) does not appear. Therefore we can take everywhere, and in particular in (5.3.15), an $R_* \leq 1$ independent of the value of p. It follows we can let $p \to \infty$ in (5.3.15), and recalling (5.3.50) we conclude with

Proposition 5.3.2 *Let $u \in W^{1,\gamma}(\mathcal{B},\mathbb{R}^N)$ be a solution to (5.3.1), under assumptions $\mathtt{set}_{\mathrm{m}}$ for $n \geq 2$ and with $f \equiv 0$. If $B_\varsigma \Subset B_\varrho$ are concentric balls contained in $\mathcal{B}$, such that $\varrho \leq 1$, then*

$$\|G_\varepsilon(\cdot, |Du|)\|_{L^\infty(B_\varsigma)} \leq \frac{c}{(\varrho-\varsigma)^{\tilde{\beta}}}\left[\|F_\varepsilon(\cdot, Du)\|_{L^1(B_\varrho)}^{\tilde{\theta}} + 1\right], \tag{5.3.51}$$

holds with $c \equiv c(\mathtt{data}_m, \|h\|_{L^d(\mathcal{B})}) \geq 1$, $\tilde{\beta}, \tilde{\theta} \equiv \tilde{\beta}, \tilde{\theta}(n, d, \sigma, \hat{\sigma}) > 0$.

In order to get Lipschitz estimates when $f \neq 0$, we shall use Proposition 5.3.1 as a preliminary ingredient. This will be done in the next Sections. We notice that Proposition 5.3.2 is stated for the case $n \geq 2$ whilst Proposition 5.3.1 refers to the case $n \geq 3$. The remaining two dimensional case can be obtained via minor modifications to the proof of Proposition 5.3.1, by choosing $2^*/2$ large enough (see (5.3.24)) in order to get $\chi > 1$ in (5.3.31). Anyway, the two dimensional case $n = 2$ will be treated in Section 5.3.7 directly for the general case $f \neq 0$. In that situation the proof cannot be readapted from the one of Proposition 5.3.1 as for Proposition 5.3.2.

5.3.5 Caccioppoli inequality on level sets

Before proceeding further, let us notice that, since the definitions in (5.2.10) yield that $\bar{G}_\varepsilon(x,t) \geq 1$ for all $(x,t) \in \Omega \times [0,\infty)$, therefore (5.1.18), and, as a direct consequence, (5.2.21), implies that:

$$\frac{g_{2,\varepsilon}(x,t)}{g_{1,\varepsilon}(x,t)} \leq c\bar{G}_\varepsilon(x,t)^{\bar{\sigma}} \qquad \bar{\sigma} := \min\{\sigma, \sigma'\}, \tag{5.3.52}$$

and this form better fits our scopes now. In (5.3.52), as in (5.2.21), $c \equiv c(n, N, \nu, \gamma, c_b)$. Now we are ready to prove the main result of this section.

Lemma 5.3.2 (Caccioppoli inequality) *Let $u \in W^{1,\gamma}(\mathcal{B},\mathbb{R}^N)$ be a solution to (5.3.1), under assumptions $\mathtt{set}_{\mathrm{m}}$ for $n \geq 2$. Let $B_r(x_0) \Subset B$ be a ball with $r \leq 1$. Then the inequality*

$$\begin{aligned}
&\int_{B_{r/2}(x_0)} |D(G_\varepsilon(x,|Du|) - \kappa)_+|^2 \, \mathrm{d}x \\
&\quad \leq \frac{c}{r^2}\|\bar{G}_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_r(x_0))}^{\vartheta\bar{\sigma}} \int_{B_r(x_0)} (G_\varepsilon(x,|Du|) - \kappa)_+^2 \, \mathrm{d}x \\
&\qquad + c\int_{B_r(x_0)} [h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\hat{\sigma}+\sigma)} \, \mathrm{d}x + c\|Du\|_{L^\infty(B_r(x_0))}^2 \int_{B_r(x_0)} |f|^2 \, \mathrm{d}x
\end{aligned} \tag{5.3.53}$$

holds whenever $\kappa \geq 0$, with $c \equiv c(\mathtt{data}_\infty)$ and ϑ as in (5.1.18).

Proof. For $s \in \{1, \ldots, n\}$, we take $\varphi_s := \eta^2(G_\varepsilon(x,|Du|) - \kappa)_+ D_s u$ in (5.3.16), where $\eta \in C^1_c(\mathcal{B})$ satisfies $\mathbb{1}_{B_{r/2}(x_0)} \leq \eta \leq \mathbb{1}_{B_r(x_0)}$ and $|D\eta| \lesssim 1/r$. Notice that all the integrals stemming from (5.3.16) extend over $\mathcal{B}^\kappa := \mathcal{B} \cap \{G_\varepsilon(\cdot,|Du|) > \kappa\}$; in particular, we can always restrict to the case it is $|Du| > T$. We start expanding the terms resulting from (5.3.16)

$$\begin{aligned}
&\sum_{s=1}^n \int_{\mathcal{B}^\kappa} \partial_z a_\varepsilon(x,Du)DD_s u \cdot D\varphi_s \, \mathrm{d}x \\
&\quad = \sum_{s=1}^n \int_{\mathcal{B}^\kappa} \eta^2(G_\varepsilon(x,|Du|) - \kappa)_+ \partial_z a_\varepsilon(x,Du)DD_s u \cdot DD_s u \, \mathrm{d}x \\
&\qquad + \sum_{s=1}^n \int_{\mathcal{B}^\kappa} \eta^2 \partial_z a_\varepsilon(x,Du)DD_s u \cdot \left(D_s u \otimes D(G_\varepsilon(x,|Du|) - \kappa)_+\right) \, \mathrm{d}x \\
&\qquad + 2\sum_{s=1}^n \int_{\mathcal{B}^\kappa} \eta(G_\varepsilon(x,|Du|) - \kappa)_+ \partial_z a_\varepsilon(x,|Du|)DD_s u \cdot D_s u \otimes D\eta \, \mathrm{d}x
\end{aligned}$$

$$=: (\mathrm{IV})_z + (\mathrm{V})_z + (\mathrm{VI})_z \ . \tag{5.3.54}$$

Moreover, it is

$$\begin{aligned}\sum_{s=1}^{n}\int_{\mathcal{B}^\kappa}\partial_{x_s}a_\varepsilon(x,Du)\cdot D\varphi_s \ \mathrm{d}x &= \sum_{s=1}^{n}\int_{\mathcal{B}^\kappa}\eta^2(G_\varepsilon(x,|Du|)-\kappa)_+\partial_{x_s}a_\varepsilon(x,Du)\cdot DD_s u \ \mathrm{d}x\\ &+\sum_{s=1}^{n}\int_{\mathcal{B}^\kappa}\eta^2\partial_{x_s}a_\varepsilon(x,Du)\cdot\big(D_s u\otimes D(G_\varepsilon(x,|Du|)-\kappa)_+\big) \ \mathrm{d}x\\ &+2\sum_{s=1}^{n}\int_{\mathcal{B}^\kappa}\eta(G_\varepsilon(x,|Du|)-\kappa)_+\partial_{x_s}a_\varepsilon(x,Du)\cdot(D_s u\otimes D\eta) \ \mathrm{d}x\\ &=: (\mathrm{IV})_x + (\mathrm{V})_x + (\mathrm{VI})_x \ .\end{aligned} \tag{5.3.55}$$

By (5.2.12)$_2$ we have

$$(\mathrm{IV})_z \geq \int_{\mathcal{B}^\kappa}\eta^2(G_\varepsilon(x,|Du|)-\kappa)_+ g_{1,\varepsilon}(x,|Du|)|D^2u|^2 \ \mathrm{d}x =: \mathcal{S}_3 \tag{5.3.56}$$

and for later use we also define

$$\mathcal{S}_4 := \int_{\mathcal{B}^\kappa}\eta^2|D(G_\varepsilon(x,|Du|)-k)_+|^2 \ \mathrm{d}x \ . \tag{5.3.57}$$

We then consider two different cases.

Case 1: $1<\gamma<2$ in (5.1.4).

We proceed estimating terms $(\mathrm{V})_z$ and $(\mathrm{VI})_z$. The estimate for the term $(\mathrm{V})_z$ is similar to the one for $(\mathrm{II})_z$ in (5.3.20); indeed, again using (5.2.21),(5.2.23),(5.3.7) and (5.3.14), we have

$$\begin{aligned}(\mathrm{V})_z &\geq \mathcal{S}_4 - c\int_{\mathcal{B}^\kappa}\eta^2 h(x)[\bar{G}_\varepsilon(x,s)]^{1+\sigma+\hat{\sigma}}|D(G_\varepsilon(x,|Du|)-k)_+| \ \mathrm{d}x\\ &\geq \frac{1}{2}\mathcal{S}_4 - c\int_{B_r(x_0)}[h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\sigma+\hat{\sigma})} \ \mathrm{d}x \ ,\end{aligned} \tag{5.3.58}$$

with $c\equiv c(n,N,\nu,\gamma,c_b)$. In turn, as in (5.3.22), we then have

$$\begin{aligned}(\mathrm{VI})_z &= 2\int_{\mathcal{B}^\kappa}\eta(G_\varepsilon(x,|Du|)-\kappa)_+\frac{\bar{a}_\varepsilon(x,|Du|)}{g_{1,\varepsilon}(x,|Du|)}DG_\varepsilon(x,|Du|)\cdot D\eta \ \mathrm{d}x\\ &\quad+2\sum_{\alpha=1}^{N}\int_{\mathcal{B}^\kappa}\mathbb{1}_{\mathcal{D}}(|Du|)(G_\varepsilon(x,|Du|)-\kappa)_+\frac{\bar{a}'_\varepsilon(x,|Du|)|Du|}{g_{1,\varepsilon}(x,|Du|)|Du|^2}\\ &\qquad\qquad\cdot(DG_\varepsilon(x,|Du|)\cdot Du^\alpha)(Du^\alpha\cdot D\eta) \ \mathrm{d}x\\ &\quad-2\int_{\mathcal{B}^\kappa}\eta(G_\varepsilon(x,|Du|)-\kappa)_+\frac{\bar{a}_\varepsilon(x,|Du|)}{g_{1,\varepsilon}(x,|Du|)}\left(\int_T^{|Du|}\partial_x g_{1,\varepsilon}(x,t)t\,\mathrm{d}t\cdot D\eta\right) \ \mathrm{d}x\\ &\quad-2\sum_{\alpha=1}^{N}\int_{\mathcal{B}^\kappa}\mathbb{1}_{\mathcal{D}}(|Du|)(G_\varepsilon(x,|Du|)-\kappa)_+\frac{\bar{a}'_\varepsilon(x,|Du|)|Du|}{g_{1,\varepsilon}(x,|Du|)|Du|^2}\\ &\qquad\qquad\cdot\left(\int_T^{|Du|}\partial_x g_{1,\varepsilon}(x,t)t \ \mathrm{d}t\cdot Du^\alpha\right)(Du^\alpha\cdot D\eta) \ \mathrm{d}x \ .\end{aligned}$$

Using (5.2.21), (5.2.23), (5.3.7)-(5.3.8), (5.3.52) and Young inequality, we have

$$|(\mathrm{VI})_z| \leq \frac{1}{4}\mathcal{S}_4 + \frac{c}{r^2}\|\bar{G}_\varepsilon(\cdot,|Du|)\|^{2\bar{\sigma}}_{L^\infty(B_r(x_0))}\int_{B_r(x_0)}(G_\varepsilon(x,|Du|)-\kappa)_+^2 \ \mathrm{d}x$$

$$+ c\int_{B_r(x_0)}[h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\hat{\sigma})}\ \mathrm{d}x\ , \tag{5.3.59}$$

where $c \equiv c(n,N,\nu,\gamma,c_a,c_b)$. Gathering (5.3.54)-(5.3.59), and using them in (5.3.16), after some further elementary estimations we have

$$\begin{aligned} \mathcal{S}_3+\mathcal{S}_4 \quad \leq \quad & c|(\mathrm{IV})_x| + c|(\mathrm{V})_x| + c|(\mathrm{VI})_x| \\ &+\frac{c}{r^2}\|\bar{G}_\varepsilon(\cdot,|Du|)\|^{\vartheta\bar{\sigma}}_{L^\infty(B_r(x_0))}\int_{B_r(x_0)}(G_\varepsilon(x,|Du|)-\kappa)^2_+\ \mathrm{d}x \\ &+c\int_{B_r(x_0)}[h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\sigma+\hat{\sigma})}\ \mathrm{d}x + c\sum_{s=1}^n\int_{\mathcal{B}^\kappa}|f\cdot D_s\varphi_s|\ \mathrm{d}x, \end{aligned} \tag{5.3.60}$$

where $\vartheta = 2$ and $c \equiv c(n,N,\nu,\gamma,c_a,c_b)$.

Case 2: $\gamma \geq 2$ in (5.1.4).

In this case it is $\vartheta = 1$ in (5.2.21) and moreover the function $t \mapsto \bar{a}_\varepsilon(\cdot,t)$ is non-decreasing, so that $\bar{a}'_\varepsilon(\cdot)$ is non-negative (when it exists). We notice that

$$\begin{aligned} (\mathrm{V})_z + (\mathrm{VI})_z \overset{(5.3.12)}{=} & \sum_{s=1}^n\int_{\mathcal{B}^\kappa}\eta^2 g_{1,\varepsilon}(x,|Du|)|Du|\partial_z a_\varepsilon(x,Du)DD_su\cdot D_su\otimes D|Du|\ \mathrm{d}x \\ &+2\sum_{s=1}^n\int_{\mathcal{B}^\kappa}\eta(G_\varepsilon(x,|Du|)-k)_+\partial_z a_\varepsilon(x,Du)DD_su\cdot D_su\otimes D\eta\ \mathrm{d}x \\ &+\sum_{s=1}^n\int_{\mathcal{B}^\kappa}\eta^2\partial_z a_\varepsilon(x,Du)DD_su\cdot D_su\otimes\int_T^{|Du|}\partial_x g_{1,\varepsilon}(x,t)t\ \mathrm{d}t\,\mathrm{d}x \\ =: & (\mathrm{V})_{z,1} + (\mathrm{VI})_z + (\mathrm{V})_{z,2}\ . \end{aligned} \tag{5.3.61}$$

In turn we have

$$\begin{aligned} (\mathrm{V})_{z,1} \overset{(5.3.10)}{=} & \int_{\mathcal{B}^\kappa}\eta^2 g_{1,\varepsilon}(x,|Du|)\bar{a}_\varepsilon(x,|Du|)|D|Du||^2|Du|^2\ \mathrm{d}x \\ &+\sum_{\alpha=1}^N\int_{\mathcal{B}^\kappa}\eta^2 g_{1,\varepsilon}(x,|Du|)\mathbb{1}_{\mathcal{D}}(|Du|)\bar{a}'_\varepsilon(x,|Du|)|Du|(D|Du|\cdot Du^\alpha)^2\ \mathrm{d}x \\ \overset{(5.3.5),\bar{a}'_\varepsilon(\cdot)\geq 0}{\geq} & \int_{\mathcal{B}^\kappa}\eta^2[g_{1,\varepsilon}(x,|Du|)|Du|]^2|D|Du||^2\ \mathrm{d}x\ . \end{aligned} \tag{5.3.62}$$

For $(\mathrm{VI})_z$, we again use (5.3.10), (5.3.52) and the fact that $\bar{a}'_\varepsilon(\cdot) \geq 0$, as follows:

$$\begin{aligned} |(\mathrm{VI})_z| \leq\ & 2\int_{\mathcal{B}^\kappa}\eta(G_\varepsilon(x,|Du|)-k)_+\bar{a}_\varepsilon(x,|Du|)|Du||D|Du|\cdot D\eta|\ \mathrm{d}x \\ &+2\sum_{\alpha=1}^N\int_{\mathcal{B}^\kappa}\eta(G_\varepsilon(x,|Du|)-k)_+\mathbb{1}_{\mathcal{D}}(|Du|)\bar{a}'_\varepsilon(x,|Du|)|(D|Du|\cdot Du^\alpha)(Du^\alpha\cdot D\eta)|\ \mathrm{d}x \\ \leq\ & \frac{1}{4}(\mathrm{V})_{z,1} + c\int_{\mathcal{B}^\kappa}\frac{(G_\varepsilon(x,|Du|)-k)^2_+}{g_{1,\varepsilon}(x,|Du|)}\bar{a}_\varepsilon(x,|Du|)|D\eta|^2\ \mathrm{d}x \\ &+c\sum_{\alpha=1}^N\int_{\mathcal{B}^\kappa}\frac{(G_\varepsilon(x,|Du|)-k)^2_+}{g_{1,\varepsilon}(x,|Du|)}\mathbb{1}_{\mathcal{D}}(|Du|)\bar{a}'_\varepsilon(x,|Du|)\frac{(Du^\alpha\cdot D\eta)^2}{|Du|}\ \mathrm{d}x \\ \leq\ & \frac{1}{4}(\mathrm{V})_{z,1} + \frac{c}{r^2}\|\bar{G}_\varepsilon(\cdot,|Du|)\|^{\bar{\sigma}}_{L^\infty(B_r(x_0))}\int_{B_r(x_0)}(G_\varepsilon(x,|Du|)-\kappa)^2_+\ \mathrm{d}x\ . \end{aligned} \tag{5.3.63}$$

As for $(\mathrm{V})_{z,2}$, by letting

$$\mathcal{G} := \int_T^{|Du|} \partial_x g_{1,\varepsilon}(x,t)t \ \mathrm{d}t \Longrightarrow |\mathcal{G}| \overset{(5.2.23)}{\leq} h(x)[\bar{G}_\varepsilon(x,|Du|)]^{1+\hat{\sigma}} \tag{5.3.64}$$

we have

$$\begin{aligned}
|(\mathrm{V})_{z,2}| \quad & \overset{(5.3.10)}{\leq} \quad 2\int_{\mathcal{B}^\kappa} \eta\bar{a}_\varepsilon(x,|Du|)|Du| \left|D|Du|\cdot\mathcal{G}\right| \ \mathrm{d}x \\
& \qquad +2\sum_{\alpha=1}^N \int_{\mathcal{B}^\kappa} \eta \mathbb{1}_{\mathcal{D}}(|Du|)\bar{a}'_\varepsilon(x,|Du|)|(D|Du|\cdot Du^\alpha)(Du^\alpha\cdot\mathcal{G})| \ \mathrm{d}x \\
& \leq \quad \frac{1}{4}(\mathrm{V})_{z,1} + c\int_{B_r(x_0)} \frac{\bar{a}_\varepsilon(x,|Du|)+\bar{a}'_\varepsilon(x,|Du|)|Du|}{g_{1,\varepsilon}(x,|Du|)}|\mathcal{G}|^2 \ \mathrm{d}x \\
& \overset{(5.3.6)_2}{\leq} \quad \frac{1}{4}(\mathrm{V})_{z,1} + c\int_{B_r(x_0)} \frac{g_{2,\varepsilon}(x,|Du|)}{g_{1,\varepsilon}(x,|Du|)}|\mathcal{G}|^2 \ \mathrm{d}x \\
& \overset{(5.2.21),(5.3.64)}{\leq} \quad \frac{1}{4}(\mathrm{V})_{z,1} + c\int_{B_r(x_0)} [h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{\sigma+2(1+\hat{\sigma})} \ \mathrm{d}x\,.
\end{aligned} \tag{5.3.65}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{S}_4 \quad & \overset{(5.3.12)}{\leq} \quad c\int_{\mathcal{B}^\kappa} \eta^2\left([g_{1,\varepsilon}(x,|Du|)|Du|]^2|D|Du||^2+|\mathcal{G}|^2\right) \mathrm{d}x \\
& \overset{(5.3.64)}{\leq} \quad c\int_{\mathcal{B}^\kappa} \eta^2\left([g_{1,\varepsilon}(x,|Du|)|Du|]^2|D|Du||^2+|h(x)|^2[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\hat{\sigma})}\right) \mathrm{d}x \\
& \overset{(5.3.62)}{\leq} \quad c(\mathrm{V})_{z,1} + c\int_{B_r(x_0)} [h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{\sigma+2(1+\hat{\sigma})} \, \mathrm{d}x\,.
\end{aligned} \tag{5.3.66}$$

Assembling the content of displays (5.3.61)-(5.3.66), we again conclude with (5.3.60), but this time with $\vartheta = 1$. We proceed estimating the x-terms coming from (5.3.60). By $(5.2.12)_3$, (5.2.30), and Young inequality, we get

$$\begin{aligned}
& |(\mathrm{IV})_x| + |(\mathrm{V})_x| \leq \bar{\varepsilon}\mathcal{S}_3 + \bar{\varepsilon}\mathcal{S}_4 + \frac{c}{\bar{\varepsilon}}\int_{B_r(x_0)} \eta^2[h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\sigma)} \, \mathrm{d}x\,, \\
& |(\mathrm{VI})_x| \leq \frac{c}{r^2}\int_{B_r(x_0)} (G_\varepsilon(x,|Du|)-\kappa)_+^2 \, \mathrm{d}x + c\int_{B_r(x_0)} [h(x)]^2[\bar{G}_\varepsilon(x,|Du|)]^{2(1+\sigma)} \, \mathrm{d}x\,,
\end{aligned}$$

with $c \equiv c(\mathtt{data}_\infty)$ and arbitrary $\bar{\varepsilon} \in (0,1)$. Finally, the estimate of the terms involving f can be done by using Young's inequality and (5.2.28) as follows:

$$\begin{aligned}
\sum_{s=1}^n \int_{\mathcal{B}} |f\cdot D_s\varphi_s| \, \mathrm{d}x & \leq \bar{\varepsilon}\mathcal{S}_3 + \bar{\varepsilon}\mathcal{S}_4 + \frac{c}{\bar{\varepsilon}}\int_{\mathcal{B}} |D\eta|^2(G_\varepsilon(x,|Du|)-\kappa)_+^2 \, \mathrm{d}x \\
& \quad +\frac{c}{\bar{\varepsilon}}\int_{\mathcal{B}} \eta^2\left[\frac{(G_\varepsilon(x,|Du|)-\kappa)_+}{g_{1,\varepsilon}(x,|Du|)}+|Du|^2\right]|f|^2 \, \mathrm{d}x \\
& \leq \bar{\varepsilon}\mathcal{S}_3 + \bar{\varepsilon}\mathcal{S}_4 + \frac{c}{\bar{\varepsilon}}\int_{\mathcal{B}} |D\eta|^2(G_\varepsilon(x,|Du|)-\kappa)_+^2 \, \mathrm{d}x \\
& \quad +\bar{\varepsilon}^{-1}c\|Du\|^2_{L^\infty(B_{r_0}(x_0))}\int_{\mathcal{B}} \eta^2|f|^2 \, \mathrm{d}x\,,
\end{aligned} \tag{5.3.67}$$

for $c \equiv c(\mathtt{data}_n)$ and arbitrary $\bar{\varepsilon} \in (0,1)$. Collecting the estimates in the last three displays to (5.3.60), recalling that $\bar{G}_\varepsilon(\cdot) \geq 1$, and selecting $\bar{\varepsilon} > 0$ sufficiently small in order to reabsorb terms, we complete the proof of Lemma 5.3.2. □

5.3.6 Nonlinear iterations

In this section we finally derive pointwise gradient bounds. This goes via Lemma 5.3.3 and Proposition 5.3.3 below.

Lemma 5.3.3 *Let $u \in W^{1,\gamma}(\mathcal{B}, \mathbb{R}^N)$ be a solution to (5.3.1), under assumptions $\texttt{set}_\infty$ for $n \geq 3$ and let $B_{2r_0}(x_0) \Subset B$ be a ball with $r_0 \leq 1$. If x_0 is a Lebesgue point of both $|Du|$ and $h(\cdot)$, then*

$$G_\varepsilon(x_0, |Du(x_0)|) \leq \kappa + c\|\bar{G}_\varepsilon(\cdot, |Du|)\|_{L^\infty(B_r(x_0))}^{\frac{n\vartheta\bar{\sigma}}{4}} \left(\fint_{B_{r_0}(x_0)} (G_\varepsilon(x, |Du|) - \kappa)_+^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$
$$+ c\|\bar{G}_\varepsilon(\cdot, |Du|)\|_{L^\infty(B_r(x_0))}^{\frac{(n-2)\vartheta\bar{\sigma}}{4}} \left[\mathbf{P}_1^{\mathfrak{h}}(x_0, 2r_0) + \|Du\|_{L^\infty(B_{r_0}(x_0))} \mathbf{P}_1^f(x_0, 2r_0) \right] , \quad (5.3.68)$$

holds for all $\kappa \geq 0$, with $c \equiv c(\texttt{data}_m)$, where ϑ is as in (5.1.18), and

$$\mathfrak{h}(x) := h(x)[\bar{G}_\varepsilon(x, |Du|)]^{(1+\sigma+\hat{\sigma})} . \quad (5.3.69)$$

Proof. Notice that we can assume that $|Du(x_0)| > T$, otherwise (5.3.68) is trivial. Let us first notice that x_0 is also a Lebesgue point of $x \mapsto G_\varepsilon(x, |Du(x)|)$ and it is

$$\lim_{r \to 0} \fint_{B_r(x_0)} G_\varepsilon(x, |Du(x)|) \, \mathrm{d}x = G_\varepsilon(x_0, |Du(x_0)|) , \quad (5.3.70)$$

i.e., the right-hand side denotes the precise representative of $G_\varepsilon(\cdot, |Du(\cdot)|)$ at the point x_0. Indeed, notice that

$$\begin{aligned}
&\lim_{r \to 0} \fint_{B_r(x_0)} |G_\varepsilon(x, |Du(x)|) - G_\varepsilon(x_0, |Du(x_0)|)| \, \mathrm{d}x \\
&\quad \leq \lim_{r \to 0} \fint_{B_r(x_0)} |G_\varepsilon(x, |Du(x)|) - G_\varepsilon(x_0, |Du(x)|)| \, \mathrm{d}x \\
&\qquad + \lim_{r \to 0} \fint_{B_r(x_0)} |G_\varepsilon(x_0, |Du(x)|) - G_\varepsilon(x_0, |Du(x_0)|)| \, \mathrm{d}x = \mathcal{C}_1(r) + \mathcal{C}_2(r) . \quad (5.3.71)
\end{aligned}$$

As x_0 is a Lebesgue point for Du, $t \mapsto G_\varepsilon(x_0, t)$ is locally Lipschitz-regular, and Du is locally bounded, we have

$$\lim_{r \to 0} \mathcal{C}_2(r) \lesssim \lim_{r \to 0} \fint_{B_r(x_0)} ||Du(x)| - |Du(x_0)|| \, \mathrm{d}x = 0 . \quad (5.3.72)$$

As for the term $C_1(\cdot)$, we have

$$\begin{aligned}
\mathcal{C}_1(r) \quad &\leq \quad \fint_{B_r(x_0)} \int_T^{\max\{|Du(x)|, T\}} |g_{1,\varepsilon}(x, s) - g_{1,\varepsilon}(x_0, s)|s \, \mathrm{d}s \, \mathrm{d}x \\
&\leq \quad \fint_{B_r(x_0)} \int_T^{\|Du\|_{L^\infty(B_r(x_0))}} |g_{1,\varepsilon}(x, s) - g_{1,\varepsilon}(x_0, s)|s \, \mathrm{d}s \, \mathrm{d}x \\
&\leq \quad \int_T^{\|Du\|_{L^\infty(B_r(x_0))}} \fint_{B_r(x_0)} |g_{1,\varepsilon}(x, s) - g_{1,\varepsilon}(x_0, s)|s \, \mathrm{d}x \, \mathrm{d}s \\
&\leq \quad \int_T^{\|Du\|_{L^\infty(B_r(x_0))}} \fint_{B_r(x_0)} \left(\int_0^1 |\partial_x g_{1,\varepsilon}(x_0 + \lambda(x - x_0), t) \cdot (x - x_0)|s \, \mathrm{d}\lambda \right) \mathrm{d}x \, \mathrm{d}s \\
&\overset{(5.2.23)}{\leq} \quad cr \fint_{B_r(x_0)} \left(\int_0^1 |h(x_0 + \lambda(x - x_0))| \, \mathrm{d}\lambda \right) \mathrm{d}x \leq cr \fint_{B_{2r}(x_0)} |h| \, \mathrm{d}x ,
\end{aligned}$$

where $c \equiv c(\|Du\|_{L^\infty(B_r(x_0))})$. Therefore, being x_0 a Lebesgue point of $h(\cdot)$, we infer $\mathcal{C}_1(r) \to 0$ as $r \to 0$. This fact, together with (5.3.71)-(5.3.72), yields (5.3.70). Thanks to (5.3.53) we can verify (2.3.4) with the choices $v(\cdot) \equiv G_\varepsilon(\cdot, |Du(\cdot)|)$, $f_1 \equiv \mathfrak{h}$, $f_2 \equiv f$, $M_1 \equiv \|\bar{G}_\varepsilon(\cdot, |Du|)\|^{\vartheta\sigma/2}_{L^\infty(B_r(x_0))}$, $M_2 \equiv 1$ and $M_3 \equiv \|Du\|_{L^\infty(B_{r_0}(x_0))}$. Applying Lemma 2.3.1 yields (5.3.68). □

Proposition 5.3.3 *Let $u \in W^{1,\gamma}(\mathcal{B}, \mathbb{R}^N)$ be a solution to (5.3.1), under assumptions $\texttt{set}_\infty$ for $n \geq 3$. There exists a positive radius $R_* \equiv R_*(\texttt{data}_m, \mathfrak{f}(\cdot), p) \leq 1$ such that if $\mathfrak{r}(\mathcal{B}) \leq R_*$ and $B_\varsigma \Subset B_\varrho$ are concentric balls contained in $\mathcal{B}$, then*

$$
\begin{aligned}
E_\varepsilon(\|Du\|_{L^\infty(B_\varsigma)})^\gamma + \left\| \int_T^{|Du|} g_{1,\varepsilon}(\cdot, s)s \ \mathrm{d}s \right\|_{L^\infty(B_\varsigma)} \\
\leq \frac{c}{(\varrho - \varsigma)^\beta} \left[\|F_\varepsilon(\cdot, Du)\|_{L^1(B_\varrho)} + \|f\|_{L(n,1)(B_\varrho)} + 1 \right]^\theta
\end{aligned} \tag{5.3.73}
$$

holds with $c \equiv c(\texttt{data}_\infty, \|h\|_{L^d(\mathcal{B})}) \geq 1$ and $\beta, \theta \equiv \beta, \theta(n, d, \gamma, \sigma, \hat{\sigma}) > 0$.

