

On the Equivalence of Youla, System-level and Input-output Parameterizations

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Abstract—A convex parameterization of internally stabilizing controllers is fundamental for many controller synthesis procedures. The celebrated Youla parameterization relies on a doubly-coprime factorization of the system, while the recent system-level and input-output parameterizations require no doubly-coprime factorization, but a set of equality constraints for achievable closed-loop responses. In this paper, we present explicit affine mappings among Youla, system-level and input-output parameterizations. Two direct implications of these affine mappings are 1) any convex problem in the Youla, system level, or input-output parameters can be equivalently and convexly formulated in any other one of these frameworks, including the convex system-level synthesis (SLS); 2) the condition of Quadratic Invariance (QI) is sufficient and necessary for the classical distributed control problem to admit an equivalent convex reformulation in terms of either Youla, system-level, or input-output parameters.

Index Terms—Stabilizing controller, Youla parameterization, System-level synthesis, Quadratic invariance.

I. INTRODUCTION

One of the most fundamental problems in control theory is to design a feedback controller that stabilizes a dynamical system. Additionally, one can further design an optimal controller by optimizing a certain performance measure [1]. It is well-known that the set of stabilizing controllers is in general non-convex, and hence, hard to optimize directly over. Many optimal controller synthesis procedures first parameterize all stabilizing controllers and the corresponding closed-loop responses in a convex way, and then minimize relevant performance measures over the new parameter(s).

For finite dimensional linear-time-invariant (LTI) systems, the set of LTI stabilizing feedback controllers is fully characterized by the celebrated *Youla parameterization* [2], where a doubly coprime factorization of the system is used. In [2], it is shown that the Youla parameterization allows for optimizing the Youla parameter (or system response) directly, instead of the controller itself, leading to a convex problem. Also, customized performance specifications on the closed-loop system can be incorporated with the Youla parameterization via convex optimization [3]. Moreover, the foundational results of robust and optimal control are built on the Youla parameterization [4], [1]. Note that a doubly-coprime factorization of the system must be computed as a preliminary step in the Youla parameterization. Recently, a system-level parameterization [5] and an input-output parameterization [6] were introduced to characterize the set of all LTI

stabilizing controllers, with no need of computing a doubly-coprime factorization of the system *a priori*. Similar to Youla, the system-level and input-output parameterizations treat certain closed-loop responses as design variables. The controller synthesis is thus shifted from designing a controller to designing the closed loop responses directly. We note that this idea of synthesizing closed-loop responses in a convex way was extensively discussed as *closed-loop convexity* in [3, Chapter 6].

The Youla [2], system-level [5] and input-output [6] parameterizations are equivalent since they characterize the same set of stabilizing controllers. However, their explicit relationships have not been fully established before. The main objective of this paper is to reveal an explicit equivalence of Youla, system-level, and input-output parameterizations. In particular, we present explicit *affine mappings* between the Youla parameter, system-level parameters, and input-output parameters. One direct consequence is that any convex problem in terms of the Youla, system-level, input-output parameters can be equivalently and convexly formulated into any other one of these three frameworks. Therefore, the so-called convex system-level synthesis (SLS) [5], [7] admits an equivalent convex formulation in terms of the Youla or input-output parameters. Another consequence is that if one controller synthesis task does not allow for an equivalent convex reformulation in Youla, a convex reformulation in the system-level or input-output parameterizations is not possible either. Consider the classical distributed controller synthesis task where a subspace constraint is imposed on the controller [8]. It has been shown that a notion of quadratic invariance (QI) is sufficient and necessary for the distributed control problem to admit an equivalent convex reformulation in the Youla parameter [8], [9]. Accordingly, the QI condition is also sufficient and necessary when using the system-level and input-output parameterizations. For systems with constraints beyond QI, we highlight that a notion of sparsity invariance (SI) [10] can be used to derive convex inner-approximations using the Youla, system-level, or input-output characterizations. We note that the SLS in [5], [7] imposes structural constraints on certain system responses directly, instead of the controller, leading to a different problem compared to the classical decentralized controller synthesis.

The rest of this paper is organized as follows. We introduce some preliminaries on LTI systems and optimal control in Section II, and review the Youla, system-level, and input-output parameterizations in Section III. Explicit affine relationships and their implication with QI are presented in Section IV. We discuss distributed controller synthesis with non-QI constraints in Section V, and conclude the paper in Section VI.

Notation: We use lower and upper case letters (e.g. x and A) to denote vectors and matrices, respectively. Lower and upper case boldface letters (e.g. $\mathbf{x}$ and $\mathbf{G}$) are used to denote signals and transfer matrices, respectively. For clarity, we consider discrete-time LTI systems only, but unless stated otherwise, all results can be extended to the continuous-time setting. We denote the set of real-rational proper stable transfer matrices as $\mathcal{RH}_\infty$. Finally, $\mathbf{G} \in \frac{1}{z}\mathcal{RH}_\infty$ means $\mathbf{G}$ is stable and strictly proper.

II. PRELIMINARIES

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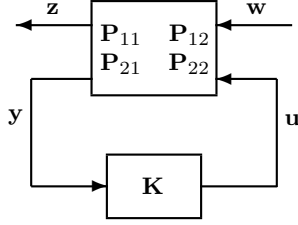


Fig. 1: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

A. System model

We consider discrete-time LTI systems of the form

$$\begin{aligned} x[t+1] &= Ax[t] + B_1w[t] + B_2u[t], \\ z[t] &= C_1x[t] + D_{11}w[t] + D_{12}u[t], \\ y[t] &= C_2x[t] + D_{21}w[t] + D_{22}u[t], \end{aligned} \quad (1)$$

where $x[t]$, $u[t]$, $w[t]$, $y[t]$, $z[t]$ are the state vector, control action, external disturbance, measurement, and regulated output at time t , respectively. System (1) can be written as

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix},$$

where $\mathbf{P}_{ij} = C_i(zI - A)^{-1}B_j + D_{ij}$. We refer to $\mathbf{P}$ as the open-loop plant model.

Consider a dynamic output feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K}$ has a state-space realization

$$\begin{aligned} \xi[t+1] &= A_k\xi[t] + B_ky[t], \\ u[t] &= C_k\xi[t] + D_ky[t], \end{aligned} \quad (2)$$

where ξ is the internal state of controller $\mathbf{K}$. We have $\mathbf{K} = C_k(zI - A_k)^{-1}B_k + D_k$. Figure 1 shows a schematic diagram of the interconnection of plant $\mathbf{P}$ and controller $\mathbf{K}$. Throughout the paper, we make the following standard assumptions.

Assumption 1: Both the plant and controller realizations are stabilizable and detectable, i.e., (A, B_2) and (A_k, B_k) are stabilizable, and (A, C_2) and (A_k, C_k) are detectable.

Assumption 2: The interconnection in Fig. 1 is well-posed, i.e., $I - D_{22}D_k$ is invertible.

B. Stabilization and optimal control

We now introduce the definition of internal stability of the system in Fig. 1.

