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**INSTANT EXIT FROM THE ASYMMETRIC WAR OF
ATTRITION**

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Instant Exit from the Asymmetric War of Attrition

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Abstract. In an asymmetric war of attrition the players' prize valuations are drawn from different distributions. A "stochastic strength" ordering, based upon relative hazard rates, is used to rank these distributions. The stochastically stronger player is perceived to be stronger *ex ante*, even though her realized valuation may be lower *ex post*. Since the classic war of attrition exhibits multiple equilibria, the game is perturbed; for instance, by imposing an arbitrarily large time limit, or allowing for the arbitrarily small probability of players that are restricted to fighting forever. In the unique equilibrium of the perturbed game, a stochastically weaker player almost always "instantly exits" at the beginning, even though her valuation may be higher.

1. WARS OF ATTRITION

In a classic war of attrition, the first player to quit concedes a prize to her opponent. Costly fighting is worthwhile only if a player expects her opponent to quit in the near future. These features are common to many important phenomena, including labor-market negotiations, the voluntary provision of public goods, macroeconomic stabilization, the adoption of technological standards, and political lobbying.² Who will win the war? When will it end?

A simple hypothesis is that the "stronger" player will win. To evaluate such a hypothesis, however, a notion of strength is required. Suppose that a player's prize valuation is the privately observed realization of a random variable; this yields a game of incomplete information. A player may be described as stronger *ex post* if her realized prize valuation is higher. If both players valuations are drawn from the same distribution (a symmetric war of attrition) then the *ex post* ranking is the only available notion of strength. As is well known,

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²Workers and employers may prolong a strike in order to obtain a preferred resolution (Kennan and Wilson, 1989). Potential suppliers may delay public-good provision in an effort to free ride on others (Bliss and Nalebuff, 1984; Bilodeau and Slivinski, 1996). Oligopolists in a declining industry may incur losses in anticipation of profitability following a competitor's exit (Fudenberg and Tirole, 1986; Ghemawat and Nalebuff, 1985, 1990). Macroeconomic stabilization may be delayed in order to push the burden of an agreement towards others (Alesina and Drazen, 1991; Casella and Eichengreen, 1996). The sponsor of a technological standard may continue its promotion in the hope that a competitor will adopt it (Farrell and Saloner, 1988; David and Monroe, 1994; Farrell, 1996). Political actors may expend irrecoverable lobbying costs in exchange for political influence (Hillman and Samet, 1987; Hillman and Riley, 1989). A war of attrition is also a bargaining game in which proposals are fixed and agreement requires the acquiescence of one participant (Osborne, 1985; Ordoover and Rubinstein, 1986; Chatterjee and Samuelson, 1987; Abreu and Gul, 2000; Kambe, 1999).

the symmetric equilibrium of such a game will yield a win for the *ex post* strongest player, and the war will end at the chosen stopping time of her *ex post* weaker opponent.³

In an asymmetric war of attrition, however, the players' prize valuations are drawn from different distributions. If these distributions can be ranked (using an appropriate stochastic dominance ordering) then a player may be described as *ex ante* stronger if her valuation distribution dominates that of her opponent. An *ex ante* stronger player may, nevertheless, have a realized prize valuation that is lower than her opponent's; she may well be weaker *ex post*. An open question, therefore, is this: is the outcome of a war of attrition determined by players' *real* strengths (their *ex post* prize valuations) or their *perceived* strengths (derived from a ranking of prize-valuation distributions)? A dramatic answer provided by the present paper is this: a player who is merely perceived to be weaker *ex ante* will exit *immediately*. Before discussing the implications of this claim, two issues are addressed.

One issue is equilibrium selection: the classic war of attrition exhibits multiple (Bayesian Nash) equilibria.⁴ In response, the specification is "perturbed" to yield a game with a unique equilibrium. To ensure that the results are not sensitive to the exact perturbation mechanism, three approaches are taken; the results of the paper hold for each of them. Firstly, the game is modified so that each player suffers *exit failure* and is forced to fight forever with some exogenous probability. Secondly, a *hybrid all-pay auction* is considered, in which a winner's fighting costs respond positively to her own planned stopping time as well as the loser's exit time.⁵ Thirdly, a *time limit* is imposed, after which the prize is awarded at random. Methods such as these have been established elsewhere in the literature.⁶ Furthermore, a unique equilibrium of the classic war of attrition may be "selected" by letting the perturbation vanish (for instance, by allowing the time limit to grow arbitrarily large). These different perturbations are subject to a common interpretation: they force threats of great aggression to be *credible*, by ruling out equilibria in which very weak players are able credibly to commit to fighting for very long periods of time.⁷

A second issue is that a notion of *ex ante* strength must be developed. To do this, the distributions from which prize valuations are drawn are ranked in three different ways. First, a

³Many theorists (Bulow and Klemperer, 1999; Krishna and Morgan, 1997, for example) have focused upon symmetric equilibria. Alas, applications will often involve some (perhaps small) asymmetry; when there are multiple equilibria, there is no symmetric equilibrium on which to focus. It is unclear why a selected equilibrium in an asymmetric game should, necessarily, be nearly symmetric.

⁴This is also true in complete-information wars of attrition: there are equilibria with the "instant exit" of a player, as well as equilibria involving constant hazard-rate exit. An atom-free mixed equilibrium is quirky: a *stronger* player exits *more quickly* than her weaker opponent. Kornhauser, Rubinstein, and Wilson (1989) found this troubling, and suggested "that the weaker player [...] should concede immediately."

⁵The war of attrition is a second-price all-pay auction; the highest bidder wins the prize, and both players pay the loser's bid. In first-price all-pay auction, the winner and loser pay their own bids. In a hybrid all-pay auction, the winner pays a convex combination of the winning and losing bid.

⁶Fudenberg and Tirole (1986), and Kornhauser, Rubinstein, and Wilson (1989) ensured that players fight forever with positive probability; Amann and Leininger (1996) and Riley (1999) considered all-pay auctions.

⁷For instance, if her opponent fails to exit, then a player must carry out a costly threat to fight forever.

player *stochastically dominates* her opponent if, *ex ante*, her valuation is more likely to exceed any benchmark level. Second, she *hazard-rate dominates* her opponent if this likelihood ratio is increasing in the benchmark. Third, she is *stochastically stronger* than her opponent if the likelihood ratio increases without bound as the benchmark grows large.⁸

These three orderings lead to the three propositions that form the contribution of the paper. First, if an *ex ante* weaker player is stochastically dominated by her opponent, then she will exit at the beginning of the game with positive probability: an initial exit from the war of attrition. Second, if the weaker player is hazard-rate dominated, then her stopping function (the strategy mapping her realized valuation to a planned exit time) is everywhere below that of her opponent: the outcome is biased toward the *ex ante* stronger player. Third, if a player is stochastically weaker *ex ante* then, in the limit as the appropriate perturbation is allowed to vanish, she will always “instantly exit” at the beginning of the game: a stochastically weaker player never even tries to fight, even though her valuation may be the highest.

The third result is bold, and implications are immediate. First, players’ true prize-valuations (their real strengths) play no role in determining the outcome of a war of attrition. Second, the allocation of the prize may, therefore, be dramatically inefficient. Third, since the game ends immediately, the war of attrition cannot, by itself, explain the existence of delay in concessionary environments. Fourth, any perceived asymmetries *ex ante* may be critical to a player’s likely success. Fifth, players may have an incentive, therefore, to engage in activities that enhance their perceived valuation of a prize, rather than the value itself.

Of course, the “instant exit” claim may be sensitive to the size of the perturbation used. Here, this sensitivity is assessed using closed-form solutions for specific examples. As might be expected, the perturbation must become very small indeed; if a non-negligible perturbation is present, then the *ex ante* weaker player’s exit is rapid, rather than instant. Similarly, fixing the perturbation, fighting occurs when players are close to being symmetric *ex ante*.

In summary, this paper is uniquely able to determine the role of real and perceived strengths. Earlier studies considered *ex ante* symmetric players, or restricted to environments in which valuations and costs are commonly known, so that there is no difference between *ex ante* and *ex post* notions of strength.⁹ One view, therefore, is that the paper’s role is to complete a branch of literature that analyzes unique equilibrium behavior in the war of attrition.

⁸Thus stochastic dominance is the textbook first-order ranking, hazard-rate dominance corresponds to the conditional stochastic-dominance ranking of Maskin and Riley (2000), and stochastic strength is related to the unbounded likelihood-ratio property required for the implementation of Mirrlees (1999) contracts in moral-hazard problems. These conditions hold for a wide range of specifications.

⁹Kornhauser, Rubinstein, and Wilson (1989), Riley (1999), Abreu and Gul (2000), and Kambe (1999) studied models in which prize valuations and fighting costs are certain: at the start of the game, it is common knowledge that one player is stronger than her opponent. Although they demonstrated “instant exit” of the weaker player, they did not separate the different roles of *ex ante* and *ex post* strength. Allowing a player’s prize valuation to be private information is important since it permits this separation. Fudenberg and Tirole (1986), Ponsati and S kovics (1995), and Amann and Leininger (1996) all offered characterizations of a unique equilibrium, but did not examine the implications of *ex ante* asymmetry.

A second view is that the results offer a contribution that is of wider interest to economists: wars of attrition are important, simply because they are common, and yet (or so it is argued here) conventional approaches to modelling simply cannot provide an effective explanation for delay in concessionary settings. To justify this claim, the formal propositions are followed by a discussion relating the results to the aforementioned applications.

The paper proceeds as follows. A war of attrition with privately realized valuations is “perturbed” to yield a unique equilibrium (§2). The three definitions of *ex ante* strength are developed (§3) and then deployed (§4) to yield the three key propositions. Specific examples are used to assess the sensitivity of the results (§5). The implications are considered in the context of the leading applications of the war of attrition (§6) before concluding (§7).

2. PERTURBING THE WAR OF ATTRITION

This section begins with a review of the classic war of attrition (§2.1). Since this game exhibits multiple equilibria, three different modifications are considered (§2.2–2.4). The unique equilibrium of each modified game is then characterized (2.5–2.6).

2.1. The Classic War of Attrition. In a classic war of attrition two players $i \in \{1, 2\}$ each choose a stopping time $t_i \in \mathbb{R}_+ \cup \{\infty\}$, which may be revised at any time $t \leq t_i$. Player i is characterized by a *fighting cost* of $c_i > 0$ and a *valuation* u_i for a prize, where $u_i \in (\underline{u}_i, \bar{u}_i) \subseteq \mathbb{R}$ and $\bar{u}_i > \underline{u}_i$. Only the ratio u_i/c_i will be of interest to i , and hence it is without loss of generality to adopt the normalization $c_i = 1$. Realized payoffs are

$$\pi_i(t_i, t_j) = u_i \left[\mathbb{I}\{t_i > t_j\} + \frac{\mathbb{I}\{t_i = t_j\}}{2} \right] - \min\{t_i, t_j\},$$

where $\mathbb{I}\{\cdot\}$ is the indicator function.¹⁰ Following Maynard Smith (1974), costs are directly proportional to the time elapsed. Other possible formulations, in which “leader” and “follower” payoffs are general functions of time lead to similar insights (Bishop and Cannings, 1978; Hendricks, Weiss, and Wilson, 1988).¹¹ The “linear costs” approach is convenient in that the war of attrition may be interpreted as an ascending-price all-pay auction (Klemperer, 1999): the price t rises until a player concedes, and both players pay the exit price.¹²

Following Bishop, Cannings, and Maynard Smith (1978) and Riley (1979, 1980) information is incomplete: players observe only their own valuations. It is commonly known that u_i is

¹⁰The prize is awarded at random if both players exit simultaneously ($t_i = t_j$). Other tie-break rules may be employed while retaining most of the results, so long as no player wins a tie with probability one.

¹¹The exiting player enjoys a leader payoff of $L_i(t)$ whilst the follower receives $F_i(t)$. Insisting that $L'_i(t) < 0$ ensures that the leader would rather quit sooner. When $F_i(t) > L_i(t)$, however, a player is willing to wait for the anticipated exit of her opponent. A special case is when a “fighting cost” corresponds to the delay before the award of a second prize. For $A > B > 0$ this might be implemented via $L_i(t) = Be^{-\delta_i t}$ and $F_i(t) = Ae^{-\delta_i t}$, where players differ via patience δ_i . Alternatively (Ponsati and Sákovic, 1995) it may be implemented via $L_i(t) = B_i e^{-t}$ and $F_i(t) = e^{-t}$ where players differ via reservation payoff B_i .

¹²Other formats include “first-price all-pay” auctions (Baye, Kovenock, and de Vries, 1993, 1996), where (in contrast to the second-price case) it is costly to raise a bid even if that bid is already the highest.

drawn from the distribution $F_i(u)$ with strictly-positive continuous density $f_i(u)$.¹³ Under the all-pay auction interpretation, the bidders have independent private values.

The classic war of attrition exhibits multiple equilibria. For instance, when players are symmetric, there is a symmetric equilibrium in which both players use the same strategy, and where a planned exit time is a smoothly increasing function of a player's prize valuation. However, when $\min\{\underline{u}_1, \underline{u}_2\} > 0$, it is also an equilibrium for one player to be infinitely aggressive and fight forever (i chooses $t_i = \infty$) and her opponent to quit immediately (j chooses $t_j = 0$).¹⁴ This extreme equilibrium is, however, particularly sensitive to the exact specification, in that i 's own payoff does not respond to her exact choice of t_i , and hence she is happy to wait forever: the threat of limitless aggression is completely costless.

The fact that credible threats of unbounded aggression may form part of an equilibrium is an unfortunate feature of the classic war of attrition. Happily, removing this feature (by “perturbing” the specification) yields a game with a unique equilibrium.¹⁵ In this paper, three perturbations are considered: exit failure, hybrid all-pay auctions, and time limits.¹⁶

2.2. Exit Failure. In the classic war of attrition a player successfully exits at her chosen time. In a first change to the basic specification, each player fails to exit with commonly known probability $\xi > 0$. Exit failures are independent events for the two players and are independent of the players' prize valuations.¹⁷ If exit failure occurs, then a player is forced to fight forever. This possibility (of *forced* infinite aggression) means that a player never *intentionally* fights forever: if i were to choose $t_i = \infty$, then with probability $\xi > 0$ she would face infinite costs (following an exit failure by j). Thus the possibility of exit failure will automatically eliminate the “extreme” equilibria discussed above.¹⁸

Other interpretations of this specification, other than exit failure, are possible. For instance, with probability ξ a player might be “crazy” and insist on fighting forever. This corresponds to the approach taken by Kornhauser, Rubinstein, and Wilson (1989), who, following Kreps and Wilson (1982a), Milgrom and Roberts (1982) and Kreps, Milgrom, Roberts, and Wilson (1982), introduced “irrationality” into a complete-information model.¹⁹

¹³Furthermore, when $\bar{u}_i < \infty$ it will be assumed that $\lim_{u \rightarrow \bar{u}_i} f_i(u) \in (0, \infty)$. This technical restriction is required since the support $(\underline{u}_i, \bar{u}_i)$ is open. With compact support this requirement can be dropped.

¹⁴Behavior “off the equilibrium path” must also be specified: i always stays in forever and, off the equilibrium path, j quits whenever she can. Players retain their prior beliefs over opposing valuations.

¹⁵Ponsati and Sákovic (1995) offered the following succinct summary: “There are a continuum of equilibria characterized by a system of ordinary differential equations. Uniqueness may be achieved by perturbing the game, imposing that for a positive measure of types it is a dominant strategy not to concede.”

¹⁶Other approaches might involve the addition of noise to actions. Anderson, Goeree, and Holt (1998a,b) studied versions of the all-pay auction and war of attrition, in which decisions are made by a logit rule. They identified logit equilibria (McKelvey and Palfrey, 1995, 1996) which exhibit “sensible” comparative statics.

¹⁷The model and analysis may be extended to incorporate asymmetric exit-failure probabilities ξ_1 and ξ_2 . The assumption $\xi_1 = \xi_2$ is imposed in order to simplify the statement of the results.

¹⁸Although (Proposition 3, §4) one of the extreme equilibria is selected in the limit as $\xi \rightarrow 0$.

¹⁹A player need not be insane in order to fight forever: a player may have good reason to do so. This latter approach was taken by Fudenberg and Tirole (1986) and Ponsati and Sákovic (1995), among others. In their

2.3. Hybrid All-Pay Auctions. The war of attrition is an ascending-price all-pay auction. This is similar to a second-price sealed-bid all-pay auction, where t_1 and t_2 are interpreted as sealed bids.²⁰ The highest bidder wins the prize, and both bidders pay the lowest (second price) bid. Of course, the class of all-pay auctions extends beyond this; in a first-price all-pay auction, each player pays their bid, with the prize once again awarded to the highest bidder.

The first-price and second-price all-pay auctions combine in a “hybrid” format. This mechanism, used by Güth and van Damme (1986), Amann and Leininger (1996) and Riley (1999), works as follows. The prize is won by the highest bidder. The loser pays her bid. The winner pays a convex combination of her own and the loser’s bid. Specifically, for some $\beta \in [0, 1]$, when $t_1 > t_2$, Player 1 wins the object and pays a price $\beta t_1 + (1 - \beta)t_2$. The first-price all-pay auction is obtained when $\beta = 1$, and the war of attrition when $\beta = 0$. Formally

$$\pi_i(t_i, t_j) = u_i \left[\mathbb{I}\{t_i > t_j\} + \frac{\mathbb{I}\{t_i = t_j\}}{2} \right] - [\min\{t_i, t_j\} + \max\{\beta(t_i - t_j), 0\}].^{21}$$

Setting $\beta > 0$ ensures that a finite-valuation player will never choose $t_i = \infty$. This is because (as in the exit-failure case) the marginal cost of fighting a little longer is always bounded away from zero; a player always faces a strict incentive to moderate her aggression.

