Modelling Preferences in Economics

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Dedicated to my dearest Duncan,
in deference to my disgraceful disregarding of this delightful duty
in a differing document.
Your dependable discernment and dauntless drive
deserve my undeviating devotion.

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Statement of Originality

Chapter 1 is entirely my original work.

Chapter 2 makes heavy use of a Matlab climate model developed by Allen et al. (2009) and modified by Bronselaer et al. (2013). Their code forms the 'geophysical' modules of the computer model discussed. The computer modules regarding the economy and utilising Monte Carlo methods are entirely my original work, as is the chapter itself.

Chapter 3 is joint work with Paul Klemperer. As we have worked jointly on this work from the outset it is not possible to fully delineate all the contributions, but I affirm that many of the ideas, and a large part of the technical proofs, were mine.

Abstract

This thesis considers the economics of preferences in two different contexts.

First it examines damages from climate change. I argue that our ignorance of the welfare implications of higher levels of warming, as well as scientific uncertainty in precisely what might trigger these scenarios, imply that our tastes and beliefs are *incomplete* (in the sense of Galaabaatar and Karni 2013). That is, there are many 'plausible' ways to evaluate a given scenario.

In Chapter 1, then, I develop this theory, and use it to formally separate climate impacts into three sorts: those understood well, those understood badly, and those representing the worst possible scenario. I provide a generalisation of the 'dismal theorem' of Weitzman (2009a), and address the question of policy choice: prices versus quantities (cf. Weitzman, 1974).

Chapter 2 is an example of the analysis propounded in Chapter 1. I explore the sensitivity of the social cost of carbon to assumed damages from 4°C warming, to the assumed extent of CO_2 emissions, and to the modelling of the climate and carbon cycles. The analysis shows that differing prior assumptions can alter our evaluation of policy by orders of magnitude.

The second part of this thesis regards preferences for indivisible goods. In Chapter 3, which is joint work with Paul Klemperer, I introduce to this field the 'tropical hypersurface', being those prices at which an agent's demand changes. Simple geometric features of this set tell us the precise trade-offs that interest the agent. Thus we develop a new taxonomy of valuations, 'demand types'; familiar notions such as substitutes and complements are examples. Finally, we provide a necessary and sufficient condition on these 'demand types' for existence of competitive equilibrium, which implies several existing results, as well as new and quite different examples.

Contents

Chapter 1

Climate Policy with Incomplete Knowledge

Abstract

We provide a framework for modelling climate change under uncertainty, that is both more general than the seminal paper of Weitzman (2009a) and, hopefully, more amenable to policy analysis. We first justify certain assumptions on the importance of 'multi-priors' and 'multi-utilities' in evaluating damages from climate change, which give rise to a structure of incomplete preferences. Climate impacts then naturally separate into three sorts: those that are understood well; a final 'catastrophe' at the other extreme; and poorly understood damages, falling short of this final level, in between. By understanding these distinctions, we can better frame the question of how to set the level of climate policy. And the latter two terms have opposing effects on the question of whether, in the long term, price or quantity considerations should guide legislation.

1.1 Introduction

When plausible outcomes are well beyond our range of experience, and may have profound and lasting effects on society as a whole, standard models for dealing with uncertainty may not suffice.

This paper is written in the context of climate change. Extreme uncertainties in the chain of causality from emissions to welfare impacts mean that very high impact events have a sufficiently high probability that they cannot be disregarded in the analysis. The potentially dramatic impacts of climate change on water availability, ecosystems, food, coasts and health are laid out by, for example, the IPCC (2007, Figure SPM.2).

This paper provides explicit assumptions giving rise to a model of *incomplete preferences and beliefs*, suitable to such circumstances. It shows how, when this model applies, expected damages split into three qualitatively different parts. And it discusses the policy relevance of this model, and in particular the implications of the model structure for the question of whether long-term policy targets should be price- or quantity-based.

The assumptions which underlie the model are firstly that a good approximation of damages from climate change cannot be made by only considering those impacts at the low level we understand well. And secondly, we assume that, beyond some threshold level, our estimations of damages from climate change are not objective, but are instead influenced heavily by the *priors* we bring to the analysis, and by how we believe outcomes should be *valued*.[1]

These assumptions are defended by a study of the literature on climate change impacts. Concerns over the extent of uncertainty in these impacts were first raised by Weitzman (2007, 2009a) and similar points have been made recently by other writers.[2] The fundamental points are that we do not yet know how likely various 'bad outcomes' might be – even if we were to know, for example, what temperature changes we should expect – and that we cannot robustly ascribe values to future scenarios very different from our collective experience. (Even if we hypothetically could make such valuations, we must recognise that economists have not yet attempted directly to do so, instead calibrating their 'damage functions' at relatively low levels of warming.) In this context, it is misleading to give a single figure for the social cost of carbon. Any such estimate is heavily dependent on the presumptions we bring to our analysis.

We address these concerns by using a model of incomplete preferences and beliefs (see Bewley, 1986, Galaabaatar and Karni, 2013). No single valuation function and probability density function can capture our knowledge and preferences; we have instead a *set* of such functions, indexed by our priors and tastes. One policy is only robustly preferred to another if it is preferred under all possible such priors and tastes. Frequently this will leave two possible policies incomparable in these terms. Instead of imposing an arbitrary comparability, we should be aware of this issue and re-examine in this light the plausibility of our priors, and what further we can understand about our tastes.

The point is not that recognising this incompleteness tells us how to make a decision; it does not. The model instead reflects the *extent* to which any decision must rest on subjective beliefs and tastes. For we do not assume that preferences are equally 'incomplete' for any possible scenario. On the contrary, we assume that one *can* make objective judgements on risks, up to some threshold level: the more familiar scenarios are more complete.[3] Thus we break down damages into three ranges.

First there are the 'conventional' damages, for which we can sensibly make predictions and valuations: here, the 'priors' we take have little effect. Next, there are 'indeterminate damages' corresponding to climatic changes whose impacts and probabilities we cannot claim to understand well, but which are unlikely to be 'so bad that they cannot get any worse'. These are our

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[1] Throughout, 'prior' is used as in the recent literature on decision-making under uncertainty, to refer to subjectively assumed probability distributions; see Gilboa and Schmeidler (1989), Klibanoff et al. (2005), Galaabaatar and Karni (2013), etc.

[2] See e.g. Millner et al. (2013), Pindyck (2013a), Stern (2013), Heal and Millner (2014).

[3] See Karni (2014).

'indeterminate' damages.

On the other hand, there is a limit to loss that climate change might cause. We will refer to this limiting damage level as 'endgame' damages. Computing this limit requires both understanding how bad things can objectively get, and evaluating the subjective level of damages one should assign to such a scenario. Of course, there is scope for much debate – both scientific and philosophical – on the the right way to understand these issues.

Once these assumptions and interpretations are established, a 'dismal theorem' is that we cannot make good estimates of damages from climate change without understanding the latter two terms – and that the latter two terms depend on the priors that we take for the problem. Moreover, detailed analyses of the 'conventional' impacts we can expect and understand become less and less relevant as the quantity of GHGs increases, raising with it the probability of bad outcomes.

We thus remove Weitzman's 'infinity' from the analysis, but in fact (as we show) this theorem is a generalisation of his. It is also more widely applicable, to more situations of 'deep' or 'Knightian' uncertainty, and in particular it does not require assumptions about functional forms of damages and uncertainty at the very high end, but weaker assumptions on the importance of priors in estimating the welfare consequences of situations very different from current experience.

It is not *ex ante* clear which of our three terms is dominant, and indeed the answer will depend on our priors and tastes, on the final atmospheric concentration of greenhouse gases, as well as on the explicit limit on catastrophe built into the model. So our model shows explicitly that a response to this uncertainty is *not* necessarily to view the problem as one of insurance against 'rare disaster' (as argued by, e.g., Pindyck, 2013a,b). Events that are less bad, but also less unlikely, may be the relevant factor – for example, the possibility of serious welfare damage from 4°C of warming (see also Newbold and Marten, 2014).

Moreover, the relative sizes of these terms has a real policy implication, as is shown by applying the model to the 'prices versus quantities' work of Weitzman (1974). We ask whether, in setting the long-term strategy of our response to climate change, we should let the 'carbon price' or the quantity of greenhouse gases accrued in the the atmosphere guide our decision-making: what should determine the targets? We focus on the single price versus quantity comparison as it provides clear counterpoints, but it also is relevant for our analysis which focuses on the large scale and long term: a body such as the UNFCCC seems unlikely to set a nuanced final target in the spirit of Roberts and Spence (1976), or to change their policy frequently.

Whatever their relative sizes, large 'non-conventional' damage terms have the effect of increasing total damages. However, on the question of 'prices versus quantities', the two terms have opposite effects. A quantity-based mechanism is preferred when 'indeterminate damages' are the most significant; in such cases, the risk of additional pollutants is sufficiently unpalatable that one is prepared to countenance their mitigation being more expensive than expected.

However, additional emissions make no difference once the 'endgame' is reached. So when this term is significant in the analysis, and when an increase in quantities makes a significant difference to this threshold being passed, there is a relatively smaller downside to overshooting our emissions target. Hence we do not necessarily need the certainty that the quantity tool provides; we may be better off setting a carbon price and avoiding damaging the economy if emission abatement is more expensive than expected. *In extremis*, it is better to choose the price instrument even if we ignore the latter avoided risk: the possible upside of a price instrument giving rise to greater-than-expected abatement might be more significant than the downside of intended targets being missed.

Note that the effect described occurs only when increased quantities do indeed alter the probability of passing the catastrophic threshold. If we are entering the territory of significantly different climate systems, to which our geophysical models are simply not calibrated, we may have to admit that we do not really know what will happen, and that possible outcomes are spread thinly over a wide range. In such circumstances, the effect of the 'endgame' damage term ceases to dominate, and 'indeterminate' damages become relatively more significant, driving us back towards a quantity-based approach.

Moreover, it does not seem palatable to choose a price-based policy because "the risk from 'endgame' is significant". If this term is very significant in the estimation of expected damages, a natural response is to ask if the quantity of emissions allowed should have been reduced. And a lower quantity of emissions, in reducing the dominance of 'endgame' damages in the analysis, shifts favour back in the direction of the quantity mechanism.

This work originated in a desire to apply Weitzman's (2009a) 'dismal theorem' to the question of 'prices versus quantities' (Baldwin, 2010). The model has developed and moved further from Weitzman's original formulation (see also Baldwin, 2013); many other authors have simultaneously addressed the same issues.

The multiple scientific and socio-economic uncertainties in the problem of climate change are discussed in depth and brought up to date by Heal and Millner (2014). A thorough discussion of climate sensitivity (the long-term response of the environment to a doubling in greenhouse gas concentrations) is also given by Millner et al. (2013).

Millner (2013) summarises the debate in the literature on whether the formulation and 'infinity' of Weitzman (2009a) is appropriate (see Nordhaus, 2009, Horowitz and Lange, 2009, Weitzman, 2009c, Costello et al., 2010, Roe and Bauman, 2013, and others). By abstracting from the question of precise functional forms and 'infinity', this paper attempts to regain the spirit of Weitzman's original insight, leaving these debates aside. Weitzman's own subsequent work (Weitzman, 2011) also attempts to move on from some of these technicalities. His work on damages (Weitzman, 2009d, 2010) suggests new models, but his focus remains the tails (see also Weitzman, 2012, 2013a,b) whereas the model of this paper emphasises the importance of the badly-understood mid-range.

Pindyck (2013a)'s damning indictment of integrated assessment models, on the grounds that "IAM damage functions are completely made up", is in close agreement with the premise of this paper. However, his prescription, developed in more detail in Pindyck (2013b), that we must focus on scenarios of rare catastrophes to build a "case for a stringent GHG abatement policy" appears to take those damage functions as a "most likely" starting point. By rejecting such functions, certainly beyond the range for which they are calibrated, the model here illuminates that bad outcomes from plausible environmental scenarios, such as 4 degrees of warming, may neither be unlikely nor represent a 'worst-case' catastrophe, and yet may dominate the analysis. Thus, closest to the conclusions of the first parts of this paper, is the (concurrent) essay of Stern (2013).

Previous authors have applied the model of Weitzman (1974) to climate change, mostly in distinguishing between 'cap and trade' and 'carbon tax' style policy approaches: Newell and Pizer (2003), Hoel and Karp (2001, 2002) provide multi-period analyses, arguing that in the short term, a price-based mechanism is more appropriate when the effects of the pollutant are cumulative, as with climate change. Hepburn (2006) gives a detailed summary of the arguments, and Goulder and Schein (2014) provide an update. Parsons and Taschini (2013) consider the importance of understanding whether shocks are temporary or permanent for choosing between short-term price and quantity instruments. Kornek and Marschinski (2013) consider the strategic consequences of combining this model with the question of coalition formation.

Regarding longer-term strategy as we do, the 'price versus quantity' net present value analysis of Keohane (2009) showed preference for quantity-based targets. Dietz and Fankhauser (2010) also argue that deep uncertainty implies we should set quantity-based targets. We believe this is the first extension of the model to incorporate a model of deep, 'Knightian' uncertainty as well as 'indecisiveness' in tastes.

The chapter is organised as follows. In Section 1.2 we provide background on the uncertainties which make climate change such a hard problem. Section 1.3 introduces a standard model (Section 1.3.1), as well as a simplified version of Weitzman (2009a) in Section 1.3.2 and the theories of ambiguity aversion and incomplete preferences (Gilboa and Schmeidler, 1989, Klibanoff et al., 2005, Galaabaatar and Karni, 2013) in Section 1.3.3. In Section 1.4 the new model for this paper is laid out: the key assumptions are given and discussed in Section 1.4.1 and the generalised 'dismal theorem' is presented in Section 1.4.2, followed by a discussion of the relative significance of the new, 'unconventional' damage terms (Section 1.4.3) and an illustrative calibration made using the Matlab model of Chapter 2 (Section 1.4.4). Section 1.5 provides the application of this model to the question of 'prices versus quantities'. Section 1.6 concludes.

1.2 Sources of uncertainty in the damages from CO_2 emissions

There is considerable uncertainty in what might be the future damages from climate change. A wide range of aspects are covered by Heal and Millner (2014); here we summarise some key points.

One reason that climate change is such a difficult problem to address is that the path of causality from emissions to damages is relatively long. There is, roughly speaking, a five-stage chain of processes, with considerable uncertainty in each stage of the chain. The stages are as follows:

1. human activities emit greenhouse gases (GHGs);
2. emissions increase atmospheric concentrations of GHGs;
3. atmospheric concentrations of GHGs change the radiative forcing of the planet, and are added to by feedbacks from the water vapor, albedo changes, clouds and changes in the lapse rate;
4. changes in forcings and feedbacks give rise to local climatic and environmental change;
5. local changes have impacts on human lives.

In each of these stages, there is uncertainty – both in the level of the damages, and the time-scale over which they take place. The majority of economic models focus on the aggregate effect of Stage 4: average global temperature change. Although this interests us, we emphasise that it is not the deepest source of uncertainty.

What one might call the 'dismal' aspect is that very bad outcomes cannot be ruled out. Here, we distinguish two different questions: firstly the probability that an outcome in the 'very bad' range occurs; and secondly how sharply probabilities decline with increasing temperature, and whether the tail of the pdf of climate sensitivity should be considered formally 'fat'.[4] The latter is a question of functional form and limiting behaviour (see Section 1.3.2), rather than to the total probability accorded to the tail. The arguments of Weitzman (2009a) focus on this question; we consider the former as well in this paper.

Consider our identified uncertainties in these contexts. The total volume of emissions is imperfectly measured: discrepancies in data from China alone may have amounted to 5% of global emissions in 2010 (Guan et al., 2012).

Currently around 60% of GHG emissions are re-absorbed by the oceans and terrestrial ecosystems. This fraction may decrease with rising temperatures (see for example Friedlingstein et al., 2006, Knorr, 2009) and there are concerns about more severe positive feedbacks, for example from melting permafrost (see, for example Shakhova et al., 2010, Schaefer et al., 2012). Such risks are not quantified as yet.

'Climate sensitivity', the parameter highlighted by Weitzman (2009a) is the equilibrium average global temperature change resulting from a doubling of atmospheric concentrations of carbon dioxide (including the associated feedbacks).[5] Regarding the probability that climate sensitivity exceeds some 'bad' level, Solomon et al. (2007, Box 10.2, Figure 2) present a range of studies giving a probability of between 15% and 35% that climate sensitivity exceeds 4.5°C, but note

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[4] The technical meaning of 'fat' is either that not all moments $E_s[s^n]$ exist, or that all the moments exist, but the moment generating function $E_s[e^s]$ does not. An example of the former situation is the log-logistic distribution; climate sensitivity is modelled in this way by, for example, Dietz (2011). An example of the latter situation is the lognormal distribution; climate sensitivity is modelled in this way by, for example, Golub et al. (2009).

[5] More generally, it drives the logarithmic relationship between stable CO_2 concentrations and long-term temperature change.

that 'there is no well-established formal way of estimating a single PDF from the individual results'.[6] The 'fat tail' of climate sensitivity, and its importance for policy, is the subject of much debate.[7]

One way to reduce the number of uncertainties to consider is to combine stages 2 and 3, and view warming as a function simply of *cumulative emissions*. Strikingly, it appears that "the peak warming caused by a given cumulative carbon dioxide emission is better constrained than the warming response to a stabilisation scenario" (Allen et al. 2009, see also Stocker et al. 2013). Moreover, this peak warming is found to be "remarkably insensitive" to the emission pathway, and so cumulative emissions are an attractive parameter for economic theorists.

Note that climate sensitivity is not the only uncertain parameter governing the relationship between emissions and temperature pathways. The 'simple coupled carbon cycle and climate' model of Allen et al. (2009, see also Chapter 2 of this thesis) additionally allows uncertainty in the diffusion into the ocean of both carbon and surface temperature anomalies, as well as the positive feedback of additional emissions from soils, vegetation and the upper ocean. Incorporating such a suite of uncertainties gives wide spread of possible temperature changes over time.

The local and environmental consequences of a given level of global warming are also hard to predict. The damages themselves are likely to be experienced not simply as a result in a change in mean temperature; changes in the frequency and severity of extreme events, and particularly changes in patterns of precipitation, are more pertinent, but harder to predict. Attempts have been made to quantify plausible impacts on biodiversity, and show great uncertainty: Thomas et al. (2004) produce estimates of biodiversity loss ranging from 9% to 52% of species being 'committed to extinction' by 2050. The passing of 'tipping points', at which a large qualitative change takes place (such as the loss of the Indian summer monsoon the Greenland ice sheet) are intrinsically hard to model, but an attempt to rank the uncertainties of such events is made by Lenton et al. (2008).

Finally, we require an estimate of the socio-economic damages associated to a given level of global warming. Tol (2014, Table 1) lists the main studies in this area; most of these are concerned with damages at lower levels of warming, below around 2.5°C. Modelled damages at higher temperatures are purely extrapolations – with little justification of their functional form.[8] Damages from 5 degrees of warming are typically modelled in the range of 1 to 7%

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[6] One of the tightest constraints on high end probability is provided by Annan and Hargreaves (2006), who estimate that $P(s > 4.5) \approx 5\%$ by using Bayesian methods and a combination of recent and paleo-climate data. However, others argue that these methods are not relevant in this context (see Henriksson et al., 2009).

[7] Roe and Baker (2007) argue that $\mathrm{var}_S(s)$ is infinite (and so that the 'fatness' of the tail is almost as great as could be possible). However, not all are convinced of these arguments: see, for example, Hannart et al., 2009 and Urban and Keller, 2009. That the fat tail in climate sensitivity gives rise to 'dismal' climate outcomes is the heart of Weitzman (2009a)'s argument, but it is argued by Otto et al. (2013), Roe and Bauman (2013), Millner (2013) that this is not necessarily the case.

[8] A convention has developed in the literature that damages have the from γT^n where n lies (usually) between 1 and 3 and the function is calibrated using γ at a point estimate. But in fact very little is know about the actual shape of the damage function.

Nordhaus (1993) sets γ on the basis of calibration at 2.5 degrees, and n to be 2 because 'there is evidence that the impact increases non-linearly as the temperature increases, and we assume that the relationship is quadratic'. This choice of a quadratic relationship was *ex post* justified because it matches well the median estimate for 6 of global consumption (see Stern, 2006, Figure 6.2). Stern himself (Stern, 2010) describes as 'ludicrously small'; what we wish to emphasise is that such numbers are based on *almost no actual calculations*, and so the uncertainty surrounding them must be vast.[9]

Although some models incorporate uncertainty in the damage function (see for example Hope, 2011b), there have been few attempts to formally estimate a probability distribution of damages.[10] Nordhaus' (1994) survey of experts (the only study to directly consider higher levels of warming) highlights the 'vast disparities' in estimates obtained, but only the median in only one set of responses was used by Nordhaus and Boyer (2003) to calibrate their damage function.[11]

Further criticisms have been made, both of the calibration provided for the damage function at 2.5 degrees[12] and of the functional form typically associated with damages.[13] Climate change may eventually be associated with mass migration and war; such impacts are difficult to attribute precisely or estimate, but this does not mean they should be ignored in economic models.[14]

In conclusion, the absence of good data means that there is vast uncertainty in the damages that may accrue from climate change, especially at higher levels. And this uncertainty is typically neither estimated nor modelled. There are of course also many uncertainties on the other side, in the cost of mitigating climate change. They are not the main focus of this work, but a summary of recent research can be found in Clarke and Jiang (2014, Section 6.3.6).

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degrees of warming obtained by Nordhaus (1994, a survey of a small number of experts which does look at higher levels of warming). Other authors follow Nordhaus in taking $n = 2$ or n uncertain but centred around 2.

[9] See also Weitzman (2011, 2012), Pindyck (2013a), Stern (2013).

[10] Hope (2011b) calibrates the probability of 'catastrophe' against Lenton et al. (2008), but takes the *level* of such a catastrophe as 25% from Nordhaus (1994), in which it featured as a question, not survey response (see Footnote 11).

[11] Three scenarios were considered, one incorporating 6 degrees of warming by the end of the century. 19 experts were asked to give the probability of a 'high-consequence' (25% consumption loss) outcome from each scenario. The median of the estimates is a 5.0% probability; this is the figure that Nordhaus and Boyer (2003) use (in an adjusted form). However, as is made clear in Figure 3 of Nordhaus (1994), the data has a strong right skew; the mean of the probability estimates is 17.5% and the range is 0.3-95%.

[12] See, for example, Freeman and Guzman (2009) and Ackerman et al. (2008, 2009), who note that it fails to account for catastrophic events, non-market costs and cross-sectoral impacts, and that there is great uncertainty in future growth and productivity, which affect estimates of damages which will principally be in the future.

[13] See in particular Pindyck (2013a). Most modellers of climate change work with a per-period damage function d_t, acting multiplicatively on utility; so if consumption is X_t and temperature change is T_t in period t, then

$$U_t(X_t, T_t) = (1 + d_t(T_t))U_t(X_t)$$

where U_t is the utility function for time t. Weitzman (2010, 2009d) disputes this assumption, on the grounds that it implies damages from climate change would be greater to a rich society than to a poor one. Sterner and Persson (2008) take this view, and consider potential impacts of climate change on relative prices. Pindyck (2012) argues that climate change should be modelled via its effect on the growth rate, rather than consumption level.

[14] US Military, in their recent review (United States of America Department of Defense, 2010), state that 'climate change could have significant geopolitical impacts around the world [...] While climate change alone does not cause conflict, it may act as an accelerant of instability or conflict'.

1.3 Models of Damages from Climate Change

A typical model of climate change considers a social welfare function

$$W(\mathbf{X}, \mathbf{q}, \theta, \phi) = \sum_{t=1}^{\infty} \beta^t U(X_t, \mathbf{q}_{s \leq t}, \theta, \phi)$$

where $\mathbf{X}$ is a global consumption pathway over time, $\mathbf{q}$ is the pathway of emissions of GHGs over time, with $\mathbf{q}_{s \leq t}$ being those components of $\mathbf{q}$ up to time t, while β is the discount factor, U is a utility function assumed to take negative values (such as greater than unity constant relative risk aversion) and θ and ϕ are realisations of random variables respectively governing uncertainty in the cost of emission reductions and damages from climate change.[15]

1.3.1 The basic model

To focus on the parts of this welfare of interest to this paper, we first assume that for any fixed cumulative emissions $q := \sum_{t \geq 1} q_t$, the precise pathway $\mathbf{q}$ of emissions is chosen to be welfare optimising.[16] The welfare function W^* subject to this constraint may thus be treated as simply dependent on cumulative emissions q. Then we may focus on welfare damages $d(\mathbf{X}, q, \phi)$ and benefits $b(\mathbf{X}, q, \theta)$ (that is, the welfare gain from undertaking polluting activity, or equivalently the cost of emission reductions) such that

$$W^*(\mathbf{X}, q, \theta, \phi) = [1 + d(\mathbf{X}, q, \phi) - b(\mathbf{X}, q, \theta)] W^*(\mathbf{X}, q, 0, 0).$$

where $\theta, \phi = 0$ respectively denote that benefits from emissions, or abatement costs, are zero.[17]

We will suppress the dependence of b and d on the consumption pathway $\mathbf{X}$ for notational convenience alone; they should be understood to depend on this and, for example, on the discount rate, in the standard way. But this notational choice allows us to treat the problem *mathematically* as if it were a one-period model (although it is, of course, assuredly not).

The principal 'choice variable' is the quantity of greenhouse gases. We typically think of cumulative emissions; as emphasised by Allen et al. (2009), modelled temperature pathways are very much more sensitive to the total amount of CO_2 emitted, than to the temporal pathway of emissions, and so this net present value presentation makes scientific sense. Similarly, statements

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[15] One might also take into account, for example, distributional, population and regional effects. These are not emphasised here for notational simplicity, but the subsequent arguments would apply if they were included.

[16] Damages from climate change may in any case be rather insensitive to this choice; Allen et al. (2009, Figure S3 of Supplementary Information) show almost indistinguishable temperature responses for a range of emission pathways with the same cumulative total.

[17] Note that damages are positive, and benefits negative, in this expression because we use a negative-valued utility function to define W. To obtain b and d in this form, one may write $d(\mathbf{X}, q, \phi) = \frac{W^*(\mathbf{X},q,0,0) - W^*(\mathbf{X},q,0,\phi)}{-W^*(\mathbf{X},q,0,0)}$ for relative damages and $b(\mathbf{X}, q, \theta, \phi) = \frac{W^*(\mathbf{X},q,\theta,\phi) - W^*(\mathbf{X},q,0,\phi)}{-W^*(\mathbf{X},q,0,0)}$ and make the additional assumption that benefits b are independent of uncertainty ϕ in damages; this seems reasonable as the greater part of any economic shift away from fossil fuels may take place in the first half of the 21st century, whereas the more serious damages will manifest later than this.

on climate policy and the cost of emission reductions tend to be framed around stabilisation concentrations of CO_2, rather than the route to such a stabilisation.

Additionally, assume that $b_1(q,\theta)$ and $b_{11}(q,\theta)$ exist and satisfy $b_1(q,\theta) > 0$ and $b_{11}(q,\theta) \leq 0$, for all (q,θ) with $\theta > 0$.[18] Assume similarly that $d_1(q,\phi)$ and $d_{11}(q,\phi)$ exist. The random variable Θ, parametrising uncertainty in the cost of emission reductions, is assumed to take values in $\mathbb{R}$, whereas the random variable Φ, parametrising uncertainty in damages, may take values in $\mathbb{R}^n$ for any fixed $n \geq 1$ (that is, Φ is a vector of n random variables). That is, uncertainty in damages is allowed to be multi-dimensional, reflecting the many issues summarised in Section 1.2.

Finally, expected (net present value) damages from a stabilisation concentration q are denoted $\mathrm{TD}(q)$, i.e. $\mathrm{TD}(q) := E_\Phi[d(q,\phi)]$.

1.3.2 The Dismal Theorem of Weitzman, 2009a

In proposing a 'dismal' theorem, Weitzman (2007, 2009a) brought the importance of fat-tailed uncertainty in climate sensitivity to the attention of climate change economics.[19] We describe a version of the 'Dismal Theorem' applied to expected damages and their derivatives.[20]

A typical damage function $d(q,\phi)$ satisfies $d_1 > 0, d_{11} \geq 0$, so damages $d(q,\phi)$ tend to infinity with q. However, there must be some finite (though admittedly very large) value to the sum of everything in this finite world, and so some bound λ on damages – a "value of statistical life" of civilisation, or of life on Earth. This anomaly does not matter in standard problems, because the probability of very high damages is usually so low; whenever this insignificance is the dominating factor, then

$$E_\Phi[d(q,\phi)\,|\,d(q,\phi) \geq \lambda]P(d(q,\phi) \geq \lambda) \to 0 \text{ as } \lambda \to \infty \tag{1.1}$$

and the expectation $\mathrm{TD}(q) = E_\Phi[d(q,\phi)]$ is finite.

However, if Φ has infinite support, there is no *ex ante* reason to believe this to hold. But we may explicitly impose some bound λ:

$$d_\lambda(q,\phi) := \begin{cases} d(q,\phi) & d(q,\phi) < \lambda \\ \lambda & d(q,\phi) \geq \lambda. \end{cases} \tag{1.2}$$

Now write $\mathrm{TD}_\lambda(q) := E_\Phi[d_\lambda(q,\phi)]$. If (1.1) fails to hold, it follows that without truncation at λ,

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[18] To distinguish derivatives from subscripts which index a function, we use the notation b_1 to denote the partial derivative with respect to the first argument, etc.

[19] The idea that CRRA utility gives rise to 'negatively infinite' expected utility in certain cases goes back to the 'St. Petersberg paradox' and was previously explored by Geweke (2001).

[20] Weitzman (2009a) works with the expectation of the 'stochastic discount factor' – that is, the marginal willingness to pay for an additional infinitesimal transfer from 'now' to 'the future', given that only this infinitesimal transfer takes place. As emphasised by Horowitz and Lange (2009) and Karp (2009), a 'dismal' result in the stochastic discount factor can be consistent with the optimal transfer from now to the future being a fraction less than one of our entire present wealth, although, as emphasised by Millner (2013), it is also consistent with the optimal transfer *being* our entire wealth. It is useful to understand this subtlety, but we do not wish to dwell on it, preferring to focus more directly on how to understand expectations of damages in these contexts.

expected damages TD(q) are infinite and that:

$$\lim_{\lambda \to \infty} \mathrm{TD}_\lambda(q) = \infty. \tag{1.3}$$

In words, Equation (1.3) states that *expected damages depend fundamentally on the bounds you place on maximal possible losses* - and on the probability of such a loss.

The "Dismal Theorem" of Weitzman (2009a) is that this is indeed the case for climate change. By considering that our modelling must be based on a finite number of observations, he concluded that the appropriate probability density function as damages become large (or, for him, as consumption approaches zero) was such that Equation (1.3) must hold.

However, the point of Equation (1.3) is not that losses are unbounded: the limit λ is assumed to exist. Instead, the point is that the expectation of losses *depends* on the value we choose for λ; they cannot be approximated by concentrating on the more moderate outcomes.

This presentation of the problem of climate change damages is highly dependent on the functional form of both damages and uncertainty at the very high end (as consumption approaches zero). There has been much discussion of valid forms to use (see Millner, 2013, for a summary) and on the question of whether the relevant probability density really has unbounded domain (see Roe and Bauman, 2013) but ultimately one may question whether an esoteric discussion of the precise way in which a probability distribution declines is the most important point to emphasise. That expected damages are highly dependent on what we view as the 'worst possible' outcome can remain true even if that outcome is strictly bounded away from zero consumption. And the sensitivity of our expectation to highly uncertain, very bad events need not be limited to those events being the limit of what is possible.

1.3.3 Ambiguity Aversion and Incomplete Preferences

Before introducing incomplete preferences to the model, we give a brief survey of recent work dealing with 'Knightian uncertainty'.

"A paradigm of rational belief should allow a distinction between assessments that are well-founded and those that are arbitrary" (Gilboa et al., 2009) – as preceding sections have discussed, this distinction which is all too relevant in questions of climate change. Thus recent models of decision theory distinguish between risk, in the form of lotteries, and subjective uncertainty over the states of the world, as well as 'indecisiveness' as to our own tastes. The typical framework encompasses a set S of 'states', and the set $\Delta(X)$ of 'lotteries' over 'outcomes' X. States are characteristics of the world which we cannot measure, whereas a lottery comes with a prescribed probability for each outcome. An 'act' is a function $h : S \to \Delta(X)$. That is, our actions will give rise to lotteries over outcomes, but which lottery we play depends on the state of the world.

The original model is Anscombe and Aumann (1963)'s model of subjective expected utility. Here, a preference relation $\succcurlyeq$ satisfies the axioms given for subjective expected utility,[21] if and

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[21] These axioms are weak order, independence, continuity, monotonicity and non-degeneracy.

only if there exists a function $u : \Delta(X) \to \mathbb{R}$, linear across probabilities and unique up to affine transformation, and an unique probability $p \in \Delta(S)$, such that $f \succcurlyeq g$ if and only if $\sum_{s\in S} p(s)u(f(s)) \geq \sum_{s\in S} p(s)u(g(s))$. (The agent is risk-averse if u is concave in outcomes). Thus, an agent must behave as if they are able to ascribe a fixed probability to each of the unknown states. It is this model which Gilboa et al. (2009) criticise.

By weakening the axioms, Gilboa and Schmeidler (1989) provide the model of maxmin expected utility.[22] A preference relation $\succcurlyeq$ satisfies this model if and only if there exists a linear utility $u : \Delta(X) \to \mathbb{R}$, as before, and a unique, non-empty closed convex set $\mathcal{P} \subset \Delta(S)$ such that the acts f are ordered according to the magnitude of $\min_{p\in\mathcal{P}} \sum_s p(s)u(f(s))$. Thus, in this model, the agent only takes into account the worst possible interpretation of the distribution of states of the world – although what is worst may vary from situation to situation.

But focusing only on the very worst state and throwing away all information about others may seem too extreme, and so Klibanoff et al. (2005) propose an alternative model. For simplicity, we describe it here as similar the model of Gilboa and Schmeidler (1989), but with a finite relevant subset $\mathcal{P} \subset \Delta(S)$, an unique probability distribution μ on $\mathcal{P}$ and an unique strictly increasing function ϕ such that acts f are ordered according to the magnitude of $\sum_{p\in\mathcal{P}} \mu(p)\phi\left(\sum_{s\in S} p(s)u(f(s))\right)$.[23] We typically expect a decision-maker to be both risk and ambiguity aversion, in which case both u and ϕ are concave. This is the model of 'smooth ambiguity aversion'.

Millner et al. (2013) apply this model to the 'climate sensitivity' parameter, but discuss the difficulty in applying it across all the uncertainties of climate change: it requires a range of possible probability density functions of damages, and some sense of the relative weights one should place on these. Such information is available in the case of the climate sensitivity parameter, but it seems harder to proceed in this way when we consider the welfare effects of higher levels of warming.

Thus, we use an alternative way to handle Knightian uncertainty: by generalising Anscombe and Aumann (1963) in a different way, abandoning the completeness axiom which requires comparability of any two acts. Bewley (1986) first introduced this idea, but as we see incompleteness of information both in *ascribing probabilities* to states of the world, and in *valuing* these outcomes, we follow the most general model, that of Galaabaatar and Karni (2013). A (strict) preference relation $\succ$ satisfies this model if there exists a nonempty *set* Π of pairs (p, u) such that

$$f \succ g \Longleftrightarrow \sum_s p(s)u(f(s)) > \sum_s p(s)u(g(s)) \text{ for all } (p,u) \in \Pi.$$

If neither the strict inequality, nor its converse, holds for all $(p, u) \in \Pi$, then the decision-maker cannot robustly provide a preference over f and g. Each p is referred to as a belief, or 'prior'; each u is a taste or utility. Thus this is a multi-prior multi-utility model of incomplete beliefs

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[22] The axiom of independence is rejected, in favour of 'weak constant independence' and 'uncertainty aversion'.

[23] The model of Klibanoff et al. (2005) give the model in the continuous case, but we present a discrete version here for simplicity.

and preferences.

The incompleteness of *beliefs* arises from our lack of knowledge as to what the *outcome* of various levels of GHGs emissions might be (see Section 1.2); under this incompleteness, different subjective judgements might give very different weights to different damage functions. Conversely, the incompleteness of *tastes* arises from not knowing how to *value* such outcomes. While society may understand well its own relative risk / inequality aversion and pure rate of time preference within well-understood scenarios, these parameters need not be revealed as constants in very different scenarios; valuations may require familiarity to become well-defined (see Karni, 2014).[24]

The contention of this paper is that, absent good information on the potential damages from higher levels of warming, we are unable to provide fully robust arguments for any one mitigation level. Instead of attempting to smooth over this by imposing ad-hoc assumptions, we should instead be explicit about this incompleteness of knowledge, and hence of preferences.

1.4 A New Model for Climate Damages

We seek here to model our failure to accurately ascribe values to outcomes, beyond a certain level. In particular, recall that the 'global damage functions' used in integrated assessment models of climate change are almost exclusively calibrated at 2.5°C of warming, or less. Those evaluations which we do have must be highly subjective, as outlined in Section 1.2, but for warming beyond this point we essentially have nothing to work with.

1.4.1 The Model Extended

We treat unknowns as 'states' rather than lotteries, following the distinction of Anscombe and Aumann (1963). We use a framework of incomplete preferences and beliefs, as in Galaabaatar and Karni (2013, see Section 1.3.3 above).[25]

We represent the set of possible beliefs by Π and tastes by Ξ. Thus, for each $\pi \in \Pi$ we have a probability density function $f_{\pi,\Phi}(\phi)$ on the set of states, and for each $\xi \in \Xi$ we have a valuation $d_\xi(q, \phi)$ of the outcome under each state. Both π and ξ are allowed to be multi-dimensional, with components distinguishing the multiple distinct deep uncertainties, and the multiple possible choices of taste – for example governing the choice of utility function and discount rate.

Additionally, we use Λ to denote the set of possible upper bounds for damages (as in Section 1.3.2). Thus, the final damage function used is the truncated $d_{\xi,\lambda}(q, \phi)$ for some $\xi \in \Xi$ and some $\lambda \in \Lambda$. This bound, this 'value of statistical life of civilisation' has a dual belief-taste nature: to

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[24] Indeed there is evidence that risk aversion is altered in the wake of extreme events; see Cameron and Shah (2013).

[25] This model is not precisely an implementation of Galaabaatar and Karni (2013) as that model has so far only been axiomatised with a discrete state and outcome space (see also Riella, 2013). The model here is a natural extension, and it seems unimportant for our purposes that it has not yet been shown to be equivalent to a set of Savage-style set of axioms. The principle remains that one may recognise a range of plausible valuations and subjective probabilities on states of the world, without retaining any further way to distinguish between them.

evaluate it, we must first consider what the worst-case scenario of climate change might *be like*, and second consider how we would *value* that scenario in welfare terms. Since additionally this parameter will play a special role in distinguishing our 'parts' of expected damages, we thus keep it distinct all other considerations in Π and Ξ.

For brevity, we refer to prior-taste combinations (π, ξ, λ) as 'priors'; they are called 'plausible' if $(\pi, \xi, \lambda) \in \Pi \times \Xi \times \Lambda$. We make additional assumptions on the coherence of what is considered plausible, as follows.

Assumption 1.4.1.

1. If $\xi, \xi' \in \Xi$ then there exists $\xi'' \in \Xi$ such that $d_{\xi''}(q, \phi) = \max\{d_{\xi}(q, \phi), d_{\xi'}(q, \phi)\}$.
2. If $\pi, \pi' \in \Pi$ then for every function $f : \mathbb{R}^n \to \mathbb{R}$ with $\min\{f_{\pi,\Phi}(\phi), f_{\pi',\Phi}(\phi)\} \leq f(\phi) \leq \max\{f_{\pi,\Phi}(\phi), f_{\pi',\Phi}(\phi)\}$ and with $\int_{\mathbb{R}^n} f(\phi) d\phi = 1$ there exists $\pi'' \in \Pi$ such that $f = f_{\pi''}$.

That is, for any two plausible damage valuations, it is plausible to believe that the worse of them pertains under any emission scenario or state; and for any two plausible sets of beliefs, one may ascribe any intermediate well-defined probability density to each state.

Thus we obtain a set of expected valuations of a choice of emission level q, one for each 'prior' (π, ξ, λ). We can only 'robustly' recommend one policy above another if it is preferred for *every* choice of (π, ξ, λ).

In many cases this will not be possible. But rather than imposing a preference in some case that it itself derived from a prior (for example, a prior over how to aggregate over or pick our priors) it could be more helpful to give the full information. It also focuses attention on the importance of understanding and narrowing our set of priors. Finally, we seek to distinguish and model the increasingly subjective nature of valuations, rather than espousing a model in which everything is presented as if it can be decided in an equally objective fashion.

Our λ represents a worst-case, as used by Weitzman (2009a), but it need not represent exactly a civilisation-wide 'value of statistical life': perhaps even the most extreme climate change will not end civilisation; perhaps, even after civilisation ends, there continue to be changes which we currently value as 'worse'. Instead, λ should be found from considering the following questions: 'what is the objective limit to the damages that could be accrued from climate change?'; 'what is the disutility we place on such an eventuality?'; and 'is this the same disutility as we would place now on a less extreme outcome?'. The answers to these questions will depend on value judgements such as inequality aversion and the discount rate.

Our use of explicit priors in this way follows from the situation outlined in Section 1.2: we cannot make objective estimates of the welfare costs or risks of climate damages when we consider high level damages. However, we allow that it may be possible to estimate climate damages with greater credibility when they are at a lower level: 'familiar' scenarios are more likely to correspond to complete preferences (as with Karni, 2014). Accordingly we introduce a new constant, the 'limit to conventional damages', as well as clarifying the model in the new context:

The limit to conventional damages is l, a known constant; assume $l \leq \lambda$ for all $\lambda \in \Lambda$.

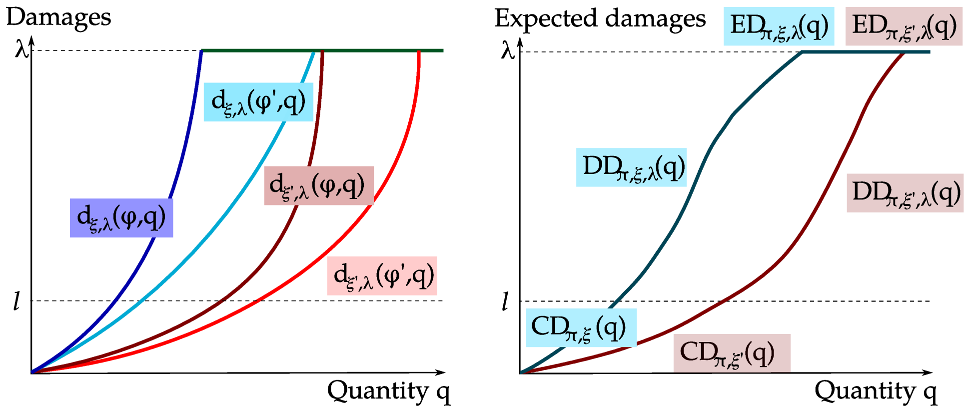
The state of the world is denoted by the previously used 'random variable' Φ.

Damages from climate change are now denoted $d_{\xi,\lambda}(q, \phi)$. The functional form of $d_{\xi,\lambda}$ depends on ξ, and it is truncated at λ as in Section 1.3.2.

A pdf on the state space induced by a prior π, is denoted $f_{\pi,\Phi}(\phi)$. Its functional form again depends on π.

Total expected damages will henceforth be written $\mathrm{TD}_{\pi,\xi,\lambda}(q) := E_\Phi[d_{\xi,\lambda}(q, \phi) \mid \pi]$. We similarly abbreviate $P(d_{\xi,\lambda}(q, \phi) < a \mid \pi)$ as $P_\pi(d_{\xi,\lambda}(q, \phi) < a)$ for any $a \in \mathbb{R}$.

The relationship between these terms is depicted in Figure 1.1a.



(a) *Damages $d_{\xi,\lambda}(q, \phi)$ as a function of quantity q, with possible realisations of ϕ, given different values of ξ and ϕ.*

(b) *The parts of expected damages assigned to be 'conventional', 'indeterminate' or 'dismal', for two realisations of ξ.*

Figure 1.1: *Damages $d_{\xi,\lambda}(q, \phi)$ under the new model of Section 1.4.1.*

Models of decision theory (see Section 1.3.3) describe how agents should rank 'acts', which map states into lotteries over outcomes. Here, an 'act' is a choice of q. For any plausible priors (π, ξ, λ) an action of q maps any state ϕ to the 'lottery' that pays off $d_{\xi,\lambda}(q, \phi)$ with certainty.[26]

We wish to make assumptions as to the relevance of the non-uniqueness of priors for the model. First define what will be meant by a 'reasonable' approximation:

Definition 1.4.2. For any $A, B \in \mathbb{R}_+$, with $B > A$, and any $K > 0$, say that A *is a K-good approximation of* B if $B - A \leq KA$ and A *is not a K-good approximation of* B if $B - A > KA$.

Now,

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[26] We could incorporate lotteries in addition by splitting the m components of ϕ into a 'state' part ψ and a 'lottery' part χ, so that an act q on state ψ gives rise to a lottery of payoffs $d_{\xi,\lambda}(q, \psi, \chi)$ such that the distribution of these payoffs is independent of ψ. Doing so simply imposes extra structure on $f_\pi(\phi)$ and adds complexity which seems unnecessary in this context.

Assumption 1.4.3. For our fixed l, and for some $K > 1$, assume:

1. there exist plausible π, ξ, λ such that $\mathrm{TD}_{\pi,\xi,l}(q)$ is not a K-good approximation for $\mathrm{TD}_{\pi,\xi,\lambda}(q)$;
2. there exist plausible π, ξ, λ and π', ξ', λ' such that $E[d_{\xi,\lambda}(q,\phi) \mid d_{\xi,\lambda}(q,\phi) > l]$ is not a K-good approximation for $E[d_{\xi',\lambda'}(q,\phi) \mid d_{\xi',\lambda'}(q,\phi) > l]$.

These assumptions can be jointly summarised as saying that the priors (which include our tastes as well as beliefs) *matter* for the analysis. The first says that expected damages are not well estimated by considering only those up to the level l which we understand well; the second says that, beyond the limit to our good understanding, priors are critical to estimating the damage outcome of a state.

Assumption 1.4.3 is closely connected with Weitzman's 'dismal theorem' in the form of Equation (1.3). An alternative way to frame that equation is: there exists no level of damages l such that $\mathrm{TD}_\lambda(q)$ is approximated well by $\mathrm{TD}_l(q)$ for all $\lambda > l$.[27] The full relationship is developed in Proposition 1.4.4 below.

The question, of whether or not Assumption 1.4.3 holds, can, and in general will, depend on the concentration q in question. Low levels of emissions will result in impacts which we *can* understand. This paper does not pretend to give an upper bound for such a level. However, Section 1.2 provides evidence that it does hold for *some* level.

Expected damages $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ naturally break down as the sum of damages up to l, damages between l and λ, and damages beyond λ. It is useful to have a nomenclature to distinguish between these, which we set as follows:

Conventional expected damages are expected damages up to l:

$$\mathrm{CD}_{\pi,\xi}(q) := E_\Phi[d_{\xi,\lambda}(q,\phi)\,|\,d_{\xi,\lambda}(q,\phi) < l,\ \pi]P_\pi(d_{\xi,\lambda}(q,\phi) < l).$$

Indeterminate expected damages are the non-conventional damages before the ultimate limit λ of damages is met:

$$\mathrm{DD}_{\pi,\xi,\lambda}(q) := E_\Phi[d_{\xi,\lambda}(q,\phi)\,|\,l \le d_{\xi,\lambda}(q,\phi) < \lambda,\ \pi]P_\pi(l \le d_{\xi,\lambda}(q,\phi) < \lambda).$$

Endgame expected damages are expected damages from catastrophe.

$$\mathrm{ED}_{\pi,\xi,\lambda}(q) := \lambda P_\pi(d_{\xi,\lambda}(q,\phi) \ge \lambda).$$

These terms simply represent the natural way to split expectations over the parts of Figure 1.1a; see Figure 1.1b. Thus

$$\mathrm{TD}_{\pi,\xi,\lambda}(q) \quad = \quad \mathrm{CD}_{\pi,\xi}(q) + \mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q) \tag{1.4}$$

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[27] More formally, this equivalence is seen using Cauchy convergence: the statement '$\mathrm{TD}_\lambda(q) \to \infty$ as $\lambda \to \infty$' is equivalent to 'for all $l > 0$ we know $(\mathrm{TD}_\lambda(q) - \mathrm{TD}_l(q)) \to \infty$ as $\lambda \to \infty$'.

Consider how restrictive the assumptions are, compared with Weitzman's dismal theorem. Suppose that Assumption 1.4.3.1 fails: there exists $l > 0$ such that $\mathrm{TD}_{\pi,\xi,l}(q)$ *does* form a K-good approximation for $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ for all plausible λ. If the set Λ of plausible maximal damages λ is not bounded above, we conclude that Equation (1.3) must fail – the limiting value of $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ is close to $\mathrm{TD}_{\pi,\xi,l}(q)$.

However, if the set Λ *is* bounded above, then Equation (1.3) must also fail. For if $\lambda \leq \lambda_0$, say, for all $\lambda \in \Lambda$, then $d_{\xi,\lambda}(q,\phi) \leq d_{\lambda_0}(q,\phi)$ for any $\lambda > \lambda_0$ and so $\lim_{\lambda\to\infty} \mathrm{TD}_\lambda(q) \leq \mathrm{TD}_{\lambda_0}(q)$. And very similarly, a failure of Assumption 1.4.3.2 must also imply a failure of Equation (1.3).

So Equation (1.3) fails if Λ is bounded above. But such a bound does by no means imply failure of Assumption 1.4.3; one may easily construct examples incorporating a wide but bounded range Λ of plausible upper bounds. Note also that Equation (1.3) says little explicitly about our failure to understand damages at high levels below their absolute limit. So:

Proposition 1.4.4. *'Weitzman's Dismal Theorem' in the form of Equation (1.3) is sufficient, but not necessary, for Assumption 1.4.3 to hold.* ☐

Thus Weitzman (2009a) also implicitly shares the view that our damage estimates depend on priors and tastes. Otherwise, the 'catastrophic' level λ would be objectively known *ex ante* and the limit as $\lambda \to \infty$ would not be relevant. But it seems implausible that our failure of understanding arises precisely as damage valuations reach their maximum. Thus, we have introduced the limit l to conventional damages, and explicitly model that a prior π and taste ξ determines judgements we make beyond this level.

1.4.2 A Generalised 'Dismal Theorem'

Our 'dismal theorem', which follows very directly (see Appendix 1.A.1 for details) is now:

Theorem 1.4.5 (The 'Dismal Theorem')**.** *If Assumption 1.4.3.1 holds for* (π, ξ, λ) *then* $\mathrm{CD}_{\pi,\xi}(q)$ *is not a K-good approximation of* $\mathrm{TD}_{\pi,\xi,\lambda}(q)$.

If Assumption 1.4.3.2 also holds then there exist π', ξ', λ' *such that* $\mathrm{TD}_{\pi',\xi',\lambda'}(q)$ *is not an* $(K-1)$*-good approximation of* $\mathrm{TD}_{\pi,\xi,\lambda}(q)$.

That is, our assumptions imply that the priors π, ξ and λ, and the terms $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ and $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ must be understood in order to calculate expected damages. Though π, ξ and λ have a smaller significance when we consider very small limits K on a 'good' approximation, we need not take K as small. Indeed the arguments of Section 1.2 imply that it is reasonable to assume that $K > 2$ in Assumption 1.4.3 and hence that both the priors, and the indeterminate and endgame expected damage terms, are of considerable importance. We shall do so for the remainder of this chapter.

To calculate expected damages, then, we must consider all three of 'conventional expected damages', 'indeterminate expected damages' and 'endgame expected damages'. The relative importance of the three terms will depend on the values of l, π, ξ and λ we consider legitimate, and will also depend on the quantity q in question. Rather than attempting to answer this question, we develop the implications of its answer for policy choice.

By Proposition 1.4.4, it follows that 'Weitzman's Dismal Theorem' implies Theorem 1.4.5, but moreover, that Theorem 1.4.5 holds in a much wider range of circumstances. It is not affected by scientists being able to say with complete certainty that equilibrium climate change resulting from a doubling of GHG concentrations is bounded by (say) 40°C – if we admit that our knowledge about damages runs out at a much lower level.[28] However, such a distribution is formally not fat tailed at all.

1.4.3 Relative sizes of $\mathrm{DD}_{\pi,\xi,\lambda}$ and $\mathrm{ED}_{\pi,\xi,\lambda}$

Which of the terms $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ and $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ is more important? The answer is that we do not know – even in the limit as $\lambda \to \infty$, and even when we *do* know, for example, that the more restrictive Dismal Theorem of Weitzman (2009a, our Equation 1.3) holds.

Proposition 1.4.6. *Assume expected damages* $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ *satisfy Weitzman's Dismal Theorem (Equation 1.3).*

1. *It is possible that* $\frac{\mathrm{ED}_{\pi,\xi,\lambda}(q)}{\mathrm{DD}_{\pi,\xi,\lambda}(q)} \to 0$ *as* $\lambda \to \infty$.
2. *It is possible that* $\frac{\mathrm{ED}_{\pi,\xi,\lambda}(q)}{\mathrm{DD}_{\pi,\xi,\lambda}(q)} \to \infty$ *as* $\lambda \to \infty$.
3. *Indeterminate damages* $\mathrm{DD}_{\pi,\xi,\lambda}(q) \to \infty$ *as* $\lambda \to \infty$.

Proof. See Appendix 1.A.1. ☐

So it could be the case that cost-benefit analysis is best understood as insurance against a very rare catastrophe – but it could also be the case that the badly understood, but still plausible, damages, drive our valuations. Indeed, the latter 'indeterminate' damages will always go to infinity with λ, whereas endgame damages $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ need not necessarily do so.

Note moreover that we need not be in the limiting case; even if $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ is dominating for absolutely vast values of λ, it is perfectly consistent that $\mathrm{CD}_{\pi,\xi}(q)$ or $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ dominates for the value of λ that we consider legitimate.

In particular, the fact that our Dismal Theorem holds does *not* now imply that cost-benefit analysis is best characterised as buying insurance against very rare absolute catastrophe, as is the general interpretation of Weitzman (2009a) and also argued by Pindyck (2013a,b). It may well be the case that the term $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ that dominates, and that our analysis is driven by the risk of very large damages from a temperature change of around, say, 4°C; essentially no economic analysis has directly considered such a large temperature change (see Section 1.2), but for atmospheric concentrations that are twice pre-industrial levels, this temperature lies within the 'most likely range' characterised by Solomon et al. (2007) and so cannot be characterised as 'rare'.

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[28] See Costello et al. (2010).

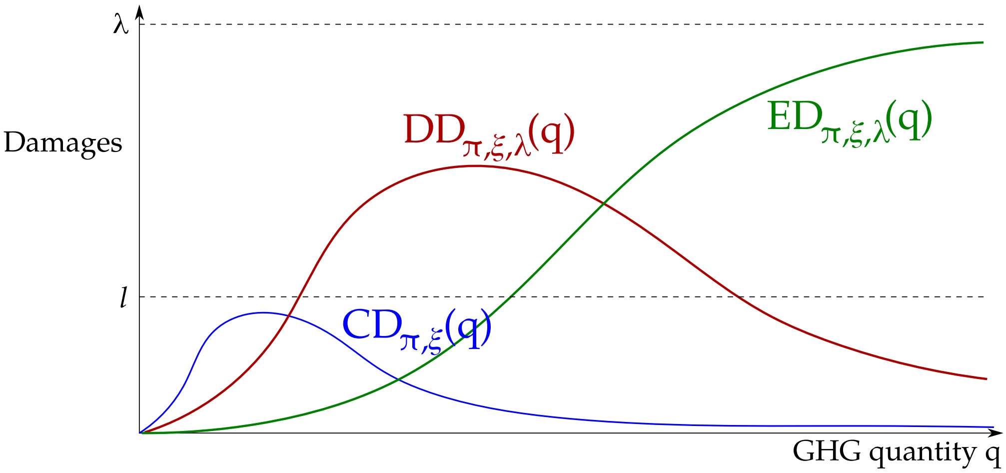

Figure 1.2: *A schematic representation of how* $\mathrm{CD}_{\pi,\xi}(q)$*,* $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ *and* $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ *vary with* q*.*

We now consider how changes in policy level, q, and in our subjective understanding of catastrophe, influence the relative importance of these terms.

First, suppose that the quantity q of GHGs increases: how does this affect the relative sizes of the three terms? While q remains small, this increases all the damage terms, though all are bounded above, by l or λ as appropriate.

However, suppose that, in the high-volume limit, the overwhelming probability is that the threshold λ has been passed. At such quantities, then, the terms $\mathrm{CD}_{\pi,\xi}(q)$ and $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ are in fact *decreasing* with q, while the 'endgame' term $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ dominates the analysis.

The evolution of $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ with q is more straightforward: it is bounded above by λ by definition, and tends to λ itself if the probability of an 'endgame' damage situation is high enough. Figure 1.2 provides a schematic representation of these relationships. We can generate examples using the Matlab model of Chapter 2; see Section 1.4.4.

Write $\pi' \geq \pi$ if the cumulative density functions of the beliefs incorporated in π, π' satisfy $F_{\pi',\Phi} \leq F_{\pi,\Phi}$ (first order stochastic dominance). Write $\xi' \geq \xi$ if tastes incorporated in ξ satisfy $d_{\xi',\lambda}(q) \geq d_{\xi,\lambda}(q)$ for all q.

Proposition 1.4.7. *Assume expected damages* $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ *satisfy Assumption 1.4.3. Then:*

1. $\mathrm{CD}_{\pi,\xi}(q)$ *and* $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ *are increasing in* q *for sufficiently small* q*.*

2. $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ *is increasing in* q, π, ξ *for all* q, π, ξ*.*

3. $\mathrm{CD}_{\pi,\xi}(q)$ *is bounded above by* l *and* $\mathrm{DD}_{\pi,\xi,\lambda}(q), \mathrm{ED}_{\pi,\xi,\lambda}(q)$ *are bounded above by* λ*.*

Suppose additionally that there exists a pathway $(q(t), \pi(t))_{t\geq 0}$ *of (weakly) increasing values of* q, π, ξ *such that* $P_{\pi(t)}(d_{\xi(t),\lambda}(q(t), \phi) \geq \lambda) \to 1$ *as* $t \to \infty$*. Then:*

4. $\mathrm{CD}_{\pi(t),\xi(t)}(q(t)), \mathrm{DD}_{\pi(t),\xi(t),\lambda}(q(t)) \to 0$ *as* $t \to \infty$*.*

5. $\mathrm{ED}_{\pi(t),\xi(t),\lambda}(q(t)) \to \lambda$ *as* $t \to \infty$.

Proof. See Appendix 1.A.1. ☐

$\mathrm{CD}_{\pi,\xi}(q)$ will in general start to decline for a smaller value of q than is the case for $\mathrm{DD}_{\pi,\xi,\lambda}(q)$: it takes a lower quantity before the declining probability of staying below the conventional threshold undermines increases in expected damage given the threshold not having been passed.

The policy implication is that detailed analyses of the impacts we *can* understand become less and less relevant as the quantity of GHGs and probability of bad outcomes increase. This observation is particularly pertinent when we consider again that damage functions for climate change are calibrated at either 1 or 2.5°C of warming (with damages at higher levels being simply extrapolations). Given recent warnings that present commitments are unlikely to constrain the global temperature increase to less than 2°C, and may leave us in 2020 unable to prevent warming that exceeds this level,[29] it seems likely that $\mathrm{CD}_{\pi,\xi}(q)$ is indeed declining in the relevant range.

1.4.4 Sample Calibration

The Matlab model of Chapter 2 provides numbers that illustrate these points. We note some of the 'priors' in using this calibration: that the model of Allen et al. (2009) simulates the climate well; that we may convert likelihoods to probability densities using a uniform prior distribution; that the economic model and damages described there are applicable. As there, we let the percentage output damages at 4°C be an explicitly varied prior, denoted α. Additionally, we set the low threshold l corresponds to a damage level $d_{\xi,\lambda}(q,\phi) = 0.01$ and that the 'endgame' catastrophic threshold λ corresponds to a damage function $d_{\xi,\lambda}(q,\phi) = 1$.

This allows us to illustrate Proposition 1.4.7, and break down damages into the terms $\mathrm{CD}_{\pi,\xi}(q)$, $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ and $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ for a range of possible priors on damages at 4 degrees: see Figure 1.3. This shows how Figure 1.2 may look in practice. We see $\mathrm{CD}_{\pi,\xi}(q)$ initially increasing before declining with q; depending on the level of assumed damages, it may dominate (Figure 1.3a) or be barely visible. Meanwhile $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ increases, on a scale determined by our prior on damages, but we only see its eventual decline when both damages and q are extremely large. $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ is negligible unless we assume that damages at 4°C are over 30–40% output, but it is only when our prior is for exceptionally high damages from 4°C, that we really see $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ take over and $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ decline. Depending on one's assumptions and the cumulative emissions relevant, any one of the terms may dominate.

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[29] See e.g. Rogelj et al. (2010a,b)

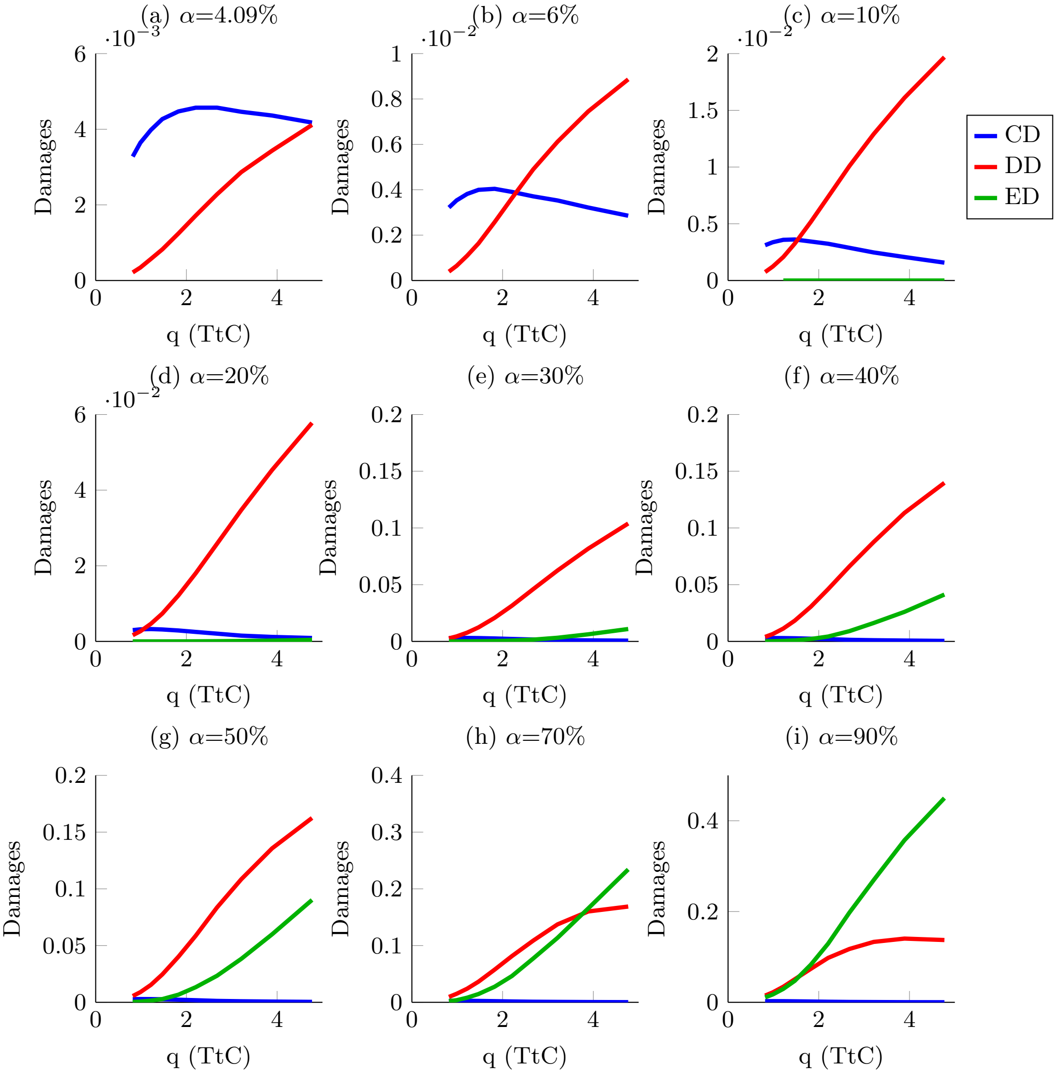

Figure 1.3: *The break down of damages into 'conventional', 'indeterminate' and 'endgame' terms, for various priors α on the damages from 4° C warming.*

1.5 Policy level and policy choice when the 'dismal theorem' holds

What does this model say about choosing our response to climate change? As ever, it would be optimal to set marginal benefits to equal marginal costs – if we knew what they were. Given the multiplicity of plausible priors, we do not expect this question to have a single answer. Certainly if one policy prescription gives a higher payoff than another under *all* plausible priors, it is robust to state that this is a better policy. More generally, considering the range of plausible marginal benefits from cutting emissions is a starting point for policy, and focuses attention more clearly on which aspects of the problem could be, and need to be, better understood.