Definition 1: The system in Fig. 1 is *internally stable* if it is well-posed, and the states $(x[t], x_k[t])$ converge to zero as $t \rightarrow \infty$ for all initial states $(x[0], x_k[0])$ when $w[t] = 0, \forall t$.

We say the controller $\mathbf{K}$ *internally stabilizes* the plant $\mathbf{P}$ if the interconnected system in Fig. 1 is internally stable. The set of all stabilizing controllers is defined as

$$\mathcal{C}_{\text{stab}} := \{\mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{P}\}. \quad (3)$$

In addition to stability, it is desirable to find a controller $\mathbf{K}$ that minimizes a suitable norm (e.g., $\mathcal{H}_2$ or $\mathcal{H}_\infty$) of the closed-loop transfer matrix from $\mathbf{w}$ to $\mathbf{z}$. This amounts to solving the following optimal control formulation [1]:

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|f(\mathbf{P}, \mathbf{K})\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}, \end{aligned} \quad (4)$$

where $f(\mathbf{P}, \mathbf{K}) = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}$. It is known that the set $\mathcal{C}_{\text{stab}}$ is non-convex. One can construct explicit examples where $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{C}_{\text{stab}}$ but $\frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2) \notin \mathcal{C}_{\text{stab}}$. Also, $f(\mathbf{P}, \mathbf{K})$

is in general a non-convex function of $\mathbf{K}$. Therefore, problem (4) is non-convex in the current form.

III. PARAMETERIZATION OF STABILIZING CONTROLLERS

To solve the optimal control problem (4), one common method is to derive an equivalent convex formulation via a suitable change of variables. A classical technique is the Youla parameterization [2]. Two recent approaches are the so-called system-level parameterization (SLP) [5], and the input-output parameterization (IOP) [6]. A common idea among these three approaches is to parameterize all stabilizing controllers in $\mathcal{C}_{\text{stab}}$ in terms of certain closed-loop responses. We review their main results in this section.

A. Youla parameterization

The classical Youla parameterization is based on a doubly-coprime factorization of the plant $\mathbf{P}_{22}$, defined as follows.

Definition 2: A collection of stable transfer matrices, $\mathbf{U}_l, \mathbf{V}_l, \mathbf{N}_l, \mathbf{M}_l, \mathbf{U}_r, \mathbf{V}_r, \mathbf{N}_r, \mathbf{M}_r \in \mathcal{RH}_\infty$ is called a doubly-coprime factorization of $\mathbf{P}_{22}$ if $\mathbf{P}_{22} = \mathbf{N}_r\mathbf{M}_r^{-1} = \mathbf{M}_l^{-1}\mathbf{N}_l$ and

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = \mathbf{I}.$$

Such doubly-coprime factorizations can always be computed if $\mathbf{P}_{22}$ is stabilizable and detectable [1]. The authors in [2] established the following equivalence

$$\mathcal{C}_{\text{stab}} = \{\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\}^1, \quad (5)$$

where $\mathbf{Q}$ is denoted as the Youla parameter. Using the change of variables $\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}$, one can derive

$$f(\mathbf{P}, \mathbf{K}) = \mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21},$$

where $\mathbf{T}_{11} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{V}_r\mathbf{M}_l\mathbf{P}_{21}$, $\mathbf{T}_{12} = -\mathbf{P}_{12}\mathbf{M}_r$, and $\mathbf{T}_{21} = \mathbf{M}_l\mathbf{P}_{21}$. Consequently, problem (4) can be equivalently reformulated in terms of the Youla parameter as

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty. \end{aligned} \quad (6)$$

One direct benefit is that (6) is convex with respect to the Youla parameter $\mathbf{Q}$.

B. System-level parameterization (SLP)

In [5], the authors proposed a system-level parameterization of $\mathcal{C}_{\text{stab}}$. This approach is based on the closed-loop maps from process and measurement disturbances to state and control action. In particular, assuming a strictly proper plant $\mathbf{P}_{22}$, i.e., $D_{22} = 0$, they use $\delta_x[t] = B_1w[t]$ to denote the disturbance on the state and $\delta_y[t] = D_{21}w[t]$ to denote the disturbance on the measurement. The dynamics of plant (1) can be accordingly written as

$$\begin{aligned} x[t+1] &= Ax[t] + B_2u[t] + \delta_x[t], \\ y[t] &= C_2x[t] + \delta_y[t]. \end{aligned}$$

Then, applying a stabilizing controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, the closed-loop system responses from perturbations (δ_x, δ_y) to $(\mathbf{x}, \mathbf{u})$ are

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}, \quad (7)$$

¹Equivalently, $\mathcal{C}_{\text{stab}} = \{(\mathbf{U}_l - \mathbf{Q}\mathbf{N}_l)^{-1}(\mathbf{V}_l - \mathbf{Q}\mathbf{M}_l) \mid \mathbf{Q} \in \mathcal{RH}_\infty\}$.

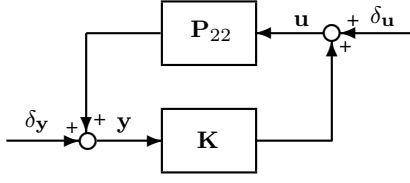


Fig. 2: Input-output stability.

where the system responses $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ lie in the following affine subspace [5]

$$[zI - A \quad -B_2] \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = [I \quad 0], \quad (8a)$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad (8b)$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \frac{1}{z}\mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty. \quad (8c)$$

In [5], it is proved that for a strictly proper² plant $\mathbf{P}_{22}$, the set of all internally stabilizing controllers can be written as

$$\mathcal{C}_{\text{stab}} = \{\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \mid \mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L} \text{ are in the affine subspace (8a)-(8c)}\}. \quad (9)$$

Also, the cost function $f(\mathbf{P}, \mathbf{K})$ can be expressed in terms of the system responses $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$. In particular, problem (4) can be equivalently reformulated as

$$\min_{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}} \left\| [C_1 \quad D_{12}] \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \quad (10)$$

subject to (8a) – (8c).

It is easy to see that (10) is convex in terms of $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$.

C. Input-output parameterization (IOP)

Recently, an input-output parameterization for $\mathcal{C}_{\text{stab}}$ was introduced in [6]. As shown in Fig. 2, the idea is based on a classical internal stability result in terms of the following closed-loop responses

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}, \quad (11)$$

where δ_u is a disturbance in the input, *i.e.*, $\mathbf{u} = \mathbf{K}\mathbf{y} + \delta_u$. Under Assumption 1, it is known that $\mathbf{K}$ internally stabilizes $\mathbf{P}$ if and only if the four transfer matrices in (11) are stable [4]. By applying a stabilizing controller $\mathbf{K}$, the closed-loop responses $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ lie in the following affine subspace [6]

$$[I \quad -\mathbf{P}_{22}] \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = [I \quad 0], \quad (12a)$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} -\mathbf{P}_{22} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad (12b)$$

$$\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \in \mathcal{RH}_\infty. \quad (12c)$$

It is shown in [6] that the set of all internally stabilizing controllers can be represented as

$$\mathcal{C}_{\text{stab}} = \{\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \mid \mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \text{ are in the affine subspace (12a)-(12c)}\}. \quad (13)$$

²The equivalence (9) only holds for strictly proper plants, *i.e.*, $D_{22} = 0$. For a general proper plant $D_{22} \neq 0$, the authors in [5] present another controller implementation that internally stabilizes the system; see [5, Section III.D]. Instead, the Youla (5) and input-output (13) parameterizations work for both strictly proper and general proper plants. Throughout the paper, we assume a strictly proper plant for the system-level parameterization.

