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PAPER

Interacting particle approximation of cross-diffusion systems

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Abstract

We prove the existence of weak solutions of a class of multi-species cross-diffusion systems as well as the propagation of chaos result by means of nonlocal approximation of the nonlinear diffusion terms, coupling methods and compactness arguments. We also prove the uniqueness under further structural assumption on the mobilities by combining the uniqueness argument for viscous porous medium equations and linear Fokker–Planck equations. We show that these equations capture the macroscopic behaviour of stochastic interacting particle systems if the localisation parameter is chosen logarithmically with respect to the number of particles.

1. Introduction

Multi-species cross-diffusion models are systems of coupled equations which describe the evolution of densities of n different species ($n \geqslant 2$). The solution of the cross-diffusion system is a vector-valued function $\rho = (\rho_1, \ldots, \rho_n)$ defined on $\mathbb{R}^d$, where ρ_k is the density of kth ($k = 1, 2, \ldots, n$) species. We consider a class of cross-diffusion systems on $\mathbb{R}^d$ as follows

$$\begin{cases} \partial_t \rho_k - \nabla \cdot (b_k \rho_k \nabla P(\rho)) = \sigma \Delta \rho_k, \\ \rho_k(0) = \rho_{k,0}, \quad \rho_{k,0} \in L^1 \cap L^\infty(\mathbb{R}^d), \end{cases} \quad k = 1, 2, \ldots, n, \tag{1.1}$$

where the parameter $b_k > 0$ denotes the mobility, $\sigma > 0$ denotes the diffusion coefficient, and the motion is driven by pressure gradients through Darcy's law, for $a_k > 0$,

$$P(\rho) = \left(\sum_{k=1}^{n} a_k \rho_k \right)^{m-1}, \quad m \geqslant 2.$$

1.1. Problem statement

In this work, we consider the well-posedness of the cross-diffusion system (1.1) as well as its particle approximation. The system can be written in gradient flow structure as

$$\partial_t \rho_k = \frac{b_k}{a_k} \nabla \cdot \left(\rho_k \nabla \frac{\delta \mathcal{A}}{\delta \rho_k} \right), \tag{1.2}$$

where the energy functional $\mathcal{A}$ is given by

$$\mathcal{A}[\rho_1, \ldots, \rho_n] = \frac{1}{m} \int_{\mathbb{R}^d} \left(\sum_{k=1}^{n} a_k \rho_k \right)^m \mathrm{d}x + \sum_{k=1}^{n} \frac{a_k}{b_k} \sigma \int_{\mathbb{R}^d} \rho_k \log \rho_k \, \mathrm{d}x. \tag{1.3}$$

The energy functional defined above can be regularised at a formal level,

$$\mathcal{A}_\varepsilon\left[\rho_1^\varepsilon,\dots,\rho_n^\varepsilon\right]=\frac{1}{m}\int_{\mathbb{R}^d}\left(\sum_{k=1}^n a_k V^\varepsilon*\rho_k^\varepsilon\right)^m \mathrm{d}x+\sum_{k=1}^n\frac{a_k}{b_k}\sigma\int_{\mathbb{R}^d}\rho_k^\varepsilon\log\rho_k^\varepsilon\,\mathrm{d}x, \tag{1.4}$$

where we denote the variable of the regularised functional as $\rho^\varepsilon=(\rho_1^\varepsilon,\dots,\rho_n^\varepsilon)$, and mollifer V^ε which is obtained from nonnegative even function $V\in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}V(x)\mathrm{d}x=1$, by scaling $V^\varepsilon(x):=\varepsilon^{-d}V(x/\varepsilon)$. As $\varepsilon\to 0$, V^ε converges to the Dirac-delta δ_0, and the regularised energy functional $\mathcal{A}_\varepsilon$ formally converges to $\mathcal{A}$. The regularised functional leads to the nonlocal equation

$$\partial_t\rho_k^\varepsilon=\nabla\cdot\left(b_k\rho_k^\varepsilon\nabla V^\varepsilon*\left(\sum_{l=1}^n a_l V^\varepsilon*\rho_l^\varepsilon\right)^{m-1}\right)+\sigma\Delta\rho_k^\varepsilon, \tag{1.5}$$

with initial data $\rho_k^\varepsilon(0)=\rho_{k,0}$, which plays an important role below as an intermediate system between the cross-diffusion system and its particle approximation.