Decisions may have to be made while better understanding is not available. This model does not guide how such decisions might be made. Rather, it makes explicit that they may *necessarily* rest on subjective judgements and cannot be objectively rationalised, and explicitly presents a subdivision of damages of differing degrees of subjectivity.

Next, the question of whether one should use the long-term carbon price, or a quantity target on emissions, may be addressed using the 'prices versus quantities' model of Weitzman (1974, see Section 1.5.1). This employs the second derivative of $\mathrm{TD}_{\pi,\xi,\lambda}(q)$. Again, multiple priors will mean multiple values for this. In this section we will thus consider the first and second derivatives of $\mathrm{TD}_{\pi,\xi,\lambda}(q)$.

Although marginal damages from climate change are in theory dependent on subsequent policy, much previous literature models or finds little sensitivity here (see, for example, Nordhaus, 2008, Table 5-1). In this case $\mathrm{TD}''_{\pi,\xi,\lambda}(q) \approx 0$. This linearity follows when one assumes that damages are approximately quadratic in temperature change (see Hope, 2006, van den Bijgaart et al., 2013). However, as discussed in Section 1.2, there is no empirical foundation to the assumption of a quadratic damage function. The sensitivity of the social cost of carbon to cumulative emissions, absent this assumption, is a key subject of discussion in Chapter 2 of this thesis; see especially Section 2.3.1.

1.5.1 The 'prices versus quantities' model (Weitzman, 1974)

We have two small generalisation of the original 'prices versus quantities' model of Weitzman (1974). Firstly, we incorporate uncertainty in benefits *ex ante*, and so working with more general forms of uncertainty in damages.[30] Secondly, we do not assume that society has chosen q to maximise benefits minus costs from emissions of GHGs. Instead, we assume that the regulator is approximately risk-neutral, which implies that for any chosen $\hat{q}$, the corresponding price to target would be $\hat{p} = E_\theta[b_1(\hat{q}, \theta)]$.[31] This $\hat{q}$ might be chosen to optimise social welfare, given the social planner's beliefs and tastes, or might be chosen according to other considerations. But the structure of incomplete beliefs and tastes means that, even if $\hat{q}$ is optimal for *some* specification of π, ξ, λ, there will be other plausible values for these priors such that $\hat{q}$ is by no means optimal; the question of whether a price- or quantity-based target is preferable can still be asked in such a case.

Following Weitzman (1974), we assume that $b_{11}(q, \theta)$ is independent of uncertainty θ in private benefits, and write it as $b''(q)$. Then:[32]

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[30] This approach is valid because the realisation of Φ plays no part in determining the outcome of either policy variable; it would not work for prices.

[31] For details on this point, see Lemma 1.A.2 and Corollary 1.A.3.

[32] We also assume also that we work in a sufficiently small neighbourhood of $\hat{q}$ that we may use quadratic approximations for costs and for expected benefits – note that this is a much weaker assumption than that $d_{\xi,\lambda}(q, \phi)$ is globally quadratic).

Theorem 1.5.1 (Weitzman 1974)**.** *Suppose that benefits $b(q,\theta)$ and damages $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ may be approximated by their second degree Taylor expansions in a neighbourhood $Q \subset \mathbb{R}$ of $\hat{q}$. Suppose that the quantity outcome of price policy, $\tilde{q}(\theta)$, lies in Q for all realisations of θ. Then the comparative advantage of prices over quantities is equal to*

$$-\frac{\mathrm{var}[b_1(\hat{q},\theta)]}{2b''(\hat{q})}(b''(\hat{q}) + \mathrm{TD}''_{\pi,\xi,\lambda}(\hat{q})). \tag{1.5}$$

A derivation of the result in this form is given in Appendix 1.A.2.[33]

Thus, the question of whether prices or quantities form a better long-term policy tool depends on understanding whether damages $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ are strongly convex, weakly convex, or even concave, in quantities.

An immediate consequence of working with expected damages in Theorem 1.5.1 is that the variance of Φ affects both marginal damages and the slope of marginal damages, effects that are lost under Weitzman's assumptions: see Example 1.A.4.

We now apply this theory to the model of Section 1.4.1. Such analysis is not concerned directly with the question of whether one should, for example, use 'cap and trade' or a carbon tax, but with long-term strategy and intended emission pathways. If a government were to decide on a cumulative emission budget over a decade, say, and implement this using a carbon tax which is regularly updated as its effects are seen, then in the context of the current paper we would regard that as a quantity-based approach.

Of course decisions may be revised over the time-scales in question. But the UNFCCC process has not proved itself fast-moving, so any decision, once taken, will not be readily revised. Moreover, if large-scale investment decisions are to be made in the short term then there must be some assurance that the policy level is being fixed for the medium to long term (see e.g. Brunner et al. 2012). So, targets applicable to the long term are what we consider here – providing an analytical framework for the discussions of Dietz and Fankhauser (2010), for example.

1.5.2 Prices versus quantities when the 'dismal theorem' holds

Now apply the model of Section 1.4.1 to the 'prices versus quantities' question. Make Assumption 1.4.3 for some $K > 1$, so that Theorem 1.4.5 (the 'dismal theorem') holds. Thus, as discussed in Section 1.4.1, we may split up expected damages into 'conventional expected damages' $\mathrm{CD}_{\pi,\xi}(q)$, 'indeterminate damages' $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ and 'endgame damages' $\mathrm{ED}_{\pi,\xi,\lambda}(q)$; the terms $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ and $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ are significant in this sum.

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[33] Note that, if we do assume that damages and benefits have a global quadratic form with uncertainty acting additively on marginal damages and benefits, then this becomes exactly the form of Weitzman (1974); the difference in sign is due to our considering quantities of a 'bad' (GHGs) rather than a 'good' (clean air).

To emphasise the pattern with the results that follow, we re-state:

$$\begin{aligned}\mathrm{TD}_{\pi,\xi,\lambda}(q) &= E_{\Phi}\big[d_{\xi,\lambda}(q,\phi) \mid d_{\xi,\lambda}(q,\phi) < \lambda,\ \pi\big] P_{\pi}(d_{\xi,\lambda}(q,\phi) < \lambda) \\ &\quad + \lambda P_{\pi}(d_{\xi,\lambda}(q,\phi) \geq \lambda) \\ &= \mathrm{CD}_{\pi,\xi}(q) + \mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q)\end{aligned} \tag{1.6}$$

The most general form of the derivatives of $\mathrm{TD}_{\pi,\xi,\lambda}$ are given as follows:

Theorem 1.5.2. *Suppose that Assumption 1.4.3 holds, along with the other assumptions of Section 1.3.1. Suppose additionally that $d_{\xi,\lambda}$ is strictly increasing and differentiable with respect to one coordinate of ϕ, which we label ϕ_n. Then:*

$$\mathrm{TD}'_{\pi,\xi,\lambda}(q) = E_{\Phi}\left[d_{\xi,\lambda,1}(q,\phi) \mid d_{\xi,\lambda}(q,\phi) \leq \lambda,\ \pi\right] P_{\pi}(d_{\xi,\lambda}(q,\phi) \leq \lambda) \tag{1.7}$$

$$\begin{aligned}\mathrm{TD}''_{\pi,\xi,\lambda}(q) &= E_{\Phi}\left[d_{\xi,\lambda,11}(q,\phi) \,|\, d_{\xi,\lambda}(q,\phi) \leq \lambda,\ \pi\right] P_{\pi}(d_{\xi,\lambda}(q,\phi) \leq \lambda) \\ &\quad - E_{\Phi_{-n}}\left[d_{\xi,\lambda,1}(q,\phi)\frac{\partial}{\partial q}P_{\pi,\tilde{\Phi}_n}(d_{\xi,\lambda}(q,\phi_{-n},\tilde{\phi}_n) \geq \lambda)\,\middle|\, d_{\xi,\lambda}(q,\phi) = \lambda,\ \pi\right]\end{aligned} \tag{1.8}$$

Proof. See Appendix 1.A.3. ☐

To understand better, it helps to work with a certain special case: all random variables ϕ acting multiplicatively on damages $d_{\xi,\lambda}$, and the probability $P_{\pi}(d_{\xi,\lambda}(q,\phi) \geq \lambda)$ declining as some power rule. Then the familiar terms $\mathrm{CD}_{\pi,\xi}(q)$, $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ and $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ re-emerge:

Corollary 1.5.3. *Suppose that both the relative response of damages to GHG quantity $\frac{d_{\xi,\lambda,1}(q,\phi)}{d_{\xi,\lambda}(q,\phi)}$ and that the relative change in probability of reaching 'endgame catastrophe' $\frac{\frac{\partial}{\partial q}P_{\pi}(d_{\xi,\lambda}(q,\phi)\geq\lambda)}{P_{\pi}(d_{\xi,\lambda}(p,\phi)\geq\lambda)}$ are independent of ϕ. Then (suppressing arguments for clarity):*

$$\mathrm{TD}'_{\pi,\xi,\lambda}(q) = \frac{d_{\xi,\lambda,1}}{d_{\xi,\lambda}}[\mathrm{CD}_{\pi,\xi} + \mathrm{DD}_{\pi,\xi,\lambda}] \tag{1.9}$$

$$\mathrm{TD}''_{\pi,\xi,\lambda}(q) = \frac{d_{\xi,\lambda,11}}{d_{\xi,\lambda}}[\mathrm{CD}_{\pi,\xi} + \mathrm{DD}_{\pi,\xi,\lambda}] - \frac{d_{\xi,\lambda,1}}{d_{\xi,\lambda}}\frac{\frac{\partial}{\partial q}P_{\pi}(d_{\xi,\lambda} \geq \lambda)}{P_{\pi}(d_{\xi,\lambda} \geq \lambda)}\mathrm{ED}_{\pi,\xi,\lambda}. \tag{1.10}$$

We discuss Theorem 1.5.2 and Corollary 1.5.3 together.

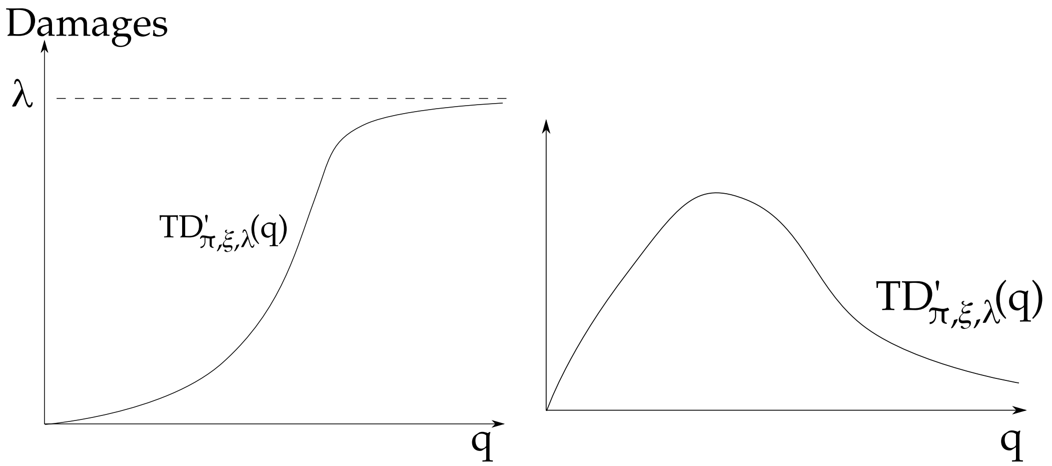
As also observed by Dietz (2011), marginal expected damages depend only on the behaviour of the damage function *up to* the 'endgame' level. Beyond that point there simply are no marginal damages. Thus we obtain (1.7) and (1.9). These terms could be split further to present expected damages in the 'conventional' range and those in the 'indeterminate' range.

The special case of (1.9) shows $\mathrm{TD}'_{\pi,\xi,\lambda}(q)$ as some multiple of the sum of conventional and indeterminate damages. So if the latter term is highly significant, due to our assumptions π, ξ, λ, then marginal expected damages will also satisfy Assumption 1.4.3 and hence our 'dismal theorem'. On the other hand, 'endgame' damages are not relevant because marginal damages have ceased from the point at which they reach λ.

The multiple acting on $\mathrm{CD}_{\pi,\xi}(q)$ and $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ is the relative response $\frac{d_{\xi,\lambda,1}}{d_{\xi,\lambda}}$ of damages to total emissions. This highlights the relatively obvious point that, if damages are more sensitive to emissions, then marginal expected damages will be greater in comparison to total damages.

Expression (1.9) also shows conventional and indeterminate damages having the same relative importance in marginal expected damages as they do in expected damages. This result is an artefact of the assumption that $\frac{d_{\xi,\lambda,1}}{d_{\xi,\lambda}}$ is independent of ϕ. Were we to assume that values of ϕ representing worse outcomes also gave rise to a greater relative responsiveness of damages to emissions, then the relative weight placed on $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ would be greater.

We turn to the second derivative. Our first observation is that ‘indeterminate’ and ‘endgame’ damages are opposing in the slope of marginal damages, and that $\mathrm{TD}_{\pi,\xi,\lambda}$ may be concave if $\mathrm{ED}_{\pi,\xi,\lambda}$ is the dominating term. The role of ‘endgame’ damages has changed dramatically as we pass from (1.6) to (1.8). This is best understood by first thinking about the shapes of the functions. See Figure 1.4a, a schematic image of how $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ varies with q in the case that large enough quantities lead to an overwhelming probability that we do indeed reach the endgame λ.[34] In this case $\mathrm{TD}_{\pi,\xi,\lambda}(q) \to \lambda$ as $q \to \infty$ (it is in general bounded above by λ).



(a) *Expected damages* $\mathrm{TD}_{\pi,\xi,\lambda}$. **(b)** *Marginal damages* $\mathrm{TD}_{\pi,\xi,\lambda}{}'(q)$

Figure 1.4: *Total and marginal expected damages, given as a function of* q.

The upper bound necessarily implies that the function must be concave for large enough q. And if, as shown, the function tends to λ in the large q limit, then marginal damages $d_{\xi,\lambda,1}(q,\phi)$ are zero if q is sufficiently large. Thus, expected marginal damages $\mathrm{TD}'_{\pi,\xi,\lambda}(q)$ must decrease with q once q is large enough, and so $\mathrm{TD}''_{\pi,\xi,\lambda}(q) < 0$ for sufficiently large q. See Figure 1.4b.

Since the upper bound λ is driving this effect, it corresponds to the situation in which $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ is significant. We turn to Equation (1.10): a large $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ term (mitigated by a multiple) does indeed bring down the curvature of total damages, whereas convexity in damages and a large $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ brings this curvature up.

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[34] For this to hold, it is sufficient to assume, as is usual, that $d_{\xi,\lambda}(q,\phi)$ is convex in the control parameter q.

To understand the these two terms, recall that a small increase in q has two potential effects on marginal damages. It increases marginal damages in the case that the catastrophic limit has not been passed – by $d_{\xi,\lambda,11}$. So a larger relative curvature in this region means a greater significance of effects of this sort. On the other hand, the opposite is possible if $d_{\xi,\lambda,11}$ is small or negative – perhaps because of diminishing sensitivity of the environment to greater accumulations of CO_2.

However, in the case that the catastrophic limit *has* been passed, there are no additional damages from a small increase in q, and so this increase brings marginal damages down. Thus an increase in the probability that we pass this threshold *decreases* marginal damages proportionately by the just-pre-catastrophe marginal damage level. And we measure this effect by looking at the effect of q on the *change in probability of passing the catastrophe point.* If increases in q make little difference to this probability then the slope of marginal damages is dominated by the positive term – and this can be the case either if the probability of passing λ is negligible, or if passing λ is a significant risk but quantities are at such a level that small changes in quantity make little difference to this risk.

The latter situation would correspond to being well in the 'tail' of the pdf of Φ – denoting a situation in which, once we are outside the realm of the well-understood, pretty much anything could happen. This may be the case for some beliefs π, and not for others.

Section 1.4.3 discussed at length the effect of our choice of priors on the relative sizes of $\mathrm{CD}_{\pi,\xi}(q)$ and $\mathrm{DD}_{\pi,\xi,\lambda}(q)$. Theorem 1.5.2 and Corollary 1.5.3 now show that the question of which term dominates is highly relevant for policy choice. We know that $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ will always dominate for large enough q, and so $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ will eventually be concave for large enough q, but the question of which dominates in the relevant region will have to be decided by a use of judgement on the plausible priors.

Priors also affect the additional terms which moderate our parts of expected damages in Corollary 1.5.3. In particular, we observe that the term $\frac{d_{\xi,\lambda,1}}{d_{\xi,\lambda}}$ is increasing in q if $d_{\xi,\lambda}$ is log convex, and decreasing in q if $d_{\xi,\lambda}$ is log concave. And if we re-write $\frac{d_{\xi,\lambda,11}}{d_{\xi,\lambda}} = \frac{d_{\xi,\lambda,1}}{d_{\xi,\lambda}} \frac{d_{\xi,\lambda,11}}{d_{\xi,\lambda,1}}$ we can also observe that $\frac{d_{\xi,\lambda,11}}{d_{\xi,\lambda,1}} > \frac{d_{\xi,\lambda,1}}{d_{\xi,\lambda}}$ if $d_{\xi,\lambda}$ is log-convex in q. So, if $d_{\xi,\lambda}$ is log-convex, the weight we place on the $\mathrm{CD}_{\pi,\xi}(q)$ and $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ in calculating $\mathrm{TD}'_{\pi,\xi,\lambda}(q)$ increases with q, relative to its importance in $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ and increases again in relevance as we move to $\mathrm{TD}''_{\pi,\xi,\lambda}(q)$. The opposite is true if $d_{\xi,\lambda}(q)$ is log-concave.

Finally, recall why a negative term increases our preference for prices. Using a price-based tool means that we may under- or overshoot our emissions target. If expected damages are in fact concave, then the upside to undershooting the target is greater than the downside to overshooting the target; the usual argument of Weitzman (1974) is reversed.

Concavities in the expected damage function $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ arise as a result of any concavity in the damage response to quantities, and as a result of the truncation of damages at the 'endgame' damage level λ. We would wish to use a price-based tool in one of three situations: the 'indeterminate' and 'endgame' damages are not especially large, and the economic downside to a quantity-based approach is significant; damages are relatively concave in quantities; or when the effect of increasing the eventual concentration is to considerably increase an already-important risk of 'endgame' damages.

So there are two fundamental reasons why we may strongly prefer prices: because damages are relatively mild, or because 'the risk of reaching final catastrophe' is high. The latter scenario, which also corresponds generally to larger values of q, seems to suggest the chosen policy target is highly questionable.

1.6 Conclusion

When there is very great uncertainty, both in what might happen, and in how society should value these outcomes, our economic models must recognise this. And the significance of poorly-understood damages may drive not only the optimal level of policy response, but also the best choice of policy tool: whether to focus on getting the right price, or the right quantity.

To summarise the argument: there is very great uncertainty in the final socio-economic damages that will result from our emissions of greenhouse gases. It follows that we do not know how to estimate their costs, and we should acknowledge this explicitly in our models. We formally model the fact that our 'priors' on the evolution of damages will determine their modelled levels, especially when damages might be large. Our tastes and beliefs are both incomplete, and hence we follow the model of Galaabaatar and Karni (2013).

We separate out damages before some 'conventional' limit, which bounds our good quality estimations, 'indeterminate' damages, whose level depends on the priors we take, and finally damages accruing at some 'endgame' damages level. In particular we are forced to consider and quantify badly-understood but plausible outcomes, and what a putative worst-case might be.

If high-level damages are sufficiently plausible then these terms will be a determining factor in the level of expected damages; one cannot assess damages or start to decide on suitable trade-offs without first considering what the effect of the 'indeterminate' or 'endgame' terms might be. And they are key not only in assessing the levels of damages, but in marginal damages and their slope.

If 'endgame' damages form the crucial term, then marginal damages may still be low. Such outcomes are 'so bad they cannot get worse' and so additional emissions do not add to the problem. However, 'indeterminate' damages do indeed add weight to marginal damages, and the greater the convexity of the damage function, the more important this effect.

We should calculate these effects across a range of plausible priors. If marginal damages could 'plausibly' be very high, society's choice of response will realistically involve and attempt to calibrate that level of 'plausibility'. As there is a limit to the quality of knowledge we can attain, so some final estimation of the right level of action might invoke the Klibanoff et al. (2005) model of smooth ambiguity aversion. However, we argue that keeping track of the range of plausible marginal values provides better information and understanding in this context than does taking such a reduced form in the first place.

Weitzman (1974) showed that ‘prices’ being better corresponds to concavities, or to convexities less extreme than the concavity of the private benefit function. Expected damages are convex in the volume of gases when damages increase steeply, when ‘endgame’ damages are not too significant in expectation, and when the probability of ‘endgame’ is only very slightly affected by a decrease the quantities emitted. It is concave if expected catastrophic damages dominate conventional ones, if quantities are very high, or if the probability of catastrophe is substantially reduced by a small decrease in emissions.

A preference for a price-based target thus corresponds to high levels of emissions, the significant risk of catastrophic damage, and the potential to do a reasonable amount about this by reducing the quantity. Such a scenario really begs the question of whether the concentration one is aiming for is indeed ‘optimal’. On the other hand, if one aims for a low concentration, convexities are more assured, and so a quantity-based tool more likely to be relevant.

The set-up of this model cannot incorporate ‘learning’ – as uncertainties are resolved, we cannot update our policy choices. In the context of climate change, the international negotiating process has proved very slow, which limits its likely ability to update decisions, and moreover, decisions taken in the short term restrict the options available later.[35] However, a dynamic version could be an interesting extension.

Appendix 1.A Proofs of results in the text

1.A.1 Proofs for Section 1.4

Proof of Theorem 1.4.5 Assumption 1.4.3.1 says that there exists $(\pi, \xi, \lambda) \in \Pi \times \Xi \times \Lambda$ such that $|\mathrm{TD}_{\pi,\xi,\lambda}(q) - \mathrm{TD}_{\pi,\xi,l}(q)| \geq K\,\mathrm{TD}_{\pi,\xi,l}(q)$. But $\mathrm{TD}_{\pi,\xi,\lambda}(q) - \mathrm{TD}_{\pi,\xi,l}(q) = \mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q) - lP_{\pi}(d_{\xi,\lambda}(q,\phi) \geq l)$ so $|\mathrm{TD}_{\pi,\xi,\lambda}(q) - \mathrm{CD}_{\pi,\xi}(q)| = \mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q) \geq K\,\mathrm{TD}_{\pi,\xi,l}(q) \geq K\,\mathrm{CD}_{\pi,\xi}(q)$, as required.

Next, if Assumption 1.4.3.2 also holds, then we can choose π, π', ξ, ξ' and λ, λ' so that $E[d_{\xi',\lambda'}(q,\phi) | d_{\xi',\lambda'}(q,\phi) > l, \pi'] > (1+K)E[d_{\xi,\lambda}(q,\phi) | d_{\xi,\lambda}(q,\phi) > l, \pi]$. Moreover, by Assumption 1.4.1, we can assume w.l.o.g. that $d_{\xi',\lambda'}(q,\phi) > d_{\xi',\lambda'}(q,\phi)$ for all (q,ϕ) and that $P_{\pi'}(d_{\xi',\lambda'} > l) > P_{\pi}(d_{\xi,\lambda} > l)$. Then, since $\mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q) = E[d_{\xi,\lambda}(q,\phi) | d_{\xi,\lambda}(q,\phi) > l, \pi]P_{\pi}(d_{\xi,\lambda} > l)$, and similarly for (π', ξ', λ'), we know

$$\mathrm{DD}_{\pi',\xi',\lambda'}(q) + \mathrm{ED}_{\pi',\xi',\lambda'}(q) > (1+K)\left(\mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q)\right).$$

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[35] See, for example, Rogelj et al. (2010a)

Now,

$$\begin{aligned}
&\mathrm{TD}_{\pi',\xi',\lambda'}(q) - \mathrm{TD}_{\pi,\xi,\lambda}(q)\\
&> \mathrm{DD}_{\pi',\xi',\lambda'}(q) + \mathrm{ED}_{\pi',\xi',\lambda'}(q) - (\mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q) + \mathrm{CD}_{\pi,\xi}(q))\\
&> K(\mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q)) - \mathrm{CD}_{\pi,\xi}(q)\\
&= (K-1)(\mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q)) + \mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q) - \mathrm{CD}_{\pi,\xi}(q)\\
&\geq (K-1)(\mathrm{DD}_{\pi,\xi,\lambda}(q) + \mathrm{ED}_{\pi,\xi,\lambda}(q)) + K\mathrm{CD}_{\pi,\xi}(q) - \mathrm{CD}_{\pi,\xi}(q)\\
&= (K-1)\mathrm{TD}_{\pi,\xi,\lambda}(q)
\end{aligned}$$

where we use the first part of the theorem in the penultimate step. □

Proof of Proposition 1.4.7.

1. Note that $\mathrm{CD}_{\pi,\xi}(0) = 0$ and that $\mathrm{CD}_{\pi,\xi}(q) \geq 0$ for $q > 0$. Similarly note $\mathrm{DD}_{\pi,\xi,\lambda}(0) = 0$ and $\mathrm{DD}_{\pi,\xi,\lambda}(q) \geq 0$ for $q > 0$.

2. Since $\mathrm{ED}_{\pi,\xi,\lambda}(q) = \lambda P_\pi(d_{\xi,\lambda}(q,\phi) \geq \lambda)$ and since $d_{\xi,\lambda\,1} \geq 0$ it follows that $P_\pi(d_{\xi,\lambda}(q,\phi) \geq \lambda)$ is weakly increasing with q, ξ and π.

3. Recall that $\mathrm{CD}_{\pi,\xi}(q) = E_\Phi[d_{\xi,\lambda}(q,\phi) \,|\, d_{\xi,\lambda}(q,\phi) < l]P_\pi(d_{\xi,\lambda}(q,\phi) < l)$ and so $\mathrm{CD}_{\pi,\xi}(q) \leq lP_\pi(d_{\xi,\lambda}(q,\phi) < l) \leq l$. Similarly, $\mathrm{DD}_{\pi,\xi,\lambda}(q) = E_\Phi[d_{\xi,\lambda}(q,\phi) \,|\, l \leq d_{\xi,\lambda}(q,\phi) < \lambda]P_\pi(l \leq d_{\xi,\lambda}(q,\phi) \leq \lambda) \leq \lambda$ and $\mathrm{ED}_{\pi,\xi,\lambda}(q) = \lambda P_\pi(d_{\xi,\lambda}(q,\phi) \geq \lambda) \leq \lambda$.

4. Tighter bounds on $\mathrm{CD}_{\pi,\xi}(q)$ and $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ are given by $lP_\pi(d_{\xi,\lambda}(q,\phi) < l))$ and $\lambda P_\pi(l \leq d_{\xi,\lambda}(q,\phi) < \lambda)$ respectively. It is clear that if $P_{\pi(t)}(d_{\xi,\lambda}(q(t),\phi) \geq \lambda) \to 1$ then these $\to 0$. 5 is clear. □

Proof of Proposition 1.4.6.

Note that, by definition

$$\mathrm{DD}_{\pi,\xi,\lambda}(q) = E_\Phi[d_{\xi,\lambda}(q,\phi) \,|\, d_{\xi,\lambda}(q,\phi) < \lambda]P_\pi(d_{\xi,\lambda}(q,\phi) < \lambda) - \mathrm{CD}_{\pi,\xi}(q)$$

and that $\mathrm{CD}_{\pi,\xi}(q)$ is independent of λ. But

$$\lim_{\lambda\to\infty} E_\Phi[d_{\xi,\lambda}(q,\phi) \,|\, d_{\xi,\lambda}(q,\phi) < \lambda]P_\pi(d_{\xi,\lambda}(q,\phi) < \lambda) = \lim_{\lambda\to\infty} \mathrm{TD}_{\pi,\xi,\lambda}(q).$$

So if $\lim_{\lambda\to\infty} \mathrm{TD}_{\pi,\xi,\lambda}(q) = \infty$ then the same is true of $\mathrm{DD}_{\pi,\xi,\lambda}(q)$.

We demonstrate that either of the dismal terms may dominate in Example 1.A.1. □

Example 1.A.1. Suppose that Φ takes values in $\mathbb{R}$. Assume that $d_{\xi,\lambda}(q,\phi) = \beta(q\phi)^r$ for some $r > 0$ and some $\beta \in \mathbb{R}_{>0}$. Write $\phi_l := \left(\frac{l}{q^r\beta}\right)^{1/r}$ for the realisation of ϕ giving rise to damages l, and $\phi_\lambda := \left(\frac{\lambda}{q^r\beta}\right)^{1/r}$ for the realisation of ϕ giving rise to damages λ. Our assumption π on the pdf $f_\Phi(\phi)$ is that, for ϕ large enough that $d_{\xi,\lambda}(q,\phi)$ is well above level l, then $\pi(\phi) = \alpha\phi^{-s}$ for some $\alpha \in \mathbb{R}_{>0}$.

Now we distinguish two cases:

1. $s = r + 1$;
2. $s < r + 1$.

1. Suppose $s = r + 1$. Now:

$$\begin{aligned} \mathrm{DD}_{\pi,\xi,\lambda}(q) &= \int_{\phi=\phi_l}^{\phi_\lambda} \alpha(q\phi)^{-r-1}\beta(q\phi)^r d\phi \\ &= \frac{\alpha\beta}{q}\big[\log\phi_\lambda - \log\phi_l\big] \to \infty \text{ as } \phi_\lambda \to \infty \end{aligned}$$

and moreover note that $\phi_\lambda \to \infty$ as $\lambda \to \infty$. Thus $\mathrm{TD}_{\pi,\xi,\lambda}(q) \to \infty$ as $\lambda \to \infty$; the 'Dismal Theorem' of Weitzman's formulation (Equation (1.3)) holds. However,

$$\mathrm{ED}_{\pi,\xi,\lambda}(q) = \lambda\int_{\phi_\lambda}^{\infty} \alpha(q\phi)^{-r-1}d\phi = \lambda\frac{\alpha q^{-r-1}}{r}\phi_\lambda^{-r} = \frac{\alpha\beta}{rq}$$

since by definition, $\beta T_\lambda^r = \lambda$. This, we see, is independent of λ (although $\mathrm{TD}_{\pi,\xi,\lambda}(q)$ is not). Thus, in the limit as λ becomes large, $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ dominates.

2. Now suppose $s < r + 1$ Now:

$$\mathrm{DD}_{\pi,\xi,\lambda}(q) = \int_{\phi_l}^{\phi_\lambda} \alpha(q\phi)^{-s}\beta(q\phi)^r d\phi = \frac{\alpha\beta q^{r-s}}{r-s+1}[\phi_\lambda^{r-s+1} - \phi_l^{r-s+1}].$$

However, noting that $\beta(q\phi_\lambda)^r = \lambda$, we see

$$\mathrm{ED}_{\pi,\xi,\lambda}(q) = \lambda\int_{\phi_\lambda}^{\infty} \alpha(q\phi)^{-s}d\phi = \frac{\lambda\alpha q^{-s}}{s-1}\phi_\lambda^{-s+1} = \frac{\alpha\beta q^{r-s}}{s-1}\phi_\lambda^{r-s+1}.$$

As ϕ_λ gets big, the term ϕ_l^{r-M+1} becomes negligible and so which out of $\mathrm{DD}_{\pi,\xi,\lambda}(q)$ and $\mathrm{ED}_{\pi,\xi,\lambda}(q)$ dominates is decided by which is greater out of $\frac{1}{r-s+1}$ and $\frac{1}{s-1}$. Thus, in the limit

$$\begin{aligned} \frac{\mathrm{DD}_{\pi,\xi,\lambda}(q)}{\mathrm{ED}_{\pi,\xi,\lambda}(q)} \to 0 \text{ as } \lambda \to \infty &\quad\Leftrightarrow\quad 2(s-1) < r \\ \frac{\mathrm{DD}_{\pi,\xi,\lambda}(q)}{\mathrm{ED}_{\pi,\xi,\lambda}(q)} \to \infty \text{ as } \lambda \to \infty &\quad\Leftrightarrow\quad 2(s-1) > r. \end{aligned}$$

1.A.2 Proofs for Section 1.5.1

We have identified a target quantity $\hat{q}$.

Suppose that the regulator uses a price instrument instead, choosing price $\hat{p}$, and write $\tilde{q}(\theta)$ for the resulting quantity, after θ is realised.

Suppose that, in a neighbourhood $Q \subset \mathbb{R}$ of q, the following second degree Taylor approximations are appropriate:

$$b(q,\theta) \approx b(\hat{q},\theta) + b_1(\hat{q},\theta)(q-\hat{q}) + \frac{1}{2}b''(\hat{q})(q-\hat{q})^2. \tag{1.11}$$

$$\mathrm{TD}(q) \approx \mathrm{TD}(\hat{q}) + \mathrm{TD}'(\hat{q})(q-\hat{q}) + \frac{1}{2}\mathrm{TD}''(\hat{q})(q-\hat{q})^2. \tag{1.12}$$

If a price p is fixed, then industry will supply the quantity $q^s(p,\theta)$ which satisfies

$$b_1(q^s(p,\theta),\theta) = p. \tag{1.13}$$

Now,

Lemma 1.A.2. *Suppose that $b''(\hat{q}) \neq \mathrm{TD}''(\hat{q})$ and $b''(\hat{q}) \neq 0$. Suppose that $q^s(\hat{p},\theta) \in Q$ for all θ, so that approximations (1.11) and (1.12) may be used. Then $\hat{p}$, the optimal choice of price instrument, is equal to $\mathrm{TD}'(\hat{q})$.*

Proof. If p satisfies $q^s(p,\theta) \in Q$ for all θ, it follows from differentiating (1.11) that

$$q^s(p,\theta) = \hat{q} + \frac{p - b_1(\hat{q},\theta)}{b''(\hat{q})}. \tag{1.14}$$

This is linear in p; the coefficient of p (namely $\frac{1}{b''(\hat{q})}$) is non-zero and independent of θ. We assume that $q^s(\hat{p},\theta) \in Q$ for the optimum price $\hat{p}$, and thus, when we optimise $E_\theta[b(q^s(p,\theta),\theta) - \mathrm{TD}(q^s(p,\theta))]$ with respect to p, cancelling a factor of q_1^s on either side,[36] we see $\hat{p}$ satisfies

$$E_\theta[\mathrm{TD}'(q^s(\hat{p},\theta))] = E_\theta[b_1(q^s(\hat{p},\theta),\theta)] = \hat{p}$$

where we have applied the defining equation for $q^s(p,\theta)$. Now, by assumption, $\mathrm{TD}'(q)$ is locally linear in q, so $E_\theta[\mathrm{TD}'(q^s(\hat{p},\theta))] = \mathrm{TD}'(E_\theta[q^s(\hat{p},\theta)])$. We may see from (1.14) that $E_\theta[q^s(\hat{p},\theta)] = \hat{q} + \frac{\hat{p}-\mathrm{TD}'(\hat{q})}{b''(\hat{q})}$. Applying assumption (1.12), it follows

$$\hat{p} = \mathrm{TD}'\left(\hat{q} + \frac{\hat{p} - \mathrm{TD}'(\hat{q})}{b''(\hat{q})}\right) = \mathrm{TD}'(\hat{q}) + \mathrm{TD}''(\hat{q})\frac{(\hat{p} - \mathrm{TD}'(\hat{q}))}{b''(\hat{q})},$$

or

$$\hat{p}(b''(\hat{q}) - \mathrm{TD}''(\hat{q})) = \mathrm{TD}'(\hat{q})(b''(\hat{q}) - \mathrm{TD}''(\hat{q})).$$

Thus, $\mathrm{TD}'(\hat{q}) = \hat{p}$ as long as $b''(\hat{q}) \neq \mathrm{TD}''(\hat{q})$, which we assumed. □

Note that $b''(\hat{q}) \neq \mathrm{TD}''(\hat{q})$ whenever the second order condition holds, and *also* whenever its converse holds.

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[36] Note that since q^s is defined via $b_1(q^s(p,\theta),\theta) = p$ it follows that $b_{11}q_1^s = 1$; hence $q_1^s \neq 0$.

Corollary 1.A.3. *Given the assumptions of Lemma 1.A.2, it follows that:*

1. *the quantity one expects from a price tool is equal to the quantity that one would have set:*

$$E_\theta[q^s(\hat{p},\theta)] = \hat{q};$$

2. *the price that one expects from a quantity tool is equal to the price that one would have set*

$$E_\theta[b_1(\hat{q},\theta)] = \hat{p}.$$

Proof. To prove 2. note the first order condition for $\hat{q}$ implies that $E_\theta[b_1(\hat{q},\theta)] = \mathrm{TD}'(\hat{q})$; the result now follows from Lemma 1.A.2. Now 1. follows from (1.14) and from $\hat{p} = \mathrm{TD}'(\hat{q}) = E_\theta[b_1(\hat{q},\theta)]$. □

We now prove Theorem 1.5.1. Recall that we do not assume that $\hat{q}$ is the optimal quantity; it might have been fixed externally by other considerations. If a price policy $\hat{p}$ is used, it satisfies $p_1 = E_\theta[b_1(\hat{q},\theta)]$. By (1.14), as shown above, it follows that $E_\theta[q^s(\hat{p},\theta)] = \hat{p}$. As in Section 1.5.1, we write $\tilde{q^s}(\theta)$ for $q^s(\hat{p},\theta)$.

Weitzman (1974) defines the ‘comparative advantage of prices over quantities’ as:

$$\Delta(\hat{q}) = E_\theta\Big[\big(b(\tilde{q}(\theta),\theta) - \mathrm{TD}_{\pi,\xi,\lambda}(\tilde{q}(\theta))\big) - \big(b(\hat{q},\theta) - \mathrm{TD}_{\pi,\xi,\lambda}(\hat{q})\big)\Big]. \qquad (1.15)$$

Proof of Theorem 1.5.1. We follow Weitzman's convention in writing

$$\alpha(\hat{q},\theta) := b_1(\hat{q},\theta) - E_\theta[b_1(\hat{q},\theta)] = b_1(q,\theta) - \hat{p};$$

this implies $E_\theta[\alpha(\hat{q},\theta)] = 0$, and, with (1.14), that

$$q^s(\hat{p},\theta) - \hat{q} = -\frac{\alpha(\hat{q},\theta)}{b''(\hat{q})}.$$

Also, we may see now that $E_\theta[\alpha(\hat{q},\theta)^2] = \mathrm{var}[b_1(\hat{q},\theta)]$.

Assuming that $q(\theta) \in Q$, we may substitute (1.11) and (1.12) into the definition (1.15) of $\Delta(\hat{q})$ to obtain

$$\Delta(\hat{q}) \quad \approx \quad E_\theta\left[-[b_1(\hat{q},\theta) - \mathrm{TD}'(\hat{q})]\frac{\alpha(\hat{q},\theta)}{b''(\hat{q})(\hat{q})} + \frac{1}{2}[b''(\hat{q}) - \mathrm{TD}''(\hat{q})]\frac{\alpha(\hat{q},\theta)^2}{b''(\hat{q})^2}\right]$$

Since $E_\theta[\alpha(\hat{q},\theta)] = 0$, this is equivalent to

$$\Delta(\hat{q}) \approx E_\theta\left[-(b_1(\hat{q},\theta) - E_\theta[b_1(\hat{q},\theta)])\frac{\alpha(\hat{q},\theta)}{b''(\hat{q})(\hat{q})} + \frac{1}{2}[b''(\hat{q}) - \mathrm{TD}''(\hat{q})]\frac{\alpha(\hat{q},\theta)^2}{b''(\hat{q})^2}\right].$$

We conclude

$$E_{s,\theta}[\Delta(\hat{q})] = -\frac{\mathrm{var}[b_1(\hat{q},\theta)]}{2b''(\hat{q})^2}(\mathrm{TD}''(\hat{q}) + b''(\hat{q}))$$

as required. $\square$

Example 1.A.4. Suppose that $\Phi \subset \mathbb{R}$, We examine the effect simply of incorporating generalised uncertainty in damages $d_{\xi,\lambda}(q,\phi)$. Assume that $d_{\xi,\lambda}(q,\phi)$ is globally quadratic in ϕ. Now, if we write $\bar{\phi} = E_\Phi[\phi]$, then

$$d_{\xi,\lambda}(q,s) = d_{\xi,\lambda}(q,\bar{\phi}) + d_2(q,\bar{\phi})(\phi - \bar{\phi}) + d_{22}(q,\bar{\phi})(\phi - \bar{\phi})^2.$$

It follows that

$$\mathrm{TD}(q) = E_\Phi[d_{\xi,\lambda}(q,\phi)] = d_{\xi,\lambda}(q,\bar{\phi}) + d_{22}(q,\bar{\phi})\mathrm{var}(\phi).$$

Jensen's inequality holds strictly: the non-linearity of d in ϕ means that the expectation over ϕ of $d_{\xi,\lambda}(q,\phi)$ is different from the damages $d_{\xi,\lambda}(q,\bar{\phi})$. Now,

$$\begin{array}{rcl} \mathrm{TD}'(q) & = & d_1(q,\bar{\phi}) + d_{221}(q,\bar{\phi})\mathrm{var}(\Phi) \\ \mathrm{TD}''(q) & = & d_{11}(q,\bar{\phi}) + d_{2211}(q,\bar{\phi})\mathrm{var}(\Phi). \end{array}$$

Both values depend on $\mathrm{var}(\Phi)$ which therefore cannot be ignored in the analysis.[37] The fourth-order derivative $\tilde{d}_{2211}(q,\bar{\phi})$ might seem far-fetched as an important coefficient, but it may be significant even when d is quadratic in ϕ. For example, if $d_{\xi,\lambda}(q,\phi) = \hat{d}(q\phi)$, then $d_{2211}(q,\bar{\phi}) = 2\hat{d}_2$, where $\hat{d}_2$ is the coefficient of the degree 2 term in the quadratic function $\hat{d}$.

1.A.3 Proofs for Section 1.5.2

We prove both parts of Theorem 1.5.2 by developing a simple rule. First we split the domain $[\underline{q},\infty) \times \mathbb{R}^{m+n}$ of our damage function into two closed subsets, which overlap when $d_{\xi,\lambda}(q,\phi) = \lambda$. Note that if $d_{\xi,\lambda}(q,\phi) \neq \lambda$ for any values of (q,ϕ) then one of these sets is empty:

$$\begin{array}{rcl} R_1 & := & \{(q,\phi) \in [\underline{q},\infty) \times \mathbb{R}^{m+n} \mid d_{\xi,\lambda}(q,\phi) \leq \lambda\} \\ R_2 & := & \{(q,\phi) \in [\underline{q},\infty) \times \mathbb{R}^{m+n} \mid d_{\xi,\lambda}(q,\phi) \geq \lambda\}. \end{array}$$

For any pair of functions $g : R_1 \to \mathbb{R}$ and $h : R_2 \to \mathbb{R}$, we write $C^{g,h} : [\underline{q},\infty) \times \phi \to \mathbb{R}$ for the function

$$C^{g,h}(q,\phi) := \left\{ \begin{array}{ll} g(q,\phi) & (q,\phi) \in R_1 \\ h(q,\phi) & (q,\phi) \in R_2 \setminus R_1 \end{array} \right. .$$

First, we assume that uncertainty is 1-dimensional. Suppose that Φ takes values in $\mathbb{R}$, and that $d_{\xi,\lambda}(q,\phi)$ is strictly increasing and differentiable with respect to both variables. In the

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[37] This is especially important when we recall that Roe and Baker (2007) argue that climate sensitivity has infinite variance.

case that there exists λ such that $d_{\xi,\lambda}(q,\phi)=\lambda$, let $\phi_\lambda:\mathbb{R}\to\mathbb{R}$ be the function which satisfies $d_{\xi,\lambda}(q,\phi_\lambda(q))=\lambda$ for all $q\in[0,\infty)$; by the implicit function theorem, ϕ_λ exists (since d is strictly increasing in ϕ) and is differentiable.

Now:

Claim 1.A.5. *Suppose that Φ takes values in $\mathbb{R}$ and that $g:R_1\to\mathbb{R}$ and $h:R_2\to\mathbb{R}$ are both continuous and differentiable on their domains. Then*

$$\begin{aligned}\frac{\partial}{\partial q}E_\Phi\left[C^{g,h}(q,\phi)\right] &= E_\Phi\left[C_1^{g,h}(q,\phi)\right]\\ &\quad+\left[h(q,\phi_\lambda(q))-g(q,\phi_\lambda(q))\right]\frac{\partial}{\partial q}P_\pi(\phi\geq\phi_\lambda(q))\end{aligned}$$

where we understand the second term on the right hand side to be zero if $d_{\xi,\lambda}(q,\phi)\neq\lambda$ for all q,ϕ.

Note that, although $C^{g,h}$ is not in general differentiable on $R_1\cap R_2$, this does not matter for its integral over $\mathbb{R}$ as the set has measure 0 (it is a single point).

Proof of Claim 1.A.5. We may elegantly obtain the result in the form stated by re-writing:

$$\begin{aligned}\int_0^\infty C^{g,h}(q,\phi)f_\phi(\phi)d\phi &= \int_{-\infty}^{\phi_\lambda(q)} g(q,\phi)f_\phi(\phi)d\phi\\ &\quad+\int_{\phi_\lambda(q)}^\infty [h(q,\phi)-h(q,\phi_\lambda(q))+g(q,\phi_\lambda(q))]f_\phi(\phi)d\phi\\ &\quad+[h(q,\phi_\lambda(q))-g(q,\phi_\lambda(q))]\,P(\phi\geq\phi_\lambda(q)).\end{aligned}$$

Then, applying Leibniz's rule,

$$\begin{aligned}&\frac{\partial}{\partial q}\int_0^\infty C^{g,h}(q,\phi)f_\phi(\phi)d\phi\\ &=\int_{-\infty}^{\phi_\lambda(q)} g_1(q,\phi)f_\phi(\phi)d\phi+\phi_\lambda'(q)g(q,\phi_\lambda(q))f_\phi(\phi_\lambda(q))\\ &\quad+\int_{\phi_\lambda(q)}^\infty\left[h_1(q,\phi)-\frac{d}{dq}h(q,\phi_\lambda(q))+\frac{d}{dq}g(q,\phi_\lambda(q))\right]f_\phi(\phi)d\phi\\ &\quad-\phi_\lambda'(q)\left[h(q,\phi_\lambda(q))-h(q,\phi_\lambda(q))+g(q,\phi_\lambda(q))\right]f_\phi(\phi_\lambda(q))\\ &\quad+\left[\frac{d}{dq}h(q,\phi_\lambda(q))-\frac{d}{dq}g(q,\phi_\lambda(q))\right]P(\phi\geq\phi_\lambda(q))\\ &\quad+\left[h(q,\phi_\lambda(q))-g(q,\phi_\lambda(q))\right]\frac{\partial}{\partial q}P(\phi\geq\phi_\lambda(q)).\end{aligned}$$

Expanding the second integral shows that almost all terms cancel, leaving the required result. ☐

We now address the general case, using a distinguished a random variable Φ_n.

Claim 1.A.6. *Let Φ take values in $\mathbb{R}^n$ for some $n \geq 1$. Suppose that $g : R_1 \to \mathbb{R}$ and $h : R_2 \to \mathbb{R}$ are both strictly increasing and differentiable with respect to q and with respect to Φ_n on their domain. Then:*

$$\begin{aligned}\frac{\partial}{\partial q} E_\Phi\left[C^{g,h}(q,\phi)\right] &= E_\Phi\left[C_1^{g,h}(q,\phi)\right] \\ &\quad + E_{\Phi_{-n}}\left[[h(q,\phi) - g(q,\phi)]\frac{\partial}{\partial q} P_{\Phi_n}(d_{\xi,\lambda}(q,\phi_{-n},\tilde{\phi}_n) \geq \lambda | \phi_{-n}) \,\middle|\, d_{\xi,\lambda}(q,\phi) = \lambda\right].\end{aligned}$$

Proof. First, fix a realisation ϕ_{-n} of the remaining random variables. Apply Claim 1.A.5 to the $C^{g,h}(q,\phi_{-n},\phi_n)$ to obtain

$$\begin{aligned}\frac{\partial}{\partial q} E_{\phi_n}\left[C^{g,h}(q,\phi_{-n},\phi_n)\right] &= E_{\phi_n}\left[C_1^{g,h}(q,\phi_{-n},\phi_n)\right] \\ &\quad + \left[h(q,\phi)\big|_{d_{\xi,\lambda}(q,\phi)=\lambda} - g(q,\phi)\big|_{d_{\xi,\lambda}(q,\phi)=\lambda}\right]\frac{\partial}{\partial q} P(d_{\xi,\lambda}(q,\phi_{-n},\phi_n) \geq \lambda | \phi_{-n}), \qquad (1.16)\end{aligned}$$

where again we understand the latter term to be zero if there do not exist q, ϕ_n such that $d_{\xi,\lambda}(q,\phi_{-n},\phi_n) = \lambda$.

The additional term in Claim 1.A.5 arises because $C^{g,h}(q,\phi)$ is not differentiable with respect to q for all q (when we fix ϕ). However, $E_{\phi_n}\left[C^{g,h}(q,\phi_{-n},\phi_n)\right]$ *is* differentiable with respect to q for all q; taking the expectation has 'smoothed' the kink and we have its derivative in Equation (1.16). It follows (using Leibniz's rule) that we may interchange differentiation with respect to q, and integration with respect to the remaining coordinates of ϕ; in other words, when we take expectations over the remaining coordinates of ϕ, we have

$$E_{\Phi_{-n}}\left[\frac{\partial}{\partial q} E_{\phi_n}\left[C^{g,h}(q,\phi_{-n},\phi_n)\right]\right] = \frac{\partial}{\partial q} E_\Phi\left[C^{g,h}(q,\phi)\right].$$

Applying $E_{\Phi_{-n}}$ to both sides of (1.16) then provides the result as stated. □

Now:

Proof of Theorem 1.5.2. First let $g(q,\phi)$ be $d(q,\phi)$ on the domain R_1 and $h(q,\phi)$ be identically equal to λ on the domain R_2. Then $d_\lambda(q,\phi) = C^{g,h}(q,\phi)$ and so, by Claim 1.A.6, we see that

$$\begin{aligned}\mathrm{TD}'_{\pi,\xi,\lambda}(q) &= E_\Phi\left[\frac{\partial}{\partial q} d_\lambda(q,\phi)\right] + (\lambda - \lambda)\frac{\partial}{\partial q} P_\pi(d(q,\phi) \geq \lambda) \\ &= E_\Phi\left[d_1(q,\phi) \mid d(q,\phi) \leq \lambda\right] P_\pi(d(q,\phi) \leq \lambda).\end{aligned}$$

Next we let $g(q,\phi)$ be $d_1(q,\phi)$ on the domain R_1 and we let $h(q,\phi)$ be identically equal to 0 on the domain R_2, apply Claim 1.A.6 again, we obtain:

$$\begin{aligned}\mathrm{TD}''_{\pi,\xi,\lambda}(q) &= E_{\Phi}\left[\frac{\partial^2}{\partial q^2} d_\lambda(q,\phi)\right] \\ &\quad - E_{\Phi_{-n}}\left[d_1(q,\phi)\frac{\partial}{\partial q}P_\pi(d(q,\phi_{-n},\tilde{\phi}_n,\phi)\geq\lambda)\,\middle|\, d(q,\phi)=\lambda\right]\end{aligned}$$

which again provides the required result. □

Chapter 2

The social cost of carbon and assumed damages from high temperatures

Abstract

There is growing concern that estimates of climate damages used in integrated assessment models lack justification, especially for higher warming. Such models also often underestimate the longevity of higher temperatures, by using overly simplified carbon cycles, and fail to accurately capture scientific uncertainty. This chapter shows how estimates of the social cost of carbon depend significantly on these imposed tastes and beliefs. A simple economic model is added to the coupled carbon and climate cycle recently developed by Allen et al. (2009), which incorporates uncertainty in five geophysical parameters. Assumed damages from 4°C warming are explicitly varied while damages up to 2.5°C are held fixed. The social cost of carbon is shown to depend fundamentally on the high temperature damage level assumed, as well as on the cumulative emissions of CO_2 which drive temperature change, and on geophysical specification and incorporation of uncertainty.

2.1 Introduction

The social cost of carbon embodies the damage caused by a single tonne of CO_2 emissions, and thus how much we should be prepared to pay to avoid this.[1]

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[1] Throughout this paper, 'the social cost of carbon' is used to refer to the marginal welfare from emitting one more tonne of CO_2 expressed in money terms, as in, e.g., Nordhaus (2008). It is thus dependent on subsequent emissions, as this paper will discuss. The term is used elsewhere in the literature to refer to the shadow price of carbon; that is, the subsequent scenario is assumed to be optimal. This paper does not make the latter restriction; it does not assume an optimal strategy will be followed.

It is well known that changing the discount rate has a dramatic effect on estimates of the social cost of carbon, as we alter the extent to which we care about the future.[2] More recent work has shown that the same is true of our estimates of the damage caused by global warming.[3] Global warming could reach 4°C above pre-industrial temperatures by the end of this century, but we have very little idea what the welfare consequences of this might be.[4] The change may superficially sound small, but it represents the same difference as that between pre-industrial times and the last ice age. Conventional 'integrated assessment models' extrapolate damages from lower calibration points, typically using variants on a 'quadratic' damage function to do so. A recent literature strongly criticises these functions, as having "no theoretical or empirical foundation",[5] and giving rise to damages that have been described as "ludicrously small".[6] The modelling of the incidence of such temperature changes is also often rather rudimentary – for example, with atmospheric CO_2 levels returning to pre-industrial levels, and bringing temperatures back down with them, much faster than scientists predict.

This paper does not propose a single new damage function, nor a distribution over such functions. Instead, it follows the philosophy of Chapter 1 of this thesis in providing estimates of the social cost of carbon as a function of our subjective beliefs. Specifically, we vary the assumed output damages from 4°C warming, while keeping the damage function up to the general maximal calibration point of 2.5°C fixed.[7] Thus the paper asks how much our ignorance of damages from higher levels of warming might matter. We show that, if one only alters hypotheses on worlds about which we have very little information, one can obtain wildly different estimates of the social cost of carbon.

The damages from high levels of warming only matter to us in current welfare term if there is also a considerable risk that this warming would take place. Instead of assuming that society will successfully apply the optimal abatement strategy (which seems unlikely in the short term), and so heading straight to finding the optimal carbon tax, we seek to understand better the relationship between the social cost of carbon, total CO_2 emissions, and assumed damages.

So there are two dimensions to the problem which are explicitly modelled: the assumptions we make about higher level damages, and those about eventual anthropogenic emissions. These are indexed by two parameters: output damages from 4°C warming, and cumulative anthropogenic emissions of CO_2. The particular functional forms of damages and the temporal pathways of emissions are ad-hoc, incorporating also strong assumptions that climate change impacts output rather than capital infrastructure or the rate of growth. Damages are modelled in this way to facilitate comparability with much of the existing literature; the range of emission scenarios are straightforwardly generated to lie between business as usual and the limits of technical feasibility.

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[2] Tol (2009, Table 2) shows changing the pure rate of time preference can change estimates of the social cost of carbon by up to an order of magnitude.

[3] See Ackerman et al. (2010), Dietz (2011), Weitzman (2012), Calel et al. (2013) and Anderson et al. (2014).

[4] See Stocker et al. (2013) for temperature projections. Several prominent experts have recently highlighted this ignorance of damages; see in particular (Stern, 2010, 2013; Pindyck, 2013a; Weitzman, 2010).

[5] Pindyck (2013a).

[6] Stern (2010).

[7] Weitzman (2012) performs a similar trick, but only considers one additional possibility.

The elasticity of the social cost of carbon to cumulative emissions is important in understanding at least three questions in climate change economics: the sensitivity of the relationship between any putative optimal carbon tax, and our estimates of the efficacy of such a tax; the choice of policy instrument, prices or quantities; and the difference between co-operative and non-cooperative outcomes and hence coalition formation.

The first novelty of this paper, then, is to estimate the social cost of carbon not as a single value, but as a function of these two parameters. In treating the social cost of carbon as a function of assumed damages, the model follows the philosophy of Chapter 1 of this thesis, regarding 'incompleteness' of preferences on climate change: when relevant information is so badly known as to be little more than guesswork, one should be explicit that guesses have been made, how they affect the outcome, and what the result would be of making different guesses.

Second, this paper also introduces a new 'simple coupled carbon cycle and climate model' to the economics literature: that of Allen et al. (2009). This differs from the geophysical components of typical economic models of climate change. Both temperatures and CO_2 levels re-equilibriate between the atmosphere, oceans and biosphere via 'diffusive', rather than 'advective' equations.[8] These account for the fact that the changes take place over multiple timescales, and result in slower returns to equilibrium. The model also explicitly accounts for the fact that increasing global temperatures will change the equilibrium carbon level in plants, soils and oceans, and so give rise to feedbacks of additional 'natural' emissions, which should be added to anthropogenic ones. And finally, the model depends on eleven parameters, five of which are taken as uncertain. The work of Allen et al. (2009) provides the joint likelihood of a parameter combination, assessing it by comparing its putative effect with data sets regarding five different geophysical constraints.[9]

The implications of using this new model are explored by re-running calculations on a variant similar to those commonly in use in the literature. The mean values we obtain for the social cost of carbon are also compared with those from the "maximum likelihood" parameter combinations, and with 15% likelihood intervals.

The economic component of this model is simple; in particular the savings rate is fixed. This assumption is made to focus computer time on Monte Carlo methods to estimate the mean of the social cost of carbon and various of its derivatives, over the five unknown geophysical parameters. It is a limitation of the model in its current form, but should not alter the results too dramatically, as dynamically optimising models tend to find a savings rate that is close to constant. The precise savings rates used are chosen to be optimal for each combination of emission trajectory and damage curve assumptions.

The results of the model are gathered into 'stylised results'.

For higher cumulative emissions, the social cost of carbon grows so rapidly with assumed

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[8] These terms are explained in Section 2.2.3.1.

[9] These likelihood are converted into a probability density function by assuming a uniform prior on the distributions of all uncertain parameters – recognising that this is an additional exogenous assumption. The robustness of results to this assumption and others on the form of data uncertainty could be tested, but the present model does handle scientific more robustly than many integrated assessment models (as lamented by Weitzman 2007, 2009a).

damages at 4°C that one would need very high assurance that such assumptions are not valid for them to be irrelevant to the analysis.

The elasticity of the social cost of carbon to assumed damages increases with both cumulative emissions and damage assumptions, being close to zero only if cumulative emissions are very low indeed (and the plausibility of higher level damages is very low). It is in the region of 0.5 to 2 for much of the range of damages and emissions considered.

Similarly, the elasticity of the social cost of carbon to cumulative emissions increases with damage assumptions, being zero only if the damage function is quadratic, or is quadratic for all plausible temperature changes (which depend on the emission scenario). It is in the region of 0.5 to 4 for much of the range of damages and emissions considered.

If the standard quadratic assumptions are correct, then the geophysical model choice is relatively unimportant. However, as assumed damages increase, the percentage additional social cost of carbon due to using the better geophysical model of Allen et al. (2009), rather than a simpler variant similar to those used in much of the literature, rises to around 200–300% for our 'lower range' of damage assumptions; damages are dramatically higher in this model for our highest damage assumptions.

When we explore the effect of incorporating parameter uncertainty, we find that it is relatively unimportant when standard damage assumptions are made; indeed in these cases it leads to a small decrease in estimates of the social cost of carbon. However, as the assumed damages increase, the discrepancy mounts up, especially for lower emission scenarios (for which incorporating uncertainty makes the greatest difference in allowing our higher assumptions on damages to matter).

Interestingly, it is the former of these two effects that matters more: under two comparisons we show that changing the choice of geophysical model makes a greater difference to the estimated social cost of carbon than does the comparison between best guess values and means: model uncertainty matters more than parameter uncertainty. This result is of course highly dependent on the precise modelling comparison being made, as well as the way uncertainty is formulated (and our prior distribution).

The extent of variability of the social cost of carbon is also explored: its probability density (given prior assumptions) is sketched, showing long right 'tails'. Its standard deviation is found to be between 50–150% of the mean for the lower damage assumptions and to increase dramatically with assumed damages.

Finally, a sensitivity analysis shows that the social cost of carbon estimates are scarcely affected by the most ad-hoc simplification of the model: the cost of abatement of CO_2 emissions.

In contrast to the results of this paper, the literature on the social cost of carbon tends to show or assume that the social cost of carbon is (close to) *independent* of cumulative emissions. This result was described as 'surprising' when first presented in detail by Hope (2006), but now appears well established and features in recent theoretical and empirical papers.[10] Hope (2006)

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[10] See, e.g., Gerlagh and Liski (2013), van den Bijgaart et al. (2013), Golosov et al. (2014). Both Golosov et al.

describes the independence as being the due to "the interplay between the logarithmic relationship between forcing and concentration, the nonlinear relationship of damage to temperature, and discounting." That this is the case of course depends on the choice of functions, and in particular the choice of the convex damage response to temperature; if one is to follow Weitzman (2010), Stern (2013), Pindyck (2013a) and others in questioning these damage functions it follows that the independence may fail to hold. It appears to be simply a coincidence that this holds of the damage function first written down by Nordhaus (1993), and since followed by so many.[11]

Uncertainty is included in much recent work on integrated assessment modelling: see e.g. Ceronsky et al. (2011), Hope (2011a), Pycroft et al. (2011), Anderson et al. (2014), Jensen (2014), Traeger (2014). And others have included it while also explicitly experimenting with different damage functions: see for example Dietz (2011), Ackerman and Stanton (2012), Tol (2012), Kopp et al. (2012), Calel et al. (2013). However, the present paper appears to be the first model systematically estimating the social cost of carbon as a function of two parameters, cumulative emissions and assumed damages, in such a way as its elasticity to these quantities may be found.

The paper is arranged as follows: Section 2.2 lays out the model, starting by explaining in more detail the slightly unusual model form and intentions (Section 2.2.1) giving the economic component of the model in Section 2.2.2 and the geophysical component in Section 2.2.3. In particular, the treatment of uncertainty and the Monte Carlo methods used are described in Section 2.2.3.2. The differences between the coupled carbon cycle and climate model here, and a more standard model from the economics literature, are discussed in Section 2.2.3.3. The results are in Section 2.3; the social cost of carbon as a function of both emissions and assumed damages is presented in Section 2.3.1; a comparison between our model and the output using a more simple geophysical component is given in Section 2.3.2, and the impact of geophysical uncertainty is explored in Section 2.3.3. In Section 2.3.4, further comparisons are made between the relative significances of choice of geophysical model and the incorporation of scientific uncertainty. A sensitivity analyses on the cost of emission reductions is performed in Section 2.3.5. Section 2.4 concludes.