Proof. Notice that it is $\texttt{set}_{\mathrm{m}} \subset \texttt{set}_\infty$, so that the result of Proposition 5.3.1 can be used here. Therefore we take exponents τ and p such that

$$
0 < \tau < d - n \quad \text{and} \quad p := \frac{(1 + \sigma + \hat{\sigma})(n + \tau)d}{d - n - \tau}, \tag{5.3.74}
$$

where $d > n$ is the exponent from (5.1.14), and fix $R_* \equiv R_*(\texttt{data}_m, \mathfrak{f}(\cdot), p) > 0$ as the radius from Proposition 5.3.1, so that (5.3.15) holds such p. With $B_\varsigma \Subset B_\varrho$ being the balls considered in the statement, with no loss of generality we can assume that $\|Du\|_{L^\infty(B_\varsigma)} \geq T$ otherwise the assertion in (5.3.73) is trivial by choosing c large enough. Let $\varrho_1 := \varsigma + (\varrho - \varsigma)/2$ and consider concentric balls $B_\varsigma \Subset B_{\tau_1} \Subset B_{\tau_2} \Subset B_{\varrho_1} \Subset B_\varrho$, a point $x_0 \in B_{\tau_1}$ which is a Lebesgue point both for $|Du|$ and for $h(\cdot)$, and $r_0 := (\tau_2 - \tau_1)/8$, so that $B_{2r_0}(x_0) \Subset B_{\tau_2}$. Needless to say, a.e. point in B_{τ_1} qualifies. In the following, whenever $B \subset B_\varrho$ is a ball, we let

$$
\mathbf{M}(B) := \|Du\|_{L^\infty(B)} \quad \text{and} \quad \mathbf{G}(B) := \|G_\varepsilon(\cdot, |Du|)\|_{L^\infty(B)} \tag{5.3.75}
$$

so that, recalling that $B_{r_0}(x_0) \subset B_\varrho$, by (5.2.21) and $(5.2.10)_2$ we find

$$
\|\bar{G}_\varepsilon(\cdot, |Du|)\|_{L^\infty(B_{r_0}(x_0))} \leq c(\gamma, c_b, T, \sigma, \sigma') \left[\mathbf{G}(B_{\tau_2}) + 1 \right]. \tag{5.3.76}
$$

Moreover, by $(5.2.10)_1$ and (5.2.17) we see that almost everywhere in $B_{r_0(x_0)}$ there holds that

$$
|Du| \leq \left[E_\varepsilon(|Du|)^\gamma - E_\varepsilon(T)^\gamma \right]^{\frac{1}{\gamma}} + E_\varepsilon(T) \leq c(\nu, \gamma, T) \left[G_\varepsilon(x, |Du|)^{\frac{1}{\gamma}} + 1 \right],
$$

thus

$$
\mathbf{M}(B_{r_0}(x_0)) \leq c \left[\mathbf{G}(B_{r_0}(x_0))^{\frac{1}{\gamma}} + 1 \right] \leq c \left[\mathbf{G}(B_{\tau_2})^{\frac{1}{\gamma}} + 1 \right], \tag{5.3.77}
$$

for $c \equiv c(\nu, \gamma, T)$. We then apply (5.3.68) on $B_{r_0}(x_0)$, and also using (5.3.76)-(5.3.77) and Hölder inequality (by (5.3.74) it is $p \geq 2$), we obtain

$$
\begin{aligned}
G_\varepsilon(x_0, |Du(x_0)|) &\leq c r_0^{-n/p} \|\bar{G}_\varepsilon(\cdot, |Du|)\|^{\frac{n\vartheta\bar{\sigma}}{4}}_{L^\infty(B_{r_0}(x_0))} \|G_\varepsilon(\cdot, |Du|)\|_{L^p(B_{r_0}(x_0))} \\
&\quad + c \|\bar{G}_\varepsilon(\cdot, |Du|)\|^{\frac{(n-2)\vartheta\bar{\sigma}}{4}}_{L^\infty(B_{r_0}(x_0))} \left[\mathbf{P}_1^{\mathfrak{h}}(x_0, 2r_0) + \mathbf{M}(B_{r_0}(x_0)) \mathbf{P}_1^f(x_0, 2r_0) \right] \\
&\leq c r_0^{-n/p} \left[\mathbf{G}(B_{\tau_2})^{\frac{n\vartheta\bar{\sigma}}{4}} + 1 \right] \|G_\varepsilon(\cdot, |Du|)\|_{L^p(B_{r_0}(x_0))}
\end{aligned}
$$

$$+ c\left[\mathbf{G}(B_{\tau_2})^{\frac{(n-2)\vartheta\bar\sigma}{4}}+1\right]\left[\mathbf{P}_1^{\mathfrak{h}}(x_0,2r_0)+\mathbf{M}(B_{r_0}(x_0))\mathbf{P}_1^{f}(x_0,2r_0)\right], \tag{5.3.78}$$

with $c \equiv c(\mathtt{data}_\infty)$. With (5.3.15) we further bound

$$\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_{r_0}(x_0))} \le c r_0^{-\beta_p}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)}+1\right]^{\theta_p},$$

for $c \equiv c(\mathtt{data}_m, \|h\|_{L^d(B_\varrho)})$. Using also (5.3.77) in (5.3.78), recalling that $x_0 \in B_{\tau_1}$ is arbitrary, and recovering a full notation, we obtain

$$\begin{aligned}
\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_1})} \le & \frac{c}{(\tau_2-\tau_1)^{n/p+\beta_p}}\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_2})}^{\frac{n\vartheta\bar\sigma}{4}}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)}+1\right]^{\theta_p}\\
& + c\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_2})}^{\frac{(n-2)\vartheta\bar\sigma}{4}}\left\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}\\
& + c\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_2})}^{\frac{(n-2)\vartheta\bar\sigma}{4}+\frac{1}{\gamma}}\left\|\mathbf{P}_1^{f}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}\\
& + \frac{c}{(\tau_2-\tau_1)^{n/p+\beta_p}}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)}+1\right]^{\theta_p}\\
& + c\left[\left\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}+\left\|\mathbf{P}_1^{f}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}\right],
\end{aligned} \tag{5.3.79}$$

where $c \equiv c(\mathtt{data}_\infty, \|h\|_{L^d(B_\varrho)})$. Observe now that (5.1.14), (5.1.17) and the definition in (5.3.52) render $n\bar\sigma\vartheta/4 < 1$, and $(n-2)\bar\sigma\vartheta/4 < 1-\gamma^{-1}$, therefore we can apply Young inequality in the first three lines of (5.3.79) with conjugate exponents

$$\left(\frac{4}{\bar\sigma n\vartheta},\frac{4}{4-n\bar\sigma\vartheta}\right), \quad \left(\frac{4}{(n-2)\bar\sigma\vartheta},\frac{4}{4-(n-2)\bar\sigma\vartheta}\right),$$
$$\left(\frac{4\gamma}{\vartheta\bar\sigma(n-2)\gamma+4},\frac{4\gamma}{4(\gamma-1)-\gamma(n-2)\vartheta\bar\sigma}\right)$$

to end up with

$$\begin{aligned}
\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_1})} \le & \frac{1}{2}\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_2})}+\frac{c}{(\tau_2-\tau_1)^{\beta'}}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)}+1\right]^{\theta'}\\
& + c\left[\left\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}+\left\|\mathbf{P}_1^{f}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}+1\right]^{\theta'},
\end{aligned} \tag{5.3.80}$$

where $c \equiv c(\mathtt{data}_\infty, \|h\|_{L^d(B_\varrho)})$ and we set

$$\begin{aligned}
\beta' &:= \left(\frac{n}{p}+\beta_p\right)\frac{4}{4-n\bar\sigma\vartheta}\\
\theta' &:= \max\left\{\frac{4\theta_p}{4-n\vartheta\bar\sigma},\frac{4}{4-(n-2)\vartheta\bar\sigma},\frac{4\gamma}{4(\gamma-1)-\gamma(n-2)\vartheta\bar\sigma}\right\}.
\end{aligned}$$

Inequality (5.3.80) allows to apply Lemma 2.4.2 with the choice $\mathcal{Z}(t) := \|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_t)}$, and this leads to

$$\begin{aligned}
\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_\varsigma)} \le & \frac{1}{2}\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_\varrho)}+\frac{c}{(\varrho-\varsigma)^{\beta'}}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)}+1\right]^{\theta'}\\
& + c\left[\left\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\varrho-\varsigma)/4)\right\|_{L^\infty(B_{\varrho_1})}+\left\|\mathbf{P}_1^{f}(\cdot,(\varrho-\varsigma)/4)\right\|_{L^\infty(B_{\varrho_1})}+1\right]^{\theta'},
\end{aligned} \tag{5.3.81}$$

for $c \equiv c(\mathtt{data}_\infty, \|h\|_{L^d(B_\varrho)})$. By using (2.3.3)$_1$ we infer

$$\|\mathbf{P}_1^f(\cdot,(\varrho-\varsigma)/4)\|_{L^\infty(B_{\varrho_1})} \le c\|f\|_{L(n,1)(B_\varrho)} .$$

Moreover, with $\varrho_2 := \varrho_1 + (\varrho-\varsigma)/4$; also using (5.3.74) and Hölder inequality yields

$$\begin{aligned}\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\varrho-\varsigma)/4)\|_{L^\infty(B_{\varrho_1})} &\overset{(2.3.3)_1}{\le} c\|\mathfrak{h}\|_{L(n,1)(B_{\varrho_2})} \le c(\tau)\|\mathfrak{h}\|_{L^{n+\tau}(B_{\varrho_2})}\\ &\overset{(5.3.69)}{\le} c\|h\|_{L^d(B_\varrho)}\|\bar{G}_\varepsilon(\cdot,|Du|)\|_{L^p(B_{\varrho_2})}^{(1+\sigma+\hat\sigma)}\\ &\overset{(5.3.15)}{\le} \frac{c\|h\|_{L^d(B_\varrho)}}{(\varrho-\varsigma)^{\beta_p(1+\sigma+\hat\sigma)}}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)}^{\theta_p(1+\sigma+\hat\sigma)}+1\right] ,\end{aligned}$$

where $c \equiv c(\mathtt{data}_\infty, \|h\|_{L^d(B_\varrho)})$. Inserting the above two estimates in (5.3.81) and recalling also (5.2.10) and (5.2.17) we finally end up with (5.3.73), where $\beta := \max\{\beta', \beta_p(1+\sigma+\hat\sigma)\delta'\}$ and $\theta := \max\{\theta', \theta_p(1+\sigma+\hat\sigma)\}$. Notice that the constant c also depends on a positive power of T, therefore it is stable as $T \to 0$. □

5.3.7 The case $n = 2$

Here we consider the missing two-dimensional case. We start with the following lemma, which is a hybrid counterpart of Proposition 5.3.1, in the sense that the a priori estimate involved still contains the L^∞-norm of Du.

Lemma 5.3.4 *Let $u \in W^{1,\gamma}(\mathcal{B},\mathbb{R}^N)$ be a solution to (5.3.1) under assumptions $\mathtt{set}_{two}$ for $n = 2$ and $B_\varsigma \Subset B_\varrho$ be concentric balls contained in $\mathcal{B}$ with $\varrho \le 1$. Then, for every $p \ge 1$, there holds*

$$\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_\varsigma)} \le \frac{c}{(\varrho-\varsigma)^{\beta_p}}\left[\|G_\varepsilon(\cdot,|Du|)\|_{L^1(B_\varrho)}^{\theta_p}+1\right] + c\|Du\|_{L^\infty(B_\varrho)}\|f\|_{L^2(B_\varrho)} \quad (5.3.82)$$

with $c \equiv c(\mathtt{data}_{\rm two}, \|h\|_{L^d(\mathcal{B})}, p) \ge 1$, $\beta_p, \theta_p \equiv \beta_p, \theta_p(d,\sigma,\hat\sigma,p) > 0$.

Proof. It is clear that we can confine ourselves to prove (5.3.82) for sufficiently large p. Therefore, for reasons that will be clear in a few lines, we consider

$$p > \max\left\{2m(1+\sigma+\hat\sigma), \frac{2m}{1-2m(\sigma+\hat\sigma)}\right\} = \frac{2m}{1-2m(\sigma+\hat\sigma)} , \quad m = \frac{d}{d-2}. \quad (5.3.83)$$

As usual, in the following lines all the balls will be concentric. We look back at the proof of Proposition 5.3.1, take $\kappa = 0$ in the test function φ_s, and perform exactly the same calculations made there up to (5.3.23). For the terms $(\mathrm{I})_f$-$(\mathrm{III})_f$ involving the right-hand side f, we notice that as $\kappa = 0$ the test functions φ_s used in the proof of Propositions 5.3.1 and Lemma 5.3.2 do coincide. Therefore we can use estimate (5.3.67), where $c \equiv c(\mathtt{data}_{\rm two})$ and $\bar\varepsilon \in (0,1)$; here $\mathcal{S}_3$ and $\mathcal{S}_4$ have been defined in (5.3.56) and (5.3.57), respectively (again $\kappa = 0$). All together, choosing $\bar\varepsilon > 0$ small enough and re-absorbing terms in a standard way, we obtain

$$\begin{aligned}\mathcal{S}_3 + \mathcal{S}_4 &\le c\left(\|h\|_{L^d(\mathcal{B})}^2+1\right)\left(\int_{\mathcal{B}}\left(\eta^{2m}+|D\eta|^{2m}\right)\left[[G_\varepsilon(x,|Du|)]^{2m(1+\sigma+\hat\sigma)}+1\right]\,\mathrm{d}x\right)^{1/m}\\ &\quad + c\mathbf{M}(B_\varrho)^2\int_{\mathcal{B}}\eta^2|f|^2\,\mathrm{d}x , \end{aligned} \quad (5.3.84)$$

where $c \equiv c(\mathtt{data}_{\rm two})$ and $\mathbf{M}(\cdot)$ has been defined in $(5.3.75)_1$. As it is $|D\eta| \lesssim 1/(\tau_2-\tau_1)$, elementary manipulations on (5.3.84) give

$$\|D(\eta G_\varepsilon(\cdot,|Du|))\|_{L^2(B_{\tau_2})}^2 \le \frac{c(\|h\|_{L^d(\mathcal{B})}+1)^2}{(\tau_2-\tau_1)^2}\|G_\varepsilon(\cdot,|Du|)\|_{L^{2m(1+\sigma+\hat\sigma)}(B_{\tau_2})}^{2(1+\sigma+\hat\sigma)}$$

$$+\frac{c(\|h\|_{L^d(\mathcal{B})}+1)^2}{(\tau_2-\tau_1)^2}+c\mathbf{M}(B_\varrho)^2\int_{B_{\tau_2}}|f|^2\,\mathrm{d}x$$

so that Sobolev embedding gives

$$\begin{aligned}\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_{\tau_1})} &\leq c_p\tau_1^{2/p}\|D(\eta G_\varepsilon(\cdot,|Du|))\|_{L^2(B_{\tau_2})}\\ &\leq \frac{c}{(\tau_2-\tau_1)}\|G_\varepsilon(\cdot,|Du|)\|^{(1+\sigma+\hat{\sigma})}_{L^{2m(1+\sigma+\hat{\sigma})}(B_{\tau_2})}+\frac{c}{(\tau_2-\tau_1)}+c\mathbf{M}(\varrho)\|f\|_{L^2(B_{\tau_2})}\,, \end{aligned}\tag{5.3.85}$$

with $c\equiv c(\mathtt{data}_{\rm two},\|h\|_{L^d(\mathcal{B})})$. Notice that $1-2m(\sigma+\hat{\sigma})>0$ is equivalent to the last inequality in (5.1.14) for $n=2$. With $\lambda_p\in(0,1)$ being defined through

$$\frac{1}{2m(1+\sigma+\hat{\sigma})}=1-\lambda_p+\frac{\lambda_p}{p}\Rightarrow\lambda_p=\frac{p[2m(1+\sigma+\hat{\sigma})-1]}{2m(p-1)(1+\sigma+\hat{\sigma})}\,,$$

using the interpolation inequality

$$\|G_\varepsilon(\cdot,|Du|)\|_{L^{2m(1+\sigma+\hat{\sigma})}(B_{\tau_2})}\leq\|G_\varepsilon(\cdot,|Du|)\|^{1-\lambda_p}_{L^1(B_{\tau_2})}\|G_\varepsilon(\cdot,|Du|)\|^{\lambda_p}_{L^p(B_{\tau_2})}\,,$$

in (5.3.85), we get

$$\begin{aligned}\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_{\tau_1})} &\leq \frac{c}{(\tau_2-\tau_1)}\|G_\varepsilon(\cdot,|Du|)\|^{(1-\lambda_p)(1+\sigma+\hat{\sigma})}_{L^1(B_{\tau_2})}\|G_\varepsilon(\cdot,|Du|)\|^{\lambda_p(1+\sigma+\hat{\sigma})}_{L^p(B_{\tau_2})}\\ &\quad+\frac{c}{(\tau_2-\tau_1)}+c\mathbf{M}(B_\varrho)\|f\|_{L^2(B_{\tau_2})}\,.\end{aligned}$$

The lower bound on p in (5.3.83) yields $\lambda_p(1+\sigma+\hat{\sigma})<1$, thus Young inequality gives

$$\begin{aligned}\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_{\tau_1})} &\leq \frac{1}{2}\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_{\tau_2})}\\ &\quad+\frac{c}{(\tau_2-\tau_1)^{\beta_p}}\left[\|G_\varepsilon(\cdot,|Du|)\|^{\theta_p}_{L^1(B_{\tau_2})}+1\right]+c\mathbf{M}(B_\varrho)\|f\|_{L^2(B_{\tau_2})}\,,\end{aligned}$$

with $c\equiv c(\mathtt{data}_{\rm two},\|h\|_{L^d(\mathcal{B})})$ and

$$\beta_p:=\frac{2m(p-1)}{p-2m[1+p(\sigma+\hat{\sigma})]}\quad\text{and}\quad\theta_p:=\frac{p-2m(1+\sigma+\hat{\sigma})}{p-2m[1+p(\sigma+\hat{\sigma})]}\,.\tag{5.3.86}$$

Lemma 2.4.2 with the choice $\mathcal{Z}(t)\equiv\|G_\varepsilon(\cdot,|Du|)\|_{L^p(B_t)}$ gives (5.3.82). □

We finally come to the a priori gradient bound in the two dimensional case.

Proposition 5.3.4 *Let $u\in W^{1,\gamma}(\mathcal{B},\mathbb{R}^N)$ be a solution to (5.3.1) under assumptions $\mathtt{set}_{two}$ for $n=2$. If $B_\varsigma\Subset B_\varrho$ are concentric balls contained in $\mathcal{B}$ with $\varrho\leq 1$, then*

$$\begin{aligned}E_\varepsilon(\|Du\|_{L^\infty(B_\varsigma)})^\gamma &+\left\|\int_T^{|Du|}g_{1,\varepsilon}(\cdot,s)s\ \mathrm{d}s\right\|_{L^\infty(B_\varsigma)}\\ &\leq\frac{c}{(\varrho-\varsigma)^\beta}\left[\|F_\varepsilon(\cdot,Du)\|_{L^1(B_\varrho)}+\|f\|_{L^2(\mathrm{LogL})^\alpha(B_\varrho)}+1\right]^\theta\,,\end{aligned}\tag{5.3.87}$$

with $c\equiv c(\mathtt{data}_{\rm two},\|h\|_{L^d(\mathcal{B})})\geq 1$ and $\beta,\theta\equiv\beta,\theta(d,\sigma,\hat{\sigma},\gamma)>0$.

Proof. We proceed as for the proof of Proposition 5.3.3, keeping the notation used there. In particular, we make the same choices done from display (5.3.74) to display (5.3.75), while, as a consequence of (5.1.19)-(5.1.20), estimate (5.3.76) holds with the limitations on σ imposed by

(5.1.19). Keeping this last fact in mind, thanks to Lemma 5.3.2 we use Lemma 2.3.1, that for $n=2$ gives

$$G_{\varepsilon}(x_0,|Du(x_0)|) \leq c\left[\mathbf{G}(B_{\tau_2})^{\frac{(1+\delta)\vartheta\sigma}{2}}+1\right]\left(\fint_{B_{r_0}(x_0)}[G_{\varepsilon}(x,|Du|)]^2\,\mathrm{d}x\right)^{1/2} + c\left[\mathbf{G}(B_{\tau_2})^{\frac{\delta\vartheta\sigma}{2}}+1\right]\left[\mathbf{P}_1^{\mathfrak{h}}(\cdot,2r_0)+\mathbf{M}(B_{\tau_2})\mathbf{P}_1^{f}(\cdot,2r_0)\right], \tag{5.3.88}$$

for every $\delta\in(0,1/2)$, where $c\equiv c(\mathtt{data}_{\mathrm{two}},\|h\|_{L^d(\mathcal{B})},\delta)$ and $\mathbf{M}(\cdot)$ and $\mathbf{G}(\cdot)$ have been defined in (5.3.75). With (5.3.82) we estimate

$$\left(\fint_{B_{r_0}(x_0)}[G_{\varepsilon}(x,|Du|)]^2\,\mathrm{d}x\right)^{1/2} \leq cr_0^{-2/p}\|G_{\varepsilon}(\cdot,|Du|)\|_{L^p(B_{r_0}(x_0))} \leq cr_0^{-(2/p+\beta_p)}\left[\|G_{\varepsilon}(\cdot,|Du|)\|_{L^1(B_{2r_0})}^{\theta_p}+1\right]+cr_0^{-2/p}\mathbf{M}(B_{\tau_2})\|f\|_{L^2(B_{2r_0})} \tag{5.3.89}$$

where p is as in (5.3.74) with $n=2$, $c\equiv c(\mathtt{data}_{\mathrm{two}},\|h\|_{L^d(\mathcal{B})})$ and β_p,θ_p are as in (5.3.86). Using (5.3.88)-(5.3.89), that hold for a.e. $x_0\in B_{\tau_1}$, we have

$$\begin{aligned}
\|G_{\varepsilon}(\cdot,|Du|)\|_{L^\infty(B_{\tau_1})} \leq\ & \frac{c}{(\tau_2-\tau_1)^{2/p}}\left[\mathbf{G}(B_{\tau_2})+1\right]^{(1+\delta)\vartheta\sigma/2}\mathbf{M}(B_{\tau_2})\|f\|_{L^2(B_{\varrho})}\\
&+\frac{c}{(\tau_2-\tau_1)^{2/p+\beta_p}}\left[\mathbf{G}(B_{\tau_2})+1\right]^{(1+\delta)\vartheta\sigma/2}\left[\|G_{\varepsilon}(\cdot,|Du|)\|_{L^1(B_{\varrho})}^{\theta_p}+1\right]\\
&+c\left[\mathbf{G}(B_{\tau_2})+1\right]^{\delta\vartheta\sigma/2}\mathbf{M}(B_{\tau_2})\left\|\mathbf{P}_1^{f}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}\\
&+c\left[\mathbf{G}(B_{\tau_2})+1\right]^{\delta\vartheta\sigma/2}\left\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}\\
&+c\left[\left\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}+\left\|\mathbf{P}_1^{f}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}\right]\\
&=: T_1+T_2+T_3+T_4+T_5\,,
\end{aligned} \tag{5.3.90}$$

for $c\equiv c(\mathtt{data}_n,\|h\|_{L^d(\mathcal{B})},\delta)$. To estimate the T-terms we take δ such that

$$(1+\delta)\sigma+1/\gamma<1 \quad\text{and}\quad \delta\sigma+(1+\sigma+\hat{\sigma})/\gamma<1 \tag{5.3.91}$$

hold, which is in turn possible by (5.1.19); this fixes δ as a function of σ,γ. We have

$$\begin{aligned}
T_1 &\overset{(5.3.77)}{\leq} \frac{c}{(\tau_2-\tau_1)^{2/p}}[\mathbf{G}(B_{\tau_2})+1]^{(1+\delta)\sigma+1/\gamma}\|f\|_{L^2(B_{\varrho})}\\
&\overset{(5.3.91)}{\leq} \frac{1}{10}\mathbf{G}(B_{\tau_2})+\frac{c\|f\|_{L^2(\mathrm{LogL})^\alpha(B_{\varrho})}^{\iota_*}+c}{(\tau_2-\tau_1)^{\beta_*}}\,,\\
T_2 &\overset{(5.3.91)}{\leq} \frac{1}{10}\mathbf{G}(B_{\tau_2})+\frac{c\|G_{\varepsilon}(\cdot,|Du|)\|_{L^1(B_{\varrho})}^{\theta_*}+c}{(\tau_2-\tau_1)^{\beta_*}}\,,\\
T_3 &\overset{(5.3.77)}{\leq} c[\mathbf{G}(B_{\tau_2})+1]^{\delta\vartheta\sigma/2+1/\gamma}\|f\|_{L^2(\mathrm{LogL})^\alpha(B_{\varrho})}\\
&\overset{(5.3.91)}{\leq} \frac{1}{10}\mathbf{G}(B_{\tau_2})+c\|f\|_{L^2(\mathrm{LogL})^\alpha(B_{\varrho})}^{\iota_*}+1\,,
\end{aligned}$$

with $c\equiv c(\mathtt{data}_n,\|h\|_{L^d(\mathcal{B})})$ (again, c depends on a positive power of T, so it is stable as $T\to 0$). Here, as in the following lines, θ_*,β_*,ι_* denote positive exponents depending on $d,\sigma,\hat{\sigma},\gamma$; they might change from line to line according to the same convention used to denote a generic constant *c.* Now, we set $\tau_3 := (\tau_1+\tau_2)/2 > \tau_1 + (\tau_1+\tau_2)/4$ so that $\tau_2 - \tau_3 = (\tau_2-\tau_1)/2$; next, recalling (5.3.74) for $n = 2$ we have

$$
\begin{aligned}
\left\|\mathbf{P}_1^{\mathfrak{h}}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})} &\overset{(2.3.3)_2}{\leq} c\|\mathfrak{h}\|_{L^2(\mathrm{LogL})^\alpha(B_{\tau_3})} \\
&\leq c(\tau)\|\mathfrak{h}\|_{L^{2+\tau}(B_{\tau_3})} \overset{(5.3.69)}{\leq} c\|h\|_{L^d(B_{\tau_3})}\|\bar{G}_\varepsilon(\cdot,|Du|)\|^{1+\sigma+\hat{\sigma}}_{L^p(B_{\tau_3})} \\
&\leq \frac{c}{(\tau_2-\tau_1)^{\beta_*}}\left[\|G_\varepsilon(\cdot,|Du|)\|^{\theta_*}_{L^1(B_{\tau_2})}+1\right] + c\left[\mathbf{M}(B_{\tau_2})\|f\|_{L^2(B_{\tau_2})}\right]^{1+\sigma+\hat{\sigma}} ,
\end{aligned}
$$

for $c \equiv c(\mathtt{data}_{\mathrm{two}}, \|h\|_{L^d(\mathcal{B})})$. Therefore, using the above inequality, (5.3.77), (5.3.91) and Young inequality, we end up with

$$
\begin{aligned}
T_4 \ &\leq\ \frac{c}{(\tau_2-\tau_1)^{\beta_*}}[\mathbf{G}(B_{\tau_2})+1]^{\delta\vartheta\sigma/2}\left[\|G_\varepsilon(\cdot,|Du|)\|^{\theta_*}_{L^1(B_\varrho)}+1\right] \\
&\qquad + c[\mathbf{G}(B_{\tau_2})+1]^{\delta\vartheta\sigma/2+(1+\sigma+\hat{\sigma})/\gamma}\|f\|^{1+\sigma+\hat{\sigma}}_{L^2(\mathrm{LogL})^\alpha(B_\varrho)} \\
&\leq\ \frac{1}{10}\mathbf{G}(B_{\tau_2}) + \frac{c\left(\|G_\varepsilon(\cdot,|Du|)\|^{\theta_*}_{L^1(B_\varrho)}+1\right)}{(\tau_2-\tau_1)^{\beta_*}} + c\|f\|^{\iota_*}_{L^2(\mathrm{LogL})^\alpha(B_\varrho)} ,
\end{aligned}
$$

and, by (5.3.77) we have

$$
\begin{aligned}
T_5 \ &\leq\ \frac{1}{10}\mathbf{G}(B_{\tau_2}) + \frac{c}{(\tau_2-\tau_1)^{\beta_*}}\left[\|G_\varepsilon(\cdot,|Du|)\|^{\theta_*}_{L^1(B_{\tau_2})}+1\right] \\
&\qquad + c\|f\|^{\iota_*}_{L^2(\mathrm{LogL})^\alpha(B_\varrho)} + c\left\|\mathbf{P}_1^{f}(\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})} ,
\end{aligned}
$$

with $c \equiv c(\mathtt{data}_{\mathrm{two}}, \|h\|_{L^d(B_\varrho)})$. Plugging the estimates for the T's in (5.3.90) and recalling the definition given in $(5.3.75)_2$ yields

$$
\begin{aligned}
\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_1})} \leq \frac{1}{2}\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_{\tau_2})} \\
+ \frac{c\left[\|G_\varepsilon(\cdot,|Du|)\|^{\theta}_{L^1(B_\varrho)} + \|f\|_{L^2(\mathrm{LogL})^\alpha(B_\varrho)}+1\right]^\theta}{(\tau_2-\tau_1)^\beta}
\end{aligned}
$$

for new exponents β,θ as in the statement, clearly larger than those appearing in the above estimates. Lemma 2.4.2 allows now to conclude with

$$
\|G_\varepsilon(\cdot,|Du|)\|_{L^\infty(B_\varsigma)} \leq \frac{c\left[\|G_\varepsilon(\cdot,|Du|)\|_{L^1(B_\varrho)} + \|f\|_{L^2(\mathrm{LogL})^\alpha(B_\varrho)}+1\right]^\theta}{(\varrho-\varsigma)^\beta} ,
$$

where $c \equiv c(\mathtt{data}_{\mathrm{two}}, \|h\|_{L^d(\mathcal{B})})$. Finally, (5.3.87) follows from this last estimate and (5.2.17). The proof is complete. □

5.4 Proofs of Theorems 17, 18 and 19

We start with the proof of Theorem 19. We fix p as in statement and without loss of generality, we assume that that $p > 1+\sigma$ (σ being as in (5.1.14)). Then, for every integer $j \geq 1$, we define $f_j \in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ as $f_j(x) := f(x)$ if $|f(x)| \leq j$, and $f_j(x) := j|f(x)|^{-1}f(x)$ otherwise. It clearly follows that

$$
|f_j| \leq |f| \quad \text{for every } j \geq 1\,, \qquad f_j \to f \text{ in } L^n_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\,. \tag{5.4.1}
$$

Next, we determine $R_* \equiv R_*(\texttt{data}_m, f(\cdot), p) \leq 1$ from Proposition 5.3.1. Pay attention here; with some abuse of notation, the f used here is not the same from Proposition 5.3.1, but rather corresponds to $\mathfrak{f}$ from (5.3.1) in the context of Proposition 5.3.1 (and thanks to (5.4.1), f_j corresponds to f in Proposition 5.3.1). In fact, here we are assuming that $f \in L^n_{\rm loc}(\Omega, \mathbb{R}^N)$. Accordingly, we fix a ball $\mathcal{B} \Subset \Omega$ such that $\mathrm{rad}(\mathcal{B}) \leq R_*$. We consider a decreasing sequence of positive numbers $\{\varepsilon_j\}$ such that $\varepsilon_j \leq \min\{1, T\}/4$ for every $j \in \mathbb{N}$, and, accordingly, we consider the family of approximating integrands $\{F_j\} \equiv \{F_{\varepsilon_j}\}$, $\{G_j\} \equiv \{G_{\varepsilon_j}\}$ constructed in (5.2.4). Notice now that any local minimizer u of the functional $\mathcal{F}(\cdot)$ in (5.0.1) belongs to $W^{1,\gamma}_{\rm loc}(\Omega, \mathbb{R}^N)$ by $(5.2.16)_1$. This allows to define $u_j \in u + W^{1,\gamma}_0(\mathcal{B}, \mathbb{R}^N)$ as the solution to

$$u_j \mapsto \min_{w \in u + W^{1,\gamma}_0(\mathcal{B}, \mathbb{R}^N)} \mathcal{F}_j(w, \mathcal{B}) =: \min_{w \in u + W^{1,\gamma}_0(\mathcal{B}, \mathbb{R}^N)} \int_{\mathcal{B}} \left[F_j(x, Dw) - f_j \cdot w\right] \, dx\,. \tag{5.4.2}$$

Directs Methods of the Calculus of Variations apply here and ensure existence (see for instance [16, Section 4.4]). As for [16, (4.55)], and recalling $(5.2.16)_2$ and (5.2.18), we find

$$\|F_j(\cdot, Du_j)\|_{L^1(\mathcal{B})} + \|Du_j\|^\gamma_{L^\gamma(\mathcal{B})} \leq c\left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|^{\gamma/(\gamma-1)}_{L^n(\mathcal{B})} + T^\gamma + \mu^\gamma\right] \tag{5.4.3}$$

for every $j \geq 1$, where $c \equiv c(n, N, \nu, \gamma)$. This implies that we can assume that, up to a not relabelled subsequence, $Du_j \rightharpoonup D\tilde{u}$ weakly in $L^\gamma(\mathcal{B}, \mathbb{R}^{N\times n})$ and $u_j \to \tilde{u}$ strongly in $L^{\frac{n}{n-1}}(\mathcal{B}, \mathbb{R}^N)$, for some $\tilde{u} \in u + W^{1,\gamma}_0(B; \mathbb{R}^N)$. Notice that Proposition 5.3.1 applies to u_j, and gives

$$\|G_j(\cdot, |Du_j|)\|_{L^p(t\mathcal{B})} \leq \frac{c}{(1-t)^{\beta_p}[\mathrm{r}(\mathcal{B})]^{\beta_p}}\left[\|F_j(\cdot, Du_j)\|^{\theta_p}_{L^1(\mathcal{B})} + 1\right] \tag{5.4.4}$$

for every $t \in (0,1)$. With $j_0 \in \mathbb{N}$, we apply (5.2.25) with $j \geq j_0$ in (5.4.4), this yields, together with (5.4.3)

$$\|G_{j_0}(\cdot, |Du_j|)\|_{L^p(t\mathcal{B})} \leq \frac{c}{(1-t)^{\beta_p}[\mathrm{r}(\mathcal{B})]^{\beta_p}}\left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|^{\gamma/(\gamma-1)}_{L^n(\mathcal{B})} + 1\right]^{\theta_p}\,.$$

Letting $j \to \infty$ in the above inequality, and using weak lower semicontinuity (recall that G_{j_0} is convex by $(5.2.20)_2$), yields

$$\|G_{j_0}(\cdot, |D\tilde{u}|)\|_{L^p(t\mathcal{B})} \leq \frac{c}{(1-t)^{\beta_p}[\mathrm{r}(\mathcal{B})]^{\beta_p}}\left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|^{\gamma/(\gamma-1)}_{L^n(\mathcal{B})} + 1\right]^{\theta_p}\,. \tag{5.4.5}$$

This holds for every $j_0 \in \mathbb{N}$, where c, β_p, θ_p are as in (5.3.15). In particular, (5.2.32) is satisfied, thus $(5.2.33)_2$ in yields that $G_{j_0}(x, |D\tilde{u}|) \to \mathfrak{g}_1 G_T(x, |D\tilde{u}|)$ for a.e. $x \in \mathcal{B}$, so we can apply Fatou's Lemma on the right-hand side of (5.4.5) to get

$$\|G_T(\cdot, |D\tilde{u}|)\|_{L^p(t\mathcal{B})} \leq \frac{c}{(1-t)^{\beta_p}[\mathrm{r}(\mathcal{B})]^{\beta_p}}\left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|^{\gamma/(\gamma-1)}_{L^n(\mathcal{B})} + 1\right]^{\theta_p}. \tag{5.4.6}$$

Next, we trivially write

$$\mathcal{F}_{j_0}(u_j, s\mathcal{B}) \leq \mathcal{F}_j(u_j, s\mathcal{B}) + \|F_j(x, Du_j) - F_{j_0}(x, Du_j)\|_{L^1(s\mathcal{B})} + \|(f_{j_0} - f_j)\cdot u_j\|_{L^1(s\mathcal{B})} \tag{5.4.7}$$

whenever $s \in (0,1)$. Properties (5.4.1)-(5.4.5), Hölder and Sobolev-Poincaré inequalities give

$$\mathcal{F}_j(u_j, s\mathcal{B}) \leq \mathcal{F}_j(u_j, \mathcal{B}) + \|f_j \cdot u_j\|_{L^1(\mathcal{B}\setminus s\mathcal{B})} \leq \mathcal{F}_j(u, \mathcal{B}) + c\|f\|_{L^n(\mathcal{B}\setminus s\mathcal{B})}\,,$$

where c is independent of s, j. Using (5.2.36), which is legal because of (5.4.6) (recall it is $p > 1+\sigma$) and (5.4.4), we have $\|F_j(x, Du_j) - F_{j_0}(x, Du_j)\|_{L^1(s\mathcal{B})} \leq c(1-s)^{-\tilde{\beta}}\mathtt{o}(j_0)$, where again c is independent of s, j, j_0 and $\tilde{\beta} = \beta_p(p+1+\sigma)/p$. Sobolev-Poincaré inequality and (5.4.1) give

$\|(f_{j_0} - f_j) \cdot u_j\|_{L^1(s\mathcal{B})} \leq c\|f_{j_0} - f_j\|_{L^n(\mathcal{B})}$, again with c independent of s, j, j_0. Using these last three inequalities in (5.4.7), and finally letting $j \to \infty$, lower semicontinuity and $(5.2.33)_1$ yield

$$\mathcal{F}_{j_0}(\tilde{u}, s\mathcal{B}) \leq \mathcal{F}(u, \mathcal{B}) + c\|f\|_{L^n(\mathcal{B}\setminus s\mathcal{B})} + c\|f_{j_0} - f\|_{L^n(\mathcal{B})} + c(1-s)^{-\tilde{\beta}}\mathtt{o}(j_0) \ . \tag{5.4.8}$$

In turn, notice that (5.4.6) allows to apply (5.2.37); this yields $\|F(x, D\tilde{u}) - F_{j_0}(x, D\tilde{u})\|_{L^1(s\mathcal{B})} \leq c(1-s)^{-\tilde{\beta}}\mathtt{o}(j_0)$, so that $\mathcal{F}_{j_0}(\tilde{u}, s\mathcal{B}) \to \mathcal{F}(\tilde{u}, s\mathcal{B})$ as $j_0 \to \infty$, where we also use (5.4.1). In view of this, letting first $j_0 \to \infty$ and then $s \to 1$ in (5.4.7), yields $\mathcal{F}(\tilde{u}, \mathcal{B}) \leq \mathcal{F}(u, \mathcal{B})$. This and the minimality of u finally give $\mathcal{F}(u, \mathcal{B}) = \mathcal{F}(\tilde{u}, B)$, therefore, by standard convexity arguments, see for instance [16, Section 4.4], we end up with

$$\text{either } \max\{|Du(x)|, |D\tilde{u}(x)|\} \leq T \quad \text{or} \quad Du(x) = D\tilde{u}(x) \ . \tag{5.4.9}$$

Using this information and (5.4.6) we can conclude with (5.1.27). The proof of Theorem 19 is complete. We now come to the proof of Theorem 17. We can use the same approximation employed for Theorem 19 (that uses weaker assumptions than those of Theorem 17). Using this time estimate (5.3.73), together with (5.4.5), we find

$$\begin{aligned} E_\varepsilon(\|Du_j\|_{L^\infty(t\mathcal{B})}) &+ \|G_j(\cdot, |Du_j|)\|_{L^\infty(t\mathcal{B})} \\ &\leq \frac{c}{(1-t)^\beta [\mathtt{r}(\mathcal{B})]^\beta} \left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|_{L(n,1)(\mathcal{B})} + 1\right]^\theta \end{aligned} \tag{5.4.10}$$

for very $s \in (0,1)$, where c, β, θ are as in (5.3.73). By (5.2.17) it follows that $\{Du_j\}$ is bounded in $L^\infty(t\mathcal{B})$ for every $t \in (0,1)$ and, in particular, for every $t \in (0,1)$ there exists M_t such that $\|Du_j\|_{L^\infty(t\mathcal{B})} \leq M_t$ for every $j \in \mathbb{N}$. Using a standard diagonalization argument we infer that, up to a not relabelled subsequences, we have $u_j \rightharpoonup^* \tilde{u}$ in $W^{1,\infty}_{\rm loc}(\mathcal{B}, \mathbb{R}^N)$ for $\tilde{u} \in u + W^{1,\gamma}_0(B; \mathbb{R}^N)$ and $\|D\tilde{u}\|_{L^\infty(t\mathcal{B})} \leq M_t$. Moreover, we can repeat verbatim the argument of Theorem 19 leading to (5.4.9). Now, denoting j_M the first integer such that $1/\varepsilon_{j_M} > M_t$, from the very definition of G_j in (5.2.10), it follows that

$$\|G_j(\cdot, |Du_j|)\|_{L^\infty(t\mathcal{B})} = \left(\sup_{x \in t\mathcal{B}} \int_T^{\max\{Du_j(x), T\}} g_{1,j}(x,s)s \ \mathrm{d}s\right) = \mathfrak{g}_1 \|G_T(\cdot, |Du_j|)\|_{L^\infty(t\mathcal{B})}$$

for every $j \geq j_M$. Moreover,

$$\begin{aligned} \left(\fint_{t\mathcal{B}} G_T(x, |Du_j|)^p \ \mathrm{d}x\right)^{\frac{1}{p}} &\leq \mathfrak{g}_1^{-1} \|G_j(\cdot, |Du_j|)\|_{L^\infty(t\mathcal{B})} \\ &\leq \frac{c}{(1-t)^\beta [\mathtt{r}(\mathcal{B})]^\beta} \left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|_{L(n,1)(\mathcal{B})} + 1\right]^\theta , \end{aligned} \tag{5.4.11}$$

for all $p \in (1, \infty)$. We stress that the constant appearing in (5.4.11) *does not* depend on p. Now we first use weak*-lower semicontinuity on the left-hand side of (5.4.11) to get

$$\left(\fint_{t\mathcal{B}} G_T(x, |Du|)^p \ \mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{c}{(1-t)^\beta [\mathtt{r}(\mathcal{B})]^\beta} \left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|_{L(n,1)(\mathcal{B})} + 1\right]^\theta$$

and then send $p \to \infty$ in the above display to end up with

$$\|G_T(\cdot, |D\tilde{u}|)\|_{L^\infty(t\mathcal{B})} \leq \frac{c}{(1-t)^\beta [\mathtt{r}(\mathcal{B})]^\beta} \left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})} + \|f\|_{L(n,1)(\mathcal{B})} + 1\right]^\theta ,$$

from which (5.1.25) follows using also (5.4.9), $(5.1.4)_2$ and the position in $(5.1.11)_1$. The proof of Theorem 17 is complete. The proof of Theorem 18 is completely similar, taking into account the content of Section 5.3.7.