Furthermore, we have $f(\mathbf{P}, \mathbf{K}) = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}$ [6]. Accordingly, problem (4) can be equivalently reformulated in terms of the system responses as

$$\min_{\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}} \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\| \quad (14)$$

subject to (12a) – (12c).

It is easy to see that (14) is convex in terms of $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$.

Remark 1: Both the SLP and the IOP can avoid the explicit computation of a doubly-coprime factorization of the plant $\mathbf{P}_{22}$. This feature offers potential convenience in some theoretical analysis, but may also introduce additional challenges in numerical computation (see Remark 3 for details). Compared to the SLP, two unique features of the IOP are: 1) the IOP (13) is directly suitable for general proper plants, while the SLP (9) works for strictly proper plants only; 2) the IOP works in frequency domain entirely, requiring no state-space realization of the plant. This second feature may allow the IOP to deal with non-rational and infinite-dimensional plants, for which a standard state-space representation required in the SLP is not available and a doubly-coprime factorization might not exist [11]; we refer the interested reader to [6] for more discussions.

IV. EXPLICIT EQUIVALENCE OF THE YOUNA PARAMETERIZATION, SLP, AND IOP

As discussed in the last section, the set of all stabilizing controllers $\mathcal{C}_{\text{stab}}$ can be parameterized in three different ways, *i.e.*, (5), (9), and (13), and the optimal control problem (4) admits three equivalent convex reformulations, *i.e.*, (6), (10), and (14). Implicitly, (5), (9), and (13) are equivalent. However, an explicit relationship between the Youla parameterization, the SLP, and the IOP is not clear from their definitions.

In this section, we present explicit *affine mappings* between the Youla parameterization, the SLP and the IOP. Important consequences are as follows: 1) any convex system-level synthesis (SLS) problem introduced by [5], [7] can be equivalently reformulated into a convex problem in terms of the Youla parameter $\mathbf{Q}$ or the input-output parameters $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$, and vice versa; 2) building on the explicit affine mappings, we show that the notion of quadratic invariance [8] allows for equivalent convex reformulations of classical distributed optimal control in either the Youla parameterization, SLP, or IOP.

A. Explicit equivalence

The explicit equivalence between the Youla parameterization and the IOP is presented in [6]:

Theorem 1 ([6]): Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of $\mathbf{P}_{22}$. The following statements hold.

- 1) For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\mathbf{Y} = (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l, \quad (15a)$$

$$\mathbf{U} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l, \quad (15b)$$

$$\mathbf{W} = (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{N}_l, \quad (15c)$$

$$\mathbf{Z} = I + (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{N}_l, \quad (15d)$$

belong to the affine subspace (12a)-(12c) and are such that $\mathbf{U}\mathbf{Y}^{-1} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}$.

- 2) For any $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$ in the affine subspace (12a)-(12c), the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l\mathbf{Y}\mathbf{U}_r - \mathbf{U}_l\mathbf{U}\mathbf{U}_r - \mathbf{V}_l\mathbf{W}\mathbf{V}_r + \mathbf{U}_l\mathbf{Z}\mathbf{V}_r - \mathbf{V}_l\mathbf{U}_r, \quad (16)$$

is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and $(\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} = \mathbf{U}\mathbf{Y}^{-1}$.

Theorem 1 presents explicit affine mappings between the Youla parameterization and the IOP: any element in the Youla parameterization (5) corresponds to an element in the IOP (13), and they represent the same controller. The following result presents explicit affine mappings between SLP and IOP.

Theorem 2: Consider a strictly proper plant $\mathbf{P}_{22}$, i.e., $D_{22} = 0$. The following statements hold.

- 1) For any $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ satisfying the affine subspace (8a) – (8c), the transfer matrices

$$\mathbf{Y} = \mathbf{C}_2 \mathbf{N} + \mathbf{I}, \quad (17a)$$

$$\mathbf{U} = \mathbf{L}, \quad (17b)$$

$$\mathbf{W} = \mathbf{C}_2 \mathbf{R} \mathbf{B}_2, \quad (17c)$$

$$\mathbf{Z} = \mathbf{M} \mathbf{B}_2 + \mathbf{I}, \quad (17d)$$

belong to the affine subspace (12a) – (12c) and are such that $\mathbf{L} - \mathbf{M} \mathbf{R}^{-1} \mathbf{N} = \mathbf{U} \mathbf{Y}^{-1}$.

- 2) For any $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ satisfying the affine subspace (12a) – (12c), the transfer matrices

$$\mathbf{R} = (z\mathbf{I} - \mathbf{A})^{-1} + (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_2 \mathbf{U} \mathbf{C}_2 (z\mathbf{I} - \mathbf{A})^{-1}, \quad (18a)$$

$$\mathbf{M} = \mathbf{U} \mathbf{C}_2 (z\mathbf{I} - \mathbf{A})^{-1}, \quad (18b)$$

$$\mathbf{N} = (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_2 \mathbf{U}, \quad (18c)$$

$$\mathbf{L} = \mathbf{U}, \quad (18d)$$

belong to the affine subspace (8a) – (8c) and are such that $\mathbf{U} \mathbf{Y}^{-1} = \mathbf{L} - \mathbf{M} \mathbf{R}^{-1} \mathbf{N}$.

Proof: *Statement 1:* considering the affine relationships (8a) – (8c) and $\mathbf{P}_{22} = \mathbf{C}_2 (z\mathbf{I} - \mathbf{A})_2^{-1}$, we have the following algebraic equalities:

$$\begin{aligned} \mathbf{Y} - \mathbf{P}_{22} \mathbf{U} &= \mathbf{C}_2 \mathbf{N} + \mathbf{I} - \mathbf{P}_{22} \mathbf{L} \\ &= \mathbf{C}_2 (\mathbf{N} - (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_2 \mathbf{L}) + \mathbf{I} = \mathbf{I}, \\ \mathbf{W} - \mathbf{P}_{22} \mathbf{Z} &= \mathbf{C}_2 \mathbf{R} \mathbf{B}_2 - \mathbf{P}_{22} (\mathbf{M} \mathbf{B}_2 + \mathbf{I}) \\ &= \mathbf{C}_2 \left(\mathbf{R} - (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_2 \mathbf{M} - (z\mathbf{I} - \mathbf{A})^{-1} \right) \mathbf{B}_2 \\ &= \mathbf{0}, \\ \mathbf{Y} \mathbf{P}_{22} - \mathbf{W} &= (\mathbf{C}_2 \mathbf{N} + \mathbf{I}) \mathbf{P}_{22} - \mathbf{C}_2 \mathbf{R} \mathbf{B}_2 \\ &= \mathbf{C}_2 \left(\mathbf{N} \mathbf{C}_2 (z\mathbf{I} - \mathbf{A})^{-1} + (z\mathbf{I} - \mathbf{A})^{-1} - \mathbf{R} \right) \mathbf{B}_2 \\ &= \mathbf{0}, \\ -\mathbf{U} \mathbf{P}_{22} + \mathbf{Z} &= -\mathbf{L} \mathbf{P}_{22} + \mathbf{M} \mathbf{B}_2 + \mathbf{I} \\ &= (-\mathbf{L} \mathbf{C}_2 (z\mathbf{I} - \mathbf{A})^{-1} + \mathbf{M}) \mathbf{B}_2 + \mathbf{I} \\ &= \mathbf{I}. \end{aligned}$$