2.2 The Model

2.2.1 The philosophy of the model

The premise of this paper is that climate damages, as modelled in typical integrated assessment models, lack detailed justification and could in particular be much too low at higher

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(2014) and Gerlagh and Liski (2013) tune their assumptions to fit with DICE damage functions models, and so the independence they find follows from the independence there. van den Bijgaart et al. (2013) develop a 'simple formula' for the social cost of carbon in which the atmospheric stock of emissions does not feature, and show that their formula accurately predicts the social cost of carbon of the DICE-2007 model (Nordhaus, 2008) for many possible parameter assumptions.

[11] Nordhaus (1993) writes "there is evidence that the impact increases non-linearly as the temperature increases, and we assume that the relationship is quadratic." All subsequent work appears to trace back to this.

temperatures.[12] Its principal output is the social cost of carbon (SCC) *today* – that is, in 2010. We show how that varies with cumulative emissions, damage function, and physical modelling choices.

Despite the extensive literature on calculations of the social cost of carbon, there is in fact very little work on the direct welfare costs of climate change, and essentially none on higher levels (see Tol, 2013, Table 1). This has been emphasised by several important recent papers (see, e.g. Stern, 2010, 2013, Heal and Millner, 2014, Pindyck, 2013a). Damages are certainly far more uncertain than is typically modelled, with even the proponents of traditional damage functions agreeing that they need "to be examined carefully or re-specified in cases of higher warming or catastrophic damages" (Nordhaus and Sztorc, 2013).

One may criticise the typical form of the damage function, acting multiplicatively on output: some argue that damages should act on capital and affect the growth rate, (Stern, 2013, Pindyck, 2012) that there is limited substitutability for damaged sectors, (Sterner and Persson, 2008) that material damages are typically modelled as being worse in a richer world, whereas the converse is more likely true, (Weitzman, 2010) or that they are simply "ludicrously small" (Stern, 2010). Instead of rehearsing these arguments in detail, we refer to the growing literature on climate change economic damages and their uncertainty.[13]

This paper does not propose a new damage function, or state a view about how high damages from higher levels of warming might be. Nor does it address most of the concerns listed above. It simply explores the effects of different assumptions of damages from higher levels of warming, to illustrate the extent to which this ignorance matters.

The importance of assumed damages from higher levels of warming depends on the temperature changes that might take place, and so on anthropogenic emissions. There are of course many degrees of freedom in how emissions may evolve over time. One route to reducing the dimensionality of this problem is provided by Allen et al. (2009) who show (Supplementary Material Figure S3) that the temperature pathway predicted by a coupled climate and carbon cycle model is close to independent of the precise *pathway* of emissions, depending only on their *cumulative total*. They only explore this relationship for a low cumulative emissions scenario, but their work implies that it is reasonable to index a range of possible emission pathways by their cumulative total.

Therefore, we estimate the mean welfare cost of climate change as a function of two parameters: "q", the cumulative emissions (see Section 2.2.2.2), and "α", assumed output damages at 4°C (see Section 2.2.2.3).

The computer model is implemented in Matlab. It does not perform an optimisation, focusing processor time on a Monte Carlo analysis of five uncertain parameters. So, in contrast to many 'integrated assessment models', both the savings rate and the emissions pathway are treated as exogenous. (Additionally, total factor productivity growth and global population are exogenous;

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[12] See e.g. Stern (2010, 2013), Weitzman (2012).

[13] See Sterner and Persson (2008), Freeman and Guzman (2009), Weitzman (2009a,b,d, 2010, 2011), Pindyck (2011, 2013a,b), Heal and Millner (2014), Millner et al. (2013), Stern (2013).

these assumptions are usual in this context; see, for example, Nordhaus and Sztorc 2013).

Exogenous emissions are used because the model seeks to provide the social cost of carbon (now) as a *function* of the emission scenario (as well as the damage assumption). This allows us to map out and understand the relationship between the social cost of carbon today and those policy choices; if we fail to set an 'optimal' carbon price today, damages tomorrow will be greater, and so the social cost now will in general be affected.

A fixed savings rate is also imposed. This choice is less 'virtuous', being a simplification to avoid the need for intertemporal optimisation and allow computer time to be focused on the five uncertain geophysical parameters. However, it is not unique to this paper (see, for example, Golosov et al., 2014, Acemoglu, 2008) and seems innocuous in light of the fact that, for example the savings rate is calculated as close to constant in an optimising run of DICE 2013. The precise constant savings rate used is chosen to maximise NPV welfare, given the emission and damage scenario.

There is no attempt to find the optimal social cost of carbon or emission trajectory; society seems unlikely to coordinate on an optimal level soon, so it is instructive instead to focus on how the social cost of carbon varies with the emission scenario and hence plot the whole 'supply curve' for environmental services, rather than to calculate a single point on this curve. Different modelling assumptions will provide different isolated optimal points; by plotting the entire curve we can learn more.

2.2.2 The Economic Component

2.2.2.1 The Growth Model

The economy is simulated using a simplified partial-equilibrium version of DICE 2013 (Nordhaus and Sztorc, 2013). The principal differences are in the fixed savings rate (as discussed in Section 2.2.1) meaning that no intertemporal optimisation takes place, in the exogenous emission and simplified emission abatement function, and in the range of damage functions. The economic component of the model runs from 2010 to 2500. The time step is one year.

Welfare is

$$W(q,\alpha) = \sum_{t=1}^{491} \frac{1}{(1+\rho)^t} L(t) U\left(\frac{M(q,d_\alpha,t)}{L(t)}\right).$$

Here, q is cumulative emissions of CO_2, which drive temperature change, and α specifies the damage function. Both of these are varied exogenously; see Sections 2.2.2.3 and 2.2.2.2. $M(q,d_\alpha,t)$ is total global consumption, which depends on the emission pathway, which we index by cumulative emissions q, as well as on the damage they cause d_α, and evolves with time t.[14] Further, ρ is the pure rate of time preference, U is the utility function, and $L(t)$ is population (we thus implicitly assume that consumption is evenly spread). World population is a cubic interpolation of the

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[14] We use M instead of C to denote consumption because we follow the notation of Allen et al. (2009), in which C represents concentrations of CO_2.

'medium' population growth scenario from United Nations (2013), which goes up to 2100, and is taken as constant thereafter, the projection having nearly stabilised by this date.

Output net of damages or abatement is

$$Y(t) = A(t)K(t)^{\gamma}L(t)^{1-\gamma}(1-\Lambda(t))\left((1-d_{\alpha}(T(t))\right)$$

where $A(t)$ is total factor productivity, and $K(t)$ is capital stock and services, the capital elasticity in output γ is 0.3, abatement costs are $\Lambda(t)$ and climate damages d_{α} depend on temperature T, which is a function of time t (and, implicitly, emission scenario). Abatement costs are described in Section 2.2.2.2; damage costs are explained in Section 2.2.2.3; they depend on temperature change $T(q,t)$, described in detail in Section 2.2.3.1. Of this output, a fraction $s(q,\alpha)$ forms investment $I(t)$ and the remainder is consumption $M(t)$; capital depreciates at rate δ_K, so that:

$$\begin{aligned} M(t) &= (1-s(q,\alpha))Y(t) \\ I(t) &= s(q,\alpha)Y(t) \\ K(t+1) &= I(t)+(1-\delta_K)K(t). \end{aligned}$$

We take $\delta_K = 0.1$. The savings rate $s(q,\alpha)$ is a constant, but the constant used depends on cumulative emissions q (see Section 2.2.2.2) and damage assumptions α (see Section 2.2.2.3): it is the rate which optimises welfare W, given these choices, when the 'best guess' geophysical parameters are used (see Section 2.2.3.2).

Growth $g_A(t)$ in total factor productivity $A(t)$ is exogenous, and set to correspond to DICE 2013. Conversion is required because DICE uses a five-year times step, whereas this model works on an annual basis. Initially g_A is at 1.536% and it depreciates at $\delta_A = 0.1197$ each year. Thus:

$$\begin{aligned} A(t) &= A(t-1)(1+g_A(t)) \\ g_A(t) &= \frac{1}{1+\delta_A}g_A(t-1). \end{aligned}$$

The output variable of interest throughout the paper is the social cost of carbon (SCC) in 2010 (and not its pathway over time) in 2005 US\$/tCO$_2$. To find this we impose an additional pulse of 10 gigatonnes of carbon (GtC) in 2010 and take differences to find $W_1(q,\alpha)$. Then the social cost of carbon is

$$SCC(q,\alpha) = \frac{\frac{12}{44}W_1(q,\alpha)}{U'\left(M(1)/L(1)\right)}$$

where the factor of $\frac{12}{44}$ converts from tC/\$, the unit of emissions, to tCO$_2$/\$, the more commonly-used unit in discussions of carbon pricing.

To numerically differentiate the SCC with respect to q and α we impose a larger pulse, or sightly larger choice of α, and take differences as relevant.

The model has been run with a pure rate of time preference $\rho = 1.5\%$ and with constant relative risk aversion $\eta = 1.45$, as in DICE 2013 (Nordhaus and Sztorc, 2013).

2.2.2.2 Emission Scenarios and Abatement Costs

One aim of the model is to map the effect of different emission policies. So emission pathways over time are treated as exogenous; they are linked back to the economy first by the climatic damages they give rise to, and secondly via an abatement cost function.

Historical emissions from Meinshausen et al. (2011) are used up to 2005.[15] After this we use idealised emissions scenarios generated in the same way as Allen et al. (2009), as follows. Emission growth g_E is piecewise-linear in time; when $g_E < 0$, emissions decline. This growth is initially 3%.[16] It continues to be 3% until some date t_1 between 2015 and 2050. From time t_1 to $t_1 + t_3$ the growth g_E declines linearly, passing zero (peak emissions) at $t_1 + t_2$, and reaching its minimum $\underline{g_E} < 0$ by t_3. Emissions decline at this constant rate thereafter. That is,

$$\begin{aligned} E(t+1) &= \left(1 + \frac{g_E(t)}{100}\right) E(t) \\ g_E(t) &= \begin{cases} 3 & t \leq t_1 \\ 3 - \frac{3}{t_2}(t - (t_1 + t + 2)) & t_1 < t \leq t_1 + t_3 \\ \underline{g_E} & t_1 + t_3 < t \end{cases} \end{aligned}$$

where t_3 is fixed to be the maximal integer such that $3 - \frac{3}{t_2}(t_1 + t_3 - (t_1 + t + 2)) > \underline{g_E}$.

The model indexes climate damages against cumulative emissions, under the assumption that this encapsulates the most important distinctions (see Allen et al., 2009, Supplementary Information Figure S3). To obtain a set of emission scenarios, each one reasonably representative of scenarios giving rise to a given cumulative total, we first fix bounds for t_1, t_2 and $\underline{g_E}$ as follows: $2015 \leq t_1 \leq 2050$, $5 \leq t_2 \leq 25$, $-10 \leq \underline{g_E} \leq -2$. The lower bounds reflect the most dramatic emission abatement that is at all realistic; the upper bounds are set to obtain a similar scenario in terms of best guess concentration pathway and temperature change to representative concentration pathway "RCP 8.5", the highest emission scenario considered by the IPCC's Fifth Assessment Report (see Riahi et al., 2007, IPCC, 2013). We then vary the three variables in sync between these lower and upper bounds, at such a rate as to obtain a range of scenarios within which the cumulative emissions increase geometrically at a constant rate.

The cost $\Lambda(t)$ of abatement is assumed to be linear in the extent $3 - g_E(t)$ of this abatement:

$$\Lambda(t) = p_E(3 - g_E(t)) \tag{2.1}$$

The linear factor p_E is 0.5, so that reducing emissions by 1% below their trend of 3% costs 0.5% of output. The principle is that the cost depends on the rate of reduction rather than absolute reduction levels because cheaper abatement will happen first. Linearity is assumed for

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[15] Historical emissions are needed because the evolution of temperature, carbon concentrations and natural emission feedbacks all depend on all the pathway of variables up to the present time, rather than just the previous period; see Section 2.2.3.1.

[16] Olivier et al. (2013) gives the average annual increase in emissions since 2000 as 2.9%. The figure for 2012 is much lower, at 1.1%, but it seems too soon to call this a trend.

simplicity. If the carbon price were linear in cumulative emission reductions by 2050 then the cost of abatement should be concave in the annual rate of reductions. Clarke et al. (2007, Figure TS.13) suggests the carbon price is slightly convex in cumulative emission reductions by 2050, so linearity may be a reasonable assumption.

This extreme simplicity is innocuous because the model does not seek to find the optimal carbon price, balancing costs and benefits of emission reductions, but simply to better understand the factors influencing the social cost of carbon – in which the cost of abatement is of second order when cumulative emissions are held constant. In Section 2.3.5 we undertake analysis of the effect on estimated damages of this cost p_E.

2.2.2.3 Damage costs

As described in Section 2.2.1, there are many ways in which one might wish to change the damage function in the economics of climate change. We start such a study here by focusing on one dimension: damages from warming beyond the calibration point of these damage functions. The focal temperature of this paper is 4°C warming; the recent IPCC assessment report emphasised that this is a possibility this century (IPCC, 2013) and its potential impacts are just beginning to be explored (see, e.g. Schellnhuber et al., 2012).

We assume damages d_α are a function of temperature change T, which is in turn a function time t and of emission scenario, indexed by cumulative emissions q; see Section 2.2.3.1. We take as a starting point the DICE 2013 damage function; we assume this to be correct up to temperature change $T = 2.5°C$ (the upper bound for most direct calculations of climate damages). We calibrate damages above this level by explicitly setting the damages from 4°C warming to be $\alpha\%$, with functional form as follows:

$$d_\alpha(T) = \begin{cases} 1 - \dfrac{1}{1+\zeta T^2} & 0 < T \leq 2.5 \\ 1 - \dfrac{1}{1+\eta T^\kappa} & T > 2.5. \end{cases}$$

Here, $\zeta = 0.00266375$ (as in DICE 2013) so that $d_\alpha(2.5) = 0.0164$ in every case. Parameters η and κ are set so that the function is continuous at $T = 2.5$ and so that $d_\alpha(4) = \frac{\alpha}{100}$.[17] In particular, note that if $\alpha = 4.09\%$ then $d_\alpha(T) = 1 - \frac{1}{1+\zeta T^2}$ for all $T > 0$; this is the 'quadratic' damage function whose form was introduced in Nordhaus (1993).

There being so little information about climate damages from 4°C, there is no *ex ante* upper bound for α below 100%. Thus the model is run for $\alpha = 10\%, \ldots, 90\%$ as well as $\alpha = 4.09\%$. However, some may consider the higher values as unreasonably high and so the results as without interest; hence we repeat modelling experiments using $\alpha = 2\%, 4\%, \ldots, 20\%$. The important benchmark of $\alpha = 4.09\%$ is generally plotted with both sets of data. The damage functions we derive are shown in Figure 2.1.

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[17] Explicitly, $\kappa = 2.1276 \log\left(60.0657\frac{\alpha}{1-\alpha}\right)$ and $\eta = \frac{\alpha}{4^r(1-\alpha)}$.

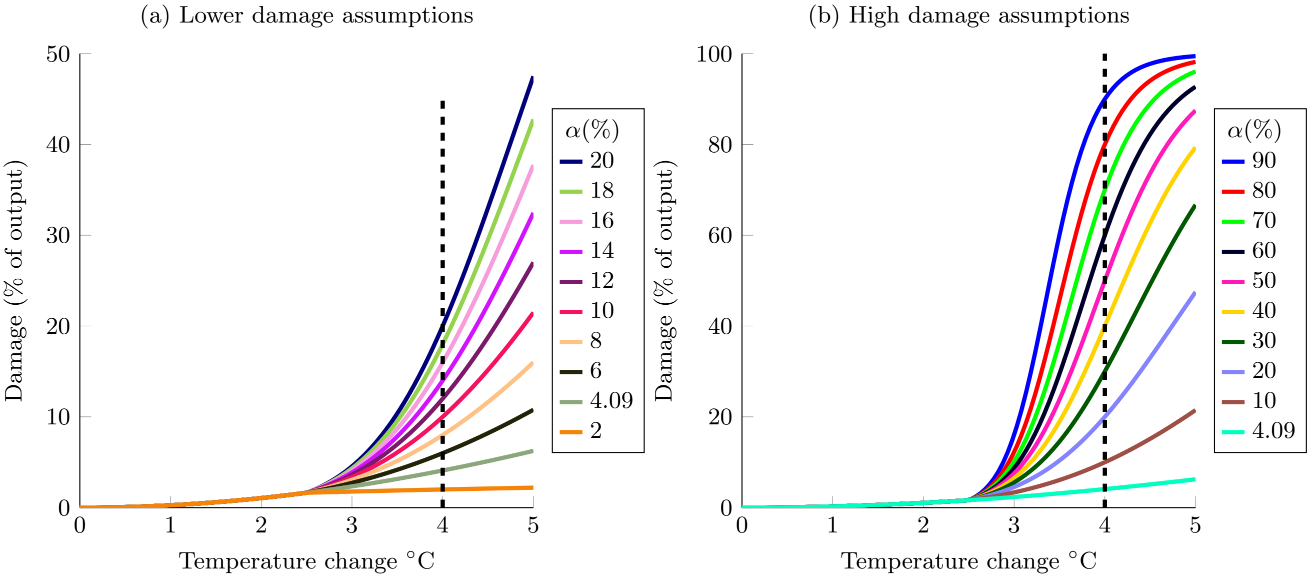

Figure 2.1: *The damage functions in use. The legends give the corresponding values of α, that is, output damages from $4^\circ C$ warming.*

The idea that the exponent of the damage function might change after a certain temperature change is not a new one, but this paper appears to the first explicitly parametrising many possible such changes by the induced damages at a higher temperature – a more easily interpreted quantity than a functional exponent.

2.2.3 The geophysical component

2.2.3.1 The coupled carbon cycle and climate model

This paper uses the full model of Allen et al. (2009).[18]

The temperature change T is driven by forcings and the diffusion of heat downwards to the oceans. Write $C(t)$ for atmospheric CO_2 and C_0 for its pre-industrial level. The model works with one-year time steps, and the discrete form of the 'diffusive' equation is:[19]

$$T(t+1) = T(t) + \frac{a_3}{a_1}\log\left(\frac{C(t)}{C_0}\right) - \frac{a_0}{a_1}T(t) - a_2\sum_{s=2}^{t}\frac{T(s)-T(s-1)}{\sqrt{(t-s+1)}}. \qquad (2.2)$$

Parameters $a_0, \ldots, a_3$ are all presented in Table 2.1.

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[18] I am very grateful to Myles Allen and Benjamin Bronselaer for sharing their Matlab code with me.

[19] Allen et al. (2009) give the following continuous version in their Supplementary Material:

$$a_1\frac{dT(t)}{dt} = a_3\log\left(\frac{C(t)}{C_0}\right) - a_0T(t) - a_2a_1\int_0^t\frac{dT(t')}{dt'}\frac{dt'}{\sqrt{(t-t')}}.$$

The discrete version is used in the Matlab code and seems easier to understand.

Parameter	Best Guess (Range if variable)	Interpretation
a_0	1.1307 (0.3574, 2.5945)	Feedback determining equilibrium climate sensitivity
a_1	7.226	Diffusion parameter for 75m ocean mixed layer and 70% ocean coverage.
a_2	0.4077 (0.1419, 0.9014)	Downward diffusion of surface temperature anomalies
a_3	5.396	Forcing parameter (with a_1 ensures forcing of $3.74Wm^{-2}$ from doubling CO_2).
b_0	0.0050 (0.0023, 0.0105)	b_0^{-1} is a time-constant for advection to the deep ocean
b_1	0.0942	$b_1 + b_3 = 0.3n$ is the fraction of a pulse of CO_2 which would remain in the atmosphere after a pulse injection, in absence of deep-ocean advection; $n = 0.4710$ is conversion factor ppmv (concentrations) to GtC (emissions).
b_2	0.4412 (0.2127, 1.3396)	Diffusitivity of the land-biosphere and ocean thermocline.
b_3	0.0471	Revelle Buffer Factor (resistance to atmospheric CO_2 absorption by the ocean surface layer due to bicarbonate chemistry).
b_4	0.2355	$b_1+b_3+b_4 = 0.8n$ is the fraction of emitted CO_2 remaining in the atmosphere after 1 year, absent any diffusion; n as above.
b_5	5.1953 (1.7047 10.2695)	Rate of feedback of natural emissions.
b_6	100	Carbon cycle lag factor determining feedbacks.

Table 2.1: *Parameters in the coupled carbon cycle and climate model*

The terms on the right-hand side of (2.2) are kept separate to clarify their different interpretations. The equation models the way in which temperature changes over time, and so $T(t)$ occurs for the first time as a baseline. The second term is the forcing due to increased concentrations of CO_2, relative to their pre-industrial level. The third term represents heat radiation away from the planet, which increases with temperature; when temperatures in equilibrium, the second and third terms jointly determine climate sensitivity.

The final term on the right-hand side of (2.2) has a different form from those typically used in geophysical components of integrated assessment models of climate change. Such models separate space into 'boxes', in which the temperature is uniform, and model the transfer of heat as proportional to the temperature difference between these 'boxes'. For example, the atmosphere might be treated as one 'box', and the ocean as another. This is of course a simplistic view, and does not take into account the fact that the distribution of heat within each box will depend on events at earlier times. Sophistication may be added by incorporating a 'deep ocean', and more additional 'boxes', but each one adds to the number of equations in use and requires additional parameters.

One may get around this problem by modelling, not a series of boxes, but rather a semi-infinite slab, down which the heat diffuses. The form of (2.2) provides a standard (to physicists) model of this situation. A small step change to the temperature just above the ocean, from time $s-1$ to time s, means that instantaneously the difference $T(s) - T(s-1)$ is driving the heat transfer – but in doing so it warms the very top part of the ocean and so slows down heat transfer to this part. The transfer continues on down, but as temperatures in the higher parts warm, the heat exchange is slowed. The pulse of heat spreads down as a Gaussian distribution with peak proportional to $\frac{1}{\surd(t-s+1)}$, this being the time elapsed since the initial pulse. The sum of all such terms provides the final term on the right-hand side of (2.2).

CO_2 concentrations $C(t)$ break down as $C(t) = C_1(t) + C_2(t) + C_3(t)$, where $C_3(t)$ is the putative long-term equilibrium (millennial timescale) CO_2 level, once equilibrated with the deep ocean, if there were to be no additional emissions; $C_2(t)$ is the excess above C_3 in the short-term equilibrium between the atmosphere, land-biosphere and upper-ocean, and $C_1(t)$ is atmospheric excess above $C_2(t) + C_3(t)$. If net emissions (anthropogenic and natural; see below) are $E(t)$ then these concentrations are related by equations:

$$C_3(t+1) = C_3(t) + b_3E(t) \tag{2.3}$$

$$C_2(t+1) = C_2(t) + b_1E(t) - b_0C_2(t) \tag{2.4}$$

$$C_1(t+1) = C_1(t) + b_4E(t) - b_2\sum_{s=2}^{t}\frac{C_1(s) - C_1(s-1)}{\surd(t-s+1)}. \tag{2.5}$$

Here, (2.3) says that a certain fraction (b_3) of emissions add to long-term atmospheric concentrations (C_3). Another fraction (b_1) of emissions add to the shorter-term concentrations C_2, but this quantity is also in the process of relaxing to zero in the absence of additional emissions, as carbon transfers from the upper-ocean to the deep ocean. This process is modelled by 'advection' and not the diffusive form discussed above because it is reasonable to assume on the relevant time-scale that the respective 'boxes' have mixed well. However, the same cannot be said of the atmospheric excess C_1 above this shorter-term equilibrium, and hence (2.5) provides another 'diffusive' process. This transfer of carbon from the atmosphere to the land-biosphere and upper ocean can be understood in the same way as the transfer of heat in Equation (2.2) that was discussed above.

The emissions $E(t)$ consist of anthropogenic emissions $E_a(t)$, but also have a feedback component, due to a change in the carbon balance of oceans, soils, forests, etc. at higher temperatures. This depends not only on current temperatures, but on the extent to which they exceed an exponentially-weighted running mean of the temperature over the preceding century (being the timescale over which these sources and sinks re-equilibriate). So:

$$E(t) = E_a(t) + b_5 \left(T(t) - T'(t)\right) \tag{2.6}$$

where b_5 is a variable parameter and[20]

$$T'(t+1) = \left(1 - \frac{1}{b_6}\right) T'(t) + \frac{1}{b_6} T(t+1). \tag{2.7}$$

More detail on the difference between this model and those incorporated in other economic models is given in Section 2.2.3.3.

2.2.3.2 Treatment of uncertainty in physical parameters

Weitzman (2007, 2009a) brought to prominence the importance of addressing geophysical uncertainty in economic analyses of climate change. Although the PAGE and FUND models incorporate multiple such uncertainties, analyses using DICE typically focus on uncertainty in climate sensitivity, the parameter emphasised most by Weitzman (2009a).[21] However, another critical parameter is heat uptake by the ocean, which determines the *rate* (rather than the long-term *level*) of temperature change. Calel et al. (2013) show the potential economic importance of uncertainty here and warming being faster than expected; conversely, Roe and Bauman (2013) argue that, since high values of climate sensitivity ought to be associated with large values for ocean heat uptake, the importance of climate sensitivity to economic analyses may have been overstated. Thus, qualitatively accurate models should not treat these variables as independent.

Uncertainty in the carbon cycle has received less discussion. However, it is also influential to long-term warming: if there are few temperature feedbacks, and emissions are easily re-absorbed, then temperatures will not be so high for so long. As emphasised by Gerlagh and Liski (2013), the carbon cycle of DICE may be too optimistic as to the long-term uptake of carbon by the oceans.

Allen et al. (2009) model five of their eleven parameters as uncertain: a_0, a_2, b_0, b_2 and b_5; these are the parameters whose values are least pinned down by other data. That is, the feedbacks determining equilibrium climate sensitivity, the thermal diffusitivity of the ocean, the rates of carbon uptake by both the land-biosphere and ocean thermocline, and the deep ocean, and the feedback of natural emissions, are all variable.

Moreover, the distributions of these parameters are not independent. Instead, Allen et al. (2009) identify five independent observable quantities, estimates of which are used to constrain the model. For each of the observables X, the model assumes that errors in $\log(X)$ are normally

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[20] In continuous time, interpret $T'(t)$ as $\frac{1}{b_6}\int_0^{t'} \left(1 - \frac{1}{b_6}\right)^{(t-t')} T(t')dt'$.

[21] See for example Ackerman et al. (2010), Jensen and Traeger (2013); note that DICE itself incorporates no scientific uncertainty.

distributed with mean 0 and standard error $\sigma_{\log(X)}$, where both the mean and standard error are approximated from published data. Then any (a_0, a_2) and any (b_0, b_2, b_5) combinations are used to generate temperature paths, whose likelihoods are assessed against these distributions.

The climate feedback parameter a_0 and the temperature diffusion parameter a_2 are jointly constrained by the "fingerprint" attribution of 20^{th} century warming to greenhouse gases and the effective heat capacity of the ocean.

The advection time constant, b_0^{-1}, is calibrated to current Earth system models. The feedback parameter b_5 is constrained by running the model with a standard emissions scenario from 1750 to 2100 both with, and without, the temperature feedback term, and hence calculating the contribution of the temperature feedback to the net airborne fraction. This is compared to corresponding values from a host of more sophisticated models. Finally, given b_0 and b_5, diffusivity b_2 is constrained by the net airborne fraction of CO_2 emissions from 1961 to 2000 (the observed increase in atmospheric CO_2 divided by total anthropogenic emissions over the same period). For more details, see the supplementary information provided by Allen et al. (2009).

The model of the present paper converts these likelihoods into probability densities by assuming a uniform prior for all five; thus they are simply re-normalised by an estimate of the integral of the likelihoods over their domain. This prior choice is defended only for its simplicity; if an economic analyst wishes to estimate a mean then some prior must be taken.[22]

The integrals required for this normalisation, as well as the means themselves, are computed using Monte Carlo methods. 'Importance sampling' reduces variance by disproportionately sampling from parameter combinations which make the largest contribution to the mean, and then re-weights the contribution of each point so the estimator remains unbiased; see Fishman (2003). Many methods of importance sampling exist; this paper uses the 'Vegas' algorithm of Lepage (1978) which is straightforward to implement in Matlab, thanks to the GNU Scientific Library (Galassi et al., 2009) and the work of Cooijmans (2012).

The Vegas algorithm assumes that the different dimensions of uncertainty are independent, and automatically tunes a histogram on each; although our uncertain variables are not in fact independent, the 'a' variables are independent from the 'b' and all distributions are uni-modal; the method has proved to successfully achieve estimators with very small standard errors. Moreover, Vegas avoids the need to correctly estimate the jumping distribution, burn-in period or auto-correlation between adjacent samples, as with Markov chain methods, and so provides a straightforward way to quickly introduce importance sampling to a model.

However, many different integrals are required for this paper, because we seek the SCC for multiple (q, α) combinations, and this complicates the situation. The optimal importance weights depend not only on the probability density function against which one integrates, but the function whose mean one is taking; 'tail' parameter combinations start to matter more if they correspond to extreme values of the objective. However, applying Vegas repeatedly for every

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[22] The use of a uniform prior in this context is standard (see Solomon et al., 2007) but problematic (see Frame et al., 2005). Pueyo (2012) identifies that the unique non-informative prior distribution for the climate sensitivity parameter, i.e. $\frac{a_3}{a_0}$, is the log-uniform distribution. Such analysis could inform future runs of the model.

cumulative emission and damage scenario would be extremely computationally inefficient: the Matlab computation of the joint likelihood of a parameter combination is resource expensive, so it makes sense to re-use samples once they have been generated. The computation of a temperature pathway from a parameter combination and emission scenario is similarly expensive, so it is sensible to calculate the SCC for every damage assumption once this pathway has been obtained.

We thus develop a novel two-step process. First, the Vegas algorithm is performed to estimate the mean of the SCC on a six (q, α) combinations, which span the range of values of interest. Vegas runs the main algorithm five times, each time with 5000 calls, to tune the importance sampling; the parameter combinations that Vegas selects in its 5th run are saved, along with their importance weight and their joint likelihood. The different sets of parameters, from each (q, α) combination, are kept distinct. Then, the SCC and its derivatives are estimated using each of these sets of parameters and importance weights (separately). Each of these sets provides an unbiased estimator, whose standard error we can also estimate. We can form an new unbiased estimator by taking the weighted combination of these estimators which minimises the overall standard error.

Finally, all such integrals are normalised against an estimate of the integral of the likelihood functions alone – which is performed with a simple implementation of Vegas.

Vegas must work on a product of bounded intervals. In each case the lower and upper bounds of a variable are set such that the joint likelihood of this bound value with the best guess for the other parameters is 0.01. These numbers are the ranges given in Table 2.1.

2.2.3.3 Comparison of the physical element with leading economic models

Economic work on climate change uses a variety of climate and carbon cycle components, of varying complexities. For example the PAGE models are highly complex, considering other greenhouse gases and sulphate aerosols. Other integrated assessment models, especially those based on DICE, consider only CO_2 and are more similar in structure to the model of Allen et al. (2009), but do not use 'diffusive' dynamic for temperature or carbon transfers, instead using the 'advective' alternatives (which model flows between a series of 'boxes'). They also tend not to include feedbacks in emissions due to temperature changes.[23]

The work of Bronselaer et al. (2013) compares the model of Allen et al. (2009) to variants which replace the 'diffusive' equations with 'advective' equivalents. By also removing the feedback component, we obtain a model which is similar in structure to that in a standard integrated assessment model. We refer to this as 'Model B'. We shall compare this results from this model to those we obtain using the full diffusive model of Allen et al. (2009), henceforth referred to as 'Model A'.[24]

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[23] PAGE and FUND models do contain feedbacks but are in other ways very different from the model in use here.

[24] This comparison model lacks the 'deep ocean' temperature component used by DICE, and so is most similar to the coupled climate and carbon cycle of Gerlagh and Liski (2013). To aid comparability, however, we shall not

To give the details, 'Model B' differs from 'Model A' in two important ways. Firstly, transfer of both heat to the ocean and carbon land-biosphere and upper ocean are advective rather than diffusive. That is, instead of Equations (2.2) and (2.5), they use the forms:

$$\begin{aligned} T(t+1) &= T(t) + \frac{a_3}{a_1}\log\left(\frac{C}{C_0}\right) - \frac{a_0}{a_1}T - a_2T(t) \qquad (2.8)\\ C_1(t+1) &= C_1(t) + b_4E - b_2C_1(t). \qquad (2.9)\end{aligned}$$

The use of such advective equations seems nearly universal in economics. (Making this change involves re-tuning all the variable parameters a_0, a_3, b_0, b_2, b_5 to match the data).

Secondly, there is no positive feedback in natural emissions due to temperature change (Equation 2.6). Such feedbacks are typically lacking in economic analyses of climate change (although the PAGE models do include variants).

Bronselaer et al. (2013) focus on the distinction between the 'diffusive' and 'advective' variants of the model: four models are considered, corresponding to the the combinations one may take: Equation (2.2) or (2.8) and Equation (2.5) or (2.9).[25]

Bronselaer et al. (2013) do not consider a model without temperature feedbacks. We briefly outline here the distinctions between possible model variants before restriction attention to 'Model A' and 'Model B'.

The conclusions of Bronselaer et al. (2013) regarding model comparison are broadly as follows: the diffusive forms lead to greater warming, but this takes the form of the system continuing to warm for longer, rather than faster warming in the short term; a diffusive form for temperature makes more difference than a diffusive form for carbon; the diffusive forms lead to a greater envelope of uncertainty; at least for the temperature equations (where the difference is greater), the diffusive form better fits the output of more complex models.

The temperature responses to three possible emission scenarios under these four models, using the most likely parameter combinations, are plotted in Figure 2.2. The key observation is that in the near to medium term, warming depends mainly on the cumulative emission scenario, but by 2100 (sooner for a lower emission scenario) the distinction due to model choice is at least 1°C, which persists. So if long-term damages are important for welfare now, the model chosen is important.

In the most recent IPCC report, Stocker et al. (2013) report a wide range of possible temperature evolution pathways post 2100 (see Figure TS.15). Bronselaer et al. (2013) do not provide conclusive evidence that the diffusive forms provide the best approximation to more complex models in the long term. However, Allen et al. (2009) show, in Figure 2, that the peak warming simulated by their model lies slightly below the centre of the range of outputs from more complex

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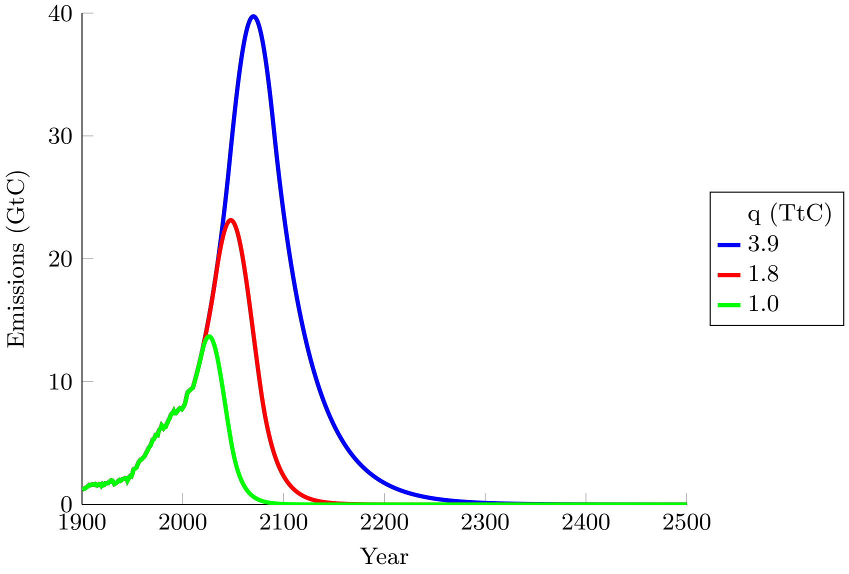
use the parameter values of Gerlagh and Liski (2013) but the values and uncertainties built in by the physicists. Future work will also consider the comparison with the DICE module itself.

[25] As Ben Bronselaer has kindly shared with me an early version of his code, it is straightforward to investigate the economic implications of these difference. Bronselaer et al. (2013) also make other changes, principally to the likelihood profiles for the constraints on the model, but as these are under revision they not yet been incorporated in the present model, whose responses remain more closely in line with others in the literature.

(a) Sample emission scenarios



(b) Resulting temperature pathways

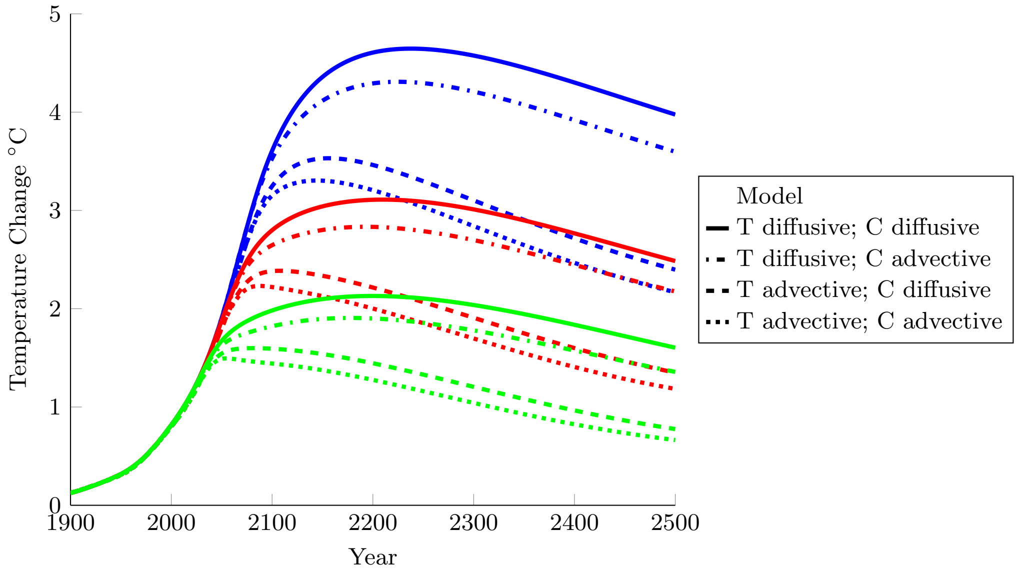


Figure 2.2: *(a) Three emission scenarios used (cumulative totals given in legend). (b) the temperature responses to each (colours correspond) using the four models of Bronselaer et al. (2013) and the 'best guess' geophysical parameter combinations.*

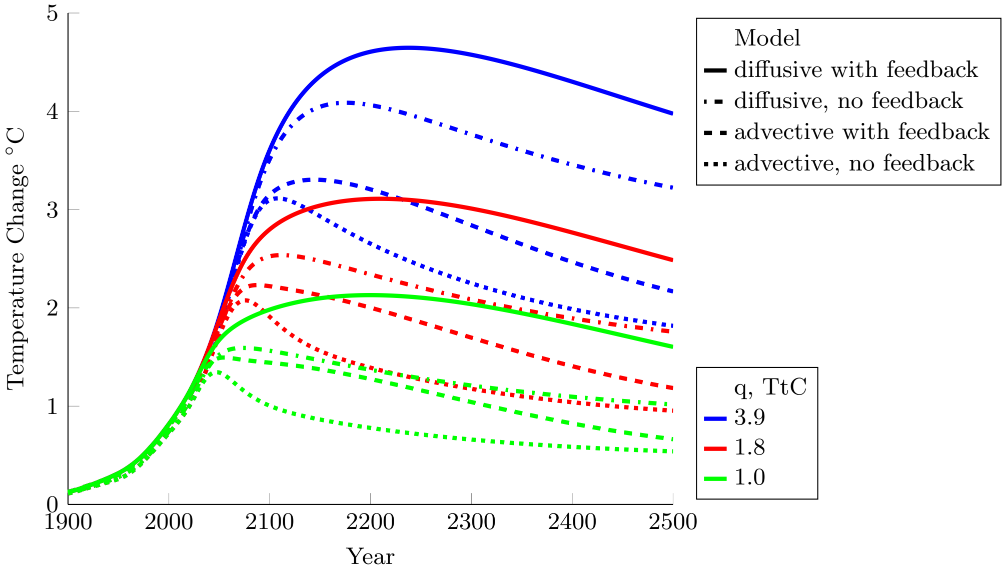


Figure 2.3: *The temperature response to three emission scenarios, using the entirely diffusive and entirely advective variants from Bronselaer et al. (2013), and two additional variants in which the temperature feedbacks are removed. The model variant is as given in the first legend; cumulative emissions are as given in the second legend.*

models; the advective model form gives rise to considerably lower peak warming. It is certainly worth investigating the economic impacts of using the fully diffusive model of Allen et al. (2009) in place of the advective version.

We now consider the effect of setting the feedback coefficient b_5 to zero. To avoid proliferation of models, we restrict attention to the model in which both carbon and temperature are diffusive (as in Allen et al., 2009, the main model for the current paper) with the case in which they are both advective (as in most climate modules in economic models), and experiment in each case with and without the temperature feedbacks: see Figure 2.3.[26] Note that, without temperature feedbacks, temperatures do not reach as high and drop much more rapidly after their peak; for the lower emission scenarios temperatures are falling between 2100 and 2200 without these feedbacks, but still rising if they are included and we use the diffusive model.

It is instructive to note in Figure 2.3 that the choice of a diffusive model over an advective form makes a greater difference than the inclusion of feedbacks.

The most dramatic comparison is also the relevant one: the solid line represents the model of Allen et al. (2009), our 'Model A', and the dotted line, our 'Model B', which approximates a standard geophysical component in an integrated assessment model.

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[26] In each case, the parameter choices are those giving the best fit with the data according to the model of Bronselaer et al. (2013), except that without feedbacks, the data regarding feedbacks has been ignored (as setting the feedback parameter to 0 is otherwise 'too unlikely' in the model). Thus the 'advective, no feedbacks' variant is structurally the same as the model of Gerlagh and Liski (2013) but may incorporate slightly different parameters; this facilitates comparison between the structural forms of models rather than interpretations of data.

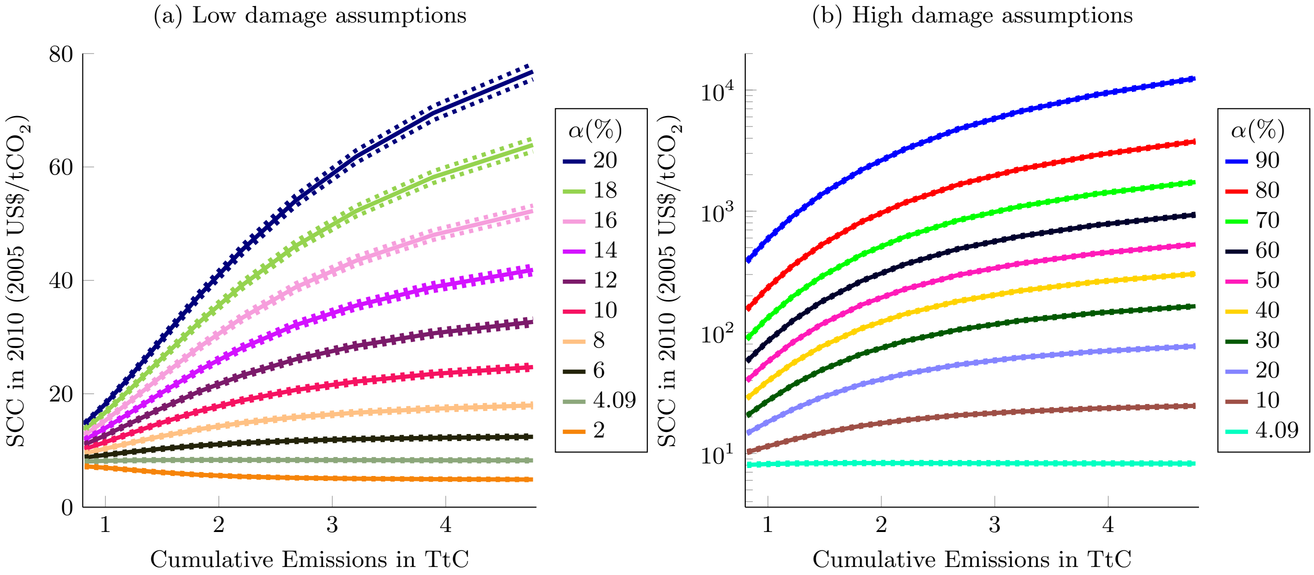

Figure 2.4: *The social cost of carbon as a function of cumulative emissions, for multiple damage function assumptions, that is, multiple values of* α.

The question remains of how much the discrepancy visible in Figure 2.3 matters economically. We come to this in Section 2.3.2, where we shall refer to the diffusive model with feedbacks (Allen et al., 2009) as 'Model A', and the advective model without feedbacks (as in Gerlagh and Liski, 2013), as 'Model B'.

2.3 Results

2.3.1 Main results: the SCC as a function of emissions and assumed damages

The principal output of the model is the social cost of carbon, as a function of cumulative emissions and assumed damages at 4°C, using the diffusive model of Allen et al. (2009).

The output is given in Figure 2.4. In Figure 2.4(b), in which damages at 4°C are allowed to vary all the way up to 90% of consumption, the social cost of carbon increases so dramatically with assumed damages that the vertical axis is given in a log scale. The standard errors of the estimators of these means is very low, as shown by the 95% confidence intervals given in each case (in Figure 2.4(b) these are so relatively small as to be barely visible).

The basic lesson from Figures 2.4 is clear:

Stylised Result 1.

(i) The SCC is sensitive to assumed damages at 4°C, and this sensitivity increases with cumulative emissions.

(ii) When damage assumptions are above those of DICE 2013, the SCC is sensitive to cumulative emissions, and this sensitivity increases with assumed damages at 4°C.

(iii) The SCC gets so large with high α that one would need very strong assurance that such assumptions are not valid for them to be irrelevant to the analysis.

Figure 2.4(a) shows clearly the insensitivity of the SCC to cumulative emissions when damages are close to the 'quadratic' damage function (which corresponds to $\alpha = 4.09\%$). This is the result of Hope (2006). But Figure 2.4(a) also shows that this result is not robust to different damage assumptions. So the 'independence' should be considered a special case when damages are quadratic at all temperatures (recall that, for every damage function presented here, damages are quadratic up to 2.5°C). However, Figure 2.4(a) implies some sensitivity of the SCC to damage assumptions even at the 'quadratic' damage level. This is explored further in Figure 2.5.

Figure 2.4(b) presents some remarkable numbers. Their relevance depends very much on the plausibility of the corresponding damage functions; we simply model the full range of damages at 4°C without prejudice as to what a good model might be. But recall (Section 2.2.1) that we have very little idea what the world might look like after 4 degrees of warming. Because the SCC grows so fast with assumed damages at this level, any probability density function assigned to damages from 4 degrees of warming would have to have an extremely 'thin' tail for no damages over, say, 30% output to have relevance. The only study on expert opinion as to damages from higher levels of warming, Nordhaus (1994), finds "vast disparities in estimates of the economic impact of potential greenhouse warming".

We investigate further the sensitivity of the SCC to damage assumptions and cumulative emissions by plotting its elasticity to the these two variables: Figures 2.5 and 2.6. There are slightly larger standard errors in the estimators due to the fact that the Monte Carlo importance sampling was focused on the social cost of carbon.

Figure 2.5 gives our clearest depiction of the sensitivity of the SCC to assumed damages at 4°C. Elasticities are small (unless assumed damages at 4°C are dramatic) at very low emissions. This should not be a surprise, given the way the range of damages functions was structured: they do not vary up to 2.5°C, and it is not very likely that this temperature is passed under the very lowest emission scenarios. However, the elasticity increases fast with cumulative emissions – even for very low damage assumptions. In particular, the quadratic damage curve ($\alpha = 4.09\%$) has an elasticity of 0.43 at $q = 1.8$ TtC, and of 0.60 at $q = 2.7$ TtC.[27] Thus, even for the standard assumptions of DICE and similar models, one can expect a noticable effect from changing assumptions about damages which are beyond the calibration points of these functions.

As is clearly seen in Figure 2.5, elasticities rise consistently with cumulative emissions, and with damage assumptions. Values hover between 0.5 and 1.5 for lower damage assumptions, and between 1 and 2 for higher damage assumptions, but higher values are possible for the most

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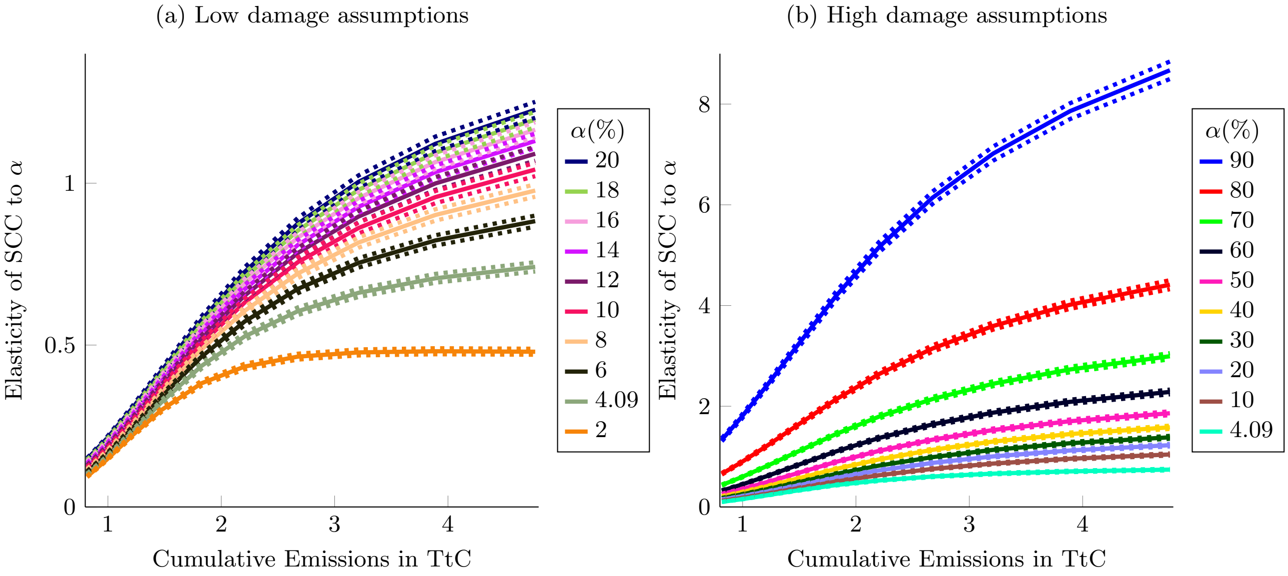
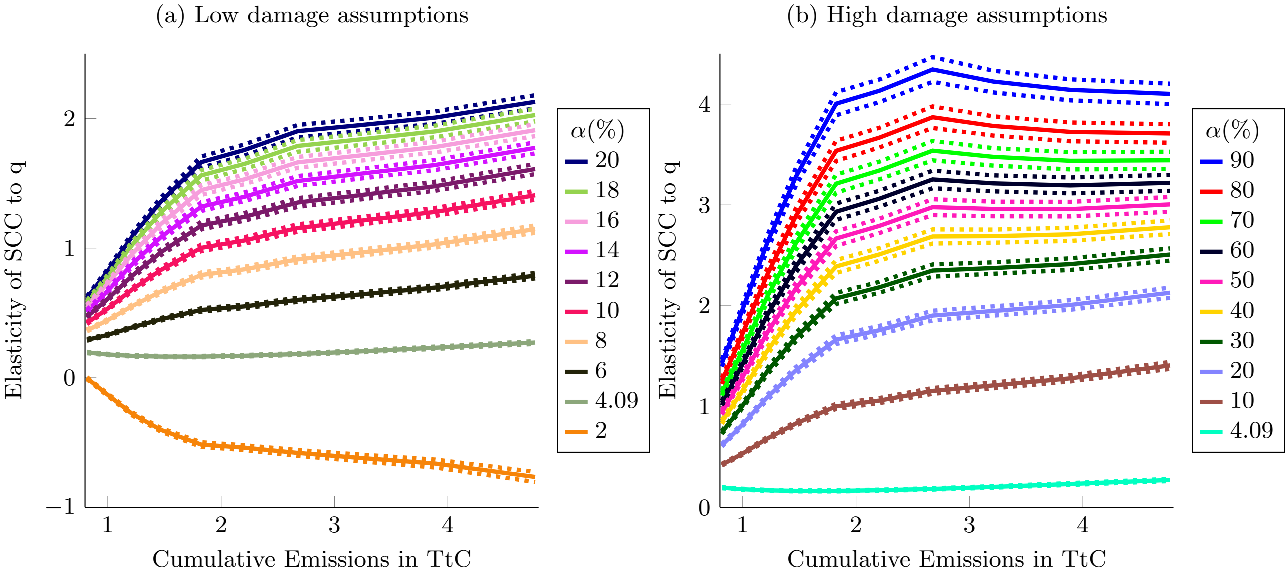
[27] These quantities gives rise to peak concentrations of around 550ppm CO_2, and 660ppm CO_2 respectively, under Model B, and so will correspond to the range of emission volumes considered optimal by many integrated assessment models.



(a) Low damage assumptions

(b) High damage assumptions

Figure 2.5: *Elasticity of the SCC to assumed damages at 4 degrees, α, plotted for multiple such damage assumptions α. The dotted lines denote the 95% confidence intervals of the estimators.*



(a) Low damage assumptions

(b) High damage assumptions

Figure 2.6: *Elasticity of the SCC to cumulative emissions q, plotted for multiple damage assumptions α. The dotted lines denote the 95% confidence intervals of the estimators.*

extreme assumptions.

As Figure 2.6 shows, once cumulative emissions pass a threshold, the elasticity of the social cost of carbon to cumulative emissions is rather high; it is in the range of 1–2 for most of the lower damage assumptions considered and reaches 2–4 and above as we increase these assumptions. For $\alpha = 2\%$ it is negative, and only if we assume quadratic damages ($\alpha = 4.09\%$) is the elasticity close to zero. Thus Figure 2.6(a) provides solid evidence that, if there is an independence of the SCC to cumulative emissions (as some of the literature seems to show, or builds into models), it applies only for the particular quadratic damage function assumption.

The shape of the curves in Figure 2.6 is interesting. Elasticities are steeply rising for low cumulative emissions, but once cumulative emissions are high enough, the elasticity is nearly constant – and even decreases for the highest damage assumptions. This is explained by recalling that damages are assumed to be quadratic up to 2.5°C, and that quadratic damages have near zero elasticity to q; the temperature change is likely to be below 2.5°C for the lowest emission scenarios.

Figures 2.5 and 2.6 show that the SCC is in general a little more sensitive to cumulative emissions than it is to damage assumptions, as long as damage assumptions are above the 'quadratic' level. However, one should recall that a very particular way for damages to enter the model, and for their level to vary, has been assumed; it would be interesting to investigate further whether this finding is robust.

We may thus refine and add to Stylised Result 1:

Stylised Result 2.

(i) The SCC is inelastic to assumed damages at 4°C only if cumulative emissions are very low and assumed damages are not too high. Otherwise the elasticity is broadly in the range of 0.5–2, and rises with q and α.

(ii) The SCC is inelastic to cumulative emissions q only if climate damages are closely approximated by a quadratic function for all plausible temperature changes. Otherwise the elasticity is broadly in the range of 0.5–4, and rises with α.

2.3.2 The effect of model choice: geophysical form

The analysis of Section 2.3.1 is repeated for 'Model B': the model with advective temperature and carbon cycles, and no feedback in emissions.[28] See Figures 2.7, 2.8 and 2.9. These show our stylised results are fairly robust to geophysical model choice, though some are more muted in the case of Model B.

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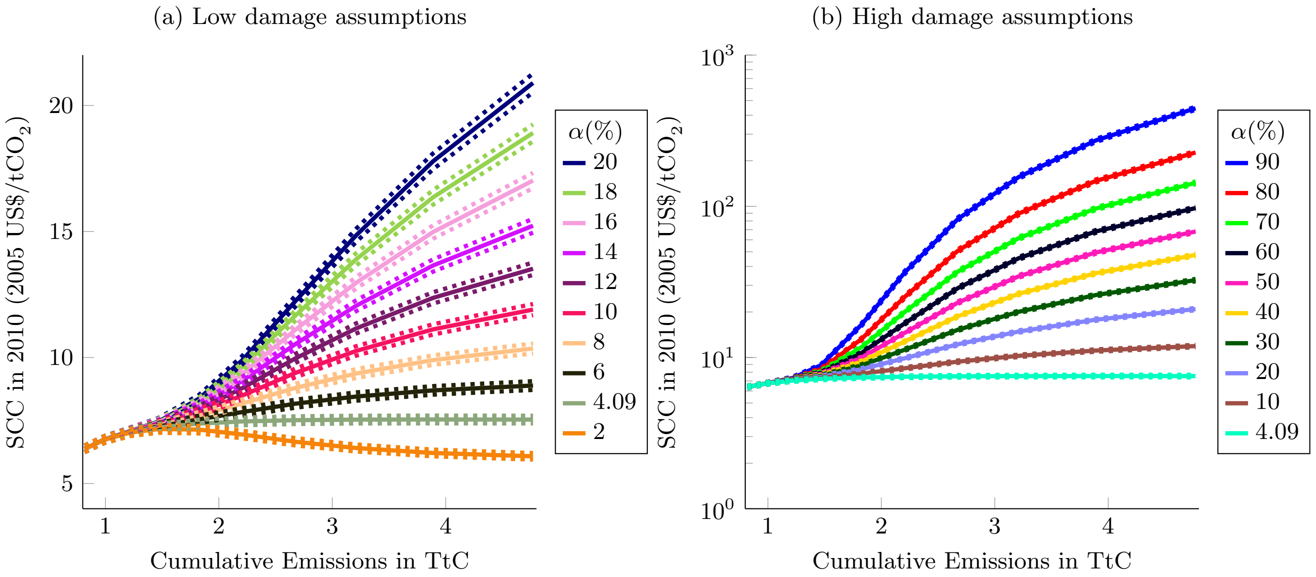
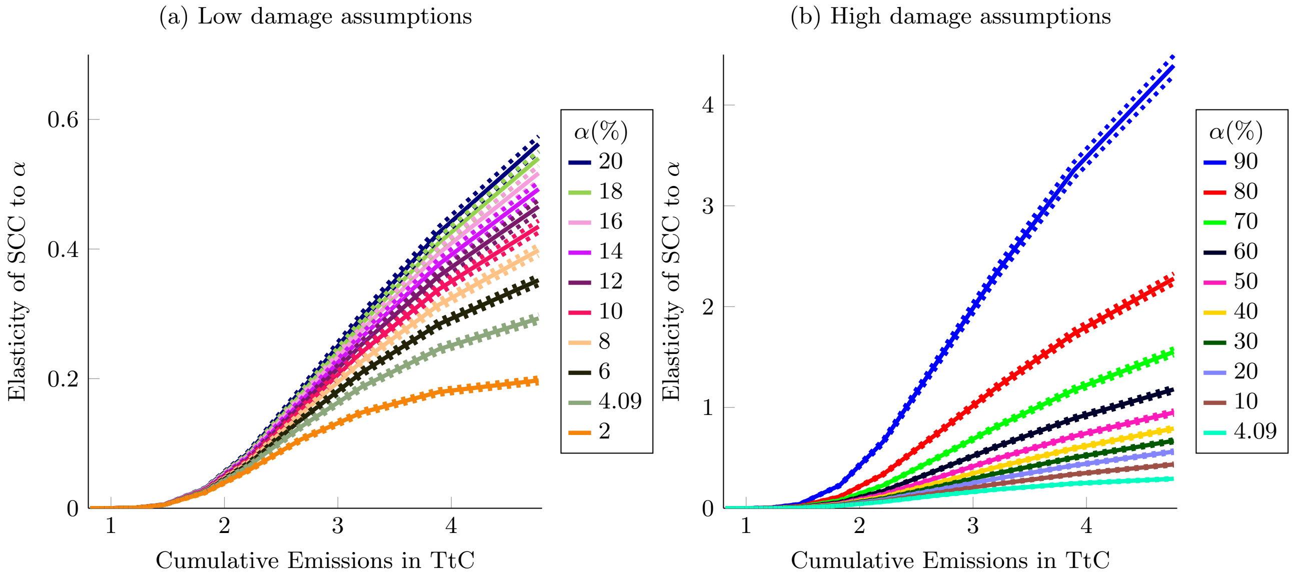
[28] As there are fewer uncertain parameters and a narrower range of likely temperature responses to a given emission scenario in Model B, fewer Monte Carlo calls are needed to obtain reasonably effective estimators of the means. Thus the model runs the Vegas algorithm against six choices of (q, α) using only approximately 1500 calls in each case.


(a) Low damage assumptions

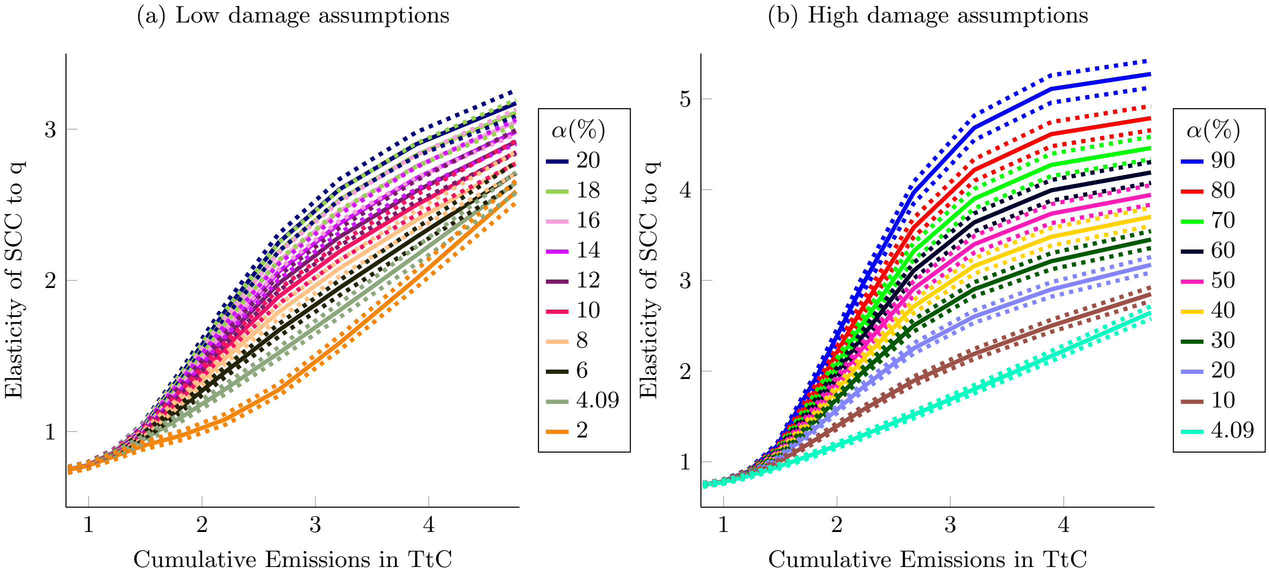
(b) High damage assumptions

Figure 2.7: *The SCC in Model B (advective temperature and carbon cycles, and no feedback in emissions).*


(a) Low damage assumptions

(b) High damage assumptions

Figure 2.8: *The elasticity of the SCC to damage assumptions α in Model B (advective temperature and carbon cycles, and no feedback in emissions).*


(a) Low damage assumptions

(b) High damage assumptions

Figure 2.9: *The elasticity of the SCC to cumulative emissions q in Model B (advective temperature and carbon cycles, and no feedback in emissions). The dotted lines denote the 95% confidence intervals of the estimators.*

The fact that the SCC is lower in Model B should come as no surprise. Recall from Figure 2.3 that temperature changes are higher and more persistent when the temperature and carbon cycles are diffusive and when emissions are added to at higher temperatures by natural feedbacks.