5.5 Uniform ellipticity and Theorem 20

Here we aim at proving Theorem 20, therefore we shall work under its assumptions. In particular, assumptions $\texttt{set}_{\rm uni}$ are in force. We first derive a suitable analog of the results of Section 5.2 in the case of uniformly elliptic setting. With the current choice of $g_1(\cdot)$ and $g_2(\cdot)$ as in (5.1.21), we apply the constructions laid down in Section 5.2, thereby obtaining, in particular, the new functions $\tilde a_\varepsilon(\cdot)$, $g_{1,\varepsilon}(\cdot)$, $g_{2,\varepsilon}(\cdot)$, $g_{3,\varepsilon}(\cdot)$; these are independent of x. In the following we shall keep the full notation introduced in Sections 5.2 and 5.3. From the definitions (5.2.6)-(5.2.8), and the assumptions (5.1.21), it follows that, in addition to the properties explained in Section 5.2, the following inequalities holds

$$g_{1,\varepsilon}(t) \le g_{2,\varepsilon}(t) \le \tilde K g_{1,\varepsilon}(t) \quad \text{and} \quad g_{3,\varepsilon}(t) \le \tilde K g_{1,\varepsilon}(t)\sqrt{t^2+\mu_\varepsilon^2} \quad \text{for all} \quad t\in[T,\infty)\ , \tag{5.5.1}$$

where $\tilde K \equiv \tilde K(n,N,\nu,\gamma,K)$. Moreover, again for $c\equiv c(n,N,\nu,\gamma,K)\ge 1$ it holds that

$$\frac{1}{2\tilde K} g_{1,\varepsilon}(t)t^2 \le G_\varepsilon(t) + g_1(T)T^2 \le \tilde K g_{1,\varepsilon}(t)t^2\ . \tag{5.5.2}$$

for all $t\in[T,\infty)$. As for the right-hand side of (5.5.2), integration by parts yields

$$\int_T^t \left[\bar a_\varepsilon(x,s)+\bar a_\varepsilon'(x,s)s\right] s\ {\rm d}s = -\int_T^t \bar a_\varepsilon(x,s)s\ {\rm d}s + \bar a_\varepsilon(x,t)t^2 - \bar a_\varepsilon(x,T)T^2 \tag{5.5.3}$$

and therefore

$$G_\varepsilon(t) \overset{(5.3.6)}{\le} \int_T^t \left[\bar a_\varepsilon(x,s)+\bar a_\varepsilon'(x,s)s\right] s\ {\rm d}s \overset{(5.3.5),(5.5.1)}{\le} \tilde K g_{1,\varepsilon}(t)t^2 - g_1(T)T^2.$$

As for the left-hand side, we similarly have

$$\begin{aligned}
G_\varepsilon(t) &\overset{(5.3.6),(5.5.1)}{\ge} \frac{1}{\tilde K}\int_T^t \left[\tilde a_\varepsilon(x,s)+\tilde a_\varepsilon'(x,s)s\right] s\ {\rm d}s\\
&\overset{(5.3.5),(5.5.3)}{\ge} -\frac{1}{\tilde K}\int_T^t g_{2,\varepsilon}(x,s)s\ {\rm d}s + \frac{1}{\tilde K}\left[\tilde a_\varepsilon(x,t)t^2 - \tilde a_\varepsilon(x,T)T^2\right]\\
&\overset{(5.5.1)}{\ge} -G_\varepsilon(t) + \frac{1}{\tilde K} g_{1,\varepsilon}(t)t^2 - g_1(T)T^2.
\end{aligned}$$

We also notice that under assumptions $\texttt{set}_{\rm uni}$, the following double sided inequality holds:

$$G_\varepsilon(t) \le F_\varepsilon(t) \le c[G_\varepsilon(t) + g_1(T)(\mu_\varepsilon^2+T^2)]\ , \qquad c\equiv c(\texttt{data}_{\rm uni})\ , \tag{5.5.4}$$

for all $t\in[T,\infty)$. The left hand side inequality is (5.2.15). As for the right hand side, this is a consequence of (5.1.4) and (5.5.1).

Proposition 5.5.1 *Let $u\in W^{1,\gamma}(\mathcal{B},\mathbb{R}^N)$ be a solution to (5.3.1), under assumptions $\texttt{set}_{\rm uni}$ for $n\ge 2$. There exists a positive radius $R_*\equiv R_*(\texttt{data}_{\rm uni}, h(\cdot))\le 1$ such that if $\mathbf{r}(\mathcal{B})\le R_*$ and $B_\varsigma \Subset B_\varrho$ are concentric balls contained in $\mathcal{B}$, then*

$$\|F_\varepsilon(|Du|)\|_{L^\infty(B_\varsigma)} \le \frac{c}{(\varrho-\varsigma)^n}\left[\|F_\varepsilon(|Du|)\|_{L^1(B_\varrho)}+1\right] + \|f\|_{X(B_\varrho)}^{\gamma/(\gamma-1)} \tag{5.5.5}$$

holds with $c\equiv c(\texttt{data}_{uni})$, where $X(B_\varrho)$ has been defined in (5.1.24).

Proof. We start considering a ball $B_r(x_0)\Subset B$, with $r\le 1$, and a number M such that

$$\max\{\|Du\|_{L^\infty(B_r(x_0))}, T\} \le M \tag{5.5.6}$$

holds and prove that the inequality

$$\int_{B_{r/2}(x_0)} |D(G_\varepsilon(|Du|)-\kappa)_+|^2 \, dx \le \frac{c}{r^2}\int_{B_r(x_0)} (G_\varepsilon(|Du|)-\kappa)_+^2 \, dx \\ + c[G_\varepsilon(M)+1]^2 \int_{B_r(x_0)} |h|^2 \, dx + cM^2 \int_{B_r(x_0)} |f|^2 \, dx \tag{5.5.7}$$

holds whenever $\kappa \ge 0$, with $c \equiv c(\mathtt{data}_{\mathrm{uni}})$. This is the analogue of (5.3.53) and to get it we modify the proof of Lemma 5.3.2, keeping the notation used there. As for the bounds for $(\mathrm{IV})_z$-$(\mathrm{VI})_z$, we have

$$\mathcal{S}_3 + \mathcal{S}_4 \le c|(\mathrm{IV})_x| + c|(\mathrm{V})_x| + c|(\mathrm{VI})_x| + \frac{c}{r^2}\int_{B_r(x_0)} (G_\varepsilon(|Du|)-\kappa)_+^2 \, dx \\ + c\sum_{s=1}^{n}\int_{\mathcal{B}^\kappa} |f\cdot D_s\varphi_s| \, dx \, , \tag{5.5.8}$$

with $c \equiv c(\mathtt{data}_{uni})$. This estimate can be obtained by adapting (5.3.56)-(5.3.59) (and also (5.3.20)-(5.3.21)) or directly taking $h(\cdot) \equiv 0$ as $g_{1,\varepsilon}(\cdot)$ is independent of x, and using $(5.5.1)_1$. In turn, the last term in (5.5.8) involving the right-hand side f can be treated exactly as in (5.3.67), and it remains to deal with the x-terms. With the help of $(5.2.12)_3$, (5.5.1) and (5.5.2) we estimate

$$c|(\mathrm{IV})_x| + c|(\mathrm{V})_x| + c|(\mathrm{VI})_x| \le \bar\varepsilon\mathcal{S}_3 + \bar\varepsilon\mathcal{S}_4 + \frac{c}{r^2}\int_{B_r(x_0)} (G_\varepsilon(|Du|)-\kappa)_+^2 \, dx \\ + \frac{c}{\bar\varepsilon}\int_{\mathcal{B}} \eta^2|h|^2[G_\varepsilon(|Du|)+1]^2 \, dx \, ,$$

for $c \equiv c(\mathtt{data}_{uni})$. Merging the content of the above three displays, choosing $\bar\varepsilon > 0$ small enough and reabsorbing terms we end up with (5.5.7), where $c \equiv c(\mathtt{data}_{uni})$. As a consequence, proceeding as for the proof of Lemma 5.3.3, we have, by an application of Lemma 2.3.1, that, if $B_{r_0}(x_0) \Subset B$ is any another ball such that $r_0 \le 1$, and M be such that (5.5.6) holds, then

$$G_\varepsilon(|Du(x_0)|) \le cr_0^{-n/2}\left[G_\varepsilon(M)\right]^{1/2}\|G_\varepsilon(|Du|)\|_{L^1(B_{\tau_2})}^{1/2} \\ + c[G_\varepsilon(M)+1]\mathbf{P}_1^h(x_0,2r_0) + c[G_\varepsilon(M)+1]^{1/\gamma}\mathbf{P}_1^f(x_0,2r_0) \tag{5.5.9}$$

holds for every x_0 which is a Lebesgue point for $|Du|$, where $c \equiv c(\mathtt{data}_{uni})$, and we have also used (5.2.17). Next, (2.3.3) gives $\|\mathbf{P}_1^h(\cdot,2r_0)\|_{L^\infty(B_{\tau_1})} \lesssim c\|h\|_{X(B_\varrho)}$ and $\|\mathbf{P}_1^f(\cdot,2r_0)\|_{L^\infty(B_{\tau_1})} \lesssim \|f\|_{X(B_\varrho)}$. Using these informations in (5.5.9) yields

$$G_\varepsilon(\|Du\|_{L^\infty(B_{\tau_1})}) \le cr_0^{-n/2}\left[G_\varepsilon(M)\right]^{1/2}\|G_\varepsilon(|Du|)\|_{L^1(B_{\tau_2})}^{1/2} \\ + c_*[G_\varepsilon(M)+1]\|h\|_{X(B_\varrho)} + c[G_\varepsilon(M)+1]^{1/\gamma}\|f\|_{X(B_\varrho)} \, , \tag{5.5.10}$$

where $c, c_* \equiv c, c_*(\mathtt{data}_{uni})$. We now determine the radius $R_* \equiv R_*(\mathtt{data}_{\mathrm{uni}}, h(\cdot))$ such that

$$\varrho \le R_* \Longrightarrow c_*\|h\|_{X(B_\varrho)} \le \frac{1}{6} \, . \tag{5.5.11}$$

Using this information and Young's inequality in (5.5.10) leads to

$$G_\varepsilon(\|Du\|_{L^\infty(B_{\tau_1})}) \le \frac{1}{2}G_\varepsilon(\|Du\|_{L^\infty(B_{\tau_2})}) + \frac{c}{(\tau_2-\tau_1)^n}\left[\|G_\varepsilon(|Du|)\|_{L^1(B_\varrho)} + 1\right] + \|f\|_{X(B_\varrho)}^{\gamma/(\gamma-1)} \, .$$

The final inequality (5.5.5) now follows using Lemma 2.4.2 with $\mathcal{Z}(t) := G_\varepsilon(t)$ and (5.5.4). □

With Proposition 5.5.1 available, we can now complete the proof of Theorem 19. Arguing as in the proof of Theorem 17 in Section 5.4, and arrive up to (5.4.5). This time we use estimate (5.5.5) instead of (5.4.10), and this yields

$$\|F_j(|Du_j|)\|_{L^\infty(B_\xi)} \le \frac{c}{(\varrho-\xi)^n}\left[\|F(|Du|)\|_{L^1(B_\varrho)}+1\right]+\|f\|_{X(B_\varrho)}^{\gamma/(\gamma-1)} \tag{5.5.12}$$

where c depends on $\texttt{data}_{\mathrm{uni}}$, which, again, depends on a positive power of T, so it is stable as $T\to 0$. Starting from (5.5.12), the rest of the proof follows with minor modifications of the proof for Theorem 17.

5.6 Applications and proof of Theorems 12-16

Here we show how to derive the results in Theorems 12-16 from Theorems 17, 18 and 20.

5.6.1 Proof of Theorem 12

Since we are considering the class of functionals with (p,q)-growth, we shall take $g_i(x,t)\equiv g_i(t)$, $i\in\{1,2,3\}$. Recalling (5.0.8), (5.1.1) and the orthogonal decomposition

$$\partial_{zz}F(x,z)=\tilde F''(x,|z|)\frac{z\otimes z}{|z|^2}+\frac{\tilde F'(x,|z|)}{|z|}\left[\mathbb{I}_{N\times n}-\frac{z\otimes z}{|z|^2}\right],$$

which implies

$$|\partial_{zz}F(x,z)|^2=[\tilde F''(x,|z|)]^2+\left[\frac{\tilde F'(x,|z|)}{|z|}\right]^2(Nn-1), \tag{5.6.1}$$

we can permanently work with the choice

$$\begin{cases} g_1(t)=\nu(\mu^2+t^2)^{\frac{p-2}{2}}\\ g_2(t)=\Lambda\sqrt{Nn}\left[(\mu^2+t^2)^{\frac{q-2}{2}}+(\mu^2+t^2)^{\frac{p-2}{2}}\right]\\ g_3(t)=\left[(\mu^2+t^2)^{\frac{q-1}{2}}+(\mu^2+t^2)^{\frac{p-1}{2}}\right], \end{cases}$$

therefore

$$G_T(t)=\frac{\nu}{p}\left[(\mu^2+t^2)^{\frac{p}{2}}-(T^2+\mu^2)^{\frac{p}{2}}\right]\quad\text{and}\quad \bar G_T(t)=G_T(t)+(T^2+1)^{\frac{p}{2}}.$$

Here, any $T\in(0,\infty)$ is fine. In this case, by (5.1.1), (5.1.2) and (5.0.8)-(5.0.10) we see that $F(\cdot)$ satisfies the growth, regularity and structural requirements of Theorems 17-18 for all $T>0$. Moreover, the very definition of g_1 assures that (5.1.4) is verified with $\gamma=p$. Moreover, since

$$\frac{g_2(t)}{g_1(t)}=\frac{\Lambda\sqrt{Nn}}{\nu}\left[1+(\mu^2+t^2)^{\frac{q-p}{2}}\right]\quad\text{and}\quad g_1(t)t=\nu(\mu^2+t^2)^{\frac{p-2}{2}}t,$$

and $p>1$, we see that (1.1.3) is satisfied for $c_a=1$. Notice that (1.1.10) holds with $\hat\sigma=0$ since g_1 does not depend on the space variable. Now set

$$c_b=2^{8q}\left(\frac{p(1+\Lambda\sqrt{Nn})}{\min\{1,\nu\}}\right)^4\left(1+\frac{p}{\nu}+\frac{\nu}{p}\right)^4 \tag{5.6.2}$$

and

$$\sigma=\sigma'=\frac{q-p}{p}, \tag{5.6.3}$$

where ϑ is defined in (5.1.17)$_2$. Given that

$$g_3(t)\sqrt{t^2+\mu^2} = \left[(\mu^2+t^2)^{\frac{q}{2}} + (\mu^2+t^2)^{\frac{p}{2}}\right],$$

$$\frac{g_3(t)^2}{g_1(t)} = \frac{\left[(\mu^2+t^2)^{\frac{p-1}{2}} + (\mu^2+t^2)^{\frac{q-1}{2}}\right]^2}{\nu(\mu^2+t^2)^{\frac{p-2}{2}}},$$

we see that assumption (1.1.11) is satisfied with the choices made in (5.6.2)-(5.6.3). In fact,

$$q = p\left(1+\frac{q-p}{p}\right) \qquad \text{and} \qquad 2q-p = p\left(1+\frac{2(q-p)}{p}\right),$$

thus

$$g_3(t)\sqrt{t^2+\mu^2} \le 2^q + 2\left(1+\frac{p}{\nu}\right)^2\left[\frac{\nu}{p}\left[(\mu^2+t^2)^{\frac{p}{2}}-\mu^p\right]+\frac{\nu}{p}\right]^{1+\sigma} \le c_b\bar{G}_T(t)^{1+\sigma},$$

$$\frac{g_3(t)^2}{g_1(t)} \le \frac{2^{q+2}}{\nu} + \left(\frac{2p}{\min\{\nu,1\}}\right)^4\left[\frac{\nu}{p}\left[(\mu^2+t^2)^{\frac{p}{2}}-\frac{1}{p}\mu^p\right]+\frac{\nu}{p}\right]^{1+2\sigma} \le c_b\bar{G}_T(x,t)^{1+2\sigma}.$$

Finally, we check on (5.1.18)-(5.1.20). Keeping in mind the value of c_b fixed by (5.6.2), we estimate

$$\frac{g_2(t)}{g_1(t)} \le \frac{2^{2q}\Lambda\sqrt{Nn}}{\nu} + \frac{p\Lambda\sqrt{Nn}}{\min\{\nu,1\}^2}\left[\frac{\nu}{p}\left[(\mu^2+t^2)^{\frac{p}{2}}-\mu^p\right]+\frac{\nu}{p}\right]^{\tilde{\sigma}} \le c_b\bar{G}_T(t)^{\tilde{\sigma}}, \qquad \tilde{\sigma}\in\{\sigma,\sigma'\} \tag{5.6.4}$$

and (5.1.18)-(5.1.20) hold thanks to the limitations imposed by (5.1.24). Hence, Theorems 17-18 apply and (5.0.12) follows.

5.6.2 Proof of Theorem 13

Before entering into the proof of Theorem 13, let us recall some notation. As usual, we denote by $H(\cdot)$ the integrand appearing in (5.0.6), i.e.: $H(x,z) := [|z|^p + a(x)|z|^q]$ and, for simplicity, define

$$\texttt{data}_{\mathrm{H}} := \left(n, N, p, q, \nu, L, d, [a]_{0,1-\frac{n}{d};B}, \|a\|_{L^\infty(B)}, \|H(\cdot,Du)\|_{L^1(B)}, \|f\|_{L^n(B)}\right),$$

where, being our results of local nature, we localized all the quantities appearing in $\texttt{data}_{\mathrm{H}}$ on balls $B \Subset \Omega$.
For the sake of clarity, we split the proof of Theorem 13 into five steps.

Step 1: Quantification of ellipticity

Let us frame the Double Phase energy (5.0.6) into the setting described in Section 5.1. Using (5.6.1) and the structure of the integrand in (5.0.6), we define

$$\begin{cases} g_1(x,t) = \min\{p-1,1\}[t^{p-2}+a(x)t^{q-2}] \\ g_2(x,t) = 2q\sqrt{nN}[t^{p-2}+a(x)t^{q-2}] \\ g_3(t) = qt^{q-1}, \end{cases} \tag{5.6.5}$$

thus

$$G_T(x,t) := \min\{p-1,1\}\left[\left(\frac{1}{p}t^p+\frac{1}{q}a(x)t^q\right)-\left(\frac{1}{p}T^p+\frac{1}{q}a(x)T^q\right)\right]$$

$$\bar{G}_T(x,t) = H_T(x,t) + (T^2+1)^{\frac{p}{2}},$$

for arbitrary $T \in (0,\infty)$. By very definition of the integrand appearing in (5.0.6), the growth, regularity and structural requirements of Theorem 17-18 are satisfied. Moreover, since

$$\frac{g_2(x,t)}{g_1(x,t)} = \frac{2q\sqrt{nN}}{\min\{p-1,1\}} \quad \text{and} \quad g_1(x,t)t = \min\{p-1,1\}[t^{p-1} + a(x)t^{q-1}], \tag{5.6.6}$$

hypotheses (1.1.3) is obviously satisfied. Let us take care of the mutual relations among the $g_1(\cdot)$, $g_2(\cdot)$ and $g_3(\cdot)$. Set

$$\hat{\sigma} = \sigma_3 = \frac{q-p}{p}, \qquad \sigma = 0, \qquad c_b = \frac{q^2p^2}{\min\{p-1,1\}^4}, \tag{5.6.7}$$

where we renamed σ_3 the exponent appearing in (1.1.11) to distinguish it from the one in (1.1.12): as we will see in a few lines, it is fundamental keeping them separated. A straightforward computation shows that $|\partial_x g_1(x,t)| = q|\partial_x a(x)|t^{q-2}$ therefore, (1.1.10) holds with $h(x) = |\partial_x a(x)|$, $\hat{\sigma}$ and c as in (5.6.7). In fact

$$|\partial_x g_1(x,t)| \le h(x)g_1(x,t)t^{p\left(\frac{q-p}{p}\right)} \le \frac{p^2h(x)}{\min\{p-1,1\}^2}g_1(x,t)\bar{G}_T(x,t)^{\hat{\sigma}}.$$

Now we consider (1.1.11) and let $\sigma \equiv \sigma_3$ be as in (5.6.7) and h as above. We get:

$$g_3(t)t = qt^q \le \frac{qp^2}{\min\{p-1,1\}^2}\left[\min\{p-1,1\}t^{p\left(\frac{q-p}{p}+1\right)}\right] \le \frac{qp^2}{\min\{p-1,1\}^2}\bar{G}_T(x,t)^{1+\sigma_3}$$

$$\frac{g_3(t)^2}{g_1(x,t)} \le \frac{q^2}{p\min\{p-1,1\}}t^{p\left(1+\frac{2(q-p)}{p}\right)} \le \frac{q^2p^2}{\min\{p-1,1\}^4}\bar{G}_T(x,t)^{1+2\sigma_3}.$$

Finally, let us have a look to (1.1.12). From $(5.6.6)_1$, we see that $g_2(x,t)/g_1(x,t) \equiv$ const, therefore (1.1.12) hold with σ and c_b as in $(5.6.7)_{2,3}$, which are admissible choices in the light of (5.1.14), (5.1.19) and (5.0.14).

Step 2: General considerations

At this point, it seems that assumptions (5.1.14)-(5.1.19) of Theorems 17-18 are not satisfied, given the values of $\hat{\sigma}$ and σ_3 in (5.6.7). However, a closer inspection of the proof of Theorems 19-18 reveals that the sum $\hat{\sigma} + \sigma$ (in our case it is $\hat{\sigma} + \sigma_3$) appears only if both $g_2(\cdot)/g_1(\cdot)$ and $g_3(\cdot)$ are non constant. Since in the present situation $g_2(\cdot)/g_1(\cdot) \equiv$ const, the exponent appearing in Sections 5.3.1-5.3.7 will multiply $1+\hat{\sigma}$ (or $1+\sigma_3$, which, by (5.6.7) is the same), rather than $(1+\sigma+\hat{\sigma})$, so, recalling also (5.0.14), we see that the quantity $1+\hat{\sigma}$ remains almost under control. We said "almost" since in (5.1.14)-(5.1.19) the strict inequality is prescribed, while (5.0.11) allows also for the equality, at least for what concerns the part involving $(n^{-1}-d^{-1})$. We can treat the borderline case by using some higher integrability result of Gehring type, which will be proved in the next step.

Step 3: Local higher integrability results

We record some of auxiliary lemmas, whose proof in the homogeneous case $f \equiv 0$ can be found in [12, 79].

Lemma 5.6.1 *Under assumptions (5.0.2), (5.0.13) and (5.0.14), let $u \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local minimizer of (5.0.6). Then there exists a positive integrability exponent $\delta_g \equiv \delta_g(\texttt{data}_{\mathrm{H}})$ so that whenever $B \Subset \Omega$ is an open ball with $\mathbf{r}(B) \in (0,1]$, the following reverse Hölder inequality holds,*

$$\left(\fint_{B/2}[1+H(x,Du)]^{1+\delta_g}\ \mathrm{d}x\right)^{\frac{1}{1+\delta_g}} \le c\fint_B \left[1+H(x,Du)\right]\ \mathrm{d}x, \tag{5.6.8}$$

with $c \equiv c(\texttt{data}_{\mathrm{H}})$. In particular, $H(\cdot,Du) \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. Let $B_\varrho \Subset B \Subset \Omega$ be two balls with $\varrho, \mathtt{r}B \le 1$, set parameters $\varrho/2 \le s < t \le \varrho$ and pick a cut-off function $\eta \in C^1_c(B_\varrho)$ so that $\mathbb{1}_{B_s} \le \eta \le \mathbb{1}_{B_t}$ and $|D\eta| \lesssim (t-s)^{-1}$. We compare by minimality u to the map $w_\varrho := u - \eta(u-(u)_{B_\varrho})$ for getting, after a few standard manipulations,

$$\int_{B_t} H(x,Du)\ \mathrm{d}x \le \int_{B_t} H(x,Dw_\varrho)\ \mathrm{d}x + \int_{B_t} \eta f\cdot(u-(u)_{B_\varrho})\ \mathrm{d}x =: \mathrm{(I)} + \mathrm{(II)}.$$

Term (I) can be estimated as in [12, 79]:

$$\mathrm{(I)} \le c\left[\int_{B_t\setminus B_s} H(x,Du)\ \mathrm{d}x + \int_{B_t} H\left(x, \frac{u-(u)_{B_\varrho}}{t-s}\right)\ \mathrm{d}x\right],$$

with $c \equiv c(n,p,q)$. By Sobolev-Poincaré and Young inequalities we bound

$$\begin{aligned}|\mathrm{(II)}| \le & \left(\int_{B_t} |f|^n\ \mathrm{d}x\right)^{\frac{1}{n}}\left(\int_{B_t}[\eta|u-(u)_{B_\varrho}|]^{\frac{n}{n-1}}\ \mathrm{d}x\right)^{\frac{n-1}{n}}\\ \le & ct^{n(1-1/p)}\|f\|_{L^n(B)}\left(\int_{B_t}|Du|^p + \left|\frac{u-(u)_{B_\varrho}}{t-s}\right|^p\ \mathrm{d}x\right)^{\frac{1}{p}}\\ \le & \frac{1}{4}\int_{B_t} H(x,Du)\ \mathrm{d}x + c\int_{B_t} H\left(x,\frac{u-(u)_\varrho}{t-s}\right)\ \mathrm{d}x + c\varrho^n,\end{aligned}$$

for $c \equiv c(n,p,q,\|f\|_{L^n(B)})$. Merging the content of the three above displays we have

$$\int_{B_t} H(x,Du)\ \mathrm{d}x \le c\int_{B_t\setminus B_s} H(x,Du)\ \mathrm{d}x + c\int_{B_t} H\left(x,\frac{u-(u)_{B_\varrho}}{t-s}\right)\ \mathrm{d}x + c\varrho^n$$

with $c \equiv c(n,p,q,\|f\|_{L^n(B)})$. The hole filling technique, Lemma 2.4.2 with the choice $\mathcal{Z}(t) := \|H(\cdot,Du)\|_{L^1(B_t)}$ and the intrinsic Sobolev-Poincaré inequality in [218] eventually render

$$\fint_{B_{\varrho/2}} H(x,Du)\ \mathrm{d}x \le c\left(\fint_{B_\varrho} H(x,Du)^\tau\ \mathrm{d}x\right)^{\frac{1}{\tau}} + c,$$

where $c \equiv c(\mathtt{data}_{\mathrm{H}})$ and $\tau \equiv \tau(n,p,q) \in (0,1)$. Now we can apply a suitable localized version of Gehring Lemma [123, Chapter 6] for getting (5.6.8) and, after a standard covering argument we can conclude that $H(\cdot,Du) \in L^1_{\mathrm{loc}}(\Omega)$. □

Let $B \Subset \Omega$ be a ball with $\mathtt{r}(B) \le 1$. As by now customary in the Double Phase setting we make in some sense quantitative the behavior of the modulating coefficient $a(\cdot)$ by introducing its phases: precisely, we say that $a(\cdot)$ degenerates on B, when $a_i(B) := \inf_{x\in B} a(x) \le 4\mathtt{r}(B)^{1-\frac{n}{d}}[a]_{0,1-\frac{n}{d};B}$, while we say that $a(\cdot)$ does not degenerate if the complementary condition holds: $a_i(B) > 4\mathtt{r}(B)^{1-\frac{n}{d}}[a]_{0,1-\frac{n}{d};B}$. This distinction is useful to localize the integrand by considering the auxiliary Young function $H^-_B(z) := [|z|^p + a_i(B)|z|^q]$. Finally, let $H_\varepsilon(\cdot)$, $H^-_{\mathtt{r}(B),\varepsilon}(\cdot)$ be the integrands defined by applying the construction in Section 5.2.1 (with $\gamma = p$ and any $T \in (0,\infty)$) to $H(\cdot)$ and to $H^-_{\mathtt{r}(B)}(\cdot)$ respectively. Now we are ready to prove a suitable, intrinsic Sobolev-Poincaré inequality for $H_\varepsilon(\cdot)$.

Lemma 5.6.2 *Under assumptions* (5.0.13) *and* (5.0.14)*, let $B \Subset \Omega$ be a ball with $\mathtt{r}(B) \le 1$ and $w_1 \in W^{1,p}_0(B,\mathbb{R}^N)$ and $w_2 \in W^{1,p}(B,\mathbb{R}^N)$ be two functions with $H_\varepsilon(\cdot,Dw_1), H_\varepsilon(\cdot,Dw_2) \in L^1(B)$. Then, the following Sobolev-Poincaré inequalities holds*

$$\left(\fint_B H_\varepsilon(x,w_1/\mathtt{r}(B))\ \mathrm{d}x\right) \le c\left(\fint_B H_\varepsilon(x,Dw_1)^\tau\ \mathrm{d}x\right)^{\frac{1}{\tau}}, \tag{5.6.9}$$

for $c \equiv c(n, N, p, q, d, [a]_{0,1-\frac{n}{d};B}, \|Dw_1\|_{L^p(B)})$ *and* $\tau \equiv \tau(n, p, q) \in (0,1)$ *and*

$$\left(\fint_B H_\varepsilon \left(x, \frac{w_2 - (w_2)_B}{r(B)} \right) \ \mathrm{d}x \right) \le c \left(\fint_B H_\varepsilon(x, Dw_2)^\tau \ \mathrm{d}x \right)^{\frac{1}{\tau}} \tag{5.6.10}$$

with $c \equiv c(n, N, p, q, d, [a]_{0,1-\frac{n}{d};B}, \|Dw_2\|_{L^p(B)})$ *and* $\tau \equiv \tau(n, p, q) \in (0,1)$.

Proof. We prove (5.6.9), the proof of (5.6.10) being totally analogous. Let $B \Subset \Omega$ and $w_1 \in W^{1,p}_0(B, \mathbb{R}^N)$ be as in the statement. For the ease of exposition, as in [218], we study separately the two phases of the modulating coefficient.

Degenerate phase

In this case, recalling the definition in $(5.6.5)_1$ and in (5.2.6), we bound

$$\begin{aligned}
\int_B H_\varepsilon \left(x, \frac{w_1}{\mathtt{r}(B)} \right) \ \mathrm{d}x \le\ & 3 \int_{B \cap \{\mathtt{r}(B)^{-1}|w_1| < \varepsilon\}} \left(\int_0^\varepsilon g_{1,\varepsilon}(x,s)s \ \mathrm{d}s \right) \ \mathrm{d}x \\
& + \int_{B \cap \{\varepsilon \le \mathtt{r}(B)^{-1}|w_1| < T_\varepsilon\}} \left(\int_\varepsilon^{\mathtt{r}(B)^{-1}|w_1|} g_{1,\varepsilon}(x,s)s \ \mathrm{d}s \right) \ \mathrm{d}x \\
& + \int_{B \cap \{\mathtt{r}(B)^{-1}|w_1| \ge T_\varepsilon\}} \left(\int_\varepsilon^{\mathtt{r}(B)^{-1}|w_1|} g_{1,\varepsilon}(x,s)s \ \mathrm{d}s \right) \ \mathrm{d}x =: \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)}.
\end{aligned}$$

A direct computation renders that

$$\mathrm{(I)} \le c(n, p, [a]_{0,1-\frac{n}{d};B}) \varepsilon^p \mathtt{r}(B)^n.$$

Now notice that, by (5.0.14) we have that $p_* \le q_* < p$, where

$$b_* := \max \left\{ 1, \frac{nb}{n+b} \right\}, \qquad b \in \{p, q\},$$

therefore using Sobolev-Poincaré inequality as in [218, Theorem 2.13] and keeping in mind (5.0.14), we estimate

$$\begin{aligned}
\mathrm{(II)} + \mathrm{(III)} \le & c\mathtt{r}(B)^n \fint_B \left[(w_1/\mathtt{r}(B))^p + \mathtt{r}(B)^{1-\frac{n}{d}} (w_1/\mathtt{r}(B))^q \right] \ \mathrm{d}x \\
& + c \int_{B \cap \{\mathtt{r}(B)^{-1}|w_1| \ge T_\varepsilon\}} a(x) T_\varepsilon^{q-p} (w_1/\mathtt{r}(B))^p \ \mathrm{d}x \\
\le & c\mathtt{r}(B)^n \fint_B \left[(w_1/\mathtt{r}(B))^p + \mathtt{r}(B)^{1-\frac{n}{d}} (w_1/\mathtt{r}(B))^q \right] \ \mathrm{d}x \le c\mathtt{r}(B)^n \left(\fint_B |Dw_1|^{p_*} \ \mathrm{d}x \right)^{\frac{p}{p_*}} \\
& + c\mathtt{r}(B)^{n+1-\frac{n}{d}} \left(\fint_B |Dw_1|^p \ \mathrm{d}x \right)^{\frac{q-p}{p}} \left(\fint_B |Dw_1|^{p_*(q_*/p)} \ \mathrm{d}x \right)^{\frac{p}{q_*}} \\
\le & c\mathtt{r}(B)^n \left(1 + \|Dw_1\|_{L^p(B)}^{q-p} \right) \left(\fint_B |Dw_1|^{p\tau'} \ \mathrm{d}x \right)^{\frac{1}{\tau'}},
\end{aligned}$$

with $c \equiv c(n, N, p, q, d, [a]_{0,1-\frac{n}{d};B})$ and $\tau' := q_*/p < 1$. Merging the content of the two above displays and recalling (5.2.17) and the bound imposed on the size of ε in Section 5.2, we have

$$\fint_B H_\varepsilon \left(x, \frac{w_1}{\mathtt{r}(B)} \right) \ \mathrm{d}x \le c \left[T^p + \left(\fint_B H_\varepsilon(x, Dw_1)^{\tau'} \ \mathrm{d}x \right)^{\frac{1}{\tau'}} \right], \tag{5.6.11}$$

for $c \equiv c(n, N, p, q, d, [a]_{0,1-\frac{n}{d};B}, \|Dw_1\|_{L^p(B)})$.