2.4. Time Limits. A time limit directly tames the aggression of players and yields a finite-horizon war of attrition (Cannings and Whittaker, 1995).²² To implement this, suppose that players may only fight up to time T .²³ If they both fight until this time, then the prize is allocated randomly. So long as T is not too large, a player will fight until the T with positive probability: doing so guarantees a payoff of at least $(u_i/2) - T$. To ensure that T is neither too large nor too small to have an effect, two simplifying assumptions are imposed. First, whenever $T < \infty$, so that a limit is in place, assume that $\bar{u}_1 = \bar{u}_2 = \infty$. This ensures that there is always positive probability that a player has a dominant strategy to fight until the time limit, and that this remains true as $T \rightarrow \infty$. Second, assume that $T > \max\{u_1, u_2\}/2$ so that a player does not always have a dominant strategy to fight until the limit.

models, valuations are fixed. Instead, the fighting cost of a player is unknown to her opponent. Crucially, it is sometimes negative. Fudenberg and Tirole’s (1986) justification for this assumption was based upon their model’s application. They considered exit from a declining industry, where a firm’s characteristics might enable it to remain profitable forever, even when facing a competitor. These interpretations of exit failure lead to an equilibrium in which, as in Abreu and Gul (2000) and Kambe (1999), “reputation” is important.

²⁰In the sealed-bid interpretation players choose their stopping times simultaneously at the beginning of the game, rather than revising them as the game progresses. Assuming that a player pays the second-highest bid is equivalent to assuming that, following the exit of her opponent, she revises her bid and stops immediately.

²¹This has a direct war-of-attrition interpretation. Hendricks, Weiss, and Wilson (1988) and Pitchik (1982) referred to wars of attrition as “noisy” games; a player can “hear” her opponent’s exit. In a “silent” game a player does not observe such an exit; this is an all-pay auction. The hybrid model has this interpretation: a player fails to observe her opponent’s exit with probability β , and must continue to her planned exit time.

²²Bilodeau and Slivinski (1996) follow Bliss and Nalebuff (1984) in modelling voluntary public-good provision as a war of attrition. Their time-limit mechanism is subtly different to the one considered here. A player’s (commonly known) valuation is the present-discounted value of the prize’s flow utility until the time limit. Hence, as the time limit is reached, lower valuation players will fight until the end.

²³A player suffering exit failure (§2.2) is assumed to exit at time T .

2.5. Equilibrium. A pure strategy $t_i(u_i)$ maps a player's valuation into the extended real line, subject to any time limit. For a pure-strategy Bayesian-Nash equilibrium (the solution concept employed in this paper) stopping rules need to be mutually optimal.²⁴ In this section, equilibrium characteristics are described (Lemmas 1–2) and, since it is assumed that either $\max\{\xi, \beta\} > 0$ or $T < \infty$, the existence of a unique equilibrium is confirmed (Lemma 3).²⁵

Equilibrium stopping rules must exhibit certain basic properties. First, the expected fighting costs of a player are strictly increasing in her stopping time, and hence equilibrium stopping rules are monotonic: a higher valuation is necessary to justify fighting for longer.²⁶ Second, a player will never exit with positive probability at any time $t \in (0, T)$: an “atom” at time t makes a player predictable. Third, players must begin exiting at time zero, and will stop exiting at the same time: if not, then there are opportunities for a player to reduce her exit time without harming her chance of winning. Finally, when a time limit is in place, there can be a single discontinuity in a player's stopping rule: at a critical time, a player finds it worthwhile to stay in until the time limit, hence guaranteeing an expected benefit of $u_i/2$. These features are described rather more formally in the following lemma.

Lemma 1. *In equilibrium, a player's stopping rule is weakly increasing in her valuation. There exist $\underline{u}_i^*$ and $\bar{u}_i^*$ satisfying $\underline{u}_i \leq \underline{u}_i^* < \bar{u}_i^* \leq \bar{u}_i$ for each $i \in \{1, 2\}$, and $\bar{t} > 0$, such that (1) $t_i(u) = 0$ for all $u \in (\underline{u}_i, \underline{u}_i^*)$ and $t_i(u) = T$ for all $u \in (\bar{u}_i^*, \bar{u}_i)$; (2) $t_i(u)$ is strictly increasing and continuous for $u \in (\underline{u}_i^*, \bar{u}_i^*)$; (3) $\lim_{u \downarrow \underline{u}_i^*} t_i(u) = 0$ and $\lim_{u \uparrow \bar{u}_i^*} t_i(u) = \bar{t}$; and (4) if $T = \infty$ then $\bar{u}_i^* = \bar{u}_i$, and if $T < \infty$ then $\bar{u}_i^* < \bar{u}_i$. Summarizing, equilibrium stopping rules ensure that players exit continuously over $(0, \bar{t})$, with possible atoms at $t = 0$ and $t = T < \infty$.*

Lemma 1 ensures that the inverses of the players' stopping rules are well defined for $t \in (0, \bar{t})$; write $v(t)$ and $w(t)$ for the inverses corresponding to Player 1 and Player 2 respectively, so that $t = t_1(v(t)) = t_2(w(t))$.²⁷ If these inverses are differentiable (Lemma 2 proves that they are) a pair of first-order conditions may be used to characterize an equilibrium.

To do this, write $G_i(t) \equiv \Pr[t_i \leq t]$ for the cumulative distribution function of i 's exit time, and $g_i(t)$ for its corresponding density. Suppose that Player 2 is considering exiting at time t . By delaying by dt , she increases the probability that Player 1 exits before her by $g_1(t) dt$: an expected benefit of $g_1(t)w(t) dt$. She also increases her fighting costs. With probability $G_1(t)$ she is the winner, yielding an increase of βdt . With probability $1 - G_1(t)$ she is the loser, generating additional costs of dt . Hence, by waiting a little longer her expected costs rise by $[1 - (1 - \beta)G_1(t)] dt$. She will be indifferent at time t when these costs and benefits

²⁴If $\xi > 0$ or $T < \infty$ all information sets are reached with positive probability, so that any Bayesian Nash equilibrium (Harsanyi, 1967a,b, 1968) is a sequential equilibrium (Kreps and Wilson, 1982b).

²⁵Since results in this section are mild modifications of similar analyses presented by Fudenberg and Tirole (1986), Amann and Leininger (1996), Ponsati and S kovics (1995), and others, formal proofs are omitted.

²⁶This is not necessarily true in the classic war of attrition. For instance, when $\bar{u}_i < \infty$, i could choose $t_i = 0$ for all valuations while her opponent j chooses any particular stopping time greater than $\bar{u}_i$.

²⁷Following from Lemma 1, $v(0) \equiv \lim_{t \downarrow 0} v(t)$ and $v(\bar{t}) \equiv \lim_{t \uparrow \bar{t}} v(t)$ are both well defined, and satisfy $v(0) = \underline{u}_1^*$ and $v(\bar{t}) = \bar{u}_1^*$. Similarly, $w(0) = \underline{u}_2^*$ and $w(\bar{t}) = \bar{u}_2^*$.

balance, so that $g_1(t)w(t)dt = [1 - (1 - \beta)G_1(t)]dt$. A similar (first order) condition may be obtained for Player 1. Now, observe that the distribution of i 's exit time will satisfy $G_i(t) = (1 - \xi)F_i(v(t))$, so that $g_1(t) = v'(t)(1 - \xi)f_1(v(t))$ and $g_2(t) = w'(t)(1 - \xi)f_2(w(t))$. Substituting and re-arranging, the first-order conditions are simply

$$v'(t) = \frac{1 - (1 - \beta)(1 - \xi)F_1(v(t))}{(1 - \xi)f_1(v(t))w(t)} \quad \text{and} \quad w'(t) = \frac{1 - (1 - \beta)(1 - \xi)F_2(w(t))}{(1 - \xi)f_2(w(t))v(t)}. \quad (1)$$

This pair of differential equations may be “integrated up” to generate the following lemma.

Lemma 2. *The inverse stopping rules are differentiable for $t \in (0, \bar{t})$. Defining*

$$\Lambda_i(u) \equiv \int_u^{\bar{u}_i^*} \frac{(1 - \beta)(1 - \xi)f_i(x)}{x(1 - (1 - \beta)(1 - \xi)F_i(x))} dx, \quad (2)$$

then, for any $t \in (0, \bar{t})$, the inverses $v(t)$ and $w(t)$ satisfy $\Lambda_1(v(t)) = \Lambda_2(w(t))$.

The integral on the right-hand side of (2) exists only because either $\max\{\xi, \beta\} > 0$ or $T < \infty$, so that the game is perturbed; this would fail for the classic war of attrition.

2.6. Uniqueness. The pair of differential equations (1) yield a family of solutions. For a unique solution, boundary conditions are needed. Such boundary conditions may be obtained by considering behavior at the beginning of the game ($t = 0$) or the end ($t = \bar{t}$).

Consider behavior at $t = 0$, and suppose that prize valuations are bounded away from zero ($\min\{\underline{u}_1, \underline{u}_2\} > 0$). Neither player has a dominant strategy to exit at the beginning of the game. In this case, a single boundary condition is available. If a player exits with positive probability at the beginning of the game, then her opponent will always find it profitable to fight for some period of time: there cannot be “instant exit” by both players. Formally, this means that either $v(0) = \underline{u}_1^* = \underline{u}_1$ or $w(0) = \underline{u}_2^* = \underline{u}_2$. This argument leads, therefore, to a single boundary condition, and this is not enough to tie down a unique equilibrium.

Next, suppose suppose that prize valuations extend below zero, so that $\max\{\underline{u}_1, \underline{u}_2\} < 0$. This ensures that, with probabilities $F_1(0)$ and $F_2(0)$ respectively, the players do not wish to win the prize: A player with a valuation $u_i \leq 0$ has a strictly dominant strategy to exit at time $t = 0$. Thus there is (trivially) instant exit from players with negative valuations. This means, in turn, that players with strictly positive valuations will always fight for some period of time. Formally, $v(0) = \underline{u}_1^* = 0$ and $w(0) = \underline{u}_2^* = 0$.

This pair of boundary conditions would appear to suggest a unique solution. Unfortunately, this is false since (Fudenberg and Tirole, 1986) the differentiable equations are not Lipschitz continuous at $t = 0$.²⁸ In both of the cases considered here, an additional boundary condition is needed—other cases (e.g. $\underline{u}_1 > 0 > \underline{u}_2$) yield similar conclusions. The search for a further boundary condition leads to an examination of behavior as $t \rightarrow \bar{t}$. As is well known, this

²⁸Examining (1), notice that the denominators of both expressions tend to zero as $t \rightarrow 0$.

attempt fails for the classic war of attrition. To see why, first integrate the first-order conditions (1) from times t_L up to time t_H and combine to obtain

$$\int_{v(t_L)}^{v(t_H)} \frac{(1-\beta)(1-\xi)f_1(x)}{x(1-(1-\beta)(1-\xi)F_1(x))} dx = \int_{w(t_L)}^{w(t_H)} \frac{(1-\beta)(1-\xi)f_2(x)}{x(1-(1-\beta)(1-\xi)F_2(x))} dx. \quad (3)$$

Next, set $\xi = \beta = 0$ and $T = \infty$. For simplicity suppose that the hazard of F_1 is increasing.²⁹ Further suppose that $\bar{u}_1 = \infty$. Letting $t_H \rightarrow \bar{t}$, the left-hand side of (3) satisfies

$$\int_{v(t_L)}^{v(t_H)} \frac{f_1(x)}{x(1-F_1(x))} dx \geq \frac{f(v(t_L))}{(1-F(v(t_L)))} \int_{v(t_L)}^{v(t_H)} \frac{1}{x} dx = \frac{f(v(t_L))}{(1-F(v(t_L)))} \log \left[\frac{v(t_H)}{v(t_L)} \right] \rightarrow \infty.$$

This follows since $v(t_H) \rightarrow \infty$ as $t_H \rightarrow \bar{t}$. The same is true, following a slightly different argument, when $\bar{u}_1 < \infty$.³⁰ Similarly, the right-hand side of (3) also diverges.

When either $\beta > 0$, $\xi > 0$, or $T < \infty$, so that the marginal cost of increased aggression is bounded away from zero, a consideration of $t \rightarrow \bar{t}$ is successful. Consider first the case where $T = \infty$, and for notational simplicity set $\xi = 0$ but $\beta > 0$. Allowing $t_H \rightarrow \bar{t}$,

$$\begin{aligned} \int_{v(t_L)}^{v(t_H)} \frac{(1-\beta)f_1(x)}{x(1-(1-\beta)F_1(x))} dx &\leq \frac{1}{v(t_L)} \int_{v(t_L)}^{v(t_H)} \frac{(1-\beta)f_1(x)}{1-(1-\beta)F_1(x)} dx \\ &= \frac{1}{v(t_L)} \log \left[\frac{1-(1-\beta)F_1(v(t_L))}{1-(1-\beta)F_1(v(t_H))} \right] \rightarrow \frac{1}{v(t_L)} \log \left[\frac{1-(1-\beta)F_1(v(t_L))}{\beta} \right]. \end{aligned}$$

Similarly, the right hand side of (3) also converges as $t_H \rightarrow \bar{t}$. When $T < \infty$, $v(t) \rightarrow \bar{u}_i^* < \bar{u}_i$ as $t \rightarrow \bar{t}$, and once again there is convergence. Such convergence yields a terminal boundary condition, as described in the final part of Lemma 2. This condition is obtained by “integrating up” the first-order conditions from time t until the last exit time $\bar{t}$. Equivalently, it may be obtained by integrating backwards from $\bar{t}$ until t . In essence, therefore, it is a consequence of *backward induction*. Utilizing boundary conditions at both $t = 0$ and as $t \rightarrow \bar{t}$ ties down a unique equilibrium. The only exception is that, when $T < \infty$, a player with valuation $\bar{u}_i^*$ is exactly indifferent between exiting at $\bar{t}$ and fighting until the time limit T . Thus, I describe the equilibrium as “essentially unique.”

Lemma 3. *There is an essentially-unique pure-strategy Bayesian-Nash equilibrium.*

Lemmas 1–3 confirm the existence of a unique equilibrium, and characterize its basic properties. This permits an unambiguous analysis of the relative exit times of the two players for the selected equilibrium; identifying “who wins the war” is now possible. Before doing this, however, an *ex ante* ordering of the players must be developed.

²⁹It would be sufficient to assume that the hazard rate is bounded away from zero for large valuations.

³⁰When Player 1’s valuation is bounded above

$$\int_{v(t_L)}^{v(t_H)} \frac{f_1(x)}{x(1-F_1(x))} dx \geq \frac{1}{\bar{u}_1} \int_{v(t_L)}^{v(t_H)} \frac{f_1(x)}{1-F_1(x)} dx = \frac{1}{\bar{u}_1} \log \left[\frac{1-F_1(v(t_L))}{1-F_1(v(t_H))} \right] \rightarrow \infty,$$

since $1-F_1(v(t_H)) \rightarrow 0$ as $t_H \rightarrow \bar{t}$.

3. *Ex Ante* STRENGTH

When the players share common beliefs over their valuations, so that $F_1(\cdot) \equiv F_2(\cdot)$, the game is symmetric, and so a unique equilibrium must also be symmetric; usually, the player with the highest valuation *ex post* will win the war.³¹ Symmetry, however, is a strong assumption, since it does not allow *ex ante* differences (however small) to influence play.

3.1. Orderings of Strength. To aid analysis, the two players must be ordered *ex ante*; this formalizes the idea of *perceived strength*. Three partial orderings are considered.

Definition 1. *Player 1 stochastically dominates Player 2, denoted $F_1 \succ_{FSD} F_2$, if F_1 first-order stochastically dominates F_2 . Formally $F_1(u) < F_2(u)$ for all $u \in (\underline{u}_1, \bar{u}_1) \cap (\underline{u}_2, \bar{u}_2)$.*

When this familiar dominance condition holds, for any increasing function $h(u)$, the expectation $E[h(u)]$ is higher under F_1 than under F_2 . A trivial corollary is that $E[u_1] > E[u_2]$ so that Player 1 has a higher expected valuation *ex ante*: absent any other information or feasible allocation mechanism a social planner might choose to give the prize to Player 1.

More stringent criteria have also been used by a number of authors to characterize a “stronger” player. Angeles de Frutos (2000), for instance, analyzed auction procedures for allocating the assets of a dissolving partnership. She used a ranking of the hazard rates of competing distributions to order competing bidders. This is *hazard-rate dominance*.

Definition 2. *Player 1 hazard-rate dominates Player 2, denoted $F_1 \succ_{HRD} F_2$, if $\bar{u}_1 \geq \bar{u}_2$, $\underline{u}_1 \geq \underline{u}_2$, and for all u satisfying $u \geq 0$ and $\bar{u}_2 > u > \underline{u}_1$, the hazard rates satisfy*

$$\frac{f_1(u)}{1 - F_1(u)} < \frac{f_2(u)}{1 - F_2(u)}.$$

First-order dominance says that, for any u , the event $u_1 > u$ is more likely than $u_2 > u$. Hazard-rate dominance goes further, stipulating that the odds ratio of $u_1 > u$ versus $u_2 > u$ is strictly increasing in u .³² This means that, conditional on $u_1 > u$ and $u_2 > u$, the conditional distribution of u_1 continues to stochastically dominate that of u_2 .³³ For this reason Maskin and Riley (2000) referred to such a ranking as *conditional stochastic dominance*.³⁴

³¹Taking the limit as the perturbation vanishes (for instance, $\xi \rightarrow 0$ or $T \rightarrow \infty$) this is precisely true. Furthermore, the revenue-equivalence theorem applies (Vickrey, 1961; Myerson, 1981; Riley and Samuelson, 1981), and the second-price all-pay auction will raise the same expected revenue as other standard formats.

³²It is related to Milgrom’s (1981) notions of good and bad news. Consider the hypothesis that an individual i is Player 1, given only that $u_i > u$. An increase in u is “good news” for the hypothesis.