The most striking difference between Figure 2.4 and Figure 2.7 is that, in the former case, the SCC is seen as being sensitive to α even for the tightest emission scenarios, whereas in the latter, cumulative emissions need to pass about 1.5 TtC for these assumptions to have any effect. This difference is due to the very much lower temperatures predicted by Model B, as well as the smaller range due to uncertain geophysical parameters, and the fact that we only vary the damage function beyond the temperature 2.5°C. This same point is seen again in Figure 2.9: elasticities to α are roughly half what they are in the Model A case, and are greater than zero only once $q > 1.5$ TtC.

Interestingly, the elasticity of the SCC to cumulative emissions can be *higher* in Model B than in Model A, especially with low damage assumptions. The SCC is generally lower for this model and so the same rate of change will correspond to a greater elasticity.

To quantify the economic consequences of the using the more geophysically sophisticated Model A, instead of the more commonly used Model B, we plot the percentage additional social cost of carbon, obtained by using the former model instead of the latter, in Figure 2.10. The percentage increase in social cost of carbon is remarkable, even for lower damage function assumptions (Figure 2.10a), and astronomical for high damage function assumptions (Figure 2.10b).

Recall from Figure 2.3 that, in the short term, the temperature paths are the same in either

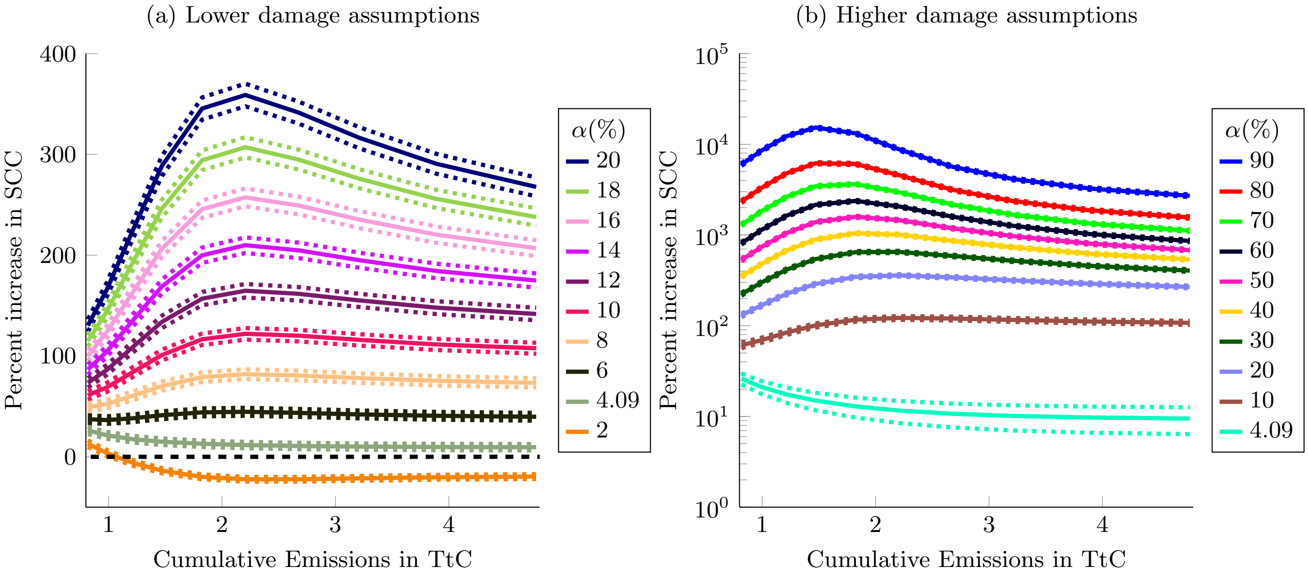


Figure 2.10: *The percent additional SCC due to using Model A instead of Model B. Dotted lines give the 95% confidence intervals of the estimators.*

model. But in Model A, temperatures continue to rise after 2100 (even though emissions may have come down well before this date) and, once they do decline, they do so much more slowly. Figure 2.10 gives a clear economic verdict to this.

Stylised Result 3. If the standard quadratic assumptions are correct, then geophysical model choice is relatively unimportant, but if there is even a relatively small chance that they do indeed seriously underestimate damages from 4°C warming, then these modelling assumptions are critical.

However, to fully quantify the difference made by using the 'diffusive' model, one should run a comparison with the geophysical component of an integrated assessment model in common usage; future work will complete this.

2.3.3 The effect of modelled uncertainty

All results presented so far give the mean SCC, over five uncertain geophysical parameters, as the sense described in Section 2.2.3.2. A simple non-parametric density estimator allows us to plot the probability density function of the SCC for every (q, α) combination: see Figure 2.11. They give some idea of the 'long tail' of high values for the SCC. Only the figures corresponding to cumulative emissions q at the lower end of the range considered have been plotted; for higher values of q and α the plots are qualitatively similar to Figure 2.11(i).

As described in Section 2.2.3.2, all estimates of means have been made by taking a uniform prior to develop a probability density function for the geophysical parameters, from their relative

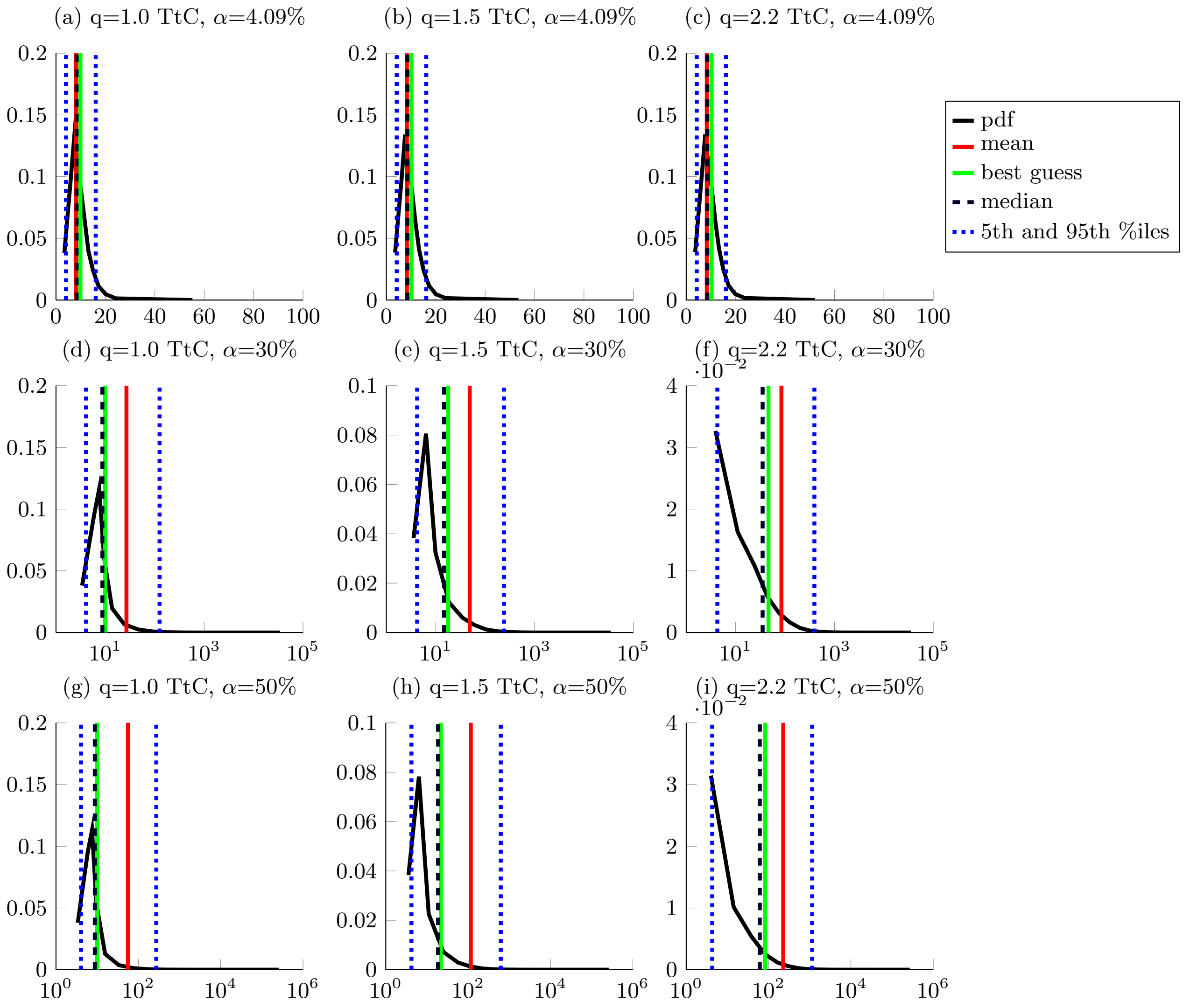


Figure 2.11: *Estimated SCC probability density functions, conditional on cumulative emission and damage assumptions as given in the subplot titles. In each case the vertical axis represents probability density and the horizontal axis represents the SCC (2005 US\$/tCO_2); subplots (d)–(i) use a log scale for this. Also plotted in each case are the mean, median, 5^{th} and 95^{th} percentiles, and the SCC using the 'most likely' geophysical parameters.*

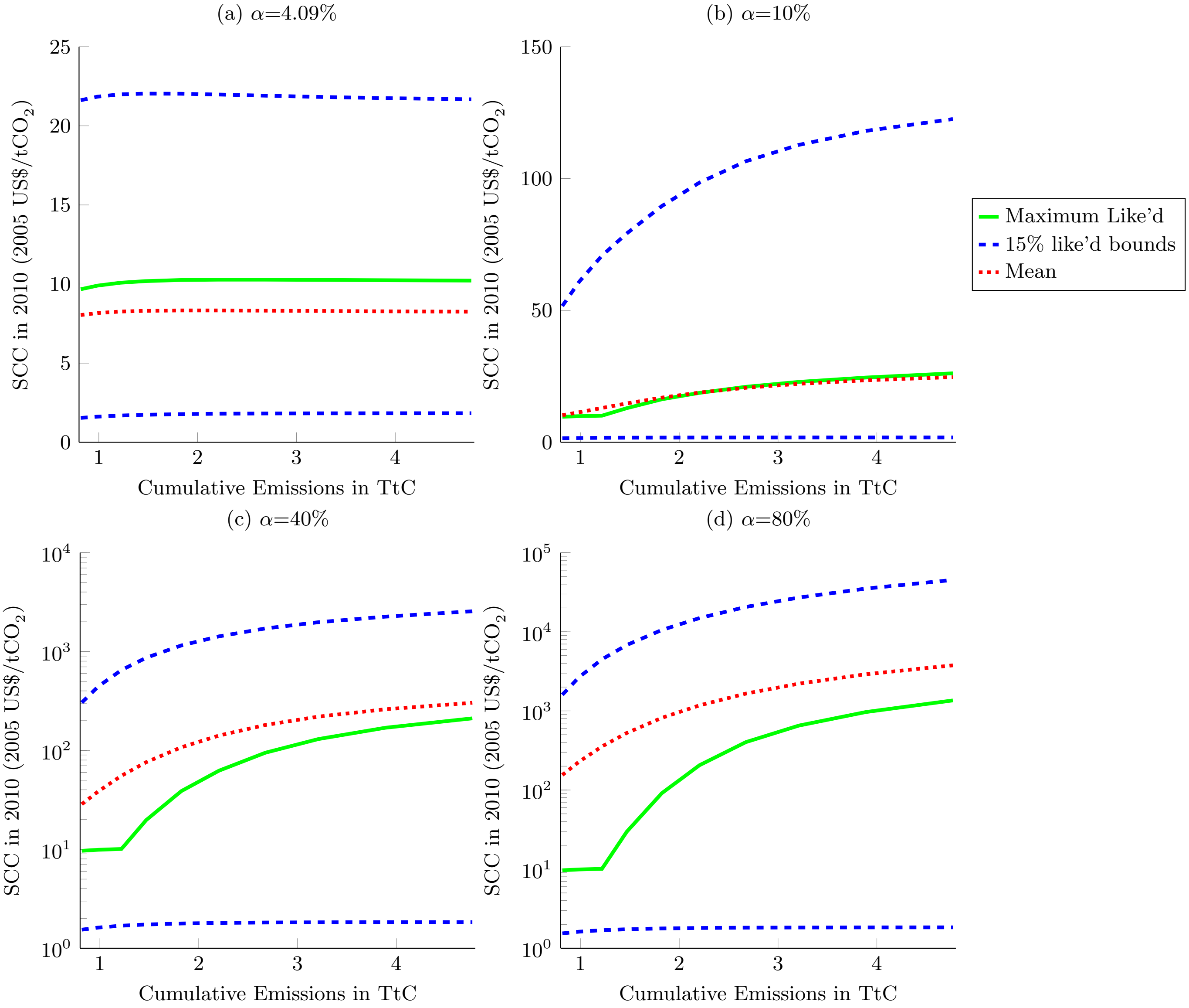


Figure 2.12: *15% likelihood intervals on the social cost of carbon, with the mean value also plotted for comparison. The legends give the damage parameter α, that is, output damages from 4° C.*

likelihood functions. This uniform prior should be regarded as another exogenous assumption we have imposed on the model. A statistically more robust approach is to draw likelihood intervals; we do this in Figure 2.12 for a selection of damage functions (plotting these for all all the possible damage functions makes the figures hard to look at). For comparison, the calculated means are also included.

Figure 2.12 makes two things clear. First, a wide range of values for the social cost of carbon

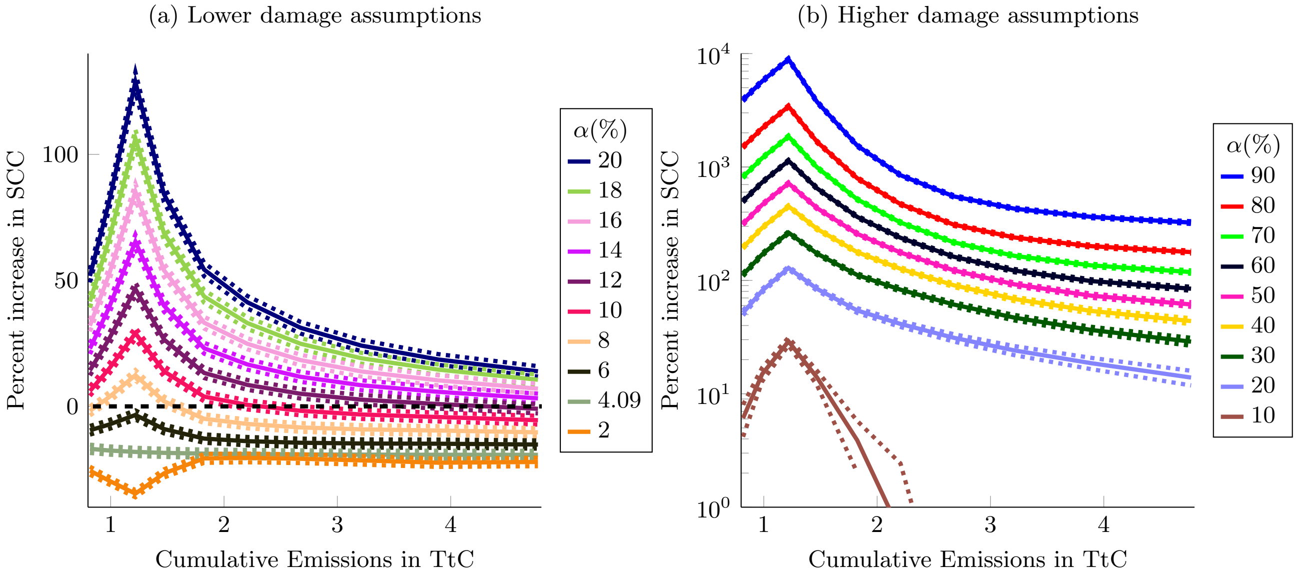


Figure 2.13: *The percent additional SCC due to using means instead of the best guesses of parameters. The legends give the damage parameter α.*

are reasonably consistent with the geophysical data, the more so when assumed damages from higher levels of warming are high. Second, for higher assumptions on damages, the estimated mean is considerably above the value we obtain if we use the most likely geophysical parameters – but for lower damage assumptions, the difference is much smaller and the estimated mean may even be beneath the best guess.

The second of these points is no surprise, given Jensen's inequality: the higher damage assumptions make damages very convex in temperature change, and so an expectation would be expected to be higher than the function of best guess. We can investigate this further by plotting the percent additional social cost of carbon we obtain by using means instead of maximum likelihood parameter choices: see Figure 2.13.[29]

Here we may test the importance of the insight of Weitzman (2007, 2009a) that economic models of climate change do not correctly reflect scientific uncertainty in geophysical parameters. We see that the mean is close to or below the maximum likelihood value for $\alpha \leq 8\%$ output damages, but can be very substantially higher as the damage assumptions increase.

Stylised Result 4. If standard assumptions about damages are correct, then incorporating scientific uncertainty slightly *decreases* the social cost of carbon. However, if there is even a relatively small chance that such assumptions do seriously underestimate damages from 4°C warming, then incorporating scientific uncertainty is critical.

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[29] Both Figures show a spike at $q = 1.22$ TtC. This is because the best guess geophysical parameters lead to a temperature pathway which peaks at just below the crucial level 2.5°C; of course there is in fact a substantial probability that this threshold is passed. This feature, then, is just an artefact of the particular damage assumptions made.

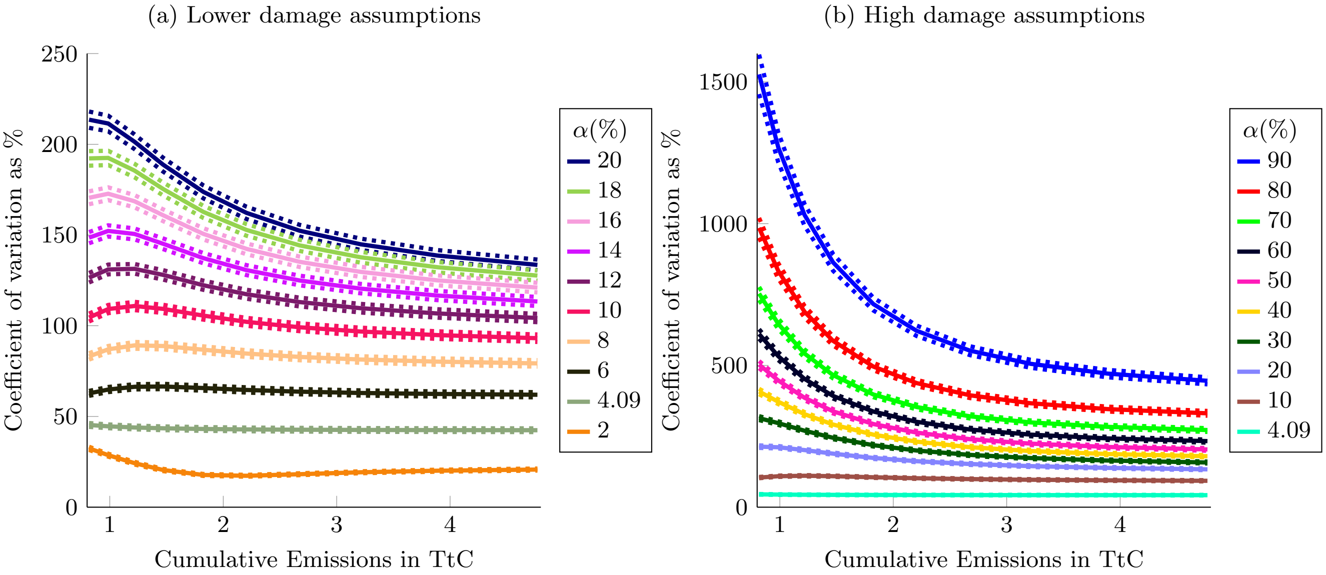


Figure 2.14: *The coefficient of variation of the social cost of carbon. The legends give the damage parameter* α*.*

A very interesting comparison is between Figures 2.10 and 2.13. The relative effect of geophysical model choice appears to be more significant than the inclusion of uncertainty, especially if we consider output damages at 4°C in the range of 6–20% (which may well be the most important range). However, we have so far started with Model A and uncertainty, and considered the relative effects of 'downgrading' either to remove uncertainty, or to switch to Model B. We consider the relative effects of either 'upgrade' from Model B without uncertainty in Section 2.3.4.

First, we explore variability in the SCC further. The standard deviation of the SCC can also be estimated using the Monte Carlo samples. The statistic that enables us most clearly to see this most clearly, while changing both q and α, is the coefficient of variation (the ratio of the population standard deviation to the mean): see Figure 2.14. Thus we see that, under the quadratic assumption, the standard deviation is consistently close to 40% of the mean, but that it rises to being as large as the mean itself as we increase damages to 10% of output at 4°C. And increasing assumptions on damages further only exacerbates this picture. (Of course the skew of the distributions, clearly visible in Figure 2.11, shows that we cannot interpret a standard deviation which is larger than the mean itself as saying that the SCC may very well be negative!)

The fact that the coefficient of variation is highest for the *lowest* cumulative emission scenarios reflects the fact that the SCC is lowest there too; the standard deviation itself increases with q.

Stylised Result 5. The standard deviation of the SCC, due to variation in geophysical parameters, is between 50–150% of the mean for low to moderate damage assumptions and increases rapidly with assumed damages.

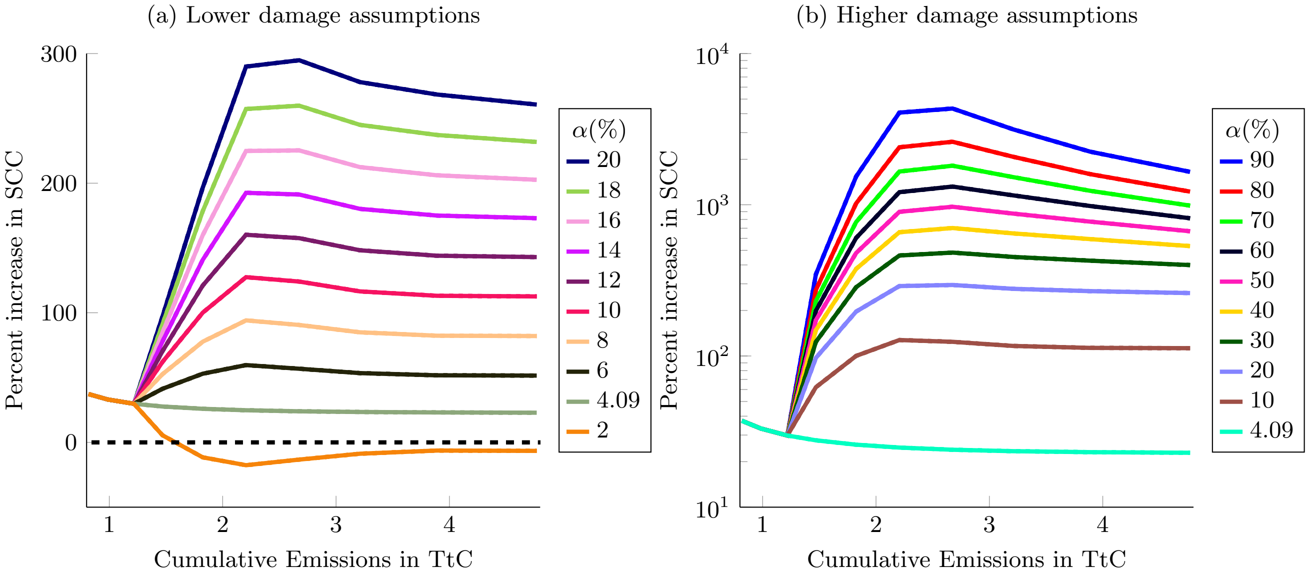


Figure 2.15: *The percent increase in SCC due to using Model A instead of Model B, when we assume all geophysical parameters are fixed at their 'best guess' values.*

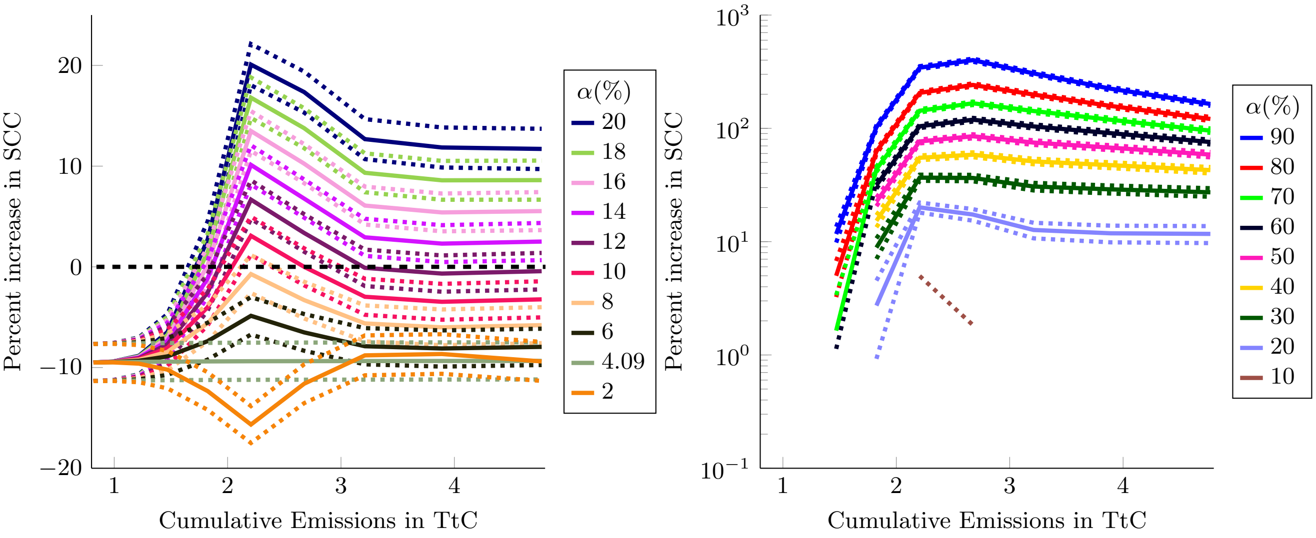
2.3.4 The relative effects of intra- versus inter-model variability

Weitzman (2007, 2009a) and others have written at length about the importance of accurately reflecting uncertainty. This is typically taken to mean scientific uncertainty in geophysical parameters, but another important source of uncertainty will always be uncertainty in the mathematical form of the model. This comparison is already visible in Figures 2.10 and 2.13. There, however, one is making the comparison from the point of view of *starting* from the sophisticated model and removing the two aspects. Here we complement them by assuming that one starts with the *less* sophisticated geophysical component, Model B, and *without* accounting for uncertainty. We now plot the proportional increase in SCC due to switching to Model A (Figure 2.15) or to incorporating uncertainty in geophysical parameters (Figure 2.16).

From both comparisons, the conclusion is clear:

Stylised Result 6. If damages from higher levels of global warming are higher than conventionally modelled, then using a geophysical model in which temperatures continue to rise longer and then fall less quickly, makes substantially more difference than does incorporating geophysical uncertainty. If standard assumptions about damages are correct, then neither change makes a very great difference.

A very strong caveat to this result is that this paper uses scientific uncertainty as estimated only by one research group. Although their calibration does use data from several difference sources, they have imposed a lognormal distribution in errors in all of these data streams (see Section 2.2.3.2). This 'Stylised Result' should be tested for robustness against other specifi-



(a) Lower damage assumptions (b) Higher damage assumptions

Figure 2.16: *The percent increase in SCC due to incorporating uncertainty in geophysical parameters in Model B. Dotted lines give the 95% confidence intervals of the estimators.*

cations. The uniform prior imposed in this paper also modifies this result, although one may hypothesise that it leads to *too* diffuse a posterior pdf and so may *overestimate* the relative effect of uncertainty.

2.3.5 Sensitivity of results to abatement costs

The structural form of abatement costs is highly simplified: recall (Equation 2.1) that they are linear in the rate of emission reduction below a trend of 3% growth per year. The justification for this is that these costs should not be critical when calculating the social cost of carbon under a fixed emission pathway. (Of course, they become critical if one wishes to calculate the *optimal* emission trajectory and social cost of carbon, but that is explicitly not the intention of this paper.) But this justification should be tested with a sensitivity analysis that varies the linear factor in question, p_E.

It is only possible to perform such a analysis for the 'best guesses' of the geophysical parameters; calculating the mean across these parameters for a whole range of potential abatement cost parameters would be computationally infeasible. See Figure 2.17 for the output. As shown in Figure 2.17(a), changing the abatement cost parameter does affect the social cost of carbon, to a minor extent.

It is useful to see how abatement costs in this model compare to the values found in the literature. Edenhofer et al. (2010) provide one of the first studies considering the economics of low stabilisation levels of greenhouse gases; they use and compare five 'global regionalized energy-environment-economy models' (E3) models. The numbers they give which are most easily

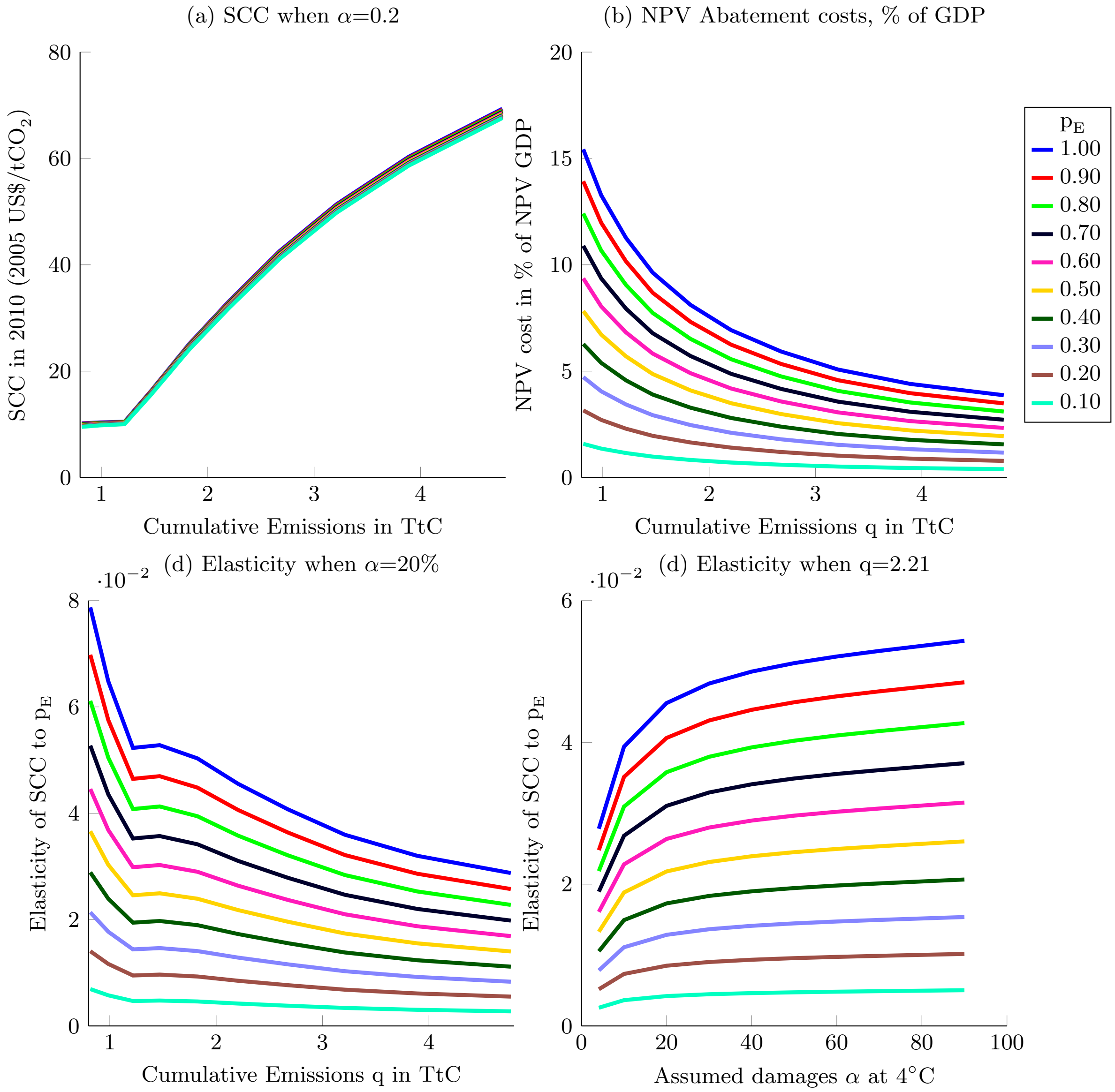


Figure 2.17: *Figures illustrating the sensitivity analysis on the abatement cost parameter p_E. The legend gives the corresponding values of p_E. Part (a) shows the SCC as a function of q, when $\alpha = 20\%$, for each of the abatement costs p_E. Part (b) shows NPV abatement costs, expressed as a % of GDP, for each p_E. Parts (c) and (d) show the elasticity of the SCC to p_E; (c) fixes $\alpha = 20\%$ and allows q to vary, while (d) fixes $q = 2.21$ and allows α to vary.*

comparable with the output of the present model are cumulative GDP losses relative to baseline, in percent of baseline cumulative GDP, using a 3% discount rate (their Figure 7).[30] They find mitigation costs of 1-2.5% of cumulative GDP for their most ambitious scenario, stabilisation at 400ppm; stabilisation at 550ppm costs 0.5-0.8% of cumulative GDP. However, these numbers are discounted back to 2000; the authors suggest a 5% rise for a base year of 2005 but do not suggest a figure for 2010. Delays in abatement compound the costs. These costs are low in comparison with others in the literature; Fisher et al. (2007, Figure 3.25) synthesises the results from many papers on less stringent stabilisation scenarios; unfortunately it does not express the cost of abatement in the same terms, but it does show annual output losses in the full range from 0 to 5% GDP from 2030 to 2100, implying similar figures for cumulative GDP.

Performing the same calculation with the present model provides Figure 2.17(b). The cost parameter used throughout the paper is 0.5, and so costs are 7.5% cumulative GDP output for tight abatement scenarios and 3–2% for less ambitious strategies. Comparison with Edenhofer et al. (2010) suggests that this is slightly high, but it is within the range that seems reasonable from Fisher et al. (2007). Values of p_E between 0.1 and 0.6 seem consistent with the literature.

In general, estimates SCC might either increase or decrease with abatement costs. Increases are due to the increase in marginal utility in a poorer world; decreases are due to modelling per-period damages as multiplicative in output, so that a decrease in output correspondingly reduces loss. In fact the particular forms used in the present model ensure that the SCC will always increase with abatement costs. Figures 2.17(c) and 2.17(d) show that this increase is not large; the elasticity is always below 0.08 and is below 0.05 for the values of p_E we have identified as being of interest. Thus the assumption that abatement costs are of second order is sound: any inaccuracy in the choice of p_E or, by extension, the functional form of abatement costs, will not have greatly affected the results of the remainder of this paper.

2.4 Conclusion

This paper asks first whether the social cost of carbon depends on the assumptions we make on damages from higher levels of climate change. The answer is a resounding yes. And although the elasticity of the social cost of carbon to cumulative emissions is close to zero for the traditional ‘quadratic’ damage function, it increases the further we deviate from this assumption. Similarly, the use of a ‘diffusive’ coupled carbon and climate cycle mode, which also accounts for feedbacks in natural emissions, and the incorporation of uncertainty in geophysical parameters, both substantially increase estimates of the social cost of carbon unless the standard ‘quadratic’ assumption holds. Perhaps surprisingly, it is the use of the more sophisticated model form which makes the greater difference, and not the incorporation of uncertainty. None the less, the standard deviation of the social cost of carbon is high, possibly several times the mean, and its

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[30] This incorporates the output from four models: MERGE-ETL, (Kypreos and Bahn, 2003, Kypreos, 2005), REMIND-R, Leimbach et al. (2010), POLES, European Commission (1996) and IMAGE/TIMER, Bouwman et al. (2006).

distribution has a long right tail: the importance of uncertainty should not be understated.

If the SCC is indeed rather sensitive to cumulative emissions, and hence to final policy outcomes, this has important implications for climate policy. If the social cost of carbon were indeed independent of our final abatement outcome then we need only set a carbon tax at this level; without this independence our policy choice will require more information about what the outcome of a given carbon tax might be.

Also, when we consider our choice of policy instrument under uncertainty, our choice between focusing on the carbon price or a quantity restriction is influenced by the relative slopes of marginal cost and marginal benefits of mitigation; see Weitzman (1974) and Chapter 1 of this thesis.

This paper has explored the impacts of several aspects of the modelling. However, other parameters will influence the results. A further exploration of our 'incomplete valuation' of climate change would be interesting. The growth in total factor productivity, the pure rate of time preference and the elasticity of marginal utility are obvious candidates; the 'high' and 'low' scenarios of population growth from United Nations (2013) could also be used; different choices in prior distribution should be used in the conversion from likelihoods to probability density function.

Criticism has been levelled at the way damages enter the model in DICE, a form copied here. Pindyck (2012) shows how climate change can instead be modelled as acting on growth, and Stern (2013) argues that they should additionally act on the capital stock. Sterner and Persson (2008) work with environmental and material goods; climate damages disproportionately affect the former, and so give rise to large changes in relative prices. More broadly, one could criticise the economic component of this model for its excessive simplicity. Many such changes and refinements could be incorporated into the Matlab model which has been built for this paper; it is hoped that future work will do so.

Chapter 3

Tropical Geometry to Analyse Demand

Elizabeth Baldwin and Paul Klemperer

Abstract

Duality techniques from convex geometry, extended by the recently-developed mathematics of tropical geometry, provide a powerful lens to study demand. We propose a new framework of "demand types", for categorising and understanding demand; our classification both incorporates existing definitions (such as substitutes, complements, "strong substitutes", etc.) and permits additional distinctions. We obtain easy-to-check necessary and sufficient conditions for the existence of a competitive equilibrium for indivisible goods. Our techniques also underpin Klemperer's (2008) *Product-Mix Auction*, introduced by the Bank of England in the financial crisis.

3.1 Introduction

This paper introduces a new way to think about economic agents' individual and aggregate demands for indivisible goods,[1] and provides a new set of geometric tools to use for this. Our model applies to agents who buy and/or sell, as well as to some matching models.

Economists mostly think about agents' demands by focusing on the direct utility functions. We instead begin by focusing on the geometric structure of the regions of price space in which an agent demands different bundles. Our crucial observation is that dividing price space in this

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[1] Baldwin and Klemperer (in preparation) show the relevance of our techniques to analysing *divisible* goods also, in contexts such as the Product-Mix Auction.

way creates *precisely* the geometric structure which is studied in the recently-developed, non-Euclidean, branch of algebraic geometry called "tropical geometry".[2] We can therefore use the tools of convex and tropical geometry, such as the duality between the geometric structure of an agent's demand in price space and the same agent's demand in quantity space, to obtain new insights about demand. Moving backwards and forwards between the dual representations of demand in price space and quantity space improves our understanding of both.

For example, it is much easier to aggregate individual demands in price space, but translating aggregate demand back into quantity space allows a strong theorem that encompasses and extends many existing results about when a competitive equilibrium exists.

On the other hand, if we start from the (direct) valuation function in quantity space, our methods for translating to the dual in price space quickly reveal the key properties of demand. Many existing results in demand theory can be understood more readily, and developed more efficiently, using our tropical-geometric perspective than using traditional methods.

Individual demand

Examining these geometric structures also suggests a natural way of classifying demand: we say two valuations have the same "type" if certain sets of vectors associated with the geometric structures are the same.

Importantly, "types" are not merely a mathematically convenient way to categorise demands. The list of vectors in the demand type is a list of possible ways an agent's demand can change when prices change. So it specifies the possible comparative statics of demand, and thus much of what economists think important about valuations. Familiar concepts such as substitutes and complements are examples of "types".

For example, a purchaser of new spectacles who is interested in having spare pairs might always buy lenses and frames in the ratio 2:1, whatever the individual prices of the goods. So when running an auction in which goods' characteristics suggest natural rates of substitution, bidders can be asked to express preferences that come from the corresponding demand type.[3]

Our classification into "demand types" makes it elementary to check, using simple rules about the signs and magnitudes of the entries in these sets of vectors, whether a demand type is, for example, substitutes, or complements, or "strong substitutes" (Milgrom and Strulovici, 2009), or "gross substitutes and complements" (Sun and Yang, 2006), etc., etc. So this approach provides an easy test of the nature of preferences.

Moreover, understanding the nature of "types" allows us to develop general results about such preferences. The comparative statics encoded in a type refer to changes in demand between

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[2] This assumes, as is standard in the indivisible-goods literature, that preferences are quasilinear. Tropical geometry was developed by, among others, Mikhalkin (2004, 2005). We believe it has not previously been applied to economics. Goeree and Kushnir (2012) have recently used techniques of convex analysis (see, e.g., Rockafellar, 1970), on which tropical geometry builds, in a very different context. However, Danilov and Koshevoy, with their coauthors (see, in particular, Danilov et al., 2001, Danilov et al., 2003 and Danilov and Koshevoy, 2004) have developed methods of discrete convex analysis with closer connections to ours which we discuss later in the Introduction, and in detail in Sections 3.4 and 3.6.3.

[3] Indeed the version of the Product-Mix Auction now being used by the Bank of England has one-for-one substitution built into its design (see below and Klemperer, 2008, 2010).

generic prices, at which demand is unique–this is the set of demand changes that the literature often restricts to, and is used, for example, in Ausubel and Milgrom's (2002) definition of substitutes. However, we develop a sufficient condition under which the set of changes that arise generically is the complete set of changes that can arise anywhere (the set of demand changes used in the definition of substitutes of Kelso and Crawford, 1982). We call demand types for which these sets are the same "complete"; our sufficient criterion for completeness can quickly and easily be checked using the determinants of sets of the vectors describing the demand. Furthermore, we show completeness is equivalent to a generalisation of Gul and Stacchetti's (1999) "single-improvement property".[4]

Examining the vectors of "demand types" also clarifies the relationships among categories of demands. For example, it makes clear why the conditions for all of three or more indivisible goods to be (ordinary) substitutes are far more restrictive than the conditions for all of them to be complements – although these conditions are of course symmetric in the two-good case.

Aggregate demand and the existence of equilibrium

Understanding the *aggregate* demand of multiple agents allows us to develop a simple necessary and sufficient condition on preferences that guarantees the existence of a competitive equilibrium for indivisible goods. The condition is the *same* condition as that providing "completeness" above. So we can quickly see whether any demand structure guarantees equilibrium existence, simply by checking the determinants of sets of the vectors describing the demand. For example, we exhibit a demand type involving only complementary relationships between goods, for which equilibrium always exists.[5]

Furthermore, it is straightforward in our framework that properties such as the existence of equilibrium are preserved under (unimodular) basis changes of these same vectors.[6] Using this observation reveals when important properties of demands are the same.

The same observation demonstrates that the existence of equilibrium is – contrary to popular belief – *not* associated with substitutes relationships. Not only are there demand types which involve only complements and for which equilibrium always exists, but every demand type for which equilibrium is guaranteed can be obtained as a basis change of a demand type involving only complementary relationships – and this is *not* true of substitutes.

Understanding when equilibrium exists with complementary relationships also allows us to obtain new results about matching models with many-player matches since, we show, stable matchings in such models correspond to the competitive equilibria of "markets" in which people are complements to those they can be matched with.

Auctions

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[4] This has previously been extended to the "gross substitutes and complements" case by Sun and Yang (2009).

[5] The demand type is fundamentally different from (i.e., not simply a basis change of) strong substitutes, unlike, e.g., "gross substitutes and complements"—see below.

[6] A unimodular matrix is an integer square matrix with integer inverse (i.e., with determinant ±1).

Finally, our understanding of the convex- and tropical-geometric structure of agents' preferences facilitates the analysis of "Product-Mix Auctions" (Klemperer, 2008, 2010; Baldwin and Klemperer, in preparation).[7] In these auctions–introduced by the Bank of England in response to the 2007 Northern Rock bank run and the subsequent financial crisis–bidders offer prices for alternative bundles of goods, so their bids can be represented geometrically as sets of points in multi-dimensional price space.[8] Our geometric techniques tell us what kinds of bids are needed to represent different kinds of preferences, what "coherent" bids look like, how to efficiently solve for equilibrium (and when it exists[9]), etc.[10]

Organisation of this paper

We begin, therefore, by explaining the basic concepts of tropical geometry. Section 3.2 describes the properties of a "tropical hypersurface", a geometric object which contains precisely those points at which the agent is indifferent between two or more bundles. Moreover, we observe that any geometric structure of this kind corresponds to a valuation function, so we can develop our understanding of demand by working directly with these geometric objects; we believe this is the first paper to do this.

A tropical hypersurface is composed of linear pieces known as "facets" which separate the regions of price space in which an agent's demand is for some specific unique bundle.

Section 3.3 explores duality in our context. The same set of vectors that are orthogonal to the facets of the tropical hypersurface also generates the surface of the agent's valuation function in quantity space (strictly, it generates the convex hull of that surface). So there is a precise correspondence between classes of tropical hypersurfaces (in price space) and subdivisions of "Newton polytopes" (in quantity space).

Section 3.4 focuses further on the structure of individual demand, by defining a "type" of demand by the same set of vectors as above. Since these vectors describe the ways in which

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[7] Product-mix auctions are "one-shot" auctions for allocating heterogeneous goods. Their equilibrium allocations and prices are similar to those of Simultaneous Multiple-Round Auctions in private-value contexts, but they permit the bid-taker to express richer preferences, are more robust against collusive and/or predatory behaviour, and are, of course, much faster.

[8] Bids are made as lists of coordinates in implementations like the Bank of England's; the Bank itself (the bid-taker) depicts these bids, and also its own preferences, geometrically.

Shortly after introducing the auction, an Executive Director of the Bank of England noted that it was "a world first in central banking", and hailed it as "potentially a major step forward in practical policies to support financial stability". And after regularly using the new design, and having auctioned over £100 billion in funds, the Governor of the Bank (Mervyn King) told *The Economist* that the Product-Mix Auction "is a marvellous application of theoretical economics to a practical problem of vital importance to financial markets". (See Bank of England, 2010, 2011, Milnes, 2010, Fisher, 2011, Fisher et al., 2011, and The Economist, 2012.) In principle, of course, funds are almost continuously divisible, but we can apply our same indivisible-good duality techniques.

[9] Klemperer (2008, 2010) explains that equilibrium always exists in the Bank of England's auction and simple extensions of it. It follows from Theorem 3.6.8 below that equilibrium exists for broader classes of preferences (see Baldwin and Klemperer, in preparation).

[10] Expressing even richer preferences, and over more goods, than the Bank of England's current implementation permits may in some circumstances be important to this or other Central Banks who have shown interest in using the auction, or for other applications such as the sale of related products by a manufacturer, the purchase of electricity generated in different locations, the trading of permits for emission reductions relating to different kinds of deforestation, etc. Our geometric methods also permit easy alternative ways of representing preferences as bids.

the bundles demanded by the agent change with prices, they identify the key characteristics of demand. Our representation permits the easy proof and interpretation of existing results in the theory of demand. We also introduce generalisations of Gul and Stacchetti (1999)'s "single improvement property" as an alternative way to understand "types" and their properties.

Section 3.5 focuses on the important class of "complete" demand types, for which the set of changes in demand from all possible prices is the same as the set of changes in demand from the (generic) prices at which demand is unique. We show that a demand type on n goods is always complete if it is "unimodular": every subset of n of the vectors that define the type has determinant 0, +1, or −1 (plus an additional condition for cases in which type of the demand in fewer dimensions than the number of goods). We also connect "completeness" to the "single improvement property".

Section 3.6 turns to the analysis of *aggregate* demand. Working in price space makes aggregating agents' valuations easy. The tropical hypersurface of aggregate demand is simply the union of the tropical hypersurfaces of the individual demands – so it is also obvious that an aggregate valuation has demand of a certain "type" if and only if all individual agents do too.

Whether or not *equilibrium* exists depends on the nature of the intersections of agents' tropical hypersurfaces. So the theory of tropical intersection multiplicities inspires our proofs that a competitive equilibrium always exists if all agents have concave demands of a given type, if and only if the *same* condition as discussed in the previous section–unimodularity–is satisfied by this type.

Although Bikhchandani and Mamer (1997) and Ma (1998) have previously given conditions for existence of equilibrium for a set of agents, their conditions are imposed upon the *aggregate* behaviour of all the agents in the economy, so must be checked against every possible combination of agents which, in many cases, seems neither practical nor to provide great insight into why agents' demands do or do not permit equilibrium. By contrast, our theorem gives a necessary and sufficient criterion on the conditions that, when imposed upon each agent *individually*, guarantee competitive equilibrium.

Our theorem tells us first, that if each valuation individually has a certain property, and that property satisfies our *sufficiency* criterion, then competitive equilibrium always exists. That is, our theorem provides a class of results, each result stating that competitive equilibrium always exists when every valuation has a certain property. An example of such a result is that competitive equilibrium always exists if all valuations satisfy the "strong substitutes" property of Milgrom and Strulovici (2009).[11] We show that other, new, examples are easy to generate. For example, we exhibit a family of valuations on four goods involving *only* complementarities, and which is *not* a unimodular basis change of strong substitutes, and for which equilibrium

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[11] This example is not a new result, but it follows immediately from our theorem, as do several other existing results including Sun and Yang's (2006) result about the existence of equilibrium in their "two-group gross substitutes and complements" economy (which, like Milgrom and Strulovici's result, is a generalisation of Kelso and Crawford's (1982) results); Hatfield et al. (2013)'s result about when a stable outcome is not guaranteed in a trading network; and Teytelboym (2014)'s proof of equilibrium existence in his model of contracts and trading on networks; as well as extensions of many of these results.

always exists.

Our theorem also gives *necessity.* So we can quickly check, for any demand type, whether equilibrium will always exist if agents' valuations are of that demand type, or whether equilibrium must sometimes fail. For example, it follows easily from our theorem that with three or fewer goods, a competitive equilibrium always exists if *and only if* goods are "strong substitutes", or are a (unimodular) basis change of strong substitutes.[12] Although this is not true in higher dimensions (see previous paragraph), competitive equilibrium cannot be guaranteed for most demand types. A benefit of our geometric approach is that our necessity result immediately provides an example of failure of equilibrium from every instance of failure of our criterion.

Our theorem is closely related to the work in a remarkable series of papers by Danilov and Koshevoy and their co-authors. In particular, Danilov et al. (2001) also provide a sufficient condition for equilibrium, which is similar to ours. However, our development of the theory illuminates the nature of individual level conditions which guarantee equilibrium. Moreover, our framework also leads to our result that the condition is also *necessary* for equilibrium, which is not shown by Danilov et al. (2001).[13]

Thus our results in Sections 6.1-6.4 show how our geometric methods improve on existing results in determining whether any set of bidders whose preferences are drawn from a class of value functions (i.e., a "demand type") are guaranteed to always have equilibrium, whatever the bidders' precise value functions, and whatever the market supply. However, we show in Section 6.5 that tropical intersection theory also improves on existing techniques for understanding whether combinations of specific value functions also always yield equilibria.

We also show how our analysis can obtain new results about stable matchings in models with many-player matches.

Finally, we observe that since it is straightforward to "add" tropical hypersurfaces in price space, a natural and easy way to compute aggregate demand from agents' direct utility functions is by first computing each agent's tropical hypersurface. This is essentially a generalisation of the point that it is easy to compute total demand from individuals' bids in a Product-Mix Auction. However, we defer substantive discussion of the application of tropical geometry to Product-Mix Auctions to Baldwin and Klemperer (in preparation). So we conclude in Section 3.7.

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[12] Thus our necessity theorem identifies the *classes* of valuation functions for which competitive equilibrium is guaranteed. This contrasts with "necessity" results of the kind given in several of the works of Footnote 11, which show only that equilibrium always exists if all agents' valuation functions have a certain property, but may fail if exactly one valuation function does not have this property.

[13] Our theory of demand types makes clear the usefulness of Danilov et al.'s results. We discuss the relationships to, and other distinctions from, Danilov and Koshevoy and their co-authors' work in Sections 3.4 and 3.6.3. We develop economic implications further than they do, aided by our concept of "demand types". Moreover, though our techniques are novel, they are more straightforward than theirs. However, their work deserves *far* more attention than it has thus far received–it seems to have been largely overlooked by the existing literature (such as that in Footnote 11).

3.2 Representing Demand in Tropical Geometry

3.2.1 Assumptions and Motivation

There are n goods, which come in indivisible units. Each agent has a valuation function $u : A \to \mathbb{R}$ over a finite set $A \subsetneq \mathbb{Z}^n$ of possible bundles, which we call the *domain* of the valuation. We permit negative bundles to allow consideration of sellers as well as buyers. Agents have quasilinear preferences (and so, for example, no budget constraints). The price vector is $\mathbf{p} \in \mathbb{R}^n$, so different units of the same good always have the same price.[14] So the agent's demand set is

$$D_u(\mathbf{p}) := \arg\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\}.$$

We are interested in how $D_u(\mathbf{p})$ varies with $\mathbf{p}$. It is of course constant while it is single-valued. All the action takes place at those $\mathbf{p}$ at which more than one bundle is demanded. So this set of prices is our principal object of study. We write this set of prices as[15]

$$\mathcal{T}_u := \{\mathbf{p} \in \mathbb{R}^n \mid \#D_u(\mathbf{p}) > 1\}. \tag{3.1}$$

The object $\mathcal{T}_u$ (with some additional structure – see Definition 3.2.3) is a convex-geometric object, known as a 'tropical hypersurface' (TH) in the new sub-discipline of algebraic geometry known as tropical geometry.[16] We believe this paper represents the first use of this structure in economics. The next two Sections (Sections 2 and 3) translates the relevant mathematics literature into our economic context.

A simple example is shown in Figure 3.1. The agent's valuations are $u(0,0) = 0$, $u(1,0) = 5$ and $u(0,1) = 4$. So its demand is for precisely one of these bundles in each of the three regions labelled, but switches from one bundle to another along the lines drawn.

The following subsections describe properties of THs, and also how the structure of the agents' demands can be recovered from them. The 'tropical' concepts may at first sound alien, but many aspects of working in price space should in fact be very natural to economists.

3.2.2 The Tropical Hypersurface: associating geometric objects with demand

We start by considering the local structure of a TH. Given a price $\mathbf{p}$ and its demand set $D_u(\mathbf{p})$, we ask for what other prices $\mathbf{p}'$ the demand set is the same, or closely related.

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[14] We can, of course, model different units of a homogeneous good which are priced independently, by simply treating them as different goods.

[15] We follow the mathematical literature in this slight abuse of notation.

[16] See Mikhalkin (2004) and others. In fact, Mikhalkin (2004) takes the tropical hypersurface associated to u to be the non-smooth locus of $\mathbf{p} \mapsto \max_{\mathbf{x}\in A}\{\mathbf{x}.\mathbf{p} - u(\mathbf{x})\}$. Thus our tropical hypersurfaces are 'upside down' compared with his. Mikhalkin's convention is not universal; Maclagan and Sturmfels (2009) take the non-smooth locus of $\mathbf{p} \mapsto \min_{\mathbf{x}\in A}\{u(\mathbf{x}) + \mathbf{x}.\mathbf{p}\}$, which defines tropical hypersurfaces the 'same way up' as ours, albeit shifted. Our convention seems the natural one from an economic point of view: we maximise surplus, that is, the value of a bundle minus its cost.

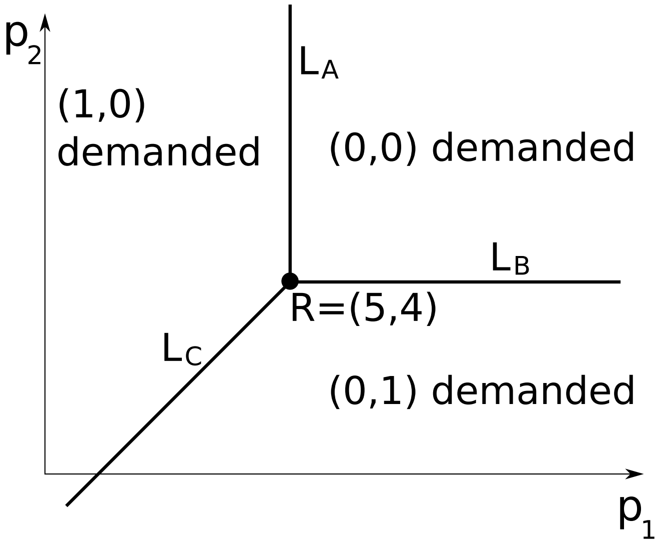

Figure 3.1: *A simple tropical hypersurface (TH). The bundle demanded on each side of the TH is labelled.*

Definition 3.2.1.

1. The *cell interior* of the TH $\mathcal{T}_u$ at a price $\mathbf{p}$ consists of points $\mathbf{p}'$ such that $D_u(\mathbf{p}) = D_u(\mathbf{p}')$.[17] A subset of $\mathcal{T}_u$ is a cell interior if it is the cell interior at some point in $\mathcal{T}_u$.
2. A subset of $\mathcal{T}_u$ is a *cell* if it the closure of a cell interior of $\mathcal{T}_u$.
3. The *affine span* of a cell of $\mathcal{T}_u$ is the smallest affine space containing the cell.[18]
4. The *boundary* of a cell of $\mathcal{T}_u$ consists of those points in the cell that are not in its cell interior.

Note that the cell interior is the largest set that is both contained in the cell and open in the affine span of the cell.[19]

We call a cell of dimension k a *k-cell*,[20] and call an $(n-1)$-cell a *facet*.

Figure 3.1 illustrates these concepts. The three line-segments L_A, L_B and L_C in the figure do *not* include the point R. Each of these line-segments is a cell interior: $D_u(\mathbf{p}) = \{(0,0),(1,0)\}$ in L_A, $D_u(\mathbf{p}) = \{(0,0),(0,1)\}$ in L_B, and $D_u(\mathbf{p}) = \{(1,0),(0,1)\}$ in L_C. The point R is also a cell interior: $D_u(R) = \{(0,0),(1,0),(0,1)\}$. The corresponding cells are the unions of these cell

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[17] Note that cells are subsets of the TH $\mathcal{T}_u$, and not, as one might intuitively guess from looking at Figure 3.1, the open areas around the sides of the TH; these are the 'unique demand regions'.

[18] Recall that an affine space in $\mathbb{R}^n$ is a parallel shift of a linear subspace, that is, a set $\{\mathbf{v}+\mathbf{c} \mid \mathbf{v} \in U\}$ for some linear subspace $U \leq \mathbb{R}^n$ and some fixed vector $\mathbf{c}$.

[19] See the equations for the three objects, given below. One might strictly refer to the 'cell interior' as the *relative interior* of the cell.

[20] To be precise, the dimension of a cell is the dimension of its affine span.

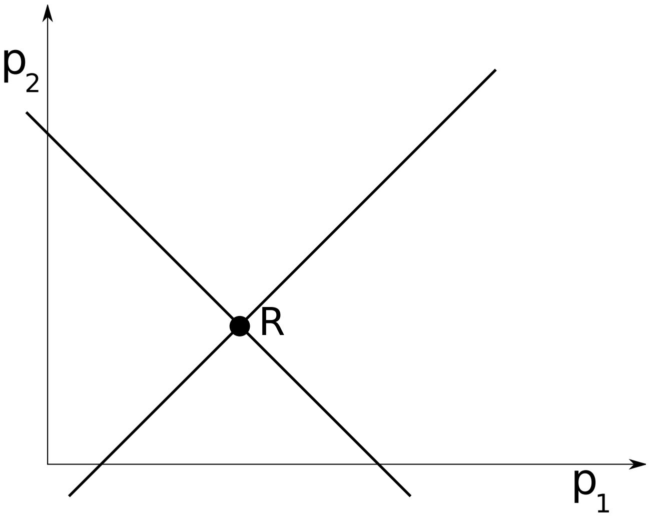

Figure 3.2: *Cell interiors do not intersect; the line segments on either side of R are distinct cells.*

interiors with their limit points: $L_A \cup R$ is thus a cell, and indeed a facet; so are $L_B \cup R$ and $L_C \cup R$. Finally, R itself is a 0-cell.

The price R is also the boundary of each of the 1-cells $L_A \cup \{R\}, L_B \cup \{R\}, L_C \cup \{R\}$. (The 0-cell R has no boundary.) Note that the price R is contained in four cells, but each price in the TH is contained in precisely one cell interior. Finally, the affine span of any cell is the set of all prices at which the agent is indifferent between all the bundles in the cell, so the affine spans of $L_A \cup R$, $L_B \cup R$, and $L_C \cup R$, are the entire lines containing those line-segments, while the affine span of R is the point R itself.

It is immediate that:

I There are finitely many distinct cells, and the TH is the union of these.

II The cell interiors do not intersect.

Figure 3.2 illustrates the latter point: although the TH is 'two line segments crossing at a point', it has *four* 1-cells with distinct interiors (and also a single 0-cell at R).

Furthermore Definition 3.2.1 implies that for a price $\mathbf{p}'$ to be in the cell interior corresponding to a set of bundles $D_u(\mathbf{p})$, the agent must be indifferent between those bundles, that is, $\mathbf{p}'.(\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, and the agent must strictly prefer these bundles to all others, that is, $\mathbf{p}'.(\mathbf{x} - \mathbf{x}'') < u(\mathbf{x}) - u(\mathbf{x}'')$ for all $\mathbf{x} \in D_u(\mathbf{p})$ and $\mathbf{x}'' \in A \backslash D_u(\mathbf{p})$. The cell corresponding to this cell interior contains its limit points, so a price $\mathbf{p}'$ is in the cell if the bundles in $D_u(\mathbf{p})$ are *weakly* preferred to all others at this price; that is, we weaken the strict inequality above to a weak inequality (while maintaining the indifference between bundles in $D_u(\mathbf{p})$).[21] So

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[21] It follows that we could alternatively define a cell as those points $\mathbf{p}'$ such that $D_u(\mathbf{p}) \subseteq D_u(\mathbf{p}')$ for some demand set $D_u(\mathbf{p})$.

a cell is the intersection of a finite number of half-spaces (sets $\{\mathbf{p}' \in \mathbb{R}^n \mid \mathbf{p}'.\mathbf{v} \leq \alpha\}$ for some $\mathbf{v} \in \mathbb{R}^n$ and some $\alpha \in \mathbb{R}$). Thus:

III Each cell is a closed convex polyhedral set in $\mathbb{R}^n$.

The affine span of the cell corresponding to $D_u(\mathbf{p})$ is simply those $\mathbf{p}'$ such that $\mathbf{p}'.(\mathbf{x}-\mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$. So the affine span of the cell is parallel to a linear subspace of $\mathbb{R}^n$, and, since $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}^n$, we have:

IV The slope of the affine span of each cell is rational.

Finally, the boundary of the cell corresponding to $D_u(\mathbf{p})$ is those $\mathbf{p}'$ such that at least one of the weak inequalities $\mathbf{p}'.(\mathbf{x}-\mathbf{x}'') \leq u(\mathbf{x}) - u(\mathbf{x}'')$ for $\mathbf{x} \in D_u(\mathbf{p})$, $\mathbf{x}'' \in A \backslash D_u(\mathbf{p})$ holds with equality. Such points therefore also lie in a lower dimensional cell, so by restricting a suitable choice of inequalities to be equalities, we have:

V The boundary of a k-cell is a union of a finite number of $(k-1)$-cells.

On the other hand, any $(k-1)$-cell lies in the boundary of some k-cell (since, from the equations defining any $(k-1)$-cell, we can obtain the equations defining some k-cell by weakening one or more of the equalities). It follows that a TH is contained in the union of its facets.