From a physical perspective, each subpopulation consists of a large number of interacting particles, which can represent molecules, cells, individuals, and so on depending on the application. Our motivation is to derive the cross-diffusion system (1.1) from stochastic many-particle systems. For the sake of notational simplicity, we take the same numbers of particles in each species as $N\in\mathbb{N}$. Let $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P}\right)$ be a filtered probability space, for any $k=1,\dots,n$, $\left(\xi_k^i\right)_{i\geqslant 1}$ is a sequence of i.i.d random variables on $\mathbb{R}^d$ with the common law $\rho_{k,0}$, $(B_k^i)_{i\geqslant 1}$ are i.i.d d-dimensional $\mathcal{F}_t$-Brownian motions that are independent of ξ_k^i. The dynamics of particle system of kth species is described by the following SDEs. For $k=1,\dots,n$ and $i=1,\cdots,N$, it is written

$$\begin{cases}\mathrm{d}X_k^{i,\varepsilon}(t)=-b_k\left[\nabla V^\varepsilon*\left(\sum_{l=1}^n\frac{a_l}{N}\sum_{j=1}^N V^\varepsilon\left(\cdot-X_l^{j,\varepsilon}(t)\right)\right)^{m-1}\right]\left(X_k^{i,\varepsilon}(t)\right)\mathrm{d}t+\sqrt{2\sigma}\mathrm{d}B_k^i(t),\\ X_k^{i,\varepsilon}(0)=\xi_k^i,\end{cases} \tag{1.6}$$

where all coefficients are the same as in (1.1) and potential V^ε is the same as in (1.4). The map $s\mapsto s^{m-1}$ is Lipschitz continuous when $m\geqslant 2$, and V^ε is bounded when ε is fixed. Then the existence and uniqueness of strong solution of (1.6) follow by standard SDE theory [36, theorem 3.1.1]. The distribution of particles $X_k^{i,\varepsilon}(t)$ is represented by $\rho_k^{(1),N,\varepsilon}(t)$, which is the first marginal of the joint law of N particles in the kth species. We will show that $\rho_k^{(1),N,\varepsilon}$ converges to ρ_k^ε which is a measure-valued solution of (1.5) when $N\to\infty$, and then ρ_k^ε converges to ρ_k which is the weak solution of (1.1) when $\varepsilon\to 0$.

In addition, we introduce the regularised McKean–Vlasov type nonlinear process $Y^{i,\varepsilon}=(Y_1^{i,\varepsilon},\dots,Y_n^{i,\varepsilon})$ satisfying the SDE below, for any k,

$$\begin{cases}\mathrm{d}Y_k^{i,\varepsilon}(t)=-b_k\left[\nabla V^\varepsilon*\left(\sum_{l=1}^n a_l V^\varepsilon*\rho_l^\varepsilon\right)^{m-1}\right]\left(Y_k^{i,\varepsilon}(t)\right)\mathrm{d}t+\sqrt{2\sigma}\mathrm{d}B_k^i(t),\\ Y_k^{i,\varepsilon}(0)=\xi_k^i,\\ \mathrm{Law}\left(Y_k^{i,\varepsilon}(t)\right)=\rho_k^\varepsilon(t),\end{cases} \tag{1.7}$$

where we choose the random variables ξ_k^i and Brownian motion $B_k^i(t)$ the same as in (1.6). We abused the notation a bit that we use $(\rho_1^\varepsilon,\dots,\rho_n^\varepsilon)$ denoting the distribution of solution of SDE $(Y_1^{i,\varepsilon},\dots,Y_n^{i,\varepsilon})$ in (1.7). But we notice that, fixing ε for any k and applying Itô's formula, the distribution ρ_k^ε formally coincides with the solution of the nonlocal equation (1.5) as

$$\frac{\partial\rho_k^\varepsilon}{\partial t}=\nabla\cdot\left(b_k\rho_k^\varepsilon\nabla V^\varepsilon*\left(\sum_{l=1}^n a_l V^\varepsilon*\rho_l^\varepsilon\right)^{m-1}\right)+\sigma\Delta\rho_k^\varepsilon.$$