³³To see this formally, observe that:

$$\frac{\Pr[u_1 > u \mid u_1 > \tilde{u}]}{\Pr[u_2 > u \mid u_2 > \tilde{u}]} = \frac{[1 - F_1(u)]/[1 - F_1(\tilde{u})]}{[1 - F_2(u)]/[1 - F_2(\tilde{u})]} > 1 \quad \Leftrightarrow \quad \frac{1 - F_1(u)}{1 - F_2(u)} > \frac{1 - F_1(\tilde{u})}{1 - F_2(\tilde{u})}.$$

³⁴They compared first-price and second-price sealed-bid auctions in an independent-private-value setting. Departing from many of the classic studies, they allowed two bidders to be asymmetric; a “strong” bidder conditionally stochastically dominates her opponent.

For the purposes of Proposition 3 (§4) the hazard rates of the distributions will not need to be ranked everywhere, but only for high valuations. This is *stochastic strength*.

Definition 3. *Player 1 is stochastically stronger than Player 2 if $\bar{u}_1 > \bar{u}_2$ (the upper bound to the support of u_1 is greater than that of u_2), or $\bar{u}_1 = \bar{u}_2 = \infty$ (unbounded support) and*

$$\liminf_{u \rightarrow \infty} \left[\frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} \right] > 0. \quad (4)$$

This is sometimes denoted $F_1 \succ_{\text{AHRD}} F_2$, indicating asymptotic hazard-rate dominance.

Notice that (4) is automatically true when $\bar{u}_1 > \bar{u}_2$, but that Definition 3 cannot be applied when $\bar{u}_1 = \bar{u}_2 < \infty$.³⁵ Furthermore, it does not apply when the hazard rates converge in the upper tails, or when the ranking of the hazard rates switches repeatedly.

A stochastically stronger player is far more likely (in a relative sense) to experience very high valuations. In fact, when Player 1 is stochastically stronger than Player 2,

$$\lim_{u \rightarrow \bar{u}_2} \frac{1 - F_1(u)}{1 - F_2(u)} = \infty. \quad (5)$$

This helps to clarify the relationship between the different competing notions of *ex ante* strength considered here. If $F_1 \succ_{\text{FSD}} F_2$, then $u_1 > u$ is more likely than $u_2 > u$; if $F_1 \succ_{\text{HRD}} F_2$, then the relatively likelihood of $u_1 > u$ versus $u_2 > u$ is strictly increasing in u ; and finally, if $F_1 \succ_{\text{AHRD}} F_2$ then this likelihood ratio is unbounded for large u .

Of course, $F_1 \succ_{\text{AHRD}} F_2$ places no restriction on hazard rates for smaller u . Furthermore, it does not imply $F_1 \succ_{\text{FSD}} F_2$; as shown below, the distributions may instead be ranked by second-order dominance (Rothschild and Stiglitz, 1970), so that F_1 is “riskier” than F_2 .

3.2. Illustrative Examples. Although a partial ordering, stochastic strength may be used for a wide range of specifications. An example is the use of uniform distributions.

Example 1 (Uniform distributions). $u_i \sim U(\underline{u}_i, \bar{u}_i)$, so that $F_i(u_i) = (u_i - \underline{u}_i)/(\bar{u}_i - \underline{u}_i)$

Here, Player 1 is stochastically stronger whenever $\bar{u}_1 > \bar{u}_2$, so that the distributions have “shifted support.” If, in addition, $\underline{u}_1 \geq \underline{u}_2$ then Player 1 is hazard-rate dominant ($F_1 \succ_{\text{HRD}} F_2$) and stochastically dominant ($F_1 \succ_{\text{FSD}} F_2$). This example is used for sensitivity analysis (§5) since closed form results are available. For $\underline{u}_1 = \underline{u}_2 = 0$, it is equivalent to Maskin and Riley’s (2000, p. 416) Example 2. F_1 is, in their parlance, a “stretched” version of F_2 . For a second example, consider distributions with unbounded support.

Example 2 (Exponential distributions). $f_i(u)/(1 - F_i(u)) = \lambda_i$ for all $u \in (0, \infty)$.

³⁵Suppose that $\bar{u}_2 < \bar{u}_1$. As $u \rightarrow \bar{u}_2$ the hazard rate of F_1 remains finite while the hazard rate of F_2 diverges to ∞ , since $f_2(u) \rightarrow f_2(\bar{u}_2) \in (0, \infty)$ and $1 - F_2(u) \rightarrow 0$. Suppose instead that $\bar{u}_1 = \bar{u}_2 < \infty$. Then (5) fails, following an application of l’Hôpital’s rule.

For Example 2, Player 1 is stochastically stronger whenever $\lambda_2 > \lambda_1$. The same inequality ensures that she is hazard-rate dominant and stochastically dominant.

Fixing the bottom of the support at $\underline{u}_1 = \underline{u}_2 = 0$, changes in $\bar{u}_1$ and $\bar{u}_2$ for Example 1 change the relative means and variances of the two distributions in the same direction. Similarly, changes in λ_1 and λ_2 for Example 2 have the same effect. Employing normal distributions, the means and variances of the competing distributions may be changed independently.

Example 3 (Normal distributions). $u_i \sim N(\mu_i, \sigma_i^2)$, so that $F_i(u) = \Phi((u - \mu_i)/\sigma_i)$.

Here, $\Phi(z)$ represents the distribution function of the standard normal and $\phi(z)$ the corresponding density. The asymptotic linearity of the normal distribution's hazard rate yields

$$\left[\frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} \right] \longrightarrow \left[\frac{u - \mu_2}{\sigma_2^2} - \frac{u - \mu_1}{\sigma_1^2} \right] \quad \text{as } u \rightarrow \infty.$$

When $\sigma_1 = \sigma_2$, Player 1 is stochastically stronger if and only if $\mu_1 > \mu_2$; an upward shift in valuations can create a stochastic strength advantage. However, if $\sigma_1 > \sigma_2$ then Player 1 is stochastically stronger even when $\mu_1 < \mu_2$; variability creates stochastic strength, by increasing the probability that a player will experience very high valuations.

4. INSTANT EXIT

This section contains the substantive results of the paper.³⁶ Throughout, it is assumed that either $\max\{\xi, \beta\} > 0$ or $T < \infty$ (but not both), yielding a perturbed war of attrition with a unique equilibrium (Lemma 3); the case when both inequalities hold is omitted for simplicity. There are three propositions, which correspond to the three different *ex ante* orderings (§3). Propositions 1 and 2 provide orderings of the players' stopping rules. For Proposition 3, the perturbation of the game is allowed to vanish; this selects a unique equilibrium of the classic war of attrition in which the stochastically weaker always exists immediately.

4.1. Initial Exit and First-Order Dominance. Consider the case $\min\{\underline{u}_1, \underline{u}_2\} > 0$, so that valuations are bounded away from zero. Will one of the players exit with positive probability at $t = 0$, and if so, which player? Lemma 2 provides an answer. Evaluation of $\Lambda_1(v(t)) = \Lambda_2(w(t))$ at $t = 0$ leads to $\Lambda_1(\underline{u}_1^*) = \Lambda_2(\underline{u}_2^*)$. Using this equality and manipulations of (2), time-zero exit probabilities are identified: when a player is perceived to be weaker in the sense of stochastic dominance, she will engage in initial concession.

Proposition 1. *If $\min\{\underline{u}_1, \underline{u}_2\} > 0$ and $F_1 \succ_{FSD} F_2$ then $\Pr[t_2 = 0] > 0$ and $\Pr[t_1 = 0] = 0$.*

A rough intuition is this. For the same stopping rule, Player 1 is more likely to fight beyond any particular time t . This means that Player 2 finds it relatively more costly to extend her stopping time, and is also relatively less likely to win. Thus, Player 1 will adopt a relatively

³⁶The formal proofs of the propositions are contained in Appendix A.

aggressive stance. This means that, when her valuation is low, Player 2 will prefer not to fight—in other words, there will positive probability of her exit at $t = 0$.

A more precise explanation emerges when Player 1 is stronger in the sense of hazard-rate dominance. (Of course, $F_1 \succ_{\text{HRD}} F_2$ implies that $F_1 \succ_{\text{FSD}} F_2$, and Proposition 1 does hold.) The simplest perturbed war of attrition to consider is one with a time limit $T < \infty$. It is straightforward to establish that $\bar{u}_1^* = \bar{u}_2^* = \bar{u}^*$, so that when the game ends at time $\bar{t}$ the updated lower bounds to the players' valuations are the same.³⁷ Equivalently, the inverse stopping rules cross: $v(\bar{t}) = w(\bar{t}) = \bar{u}^*$. Taking the ratio of $v'(t)$ and $w'(t)$ from (1),

$$\frac{t'_1(\bar{u}^*)}{t'_2(\bar{u}^*)} = \frac{w'(\bar{t})}{v'(\bar{t})} = \frac{1 - F_2(\bar{u}^*)}{f_2(\bar{u}^*)} \times \frac{f_1(\bar{u}^*)}{1 - F_1(\bar{u}^*)} < 1,$$

where the strict inequality follows from $F_1 \succ_{\text{HRD}} F_2$. Hence $t_1(u)$ is steeper than $t_2(u)$ as $u \rightarrow \bar{u}^*$, and thus $t_2(u) < t_1(u)$ for u just below $\bar{u}^*$. In fact, the same argument may be employed whenever the stopping rules cross. This implies that they cross only once, and hence $t_2(u)$ lies everywhere below $t_1(u)$. This is sufficient for the conclusion of Proposition 1.³⁸

Of course, Proposition 1 is limited in two ways. First, it applies only when valuations are bounded away from zero. Second, it establishes that instant exit occurs with *some* probability, but does not show that a weaker player will almost always give up immediately.

4.2. Quicker Exit and Hazard-Rate Dominance. To address the first limitation, turn to the case $\max\{\underline{u}_1, \underline{u}_2\} < 0$. Trivially, both players “instantly exit” when their valuations are negative, and a player with a positive valuation is always willing to fight for some time; there is no instantaneous exit from a player who positively values the prize. Nevertheless, it is still possible to order the behavior of the players. So long as the Player 1 is stronger than Player 2 in the sense of hazard-rate dominance ($F_1 \succ_{\text{HRD}} F_2$), the argument given above ensures that $t_1(u) > t_2(u)$ whenever a time limit is in place ($T < \infty$). In fact, a similar argument may be employed when the classic war of attrition is subject to the other perturbations. For instance, set $\bar{u}_1 = \bar{u}_2 = \infty$, $\xi = 0$ and $T = \infty$. Thus the only perturbation is the “first-price effect” $\beta > 0$. For this configuration, $\bar{u}_1^* = \bar{u}_2^* = \infty$, and Lemma 2 yields

$$\int_{v(t)}^{\infty} \frac{1}{x} \frac{f_1(x)}{1 - (1 - \beta)F_1(x)} dx = \int_{w(t)}^{\infty} \frac{1}{x} \frac{f_2(x)}{1 - (1 - \beta)F_2(x)} dx.$$

³⁷Recall that $\lim_{u \uparrow \bar{u}_i^*} t_i(u) = \bar{t}$, but that $t_i(u) = T$ for all $u \in (\bar{u}_i^*, \bar{u}_i)$. Hence, at a valuation $u = \bar{u}_i^*$, i 's exit time jumps from $\bar{t}$ to T . Thus, i must be just indifferent between exiting at $\bar{t}$ and remaining until T . Continued fighting from $\bar{t}$ until T involves additional costs of $T - \bar{t}$. When the war is over at T , i receives the prize with probability $1/2$, and hence $\bar{u}_i^*/2 = T - \bar{t}$. This means that $\bar{u}_1^* = \bar{u}_2^* = \bar{u}^* = 2(T - \bar{t})$.

³⁸ $\Lambda_1(\underline{u}_1^*) = \Lambda_2(\underline{u}_2^*)$ is explicitly $\int_{\underline{u}_1^*}^{\bar{u}_1^*} \frac{f_1(x)}{x(1-F_1(x))} dx = \int_{\underline{u}_2^*}^{\bar{u}_2^*} \frac{f_2(x)}{x(1-F_2(x))} dx$. Given $F_1 \succ_{\text{HRD}} F_2$, for each x the integrand on the right-hand side exceeds the integrand on the left. Hence, if the equality is to be maintained, the range of integration on the left must be larger: $\underline{u}_1^* < \underline{u}_2^*$. But, since $\underline{u}_1 \geq \underline{u}_2$ (a requirement of Definition 2), this implies that $\underline{u}_2^* > \underline{u}_2$: exactly the initial-exit predicted by Proposition 1. The formal proof of the proposition shows that first-order stochastic dominance is sufficient for the result to obtain, and that the result holds for $T = \infty$ with $\max\{\xi, \beta\} > 0$.

For any x , and β sufficiently small, the integrand of the right hand expression exceeds that of the left, following from $F_1 \succ_{\text{HRD}} F_2$, and hence $v(t) < w(t)$. The only qualification here is that the perturbation must not be too large. Formally:

Proposition 2. *Suppose that $\bar{u}_1 = \bar{u}_2 = \infty$ and $F_1 \succ_{\text{HRD}} F_2$. For $u > \max\{\underline{u}_1, 0\}$, if either $T < \infty$ and $u < \bar{u}^*$, or $T = \infty$ and for $\max\{\xi, \beta\}$ sufficiently small, then $t_1(u) > t_2(u)$.*

Hence the *ex ante* stronger player is willing to fight for longer given any particular valuation.

4.3. Stochastic Strength and Instant Exit. Turning to the second limitation, the notion of stochastic strength is associated with a more striking result, despite the fact that the stochastic-strength ordering limits only the upper tail behavior of the distributions.

For ease of exposition, suppose that Proposition 1 holds, so that $\underline{u}_2^* > \underline{u}_1^*$. When $T < \infty$,

$$\Lambda_1(\underline{u}_1^*) = \Lambda_2(\underline{u}_2^*) \quad \Rightarrow \quad \int_{\underline{u}_1^*}^{\underline{u}_2^*} \frac{f_1(x)}{x(1 - F_1(x))} dx = \int_{\underline{u}_2^*}^{\bar{u}^*} \frac{1}{x} \left[\frac{f_2(x)}{1 - F_2(x)} - \frac{f_1(x)}{1 - F_1(x)} \right] dx. \quad (6)$$

It is straightforward to show (in the formal proofs) that $\bar{u}^* \rightarrow \infty$ as $T \rightarrow \infty$. This means that, if the hazard rates of F_1 and F_2 are bounded apart in the upper tails, then the right-hand side of (6) tends to ∞ as $T \rightarrow \infty$. To offset this effect, and maintain the equality, $\underline{u}_2^*$ must satisfy $\underline{u}_2^* \rightarrow \infty$ as $T \rightarrow \infty$. In the limit, Player 2 always exits at the beginning of the game, automatically conceding the prize to her (stochastically stronger) opponent.

Proposition 3. *If Player 1 is stochastically stronger than Player 2 and $T < \infty$, then*

- (1) *if $\min\{\underline{u}_1, \underline{u}_2\} > 0$ then $\Pr[t_1 = 0] = 0$ for large T and $\Pr[t_2 = 0] \rightarrow 1$ as $T \rightarrow \infty$,*
- (2) *if $\max\{\underline{u}_1, \underline{u}_2\} < 0$ then $\Pr[t_2 < t_1 \mid \min\{u_1, u_2\} > 0] \rightarrow 1$ as $T \rightarrow \infty$, and*
- (3) *for all $u \in (\underline{u}_2, \bar{u}_2)$, $t_2(u) \rightarrow 0$ as $T \rightarrow \infty$.*

Hence the stochastically weaker player almost always loses the war of attrition. If, instead, $T = \infty$ and $\max\{\xi, \beta\} > 0$, then the same results hold as $\max\{\xi, \beta\} \rightarrow 0$ instead of $T \rightarrow \infty$.

Heuristically, a stochastically stronger player's threat to fight for very long periods is more credible. To see this, note that, since such threats are always costly, a player must have a sufficiently high valuation with sufficiently high probability in order to make them. As the threat grows, the probability that the valuation is sufficiently high shrinks. However, the likelihood ratio that Player 1 is able to execute the threat versus Player 2 diverges: it is many times more likely that Player 1 has a very high valuation—this is the content of (5).³⁹

³⁹There are parallels between Proposition 3 and the “unpleasant theorem” of Mirrlees (1999). A special case was this: a principal wishes to induce action $i = 1$ rather than $i = 2$, and bases compensation on $y \sim F_i(y)$. There are two critical assumptions: the agent's utility is unbounded below—hence large punishments are possible; and $F_1(y)/F_2(y) \rightarrow 0$ as $y \rightarrow -\infty$. This unbounded likelihood ratio means that the principal can threaten the agent with a very large punishment if $y \leq y^*$. With y^* suitably chosen, the punishment takes place with arbitrarily low probability under $i = 1$, but is (although still unlikely) far more more likely under $i = 2$. These features are present in the current model: the unbounded likelihood ratio stems from stochastic

Notice that, as a corollary of Proposition 3, and in the limit, the war of attrition will end at time $t = 0$. Thus the result implies that the selected equilibrium in a classic war of attrition might not explain the existence of delay in concessionary settings.

5. SENSITIVITY ANALYSIS

The “almost-always instant-exit” result of Proposition 3 relies upon the stochastic strength of one of the players, and an arbitrarily small perturbation. In this section, Examples 1 and 2 are used to ascertain the sensitivity of the result to these requirements.

5.1. Uniform Distributions (Example 1). The uniform-distribution specification allows closed-form solutions and simple illustrations. Set $T = \infty$ (since $\bar{u}_1 < \infty$ and $\bar{u}_2 < \infty$) and, for simplicity, $\xi = 0$, so that the only perturbation is the “first-price effect” $\beta > 0$.

Proposition 4. *If $\bar{u}_1 > \bar{u}_2$, then as $\beta \rightarrow 0$ Player 1 almost always wins almost immediately. If $\bar{u}_1 = \bar{u}_2 = \bar{u}$ and $\underline{u}_1 > \underline{u}_2$ with $\underline{u}_1 > 0$, then $\beta \rightarrow 0$ yields an equilibrium in which*

$$\frac{w(t)}{\bar{u} - w(t)} = \frac{v(t)}{\bar{u} - v(t)} \times \frac{\bar{u} - \underline{u}_1}{\bar{u} - \underline{u}_2}, \quad \text{and hence} \quad w(0) = \frac{\underline{u}_1}{1 + (\underline{u}_1 - \underline{u}_2)/\bar{u}}$$

Hence Player 2 exits immediately with positive probability, but $w(t) < v(t)$.