We can therefore conclude that every TH for demand over n distinct goods can be understood as an $(n-1)$-dimensional *rational polyhedral complex*:

Definition 3.2.2 (Mikhalkin, 2004, Definitions 1 and 2)**.** A subset $\Pi \subsetneq \mathbb{R}^n$ is a *rational polyhedral complex* if it is a finite union of closed sets in $\mathbb{R}^n$ called cells which satisfy properties **I-V** above. Π is *k-dimensional* if it is contained in the union of its k-cells.

By definition, demand in the complement of a TH is unique. We call a connected component of the complement of a TH a *unique demand region* (UDR). Demand is constant in each UDR, since the bundle demanded cannot change without the price crossing the TH. But to understand how demand changes as we move between UDRs, we need one additional type of information: 'weightings' on the facets.

Let F be a facet and let bundles $\mathbf{x}$ and $\mathbf{x}'$ be demanded in the UDRs on either side. So at prices $\mathbf{p} \in F$, the agent is indifferent between $\mathbf{x}$ and $\mathbf{x}'$, that is, $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$. The crucial point is that because $\mathbf{p}.(\mathbf{x}'-\mathbf{x})$ is therefore a constant for these prices, the vector $\mathbf{x}'-\mathbf{x}$ is normal to F. Call the greatest common divisor of the entries of $\mathbf{x}'-\mathbf{x}$ the *weight* of the facet, $w(F)$. So $\mathbf{v}_F := \frac{1}{w(F)}(\mathbf{x}'-\mathbf{x})$ is a primitive integer vector (that is, the greatest common divisor of its entries is 1), and it points from the UDR where $\mathbf{x}'$ is demanded to the UDR where $\mathbf{x}$ is. But since F is $(n-1)$ dimensional, its normal direction is unique, so there is a unique primitive integer normal vector pointing from the UDR of $\mathbf{x}'$ to that of $\mathbf{x}$. Thus knowing only F, $w(F)$ and $\mathbf{x}$ allows us to derive $\mathbf{v}_F$, and hence $\mathbf{x}'$. It therefore follows that if we know demand in any one

UDR, we can find demand everywhere from knowing the set of facets (and hence their primitive integer normal vectors) and their weights.

A rational polyhedral complex is described as *weighted* if a positive integer weight is attached to each facet. We provide examples in 3.2.4.

Understanding these weightings allows us to now give the full, formal definition of a tropical hypersurface – recall that we have so far worked only with the underlying set. For completeness we repeat the definition of that set here, and so:[22]

Definition 3.2.3 (Mikhalkin, 2004, Example 2)**.** Let $A \subsetneq \mathbb{Z}^n$ be a finite set and let $u : A \to \mathbb{R}$ be any function. Then the *tropical hypersurface* $\mathcal{T}_u$ associated with u is the weighted rational polyhedral complex such that:

1. its underlying set is $\{\mathbf{p} \in \mathbb{R} \mid \# D_u(\mathbf{p}) > 1\}$;
2. the weight $w(F)$ of the facet F is the integer defined by $w(F)\mathbf{v}_F = \mathbf{x}' - \mathbf{x}$, in which $\mathbf{x}'$ is demanded in the UDR on one side of F, and $\mathbf{x}$ is demanded in the UDR on the other side, while $\mathbf{v}_F$ is the primitive integer normal vector pointing from the former to the latter.

We will see that the TH captures all the information we might ever need to know about an agent's demand and valuation function, if the latter is concave in the standard sense:

Definition 3.2.4. A function $u : A \to \mathbb{R}$ is *concave* if $\mathrm{Conv}_{\mathbb{R}}(A) \cap \mathbb{Z}^n = A$ and if u can be extended to a weakly concave function on $\mathbb{R}^n$.

It is a standard result that concave functions are precisely those for which there are no bundles in A that are never demanded (see, e.g., Milgrom and Strulovici, 2009, Theorem 1). That is:

Lemma 3.2.5. *Let $A \subset \mathbb{Z}^n$. A function $u : A \to \mathbb{R}$ is concave iff, for all $\mathbf{x} \in A$, there exists $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} \in D_u(\mathbf{p})$.*

We do not assume that all valuations are concave. It will be very important in our considerations of equilibrium (see Section 3.6) to know that, if bundles are demanded, they are demanded at the 'natural' price:

Lemma 3.2.6 (Pseudo-equilibrium Prices Lemma, Milgrom and Strulovici, 2009, Proposition 2)**.** *Let u be **any** valuation function. Suppose $\mathbf{p}$ is any price vector, and $\mathbf{x}$ is an integer bundle in $\mathrm{Conv}\, D_u(\mathbf{p})$. If there exists any price vector $\mathbf{p}'$ such that $\mathbf{x} \in D_u(\mathbf{p}')$, then $\mathbf{x} \in D_u(\mathbf{p})$.*

Proof. For all $\mathbf{x}^\beta \in D_u(\mathbf{p})$, we know $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} \leq u(\mathbf{x}^\beta) - \mathbf{p}.\mathbf{x}^\beta$, with equality only if $\mathbf{x} \in D_u(\mathbf{p})$. So if $\mathbf{x} \in \mathrm{Conv}\, D_u(\mathbf{p})$, i.e., $\mathbf{x} = \sum_\beta \lambda_\beta \mathbf{x}^\beta$ for some $\lambda_\beta \in [0,1]$ with $\sum_\beta \lambda_\beta = 1$, then it follows that $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = \sum_\beta \lambda_\beta \left(u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\right) \leq \sum_\beta \lambda_\beta u(\mathbf{x}^\beta) - \sum_\beta \lambda_\beta \mathbf{p}.\mathbf{x}^\beta = \sum_\beta \lambda_\beta u(\mathbf{x}^\beta) - \mathbf{p}.\mathbf{x}$ and so, simplifying, that $u(\mathbf{x}) \leq \sum_\beta \lambda_\beta u(\mathbf{x}^\beta)$, with equality only if $\mathbf{x} \in D_u(\mathbf{p})$.

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[22] These definitions are mathematically identical to those of Mikhalkin (2004 and subsequent work), but the mathematical literature has not, of course, interpreted them in an economic context (that is, understood the $D_u(p)$ as demand sets).

Now suppose $\mathbf{x} \in D_u(\mathbf{p}')$. Then $u(\mathbf{x}) - \mathbf{p}'.\mathbf{x} \geq u(\mathbf{x}^\beta) - \mathbf{p}'.\mathbf{x}^\beta$ for all $\mathbf{x}^\beta$ so we similarly show that $u(\mathbf{x}) \geq \sum_\beta \lambda_\beta u(\mathbf{x}^\beta)$. Hence, if $\mathbf{x} \in D_u(\mathbf{p}')$ for any $\mathbf{p}'$, then $\mathbf{x} \in D_u(\mathbf{p})$. ☐

3.2.3 Associating demand with geometric objects

When does a weighted rational polyhedral complex depict a valid demand of some agent?

If we construct a TH by starting from some valuation function u, then the weights we attach will necessarily be coherent, in the sense that if we cross facets by passing through a sequence of UDRs that ends where it started, we must demand at the end precisely what we demanded at the beginning. In particular, the TH will satisfy the *balancing condition*:

Definition 3.2.7 (Mikhalkin, 2004, Definition 3)**.** An $(n-1)$-dimensional weighted rational polyhedral complex $\Pi \subsetneq \mathbb{R}^n$ is *balanced* if for every for every $(n-2)$-cell $G \subsetneq \Pi$, the weights $w(F_j)$ on the facets $F_1, \ldots, F_l$ that are adjacent to G, and primitive integer normal vectors $\mathbf{v}_{F_j}$ for these facets that are defined by a rotational direction about G, satisfy $\sum_{j=1}^{l} w(F_j)\mathbf{v}_{F_j} = 0$.[23]

Note that there do not necessarily exist weights to balance a general rational polyhedral complex.[24] However, the balancing condition is in fact the *only* condition that a weighted rational polyhedral complex has to satisfy to be the TH of some valuation function:

Theorem 3.2.8 (Mikhalkin, 2004, Proposition 2.4; also Mikhalkin, 2005, Theorem 3.15)**.** *Suppose that $\Pi \subsetneq \mathbb{R}^n$ is an $(n-1)$-dimensional balanced weighted rational polyhedral complex.*[25] *Then there exists a finite set $A \subsetneq \mathbb{Z}^n$ and a function $u : A \to \mathbb{R}$ such that Π is the TH, $\mathcal{T}_u$.*

The correspondence between a TH and its associated set A and function u is not unique, but the ambiguities are trivial if u is concave. Clearly, adding a constant to $u(\mathbf{x})$ leaves the TH unchanged, as does increasing every available bundle by a fixed bundle and making a corresponding shift in the valuation (though the bundle demanded in each UDR will then also be increased by the fixed bundle). That is, if $A' = \{\mathbf{x} + \mathbf{c} \mid \mathbf{x} \in A\}$ and $u'(\mathbf{x} + \mathbf{c}) = u(\mathbf{x}) + \alpha$ for all $\mathbf{x} \in A$, some $\mathbf{c} \in \mathbb{Z}^n$, and some $\alpha \in \mathbb{R}$, then $\mathcal{T}_{u'} = \mathcal{T}_u$. (See Example 3.2.12 for an example of such a shift).

Furthermore, any non-concave u has the same TH as the minimal weakly-concave function that weakly exceeds it everywhere on A. To see this, observe that if a bundle is never demanded then its precise value to the agent is immaterial, so we can increase its value up to the threshold at which it is just marginally demanded for some price(s) without altering the shape or properties of

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[23] To choose a rotational direction around G, pick a 2-dimensional affine subspace H of $\mathbb{R}^n$ orthogonal to G, such that the intersection of each F_j with H is 1-dimensional. The intersection of H with the TH is then a collection of 1-cells meeting at the 0-cell which is $G \cap H$. An ordinary choice of rotational direction in this two-dimensional picture gives a rotational direction around G in $\mathbb{R}^n$.

[24] For example, in two dimensions, consider three 0-cells, each with three adjacent facets. If each pair of 0-cells has an adjacent facet in common, the six weights must satisfy six balancing conditions (that is, three equations in each of the two dimensions). But since the balancing conditions are trivially satisfied by setting all weights equal to zero, the conditions can only be satisfied by positive integer weights if the conditions are not linearly independent–which is non-generic.

[25] Strictly speaking, of course, Π is a subset of the space $\mathbb{R}^n$ *and* has weights. As before, we follow Mikhalkin and the mathematical literature in our presentation.

the TH. Doing this for all never-demanded bundles removes any non-concavities in the valuation function. It is also now clear that if two agents have valuations u and u', respectively on different sets of bundles A and A', but their convex hulls in $\mathbb{R}^n$, which we write $\operatorname{Conv} A$ and $\operatorname{Conv} A'$, coincide; and if $\widehat{u}$ is the minimal concave function on $\operatorname{Conv} A$ such that $\widehat{u} \geq u$ on A, and is also the minimal concave function on $\operatorname{Conv} A$ such that $\widehat{u} \geq u'$ on A'; then $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u = \mathcal{T}_{u'}$.[26]

Summing up:

Theorem 3.2.9 (Mikhalkin, 2004, Remark 2.3)**.** *There is a 1-1 correspondence between THs with an identified 'demand 0' UDR, and pairs* (u, A)*, where* $A \subsetneq \mathbb{Z}^n$ *is finite and convex in* $\mathbb{Z}^n$*,* u *is a weakly-concave, function* u *on* A*, for which* $u(0) = 0$ *and* 0 *is demanded where specified.*

Thus we have full equivalence between THs and weakly-concave valuation functions (such that $u(0) = 0$ and 0 is demanded in a specified UDR). Note, in particular, that a given set in $\mathbb{R}^n$ is the TH of some quasilinear demand if and only if it is a rational polyhedral complex and there exist weights for the facets such that it is balanced. Although we do *not* restrict attention to concave valuation functions–indeed Section 3.6.3 will ask when an aggregate valuation is concave–understanding of the concave case is important.

Similarly, we do *not* restrict attention to what is demanded in UDRs, but doing so is an important first step. Generically all prices are in a UDR so, as noted above Definition 3.2.3, given any TH and a specified 'zero demand' UDR we can easily work out what is demanded for a generic price. And it is particularly straightforward to relate properties of demand such as substitutes or complements to the geometry of the TH; see Section 3.4.

3.2.4 Demand examples

Example 3.2.10. Let $A = \{\mathbf{x} \in \mathbb{Z}^2_{\geq 0} \mid x_1 + x_2 \leq 2\}$ and let $u : A \to \mathbb{R}$ be as follows (we arrange the terms in this "back-to-front" way to correspond to the fact that *smaller* quantities will appear *higher* in, and further *right* in, the TH; this convention will be particularly helpful later):

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
7	6	0	$x_2 = 0$
	9	4	$x_2 = 1$
		8	$x_2 = 2$

The TH associated with the agent's valuation, u, is shown in Figure 3.3, in which we have additionally marked in red the bundle demanded by this agent in each UDR. The facet between the UDRs in which $(0, 0)$ and $(0, 2)$ are demanded has weight 2. For $\mathbf{p}$ in this facet (that is, for $p_2 = 4$ and $p_1 > 6$) we have $D_u(\mathbf{p}) = \{(0, 0), (0, 1), (0, 2)\}$; in particular the bundle $(0, 1)$ *is* demanded for some price and the function *is* concave. An otherwise-identical valuation u' in which $u'(0, 1) < 4$ would give rise to the *same* TH, but would not be concave; $(0, 1)$ would not be demanded for any price.

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[26] We defined $\widehat{u}$ on $\operatorname{Conv} A \subsetneq \mathbb{R}^n$, but it still defines a TH if it is restricted to $\operatorname{Conv} A \cap \mathbb{Z}^n$.

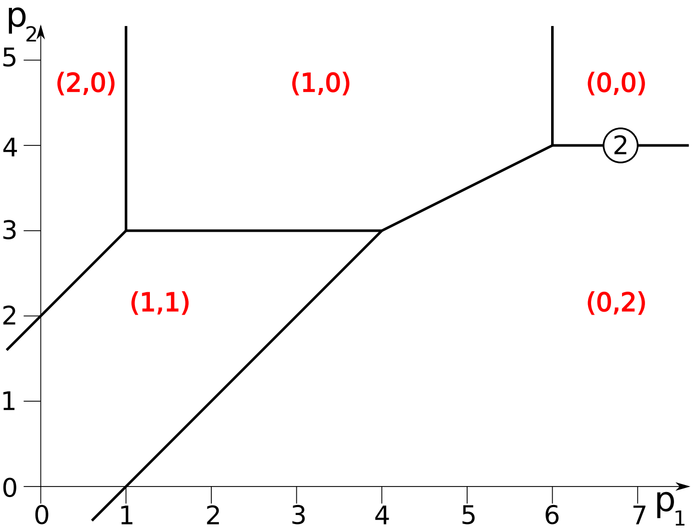

Figure 3.3: *The TH of Example 3.2.10, with the bundle demanded in each UDR marked in red.*

It is easy to work out, from the TH, which bundle is demanded in each UDR, if one already knows what is demanded in any one UDR. If $x_1 = x_2 = 0$ in the top right UDR we can simply "walk around" the diagram, adding to x_1 (x_2) the weight of any facet crossed times the first (second) coordinate of the primitive integer facet normal. Thus starting from the top right UDR, crossing the vertical facet with normal $(1, 0)$, that is, $\{\mathbf{p} \in \mathbb{R}^2 \mid p_1 = 6, p_2 > 4\}$, changes demand from $(0, 0)$ to $(1, 0)$; from there, crossing the facet with normal $(-1, 2)$ changes demand to $(0, 2)$, as may also be seen by crossing the weight-2 horizontal from $(0, 0)$ downwards; and so on.

Example 3.2.11. It will be useful later to discuss very simple examples of substitutes and complements demands: if $A = \{0, 1\}^2$, then $u^1 : A \to \mathbb{R}$ and $u^2 : A \to \mathbb{R}$ defined as follows are demands for substitutes and complements, respectively, and their THs are shown in Figures 3.4a and 3.4b.[27]

$$\begin{array}{cc|c} x_1 = 1 & x_1 = 0 & u^1 \\ \hline 1 & 0 & x_2 = 0 \\ 1 & 1 & x_2 = 1 \end{array} \quad \text{and} \quad \begin{array}{cc|c} x_1 = 1 & x_1 = 0 & u^2 \\ \hline 0 & 0 & x_2 = 0 \\ 1 & 0 & x_2 = 1 \end{array} .$$

Clearly each TH has four UDRs in which these agents demand the bundles $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$, respectively, as one moves clockwise around the UDRs starting at the top right–as is also easily confirmed by adding the appropriate primitive integer facet normal on every crossing between UDRs.

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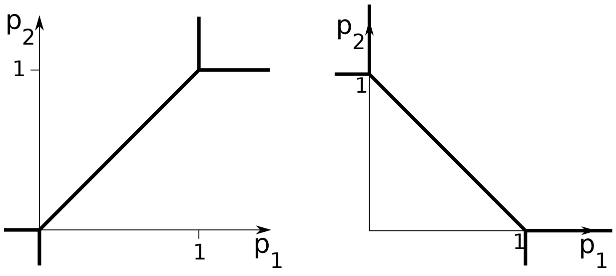
[27] The TH of Figure 3.4a appears to be a translation of Figure 3.1, but there is an important distinction. In Figure 3.1 the domain is $\{(0, 0), (0, 1), (1, 0)\}$, so the TH has only one 0-cell; here, u^1 has domain $\{0, 1\}^2$, and its TH has two 0-cells. (If we restricted u^1 to the domain $\{(0, 0), (0, 1), (1, 0)\}$ its TH would coincide with $\mathcal{T}_{u^1}$ on $\mathbb{R}^2_{\geq 0}$ but have only one 0-cell.)



(a) $\mathcal{T}_{u^1}$. **(b)** $\mathcal{T}_{u^2}$.

Figure 3.4: *The THs of Example 3.2.11.*

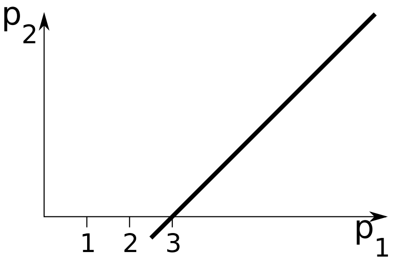


Figure 3.5: *The TH of Example 3.2.12.*

Example 3.2.12. To illustrate the case in which an agent both buys and sells goods, let $A = \{(0,0), (-1,1)\}$ and let $u(0,0) = 0$, and $u(-1,1) = -3$. This corresponds to an agent who can convert one unit of good 2 into one unit of good 1 at a cost of 3; the agent will therefore buy one unit of good 2 and sell one unit of good 1 if $p_1 - p_2 > 3$, and do nothing if $p_1 - p_2 < 3$. See Figure 3.5.

Observe that it would be economically identical if the agent were initially endowed with one unit of good 1 which it would be prepared to trade for a unit of good 2 if the price difference were at least 3–the agent's choices of what to buy and sell would depend on the prices in exactly the same way. This corresponds to simply shifting the valuation to the right so $A = \{(1,0), (0,1)\}$ with $u(1,0) = 0$ and $u(0,1) = -3$. Note that in this case $(0,0) \notin A$. We need not (and do not) prescribe that the zero bundle has to be an available option.

Example 3.2.13. For a simple 3-dimensional example, let $A = \{\mathbf{x} \in \mathbb{Z}^3_{\geq 0} \mid x_1 + x_2 + x_3 \leq 1\}$ and let $u(0,0,0) = 0$ and $u(1,0,0) = u(0,1,0) = u(0,0,1) = 1$. The TH is given in Figure 3.6. Now, the facets are 2-dimensional (pieces of planes), there are additionally 1-cells (lines along which these facets meet), and a 0-cell (point at which these lines meet). Three of these facets, having normals $(1,0,0)$ (dark-green facet), $(0,1,0)$ (red facet), and $(0,0,1)$ (turquoise facet), border the UDR in which $(0,0,0)$ is demanded; this UDR is of course $\{\mathbf{p} \in \mathbb{R}^3 \mid p_1, p_2, p_3 > 1\}$. Crossing any one of these facets takes us to the UDR in which the corresponding bundle is demanded. We swap between the latter UDRs by crossing the remaining three facets, which have normals $(1,-1,0)$ (orange facet), $(0,1,-1)$ (bluish-purple facet) and $(1,0,-1)$ (yellow facet).

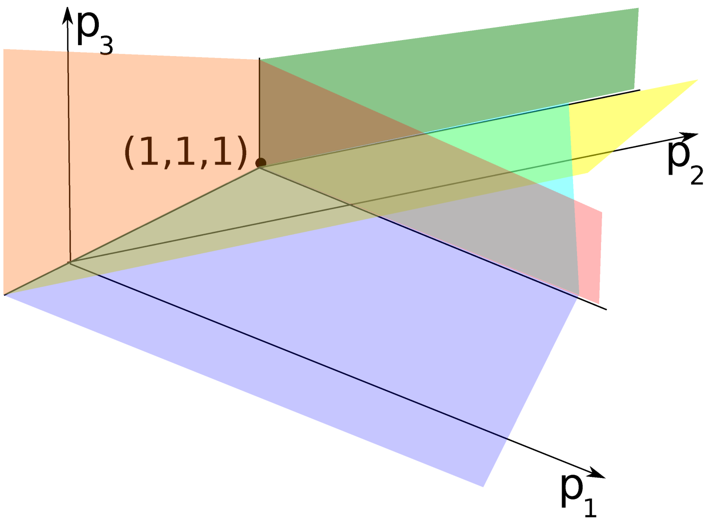


Figure 3.6: *The TH of Example 3.2.13.*

3.2.5 Classic models interpreted in our framework

Many classic models in which agents have quasi-linear demands for indivisible goods are special cases of our framework:

First, it is trivial that Bikhchandani and Mamer (1997) is the restriction of our model to $A = \{0,1\}^n$.

Example 3.2.14 (Workers and Firms–Kelso and Crawford, 1982, and Hatfield and Milgrom, 2005)**.** Kelso and Crawford (1982) model matching between workers, desiring at most one job, and firms, interested in hiring many workers, who they regard as substitutes. Thus each 'good' is a contract between a worker and a firm, and its 'price' is the salary.

To represent this in our framework, let $i \in \{1, \ldots, m_1\}$ be the workers, and $j \in \{1, \ldots, m_2\}$ be the firms, so there are $n = m_1 m_2$ contracts which we can index as (i, j). Then worker i has valuation $u^i : A^i \to \mathbb{R}$ with domain $A^i := \{\mathbf{0}, -\mathbf{e}^{(i,j)} \mid j = 1, \ldots, m_2\} \subsetneq \{-1, 0\}^n$. That is, we regard it as a seller of its labour, and it has no preferences over the sale of other workers' labour. On the other hand, firm j has valuation $u^j : A^j \to \mathbb{R}$ with domain $A^j := \{\mathbf{x} \in \{0,1\}^n \mid x_{(i,j')} = 0 \text{ for } j' \neq j\} \subsetneq \{0,1\}^n$. That is, it is only able to buy workers, and only has preferences over the workers it 'buys' itself. Note that the total set of meaningful bundles $\{-1,0,1\}^n$ is the (Minkowski) sum of all the sets A^i and A^j; in Section 3.6 we will refer to this set as the domain of the aggregate valuation. We discuss Kelso and Crawford's 'gross substitutes' condition in Section 3.5.3.

Hatfield and Milgrom (2005) consider matchings between firms and workers with more general 'contracts' than just salaries, but their model can be embedded in Kelso and Crawford (1982).[28]

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[28] Echenique (2012) shows this. Hatfield and Kojima (2010) does not fit into our framework, since it is inconsistent with quasi-linear preferences (see the discussion in Echenique, 2012).

so can also be presented in our framework.

Example 3.2.15 (General 'Trading Networks'–Hatfield et al., 2013, Ostrovsky, 2008)**.** Models such as Hatfield et al. (2013) consider agents each of whom can both buy and sell. Each 'good' in these models is the trade of a single unit of a product between a specified buyer and a specified seller; additional units of the identical product are treated as separate trades and may have differing prices.[29]

To embed this in our framework, let n be the total number of feasible trades, and for $j = 1, \dots, n$ let $b(j)$ be the buyer and $s(j)$ be the seller in the (potential) trade. Then agent i has valuation $u^i : A^i \to \mathbb{R}$ with domain $A^i \subseteq \{\mathbf{x} \in \{-1, 0, 1\}^n \mid x_j < 0 \Rightarrow s(j) = i;\ x_j > 0 \Rightarrow b(j) = i\}$. That is, agent i considers bundles in which the goods it sells are in non-positive quantities, and the goods it buys are in non-negative quantities. However, agent i need not consider the *whole* of this set; there may be bundles that are infeasible (for example, if it cannot sell good 1 unless it also buys one of goods 2, 3 or 4, then bundle $-\mathbf{e}^1$ is not in the domain A^i). Example 3.2.12 is a special case of this model.

We discuss Hatfield et al. (2013)'s 'full substitutability' condition in Section 3.5.3.

Example 3.2.16 (Coalition Formation with Transferable Utility)**.** [30] A classic literature models coalition formation – typically in a bipartite context. Each person (we do not refer to people as 'agents', since they will not take the roles of 'agents' in our framework) gets intrinsic value from being a member of a coalition. However, the surplus of any coalition can be transferred among people within that coalition, in the form of side-payments. Each person has quasi-linear utility in the intrinsic value of the coalition and this side-payment. We typically seek a 'stable' outcome, in which every person is assigned to precisely one coalition (perhaps entailing them being alone) and no subset of people can all (strictly) gain by deviating from their prescribed coalition and forming a new one.[31]

We model this in our framework as follows: the 'agents' are potential *coalitions*, who buy 'goods', which are *people*. The 'price' paid by an 'agent' for a good is the surplus (including any side payments) the person receives in that coalition.

So the goods available are 'person-goods' indexed $i = 1, \dots, n$. The bundles $\mathbf{x} \in \{0, 1\}^n$ denote sets of 'person-goods', where $x_i = 1$ iff i is included in the set. We let B be the set of feasible coalitions; in general, not every set of people is a feasible coalition, but we include a distinct 'coalition-agent' for every coalition that is feasible, including any feasible coalition that yields zero utility (for example, a given person being alone may be a feasible coalition that yields zero utility).

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[29] Hatfield et al. (2013) impose no restrictions on the shape of the 'trading network' formed by the feasible trades, so thus generalise the 'two-sided matching literature' started by Gale and Shapley (1962) in the case in which all preferences are quasilinear. (Ostrovsky, 2008 is also related, but does not require quasi-linearity and has discrete prices.)

[30] Cf. Koopmans and Beckmann (1957), Shapley and Shubik (1971), Kaneko and Wooders (1982), Eriksson and Karlander (2001), Talman and Yang (2011) and Chiappori et al. (2012).

[31] We will see (in Section 3.6.2) that in this setting the stable outcomes will coincide with the core allocations (and also coincide with the core allocations of a game with fully transferable utility, i.e., across as well as within coalitions).

We consider the people as each being assigned to a coalition and immediately handing their value in that coalition to the coalition-agent itself. Some of this money is then transferred back to the people in the coalition: p_i is the *price* the coalition-agent pays for person-good i. If a person-good is priced at p_i then the *net* side-payment to this person from coalition-agent $\mathbf{x}$ is thus $p_i - u^i(\mathbf{x})$, where $u^i : \{\mathbf{x} \in B \mid x_i = 1\} \to \mathbb{R}$ is the individual's intrinsic valuation function on coalitions. Hence, the *net utility* to person i at this stage is simply $u^i(\mathbf{x}) + p_i - u^i(\mathbf{x}) = p_i$. (There will in general be additional surplus to distribute among the coalition members at a later stage; we think of the 'price' as the minimum that needs to be offered to buy the person-good.)

Thus we think of each person as stating a price for himself[32] and seeing which coalition will buy; although his intrinsic values for the different coalitions may differ, his net utilities when he receives this price are all the same.

The 'coalition-agent' corresponding to coalition $\mathbf{x}$ obtains the sum of the individual values of the people in that coalition *minus* the 'prices' it pays for those people, if they are all assigned to it. So the domain of the coalition-agent's valuation is $A^{\mathbf{x}} := \{\mathbf{0}, \mathbf{x}\}$ and we define $u^{\mathbf{x}}(\mathbf{0}) = 0$ and $u^{\mathbf{x}}(\mathbf{x}) = \sum_{i\,:\,x_i=1} u^i(\mathbf{x})$. If the vector of prices for person-goods is $\mathbf{p}$, the coalition-agent's net utility from bundle $\mathbf{y}$ is $u^{\mathbf{x}}(\mathbf{y}) - \mathbf{p}.\mathbf{y}$. So if the sum of the 'prices' of all the people is at most the total surplus from the coalition, that is, the coalition-agent can make side-payments that give each person the utility he demands, then the coalition-agent's maximising bundle is $\mathbf{x}$; otherwise it is $\mathbf{0}$. Thus the formation of coalitions is just the maximising behaviour of coalition-agents.

We discuss the formation of stable coalitions in equilibrium in Section 3.6.2.

3.3 Duality in Tropical Geometry

The previous section demonstrated the equivalence between THs and specific valuation functions. However, we now describe a coarser correspondence between a *set* of THs that are "essentially" the same as one another, and *sets* of valuation functions which–we will see–have the same fundamental properties.

Looking, e.g., at Figure 1, the important structure is that there are particular UDRs and particular sets of prices at which the agent is indifferent between the bundles of those UDRs. So we say that two THs have the same *combinatorial type* if there is a 1-1 correspondence between the cells of the THs which have the same dimension and slope, and these cells connect to one another in the same way. Demands corresponding to THs of the same combinatorial type are "essentially" the same in that they represent agents who make the same trade-offs between additional units of goods, even if not always at the same prices. We will show that all THs of the same combinatorial type are, in a precise way, dual to a particular subdivision of Conv A.

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[32] We prefer the use of the female pronoun for people, except where–as here–they are treated as goods to be priced and traded.

3.3.1 Duality between convex polytopes and cells

Although we assume that goods are indivisible, we now develop a structure of convex polytopes and their faces in quantity space, so extend our focus from A to $\operatorname{Conv} A \subsetneq \mathbb{R}^n$. We first show that this extension has no effect on the way we separate prices into different cell interiors by showing that $\operatorname{Conv} D_u(\mathbf{p}) = \operatorname{Conv} D_u(\mathbf{p}') \iff D_u(\mathbf{p}) = D_u(\mathbf{p}')$, for any prices $\mathbf{p}$ and $\mathbf{p}'$. This is an immediate corollary of Lemma 3.2.6.

Corollary 3.3.1. *For **any** valuation function, u, if $\mathbf{p}$ and $\mathbf{p}'$ are any two price vectors, then* $\operatorname{Conv} D_u(\mathbf{p}) = \operatorname{Conv} D_u(\mathbf{p}') \iff D_u(\mathbf{p}) = D_u(\mathbf{p}')$.

Proof. It is immediate from Lemma 3.2.6 that if $\mathbf{x} \in D_u(\mathbf{p}') \subseteq \operatorname{Conv} D_u(\mathbf{p}') = \operatorname{Conv} D_u(\mathbf{p})$ then $\mathbf{x} \in D_u(\mathbf{p})$, so the result follows. ☐

For any price, $\mathbf{p}$, we write $\Delta(\mathbf{p}) := \operatorname{Conv} D_u(\mathbf{p})$ for this polytope in (divisible) quantity space $\mathbb{R}^n$. From Definition 3.2.1, and Corollary 3.3.1 we can write the associated cell interior as $\{\mathbf{p}'' \in \mathbb{R}^n \mid \Delta(\mathbf{p}) = \Delta(\mathbf{p}'')\}$, and since it is therefore defined by the polytope $\Delta(\mathbf{p})$, we write $C_{\Delta(\mathbf{p})}$ for the corresponding cell (its closure). Recall from the discussion in Section 3.2.2 that a price $\mathbf{p}''$ is in the cell $C_{\Delta(\mathbf{p})}$ iff the bundles in $D_u(\mathbf{p})$ are weakly preferred to all others at price $\mathbf{p}''$, i.e., iff $D_u(\mathbf{p}) \subseteq D_u(\mathbf{p}'')$.[33] Applying Corollary 3.3.1 again, we conclude that $C_{\Delta(\mathbf{p})} = \{\mathbf{p}'' \in \mathbb{R}^n \mid \Delta(\mathbf{p}) \subseteq \Delta(\mathbf{p}'')\}$. It follows immediately

$$\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}') \iff C_{\Delta(\mathbf{p}')} \subsetneq C_{\Delta(\mathbf{p})}. \tag{3.2}$$

We now describe the dualities between the polytope $\Delta(\mathbf{p})$ in quantity space, and the associated cell $C_\Delta(\mathbf{p})$ in price space; we show how they extend to the global structure in the next subsection.

First, note the dimensions of $\Delta(\mathbf{p})$ and $C_{\Delta(\mathbf{p})}$ are dual. $C_{\Delta(\mathbf{p})}$ has the dimension of its affine span, that is, of that set of prices $\mathbf{p}'$ such that $\mathbf{p}'.(\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$. If $\Delta(\mathbf{p})$ is k-dimensional, these equations impose k linearly independent constraints on such $\mathbf{p}'$, so $\dim C_{\Delta(\mathbf{p})} = n - k$.

Next observe the affine spans of these sets are orthogonal: since $\mathbf{p}'.(\mathbf{x} - \mathbf{x}')$ is constant for all $\mathbf{p}' \in C_{\Delta(\mathbf{p})}$ and all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, we have $(\mathbf{p}' - \mathbf{p}'').(\mathbf{x} - \mathbf{x}') = 0$ for any $\mathbf{p}', \mathbf{p}'' \in C_{\Delta(\mathbf{p})}$ and $\mathbf{x}, \mathbf{x}' \in \Delta(\mathbf{p})$. So all prices in $C_{\Delta(\mathbf{p})}$ lie in a subspace of $\mathbb{R}^n$ orthogonal to any $\mathbf{x} - \mathbf{x}'$ where $\mathbf{x}, \mathbf{x}' \in \Delta(\mathbf{p})$, and all bundles in $\Delta(\mathbf{p})$ lie in a subspace of $\mathbb{R}^n$ orthogonal to $\mathbf{p}' - \mathbf{p}''$ for any $\mathbf{p}', \mathbf{p}'' \in C_{\Delta(\mathbf{p})}$.

Therefore, any ($n - 1$ dimensional) facet $F = C_{\Delta(\mathbf{p})}$ (in price space) corresponds to a 1-dimensional polytope, i.e., a line-segment, $\Delta(\mathbf{p})$, orthogonal to it (in quantity space). And if $\mathbf{x}$ and $\mathbf{x}'$ are the endpoints of the line-segment $\Delta(\mathbf{p})$, then $\mathbf{x} - \mathbf{x}' = w\mathbf{v}_F$ for some $w \in \mathbb{Z}$, where $\mathbf{v}_F$ is a primitive integer vector in the direction of $\Delta(p)$, i.e. in the normal direction to F; let

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[33] See also Footnote 21.

us chose $\mathbf{v}_F$ so that $w > 0$. And since the bundles demanded in the UDRs on either side of F are precisely the vertices at the endpoints of $\Delta(\mathbf{p})$, it also follows that this w is the weight of F, as defined in Section 3.2.2. In words, the "length" of the line-segment $\Delta(\mathbf{p})$ in quantity space is the weight of its corresponding facet in price space.

3.3.2 The subdivided Newton Polytope

Convex geometry now provides a clever trick to find the set of all the polytopes, $\Delta(\mathbf{p})$, very quickly, and see how they fit together in quantity space. From this it is easy to see how the cells of the TH fit together in price space.

The condition that a bundle $\mathbf{x}' \in D_u(\mathbf{p})$ maximises the agent's surplus at price $\mathbf{p}$ can be re-written using vectors in $\mathbb{R}^{n+1}$ as $(-\mathbf{p}, 1).(\mathbf{x}, u(\mathbf{x})) \leq (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ for all $\mathbf{x} \in A$. So the points $(\mathbf{x}, u(\mathbf{x}))$, for all $\mathbf{x} \in A$, must lie in a particular half-space of $\mathbb{R}^{n+1}$. Furthermore all the other bundles $\mathbf{x}''$ which are optimal at the same price $\mathbf{p}$ satisfy $(-\mathbf{p}, 1).(\mathbf{x}'', u(\mathbf{x}'')) = (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ and so all lie on the hyperplane in $\mathbb{R}^{n+1}$ bounding this half-space. Hence every set $\Delta(\mathbf{p})$ (i.e. any $\operatorname{Conv} D_u(\mathbf{p})$) is the projection to the first n coordinates of a face of the set

$$\widehat{A} := \operatorname{Conv}\{(\mathbf{x}, u(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in A\}. \tag{3.3}$$

Conversely, consider any face $\widehat{\Delta}$ of $\widehat{A}$ on the 'upper side' with respect to the final coordinate (i.e., any face such that points with a slightly lower final coordinate than those in the face are in $\widehat{A}$, and those with a slightly higher final coordinate are not). $\widehat{\Delta}$ is the intersection of $\widehat{A}$ with some hyperplane $\{\mathbf{y} \in \mathbb{R}^{n+1} \mid \mathbf{v}.\mathbf{y} = \alpha\}$ for some $\alpha \in \mathbb{R}$, and some normal vector $\mathbf{v} \in \mathbb{R}^{n+1}$. We know $\widehat{A}$ is contained in the half-space below the hyperplane with respect to the final coordinate. Renormalising so the final coordinate of $\mathbf{v}$ is 1, so $\mathbf{v} = (-\mathbf{p}, 1)$ for some vector $\mathbf{p} \in \mathbb{R}^n$, the face $\widehat{\Delta}$ is the convex hull of all points $(\mathbf{x}', u(\mathbf{x}'))$, where $\mathbf{x}'$ is in A, satisfying $(-\mathbf{p}, 1).(\mathbf{x}, u(\mathbf{x})) \leq (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ for all $\mathbf{x} \in A$; that is, $u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$ is maximal over bundles in A. Thus the projection of $\widehat{\Delta}$ to its first n coordinates is exactly $\Delta(\mathbf{p})$ for this $\mathbf{p}$.

Summarising, each upper face of $\widehat{A}$ is the set $(\mathbf{x}, u(\mathbf{x}))$ that are maximal when viewed in the direction of some vector $(-\mathbf{p}, 1)$; the face then projects to $\Delta(\mathbf{p})$. And conversely, any set $\Delta(\mathbf{p})$ is the projection of an upper face of $\widehat{A}$. So the information about the demand sets is contained in the projections of these faces, that is, in the collection of sets $\{\mathbf{x} \mid (\mathbf{x}, u(\mathbf{x})) \in \widehat{\Delta}\}$, where $\widehat{\Delta}$ is an upper face of $\widehat{A}$.

Definition 3.3.2.

1. The subdivision of $\operatorname{Conv} A$ given by the projections of the upper faces of $\widehat{A}$ onto $\operatorname{Conv} A$ is a *subdivided Newton polytope* (SNP).[34]
2. The image Δ of a k-dimensional face $\widehat{\Delta}$ of $\widehat{A}$ is a *k-face* of the SNP.

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[34] It is a subdivision of the set $\operatorname{Conv} A$ which is itself called a Newton polytope in (tropical) algebraic geometry.

We give an example of how to construct an SNP in practice in Section 3.3.3.

Since, for $k < n$, any k-face of $\widehat{A}$ is the face of an n-face of $\widehat{A}$, it is sufficient to consider only the maximal faces of $\widehat{A}$ to identify the full SNP structure.

In particular, an SNP n-face, Δ, is the projection of an upper n-face $\widehat{\Delta}$ of $\widehat{A}$. But since $\widehat{\Delta}$ is n-dimensional, there is a unique hyperplane of $\mathbb{R}^{n+1}$ passing through it, and so a unique normal vector of the form $(-\mathbf{p}, 1)$. So the projection Δ of $\widehat{\Delta}$ to $\operatorname{Conv} A$ is exactly $\Delta(\mathbf{p}) = \operatorname{Conv} D_U(\mathbf{p})$, and is *not* $\Delta(\mathbf{p}')$ for any $\mathbf{p}' \neq \mathbf{p}$. So $\mathbf{p}$ is the *only* price at which *all* these bundles are demanded, and $\{\mathbf{p}\}$ is therefore a 0-cell in the TH, i.e. $\{\mathbf{p}\} = C_{\Delta(\mathbf{p})}$.

At the other extreme, for any 0-face $\{\mathbf{x}\}$ of the SNP, there exist prices $\mathbf{p}$ at which $(\mathbf{x}, u(\mathbf{x}))$ is the unique point of $\widehat{A}$ intersecting a supporting hyperplane normal to $(-\mathbf{p}, 1)$. For any such $\mathbf{p}$ we know $\{\mathbf{x}\} = D_u(\mathbf{p})$. Furthermore, if any such $\mathbf{p}$ is changed infinitesimally in any coordinate direction, the point $\{(\mathbf{x}, u(\mathbf{x}))\}$ is still the unique point of $\widehat{A}$ intersecting the corresponding supporting hyperplane. So the UDR in which $\mathbf{x}$ is demanded, that is, the set of $\mathbf{p}$ such that $\{\mathbf{x}\} = D_u(\mathbf{p})$, is (of course) n-dimensional.

Between these extremes, any upper k-face of $\widehat{A}$, where $2 \leq k \leq n-1$, is the intersection of $\widehat{A}$ with any one of a range of hyperplanes in $\mathbb{R}^{n+1}$. The vector $(-\mathbf{p}, 1)$ normal to any such hyperplane defines a price $\mathbf{p}$ lying in the corresponding $(n-k)$-dimensional cell interior of the TH.

Note also that, since $\operatorname{Conv} A$ need *not* in general be n-dimensional (see Example 3.2.12 for an example in which it is not) the SNP need not have any n-faces; this corresponds to a TH with no 0-cells (such as that in Figure 3.5).

The fact that the SNP's faces, $\Delta(\mathbf{p})$, are the projections of faces of a convex set tells us how they fit together, and hence how the sets $D_u(\mathbf{p})$ fit together. If $\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}')$ for two faces of the SNP, then $\Delta(\mathbf{p})$ must be a face of the polytope $\Delta(\mathbf{p}')$. But recall (displayed equation 3.2) that $\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}')$ iff $C_{\Delta(\mathbf{p}')} \subsetneq C_{\Delta(\mathbf{p})}$. As discussed above (at and beneath point $\mathbf{V}$ of Section 3.2.2,) the latter holds iff $C_{\Delta(\mathbf{p}')}$ is in the boundary of $C_{\Delta(\mathbf{p})}$. Moreover, $\Delta(\mathbf{p})$ and $C_{\Delta(\mathbf{p})}$ are orthogonal, as discussed in Section 3.3.1. So knowing how the $\Delta(\mathbf{p})$ fit together in quantity space makes it immediately obvious how the $C_{\Delta(\mathbf{p})}$ fit together in price space, and vice versa.

So the SNP tells us which cells must exist in the corresponding THs, their slopes, and how they are connected. In other words

Theorem 3.3.3 (Mikhalkin, 2004, Proposition 2.1.)**.** *For a given* $\operatorname{Conv} A$ *there is a 1-1 correspondence between SNPs of THs and combinatorial types of THs.*

As noted above, this correspondence is coarser than the correspondence we described in the previous subsection (Theorem 3.2.9): different valuations correspond to the same SNP, and hence to a TH of same combinatorial type, even though the coordinates of the parts of the TH differ. However, this correspondence isolates the underlying properties of demands, specifically the sets of bundles one might ever be indifferent between, and the trade-offs one might make.

Also, starting from any SNP, it is easy to find the combinatorial type of the TH, and so see which coordinates uniquely define the TH. The TH can then be completely identified using the valuation u. We illustrate this in Section 3.3.3.

Another important point follows: if A is small, it is easy to list all the possible SNPs, and hence also list all possible combinatorial types of THs for the set A. That is, we can easily list every possible distinct structure of trade-offs that an agent might make between a given finite collection of goods.

Of course, we do not need to start with the SNP. Given the TH and an identified 'demand 0' UDR, we can easily infer both A and the full SNP using the duality described in this section.

Note, however, that if we do not know *ex ante* whether a TH is concave, then neither the TH nor the SNP can necessarily tell us which bundles are demanded in each cell of the TH. The information we do have is as follows:

Corollary 3.3.4. *Let A be convex in $\mathbb{Z}^n$, let $u : A \to \mathbb{R}$ be a valuation, and consider the corresponding SNP.*

1. *A bundle $\mathbf{x} \in A$ is a vertex of the SNP iff it is demanded in some UDR of the corresponding TH.*
2. *If every bundle $\mathbf{x} \in A$ is a vertex of the SNP, then $\widehat{u}$ is concave for every valuation $\widehat{u} : A \to \mathbb{R}$ such that $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u$.*
3. *If a bundle $\mathbf{x} \in A$ is not a vertex of the SNP, there exist valuations $\widehat{u} : A \to \mathbb{R}$ such that $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u$ but $\mathbf{x} \notin D_{\widehat{u}}(\mathbf{p})$ for any $\mathbf{p} \in \mathbb{R}$.*

Proof. 1 is clear from the previous discussion. 2 follows from Lemma 3.2.5. For 3, define $\widehat{u}$ to be equal to u on the vertices of the SNP, and to be arbitrarily large negative numbers on those bundles in A that are not vertices of the SNP. ☐

However, in quantity space we do *not* have an analogue of Theorem 3.2.8. Nor does there seem to be any simple analogue of Theorem 3.2.8's easy balancing condition that would check whether a given subdivision is the SNP of a valuation function:

Fact 3.3.5. It is *not* the case that every subdivision of every Newton polytope is the SNP of some valuation function.

Proof. A counterexample is provided by Gathmann (2006, Figure 7). ☐

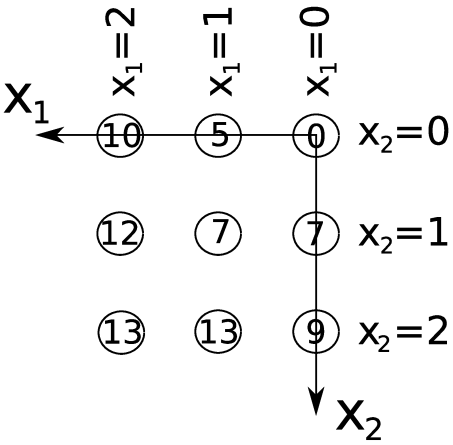
3.3.3 Examples

Example 3.3.6. Starting from a valuation function, a TH can easily be drawn by first deriving the SNP, then using the SNP to draw the shape of the TH's combinatorial type, and finally using the valuations to fix the TH's exact location in price space.

Figure 3.7 presents a valuation function u, both in the usual tabular representation, and by showing the permissible set of bundles A, as a subset of the lattice $\mathbb{Z}^n$, labelled with their valuations. As before, the quantity of good 1 increases as we move to the *left*, and the quantity of good 2 increases as we move *down*, in order to show the duality between the SNP and the TH most clearly.

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
10	5	0	$x_2 = 0$
12	8	7	$x_2 = 1$
13	13	9	$x_2 = 2$

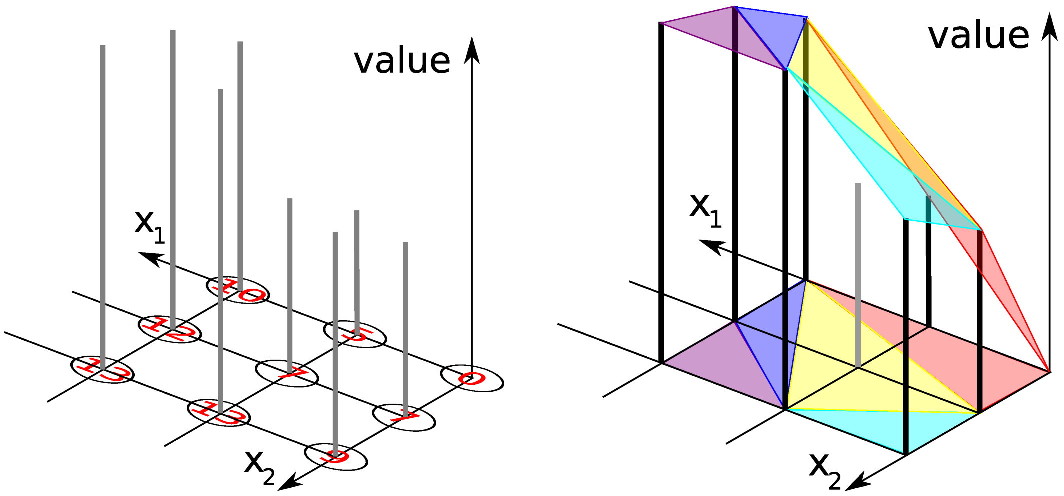
(a) *Tabular representation of the valuation.*



(b) *Each circled number gives the value of the bundle in that position.*

Figure 3.7: *Alternative representations of a valuation function.*

Figure 3.8 adds a third dimension to Figure 3.7b. Figure 3.8a shows the points $(\mathbf{x}, u(\mathbf{x}))$ for all $\mathbf{x} \in A$, with the valuations $u(\mathbf{x})$ drawn as lines connecting them to their associated bundles, $\mathbf{x}$, to make the relationships clearer. Figure 3.8b then pictures the upper surface of $\widehat{A}$, with those lines that correspond to bundles that are demanded for some price(s) in bold. Note that the



(a) *The values* $(\mathbf{x}, u(\mathbf{x}))$ *for all* $\mathbf{x} \in A$.

(b) *The upper surface of* $\widehat{A}$.

Figure 3.8: *Finding* $\widehat{A}$ *in three dimensions.*

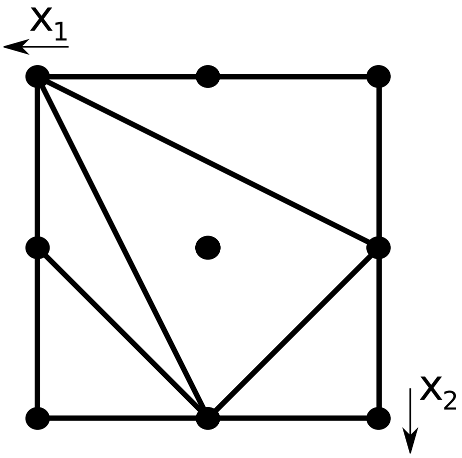
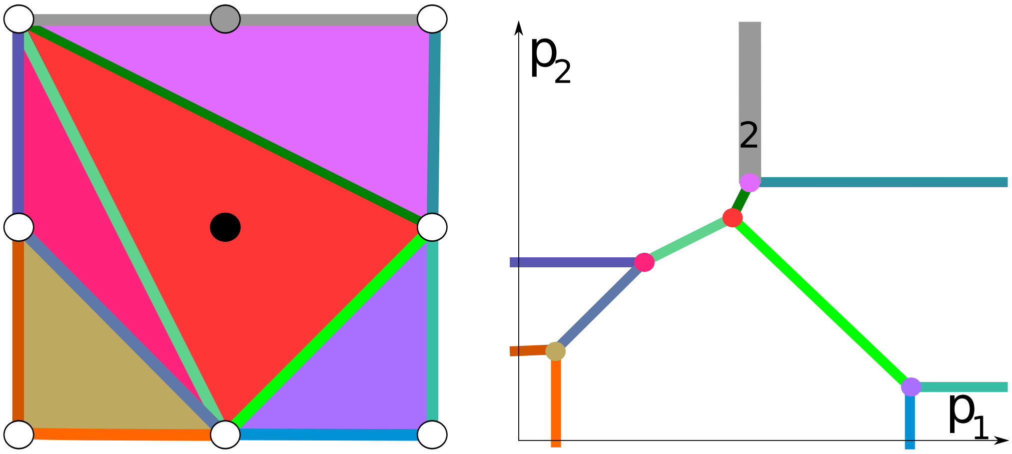


Figure 3.9: *The SNP.*

valuation is non-concave and the bundle $(1, 1)$ is never demanded.

The SNP is pictured in Figure 3.9. It is drawn *without* axes, since replacing A with $A + \mathbf{c}$ for some $\mathbf{c} \in \mathbb{Z}^n$ and re-defining u to correspond gives us the *same* SNP and TH. A depiction of the SNP and an example of a TH of the corresponding combinatorial type is given in Figure 3.10, colour-coded so that objects that are the geometric duals of each other have the same colours as each other. That is, each point in the TH has the same colour as its corresponding area in the SNP; each line-segment (facet) in the TH has the same colour as the line-segment (edge) in the SNP that it corresponds, and is *orthogonal* to; and the white areas (UDRs) in the TH correspond to the white points (bundles that are vertices) in the SNP.



(a) *The SNP, colour-coded.* **(b)** *A TH, colour-coded to correspond.*

Figure 3.10: *The SNP and a TH of the corresponding combinatorial type, colour-coded so that dual geometric objects have the same colours.*

Note that the black point in the SNP that represents the bundle $(1, 1)$ has no object cor-

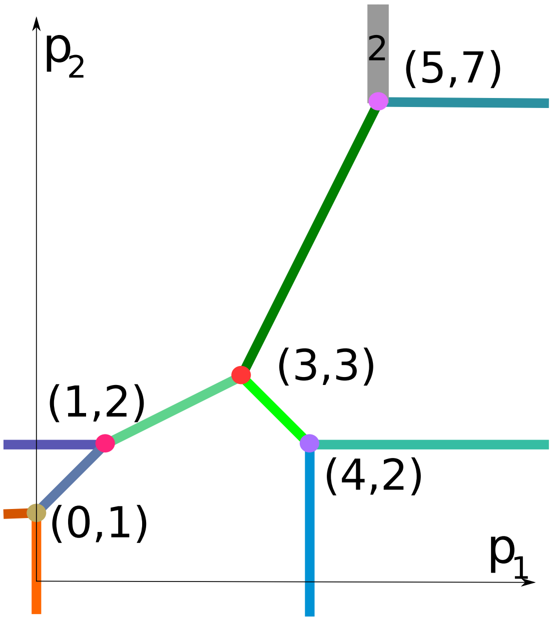


Figure 3.11: *The TH of the valuation function presented in Figure 3.7.*

responding to it in the TH–it is "hidden" inside the scarlet-coloured point in the TH. If that bundle's valuation were greater so that, rather than the line corresponding to it in Figure 3.8b lying strictly below a plane coincident with $\widehat{A}$, the line instead just touched that plane,[35] then the bundle would be demanded at the price corresponding to the scarlet-coloured point in the TH. And if the bundle had a still higher valuation, that point in the TH would "open up" to form an area corresponding to the range of prices at which the bundle would then be demanded.

The final SNP lattice point is coloured grey. It is not an SNP vertex, but lies within (horizontal) SNP edge of the same colour, which has "length" 2 (more precisely, the greatest common divisor of the differences (2, and 0) between the co-ordinates of the bundles at the ends of this edge is 2). And this corresponds to the vertical grey facet in the TH which is labelled "2", reflecting its weight.

Finally, remember that Figure 3.10b shows only one of many THs of the combinatorial type corresponding to the SNP in that figure; the SNP is silent on the lengths of the lines in its corresponding THs. However, the exact location of the TH for our specific set of valuations can easily be worked out from the valuations of different bundles: See Figure 3.11.

For example, it is clear from the valuations of bundles (1,0) and (0,1) that the top right (pinky-purple) point of the TH is at $\mathbf{p} = (5, 7)$, since 5 and 7 are the prices below which the agent will first buy any of goods 1 and 2, respectively,when the other good's price is very high. And the coordinates of the purple point at the bottom right of the TH must be (4,2) since $9 - 7 = 2$ is the incremental value of a second unit of good 2, when the agent has no unit of good

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[35] It is easy to compute that the valuation of this bundle would have to be 10 for this to happen.

1, and $13 - 9 = 4$ would be the further increment in value from then also having a unit of good 1, etc.

We discussed above (Section 3.2.2; see especially Example 3.2.10) how the demand in each UDR can easily be worked out from the TH.

Example 3.2.10 revisited. It is not hard to check that the SNP for Example 3.2.10 is as shown in Figure 3.12a. Two examples of THs of the corresponding combinatorial type are given in Figures 3.12b and 3.12c.

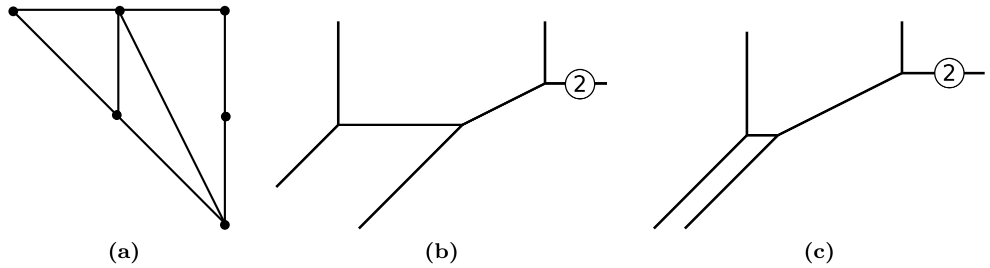

Figure 3.12: *(a) the SNP of Example 3.2.10; (b) and (c) two examples of THs of the combinatorial type of Example 3.2.10.*

Example 3.3.7. For a fixed A, it is easy to draw every possible SNP and so obtain every possible combinatorial type of TH, thus enumerating all possible "essentially-different" structures of demand. We do this for $A = \{0, 1\}^2$ in Figure 3.13.

It is not hard to see that Figure 3.13a applies when $u(0,0) + u(1,1) < u(1,0) + u(0,1)$, so represents substitutes; Figure 3.13b applies when $u(0,0) + u(1,1) = u(1,0) + u(0,1)$, so is additively separable demand; and Figure 3.13c applies when $u(0,0)+u(1,1) > u(1,0)+u(0,1)$, so is complements. (Recall Figure 3.4.) Importantly, it is clear that these are the *only* possibilities.

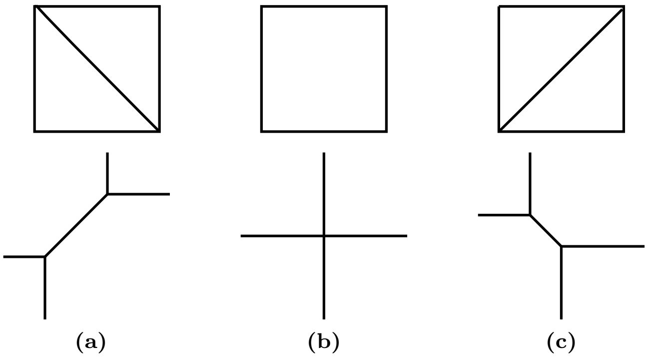

Figure 3.13: *All the possible SNPs, and examples of their corresponding combinatorial types of TH when $A = \{0, 1\}^2$.*

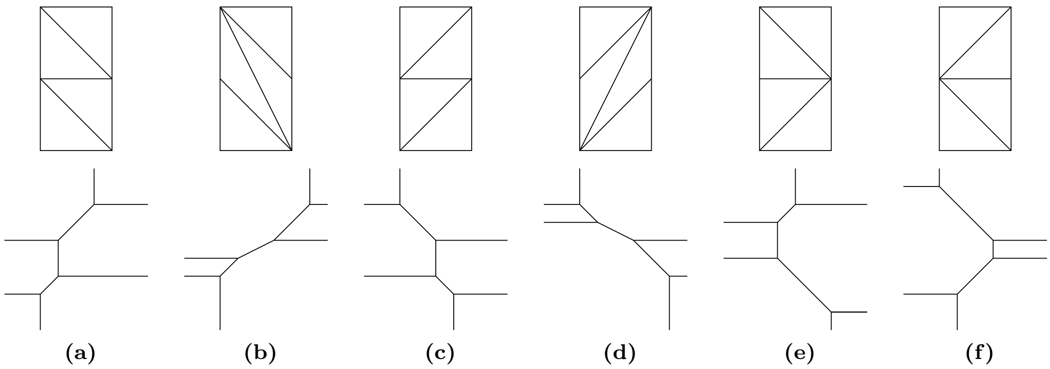

Figure 3.14: *All the possible SNPs with maximal subdivision, and examples of their corresponding combinatorial types of TH, when* $A = \{0, 1\} \times \{0, 1, 2\}$.

Observe that Figure 3.13b can be seen as a limit of Figure 3.13a (or, equivalently, Figure 3.13c). In the TH, the two 0-cells become arbitrarily close and then coincide in the limit; in the SNP, the faces of $\widehat{A}$ tilt until they are coplanar when the SNP edge distinguishing them disappears in this limit.

Likewise, any SNP in which the subdivision is not maximal (that is, additional valid $(n-1)$-faces could be added) can be recovered by deleting $(n-1)$-faces from some SNP whose subdivision *is* maximal; the corresponding TH is a limit (or 'degeneration'). Even for larger domains than $A = \{0, 1\}^2$, we can efficiently enumerate all those combinatorial types of demand for which the SNP subdivision is maximal, knowing we can recover the remainder as their limits. We do this for $A = \{0, 1, 2\} \times \{0, 1\}$ in Figure 3.14.

With a bit of practice, starting from either the TH or SNP it is easy to draw the other figure quite fast, at least in two dimensions: if we start with the TH, we know each area around the TH corresponds to a vertex in the SNP, and areas that are separated by a line-segment in the TH correspond to vertices that are connected by a line-segment in the SNP. So we can immediately draw all the vertices and lines. The remaining task is to "straighten out the SNP" without changing it topologically, noting that each line-segment in the SNP is orthogonal to its corresponding line-segment in the TH, and that where a line-segment of weight N is crossed in the TH, there are $(N - 1)$ points between the vertices of the corresponding line-segment in the SNP. (The existence of additional points in the SNP that are not on any line segment becomes apparent once the relative positions of all lines are fixed.) Going from the SNP to the TH essentially reverses the process, as we illustrated in Example 3.3.6, above.

3.4 Classifying Demands: Demand "Types"

The previous sections suggest classifying demands according to the normal vectors that determine the shapes of agents' THs. We now show that defining demand 'types' in this way does indeed provide a simple characterisation of the standard concepts of substitutes and complements, as well as (in Section 3.5) more recently developed concepts such as strong substitutes, and gross substitutes and complements, and that demand 'types' also allow us to make other useful distinctions.

We provide a theorem showing how easily a demand type translates to these concepts and, moreover, show how generalisations of the idea of the "single improvement property" (Gul and Stacchetti, 1999), which we will call the "$\mathcal{D}$- and the "$\mathbb{Z}\mathcal{D}$-Improvement Properties", help analyse these distinctions.

Our approach additionally gives a natural answer to the question of when demand "types" are similar: they share many properties when they are unimodular basis changes of each other. Furthermore, we will show in Section 3.6 that this framework also allows us to develop new results about aggregate demand, for example, about the existence of competitive equilibrium.[36]

Finally, these results also provide a quick way to check in practical applications (such as in further developments of the Product-Mix Auction) whether demands are, e.g., strong substitutes, or are such that equilibrium exists, since there are easy software solutions to calculate the normal vectors of the TH for any valuation function, u, and hence to immediately reveal the demand's 'type'.

Although we define an agent's demand type by the vectors normal to the facets of its TH (in price space), it would of course be equivalent to define the demand type of an agent by referring to the edges of its SNP (in quantity space). Danilov, Koshevoy and their co-authors focus on quantity space in the course of their impressive body of work that, we will see in Sections 3.6.3-3.6.4, has close connections to ours. However, they do not use these vectors to create a taxonomy of demand – we, by contrast, develop a general framework to understand these vectors in economic terms. Nor are Danilov et al. directly interested in price space.[37]

We, however, emphasise price space for several reasons. First, working in price space seems more intuitive and natural. An SNP in quantity space shows (only) the collections of bundles among which the agent is indifferent for some price vectors. By contrast, a corresponding TH in price space shows clearly which bundles are demanded in which regions of prices.[38] So the geometric objects in price space are easier to interpret, and working with them facilitates the

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[36] In other work, we use this framework to derive implications about the scope of possible demand functions which are substitutes; for example, various marginal valuations must be equal. See also Fujishige and Yang (2003).