Therefore, (12a) and (12b) are satisfied. Obviously, the transfer matrices in (17) are stable, i.e. (12c) is satisfied. In addition, from (8b), we have $\mathbf{R} = (\mathbf{I} + \mathbf{N} \mathbf{C}_2) (z\mathbf{I} - \mathbf{A})^{-1}$. Then,

$$\begin{aligned} \mathbf{L} - \mathbf{M} \mathbf{R}^{-1} \mathbf{N} &= \mathbf{L} - \mathbf{M} (z\mathbf{I} - \mathbf{A}) (\mathbf{I} + \mathbf{N} \mathbf{C}_2)^{-1} \mathbf{N} \\ &= \mathbf{L} - \mathbf{L} \mathbf{C}_2 (\mathbf{I} + \mathbf{N} \mathbf{C}_2)^{-1} \mathbf{N} \\ &= \mathbf{L} - \mathbf{L} \mathbf{C}_2 \mathbf{N} (\mathbf{I} + \mathbf{C}_2 \mathbf{N})^{-1} \\ &= \mathbf{L} (\mathbf{I} + \mathbf{C}_2 \mathbf{N})^{-1} \\ &= \mathbf{U} \mathbf{Y}^{-1}. \end{aligned}$$

Statement 2: In the Appendix A, we verify algebraically that the transfer matrices $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ defined in (18) are exactly the closed-loop responses in (7) with controller $\mathbf{K} = \mathbf{U} \mathbf{Y}^{-1}$.

Since $\mathbf{K} = \mathbf{U} \mathbf{Y}^{-1}$ is internally stabilizing $\mathbf{P}_{22}$, the transfer matrices $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ defined in (18) naturally satisfy the constraints (8a)–(8c). Finally, we can check that the transfer matrices $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$

defined in (18) satisfy $\mathbf{U} \mathbf{Y}^{-1} = \mathbf{L} - \mathbf{M} \mathbf{R}^{-1} \mathbf{N}$. This completes the proof. ■

Combining Theorems 1 and 2, we arrive at the explicit affine mappings between Youla parameterization and SLP, which was not provided in [5] and subsequent work [7].

Theorem 3: Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of the strictly proper system $\mathbf{P}_{22}$, i.e., $D_{22} = 0$. The following statements hold.

- 1) For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\mathbf{R} = (z\mathbf{I} - \mathbf{A})^{-1} + (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l \mathbf{C}_2 (z\mathbf{I} - \mathbf{A})^{-1}, \quad (19a)$$

$$\mathbf{M} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l \mathbf{C}_2 (z\mathbf{I} - \mathbf{A})^{-1}, \quad (19b)$$

$$\mathbf{N} = (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l, \quad (19c)$$

$$\mathbf{L} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l, \quad (19d)$$

belong to the affine subspace (8a) – (8c) and are such that $\mathbf{L} - \mathbf{M} \mathbf{R}^{-1} \mathbf{N} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) (\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1}$.

- 2) For any $(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L})$ in the affine subspace (8a) – (8c), the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l \mathbf{C}_2 \mathbf{N} \mathbf{U}_r - \mathbf{U}_l \mathbf{L} \mathbf{U}_r - \mathbf{V}_l \mathbf{C}_2 \mathbf{R} \mathbf{B}_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{M} \mathbf{B}_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{V}_r \quad (20)$$

is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and $(\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) (\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} = \mathbf{L} - \mathbf{M} \mathbf{R}^{-1} \mathbf{N}$.

Proof: Statement 1 directly follows by combining the statement 1 of Theorem 1 with the statement 2 of Theorem 2.

Combining the statement 2 of Theorem 1 with the statement 1 of Theorem 2 leads to

$$\begin{aligned} \mathbf{Q} &= \mathbf{V}_l (\mathbf{C}_2 \mathbf{N} + \mathbf{I}) \mathbf{U}_r - \mathbf{U}_l \mathbf{L} \mathbf{U}_r - \mathbf{V}_l \mathbf{C}_2 \mathbf{R} \mathbf{B}_2 \mathbf{V}_r \\ &\quad + \mathbf{U}_l (\mathbf{M} \mathbf{B}_2 + \mathbf{I}) \mathbf{V}_r - \mathbf{V}_l \mathbf{U}_r, \\ &= \mathbf{V}_l \mathbf{C}_2 \mathbf{N} \mathbf{U}_r - \mathbf{U}_l \mathbf{L} \mathbf{U}_r - \mathbf{V}_l \mathbf{C}_2 \mathbf{R} \mathbf{B}_2 \mathbf{V}_r \\ &\quad + \mathbf{U}_l \mathbf{M} \mathbf{B}_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{V}_r. \end{aligned}$$

This completes the proof. ■

An overview of the equivalence of Youla parameterization, SLP, and IOP is shown in Fig. 3.

Remark 2 (Closed-loop convexity): In both SLP and IOP, the parameters $(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L})$ and $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$ have explicit and distinct physical interpretations as corresponding closed-loop transfer matrices. Specifically, $(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L})$ are the closed-loop responses from (δ_x, δ_y) to $(\mathbf{x}, \mathbf{u})$, and $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$ are the closed-loop responses from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$. Also, the Youla parameter $\mathbf{Q}$ can be viewed as a closed-loop transfer matrix when the plant $\mathbf{P}_{22}$ is stable (see Remark 6). In this sense, the Youla parameterization, the SLP, and the IOP all shift the controller synthesis task from the design of a controller in (4), which is non-convex, to the design of closed-loop responses, resulting in convex formulations (6), (10), and (14). Note that this idea of closed-loop convexity has been extensively discussed in the book [3], and a comprehensive historical note is given in [3, Chapter 16.3].