For every fixed ε, $\sum_{l=1}^{n} a_l V^\varepsilon * \rho_l^\varepsilon$ is a bounded finite measure and ∇V^ε is compactly supported, which implies that the Lipschitz continuity holds

$$\begin{aligned}&\left|\nabla V^\varepsilon * \left(\sum_{l=1}^{n} a_l V^\varepsilon * \rho_l^\varepsilon\right)^{m-1}(x) - \nabla V^\varepsilon * \left(\sum_{l=1}^{n} a_l V^\varepsilon * \rho_l^\varepsilon\right)^{m-1}(y)\right|\\ &\leqslant \int_{\mathbb{R}^d} \left|\nabla V^\varepsilon (x-z) - \nabla V^\varepsilon (y-z)\right| \left(\sum_{l=1}^{n} a_l V^\varepsilon * \rho_l^\varepsilon\right)^{m-1} \mathrm{d}z\\ &\leqslant C_\varepsilon |x-y|.\end{aligned}$$

Then the existence and uniqueness hold for solutions of (1.7), both trajectorially and in law [39, theorem 1.1].

1.2. State of the art

Cross-diffusion systems have many applications in various fields, including biology, chemistry and population dynamics. We refer to [29] for a detailed discussion of cross-diffusion systems, particularly those admitting a gradient flow structure. There, the functional $\mathcal{A}$ is a functional of $\rho = (\rho_1, \ldots, \rho_n)$ given by

$$\mathcal{A}[\rho] = \int_{\mathbb{R}^d} e(\rho)\,\mathrm{d}x,$$

for some function $e: [0,+\infty)^n \to [0,+\infty)$. The corresponding system can then be written in the form

$$\partial_t \rho = \nabla \cdot \left(B(\rho)\nabla \frac{\delta \mathcal{A}}{\delta \rho}\right),$$

where the diffusion matrix $B(\rho)$ is positive and semidefinite. We have adopted this gradient flow structure and chosen a suitable functional $\mathcal{A}$ representing nonlinear diffusion modelling cell dynamics with volume exclusion [7, 11, 19] in tissue growth. In this work, we consider cross-diffusion models with more general pressure $P(\rho) = \left(\sum_{k=1}^{n} a_k \rho_k\right)^{m-1}$ where $m \geqslant 2$ compared to [17, 18, 30] with the addition of linear diffusion. These cross-diffusion systems are also related to aggregation-diffusion used in mathematical biology [1, 5, 10].

To derive macroscopic models from microscopic dynamics, one way is to take suitable scaling limit as the number of particles diverges. The mean field limit is one of the widely considered regimes. For deterministic cases, [24] provides a comprehensive review, while stochastic cases are discussed in [27, 39]. In the stochastic case, the N-particle microscopic dynamics is governed by SDEs (1st order system) as

$$\mathrm{d}X^i(t) = \frac{1}{N}\sum_{j=1}^{N} K\left(X^i(t) - X^j(t)\right)\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B^i(t),$$

where K is interaction kernel and B^i are i.i.d standard Brownian motion. Oelschläger proposed the moderate interaction scaling as

$$\mathrm{d}X^i(t) = -\frac{1}{N}\sum_{j=1}^{N} \nabla W^N\left(X^i(t) - X^j(t)\right)\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B^i(t),$$

where the kernel is written in gradient form and depends on the number of particles [32]. A usual choice is $W^N(x) = N^{\beta d} W(N^\beta x)$ where $\beta \in (0, \frac{1}{d+2})$, and W^N converges to a Dirac delta δ_0 when N goes to infinity. The term 'moderate' means that this nonlocal interaction is more local than the mean field regime, but when N goes to infinity, particles will move in an asymptotically deterministic force field. Oelschläger [32, 33] rigorously derived the viscous porous medium equation from this moderate interaction. This interaction regime has recently seen rising interest, see for example [4, 22, 25, 28] and its application on cell-cell adhesion [11, 12, 16, 20].

In the realm of deriving cross-diffusion systems from interacting particle systems, the literature is growing. Notably, [38] focuses on the chemotaxis models and [26] deals with reaction-diffusion equations. It considers Maxwell–Stefan equation as the hydrodynamic limit of two-component Brownian particles in [37], while [23] derives nonlocal Lotka–Volterra cross diffusion system as large population limit of point measure-value Markov processes. [15] derives cross-diffusion systems of Shigesada–Kawasaki–Teramoto (SKT) type from Markov processes with mean-field scaling. Moreover, [14] adopts the idea of moderate interaction, where they prove the many-particle system converges to an intermediate nonlocal diffusion system ($N \to \infty$), and then obtain local cross-diffusion system when interaction potentials approach the Dirac delta distribution ($\varepsilon \to 0$). Further work [13] derives SKT type cross-diffusion system from stochastic particle system, where a two-step limit is also applied. The cross-diffusion systems considered in [14] are of the form

$$\frac{\partial \rho_k}{\partial t} = \nabla \cdot \left(\sum_{l=1}^{n} a_{kl} \rho_k \nabla \rho_l \right) + \sigma_k \Delta \rho_k, \tag{1.8}$$

for smooth initial data, $\rho_{k,0} \in H^s(\mathbb{R}^d)$ with $s > d/2 + 1$, sufficiently small.