[37] See Danilov et al. (2001) and Danilov et al. (2003, 2008, 2013). Working almost exclusively in quantity space means that they do not consider these vectors as defining the boundaries of the UDRs in price space, or equivalently giving the changes in demand as we move between generic points in price space (see Theorems 3.4.4 and 3.4.5, and Corollary 3.5.5).

In terms of the generality of the discussion, their principal interest is in the existence of competitive equilibrium (see Section 3.6.3) and so they almost exclusively study what we call 'unimodular demand types' (see Definition 3.5.9). As we show in this Section, more general demand types are also of considerable economic interest. Additionally, they focus on examples containing all the coordinate vectors; not every economically interesting demand need satisfy this restriction (see, e.g. Example 3.2.12). We, therefore, do not make these restrictions.

[38] With n goods, a TH is naturally an $(n-1)$-dimensional object in n-dimensional space, whereas a SNP is best understood as the n-dimensional projection of a collection of related $(n+1)$-dimensional objects. Of course, a specific TH depends on specific details of the valuation, whereas a SNP describes a class of valuations. However, examining any one TH gives the flavour of–and is enough for many purposes to develop intuition and understanding about–the entire class of THs that correspond to any single SNP.

development of intuition and understanding.

Second, recall from Theorem 3.2.8 that any geometric object satisfying the simple 'balancing condition' of Definition 3.2.7 *is* the TH of some valuation u, but (Fact 3.3.5) *not* every subdivision of every Newton polytope is induced by some valuation. So we can easily recover the full set of valuations satisfying an additional condition (for example, on their facet normals) from the set of THs in price space, and we can also specify all valuations with a particular property by referring to all THs with the corresponding property in price space–but there are no obvious corresponding procedures to do these things in quantity space.

A further advantage of our approach will become apparent in Section 3.6.1: it is straightforward to aggregate agents' demands in price space, and it will follow immediately that if two agents have demand of the same demand 'type', then aggregate demand will also be of the same demand 'type'.

3.4.1 Introducing Demand Types

Let $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ be a set of primitive integer vectors in $\mathbb{Z}^n$, such that if $\mathbf{v} \in \mathcal{D}$ then $-\mathbf{v} \in \mathcal{D}$. (We will often abuse notation by writing the set to include just one representative of each such pair).

Definition 3.4.1. A valuation is of *demand type* $\mathcal{D}$ if all the primitive integer normals to the facets of the associated TH lie in the set $\mathcal{D}$.

A valuation is of *concave demand type* $\mathcal{D}$ it is of demand type $\mathcal{D}$ and is concave.

The geometric meaning of these sets is that they give the possible slopes of the facets of the THs. But they also have an important economic meaning: recall that each *facet normal gives the direction of change in demand as we cross the facet.* We will see that this combination of being both mathematically tractable and economically intuitive makes them powerful. As noted above, it would be equivalent to define the demand type of an agent by referring to the edges of its SNP (in quantity space). However, price space is in general more intuitive to work with.

We will represent $\mathcal{D}$ by any $n \times r$ matrix D whose columns contain one representative of each ± pair in $\mathcal{D}$. Of course, D is not unique, since its can take either sign columns can be in any order, whereas the set $\mathcal{D}$ *is* unique.[39] For example, any of a number of matrices including, for example,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

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[39] Note our definition does not consider the weights on facets. We could take these into account, by relaxing the condition that all vectors in $\mathcal{D}$ be primitive. Then, for every facet F (with weight ω_F and primitive integer normal $\mathbf{v}_F$), we could require either that $\omega_F \mathbf{v}_F \in \mathcal{D}$, or that $\omega_F \mathbf{v}_F = k\mathbf{v}$, for some $k \in \mathbb{Z}$ and some $\mathbf{v} \in \mathcal{D}$. The former approach would allow us to specify the precise weights that facets may possess; this may seem unnatural, since a higher weight facet can be considered as the limit of two lower weight facets as they come arbitrarily close together, and thus very similar agent demands would be classified differently. The latter approach would allow us to insist on certain weak- or non-concavities of demand, and is a more straightforward generalisation of our definition.

represent the demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ of the THs in Figures 3.1, 3.4a, 3.13a, 3.13b, and 3.14a.[40] Note that a TH has any demand type which contains its facet normals; we do not restrict to the minimal such set. (So, for example, any of the THs listed earlier in this paragraph are *also* of type $\mathcal{D} \cup \{\pm(N_1, N_2)\}$, for any primitive integer (N_1, N_2).)

The set of vectors in a demand type determines the complete set of ways in which demand can change as we move between adjacent UDRs, and thus the possible changes in demands that can generically result from a small change in prices (since the UDRs are dense in price space). So we can immediately identify properties such as substitutes or complements.

For example, in Figure 3.4a–the simplest case of substitutes–an increase in any good's price that moves between UDR's can result in the agent swapping that good for another good, but can never result in the agent decreasing its demand for another good. That is, if the demand for one good changes when its own price does, then the change in demand for another good cannot be in the same direction–and this is precisely reflected in the fact that the vectors that are normal to the facets may have two non-zero entries of opposite signs, but never have two non-zero entries of the same sign. Likewise, in Figure 3.4b–the simplest case of complements–if either good's price increases to move across the downward-sloping diagonal facet, then the agent reduces its quantity of both goods, precisely because both components of the vector normal to this facet are of the same sign. Moreover, in this case there is no facet whose normal vector has two non-zero entries of different signs.

So we can distinguish between substitutes and complements valuations by examining the coordinate entries of the vectors in demand types. But to make the distinctions precise and, in particular, to deal with some subtleties involving changes in prices at which demand is non-unique, it is helpful to first develop the additional analytical tool of "$\mathbb{Z}\mathcal{D}$ steps".

3.4.2 $\mathcal{D}$- and $\mathbb{Z}\mathcal{D}$-Steps, and the $\mathcal{D}$- and $\mathbb{Z}\mathcal{D}$-Improvement Properties

We now show that the change in demand along the (straight) line joining any pair of prices at which demands are unique can be broken down into changes that are 'improving $\mathbb{Z}\mathcal{D}$-*steps*', and the change in demand from the first price to the last price satisfies the 'ordinary $\mathbb{Z}\mathcal{D}$-improvement property'.

These concepts are introduced to show that 'demand types' generalise the way in which we normally think about, for example, a substitutes valuation. If prices change from $\mathbf{p}$ to $\mathbf{p}' \geq \mathbf{p}$, then we think of an agent as viewing these goods as substitutes if their demand weakly increases for all goods whose price has not changed. When goods are indivisible, their demand will change from that at $\mathbf{p}$ to that at $\mathbf{p}'$ in discrete steps. By understanding the nature of these steps, we may understand all ways in which demand may change for a valuation with the property we are studying.

There are two ways to construct such steps. We may either consider a gradual change in price

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[40] We will see later (Section 3.5.3) that this demand type is "strong substitutes" in the two-good case, which we will label $\mathcal{D}^2_{ss}$.

from $\mathbf{p}$ to $\mathbf{p}'$, and look at the step changes in demand that will be triggered *en route*. Or we may think of the price change as having taken place all in one go, but that the agent is incrementally swapping bundles for ones that are preferred at the new and final prices. In either case, if the agent sees goods as 'substitutes', we expect this property to be evident at every one of the steps.

By treating such steps as our primitives, we can build up a more general understanding of the economic nature of valuations for goods, be they substitutes, complements, or some combination.

By '$\mathbb{Z}\mathcal{D}$-*steps*', then, we mean a sequence of bundles demanded on the straight line between $\mathbf{p}$ and $\mathbf{p}'$ such that the differences between consecutive bundles in this sequence are vectors in $\mathcal{D}$ or integer multiples thereof (we denote these vectors as $\mathbb{Z}\mathcal{D} := \{w\mathbf{v} \mid w \in \mathbb{Z}, \mathbf{v} \in \mathcal{D}\}$); by 'improving', we mean that the bundle at the end of each step is preferred to the bundle at the beginning of each step, at the final price vector.

It follows immediately that *any* valuation of *any* demand type $\mathcal{D}$ must satisfy a property we will call the '*ordinary* $\mathbb{Z}\mathcal{D}$*-improvement property*': given any starting bundle, $\mathbf{x}$, which is the unique demand at some price, $\mathbf{p}$, and any price $\mathbf{p}'$ at which $\mathbf{x}$ is not demanded, there exists a bundle $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at $\mathbf{p}'$, and such that $\mathbf{x}'' - \mathbf{x} \in \mathbb{Z}\mathcal{D}$.[41]

This property is closely connected to Gul and Stacchetti's (1999) result that a specific set of valuations satisfies their "single improvement property".[42] This holds if, given any starting bundle, $\mathbf{x}$, and any price $\mathbf{p}'$ at which $\mathbf{x}$ is not demanded, there exists a bundle $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at $\mathbf{p}'$, and such that $\mathbf{x}'' - \mathbf{x}$ is a vector whose entries include include at most one $+1$, at most one -1, and all others zero. Since the ordinary $\mathbb{Z}\mathcal{D}$-improvement property applies only to "starting bundles", $\mathbf{x}$, corresponding to UDR prices, it is not a strict generalisation of the "single improvement property" which applies to *all* starting bundles for those valuations for which it holds. However, we will introduce a refinement of our property which strictly generalises the "single improvement property", and so allows strict generalisations of Gul and Stacchetti's (1999) results, in Section 3.5.

The importance of our definitions and results is that we can use them to show how our "demand types" correspond precisely to interesting properties of demand such as whether agents view goods as substitutes or complements; see Subsections 3.4.3.1 and 3.4.3.2.

We proceed by first giving a formal definition of improving $\mathbb{Z}\mathcal{D}$-*steps* that is easily shown to be equivalent to the informal definition given above (see the discussion of Theorem 3.4.4), and will be easier to work with. Specifically we will require that an *improving* $\mathbb{Z}\mathcal{D}$*-step* is a $\mathbb{Z}\mathcal{D}$*-step* which "satisfies the strict law of demand" with respect to the *overall* price change:[43]

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[41] It obviously suffices to let $\mathbf{x}''$ be the bundle at the end of the first of any set of improving $\mathbb{Z}\mathcal{D}$-steps into which the demand change is broken down. Note, however, that the way we define our 'ordinary $\mathbb{Z}\mathcal{D}$-improvement property' does *not* require that $\mathbf{x}''$ be demanded at some price on the straight line between $\mathbf{p}$ and $\mathbf{p}'$. We do this for consistency with Gul and Stacchetti's (1999) related definition (see below). But the stronger concept of a $\mathbb{Z}\mathcal{D}$-step will be easier to use to characterise properties that valuations may have.

[42] Gul and Stacchetti (1999) restrict their attention to $A = \{0, 1\}^n$. We will see later (Section 3.5.3) that this set of valuations corresponds to our concave demand type $\mathcal{D}^n_{ss}$.

[43] That is, for an *overall* price change from $\mathbf{p}$ to $\mathbf{p}'$, and a demand *step* from $\widetilde{\mathbf{x}}$ to $\widetilde{\widetilde{\mathbf{x}}}$, we require $(\mathbf{p}'-\mathbf{p}).(\widetilde{\widetilde{\mathbf{x}}}-\widetilde{\mathbf{x}}) < 0$, with the exception that the change in demand is, of course, zero if $\mathbf{p}$ and $\mathbf{p}'$ are within the same UDR. (The strict law of demand generalises the observation that demand must go down for goods whose prices go up–see e.g. Mas-Colell et al., 1995, Proposition 2.F.1.)

Definition 3.4.2. For prices $\mathbf{p}$ and $\mathbf{p}'$ such that $D_u(\mathbf{p}) \neq D_u(\mathbf{p}')$, we say we can *break down the demand change from* $\mathbf{p}$ *to* $\mathbf{p}'$ *into* $\mathbb{Z}\mathcal{D}$*-steps* if for all $\mathbf{x} \in D_u(\mathbf{p})$ there exists $\mathbf{x}' \in D_u(\mathbf{p}')$, and a series of bundles $\mathbf{x} = \mathbf{x}^0, \ldots, \mathbf{x}^l = \mathbf{x}'$, demanded respectively at prices $(1 - \lambda_j)\mathbf{p} + \lambda_j \mathbf{p}'$, $j = 0, \ldots, l$, for some $0 = \lambda_0 \leq \cdots \leq \lambda_l = 1$, such that $\mathbf{x}^j - \mathbf{x}^{j-1} \in \mathbb{Z}\mathcal{D}$.

We call these $\mathbb{Z}\mathcal{D}$-steps *improving* if each demand change additionally "satisfies the strict law of demand" with respect to the *whole* price change, that is, $(\mathbf{p}' - \mathbf{p}).(\mathbf{x}^j - \mathbf{x}^{j-1}) < 0$.

We say we can *break down the demand change into improving* $\mathcal{D}$*-steps* if there always exists $\mathbf{x}^0, \ldots, \mathbf{x}^l = \mathbf{x}' \in D_u(\mathbf{p}')$ satisfying these conditions and $\mathbf{x}^j - \mathbf{x}^{j-1} \in \mathcal{D}$ for all j.

Definition 3.4.3. We say a valuation u satisfies the *ordinary* $\mathbb{Z}\mathcal{D}$*-improvement property* if, for any bundle $\mathbf{x}$ which is the unique demand at some price, $\mathbf{p}$, i.e., satisfying $D_u(\mathbf{p}) = \{\mathbf{x}\}$, and any price $\mathbf{p}'$ such that $\mathbf{x} \notin D_u(\mathbf{p}')$, there exists $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$, and such that $\mathbf{x}'' - \mathbf{x} \in \mathbb{Z}\mathcal{D}$.

We say the valuation u satisfies the *ordinary* $\mathcal{D}$*-improvement property* if there always exists $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at $\mathbf{p}'$ and $\mathbf{x}'' - \mathbf{x} \in \mathcal{D}$.

Theorem 3.4.4. *If* $\mathcal{D}$ *is any demand type, the following are equivalent for a valuation* u*:*

1. u *is of demand type* $\mathcal{D}$*;*

2. *for any* $\mathbf{p}$ *such that* $\#D_u(\mathbf{p}) = 1$ *and any* $\mathbf{p}'$ *we can break down the demand change from* $\mathbf{p}$ *to* $\mathbf{p}'$ *in improving* $\mathbb{Z}\mathcal{D}$*-steps;*

3. *for any* $\mathbf{p}$ *such that* $\#D_u(\mathbf{p}) = 1$*, and any* $i \in \{1, \ldots, n\}$ *and any* $\epsilon > 0$*, we can break down the demand change from* $\mathbf{p}$ *to* $\mathbf{p} + \epsilon\mathbf{e}^i$ *in* $\mathbb{Z}\mathcal{D}$*-steps;*

4. u *satisfies the ordinary* $\mathbb{Z}\mathcal{D}$*-improvement property.*

Proof. See Appendix 3.A.1. □

Condition 2 implies 4 straightforwardly, because the fact that each separate step satisfies the "strict law of demand" with respect to the overall price change, means that each successive bundle in the sequence must be strictly preferred to its predecessor at the final price. So, in particular, the second bundle in the sequence is a bundle that is strictly preferred to the starting bundle at the final price. Condition 4 implies 1 because the only bundles preferred to a bundle demanded in a UDR, at any price that is just on the other side of any facet bounding the UDR, are demanded on the facet itself, so the ordinary $\mathbb{Z}\mathcal{D}$-improvement property implies that the primitive integer facet normal must lie in $\mathcal{D}$. Condition 2 clearly implies 3, and 3 implies 1 because any facet whose normal was not in $\mathcal{D}$ would not satisfy property 3 for some good i. Finally, the relationship between 1 and 2 is straightforward for any pair of prices at which the demands are unique, and for which the straight line joining them crosses only facet interiors, although it needs more careful argument when the price path crosses lower dimensional cells of the TH.

The inclusion of 3 shows that studying demand changes arising from a change in a single price suffices to understand the shapes of the trade-offs an agent might make and, moreover, that we need not assume the 'improving' property, since this will hold automatically.

It is, however, very useful to know that we can always choose 'improving' $\mathbb{Z}\mathcal{D}$-steps, since this greatly restricts the range of possible steps for any price change. We will see this when we prove the relationships between certain demand types and the concepts of substitutes and complements (see Propositions 3.4.8 and 3.4.11).

The reason we need $\mathbb{Z}\mathcal{D}$-steps, rather than only $\mathcal{D}$-steps, is because of possible failures of concavity–consider, for example, the function of one variable $v(0) = v(1) = 0$, $v(2) = 2$ for which demand jumps from zero to 2 as price rises above 1. If we additionally assume that u is concave throughout, we can strengthen Theorem 3.4.4 to deal with improving $\mathcal{D}$-steps and the ordinary $\mathcal{D}$-improvement property:

Theorem 3.4.5. *If $\mathcal{D}$ is any concave demand type, the following are equivalent for a concave valuation u:*

1. *u is of (concave) demand type $\mathcal{D}$;*
2. *for any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$ and any $\mathbf{p}'$ we can break down the demand change from $\mathbf{p}$ to $\mathbf{p}'$ in improving $\mathcal{D}$-steps;*
3. *for any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$, and any $i \in \{1, \ldots, n\}$ and any $\epsilon > 0$, we can break down the demand change from $\mathbf{p}$ to $\mathbf{p} + \epsilon \mathbf{e}^i$ in improving $\mathcal{D}$-steps;*
4. *u satisfies the ordinary $\mathcal{D}$-improvement property.*

Proof. See Appendix 3.A.1. ☐

On the other hand, we might ask whether concavity need be explicitly assumed in 2, 3 and 4. However, Example 3.A.1 in the appendix shows the fact that we can always break down the demand change between any UDR prices in improving $\mathcal{D}$-steps does not imply u is concave, so we must assume concavity for 2 and 3. Similarly, Example 3.A.2 shows that the ordinary $\mathcal{D}$-improvement property on its own does not imply concavity. Thus Theorem 3.4.5 is indeed stated in its most general form.

We now illustrate the application of improving $\mathbb{Z}\mathcal{D}$-steps, and Theorem 3.4.4, in the cases of 'ordinary' substitutes and complements:

3.4.3 Examples

3.4.3.1 Ordinary Substitutes

We use the standard definitions of "(ordinary) substitutes" as in Ausubel and Milgrom (2002).[44] An appealing aspect of this definition is that, as they show (their Theorem 10), it is

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[44] That is, we call "ordinary substitutes", precisely what Ausubel and Milgrom (2002) simply call "substitutes". We hope this increases clarity (since others loosely refer to substitutes in other ways). Note, in particular, that

equivalent to the submodularity of the dual profit function.

Definition 3.4.6 (Ausubel and Milgrom, 2002)**.** Let $A \subsetneq \mathbb{Z}^n$ be finite, and $u : A \to \mathbb{R}$ be a valuation. Goods are *ordinary substitutes* if for any prices $\mathbf{p}' \geq \mathbf{p}$ such that $\#D_u(\mathbf{p}) = \#D_u(\mathbf{p}') = 1$, if $\{\mathbf{x}\} = D_u(\mathbf{p})$ and $\{\mathbf{x}'\} = D_u(\mathbf{p}')$ then $x'_k \geq x_k$ for all k such that $p_k = p'_k$.

We define correspondingly the demand type, $\mathcal{D}^n_{os}$.

Definition 3.4.7. $\mathcal{D}^n_{os}$ consists of those primitive integer vectors in $\mathbb{Z}^n$ with at most one positive and at most one negative coordinate entry, and all others zero.[45]

Theorem 3.4.4 enables us to easily relate $\mathcal{D}^n_{os}$ to ordinary substitutes; we retain the proof in the body text as it provides a powerful illustration of how useful it is to 'break down demand changes in improving $\mathbb{Z}\mathcal{D}$-steps'.[46]

Proposition 3.4.8. *A valuation is of demand type $\mathcal{D}^n_{os}$ iff it is an ordinary substitutes valuation.*

Proof. We apply Theorem 3.4.4, as follows. If valuation is of demand type $\mathcal{D}^n_{os}$ then we can break down the demand change from any UDR price $\mathbf{p}$ to any UDR price $\mathbf{p}' \geq \mathbf{p}$ in improving $\mathbb{Z}\mathcal{D}$-steps. Since each step is in $\mathcal{D}^n_{os}$, demand strictly decreases for at most one good at each step, and by the strict law of demand, that good must be one whose price has increased. Thus at each step, demand weakly increases for all goods whose prices have remained constant. Hence this holds overall.

Conversely, if the valuation is not of demand type $\mathcal{D}^n_{os}$ then there exists a facet F with normal $\mathbf{v}$ where $v_i, v_j < 0$ for some $i \neq j$. Then $\mathbf{e}^i.\mathbf{v} \neq 0$ so we may choose UDR prices $\mathbf{p}, \mathbf{p} + \epsilon\mathbf{e}^i$ on either side of this facet. We know the demand change is a positive integer multiple of $\mathbf{v}$, and so demand for good j decreases: goods i and j are not substitutes. ☐

So we can straightforwardly identify whether goods are ordinary substitutes from their 'demand type'.

It is immediate, for example, that the examples of Figures 3.1, 3.4a, 3.5, 3.13a, 3.13b, 3.14a and 3.14b, are all of type $\mathcal{D}^2_{os}$, as is Example 3.2.10 (Figure 3.3), while our 3-dimensional example, Figure 3.6, has demand type $\mathcal{D}^3_{os}$, and we now show Example 3.2.14 and Example 3.2.15 have demand type $\mathcal{D}^n_{os}$.

Ausubel and Milgrom's (2002) definition (our Definition 3.4.6) is *not* identical to that of Kelso and Crawford (1982) when there are multiple units of three or more goods. (See Danilov et al. 2003 Example 6 and Theorem 1); the definitions *are* equivalent in the simpler cases $n = 2$ (see Baldwin, Klemperer and Milgrom, in preparation) and $A = \{0, 1\}^n$ (Danilov et al. 2003, Corollary 5; see also Hatfield et al. 2011 Theorem A.1). Milgrom and Strulovici (2009) call Kelso and Crawford's original definition "weak substitutes", but this is in fact a *stronger* definition of substitutes than Ausubel and Milgrom's. The latter definition (that we follow) seems most natural in the general case, and is also equivalent to several properties that seem to naturally characterise "standard" substitutes, and to the indirect utility function ($\max_{\mathbf{x} \in A}\{u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\}$) being submodular. We discuss these issues in detail in Baldwin, Klemperer and Milgrom (in preparation).

[45] Danilov et al. (2003) say 'each cell of a valuation's parquet is a polymatroid' where we say that a valuation has demand type $\mathcal{D}^n_{os}$.

[46] This result also follows if we combine Ausubel and Milgrom (2002, Theorem 10) and Danilov et al. (2003, Theorem 1); we provide this alternative proof to illustrate the use of Theorem 3.4.4 in understanding demand types.

Example 3.2.14 revisited. Because the valuations of the ‘workers’ in Kelso and Crawford’s (1982) matching model have domain $\{\mathbf{0}, -\mathbf{e}^{(i,j)} \mid j = 1, \ldots, m_2\}$, their only possible SNP edges are in $\mathcal{D}^n_{os}$, so they are of demand type $\mathcal{D}^n_{os}$. Since the ‘firms’ in this model have valuations with domain $\{0,1\}^{m_1}$ which are assumed to satisfy the conditions for ordinary substitutes, they are also of demand type $\mathcal{D}^n_{os}$.

Example 3.2.15 revisited. Hatfield et al. (2013) describe the goods to be sold in their model as *complements* of goods to be bought, because they measure both buying and selling as non-negative quantities. But if we instead think of selling as just “negative buying” then the “complementarities” in their model disappear, and it is then clear that their condition of ‘full substitutability’ is precisely the ordinary substitutes condition of, e.g., Ausubel and Milgrom (2002). That is, an agent whose valuation domain is as described in Example 3.2.15 has ‘fully substitutable’ preferences iff the valuation is of type $\mathcal{D}^n_{os}$.[47]

Example 3.2.12 illustrates a special case of this model: the agent regards the actions of buying good 2 and selling good 1 as complements, but regards the *buying* of both goods (possibly in negative quantities) as substitutes. The point is easily seen geometrically–Figure 3.5 clearly represents substitutes preferences, and not the complements preferences of Figure 3.4b, which we formally introduce in Section 3.4.3.2.

3.4.3.2 Ordinary Complements

“Complements” can be defined analogously to the Definition 3.4.6 of “ordinary substitutes”:

Definition 3.4.9. Let $A \subsetneq \mathbb{Z}^n$ be finite, and let $u : A \to \mathbb{R}$ be a valuation. Goods are *ordinary complements* if, for any prices $\mathbf{p}' \geq \mathbf{p}$ such that $\#D_u(\mathbf{p}) = \#D_u(\mathbf{p}') = 1$, if $\{\mathbf{x}\} = D_u(\mathbf{p})$ and $\{\mathbf{x}'\} = D_u(\mathbf{p}')$ then $x'_k \leq x_k$ for all k such that $p_k = p'_k$.

Similarly to Definition 3.4.7 we define a corresponding demand type:

Definition 3.4.10. $\mathcal{D}^n_{oc}$ consists of those primitive integer vectors in $\mathbb{Z}^n$ whose non-zero coordinate entries are all of the same sign.

As in Proposition 3.4.8, it is an elementary consequence of Theorem 3.4.4 that

Proposition 3.4.11. *A valuation is of demand type $\mathcal{D}^n_{oc}$ iff it is an ordinary complements valuation.*

Proof. As with Proposition 3.4.8, this follows immediately from Theorem 3.4.4: we break down the change in demand from some UDR price $\mathbf{p}$ to $\mathbf{p}' \geq \mathbf{p}$ into improving $\mathbb{Z}\mathcal{D}$-steps. As before,

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[47] The precise ‘choice language’ definition of Hatfield et al. (2013) is superficially different from our Definition 3.4.6, but an earlier version of their paper presents an alternative ‘demand-language’ definition (Hatfield et al., 2011, Definition 4) which corresponds precisely to the Ausubel and Milgrom (2002) definition, and moreover they confirm (Hatfield et al., 2011, Theorem A.1) that this definition is equivalent to the ‘choice-language’ definition of Hatfield et al. (2013). They define “full substitutability” using Sun and Yang’s (2006) ‘gross substitutes and complements’ ideas; see Section 3.5.5.

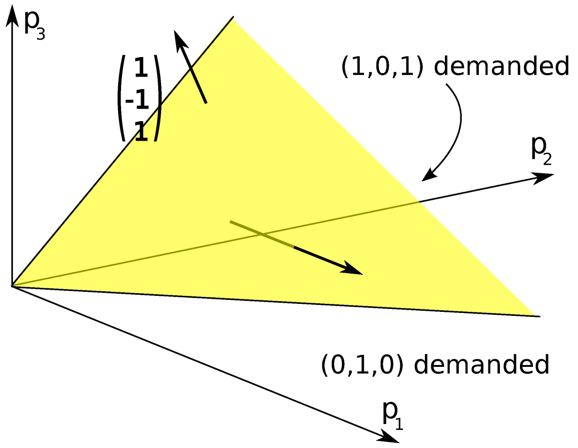

Figure 3.15: *A facet with normal* $(1,-1,1)$*: increasing either* p_1 *(as shown with an arrow) or* p_3 *demonstrates complementarities between goods 1 and 3.*

at each step demand must strictly decrease for at least one good whose price has increased; this time, the nature of $\mathcal{D}^n_{oc}$ implies that demand weakly decreases for all goods. Conversely, if there exists a facet with normal $\mathbf{v}$ such that $v_i > 0, v_j < 0$ then we may pick prices $\mathbf{p}, \mathbf{p} + \epsilon\mathbf{e}^j$ on either side of this facet and demonstrate failure of complements. ☐

The examples of Figures 3.4b, 3.13b, 3.13c, 3.14c and 3.14d are all of type $\mathcal{D}^2_{oc}$.

Note that although complements are often thought of as directly analogous to (ordinary) substitutes–as they are in two dimensions–this is not true if there are more than two goods. The case of complements permits facet normals with *any* number of non-zero entries, whereas substitutes permits at most two non-zero entries.

The reason is that with substitutes, if any one good could trade-off against two others at the same price, it would necessarily follow that the two other goods were complementary. Even when all goods are mutual substitutes, there can never be trade-offs between more than two of them across a single facet: if more than two facet normal coordinate entries are non-zero, then at least two must have the same sign, so there are complementarities between the corresponding goods.

Consider, for example, Figure 3.15, in which there is a facet with normal (1,-1,1), defined by $\{\mathbf{p} \in \mathbb{R}^3 \mid p_1 + p_3 = p_2;\ p_1, p_2, p_3 \geq 0\}$: an increase in the price of *either* good 1 or good 3 that moves from the UDR with $p_1 + p_3 < p_2$ to the UDR with $p_1 + p_3 > p_2$ reduces demand for *both* goods. So, despite the symmetry between Definitions 3.4.6 and 3.4.9, complements allows far more degrees of freedom than does substitutes.[48] One benefit of our way of classifying demand

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[48] To illustrate why the conditions for indivisible goods to be substitutes are so restrictive, consider a consumer who regularly makes three kinds of trips: journey A can be made only by bus or train; journey B can be made only by car or train; journey C can be made only by car or bus. Thought of as divisible goods, the three modes of transport are clearly always mutual substitutes. But if bus tickets, train tickets, and cars are all *in*divisible, there are typically price vectors at which two of the goods are locally complements. Start at any prices at which the consumer just prefers to use only public transport, i.e., has no car. Then if the price of *either* of the forms of

"types" is that it makes this lack of symmetry between substitutes and complements very clear.

Example 3.2.16 revisited. Recall that we can embed a model of 'coalition formation with transferable utilities' in our framework by considering each feasible coalition as a separate agent. The agent corresponding to any feasible coalition $\mathbf{x}$ has valuation with domain $\{\mathbf{0}, \mathbf{x}\}$, so the only facet of the corresponding TH has normal $\mathbf{x}$: the agent views the people who would form this coalition as perfect complements, and considers only the trade-off between all of them and none of them.

Consistent with our terminology, we say a coalition-formation problem with transferable utility is of 'type $\mathcal{D}$', for any $\mathcal{D}$ containing all the feasible coalitions in the problem. (We would normally consider 'complements' demand types, $\mathcal{D}$.)

3.4.3.3 Additively Separable Demand

Additively separable demand corresponds to an extremely simple demand type:

Definition 3.4.12. $\mathcal{D}_a^n$ consists of the coordinate vectors $\{\mathbf{e}^i \mid i = 1, \ldots, n\}$ in $\mathbb{Z}^n$.

In the additively separable case, a change in the price of one good will never affect demand for any other good. So it is not hard to show:

Proposition 3.4.13. *A valuation is of concave demand type $\mathcal{D}_a^n$ iff it is concave and additively separable.*

Proof. Recall that demand is additively separable iff a change in the price for any one good has no effect on the demand for other goods; if demand is additionally concave, then it is also true that demand is additively separable iff a change in price between UDR prices for any one good has no effect on the demand for other goods. Referring to Theorem 3.4.4, we see that this holds iff the change in demand at each step must only affect one good – that is, $\mathcal{D} = \mathcal{D}_a^n$. □

Note that being additively separable is a more stringent condition than being both substitutes and complements: we can only guarantee such a valuation is additively separable if it is also concave. A simple example of a valuation of type $\mathcal{D}_a^2$ which is not concave, and not additively separable is: $A = \{0, 1, 2\}^2$, and

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & (x_1, x_2) \neq (1, 1) \\ 0 & (x_1, x_2) = (1, 1). \end{cases}$$

---

public transport is slightly raised, the consumer buys a car and in general reduces her use of *both* forms of public transport. Qualitatively the situation is locally exactly that pictured in Figure 3.15, in which the car takes the role of good 2, and the two forms of public transport take the roles of goods 1 and 3.

3.4.4 Changes of basis

It is straightforward that two demands share many properties if one can be transformed into the other by a unimodular basis change.[49],[50] Such a basis change is equivalent to re-packaging the goods so that any integer bundle can still be obtained by buying and selling an (integer) selection of the new packages; and any integer selection of the new packages was available as an integer combination of the original goods. So such a basis change leaves many important properties of demand–including, we will see in Section 3.6, the existence of competitive equilibrium–unaffected.

Likewise, such a basis change simply distorts the TH by a linear transformation which leaves its important structure unaffected:

Proposition 3.4.14 (cf. Gorman, 1976, p. 219)**.** *For* $A \subsetneq \mathbb{Z}^n$ *and* $u : A \to \mathbb{R}$ *and a unimodular* $n \times n$ *matrix* G*, define the ("pullback") basis change of* u *by* G *to be* $G^*u : G^{-1}A \to \mathbb{R}$ *via* $G^*u(\mathbf{y}) := u(G\mathbf{y})$*. Then*

1. *A bundle is demanded under the original demand at a certain price iff an associated bundle is demanded under the transformed demand at an associated price; specifically:* $\mathbf{x} \in D_u(\mathbf{p}) \iff G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p})$.
2. *The TH of the transformed demand is given by a linear transformation of the original demand:* $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$*;*
3. *The inverse transformation to* G *applies to demand types:* $u(\cdot)$ *is of (concave) demand type* $\mathcal{D}$ *iff* $G^*u(\cdot)$ *is of (concave) demand type* $G^{-1}\mathcal{D} = \{G^{-1}\mathbf{v} \mid \mathbf{v} \in \mathcal{D}\}$*.*

Proof. See Appendix 3.A.1. □

As a simple example, if $n = 2$ then ordinary substitutes $\mathcal{D}^n_{os}$ are a unimodular basis change of ordinary complements $\mathcal{D}^n_{oc}$, via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.[51] As discussed in Section 3.4.3.2, this does not hold for $n \geq 3$.

3.5 Complete Demand Types and Unimodular Demand Types

The previous section showed that the change in demand between any two UDR prices could be broken down into improving $\mathbb{Z}\mathcal{D}$-steps and, for concave valuations, into improving $\mathcal{D}$-steps.

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[49] A unimodular matrix G is an integer matrix with integer inverse; an action of G on bundles of goods corresponds to an action of G^T on prices.

[50] Specific cases of this observation have been made before (see e.g. Sun and Yang 2006, and a more general treatment for substitutes in Sun and Yang 2008, and Hatfield et al., 2013); we lay out the general behaviour here.

[51] So, for example, we will see that with two goods (in indivisible units), competitive equilibrium fails "as often" for sets of agents with ordinary substitutes demands, as for sets of agents with ordinary complements demands, even though the economic properties of substitutes and complements are, of course, very different.

Since UDR prices are dense in the set of all prices, this tells us a great deal about the structure of demand.

However, one may wish for results relating to every possible price and starting bundle. For example, the standard definition of 'ordinary substitutes' for indivisible goods (see, e.g., Ausubel and Milgrom, 2002) that we use only considers demand changes between UDR prices, but others have used a definition of substitutes that compares demands at any pair of prices (see, for example, Kelso and Crawford 1982).[52] As is shown by Danilov et al. (2003, Example 6), for $n \geq 3$ there exist valuations which satisfy the 'ordinary substitutes' property but do not satisfy the stronger requirement. Similar remarks apply to 'ordinary complements'.

In this section we clarify this distinction, and introduce a property which a valuation might have: $\mathcal{D}$-completeness, which means we that *can* break down the change in demand between any two prices into improving $\mathcal{D}$-steps. So if a valuation is $\mathcal{D}^n_{os}$- or $\mathcal{D}^n_{oc}$-complete then it does indeed satisfy the corresponding stronger substitute or complement conditions.

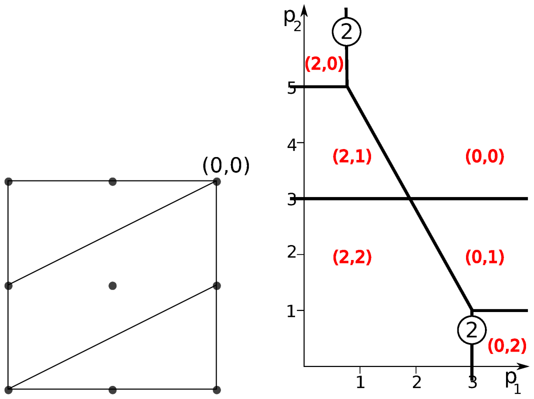
We correspondingly identify a new class of 'complete' demand types: those types $\mathcal{D}$ for which every valuation of type $\mathcal{D}$ is $\mathcal{D}$-complete.

We begin with an illustration of a demand type $\mathcal{D}$ and a concave valuation of type $\mathcal{D}$ which is *not* $\mathcal{D}$-complete: we cannot break down the change in demand between any two prices into improving $\mathcal{D}$-steps, or even $\mathbb{Z}\mathcal{D}$-steps, even for concave valuations.[53]

Example 3.5.1. Consider the following (concave) valuation given, with its SNP, in Figure 3.16. This is of demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(2,1)\}$. Note that $(1,1) \in D_u(2,3)$ but that

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
2	1	0	$x_2 = 0$
7	5	3	$x_2 = 1$
10	7	4	$x_2 = 2$

(a) *The concave valuation u.*



(b) *The SNP of u.* **(c)** *The TH of u.*

Figure 3.16: *The valuation, SNP, and TH of Example 3.5.1.*

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[52] These two definitions are equivalent when $A = \{0,1\}^n$ (see Corollary 3.5.17) and almost all preceding work has been restricted to this case, so the distinction between these definitions is sometimes blurred. We discuss the relationships between alternative definitions of substitutes at length in Baldwin, Klemperer and Milgrom (in preparation).

[53] When $n = 2$ the ordinary substitutes demand type $\mathcal{D}^n_{os}$ is in fact 'complete', in our terminology (see Baldwin, Klemperer and Milgrom, in preparation). The 3-dimensional example of Danilov et al. (2006) is more complicated than is needed here, and so we use a very restrictive demand type.

$D_u(2, 3 - \epsilon) = \{(2, 2)\}$ for any $\epsilon > 0$. Any series of improving $\mathbb{Z}\mathcal{D}$-steps from $(1, 1)$ to $(2, 2)$ would thus have to increase the demand for the first good, as well as the second – but a step in direction $(1, 0)$ does not satisfy the strict law of demand for the price change we consider, and a step in direction $(2, 1)$ increases demand for the first good by too much. So we cannot break down the change in demand from $\mathbf{p} = (2, 3)$ to $\mathbf{p}' = (2, 2.9)$ in improving $\mathbb{Z}\mathcal{D}$-steps.

We now turn to developing exactly what we mean by 'completeness':

3.5.1 Completeness of a Demand Type

Recall that Definition 3.4.3 was not exactly a generalisation of Gul and Stacchetti's (1999) "single-improvement property": a true generalisation may be given as follows:[54]

Definition 3.5.2. We say a valuation u satisfies the *complete $\mathcal{D}$-improvement property* if, for any bundle $\mathbf{x}$ and for any price $\mathbf{p}'$ such that $\mathbf{x} \notin D_u(\mathbf{p}')$, there exists $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$ and such that $\mathbf{x}'' - \mathbf{x} \in \mathcal{D}$.

Note how this differs from Definition 3.4.3: for the 'complete' property we do not insist that the bundle $\mathbf{x}$ be demanded at any price at all, whereas the 'ordinary' property requires that $\mathbf{x}$ be demanded *uniquely* at some price.

We show that, given concavity, the complete $\mathcal{D}$-improvement property is equivalent to our being able to break down any change in demand into improving $\mathcal{D}$-steps:

Proposition 3.5.3. *The following are equivalent for a concave valuation u of type $\mathcal{D}$:*

1. *we can break down the demand change from any $\mathbf{p}$ to any $\mathbf{p}'$ in improving $\mathcal{D}$-steps;*
2. *u satisfies the complete $\mathcal{D}$-improvement property.*

Proof. See Appendix 3.A.2. □

The assumption of concavity is needed to ensure that every bundle $\mathbf{x}$ is demanded at *some* price – and hence that we can use improving $\mathcal{D}$-steps to obtain $\mathcal{D}$-improvements.[55] Since concavity is also needed for the existence of improving $\mathcal{D}$-steps and the ordinary $\mathcal{D}$-improvement property (as distinct from $\mathbb{Z}\mathcal{D}$-steps and the ordinary $\mathbb{Z}\mathcal{D}$-improvement property) we simply work under this assumption for this section.

Definition 3.5.4.

1. We say a concave valuation u is *$\mathcal{D}$-complete* if the equivalent conditions of Proposition 3.5.3 hold.

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[54] This also generalises Sun and Yang (2009), which generalised Gul and Stacchetti (1999) to the case of "gross substitutes and complements".

[55] In fact, 3.5.3.2⇒3.5.3.1 when u is not concave; one may see that the proof makes no use of concavity. However, a non-concave valuation satisfying 3.5.3.1 need not satisfy 3.5.3.2; consider Example 3.5.1 with the modification $u(1, 1) = 4$.

2. We say a concave demand type $\mathcal{D}$ is a *complete* demand type if every concave valuation of type $\mathcal{D}$ is $\mathcal{D}$-complete.

It is straightforward now to provide the following analogue to Theorems 3.4.4 and 3.4.5:

Corollary 3.5.5. *If $\mathcal{D}$ is any complete demand type, then the following are equivalent for a concave valuation u:*

1. *u is of concave demand type $\mathcal{D}$ (and the equivalent conditions of Theorem 3.4.4 hold);*
2. *we can break down the demand change from any $\mathbf{p}$ to any $\mathbf{p}'$ in improving $\mathcal{D}$-steps;*
3. *for any $\mathbf{p}$ and any $i \in \{1, \ldots, n\}$ and any $\epsilon > 0$, we can break down the demand change from $\mathbf{p}$ to $\mathbf{p} + \epsilon \mathbf{e}^i$ in improving $\mathcal{D}$-steps;*
4. *u satisfies the complete $\mathcal{D}$-improvement property.*

Proof. Given that $\mathcal{D}$ is complete, 1⇔2 and 1⇔4 by Proposition 3.5.3 and Definition 3.5.2. That 2⇒3 is clear; that 3⇒1 follows by the corresponding argument of Theorem 3.4.5. □

Thus, once completeness of a demand type is known, the weaker checks of Theorem 3.4.4 are sufficient to provide their 'complete' counterparts given here. Moreover, as is intuitive, completeness is preserved under unimodular basis changes:

Proposition 3.5.6. *Suppose G is a unimodular $n \times n$ matrix.*

1. *If $u : A \to \mathbb{R}$ is concave and $\mathcal{D}$-complete, then $G^* u$ is concave and is $G^{-1}\mathcal{D}$-complete.*
2. *If $\mathcal{D}$ is complete then $G^{-1}\mathcal{D}$ is complete.*

Proof. See Appendix 3.A.2. □

The question remains of which demand types are complete. In Section 3.5.2 we develop a sufficient condition: unimodularity. However, this condition is not necessary. Here we give two examples, one unimodular and one not.

Example 3.5.7. The set $\mathcal{D}^n_{all}$ of *all* primitive integer n-vectors is complete. This is trivial: given any concave valuation and any prices $\mathbf{p}, \mathbf{p}'$ we can simply note the bundles demanded in improving steps on the straight line from $\mathbf{p}$ to $\mathbf{p}'$, interpolating additional primitive integer steps if necessary. It is of interest to emphasise this point, because many sub-types $\mathcal{D} \subsetneq \mathcal{D}^n_{all}$ are *not* complete. This fact stands in contrast to the property of unimodularity, which we shall come to in Section 3.5.2.

Example 3.5.8. The set $\mathcal{D}^n_a$ of the coordinate vectors is a complete demand type. As shown in Proposition 3.4.13, any concave valuation u of type $\mathcal{D}^n_a$ is additively separable. Thus, for any good i, the quantity demanded of any single good depends only on the price for that good. So if $\mathbf{x} \notin D_u(\mathbf{p})$ for any bundle $\mathbf{x}$ and price vector $\mathbf{p}$, then we may adjust the quantities of all goods independently, towards the desired levels. That is, we may make a series of $\mathcal{D}^n_a$-improvements given any starting bundle and price.

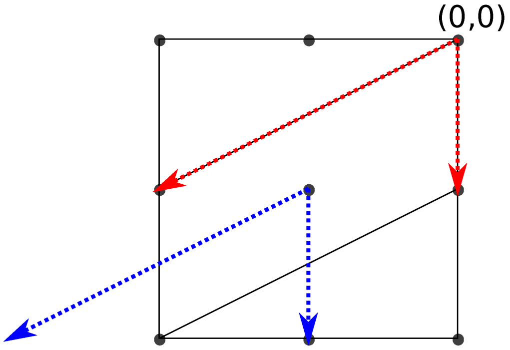

Figure 3.17: *The SNP of Example 3.5.1, with the relevant vectors highlighted.*

3.5.2 Unimodularity: a sufficient condition for completeness

We now introduce a condition on demand types which is sufficient for their completeness; strikingly, the same condition is also necessary and sufficient for existence of competitive equilibrium (see Section 3.6.3).

Throughout, we write "the *determinant* of vectors $\mathbf{w}^1, \ldots, \mathbf{w}^n$" to mean the determinant of the $n \times n$ matrix which has these vectors as its columns.[56] We say that a linearly independent set $\{\mathbf{w}^1, \ldots, \mathbf{w}^s\}$ of vectors is "an *integer basis* for the subset they span" if, whenever $\mathbf{y} \in \mathbb{Z}^n$ can be written as $\sum_{i=1}^{s} a_i \mathbf{w}^i$, in which $a_i \in \mathbb{R}$, then in fact $a_i \in \mathbb{Z}$ for $i = 1, \ldots, s$.

Definition 3.5.9. We say a demand type $\mathcal{D}$ is *unimodular* if any linearly independent set of vectors in $\mathcal{D}$ is an integer basis for the subspace they span.

It is clear that the set $\mathcal{D}^n_{all}$ of Example 3.5.7 does not have this property, whereas the set $\mathcal{D}^n_a$ of Example 3.5.8 does have it. We now clarify the implications of unimodularity by considering an example in which both unimodularity and completeness fail:

Example 3.5.1 revisited. Recall we considered a particular valuation u of demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(2,1)\}$. This demand type is not unimodular. The vector $(1,1)$ is in the span of $(0,1)$ and $(2,1)$ but it is not possible to express it as an integer combination of these vectors. This impossibility is illustrated in Figure 3.17, in which the vectors $(0,1)$ and $(2,1)$, starting from the origin, are highlighted in red.

It is similarly impossible to move on from the bundle $(1,1)$ to the bundle $(2,2)$ using this pair of directions, as highlighted in blue. Since these two vectors are the only two in $\mathcal{D}$ that change the quantity of Good 2 demanded, the failure of $\mathcal{D}$-completeness follows.[57]

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[56] Changing the order of the vectors may change the sign of the determinant, so strictly speaking the determinant is a property of an ordered n-tuple of vectors. This detail does not concern us as we are only ever interested in the absolute values of determinants.

[57] If $(1,1)$ were in the demand type, then it would be possible to make a suitable step, confirming (as is already clear from Example 3.5.7) that unimodularity is not necessary.

To understand the relevance of unimodularity in general, consider a set of s linearly independent vectors from a demand type $\mathcal{D}$. These are the edges of an s-dimensional parallelepiped. This shape will contain no integer point (either in its boundary or in its interior) aside from its vertices, iff, our set of vectors is an integer basis for the subspace they span. Thus the scenario of Figure 3.17, in which the bundle $(1,1)$ gives difficulties, simply cannot arise if the demand type is unimodular.

This condition may appear difficult to check. However, returning to the parallelepiped, it is not too difficult to see that unimodularity holds iff the s-dimensional *volume* of the parallelepiped is 1. When $s = n$, this volume is simply the (absolute value of the) determinant of the vectors along its edges. When $s < n$, unimodularity is equivalent to the ability to add additional integer vectors that build the s-dimensional volume out into an n-dimensional volume of 1, i.e., so that the determinant is again ± 1. (In the case of Figure 3.17, the determinant in question is 2).

We gather these conditions with an alternative check for unimodularity when $s < n$, via appropriate determinants of submatrices, as follows:

Remark 3.5.10. The following are equivalent, for a set of s linearly independent vectors in $\mathbb{Z}^n$:

1. they are an integer basis for the subspace they span;
2. A s-dimensional parallelepiped in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$ and these vectors as edges contains no point in $\mathbb{Z}^n$ except its vertices;
3. they can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1;
4. among the determinants of all the $s \times s$ matrices consisting of s rows of the $n \times s$ matrix whose columns are these s vectors, the greatest common factor is 1.

Proofs of these facts may be found in Cassels (1959).[58] Note that when $s = n$ then both 2 and 3 remind us that the determinant of the vectors is ± 1. We refer to a set of vectors as unimodular if every linearly independent subset has these properties.

Our result is now:

Theorem 3.5.11. *If a concave demand type is unimodular, then it is complete.*

Proof. See Appendix 3.A.2.[59] ☐

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[58] 1 ⇔ 3 follows from Cassels (1959) Lemma I.1 and Corollary I.3. 1 ⇔ 4 is Cassels (1959) Lemma I.2. For 1 ⇔ 2 consider a parallelepiped P whose vertices are $\mathbf{y} + \sum_{i=1}^{s} a_i \mathbf{w}^i$ for $a_i \in \{0,1\}$. If $\mathbf{z}$ is a non-vertex integer point in P, then $\mathbf{z} - \mathbf{y}$ exhibits the failure of 1. Conversely, if failure of 1 is exhibited by an integer $\sum_{i=1}^{s} b_i \mathbf{w}^i$ where b_i are not all integers, then $\mathbf{y} + \sum_{i=1}^{s} a_i \mathbf{w}^i$ exhibits failure of 2, where a_i is the non-integer part of b_i in each case.

[59] As shown there, in fact we prove a stronger result: we need only assume that, for any vectors $\mathcal{V} \subseteq \mathcal{D}$, there exist linearly independent vectors $\mathbf{w}^1, \ldots, \mathbf{w}^s \in \mathcal{D}$, whose span over $\mathbb{R}$ coincides with the span over $\mathbb{R}$ of $\mathcal{V}$ and such that, for any $\mathbf{v}^1, \ldots, \mathbf{v}^{s-1} \in \mathcal{V}$ and any $i = 1, \ldots, s$, the vectors $\mathbf{v}^1, \ldots, \mathbf{v}^{s-1}, \mathbf{w}^i$ are either linearly dependent or are a unimodular set. This property is clearly implied by unimodularity of $\mathcal{D}$, as we may take the set $\mathbf{w}^1 \ldots, \mathbf{w}^s$ to be any linearly independent and spanning subset of $\mathcal{V}$. However, it is not sufficient for unimodularity; the set $\mathcal{D} = \pm\{0,1\}^3$ is not unimodular but does have this property.

Using Remark 3.5.10.3 we can state a slightly weaker result that is more intuitive and very easy to check:

Corollary 3.5.12. *With* n *goods, a concave demand type* $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$*, in which* $\mathbf{v}^1, \ldots, \mathbf{v}^r$ *span* $\mathbb{R}^n$*, is complete if every subset of* n *vectors from* $\mathcal{D}$ *has determinant* 0 *or* ± 1*.*

The intuition for Theorem 3.5.11 is as described above regarding Figure 3.17. More detail is required because not every SNP-face is a parallelepiped, and general SNP faces may contain non-vertex points, even for a unimodular demand type. However, we show that, when unimodularity holds, we may step from a non-vertex point to another point in the SNP face, in the direction of a vector in $\mathcal{D}$. The additional assumption of concavity is required to ensure that this new point in the SNP face is indeed demanded at relevant prices; see Example 3.A.3 for an example of what happens when it fails.

We will see below that a very nice example of a concave unimodular demand type is 'strong substitutes'. The relationship between this and the corresponding 'single' improvement property is well known (Gul and Stacchetti, 1999; Milgrom and Strulovici, 2009). Corollary 3.5.5 and Theorem 3.5.11 together strictly generalise this result by showing that the complete $\mathcal{D}$-improvement property is satisfied by all valuations of type $\mathcal{D}$ for any unimodular concave demand type $\mathcal{D}$.[60] It follows from Seymour (1980) that there are many such demand types that are *not* a unimodular basis change of strong substitutes.

3.5.3 Unimodular Demand Types for Substitute Goods

Following from Theorem 3.5.11 and the arguments of Section 3.4.3.1 it is clear that a valuation will satisfy the following definition of substitutes if it is $\mathcal{D}^n_{os}$-complete:

Proposition 3.5.13. *If a concave valuation* u *is* $\mathcal{D}^n_{os}$*-complete then, for any prices* $\mathbf{p}' \geq \mathbf{p}$ *and any* $\mathbf{x} \in D_u(\mathbf{p})$*, there exists* $\mathbf{x}' \in D_u(\mathbf{p}')$ *such that* $x'_k \geq x_k$ *for all* k *such that* $p'_k = p_k$*.*

Proof. One may make use of $\mathcal{D}^n_{os}$-steps exactly as in Proposition 3.4.8, but now starting at any price $\mathbf{p}$. □

Note that the latter condition is provided as the definition of indivisible substitutes by Kelso and Crawford (1982).[61]

Following Theorem 3.5.11, then, we see:

Proposition 3.5.14. *If* $\mathcal{D} \subsetneq \mathcal{D}^n_{os}$ *is unimodular then every concave valuation of type* $\mathcal{D}$ *is a* $\mathcal{D}^n_{os}$*-complete valuation (and so satisfies the Kelso-Crawford definition).*

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[60] It is also observed by Danilov et al. (2008, 2013) that 'interval concave functions' can equivalently be characterised by 'interval package improvements' – see our Example 3.5.24. However, as demonstrated in Example 3.5.24, the interval package demand type is a unimodular basis change of the strong substitute demand type, and so their result follows immediately from the Gul and Stacchetti result.

[61] There are various nomenclatures in use for this concept; Danilov et al. (2003) call them 'gross substitutes' following Kelso and Crawford (who strictly speaking deal only with the $\{0,1\}^n$ case); Milgrom and Strulovici (2009) call them 'weak substitutes'.

On the other hand, for *ordinary* substitutes the $\mathcal{D}^n_{os}$-complete property may fail: see Danilov et al. (2003, Example 6).[62] Note also that, in order to apply the proposition, the demand type with respect to which u is complete must itself be a subset of $\mathcal{D}^n_{os}$; since (as just noted) there are *ordinary* substitutes which are not $\mathcal{D}^n_{os}$-complete, and since every valuation is of the complete demand type $\mathcal{D}^n_{all}$ (see Example 3.5.7), there clearly exist ordinary substitute valuations that are $\mathcal{D}$-complete for some $\mathcal{D} \not\subseteq \mathcal{D}^n_{os}$, but are *not* $\mathcal{D}^n_{os}$-complete.

We refer to unimodular demand types $\mathcal{D} \subsetneq \mathcal{D}^n_{os}$ as 'unimodular substitute' demand types.

One example of a 'unimodular substitute' demand type is $\{\pm(-1,1), \pm(-1,2), \pm(0,1)\}$. Note that this demand type does not contain both distinct coordinate vectors – an agent *cannot* be indifferent about whether to buy or sell an additional unit of good 1, while maintaining constant demand for good 2. But this is still a natural model if, for example, agents are manufacturers, selling good 1, which they can manufacture from different quantities of good 2, depending perhaps on the technology in use–indeed this is the natural generalisation of Hatfield et al. (2013) to multiple units of goods, see Examples 3.2.12 and 3.2.15.

The best-known example of a unimodular set of vectors also gives rise to a substitute demand type:

Definition 3.5.15. $\mathcal{D}^n_{ss}$ consists of those vectors in $\mathbb{Z}^n$ with at most one $+1$ and at most one -1 coordinate entry, and all others zero.[63]

Checking that this is a unimodular demand type is an application of a well-known result:

Theorem 3.5.16. *$\mathcal{D}^n_{ss}$ is a unimodular demand type, and is* maximal*: if $\mathcal{D}^n_{ss} \subsetneq \mathcal{D}$ then $\mathcal{D}$ is not unimodular.*

Proof. That the set of vectors that $\mathcal{D}^n_{ss}$ comprises is unimodular was first shown by Poincaré (1900); an attractive inductive proof was provided by Veblen and Franklin (1921), which we reproduce in Appendix 3.A.2. Its maximality is easy to show. If a vector $\mathbf{w}$ were introduced with $|w_j| > 1$ for some j then the determinant of $\mathbf{w}$ with all $\mathbf{e}^i$ such that $i \neq j$ is w_j, contradicting unimodularity by assumption. If a vector $\mathbf{w}$ has two $+1$ coordinate entries, say in i and j, then the determinant of it with $\mathbf{e}^i - \mathbf{e}^j$ and all $\mathbf{e}^k$ such that $k \neq i, j$, has absolute value 2. Thus no additional vectors may be introduced without contradicting unimodularity. $\square$

From the unimodularity of $\mathcal{D}^n_{ss}$ immediately follows:

Corollary 3.5.17. [64] *A concave ordinary substitute valuation with domain $A = \{0,1\}^n$ is a $\mathcal{D}^n_{os}$-complete valuation.*

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[62] Thus, Ausubel and Milgrom's (2002) definition (our Definition 3.4.6) is *not* identical to that of Kelso and Crawford (1982) when there are multiple units of three or more goods. The definitions *are* equivalent in the simpler cases $n = 2$ (see Baldwin, Klemperer and Milgrom, in preparation) and $A = \{0,1\}^n$ (Danilov et al. 2003, Corollary 5; see also Hatfield et al. 2011 Theorem A.1). See Baldwin, Klemperer and Milgrom (in preparation) for further discussion.

[63] Danilov et al. (2003) say a valuation is a 'PM-function' (where PM stands for polymatroid) in this case.

[64] An analogous result is Danilov et al. (2003) Corollary 5.

Proof. By Proposition 3.4.8 such a valuation is of demand type $\mathcal{D}^n_{os}$. If $A = \{0,1\}^n$ then the only possible SNP edges in $\mathcal{D}^n_{os}$, and hence the only possible facet normals, are those in $\mathcal{D}^n_{ss}$. $\square$

The vectors in $\mathcal{D}^n_{ss}$ were first related to substitutes by Tomizawa (1983), but without proof; proofs were offered by Danilov et al. (2003), and Fujishige and Yang (2003). An equivalent formulation was given by Milgrom and Strulovici (2009), who also show that there are several additional equivalent characterisations. Here we gather several relevant definitions:

Definition 3.5.18.

1. A valuation u is a *step-wise gross substitute valuation* if for any $\mathbf{p} \in \mathbb{R}^n$, any $\mathbf{x} \in D_u(\mathbf{p})$ and any $i \in \{0, \ldots, n\}$, either $\mathbf{x} \in D_u(\mathbf{p} + \epsilon \mathbf{e}^i)$ for all $\epsilon \geq 0$ or there exists $\epsilon \geq 0$ and $\mathbf{x}' \in D_u(\mathbf{p} + \epsilon \mathbf{e}^i)$ such that $x'_i = x_i - 1$ and $x'_{-i} \geq x_{-i}$.[65]
2. A valuation u is a *strong substitute valuation* if, when we consider every unit of every good as a separate good, then they are ordinary substitutes.[66]
3. A valuation u satisfies the *ordinary (complete) law of aggregate demand* if for $k \in \{1, \ldots, n\}$, any $\epsilon > 0$ any $\mathbf{p} \in \mathbb{R}^n$ and any $\mathbf{x} \in \mathbb{Z}^n$ such that $\{\mathbf{x}\} = D_u(\mathbf{p})$ (resp. $\mathbf{x} \in D_u(\mathbf{p})$) there exists $\mathbf{x}' \in D_u(\mathbf{p} + \epsilon \mathbf{e}^k)$ such that $\sum_i x'_i \leq \sum_i x_i$.[67]
4. A valuation u satisfies the *consecutive integer property* if, for every $\mathbf{p} \in \mathbb{R}^n$ and every $i \in \{1, \ldots, n\}$, the set $\{x_i \mid \mathbf{x} \in D_u(\mathbf{p})\}$ consists of consecutive integers.[68]

Demand types provide nice ways to characterise the “law of aggregate demand” and the consecutive integer property, and these characterisations in turn make it much easier to demonstrate alternative characterisations of strong substitutes than is possible using traditional techniques. It is both intuitively clear and straightforward that (details are in Appendix 3.A.2):

Lemma 3.5.19.

1. *A valuation u satisfies the ordinary (complete) law of aggregate demand iff it is of some (unimodular concave) demand type $\mathcal{D}$ such that, for all $\mathbf{v} \in \mathcal{D}$, either $\mathbf{v} \geq \mathbf{0}$, $-\mathbf{v} \geq \mathbf{0}$, or $\sum_{i=1}^n v_i = 0$.*
2. *A concave valuation u satisfies the consecutive integer property iff it is of some concave demand type $\mathcal{D} \subseteq \{-1, 0, 1\}^n$.*

It is then easy to use this Lemma to show the following, which slightly generalises Milgrom and Strulovici (2009, Theorem 13):[69]

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[65] See Danilov et al. (2003).

[66] See Milgrom and Strulovici (2009). Note that, by Corollary 3.5.17, it is equivalent whether we define strong substitutes as all units of all goods being $\mathcal{D}^n_{os}$-complete substitutes, or just as all units of all goods being ordinary substitutes.

[67] See Hatfield and Milgrom (2005) for the ‘complete’ version.

[68] See Milgrom and Strulovici (2009).

[69] Milgrom and Strulovici (2009, Theorem 13) does not mention step-wise gross substitutes and assumes the $\mathcal{D}^n_{os}$-complete property in 5 and 6.

Corollary 3.5.20. *For a valuation u, the following are equivalent:*

1. *u is of concave demand type $\mathcal{D}^n_{ss}$;*
2. *u is a strong substitute valuation;*
3. *u is a concave step-wise gross substitute valuation;*
4. *u is concave and satisfies the complete $\mathcal{D}^n_{ss}$-improvement property;*
5. *u is a concave ordinary substitute valuation and satisfies the ordinary law of aggregate demand;*
6. *u is a concave ordinary substitute valuation and satisfies the consecutive integer property;*

Proof. 1⇔4 is an application of Theorem 3.5.11. The complete $\mathcal{D}^n_{ss}$-improvement property is precisely the single-improvement property of Gul and Stacchetti (1999), so 4⇔2 follows from Milgrom and Strulovici (2009, Theorem 13).[70] That 1⇔3 is clear from Corollary 3.5.5.1⇔3 (and also given by Danilov et al. 2003, Proposition 7). That 1⇔5 follows since by Proposition 3.4.8 we know ordinary substitutes to be equivalent to demand type $\mathcal{D}^n_{os}$ and since by Lemma 3.5.19.1, for the ordinary law of aggregate demand to additionally hold, the only vectors in $\mathcal{D}^n_{os}$ we can allow are those in $\mathcal{D}^n_{ss}$. That 1⇔6 similarly follows from Lemma 3.5.19.2 since the vectors in $\mathcal{D}^n_{os}$ satisfying the consecutive integer property are precisely the vectors of $\mathcal{D}^n_{ss}$. ☐

We will in general refer to the valuations satisfying these equivalent conditions as 'strong substitutes' as this terminology appears to have become more widely used (and is briefer than 'concave step-wise gross substitutes'). We can now quickly and easily identify such valuations. It is immediate, for example, that the examples of Figures 3.1, 3.4a, 3.5, 3.13a, 3.13b, and 3.14a are all of type $\mathcal{D}^2_{ss}$, while our 3-dimensional example, Figure 3.6, has demand type $\mathcal{D}^3_{ss}$. However, Example 3.2.10 (Figure 3.3) has a facet with normal $(-1, 2)$ (the line segment between the prices (4,3) and (6,4)), in addition to facets with normals $(1, 0), (0, 1)$, and $(1, -1)$, and so is not of type $\mathcal{D}^2_{ss}$, but *is* of type $\mathcal{D}^2_{os}$, as is the example of Figure 3.14b.[71]

Examples 3.2.14 and 3.2.15 revisited again. Recall we saw in Section 3.4.3.1 that the models of Kelso and Crawford (1982) and Hatfield et al. (2013) could be understood as ordinary substitute demand types. Now note that they are both also of demand type $\mathcal{D}^n_{ss}$. In the case of Kelso and Crawford (1982), all workers and all firms have valuations with domain contained in $\{0, 1\}^{m_1+m_2}$. In the case of Hatfield et al. (2013), since each agent is restricted to *either* selling

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[70] Strictly speaking, the proof of Milgrom and Strulovici (2009, Theorem 13) is incomplete, as the connection between the single-improvement property and the strong substitute property relies on their Theorem 2, whose proof is incomplete. However, Danilov et al. (2003, Corollary 5) provide the missing piece.