Remark 3 (Numerical computation): After computing a doubly-coprime factorization of the plant, the Youla parameter $\mathbf{Q}$ is free in $\mathcal{RH}_\infty$ for parameterizing $\mathcal{C}_{stab}$, and there are no equality constraints for achievable closed-loop responses. This feature allows to reformulate problem (4) as a model matching problem (6), which can be reduced to the Nehari problem and then solved via the state-space method in [4]. Instead, both the SLP and the IOP do not require to compute a doubly-coprime factorization *a priori*, but have explicit affine constraints for achievable closed-loop responses. Since the decision variables in constraints (8a) – (8c) and (12a) – (12c) are infinite dimensional, there is no immediately efficient numerical

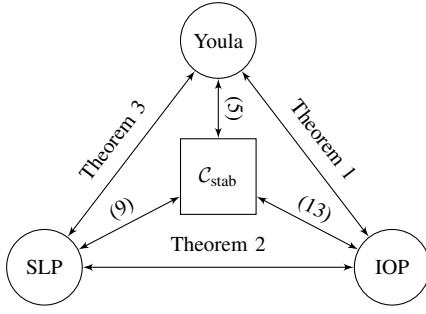


Fig. 3: Equivalence of Youla parameterization, System-level parameterization (SLP), and Input-output parameterization (IOP).

method for solving (10) or (14). The Ritz approximation [3, Chapter 15] is one method for solving infinite dimensional optimization problems. Specifically, for discrete-time systems, the finite impulse response (FIR) approximation is a practical choice [5], [6].

B. Convex system-level synthesis

In [5], [7], the authors introduced a general framework of system-level synthesis (SLS), which revisits “the broadest known class of constrained optimal control problems that can be solved using convex programming” (cf. [7]). Thanks to the full equivalence in Theorems 1-3, we can show that 1) any SLS problem can be equivalently formulated in the Youla or input-output framework, 2) any convex SLS can be addressed by solving a convex problem in terms of the Youla parameter $\mathbf{Q}$ or the input-output parameters $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$.

Let $g(\cdot)$ be a functional capturing a desired measure of the performance of the plant $\mathbf{P}_{22}$, and let $\mathcal{S}$ be a system-level constraint. The SLS problem in [5] is posed as

$$\begin{aligned} \min_{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}} \quad & g(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}) \\ \text{subject to} \quad & (8a) - (8c), \\ & \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \mathcal{S}. \end{aligned} \quad (21)$$

We refer the interested reader to [5] for a detailed discussion of SLS. Then, we have the following result.

Theorem 4: Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of the strictly proper system $\mathbf{P}_{22}$, i.e., $D_{22} = 0$. The following statements hold.

- 1) The SLS problem (21) is equivalent to the following problem in the Youla parameter $\mathbf{Q}$,

$$\begin{aligned} \min_{\mathbf{Q}} \quad & g_1(\mathbf{Q}) \\ \text{subject to} \quad & \begin{bmatrix} f_1(\mathbf{Q}) & f_3(\mathbf{Q}) \\ f_2(\mathbf{Q}) & f_4(\mathbf{Q}) \end{bmatrix} \in \mathcal{S}, \end{aligned} \quad (22)$$

where $f_1(\mathbf{Q}), f_2(\mathbf{Q}), f_3(\mathbf{Q}), f_4(\mathbf{Q})$ are defined by (19a) – (19d), respectively, and

$$g_1(\mathbf{Q}) := g(f_1(\mathbf{Q}), f_2(\mathbf{Q}), f_3(\mathbf{Q}), f_4(\mathbf{Q})).$$

- 2) The SLS problem (21) is equivalent to the following problem in the input-output parameters $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$,

$$\begin{aligned} \min_{\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}} \quad & \hat{g}_1(\mathbf{U}) \\ \text{subject to} \quad & (12a) - (12c) \\ & \begin{bmatrix} \hat{f}_1(\mathbf{U}) & \hat{f}_3(\mathbf{U}) \\ \hat{f}_2(\mathbf{U}) & \hat{f}_4(\mathbf{U}) \end{bmatrix} \in \mathcal{S}, \end{aligned} \quad (23)$$

where $\hat{f}_1(\mathbf{U}), \hat{f}_2(\mathbf{U}), \hat{f}_3(\mathbf{U}), \hat{f}_4(\mathbf{U})$ are defined by (18a) – (18d), respectively, and

$$\hat{g}_1(\mathbf{U}) := g(\hat{f}_1(\mathbf{U}), \hat{f}_2(\mathbf{U}), \hat{f}_3(\mathbf{U}), \hat{f}_4(\mathbf{U})).$$

- 3) If the SLS problem (21) is convex, then problems (22) and (23) are both convex.

Proof: The first two statements directly follow from Theorems 2 and 3. The last statement follows from the facts that $f_i(\mathbf{Q})$ and $\hat{f}_i(\mathbf{Q}), i = 1, \dots, 4$, are all affine. Then, if $\mathcal{S}$ is a convex set and $g(\cdot)$ is a convex functional, the constraint in (22) (resp. (23)) defines a convex set in $\mathbf{Q}$ (resp. $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$), and $g_1(\cdot)$ (or $\hat{g}_1(\cdot)$) is convex. Therefore, both problem (22) and problem (23) are convex. ■

C. Distributed optimal control and quadratic invariance (QI)

Unlike the SLS, which impose constraints on closed-loop responses (see (21)), the classical distributed optimal control problem typically considers a subspace constraint $\mathcal{L}$ on the controller $\mathbf{K}$, which is formulated as [8], [12], [13]

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|f(\mathbf{P}, \mathbf{K})\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}} \cap \mathcal{L}. \end{aligned} \quad (24)$$

It is shown in [8], [12] that if the subspace constraint $\mathcal{L}$ is *quadratically invariant* (QI) under $\mathbf{P}_{22}$ (i.e., $\mathbf{K}\mathbf{P}_{22}\mathbf{K} \in \mathcal{L}, \forall \mathbf{K} \in \mathcal{L}$), then we have

$$\begin{aligned} \mathcal{C}_{\text{stab}} \cap \mathcal{L} = \{ \mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} \mid \\ (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})\mathbf{M}_l \in \mathcal{L}, \mathbf{Q} \in \mathcal{RH}_{\infty} \}. \end{aligned}$$

Problem (24) can thus be equivalently formulated as a convex problem in $\mathbf{Q}$ [8], [12],

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})\mathbf{M}_l \in \mathcal{L}, \\ & \mathbf{Q} \in \mathcal{RH}_{\infty}. \end{aligned} \quad (25)$$

Considering the equivalence shown in Theorems 1 and 3, the following corollaries are immediate.