The first part of Proposition 4 follows straightforwardly from Proposition 3. For the second claim, however, $\bar{u}_1 = \bar{u}_2$. Thus Player 1 is *not* stochastically stronger than Player 2, and Proposition 3 does not apply. Nevertheless, it is straightforward to observe that $F_1 \succ_{\text{FSD}} F_2$ and hence Proposition 1 *does* apply. Thus Player 2 exits at time $t = 0$ with positive probability. Interestingly, however, for this case $w(t) < v(t)$. This means that, valuation for valuation, Player 1 is *less likely* to win the war of attrition—it is biased against her.

To see why, return to the environment in which $\xi > 0$, and set $\beta = 0$. (Proposition 4 continues to apply, with $\xi \rightarrow 0$ replacing $\beta \rightarrow 0$.) Player 2 instantly exits at time zero with positive probability. Just after $t = 0$, the probability that Player 1 suffers exit failure is close to ξ . The probability that Player 2 suffers exit failure, *conditional on this time being reached*, is *greater* than ξ . If exit failure is interpreted as “craziness,” then the initial exit of Player 2 enhances her reputation for such craziness whenever she remains in the game.

As a counterpoint to Proposition 3, the $\bar{u}_1 = \bar{u}_2$ case establishes that fighting does occur.⁴⁰ It might be argued, however, that this is a knife-edge case. Hence, if $\bar{u}_2 \leq \bar{u}_1 < \infty$, one might normally expect $\bar{u}_2 < \bar{u}_1$ and thus the stochastic-strength advantage of Player 1.

strength, and infinite punishments are possible since the war of attrition can go on forever, at least when $T = \infty$. When $T \rightarrow \infty$, the “maximum punishment” is allowed to become arbitrarily large.

⁴⁰Propositions 1–3 fail to apply when the players are symmetric. Indeed, the uniqueness result of Proposition 3 ensures that, with symmetric players, the unique equilibrium will be symmetric and the *ex post* stronger player will win.

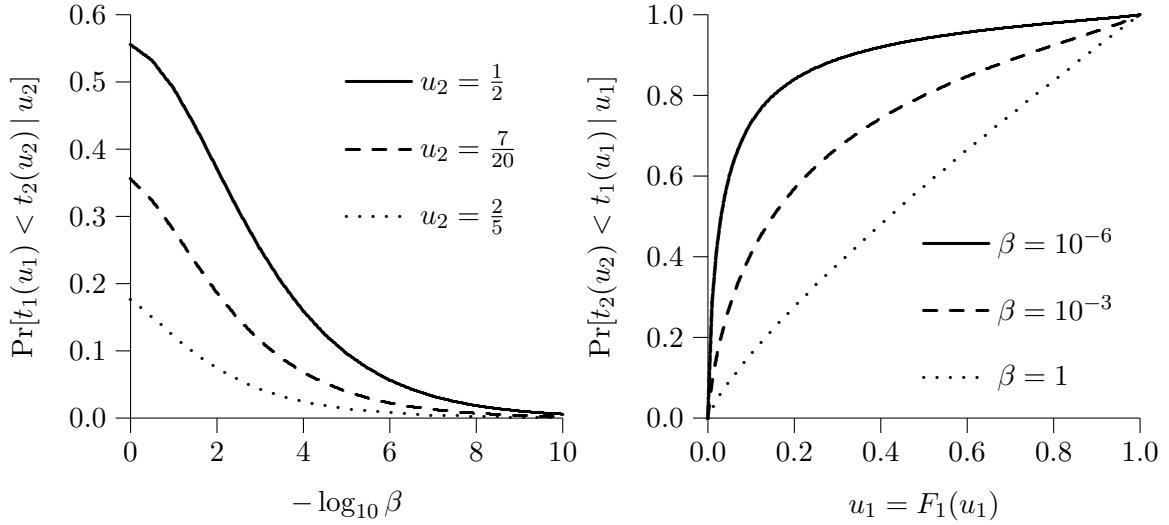


FIGURE 1. Uniform Distributions (Example 1, $\bar{u}_1 = 1$ and $\bar{u}_2 = \frac{4}{5}$)

Of course, when the players are ranked by stochastic strength, the immediate exit of the stochastically weaker player only happens in the limit as $\beta \rightarrow 0$ (or, equivalently, $\xi \rightarrow 0$). When $\beta > 0$ is fixed, taking away any asymmetry reverses the result. It is interesting to calculate, therefore, the degree to which “instant exit” is sensitive to the exact choice of β . To do this, set $\underline{u}_1 = \underline{u}_2 = 0$. Appendix A.3 solves for $v(t)/\bar{u}_1$ in terms of $w(t)$. Of course, $v(t)/\bar{u}_1 = \Pr[u_1 \leq v(t)]$. This is the probability that Player 1 loses the war of attrition, given that Player 2 has a valuation $u_2 = w(t)$. Equivalently, it is the probability Player 2 wins the war of attrition given that she has a valuation $u_2 = w(t)$. Fixing u_2 , this probability vanishes as $\beta \rightarrow 0$. Away from the limit, however, this probability may be calculated for a range of different parameter values.

Figure 1 displays the results of this exercise. When u_2 is moderately large, Player 2 retains a “fighting chance” of winning until β vanishes completely (notice the log scale for β). The lesson is that the perturbation has to be very small indeed for the instant-exit result to have bite. When the perturbation is taken seriously—small, but not taken to the limit of zero—then the exit of the weaker player is relatively rapid, rather than always instant.

5.2. Exponential Distributions (Example 2). To investigate this point further, consider Example 2. Set $\xi = \beta = 0$, but impose a time limit $T < \infty$. Algebraically, this example is relatively straightforward, since $\Lambda_i(u) = \lambda_i \log[\bar{u}^*/u]$. For the illustrations of Figure 2, set $\lambda_1 = 1$ and $\lambda_2 = 1\frac{1}{2}$, so that $E[u_1] = 1$ and $E[u_2] = \frac{2}{3}$. This means that $\Pr[u_1 > u_2] = \frac{3}{5}$, and Player 1 is stronger *ex ante* in all senses (Definitions 1–3).

Figure 2 displays the results. Once again, the perturbation must be very small before instant exit bites. In this case, a small perturbation corresponds to a large time limit T . For instance,

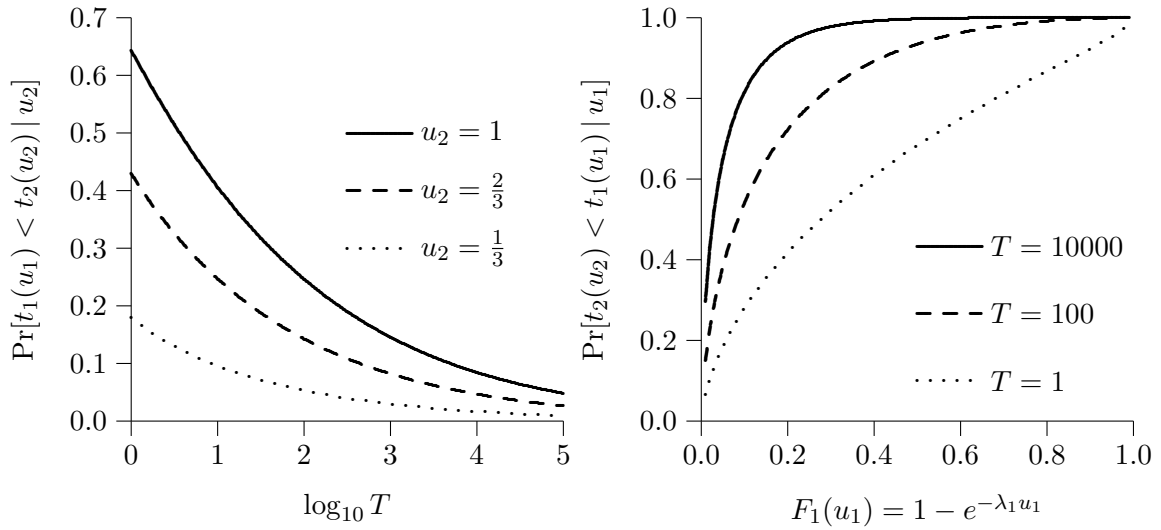


FIGURE 2. Exponential Distributions (Example 2, $\lambda_1 = 1$ and $\lambda_2 = 1\frac{1}{2}$)

when u_2 takes its median value of $u_2 = \log 2/\lambda_2 \approx 0.46$ and $T = 10$, Player 2 retains a 15% chance of winning, compared to a 37% chance of having the highest valuation.

Taken together, these sensitivity analyses suggest that the selection of the “instant exit” equilibrium is an extreme case. In any application of the war of attrition, therefore, the analyst may wish to take seriously the different perturbations that may be present, and hence assess the exact speed of the “rapid exit” that may occur.

On the other hand, if an equilibrium of the classic war of attrition is to be selected, then Proposition 3 suggests that the only candidate equilibrium must, it seem, involve instant exit. Thus, if attention is to be restricted to the symmetric equilibrium of a symmetric game, then the intended application must involve players that are exceptionally similar *ex ante*.

6. DISCUSSION

The war of attrition lends itself to a wide range of social-scientific applications. This sections offers discussion of the implications of the results for a number of these.

6.1. Rent Dissipation. In a classic paper, Posner (1975) argued that the social costs of monopoly should include monopoly profits.⁴¹ His idea was that such profits would be dissipated by rent-seeking activities. Wars of attrition may be used to model rent-seeking. For instance, Fudenberg and Tirole (1987) used “game theory to shed light on a particular

⁴¹Posner (1975) went on to combine monopoly profits with the usual Harberger (1954) deadweight loss triangles, and turned to appropriate data to obtain a relatively large estimate of the social costs of monopoly. Wenders (1987) went further by adding “rent-defending” to the “rent-seeking” equation.

issue—the hypothesis of monopoly rent dissipation.” In their complete-information model, the symmetric equilibrium succeeds in dissipating all rents.⁴²

Posner’s (1975) claim is subject to a number of critiques. Fisher (1985), for instance, argued that the rents attributed to an initial advantage should not be counted as a social cost. An interpretation of this is that *asymmetries* between the participants may prevent the complete dissipation of the prize. Others have considered the actual size of rent-seeking costs. Tullock (1967), for instance, recognized that political actors will expend resources in order to obtain political rents. Later (Tullock, 1980), however, he noted that the costs incurred by participants appear to be relatively low compared to the prizes involved. Examining this “Tullock paradox,” Riley (1999) demonstrated that asymmetries in a complete-information war of attrition may dramatically reduce rent-seeking costs. The present paper, therefore, extends Riley’s (1999) explanation to the incomplete-information (or random rewards) case: merely the *perception* of an initial advantage may dramatically reduce Tullock costs.

6.2. All-Pay Auctions. Wars of attrition have attracted the attention of leading auction theorists. Bulow and Klemperer (1999), for instance, considered a *generalized* war of attrition, in which $N + K$ symmetric bidders compete for N privately valued prizes. As is well known (Haigh and Cannings, 1989), when $K > 1$ such a game has no symmetric equilibrium; $K - 1$ players must exit immediately, yielding a war in which the $N + 1$ remaining bidders compete for N prizes. To circumvent this, Bulow and Klemperer (1999) perturbed the game in an ingenious manner: a conceding player continues to pay a (perhaps small) fraction of her fighting costs until the game ends. The perturbed game has a symmetric equilibrium involving rapid exit until the $N + 1$ highest-valued players remain.⁴³

Unfortunately, their approach does not solve the equilibrium-selection problem, as symmetry is used as an *ad hoc* selection criterion.⁴⁴ In an (even slightly) asymmetric war of attrition the selected equilibrium may be very far from symmetric.⁴⁵ The revenue rankings of different auction formats may then be overturned. For instance, Krishna and Morgan (1997) considered the symmetric equilibrium of a model with affiliated values, and found that (from a revenue perspective) the war of attrition outperforms other standard auction formats. In contrast, the present paper suggests that the war of attrition may raise no revenue at all.

⁴²Since the symmetric mixed-strategy equilibrium of such a complete-information war of attrition yields no net benefits to either player, Maynard Smith (1974) described this quirky feature as an apparent “absurdity.”

⁴³They refer to this as “instant sorting” (exit of $K - 1$) until “one too many” ($N + 1$) remain. In contrast, the present paper suggests “complete *ex ante* sorting” (exit of the K *ex ante* weakest) until N players remain. Bulow and Klemperer (1999, p. 178, note 15) acknowledge this possibility, noting that “[a]symmetric perfect-Bayesian equilibria include those in which K (pre-identified) firms quit in zero time ... [e]quilibria of this kind seem particularly natural if (in contrast to our model) there any asymmetries between players.” Here I suggest that stochastic strength is exactly the kind of asymmetry to generate this result.

⁴⁴For instance, when $N = K = 1$ their model reduces to a classic war of attrition. This classic war of attrition has multiple equilibria. Their perturbation has no bite in the last stage of their game.

⁴⁵In other papers, Klemperer (1998) and Bulow, Huang, and Klemperer (1999) paid great heed to the potential influence of small asymmetries between bidders in common-value settings.

6.3. Macroeconomic Stabilization. Alesina and Drazen (1991) used the war of attrition to model a process of macroeconomic stabilization. They considered an economy with a rising debt-to-GNP ratio and hence a need for fiscal action (Drazen and Helpman 1987, 1990). This requires the consent of two socioeconomic groups, and a concession by one side consists of an agreement “to bear a disproportionate share of the tax increase necessary to effect a stabilization.” Crucially, however, the Alesina-Drazen model is *symmetric*: the utility losses suffered due to the tax increases are drawn from the same distribution for both socioeconomic groups. They (and indeed subsequent authors, such as Casella and Eichengreen 1996) focused, therefore, on a symmetric equilibrium. In some situations, players might be expected to be “approximately” symmetric. In the Alesina-Drazen application, however, the different socioeconomic interest groups are likely to be asymmetric *ex ante*, and hence stabilization is likely to occur far more rapidly.⁴⁶

6.4. Technological Standards. The use of a symmetric specification, and a focus on symmetric equilibria, is also present in studies of standards adoption. Farrell and Saloner (1988) viewed the adoption of a technological standard as a “battle of the sexes” game: firms wish to coordinate on a some common standard, but prefer their own standard. They then modelled a standards committee as a war of attrition, in which each participant “holds out,” hoping that the opponent will agree to coordinate on the preferred standard. Crucially, they focus on the symmetric mixed-strategy equilibrium to this game. In an extension, Farrell (1996) considered an incomplete-information version of the same standards-adoption game. Once again, however, he chose the symmetric equilibrium as the vehicle for his analysis. With *ex ante* asymmetries, the situation may change: there may be immediate concession in favor of the stochastically stronger standard, even if it is suboptimal *ex post*.

6.5. Private Provision of a Public Good. An early economic application of the war of attrition was provided by Bliss and Nalebuff (1984). They considered N impatient individuals, each of whom is capable of providing a public good. The first person to “concede” pays a private cost to supply the good, yielding a prize (the opportunity to free ride) for each of the $N - 1$ remaining players. The authors’ specification is, once again, symmetric, and they examined a symmetric equilibrium. The player with the lowest private cost of provision, relative to her benefit from the good, will provide the good following some delay. The results of the present paper suggest that, in an asymmetric version of the same game, the public good would be provided relatively quickly (hence increasing efficiency) by the *ex ante* most efficient player (who might be less efficient *ex post*, and hence this may reduce efficiency).⁴⁷

⁴⁶In their discussion of European deficits following World War I, Alesina and Drazen (1991) noted the “dominant position of the Conservatives” in Britain, which they claimed “led to a rapid stabilization by means that favored the Conservatives’ traditional constituencies.” The interpretation offered here is that the Conservatives were stochastically stronger in that situation.

⁴⁷In a complete-information model, Bilodeau and Slivinski (1996) obtained a unique equilibrium by imposing a time limit. The public good is provided immediately by the most efficient player. Their complete-information assumption, however, meant that they did not distinguish between *ex ante* and *ex post* strength.

7. CONCLUDING REMARKS

It is suggested that the war of attrition may well end with the instant exit of one participant. A player is stochastically stronger when she is much more likely to experience very large valuations. When this ordering holds, and the classic war of attrition is perturbed using a “credibility” device, the selected equilibrium involves the instant exit of the stochastically weaker player. Important lessons emerge. First, *ex ante* perceptions of “strength” may be more important than a player’s *ex post* valuation for a prize. Second, a focus on the symmetric equilibrium of a symmetric game may be misleading, since small asymmetries may dramatically affect the outcome.

The sensitivity analysis (§5) addresses two possible problems with the “instant exit” proposition. A third problem is this: the results suggest that wars of attrition should end relatively quickly, and thus fighting will not be seen. In contrast, the classic war of attrition is used to model situations in which delays are actually observed. If this paper is to be believed, then why might delay occur in a concessionary environment?

One response is to embed the war of attrition into a larger game. The key determinant of the outcome here is the stochastic strength of a player. Each participant, therefore, has a strong incentive to appear stronger *ex ante*. Thus a stochastically weaker player may wish to delay concession in the hope that a change in the players’ perceived relative strengths may leave her in the stochastically stronger position.

To flesh out this idea, suppose that the war of attrition takes place in two stages. In the first stage, if both players fight, then a signal of the relative strengths emerge. This might be generated from conflict between them,⁴⁸ and the players may then update to obtain an interim assessment of each others’ strength. If the signal is sufficiently precise to allow the overturning of an initial stochastic-strength ranking, then a stochastically weaker player has an incentive to participate in this first stage. Notice that this story involves *direct learning*: an additional signal changes the players’ beliefs. In contrast, the classic war of attrition involves only learning by *revelation*: a player updates her belief based upon her opponent’s continued presence in the game. Of course, a complete story would combine both direct learning and revelation in a unified model. The results offered here suggest that such a unified model may be needed to explain some instances of costly delay.