In [21], the authors consider the single-species viscous porous medium equation with exponent $m > 1$:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \rho^{m-1} \right) + \Delta \rho. \tag{1.9}$$

They generalised the result of [35] and [34] for $m = 2$ and proved that the (very) weak solution of the viscous porous medium equation can be obtained by the limit of solutions of the following nonlocal equations

$$\frac{\partial \rho^\varepsilon}{\partial t} = \nabla \cdot \left(\rho^\varepsilon \nabla \left(V^\varepsilon * \left(V^\varepsilon * \rho^\varepsilon \right)^{m-1} \right) \right) + \Delta \rho^\varepsilon.$$

The connection between nonlocal equations and porous-medium type equations sheds light on the particle approximation (see [6, 8, 9, 31]). The authors also derive the above nonlocal equation from stochastic particle system, thereby showing the convergence of the particle approximation of viscous porous medium equation.

Comparing to the system (1.8) and (1.9), we study a class of multi-species cross-diffusion system with the parameter of nonlinearity $m \geqslant 2$:

$$\partial_t \rho_k^\varepsilon = \nabla \cdot \left(b_k \rho_k^\varepsilon \nabla \left(\sum_{l=1}^{n} a_l \rho_l^\varepsilon \right)^{m-1} \right) + \sigma \Delta \rho_k^\varepsilon.$$

The main difficulty lies in obtaining higher integrability and regularity for each species due to the coupled structure. To prove the existence, we use nonlocal to local approximation, which can cover the cases with different mobility b_k. The higher regularity of each density is obtained by the linear diffusion, which is crucial to show the strong convergence in L^1. While the proof of uniqueness is more delicate where we have to assume the same mobility for different species. The result comes from the important observation that the sum of the densities satisfies a viscous porous medium equation, and the evolution of each species can be considered as a linear Fokker–Planck equation with the fixed pressure. In terms of particle approximation, as in [21], we also obtain a logarithmic scale relation between the number of particles N and the localisation parameter ε, and present it under the framework of propagation of chaos.

1.3. Notations

Throughout the paper, we use the following notations. Let $\mathcal{M}(\mathbb{R}^d)$ be the space of probability measure equipped with the following metric which measures the weak convergence in $\mathcal{M}(\mathbb{R}^d)$, for $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^d)$

$$d(\mu_1, \mu_2) := \sup_{f \in BL} \left| \int_{\mathbb{R}^d} f(x) \mu_1(\mathrm{d}x) - \int_{\mathbb{R}^d} f(x) \mu_2(\mathrm{d}x) \right|,$$

where the function space BL denotes the set of functions which are bounded with Lipschitz constant 1. Let $C_b^2(\mathbb{R}^d)$ be the space of bounded and twice differentiable functions, which is common to be used as the space for test functions. We denote the weighted space-time L^p-function space by $L^p([0,T] \times \mathbb{R}^d, \mu \mathrm{d}x\mathrm{d}t)$, where the weight μ on $[0,T] \times \mathbb{R}^d$ is given in the form $\mu = \mu(t)\mathrm{d}t$ by means of a family of nonnegative measures $\left(\mu(t)\right)_{t \in [0,T]}$ on $\mathbb{R}^d$ and the norm is given as

$$\| \cdot \|^p_{L^p([0,T] \times \mathbb{R}^d, \mu \mathrm{d}x\mathrm{d}t)} := \int_0^T \int_{\mathbb{R}^d} | \cdot |^2 \mu(\mathrm{d}x\mathrm{d}t).$$

B_R denotes the closed ball in $\mathbb{R}^d$ centred at origin with radius R, while B_R^c is its complementary set. Also, we denote the M-fold tensorisation $f^{\otimes M}$ on $\mathbb{R}^{dM}$ by

$$f^{\otimes M}\left(x^1, x^2, \ldots, x^M\right) := f\left(x^1\right) f\left(x^2\right) \cdots f\left(x^M\right),$$

where f is a function on $\mathbb{R}^d$.

1.4. Main results

The well-posedness of nonlinear processes (1.7) implies the following proposition.