Corollary 1 (QI with the IOP): If $\mathcal{L}$ is QI under $\mathbf{P}_{22}$, then

- 1) We have
$$\mathcal{C}_{\text{stab}} \cap \mathcal{L} = \{ \mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \mid \mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \text{ are in the affine subspace (12a) – (12c), } \mathbf{U} \in \mathcal{L} \}.$$
- 2) Problem (24) can be equivalently formulated as a convex problem
$$\begin{aligned} \min_{\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\| \\ \text{subject to} \quad & (12a) - (12c), \\ & \mathbf{U} \in \mathcal{L}. \end{aligned} \quad (26)$$

Corollary 2 (QI with the SLP): If $\mathcal{L}$ is QI under a strictly proper plant $\mathbf{P}_{22}$ (where $D_{22} = 0$), then

- 1) We have
$$\mathcal{C}_{\text{stab}} \cap \mathcal{L} = \{ \mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \mid \mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L} \text{ are in the affine subspace (8a) – (8c), } \mathbf{L} \in \mathcal{L} \}.$$
- 2) Problem (24) can be equivalently formulated as a convex problem
$$\begin{aligned} \min_{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}} \quad & \left\| [C_1 \quad D_{12}] \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \\ \text{subject to} \quad & (8a) - (8c), \\ & \mathbf{L} \in \mathcal{L}. \end{aligned} \quad (27)$$

Corollary 1 is the same as Theorem 3 of [6] and Corollary 2 is consistent with Theorem 3 of [5]. One main insight is that the specialized proofs in [6], [5] may be not needed anymore, thanks to the explicit affine mappings between Youla, SLP and IOP. We also note that the original proof of Theorem 3 in [5] is not complete: it relies on the fact the affine mapping $\mathbf{L} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l$ is invertible. However, given $\mathbf{L} \in \mathcal{RH}_\infty$, it is not immediate to see that $\mathbf{Q} = \mathbf{M}_r^{-1}(\mathbf{V}_r - \mathbf{L} \mathbf{M}_l^{-1})$ is stable. We complete this fact via the construction of $\mathbf{Q}$ in (20).

Remark 4 (SLS and classical distributed control): It should be noted that the SLS (21) and the classical distributed control problem (24) are two distinct formulations: 1) the former imposes constraints on closed-loop responses while the latter imposes a constraint on controller $\mathbf{K}$; 2) feasibility of the former does not imply feasibility of the latter, and vice-versa. Only when the QI property holds, can problem (24) be equivalently reformulated into a convex problem in terms of the Youla, system-level, or input-output parameters. Based on the results in [9], QI is necessary for the existence of such equivalent convex reformulation. For systems with QI constraints, the SLS (21) can be equivalent to the classical problem (24), as shown in Corollary 2; for the cases beyond QI, they are not directly comparable.

V. DISTRIBUTED OPTIMAL CONTROL WITH NON-QI CONSTRAINTS

In this section, we highlight that for systems with non-QI constraints, we may derive convex approximations of (24) using the Youla, system-level, or input-output parameters. In certain cases, a globally optimal solution can still be obtained. Our approximation procedure is consistent with the idea of sparsity invariance (SI) [10]. In particular, we consider Example 1 in [5], [7]. We first present simplified versions of the Youla, system-level, and input-output parameterizations for special cases of state feedback (for completeness, other simplified versions for stable plants are presented in Appendix C). Then, we show that Example 1 can be solved exactly using the Youla, system-level, or input-output parameters via convex optimization.

A. Simplified parameterizations for state feedback

In [5], it is shown that for state feedback where $C_2 = I, D_{22} = 0$, the set of internally stabilizing controllers is

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} = \mathbf{M} \mathbf{R}^{-1} \mid \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \begin{bmatrix} zI - A & -B_2 \end{bmatrix} = I, \right. \\ \left. \mathbf{M}, \mathbf{R} \in \frac{1}{z} \mathcal{RH}_\infty \right\}. \quad (28)$$

The proof in [5] is directly based on the definition of internal stability. As expected, this special case (28) can be reduced from the general case (9) from purely algebraic operations. We provide this alternative proof in Appendix B.

For IOP and Youla parameterization, simplifications are possible with further assumptions.

Corollary 3 (Input-output parameterization): Suppose $C_2 = I, D_{22} = 0$ and B_2 is invertible. We have

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} = (\mathbf{Z} - I) \mathbf{W}^{-1} \mid \begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{Z} \end{bmatrix} = 0 \right. \\ \left. \mathbf{Z} \in \mathcal{RH}_\infty, \mathbf{W} \in \frac{1}{z} \mathcal{RH}_\infty \right\}. \quad (29)$$

Proof: We show that any controller in (29) is an internally stabilizing controller in (13). The other direction is similar.

Given any $\mathbf{W}, \mathbf{Z}$ satisfying the constraints in (29), we define

$$\mathbf{U} = (\mathbf{Z} - I) B_2^{-1} (zI - A) \in \mathcal{RH}_\infty, \\ \mathbf{Y} = \mathbf{W} B_2^{-1} (zI - A) \in \mathcal{RH}_\infty.$$

Then, we can easily verify

$$\mathbf{Z} - \mathbf{U} \mathbf{P}_{22} = I, \\ \mathbf{W} - \mathbf{Y} \mathbf{P}_{22} = 0, \\ \mathbf{Y} - \mathbf{P}_{22} \mathbf{U} = I.$$

Thus, $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ above satisfy (12a)-(12c). We also have

$$\mathbf{U} \mathbf{Y}^{-1} = (\mathbf{Z} - I) B_2^{-1} (zI - A) (\mathbf{W} B_2^{-1} (zI - A))^{-1} \\ = (\mathbf{Z} - I) \mathbf{W}^{-1}.$$

This completes the proof. $\blacksquare$

Corollary 4 (Youla parameterization): Suppose $C_2 = B_2 = I, D_{22} = 0$. We have³

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} = \left(-A - \left(I - \frac{1}{z} A \right) \mathbf{Q} \right) \left(I - \frac{1}{z} \mathbf{Q} \right)^{-1}, \right. \\ \left. \mathbf{Q} \in \mathcal{RH}_\infty \right\}$$

Proof: The proof directly follows by choosing the following doubly-coprime factorization:

$$\mathbf{U}_l = I, \mathbf{V}_l = -A, \mathbf{N}_l = \frac{1}{z} I, \mathbf{M}_l = I - \frac{1}{z} A, \\ \mathbf{U}_r = I, \mathbf{V}_r = -A, \mathbf{N}_r = \frac{1}{z} I, \mathbf{M}_r = I - \frac{1}{z} A. \quad (30)$$

Substituting (30) into (5) completes the proof. $\blacksquare$

B. Example 1 in [5], [7]

Consider the following optimal control problem, which is Example 1 in [5], [7],

$$\min_{\mathbf{K}} \quad \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^T \mathbb{E} \|x[t]\|^2 \\ \text{subject to} \quad x[t+1] = Ax[t] + u[t] + w[t], \\ \mathbf{u} = \mathbf{K} \mathbf{x}, \quad (31)$$

where disturbance $w[t] \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, I)$. It can be verified (e.g., via solving the discrete-time algebraic Riccati equation) that the globally optimal solution is the static feedback given by $\mathbf{K} = -A$. Assume that A is sparse and let its sparsity pattern define the adjacency matrix of a graph $\mathcal{G}$. Then, the optimal controller has a particular structure according to $\mathcal{G}$. In this case, the optimal controller can be implemented in a localized manner according to $\mathcal{G}$.