Non-degenerate phase

When $a_i(B) > 4\mathtt{r}(B)^{1-\frac{n}{d}}[a]_{0,1-\frac{n}{d};B}$, it is easy to see that

$$H_\varepsilon(x,z) \sim H^-_{\varrho,\varepsilon}(z) \quad \text{for all} \quad (x,z) \in B \times \mathbb{R}^{N\times n}, \tag{5.6.12}$$

up to constants depending on (n,d). Moreover, recalling the discussion made at the beginning of Section 5.3, see also [16, Remark 5.5], we see that $(H^-_{\varrho,\varepsilon})''(t)$ makes sense and that $t(H^-_{\varrho,\varepsilon})''(t) \sim (H^-_{\varrho,\varepsilon})'(t)$ with constants implicit in "$\sim$" depending only from (p,q). This means that [87, Theorem 7] applies, so we have

$$\left(\fint_B H^-_{\mathtt{r}(B),\varepsilon}(w_1/\mathtt{r}(B)) \ \mathrm{d}x\right) \le c\left(\fint_B H^-_{\mathtt{r}(B),\varepsilon}(Dw_1)^{\tau''} \ \mathrm{d}x\right)^{\frac{1}{\tau''}}, \tag{5.6.13}$$

with $c \equiv c(n,N,p,q)$ and $\tau'' \equiv \tau''(n,p,q) \in (0,1)$. By (5.6.12) and (5.6.13) we obtain

$$\begin{aligned}\fint_B H_\varepsilon\left(x, \frac{w_1}{\mathtt{r}(B)}\right) \ \mathrm{d}x \le & c\fint_B H^-_{\mathtt{r}(B),\varepsilon}\left(w_1/\mathtt{r}(B)\right) \ \mathrm{d}x \\ \le & c\left(\fint_B H^-_{\mathtt{r}(B),\varepsilon}(Dw_1)^{\tau''} \ \mathrm{d}x\right)^{\frac{1}{\tau''}} \le c\left(\fint_B H_\varepsilon(x,Dw_1)^{\tau''} \ \mathrm{d}x\right)^{\frac{1}{\tau''}},\end{aligned} \tag{5.6.14}$$

for $c \equiv c(n,N,p,q)$. Combining (5.6.11)-(5.6.14) and setting $\tau := \min\{\tau', \tau''\}$, we end up with (5.6.9) and the proof is complete. □

Finally, a Gehring type higher integrability result up to the boundary.

Lemma 5.6.3 *Under assumptions (5.0.2), (5.0.13) and (5.0.14), let $B \Subset \Omega$ be a ball with $\mathtt{r}(B) \le 1$, $\tilde{v} \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a map so that $H(\cdot, D\tilde{v}) \in L^{1+\delta}_{\mathrm{loc}}(\Omega)$ for some $\delta > 0$ and $v_\epsilon \in \tilde{v} + W^{1,p}_0(B,\mathbb{R}^N)$ be the solution of Dirichlet problem (5.3.4) with $F_\varepsilon(\cdot) \equiv H_\varepsilon(\cdot)$. Then there exists a positive constant $c \equiv c(\mathtt{data}_{\mathrm{H}})$ and a positive higher integrability exponent $\sigma_g \equiv \sigma_g(\mathtt{data}_{\mathrm{H}}, \delta) \in (0,\delta)$ so that the following inequality*

$$\left(\fint_B [1 + H_\varepsilon(x, Dv_\varepsilon)]^{1+\sigma_g} \ \mathrm{d}x\right)^{\frac{1}{1+\sigma_g}} \le c\left(\fint_B [1 + H_\varepsilon(x, D\tilde{v})]^{1+\sigma_g} \ \mathrm{d}x\right)^{\frac{1}{1+\sigma_g}},$$

holds true.

Proof. Once inequalities (5.6.9)-(5.6.10) are available, we can proceed exactly as in [69, Lemma 10] to conclude with (5.6.15). □

To summarize, if $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a minimum of the functional in (5.0.6), then $H(\cdot, Du) \in L^{1+\delta_g}_{\mathrm{loc}}(\Omega)$ for some positive $\delta_g \equiv \delta_g(\mathtt{data}_{\mathrm{H}})$. This means that, by Lemma 5.6.3, the solution v_ε of problem (5.3.4) (with $H_\varepsilon(\cdot)$ replacing $F_\varepsilon(\cdot)$ of course) satisfies:

$$\begin{aligned}\left(\fint_B [1 + H_\varepsilon(x, Dv_\varepsilon)]^{1+\sigma_g} \ \mathrm{d}x\right)^{\frac{1}{1+\sigma_g}} \le & c\left(\fint_B [1 + H_\varepsilon(x, Du)]^{1+\sigma_g} \ \mathrm{d}x\right)^{\frac{1}{1+\sigma_g}} \\ \le & c\left(\fint_B [1 + H(x, Du)]^{1+\sigma_g} \ \mathrm{d}x\right)^{\frac{1}{1+\sigma_g}},\end{aligned} \tag{5.6.15}$$

where $c \equiv c(\mathtt{data}_{\mathrm{H}})$, $\sigma_g \equiv \sigma_g(\mathtt{data}_{\mathrm{H}}) \in (0, \delta_g)$ and we also used that, by definition,

$$H_\varepsilon(x,t) \le c(p,q,\|a\|_{L^\infty(B)})[1 + H(x,t)] \quad \text{for all} \quad (x,t) \in \Omega \times [0,\infty). \tag{5.6.16}$$

We finally remark that we should expect that the constants appearing in the previous display depend also on $\|Dv_\varepsilon\|_{L^p(B)}$. However, such a dependency comes from the application of Sobolev-Poincaré inequality: in fact a quick inspection of the proof of Lemma 5.6.2 shows that c is an increasing function of $\|Dv_\varepsilon\|_{L^p(B)}$, and $\|Dv_\varepsilon\|_{L^p(B)} \le c(n,N,p,T,\|H(\cdot,Du)\|_{L^1(B)},\|f\|_{L^n(B)})$, cf. (5.4.3), therefore the dependency from $\mathtt{data}_{\mathrm{H}}$ is the right one.

Step 4: The case $n \geq 3$

We jump back to Section 5.3.3 and, instead of the interpolation inequality (5.3.43), as in Theorem 8 of Chapter 4, we use the following

$$\begin{aligned}\|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|_{L^{s_1}(B_{\tau_2})} &\leq \|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|^{\lambda_{i+1}}_{L^{s_{i+1}}(B_{\tau_2})}\|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|^{1-\lambda_{i+1}}_{L^{1+\sigma_g}(B)} \\ &\leq c\|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|^{\lambda_{i+1}}_{L^{s_{i+1}}(B_{\tau_2})}\|1+H(\cdot,Du)\|^{1-\lambda_{i+1}}_{L^{1+\sigma_g}(B)},\end{aligned}$$

where $c \equiv c(\texttt{data}_{\mathrm{H}})$, $\lambda_{i+1} = \frac{s_{i+1}(s_1-(1+\sigma_g))}{s_1(s_{i+1}-(1+\sigma_g))}$ and we clearly used (5.6.15), $(5.2.15)_3$ and (5.6.16). In these terms, verifying (5.3.45)-(5.3.46) amounts to show that

$$\lim_{i\to\infty}\frac{\lambda_{i+1}\chi^i s_1}{s_{i+1}} = \lim_{i\to\infty}\frac{\chi^i(s_1-(1+\sigma_g))}{s_{i+1}-(1+\sigma_g)} = \tilde{l} < 1 \iff \frac{2m(1+\hat{\sigma})-(1+\sigma_g)}{2m\left(1-\frac{\hat{\sigma}}{\chi-1}\right)} < 1,$$

which is verified also when $\hat{\sigma} \equiv (n^{-1}-d^{-1})$, being $\sigma_g > 0$. With the content of the two above displays at hand, Theorem 19 follows at once. We stress that so far we only needed $\frac{q}{p} \leq 1+\frac{1}{n}-\frac{1}{d}$. Now we can proceed further and look at Section 5.3.6. Precisely, by $(5.6.6)_1$ and $(5.6.7)_2$ we see that the exponent $\bar{\sigma}$, defined in (5.3.52), appearing in (5.3.68) is equal to zero, therefore the rest of the proof of Proposition 5.3.3 adjusts accordingly without requiring other restrictions on the size of q/p. In particular, the second condition on σ appearing in (5.1.14) can be neglected.

Step 5: The case $n = 2$

Notice that the discussion outlined in *Step 3* is still true when $n = 2$. We fix

$$\hat{p} > \max\left\{1+\sigma_g, 2m(1+\hat{\sigma}), \frac{2m(1+\sigma_g)}{\sigma_g}\right\}$$

and substitute the interpolation inequality appearing in Section 5.3.7 with the following

$$\begin{aligned}\|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|_{L^{2m(1+\hat{\sigma})}(B_{\tau_2})} &\leq \|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|^{\lambda_{\hat{p}}}_{L^{\hat{p}}(B_{\tau_2})}\|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|^{1-\lambda_{\hat{p}}}_{L^{1+\sigma_g}(B)} \\ &\leq c\|G_\varepsilon(\cdot,|Dv_\varepsilon|)\|^{\lambda_{\hat{p}}}_{L^{\hat{p}}(B_{\tau_2})}\|1+H(\cdot,Du)\|^{1-\lambda_{\hat{p}}}_{L^{1+\sigma_g}(B)},\end{aligned}$$

where we used (5.6.15), (5.6.16) and $(5.2.15)_3$ to control the $L^{1+\sigma_g}$-norm of $G_\varepsilon(\cdot, Dv_\varepsilon)$ and $c \equiv c(\texttt{data}_{\mathrm{H}})$. In the previous display, $\lambda_{\hat{p}} = \frac{\hat{p}[2m(1+\hat{\sigma})-(1+\sigma_g)]}{2m(1+\hat{\sigma})[\hat{p}-(1+\sigma_g)]}$ and we replaced the exponent p appearing in Section 5.3.7 with $\hat{p}$, otherwise some confusion may arise. Finally we see that

$$\frac{\hat{p}[2m(1+\sigma_g)-(1+\sigma_g)]}{2m[\hat{p}-(1+\sigma_g)]} < 1 \iff 2m(1+\sigma_g) < \hat{p}[1+\sigma_g-2m\hat{\sigma}],$$

which is possible also when $\hat{\sigma} = (2^{-1}-d^{-1})$ since $\sigma_g > 0$. We then look at the proof of Proposition 5.3.4 and, being $g_2(\cdot)/g_1(\cdot) \equiv \text{const}$, it goes through with the additional restriction $\hat{\sigma} < p-1$, anyway taken into account by (5.0.11).

Merging the content of the above five steps, we obtain (5.0.15) and the proof is complete.

5.6.3 Proof of Theorem 14

It is a corollary of Theorem 15.

5.6.4 Proof of Theorem 15

We deduce Theorem 15 from Theorems 17-18, by making a suitable choice of the functions $g_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$ and of the parameters $\sigma, \hat{\sigma}, \sigma', c_a, c_b, \gamma$ with $T = e$ and $\mu = 0$. For the sake of clarity, we split the proof of Theorem 15 into three steps.

Step 1: Computation of $g_1(\cdot)$ and $g_2(\cdot)$

With minor changes to [16, Section 6.2] we see that

$$\begin{cases} \mathbf{e}_0'(x,t) = c_0(x)p_0(x)t^{p_0(x)-1}\mathbf{e}_0(x,t) \\ \mathbf{e}_k'(x,t) = p_k(x)c_k(x)t^{p_0(x)-1}\Pi_k(x,t)\mathbf{e}_k(x,t) \quad \text{for} \quad k \geq 1, \end{cases} \tag{5.6.17}$$

where we set

$$\Pi_k(x,t) := \prod_{j=0}^{k-1} p_j(x)c_j(x)[\mathbf{e}_j(x,t)]^{p_{j+1}(x)} \quad \text{for } k \geq 1 \quad \text{and} \quad \Pi_0(x,t) := 1,$$

therefore

$$\begin{aligned} \mathbf{e}_k''(x,t) =& t^{p_0(x)-1}\mathbf{e}_k'(x,t)\left[p_k(x)c_k(x)\Pi_k(x,t) + \sum_{j=0}^{k-2}\left(p_{j+2}(x)p_{j+1}(x)c_{j+1}(x)\Pi_{j+1}(x,t)\right)\right] \\ &+ t^{p_0(x)-1}\mathbf{e}_k'(x,t)\left[\frac{p_0(x)-1}{t^{p_0(x)}} + p_1(x)p_0(x)c_0(x)\right] \quad \text{for} \quad k \geq 2 \end{aligned}$$

and

$$\begin{cases} \mathbf{e}_0''(x,t) = t^{p_0(x)-1}\mathbf{e}_0'(x,t)\left[p_0(x)c_0(x) + \frac{p_0(x)-1}{t^{p_0(x)}}\right] \\ \mathbf{e}_1''(x,t) = t^{p_0(x)-1}\mathbf{e}_1'(x,t)\left[p_1(x)p_0(x)c_0(x) + \mathbf{e}_0(x,t)^{p_1(x)}p_1(x)p_0(x)c_1(x)c_0(x) + \frac{p_0(x)-1}{t^{p_0(x)}}\right]. \end{cases}$$

Now, let $\phi \in C([0,\infty),[0,1])$ be a non-decreasing function so that $\phi(t) = 0$ when $t \in [0, e/2]$ and $\phi(t) = 1$ for $t \in [e,\infty)$. The previous computations prove that the functions $g_1(\cdot)$, $g_2(\cdot)$ bounding from below and above second derivatives of the integrands $\mathbf{e}_k(\cdot)$ can be defined as

$$\begin{cases} g_1(x,t) := \phi(t)(p_m-1)\mathbf{e}_k'(x,t)t^{-1} \\ g_2(x,t) := \phi(t)\left[\frac{\sqrt{Nn-1}}{\nu^k} + kp_M L\left(1 + \frac{2p_M+1}{\nu^k}\right)\right]\Pi_k(x,t)\mathbf{e}_k'(x,t)t^{p_0(x)-1} \end{cases} \quad k \geq 1, \tag{5.6.18}$$

while for $k = 0$ we get

$$\begin{cases} g_1(x,t) := \phi(t)(p_m-1)\mathbf{e}_0'(x,t)t^{-1} \\ g_2(x,t) := \phi(t)\left[\sqrt{Nn-1} + p_M(L+1)\right]\mathbf{e}_0'(x,t)t^{p_0(x)-1}, \end{cases} \tag{5.6.19}$$

where p_m is defined in (5.0.19). With the definition given in (5.6.19) we immediately get

$$G_e(x,t) = (p_m-1)\left[\mathbf{e}_k(x,t) - \mathbf{e}_k(x,e)\right] \quad \text{and} \quad \bar{G}_e(x,t) = G_e(x,t) + (e^2+1)^{\frac{p_m}{2}}.$$

Step 2: Determining $g_3(\cdot)$

Now we prove that

$$\begin{cases} g_3(x,t) = (1-\phi(t))\mathcal{M}_k + \phi(t)\mathcal{M}_k\mathfrak{e}_k\Pi_k(x,t)\mathbf{e}_k'(x,t)t^{p_0(x)}\log t & \text{if} \quad k \geq 1 \\ g_3(x,t) = (1-\phi(t))\mathcal{M}_0 + \phi(t)\mathcal{M}_0\mathfrak{e}_0\mathbf{e}_0'(x,t)t^{p_0(x)}\log t & \text{if} \quad k = 0, \end{cases} \tag{5.6.20}$$

where $\phi(\cdot)$ is the same cut-off function appearing in (5.6.18)-(5.6.19) and, for all $k \in \mathbb{N}$, $\mathcal{M}_0, \mathcal{M}_k \equiv \mathcal{M}_0, \mathcal{M}_k(n,k,\nu,L,p_M) \geq 1$, $\mathfrak{e}_0 := (\nu\log(e/2))^{-1}$, $\mathfrak{e}_k := \left(\min\{1,\nu\}^{2(k+1)}\log(e/2)\right)^{-1}$ are constants. A direct computation shows that

$$|\partial_{x,z}\mathbf{e}_k(x,|z|)| = |\partial_x\mathbf{e}_k'(x,|z|)|, \tag{5.6.21}$$

therefore if we are able to suitably bound $\partial_x \mathbf{e}'_k(x,|z|)$, we are done. To derive an explicit expression of $\partial_x \mathbf{e}'_k(x,|z|)$, we set $\mathbf{e}_{-1}(x,t) := t$ and introduce the auxiliary vector fields $\mathscr{D}_k, \mathscr{L}_k \colon \Omega \times (0,\infty) \to \mathbb{R}^n$ to quantify the space derivatives of $\mathbf{e}_k(\cdot)$ and $\Pi_k(\cdot)$. By induction, we have for all $k \geq 1$

$$\begin{cases} \partial_x \mathbf{e}_k(x,t) = \mathbf{e}_k(x,t)[\mathbf{e}_{k-1}(x,t)]^{p_k(x)}\mathscr{D}_k(x,t) \\ \partial_x[\mathbf{e}_{k-1}(x,t)]^{p_k(x)} := [\mathbf{e}_{k-1}(x,t)]^{p_k(x,t)}[\mathbf{e}_{k-2}(x,t)]^{p_{k-1}(x,t)} \\ \qquad\qquad \cdot \left[c_{k-1}(x)\partial_x p_k(x) + p_k(x)\mathscr{D}_{k-1}\right] \\ \mathscr{D}_k(x,t) := \partial_x c_k(x) + c_k(x)c_{k-1}(x)[\mathbf{e}_{k-2}(x,t)]^{p_{k-1}(x)}\partial_x p_k(x) \\ \qquad\qquad +c_k(x)p_k(x)[\mathbf{e}_{k-2}(x,t)]^{p_{k-1}(x)}\mathscr{D}_{k-1}(x,t) \\ \mathscr{D}_0(x,t) := \left[\partial_x c_0(x) + c_0(x)\log t\, \partial_x p_0(x)\right]. \end{cases} \tag{5.6.22}$$

Again, by induction and (5.6.22), for $k \geq 1$ we have

$$\begin{cases} \partial_x \Pi_0(x,t) = 0 =: \mathscr{L}_0(x,t) \\ \partial_x \Pi_k(x,t) = \Pi_k(x,t)\mathscr{L}_k(x,t) \\ \mathscr{L}_k(x,t) := \Big[\partial_x \log\left(c_{k-1}(x)p_{k-1}(x)\right) \\ \qquad +[\mathbf{e}_{k-2}(x,t)]^{p_{k-1}(x)}\left(c_{k-1}(x)\partial_x p_k(x) + p_k(x)\mathscr{D}_{k-1}(x,t)\right) + \mathscr{L}_{k-1}(x,t)\Big]. \end{cases} \tag{5.6.23}$$

Combining (5.6.22)-(5.6.23) with (5.6.17), for all $k \in \mathbb{N}\cup\{0\}$ we deduce that

$$\begin{aligned} \partial_x \mathbf{e}'_k(x,t) &= \mathbf{e}'_k(x,t) \\ &\cdot \left[\partial_x \left(\log(c_k(x)p_k(x))\right) + \log t \partial_x p_0(x) + \mathscr{L}_k(x,t) + [\mathbf{e}_{k-1}(x,t)]^{p_k(x)}\mathscr{D}_k(x,t)\right]. \end{aligned} \tag{5.6.24}$$

Define

$$\hat{h}_k := \max_{i\in\{0,\cdots,k\}} \left\{\max\{1, |\partial_x p_i|, |\partial_x c_i|\}\right\} \overset{(5.0.19)}{\in} L^d_{\mathrm{loc}}(\Omega). \tag{5.6.25}$$

Recalling that by (5.0.19) it is $p_m > 1$ and looking at the explicit expansion in (5.6.24), we immediately deduce that for all $k \in \mathbb{N}\cup\{0\}$ there exits a constant $\mathscr{M}_k \equiv \mathscr{M}_k(n,k,\nu,L,p_M) \geq 1$ verifying

$$|\partial_x \mathbf{e}'_k(x,t)| \leq \mathscr{M}_k \hat{h}_k(x) \quad \text{for all} \quad (x,t) \in \Omega \times [0,e]. \tag{5.6.26}$$

Now we shall prove that for all $k \in \mathbb{N}$ there exists a function $0 \leq h_k(\cdot) \in L^d_{\mathrm{loc}}(\Omega)$ such that

$$\begin{cases} |\partial_x \mathbf{e}'_k(x,t)| \leq h_k(x)\mathbf{e}'_k(x,t)\Pi_k(x,t)t^{p_0(x)}\log t & \text{for } \ k \geq 1 \\ |\partial_x \mathbf{e}'_0(x,t)| \leq h_0(x)\mathbf{e}'_0(x,t)t^{p_0(x)}\log t & \text{for } \ k = 0. \end{cases} \tag{5.6.27}$$

for all $(x,t) \in \Omega \times [e,\infty)$. The estimates in (5.6.27) are actually a consequence of the following lemma.

Lemma 5.6.4 *For any $k \in \mathbb{N}$, there exist functions $0 \leq \tilde{h}_0(\cdot),\ h_k(\cdot) \in L^d_{\mathrm{loc}}(\Omega)$ so that*

$$\begin{cases} |\mathscr{D}_0(x,t)| \leq \tilde{h}_0(x)\log t \quad \textit{and} \quad |\mathscr{L}_0(x,t)| = 0 \\ |\mathscr{D}_k(x,t)|, |\mathscr{L}_k(x,t)| \leq \tilde{h}_k(x)t^{p_0(x)}\log t\Pi_{k-1}(x,t) \end{cases} \tag{5.6.28}$$

hold for all $(x,t) \in \Omega \times [e,\infty)$.

Proof. When $k = 0$, we easily have:

$$|\mathscr{D}_0(x,t)| \overset{(5.6.22)_3}{\leq} \hat{h}_0(x)(1+L)\log t \quad \text{and} \quad |\mathscr{L}_0(x,t)| \overset{(5.6.23)_1}{=} 0, \tag{5.6.29}$$

for all $(x,t) \in \Omega \times [e,\infty]$, which is $(5.6.28)_1$ with

$$\tilde{h}_0 := (1+L)\hat{h}_0 \overset{(5.6.25)}{\in} L^d_{\text{loc}}(\Omega). \tag{5.6.30}$$

We then proceed by induction. For $k = 1$, by (5.6.22)-(5.6.23) and (5.6.29) we get

$$|\mathscr{D}_1(x,t)| \leq t^{p_0(x)} \log t \left[(1+L^2)\hat{h}_1(x) + Lp_M\tilde{h}_0(x)\right] \quad \text{for all} \quad (x,t) \in \Omega \times [e,\infty)$$

and

$$|\mathscr{L}_1(x,t)| \leq t^{p_0(x)} \log t \left[(p_m^{-1} + \nu^{-1} + L)\hat{h}_1(x) + p_M\tilde{h}_0(x)\right],$$

thus $(5.6.28)_2$ follows when $k = 1$ by setting

$$\tilde{h}_1 := \max\left\{\left[(1+L^2)\hat{h}_1 + Lp_M\tilde{h}_0\right], \left[(p_m^{-1} + \nu^{-1} + L)\hat{h}_1 + p_M\tilde{h}_0\right]\right\} \overset{(5.6.25),(5.6.30)}{\in} L^d_{\text{loc}}(\Omega).$$

Assume now that there exists a function $0 \leq \tilde{h}_k(\cdot) \in L^d_{\text{loc}}(\Omega)$ such that

$$|\mathscr{D}_k(x,t)|, |\mathscr{L}_k(x,t)| \leq \tilde{h}_k(x)t^{p_0(x)} \log t\Pi_{k-1}(x,t) \tag{5.6.31}$$

for all $(x,t) \in \Omega \times [e,\infty)$. By means of (5.6.22)-(5.6.23) and (5.6.31) we bound

$$|\mathscr{D}_{k+1}(x,t)| \leq t^{p_0(x)} \log t\Pi_k(x,t) \left[\frac{Lp_M}{\nu}\tilde{h}_k(x) + \frac{1+L^2}{\min\{1,\nu\}^{k+1}}\hat{h}_{k+1}(x)\right]$$

and

$$|\mathscr{L}_{k+1}(x,t)| \leq \frac{1}{\min\{1,\nu\}^{k+4}} \left[(\nu^{-1} + p_m^{-1} + L)\hat{h}_{k+1}(x) + (p_M+1)\tilde{h}_k(x)\right] t^{p_0(x)} \log t\Pi_k(x,t)$$

so we can conclude by choosing

$$\begin{aligned}\tilde{h}_{k+1} :=& \frac{1}{\min\{1,\nu\}^{k+4}} \max\left\{\left[\frac{Lp_M}{\nu}\tilde{h}_k + (1+L^2)\hat{h}_{k+1}\right],\right.\\ &\left.\left[(\nu^{-1} + p_m^{-1} + L)\hat{h}_{k+1} + (p_M+1)\tilde{h}_k\right]\right\} \overset{(5.6.25),(5.6.31)}{\in} L^d_{\text{loc}}(\Omega).\end{aligned}$$

□

With (5.6.28) at hand, (5.6.27) follows from (5.6.24) by choosing

$$h_k := \frac{1}{\min\{1,\nu\}^{k+4}} \left[(p_m^{-1} + \nu^{-1} + 1)\hat{h}_k + \tilde{h}_k + 1\right] \overset{(5.6.25),(5.6.28)}{\in} L^d_{\text{loc}}(\Omega).$$

This means that $g_3(\cdot)$ is exactly the function appearing in (5.6.20).

Step 3: Conclusions

Before verifying that $g_1(\cdot)$, $g_2(\cdot)$ and $g_3(\cdot)$ satisfy the mutual relations prescribed in Sections 5.1.1-5.1.3, let us prove another auxiliary result.

Lemma 5.6.5 *For any $k \in \mathbb{N}$, $\delta \in (0,1)$, there exists a positive constant $c \equiv c(\nu, L, p_M, \delta, k)$ such that*

$$\Pi_k(x,t)t^{p_0(x)+1}\log t \le c\mathbf{e}_k(x,t)^{\delta} \quad \textit{for all} \quad (x,t) \in \Omega \times [e,\infty). \tag{5.6.32}$$

Proof. It is well known that

$$t \ge 1 \implies t^{\delta'} \ge \delta' \log t \quad \text{for all} \quad \delta' > 0, \tag{5.6.33}$$

therefore, using the definition in (5.0.20) we have

$$\begin{aligned}\mathbf{e}_k(x,t) &= \left[\frac{1}{c_{k+1}(x)}\log(\mathbf{e}_{k+1}(x,t))\right]^{\frac{1}{p_{k+1}(x)}} \\ &\le \frac{1}{\min\{\nu,1\}}\max\left\{1, \log(\mathbf{e}_{k+1}(x,t))\right\} \le \frac{2}{\min\{\nu,1\}\delta'}\mathbf{e}_{k+1}(x,t)^{\delta'},\end{aligned} \tag{5.6.34}$$

for all $(x,t) \in \Omega \times [e,\infty)$. Fix any $\delta \in (0,1)$. We use the fact that

$$\mathbf{e}_l(x,t) \le \min\{1,\nu\}^{-1}\mathbf{e}_{l+1}(x,t) \quad \text{for all} \quad l \in \mathbb{N}\cup\{0\}, \tag{5.6.35}$$

for all $l \ge 0$, (5.6.34) and (5.6.33) with $\delta' := \frac{\delta}{4(k+1)}$ to get

$$\Pi_k(x,t) \le \left(\frac{p_M^k L^k}{\min\{1,\nu\}^k}\right)\prod_{l=0}^{k-1}\max\left\{1, \log(\mathbf{e}_{l+1})\right\} \le \left(\frac{16kp_M L}{\min\{\nu,1\}\delta}\right)^{4k}\mathbf{e}_k(x,t)^{\frac{\delta}{4}}. \tag{5.6.36}$$

Moreover, by (5.6.33) for $\delta' = \frac{\delta}{2}$ we get, after a straightforward manipulation

$$t^{p_0(x)+1}\log t \le \frac{t^{p_0(x)+1+\delta'}}{\delta'} \le \frac{1}{c_e}\mathbf{e}_0(x,t)^{\delta'}, \qquad c_e \equiv c_e(\nu, p_M, \delta') << 1$$

so applying (5.6.35) and (5.6.36) we end up with

$$t^{p_0(x)+1}\log t \le c\mathbf{e}_k(x,t)^{\frac{\delta}{2}}, \tag{5.6.37}$$

with $c \equiv c(\nu, L, p_M, \delta, k)$. Merging (5.6.36)-(5.6.37) we obtain (5.6.32). □

We notice that (5.1.1)-(5.1.3) are satisfied and $t \mapsto g_2(x,t)/g_1(x,t)$, $t \mapsto g_1(x,t)t$ are both increasing for all $(x,t) \in \Omega \times [e,\infty)$, so (1.1.3) holds with $c_a = 1$. Concerning (5.1.4), it is satisfied by choosing $\gamma = p_m$. Conditions (5.1.15)-(5.1.18) (and (5.1.19) when $n = 2$) follow from the definitions of $g_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$ and (5.6.32) for $h \equiv h_k$, all $\sigma, \hat{\sigma} > 0$ and $c_b \equiv c_b(n, N, \nu, L, p_m, p_M, \sigma, \hat{\sigma}, k)$. This means that the assumptions of Theorems 17-18 are satisfied, so (5.0.22) is true.

5.6.5 Proof of Theorem 16

From (1.2.20) we readily see that (5.1.1) and (5.1.2) are verified. A straightforward computation shows that

$$\begin{aligned}&|\partial_z^2 A(x,|z|)| \le L\left[1+\sqrt{Nn-1}\right]\max\{1, 1+s_a\}\tilde{a}(|z|), \\ &\partial_z^2 A(x,|z|)\xi\cdot\xi \ge \min\{1, 1+i_a\}\tilde{a}(|z|)|\xi|^2,\end{aligned}$$

$$|\partial_{x,z}A(x,|z|)| \leq |Dc(x)|\tilde{a}(|z|)|z|,$$

so we can define:

$$g_1(t) := \nu \min\{1, 1+i_a\}\tilde{a}(t) \quad \text{and} \quad g_2(t) := L\left[1+\sqrt{Nn-1}\right]\max\{1,1+s_a\}\tilde{a}(t),$$
$$g_3(t) := \tilde{a}(t)t, \qquad h := |Dc| \in X_{\mathrm{loc}}(\Omega),$$

with $X_{\mathrm{loc}}(\Omega)$ as in (5.1.24). A consequence of (1.2.20) is that $t^{-i_a}\tilde{a}(t)$ is non-decreasing and $t^{-s_a}\tilde{a}(t)$ is non-increasing, see [9, Section 3], therefore it is easy to see that (5.1.4) is verified with $\gamma = i_a + 2$, $\mu = 0$ and $g_1(t)t$ is non-decreasing. The definitions given above assure that conditions in (5.1.21) are satisfied with

$$K = \frac{L\max\{1,1+s_a\}\left(1+\sqrt{Nn-1}\right)}{\min\{1,1+i_a\}}.$$

This means that Theorem 20 applies and the Lipschitz estimate (5.0.24) holds true.

5.7 Appendix: Functionals with standard growth

In this final section we justify the claim in (5.3.9), which is necessary to carry out the rest of the estimates in Sections 5.3 and 5.5. We therefore consider a functional like (5.0.1), with the integrand $F(\cdot)$ satisfying the structure condition $(5.1.2)_1$, where $\tilde{F}\colon \Omega\times[0,\infty)\to\mathbb{R}$ is such that

$$\begin{cases} t \mapsto \tilde{F}(x,t) \in C^1_{\mathrm{loc}}[0,\infty)\cap W^{2,\infty}_{\mathrm{loc}}[0,\infty) & \text{for all } \ x\in\Omega \\ x \mapsto \tilde{F}'(x,t) \in W_{\mathrm{loc}}(1;X)(\Omega) & \text{for every } \ t>0\,. \end{cases} \tag{5.7.1}$$

The last condition means that $|\partial_x \tilde{F}'(\cdot,t)| \in X_{\mathrm{loc}}(\Omega)$ for every $t>0$, where $X_{\mathrm{loc}}(\Omega)$ has been defined in (5.1.24). We assume that

$$\begin{cases} \nu(|z|^2+\mu^2)^{\gamma/2} - \Lambda\mu^\gamma \leq F(x,z) \leq \Lambda(|z|^2+\mu^2)^{\gamma/2} \\ |\partial_{zz}F(x,z)| \leq \Lambda(|z|^2+\mu^2)^{(\gamma-2)/2} \\ \nu(|z|^2+\mu^2)^{(\gamma-2)/2}|\xi|^2 \leq \partial_{zz}F(x,z)\xi\cdot\xi \\ |\partial_{xz}F(x,z)| \leq \Lambda h(x)(|z|^2+\mu^2)^{(\gamma-1)/2} \\ |f|, h \in X(\Omega) \end{cases} \tag{5.7.2}$$

hold for $x\in\Omega$ and $z,\xi\in\mathbb{R}^{N\times n}$ (provided $\partial_{zz}F(x,z)$ exists). In (5.7.2), it is $\gamma>1$, $0<\nu\leq 1\leq\Lambda$, $0<\mu\leq 1$. This functional is of the type considered in (5.3.4) by (5.2.13) and (5.2.19), so that the claim in (5.3.9) is justified by the following:

Theorem 21 *Let $u\in W^{1,\gamma}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a minimizer of $\mathcal{F}$ in (5.0.1), under assumptions (5.7.1)-(5.7.2). Then $Du\in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$, $u\in W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and $a(\cdot,Du)\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$.*

The proof of Theorem 21 goes now in four different steps, where we essentially revisit and readapt a few hidden facts in the literature.