Proposition 1.1. *Assume initial data $\rho_{k,0}$ is a probability measure and with density $\rho_{k,0} \in L^1 \cap L^\infty(\mathbb{R}^d)$, then there exists a measure-valued solution $\rho_k^\varepsilon \in C([0,T], \mathcal{M}(\mathbb{R}^d))$ of (1.5).*

Actually, we can obtain higher regularity of solutions of the nonlocal intermediate PDE (1.5), but the statement in the proposition above is enough for our argument in this paper.

The quantitative error estimate between particles and nonlinear process is as follows.

Proposition 1.2 (error estimate of the stochastic systems). *Under the assumptions above, the distance between the strong solutions of SDEs (1.6) and (1.7) can be estimated as, for fixed $\varepsilon > 0$*

$$\sum_{k=1}^n \mathbb{E}\left[\sup_{0 \leqslant s \leqslant t} \left|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)\right|^2\right] \leqslant \frac{C(\varepsilon, t)}{N},$$

where the constant $C(\varepsilon, t)$ can be made explicitly.

See section 2 for the proof of this proposition. In terms of the distribution of particles, we have the following remark.

Remark 1.3. By the definition of 2-Wasserstein metric, for any kth species, the distance between the one-particle distribution $\rho_k^{(1),N,\varepsilon}(t) = \mathrm{Law}(X_k^{i,\varepsilon}(t))$ and $\rho_k^\varepsilon(t) = \mathrm{Law}(Y_k^{i,\varepsilon}(t))$ can be estimated as follows, for $t \in [0,T]$,

$$\begin{aligned}
W_2^2\left(\rho_k^{(1),N,\varepsilon}(t), \rho_k^\varepsilon(t)\right) &\leqslant \sum_{k=1}^n W_2^2\left(\rho_k^{(1),N,\varepsilon}(t), \rho_k^\varepsilon(t)\right) \leqslant \sum_{k=1}^n \mathbb{E}\left[\left|X_k^{i,\varepsilon}(t) - Y_k^{i,\varepsilon}(t)\right|^2\right] \\
&\leqslant \sum_{k=1}^n \mathbb{E}\left[\sup_{0 \leqslant s \leqslant T}\left|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)\right|^2\right] \leqslant \frac{C(\varepsilon, T)}{N}.
\end{aligned}$$

Remark 1.4. According to proposition 1.2 and the expression of $C(\varepsilon, t)$ (see (2.2)), we can take suitable logarithmic dependence of ε and N as $\varepsilon = \varepsilon(N)$ which goes to 0 when N goes to ∞. Then it holds for $t \in [0,T]$,

$$W_2\left(\rho_k^{(1),N,\varepsilon(N)}(t), \rho_k^{\varepsilon(N)}(t)\right) \to 0, \text{as } N \to \infty.$$

Now we define the weak solution of the cross-diffusion system (1.1):

Definition 1.5. A weak solution $\rho = (\rho_1, \ldots, \rho_n)$ of the cross-diffusion system (1.1) on the time interval $[0,T]$ satisfies that, for each species k,

(1) $\rho_k \in C([0,T], \mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution with initial data $\rho_{k,0} \in L^1 \cap L^\infty(\mathbb{R}^d)$;
(2) for almost every $t \in [0,T]$, $\rho_k(t)$ is absolutely continuous with respect to Lebesgue measure (for simplicity which is also denoted by $\rho_k(t)$), and $\rho_k \in L^m([0,T] \times \mathbb{R}^d)$;
(3) $\left(\sum_{l=1}^n a_l \rho_l\right)^{m-1} \in L^{\frac{m}{m-1}}(0,T; W^{1,\frac{m}{m-1}}(\mathbb{R}^d))$;
(4) for almost any $t \in [0,T]$ and $f \in C^1([0,T], C_b^2(\mathbb{R}^d))$, it holds

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(t,x)\,\rho_k(t,x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} \partial_s f(s,x)\,\rho_k(s,x)\,\mathrm{d}x\mathrm{d}s = \int_{\mathbb{R}^d} f(0,x)\,\rho_{k,0}(x)\,\mathrm{d}x \\
&\quad + \sigma \int_0^t \int_{\mathbb{R}^d} \Delta f(s,x)\,\rho_k(s,x)\,\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} b_k \rho_k(s,x)\,\nabla f(s,x) \cdot \nabla\left(\sum_{l=1}^n a_l \rho_l(s,x)\right)^{m-1} \mathrm{d}x\mathrm{d}s. \qquad (1.10)
\end{aligned}$$

The following theorems give the well-posedness of the cross-diffusion system (1.1).