⁴⁸Using the terminology of Maynard Smith (1974), this initial stage is a “tournament” rather than a “display.” In a recent working paper, Lee (2002) models direct learning by firms competing to obtain a monopoly position, using continuous time filtering techniques (Bolton and Harris, 1999; Keller and Rady, 1999; Moscarini and Smith, 2001). As time goes on, participants observe a Wiener process, with a drift determined by the relative profitability of the two firms, and thus learn directly, rather than by revelation.

APPENDIX A. PROOFS

Lemmas 1–3 are somewhat standard, and hence their proofs are omitted from the main paper. For completeness, they are collected together in the not-for-publication Appendix B.

Propositions 1–3 are proved separately for the case $T < \infty$ and $\xi = \beta = 0$ and the case $T = \infty$ and $\max\{\xi, \beta\} > 0$. The proof of Proposition 4 and further details of the sensitivity analysis follow.

A.1. Proofs of Propositions 1–3 for $T < \infty$ and $\xi = \beta = 0$. Recall that, when $T < \infty$, it is assumed that $\bar{u}_1 = \bar{u}_2 = \infty$. Furthermore, following Footnote 37, $\bar{u}_1^* = \bar{u}_2^* = \bar{u}^* = 2(T - \bar{t}) < \infty$.

Proof of Proposition 1 for $T < \infty$. Take (2) and integrate by parts:

$$\begin{aligned} \Lambda_i(u) &= \int_u^{\bar{u}^*} \frac{f_i(x)}{x(1 - F_i(x))} dx = - \int_u^{\bar{u}^*} \frac{1}{x} d\log[1 - F_i(x)] \\ &= - \frac{\log[1 - F_i(x)]}{x} \Big|_u^{\bar{u}^*} + \int_u^{\bar{u}^*} \log[1 - F_i(x)] d(1/x) \\ &= \frac{\log[1 - F_i(u)]}{u} - \frac{\log[1 - F_i(\bar{u}^*)]}{\bar{u}^*} - \int_u^{\bar{u}^*} \frac{\log[1 - F_i(x)]}{x^2} dx. \end{aligned}$$

Suppose, contrary to Proposition 1, that $\underline{u}_2^* = \underline{u}_2$, so that $F_2(\underline{u}_2^*) = 0$ and $\log[1 - F_2(\underline{u}_2^*)] = 0$. Since $F_1 \succ_{\text{FSD}} F_2$ it must be that $\underline{u}_1 \geq \underline{u}_2$, and hence $\underline{u}_1^* \geq \underline{u}_1 \geq \underline{u}_2 = \underline{u}_2^*$. $\Lambda_i(\underline{u}_i^*)$ is well defined for $i \in \{1, 2\}$, and evaluation of $\Lambda_2(\underline{u}_2^*)$ yields

$$\Lambda_2(\underline{u}_2^*) = - \frac{\log[1 - F_2(\bar{u}^*)]}{\bar{u}^*} - \int_{\underline{u}_2^*}^{\underline{u}_1^*} \frac{\log[1 - F_2(x)]}{x^2} dx - \int_{\underline{u}_1^*}^{\bar{u}^*} \frac{\log[1 - F_2(x)]}{x^2} dx,$$

while the evaluation of $\Lambda_1(\underline{u}_1^*)$ yields

$$\Lambda_1(\underline{u}_1^*) = \frac{\log[1 - F_1(\underline{u}_1^*)]}{\underline{u}_1^*} - \frac{\log[1 - F_1(\bar{u}^*)]}{\bar{u}^*} - \int_{\underline{u}_1^*}^{\bar{u}^*} \frac{\log[1 - F_1(x)]}{x^2} dx.$$

$\Lambda_1(v(t)) = \Lambda_2(w(t))$ holds in the limit as $t \rightarrow 0$. Equate $\Lambda_1(\underline{u}_1^*)$ and $\Lambda_2(\underline{u}_2^*)$ to obtain

$$\begin{aligned} \frac{1}{\bar{u}^*} \log \left[\frac{1 - F_1(\bar{u}^*)}{1 - F_2(\bar{u}^*)} \right] - \int_{\underline{u}_2^*}^{\underline{u}_1^*} \frac{\log[1 - F_2(x)]}{x^2} dx = \\ \frac{\log[1 - F_1(\underline{u}_1^*)]}{\underline{u}_1^*} - \int_{\underline{u}_1^*}^{\bar{u}^*} \frac{1}{x^2} \log \left[\frac{1 - F_1(x)}{1 - F_2(x)} \right] dx. \end{aligned}$$

Since $F_1(u) < F_2(u)$ for all $u > \underline{u}_1$, the left-hand side of the equation is strictly positive and the right-hand side is strictly negative. This is a contradiction, and hence $\underline{u}_2^* > \underline{u}_2$. $\square$

Proof of Proposition 2 for $T < \infty$. It is claimed that $\underline{u}_2^* \geq \underline{u}_1^* \geq 0$. To prove this, note that $t_i(u) = 0$ for all $u \leq 0$, and hence $\underline{u}_i^* \geq 0$ for $i \in \{1, 2\}$. Suppose, contrary to the stated claim, that $\underline{u}_1^* > \underline{u}_2^* \geq 0$. Since $\underline{u}_1^* > 0$, both $\Lambda_1(\underline{u}_1^*)$ and $\Lambda_2(\underline{u}_1^*)$ are well defined. Lemma 2 then implies that $\Lambda_2(\underline{u}_1^*) - \Lambda_1(\underline{u}_1^*) < 0$. This is inconsistent with $F_1 \succ_{\text{HRD}} F_2$, since

$$\Lambda_2(\underline{u}_1^*) - \Lambda_1(\underline{u}_1^*) = \int_{\underline{u}_1^*}^{\bar{u}^*} \frac{1}{x} \left[\frac{f_2(x)}{1 - F_2(x)} - \frac{f_1(x)}{1 - F_1(x)} \right] dx > 0.$$

This is a contradiction, and so the claim is true. With the claim in hand, turn to the proposition. Suppose that $u > \underline{u}_2^* \geq \underline{u}_1^* \geq 0$. Both $\Lambda_1(u)$ and $\Lambda_2(u)$ are well defined. Since $F_1 \succ_{\text{HRD}} F_2$, and following the logic employed above, $\Lambda_2(u) - \Lambda_1(u) > 0$. Set $t = t_1(u)$, so that $v(t) = u$. Lemma 2 insists that $\Lambda_2(w(t)) - \Lambda_1(v(t)) = 0$. This means that $w(t) > u$, and that $t_2(u) < t = t_1(u)$, which is the desired result. Next, suppose that $\underline{u}_2^* \geq u > \underline{u}_1^* \geq 0$. This implies that $t_2(u) = 0$, $t_1(u) > 0$, and hence $t_1(u) > t_2(u)$. Finally, suppose that $\underline{u}_2^* \geq \underline{u}_1^* \geq u > 0$, so that $t_1(u) = t_2(u) = 0$. Since $u > \underline{u}_1 \geq \underline{u}_2$, this means that both players exit at time zero with positive probability, and with positive valuations. This cannot be optimal for either player; a contradiction $\square$

Proof of Proposition 3 for $T < \infty$. First, it is shown that $\bar{u} \rightarrow \infty$ as $T \rightarrow \infty$. Second, it is shown that when Player 1 is stochastically stronger than Player 2, $\Lambda_2(u) - \Lambda_1(u) \rightarrow \infty$ as $T \rightarrow \infty$. This fact is used to prove the statements given in the proposition.

The first step must show that, as the time limit grows large, the valuation at which the equilibrium stopping rules jump up to the time limit grows without bound. Integrating first-order conditions,

$$t_1(u) = \int_{\underline{u}_1^*}^u \frac{f_1(x)w(t_1(x))}{1 - F_1(x)} dx \leq \bar{u}_2^* \log \left[\frac{1 - F_1(\underline{u}_1^*)}{1 - F_1(u)} \right] = \bar{u}^* \log \left[\frac{1 - F_1(\underline{u}_1^*)}{1 - F_1(u)} \right],$$

where the second equality follows from $\bar{u}_2^* = \bar{u}_1^* = \bar{u}^*$. Letting $u \rightarrow \bar{u}^*$, $t_1(u) \rightarrow \bar{t}$ and hence

$$\bar{t} \leq \bar{u}^* \log \left[\frac{1 - F_1(\underline{u}_1^*)}{1 - F_1(\bar{u}^*)} \right] \leq -\bar{u}^* \log[1 - F_1(\bar{u}^*)]$$

If the claim ($\bar{u}^* \rightarrow \infty$) is false then a sequence of time limits $\{T\}$ maybe constructed satisfying $T \rightarrow \infty$ such that, in the equilibria, $\bar{u}^*$ remains bounded throughout the sequence. It follows from the inequality above that $\bar{t}$ remains bounded throughout the sequence. But if $\bar{t}$ remains bounded, then $\bar{u}^* = 2(T - \bar{t}) \rightarrow \infty$ as $T \rightarrow \infty$: A contradiction.

For the second step, recall that Player 1 is stochastically strong, so that the hazards of F_1 and F_2 are bounded apart for large valuations. Hence, for some $\lambda > 0$ and $U > \max\{\underline{u}_1, \underline{u}_2, 0\}$,

$$\frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} > \lambda \quad \text{for all } u \geq U.$$

Following from the first step in the proof, $\bar{u}^* = \bar{u}_1^* = \bar{u}_2^* > U$ for all T sufficiently large. Hence restrict attention to such T . For any $u \in [\max\{\underline{u}_1, \underline{u}_2\}, \bar{u}^*)$ satisfying $u > 0$,

$$\begin{aligned} \Lambda_2(u) - \Lambda_1(u) &= \int_u^{\bar{u}^*} \frac{1}{x} \left[\frac{f_2(x)}{1 - F_2(x)} - \frac{f_1(x)}{1 - F_1(x)} \right] dx \\ &= \int_{\min\{u, U\}}^{\max\{u, U\}} \frac{1}{x} \left[\frac{f_2(x)}{1 - F_2(x)} - \frac{f_1(x)}{1 - F_1(x)} \right] dx + \int_{\max\{u, U\}}^{\bar{u}^*} \frac{1}{x} \left[\frac{f_2(x)}{1 - F_2(x)} - \frac{f_1(x)}{1 - F_1(x)} \right] dx. \end{aligned}$$

Fix u and U , and allow $T \rightarrow \infty$. The first term remains constant. $\bar{u}^* \rightarrow \infty$, and the second satisfies

$$\int_{\max\{u, U\}}^{\bar{u}^*} \frac{1}{x} \left[\frac{f_2(x)}{1 - F_2(x)} - \frac{f_1(x)}{1 - F_1(x)} \right] dx \geq \lambda \log \left[\frac{\bar{u}^*}{\max\{u, U\}} \right] \rightarrow \infty.$$

This completes the second step in the proof.

For the third step, begin with the case where $\min\{\underline{u}_1, \underline{u}_2\} > 0$. If $\underline{u}_1^* > \underline{u}_2^* > 0$, then

$$\Lambda_2(\underline{u}_1^*) - \Lambda_1(\underline{u}_1^*) = \Lambda_2(\underline{u}_1^*) - \Lambda_2(\underline{u}_2^*).$$

If $\underline{u}_1^*$ remains bounded as $T \rightarrow \infty$, then the right-hand side remains bounded, whereas the left-hand side diverges to ∞ from the proof's second step. This is a contradiction, and hence $\underline{u}_1^* \rightarrow \infty$. Thus, $\Lambda_2(\underline{u}_1^*) - \Lambda_1(\underline{u}_1^*)$ is positive for T sufficiently large. But the left-hand side is negative: a contradiction. Thus $\underline{u}_2^* \geq \underline{u}_1^* > 0$, and $\Lambda_2(\underline{u}_2^*) - \Lambda_1(\underline{u}_2^*) = \Lambda_1(\underline{u}_1^*) - \Lambda_1(\underline{u}_2^*)$. Once again, if $\underline{u}_2^*$ remains bounded as $T \rightarrow \infty$, then a contradiction is obtained: the right-hand side remains bounded, and the left-hand side diverges. Conclude that $\underline{u}_2^* \rightarrow \infty$, and hence, for T sufficiently large, $\underline{u}_1^* = \underline{u}_1$. This establishes the first statement of Proposition 3.

For the second statement, note first that $\underline{u}_1^* = \underline{u}_2^* = 0$. Fix $u > 0$, and restrict to large enough T so that $\bar{u}^* > u$. Lemma 2 implies that $\Lambda_1(u) = \Lambda_2(w(t_1(u)))$ and hence

$$\Lambda_2(u) - \Lambda_1(u) = \Lambda_2(u) - \Lambda_2(w(t_1(u))) = \int_u^{w(t_1(u))} \frac{f_2(x)}{x(1 - F_2(x))} dx.$$

The left hand side diverges to ∞ as $T \rightarrow \infty$, and hence $w(t_1(u)) \rightarrow \infty$. Thus, conditional on $u_1 = u > 0$, $\Pr[t_2 \geq t_1] = \Pr[u_2 \geq w(t_1(u))] \rightarrow 0$ as $T \rightarrow \infty$. This establishes the second statement.

Finally, turn to the third statement. If $u \leq 0$, then the statement is true. Hence restrict to $u > 0$. If the statement is false, then there is a sequence $T \rightarrow \infty$ such that $t_2(u) \geq \varepsilon > 0$ throughout the sequence, and hence $u > \underline{u}_2^*$. The stopping rule for Player 2 satisfies

$$t_2(u) = \int_{\underline{u}_2^*}^u \frac{f_2(x)\Lambda_1^{-1}(\Lambda_2(x))}{1 - F_2(x)} dx \leq v(t_2(u)) \int_{\underline{u}_2^*}^u \frac{f_2(x)}{1 - F_2(x)} dx \leq -v(t_2(u)) \log[1 - F_2(u)].$$

Employing Lemma 2 once more,

$$\Lambda_2(u) - \Lambda_1(u) = \Lambda_1(v(t_2(u))) - \Lambda_1(u) = \int_{v(t_2(u))}^u \frac{f_1(x)}{x(1 - F_1(x))} dx.$$

Once again, the left-hand side diverges as $T \rightarrow \infty$. For the equality to be maintained, it must be the case that $v(t_2(u)) \rightarrow 0$. But this implies that, for sufficiently large T , $t_2(u) < \varepsilon$. This is a contradiction, and hence $t_2(u) \rightarrow 0$ as $T \rightarrow \infty$. This proves the third statement. $\square$

A.2. Proofs of Propositions 1–3 for $T = \infty$ and $\max\{\xi, \beta\} > 0$. Without loss of generality, set $\xi = 0$ and $\beta > 0$ for all of these proofs. Recall that, since $T = \infty$, $\bar{u}_i^* = \bar{u}_i$.

Proof of Proposition 1 for $T = \infty$. When $\bar{u}_i < \infty$, integrate by parts to obtain

$$\Lambda_i(u) = \frac{\log[1 - (1 - \beta)F_i(u)]}{u} - \frac{\log \beta}{\bar{u}_i} - \int_u^{\bar{u}_i} \frac{\log[1 - (1 - \beta)F_i(x)]}{x^2} dx.$$

When $\bar{u}_i = \infty$, the same equality holds so long as the term $\log \beta / \bar{u}_i$ is set to zero. Suppose, contrary to the proposition, that $\underline{u}_2^* = \underline{u}_2$. This means that $F_2(\underline{u}_2^*) = 0$ and hence $\log[1 - (1 - \beta)F_2(\underline{u}_2^*)] = 0$. Since $F_1 \succ_{\text{FSD}} F_2$ it must be the case that $\underline{u}_1 \geq \underline{u}_2$, and hence $\underline{u}_1^* \geq \underline{u}_1 \geq \underline{u}_2 = \underline{u}_2^*$, and also $\bar{u}_1 \geq \bar{u}_2$. With these observations in hand, and for $\underline{u}_1^* < \bar{u}_2$,

$$\Lambda_2(\underline{u}_2^*) = -\frac{\log \beta}{\bar{u}_2} - \int_{\underline{u}_2^*}^{\underline{u}_1^*} \frac{\log[1 - (1 - \beta)F_2(x)]}{x^2} dx - \int_{\underline{u}_1^*}^{\bar{u}_2} \frac{\log[1 - (1 - \beta)F_2(x)]}{x^2} dx,$$

while the evaluation of $\Lambda_1(\underline{u}_1^*)$ yields

$$\begin{aligned} \Lambda_1(\underline{u}_1^*) &= \frac{\log[1 - (1 - \beta)F_1(\underline{u}_1^*)]}{\underline{u}_1^*} - \frac{\log \beta}{\bar{u}_1} \\ &\quad - \int_{\underline{u}_1^*}^{\bar{u}_2} \frac{\log[1 - (1 - \beta)F_1(x)]}{x^2} dx - \int_{\bar{u}_2}^{\bar{u}_1} \frac{\log[1 - (1 - \beta)F_1(x)]}{x^2} dx. \end{aligned}$$

$\Lambda_1(\underline{u}_1^*)$ and $\Lambda_2(\underline{u}_2^*)$ are well defined and equal. Equating them, obtain

$$\begin{aligned} \int_{\underline{u}_1^*}^{\bar{u}_2} \frac{1}{x^2} \log \left[\frac{1 - (1 - \beta)F_1(x)}{1 - (1 - \beta)F_2(x)} \right] dx &= \int_{\underline{u}_2^*}^{\underline{u}_1^*} \frac{\log[1 - (1 - \beta)F_2(x)]}{x^2} dx \\ &\quad + \frac{\log[1 - (1 - \beta)F_1(\underline{u}_1^*)]}{\underline{u}_1^*} + \log \beta \left[\frac{1}{\bar{u}_2} - \frac{1}{\bar{u}_1} \right] - \int_{\bar{u}_2}^{\bar{u}_1} \frac{\log[1 - (1 - \beta)F_1(x)]}{x^2} dx. \end{aligned} \quad (7)$$