Step 1: Introduction of approximating problems

We revisit the procedure we used in [77, Theorem 4, Step 1] and start fixing a ball $\mathcal{B}\Subset\Omega$ such that $\mathtt{r}(\mathcal{B})\leq 1$. For this, we first extend $\tilde{F}$ by reflection, i.e., $\tilde{F}(x,t) := \tilde{F}(x,-t)$, and then we consider standard, radially symmetric mollifiers $\phi_1\in C^\infty_c(B_1)$, $\phi_2\in C^\infty_c(-1,1)$, $\|\phi_1\|_{L^1(\mathbb{R}^n)} = \|\phi_2\|_{L^1(\mathbb{R})} =$

$1, \phi_{1,\delta}(x) := \delta^{-n}\phi_1(x/\delta)$, $\phi_{2,\delta}(x) := \delta^{-1}\phi(x/\delta)$, $B_{3/4}(0) \subset \operatorname{supp}\phi_1, (-3/4, 3/4) \subset \operatorname{supp}\phi_2$. With $\delta \in \big(0, \operatorname{dist}(\mathcal{B}, \partial\Omega)/2\big)$, define

$$\tilde{F}_\delta(x,t) := \fint_{(-1,1)} \fint_{B_1} \tilde{F}(x+\delta y, t+\delta s)\phi_1(y)\phi_2(s)\,dy\,ds\,, \tag{5.7.3}$$

for all $(x,t) \in \mathcal{B}\times\mathbb{R}$, and $h_\delta(x) := (h * \phi_{1,\delta})(x)$. By setting $F_\delta(x,z) := \tilde{F}_\delta(x,|z|)$, we obtain a family of smooth integrands satisfying

$$\begin{cases} \frac{1}{\tilde{c}}(|z|^2+\mu_\delta^2)^{\gamma/2} - \tilde{c}\mu_\delta^\gamma \le F_\delta(x,z) \le \tilde{c}(|z|^2+\mu_\delta^2)^{\gamma/2} \\ |\partial_{zz}F_\delta(x,z)| \le \tilde{c}(|z|^2+\mu_\delta^2)^{(\gamma-2)/2} \\ \frac{1}{\tilde{c}}(|z|^2+\mu_\delta^2)^{(\gamma-2)/2}|\xi|^2 \le \partial_{zz}F_\delta(x,z)\xi\cdot\xi \\ |\partial_{xz}F_\delta(x,z)| \le \tilde{c}h_\delta(x)(|z|^2+\mu_\delta^2)^{(\gamma-1)/2}\,, \end{cases} \tag{5.7.4}$$

for every $x \in \Omega'$, $z, \xi \in \mathbb{R}^{N\times n}$, where $\tilde{c} = \tilde{c}(n, N, \nu, \Lambda, \gamma)$ is a positive constant, and, as usual, it is $\mu_\delta := \mu + \delta$. The verification of (5.7.4) follows straightaway from the definition in (5.7.3), but the one of $(5.7.4)_3$, that maybe deserves some more explanation. For this, denote as usual $\tilde{a}_\delta(x,t) := \tilde{F}'_\delta(x,t)/t$, and notice that $(5.7.4)_3$ is equivalent to

$$(t^2+\mu_\delta^2)^{(\gamma-2)/2} \lesssim \tilde{a}_\delta(x,t) \quad \text{and} \quad (t^2+\mu_\delta^2)^{(\gamma-2)/2} \lesssim \tilde{a}_\delta(x,t) + \tilde{a}'_\delta(x,t)t = \tilde{F}''_\delta(x,t)\,. \tag{5.7.5}$$

See also the arguments for (5.3.5)-(5.3.6) and Lemma 5.3.1; here all the implied constants in the symbol $\lesssim$ depend only on n, N, ν, Λ and γ. To prove the first inequality in (5.7.5) notice that $\tilde{F}_\delta(x,t)$ is still such that $\tilde{F}_\delta(x,t) = \tilde{F}_\delta(x,-t)$ for every $t \in \mathbb{R}$ so that $\tilde{F}'_\delta(x,0) = 0$. Moreover, from $(5.7.2)_3$ and again the equivalence in (5.7.5) applied this time to the original integrand $\tilde{F}(\cdot)$, it follows that $(t^2+\mu^2)^{(\gamma-2)/2} \lesssim \tilde{F}''(x,t)$. From this and the definition in (5.7.3), following the same argument in [77, Section 4.5] we gain $(t^2+\mu_\delta^2)^{(\gamma-2)/2} \lesssim \tilde{F}''_\delta(x,t)$, which is the second relation in (5.7.5). In turn, integrating this last inequality and using $\tilde{F}'_\delta(x,0) = 0$, (implied by $\tilde{F}(x,t) = \tilde{F}(x,-t)$), yields $(t^2+\mu_\delta^2)^{(\gamma-2)/2}t \lesssim \tilde{F}'_\delta(x,t)$, which is in fact the first inequality in (5.7.5) and $(5.7.4)_3$ is verified. By the very definitions of $\tilde{F}_\delta(\cdot)$ and h_δ, we also have

$$\begin{cases} F_\delta(x,z) \to F(x,z) & \text{uniformly on compact subsets of } \overline{\mathcal{B}}\times\mathbb{R}^n \text{ as } \delta \to 0 \\ & \|h_\delta\|_{\mathfrak{X}(B)} \le c\|h\|_{\mathfrak{X}(B+\delta B_1(0))} \end{cases} \tag{5.7.6}$$

whenever $B \subset \mathcal{B}$ is a ball such that $B + \delta B_1(0) \subset \Omega$, where c is independent of δ and $h(\cdot)$. Next, we set $f_\delta \in L^\infty(\Omega, \mathbb{R}^N)$ as $f_\delta(x) := f(x)$ if $|f(x)| \le 1/\delta$, and $f_\delta(x) := \delta^{-1}|f(x)|^{-1}f(x)$ otherwise. Finally, we define $u_\delta \in W^{1,\gamma}(\mathcal{B}, \mathbb{R}^{N\times n})$ as the unique solution of the Dirichlet problem

$$u + W_0^{1,\gamma}(\mathcal{B}, \mathbb{R}^N) \ni w \mapsto \min \int_{\mathcal{B}} \left[F_\delta(x, Dw) - f_\delta\cdot w\right]\,dx\,. \tag{5.7.7}$$

Up to now, we have just required that δ is small enough to satisfy $\delta < \operatorname{dist}(\mathcal{B}, \partial\Omega)/2$. In the next step we shall choose additional smallness conditions on δ.

Step 2: $Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$

Thanks to (5.7.4), standard regularity theory yields

$$Du_\delta \in L^\infty_{\mathrm{loc}}(\mathcal{B}, \mathbb{R}^{N\times n}), \quad u_\delta \in W^{2,2}_{\mathrm{loc}}(\mathcal{B}, \mathbb{R}^N), \quad \partial_z F_\delta(x, Du_\delta) \in W^{1,2}_{\mathrm{loc}}(\mathcal{B}, \mathbb{R}^{N\times n})\,. \tag{5.7.8}$$

We can therefore proceed exactly as in the proof of Proposition 5.5.1. This yields the existence of $\delta_0 \equiv \delta_0(n, N, \nu, \gamma, \Lambda, h(\cdot)) \in (0,1)$ and $R_* \equiv R_*(n, N, \nu, \gamma, \Lambda, h(\cdot), \alpha) \le 1$ such that the estimate

$$\|Du_\delta\|_{L^\infty(t\mathcal{B})} \le \frac{c}{[(1-t)\mathtt{r}(\mathcal{B})]^{n/\gamma}}\left[\|F_\delta(\cdot, Du_\delta)\|^{1/\gamma}_{L^1(\mathcal{B})} + 1\right] + c\|f_\delta\|^{1/(\gamma-1)}_{\mathfrak{X}(\mathcal{B})} \tag{5.7.9}$$

holds whenever $t \in (0,1)$, provided $\delta \leq \min\{\delta_0, \operatorname{dist}(\mathcal{B}, \partial\Omega)/2\}$ and $\mathtt{r}(\mathcal{B}) \leq R_*$, where $c \equiv c(n, N, \nu, \Lambda, \gamma, \alpha) \geq 1$ is independent of δ. Indeed, the setting of Proposition 5.5.1 applies with the obvious choices $g_{1,\varepsilon}(t) \equiv (t^2 + \mu_\delta^2)^{(\gamma-2)/2}/\tilde{c}$, $g_{2,\varepsilon}(t) \equiv \tilde{c}(t^2 + \mu_\delta^2)^{(\gamma-2)/2}$, $g_{3,\varepsilon}(t) \equiv \tilde{c}(t^2 + \mu_\delta^2)^{(\gamma-1)/2}$ and $h(\cdot) \equiv h_\delta(\cdot)$; moreover, the only qualitative properties needed to argue as in Proposition 5.5.1 are those in (5.3.9), that are exactly those in (5.7.8). Therefore the whole bunch of estimates developed there applies here verbatim. Notice that, proceeding as in Proposition 5.5.1, and recalling (5.5.11), the radius R_* here should exhibit a dependence on $h_\delta(\cdot)$, and therefore ultimately on δ. However, R_* can be made independent of δ, thanks to $(5.7.6)_2$ by further taking δ small enough, and without creating vicious circles. Specifically, we arrive at (5.5.10) with the above choice of $g_{1,\varepsilon}(\cdot), g_{2,\varepsilon}(\cdot), g_{3,\varepsilon}(\cdot)$ and (5.5.11) turns out to be $c_*\|h_\delta\|_{\mathfrak{X}(\mathcal{B})} \leq 1/6$, where c_* is independent of δ but only depends of $\mathtt{data}_{\mathrm{uni}}$. Given the actual choice of parameters, c_* depends only on $n, N, \nu, \Lambda, \gamma, \alpha$. We use $(5.7.6)_2$ to reduce the last condition to $cc_*\|h\|_{\mathfrak{X}(\mathcal{B}+\delta B_1(0))} \leq 1/6$, where c is the constant appearing in $(5.7.6)_2$ and it is independent of δ. Therefore, by absolute continuity we find $\delta_0, R_* \equiv \delta_0, R_*(n, N, \nu, \gamma, \Lambda, \alpha, h(\cdot))$ as described above, such that the last inequality is satisfied. This allows to set inequality (5.7.9) free from any dependence on δ. Next, observe that, using $(5.7.2)_1$, $(5.7.4)_1$ and finally the minimality of u_δ in (5.7.9), we gain

$$\|Du_\delta\|_{L^\infty(t\mathcal{B})} \leq \frac{c}{[(1-t)\mathtt{r}(\mathcal{B})]^{n/\gamma}} \left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})}^{1/\gamma} + 1\right] + c\|f\|_{\mathfrak{X}(\mathcal{B})}^{1/(\gamma-1)} \tag{5.7.10}$$

with c being independent of δ. From this, a standard convergence argument based on $(5.7.6)_1$ (see again the proof of Theorem 19) extracting a subsequence $\{u_\delta\}$ such that $u_\delta \rightharpoonup^* u$ weakly in $W^{1,\infty}(t\mathcal{B}; \mathbb{R}^N)$, leads to

$$\|Du\|_{L^\infty(t\mathcal{B})} \leq \frac{c}{[(1-t)\mathtt{r}(\mathcal{B})]^{n/\gamma}} \left[\|F(\cdot, Du)\|_{L^1(\mathcal{B})}^{1/\gamma} + 1\right] + c\|f\|_{\mathfrak{X}(\mathcal{B})}^{1/(\gamma-1)}$$

and a standard covering argument gives that Du is locally bounded.

Step 3: $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$

For this we shall reuse some arguments from [164, Theorems 4.5-4.6]. We test the weak formulation of the Euler-Lagrange system $-\operatorname{div} \partial_z F_\delta(x, Du_\delta) = f_\delta$ by $D_s\varphi$, for $s \in \{1, \ldots, n\}$ and $\varphi \in C_0^\infty(\mathcal{B})$; integration by parts yields

$$\int_{\mathcal{B}} D_s[\partial F_\delta(x, Du_\delta)] \cdot D\varphi \,\mathrm{d}x = -\int_{\mathcal{B}} f_\delta \cdot D_s\varphi \,\mathrm{d}x\,. \tag{5.7.11}$$

We then take $\eta \in C_c^\infty(\mathcal{B}/2; [0,1])$ with $\eta \equiv 1$ in $\mathcal{B}/4$, $\|D\eta\|_{L^\infty(\mathcal{B})} \lesssim 1/[\mathtt{r}(\mathcal{B})]$, and we define $\varphi := \eta^2 D_s u_\delta$ so that $\varphi \in W_0^{1,2}(\mathcal{B}, \mathbb{R}^N)$. Using φ as test function, summing over $s \in \{1, \cdots n\}$ and using (5.7.4), yields

$$\begin{aligned}\int_{\mathcal{B}} (|Du_\delta|^2 + \mu_\delta^2)^{\frac{\gamma-2}{2}} |D^2 u_\delta|^2 \eta^2 \,\mathrm{d}x &\leq c\int_{\mathcal{B}} \eta(|Du_\delta|^2 + \mu_\delta^2)^{\frac{\gamma-2}{2}} |D^2 u_\delta||Du_\delta||D\eta| \,\mathrm{d}x \\ &+c\int_{\mathcal{B}} h_\delta(|Du_\delta|^2 + \mu_\delta^2)^{\frac{\gamma-1}{2}} [\eta^2|D^2 u_\delta| + \eta|D\eta||Du_\delta|] \,\mathrm{d}x + \sum_{s=1}^n \int_B |f_\delta||D\varphi_s| \,\mathrm{d}x\,.\end{aligned}$$

Estimating the third integral in a standard way (see for instance in [164, pag. 395]) we get

$$\begin{aligned}\int_{\mathcal{B}} (|Du_\delta|^2 + \mu_\delta^2)^{\frac{\gamma-2}{2}} |D^2 u_\delta|^2 \eta^2 \,\mathrm{d}x &\leq c[\mathtt{r}(\mathcal{B})]^{-2} \int_{\mathcal{B}} \eta[1 + h_\delta^2](|Du_\delta|^2 + \mu_\delta^2)^{\gamma/2} \,\mathrm{d}x \\ &+ c[\mathtt{r}(\mathcal{B})]^{-1}\|f\|_{L^2(\mathcal{B})}\|Du_\delta\|_{L^2(\mathcal{B}/2)} + c\|f\|_{L^2(\mathcal{B})}\|\eta^2 D^2 u_\delta\|_{L^2(\mathcal{B})}.\end{aligned} \tag{5.7.12}$$

The involved constant c only depends on n, N, ν, Λ and γ and is otherwise independent of $\delta \in (0,1)$. We now set $M := \sup_\delta \|Du_\delta\|_{L^\infty(\mathcal{B}/2)} + 1$, which is a finite quantity by (5.7.10). We start considering the case $\gamma \geq 2$, where we have

$$\mu^{\gamma-2}\|\eta D^2 u_\delta\|^2_{L^2(\mathcal{B})} \leq c\|1+h_\delta\|^2_{L^2(\mathcal{B})}(M^2+\mu_\delta^2)^{\gamma/2} + c\|f\|_{L^2(\mathcal{B})}M + c\|f\|_{L^2(\mathcal{B})}\|\eta^2 D^2 u_\delta\|_{L^2(\mathcal{B})}$$

where $c \equiv c(n, N, \nu, \Lambda, \gamma, \mathtt{r}(\mathcal{B}))$ and therefore, via Young's inequality, we get

$$\|D^2 u_\delta\|^2_{L^2(\mathcal{B}/4)} \leq c\mu^{2-\gamma}\|1+h_\delta\|^2_{L^2(\mathcal{B})}[M^2+1]^{\gamma/2} + c\mu^{2-\gamma}\|f\|_{L^2(\mathcal{B})}M + c\mu^{2(2-\gamma)}\|f\|^2_{L^2(\mathcal{B})}$$

which is a uniform (with respect to δ) local bound for $\{D^2 u_\delta\}$:

$$\|D^2 u_\delta\|_{L^2(\mathcal{B}/4)} \leq c(n, N, \nu, \gamma, \Lambda, \|h\|_{L^2(\mathcal{B})}, \|f\|_{L^2(\mathcal{B})}, \mathtt{r}(\mathcal{B}), M, \mu)\,. \tag{5.7.13}$$

In the case $1 < \gamma < 2$, we can argue exactly as after (5.7.12), but replacing μ by M, thereby getting again (5.7.13). Starting from (5.7.13), using the same approximation argument for the proof of Theorem 19 and in Step 2 here, we can let $\delta \to 0$ (via a subsequence) in (5.7.13) finally getting a local upper bound for D^2u in L^2. The assertion then follows via the usual covering argument.

Step 4: $a(\cdot, Du) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$

The claim follows by using the content of the previous two steps and the non-autonomous chain rule (cf. the discussion made at the beginning of Section 5.3).

Chapter 6

Higher differentiability for minimizers of variational integrals

Joint work with L. Koch (University of Oxford) and J. Kristensen (University of Oxford)
***Preprint* (2020)**

We prove higher integrability and weak differentiability results for vector-valued local minima of autonomous integrals of the calculus of variations

$$W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \ni w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega F(Dw)\ \mathrm{d}x, \tag{6.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded set, $n \geq 2$, $N \geq 1$ and $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous integrand satisfying (p,q)-growth, in the sense that

$$|z|^p \lesssim F(z) \lesssim 1 + |z|^q \qquad \text{for} \ \ 1 < p \leq q, \tag{6.0.2}$$

for all $z \in \mathbb{R}^{N\times n}$, where "$\lesssim$" means that the inequality displayed above holds up to absolute constants. A systematic study of the regularity of minimizers of these variational integrals started with the fundamental papers [181,183,184] and, subsequently, has undergone an intensive development over the last years, see [9,16–18,25,28,29,44–46,74,89,99,100,105,144] and references therein for a list of the most recent advances in the field. As pointed out by the counterexamples contained in [114, 182, 184], a necessary and sufficient condition for the regularity of minima of the functional in (6.0.1) is that the exponents (p,q) cannot be too far apart from each other. Precisely, it turns out that (1.2.6) guarantees regularity. However, the optimal expression of the quantity implicit in $\mathsf{o}(n)$ is still missing. In fact, the classical bound in force for proving $W^{1,q}$-local regularity of $W^{1,p}$-minima of (6.0.1) is (1.2.9), see [16, 99, 100, 183]. Recently, in [18] condition (1.2.6) was improved in the scalar case $N = 1$ to (1.2.10) by using a refinement of Moser's iteration technique via optimization on radial cut-off functions, allowing the use of Sobolev inequality on spheres rather than on balls. In this respect, we propose a new approach to the regularity theory for minima of functionals with (p,q)-growth based on convex duality. Our methods find their roots in [45], where convex duality methods are employed to prove that global minima of (6.0.1) belong to $W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ provided that

$$1 \leq \frac{q}{p} < \frac{n}{n-1},$$

assuming only the natural growth condition (6.0.2). The procedure developed in [45] plays on the duality between the gradient of the minimum Du and the stress tensor $\partial_z F(Du)$. In fact, it turns out that if $u \in g + W_0^{1,p}(\Omega, \mathbb{R}^N)$ is a global minimizer of (6.0.1) for an assigned boundary datum $g \in W^{1,q}(\Omega, \mathbb{R}^N)$, then $\partial_z F(Du)$ is the unique maximizer of the functional

$$\zeta \mapsto \int_\Omega \left[\zeta \cdot Dg - F^*(\zeta)\right] \, \mathrm{d}x$$

defined over all the row-wise solenoidal vector fields $\zeta \in L^{q'}(\Omega, \mathbb{R}^{N\times n})$. In the above display, $F^*(\cdot)$ is the Fenchel conjugate of $F(\cdot)$. We further elaborate on such arguments and introduce the dual controlled growth condition (6.1.4) below. It is slightly stricter than the usual controlled (p, q) growth conditions appearing e.g. in [16, 99, 100, 183], nonetheless it is satisfied by the main autonomous models with (p, q)-growth, such as

$$\mathcal{F}_1(w, \Omega) := \int_\Omega \left[|Dw|^p + \sum_{i=1}^{n} |D_i w|^{p_i}\right] \, \mathrm{d}x,$$

where $p \le p_i$ for all $i \in \{1, \cdots, n\}$ and $q := \max_{i\in\{1,\cdots,n\}} p_i$ and

$$\mathcal{F}_2(w, \Omega) := \int_\Omega \sum_{i=1}^{k} |Dw|^{p_i} \, \mathrm{d}x,$$

with $p := \min_{i\in\{1,\cdots,n\}} p_i$ and $q := \max_{i\in\{1,\cdots,n\}} p_i$. As remarked by the example in [74], employing the full structure of the integrand leads to better results then those obtained retaining only the growth from above and below of its second derivatives, which could lead to a severe loss of information. This is precisely what we do, as (6.1.4) below allows us to fully exploit the strong convexity of the integrand $F(\cdot)$. In this perspective, our main result is the following theorem.

Theorem 22 *Under assumptions (6.1.1)-(6.1.4), let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of functional (6.2.1). Then,*

$$V_{\mu,p}(Du), V_{\mu,q'}(\partial_z F(Du)) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}).$$

As a consequence,

$$u \in W^{1,d}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \text{for all } \ d \in [1, m(n)],$$

where

$$m(n) := \begin{cases} \frac{np}{n-2} & \text{if } \ n \ge 3 \\ \text{any number in } \ [1, \infty) & \text{if } \ n = 2. \end{cases} \tag{6.0.3}$$

Moreover, if $B_\varrho \Subset B_r \Subset \Omega$ are concentric balls with $0 < \varrho < r \le 1$ there holds that

$$\|Du\|_{L^d(B_\varrho)} \le \frac{c}{(r-\varrho)^\gamma} \left[1 + \|u\|_{W^{1,p}(B_r)} + \mathcal{F}(u, B_r)\right]^{\hat{\gamma}} \quad \text{for all } \ d \in [1, m(n)],$$

with $c \equiv c(n, N, \nu, L, p, q, d)$, and $\gamma, \hat{\gamma} \equiv \gamma, \hat{\gamma}(n, p, q)$.

If the bound prescribed by (6.1.2) is violated in the sense of (6.0.4) below, we can still recover some improvement in integrability.

Theorem 23 *Under hypotheses (6.1.1), (6.1.3) and (6.1.4), let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of functional (6.2.1). Assume further that the exponents (p, q) verify*

$$2 < p < \frac{np}{n-2} \le q < p^* \qquad \text{with } \ n \ge 3. \tag{6.0.4}$$

Then

$$u \in W^{1,d}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \qquad \textit{for all} \;\; d \in \left[1, \frac{n(p-2)}{n-2-\frac{2n}{q}}\right)$$

and if $B_\varrho \Subset B_r \Subset \Omega$ *are concentric balls with* $0 < \varrho < r \leq 1$ *there holds that*

$$\|Du\|_{L^d(B_\varrho)} \leq \frac{c}{(r-\varrho)^{\gamma_d}}\left[1 + \|u\|_{W^{1,p}(B_r)} + \mathcal{F}(u, B_r)\right]^{\hat{\gamma}_d} \qquad \textit{for all} \;\; d \in \left[1, \frac{n(p-2)}{n-2-\frac{2n}{q}}\right),$$

where $c \equiv c(n, N, \nu, L, p, q, d)$ *is a positive constant so that* $c \to \infty$ *as* $d \to \frac{n(p-2)}{n-2-\frac{2n}{q}}$ *and* $\gamma_d, \hat{\gamma}_d \equiv \gamma_d, \tilde{\gamma}_d(n, p, q, d)$.

We stress that, coupling the result in Theorem 22 with the Moser's iteration presented for instance in [183] (and additionally assuming that $F(\cdot)$ has radial structure $F(z) \equiv \tilde{F}(|z|)$ in the purely vectorial framework $N \geq 2$), then we can prove Lipschitz continuity for minima under the bound (6.1.2). Finally, Theorem 22 allows us to recover a Morrey-type regularity result in two space dimensions for local minima of non-degenerate functionals.

Theorem 24 *Let* $n = 2$ *and* $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *be a local minimizer of functional* (6.2.1)*, under hypotheses* (6.1.1)*,* (6.1.2)*,* (6.1.3) *and* (6.1.4) *with* $\mu = 1$*. Then* $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *for all* $\beta \in (0, 1)$*.*

The result in Theorem 24 is another instance of the potential of the controlled dual growth conditions (6.1.4). Indeed, we obtain the same result as in [25], without imposing any restriction on the size of $q - p$.

6.1 Growth conditions and duality

Let $F \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying (p, q)-growth:

$$\nu|z|^p - L \leq F(z) \leq L(1 + |z|^q), \tag{6.1.1}$$

for all $z \in \mathbb{R}^{N\times n}$. The exponents (p, q) verify

$$2 \leq p < q < \frac{np}{n-2} \;\; \text{if} \;\; n > 2 \qquad \text{and} \qquad 2 \leq p < q \;\; \text{if} \;\; n = 2. \tag{6.1.2}$$

In the above display, we took strict inequality between p and q, since when $p \equiv q$ there is nothing new to prove. We shall also assume some regularity on $F(\cdot)$:

$$F \in C^2(\mathbb{R}^{N\times n}) \tag{6.1.3}$$

together with the dual controlled growth conditions:

$$\nu(\mu^2 + |z_0|^2)^{\frac{p-2}{2}} \leq \frac{\big(F(z) - F(z_0) - \partial_z F(z_0)\cdot(z - z_0)\big)}{|z - z_0|^2} \leq L(\mu^2 + |\partial_z F(z_0)|^2)^{\frac{q-2}{2(q-1)}}, \tag{6.1.4}$$

for all $z, z_0 \in \mathbb{R}^{N\times n}$. Condition (6.1.4) clearly implies that $F(\cdot)$ is strictly convex. In the light of (6.1.4) it is natural to additionally require thet

$$|\partial_z F(z)| \leq L(1 + F(z)) \quad \text{for all} \;\; z \in \mathbb{R}^{N\times n}, \tag{6.1.5}$$

which is, by the way satisfied by the main models with (p, q) growth appearing in the literature, [89, 183]. In the previous displays, $0 < \nu < L$ are finite, absolute constants and $\mu \in [0, 1]$. The Fenchel conjugate integrand of $F(\cdot)$ is defined by

$$F^*(\xi) := \sup_{z\in\mathbb{R}^{N\times n}} \big(\xi\cdot z - F(z)\big) \quad \text{for all} \;\; \xi \in \mathbb{R}^{N\times n}. \tag{6.1.6}$$

Being the supremum of affine functions, $F^*(\cdot)$ is convex and it can be easily checked that, being (6.1.1) in force, $F^*(\cdot)$ features (p', q')-growth conditions, i.e.:

$$\nu^*|\xi|^{q'} - L \le F^*(\xi) \le L^*|\xi|^{p'} \quad \text{for all } \ \xi \in \mathbb{R}^{N\times n}, \tag{6.1.7}$$

where $\nu^* := (Lq)^{-\frac{1}{q-1}}(1-q^{-1})$ and $L^* := (p\nu)^{-\frac{1}{p-1}}(1-p^{-1})$, see [45, Section 2] for more details on this matter. Since $F(\cdot)$ is convex and real-valued, it is lower-semicontinuous, thus $F^{**}(\cdot) \equiv F(\cdot)$ by Fenchel-Moreau theorem. The definition of polar integrand yields that Young inequality holds:

$$\xi \cdot z \le F^*(\xi) + F^{**}(z) \quad \text{for all } \ \xi, z \in \mathbb{R}^{N\times n}. \tag{6.1.8}$$

We stress that equality holds in (6.1.8) whenever ξ belongs to the subgradient of $F^{**}(\cdot)$ in z, in other terms, keeping in mind the convexity and the additional regularity assumed on $F(\cdot)$, whenever $\xi_0 := \partial_z F(z_0)$ for some $z_0 \in \mathbb{R}^{N\times n}$, we have

$$\xi_0 \cdot z_0 = F^*(\xi_0) + F(z_0). \tag{6.1.9}$$

Furthermore, we record that $F(\cdot)$ is strictly convex if and only if $F^* \in C^1(\mathbb{R}^{N\times n})$, and in this case we have

$$\partial_z F^*(\partial_z F(z)) = z \quad \text{for all } \ z \in \mathbb{R}^{N\times n}. \tag{6.1.10}$$

However, being (6.1.3)-(6.1.4) in force, we can say more: in fact $F^* \in C^2(\mathbb{R}^{N\times n})$ and it is strongly convex, cf. [143, Chapter X]. Let us be more precise in this respect.

Lemma 6.1.1 *If $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is an integrand satisfying* (6.1.3)-(6.1.4)*, then for all $z_0, \xi \in \mathbb{R}^{N\times n}$ its Fenchel conjugate $F^*\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ verifies*

$$\frac{1}{4L}(\mu^2+|\xi_0|^2)^{\frac{q'-2}{2}} \le \frac{\big(F^*(\xi)-F^*(\xi_0)-\partial_z F^*(\xi_0)\cdot(\xi-\xi_0)\big)}{|\xi-\xi_0|^2} \le \frac{1}{4\nu}(\mu^2+|z_0|^2)^{\frac{2-p}{2}}, \tag{6.1.11}$$

where $\xi_0 := \partial_z F(z_0)$.

Proof. Let $\xi, \xi_0 \in \mathbb{R}^{N\times n}$ be as in the statement. Rearranging the upper bound in (6.1.4) we find

$$\Big((\xi-\xi_0)\cdot z - L(\mu^2+|\xi_0|^2)^{\frac{q-2}{2(q-1)}}|z-z_0|^2\Big) + \big(\xi_0\cdot z_0 - F(z_0)\big) \le \xi\cdot z - F(z).$$

Using the definition in (6.1.6), the extremality relation (6.1.9) and taking the supremum with respect to $z \in \mathbb{R}^{N\times n}$ on the left-hand side we end up with

$$\begin{aligned}(\xi-\xi_0)\cdot z_0 &+ \frac{1}{4L}(\mu^2+|\xi_0|^2)^{\frac{2-q}{2(q-1)}}|\xi-\xi_0|^2 \\ &= \sup_{z\in\mathbb{R}^{N\times n}}\Big((\xi-\xi_0)\cdot z - L(\mu^2+|\xi_0|^2)^{\frac{q-2}{2(q-1)}}|z-z_0|^2\Big) \le F^*(\xi)-F^*(\xi_0).\end{aligned}$$

Rearranging the inequality in the previous display via (6.1.10), we obtain the lower bound in (6.1.11). The upper bound can be derived in a totally similar way. □

A straightforward consequence of (6.1.4) combined with (6.1.11) is presented in the next corollary.

Corollary 6.1.1 *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *be an integrand satisfying* (6.1.3) *and* (6.1.4). *Then there exists a positive constant* $c \equiv c(n, N, \nu, L, p, q)$ *such that for all* $z_1, z_2 \in \mathbb{R}^{N\times n}$ *the following inequality holds true:*

$$\begin{aligned}\big(\partial_z F(z_1) - \partial_z F(z_2)\big) \cdot (z_1 - z_2) &\geq c(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2 \\ &\quad + c(\mu^2 + |\partial_z F(z_1)|^2 + |\partial_z F(z_2)|^2)^{\frac{q'-2}{2}} |\partial_z F(z_1) - \partial_z F(z_2)|^2. \end{aligned} \tag{6.1.12}$$

Proof. Pick any $z_1, z_2 \in \mathbb{R}^{N\times n}$ and set $\xi_1 := \partial_z F(z_1)$ and $\xi_2 := \partial_z F(z_2)$. A direct manipulation of (6.1.11) renders that

$$\big(\partial_z F^*(\xi_1) - \partial_z F^*(\xi_2)\big) \cdot (\xi_1 - \xi_2) \geq \frac{1}{2L}(\mu^2 + |\xi_1|^2 + |\xi_2|^2)^{\frac{q'-2}{2}} |\xi_1 - \xi_2|^2,$$

where we also used that $q' < 2$, see (6.1.2). Notice that, by (6.1.3), (6.1.4) and (6.1.11), both $\partial_z F(\cdot)$ and $\partial_z F^*(\cdot)$ have domain coinciding with $\mathbb{R}^{N\times n}$ and they are mutually inverse, cf. (6.1.10). This comes via a straightforward variant of the arguments presented in [143, Chapter X, Sections 4.1-4.2]. The previous discussion, (6.1.3) and the content of the above display yield in particular that

$$\partial_{zz} F^*(\xi)\zeta \cdot \zeta \geq \frac{1}{8L}(\mu^2 + |\xi|^2)^{\frac{q'-2}{2}} |\zeta|^2, \tag{6.1.13}$$

for all $\xi, \zeta \in \mathbb{R}^{N\times n}$. Combining (6.1.10), (6.1.13) and Lemma 2.4.1 we obtain

$$\begin{aligned}(\partial_z F(z_1) - \partial_z F(z_2)) \cdot (z_1 - z_2) &= (\xi_1 - \xi_2) \cdot (\partial_z F^*(\xi_1) - \partial_z F^*(\xi_2)) \\ &= \left(\int_0^1 \partial_{zz} F^*(\xi_2 + \lambda(\xi_1 - \xi_2)) \, \mathrm{d}\lambda\right) (\xi_1 - \xi_2) \cdot (\xi_1 - \xi_2) \\ &\geq \frac{1}{8L}\left(\int_0^1 (\mu^2 + |\xi_2 + \lambda(\xi_1 - \xi_2)|^2)^{\frac{q'-2}{2}} \, \mathrm{d}\lambda\right) |\xi_1 - \xi_2|^2 \\ &\geq c(\mu^2 + |\xi_1|^2 + |\xi_2|^2)^{\frac{q'-2}{2}} |\xi_1 - \xi_2|^2, \end{aligned}$$

with $c \equiv c(n, N, L, q)$. Recalling the definition of ξ_1, ξ_2 and (6.1.4) we obtain (6.1.12). □

Finally, we record that, combining (6.1.1) and the convexity implied by (6.1.4) we have

$$|\partial_z F(z)| \leq c(L, q)(1 + |z|^{q-1}) \quad \text{for all} \;\; z \in \mathbb{R}^{N\times n}, \tag{6.1.14}$$

see [183, Lemma 2.1].

Remark 6.1.1 If instead of local minima of (6.0.1) we consider solutions to the Dirichlet problem

$$g + W^{1,p}_0(\Omega, \mathbb{R}^N) \ni w \mapsto \min \mathcal{F}(w, \Omega)$$

for some assigned boundary datum $g \in W^{1,p}(\Omega, \mathbb{R}^N)$ so that $F(tDg) \in L^1_{\mathrm{loc}}(\Omega)$ for some $t > 1$, we can omit assumption (6.1.5), cf. [46].