Now suppose that we attempt to solve problem (31) by converting it to its equivalent $\mathcal{H}_2$ optimal control problem in the form of (24), where the constraint $\mathcal{L}$ corresponds to the sparsity pattern of A (see the Example 1 in [5], [7] for a precise definition). Since (24) is not convex in its present form, a certain reformulation is required for numerical computation, e.g., using the Youla parameterization, the SLP, or the IOP. Building on the results in [9], we have the following proposition.

Proposition 1: If the graph $\mathcal{G}$ is strongly connected, then problem (31) with a sparsity constraint $\mathbf{K} \in \mathcal{L}$ in the form of (24) does not admit any equivalent convex reformulation in the Youla parameterization, or the SLP, or the IOP.

Proof: If $\mathcal{G}$ is strongly connected, then the sparsity constraint $\mathcal{L}$ is not QI under $\mathbf{P}_{22} = (zI - A)^{-1}$, since $\mathbf{P}_{22}$ is a dense transfer

³Note that Corollary 4 is only valid in discrete-time systems, since the doubly-coprime factorization (30) has no counterpart in continuous time.

matrix and it fails to satisfy $\mathbf{K}\mathbf{P}_{22}\mathbf{K} \in \mathcal{L}, \forall \mathbf{K} \in \mathcal{L}$. According to [9], QI is necessary for the existence of an *equivalent* convex reformulation in the Youla parameter $\mathbf{Q}$ for (24). The equivalence in Theorem 4 prevents any *equivalent* convex reformulation via the SLP or the IOP as well. ■

Although there is no equivalent convex reformulation when $\mathcal{G}$ is strongly connected, we could still develop a certain *convex approximation* of (24) in the Youla parameterization, the SLP, or the IOP. In the following, we use $\mathcal{I}$ to denote a diagonal structure.

- 1) *SLP*: As suggested by [5], we can add the constraints

$$\mathbf{M} \in \mathcal{L}, \mathbf{R} \in \mathcal{I}$$

to problem (10), leading to a convex approximation of (24). It can be checked that $\mathbf{R} = \frac{1}{z}I$ and $\mathbf{M} = -\frac{1}{z}A$ is the optimal solution, recovering the globally optimal controller

$$\mathbf{K} = \mathbf{M}\mathbf{R}^{-1} = -A.$$

- 2) *Youla parameterization*: We use the simplified Youla parameterization in Corollary 4, and add the following constraints

$$-A - (I - \frac{1}{z}A)\mathbf{Q} \in \mathcal{L}, I - \frac{1}{z}\mathbf{Q} \in \mathcal{I},$$

to problem (6). This leads to a convex program. We can check that the optimal solution is $\mathbf{Q} = 0$, leading to

$$\mathbf{K} = \left(-A - (I - \frac{1}{z}A)\mathbf{Q}\right) \left(I - \frac{1}{z}\mathbf{Q}\right)^{-1} = -A.$$

- 3) *IOP*: Since $C_2 = I, B_2 = I$ is invertible, we can use the result in Corollary 3. Then, we introduce constraints

$$\mathbf{Z} - I \in \mathcal{L}, \mathbf{W} \in \mathcal{I}$$

to problem (14), leading to a convex program. We can check that the solution $\mathbf{W} = \frac{1}{z}I$ and $\mathbf{Z} = I - \frac{1}{z}A$ is optimal. Then,

$$\mathbf{K} = (\mathbf{Z} - I)\mathbf{W}^{-1} = -A.$$

Remark 5 (Sparsity invariance and beyond QI): In the aforementioned procedures, we choose separate subspace constraints for the factors of $\mathbf{K}$ in the following form

$$\mathbf{S} \in \mathcal{L}, \mathbf{T} \in \mathcal{I} \Rightarrow \mathbf{K} = \mathbf{S}\mathbf{T}^{-1} \in \mathcal{L}, \quad (32)$$

where $\mathbf{S}, \mathbf{T}$ denote appropriate transfer matrices in the Youla, system-level, and input-output parameterizations. Obviously, this choice leads to a convex inner-approximation of (24) since the feasible region of $\mathbf{K}$ is narrowed. For this simple instance, the globally optimal solution is parameterized when using (32). Thus, the globally optimal controller can be found using the Youla, system-level or input-output parameters via convex optimization. However, as observed in [10], this procedure has no guarantee of optimality for general constraints beyond QI using either the Youla parameterization, the SLP, or the IOP. The property of QI is not needed for this simple instance since we already know its globally optimal solution analytically. But QI can serve as a certificate of the global optimality for solving general problem (24).

We note that the property (32) is a special case of sparsity invariance (SI) [10]. There may exist other subspace choices for $\mathbf{S}, \mathbf{T}$ satisfying $\mathbf{S}\mathbf{T}^{-1} \in \mathcal{L}$, which still return a structured controller $\mathbf{K} \in \mathcal{L}$. Indeed, the notion of SI goes beyond QI for sparsity constraints, as it includes QI as a special case. We refer the interested reader to [10] for details.

VI. CONCLUSION

In this paper, we have presented an explicit equivalence between the Youla, system-level, and input-output parameterizations for the set of internally stabilizing controllers. A doubly-coprime factorization of the system can be considered as a way to eliminate the explicit equality constraints in the SLP and the IOP. Indeed, both the SLP and the IOP involve four parameters; but due to the equality constraints, the SLP and the IOP present the same degrees of freedom as the Youla parameterization.

We remark that the equivalence between the Youla parameterization, the SLP, and the IOP does not indicate they offer the same computational features. One parameterization may be better suited for a particular context. For instance, it seems that the SLP is more convenient for the case of state feedback and has found applications in quantifying the sample complexity of LQR problems [14]; the IOP seems to be better suited for the case of output feedback as it exclusively deals with the maps from inputs to outputs without explicitly touching the system state; and the Youla parameterization may be more convenient when a doubly-coprime factorization is available *a priori*. It is interesting to investigate whether there exist other parameterizations of stabilizing controllers that suit a particular control application. Finally, we note that the Youla, the SLP, and the IOP are naturally suited for parameterizing dynamical controllers, but none of them can parameterize the set of static stabilizing controllers in a convex way. Thus, QI is not relevant for structured static controller synthesis, and this problem deserves further investigations.

APPENDIX

A. Proof of Statement 2 in Theorem 2

Given any $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ satisfying the affine subspace (12a) – (12c), we know that $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$ internally stabilizes the plant $\mathbf{P}_{22}$. In the following, we verify that the transfer matrices $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ defined in (18) are exactly the closed-loop responses in (7) with controller $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$.