Since $F_1 \succ_{\text{FSD}} F_2$ the left-hand side of (7) is strictly positive:

$$F_1(x) < F_2(x) \quad \Rightarrow \quad \log \left[\frac{1 - (1 - \beta)F_1(x)}{1 - (1 - \beta)F_2(x)} \right] > 0 \quad \forall x \in (\underline{u}_1^*, \bar{u}_2).$$

Both of the first two terms on the right of (7) are (at least weakly) negative. Hence, if the equation is to hold, the remaining terms must be strictly positive. Hence,

$$\log \beta \left[\frac{1}{\bar{u}_2} - \frac{1}{\bar{u}_1} \right] > \int_{\bar{u}_2}^{\bar{u}_1} \frac{\log[1 - (1 - \beta)F_1(x)]}{x^2} dx \geq \log \beta \int_{\bar{u}_2}^{\bar{u}_1} \frac{1}{x^2} dx = \log \beta \left[\frac{1}{\bar{u}_2} - \frac{1}{\bar{u}_1} \right].$$

Clearly, this is a contradiction, and (7) cannot hold. The derivations above were for the case $\bar{u}_2 > \underline{u}_1^*$. Suppose instead that $\underline{u}_1^* \geq \bar{u}_2$. I may write $\Lambda_2(\underline{u}_2^*)$ and $\Lambda_1(\underline{u}_1^*)$ as

$$\begin{aligned} \Lambda_2(\underline{u}_2^*) &= -\frac{\log \beta}{\bar{u}_2} - \int_{\underline{u}_2^*}^{\bar{u}_2} \frac{\log[1 - (1 - \beta)F_2(x)]}{x^2} dx \\ \text{and } \Lambda_1(\underline{u}_1^*) &= \frac{\log[1 - (1 - \beta)F_1(\underline{u}_1^*)]}{\underline{u}_1^*} - \frac{\log \beta}{\bar{u}_1} - \int_{\underline{u}_1^*}^{\bar{u}_1} \frac{\log[1 - (1 - \beta)F_1(x)]}{x^2} dx. \end{aligned}$$

Equating these two, as before, obtain

$$\begin{aligned} \int_{\underline{u}_2^*}^{\bar{u}_2} \frac{\log[1 - (1 - \beta)F_2(x)]}{x^2} dx + \frac{\log[1 - (1 - \beta)F_1(\underline{u}_1^*)]}{\underline{u}_1^*} \\ = \int_{\underline{u}_1^*}^{\bar{u}_1} \frac{\log[1 - (1 - \beta)F_1(x)]}{x^2} dx - \log \beta \left[\frac{1}{\bar{u}_2} - \frac{1}{\bar{u}_1} \right]. \end{aligned}$$

The left hand side is strictly negative. Hence,

$$\log \beta \left[\frac{1}{\bar{u}_2} - \frac{1}{\bar{u}_1} \right] > \int_{\underline{u}_1^*}^{\bar{u}_1} \frac{\log[1 - (1 - \beta)F_1(x)]}{x^2} dx \geq \log \beta \left[\frac{1}{\underline{u}_1^*} - \frac{1}{\bar{u}_1} \right] \quad \Rightarrow \quad \underline{u}_1^* < \bar{u}_2,$$

which is, of course, a contradiction. A contradiction has been reached for all cases, and hence the original supposition that $\underline{u}_2^* = \underline{u}_2$ is false. This completes the proof. $\square$

Proposition 2 gives conditions under which $t_1(u) > t_2(u)$, so that, when the players share the same prize valuation u , Player 1 fights for longer and wins the prize. For $u \in (\underline{u}_1^*, \bar{u}_1^*) \cap (\underline{u}_2^*, \bar{u}_2^*)$ it is clear that this inequality will hold if and only if $\Lambda_2(u) - \Lambda_1(u) > 0$. Thus, in the following lemmas, the properties of $\Lambda_2(u) - \Lambda_1(u)$ when $T = \infty$ may be established.

Lemma 4. If $\bar{u}_1 > \bar{u}_2 > u > 0$ and $u \geq \max\{\underline{u}_1, \underline{u}_2\}$, then $\lim_{\beta \rightarrow 0} [\Lambda_2(u) - \Lambda_1(u)] = \infty$.

Proof. Since $\bar{u}_1 > \bar{u}_2$, $\tilde{u}$ may be chosen satisfying $\bar{u}_2 < \tilde{u} < \bar{u}_1$. Write $\Lambda_2(u) - \Lambda_1(u) = \Lambda_2(u) - \Lambda_1(\tilde{u}) \times [\Lambda_1(u)/\Lambda_1(\tilde{u})]$. A lower bound for $\Lambda_2(u)$ may be constructed:

$$\begin{aligned} \Lambda_2(u) &= \int_u^{\bar{u}_2} \frac{(1-\beta)f_2(x)}{x(1-(1-\beta)F_2(x))} dx \\ &> \frac{1}{\bar{u}_2} \int_u^{\bar{u}_2} \frac{(1-\beta)f_2(x)}{1-(1-\beta)F_2(x)} dx = \frac{1}{\bar{u}_2} \log \left[\frac{1-(1-\beta)F_2(u)}{\beta} \right]. \end{aligned}$$

Similarly, an upper bound for $\Lambda_1(\tilde{u})$ may be constructed:

$$\begin{aligned} \Lambda_1(\tilde{u}) &= \int_{\tilde{u}}^{\bar{u}_1} \frac{(1-\beta)f_1(x)}{x(1-(1-\beta)F_1(x))} dx \\ &< \frac{1}{\tilde{u}} \int_{\tilde{u}}^{\bar{u}_1} \frac{(1-\beta)f_1(x)}{1-(1-\beta)F_1(x)} dx = \frac{1}{\tilde{u}} \log \left[\frac{1-(1-\beta)F_1(\tilde{u})}{\beta} \right]. \end{aligned}$$

Employing both of these inequalities, obtain

$$\begin{aligned} \Lambda_2(u) - \Lambda_1(u) &> \frac{1}{\bar{u}_2} \log \left[\frac{1-(1-\beta)F_2(u)}{\beta} \right] - \frac{1}{\tilde{u}} \log \left[\frac{1-(1-\beta)F_1(\tilde{u})}{\beta} \right] \times \frac{\Lambda_1(u)}{\Lambda_1(\tilde{u})} \\ &= \frac{\log[1-(1-\beta)F_2(u)]}{\bar{u}_2} - \frac{\Lambda_1(u)}{\Lambda_1(\tilde{u})} \frac{\log[1-(1-\beta)F_1(\tilde{u})]}{\tilde{u}} - \log \beta \left[\frac{1}{\bar{u}_2} - \frac{\Lambda_1(u)}{\Lambda_1(\tilde{u})} \frac{1}{\tilde{u}} \right]. \end{aligned}$$

Turn now to the ratio $\Lambda_1(u)/\Lambda_1(\tilde{u})$. Since $u < \bar{u}_2 < \tilde{u}$ this satisfies

$$\frac{\Lambda_1(u)}{\Lambda_1(\tilde{u})} = 1 + \frac{\Lambda_1(u) - \Lambda_1(\tilde{u})}{\Lambda_1(\tilde{u})} \rightarrow 1 \quad \text{as } \beta \rightarrow 0.$$

To verify this claim, notice that $\Lambda_1(\tilde{u}) \rightarrow \infty$ as $\beta \rightarrow 0$ and

$$\Lambda_1(u) - \Lambda_1(\tilde{u}) = \int_u^{\tilde{u}} \frac{(1-\beta)f_1(x)}{x(1-(1-\beta)F_1(x))} dx \rightarrow \int_u^{\tilde{u}} \frac{f_1(x)}{x(1-F_1(x))} dx < \infty.$$

This means that

$$\liminf_{\beta \rightarrow 0} [\Lambda_2(u) - \Lambda_1(u)] \geq \frac{\log[1-F_2(u)]}{\bar{u}_2} - \frac{\log[1-F_1(\tilde{u})]}{\tilde{u}} + \lim_{\beta \rightarrow 0} \log \left[\frac{1}{\beta} \right] \left[\frac{1}{\bar{u}_2} - \frac{1}{\tilde{u}} \right] = \infty.$$

This completes the proof. $\square$

Lemma 5. Suppose that $\bar{u}_1 = \bar{u}_2 = \infty$. Suppose that, for some pair U_H and $U_L > 0$ satisfying $U_H \geq U_L \geq \max\{\underline{u}_1, \underline{u}_2\}$, $F_1(x) < F_2(x)$ for all $x \geq U_H$ and $f_1(x)/(1-F_1(x)) < f_2(x)/(1-F_2(x))$ for all $x \geq U_L$. Then, for all $u \geq U_L$, $\liminf_{\beta \rightarrow 0} [\Lambda_2(u) - \Lambda_1(u)] > 0$.

Thus, for this lemma, $F_1 \succ_{\text{FSD}} F_2$ and $F_1 \succ_{\text{HRD}} F_2$ in the upper tails of the distributions.

Proof. Fix $u \geq U_L$ and $U > \max\{u, U_H\}$. Integration by parts yields

$$\begin{aligned} \Lambda_2(u) - \Lambda_1(u) &= -\frac{1}{u} \log \left[\frac{1-(1-\beta)F_1(u)}{1-(1-\beta)F_2(u)} \right] + \int_u^\infty \frac{1}{x^2} \log \left[\frac{1-(1-\beta)F_1(x)}{1-(1-\beta)F_2(x)} \right] dx \\ &> -\frac{1}{u} \log \left[\frac{1-(1-\beta)F_1(u)}{1-(1-\beta)F_2(u)} \right] + \int_u^U \frac{1}{x^2} \log \left[\frac{1-(1-\beta)F_1(x)}{1-(1-\beta)F_2(x)} \right] dx \end{aligned}$$

where the inequality follows from the fact that the integrand of the second term on the right hand side of the equality is strictly positive for all $x \geq U_H$ and hence all $x \geq U$. For the terms on the right-hand side of the inequality, take limits as $\beta \rightarrow 0$. First,

$$\frac{1}{u} \log \left[\frac{1 - (1 - \beta)F_1(u)}{1 - (1 - \beta)F_2(u)} \right] \rightarrow \frac{1}{u} \log \left[\frac{1 - F_1(u)}{1 - F_2(u)} \right],$$

and for the second term,

$$\begin{aligned} \int_u^U \frac{1}{x^2} \log \left[\frac{1 - (1 - \beta)F_1(x)}{1 - (1 - \beta)F_2(x)} \right] dx &\longrightarrow \int_u^U \frac{1}{x^2} \log \left[\frac{1 - F_1(x)}{1 - F_2(x)} \right] dx \\ &> \log \left[\frac{1 - F_1(u)}{1 - F_2(u)} \right] \int_u^U \frac{1}{x^2} dx = \log \left[\frac{1 - F_1(u)}{1 - F_2(u)} \right] \left[\frac{1}{u} - \frac{1}{U} \right]. \end{aligned}$$

To justify the limit as $\beta \rightarrow 0$, notice that the integrand is continuous in x and β and converges to a continuous limit. The range of integration is the compact set $[u, U]$, and the order of limit and integration may be interchanged. The inequality follows from the fact that $(1 - F_1(x))/(1 - F_2(x))$ is strictly increasing for all $x \geq u$, following from the assumption that the hazard of F_1 is strictly lower than that of F_2 for such x . Conclude that, for fixed u and any U chosen sufficiently large,

$$\liminf_{\beta \rightarrow 0} [\Lambda_2(u) - \Lambda_1(u)] > -\frac{1}{U} \log \left[\frac{1 - F_1(u)}{1 - F_2(u)} \right].$$

Of course, this holds for all larger U as well. Hence, allowing $U \rightarrow \infty$, conclude that

$$\liminf_{\beta \rightarrow 0} [\Lambda_2(u) - \Lambda_1(u)] \geq 0.$$

Notice that this weak inequality continues to hold for $\tilde{u} > u$. Of course,

$$\Lambda_2(u) - \Lambda_1(u) = [\Lambda_2(\tilde{u}) - \Lambda_1(\tilde{u})] + \int_u^{\tilde{u}} \frac{1}{x} \left[\frac{(1 - \beta)f_2(x)}{1 - (1 - \beta)F_2(x)} - \frac{(1 - \beta)f_1(x)}{1 - (1 - \beta)F_1(x)} \right] dx,$$

and the second term is strictly positive in the limit as $\beta \rightarrow 0$. Hence the weak inequality derived above may be replaced with a strict inequality. This concludes the proof. $\square$

Lemma 6. *Suppose that Player 1 is stochastically stronger than Player 2, and fix $u > 0$ satisfying $\max\{\bar{u}_1, \bar{u}_2\} > u \geq \min\{\underline{u}_1, \underline{u}_2\}$. Then $\lim_{\beta \rightarrow 0} [\Lambda_2(u) - \Lambda_1(u)] = \infty$.*

Proof. If $\bar{u}_1 > \bar{u}_2$, then the claim follows directly from Lemma 4. Suppose instead that $\bar{u}_1 = \bar{u}_2 = \infty$. Player 1 is stochastically stronger than Player 2 by assumption $(F_1 \succ_{\text{AHRD}} F_2)$ and hence there exists $U > 0$ such that, for all $x \geq U$ and some $\lambda > 0$,

$$\frac{f_2(x)}{1 - F_2(x)} - \frac{f_1(x)}{1 - F_1(x)} > \lambda.$$

Set $U_L = \max\{u, U\}$. Since the hazards are bounded apart in the tails, there exists some $U_H \geq U_L$ such that $F_1(x) < F_2(x)$ for all $x \geq U_H$. This continues to be true for larger U_H . Now,

$$\begin{aligned} \Lambda_2(u) - \Lambda_1(u) &= \int_u^{U_L} \frac{1}{x} \left[\frac{(1 - \beta)f_2(x)}{1 - (1 - \beta)F_2(x)} - \frac{(1 - \beta)f_1(x)}{1 - (1 - \beta)F_1(x)} \right] dx \\ &\quad + \int_{U_L}^{U_H} \frac{1}{x} \left[\frac{(1 - \beta)f_2(x)}{1 - (1 - \beta)F_2(x)} - \frac{(1 - \beta)f_1(x)}{1 - (1 - \beta)F_1(x)} \right] dx + \Lambda_2(U_H) - \Lambda_1(U_H) \end{aligned}$$

Taking limits as $\beta \rightarrow \infty$, the first term satisfies,

$$\int_u^{U_L} \frac{1}{x} \left[\frac{(1-\beta)f_2(x)}{1-(1-\beta)F_2(x)} - \frac{(1-\beta)f_1(x)}{1-(1-\beta)F_1(x)} \right] dx \rightarrow \int_u^{U_L} \frac{1}{x} \left[\frac{f_2(x)}{1-F_2(x)} - \frac{f_1(x)}{1-F_1(x)} \right] dx$$

which is a finite limit, and independent of U_H . This follows, since the integrand is continuous in x and β , converges to a well-defined and continuous limit as $\beta \rightarrow 0$, and the range of integration is a compact set. Thus the interchange of limit and integral is valid. Similarly,

$$\begin{aligned} \int_{U_L}^{U_H} \frac{1}{x} \left[\frac{(1-\beta)f_2(x)}{1-(1-\beta)F_2(x)} - \frac{(1-\beta)f_1(x)}{1-(1-\beta)F_1(x)} \right] dx &\rightarrow \int_{U_L}^{U_H} \frac{1}{x} \left[\frac{f_2(x)}{1-F_2(x)} - \frac{f_1(x)}{1-F_1(x)} \right] dx \\ &\geq \lambda \int_{U_L}^{U_H} \frac{1}{x} dx = \lambda \log \left[\frac{U_H}{U_L} \right]. \end{aligned}$$

Hence,

$$\begin{aligned} \liminf_{\beta \rightarrow 0} [\Lambda_2(u) - \Lambda_1(u)] &\geq \int_u^{U_L} \frac{1}{x} \left[\frac{f_2(x)}{1-F_2(x)} - \frac{f_1(x)}{1-F_1(x)} \right] dx \\ &\quad + \lambda \log \left[\frac{U_H}{U_L} \right] + \liminf_{\beta \rightarrow 0} [\Lambda_2(U_H) - \Lambda_1(U_H)] \end{aligned}$$

The first term is finite and independent of U_H . The third term is positive, following an application of Lemma 5. The third term may be made arbitrarily large via a sufficiently large choice of U_H . Hence $\Lambda_2(u) - \Lambda_1(u) \rightarrow \infty$ as $\beta \rightarrow 0$. $\square$

With these lemmas in hand, the proofs of Propositions 2 and 3 may proceed.

Proof of Proposition 2 for $T = \infty$. Set $U_H = U_L = u$. Since $F_1 \succ_{\text{HRD}} F_2$, the conditions of Lemma 5 are met. Hence $\Lambda_2(u) - \Lambda_1(u) > 0$ for β sufficiently small. Following the arguments given in the proof for $T < \infty$, it must be the case that $t_1(u) > t_2(u)$. $\square$

Proof of Proposition 3 for $T = \infty$. The proof mirrors the third step in the proof for the case $T < \infty$. For instance, consider the case where $\min\{\underline{u}_1, \underline{u}_2\} > 0$. If $\underline{u}_1^* > \underline{u}_2^*$, then

$$\Lambda_2(\underline{u}_1^*) - \Lambda_1(\underline{u}_1^*) = \Lambda_2(\underline{u}_1^*) - \Lambda_2(\underline{u}_2^*).$$

If $\underline{u}_1^*$ remains bounded away from $\bar{u}_1$ as $\beta \rightarrow 0$, then the right hand side remains bounded, whereas the left hand side diverges to ∞ from Lemma 6. This is a contradiction, and hence $\underline{u}_1^* \rightarrow \infty$. Thus, for β sufficiently small, $\Lambda_2(\underline{u}_1^*) - \Lambda_1(\underline{u}_1^*)$ is positive. But the left-hand side is negative, hence this is a contradiction. Thus $\underline{u}_2^* \geq \underline{u}_1^* > 0$. Continuing in this manner, in an identical manner to the case $T < \infty$, it may be established that $\underline{u}_2^* \rightarrow \bar{u}_2$ and hence, for β sufficiently small, that $\underline{u}_1^* = \underline{u}_1$. This yields the first claim in Proposition 3. Other claims follow in a similar fashion. $\square$

A.3. Sensitivity Analysis. Calculations for the uniform case are based on the following lemma.