6.2 Higher differentiability

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the variational integral

$$W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \ni w \mapsto \mathcal{F}(w, \Omega) := \int_\Omega F(Dw) \ \mathrm{d}x, \tag{6.2.1}$$

where the integrand $F(\cdot)$ satisfies (6.1.1)-(6.1.5). As usually done in the framework of problems with (p, q)-growth, see e.g. [101], we regularize u via $\{\phi_\varepsilon\}$, family of nonnegative, radially symmetric mollifiers of $\mathbb{R}^n$ with unitary mass, thus obtaining a sequence $\{u_\varepsilon\} \subset C^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ so that

$$u_\varepsilon \to u \quad \text{strongly in} \ \ W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N). \tag{6.2.2}$$

Moreover, via Jensen inequality we see that

$$F(Du_\varepsilon) \le F(Du) * \phi_\varepsilon \to F(Du) \quad \text{strongly in} \ \ L^1_{\mathrm{loc}}(\Omega). \tag{6.2.3}$$

Now, fix any ball $B_\varrho \Subset \Omega$ with radius $\varrho \le 1$ and define the auxiliary functionals

$$W^{1,p}(B_\varrho, \mathbb{R}^N) \ni w \mapsto \mathcal{F}_\varepsilon(w, B_\varrho) := \int_{B_\varrho} F_\varepsilon(Dw) \ \mathrm{d}x,$$

where

$$F_\varepsilon(z) := \left[F(z) + \kappa_\varepsilon(\mu^2 + |z|^2)^{\frac{q}{2}}\right] \tag{6.2.4}$$

and $\kappa_\varepsilon \in (0, 1)$ is defined as

$$\kappa_\varepsilon := \left(\varepsilon^{-1} + 1 + \|Du_\varepsilon\|^q_{L^q(B_\varrho)}\right) \to 0. \tag{6.2.5}$$

By direct methods and strict convexity, we have that there exists a unique solution $v_\varepsilon \in W^{1,q}(B_\varrho, \mathbb{R}^N)$ to the Dirichlet problem

$$u_\varepsilon + W^{1,q}_0(B, \mathbb{R}^N) \ni w \mapsto \min \mathcal{F}_\varepsilon(w, B)$$

satisfying the integral identity:

$$0 = \int_{B_\varrho} \partial_z F_\varepsilon(Dv_\varepsilon) \cdot D\varphi \ \mathrm{d}x \quad \text{for all} \ \ \varphi \in W^{1,q}_0(B, \mathbb{R}^N). \tag{6.2.6}$$

Notice that, by the minimality of v_ε in class $u_\varepsilon + W^{1,q}_0(B_\varrho, \mathbb{R}^N)$ we have

$$\begin{aligned} \nu \int_B |Dv_\varepsilon|^p \ \mathrm{d}x \quad &+ \quad \kappa_\varepsilon \int_B (\mu^2 + |Dv_\varepsilon|^2)^{\frac{q}{2}} \ \mathrm{d}x \\ &\overset{(6.1.1),(6.2.4)}{\le} \int_B \left[F_\varepsilon(Dv_\varepsilon) + L\right] \ \mathrm{d}x \\ &\le \int_B \left[F(Du_\varepsilon) + L\right] \ \mathrm{d}x + \kappa_\varepsilon \int_B (\mu^2 + |Dv_\varepsilon|^2)^{\frac{q}{2}} \ \mathrm{d}x \\ &\overset{(6.2.3),(6.2.5)}{\le} \mathcal{F}(u, B_\varrho) + L|B_\varrho| + \mathsf{o}(\varepsilon) =: \mathcal{M}. \end{aligned} \tag{6.2.7}$$

Moreover, by (6.2.7) and Poincaré inequality we also get

$$\|v_\varepsilon\|_{W^{1,p}(B_\varrho)} \le c\left[\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{M}\right] + \mathsf{o}(\varepsilon), \tag{6.2.8}$$

for $c \equiv c(n, N, \nu, L, p)$. Let us recover a bound on the L^1-norm of $\partial_z F(v_\varepsilon)$:

$$\begin{aligned}\int_{B_\varrho} |\partial_z F(Dv_\varepsilon)| \ \mathrm{d}x &\stackrel{(6.1.5)}{\leq} L \int_{B_\varrho} \left[F(Dv_\varepsilon)+1\right] \ \mathrm{d}x \leq L \int_{B_\varrho} \left[F_\varepsilon(Dv_\varepsilon)+1\right] \ \mathrm{d}x \\ &\stackrel{(6.2.7)}{\leq} c\left[\mathcal{F}(u,B_\varrho)+1\right], \end{aligned} \tag{6.2.9}$$

with $c \equiv c(n, L)$. Now we are ready for proving the higher integrability result in Theorem 22. We select parameters $\varrho/2 \leq \tau_1 < \tau_2 \leq \varrho$, a cut-off function $\eta \in C^1_c(B)$ so that

$$\mathbb{1}_{B_{\tau_1}} \leq \eta \leq \mathbb{1}_{B_{(\tau_2+\tau_1)/2}} \quad \text{and} \quad |D\eta| \leq \frac{8}{\tau_2-\tau_2}, \tag{6.2.10}$$

a vector $h \in \mathbb{R}^n$ with $|h| \leq \frac{1}{10000}\min\left\{1, \operatorname{dist}(\operatorname{supp}(\eta), \partial B)\right\}$ and test (6.2.6) against $\varphi := \tau_{-h}(\eta^2\tau_h v_\varepsilon)$. Using in (6.2.6) the integration by parts technique for finite difference operators, we then obtain

$$0 = \int_B \tau_h \partial F_\varepsilon(Dv_\varepsilon) \cdot \left[\eta^2\tau_h Dv_\varepsilon + 2\eta D\eta \otimes \tau_h v_\varepsilon\right] \ \mathrm{d}x =: \text{(I)} + \text{(II)}. \tag{6.2.11}$$

For the sake of clarity, given any map $w \in W^{1,q}(B_\varrho, \mathbb{R}^N)$ we define

$$\mathcal{D}(h, Dw) := \left(\mu^2 + |Dw(x+h)|^2 + |Dw(x)|^2\right).$$

Let us estimate term (I). We have, via Lemma 2.4.1:

$$\begin{aligned}\text{(I)} \quad &= \quad \int_{B_\varrho} \eta^2\tau_h\partial_z F(Dv_\varepsilon)\cdot\tau_h Dv_\varepsilon \ \mathrm{d}x + q\kappa_\varepsilon \int_{B_\varrho} \eta^2\tau_h\left((\mu^2+|Dv_\varepsilon|^2)^{\frac{q-2}{2}}Dv_\varepsilon\right)\cdot\tau_h Dv_\varepsilon \ \mathrm{d}x \\ &\stackrel{(6.1.12)}{\geq} \quad c\int_{B_\varrho}\eta^2\mathcal{D}(h,Dv_\varepsilon)^{\frac{p-2}{2}}|\tau_h Dv_\varepsilon|^2 \ \mathrm{d}x + c\int_{B_\varrho}\eta^2\mathcal{D}(h,\partial_z F(Dv_\varepsilon))^{\frac{q'-2}{2}}|\tau_h\partial_z F(Dv_\varepsilon)|^2 \ \mathrm{d}x \\ &+ \quad c\kappa_\varepsilon\int_{B_\varrho}\eta^2\mathcal{D}(h,Dv_\varepsilon)^{\frac{q-2}{2}}|\tau_h Dv_\varepsilon|^2 \ \mathrm{d}x \\ &\stackrel{(2.4.2)}{\geq} \quad c\int_{B_\varrho}\eta^2|\tau_h V_{\mu,p}(Dv_\varepsilon)|^2 \ \mathrm{d}x + c\int_{B_\varrho}\eta^2|\tau_h V_{\mu,q'}(\partial_z F(Dv_\varepsilon))|^2 \ \mathrm{d}x \\ &+ \quad c\kappa_\varepsilon\int_{B_\varrho}\eta^2|\tau_h V_{\mu,q}(Dv_\varepsilon)|^2 \ \mathrm{d}x. \end{aligned}$$

with $c \equiv c(n, N, \nu, L, p, q)$. Now we take care of term (II). By (2.4.2), Lemma 2.4.1, the mean value theorem, Hölder and Young inequalities we obtain

$$\begin{aligned}|\text{(II)}| \quad &\leq \quad c\int_{B_\varrho}\eta|D\eta|\mathcal{D}(h,\partial_z F(Dv_\varepsilon))^{\pm\frac{q'-2}{4}}|\tau_h\partial_z F(Dv_\varepsilon)||\tau_h v_\varepsilon| \ \mathrm{d}x \\ &+ \quad c\kappa_\varepsilon\int_{B_\varrho}\eta^2\mathcal{D}(h,Dv_\varepsilon)^{\frac{q-2}{2}}|\tau_h Dv_\varepsilon||D\eta||\tau_h v_\varepsilon| \ \mathrm{d}x \\ &\leq \quad \iota\int_{B_\varrho}\eta^2\mathcal{D}(h,\partial_z F(Dv_\varepsilon))^{\frac{q'-2}{2}}|\tau_h\partial_z F(Dv_\varepsilon)|^2 \ \mathrm{d}x \\ &+ \quad \frac{c}{\iota}\int_{B_\varrho}|D\eta|^2\mathcal{D}(h,\partial_z F(Dv_\varepsilon))^{\frac{2-q'}{2}}|\tau_h v_\varepsilon|^2 \ \mathrm{d}x \\ &+ \quad \iota\kappa_\varepsilon\int_{B_\varrho}\eta^2\mathcal{D}(h,Dv_\varepsilon)^{\frac{q-2}{2}}|\tau_h Dv_\varepsilon|^2 \ \mathrm{d}x \\ &+ \quad \frac{c\kappa_\varepsilon}{\iota}\int_{B_\varrho}|D\eta|^2\mathcal{D}(h,Dv_\varepsilon)^{\frac{q-2}{2}}|\tau_h v_\varepsilon|^2 \ \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leq \iota c \int_{B_\varrho} \eta^2 |\tau_h V_{\mu,q'}(\partial_z F(Dv_\varepsilon))|^2 \ \mathrm{d}x \\
&+ \frac{c}{\iota}\left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, \partial_z F(Dv_\varepsilon))^{\frac{q'}{2}} \ \mathrm{d}x\right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 |\tau_h v_\varepsilon|^q \ \mathrm{d}x\right)^{\frac{2}{q}} \\
&+ \iota\kappa_\varepsilon c \int_{B_\varrho} \eta^2 |\tau_h V_{\mu,q}(Dv_\varepsilon)|^2 \ \mathrm{d}x + \frac{c\kappa_\varepsilon |h|^2}{\iota} \int |D\eta|^2 \mathcal{D}(h, Dv_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \ \mathrm{d}x,
\end{aligned}
$$

for $c \equiv c(n, N, L, p, q)$ and arbitrary $\iota \in (0,1)$. Now we can merge the two above displays, choose $\iota > 0$ sufficiently small and obtain

$$
\begin{aligned}
\int_{B_\varrho} \eta^2 |\tau_h V_{\mu,p}(Dv_\varepsilon)|^2 \ \mathrm{d}x &+ \int_{B_\varrho} \eta^2 |\tau_h V_{\mu,q'}(\partial_z F(Dv_\varepsilon))|^2 \ \mathrm{d}x + \kappa_\varepsilon \int_{B_\varrho} \eta^2 |\tau_h V_{\mu,q}(Dv_\varepsilon)|^2 \ \mathrm{d}x \\
&\leq c\left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, \partial_z F(Dv_\varepsilon))^{\frac{q'}{2}} \ \mathrm{d}x\right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 |\tau_h v_\varepsilon|^q \ \mathrm{d}x\right)^{\frac{2}{q}} \\
&+ c|h|^2 \kappa_\varepsilon \int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, Dv_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \ \mathrm{d}x, \qquad (6.2.12)
\end{aligned}
$$

where $c \equiv c(n, N, \nu, L, p, q)$. Once the previous inequality is available, we split the rest of the proof into four steps.

6.2.1 Step 1: Higher integrability of $\partial_z F(Dv_\varepsilon)$

Now, let α be as in (2.2.1) and notice that, because of the restrictions imposed by (6.1.2), there holds that

$$
\alpha \in (0,1). \qquad (6.2.13)
$$

Dividing both sides of the inequality in (6.2.12) by $|h|^{2\alpha}$ we get

$$
\begin{aligned}
\mathcal{I}(h) := &\int_{B_\varrho} \eta^2 \left| \frac{\tau_h V_{\mu,p}(Dv_\varepsilon)}{|h|^\alpha} \right|^2 \ \mathrm{d}x + \int_{B_\varrho} \eta^2 \left| \frac{\tau_h V_{\mu,q'}(\partial_z F(Dv_\varepsilon))}{|h|^\alpha} \right|^2 \ \mathrm{d}x \\
&+ \kappa_\varepsilon \int_{B_\varrho} \eta^2 \left| \frac{\tau_h V_{\mu,q}(Dv_\varepsilon)}{|h|^\alpha} \right|^2 \ \mathrm{d}x \\
\leq &c\left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, \partial_z F(Dv_\varepsilon))^{\frac{q'}{2}} \ \mathrm{d}x\right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|^\alpha} \right|^q \ \mathrm{d}x\right)^{\frac{2}{q}} \\
&+ c|h|^{2(1-\alpha)} \kappa_\varepsilon \int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, Dv_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \ \mathrm{d}x =: \mathcal{S}_1(h) + \mathcal{S}_2(h). \qquad (6.2.14)
\end{aligned}
$$

Let us estimate separately the two terms appearing on the right-hand side of (6.2.14). By Hölder inequality, (6.2.7), (6.2.13) and Lemmas 2.2.4-2.2.5 we have that

$$
\mathcal{S}_2(h) \leq c\kappa_\varepsilon \left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, Dv_\varepsilon)^{\frac{q}{2}} \ \mathrm{d}x\right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 |\Delta_h v_\varepsilon|^q \ \mathrm{d}x\right)^{\frac{2}{q}} \leq \frac{c\mathcal{M}}{(\tau_2 - \tau_1)^2}, \qquad (6.2.15)
$$

with $c \equiv c(n, N, \nu, L, p, q)$. Moreover, using the embedding in Lemma 2.2.2, Lemma 2.2.4 and (6.2.8) we obtain

$$
\mathcal{S}_1(h) \leq \frac{c}{(\tau_2 - \tau_1)^4} \left(\int_{B_{(\tau_2+\tau_1)/2}} (\mu^2 + |\partial_z F(Dv_\varepsilon)|^2)^{\frac{q'}{2}} \ \mathrm{d}x\right)^{\frac{q-2}{q}} \|v_\varepsilon\|^2_{W^{1,p}(B_{(\tau_2+\tau_1)/2})}
$$

$$\leq \frac{c\left(1+\|u\|_{W^{1,p}(B_\varrho)}+\mathcal{F}(u,B_\varrho)\right)^2}{(\tau_2-\tau_1)^4}\left(\int_{B_{\tau_2}}(\mu^2+|\partial_z F(Dv_\varepsilon)|^2)^{\frac{q'}{2}}\,\mathrm{d}x\right)^{\frac{q-2}{q}} \tag{6.2.16}$$

for $c \equiv c(n,N,\nu,L,p)$. Inserting the content of the previous two displays into (6.2.14) we obtain

$$\mathcal{G}(h) \leq \frac{c}{(\tau_2-\tau_1)^4}\left(\int_{B_{\tau_2}}(\mu^2+|\partial_z F(Dv_\varepsilon)|^2)^{\frac{q'}{2}}\,\mathrm{d}x\right)^{\frac{q-2}{q}}+\frac{c\mathcal{M}}{(\tau_2-\tau_1)^2},$$

with $c \equiv c(n,N,\nu,L,p,q,\mathcal{F}(u,B_\varrho),\|u\|_{W^{1,p}(B_\varrho)})$, therefore for any $\beta \in (0,\alpha)$, we can apply Lemma 2.2.3 to get

$$\begin{aligned}
&\|V_{\mu,q'}(\partial_z F(Dv_\varepsilon))\|_{W^{\beta,2}(B_{\tau_1})}+\|V_{\mu,p}(Dv_\varepsilon)\|_{W^{\beta,2}(B_{\tau_1})}\\
&\quad\leq \frac{c}{(\alpha-\beta)^{\frac{1}{2}}(\tau_2-\tau_1)^{2-\beta+\alpha}}\left[\|V_{\mu,q'}(\partial_z F(Dv_\varepsilon))\|_{L^2(B_{\tau_2})}^{\frac{q-2}{q}}+\mathcal{M}^{\frac{1}{2}}\right]\\
&\quad\quad+\frac{c}{(\tau_2-\tau_1)^{n/2+\beta}}\|V_{\mu,q'}(\partial_z F(Dv_\varepsilon))\|_{L^2(B_{\tau_2})},
\end{aligned} \tag{6.2.17}$$

where $c \equiv c(n,N,\nu,L,p,q,\mathcal{F}(u,B_\varrho),\|u\|_{W^{1,p}(B_\varrho)})$. In (6.2.17), we apply the embedding in Lemma 2.2.1 combined with (2.4.2) to obtain

$$\begin{aligned}
\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})}^{\frac{q'}{2}} &\leq c\|V_{\mu,q'}(\partial_z F(Dv_\varepsilon))\|_{L^{\frac{2n}{n-2\beta}}}\\
&\leq \frac{c}{(\alpha-\beta)^{\frac{1}{2}}(\tau_2-\tau_1)^{2-\beta+\alpha}}\left[\|V_{\mu,q'}(\partial_z F(Dv_\varepsilon))\|_{L^2(B_{\tau_2})}^{\frac{q-2}{q}}+\mathcal{M}^{\frac{1}{2}}\right]\\
&\quad+\frac{c}{(\tau_2-\tau_1)^{n/2+\beta}}\|V_{\mu,q'}(\partial_z F(Dv_\varepsilon))\|_{L^2(B_{\tau_2})}\\
&\leq \frac{c}{(\alpha-\beta)^{\frac{1}{2}}(\tau_2-\tau_1)^{2+\beta-\alpha}}\left[\|\partial_z F(Dv_\varepsilon)\|_{L^{q'}(B_{\tau_2})}^{\frac{2-q'}{2}}+\mathcal{M}^{\frac{1}{2}}\right]\\
&\quad+\frac{c}{(\tau_2-\tau_1)^{n/2+\beta}}\left[1+\|\partial_z F(Dv_\varepsilon)\|_{L^{q'}(B_{\tau_2})}^{\frac{q'}{2}}\right],
\end{aligned}$$

in other terms,

$$\begin{aligned}
&\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})}\\
&\quad\leq \frac{c}{(\alpha-\beta)^{\frac{1}{q'}}(\tau_2-\tau_1)^{2(2-\beta+\alpha)/q'}}\left[\|\partial_z F(Dv_\varepsilon)\|_{L^{q'}(B_{\tau_2})}^{\frac{2-q'}{q'}}+\mathcal{M}^{\frac{1}{q'}}\right]\\
&\quad\quad+\frac{c}{(\tau_2-\tau_1)^{(n+2\beta)/q'}}\left[1+\|\partial_z F(Dv_\varepsilon)\|_{L^{q'}(B_{\tau_2})}\right].
\end{aligned} \tag{6.2.18}$$

Being $\beta > 0$, there holds that $1 < q' < \frac{nq'}{n-2\beta}$, so we can apply on the right-hand side of (6.2.18) the interpolation inequality

$$\|\partial_z F(Dv_\varepsilon)\|_{L^{q'}(B_{\tau_2})} \leq \|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}^{\theta}\|\partial_z F(Dv_\varepsilon)\|_{L^1(B_{\tau_2})}^{1-\theta}, \tag{6.2.19}$$

where $\theta \in (0,1)$ can be computed via the following identity:

$$\frac{1}{q'}=\frac{\theta(n-2\beta)}{nq'}+1-\theta \implies \theta=\frac{n(q'-1)}{n(q'-1)+2\beta}, \tag{6.2.20}$$

to end up with

$$
\begin{aligned}
&\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})}\\
&\quad\le \frac{c}{(\alpha-\beta)^{1/q'}(\tau_2-\tau_1)^{2(2-\beta+\alpha)/q'}}\left[1+\mathcal{M}^{\frac{1}{q'}}+\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}^{\frac{(2-q')\theta}{q'}}\|\partial_z F(Dv_\varepsilon)\|_{L^1(B_{\tau_2})}^{\frac{(2-q')(1-\theta)}{q'}}\right]\\
&\quad+\frac{c}{(\tau_2-\tau_1)^{(n+2\beta)/q'}}\left[1+\|\partial_z F(Dv_\varepsilon)\|^{\theta}_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\|\partial_z F(Dv_\varepsilon)\|^{1-\theta}_{L^1(B_{\tau_2})}\right], \qquad (6.2.21)
\end{aligned}
$$

with $c\equiv c(n,N,\nu,L,p,q,\mathcal{F}(u,B_\varrho),\|u\|_{W^{1,p}(B_\varrho)})$. At this point we apply Young inequality with conjugate exponents

$$
\left(\frac{q'}{(2-q')\theta},\frac{q'}{q'-(2-q')\theta}\right) \quad\text{and}\quad \left(\frac{1}{\theta},\frac{1}{1-\theta}\right)
$$

in (6.2.21) to obtain

$$
\begin{aligned}
\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})} &\le \frac{1}{2}\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\\
&+\frac{c}{(\alpha-\beta)^{\tilde\gamma_1}(\tau_2-\tau_1)^{\gamma_1}}\left[1+\mathcal{M}^{\frac{1}{q'}}+\|\partial_z F(Dv_\varepsilon)\|_{L^1(B_{\tau_2})}^{\frac{(2-q')(1-\theta)}{q'-(2-q')\theta}}\right]\\
&\overset{(6.2.9)}{\le} \frac{1}{2}\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\\
&+\frac{c}{(\alpha-\beta)^{\tilde\gamma_1}(\tau_2-\tau_1)^{\gamma_1}}\left[1+\mathcal{M}^{\frac{1}{q'}}+\mathcal{F}(u,B_\varrho)^{\frac{(2-q')(1-\theta)}{q'-(2-q')\theta}}\right],
\end{aligned}
$$

where we set

$$
\gamma_1:=\frac{\max\{2(2+\beta-\alpha),n+2\beta\}}{q'-(2-q')\theta},\qquad \tilde\gamma_1:=\frac{1}{(2-q')\theta}
$$

and $c\equiv c(n,N,\nu,L,p,q,\mathcal{F}(u,B_\varrho),\|u\|_{W^{1,p}(B_\varrho)})$. Recalling the explicit expression of $\mathcal{M}$ in (6.2.7) and from (6.2.16) how c depends on $\mathcal{F}(u,B_\varrho)$ and on $\|u\|_{W^{1,p}(B_\varrho)}$, we rearrange the previous inequality as follows:

$$
\begin{aligned}
\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_1})} \le& \frac{1}{2}\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\tau_2})}\\
&+\frac{c}{(\alpha-\beta)^{\tilde\gamma_1}(\tau_2-\tau_1)^{\gamma_1}}\left[1+\|u\|_{W^{1,p}(B_\varrho)}+\mathcal{F}(u,B_\varrho)\right]^{\hat\gamma_1},
\end{aligned}
$$

with $\hat\gamma_1:=4\max\left\{1,\frac{(2-q')(1-\theta)}{q'-(2-q')\theta}+\frac{1}{q'}\right\}$ and $c\equiv c(n,N,\nu,L,p,q)$. Finally, by means of Lemma 2.4.2, we obtain

$$
\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_{\varrho/2})}\le\frac{c}{(\alpha-\beta)^{\tilde\gamma_1}\varrho^{\gamma_1}}\left[1+\|u\|_{W^{1,p}(B_\varrho)}+\mathcal{F}(u,B_\varrho)\right]^{\hat\gamma_1}, \qquad (6.2.22)
$$

for $c\equiv c(n,N,\nu,L,p,q)$. Combining (6.2.22) with a standard covering argument we can conclude that $\partial_z F(Dv_\varepsilon)\in L^{\frac{nq'}{n-2\beta}}_{\mathrm{loc}}(B_\varrho,\mathbb{R}^{N\times n})$ and, whenever $B_s\Subset B_t\Subset B_\varrho$ are concentric balls, the estimate in (6.2.22) can be generalized to

$$
\|\partial_z F(Dv_\varepsilon)\|_{L^{q'}(B_s)} \le c\|\partial_z F(Dv_\varepsilon)\|_{L^{\frac{nq'}{n-2\beta}}(B_s)}
$$

$$\leq \frac{c}{(t-s)^{\gamma_1}} \left[1 + \|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u, B_\varrho)\right]^{\hat{\gamma}_1}, \tag{6.2.23}$$

for all $\beta \in (0, 1 - n(p^{-1} - q^{-1}))$, with $c \equiv c(n, N, \nu, L, p, q, \alpha - \beta)$ and $\gamma_1, \hat{\gamma}_1$ as in (6.2.22). Notice that $c \to \infty$ as $\beta \to \alpha$, however, sice we are taking an arbitrary value of $\beta \in (0, \alpha)$, we see that β ultimately depends on α, therefore, keeping in mind (2.2.1), we can conclude that c depends on (n, N, ν, L, p, q).

6.2.2 Step 2: Higher integrability for Dv_ε

Once (6.2.23) is available, for determining uniform higher integrability for Dv_ε we distinguish two cases: $n > 2$ and $n = 2$ according to (6.1.2).

Case 1: $n > 2$

We jump back to (6.2.12) and divide both sides of the inequality by $|h|^2$, thus getting

$$\begin{aligned}
\mathcal{G}(h) :\ &= \int_{B_\varrho} \eta^2 \left| \frac{\tau_h V_{\mu,p}(Dv_\varepsilon)}{|h|} \right|^2 \,\mathrm{d}x + \int_{B_\varrho} \eta^2 \left| \frac{\tau_h V_{\mu,q'}(\partial_z F(Dv_\varepsilon))}{|h|} \right|^2 \,\mathrm{d}x \\
&\leq c \left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, \partial_z F(Dv_\varepsilon))^{\frac{q'}{2}} \,\mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|} \right|^q \,\mathrm{d}x \right)^{\frac{2}{q}} \\
&+ c\kappa_\varepsilon \int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, Dv_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \,\mathrm{d}x =: \mathcal{T}_1(h) + \mathcal{T}_2(h).
\end{aligned} \tag{6.2.24}$$

Term $\mathcal{T}_2(h)$ can be estimated by means of (6.2.7) and Lemmas 2.2.4-2.2.5:

$$\begin{aligned}
\limsup_{|h|\to 0} \mathcal{T}_2(h) &\leq \limsup_{|h|\to 0} \left(\frac{c\kappa_\varepsilon}{(\tau_2 - \tau_1)^2} \int_{B_{(\tau_2+\tau_1)/2}} \mathcal{D}(h, Dv_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \,\mathrm{d}x \right) \\
&\leq \frac{c\kappa_\varepsilon}{(\tau_2 - \tau_1)^2} \int_{B_{(\tau_2+\tau_1)/2}} (\mu^2 + |Dv_\varepsilon|^2)^{\frac{q}{2}} \,\mathrm{d}x \leq \frac{c\mathcal{M}}{(\tau_2 - \tau_1)^2},
\end{aligned}$$

with $c \equiv c(n, N, \nu, L, p, q)$. Concerning term $\mathcal{T}_1(h)$, via Lemma 2.2.4 and (6.2.23) we have

$$\begin{aligned}
\limsup_{|h|\to 0} \mathcal{T}_1(h) &= \limsup_{|h|\to 0} c \left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, \partial_z F(Dv_\varepsilon))^{\frac{q'}{2}} \,\mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|} \right|^q \,\mathrm{d}x \right)^{\frac{2}{q}} \\
&\leq \frac{c}{(\tau_2 - \tau_1)^2} \left(\int_{B_{(\tau_2+\tau_1)/2}} (\mu^2 + |\partial_z F(Dv_\varepsilon)|^2)^{\frac{q'}{2}} \,\mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_{(\tau_2+\tau_1)/2}} |Dv_\varepsilon|^q \,\mathrm{d}x \right)^{\frac{2}{q}} \\
&\leq \frac{c}{(\tau_2 - \tau_1)^{(2-q')\gamma_1}} \|Dv_\varepsilon\|^2_{L^q(B_{\tau_2})},
\end{aligned}$$

for $c \equiv c(n, N, \nu, L, p, q, \mathcal{F}(u, B_\varrho), \|u\|_{W^{1,p}(B_\varrho)})$ depending on $\mathcal{F}(u, B_\varrho)$ and on $\|u\|_{W^{1,p}(B_\varrho)}$ as in (6.2.23) and γ_1 is the same appearing in (6.2.22)-(6.2.23). Plugging the content of the above displays into (6.2.24) and using Sobolev embedding theorem we obtain

$$\begin{aligned}
\|Dv_\varepsilon\|^{p/2}_{L^{\frac{np}{n-2}}(B_{\tau_1})} &\leq c \left[\|DV_{\mu,p}(Dv_\varepsilon)\|_{L^2(B_{\tau_1})} + \|V_{\mu,p}(Dv_\varepsilon)\|_{L^2(B_{\tau_1})} \right] \\
&\leq \frac{c}{(\tau_2 - \tau_1)^{(2-q')\gamma_1/2}} \|Dv_\varepsilon\|_{L^q(B_{\tau_2})} + \frac{c\mathcal{M}^{1/2}}{(\tau_2 - \tau_1)} + \|Dv_\varepsilon\|^{p/2}_{L^p(B_{\tau_2})},
\end{aligned} \tag{6.2.25}$$

with $c \equiv c(n, N, \nu, L, p, q, \|u\|_{W^{1,p}(B_\varrho)}, \mathcal{F}(u, B_\varrho))$. Recalling $(6.1.2)_1$, in (6.2.25) we can use the interpolation inequality

$$\|Dv_\varepsilon\|_{L^q(B_{\tau_2})} \le \|Dv_\varepsilon\|^{\theta}_{L^{\frac{np}{n-2}}(B_{\tau_2})} \|Dv_\varepsilon\|^{1-\theta}_{L^p(B_{\tau_2})} \tag{6.2.26}$$

where $\theta \in (0, 1)$ is derived via

$$\frac{1}{q} = \frac{\theta(n-2)}{np} + \frac{1-\theta}{p} \implies \theta = \frac{n(q-p)}{2q} \tag{6.2.27}$$

to get

$$\begin{aligned} \|Dv_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\tau_1})} \le & \frac{c}{(\tau_2-\tau_2)^{(2-q')\gamma_1/p}} \|Dv_\varepsilon\|^{\frac{2\theta}{p}}_{L^{\frac{np}{n-2}}(B_{\tau_2})} \|Dv_\varepsilon\|^{\frac{2(1-\theta)}{p}}_{L^p(B_{\tau_2})} \\ & + \frac{\mathcal{M}^{1/p}}{(\tau_2-\tau_1)^{2/p}} + \|Dv_\varepsilon\|_{L^p(B_{\tau_2})}. \end{aligned}$$

By (6.1.2) we readily see that $2\theta/p < 1$ so we can use Young inequality with conjugate exponents

$$\left(\frac{p}{2\theta}, \frac{p}{p-2\theta}\right)$$

and (6.2.7) to conclude with

$$\|Dv_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\tau_1})} \le \frac{1}{2}\|Dv_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\tau_1})} + \frac{c}{(\tau_2-\tau_1)^{\gamma_2}}\left[1 + \mathcal{F}(u, B_\varrho), \|u\|_{W^{1,p}(B_\varrho)}\right]^{\hat{\gamma}_2}, \tag{6.2.28}$$

where $\gamma_2 := \frac{1}{p-2\theta}\max\{\gamma_1, 2\}$, $\hat{\gamma}_2 := \frac{2p}{p-2\theta}\max\left\{(2-q')\hat{\gamma}_1, \frac{2(1-\theta)}{p}, 1\right\}$ and $c \equiv c(n, N, \nu, L, p, q)$. By Lemma 2.4.2, we obtain

$$\|Dv_\varepsilon\|_{L^{\frac{np}{n-2}}(B_{\varrho/2})} \le \frac{c}{\varrho^{\gamma_2}}\left[1 + \|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u, B_\varrho)\right]^{\hat{\gamma}_2},$$

with $c \equiv c(n, N, \nu, L, p, q)$ and $\gamma_2, \hat{\gamma}_2$ as in (6.2.28). As before, the previous inequality can be generalized to arbitrary concentric balls $B_s \Subset B_t \Subset B_\varrho$:

$$\|Dv_\varepsilon\|_{L^{\frac{np}{n-2}}(B_s)} \le \frac{c}{(t-s)^{\gamma_2}}\left[1 + \|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u, B_\varrho)\right]^{\hat{\gamma}_2}, \tag{6.2.29}$$

for $c(n, N, \nu, L, p, q)$.

Case 2: $n = 2$

We modify (6.2.25) by using the limiting case of Sobolev embedding theorem:

$$\begin{aligned} \|Dv_\varepsilon\|^{\frac{p}{2}}_{L^m(B_{\tau_1})} \le & c\left[\|DV_{\mu,p}(Dv_\varepsilon)\|_{L^2(B_{\tau_1})} + \|V_{\mu,p}(Dv_\varepsilon)\|_{L^2(B_{\tau_1})}\right] \\ \le & \frac{c}{(\tau_2-\tau_1)^{\frac{(2-q')\gamma_1}{2}}}\|Dv_\varepsilon\|_{L^q(B_{\tau_2})} + \frac{c\mathcal{M}^{\frac{1}{2}}}{(\tau_2-\tau_1)} + c\|Dv_\varepsilon\|^{\frac{p}{2}}_{L^p(B_{\tau_2})}, \end{aligned} \tag{6.2.30}$$

for all $m > p$. Specifically, we shall pick

$$m > \max\left\{\frac{p^2q}{(pq-2q+2p)}, q\right\} > 0, \tag{6.2.31}$$

which is a finite positive quantity, as, being $p \geq 2$, there holds that $q(p-2)+2p \geq 2p$. This time we use the interpolation inequality

$$\|Dv_\varepsilon\|_{L^q(B_{\tau_2})} \leq \|Dv_\varepsilon\|^{\theta}_{L^m(B_{\tau_2})} \|Dv_\varepsilon\|^{(1-\theta)}_{L^p(B_{\tau_2})} \quad \text{with} \quad \theta = \frac{m(q-p)}{q(m-p)} \tag{6.2.32}$$

for getting

$$\begin{aligned}\|Dv_\varepsilon\|_{L^m(B_{\tau_1})} \leq & \frac{c}{(\tau_2-\tau_1)^{\frac{(2-q')\gamma_1}{p}}} \|Dv_\varepsilon\|^{\frac{2\theta}{p}}_{L^m(B_{\tau_2})} \|Dv_\varepsilon\|^{\frac{2(1-\theta)}{p}}_{L^p(B_{\tau_2})} \\ & + \frac{c\mathcal{M}^{\frac{1}{p}}}{(\tau_2-\tau_1)^{\frac{1}{p}}} + c\|Dv_\varepsilon\|_{L^p(B_{\tau_2})}.\end{aligned}$$

The choice we made in (6.2.31) renders that $2\theta/p < 1$, so, by Young inequality with conjugate exponents

$$\left(\frac{p}{2\theta}, \frac{p}{p-2\theta}\right)$$

and (6.2.7) we get

$$\|Dv_\varepsilon\|_{L^m(B_{\tau_1})} \leq \frac{1}{2}\|Dv_\varepsilon\|_{L^m(B_{\tau_2})} + \frac{c}{(\tau_2-\tau_1)^{\gamma_3}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\hat{\gamma}_3},$$

with $\gamma_3 := \frac{2p}{p-2\theta}\max\left\{1, \frac{1}{p}, \frac{(2-q')\gamma_1}{p-2\theta}\right\}$, $\hat{\gamma}_3 := \frac{p}{p-2\theta}\max\left\{1, \hat{\gamma}_1, \frac{1}{p}, \frac{2(1-\theta)}{p-2\theta}\right\}$ and $c \equiv c(n,N,\nu,L,p,q)$. Lemma 2.4.2 finally leads to

$$\|Dv_\varepsilon\|_{L^m(B_{\varrho/2})} \leq \frac{c}{\varrho^{\gamma_3}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\hat{\gamma}_3}.$$

The previous inequality can be generalized to arbitrary concentric balls $B_s \Subset B_t \Subset B_\varrho$:

$$\|Dv_\varepsilon\|_{L^m(B_s)} \leq \frac{c}{(t-s)^{\gamma_3}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\hat{\gamma}_3}. \tag{6.2.33}$$

6.2.3 Step 3: Convergence

By (6.1.4), the minimality of v_ε in class $u_\varepsilon + W^{1,q}_0(B_\varrho,\mathbb{R}^N)$, (2.4.2), (6.2.5) and (6.2.3) we have

$$\begin{aligned}\int_{B_\varrho} |Dv_\varepsilon - Du|^p \,\mathrm{d}x \leq & c\int_{B_\varrho} |V_{\mu,p}(Dv_\varepsilon) - V_{\mu,p}(Du)|^2 \,\mathrm{d}x \\ \leq & c\int_{B_\varrho}\left[F_\varepsilon(Dv_\varepsilon) - F_\varepsilon(Du) - \partial_z F_\varepsilon(Dv_\varepsilon)\cdot(Du - Dv_\varepsilon)\right] \,\mathrm{d}x \\ \leq & c\int_{B_\varrho}\left[F_\varepsilon(Du_\varepsilon) - F_\varepsilon(Du)\right] \,\mathrm{d}x \to 0,\end{aligned}$$

thus $Dv_\varepsilon \to Du$ strongly in $L^p(B_\varrho,\mathbb{R}^{N\times n})$ and, up to non-relabelled subsequences, $Dv_\varepsilon \to Du$ a.e. in B_ϱ. Now fix concentric balls $B_s \Subset B_t \Subset B_\varrho$. We can use weak lower semicontinuity in (6.2.29)-(6.2.33) for getting that

$$\|Du\|_{L^q(B_s)} \leq c\|Du\|_{L^{m(n)}(B_s)} \leq \frac{c}{(t-s)^{\gamma}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right],$$

where $m(n)$ is defined as in (6.0.3), $c \equiv c(n,N,\nu,L,p,q)$, $\gamma := \max\{\gamma_2,\gamma_3\}$ and $\hat{\gamma} := \max\{\hat{\gamma}_2,\hat{\gamma}_3\}$. Now we study the convergence of $\{\partial_z F(Dv_\varepsilon)\}$. By (6.1.5), the minimality of v_ε, (6.2.3), (6.2.5) and a well-known variant of the dominated convergence theorem we have that $\partial_z F(Dv_\varepsilon) \to \partial_z F(Du)$ strongly in $L^1(B_\varrho, \mathbb{R}^{N\times n})$. This means that, up to non-relabelled subsequences, $\partial_z F(Dv_\varepsilon) \to \partial_z F(Du)$ a.e. in B_ϱ, so by Fatou lemma and (6.2.23) we can conclude that

$$\|\partial_z F(Du)\|_{L^{q'}(B_s)} \le c\|\partial_z F(Du)\|_{L^{\frac{nq'}{n-2\beta}}(B_s)} \le \frac{c}{(t-s)^{\gamma_1}} \left[1 + \|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u, B_\varrho)\right]^{\hat{\gamma}_1},$$

where $B_s \Subset B_t \Subset B_\varrho$ are concentric balls. Now, since $B_\varrho \Subset \Omega$ is arbitrary, we can use a standard covering argument to obtain that

$$Du \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}) \quad \text{and} \quad \partial_z F(Du) \in L^{q'}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}). \tag{6.2.34}$$

6.2.4 Step 4: Weak differentiability of $\partial_z F(Du)$

Once (6.2.34) is available, we can consider the Euler-Lagrange equation of functional (6.2.1) and repeat precisely the same procedure presented in *Step 1* till (6.2.12) (with $F(\cdot)$ and u instead of $F_\varepsilon(\cdot)$ and v_ε - in particular all those terms multiplying κ_ε are zero), this time picking $\eta \in C^1_c(B_\varrho)$ with $\mathbb{1}_{B_{\varrho/2}} \le \eta \le \mathbb{1}_{B_\varrho}$ and $|D\eta| \lesssim \varrho^{-1}$. We can then divide both sides of the inequality by $|h|^2$ to get

$$\begin{aligned}
&\limsup_{|h|\to 0} \int_{B_\varrho} \eta^2 \left| \frac{\tau_h V_{\mu,q'}(\partial_z F(Du))}{|h|} \right|^2 \mathrm{d}x \\
&\quad \le \limsup_{|h|\to 0} \left[c \left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, \partial_z F(Du))^{\frac{q'}{2}} \mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 \left| \frac{\tau_h u}{|h|} \right|^q \mathrm{d}x \right)^{\frac{2}{q}} \right] \\
&\quad \le \frac{c}{\varrho^2} \|\partial_z F(Du)\|^{2-q'}_{L^{q'}(B_\varrho)} \|Du\|^2_{L^q(B_\varrho)} \overset{(6.2.34)}{<} \infty,
\end{aligned}$$

with $c \equiv c(n, N, \nu, L, p, q)$. After covering, we deduce that $V_{\mu,q'}(\partial_z F(Du)) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ and Sobolev embedding finally yields that

$$\partial_z F(Du) \in L^d_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}) \qquad \text{with } d := \begin{cases} \frac{nq'}{n-2} & \text{if } n > 2 \\ \text{any } d \in (1, \infty) & \text{if } n = 2. \end{cases}$$

Remark 6.2.1 In case we know a priori that a minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ has slightly more integrable gradient, in the sense that there exists a $\delta > 0$ so that $Du \in L^{p(1+\delta)}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$, then we can assume the large inequality in (6.1.2), i.e.:

$$2 \le p \le q \le \frac{np}{n-2} \ \text{ if } n \ge 3 \quad \text{and} \quad 2 \le p \le q < \infty \ \text{ if } n = 2.$$

In this case, the interpolation procedure is analogous to the one presented in Theorem 8 of Chapter 4.