Recall that $\mathbf{P}_{22}$ is strictly proper, *i.e.*, $\mathbf{P}_{22} = C_2(zI - A)^{-1}B_2$. Then, we can verify the following equation:

$$\begin{aligned} \mathbf{R} &= (zI - A)^{-1} + (zI - A)^{-1}B_2\mathbf{U}C_2(zI - A)^{-1} \\ &= \left[(I + B_2\mathbf{U}C_2(zI - A)^{-1})^{-1}(zI - A)\right]^{-1} \\ &= \left[zI - A - (I + B_2\mathbf{U}C_2(zI - A)^{-1})^{-1}B_2\mathbf{U}C_2\right]^{-1} \\ &= \left[zI - A - B_2\mathbf{U}(I + C_2(zI - A)^{-1}B_2\mathbf{U})^{-1}C_2\right]^{-1} \\ &= \left(zI - A - B_2\mathbf{U}\mathbf{Y}^{-1}C_2\right)^{-1} \\ &= (zI - A - B_2\mathbf{K}C_2)^{-1} \end{aligned}$$

Also, we can verify

$$\begin{aligned} \mathbf{M} &= \mathbf{U}C_2(zI - A)^{-1} = (I - \mathbf{K}\mathbf{P}_{22})^{-1}\mathbf{K}C_2(zI - A)^{-1} \\ &= \mathbf{K}C_2(zI - A)^{-1}(I - B_2\mathbf{K}C_2(zI - A)^{-1})^{-1} \\ &= \mathbf{K}C_2(zI - A - B_2\mathbf{K}C_2)^{-1} \\ &= \mathbf{K}C_2\mathbf{R} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathbf{N} &= (zI - A)^{-1}B_2\mathbf{U} = \mathbf{R}B_2\mathbf{K}, \\ \mathbf{L} &= \mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}. \end{aligned}$$

Then, the transfer matrices $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ are exactly the closed-loop responses in (7) with $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$.

B. Proof of (28)

First, we show that any controller in (28) is an internally stabilizing controller in (9). Consider any $\mathbf{R}, \mathbf{M} \in \frac{1}{z}\mathcal{RH}_\infty$ satisfying

$$\begin{bmatrix} (zI - A) & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I. \quad (33)$$

Upon defining

$$\mathbf{L} = \mathbf{M}(zI - A), \quad \mathbf{N} = \mathbf{R}(zI - A) - I,$$

it is easy to see $\mathbf{N}, \mathbf{L} \in \mathcal{RH}_\infty$. Then, one can straightforwardly verify that $\mathbf{N}, \mathbf{L}$ and $\mathbf{R}, \mathbf{M}$ above satisfy (8a) – (8b) when $C_2 = I$. It is routinely to verify that

$$\begin{aligned} \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} &= \mathbf{M}(zI - A) - \mathbf{M}\mathbf{R}^{-1}(\mathbf{R}(zI - A) - I) \\ &= \mathbf{M}\mathbf{R}^{-1}. \end{aligned}$$

It remains to check that $\mathbf{N}$ defined above is strictly proper. This fact follows from (8a) that $(zI - A)\mathbf{N} - B_2\mathbf{L} = 0$, implying that $\mathbf{N} = (zI - A)^{-1}B_2\mathbf{L}$ is strictly proper. Therefore, we have shown that (8c) also holds.

Second, when $C_2 = I$, it is similar to show that any controller in (9) is an internally stabilizing controller in (28). Thus, the general parameterization (9) can be reduced to (28).

C. Stable plants

When $\mathbf{P}_{22} \in \mathcal{RH}_\infty$, we show that Youla, SLP, and IOP can be simplified, and only two parameters are required in the SLP/IOP.

Proposition 2: If $\mathbf{P}_{22} \in \mathcal{RH}_\infty$, we have:

- 1) The Youla parameterization can be reduced to

$$\mathcal{C}_{\text{stab}} = \{\mathbf{K} = -\mathbf{Q}(I - \mathbf{P}_{22}\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\}. \quad (34)$$

- 2) For strictly proper $\mathbf{P}_{22}$, the SLP can be reduced to

$$\begin{aligned} \mathcal{C}_{\text{stab}} &= \left\{ \mathbf{K} = \mathbf{L}(C_2\mathbf{N} + I)^{-1} \mid \mathbf{L} \in \mathcal{RH}_\infty, \right. \\ &\quad \left. \begin{bmatrix} (zI - A) & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{N} \\ \mathbf{L} \end{bmatrix} = 0 \right\}. \end{aligned} \quad (35)$$

- 3) The IOP can be reduced to

$$\begin{aligned} \mathcal{C}_{\text{stab}} &= \left\{ \mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \mid \begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{U} \end{bmatrix} = I, \right. \\ &\quad \left. \mathbf{U} \in \mathcal{RH}_\infty \right\}. \end{aligned} \quad (36)$$

Proof: The proof is directly from the following observations.

- 1) If $\mathbf{P}_{22} \in \mathcal{RH}_\infty$, a doubly-coprime factorization of $\mathbf{P}_{22}$ can be trivially chosen as

$$\begin{aligned} \mathbf{U}_l &= I, \mathbf{V}_l = 0, \mathbf{N}_l = \mathbf{P}_{22}, \mathbf{M}_l = I, \\ \mathbf{U}_r &= I, \mathbf{V}_r = 0, \mathbf{N}_r = \mathbf{P}_{22}, \mathbf{M}_r = I. \end{aligned}$$

Then, the parameterization (5) is reduced to (34).

- 2) Given $\mathbf{N}, \mathbf{L}$ in (35), we define

$$\begin{aligned} \mathbf{N} &= (zI - A)^{-1}B_2\mathbf{L}, \\ \mathbf{R} &= (zI - A)^{-1} + \mathbf{N}C_2(zI - A)^{-1}, \\ \mathbf{M} &= \mathbf{L}C_2(zI - A)^{-1}. \end{aligned}$$

Considering $(zI - A)^{-1} \in \mathcal{RH}_\infty$, if $\mathbf{L} \in \mathcal{RH}_\infty$, we have $\mathbf{N}, \mathbf{R}, \mathbf{M} \in \frac{1}{z}\mathcal{RH}_\infty$. It can be verified that the $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ above satisfies (8a) – (8c) when (35) holds. Also, we have

$$\begin{aligned} \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} &= \mathbf{L} - \mathbf{L}C_2(I + \mathbf{N}C_2)^{-1}\mathbf{N} \\ &= \mathbf{L}(C_2\mathbf{N} + I)^{-1}. \end{aligned}$$

Thus, (8a) – (8c) can be reduced to (35).

- 3) Upon defining

$$\begin{aligned} \mathbf{Y} &= I + \mathbf{P}_{22}\mathbf{U}, \\ \mathbf{Z} &= I + \mathbf{U}\mathbf{P}_{22}, \\ \mathbf{W} &= \mathbf{P}_{22}\mathbf{Z}, \end{aligned}$$

we have $\mathbf{Y}, \mathbf{W}, \mathbf{Z} \in \mathcal{RH}_\infty$ if $\mathbf{P}_{22}, \mathbf{U} \in \mathcal{RH}_\infty$. Also, the $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ above satisfies (12a) – (12b) if (36) holds. Thus, (12a) – (12c) can be reduced to (36).

Remark 6: The first statement in Proposition 2 is a classical result [1, Corollary 5.5]. We note that parameterizations (35) and (36) are identical to (34) by noticing that $\mathbf{L} = \mathbf{U} = -\mathbf{Q}$. They are all reduced to the same form.

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