Lemma 7. *Setting $\xi = 0$ without loss of generality, for Example 1 $\Lambda_i(u)$ satisfies*

$$\Lambda_i(u) = \frac{1-\beta}{\bar{u}_i - \beta \underline{u}_i} \log \left(\frac{\bar{u}_i \times [\bar{u}_i - \beta \underline{u}_i - (1-\beta)u]}{u\beta(\bar{u}_i - \underline{u}_i)} \right) \quad (8)$$

Proof. $F_i(x) = (x - \underline{u}_i)/(\bar{u}_i - \underline{u}_i)$ and $f_i(x) = 1/(\bar{u}_i - \underline{u}_i)$ so that

$$1 - (1 - \beta)F_i(x) = \frac{\bar{u}_i - \beta\underline{u}_i - (1 - \beta)x}{\bar{u}_i - \underline{u}_i},$$

which implies that

$$\frac{(1 - \beta)f_i(x)}{x(1 - (1 - \beta)F_i(x))} = \frac{1 - \beta}{x[\bar{u}_i - \beta\underline{u}_i - (1 - \beta)x]}$$

Write (8) as

$$\Lambda_i(u) = \frac{1 - \beta}{\bar{u}_i - \beta\underline{u}_i} \left(\log \left[\frac{\bar{u}_i}{\beta(\bar{u}_i - \underline{u}_i)} \right] + \log [\bar{u}_i - \beta\underline{u}_i - (1 - \beta)u] - \log u \right),$$

and differentiate to obtain

$$\frac{\partial \Lambda_i(u)}{\partial u} = -\frac{1 - \beta}{\bar{u}_i - \beta\underline{u}_i} \left(\frac{(1 - \beta)}{\bar{u}_i - \beta\underline{u}_i - (1 - \beta)u} + \frac{1}{u} \right) = \frac{1 - \beta}{u[\bar{u}_i - \beta\underline{u}_i - (1 - \beta)u]}.$$

Furthermore, evaluate $\Lambda_i(u)$ at $u = \bar{u}_i$ to obtain $\Lambda_i(\bar{u}_i) = 0$. Hence (8) is correct. $\square$

Proof of Proposition 4. Suppose that $\bar{u}_1 > \bar{u}_2$. Then Player 1 is stochastically stronger than Player 2, and Proposition 3 applies directly.

Next consider the case where $\bar{u}_1 = \bar{u}_2 = \bar{u}$ but $\underline{u}_2 < \underline{u}_1$. Hence u_1 and u_2 share the same distribution above $\underline{u}_1$, and F_1 is really a truncation of F_2 . Write $\Lambda_i(u)$ as

$$\begin{aligned} \Lambda_i(u) &= \frac{1 - \beta}{\bar{u} - \beta\underline{u}_i} \left[\log \left(\frac{\bar{u}(\bar{u} - \beta\underline{u}_i - (1 - \beta)u)}{u(\bar{u} - \underline{u}_i)} \right) - \log \beta \right] \\ &\rightarrow \frac{1}{\bar{u}} \log \left(\frac{\bar{u}(\bar{u} - u)}{u(\bar{u} - \underline{u}_i)} \right) - \left[\frac{1 - \beta}{\bar{u} - \beta\underline{u}_i} \right] \log \beta \end{aligned}$$

so that $\Lambda_i(u)$ tends to a finite term (the first) plus a term that diverges to ∞ (the second). Hence, take the difference $\Lambda_2(u) - \Lambda_1(u)$ to obtain

$$\Lambda_2(w) - \Lambda_1(v) \rightarrow \frac{1}{\bar{u}} \log \left[\frac{(\bar{u} - w)v}{(\bar{u} - v)w} \times \frac{\bar{u} - \underline{u}_1}{\bar{u} - \underline{u}_2} \right] - \frac{(1 - \beta)(\underline{u}_2 - \underline{u}_1)\beta \log \beta}{(\bar{u} - \beta\underline{u}_1)(\bar{u} - \beta\underline{u}_2)}.$$

The last term tends to zero, since $\beta \log \beta \rightarrow 0$. In equilibrium, $\Lambda_1(v(t)) = \Lambda_2(w(t))$. Hence

$$\frac{w(t)}{\bar{u} - w(t)} = \frac{v(t)}{\bar{u} - v(t)} \times \frac{\bar{u} - \underline{u}_1}{\bar{u} - \underline{u}_2} < \frac{v(t)}{\bar{u} - v(t)} \quad \Rightarrow \quad w(t) < v(t)$$

Notice that in this example $F_1 \succ_{\text{FSD}} F_2$, and hence Player 2 must “instantly exit” with positive probability. Thus $v(0) = \underline{u}_1$. Hence

$$\frac{w(0)}{\bar{u} - w(0)} = \frac{\underline{u}_1}{\bar{u} - \underline{u}_1} \times \frac{\bar{u} - \underline{u}_1}{\bar{u} - \underline{u}_2} = \frac{\underline{u}_1}{\bar{u} - \underline{u}_2} \quad \Leftrightarrow \quad w(0) = \frac{\underline{u}_1}{1 + (\underline{u}_1 - \underline{u}_2)/\bar{u}}$$

which completes the proof. $\square$

Next, keeping Lemma 7 in hand, with $\underline{u}_1 = \underline{u}_2 = 0$, $\Lambda_i(u)$ reduces to

$$\Lambda_i(u) = \frac{1 - \beta}{\bar{u}_i} \log \left(\frac{\bar{u}_i - (1 - \beta)u}{\beta u} \right).$$

Via Proposition 2, $v(t)$ may be solved in terms of $w(t)$. Simple algebra reveals that

$$\frac{v(t)}{\bar{u}_1} = \left[(1 - \beta) + \beta \left(\frac{\bar{u}_2 - (1 - \beta)w(t)}{\beta w(t)} \right)^{\bar{u}_1/\bar{u}_2} \right]^{-1}.$$

This is the expression used for the generation of the first pane of Figure 1. Similarly, to construct the second pane, notice that the value of $u_1 = v(t)$ will be fixed, and hence

$$\frac{w(t)}{\bar{u}_2} = \left[(1 - \beta) + \beta \left(\frac{\bar{u}_1 - (1 - \beta)v(t)}{\beta v(t)} \right)^{\bar{u}_2/\bar{u}_1} \right]^{-1}.$$

$w(t)/\bar{u}_2$ is the probability that Player 2 does not fight until time t in equilibrium, and hence is the probability that Player 1 actually wins. For the exponential distributions (Example 2), the following lemma pins down the solutions—the proof follows from simple algebra.

Lemma 8. *Setting $\xi = \beta = 0$ but $T < \infty$, for Example 2 $\Lambda_i(u) = \lambda_i \log[\bar{u}^*/u]$ and*

$$\log w(t) = \frac{\lambda_2 - \lambda_1}{\lambda_2} \log \bar{u}^* + \frac{\lambda_1}{\lambda_2} \log v(t) \quad \text{where} \quad \bar{u}^* = \frac{\sqrt{1 + 16T\lambda_1\lambda_2/(\lambda_1 + \lambda_2)} - 1}{4\lambda_1\lambda_2/(\lambda_1 + \lambda_2)}. \quad (9)$$

Proof. The first claim follows from simple integration. From $\Lambda_1(v(t)) = \Lambda_2(v(t))$ I obtain $\lambda_1 \log[\bar{u}^*/v(t)] = \lambda_2 \log[\bar{u}^*/w(t)]$, and obtain the first part of (9). Integrating up,

$$\begin{aligned} t_1(v) &= \int_0^v \frac{f_1(x)w(x; \bar{u}^*)}{1 - F_1(x)} dx = \lambda_1 \int_0^v [\bar{u}^*]^{(\lambda_2 - \lambda_1)/\lambda_2} x^{\lambda_1/\lambda_2} dx \\ &= \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} [\bar{u}^*]^{(\lambda_2 - \lambda_1)/\lambda_2} v^{(\lambda_1 + \lambda_1)/\lambda_2}, \end{aligned}$$

which evaluated at $v = \bar{u}^*$ yields $t_1(\bar{u}^*) = \lambda_1 \lambda_2 [\bar{u}^*]^2 / (\lambda_1 + \lambda_2)$. Now, $\bar{t} = t_1(\bar{u}^*)$, and also $\bar{t} = T - \bar{u}^*/2$. Combining these two equations obtain

$$\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} [\bar{u}^*]^2 + \frac{\bar{u}^*}{2} - T = 0.$$

The quadratic has a unique positive root, as given in (9). □

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Not-for-Publication Appendix

Instant Exit from the Asymmetric War of Attrition

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Lemmas 1–3 are somewhat standard, and hence their proofs are omitted from the main text. For completeness, formal proofs are collected in this not-for-publication appendix.

APPENDIX B. FURTHER OMITTED PROOFS

Lemma 1 follows from a sequence of claims that characterize the properties of equilibrium stopping rules. A first step is to show that, in a “perturbed” war of attrition, fighting is always costly.

Claim 1. *Expected fighting costs are strictly increasing in stopping times.*

Proof. Suppose that a player increases her planned stopping time. (i) If $\xi > 0$ then there is strictly positive probability that her opponent will remain in the game forever, and hence she will have to pay for this increase. (ii) If $\beta > 0$ then even a winner’s costs increase with her planned exit time. (iii) If $T < \infty$, then it is a maintained assumption (§2.4) that $\bar{u}_1 = \bar{u}_2 = \infty$. Hence each player will (endogenously) fight until T with positive probability, and the argument (i) will apply. $\square$

Optimal exit depends upon a tradeoff between costs and benefits. Since fighting is always costly (Claim 1) a player who fights for longer must have a higher valuation. Equivalently, a player’s stopping rule must be weakly increasing. Furthermore, a player will never choose to fight forever.

Claim 2. *A player’s stopping time satisfies $t_i(u) < \infty$ and is weakly increasing in her valuation.*

Proof. Fighting forever yields infinite expected costs and so $t_i(u) < \infty$. For $u_i \leq 0$ it is a dominant strategy to choose $t_i = 0$. For $u_i > 0$, $t_i(u)$ cannot be strictly decreasing. To prove this, suppose to the contrary that $t_H = t_i(u_L) > t_i(u_H) = t_L$ for $u_H > u_L > 0$. Waiting from t_L until t_H involves a strict increase in expected costs, and must be weakly outweighed by a strictly positive increase in expected benefits. This increase is strictly higher with valuation u_H . Hence, when i has a valuation u_H , she would be strictly better off playing t_H rather than t_L . This yields a contradiction. $\square$

Since a player’s stopping rule is weakly increasing, it must be continuous almost everywhere. Remaining discontinuities are, with one exception, eliminated by the following claim.

Claim 3. *An equilibrium stopping rule $t_i(u)$ has at most one discontinuity, which must involve a jump upward to an exit time of T : If there is a discontinuity at $u_i^* \in (\underline{u}_i, \bar{u}_i)$ then*

$$\lim_{u \uparrow u_i^*} t_i(u) = \bar{t}_i < T = \lim_{u \downarrow u_i^*} t_i(u)$$

Hence if $T = \infty$ (i.e. there is no time limit) then the stopping rule is continuous.

Proof. $t_i(u) = 0$ for $u \leq 0$, and hence any discontinuity must occur at some $u_i^* \geq 0$, so that $\lim_{u \uparrow u_i^*} t_i(u) = t_L < t_H = \lim_{u \downarrow u_i^*} t_i(u)$. (The limits are defined, since $t_i(u)$ is weakly increasing.)

Player $j \neq i$ will not quit in the interval (t_L, t_H) , for a small reduction in t_j would strictly reduce her expected fighting costs without reducing her chance of winning the prize.

Suppose that $t_H < T$. By definition $t_H = \lim_{u \downarrow u_i^*} t_i(u)$. Furthermore $t_i(u)$ is weakly increasing. Hence for $\varepsilon > 0$, however small, there is some $u \geq u_i^*$ close to u_i^* , satisfying $t_H \leq t_i(u) \leq t_H + \varepsilon < T$. When i has a valuation u , she increases her expected fighting costs non-negligibly by fighting to $t_i(u)$ rather than t_L . Hence there must be a non-negligible probability that $j \neq i$ exits in $[t_H, t_i(u)]$. Taking $\varepsilon \rightarrow 0$ reveals that j must exit at exactly time t_H with strictly positive probability. But this means that i will never exit at t_H , since she could wait a moment longer (since $t_H < T$) and benefit from j 's potential exit at time t_H . But this means that j could save fighting costs by reducing her stopping time from t_H , without affecting her chance of winning. A contradiction has been reached. The source of this contradiction was the supposition that there was discontinuity at u_i^* in which i 's stopping time jumped from t_L to $t_H < T$. Hence, if there is a discontinuity, it must involve a player's stopping time jumping from $t_L < T$ to $t_H = T$. Of course, stopping rules are weakly monotonic (Claim 2) and so there can be at most one such discontinuity. Furthermore, when $T = \infty$, stopping rules must be continuous everywhere, since no player will ever choose an infinite exit time (again from Claim 2). $\square$

Claim 3 establishes that a player will never exit at a particular time with positive probability.

Claim 4. *A player will not exit with positive probability at time t satisfying $0 < t < T$.*

Proof. Suppose to the contrary that $t_i(u_L) = t_i(u_H) = t \in (0, \bar{t})$ for $0 < u_L < u_H$, so that i exits at time t with probability greater than $(1 - \xi)(F_i(u_H) - F_i(u_L)) > 0$. For sufficiently small ε , $j \neq i$ will not quit in $(t - \varepsilon, t]$ since doing so is strictly dominated by remaining until just after t : the benefit to doing so is bounded away from zero, and the additional cost may be made arbitrarily small via a suitable choice of ε . But this means that i would wish to lower the exit time t , a contradiction. $\square$

Thus $t_i(u)$ is strictly increasing, except when $t_i(u) = 0$ or $t_i(u) = T$. The claims so far indicate that stopping rules are strictly increasing and continuous, save for at most a single discontinuity (when $T < \infty$) and constant portions at the beginning and end of the stopping rule. The next claim establishes the range of stopping times, when the game has no time limit (so that $T = \infty$).

Claim 5. *For $T = \infty$ and $\bar{t} > 0$, stopping rules satisfy $\lim_{u \downarrow \underline{u}_i} t_i(u) = 0$ and $\lim_{u \uparrow \bar{u}_i} t_i(u) = \bar{t}$.*

Proof. Write $\underline{t}_i = \lim_{u \downarrow \underline{u}_i} t_i(u)$. This is well defined, since $t_i(u)$ is weakly increasing (Claim 2). Suppose that $\underline{t}_i > 0$. Clearly, $j \neq i$ will never exit in $(0, \underline{t}_i)$. Since i is always prepared to incur a non-negligible fighting cost, the logic employed in the proof of Claim 3 implies that there must be a non-negligible probability that j exits at $\underline{t}_i$. This contradicts Claim 4. Hence $\underline{t}_i = 0$.

Write $\bar{t}_i = \lim_{u \uparrow \bar{u}_i} t_i(u)$. Suppose that $\bar{t}_i = 0$, so that (absent exit failure) i never fights. When $j \neq i$ has a strictly positive valuation, she will not exit at $t = 0$, since by waiting for $\varepsilon > 0$ she can benefit from the (probability $1 - \xi$) exit of i at $t = 0$. Waiting beyond this would increase her expected

costs with no benefit. It follows that, for $u > 0$, she wishes to choose $t_j(u) = \min\{t : t > 0\}$. Of course, this minimum does not exist, since $\{t : t > 0\}$ is open below.⁴⁹ Hence $\bar{t}_i > 0$.

Finally, suppose that $0 < \bar{t}_i < \bar{t}_j$. Thus, for some u , j chooses a stopping time $t_j(u) > \bar{t}_i$. She would be able to lower her exit time, and hence fighting costs, without changing her probability of winning, contradicting the optimality of $t_j(u)$. Hence $\bar{t}_1 = \bar{t}_2 = \bar{t} > 0$. $\square$

With this sequence of claims in hand, the proof of Lemma 1 may proceed.

Proof of Lemma 1. Set $T = \infty$. Claim 2 yields weak monotonicity. If $t_i(u) > 0$ for all $u \in (\underline{u}_i, \bar{u}_i)$, set $\underline{u}_i^* = \underline{u}_i$. If not, set $\underline{u}_i^* = \sup\{u \in (\underline{u}_i, \bar{u}_i) \mid t_i(u) = 0\}$. Set $\bar{u}_i^* = \bar{u}_i$. Hence Statements (1) and (4) of the lemma are true. Statement (2) follows from Claims 3–4. Statement (3) is Claim 5.

Next, set $T < \infty$. Claim 2 yields weak monotonicity. Since $T < \infty$, there is positive probability that a player has a dominant strategy to fight until T . Hence, define $\bar{u}_i^* = \inf\{u \in (\underline{u}_i, \bar{u}_i) \mid t_i(u) = T\}$. This must satisfy $\bar{u}_i^* < \bar{u}_i$. As before, if $t_i(u) > 0$ for all $u \in (\underline{u}_i, \bar{u}_i)$ then set $\underline{u}_i^* = \underline{u}_i$. If not, then set $\underline{u}_i^* = \sup\{u \in (\underline{u}_i, \bar{u}_i) \mid t_i(u) = 0\}$. Hence Statements (1) and (4) hold.

$\bar{u}_i^* > \underline{u}_i$ is now proven. If false, then i always fights until T . Her opponent's best response must be to choose either $t_j = 0$ or $t_j = T$. Since $\bar{u}_j > T/2 > \underline{u}_j$ (a consequence of the maintained assumptions), then, for some u_j^* , j chooses $t_j(u) = 0$ for $u < u_j^*$ and $t_j(u) = T$ for $u > u_j^*$. But this means that it cannot be optimal for i always to fight until T : for $u \in (\underline{u}_i, T/2)$ it is better to fight for some very small period $\varepsilon > 0$. A contradiction has been reached. Notice that the argument used here also ensures that $\bar{u}_i^* > \underline{u}_i^*$. Hence, for $u \in (\underline{u}_i^*, \bar{u}_i^*)$, i employs a stopping time $t_i(u) \in (0, T)$.