Remark 6.2.2 If $p > n-2$, by Morrey's embedding theorem we have that $u \in C^{0,\beta'}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $\beta' := 1 - \frac{n-2}{p}$.

The proof of Theorem 22 is complete.

6.3 Integrability improvement in violation of the bound

In this section we prove Theorem 23, which provides an improvement in integrability for the gradient of local minima of (6.2.1) when the restriction displayed in $(6.1.2)_1$ is violated (clearly, there is no bound to violate in the case $n = 2$, so we work with $n \geq 3$). Precisely, in this part we shall assume that (6.0.4) is in force. For reasons that will be clear in a few lines, define the following sequences of numbers:

$$\tilde{p}_0 := p, \qquad \tilde{p}_j := \frac{np}{n - 2\alpha_{j-1}},$$

$$\alpha_0 := 1 - n\left(\frac{1}{p} - \frac{1}{q}\right), \qquad \alpha_j := 1 - n\left(\frac{1}{\tilde{p}_j} - \frac{1}{q}\right).$$

By induction, when $j \in \mathbb{N} \setminus \{0\}$ there holds that

$$\alpha_j = \alpha_0 \sum_{i=0}^{j} (2/p)^i = \frac{\alpha_0 p \left(1 - (2/p)^{j+1}\right)}{p-2} \overset{(6.0.4)_1}{\in} (0,1).$$

Plugging the value of α_j determined above into the expansion of $\tilde{p}_j$, we readily see that

$$\{\tilde{p}_j\} \text{ is increasing} \quad \text{and} \quad \tilde{p}_j = \frac{np}{n - \frac{2p\alpha_0(1-(2/p)^j)}{p-2}} \to \tilde{p} := \frac{n(p-2)}{n-2-\frac{2n}{q}} > p \iff q < p^*. \tag{6.3.1}$$

Notice also that, by (6.0.4), we have that

$$q \geq \tilde{p}. \tag{6.3.2}$$

Consider now two new sequences, $\{\beta_j\}$ and $\{p_j\}$ so that:

$$\beta_j \in \left(\frac{n(\tilde{p}_j - p)}{2\tilde{p}_j}, \alpha_j\right) \qquad \text{and} \quad p_j := \frac{np}{n - 2\beta_{j-1}}.$$

Notice that numbers β_j are well-defined. In fact we have

$$\alpha_j > \frac{n(\tilde{p}_j - p)}{2\tilde{p}_j} \iff \tilde{p}_j < \frac{nq(p-2)}{nq - 2q - 2n} = \frac{n(p-2)}{n-2-\frac{2n}{q}},$$

which is the case by (6.3.1). Moreover,

$$\beta_j < \alpha_j \implies \tilde{p}_j > p_j \quad \text{and} \quad \beta_j > \frac{n(\tilde{p}_j - p)}{2\tilde{p}_j} \implies p_j > \tilde{p}_{j-1}. \tag{6.3.3}$$

Now take two integers $k > j+1$. By (6.3.3) and $(6.3.1)_1$ we get

$$0 < \tilde{p}_{k-1} - \tilde{p}_j < p_k - p_j < \tilde{p}_k - \tilde{p}_{j-1},$$

so recalling that, by $(6.3.1)_2$ the sequence $\{\tilde{p}_j\}$ is Cauchy, we can conclude that the sequence $\{p_j\}$ is Cauchy as well and, with the chain of inequalities established in (6.3.3), we can conclude that

$$\lim_{j\to\infty} \tilde{p}_j = \lim_{j\to\infty} p_j = \tilde{p}. \tag{6.3.4}$$

Finally, notice that by $(6.3.1)_1$ there holds that

$$\frac{n(\tilde{p}_j - p)}{2\tilde{p}_j} < \frac{n(\tilde{p}_{j+1} - p)}{2\tilde{p}_{j+1}}$$

therefore we can choose the numbers β_j in such a way that the sequence $\{\beta_j\}$ is increasing, so, as a consequence,

$$\text{the sequence } \ \{p_j\} \ \text{ is increasing} \tag{6.3.5}$$

by very definition. Now we borrow the same approximation scheme developed for the proof of Theorem 22, notice that the procedure exposed to prove the estimate in (6.2.23) only requires that $q < p^*$ therefore it is verified also in the present case because of (6.0.4). We consider again estimate (6.2.14):

$$\begin{aligned}
&\int_{B_\varrho} \eta^2 \left| \frac{\tau_h V_{\mu,p}(Dv_\varepsilon)}{|h|^\alpha} \right|^2 \, \mathrm{d}x \\
&\leq c \left(\int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, \partial_z F(Dv_\varepsilon))^{\frac{q'}{2}} \, \mathrm{d}x \right)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |D\eta|^2 \left| \frac{\tau_h v_\varepsilon}{|h|^\alpha} \right|^q \, \mathrm{d}x \right)^{\frac{2}{q}} \\
&\quad + c|h|^{2(1-\alpha)} \kappa_\varepsilon \int_{B_\varrho} |D\eta|^2 \mathcal{D}(h, Dv_\varepsilon)^{\frac{q-2}{2}} |\Delta_h v_\varepsilon|^2 \, \mathrm{d}x =: \mathcal{S}_1(h) + \mathcal{S}_2(h),
\end{aligned} \tag{6.3.6}$$

for $c \equiv c(n, N, \nu, L, p, q)$, $\alpha \in (0,1)$ still to be fixed and $0 \leq \eta(\cdot) \in C^1_c(B_\varrho)$. For $j \in \mathbb{N} \cup \{0\}$, we consider a sequence $\{B_{\varrho_j}\}$ of shirking balls, where $\varrho_j := \frac{\varrho}{2} + \left(\frac{3\varrho}{4} - \frac{\varrho}{2} \right) 2^{-j}$. Notice that $\{\varrho_j\}$ is a decreasing sequence such that $\varrho_0 = \frac{3\varrho}{4}$ and $\varrho_j \to \frac{\varrho}{2}$; therefore it is $\cap_j B_{\varrho_j} = B_{\varrho/2}$ and $B_{\varrho_0} = B_{3\varrho/4}$. Accordingly, we fix parameters $\varrho_{j+1} \leq \tau_{1,j} < \tau_{2,j} \leq \varrho_j$ corresponding cut-off functions $\eta_j \in C^1_c(B_{\varrho_j})$ with

$$\mathbb{1}_{B_{\tau_{1,j}}} \leq \eta \leq \mathbb{1}_{B_{(\tau_{2,j}+\tau_{1,j})/2}} \qquad \text{and} \qquad |D\eta_j| \lesssim \frac{2}{\tau_{2,j} - \tau_{1,j}} \, .$$

In (6.3.6) we choose $\eta \equiv \eta_j$, $\alpha \equiv \alpha_j$, incorporate (6.2.15), use (6.2.23) and Lemma 2.2.4 to control the $L^{q'}$-norm of $\partial_z F(Dv_\varepsilon)$ and eventually obtain

$$\int_{B_{\tau_{1,j}}} \left| \frac{\tau_h V_{\mu,p}(Dv_\varepsilon)}{|h|^{\alpha_j}} \right|^2 \, \mathrm{d}x \leq \frac{c}{(\tau_{2,j} - \tau_{1,j})^{2+\gamma_1(2-q')}} \left(\int_{B_{(\tau_{2,j}+\tau_{1,j})/2}} \left| \frac{\tau_h v_\varepsilon}{|h|^{\alpha_j}} \right|^q \, \mathrm{d}x \right)^{\frac{2}{q}}, \tag{6.3.7}$$

with $c \equiv c(n, N, \nu, L, p, q, \|u\|_{W^{1,p}(B_\varrho)}, \mathcal{F}(u, B_\varrho))$. We stress that, in the light of (6.3.2), (6.3.4) and (6.3.5), all the quantities appearing above are finite. We shall prove that

$$v_\varepsilon \in W^{1,p_j}(B_{\varrho_j}, \mathbb{R}^N) \implies v_\varepsilon \in W^{1,p_{j+1}}(B_{\varrho_{j+1}}, \mathbb{R}^N). \tag{6.3.8}$$

We proceed inductively. For $j = 0$, by Lemmas 2.2.1, 2.2.2, 2.2.3 and (6.2.8) we have:

$$\begin{aligned}
\|v_\varepsilon\|_{W^{1,p_1}(B_{\varrho_1})} &\leq c \|V_{\mu,p}(Dv_\varepsilon)\|^{\frac{2}{p}}_{W^{\beta_0,2}(B_{\varrho_1})} + c|(v_\varepsilon)_{B_{\varrho_1}}||B_{\varrho_1}|^{\frac{1}{p_1}} \\
&\leq \frac{c}{(\alpha_0 - \beta_0)^{1/p}(\varrho_0 - \varrho_1)^{\gamma_0}} \left[\|v_\varepsilon\|^{\frac{2}{p}}_{W^{1,p}(B_\varrho)} + 1 \right] \\
&\leq \frac{c}{(\alpha_0 - \beta_0)^{1/p}(\varrho_0 - \varrho_1)^{\gamma_0}} \left[1 + \|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u, B_\varrho) \right]^{\hat{\gamma}_0},
\end{aligned}$$

where

$$\gamma_0 := 2 + \max\left\{ \frac{2(2+\gamma_1(2-q'))}{p}, \frac{n+2\beta_0}{p}, \frac{n(p_1-1)}{p_1} \right\}, \qquad \hat{\gamma}_0 := \frac{2[(2-q')\hat{\gamma}_1 + 1]}{p}$$

and $\hat{\gamma}_0 := 2p^{-1}[(2-q')\hat{\gamma}_1+1]$ and $c \equiv c(n,N,\nu,L,p,q)$. Now we assume that $v_\varepsilon \in W^{1,p_j}(B_{\varrho_j},\mathbb{R}^N)$. Via (6.2.23), Lemmas 2.2.2, 2.2.3 and 2.2.1 we estimate

$$
\begin{aligned}
\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\tau_{1,j}})} \le & c\|V_{\mu,p}(Dv_\varepsilon)\|^{\frac{2}{p}}_{W^{\beta_j,2}(B_{\tau_{1,j}})} + c|(v_\varepsilon)_{B_{\tau_{1,j}}}||B_{\tau_{1,j}}|^{\frac{1}{p_{j+1}}} \\
\le & \frac{c\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\frac{2(2-q')\hat{\gamma}_1}{p}}}{(\alpha_j-\beta_j)^{1/p}(\tau_{2,j}-\tau_{\tau_{1,j}})^{\gamma'_j}}\left[\|v_\varepsilon\|^{\frac{2}{p}}_{W^{1,\tilde{p}_j}(B_{\tau_{2,j}})}+1\right], \qquad (6.3.9)
\end{aligned}
$$

with $\gamma'_j := 2+\max\left\{\frac{2(2+\gamma_1(2-q'))}{p}, \frac{n+2\beta_0}{p}, \frac{n(p_j-1)}{p_1}\right\}$ and $c \equiv c(n,N,\nu,L,p,q)$. At this point we recall that (6.3.3) legalizes an application to the right-hand side of (6.3.9) of the interpolation inequality

$$
\|v_\varepsilon\|_{W^{1,\tilde{p}_j}(B_{\tau_{2,j}})} \le \|v_\varepsilon\|^{\theta_j}_{W^{1,p_{j+1}}(B_{\tau_{2,j}})}\|v_\varepsilon\|^{1-\theta_j}_{W^{1,p_j}(B_{\tau_{2,j}})},
$$

where $\theta_j \in (0,1)$ verifies the identity

$$
\frac{1}{\tilde{p}_j} = \frac{\theta_j}{p_{j+1}} + \frac{1-\theta_j}{p_j} \implies \theta_j = \frac{p_{j+1}(\tilde{p}_j-p_j)}{\tilde{p}_j(p_{j+1}-p_j)},
$$

so we have

$$
\begin{aligned}
\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\tau_{1,j}})} \le & \frac{c\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\frac{2(2-q')\hat{\gamma}_1}{p}}}{(\alpha_j-\beta_j)^{1/p}(\tau_{2,j}-\tau_{1,j})^{\gamma'_j}} \\
& \cdot \left[\|v_\varepsilon\|^{\frac{2\theta}{p}}_{W^{1,p_{j+1}}(B_{\tau_{2,j}})}\|v_\varepsilon\|^{\frac{2(1-\theta)}{p}}_{W^{1,p_j}(B_{\tau_{2,j}})}+1\right]. \qquad (6.3.10)
\end{aligned}
$$

Then, in (6.3.10) we apply Young inequality with conjugate exponents

$$
\left(\frac{p}{2\theta_j}, \frac{p}{p-2\theta_j}\right)
$$

to get

$$
\begin{aligned}
\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\tau_{1,j}})} \le & \frac{1}{2}\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\tau_{2,j}})} \\
& + \frac{c\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\frac{2(2-q')\hat{\gamma}_1}{p-2\theta_j}}}{(\alpha_j-\beta_j)^{1/(p-2\theta_j)}(\tau_{2,j}-\tau_{1,j})^{\gamma_j}}\left[\|v_\varepsilon\|^{\frac{2(1-\theta_j)}{p-2\theta_j}}_{W^{1,p_j}B(\varrho_j)}+1\right],
\end{aligned}
$$

for $\gamma_j := \frac{p\gamma'_j}{p-2\theta_j}$ and $c \equiv c(n,N,\nu,L,p,q)$. Lemma 2.4.2 eventually leads to

$$
\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\varrho_{j+1}})} \le \frac{c\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\frac{2(2-q')\hat{\gamma}_1}{p-2\theta_j}}}{(\alpha_j-\beta_j)^{1/(p-2\theta_j)}(\varrho_j-\varrho_{j+1})^{\gamma_j}}\left[\|v_\varepsilon\|^{\frac{2(1-\theta_j)}{p-2\theta_j}}_{W^{1,p_j}B(\varrho_j)}+1\right],
$$

and (6.3.8) is proven. Now that we know that $v_\varepsilon \in W^{1,p_j}(B_{\varrho_{j+1}},\mathbb{R}^N)$ for all $j \in \mathbb{N}\cup\{0\}$ with bounding constants independent on ε, we can jump back to (6.3.9), recall (6.3.3) and use the interpolation inequality

$$
\|v_\varepsilon\|_{W^{1,\tilde{p}_j}(B_{\tau_{2,j}})} \le \|v_\varepsilon\|^{\theta}_{W^{1,p_{j+1}}(B_{\tau_{2,j}})}\|v_\varepsilon\|^{1-\theta}_{W^{1,p}(B_{\tau_{2,j}})},
$$

where $\theta \in (0,1)$ this time solves

$$\frac{1}{\tilde{p}_j} = \frac{\theta}{p_{j+1}} + \frac{1-\theta}{p} \implies \theta = \frac{p_{j+1}(\tilde{p}_j - p)}{\tilde{p}_j(p_{j+1}-p)}.$$

Keeping in mind $(6.0.4)_2$ we apply Young inequality with conjugate exponents

$$\left(\frac{p}{2\theta}, \frac{p}{p-2\theta}\right)$$

to deduce, using also (6.2.8),

$$\begin{aligned}
\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\tau_{1,j}})} \le & \frac{1}{2}\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\tau_{2,j}})} \\
& + \frac{c\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\frac{2(2-q')\hat{\gamma}_1}{p-2\theta}}}{(\alpha_j-\beta_j)^{1/(p-2\theta)}(\tau_{2,j}-\tau_{1,j})^{\frac{p\gamma_j'}{p-2\theta}}}\left[\|v_\varepsilon\|_{W^{1,p}(B_{\varrho_j})}^{\frac{2(1-\theta)}{p-2\theta}}+1\right] \\
\le & \frac{1}{2}\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\tau_{2,j}})} \\
& + \frac{c}{(\alpha_j-\beta_j)^{1/(p-2\theta)}(\tau_{2,j}-\tau_{1,j})^{\bar{\gamma}_j}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\hat{\gamma}_j}
\end{aligned}$$

for $c \equiv c(n,N,\nu,L,p,q,\|u\|_{W^{1,p}(B_\varrho)},\mathcal{F}(u,B_\varrho))$ and

$$\bar{\gamma}_j := \frac{p\gamma_j'}{p-2\theta}, \qquad \hat{\gamma}_j := \frac{2[(2-q')\hat{\gamma}_1 + (1-\theta)]}{p-2\theta}.$$

Finally, Lemma 2.4.2 leads to

$$\begin{aligned}
\|v_\varepsilon\|_{W^{1,p_{j+1}}(B_{\varrho_{j+1}})} \le & \frac{c}{(\alpha_j-\beta_j)^{1/(p-2\theta)}(\varrho_j-\varrho_{j+1})^{\bar{\gamma}_j}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\hat{\gamma}_j} \\
\le & \frac{c2^{2\bar{\gamma}_j j}}{(\alpha_j-\beta_j)^{1/(p-2\theta)}\varrho^{\bar{\gamma}_j}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\hat{\gamma}_j} \qquad (6.3.11)
\end{aligned}$$

for $c \equiv c(n,N,\nu,L,p,q)$. From (6.3.11), it is evident that we have no chances of sending $j \to \infty$ without making the bounding constants blow up. However, by (6.3.5), for any $d \in [1,\tilde{p})$ we can find $j_d \equiv j_d(n,p,q,d) \in \mathbb{N} \cup \{0\}$ so that $d < p_{j_d+1}$, so coupling such an information with (6.3.11) and Hölder inequality we deduce that

$$\|Dv_\varepsilon\|_{L^d(B_{\varrho/2})} \le c\|Dv_\varepsilon\|_{L^{p_{j_d+1}}(B_{\varrho_{j_d}})} \le \frac{c2^{2\bar{\gamma}_{j_d}j_d}}{(\alpha_{j_d}-\beta_{j_d})^{1/(p-2\theta)}\varrho^{\bar{\gamma}_{j_d}}}\left[1+\|u\|_{W^{1,p}(B_\varrho)} + \mathcal{F}(u,B_\varrho)\right]^{\hat{\gamma}}.$$

Once the estimate in the previous display is available, we can follow verbatim the convergence scheme described in *Step 3* to transfer the improvement in integrability from v_ε to u and the proof is complete.

Remark 6.3.1 The case $p = 2$ is excluded since $2^* \equiv \frac{2n}{n-2}$, therefore a violation of type $(6.0.4)_1$ does not make sense, being the interval $\left(\frac{2n}{n-2}, 2^*\right)$ empty. Moreover, the asymptotics for $\tilde{p}$ are the following:

$$\begin{cases} \tilde{p} \searrow p \ \text{ as } \ q \nearrow p^* & \text{if } \ 2 < p < n \\ \tilde{p} \searrow \frac{n(p-2)}{n-2} \ \text{ as } \ q \to \infty & \text{if } \ p \ge n. \end{cases}$$

6.4 A Morrey-type result in two space dimensions

In this section we revisit the arguments presented in [25] in the light of the result obtained in Theorem 22 for $n = 2$. In particular, we shall consider a C^2-regular integrand $F\colon \mathbb{R}^{N\times 2} \to \mathbb{R}$ satisfying (6.1.1)-(6.1.5) in the non-degenerate case $\mu = 1$. From Theorem 22 we know in particular that

$$\begin{cases} V_{1,p}(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2}) \implies u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,d}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \text{ for all } d \in [1, \infty) \\ V_{1,q'}(\partial_z F(Du)) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2}) \implies V_{1,q'}(\partial_z F(Du)) \in L^d_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2}) \text{ for all } d \in [1, \infty) \end{cases} \tag{6.4.1}$$

and so we can test the Euler-Lagrange equation of functional (6.2.1) against linear combinations of u with suitable cut-off functions. Let us select discs (non necessarily concentric) $B_\varrho \Subset B_{r/2} \Subset B_r \Subset \Omega$ with radii $0 < \varrho < r \le 1$, a map $\eta \in C^1_c(B_\varrho)$ so that $\mathbb{1}_{B_{\varrho/2}} \le \eta \le \mathbb{1}_{B_\varrho}$ and $|D\eta| \lesssim \varrho^{-1}$, a number $h \in \left(0, \min\left\{1, \frac{\mathrm{dist}(\partial B_\varrho, \mathrm{supp}(\eta))}{10000}\right\}\right)$ and define the comparison map $\varphi_s := \tau_{-h_s}\left(\eta^2(\tau_{h_s} u - (Du)_{B_\varrho} h_s)\right)$, where $h_s := h e_s$ with e_s belonging to the standard orthonormal basis of $\mathbb{R}^2$. Set

$$\Gamma(Du) := \int_0^1 \partial_{zz} F(Du + \lambda \tau_{h_s} Du) \ \mathrm{d}\lambda$$

and notice that, being $\partial_{zz} F(\cdot)$ symmetric and positive definite (recall (6.1.4)), Γ is symmetric and positive definite as well. Using the integration by parts formula for finite difference operators, the mean value theorem, Cauchy-Schwartz inequality and (6.1.4), we obtain

$$\begin{aligned}
\int_{B_\varrho} & \eta^2 \Gamma(Du) \tau_{h_s} Du \cdot \tau_{h_s} Du \ \mathrm{d}x = \int_{B_\varrho} \eta^2 \tau_{h_s} \partial_z F(Du) \cdot \tau_{h_s} Du \ \mathrm{d}x \\
&= -2 \int_{B_\varrho} \eta \tau_{h_s} \partial_z F(Du) D\eta \otimes (D_s u - (Du)_{B_\varrho} h_s) \ \mathrm{d}x \\
&= -2 \int_{B_\varrho} \eta \left(\sqrt{\Gamma(Du)} \tau_{h_s} Du \cdot \sqrt{\Gamma(Du)} D\eta \otimes (\tau_{h_s} u - (Du)_{B_\varrho} h_s)\right) \ \mathrm{d}x \\
&\le 2 \int_{B_\varrho} \left(\sqrt{\Gamma(Du)} \tau_{h_s} Du \cdot \sqrt{\Gamma(Du)} \tau_{h_s} Du\right)^{\frac{1}{2}} \left|\sqrt{\Gamma(Du)} D\eta \otimes (\tau_{h_s} u - (Du)_{B_\varrho} h_s)\right| \ \mathrm{d}x \\
&\le 2L \int_{B_\varrho} \left(\eta^2 \Gamma(Du) \tau_{h_s} Du \cdot \tau_{h_s} Du\right)^{\frac{1}{2}} (1 + |\partial_z F(Du)|^2)^{\frac{q-2}{4(q-1)}} |D\eta| |\tau_{h_s} u - (Du)_{B_\varrho} h_s| \ \mathrm{d}x \\
&\le c \int_{B_\varrho} \left(\eta^2 \Gamma(Du) \tau_{h_s} Du \cdot \tau_{h_s} Du\right)^{\frac{1}{2}} (1 + |V_{\mu,q'}(\partial_z F(Du))|)^{\frac{q-2}{q}} |D\eta| |\tau_{h_s} u - (Du)_{B_\varrho} h_s| \ \mathrm{d}x,
\end{aligned}$$

for $c \equiv c(L, q)$. Now we divide both sides of the previous inequality by h^2, sum over $s \in \{1, 2\}$ and then send $h \to 0$ to conclude, by means of (6.4.1), Lemmas 2.2.4-2.2.5, Fatou Lemma on the left-hand side, the dominated convergence theorem on the right-hand side and taking averages:

$$\fint_{B_{\varrho/2}} \Upsilon(u)^2 \ \mathrm{d}x \le \frac{c(L, q)}{\varrho} \fint_{B_\varrho} \sigma \Upsilon(u) |Du - (Du)_{B_\varrho}| \ \mathrm{d}x, \tag{6.4.2}$$

where we defined

$$\Upsilon(u) := \sqrt{\partial_{zz} F(Du) D^2 u \cdot D^2 u} \qquad \text{and} \qquad \sigma := (1 + |V_{\mu,q'}(\partial_z F(Du))|)^{\frac{q-2}{q}}.$$

Inequality (6.4.2), together with (6.4.1) and a standard covering argument imply that

$$\Upsilon(u) \in L^2_{\mathrm{loc}}(\Omega). \tag{6.4.3}$$

Now, back to (6.4.2), set $\delta := 3/4$ and apply Hölder and Sobolev-Poincaré inequalities on the right-hand side of (6.4.2) to get

$$\begin{aligned}\fint_{B_\varrho} \sigma\Upsilon(u)|Du-(Du)_{B_\varrho}| \ \mathrm{d}x &\le \left(\fint_{B_\varrho}(\sigma\Upsilon(u))^{\delta} \ \mathrm{d}x\right)^{\frac{1}{\delta}}\left(\fint_{B_\varrho}|Du-(Du)_\varrho|^{\delta'} \ \mathrm{d}x\right)^{\frac{1}{\delta'}}\\ &\le c\varrho\left(\fint_{B_\varrho}(\sigma\Upsilon(u))^{\delta} \ \mathrm{d}x\right)^{\frac{1}{\delta}}\left(\fint_{B_\varrho}|D^2u|^{\delta} \ \mathrm{d}x\right)^{\frac{1}{\delta}}, \qquad (6.4.4)\end{aligned}$$

where $c\equiv c(N)$ and we also used that in two dimensions $\delta'=\delta^*$. We stress that the content of the previous display makes sense because of (6.4.1). Notice that, by (6.1.4) and being $p\ge 2$ we have

$$|D^2u|^2\le |D^2u|^2(1+|Du|^2)^{\frac{p-2}{2}}\le c(N,\nu,p)\Upsilon(u)^2,$$

therefore we can complete estimate (6.4.4) as follows:

$$\begin{aligned}\fint_{B_\varrho} \sigma\Upsilon(u)|Du-(Du)_{B_\varrho}| \ \mathrm{d}x &\le c\varrho\left(\fint_{B_\varrho}(\sigma\Upsilon(u))^{\delta} \ \mathrm{d}x\right)^{\frac{1}{\delta}}\left(\fint_{B_\varrho}\Upsilon(u)^{\delta} \ \mathrm{d}x\right)^{\frac{1}{\delta}}\\ &\le c\varrho\left(\fint_{B_\varrho}(\sigma\Upsilon(u))^{\delta} \ \mathrm{d}x\right)^{\frac{2}{\delta}}, \qquad (6.4.5)\end{aligned}$$

where we also used that $\sigma\ge 1$ and $c\equiv c(N,\nu,p)$. Merging (6.4.2) and (6.4.5) we obtain

$$\left(\fint_{B_{\varrho/2}}\tilde{\Upsilon}(u)^{\frac{2}{\delta}} \ \mathrm{d}x\right)^{\frac{\delta}{2}}\le c\left(\fint_{B_\varrho}\tilde{\sigma}\tilde{\Upsilon}(u) \ \mathrm{d}x\right), \qquad (6.4.6)$$

where $c\equiv c(N,\nu,L,p,q)$, $\tilde{\Upsilon}(u):=\Upsilon(u)^{\delta}$, $\tilde{\sigma}:=\sigma^{\delta}$ and $2/\delta>1$. By $(6.4.1)_2$ and Trudinger's inequality [122, Theorem 7.15] we deduce that there exists a positive constant

$$\alpha_0\equiv\alpha_0(\|V_{1,q'}(\partial_zF(Du))\|_{W^{1,2}(B_r)})$$

so that

$$\fint_{B_{r/2}}\exp\left(\alpha_0\left[1+|V_{1,q'}(\partial_zF(Du))|\right]^2\right) \ \mathrm{d}x<\infty.$$

Moreover, we see that for all $\kappa\in(0,4/q)$ there holds that

$$\left[1+|V_{1,q'}(\partial_zF(Du))|\right]^2\ge\left[1+|V_{1,q'}(\partial_zF(Du))|\right]^{2-\kappa}\ge\tilde{\sigma}^{\frac{2}{\delta}},$$

so we deduce that

$$\fint_{B_{r/2}}\exp\left(\alpha\tilde{\sigma}^{\frac{2}{\delta}}\right) \ \mathrm{d}x\le c<\infty \quad \text{for all} \ \ \alpha\in(0,1), \qquad (6.4.7)$$

with $c\equiv c(\alpha,\kappa,\|V_{1,q'}(\partial_zF(Du))\|_{W^{1,2}(B_r)})$. Estimates (6.4.6)-(6.4.7) and (6.4.3) allow applying [25, Lemma 1.2] to prove that there exists a positive constant $c_0\equiv c_0(N,\nu,L,p,q)$ so that

$$\tilde{\Upsilon}(u)\in L^{\frac{2}{\delta}}_{\mathrm{loc}}(\mathrm{Log\ L})^{c_0\alpha}(B_\varrho)\implies\Upsilon(u)\in L^2_{\mathrm{loc}}(\mathrm{LogL})^{c_0\alpha}(B_\varrho) \qquad (6.4.8)$$

for all $\alpha>0$. Now, by (6.1.4) and (6.4.1) we have that

$$|D\partial_zF(Du)|\le c(L,q)\Upsilon(u)\sigma. \qquad (6.4.9)$$

Moreover, we recall that for all $\alpha > 0$ there exists a constant $c \equiv c(\alpha) > 0$ so that if a, b are real, non-negative numbers there holds that

$$(ab)^2 \log^{\alpha}(e + ab) \le 2^{\alpha} a^2 \log^{\alpha+2}(e + a) + c(\alpha) \exp(6b), \tag{6.4.10}$$

cf. [25, Inequality (2.12)]. Now, given any disc $B_{\varsigma} \Subset B_{\varrho}$, we can finally estimate

$$\begin{aligned}
&\int_{B_{\varsigma}} |D\partial_z F(Du)|^2 \log^{\alpha}\left(e + |D\partial_z F(Du)|\right) \, \mathrm{d}x \\
&\quad \overset{(6.4.9)}{\le} \int_{B_{\varsigma}} |c\Upsilon(u)\sigma|^2 \log^{\alpha}(e + |c\Upsilon(u)\sigma|) \, \mathrm{d}x \\
&\quad \overset{(6.4.10)}{\le} 2^{\alpha} \int_{B_{\varsigma}} |\Upsilon(u)|^2 \log^{\alpha+2}(e + \Upsilon(u)) \, \mathrm{d}x \\
&\qquad + c(\alpha) \int_{B_{\varsigma}} \exp(6c\sigma) \, \mathrm{d}x \overset{(6.4.7),(6.4.8)}{<} \infty,
\end{aligned}$$

for all $\alpha > 0$, where we also used that $t \mapsto t^2 \log^{\alpha}(e + t)$ is increasing and that $\tilde{\sigma}^{2/\delta} = \sigma^2$ by definition. After a standard covering argument, we just got that $|D\partial_z F(Du)| \in L^2_{\mathrm{loc}}(\mathrm{LogL})^{\alpha}(\Omega)$ for all $\alpha > 0$, which implies that, choosing $\alpha > 1$, with [159, Example 5.3] we have that $\partial_z F(Du) \in C_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$, which implies that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$. Once this last information is available, the non-uniform ellipticity of functional (6.2.1) becomes immaterial, so we can conclude by standard arguments that $Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times 2})$ for all $\beta \in (0, 1)$ and the proof of Theorem 24 is complete.

References

[1] E. Acerbi, G. Bouchitté, and I. Fonseca. Relaxation of convex functionals: the gap problem. *Ann. I. H. Poincaré - AN*, 20:359–390, 2003.

[2] E. Acerbi and N. Fusco. Regularity for minimizers of non-quadratic functionals: the case $1 < p < 2$. *J. Math. Anal. Appl.*, 140:115–135, 1989.

[3] E. Acerbi and G. Mingione. Regularity results for a class of functionals with nonstandard growth. *Arch. Ration. Mech. Anal.*, 156:121–140, 2001.

[4] E. Acerbi and G. Mingione. Regularity results for electrorheological fluids: the stationary case. *C. R. Math. Acad. Sci. Paris*, 334:817–822, 2002.

[5] Y. Ahmida, I. Chlebicka, P. Gwiazda, and A. Youssfi. Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces. *J. Functional Analysis*, 275:2538–2571, 2018.

[6] B. Avelin, T. Kuusi, and G. Mingione. Nonlinear Calderón-Zygmund theory in the limiting case. *Arch. Ration. Mech. Anal.*, 227:663–714, 2018.

[7] A. K. Balci, L. Diening, and M. Surnachev. New examples on Lavrentiev gap using fractals. *Preprint*, 2019.

[8] A. Banerjee and I. H. Munive. Gradient continuity estimates for the normalized p-Poisson equation. *Comm. Cont. Math.*, 2019.

[9] P. Baroni. Riesz potential estimates for a general class of quasilinear equations. *Calc. Var. & PDE*, 53:803–846, 2015.

[10] P. Baroni, M. Colombo, and G. Mingione. Harnack inequalities for double phase functionals. *Nonlinear Anal.*, 121:206–222, 2015.

[11] P. Baroni, M. Colombo, and G. Mingione. Non-autonomous functionals, borderline cases and related function classes. *St. Petersburg Math. J.*, 27:347–379, 2016.

[12] P. Baroni, M. Colombo, and G. Mingione. Regularity for general functionals with double phase. *Calc. Var. & PDE*, 57, 2018.

[13] D. Baruah, P. Harjulehto, and P. Hästö. Capacities in generalized Orlicz spaces. *J. Funct. Spaces*, 2018.