In the case that $n = 1$, the result that 1⇔2 follows from Kelso and Crawford (1982, Theorem 6).

[71] It can be shown that any strong substitutes preferences can be represented by a simple extension of the Bank of England's implementation of the Product-Mix Auction to allow negative bids. See Klemperer (2010, note 22) for the two-good case; Baldwin and Klemperer (in preparation) demonstrates this for the general case.

or buying any individual good, a re-ordering of goods shows their valuation has domain a subset of $\{0,1\}^{n_1} \times \{-1,0\}^{n_2}$ for some non-negative $n_1 + n_2 = n$.[72] Thus, following the same arguments as Corollary 3.5.17, every SNP edge, and so every facet normal, must be in $\mathcal{D}^n_{ss}$: the structure of preferences is again strong substitutes.

3.5.4 Unimodular Demand Types for Complementary Goods

Preferences for complementary goods are less thoroughly treated in the literature than preferences for substitutes, but we can use our techniques, and the analogy between the two cases, to develop some results for complements:

Proposition 3.5.21. *If a concave valuation is $\mathcal{D}^n_{oc}$-complete then, for any prices $\mathbf{p}' \geq \mathbf{p}$ and any $\mathbf{x} \in D_u(\mathbf{p})$ there exists $\mathbf{x}' \in D_u(\mathbf{p}')$ such that $x'_k \leq x_k$ for all k such that $p_k = p'_k$.*

Proof. One may make use of $\mathcal{D}^n_{oc}$-steps exactly as in Proposition 3.4.11, but now starting at any price $\mathbf{p}$. □

Following Theorem 3.5.11 it is now easy to provide a sufficient condition for $\mathcal{D}^n_{oc}$-completeness:

Proposition 3.5.22. *If $\mathcal{D} \subsetneq \mathcal{D}^n_{oc}$ is unimodular then every concave valuation of type $\mathcal{D}$ is a $\mathcal{D}^n_{oc}$-complete valuation.*

One example of such a demand type is $\mathcal{D} = \{\pm(1,1), \pm(1,2)\}$. But, as with substitutes, our main interest is in unimodular demand types which are also subsets of $\{-1,0,1\}^n$. Indeed such demand types are of particular interest for complements, since they correspond to coalition-formation problems with transferable utility, though note that in that context, every valuation is automatically complete, since the domain of every individual is only two points.[73] By definition of $\mathcal{D}^n_{oc}$ the vectors in such demand types are either in $\{0,1\}^n$ or $\{-1,0\}^n$; we write $\mathcal{D} \subseteq \pm\{0,1\}^n$ for brevity. By Lemma 3.5.19, the consecutive integer property is also satisfied by valuations of such demand types; by Corollary 3.5.5.3 they are also "step-wise gross complements" valuations, where we define, in analogy with Danilov et al. (2003):

Definition 3.5.23. A valuation is a *step-wise gross complements valuation* if for any $\mathbf{p} \in \mathbb{R}^n$, any $\mathbf{x} \in D_u(\mathbf{p})$ and any $i \in \{0, \ldots, n\}$, either $\mathbf{x} \in D_u(\mathbf{p} + \epsilon \mathbf{e}^i)$ for all $\epsilon \geq 0$ or there exists $\epsilon \geq 0$ and $\mathbf{x}' \in D_u(\mathbf{p} + \epsilon \mathbf{e}^i)$ such that $x'_i = x_i - 1$ and $x'_{-i} \leq x_{-i}$.

For $n \geq 3$, however, there is *no* unique maximal unimodular demand type contained in $\pm\{0,1\}^n$.[74] We explain this by developing an example.

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[72] Strictly speaking, a buyer might buy a good from one agent and sell a physically identical good to another agent in Hatfield et al. (2013) but because the transactions are independently priced we consider these goods to be distinct.

[73] See Example 3.2.16.

[74] If $n = 1$ then any valuation is a $\mathcal{D}^n_{oc}$-complete valuation; if $n = 2$ then the unique maximal unimodular complements demand type is $\pm\{0,1\}^2$ itself. For $n \geq 3$ the set $\pm\{0,1\}^n$ is not unimodular; see Example 3.6.13.

Example 3.5.24 ('Interval package' valuations, see Danilov et al. 2008, 2013)**.** Let G be the upper triangular matrix of 1s:

$$G := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then (following the notation of Section 3.4.4) the demand type $G\mathcal{D}_{ss}^n$ is unimodular. The vectors in $G\mathcal{D}_{ss}^n$ are of the forms $G\mathbf{e}^i = \sum_{k=1}^{i} \mathbf{e}^k$ and $G(\mathbf{e}^i - \mathbf{e}^j) = \sum_{k=j+1}^{i} \mathbf{e}^k$ for $i > j$ (as well as the negations of these). These are all therefore in $\pm\{0,1\}^n$, so $G\mathcal{D}_{ss}^n$ is a unimodular complements demand type.

Moreover, $G\mathcal{D}_{ss}^n$ has attractive economic properties. If the goods have a natural fixed order, then under $G\mathcal{D}_{ss}^n$, any contiguous collection of goods may be considered as complements by any agent. Such valuations may arise when agents consider, for example, bands of radio spectrum, or 'lots' of sea bed which might be developed for offshore wind (see Ausubel and Cramton, 2011).

Now, since the demand type $\mathcal{D}_{ss}^n$ is a maximal unimodular demand type (see Theorem 3.5.16) it follows that $G\mathcal{D}_{ss}^n$ is also maximal as a unimodular demand type.

However, it is easy to see that if $n \geq 3$ there exist vectors $\mathbf{v} \in \{0,1\}^n$, $\mathbf{v} \notin G\mathcal{D}_{ss}^n$ which do lie in *some* unimodular complements demand type. If we simply change the order of the goods, this corresponds to a unimodular basis change P, and $PG\mathcal{D}_{ss}^n$ has the same properties as $G\mathcal{D}_{ss}^n$. But if P is the permutation swapping coordinates 2 and n and leaving the rest unaltered, then $\mathbf{e}^1 + \mathbf{e}^n \in PG\mathcal{D}_{ss}^n$. However, $\mathbf{e}^1 + \mathbf{e}^n \notin G\mathcal{D}_{ss}^n$.

Moreover, there exist unimodular complements demand types that are not themselves unimodular basis changes of $\mathcal{D}_{ss}^n$; as sets of vectors, these were characterised by Seymour (1980). One such is as follows:

Example 3.5.25. Consider the demand type defined by the matrix

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Here, the first three goods are of value on their own, but the fourth is not; there are pairwise complementarities between any one of the first three goods together with the fourth, and there are additional complementarities between any pair of the first three goods when the fourth is also present. We interpret this as a matching model (cf. Example 3.2.16) in Section 3.6.3.1.

Claim 3.5.26. *The demand type of Example 3.5.25 is unimodular, and is not a basis change from 4-dimensional strong substitutes* $\mathcal{D}_{ss}^4$.

Proof. One may easily show this demand type is unimodular.[75]

To show it is not a basis change from 4-D strong substitutes, $\mathcal{D}^4_{ss}$, assume (for contradiction) there exists a unimodular matrix G such that $G^{-1}D$ consists entirely of distinct column vectors from $\mathcal{D}^4_{ss}$.[76] Since D has 9 columns, $G^{-1}D$ must include all but one of the 10 distinct vectors in $\mathcal{D}^4_{ss}$. Let $\mathbf{w} := (1,1,1,1,1,1,-1,-1,-1)$ and note that $D\mathbf{w}' = \mathbf{0}$, so $G^{-1}D\mathbf{w}' = \mathbf{0}$ also. It follows that every row $\mathbf{r}$ of $G^{-1}D$ satisfies $\mathbf{r}.\mathbf{w} = 0$. But there are precisely four vectors in $\mathcal{D}^4_{ss}$ with non-zero entry in any coordinate i ($\mathbf{e}^i$, and $\mathbf{e}^i - \mathbf{e}^j$ for the three values of $j \neq i$), so there are four non-zero entries in every row of the matrix whose columns are the 10 distinct vectors of $\mathcal{D}^4_{ss}$, and if we delete any one column, then at least one row must have exactly three non-zero entries. Since these three entries are ± 1, there is no way to add or subtract the three together to obtain zero; it is impossible that this row has zero dot product with $\mathbf{w}$. Thus no nine vectors of $\mathcal{D}^4_{ss}$ can form the columns of $G^{-1}D$, for any unimodular matrix G. □

Moreover, it follows from the mathematical results of Grishukhin et al. (2010) that *all* unimodular demand types are a unimodular basis change from a demand type $\mathcal{D} \subseteq \pm\{0,1\}^n$; unimodularity is preserved under unimodular basis change, and so we obtain the surprising result:

Theorem 3.5.27. *Every unimodular demand type is a unimodular basis change of a unimodular complements demand type contained in* $\pm\{0,1\}^n$.

In other work we plan to use their results to generate more examples of these demand types, such as our Example 3.5.25.

3.5.5 Generalised Gross Substitutes and Complements (cf. Sun and Yang, 2006)

We can extend Sun and Yang's (2006, see also 2009) definition of "gross substitutes and complements" to permit multiple units of goods.[77] First recall:

Definition 3.5.28 (Sun and Yang, 2006, Definition 2.1)**.** A valuation $u : \{0,1\}^{n_1+n_2} \to \mathbb{R}$ is a *gross substitutes and complements valuation (in the sense of Sun and Yang)* if, for any price $\mathbf{p}$ and any $\mathbf{p}' = \mathbf{p} + \delta\mathbf{e}^i$ where $\delta > 0$, and any $\mathbf{x} \in D_u(\mathbf{p})$: if $i \leq n_1$ then there exists $\mathbf{x}' \in D_u(\mathbf{p}')$ such that $x'_k \geq x_k$ for all $k \leq n_1$ such that $k \neq i$, and $x'_k \leq x_k$ for all $k > n_1$; and if $i > n_1$ then there exists $\mathbf{x}'' \in D_u(\mathbf{p}'')$ such that $x''_k \leq x_k$ for all $k \leq n_1$, and $x''_k \geq x_k$ for all $k > n_1$ such that $k \neq i$.

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[75] For example, it can be confirmed using Matlab that the determinant of every set of four columns of D is ± 1 or 0.

[76] Vectors which are the negation of one another are *not* considered "distinct" in this context.

[77] Shioura and Yang (2013) have independently made the same generalisation; their Theorem 2 shows, as we do below, that (in our language) "generalised gross substitutes and complements" (GGSC) satisfy the complete $D^{n_1;n_2}_{GGSC}$-improvement property.

We will write I_{n_1,n_2} for the $(n_1+n_2)\times(n_1+n_2)$ matrix $I_{n_1,n_2} := \begin{pmatrix} -I_{n_1} & \mathbf{0} \\ \mathbf{0} & I_{n_2} \end{pmatrix}$ where I_{n_i} is the $n_i \times n_i$ identity matrix, $i = 1, 2$. Recall from Proposition 3.4.14 that, if $A \subsetneq \mathbb{Z}^{n_1+n_2}$ then, for any $u : A \to \mathbb{R}$, we define the valuation $I_{n_1,n_2}^{\ *}u : I_{n_1,n_2}^{-1}A \to \mathbb{R}$ via $I_{n_1,n_2}^{\ *}u(\mathbf{y}) = u(I_{n_1,n_2}\mathbf{y})$ for all $\mathbf{y} \in I_{n_1,n_2}^{-1}A$. Now we define:

Definition 3.5.29 (Cf. Shioura and Yang, 2013, Definition 2)**.** Let $A \subsetneq \mathbb{Z}^n$ be finite, and let $u : A \to \mathbb{R}$ be a valuation. Goods are *generalised gross substitutes and complements (GGSC)* if the goods may be reordered such that, for some $n_1 + n_2 = n$, the valuation $I_{n_1,n_2}^{\ *}u$ is a strong substitute valuation.

The corresponding demand type we define is as follows:

Definition 3.5.30. $\mathcal{D}_{GGSC}^{n_1,n_2}$ is the following set of vectors in $\mathbb{Z}^{n_1+n_2}$

$$\{\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^i - \mathbf{e}^{i'}, \mathbf{e}^i + \mathbf{e}^j, \mathbf{e}^j - \mathbf{e}^{j'} \mid i, i' \in \{1, \ldots, n_1\}, j, j' \in \{n_1+1, \ldots, n_1+n_2\}\}.$$

It follows straightforwardly that:

Proposition 3.5.31. *A valuation is a GGSC valuation iff the goods may be reordered such that, for some $n_1+n_2 = n$, it is of concave type $\mathcal{D}_{GGSC}^{n_1,n_2}$. If the domain of the valuation is $\{0,1\}^n$ then this holds iff it is a gross substitutes and complements valuation in the sense of Sun and Yang.*

Proof. It is not hard to see that $\mathcal{D}_{GGSC}^{n_1,n_2} = I_{n_1,n_2}^{-1}\mathcal{D}_{ss}^n$. So the first result follows by Definition 3.5.29 and by Proposition 3.4.14. The second result is provided by Sun and Yang (2006, Section 3), who show that Definitions 3.5.28 and 3.5.29 coincide when $A = \{0,1\}^n$. □

As we know $\mathcal{D}_{ss}^n$ to be unimodular, it follows that $\mathcal{D}_{GGSC}^{n_1,n_2}$ is another unimodular demand type, and therefore (Theorem 3.5.11) it is also another complete demand type.

3.6 Aggregate Demand and Equilibrium

We now consider aggregate demand across many agents. In particular, we precisely identify the demand types for which competitive equilibrium always exists.

3.6.1 The structure of aggregate demand

First, we show that aggregate demand among agents may be understood in the way developed by Koopmans (1951): the aggregate value represents the most efficient use of the bundles available, in terms of generated welfare.

We have a finite set J of agents. Each agent j has a valuation u^j of integer bundles in a finite set A_j, so the bundles of interest on aggregate are $A := \{\sum_{j\in J}\mathbf{x}^j \mid \mathbf{x}^j \in A_j\}$, which we shall refer to as the *domain* of the aggregate valuation.[78] The aggregate demand at any price $\mathbf{p}$ is simply

$$D_{\{u^j\}}(\mathbf{p}) := \left\{ \sum_{j\in J} \mathbf{x}^j \mid \mathbf{x}^j \in D_{u^j}(\mathbf{p}) \right\}. \tag{3.4}$$

One way to find aggregate demand is to start with the valuation functions $u^j(\cdot)$, combine them to give an 'aggregate valuation function', and then proceed in exactly the same way as for individual demand. It is standard (see Appendix 3.A.3) that if agents' preferences are quasilinear then one attains an aggregate valuation function $U : A \to \mathbb{R}$ as the greatest sum of valuations that can be attained by dividing any bundle $\mathbf{y} \in A$ between the agents, that is, the most efficient division of this bundle:

$$U(\mathbf{y}) := \max \left\{ \sum_{j\in J} u^j(\mathbf{x}^j) \mid \mathbf{x}^j \in A_j, \sum_{j\in J} \mathbf{x}^j = \mathbf{y} \right\}.$$

Now:

Proposition 3.6.1. *$D_{\{u^j\}}(\mathbf{p}) = D_U(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.*

So we henceforth refer to $D_{\{u^j\}}(\mathbf{p})$ using the simpler notation $D_U(\mathbf{p})$.

However, the problem with this approach is that $U(\cdot)$ is very hard to work with–to find any value of $U(\mathbf{y})$, we need to consider all possible partitions of y among the agents, which is both time-consuming and unintuitive.

It is straightforward, on the other hand, to start with the individual THs, $\mathcal{T}_{u^j}$, combine them to form an aggregate TH, $\mathcal{T}_{\{u^j\}}$, and find information about aggregate demand from that. Recall that the underlying set of $\mathcal{T}_{u^j}$ is those prices at which demand u^j is non-unique. So, since aggregate demand $D_U(\mathbf{p})$ is unique iff all individual demands $D_{u^i}(\mathbf{p})$ are, the underlying set of $\mathcal{T}_{\{u^j\}}$ is just the union of all the $\mathcal{T}_{u^j}$. Figure 3.18 illustrates this for the aggregate of the two agents' demands in our simple substitutes and complements example, Example 3.2.11.

$\mathcal{T}_{\{u^j\}}$ inherits the structure of a proper rational polyhedral complex from the individual THs, although the cells will not in general be exactly the same: if cell interiors from two different agents intersect, the cells are split up into new, smaller cells in $\mathcal{T}_{\{u^j\}}$ with a new, lower-dimensional, cell at their intersection. For example, in Figure 3.18c, the point $(\frac{1}{2}, \frac{1}{2})$ is a 0-cell, on the boundary of four *distinct* 1-cells.

It is easy to see that $\mathcal{T}_{\{u^j\}}$ also inherits a balanced weighting from the weightings of the individual THs. For any facet F of $\mathcal{T}_{\{u^j\}}$, let its weighting $w_{\{u^j\}}(F)$ be $\sum_{j\in J} w_j(F)$, in which $w_j(F)$ is the weight of the facet $F_j \supseteq F$ of $\mathcal{T}_{u^j}$, or $w_j(F) = 0$ if no facet $F_j \supseteq F$ of $\mathcal{T}_{u^j}$ exists. Since each individual TH is balanced, adding weightings in this way creates a balanced weighting.[79]

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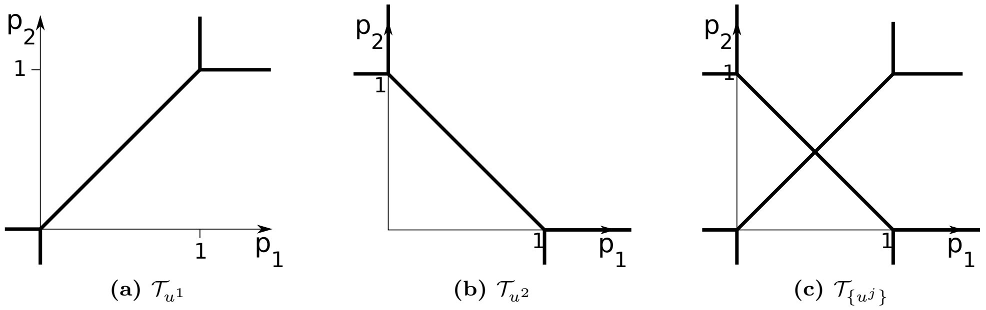
[78] We could alternatively consider each agent as having a valuation over the full domain of the aggregate valuation A by letting $u^j(\mathbf{x}) := \max\{u(\mathbf{y}) \mid \mathbf{y} \in A_j, y_i \leq x_i, i = 1, \ldots, n\}$ for any $\mathbf{x} \in A$ for which this set is non-empty, and $u^j(\mathbf{x}) = -\infty$ otherwise.

[79] In more detail: let G be a $(n-2)$-cell in $\mathcal{T}_{\{u^j\}}$, let $F_1, \ldots, F_l$ be the facets adjacent to G, and let $\mathbf{v}_{F_k}$ be



(a) $\mathcal{T}_{u^1}$ **(b)** $\mathcal{T}_{u^2}$ **(c)** $\mathcal{T}_{\{u^j\}}$

Figure 3.18: *(a) and (b) the THs of the individual demands of Example 3.2.11; (c) the TH of the aggregate of the two demands of Example 3.2.11.*

And the change in aggregate demand as we cross a facet is just the sum of changes in individual demand.

So, since the underlying sets of $\mathcal{T}_{\{u^j\}}$ and $\mathcal{T}_U$ are the same, and so are their weightings, it follows (see Appendix 3.A.3) that *as THs*,

Proposition 3.6.2. $\mathcal{T}_{\{u^j\}} = \mathcal{T}_U$.

So we will henceforth also refer to the aggregate TH, $\mathcal{T}_{\{u^j\}}$, using the simpler notation $\mathcal{T}_U$.

Thus simply "adding" the individual THs yields the aggregate TH.[80] If we know what is demanded in one UDR then, as before, we immediately know what is demanded in all the UDRs, without needing to directly consider the function U. And it is immediate that demand 'type' is preserved under aggregation:

Corollary 3.6.3. *Valuations u^j are of demand type $\mathcal{D}$ for all $j \in J$ iff the aggregate demand $\mathcal{T}_U$ is of demand type $\mathcal{D}$.*

Proof. This is immediate from Proposition 3.6.2 and the definition of $\mathcal{T}_{\{u^j\}}$. □

3.6.2 Competitive Equilibrium and Stable Matchings

It is *not* the case that concavity of each individual demand implies concavity of the aggregate demand. (We will exhibit a simple example of this failure in Example 3.6.12.) And we have seen

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primitive integer vectors for each, chosen according to a coherent orientation. Then for every agent j, the equation $\sum_{k=1}^{l} w_j(F_k)\mathbf{v}_{F_k} = 0$ holds: if G is contained in an $(n-2)$-cell of $\mathcal{T}_{u^j}$ then, this follows from $\mathcal{T}_{u^j}$ being balanced; if G is contained only in a single facet of $\mathcal{T}_{u^j}$ then the only non-zero terms in this sum are those which first add and then subtract the weight of this facet to j; if $G \cap \mathcal{T}_{u^j} = \emptyset$ then the expression is identically zero. We conclude $\sum_{k=1}^{l} w_{\{u^j\}}(F_k)\mathbf{v}_{F_k} = \sum_{j\in J}\sum_{k=1}^{l} w_j(F_k)\mathbf{v}_{F_k} = 0$. Alternatively, one can see this by appealing to Appendix 3.A.3, which confirms that the weightings are the same as those on $\mathcal{T}_U$ – being, of course, automatically balanced since it is the TH corresponding to $U(\cdot)$.

[80] This is, of course, essentially the same point as the fact that in the Product-Mix Auction we can simply "add" individual bidders' sets of bids to form a single aggregate set of bids that represents bidders' aggregate demand. See Baldwin and Klemperer (in preparation) for further discussion.

(Lemma 3.2.5) that if the function U is not concave, then there exists a bundle in A that is never demanded.

Of course, if there is a bundle which is not the aggregate demand of the agents for any price, then a competitive equilibrium does not exist when this is the bundle of goods available in the economy.

We cannot generally infer from only the aggregate TH whether there is a bundle that is never the aggregate demand–recall that the geometric construction does not tell us the precise demand set, $D_U(\mathbf{p})$, at all prices $\mathbf{p} \in \mathcal{T}_{\{u^j\}}$, so it is ambiguous from the geometry whether any integer vectors in $\operatorname{Conv} D_U(\mathbf{p})$ that are not vertices of $\operatorname{Conv} D_U(\mathbf{p})$ are in $D_U(\mathbf{p})$ (see Corollary 3.3.4). However, as we now show, we can start to answer these questions if we know not only the aggregate TH but also each individual TH.

The next subsection therefore provides conditions which guarantee that a competitive equilibrium *always* exists, by providing conditions which guarantee that the aggregate valuation $U(\cdot)$ is concave (without needing to explicitly calculate $U(\cdot)$). In particular, we are interested in the existence of equilibrium for agents with specified demand types, as defined in Section 3.4:

Definition 3.6.4. A (concave) demand type $\mathcal{D}$ *always has a competitive equilibrium* if, for every set of agents with (concave) demands of type $\mathcal{D}$, and for an economy endowed with any bundle in the domain of the aggregate valuation, a competitive equilibrium exists.

Note that there always exist *some* collections of agents with demands of type $\mathcal{D}$ which *do* have a competitive equilibrium for any supply in their domain, whether or not the type $\mathcal{D}$ 'always has a competitive equilibrium'.[81] Note also that since the demand of a single agent with non-concave valuation function fails to always have a competitive equilibrium, we are only interested in concave demand types here.

A benefit of our method of categorising demand types is that it is straightforward that:

Proposition 3.6.5. *Always having a competitive equilibrium is a property that is preserved under unimodular basis changes.*

Proof. See Appendix 3.A.3 ☐

Recall that in Example 3.2.16 we presented a model of matching with transferable utility in our framework, using 'coalition-agents' and 'person-goods'. The price paid for each person-good was precisely the net utility that person received if a matching took place. The typical question in such a matching model is whether a *stable* set of coalitions exists, i.e., a set such that no subset of people would prefer to deviate and form a new feasible coalition.

In this setting, the set of stable matchings corresponds to the set of core allocations, because if any group of people could make themselves better off by defecting and forming one or more new coalitions (with utility being transferable between people within, but not across, new coalitions)

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[81] Trivially, a set of identical agents with concave valuations of the type $\mathcal{D}$ *does* always have a competitive equilibrium.

then a subset could defect and make a single coalition better off.[82] (It is also straightforward that this set also corresponds precisely to the set of core allocations of such a game when utility is also transferable across coalitions.[83]):

Definition 3.6.6 (cf. Gale and Shapley, 1962, and Shapley and Shubik, 1971)**.** In a model of matching with transferable utility within matches, a *stable matching* is an allocation in the *core* of the game among the people, that is, an assignment of each person to exactly one coalition and a set of transfers between the people within each coalition, such that there exists no feasible coalition which is not formed but whose formation would give strictly greater utility to all those people it would comprise.

So we have:

Theorem 3.6.7. [84] *A stable matching exists in a model of coalition formation with transferable utility iff there exists a competitive equilibrium in the re-formulation of Example 3.2.16.*

Proof. If a stable matching exists, the net utility received by each person is the price of that person (good). Each coalition which has formed must correspond to a coalition-agent with non-negative net utility: these do demand their corresponding people (goods) at these prices. Each coalition which has not formed cannot offer additional surplus to the people who would form it: the coalition-agent cannot afford the person-goods at the prevailing prices. So this is a competitive equilibrium.

Suppose a competitive equilibrium exists. A subset of people would only wish to deviate from their prescribed coalitions and form a new one together, if they could achieve a strictly higher net utility in the new coalition. The corresponding coalition-agent would have to be willing to offer a higher price for each of them. But the existence of such a coalition-agent would contradict competitive equilibrium. □

Recall that a person's 'price' is the minimum a coalition needs to pay to 'buy' him, in competitive equilibrium; any excess surplus can be split in any way among the people in a coalition.

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[82] Since, in our setting, people are only ever part of a single group, questions about whether defectors remain parts of their previous groups do not arise. (Such questions are addressed in the literature on "group stability" – see Roth et al., 1992, Konishi and Ünver, 2006, etc. – and "setwise stability" – Sotomayor, 1999, Echenique and Oviedo, 2006, Klaus and Walzl, 2009, etc. – which impose additional requirement about which contracts are maintained after any deviation. See also Teytelboym, 2013.)

[83] If any group of people could make themselves better off by defecting and forming one or more new coalitions, perhaps *after* transfers between new coalitions, then at least one of the new coalitions is not receiving a positive transfer from the others, and that new coalition could have made itself better off by defecting on its own. So a stable matching must be in the core of the fully transferable utility game. (The converse is obvious.)

[84] Related literature includes Shapley and Shubik, 1971, Kaneko and Wooders, 1982, Kelso and Crawford, 1982, Chung, 2000, Eriksson and Karlander, 2001, Klaus and Nichifor, 2010, Talman and Yang, 2011, Chiappori et al., 2012, Hatfield et al., 2013, Hatfield and Kominers, 2013, etc.

3.6.3 When does Competitive Equilibrium exist?

We now state and explain a theorem which provides a necessary and sufficient condition for any set of valuations in a demand type to always have a competitive equilibrium.

Our theorem requires much weaker assumptions about agents' preferences than used in the existing leading economics literature, so our condition for equilibrium is correspondingly much more general. In particular (see Section 3.6.4.3) it is *not* necessary for all agents to have strong substitute demands (or some basis change thereof) for equilibrium to always exist.[85] Instead, concavity and the unimodularity condition of Section 3.5.2 are all that are required.

A remarkable series of papers by Danilov, Koshevoy and their coauthors, has developed results that are very closely related to ours. In particular, Theorems 1, 3 and 4[86] of Danilov et al. (2001) together provide a *sufficient* condition for equilibrium ("$\mathcal{D}$-concavity" of valuations, where $\mathcal{D}$ is unimodular), which is analogous to our condition on demand types. However, the interpretation or usefulness of their result is not made clear; by contrast, our theorem both demonstrates the applicability of the result, and clarifies the connections to existing economic results.[87]

Danilov et al. also prove no *necessity* result. Because they have not developed their definition as a taxonomy of demand, in the way we do with demand types, they do not show the necessity of unimodularity of $\mathcal{D}$ for the existence of competitive equilibrium. Once our concept is introduced, however, a necessity result can easily be developed.[88]

Danilov et al. moreover state their results under different assumptions from ours. They assume the domain, A, of every agent's valuation is $\mathbb{Z}^n_{\geq 0}$, which precludes, for example, the application to Hatfield et al.'s (2013) model which our more general assumption permits.[89]

Finally, although the techniques we use to prove our results are novel, they seem simpler and more accessible to economists than Danilov et al.'s *very* advanced mathematical techniques.

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[85] For example, results such as those of Kelso and Crawford (1982), Hatfield and Kojima (2008), and Hatfield et al. (2013) are necessary 'in the maximal domain sense', in Hatfield et al. (2013)'s words. That is, in our language, they show that equilibrium always exists for some demand type $\mathcal{D}$, but that if one agent has preferences outside of $\mathcal{D}$ then this may fail.

[86] The proof of Theorem 4 is given in Danilov and Koshevoy (2004, Theorem 2).

[87] We will see that our Theorem 3.6.8 provides generalisations of the results on competitive equilibrium in, for example, Kelso and Crawford (1982), Hatfield and Milgrom (2005), Sun and Yang (2006), Milgrom and Strulovici (2009), and Hatfield et al. (2013). The absence in Danilov et. al's work of the notion of demand types, and its presentation in relatively unfamiliar terms (namely the relationships between sets of primitive integer vectors which are parallel to edges of specific collections of integral pointed polyhedra and the "classes of discrete convexity" that they define) seems to have resulted in leading economists, and the leading related existing literature, being unaware of their work or its implications. (We were also unaware of their work until after we had developed our own results.)

[88] The sufficiency part of our theorem follows from combining Theorems 1, 3 and 4 of Danilov et al. (2001). To understand the relationship between these theorems and our Theorem 3.6.8, observe that in their Theorem 4 certain sets of "primitive [integer] vectors, which are parallel to edges of" a certain "collection of integral pointed polyhedra" are analogous to our demand types; furthermore, the "classes of discrete convexity" they define are analogous to a set of demand sets $D_u(\mathbf{p})$ such that $\operatorname{Conv} D_u(\mathbf{p}) \cap \mathbb{Z}^n = D_u(\mathbf{p})$ in every case and such that this property is preserved under aggregation. It is not hard to also show, using our Proposition 3.2.6, the necessity of a demand type being a "class of discrete convexity" for competitive equilibrium to always exist, and in this way we can also derive our *necessity* result from their work.

[89] In fact Danilov et al.'s assumption seems unnecessary for them, so we could develop our full theorem by extending their work. See our note 88, above. See also our discussion about the distinction between their approach and ours in the introduction to our Section 3.4. Their work also covers some of the examples in Section 3.6.4, as we note in that Section.

So we will prove our theorem using our alternative method, which understands the result as an application of "intersection multiplicities" in tropical geometry.[90]

Theorem 3.6.8. *A concave demand type $\mathcal{D}$ always has a competitive equilibrium iff it is unimodular.*

(Recall from Definition 3.6.4 that competitive equilibrium 'always exists' for a (concave) demand type iff, for any set of agents with (concave) valuation of that type, and any supply bundle in the convex hull of aggregate demand, there exists a price such that the market clears.)

As in Section 3.5.2, the intuition is that the volume of a SNP face which is a parallelepiped in $\mathbb{R}^n$, with vertices in $\mathbb{Z}^n$ and edges in $\mathcal{D}$, cannot exceed 1 if the set of vectors that form its edges is unimodular (see Remark 3.5.10), so if $\mathcal{D}$ is unimodular, such an SNP face of aggregate demand contains no integer points other than its vertices. So there is also no integer bundle "hidden" inside such an intersection of individual agents' THs, so no integer bundle that is never chosen at any price vector, and competitive equilibrium therefore always exists. On the other hand, if the set of edges exhibits failure of unimodularity, then such a parallelepiped's volume does exceed 1, there does exist a bundle not at a vertex, and–we will see–such bundles may not be chosen at any price vector, so competitive equilibrium may fail.

When $s = n$, the volume in question is simply the (absolute value of the) determinant of the vectors along its edges. So, as with Corollary 3.5.12, if the set of aggregate demands is in the same dimension as the number of goods, we can re-state the theorem in a form that is easier to check:

Corollary 3.6.9. *With n goods, a concave demand type $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$, in which $\mathbf{v}^1, \ldots, \mathbf{v}^r$ span $\mathbb{R}^n$, always has a competitive equilibrium iff every subset of n vectors from $\mathcal{D}$ has determinant 0 or ± 1.*

In the more general case of Theorem 3.6.8 we allow demand types $\mathcal{D}$ that ignore some directions of good availability. In such a $\mathcal{D}$ there are no collections of n linearly independent vectors, so every subset of n vectors has determinant 0, and the check of Corollary 3.6.9 tells us nothing.

In this case, however, we can use one of the equivalent conditions in Remark 3.5.10.3 and 3.5.10.4.

A demand type, $\mathcal{D}$, always has a competitive equilibrium iff all integer bundles (i.e., all lattice points) in any type-$\mathcal{D}$ SNP of aggregate demand are demanded for some price. It is immediate that any integer bundle that is at a vertex of the SNP is demanded (recall Corollary 3.3.4). On the other hand, any integer lattice point in the SNP of aggregate demand that is not a vertex is "hidden" inside the corresponding intersection of the individual agents' THs. Such a bundle is in the convex hull of the aggregate demands of the agents, at the price at which their THs intersect. Since Lemma 3.2.6 tells us that this is the only possible price at which such a bundle can be demanded, the question is, therefore, whether such a bundle is always demanded at the

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[90] It was this theory that inspired our (independent) development of our results. Full details of our proof are in Appendix 3.A.3.

intersection price. So we can prove Theorem 3.6.8 by considering the forms such intersections may take.

We begin, in the next subsection, by considering a simple special case, which both proves the necessity of unimodularity for competitive equilibrium to always exist, and is of independent interest in understanding stable matching. The subsequent subsection extends this special case to prove the general case of the Theorem.

3.6.3.1 Competitive Equilibrium with "Simple" Intersections; and Stable Matchings

The simple case we start with is that for which an intersection between individual agents' THs lies in the interior of a facet of each agent, so each agent is indifferent between precisely two bundles, and the set of vectors normal to these facets is linearly independent. We show that, for such simple intersections, competitive equilibrium exists for every possible supply bundle if and only if this set of vectors is unimodular.[91] Using this result, examples of failure of equilibrium are always easy to construct.

Proposition 3.6.10. *Consider $s \leq n$ agents each of whose demand set includes precisely 2 bundles at price $\mathbf{p}$, i.e., $\#D_{u^i}(\mathbf{p}) = 2$, for $i = 1, \ldots, s$. Write $\mathbf{v}^i$ for the difference between the two bundles demanded by agent i (so $\mathbf{v}^i$ is normal to i's facet of demand at $\mathbf{p}$). Suppose the s vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s$ are linearly independent. Write U for the aggregate valuation. There exists an integer bundle in* $\operatorname{Conv} D_U(\mathbf{p})$ *which is not demanded at any price iff vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s$ do not form a unimodular set.*

Proof. By Lemma 3.2.6, an integer bundle in $\operatorname{Conv} D_U(\mathbf{p})$ is not demanded at any price iff it is not in $D_U(\mathbf{p})$. Now, each individual agent i's demand at $\mathbf{p}$ has the form $D_{u^i}(\mathbf{p}) = \left\{\mathbf{y}^i + \delta_i \mathbf{v}^i \mid \delta_i \in \{0,1\}\right\}$, where $\mathbf{y}^i$ is the bundle demanded on the appropriate side of the TH facet. So the set of bundles demanded on aggregate at $\mathbf{p}$ is

$$D_U(\mathbf{p}) = \left\{\mathbf{y} + \delta_1 \mathbf{v}^1 + \cdots + \delta_s \mathbf{v}^s \mid \delta_i \in \{0,1\}; i = 1, \ldots, s\right\},$$

where $\mathbf{y} = \sum_i \mathbf{y}^i$. These points are precisely the vertices of an s-dimensional parallelepiped in $\mathbb{Z}^n$ (since its edges, the $\mathbf{v}^i$, are linearly independent). There exists an integer bundle in $\operatorname{Conv} D_U(\mathbf{p})$ which is not in $D_U(\mathbf{p})$ iff this parallelepiped contains an integer bundle which is not a vertex, and, by Remark 3.5.10.1 and 2, this holds iff the set $\{\mathbf{v}^1, \ldots, \mathbf{v}^s\}$ is not unimodular. □

This result tells us more than just the necessity of unimodularity for existence of competitive equilibrium. It shows us how to construct simple examples of failure of equilibrium: for any non-unimodular set of vectors, we simply need to choose a price vector, and then choose valuations so agents are all indifferent between exactly two bundles at this price vector.

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[91] Unimodularity is equivalent to the *tropical intersection multiplicity* being equal to one in such a case (see e.g. Osserman and Payne, 2013).

In particular, *every* intersection of individual THs is of the form of Proposition 3.6.10 if every agent's domain of valuations contains only two bundles, as in the application to coalition-formation, Example 3.2.16, so Proposition 3.6.10 is of particular value in developing results in this context. For example, it is immediate that a stable matching always exists for *every* set of people (not necessarily just one person of each type) iff $\mathcal{D}$ is unimodular:

Corollary 3.6.11. *For every model of coalition formation with transferable utility of type $\mathcal{D}$, a stable matching exists for every set of people iff $\mathcal{D}$ is unimodular.*

Proof. Follows immediately from Theorem 3.6.7 and Proposition 3.6.10. □

Example 3.5.25 revisited. Recall from Section 3.5.4 that the columns of the matrix

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

define a unimodular demand type. Since, also, all its entries are 0 or +1, it is an example of a coalition-formation problem for which a stable match always exists. It might, for example, model the demand for three workers (the first three goods) and a manager (the fourth good). The first three columns of D show that each of the three workers has value on his own; the manager on her own is worthless (because $\mathbf{e}^4 \notin \mathcal{D}$), but the middle three columns of D show that the manager increases the value of any one of the workers, and the last three columns of D show that there are also complementarities between any two of the workers if (but only if) the manager is also present.

3.6.3.2 Proof of the Equilibrium Existence Theorem in the General Case

The *necessity* of unimodularity for competitive equilibrium is demonstrated immediately by considering intersections of individual agents' THs that take the simple form of Proposition 3.6.10.

We now prove in two stages that unimodularity and concavity are *sufficient* for competitive equilibrium to always exist.

First, we show in Proposition 3.A.4 that all the integer bundles in the convex hull of the demands at any "nice" intersection of agents' THs are always demanded. Our definition of a "nice" intersection (see Proposition 3.A.4) covers any generic intersection at a single price. For example, in two dimensions, two lines crossing at a single point is "nice", but two coincident lines is not "nice", and nor is three lines crossing at a single point; in three dimensions, either three planes meeting in a single point, or a line meeting a plane in a single point is "nice".

The important property of "nice" intersections is that the changes in bundles considered by the different agents (as each agent crosses between different regions of its TH) are always linearly independent. This means that any change in the aggregate supply, which remains in the convex hull of aggregate demand at at this price point, can be straightforwardly and uniquely apportioned between the individual agents, by simply assigning to each individual agent that part of the aggregate change that follows its direction of change. Unimodularity of $\mathcal{D}$ implies that, if the aggregate change is by an integer bundle, then so are each of these individual changes. The concavity of each individual agent's valuation then means that each agent demands its new assigned bundle. So each separate component of the total bundle is demanded by an individual agent, and the aggregate bundle is therefore also demanded.

The second half of the proof of the sufficiency part of the theorem proceeds by showing that generically all TH intersections are "nice". That is, there always exist arbitrarily small perturbations of all agents' valuations that lead to a situation in which all intersections *are* nice, and so all bundles *are* demanded on aggregate at some price. But if, prior to these perturbations, there exists an integer bundle which is *not* demanded, then the aggregate value from this bundle must be a finite amount lower than the valuation that would be required for it to be demanded. So we can therefore make arbitrarily small perturbations in valuations that are on the one hand small enough that the integer bundle in question can still not be demanded at any price, but on the other hand mean that all bundles *are* demanded on aggregate at some price–a contradiction. We give the details in Appendix 3.A.3.

3.6.4 Examples

3.6.4.1 Examples of non-existence of equilibrium

We first illustrate our result with two simple examples of non-existence of equilibrium; as in Section 3.5.2 we can see the importance of unimodularity by examining examples in which it fails. Proposition 3.6.10 shows us how to construct a failure of competitive equilibrium in such a case. We could mirror the demand type of Example 3.5.1 again, but instead we use the even simpler Example 3.2.11: substitutes and complements.

Example 3.6.12. [92] Suppose $\mathcal{D}$ can be represented by

$$D = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

in which the first three column vectors together yield the substitutes demand, and the last three column vectors together yield complements demand. Trivially, the matrix formed by the first and last column has determinant 2, so equilibrium need not exist.

Our Example 3.2.11 is of this type: we repeat its valuation functions for the "substitutes

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[92] See Danilov et al. (2001, Example 1) and Hatfield et al. (2013, Example 2).

agent" and "complements agent" respectively, below:

$x_1 = 1$	$x_1 = 0$	u^1
1	0	$x_2 = 0$
1	1	$x_2 = 1$

and

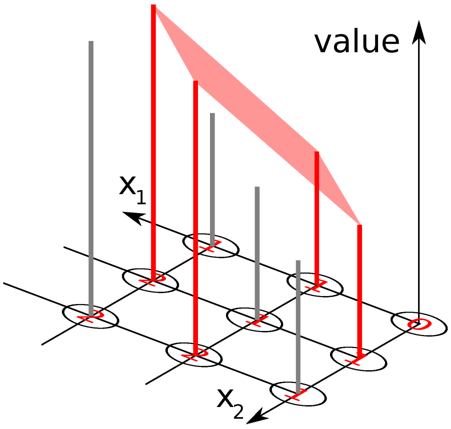
$x_1 = 1$	$x_1 = 0$	u^2
0	0	$x_2 = 0$
1	0	$x_2 = 1$

,

Note that *both* these valuation functions are concave. However, the aggregate valuation function, which we give in Figure 3.19a is *not* concave, as can be easily seen by observing that $(U(1,0) + U(0,1) + U(2,1) + U(1,2))/4 > U(1,1)$. This inequality is also apparent in Figure 3.19b which shows a 3-dimensional illustration of U together with the face of $\widehat{A}$ (see equation (3.3) and Section 3.3.2) that corresponds to the price vector $(\frac{1}{2}, \frac{1}{2})$. It follows that all the bundles $(1, 0)$, $(0, 1)$, $(2, 1)$, and $(1, 2)$ are demanded at this price, while the bundle $(1, 1)$ is "hidden" at the intersection of the diagonals of the TH at the price, $(\frac{1}{2}, \frac{1}{2})$, and is never demanded at any price. So aggregate demand is never $x_1 = x_2 = 1$. The SNP and the TH of the individual and aggregate demands are shown in Figure 3.20. Observe in Figure 3.20c that in the aggregate SNP the bundle $(1, 1)$ is not a vertex, and the area of the diamond is $\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$.

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	U
1	1	0	$x_2 = 0$
2	1	1	$x_2 = 1$
2	2	1	$x_2 = 2$

(a) *Aggregate valuation.*



(b) *3 dimensional illustration of the aggregate valuation, showing the face of $\widehat{A}$ that corresponds to the price vector $(\frac{1}{2}, \frac{1}{2})$.*

Figure 3.19: *The aggregate valuation of Example 3.6.12.*

Of course, our analysis only shows that equilibrium *may* not exist for this type of demand. Equilibrium would exist if, for example, the "complements" consumer had valuation 3 for the combination of 1 unit of each of x_1 and x_2. In that case the facets corresponding to the vectors $(1, 1)$ and $(1, -1)$ would not intersect, so Proposition 3.6.10 does not apply. We will return to this issue in Section 3.6.5.

Example 3.6.13. [93] Consider a set of "complements" consumers each of whom is only interested

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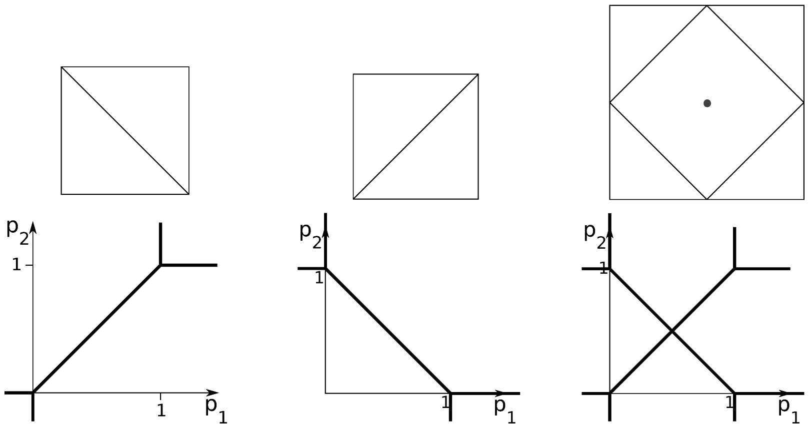
[93] Sun and Yang (2011), and also Teytelboym (2014), have independently considered the demand described in



(a) *The "substitutes" SNP and TH.* **(b)** *The "complements" SNP and TH.* **(c)** *The aggregate SNP and TH.*

Figure 3.20: *The individual and aggregate SNPs and THs for Example 3.6.12.*

in a different pair of goods. One context in which such a situation may arise is the "coalition formation" of Example 3.2.16–recall that in this case a stable matching is given by a competitive equilibrium (Theorem 3.6.7).

Moreover, assume that there is a cycle in the pairs of goods that these consumers wish for. That is, we can number both consumers and goods $1, \ldots, n$, such that every consumer $i < n$ demands goods i and $i+1$, which it sees as perfect complements, and consumer n demands goods n and 1. It is not hard to see that:[94]

$$\text{if } D = \begin{pmatrix} 1 & 0 & 0 & & & 0 & 1 \\ 1 & 1 & 0 & & & 0 & 0 \\ 0 & 1 & 1 & & & 0 & 0 \\ \cdot & 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & 1 & 0 \\ 0 & 0 & 0 & & & 1 & 1 \end{pmatrix} \text{ then } \det D = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

So if n is odd, there exist agents with demands of this type such that equilibrium does not exist.

Indeed, one can see directly that equilibrium fails in the simplest symmetric case: if each consumer has valuation 1 for any allocation that includes the pair it desires, and valuation 0 for any other allocation, aggregate demand is never exactly 1 unit of each good. To see this, note

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this example, using alternative methods that extend Sun and Yang (2006), showing as we do that equilibrium always exists iff n is even. See also Footnote 95. If we use the 'matching' interpretation of Example 3.2.16 then this example is the transferable utility version of the 'no odd rings' condition of Chung (2000).

[94] To see this easily, expand by the first row: noting the "1"s in the first and the last column of that row, we have $\det D = 1(1) + (-1)^{n-1}(1)$.

that at least one good, w.l.o.g. good 1, would not be part of a pair. So $p_1 = 0$. Therefore $p_2 \geq 1$ (else consumer 1 would demand the pair of goods 1 and 2). So $p_2 = 1$, and therefore $p_3 = 0$, since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore $p_4 \geq 1$ (else consumer 3 would demand goods 3 and 4). So $p_4 = 1$, and $p_5 = 0$, etc. In particular, $p_j = 0$ if j is odd. But in that case, consumer n wishes to buy goods n and 1, which is a contradiction.

On the other hand, if n is *even*, the columns of D are not linearly independent, but if we exclude the ith column, for any i, the remaining $n-1$ rows are linearly independent and can trivially be extended to n linearly independent vectors with determinant 1 by adding the column $\mathbf{e}^i$, so Theorem 3.6.8 then shows that equilibrium *always* exists. For example, in the simple symmetric case, $p_j = 0$ if j is odd and $p_j = 1$ if j is even, for all j, supports $q_i = 1$ for all i as an aggregate demand.

3.6.4.2 Strong substitutes and Generalised gross substitutes and complements

Recall from Section 3.5.3 that a valuation is 'strong substitutes', in the terminology introduced by Milgrom and Strulovici (2009), if every unit of every good is an ordinary substitute for every other unit of every good (including being an ordinary substitute for every other unit of the same good). We showed in Proposition 3.5.20.2 that strong substitutes are precisely are concave demand type $\mathcal{D}^n_{ss}$; the latter may be presented as $\{\mathbf{e}^i, \mathbf{e}^i - \mathbf{e}^j \mid i, j = 1, \ldots, n; i < j\}$ (see Section 3.5.3).

We know $\mathcal{D}^n_{ss}$ is unimodular (see Theorem 3.5.16), so another of the pleasing properties of 'strong substitutes' is that equilibrium always exists:

Proposition 3.6.14 (Milgrom and Strulovici, 2009, Theorem 19; cf. Danilov et al., 2003, Proposition 7 and Danilov et al., 2001, Example 4)**.** *Equilibrium always exists when agents' demands are strong substitutes.*

Recall that we showed that $\mathcal{D}^n_{ss}$ is not just unimodular, but is also maximal as a unimodular demand type (Theorem 3.5.16). This implies:

Proposition 3.6.15 (Gul and Stacchetti, 1999, Theorem 2, Milgrom and Strulovici, 2009, Theorem 16; Hatfield et al., 2013, Theorem 7)**.** *Given any one agent who does not have a strong substitute valuation, we can find strong substitute valuations for other agents such that competitive equilibrium fails to exist.*

Indeed, applying Proposition 3.6.10, we see that very simple valuations for the additional agents will suffice: each additional agent need only consider either whether or not to demand one unit of one good, or whether or not to swap one unit of one good for one unit of another.

Moreover, it is now trivial to reproduce:

Corollary 3.6.16 (Milgrom and Strulovici, 2009, Theorem 20; Danilov et al., 2003, Proposition 5)**.** *If u^j is a strong substitute valuation for all $j \in J$, then the aggregate valuation U is a strong substitute valuation.*

Proof. If u^j is of concave demand type $\mathcal{D}^n_{ss}$ for $j \in J$, then U is of type $\mathcal{D}^n_{ss}$ by Corollary 3.6.3. By Proposition 3.6.14 (and Lemma 3.2.5) U is also concave; applying Corollary 3.5.20 completes the proof. ☐

Because equilibrium existence is preserved under unimodular basis changes (the clarity of this is one of the benefits of our representation of demand), an elementary application of Proposition 3.6.14 is:

Corollary 3.6.17 (cf. Sun and Yang, 2006, Theorem 3.1 and Shioura and Yang, 2013, Theorem 5.)**.** *Equilibrium always exists for the 'generalised gross substitutes and complements' type of demand.*

Proof: Immediate from Propositions 3.6.14, 3.5.31 and 3.6.5. ☐

Note this corollary also provides another proof of Example 3.6.13's "even cycle of complements" result. If we separate the goods into two classes corresponding to the odd- and even-numbered goods, and re-order so that all the odd ones come first, demand is then of type $\mathcal{D}^{n/2,n/2}_{GGSC}$, so Corollary 3.6.17 applies.

Moreover, we can now generalise further to an even *more* general style of GGSC-like demand, in which goods are separated into an arbitrary number of groups, with goods within the same group being strong substitutes, but with 1-1 complementarities between some pairs of groups (that is, for those pairs of groups, each good in one of the groups may exhibit 1-1 complementarities with any good in the other group). If all the "cycles" formed by the sequences of "paired" groups are of even length, then we can again separate the groups of goods into two classes, so that the demand is again GGSC demand, and so always has a competitive equilibrium. But if any odd cycle exists then, just as in Example 3.6.13, competitive equilibrium may fail.[95]

3.6.4.3 When is Strong Substitutes a necessary condition for equilibrium?

Danilov and Grishukhin (1999) provided a characterisation of all of (what we call) unimodular demand types, including a list giving, up to unimodular basis change, all maximal such types up to dimension 6. From this list it is immediate that

Theorem 3.6.18. *If $n \leq 3$, equilibrium always exists for a concave demand type if and only if it is a unimodular basis change from strong substitutes, or a subset thereof.*

With $n > 3$, there exist concave demand types for which equilibrium always exists, which are not *a unimodular basis change from strong substitutes, or a subset thereof.*

That is, while if there are at most three goods, all unimodular demand types are unimodular basis changes from strong substitutes, this is far from true more generally. Indeed, we already showed this in Example 3.5.25 which, moreover, has *only* complementary relationships. This

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[95] This result has independently been established by Sun and Yang (2011), and also Teytelboym (2014); the latter paper gives fuller details.

example also provides (see Theorem 3.6.7) a concrete example of preferences for which a stable coalition structure always exists – although coalitions may consist of one, two or three agents.

Furthermore, recall (Theorem 3.5.27) that *every* unimodular demand type, that is, *every* demand type for which competitive equilibrium is guaranteed, *is* a unimodular basis change of a unimodular complements demand type–in stark contrast to conventional wisdom about the "necessity" of substitutes for competitive equilibrium.[96]

3.6.4.4 Equilibrium in extensions of the Product-Mix Auction

It is not hard to check that the bids in any Product-Mix Auction of the kind implemented by the Bank of England all represent strong substitutes preferences, so (see Section 3.6.4.2) equilibrium is guaranteed even if individual units of goods are indivisible. This remains true if the auction is augmented by permitting bidders to use "negative" bids (in which case, it can be shown that all strong substitutes preferences can be represented).[97]

In fact, the Bank's implementation of the Product-Mix Auction allows the auctioneer to ration whenever it wishes. However, there are many contexts in which rationing may not be possible. For example, a piece of radio spectrum may only be useful if it is above a certain minimum size. Similarly, bidders might make competing offers to build gas-fired plants, nuclear-power stations, wind farms, etc., to a government needing energy capacity—and nuclear-plants, at least, may be indivisible.[98] So results about equilibrium when goods are indivisible may be needed to apply the Product-Mix Auction to problems currently facing regulators such as the U.S. Federal Communications Commission, the U.K.'s Ofcom, and the U.K. Department for Energy and Climate Change. We can use Theorem 3.6.8 to guarantee that existence of competitive equilibrium is retained in extensions of the Product-Mix Auction—see Baldwin and Klemperer (in preparation).[99]

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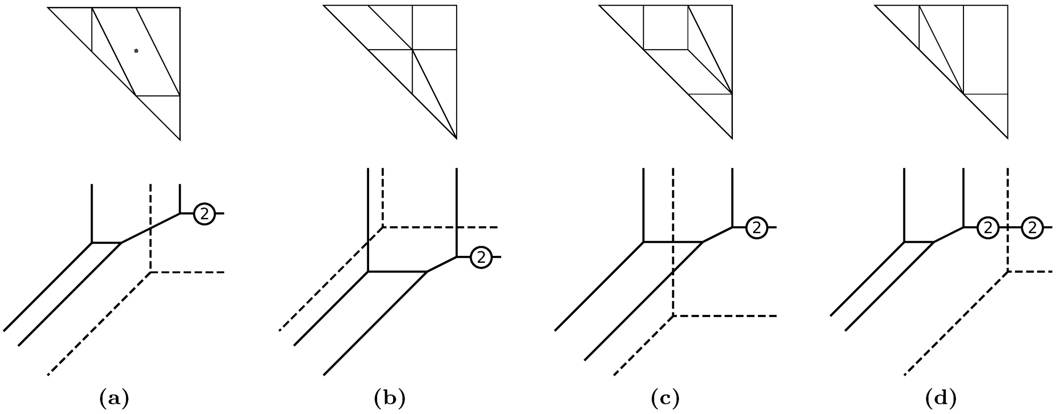
[96] For example, Gul and Stacchetti (1999, p. 96) state "in a sense, the GS [gross substitutes] condition is necessary to ensure existence of a Walrasian equilibrium". The "necessity" makes sense in their context in which any set of agents they consider may contain any agent who demands at most one unit of any good. (Since such agents have demand type $\mathcal{D}^n_{ss}$, their result is equivalent to the maximality of $\mathcal{D}^n_{ss}$ as a unimodular set (see Corollary 3.6.15).) But the specificity of the context in which claims like this makes sense often seems to be forgotten.

[97] See Klemperer (2010), and Baldwin and Klemperer (in preparation).

[98] Making matters even more complex, alternatives such as wind farms and back-up facilities for use when no wind is blowing may be complements.

[99] Product-Mix auctions can also be used to improve clock auctions: one criticism of clock auctions is that they fail to find the exact competitive equilibrium when it is unique, or the correct competitive equilibrium when it is not unique, see, for example, Harbord et al., 2011, Appendix A. (In auctions for substitutes the usual objective is to select the unique bidder-preferred competitive equilibrium in the event that competitive equilibrium is non-unique.) We can solve this problem of clock auctions by accepting Product-Mix bids either between the bid increments of a clock auction (this is a generalisation of the "intraround bidding" popularised by Ausubel and Cramton (2004)) and/or after using simpler techniques to identify the price range in which competitive equilibrium must lie. (Footnote 7 discusses other disadvantages of clock auctions and simultaneous multiple round auctions relative to (pure) Product-Mix auctions.)



(a) (b) (c) (d)

Figure 3.21: *Examples of aggregate THs and SNPs of agents with THs of the combinatorial types of Fig. 3.1 (dashed line) and Fig. 3.12 (solid line). The number of intersections of the THs, weighted by facet weights, reveals the existence or failure of equilibrium.*

3.6.5 Existence of equilibrium for specific demands

Our Theorem 3.6.8 tells us which demand types *always* have a competitive equilibrium. When the answer is negative, it does not tell us whether competitive equilibrium exists for every supply bundle, for a specific set of demands. But if all intersections are "nice" (in the sense of Section 3.6.3) then we can apply Proposition 3.A.4 to each intersection point to check for such a failure.

Take, for example, Agents 1 and 2 who have THs of the combinatorial types of Figures 3.1 and 3.12, respectively, and concave valuations. (A valuation function of the combinatorial type of Figure 3.1 *must* be concave. A valuation function of the type of Figure 3.12 need not be concave, though the specific valuation function of this type that is given in Example 3.2.10 is concave.)

The combinatorial type of aggregate demand will depend on how the agents' THs meet in price space; assume they only intersect "nicely". Applying Propositions 3.6.10 and 3.A.4, we see that there exists a supply bundle such that competitive equilibrium does not exist iff the facets with normals $(1,0)$ and $(-1,2)$ intersect (since $\det\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 2 > 1$). An example of aggregate demand of this combinatorial type is illustrated in Figure 3.21a; the bundle $(1,1)$ is in the interior of the parallelogram in the SNP of Figure 3.21a, and is never demanded on aggregate (see Proposition 3.6.10).

Combinatorial types of aggregate demand in which competitive equilibrium *does* exist for any supply bundle are illustrated in Figures 3.21b, 3.21c and 3.21d (there are others).

In Figures 3.21b and 3.21c, there are *two* intersections between the THs. In each case, the areas of the SNP faces corresponding to the intersections are 1. We call this area the 'multiplicity' of the intersection; note that it is, of course, the determinant of the (primitive integer) edges of the SNP face (and so, as we have seen, intimately connected with the existence of competitive equilibrium).

Conversely, in Figures 3.21a and 3.21d there is only *one* intersection. Now, however, the corresponding SNP face has area 2; we say the 'multiplicity' of the intersection is 2.

Observe that in each case, the number of intersections, weighted by multiplicity, is 2. It can be checked that this holds for every other aggregate of the demands of two agents whose individual THs are of the same combinatorial types as Figures 3.1 and 3.12, respectively. This is a special case of the *Tropical Bézout Theorem*.[100]

However, the natures of the multiplicity 2 intersections in Figures 3.21a and 3.21d are different. In Figure 3.21d, one of the corresponding facets is of weight 2; Agent 2 has a concave valuation and so has *three* bundles in its demand set, so Proposition 3.6.10 does not apply– the bundles 'inside' the weight-2 facet (in the centres of the long edges of the rectangle in the SNP) are both demanded at this price. The best way to understand this situation is that 'two intersections have become arbitrarily close'. By contrast, in Figure 3.21a, neither of the corresponding facets has weight 2, Proposition 3.6.10 *does* apply, and the bundle in the centre of the parallelogram is *not* demanded at any price.

Recall that the multiplicity of the intersection is the area of the SNP face, which equals the (absolute value of the) determinant of its edges. The key point is that this can be factorised into the product of the facet weights times the (absolute value of the) determinant of the *primitive integer* edge directions (that is, the primitive integer facet normals). And equilibrium fails iff the (absolute value of the) *latter* determinant exceeds 1. So the *existence* of a supply bundle for which competitive equilibrium fails is signalled by a case in which the sum of intersections, weighted *only* by facet weights, is too small.

These ideas can be applied more generally, as will be developed in future work.

3.7 Conclusion

Studying the tropical geometry of demand yields a range of insights. The structure of an agent's preferences can be efficiently summarised by a set of vectors that is orthogonal to the divisions between the regions of price space in which the agent demands different bundles. So examining these vectors is an efficient way of determining the "type" of demand, and the same set of vectors also generates the surface of the convex hull of the agent's valuation function in quantity space. The duality between these representations has powerful implications, and the pictorial representations that tropical geometry gives us generate new intuitions.[101]

We began this work while studying the properties of many-dimensional Product-Mix Auctions. Convex and tropical geometry is the key to much of our analysis in Baldwin and Klemperer (in preparation) in which we describe ways in which different preferences can be represented in

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[100] See Richter-Gebert et al (2005).

[101] These intuitions are obscured by existing pictorial representations which shoehorn indivisible demand into the standard divisible-demand framework.

these auctions, and the implications of different restrictions on bids.[102] Geometric reasoning has also helped us develop extensions to the Bank of England's original implementation of the auction,[103] and understand the connections to related auction designs.[104]

In other work, we have found that similar geometric analysis is useful in understanding results obtained by others, and that it can prove these results more quickly than currently-used techniques. So we are optimistic that tropical-geometric analysis will yield more economic insights in the future; we hope others will take up these methods.

Appendix 3.A Proofs of Results in the text

3.A.1 Proofs of Results in Section 3.4

Proof of Theorem 3.4.4. First we show 1⇒2: Suppose u is of type $\mathcal{D}$. Generically the line $[\mathbf{p}, \mathbf{p}']$ crosses only facets, not any lower dimensional cells in $\mathcal{T}_u$. Furthermore, because the UDRs are open sets and because there are only finitely many cells of lower dimension than $n-1$, we can always choose a change in price, $\mathbf{q}$, such that $D_u(\mathbf{p}-\mathbf{q}) = D_u(\mathbf{p})$ and $D_u(\mathbf{p}'-\mathbf{q}) = D_u(\mathbf{p}')$ and the line $[\mathbf{p}-\mathbf{q}, \mathbf{p}'-\mathbf{q}]$ does indeed cross only facets; we choose $\mathbf{q}$ sufficiently small that $[\mathbf{p}-\mathbf{q}, \mathbf{p}'-\mathbf{q}]$ only crosses facets that are also crossed by $[\mathbf{p}, \mathbf{p}']$, although the latter may meet these facets at their boundaries. Now let $\mathbf{x}^0, \ldots, \mathbf{x}^l$ be demanded in each UDR that $[\mathbf{p}-\mathbf{q}, \mathbf{p}'-\mathbf{q}]$ meets; by construction, each of these bundles is also demanded at some price in sequence on the line $[\mathbf{p}, \mathbf{p}']$ and, also by construction, the difference between each pair of consecutive bundles is in the direction of a facet normal, that is, a vector in $\mathcal{D}$. Since the bundle demanded changes in each case, $(\mathbf{x}^j - \mathbf{x}^{j-1}).(\mathbf{p}' - \mathbf{p}) < 0$ in each case.