It has been shown that for $\underline{u}_i^*$ and $\bar{u}_i^*$ where $\underline{u}_i \leq \underline{u}_i^* < \bar{u}_i^* < \bar{u}_i = \infty$, $t_i(u)$ satisfies $t_i(u) = 0$ for all $u < \underline{u}_i^*$ and $t_i(u) = T$ for all $u > \bar{u}_i^*$. Also, $t_i(u)$ must be strictly increasing and continuous for $u \in (\underline{u}_i^*, \bar{u}_i^*)$, following Claims 3 and 4, which verifies Statement (2). Hence $\underline{t}_i = \lim_{u \downarrow \underline{u}_i^*} t_i(u)$ and $\bar{t}_i = \lim_{u \uparrow \bar{u}_i^*} t_i(u)$ both exist and satisfy $0 \leq \underline{t}_i < \bar{t}_i \leq T$. The argument used for Claim 5 may be recycled to establish $\underline{t}_1 = \underline{t}_2 = 0$ and $\bar{t}_1 = \bar{t}_2 = \bar{t}$. Hence Statement (3) is true. $\square$

Proof of Lemma 2. For $t \in (0, \bar{t})$ the distribution of Player 1's stopping time satisfies $G_1(t) \equiv \Pr[t_1 \leq t] = (1 - \xi)F_1(v(t))$. For valuation u_2 and $t_2 > 0$, at points of differentiability, Player 2's expected payoff is $\pi_2 = u_2 G_1(t_2) - \left[\int_0^{t_2} (t + \beta(t_2 - t)) dG_1(t) \right] - t_2(1 - G_1(t_2))$, and hence

$$\frac{\partial \pi_2}{\partial t_2} = u_2 g_1(t_2) - (1 - (1 - \beta)G_1(t_2)) = u_2(1 - \xi)f_1(v(t_2))v'(t_2) - (1 - (1 - \beta)(1 - \xi)F_1(v(t_2))).$$

Setting this derivative equal to zero, $t_2 = t$, $u_2 = w(t)$, and re-arranging, yields the desired first-order condition for Player 2. The same procedure applies to Player 1. This analysis applies at points of differentiability. The monotonicity of stopping rules ensures differentiability almost everywhere.

To prove differentiability everywhere, proceed in three steps. The first is to show the Lipschitz continuity of $G_1(t)$ on $(0, \bar{t})$. Take a closed interval $C \subseteq (0, \bar{t})$. Note that $w(t)$ is continuous, C is compact, and $w(t) > 0$ for all $t > 0$. Hence $\min_{t \in C} \{w(t)\}$ exists and is strictly positive. $1/[\min_{t \in C} \{w(t)\}]$ will be the Lipschitz constant on C . To see this, take $t_L \in C$ and $t_H \in C$ where

⁴⁹For completeness, however, suppose that j did wait an arbitrarily small period of time ε before exiting. It would no longer be optimal for i to always exit at time $t = 0$: for high enough u_i (or, if u_i is bounded above so that $\bar{u}_i < \infty$, for ε sufficiently small) she could wait for a period ε in anticipation of j 's imminent exit.

$t_H > t_L$. Consider $G_1(t_H) - G_1(t_L)$. Suppose that Player 2 has valuation $w(t_L)$ and is considering an exit at t_H rather than t_L . The additional cost of doing so (absent exit failure) is bounded above by $t_H - t_L$. The increased probability of winning is $G_1(t_H) - G_1(t_L)$. Since t_L is optimal,

$$G_1(t_H) - G_1(t_L) \leq \frac{t_H - t_L}{w(t_L)} \leq \frac{t_H - t_L}{\min_{t \in C} \{w(t)\}}.$$

Taking absolute values, it follows that $G_1(t)$ is Lipschitz continuous in t . A second step toward proving differentiability is to establish the Lipschitz continuity of $v(t)$:

$$\begin{aligned} v(t_H) - v(t_L) &= F_1^{-1} \left(\frac{G_1(t_H)}{1 - \xi} \right) - F_1^{-1} \left(\frac{G_1(t_L)}{1 - \xi} \right) \\ &\leq \frac{1}{\min_{t \in C} \{f_1(v(t))\}} \times \frac{G_1(t_H) - G_1(t_L)}{1 - \xi} \leq \frac{t_H - t_L}{(1 - \xi) \times \min_{t \in C} \{f_1(v(t))\} \times \min_{t \in C} \{w(t)\}}. \end{aligned}$$

Notice that $\min_{t \in C} \{f_1(v(t))\}$ exists and is strictly positive. This is because C is a compact set and $v(t)$ is continuous, mapping C to a compact set that excludes zero. The density f_1 is strictly positive, and hence achieves a strictly positive minimum on a compact set. Introducing the Lipschitz constant $K = [(1 - \xi) \times \min_{t \in C} \{f_1(v(t))\} \times \min_{t \in C} \{w(t)\}]^{-1}$, for all $t_H, t_L \in C$

$$|v(t_H) - v(t_L)| \leq K|t_H - t_L|.$$

Hence $v(t)$ is Lipschitz continuous on C . C was arbitrary, hence $v(t)$ is Lipschitz continuous for all $t > 0$ and is differentiable almost everywhere. The same is true for $w(t)$, from a similar argument.

For the third step in proving differentiability, note that when the derivatives exist, they satisfy (1). It follows that the inverse stopping rules $v(t)$ and $w(t)$ are integrals of Lipschitz continuous functions of t . Hence they are differentiable everywhere, by the Fundamental Theorem of Calculus.

(1) holds everywhere. To show (3), differentiate the left-hand side with respect to t_H to obtain

$$\begin{aligned} \frac{\partial}{\partial t_H} \left\{ \int_{v(t_L)}^{v(t_H)} \frac{(1 - \beta)(1 - \xi)f_1(x)}{x(1 - (1 - \xi)(1 - \beta)F_1(x))} dx \right\} &= \frac{(1 - \beta)(1 - \xi)f_1(v(t_H))v'(t_H)}{v(t_H)(1 - (1 - \xi)(1 - \beta)F_1(v(t_H)))} \\ &= \frac{1 - \beta}{v(t_H)w(t_H)}. \end{aligned}$$

The second equality uses (1). Differentiating the right-hand side of (3) yields an identical expression.

For the final part of Lemma 2, consider (3), and let $t_H \rightarrow \bar{t}$ so that $v(t_H) \rightarrow \bar{u}_1^*$ and $w(t_H) \rightarrow \bar{u}_2^*$. The argument given in the main text ensures that the integrals are well defined in the limit so long as either $\max\{\xi, \beta\} > 0$ or $T < \infty$. Set $t_L = t$ to obtain the desired result. $\square$

Proof of Proposition 3 for $T = \infty$. $\Lambda_1(u)$ and $\Lambda_2(u)$ are both strictly decreasing continuous functions of u , and hence their inverses $\Lambda_1^{-1}(\cdot)$ and $\Lambda_2^{-1}(\cdot)$ are well defined. Since $\Lambda_1(v(t)) = \Lambda(w(t))$, knowledge of $v(t)$ uniquely determines $w(t)$, via $w(t) = \Lambda_2^{-1}(\Lambda_1(v(t)))$.

Now, suppose that $\min\{\underline{u}_1, \underline{u}_2\} > 0$, so that both players' valuations are bounded away from zero. Section 2.6 demonstrates that either $v(0) = \underline{u}_1^* = \underline{u}_1$ or $w(0) = \underline{u}_2^* = \underline{u}_2$, or possibly both. Take the first possibility, where $\underline{u}_1^* = \underline{u}_1$. If $\Lambda_2(\underline{u}_2) \geq \Lambda_1(\underline{u}_1)$ then there is a unique $\underline{u}_2^* \geq \underline{u}_2$ satisfying $\underline{u}_2^* = \Lambda_2^{-1}(\Lambda_1(\underline{u}_1^*))$. Furthermore, there cannot be another equilibrium: Raising $\underline{u}_1^*$ implies (via $\Lambda_1(\underline{u}_1^*) = \Lambda_2(\underline{u}_2^*)$) a strictly higher $\underline{u}_2^*$. This would yield $\underline{u}_1^* > \underline{u}_1$ and $\underline{u}_2^* > \underline{u}_2$, in violation of

boundary condition at $t = 0$. If this first possibility fails, then $\Lambda_2(\underline{u}_2) < \Lambda_1(\underline{u}_1)$. But then set $\underline{u}_2^* = \underline{u}_2$ and $\underline{u}_1^* = \Lambda_1^{-1}(\Lambda_2(\underline{u}_2^*))$ to tie down the equilibrium. Thus, when valuations are bounded away from zero, there is a unique candidate for $\underline{u}_1^*$. Of course, when valuations extend below zero, $\underline{u}_1^* = \underline{u}_2^* = 0$. Similar arguments ensure that in other cases (e.g. $\underline{u}_2 < 0 < \underline{u}_1$) $\underline{u}_1^*$ and $\underline{u}_2^*$ are uniquely defined. Next, return to the first-order conditions of (1), and in particular

$$v'(t) = \frac{1 - (1 - \beta)(1 - \xi)F_1(v(t))}{(1 - \xi)f_1(v(t))w(t)} \Rightarrow t_1(u) = \int_{\underline{u}_1^*}^u \frac{(1 - \xi)f_1(x)\Lambda_2^{-1}(\Lambda_1(x))}{1 - (1 - \beta)(1 - \xi)F_1(x)} dx.$$

The stopping rule $t_2(u)$ may be obtained via a similar procedure. By construction, this is the only pair of stopping rules that satisfies the appropriate boundary and first-order conditions, and hence is the only candidate for a pure-strategy Bayesian Nash equilibrium. It need only be checked that each player with each possible valuation is choosing an optimal stopping time. This boils down to checking that the first-order conditions for each player do yield a global optimum. Doing so,

$$\begin{aligned} \frac{\partial \pi_1}{\partial t_1} &= u_1(1 - \xi)f_2(w(t_1))w'(t_1) - (1 - (1 - \beta)(1 - \xi)F_2(w(t_1))) < 0 \\ \Leftrightarrow u_1 &< \frac{1 - (1 - \beta)(1 - \xi)F_2(w(t_1))}{(1 - \xi)f_2(w(t_1))w'(t_1)} \Leftrightarrow u_1 < v(t_1). \end{aligned}$$

The stopping rules constructed in this proof are strictly increasing. Hence, if $t_1 > t_1(u_1)$ then $v(t_1) > u_1$, and hence $\partial \pi_1 / \partial t_1 < 0$. Thus $\pi_1(t_1)$ is uniquely maximized at $t_1(u_1)$. It follows that first order conditions yield global optima. This concludes the proof. $\square$

Proof of Lemma 3 for $T < \infty$. First consider $\max\{u_1, u_2\} < 0$, so that $\underline{u}_1^* = \underline{u}_2^* = 0$. The two key parameters are $\bar{u}^*$ and $\bar{t}$. In equilibrium, $\bar{t} \in (0, \infty)$ and $\bar{u}^* \in (0, \infty)$ satisfy

$$\bar{t} = T - \frac{\bar{u}^*}{2}. \quad (10)$$

Hence a player with valuation $u_i = \bar{u}^*$ is indifferent between exiting at $\bar{t}$ and fighting until T . The right hand side is strictly decreasing and continuous in $\bar{u}^*$, strictly positive for $\bar{u}^* = 0$, and strictly negative for $\bar{u}^* > 2T$. Next, a second equation for $\bar{t}$ may be derived by “integrating up” the differential equations. Recall that for all $t \in (0, \bar{t})$, $\Lambda_1(v(t)) = \Lambda_2(w(t))$ where $\Lambda_i(u) = \int_u^{\bar{u}^*} \frac{f_i(x)}{x(1 - F_i(x))} dx$. $\Lambda_i(u)$ is strictly increasing and continuous, satisfying $\Lambda_i(\bar{u}^*) = 0$ and $\Lambda_i(u) \rightarrow \infty$ as $u \rightarrow 0$. Hence, as for the case $T = \infty$, its inverse is well defined. With $\bar{u}^*$ as a parameter, I write $w(v; \bar{u}^*) = \Lambda_2^{-1}(\Lambda_1(v))$ and $v(w; \bar{u}^*) = \Lambda_1^{-1}(\Lambda_2(w))$. Hence, for $t \in (0, \bar{t})$ and hence $v(t) \in (0, \bar{u}^*)$, $w(t) = w(v(t); \bar{u}^*)$. Furthermore, for $t \in (0, \bar{t})$ in equilibrium, $w(t) = w(v(t); \bar{u}^*)$. Now, from the first-order conditions of (1), integrate up to obtain

$$\bar{t} = \tau(\bar{u}^*) \quad \text{where} \quad \tau(\bar{u}^*) = \int_0^{\bar{u}^*} \frac{f_1(v)w(v; \bar{u}^*)}{1 - F_1(v)} dv. \quad (11)$$

Observe that $\tau(0) = 0$ and $\tau(\bar{u}^*) > 0$ for $\bar{u}^* > 0$. I wish to evaluate the slope of $\tau(\bar{u}^*)$. I claim that $\tau'(\bar{u}^*) > 0$. To verify this claim, differentiate with respect to $\bar{u}^*$ to obtain

$$\tau'(\bar{u}^*) = \frac{f_1(\bar{u}^*)\bar{u}^*}{1 - F_1(\bar{u}^*)} + \int_0^{\bar{u}^*} \frac{f_1(v)w(v; \bar{u}^*)}{1 - F_1(v)} \frac{\partial w(v; \bar{u}^*)}{\partial \bar{u}^*} dv.$$

The first term is positive. Hence, the claim is true if, for all $v \in (0, \bar{u}^*)$, $\partial w(v; \bar{u}^*)/\partial \bar{u}^*$ is positive. To address this, use the properties of $w(v; \bar{u}^*)$ to obtain

$$\begin{aligned} \int_{w(v; \bar{u}^*)}^{\bar{u}^*} \frac{f_2(x)}{x(1 - F_2(x))} dx &= \int_v^{\bar{u}^*} \frac{f_1(x)}{x(1 - F_1(x))} dx, \\ \Rightarrow \frac{\partial w}{\partial \bar{u}^*} &= \frac{w(1 - F_2(w))}{f_2(w)} \left[\frac{f_2(\bar{u}^*)}{\bar{u}^*(1 - F_2(\bar{u}^*))} - \frac{f_1(\bar{u}^*)}{\bar{u}^*(1 - F_1(\bar{u}^*))} \right] \end{aligned}$$

This is positive if $F_1(\bar{u}^*) \succ_{\text{HRD}} F_2(\bar{u}^*)$ and hence $\tau'(\bar{u}^*) > 0$. If the hazard of F_1 (weakly) exceeds that of F_2 when evaluated at $\bar{u}^*$, then it is equivalent to write $\tau(\bar{u}^*)$ as

$$\tau(\bar{u}^*) = \int_0^{\bar{u}^*} \frac{f_2(w)v(w; \bar{u}^*)}{1 - F_2(w)} dw,$$

and the same approach would ensure that $\tau'(\bar{u}^*) > 0$. (10) defines $\bar{t}$ as a strictly decreasing function of $\bar{u}^*$ that moves from positive to negative. (11) defines $\bar{t}$ as a strictly increasing function of $\bar{u}^*$ that begins at zero. Together, the equations uniquely determine $\bar{u}^*$ and $\bar{t}$, and the equilibrium stopping rules may be obtained by integrating up. It may be confirmed that each player is acting optimally.

For $\min\{\underline{u}_1, \underline{u}_2\} > 0$, and other cases, similar proofs may be employed. For instance, suppose that $F_1(\bar{u}^*) \succ_{\text{HRD}} F_2(\bar{u}^*)$. When $\underline{u}_2^* > \underline{u}_2$, then

$$\tau'(\bar{u}^*) = \frac{f_1(\bar{u}^*)\bar{u}^*}{1 - F_1(\bar{u}^*)} + \int_{\underline{u}_1}^{\bar{u}^*} \frac{f_1(v)w(v; \bar{u}^*)}{1 - F_1(v)} \frac{\partial w(v; \bar{u}^*)}{\partial \bar{u}^*} dv$$

and the same proof applies. In contrast, when $\underline{u}_1^* > \underline{u}_1$

$$\tau'(\bar{u}^*) = \frac{f_1(\bar{u}^*)\bar{u}^*}{1 - F_1(\bar{u}^*)} - \frac{f_1(\underline{u}_1^*)\underline{u}_1^*}{1 - F_1(\underline{u}_1^*)} \frac{\partial \underline{u}_1^*}{\partial \bar{u}^*} + \int_{\underline{u}_1^*}^{\bar{u}^*} \frac{f_1(v)w(v; \bar{u}^*)}{1 - F_1(v)} \frac{\partial w(v; \bar{u}^*)}{\partial \bar{u}^*} dv$$

The proof continues to work so long as $\partial \underline{u}_1^*/\partial \bar{u}^* \leq 0$. But

$$\int_{\underline{u}_1^*}^{\bar{u}^*} \frac{f_1(x)}{x(1 - F_1(x))} dx = \int_{\underline{u}_2}^{\bar{u}^*} \frac{f_2(x)}{x(1 - F_2(x))} dx,$$

so that differentiating

$$\frac{\partial \underline{u}_1^*}{\partial \bar{u}^*} = -\frac{\underline{u}_1^*(1 - F_1(\underline{u}_1^*))}{\bar{u}^* f_1(\underline{u}_1^*)} \left[\frac{f_2(\bar{u}^*)}{1 - F_2(\bar{u}^*)} - \frac{f_1(\bar{u}^*)}{1 - F_1(\bar{u}^*)} \right] \leq 0.$$

Hence the result continues to hold. $\square$