[14] L. Beck, M. Bulíčeck, and F. Gmeineder. On a Neumann problem for variational functionals of linear growth. *Ann. SNS Pisa, to appear.*

[15] L. Beck, C. De Filippis, and G. Mingione. Flow of non-uniformly elliptic problems. *In preparation.*

[16] L. Beck and G. Mingione. Lipschitz bounds and non-uniform ellipticity. *Comm. Pure Appl. Math.*, 73:944–1034, 2020.

[17] P. Bella and M. Schäffner. Local boundedness and Harnack inequality for solutions of linear non-uniformly elliptic equations. *Comm. Pure Appl. Math.*, to appear.

[18] P. Bella and M. Schäffner. On the regularity of scalar integral functionals with (p,q)-growth. *Anal. PDE*, to appear.

[19] S. Bernstein. Sur la nature analytique des solutions des équations aux derivées partielles du second ordre. *Math. Ann.*, 59:20–76, 1904.

[20] F. Bethuel. A new obstruction to the extension problem for Sobolev maps between manifolds. *J. Fixed Point Theory Appl.*, 15:155–183, 2014.

[21] F. Bethuel and D. Chiron. Some questions related to the lifting problem in Sobolev spaces. *Contemp. Math., Amer. Math. Soc., Providence, R.I.*, 446:125–152, 1995.

[22] F. Bethuel and F. Demengel. Extensions for Sobolev mappings between manifolds. *Calc. Var. & PDE*, 3:475–491, 1995.

[23] M. Bildhauer. Convex variational problems. linear, nearly linear and anisotropic growth conditions. *Lecture Notes in Mathematics, Springer-Verlag, Berlin*, 1818, 2018.

[24] M. Bildhauer and M. Fuchs. $C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems. *Calc. Var. & PDE*, 24:309–340, 2005.

[25] M. Bildhauer, M. Fuchs, and X. Zhong. A lemma on the higher integrability of functions with applications to the regularity theory of two-dimensional generalized newtonian fluids. *Manuscripta Math.*, 116:135–156, 2005.

[26] J. Bourgain, H. Brezis, and P. Mironescu. Lifting in Sobolev spaces. *J. Anal. Math.*, 80:37–86, 2000.

[27] J. Bourgain, A. C. Ponce, and J. Van Schaftingen. Strong approximation of fractional Sobolev maps. *J. Fixed Point Theory Appl.*, 15:133–153, 2014.

[28] P. Bousquet and L. Brasco. C^1-regularity of orthotropic p-harmonic functions in the plane. *Anal. PDE*, 11:813–854, 2018.

[29] P. Bousquet and L. Brasco. Lipschitz regularity for orthotropic functionals with non standard growth conditions. *Rev. Math. Iberoam.*, to appear.

[30] H. Brezis and F. Browder. Partial differential equations in the 20th century. *Adv. in Math.*, 135:76–144, 1998.

[31] H. Brezis and P. Mironescu. Density in $W^{s,p}(\omega, \mathcal{N})$. *J. Funct. Anal.*, 269:2045–2109, 2015.

[32] G. Buttazzo and V. J. Mizel. Interpretation of the Lavrentiev Phenomenon by relaxation. *J. Funct. Anal.*, 110:434–460, 1992.

[33] S. S. Byun and J. Oh. Global gradient estimates for non-uniformly elliptic equations. *Calc. Var. & PDE*, 56, 2017.

[34] S. S. Byun, J. Oh, and S. Ryu. Global gradient estimates for elliptic equations of $p(x)$-laplacian type with bmo nonlinearity. *J. Reine Angew. Math.*, 1-38, 2016.

[35] S. S. Byun and Y. Youn. Riesz potential estimates for a class of double phase problems. *J. Diff. Equations*, 264, 2016.

[36] R. Caccioppoli. Sulle equazioni ellittiche non lineari a derivate parziali. *Rend. Accad. Naz. Lincei*, 18:103–106, 1933.

[37] R. Caccioppoli. Sulle equazioni ellittiche a derivate parziali con n variabili indipendenti. *Rend. Accad. Naz. Lincei*, 19:83–89, 1934.

[38] R. Caccioppoli. Sulle equazioni ellittiche a derivate parziali con due variabili indipendenti e sui problemi regolari del calcolo delle variazioni. *Rend. Accad. Naz. Lincei*, 22:305–310, 1935.

[39] R. Caccioppoli. Limitazioni integrali per le soluzioni di un'equazione lineare ellittica a derivate parziali. *Giornale di Mat. di Battaglini*, 80:186–212, 1950-1951.

[40] L. Caffarelli and I. Peral. On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.*, 51:1–21, 1998.

[41] A. P. Calderón and A. Zygmund. On the existence of certain singular integrals. *Acta Math.*, 88:85–139, 1952.

[42] A. P. Calderón and A. Zygmund. On singular integrals. *Amer. J. Math.*, 78:289–309, 1956.

[43] S. Campanato. Equazioni ellittiche del II ordine e spazi $L(2, \lambda)$. *Annali Math. Pura Appl.*, 69:321–381, 1965.

[44] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. Higher differentiability of minimizers of convex variational integrals. *Ann. I. H. Poincaré - AN*, 28:395–411, 2011.

[45] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. Regularity of minimizers of autonomous convex variational integrals. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, XIII, 2014.

[46] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. On the validity of the Euler-Lagrange system. *Communications on Pure and Applied Analysis*, 14:51–62, 2015.

[47] M. Caselli, M. Eleuteri, and A. Passarelli di Napoli. Regularity results for a class of obstacle problems with p, q-growth conditions. *Preprint*, 2019.

[48] A. Cellina. On the validity of the Euler-Lagrange equation. *J. Differential Equations*, 171:430–442, 2001.

[49] A. Cellina and V. Staicu. On the higher differentiability of solutions to a class of variational problems of fast growth. *Calc. Var. & PDE*, 2018.

[50] M. Cencelj, V. Radulescu, and D. Repovš. Double phase problems with variable growth. *Nonlinear Anal.*, 177:270–287, 2018.

[51] I. Chlebicka and C. De Filippis. Removable sets in non-uniformly elliptic problems. *Ann. Mat. Pura Appl.*, pages 619–649, 2020.

[52] I. Chlebicka, C. De Filippis, and L. Koch. Boundary regularity for manifold constrained $p(x)$-harmonic maps. *Preprint*, 2020.

[53] H. J. Choe. Interior behaviour of minimizers for certain functionals with nonstandard growth. *Nonlinear Anal.*, 19:933–945, 1992.

[54] G. Choquet. Theory of capacities. *Ann. Inst. Fourier, Grenoble*, 5:131–295, 1955.

[55] A. Cianchi. Maximizing the L^∞-norm of the gradient of solutions to the Poisson equation. *J. Geom. Anal.*, 2:499–515, 1992.

[56] A. Cianchi and V. G. Maz'ya. Global Lipschitz regularity for a class of quasilinear equations. *Comm. PDE*, 36:100–133, 2011.

[57] A. Cianchi and V. G. Maz'ya. Global boundedness of the gradient for a class of nonlinear elliptic systems. *Arch. Ration. Mech. Anal.*, 212:129–177, 2014.

[58] A. Cianchi and V. G. Maz'ya. Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems. *J. Eur. Math. Soc. (JEMS)*, 16:571–595, 2014.

[59] A. Cianchi and V. G. Maz'ya. Global gradient estimates in elliptic problems under minimal data and domain regularity. *Comm. Pure Appl. Anal.*, 14:285–311, 2015.

[60] M. Colombo and G. Mingione. Bounded minimisers of double phase variationals integrals. *Arch. Rational Mech. Anal.*, 218:219–273, 2015.

[61] M. Colombo and G. Mingione. Regularity for double phase variational problems. *Arch. Rational Mech. Anal.*, 215:443–496, 2015.

[62] M. Colombo and G. Mingione. Calderón-Zygmund estimates and non-uniformly elliptic operators. *J. Funct. Anal.*, 270:1416–1478, 2016.

[63] A. Coscia and G. Mingione. Hölder continuity of the gradient of $p(x)$-harmonic mappings. *C. R. Acad. Sci. Paris*, 328:363–368, 1999.

[64] S. Costea. Sobolev-Lorentz spaces in the euclidean setting and counterexamples. *Nonlinear Analysis*, 152:149–182, 2017.

[65] P. Daskalopoulos, T. Kuusi, and G. Mingione. Borderline estimates for fully nonlinear elliptic equations. *Comm. Partial Differential Equations*, 39:309–363, 2014.

[66] V. De Cicco and G. Leoni. A chain rule in $\mathrm{L}^1(\mathrm{div}, \omega)$ and its applications to lower semicontinuity. *Calc. Var. & PDE*, 19:23–51, 2004.

[67] C. De Filippis. Higher integrability for constrained minimizers of integral functionals with (p,q)-growth in low dimension. *Nonlinear Analysis*, 2018.

[68] C. De Filippis. On the regularity of the ω-minima of φ-functionals. *Nonlinear Analysis*, 2019.

[69] C. De Filippis. Partial regularity for manifold constrained p(x)-harmonic maps. *Calc. Var. & PDE*, 58(2):47, Feb 2019.

[70] C. De Filippis. Regularity results for a class of non-autonomous obstacle problems with (p, q)-growth. *J. Math. Anal. Appl.*, 2019.

[71] C. De Filippis. Gradient bounds for solutions to irregular parabolic equations with (p, q)-growth. *Preprint*, 2020.

[72] C. De Filippis. Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy. *Proc. Royal Soc. Edinburgh Math.*, to appear.

[73] C. De Filippis, L. Koch, and J. Kristensen. Higher differentiability for minimizers of variational integrals. *Preprint*, 2020.

[74] C. De Filippis and F. Leonetti. Uniform ellipticity and p, q-growth. *Preprint*, 2020.

[75] C. De Filippis and G. Mingione. Manifold constrained non-uniformly elliptic problems. *J. Geometric Analysis*, pages 1661–1723, 2020.

[76] C. De Filippis and G. Mingione. Lipschitz bounds and non-autonomous functionals. *Preprint*, 2020.

[77] C. De Filippis and G. Mingione. On the regularity of minima of non-autonomous functionals. *J. Geometric Analysis*, pages 1584–1626, 2020.

[78] C. De Filippis and G. Mingione. A borderline case of Calderón-Zygmund estimates for non-uniformly elliptic problems. *St. Petersburg Math. J.*, to appear.

[79] C. De Filippis and J. Oh. Regularity for multi-phase variational problems. *J. Differential Equations*, 2019.

[80] C. De Filippis and G. Palatucci. Hölder regularity for nonlocal double phase equations. *Journal of Differential Equations*, 267(1):547–586, 2019.

[81] E. De Giorgi. Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari. *Mem. Acad. Sci. Torino*, 3:25–43, 1957.

[82] E. De Giorgi. Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. *Boll. UMI*, 4:135–137, 1968.

[83] R. A. DeVore and R. C. Sharpley. Maximal functions measuring smoothness. *Memoirs of Amer. Math. Soc.*, 47, 1984.

[84] T. Di Marco and P. Marcellini. A-priori gradient bound for elliptic systems under either slow or fast growth conditions. *Preprint*, 2020.

[85] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bulletin des Sciences Mathématiques*, 136:521–573, 2012.

[86] E. DiBenedetto and J. J. Manfredi. On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. *Amer. J. Math.*, 115:1107–1134, 1993.

[87] L. Diening and F. Ettwein. Fractional estimate for non-differentiable elliptic systems with general growth. *Forum Math.*, 20:523–556, 2008.

[88] L. Diening, P. Harjulehto, P. Hästo, and M. Ružička. Lebesgue and Sobolev spaces with a variable growth exponent. *Springer Lecture notes Math*, 2017, 2011.

[89] L. Diening, B. Stroffolini, and A. Verde. Everywhere regularity of functionals with φ-growth. *Manuscripta Math.*, 129:449–481, 2009.

[90] L. Diening, B. Stroffolini, and A. Verde. The φ-harmonic approximation and the regularity of φ-harmonic maps. *J. Differential Equations*, 253:1943–1958, 2012.

[91] D. M. Duc and J. Eells. Regularity of exponentially harmonic functions. *Internat. J. Math.*, 2:395–408, 1991.

[92] M. Duerinckx and A. Gloria. Stochastic homogenization of nonconvex unbounded integral functionals with convex growth. *Arch. Ration. Mech. Anal.*, 221:1511–1584, 2016.

[93] F. Duzaar and G. Mingione. The p-harmonic approximation and the regularity of p-harmonic maps. *Calc. Var. & PDE*, 20:235–256, 2004.

[94] F. Duzaar and G. Mingione. Local Lipschitz regularity for degenerate elliptic systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27:1361–1396, 2010.

[95] J. Eells and J. H. Sampson. Harmonic mappings of riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.

[96] M. Eleuteri. Hölder continuity results for a class of functionals with non standard growth. *Bollettino Unione Mat. Iital.*, pages 129–157, 2004.

[97] M. Eleuteri, P. Marcellini, and E. Mascolo. Lipschitz estimates for systems with ellipticity conditions at infinity. *Ann. Mat. Pura Appl.*, 195:1575–1603, 2016.

[98] M. Eleuteri, P. Marcellini, and E. Mascolo. Regularity for scalar integrals without structure conditions. *Adv. Calc. Var.*, 2018.

[99] L. Esposito, F. Leonetti, and G. Mingione. Higher integrability for minimizers of integral functionals with (p, q) growth. *J. Differential Equations*, 157:414–438, 1999.

[100] L. Esposito, F. Leonetti, and G. Mingione. Regularity results for minimizers of irregular integrals with (p, q) growth. *Forum Math.*, 14:245–272, 2002.

[101] L. Esposito, F. Leonetti, and G. Mingione. Sharp regularity for functionals with (p, q) growth. *J. Differential Equations*, 204:5–55, 2004.

[102] L. C. Evans. Some new PDE methods for weak KAM theory. *Calc. Var. & PDE*, 17:159–177, 2003.

[103] L. C. Evans and R. F. Gariepy. Partial regularity for constrained minimizers of convex or quasiconvex functionals. *Rend. Sem. Mat. Univ. Pol. Torino*, 1989.

[104] L. C. Evans and R. F. Gariepy. Measure theory and fine properties of functions. *CRC Press*, 1991.

[105] I. Fonseca and N. Fusco. Regularity results for anisotropic image segmentation models. *Ann. Sc. Norm. Sup. Pisa Cl. Sci.*, 24:463–499, 1997.

[106] I. Fonseca, J. Malý, and G. Mingione. Scalar minimizers with fractal singular sets. *Arch. Ration. Mech. Anal.*, 172:295–307, 2004.

[107] J. Frehse. A note on the holder continuity of solutions of variational problems. *Abh. Math. Semin. Univ. Hamb.*, 43:59–63, 1975.

[108] J. Frehse. Capacity methods in the theory of partial differential equations. *Jahresber. Deutsch. Math.-Verein.*, 84:1–44, 1982.

[109] M. Fuchs. Hölder continuity of the gradient for degenerate variational inequalities. *Nonlinear Anal.*, 15:85–100, 1990.

[110] M. Fuchs. p-harmonic obstacle problems. i: Partial regularity theory. *Ann. Mat. Pura Appl. (IV)*, 156:127–158, 1990.

[111] M. Fuchs. p-harmonic obstacle problems. iii. boundary regularity. *Ann. Mat. Pura Appl. (IV)*, 156:159–180, 1990.

[112] M. Fuchs and G. Mingione. Full $\mathrm{C}^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth. *Manuscripta Math..*, 102:227–250, 2000.

[113] M. Giaquinta. Multiple integrals in the calculus of variations and nonlinear elliptic systems. *Princeton Univ. Press, Princeton*, 1983.

[114] M. Giaquinta. Growth conditions and regularity. a counterexample. *Manuscr. Math.*, 59:245–248, 1987.

[115] M. Giaquinta and E. Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[116] M. Giaquinta and E. Giusti. Differentiability of minima of nondifferentiable functionals. *Invent. Math.*, 72:285–298, 1983.

[117] M. Giaquinta and E. Giusti. Global $C^{1,\alpha}$-regularity for second order elliptic equations in divergence form. *J. Reine Angew. Math.*, 351:55–65, 1984.

[118] M. Giaquinta and E. Giusti. The singular set of the minima of certain quadratic functionals. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, 11:45–55, 1984.

[119] M. Giaquinta and L. Martinazzi. An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs. *Edizioni della Normale*, 2012.

[120] M. Giaquinta and G. Modica. Almost-everywhere regularity for solutions of nonlinear elliptic systems. *Manuscripta Math.*, 28:109–158, 1979.

[121] M. Giaquinta and G. Modica. Remarks on the regularity of the minimizers of certain degenerate functionals. *Manuscripta Math.*, 57:55–99, 1986.

[122] D. Gilbarg and N. Trudinger. Elliptic partial differential equations of the second order. *Springer-Verlag, Berlin-Heidelberg.*, 1983.

[123] E. Giusti. Direct methods in the calculus of variations. *World Scientific Publishing Co., Inc., River Edge.*, 2003.

[124] E. Giusti and M. Miranda. Sulla regolaritá delle soluzioni deboli di una classe di sistemi ellittici quasilineari. *Arch. Rat. Mech. Anal.*, 31:173–184, 1968.

[125] E. Giusti and M. Miranda. Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni. *Boll. UMI*, 2:1–8, 1968.

[126] F. Gmeineder and J. Kristensen. Sobolev regularity for convex functionals on BD. *Calc. Var. & PDE*, 2019.

[127] F. Gmeineder and J. Kristensen. Partial regularity for bv minimizers. *Arch. Ration. Mech. Anal.*, 232:1429–1473, 2019.

[128] C. E. Gutiérrez and I. Peral. A harmonic analysis theorem and applications to homogenization. *Indiana Univ. Math. J.*, 50:1651–1674, 2001.

[129] C. Hamburger. Regularity of differential forms minimizing degenerate elliptic functionals. *J. Reine Angew. Math.*, 431:7–64, 1992.

[130] W. Hao, S. Leonardi, and J. Nečas. An example of irregular solution to a nonlinear euler-lagrange elliptic system with real analytic coefficients. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, IV:57–67, 1996.

[131] R. Hardt, D. Kinderlehrer, and F.-H. Lin. Stable defects of minimizers of constrained variational principles. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5:297–322, 1988.

[132] R. Hardt and F.-H. Lin. Mappings minimizing the L^p norm of the gradient. *Comm. Pure and Appl. Math.*, XL:555–588, 1987.

[133] R. Hardt, F.-H. Lin, and C. Wang. The p-energy minimality of $x/|x|$. *Comm. in Anal. and Geom.*, 6:141–152, 1998.

[134] P. Harjulehto and P. Hästö. Boundary regularity under generalized growth conditions. *Preprint.*, 2017,.

[135] P. Harjulehto and P. Hästö. Orlicz spaces and generalized Orlicz spaces. *Lecture Notes in Mathematics.*, 2236, 2019.

[136] P. Harjulehto and P. Hästö. Double phase image restoration. *J. Math. Anal. Appl.*, 2020,.

[137] P. Harjulehto, P. Hästö, and M. Lee. Hölder continuity of ω-minimizers of functionals with generalized Orlicz growth. *Preprint.*, 2019.

[138] P. Harjulehto, P. Hästö, and O. Toivanen. Hölder regularity of quasiminimizers under generalized growth conditions. *Calc. Var. & PDE*, 56, 2017,.

[139] J. Heinonen, T. Kilpeläinen, and O. Martio. Nonlinear potential theory of degenerate elliptic equations. *Dover Publications Inc., Mineola, NY*, 2006.

[140] F. Helein. Regularity and uniqueness of harmonic maps into an ellipsoid. *Manuscripta Math.*, 60:235–257, 1988.

[141] S. Hildebrandt, H. Kaul, and K. O. Widman. An existence theorem for harmonic mappings of riemannian manifolds. *Acta Math.*, 138:550–569, 1970.

[142] S. Hildebrandt and K. O. Widman. On the H ölder continuity of weak solutions of quasilinear elliptic systems of second order. *Ann. Sc. Norm. Super. Pisa*, IV:145–178, 1977.

[143] J.-B. Hiriart and C. Lemaréchal. Convex analysis and minimization algorithms ii - advanced theory and bundle methods. *Grundlehren der mathematischen Wissenschaften, Springer-Verlag Heidelberg GmbH*, 306, 1993.

[144] J. Hirsch and M. Schäffner. Growth conditions and regularity, an optimal local boundedness result. *Comm. Cont. Math.*, to appear.

[145] P. Hästö. The maximal operator on generalized Orlicz spaces. *J. Funct. Anal.*, 269:4038–4048, 2015,.

[146] P. Hästö and J. Ok. Maximal regularity for local minimizers of non-autonomous functionals. *Preprint.*, 2019.

[147] M.-C. Hong. Some remarks on the minimizers of variational integrals with nonstandard growth conditions. *Boll. Unione Mat. Ital.*, VII:91–101, 1992.

[148] M.-C. Hong. On the minimality of the p-harmonic map $\frac{x}{|x|}\colon \mathrm{B}^n \to \mathbb{S}^{n-1}$. *Calc. Var & PDE*, 13:459–468, 2001.

[149] E. Hopf. Zum analytischen Character der lösungen regulärer zweidimensionaler Variationsprobleme. *Math. Z.*, 30:404–413, 1929.

[150] E. Hopf. Uber den funktionalen, insbesondere den analytischen Character der lösungen elliptischer Differentialgleichungen zweiter Ordnung. *Math. Z.*, 34:194–233, 1932.

[151] C. Hopper. Partial regularity for holonomic minimizers of quasiconvex functionals. *Arch. Rational Mech. Anal.*, 222:91–141, 2016.

[152] C. Imbert and L. Silvestre. $\mathrm{C}^{1,\alpha}$ regularity of solutions of some degenerate, fully non-linear elliptic equations. *Adv. Math.*, 233:196–206, 2013.

[153] P.-A. Ivert. Regularitätsuntersuchungen von lösungen elliptischer Systeme von quasilinearen Differentialgleichungen zweiter Ordnung. *Manuscripta Math.*, 30:53–88, 1979/80.

[154] P.-A. Ivert. Partial regularity of vector valued functions minimizing variational integrals. *Preprint Univ. Bonn*, 1982.

[155] T. Iwaniec. On L^p-integrability in pdes and quasiregular mappings for large exponents. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 7:301–322, 1982.

[156] T. Iwaniec. Projections onto gradient fields and L^p-estimates for degenerated elliptic operators. *Studia Math.*, 75:293–312, 1983.

[157] T. Jin, V. G. Maz'ya, and J. Van Schaftingen. Pathological solutions to elliptic problems in divergence form with continuous coefficients. *C. R. Math. Acad. Sci. Paris*, 347:773–778, 2009.

[158] A. Karppinen. Global continuity and higher integrability of a minimizer of an obstacle problem under generalized Orlicz growth conditions. *Manuscripta Math.*, 2019.

[159] J. Kauhanen, P. Koskela, and J. Malý. On functions with derivatives in a Lorentz space. *Manuscripta Math.*, 100:87–101, 1999.

[160] T. Kilpeläinen and J. Malý. The wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.*, 172:137–161, 1993.

[161] J. Kinnunen and S. Zhou. A local estimate for nonlinear equations with discontinuous coefficients. *Comm. Partial Differential Equations*, 24:2043–2068, 1999.

[162] J. Kristensen and G. Mingione. The singular set of minima of integral functionals. *Arch. Rational Mech. Anal.*, 180:331–398, 2006.

[163] J. Kristensen and G. Mingione. The singular set of Lipschitzian minima of multiple integrals. *Arch. Rational Mech. Anal.*, 184:341–369, 2007.

[164] J. Kristensen and G. Mingione. Boundary regularity in variational problems. *Arch. Rational Mech. Anal.*, 198:369–455, 2010.

[165] T. Kuusi and G. Mingione. Potential estimates and gradient boundedness for nonlinear parabolic systems. *Rev. Mat. Iberoam.*, 28:535–576, 2012.

[166] T. Kuusi and G. Mingione. Linear potentials in nonlinear potential theory. *Arch. Rational Mech. Anal.*, 207:215–246, 2013.

[167] T. Kuusi and G. Mingione. Borderline gradient continuity for nonlinear parabolic systems. *Math. Ann.*, 360:937–993, 2014.

[168] T. Kuusi and G. Mingione. Guide to nonlinear potential estimates. *Bull. Math. Sci.*, 4:1–82, 2014.

[169] T. Kuusi and G. Mingione. A nonlinear Stein theorem. *Calc. Var. & PDE.*, 51:45–86, 2014.

[170] T. Kuusi and G. Mingione. Vectorial nonlinear potential theory. *J. Europ. Math. Soc. (JEMS).*, 20:929–1004, 2018.

[171] T. Kuusi, G. Mingione, and Y. Sire. Nonlocal self-improving properties. *Analysis and PDE*, 8:57–114, 2015.

[172] O. A. Ladyzhenskaya and N. N. Ural'tseva. Linear and quasilinear elliptic equations. *Mathematics in Science and Engineering*, 1968.

[173] O. A. Ladyzhenskaya and N. N. Ural'tseva. Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 23:677–703, 1970.

[174] F. Leonetti, E. Mascolo, and F. Siepe. Everywhere regularity for a class of vectorial functionals under subquadratic general growth. *J. Math. Anal. Appl.*, 287:593–608, 2003.

[175] F. Leonetti and F. Siepe. Maximum principle for vector valued minimizers. *J. Convex Anal.*, 12:267–278, 2005.

[176] G. M. Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12:1203–1219, 1988.

[177] G. M. Lieberman. The natural generalization of the natural condition of ladyzhenskaya and ural'tseva for elliptic equation. *Comm. PDE.*, 16:311–361, 1991.

[178] G. M. Lieberman. Gradient estimates for a class of elliptic systems. *Ann. Mat. Pura Appl.*, 164:103–120, 1993.

[179] S. Luckhaus. Partial H ölder continuity for minima of certain energies among maps into a riemannian manifold. *Indiana Univ. Math. J.*, 37:349–367, 1988.

[180] J. J. Manfredi. Regularity for minima of functionals with p-growth. *J. Differ. Equations.*, 76:203–212, 1988.

[181] P. Marcellini. On the definition and the lower semicontinuity of certain quasiconvex integrals. *Annales de l'I.H.P. - AN*, 3:391–409, 1986.

[182] P. Marcellini. Un example de solution discontinue d'un probléme variationnel dans le cas scalaire. *Preprint, Ist. U. Dini, Firenze.*, 1987-1988.

[183] P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions. *Arch. Ration. Mech. Anal.*, 105:267–284, 1989.

[184] P. Marcellini. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differential Equations.*, 90:1–30, 1991.

[185] P. Marcellini. Regularity for elliptic equations with general growth conditions. *J. Differential Equations.*, 105:296–333, 1993.

[186] P. Marcellini. Everywhere regularity for a class of elliptic systems without growth conditions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 23:1–25, 1996.

[187] P. Marcellini. Regularity for some scalar variational problems under general growth conditions. *J. Optim. Theory Appl.*, 90:161–181, 1996.

[188] P. Marcellini. Regularity for some scalar variational problems under general growth conditions. *J. Optimization Theory and Applications*, 90:161–181, 1996.

[189] P. Marcellini and G. Papi. Nonlinear elliptic systems with general growth. *J. Differential Equations*, 221:412–443, 2006.

[190] E. Mascolo and A. P. Migliorini. Everywhere regularity for vectorial functionals with general growth. *ESAIM: Control, Optimisation and Calculus of Variations.*, 9:399–418, 2003.

[191] K. Mazowiecka, M. Mi'skiewicz, and A. Schikorra. On the size of the singular set of minimizing harmonic maps into the 2-sphere in dimension four and higher. *Arxiv preprint, 2019.*

[192] V. G. Mazya. Sobolev spaces with applications to elliptic partial differential equations. second, revised and augmented edition. *Grundlehren der Mathematischen Wissenschaften, Springer, Heidelberg.*, 342, 2011.

[193] G. Mingione. Bounds for the singular set of solutions to non linear elliptic systems. *Calc. Var. & PDE*, 18:373–400, 2003.

[194] G. Mingione. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.*, 51:355–425, 2006.

[195] G. Mingione. The Calderón-Zygmund theory for elliptic problems with measure data. *Ann. Scu. Norm. Sup. Pisa Cl. Sci. (V)*, 6:195–261, 2007.

[196] G. Mingione. Nonlinear aspects of Calderón-Zygmund theory. *Jahresbericht der DMV*, 112:159–191, 2010.

[197] P. Mironescu. Sobolev maps on manifolds: degree, approximation, lifting. *Arithmetic, Geometry and Coding Theory - Contemporary mathematics, American Mathematical Society*, 446:413–436, 2007.

[198] P. Mironescu and J. Van Schaftingen. Trace Theory for Sobolev mappings into a manifold. *Preprint*, 2020.

[199] C. Mooney and O. Savin. Some singular minimizers in low dimensions in the calculus of variations. *Arch. Ration. Mech. Anal.*, 221:1–22, 2016.

[200] C. B. Morrey. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.*, 43:126–166, 1938.

[201] C. B. Morrey. Existence and differentiability theorems for the solutions of variational problems for multiple integrals. *Bull. Amer. Math. Soc.*, 46:439–458, 1940.

[202] C. B. Morrey. Multiple integral problems in the calculus of variations and related topics. *Univ. of California Publ. in Math., new ser.*, I:1–130, 1943.

[203] C. B. Morrey. The problem of plateau on a riemannian manifold. *Annals of Math.*, 49:807–851, 1948.

[204] C. B. Morrey. Second order elliptic systems of differential equations. *Annals of Math. Studies*, 33:101–159, 1954.

[205] C. B. Morrey. On the analyticity of the solutions of analytic nonlinear elliptic systems of partial differential equations i, ii. *Amer. J. of Math.*, 80:198–218; 219–234, 1958.

[206] C. B. Morrey. Second order elliptic equations in several variables and hölder continuity. *Math. Z.*, 72:146–164, 1959.

[207] C. B. Morrey. Multiple integral problems in the calculus of variations and related topics. *Ann. Sc. Norm. Super. Pisa*, 14:1–61, 1960.

[208] C. B. Morrey. Partial regularity results for nonlinear elliptic systems. *Journ. Math. and Mech.*, 17:649–670, 1968.

[209] J. Moser. A new proof of de giorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.*, 13:457–468, 1960.

[210] J. Moser. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961.

[211] J. Moser. A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17:101–134, 1964.

[212] J. Musielak. Orlicz spaces and Modular spaces. *Lecture Notes in Mathematics, Springer*, 1034, 1983.

[213] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. of Math.*, 80:931–954, 1958.

[214] J. Nečas. Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity. *Theor. Nonlin. Oper., Constr. Aspects. Proc. 4th Int. Summer School. Akademie-Verlag, Berlin*, pages 197–206, 1975.

[215] E. Nieminen. Hausdorff measures, capacities, and Sobolev spaces with weights. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, 81:197–206, 1991.

[216] S. M. Nikol'skii. Approximation of functions of several variables and imbedding theorems. *Springer-Verlag, New York*, 1975.

[217] J. Ok. Gradient estimates for elliptic equations with $l^{p(\cdot)}\log l$ growth. *Calc. Var. & PDE*, 55, 2016.

[218] J. Ok. Regularity of ω-minimizers for a class of functionals with non-standard growth. *Calc. Var. & PDE*, 56, 2017.

[219] N. Papageorgiou, V. Radulescu, and D. Repov. Double-phase problems with reaction of arbitrary growth. *Z. Angew. Math. Phys.*, 69, 2018.

[220] T. Rado. On the problem of plateau. *Springer, Berlin*, 1933.

[221] V. Radulescu and D. Repovš. Partial differential equations with variable exponents: variational methods and qualitative anlysis. *Chapman and Hall/CRC Monographs and Research Notes in Mathematics.*, 2015.

[222] M. A. Ragusa and A. Tachikawa. On interior regularity of minimizers of $p(x)$-energy functionals. *Nonlinear Anal.*, 93:162–167, 2013.

[223] M. A. Ragusa and A. Tachikawa. Boundary regularity of minimizers of $p(x)$-energy functionals. *Ann. IHP, Nonlinear Anal.*, 33:451–476, 2017.

[224] M. A. Ragusa, A. Tachikawa, and H. Takabayashi. Partial regularity of $p(x)$-harmonic maps. *Transaction of the AMS*, 365:3329–3353, 2013.

[225] K. R. Rajagopal and M. Ružička. Mathematical modeling of electrorheological materials. *Cont. Mech. Therm.*, 13:59–78, 2001.

[226] T. Riviere. Regularité des applications harmoniques á valeur dans les tores de révolution. *Ph.D. Thesis, Université de Paris VI*, 1993.

[227] T. Riviere. Everywhere discontinuous harmonic maps into spheres. *Acta Math.*, 175:197–226, 1995.

[228] D. Sarason. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.*, 207:391–405, 1975.

[229] J. Schauder. Uber lineare elliptische differentialgleichungen zweiter ordnung. *Math. Z.*, 38:257–282, 1934.

[230] R. Schoen and K. Uhlenbeck. A regularity theory for harmonic maps. *J. Differential Geometry*, 17:307–335, 1982.

[231] R. Schoen and K. Uhlenbeck. Boundary regularity and the dirichlet problem for harmonic maps. *J. Differential Geometry*, 18:253–268, 1983.

[232] S. Sil. Nonlinear stein theorem for differential forms. *Calc. Var. & PDE*, 58, 2019.

[233] L. Simon. Interior gradient bounds for non-uniformly elliptic equations. *Indiana Univ. Math. J.*, 25:821–855, 1976.

[234] L. Simon. Theorems on regularity and singularity of energy minimizing maps. *Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel,*, 1996.

[235] E. M. Stein. Editor's note: the differentiability of functions in $\mathbb{R}^n$. *Ann. of Math.*, 113:383–385, 1981.

[236] M. Struwe. Plateau's problem and the calculus of variations. *Mathematical Notes (Princeton University Press).*, 35:383–385, 1989.

[237] A. Tachikawa. On the singular set of $p(x)$-energy. *Calc. Var. & PDE*, 50:145–169, 2014.

[238] A. Tachikawa and K. Usuba. Regularity results up to the boundary for minimizers of $p(x)$-energy with $p(x) > 1$. *Manus. Math.*, 152:127–151, 2017.

[239] H. Triebel. Interpolation theory, function spaces, differential operators. *Johann Ambrosius Barth Verlag, Heidelberg - Leipzig*, 1995.

[240] B. O. Turesson. Nonlinear potential theory and weighted Sobolev spaces. *Lecture Notes in Math. Springer-Verlag, Berlin*, 1736, 2000.

[241] K. Uhlenbeck. Regularity for a class of non-linear elliptic sysyem. *Acta Math.*, 138:219–240, 1977.

[242] N. N. Ural'tseva. Degenerate quasilinear elliptic systems. *Semin. in Mathematics, V. A. Steklov Math. Inst., Leningrad*, 7:83–99, 1968.

[243] N. N. Ural'tseva and A. B. Urdaletova. The boundedness of the gradients of generalized solutions of degenerate quasilinear non-uniformly elliptic equations. *Vestnik Leningrad Univ. Math.*, 19:263–270, 1983.

[244] V. Šverák and X. Yan. Non-Lipschitz minimizers of smooth uniformly convex variational integrals. *Proc. Natl. Acad. Sci. USA*, 99:15269–15276, 2002.

[245] V. Šverák and X. Yan. A singular minimizer of a smooth strongly convex functional in three dimensions. *Calc. Var & PDE*, 10:213–221, 2002.

[246] V. V. Zhikov. Averaging of functionals of the calculus of variations and elasticity theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50:675–710, 1986.

[247] V. V. Zhikov. On Lavrentiev's phenomenon. *Russian J. Math. Phys.*, 3:249–269, 1995.

[248] V. V. Zhikov. On some variational problems. *Russian J. Math. Phys.*, 5:105–116, 1997.

[249] V. V. Zhikov, S. M. Kozlov, and O. A. Oleinik. Homogenization of differential operators and integral functionals. *Springer-Verlag, Berlin*, 1994.