Next show 2⇒4: Suppose that $\{\mathbf{x}\} = D_u(\mathbf{p})$ and that $\mathbf{x} \notin D_u(\mathbf{p}')$. By assumption we can break down the demand change from $\mathbf{p}$ to $\mathbf{p}'$ in improving $\mathbb{Z}\mathcal{D}$-steps. Let $\mathbf{x}''$ be the first bundle $\mathbf{x}^1$ in this sequence; we know that $\frac{1}{w}(\mathbf{x}''-\mathbf{x}) \in \mathcal{D}$ for some $w \in \mathbb{Z}$ and that $(\mathbf{p}'-\mathbf{p}).(\mathbf{x}''-\mathbf{x}) < 0$. We re-write the latter as

$$(\mathbf{p}' - \mathbf{p}).\mathbf{x}'' < (\mathbf{p}' - \mathbf{p}).\mathbf{x} \tag{3.5}$$

---

[102] In the Bank of England's implementation, the bid-taker expresses preferences through a "supply function" while bidders can make sets of "or" bids that can, if desired, be represented as sets of points on a graph. Permitting negative as well as positive bids broadens the set of preferences that can be expressed, as does permitting bidders to specify additional constraints (Klemperer, 2008, 2010). The issue is: what kinds of bids should we permit to achieve a sufficiently rich representation of preferences, while retaining a unique solution (the extent to which we can permit some degree of complements is a particular challenge), achieving an efficient outcome (in particular, not incentivising strategic behaviour), and retaining simplicity and transparency?

[103] Extensions include broadening the range of contexts to which these (or related) auctions can be applied, through a better understanding of when equilibrium is guaranteed to exist, as well as better ways of representing bidders' and bid-takers' multi-dimensional preferences.

[104] Related designs include, in particular, the Assignment Auction suggested independently by Milgrom (2009), and versions of Simultaneous Multiple Round Auction (see, e.g., Milgrom, 2000) and "Clock Auctions" (see, e.g., Ausubel and Milgrom, 2002, Gul and Stacchetti, 2000, and Milgrom and Strulovici, 2009); see also the papers in Cramton, Shoham, and Steinberg (2006). As noted in the Introduction, we are also concerned with efficient solution techniques for Product-Mix Auctions, both when we need integer solutions, and when rationing is permitted, etc.

Moreover, since $\mathbf{x}'' \in D_u((1-\lambda_1)\mathbf{p} + \lambda_1\mathbf{p}')$, we know

$$u(\mathbf{x}'') - [(1-\lambda_1)\mathbf{p} + \lambda_1\mathbf{p}'].\mathbf{x}'' \geq u(\mathbf{x}) - [(1-\lambda_1)\mathbf{p} + \lambda_1\mathbf{p}'].\mathbf{x}$$

Subtracting $(1-\lambda_1)$ times equation (3.5) we obtain

$$u(\mathbf{x}'') - \mathbf{p}'.\mathbf{x}'' > u(\mathbf{x}) - \mathbf{p}'.\mathbf{x}$$

i.e. $\mathbf{x}''$ is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$, as required.

Next we show 4⇒1. Suppose u is not of demand type $\mathcal{D}$. Then $\mathcal{T}_u$ has a facet F with primitive integer normal $\mathbf{n} \notin \mathcal{D}$. Let $\mathbf{p}^0$ be in the interior F° of this facet. For $\epsilon > 0$ sufficiently small, $\#D_u(\mathbf{p}^0 + \epsilon\mathbf{n}) = 1$; let $\{\mathbf{x}\} = D_u(\mathbf{p}^0 + \epsilon\mathbf{n})$. Then $\mathbf{x} \notin Du(\mathbf{p}^0 - \eta\mathbf{n})$ for any $\eta > 0$; set $\mathbf{p}' := \mathbf{p}^0 - \eta\mathbf{n}$ where η is sufficiently small that, at $\mathbf{p}'$, any bundle in $D_u(\mathbf{p}^0)$ is preferred to to any outside this set. The only bundles strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$ are bundles in $D_u(\mathbf{p}^0) - \{\mathbf{x}\}$. But, since $\mathbf{p}^0$ is in the interior of the facet F, we know that $D_u(\mathbf{p}^0) = \{\mathbf{x} + w\mathbf{n} \mid w \in W\}$ for some finite set $W \subsetneq \mathbb{Z}$. So since we assumed that $\mathbf{n} \notin \mathcal{D}$, the ordinary $\mathbb{Z}\mathcal{D}$-improvement property cannot hold.

It is clear that 2⇒3, so we conclude by showing that 3⇒1. So suppose that u satisfies 3. Consider a facet normal $\mathbf{v}$ of $\mathcal{T}_u$. Since $\mathbf{v} \neq 0$ we can pick i such that $v_i \neq 0$. Then it is possible to pick some $\epsilon > 0$ and prices $\mathbf{p}, \mathbf{p} + \epsilon\mathbf{e}^i$ in the UDRs either side of this facet, so that $[\mathbf{p}, \mathbf{p} + \epsilon\mathbf{e}^i]$ crosses only this facet. By property 3 we can break down the change in demand from $\mathbf{p}$ to $\mathbf{p} + \epsilon\mathbf{e}^i$ in improving $\mathbb{Z}\mathcal{D}$-steps – but it is clear that the only way to break down the change in demand from $\mathbf{p}$ to $\mathbf{p} + \epsilon\mathbf{e}^i$ is in steps in the direction of $\mathbf{v}$. We conclude that $\mathbf{v} \in \mathcal{D}$. Hence, u is of demand type $\mathcal{D}$. □

Proof of Theorem 3.4.5. This proof closely follows the proof of Theorem 3.4.4, except that in showing 1⇒2 we choose $\mathbf{x}^0, \ldots, \mathbf{x}^l$ which are demanded on $[\mathbf{p} - \mathbf{q}, \mathbf{p}' - \mathbf{q}]$ but are not necessarily demanded in any UDR; instead, we stipulate that each new bundle differs from the former by a vector in $\mathcal{D}$. Since u is concave, such bundles exist. In showing 2⇒4 we note that, if we start with a series of improving $\mathcal{D}$-steps, then $\mathbf{x}''$ will differ from $\mathbf{x}$ by a vector in $\mathcal{D}$. That 4⇒1 and 3⇒1 clearly follow from 4⇒1 and 3⇒1 of Theorem 3.4.4, respectively, under the additional assumption that u is concave. □

Example 3.A.1. Consider the unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$ and the non-concave valuation of this type:

$x_1 = 3$	$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
5	4	2	0	$x_2 = 0$
6	5	4	2	$x_2 = 1$
6	6	4	4	$x_2 = 2$
6	6	6	5	$x_2 = 3.$

We show that we *can* break down the change in demand between any $\mathbf{p}, \mathbf{p}'$ into improving $\mathcal{D}$-steps. This valuation fails to be concave at the bundle $(1,2)$; it would have the same TH but would be concave if $v(1,2)$ were equal to 5, in which case we would have $(1,2) \in D_u(1,1)$. By Theorem 3.5.11 and Corollary 3.5.5, we know that we would be able to break down the demand change from any $\mathbf{p}$ to any $\mathbf{p}'$ in improving $\mathcal{D}$-steps if it were concave. Thus there is only a question as to whether this is possible for $\mathbf{p}, \mathbf{p}'$ such that $(1,1) \in [\mathbf{p}, \mathbf{p}']$. And we need only consider $\mathbf{p}, \mathbf{p}'$ in UDRs and facets that contain $(1,1)$ in their closure.

It is easiest to break down the cases to consider by considering possible choices of $\mathbf{x}^0$. Suppose first that $(0,2) \in D_u(\mathbf{p})$. Then $(3,1) \in D_u(\mathbf{p}')$ and $p_1 - p_1' > 0$ and $p_1 - {p'}_1 \geq p_2 - {p'}_2 \geq 0$. So we may take $\mathbf{x}^1 = (1,1)$, $\mathbf{x}^2 = (2,1)$, $\mathbf{x}^3 = (3,1)$ as the demand change broken down in improving $\mathcal{D}$-steps. And if $\mathbf{x}^0 = (3,1)$ then it is easy to see that the same steps in reverse will break down the demand change in improving $\mathcal{D}$-steps. Moreover, by the symmetry of the figure, we may find an analogous way to break down the demand change if $\mathbf{x}^0 = (2,0)$ or $(1,3)$.

Suppose next that $(1,1) \in D_u(\mathbf{p})$. Then $(2,2) \in D_u(\mathbf{p}')$ and $p_1 - p_1' = p_2 - p_2' > 0$. So if $(1,1) = \mathbf{x}^0$ then we may take $\mathbf{x}^1 = (2,1)$ and $\mathbf{x}^2 = (2,2)$ as the demand change broken down in improving $\mathcal{D}$-steps. Again, if $\mathbf{x}^1 = (2,2)$ then the reverse sequence will suffice.

Suppose that $(0,3) \in D_u(\mathbf{p})$. Then $(3,0) \in D_u(\mathbf{p}')$ and $p_1' - p_1 \leq 0$, $p_2 - p_2' \leq 0$ and both cannot hold with equality. Suppose the latter holds with equality then $(0,2) \in D_u(\mathbf{p})$ also. As argued above it follows that $(3,1) \in D_u(\mathbf{p}')$. In this case we break down the demand change in improving $\mathcal{D}$ steps via $(1,3), (2,2), (3,1)$. Similarly, if $p_1' - p_1 = 0$ we can break down the demand change in improving $\mathcal{D}$-steps. Finally, if $p_1' - p_1 < 0$ and $p_2' - p_2 < 0$ then $(3,0) \in D_u(\mathbf{p}')$ and we can break down the demand change as $(0,3), (1,3), (2,2), (3,1), (3,0)$, these being improving $\mathcal{D}$-steps.

Example 3.A.2 (Milgrom and Strulovici, 2009, p224.)**.** The valuation

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
2	1	0	$x_2 = 0$
3	1	1	$x_2 = 1$
4	3	2	$x_2 = 2$

is of demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ and does satisfy the $\mathcal{D}$-improvement property, but is not concave.

Proof of Proposition 3.4.14 1. By definition, $\mathbf{x} \in D_u(\mathbf{p})$ if $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}' \in A$, with equality iff $\mathbf{x}' \in D_u(\mathbf{p})$ also. For any invertible matrix G, we may re-write

$$\mathbf{p}^T(\mathbf{x} - \mathbf{x}') = \mathbf{p}^T G G^{-1}(\mathbf{x} - \mathbf{x}') = (G^T\mathbf{p})^T(G^{-1}\mathbf{x} - G^{-1}\mathbf{x}').$$

If G is additionally unimodular, then $G^{-1}\mathbf{x}$ and $G^{-1}\mathbf{x}' \in \mathbb{Z}^n$. We define a new valuation G^*u on the finite set $G^{-1}A \subsetneq \mathbb{Z}^n$ via $G^*u(\mathbf{y}) := u(G\mathbf{y})$. If we write $\mathbf{y} = G^{-1}\mathbf{x}$ and $\mathbf{y}' = G^{-1}\mathbf{x}'$ then $(G^T\mathbf{p})^T(\mathbf{y} - \mathbf{y}') \leq G^*u(\mathbf{y}) - G^*u(\mathbf{y}')$ holds iff $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$. So we have

$$\mathbf{x} \in D_u(\mathbf{p}) \;\Leftrightarrow\; \mathbf{y} = G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p}),$$

as required.

2. Since the underlying set of $\mathcal{T}_u$ is those $\mathbf{p}$ for which $\#D_u(\mathbf{p}) > 1$ it follows immediately from 1. that $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$, as required.

3. Suppose $\mathbf{v}$ is normal to a facet F of $\mathcal{T}_u$. It follows from 2. that the facet corresponding to F in $\mathcal{T}_{G^*u}$ has the form $G^TF = \{G^T\mathbf{p} \mid \mathbf{p} \in F\}$. We know $\mathbf{p}^T\mathbf{v}$ is constant for $\mathbf{p} \in F$, from which it follows that $(G^T\mathbf{p})^TG^{-1}\mathbf{v} = \mathbf{p}^TGG^{-1}\mathbf{v}$ is constant for $G^T\mathbf{p} \in G^TF$: we see $G^{-1}\mathbf{v}$ is normal to a facet of $\mathcal{T}_{G^*u}$. As G has an integer inverse, the converse is also true. Trivially, for any unimodular matrix G, the valuation G^*u is concave iff the valuation u is. □

3.A.2 Proofs of Results in Section 3.5

Proof of Proposition 3.5.3. To show 1⇒2, take any such u, let $\mathbf{x} \in A$ and let $\mathbf{p}'$ be such that $\mathbf{x} \notin D_u(\mathbf{p}')$. Since u is concave there exists $\mathbf{p}$ such that $\mathbf{x} \in D_u(\mathbf{p})$. We can break down the demand change from $\mathbf{p}$ to $\mathbf{p}'$ in improving $\mathcal{D}$-steps; let $\mathbf{x}''$ be the first bundle $\mathbf{x}^1$ in this sequence. That $\mathbf{x}''$ has the desired properties follows exactly as in the proof that 2⇒4 in Theorem 3.4.4.

Now show that 2⇒1. Given any u of type $\mathcal{D}$, given $\mathbf{p} \in \mathbb{R}^n$, given any $\mathbf{x} \in D_u(\mathbf{p})$ and given any $\mathbf{p}' \neq \mathbf{p}$, either $\mathbf{x} \in D_u(\mathbf{p}')$, or there exists some maximal λ_1 such that $\mathbf{x} \in D_u((1-\lambda_1)\mathbf{p} + \lambda_1\mathbf{p}')$. Suppose the latter is the case and choose $\lambda' > \lambda_1$ sufficiently small that all bundles in $D_u((1-\lambda_1)\mathbf{p} + \lambda_1\mathbf{p}')$ are preferred to any bundle outside this set, at price $\mathbf{p}'' := (1-\lambda')\mathbf{p} + \lambda'\mathbf{p}'$. If the complete $\mathcal{D}$-improvement property holds then there exists $\mathbf{x}^1$ such that $\mathbf{x}^1 - \mathbf{x} \in \mathcal{D}$ and such that $\mathbf{x}^1$ is strictly preferred to $\mathbf{x}$ at $\mathbf{p}''$. But, by our choice of $\mathbf{p}''$, we know $\mathbf{x}^1 \in D_u((1-\lambda_1)\mathbf{p} + \lambda_1\mathbf{p}')$, and by the fact that $\mathbf{x}^1$ is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}''$, we know the demand change from $\mathbf{x}$ to $\mathbf{x}^1$ satisfies the strict law of demand. So $\mathbf{x}^1$ is the first improving $\mathcal{D}$-step. If $\mathbf{x}^1 \notin D_u(\mathbf{p}'')$ we set $\lambda_2 = \lambda_1$ and find $\mathbf{x}^2$ strictly preferred to $\mathbf{x}^1$ at $\mathbf{p}''$ as before; by finiteness of the set A we eventually find $\mathbf{x}^j \in D_u(\mathbf{p}'')$. Now let λ_{j+1} be maximal such that $\mathbf{x}^j \in D_u((1-\lambda_{j+1})\mathbf{p} + \lambda_{j+1}\mathbf{p}')$ and continue as before; by finiteness of A we eventually retrieve $\mathbf{x}^l \in D_u(\mathbf{p}')$. □

Proof of Proposition 3.5.6. 1. Given $\mathbf{p}'$ and $\mathbf{x} \in G^{-1}A$, $\mathbf{x} \notin D_{G^*u}(\mathbf{p}')$, we know (by Proposition 3.4.14.1) that $G\mathbf{x} \in D_u(G^{-T}\mathbf{p}')$. So there exists $\mathbf{y}''$ such that $u(\mathbf{y}''^{-T}\mathbf{p}'.\mathbf{y}'' > u(G\mathbf{x}) - G^{-T}\mathbf{p}'^{-1}\mathbf{x}$. and such that $\mathbf{y}'' - G\mathbf{x} \in \mathcal{D}$. Let $\mathbf{x}''^{-1}\mathbf{y}''$; then we have $u(G\mathbf{x}''^{-T}\mathbf{p}'^{-1}\mathbf{x}'' > u(G\mathbf{x}) - G^{-T}\mathbf{p}'^{-1}\mathbf{x}$, which says precisely that $G^*u(\mathbf{x}'') - \mathbf{p}'.\mathbf{x}''^{*}u(\mathbf{x}) - \mathbf{p}'.\mathbf{x}$. Moreover, $\mathbf{x}'' - \mathbf{x} = G^{-1}(\mathbf{y}'' - G\mathbf{x}) \in G^{-1}\mathcal{D}$. So G^*u satisfies the complete $G^{-1}\mathcal{D}$-improvement property. It is also concave (see Proposition 3.4.14.3), and so $G^{-1}\mathcal{D}$-complete, as required.

2 follows from 1 by definition. □

Proof of Theorem 3.5.11. As noted in Footnote 59, we need only assume that for any vectors $\mathcal{V} \subseteq \mathcal{D}$, there exist linearly independent vectors $\mathbf{w}^1, \ldots, \mathbf{w}^s \in \mathcal{D}$, whose span over $\mathbb{R}$ coincides with the span over $\mathbb{R}$ of $\mathcal{V}$ and such that, for any $\mathbf{v}^1, \dots, \mathbf{v}^{s-1} \in \mathcal{V}$ and any $i = 1, \dots, s$, the vectors $\mathbf{v}^1, \dots, \mathbf{v}^{s-1}, \mathbf{w}^i$ are either linearly dependent or are a unimodular set. This property is clearly implied by unimodularity of $\mathcal{D}$, as we may take the set $\mathbf{w}^1 \dots, \mathbf{w}^s$ to be any linearly independent and spanning subset of $\mathcal{V}$.

Suppose $\mathcal{D}$ is as above, and that u is concave and of demand type $\mathcal{D}$. We will show that u satisfies Condition 1 of Proposition 3.5.3. Let $\mathbf{p} \neq \mathbf{p}'$. Suppose $\mathbf{x} \in D_u(\mathbf{p})$. Let $\lambda_1 := \max\{\lambda \in [0,1] \mid \mathbf{x} \in D_u((1-\lambda)\mathbf{p} + \lambda\mathbf{p}'\}$. If $\lambda_1 = 1$ then $l = 1$ and we are done; suppose not. Let $\mathbf{p}^1 := (1-\lambda_1)\mathbf{p} + \lambda_1\mathbf{p}'$ and write $\mathbf{c}$ for $\mathbf{p}' - \mathbf{p}$.

We first argue that it is sufficient to find $\mathbf{v} \in \mathcal{D}$ such that $\mathbf{x}^1 := \mathbf{x}+\mathbf{v} \in D_u(\mathbf{p}^1)$ and such that $(\mathbf{p}' - \mathbf{p}).(\mathbf{x}^1 - \mathbf{x}) = \mathbf{c}.\mathbf{v} < 0$. Such $\mathbf{x}^1$ then satisfies the conditions to be the second improving $\mathcal{D}$-step. We may iterate the procedure; since A is finite, after a finite number of such steps find $\mathbf{x}^k$ such that $\mathbf{c}.\mathbf{x}^k$ is minimal for bundles in $D_u(\mathbf{p}^1)$. Then $\mathbf{x}^k \in D_u(\mathbf{p} + \epsilon\mathbf{c})$ for small enough $\epsilon > 0$. Setting now $\lambda_{k+1} := \max\{\lambda \in [0,1] \mid \mathbf{x}^k \in D_u((1-\lambda)\mathbf{p} + \lambda\mathbf{p}')\}$ we continue as before; again by finiteness of A, this process must terminate at price $\mathbf{p}'$ after finitely many steps.

Let Δ be a minimal SNP face for u such that $\mathbf{x} \in \Delta$ and $\Delta \subseteq \operatorname{Conv} D_u(\mathbf{p}^1)$ and such that $\mathbf{c}.\mathbf{y} < \mathbf{c}.\mathbf{x}$ for some $\mathbf{y} \in \Delta \cap \mathbb{Z}$. Such a face exists since $\operatorname{Conv} D_u(\mathbf{p}^1)$ has the required properties with the possible exception of minimality. Let $\mathbf{w}^1, \dots, \mathbf{w}^s \in \mathcal{D}$ be linearly independent vectors whose span over $\mathbb{R}$ coincides with the span over $\mathbb{R}$ of the edges of Δ and with the property as described at the beginning of this proof; since there exists $\mathbf{y} \in \Delta$ such that $\mathbf{c}.\mathbf{y} < \mathbf{c}.\mathbf{x}$ it follows that there exists i such that $\mathbf{c}.\mathbf{w}^i \neq 0$; we write $\mathbf{w}$ for whichever of $\mathbf{w}^i$ or $-\mathbf{w}^i$ provides $\mathbf{c}.\mathbf{w} < 0$ and show that $\mathbf{x} + \mathbf{w} \in D_u(\mathbf{p}^1)$.

Let $\bar{n} := \dim \Delta$ and let $\mathbb{A}^{\bar{n}}$ be the affine span of Δ. The polytope Δ is the intersection of half-spaces $H^+_{\mathbf{q},\alpha}$ and hyperplanes $H_{\mathbf{q},\alpha}$ where $(\mathbf{q}, \alpha) \in \mathbb{Z}$. We show that we may choose the normal vectors $\mathbf{q}$ so that $\mathbf{q}.\mathbf{w} = 0$ or ± 1 in every case.

If $\mathbf{q}.\mathbf{w} = 0$ then we have no problem, so assume not. For every $(\mathbf{q}, \alpha)$, there are $\bar{n} - 1$ edges of Δ embedded in $H_{\mathbf{q},\alpha}$ whose directions are $\bar{n} - 1$ linearly independent vectors in $\mathcal{D}$ (since u is of demand type $\mathcal{D}$). Let these directions be $\mathbf{v}^1, \dots, \mathbf{v}^{\bar{n}-1} \in \mathcal{D}$. By assumption, $\mathbf{w}$ is not in the span of $\mathbf{v}^1, \dots, \mathbf{v}^{\bar{n}-1}$. So $\mathbf{v}^1, \dots, \mathbf{v}^{\bar{n}-1}, \mathbf{w}$ is a unimodular set (by definition of $\mathbf{w}$) and hence we may choose vectors $\mathbf{v}^{\bar{n}}, \dots, \mathbf{v}^{n-1}$ (*not* necessarily in $\mathbf{D}$) such that $\det(\mathbf{v}^1, \dots, \mathbf{v}^{n-1}, \mathbf{w}) = \pm 1$. But also, we may choose the halfspace $H^+_{\mathbf{q},\alpha}$ defining Δ so that $\mathbf{q}$ is additionally normal to all the vectors $\mathbf{v}^{\bar{n}}, \dots, \mathbf{v}^{n-1}$: recall it was already normal to the first set of vectors identified, and the second set are by construction in directions linearly independent to Δ; we are free to choose the behaviour of $H_{\mathbf{q},\alpha}$ beyond the affine span of Δ. However, if the vector $\mathbf{q}$ is a primitive integer vector normal to this set, then it is in the direction their vector product $\mathbf{v}^1 \times \dots \times \mathbf{v}^{n-1}$; it is a theorem of linear algebra that $(\mathbf{v}^1 \times \dots \times \mathbf{v}^{n-1}).\mathbf{w} = \det(\mathbf{v}^1, \dots, \mathbf{v}^{n-1}, \mathbf{w})$. So, we can in each case choose $\mathbf{q}$ so that $\mathbf{q}.\mathbf{w} = \pm 1$.

We now wish to check that $\mathbf{x} + \mathbf{w} \in \Delta$. Since $\mathbf{x} \in \Delta$ we know that $\mathbf{q}.\mathbf{x} \geq \alpha$ and so, that $\mathbf{q}.(\mathbf{x} + \mathbf{w}) \geq \alpha + \mathbf{q}.\mathbf{w}$ for every defining halfspace $H^+_{\mathbf{q},\alpha}$ or hyperplane $H_{\mathbf{q},\alpha}$. For those $\mathbf{q}$ such that $\mathbf{q}.\mathbf{w} = 0$ we are done. Since $\mathbf{w}$ lies in the span over $\mathbb{R}$ of the edges of Δ, those $\mathbf{q}$ such that $\mathbf{q}.\mathbf{w} = \pm 1$ must define half-spaces and not hyperplanes. If $\mathbf{x} \in \Delta^\circ$ then in every such case $\mathbf{q}.\mathbf{x} > \alpha$ and hence, since the equations are integral, $\mathbf{q}.\mathbf{x} \geq \alpha + 1$; thus $\mathbf{q}.(\mathbf{x}+\mathbf{w}) \geq \alpha$. Our only difficulty arises if $\mathbf{x}$ lies in a (strict) face Δ' of Δ.

Suppose then that $\mathbf{x}$ is in such a face $\Delta' \subsetneq \Delta$. We show that $x \in \Delta'^\circ$ and that $\dim \Delta' = \dim \Delta - 1$ and $\Delta' \subset H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$, where we recall that we write $\mathbf{c}$ for the change in price $\mathbf{p}' - \mathbf{p}$ under consideration.

First note that, by minimality of Δ, we know $\mathbf{c}.\mathbf{y} \geq \mathbf{c}.\mathbf{x}$ for all $\mathbf{y} \in \Delta'$. So $\Delta' \subset H^+_{\mathbf{c},\mathbf{c}.\mathbf{x}}$; let Δ'' be the face (which we will see to be the whole of Δ') of Δ' given by $\Delta' \cap H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$.

Suppose for a contradiction that $\dim \Delta'' \leq \dim \Delta - 2$. By the standard properties of polytopes we know that Δ'' is also a face of Δ, and additionally is a face of the polytope $\overline{\Delta} := \Delta \cap H^-_{\mathbf{c},\mathbf{c}.\mathbf{x}}$. The latter polyhedron has the same dimension as Δ since, by assumption, Δ is not contained in $H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$. So $\dim \Delta'' \leq \dim \overline{\Delta} - 2$, and so Δ'' is the intersection of at least two maximal strict faces of $\overline{\Delta}$. But at most one of these may be contained in the hyperplane $H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$; the other must contain $\mathbf{y}$ such that $\mathbf{c}.\mathbf{y} \neq \mathbf{c}.\mathbf{c}$ and hence (by definition of $\overline{\Delta}$) must contain $\mathbf{y}$ such that $\mathbf{c}.\mathbf{y} < \mathbf{c}.\mathbf{x}$. But, since $\mathbf{x} \in \Delta''$ which is contained in this face, this contradicts the minimality assumption on Δ.

Thus $\dim \Delta'' \geq \dim \Delta - 1$; since $\Delta'' \subseteq \Delta' \subsetneq \Delta$ we conclude that $\Delta'' = \Delta'$, whence $\dim \Delta' = \dim \Delta - 1$ and $\Delta' \subseteq H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$. Finally, if $\mathbf{x}$ were in the boundary of Δ' then $\mathbf{x}$ would be in a lower dimensional face of Δ than Δ', which we have shown not to be the case. So $\mathbf{x} \in \Delta'^\circ$.

We may now modify the list of halfspaces describing Δ: replace the half-space defining the maximal face Δ' with $H^+_{-\mathbf{c},-\mathbf{c}.\mathbf{x}}$. By assumption $\mathbf{c}.\mathbf{w} < 0$ and so $-\mathbf{c}.(\mathbf{x}+\mathbf{w}) \geq -\mathbf{c}.\mathbf{x}$. By construction, this was the remaining half-space defining Δ which required this check, and so we know that $\mathbf{x}+\mathbf{w} \in \Delta \subseteq \operatorname{Conv} D_u(\mathbf{p}^1)$. Moreover, as both vectors are integral, $\mathbf{x}+\mathbf{w} \in \operatorname{Conv} D_u(\mathbf{p}^1) \cap \mathbb{Z}^n$, and since u is concave, it follows that $\mathbf{x}+\mathbf{w} \in D_u(\mathbf{p}^1)$, as required. ☐

Example 3.A.3. Consider the unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1)\}$ and the non-concave valuation of this type:

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
2	1	0	$x_2 = 0$
4	2	2	$x_2 = 1$

We show that we *cannot* break down the demand change from every $\mathbf{p}$ to every other $\mathbf{p}'$ in improving $\mathbb{Z}\mathcal{D}$-steps. Consider the price change from $(1,3)$ to $(1,1)$ (noting that $(1,3)$ is not a UDR price; $D_u(1,3) = \{(0,0),(1,0),(2,0)\}$). The bundle $(1,0)$ is in $D_u(1,3)$, but $D_u(1,1) = \{(0,1),(2,1)\}$. We cannot break down the change with this starting bundle without changing x_1. However, the only vectors in $\mathbb{Z}\mathcal{D}$ which would give a change in x_1 are integer multiples of $\pm(1,0)$. The price change we consider is $(0,-2)$ so a change in direction $\pm(1,0)$ would not obey the strict law of demand. Thus, it is impossible to break down the change in demand from price $(1,3)$ to $(1,1)$ in improving $\mathbb{Z}\mathcal{D}$-steps.

Proof of Theorem 3.5.16. We follow the technique of Veblen and Franklin (1921) to show that $\mathcal{D}^n_{ss}$ is unimodular. Note first that any vector $\mathbf{v} = \mathbf{e}^i - \mathbf{e}^j$ satisfies $\mathbf{v}.\mathbf{1} = 0$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$. So any set of n vectors that are all of the form $\mathbf{e}^i - \mathbf{e}^j$ does not have $\mathbf{1}$ in its span, so is not linearly independent and therefore has determinant 0. It follows that any set of linearly independent vectors in $\mathcal{D}^n_{ss}$ must include a coordinate vector $\mathbf{e}^i$ (or $-\mathbf{e}^i$). Now observe that the determinant of any matrix which has this set of vectors as its columns is non-zero (since the vectors are linearly independent), and also ± 1 times the $(n-1) \times (n-1)$ matrix formed when we delete row i and the column in which $\pm\mathbf{e}^i$ was placed. But since this $(n-1) \times (n-1)$ matrix therefore has non-zero determinant, its columns are linearly independent, and they are also vectors in $\mathcal{D}^{n-1}_{ss}$. So $\mathcal{D}^n_{ss}$ satisfies the determinant condition if $\mathcal{D}^{n-1}_{ss}$ does. But it is trivial that $\mathcal{D}^1_{ss}$ satisfies the condition so, by induction on n, $\mathcal{D}^n_{ss}$ is unimodular for all n.

We showed in the text that $\mathcal{D}^n_{ss}$ is maximal. □

Proof of Lemma 3.5.19. 1. If u is of some demand type $\mathcal{D}$ as described, then we can break down the demand change from any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$ to $\mathbf{p} + \epsilon\mathbf{e}^k$ in improving $\mathbb{Z}\mathcal{D}$-steps; that is, in each case, $\mathbf{x}^{j+1} - \mathbf{x}^j = w\mathbf{v}$ for $w \in \mathbb{Z}_+$ and $\mathbf{v} \in \mathcal{D}$, and additionally, at each such step, $(\mathbf{x}^{j+1} - \mathbf{x}^j).\mathbf{e}^k < 0$. It follows that $v_k < 0$ for this $\mathbf{v}$ and so, by assumption, that either $-\mathbf{v} \in \mathbb{Z}^n_{\geq 0}$ or $\sum_{i=1}^n v_i = 0$; we conclude that $\sum_{i=1}^n v_i \leq 0$. Thus $\sum_i x_i^{j+1} \leq \sum_i x_i^j$, as required. Applying this at each $\mathbb{Z}\mathcal{D}$-step provides the ordinary law of aggregate demand.

Conversely, if u is not of some demand type $\mathcal{D}$ as described, then u must have a facet with normal $\mathbf{v}$ not satisfying the description given. Both $\mathbf{v}$ and $-\mathbf{v}$ are facet normals so without loss of generality assume that $\sum_{i=1}^n v_i > 0$ but that there exists $k \in \{1, \ldots, n\}$ such that $v_k < 0$. It follows that $\mathbf{e}^k.\mathbf{v} \neq 0$ so there exist prices $\mathbf{p}, \mathbf{p} + \epsilon\mathbf{e}^k$ in the UDRs on either side of this facet; by construction the ordinary law of aggregate demand fails for these prices.

Finally, if $\mathcal{D}$ is unimodular and u is concave then we can break the demand change from *any* $\mathbf{p}$ to any $\mathbf{p} + \epsilon\mathbf{e}^k$ in improving $\mathcal{D}$-steps, and so the complete law of aggregate demand holds.

2. Suppose u is of concave demand type $\mathbf{D} \subset \{-1, 0, 1\}^n$. Then, for any $\mathbf{p}$ and any good i, we can move from those points in $D_u(\mathbf{p})$ minimal for i to those points maximal for i along the edge vectors; as u is concave every lattice point on the edges in also in $D_u(\mathbf{p})$, so we can move in steps entirely from $\mathbf{D}$ while staying in $D_u(\mathbf{p})$; since $\mathbf{D} \subset \{-1, 0, 1\}^n$ this illustrates the consecutive integer property.

Suppose that u is not of any demand type $\mathcal{D} \subset \{-1, 0, 1\}^n$; it follows that $\mathcal{T}_u$ has a facet with normal v such that $\|v_i\| \geq 2$ for some good i. For any $\mathbf{p}$ in the interior of this facet, the consecutive integer property fails for good i. □

3.A.3 Proofs of Results in Section 3.6.1

Proof of Propositions 3.6.1 and 3.6.2. Proposition 3.6.1 is straightforward. Note that

$$\sum_{j\in J}\max_{\mathbf{x}^j\in A}\{u^j(\mathbf{x}^j)-\mathbf{p}.\mathbf{x}^j\} = \max\left\{\sum_{j\in J}u^j(\mathbf{x}^j)-\mathbf{p}.\left(\sum_{j\in J}\mathbf{x}^j\right)\ \mid\ \mathbf{x}^j\in A^j, j\in J\right\},$$

and on the other hand (since $\mathbf{y}\in A$ iff $\mathbf{y}=\sum_{j\in J}\mathbf{x}^j, \mathbf{x}^j\in A_J$) that

$$\begin{aligned}
&\max_{\mathbf{y}\in A}\{U(\mathbf{y})-\mathbf{p}.\mathbf{y}\}\\
&= \max\left\{\max\left\{\sum_{j\in J}u^j(\mathbf{x}^j)\mid \mathbf{x}^j\in A_j, \sum_{j\in J}\mathbf{x}^j=\mathbf{y}\right\}-\mathbf{p}.\mathbf{y}\mid \mathbf{y}=\sum_{j\in J}\mathbf{x}^j, \mathbf{x}^j\in A_j j\in J\right\}\\
&= \max\left\{\sum_{j\in J}u^j(\mathbf{x}^j)-\mathbf{p}.\left(\sum_{j\in J}\mathbf{x}^j\right)\ \mid\ \mathbf{x}^j\in A, j\in J\right\},
\end{aligned}$$

and that the same arguments $\mathbf{x}^j\in A$, with $\mathbf{y}=\sum_{j\in J}\mathbf{x}^j$, are maximising in either case.

The text showed the underlying sets of $\mathcal{T}_U$ and $\mathcal{T}_{\{u^j\}}$ are the same, so completing the proof of Proposition 3.6.2 only requires checking the weightings are the same. So suppose F is a facet of $\mathcal{T}_U$ with adjacent UDRs U and U'; let $\mathbf{v}_F$ be a primitive integer vector pointing from U to U'. Suppose agent j demands $\mathbf{x}^j$ in U and $\mathbf{x}^{j\prime}$ in U' (for some agent these will be distinct, but not necessarily for all). Then $w_j(F)\mathbf{v}_F=\mathbf{x}^{j\prime}-\mathbf{x}^j$ for all j, and so

$$\sum_j w_j(F)\mathbf{v}_F=\sum_j\mathbf{x}^{j\prime}-\sum_j\mathbf{x}^j.$$

So $w_U(F)=\sum_j w_j(F)=w_{\{u^j\}}(F)$, as required. □

Proof of Proposition 3.6.5. Suppose $G^{-1}\mathcal{D}$ always has a competitive equilibrium. Consider any agent valuations $u^1,\dots,u^k$ of type $\mathcal{D}$ and let $\mathbf{x}$ be in the domain of their aggregate valuation. Then demands $G^*u^1,\dots,G^*u^k$ have type $G^{-1}\mathcal{D}$ and $\mathbf{y}:=G^{-1}\mathbf{x}$ is in the domain of their aggregate valuation. By assumption competitive equilibrium exists in the latter case: there exists a price $\mathbf{p}$ at which the agent with valuation G^*u^i demands $\mathbf{y}^i$ and $\sum_i\mathbf{y}^i=\mathbf{y}$. But then in each case we may define $\mathbf{x}^i:=G\mathbf{y}^i\in D_{u^i}(G^{-T}\mathbf{p})$ (see Proposition 3.4.14.1). At price $G^{-T}\mathbf{p}$ the market clears for $\mathbf{x}:=\sum_i\mathbf{x}^i$. So $\mathcal{D}$ has a competitive equilibrium. The converse is shown by repeating the argument, using the unimodular matrix G^{-T}. □

3.A.4 Proof of results in Section 3.6.3

This Appendix gives the additional details needed to complete the proof of Theorem 3.6.8. Proposition 3.A.4 shows that, for "nice" intersections, the condition of unimodularity is necessary and sufficient for competitive equilibrium to always exist.

The second half of the proof of the sufficiency part of the theorem shows that generically all TH intersections are "nice", and that any non-"nice" intersection is therefore close enough to being a "nice" intersection that Theorem 3.6.8's condition still suffices.

Lemmas 3.A.5, 3.A.8 and 3.A.9 demonstrate that generically all single-point intersections of the TH are "nice". The logic is as follows: first (Lemma 3.A.5), we show how to perform affine translations of agents' THs, and bound the associated change in valuation. Now consider an intersection of two cells from distinct agents' THs. Generically (in the space of affine translations) there can be no vector normal to both; if there were, a small shift of one of the agents' demands in the direction of this vector would mean the cells no longer intersected at all. We argue thus in Lemma 3.A.8.

In Lemma 3.A.9 we show how to make all intersections 'nice', while bounding the change in any agent's valuation. Begin by considering an intersection of two cells from distinct agents' THs. If necessary, make small shifts as described by Lemma 3.A.8. Now, for each of the two cells that intersect, we nominate a linearly independent set of vectors normal to adjacent facets. The fact that there is no vector normal to both the cells means that the union of these sets remains linearly independent. But the intersection of the two cells is now a cell of the TH of the aggregate demand of the two agents, and the collection of vectors we have defined so far are normal to facets in this TH whose intersection is this new cell. Continuing to add any additional agents' demands that intersect the cell generically, we can construct a set of linearly independent vectors, each normal to a facet of the TH of aggregate demand, such that the intersection of these facets locally defines the intersection of the cells in question.

After these small perturbations, any bundle is demanded at some price (by Proposition 3.A.4). We complete the proof of Theorem 3.6.8 by showing that, if a bundle is demanded following an extremely small perturbation in agents' valuations, it must have also been demanded before this perturbation.[105] This proves the sufficiency of unimodularity (with concavity) for Theorem 3.6.8.

Proposition 3.A.4. *Suppose price* $\mathbf{p}$ *is in the interior of an* $(n-k_i)$*-cell* C_i *of the TH* $\mathcal{T}_{u^i}$

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[105] In more detail: consider an integer bundle that is "hidden" in the convex hull of aggregate demand at a price point in a *not*-nice intersection. If it is not demanded at this price, agents' aggregate utility from this bundle, at this price vector, must be strictly lower than their aggregate utility from any bundle that is demanded at this price. Since this bundle is a convex combination of other bundles that are demanded at this price vector, the aggregate valuation from the bundle in question is strictly lower than the same convex combination of the aggregate valuations of these other bundles. Let this aggregate valuation difference be ϵ.

Now consider perturbing all agents' valuation functions by arbitrarily small amounts, so that their TH undergoes a small translation in price space. It is straightforward, although somewhat tedious, to show that generically all the TH intersections are now "nice". So we can choose these small perturbations so this holds; additionally, we ensure that no agent's valuation of any available bundle is affected by more than $\frac{\epsilon}{3m}$, in which m is the number of agents present.

If $\mathcal{D}$ is concave and unimodular, the bundle in question is (by Proposition 3.A.4) demanded by agents with the perturbed valuation functions at some price. But the perturbation of the valuation functions cannot change the aggregate valuation from either this bundle, or the same convex combination of the aggregate valuation of the other bundles, by more than $\epsilon/3$. So the aggregate valuation from this bundle is still below the same convex combination of the aggregate valuation of the other bundles, and therefore the aggregate utility of this bundle is also still below the same convex combination of the aggregate utility of the other bundles at any prices (since at any prices, the cost of this bundle equals this convex combination of the cost of the other bundles). So we have a contradiction, and the lattice point must have been demanded at the original price point. That is, Theorem 3.6.8's condition is also sufficient for non-"nice" intersections.

of each of s agents $i = 1, \ldots, s$, who have concave valuations u^i, and together have aggregate valuation $\tilde{U}$. Then every integer bundle in $\operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$ *is demanded at* $\mathbf{p}$ *if each C_i is a subset of the intersection of a set of facets $F^i_1, \ldots, F^i_{k_i}$ of $\mathcal{T}_{u^i}$ (not necessarily comprising all facets of $\mathcal{T}_{u^i}$ that pass through C_i) with primitive integer normal vectors $\mathbf{v}^i_1, \ldots, \mathbf{v}^i_{k_i}$ and $\{\mathbf{v}^i_j \mid i = 1, \ldots, s; j = 1, \ldots, k_i\}$ are unimodular.*

Proof. All bundles demanded by agent i at $\mathbf{p}$ are demanded throughout the $(n - k_i)$-cell C_i, which corresponds to a k_i-dimensional polytope Δ_i in the SNP of agent i. Moreover, Δ_i possesses an edge in direction $\mathbf{v}^i_j$ for $j = 1, \ldots, k_i$; each corresponds to the facet F^i_j. Thus, if $\mathbf{y}^i$ is some integer bundle in $D_{u^i}(\mathbf{p})$, then (by a dimension count) the affine span of Δ_i is precisely $\left\{\mathbf{y}^i + \sum_{j=1}^{k_i} \beta^i_j \mathbf{v}^i_j \mid \beta^i_j \in \mathbb{R} \text{ for } j = 1, \ldots, k_i\right\}$, and in particular, $D_{u^i}(\mathbf{p})$ is contained in this set.

Thus, using equation (3.4) we may express aggregate demand among these agents as $D_{\tilde{U}}(\mathbf{p}) = \left\{\mathbf{y} + \sum_{i=1}^{s} \sum_{j=1}^{k_i} a^i_j \mathbf{v}^i_j \mid \mathbf{y}^i + \sum_{j=1}^{k_i} a^i_j \mathbf{v}^i_j \in D_{u^i}(\mathbf{p}) \text{ for } i = 1, \ldots, s\right\}$, where $\mathbf{y} := \sum_{i=1}^{s} \mathbf{y}^i$.

Now, suppose $\mathbf{x}$ is an integer bundle in $\operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$. Then $\mathbf{x} - \mathbf{y}$ is in the span of the $\mathbf{v}^i_j$. But since they are an integer basis for their span, we can write $\mathbf{x} - \mathbf{y} = \sum_{i=1}^{s} \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j$, for some $b^i_j \in \mathbb{Z}$. So we can define $\mathbf{x}^i := \mathbf{y}^i + \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j$, and know that $\mathbf{x}^i \in \mathbb{Z}^n$.

But we also know $\mathbf{x}^i \in \operatorname{Conv} D_{u^i}(\mathbf{p})$. To see this, observe that since $\mathbf{x} \in \operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$, we can write $\mathbf{x} - \mathbf{y} = \sum_\beta \sum_{i=1}^{s} \sum_{j=1}^{k_i} \lambda_\beta a^i_{j,\beta} \mathbf{v}^i_j$ for some finite set of weights $\lambda_\beta \in [0, 1]$ such that $\sum_\beta \lambda_\beta = 1$ and such that $\mathbf{y}^i + \sum_{j=1}^{k_i} a^i_{j,\beta} \mathbf{v}^i_j \in D_{u^i}(\mathbf{p})$ for each agent i and for each β. But since the $\mathbf{v}^i_j$ are linearly independent, there is an unique way to write $\mathbf{x} - \mathbf{y}$ as a weighted sum of the $\mathbf{v}^i_j$, so $b^i_j = \sum_\beta \lambda_\beta a^i_{j,\beta}$, and so $\mathbf{x}^i = \mathbf{y}^i + \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j = \mathbf{y}^i + \sum_{j=1}^{k_i} \sum_\beta \lambda_\beta a^i_{j,\beta} \mathbf{v}^i_j \in \operatorname{Conv} D_{u^i}(\mathbf{p})$.

So $\mathbf{x}^i$ is an integer vector in $\operatorname{Conv} D_{u^i}(\mathbf{p})$. By concavity of u^i there exists *some* price at which $\mathbf{x}^i$ is demanded by agent i (Lemma 3.2.5), and so by Lemma 3.2.6 we know $\mathbf{x}^i \in D_{u^i}(\mathbf{p})$. Thus $\mathbf{x} = \sum_{i=1}^{s} \mathbf{x}^i \in D_{\tilde{U}}(\mathbf{p})$. That is, $\mathbf{x}$ is demanded at $\mathbf{p}$, as required. □

We introduce the affine perturbations discussed above.

Lemma 3.A.5. *Suppose an agent has valuation function $u : A \to \mathbb{R}$. For any $\mathbf{w} \in \mathbb{R}^n$, we may define a valuation function $u_{\mathbf{w}} : A \to \mathbb{R}$ such that, for all $\mathbf{p} \in \mathbb{R}^n$, we have*

1. $D_{u_{\mathbf{w}}}(\mathbf{p}) = D_u(\mathbf{p} + \mathbf{w})$;
2. $\mathcal{T}_{u_{\mathbf{w}}} = \{\mathbf{p} - \mathbf{w} \mid \mathbf{p} \in \mathcal{T}_u\}$;
3. $\|u_{\mathbf{w}}(\mathbf{x}) - u(\mathbf{x})\| \leq R\|\mathbf{w}\|$, *where R satisfies $\|\mathbf{x}\| < R$ for all $\mathbf{x} \in A$.*

Proof. Let $u_{\mathbf{w}}(\mathbf{x}) = u(\mathbf{x}) - \mathbf{x}.\mathbf{w}$. Then

$$D_{u_{\mathbf{w}}}(\mathbf{p}) = \operatorname*{arg\,max}_{\mathbf{x} \in A}\{u(\mathbf{x}) - \mathbf{x}.\mathbf{w} - \mathbf{x}.\mathbf{p}\} = \operatorname*{arg\,max}_{\mathbf{x} \in A}\{u(\mathbf{x}) - \mathbf{x}.(\mathbf{p} + \mathbf{w})\} = D_u(\mathbf{p} + \mathbf{w}).$$

The remainder of the lemma follows by definition of $\mathcal{T}_u$, and the Cauchy-Schwarz inequality. □

To prove that the hypotheses of Proposition 3.A.4 are satisfied after such perturbations, is convenient to use “annihilator spaces”. For a linear or affine subspace of $\mathbb{R}^n$, these give the linear subspace of all orthogonal vectors. We recall their definition and basic properties.

Definition 3.A.6 (See e.g. Spence et al., 2000)**.** If $C \subseteq \mathbb{R}^n$ is an affine subspace, define

$$C^\circ := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}.(\mathbf{c} - \mathbf{c}') = 0,\ \forall \mathbf{c}, \mathbf{c}' \in C\}.[106]$$

Note that if $D = C + \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^n$ then $D^\circ = C^\circ$.

We use annihilator spaces for the following results.

Lemma 3.A.7 (See e.g. Spence et al., 2000)**.** *Suppose that $C_1, C_2 \subseteq \mathbb{R}^n$ are affine subspaces.*

1. *If $C_1 \subseteq C_2$ then $C_2^\circ \subseteq C_1^\circ$*
2. *If $C_1 \cap C_2 \neq \emptyset$ then additionally $(C_1 \cap C_2)^\circ = C_1^\circ + C_2^\circ$.*
3. $\dim C_1 + \dim(C_1)^\circ = n$

Proof. Part 1 is clear. Part 2 follows from the standard result when C_1 and C_2 are linear subspaces (see, e.g. Spence et al. 2000): if $-\mathbf{w} \in C_1 \cap C_2$ then $C_1 + \mathbf{w}$ and $C_2 + \mathbf{w}$ are linear subspaces, and so $((C_1 + \mathbf{w}) \cap (C_2 + \mathbf{w}))^\circ = (C_1 + \mathbf{w})^\circ + (C_2 + \mathbf{w})^\circ$. But $(C_1 + \mathbf{w}) \cap (C_2 + \mathbf{w}) = (C_1 \cap C_2) + \mathbf{w}$ so the result follows from the note above. Part 3 similarly follows immediately from the linear case. □

Now we show that any two THs may be perturbed so that the intersection of their cells is ‘generic’ (as given in the statement of the following lemma):

Lemma 3.A.8. *Suppose we have agents* 1 *and* 2 *with valuation functions u^1 and u^2 (not necessarily concave). For any $\epsilon > 0$ we may find a vector $\mathbf{w}$ such that, if we perturb agent 2’s demand by $\mathbf{w}$ to obtain $u^2_{\mathbf{w}}$, then $\|u^2_{\mathbf{w}}(\mathbf{x}) - u^2(\mathbf{x})\| < \epsilon$ for all $\mathbf{x} \in A$, and any cells C_1 of $\mathcal{T}_{u^1}$ and $C_2^{\mathbf{w}}$ of $\mathcal{T}_{u^2_{\mathbf{w}}}$ satisfy $C_1 \cap C_2^{\mathbf{w}} \neq \emptyset \Rightarrow C_1^\circ \cap (C_2^{\mathbf{w}})^\circ = \{0\}$.*

Proof. Suppose that C_1 in $\mathcal{T}_{u^1}$ and C_2 in $\mathcal{T}_{u^2}$ satisfy $C_1 \cap C_2 \neq \emptyset$ and $C_1^\circ \cap C_2^\circ \neq \{0\}$. Choose $\mathbf{w}_1 \in C_1^\circ \cap C_2^\circ$ with $\mathbf{w}_1 \neq 0$. Then, for all $\eta > 0$, we show that $(C_2 + \eta\mathbf{w}_1) \cap C_1 = \emptyset$. For, given any $c_2 \in C_2$, if $c_1 \in C_1 \cap C_2$ then $\mathbf{w}_1.(c_1 - (c_2 + \eta\mathbf{w}_1)) = \eta\|\mathbf{w}_1\|^2 \neq 0$ (since $c_1, c_2 \in C_2$) and so, since $\mathbf{w}_1 \in C_1^\circ$, it follows that $c_2 + \eta\mathbf{w}_1 \notin C_1$.

On the other hand, recall that the cells of THs are closed objects. It follows that a sufficiently small perturbation of one of the THs will not introduce any new intersections between cells. So there exists $\eta_1 > 0$ such that if $\eta < \eta_1$ then no new intersections arise.[107]

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[106] If $D \leq \mathbb{R}^n$ is a linear subspace then this definition clearly coincides with the usual $D^\circ := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}.\mathbf{d} = 0\ \forall \mathbf{d} \in D\}$.

[107] To be precise: if $C_{2a}^{\mathbf{w}}$ in $\mathcal{T}_{u^2_{\mathbf{w}}}$ satisfies $C_{2a}^{\mathbf{w}} \cap C_{1a} \neq \emptyset$ for any cell C_{1a} in $\mathcal{T}_{u^1}$ then the corresponding C_{2a} in $\mathcal{T}_{u^2}$ satisfies $C_{2a} \cap C_{1a} \neq \emptyset$.

Since THs consist of a finite number of affine cells, we may suppose that there are in total d intersections of cells in $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ whose annihilator spaces have non-zero intersection. We find $\mathbf{w}_j$ and η_j as above for each in turn, and apply them all.[108] Thus, perturbing Agent 2 by $\mathbf{w} = \eta\mathbf{v}$, where $\mathbf{v} = \sum_{j=1}^{d} \eta_j \|\mathbf{w}_j\|$ and $\eta \in (0, 1]$, gives us the intersection properties required. To ensure that the perturbation to the agent's valuation is sufficiently small, we choose $\eta < \frac{\epsilon}{R\|\mathbf{v}\|}$ where R satisfies $\|\mathbf{x}\| < R$ for all $\mathbf{x} \in A$. By Lemma 3.A.5.3, this implies that $\|u^2_{\mathbf{w}}(\mathbf{x}) - u^2(\mathbf{x})\| < \epsilon$ for all $\mathbf{x} \in A$, as required. □

We may now take a set of m agents, and shift each agent's demand so that its valuation for any bundle is changed by at most ϵ, and nearly all the conditions of Proposition 3.A.4 are met at every intersection of the THs. The only condition we do not insist on is that the set of primitive integer facet normals are unimodular; whether or not this could possibly hold will depend on the demand types of the agents in question. What we prove is that these vectors are linearly independent.

Lemma 3.A.9. *Suppose we have m agents, with valuations u^i for $i = 1, \ldots, m$. For every $\epsilon > 0$ we may perturb each agent's valuation by a vector $\mathbf{w}^i$ such that $\|u^i(\mathbf{x}) - u^i_{\mathbf{w}^i}(\mathbf{x})\| < \epsilon$ for all $\mathbf{x}$ in $\mathbb{R}$, and such that, whenever a price point $\mathbf{p}$ is in the interior of $(n - k_{i_j})$-cell C_{i_j} of the TH $\mathcal{T}_{u^{i_j}}$ for agents $i_1, \ldots, i_s$, then each C_{i_j} is locally to $\mathbf{p}$, given by the intersection of a set of facets $F^{i_j}_1, \ldots, F^{i_j}_{k_{i_j}}$ of $\mathcal{T}_{u^{i_j}}$ (not necessarily comprising all facets of $\mathcal{T}_{u^{i_j}}$ that pass through C_{i_j}) with primitive integer normal vectors $\mathbf{v}^{i_j}_1, \ldots \mathbf{v}^{i_j}_{k_{i_j}}$, such that the full set $\{\mathbf{v}^{i_j}_l \mid j = 1, \ldots, s; l = 1, \ldots, k_{i_j}\}$ is linearly independent.*

Proof. We make a series of perturbations of agents' individual demands, as in Lemmas 3.A.5 and 3.A.8. First, we allow Agent 1 to remain unperturbed. For $i = 2, \ldots, m$ we compare:

1. the TH of aggregate demand of agents $1, \ldots, i - 1$;
2. the TH of agent i.

In each case, we apply Lemma 3.A.8 to find $\mathbf{w}^i$ with $\|u^i(\mathbf{x}) - u^i_{\mathbf{w}^i}(\mathbf{x})\| < \epsilon$, and such that, after the perturbation, $C^\circ_i \cap C^\circ = \{0\}$ whenever $C_i \cap C \neq \emptyset$, where C_i is any cell in $\mathcal{T}_{u^i}$ and C is any cell in the TH of aggregate demand of agents $1, \ldots, i - 1$.

Write U' for the new aggregate demand, after all agents have been perturbed. Now we need to see that the hypotheses of Proposition 3.A.4 are satisfied at every intersection of individual perturbed THs that make up $\mathcal{T}_{U'}$. Consider a price point $\mathbf{p}$, which lies in the interior of $(n-k_{i_j})$-cells $C_{i_1}, \ldots, C_{i_s}$ of the THs of individual demand from distinct agents $i_1, \ldots, i_s$ respectively, where we index so that $i_1 < \cdots < i_s$. From Lemma 3.A.7.3 we know that $\dim C^\circ_{i_j}$ is k_{i_j}.

Let $C := \bigcap_{j=1}^{s} C_{i_j}$. By Lemma 3.A.7.2, we know that $C^\circ = \left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ + C^\circ_{i_s}$. On the other hand, $\mathbf{p} \in \bigcap_{j=1}^{s-1} C_{i_j}$, and so there is a cell C' of the tropical variety of aggregate demand

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[108] Strictly speaking, each η_j should be found when we compare the cells after $\mathcal{T}_{u^2}$ has undergone the translations corresponding to intersections $1, \ldots, j - 1$.

of agents $1, \dots, i_s - 1$, with $\mathbf{p} \in C'$. Since demand is constant in the interior of a cell, it follows that $C' \subseteq C_{i_j}$ for $j = 1, \dots, i_{s-1}$ and so $C' \subseteq \bigcap_{j=1}^{s-1} C_{i_j}$. We know that $\mathbf{p} \in C_{i_s} \cap C'$ and so, by the construction of the perturbations, we know $C'^{\circ} \cap C_{i_s}^{\circ} = \{0\}$. As $C' \subseteq \bigcap_{j=1}^{s-1} C_{i_j}$, it follows by Lemma 3.A.7.1 that $\left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^{\circ} \subseteq C'^{\circ}$, so we may conclude that $\left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^{\circ} \cap C_{i_s}^{\circ} = \{0\}$. Thus

$$C^{\circ} = \left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^{\circ} \oplus C_{i_s}^{\circ}.$$

Proceeding inductively

$$C^{\circ} = C_{i_1}^{\circ} \oplus \cdots \oplus C_{i_s}^{\circ}.$$

We conclude in particular: if $\mathbf{v}_1^{i_j}, \dots, \mathbf{v}_{k_{i_j}}^{i_j}$ are a basis for $C_{i_j}^{\circ}$ then $\{\mathbf{v}_l^{i_j} \mid l = 1, \dots, k_{i_j}; j = 1, \dots, s\}$ is a set of linearly independent vectors.

But if $C_{i_j} = \bigcap_l F_l^{i_j}$ where $F_l^{i_j}$ are all the facets of this agent's TH of demand which contain C_{i_j} in their boundary, then applying Lemma 3.A.7.2 again, $C_{i_j}^{\circ}$ is the sum of the spaces $(F_l^{i_j})^{\circ}$. Each $(F_l^{i_j})^{\circ}$ is spanned by a single vector $\mathbf{v}_l^{i_j}$, which we may choose to be a primitive integer vector. We may select a maximal linearly independent subset of these vectors, and re-index so these are $\{\mathbf{v}_l^{i_j} \mid l = 1, \dots, k_{i_j}\}$. Then $C_{i_j}^{\circ} = \bigoplus_{l=1}^{k_{i_j}} (F_l^{i_j})^{\circ}$. We already know that $C_{i_j} \subseteq \bigcap_{l=1}^{k_{i_j}} F_l^{i_j}$ so it follows (by Lemma 3.A.7.2) that the affine spans of C_{i_j} and $\bigcap_{l=1}^{k_{i_j}} F_l^{i_j}$ coincide. It follows that C_{i_j} is given, locally around $\mathbf{p}$, by the intersection of the facets $F_1^{i_j}, \dots, F_{k_{i_j}}^{i_j}$; these facets were chosen above such that their normal vectors are linearly independent. □

We now have the technical results we need to prove Theorem 3.6.8.

Proof of Theorem 3.6.8 Proposition 3.6.10 covers the case in which condition of the theorem is not satisfied. So suppose that the condition is satisfied. Suppose we have m agents and for $j = 1, \dots, m$ their valuation is $u^j : A_j \to \mathbb{R}$; write $U : A \to \mathbb{R}$ for the aggregate valuation (as in Section 3.6.1). We have the tropical variety $\mathcal{T}_U$ of aggregate demand, and the corresponding SNP.

This SNP provides a subdivision of Conv(A). Our bundle $\mathbf{x}$ may lie at a vertex of the subdivision, in which case there exists a price vector at which it is uniquely demanded. If not, it lies in some k-face of the SNP for some $k \neq 0$. Let $\Delta_{\mathbf{x}}$ be one such k-face. Let $\mathbf{p}_{\mathbf{x}} \in \mathbb{R}^n$ be a price in the corresponding $(n-k)$-cell $C_{\mathbf{x}}$ of aggregate demand. The set $\{\mathbf{y}^{\beta} \mid \beta \in B\}$ of vertices of $\Delta_{\mathbf{x}}$ are the bundles which are uniquely demanded in an open (n-dimensional) region of $\mathbb{R}^n$ with $C_{\mathbf{x}}$ in its boundary. By assumption there exist $\lambda_{\beta} \in [0, 1]$ with $\sum_{\beta} \lambda_{\beta} = 1$ such that $\mathbf{x} = \sum_{\beta} \lambda_{\beta} \mathbf{y}^{\beta}$.

Suppose that $\mathbf{x}$ is not demanded on aggregate at any price. Then, as in the proof of Lemma 3.2.6, it must follows that $U(\mathbf{x}) < \sum_{\beta} \lambda_{\beta} U(\mathbf{y}^{\beta})$.

Pick ϵ so that

$$U(\mathbf{x}) < \sum_{\beta} \lambda_{\beta} U(\mathbf{y}^{\beta}) - \epsilon.$$

Now apply Lemma 3.A.9, perturbing agents $j = 2, \ldots, m$ so that their valuation function is altered by no more than $\frac{\epsilon}{3m}$, where we recall that m is the number of agents present. It follows, by assumption regarding the demand type $\mathcal{D}$, that the conditions of Proposition 3.A.4 are satisfied at any intersection of agents' demands. Let U' be the new aggregate demand.

Now $\mathbf{x}$ lies in some k-face of the SNP of this new aggregate demand U', which corresponds to some $(n-k)$-cell of $\mathcal{T}_{U'}$. Let price $\mathbf{p}' \in \mathbb{R}^n$ be in this cell. By Proposition 3.A.4, it follows that $\mathbf{x} \in D_{U'}(\mathbf{p}')$.

However, $\mathbf{x} \in D_{U'}(\mathbf{p}')$ means that $\mathbf{x}$ is weakly preferred on aggregate to any other bundle – including all those in our original vertex set $\{\mathbf{y}^\beta\}$. So, for each $\beta \in B$, we have

$$U'(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq U'(\mathbf{y}^\beta) - \mathbf{y}^\beta.\mathbf{p}'. \tag{3.6}$$

But $U'(\mathbf{x}) = \sum_{j=1}^m (u^j)'(\mathbf{x}^j)$, where $\mathbf{x}^j \in A_j$ is the bundle accorded to agent j under this optimal allocation (in particular $\sum_j \mathbf{x}^j = \mathbf{x}$) and $(u^j)'$ is the agent's perturbed valuation function. So

$$\|U'(\mathbf{x}) - U(\mathbf{x})\| = \|\sum_{j=1}^m [(u^j)'(\mathbf{x}^j) - u^j(\mathbf{x}^j)]\| \leq \sum_{j=1}^m \|(u^j)'(\mathbf{x}^j) - u^j(\mathbf{x}^j)\| \leq m \cdot \frac{\epsilon}{3m} = \frac{\epsilon}{3}$$

and hence $U(\mathbf{x}) + \frac{\epsilon}{3} \geq U'(\mathbf{x})$. Similarly, for all $\beta \in B$, we have $\|U'(\mathbf{y}^\beta) - U(\mathbf{y}^\beta)\| \leq \frac{\epsilon}{3}$ and so $U'(\mathbf{y}^\beta) \geq U(\mathbf{y}^\beta) - \frac{\epsilon}{3}$. Putting these facts together in line (3.6) we find:

$$U(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq U(\mathbf{y}^\beta) - \mathbf{y}^\beta.\mathbf{p}' - \frac{2\epsilon}{3}.$$

Since this holds for all vertices $\mathbf{y}^\beta$ of our original k-face $\Delta_{\mathbf{x}}$ of the SNP, it follows that we may take a weighted sum, using the same weights as originally identified:

$$U(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \sum_\beta \lambda_\beta \mathbf{y}^\beta.\mathbf{p}' - \frac{2\epsilon}{3} \implies U(\mathbf{x}) \geq \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \frac{2\epsilon}{3}.$$

But we originally chose ϵ to satisfy $U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \epsilon$. This contradiction completes the proof. □

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