Theorem 1.6 (existence). *Up to a subsequence, the solutions of nonlocal equation* (1.5) $(\rho_k^\varepsilon)_{\varepsilon>0}$ *converge in* $C([0,T],\mathcal{M}(\mathbb{R}^d))$ *to* ρ_k *a weak solution of* (1.1)*.*

See section 3 for the proof of this theorem.

Remark 1.7. We emphasise that theorem 1.6 characterise all possible adherence points of the convergent subsequences as ε goes to 0, as weak solutions of the cross-diffusion system (1.1).

Theorem 1.8 (uniqueness). *If we assume further that all the species have the same mobility, i.e.* $b_1=\ldots=b_n=b>0$*, then there exists a unique weak solution of the cross-diffusion equation* (1.1) *defined as in definition* 1.5*.*

We give the proof of the uniqueness result in section 4.

Remark 1.9. It is an open problem to show the uniqueness for (1.1) without the assumption of having the same mobility.

As the direct consequence of proposition 1.2 and theorems 1.6, 1.10 together with corollary 1.11 is our second main result concerning particle approximation.

Theorem 1.10 (particle approximation). *Under the assumptions of theorem* 1.8*, for almost any* $t\in[0,T]$ *and any species k, the distribution of the particle* (1.6) *converges to the weak solution of the cross-diffusion system* (1.1) *when N goes to infinity, and then* ε *goes to 0 that*

$$\lim_{\varepsilon\to0}\lim_{N\to\infty}\rho_k^{(1),N,\varepsilon}(t)=\rho_k(t).$$

In fact, we can take ε depending on N as in remark 1.4 and combine the two-step limit into one as $\lim_{N\to\infty}\rho_k^{(1),N,\varepsilon(N)}(t)=\rho_k(t)$.

Let M be a fixed natural number and $\rho_k^{(M),N,\varepsilon}$ is the joint law of $X_k^{i,\varepsilon}$, $i=1,2,\ldots,M$ on $\mathbb{R}^{dM}$, i.e. the M-marginal of the joint law of N particles. We denote the independently tensorised solution of cross-diffusion system on $\mathbb{R}^{dM}$ by $\rho_k^{\otimes M}$. We obtain the following propagation of chaos result, Under the assumptions of T

Corollary 1.11. *Under the assumptions of theorem* 1.10*, it holds*

$$\lim_{\varepsilon\to0}\lim_{N\to\infty}\rho_k^{(M),N,\varepsilon}(t)=\rho_k^{\otimes M}(t).$$

The paper is organised as follows. Section 2 delves into the error estimate between moderately interacting particle system and associated nonlinear nonlocal process; in section 3 we investigate the convergence from nonlocal to local cross-diffusion system, which implies the existence of the limiting cross-diffusion system. Section 4 shows the uniqueness of the cross-diffusion system.

2. Proof of error estimate for the stochastic systems

In this section, we investigate the large N limit of the particle system (1.6). In particularly, we will prove the convergence $\lim_{N\to\infty}\rho_k^{(1),N,\varepsilon}=\rho_k^\varepsilon$.

Since $X_k^{i,\varepsilon}(0)=Y_k^{i,\varepsilon}(0)=\xi_k^i$, Hölder's inequality implies

$$\begin{aligned}\left|X_k^{i,\varepsilon}(t)-Y_k^{i,\varepsilon}(t)\right|^2=&\left|\int_0^t\int_{\mathbb{R}^d}b_k\nabla V^\varepsilon(z)\left[\left(\sum_{l=1}^n\frac{a_l}{N}\sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s)-X_l^{j,\varepsilon}(s)-z\right)\right)^{m-1}\right.\right.\\&\left.\left.-\left(\sum_{l=1}^n a_lV^\varepsilon*\rho_l^\varepsilon\left(s,Y_k^{i,\varepsilon}(s)-z\right)\right)^{m-1}\right]\mathrm{d}z\mathrm{d}s\right|^2\\\leqslant&\, t\int_0^t\left(\int_{\mathbb{R}^d}b_k|\nabla V^\varepsilon(z)|\left[\left(\sum_{l=1}^n\frac{a_l}{N}\sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s)-X_l^{j,\varepsilon}(s)-z\right)\right)^{m-1}\right.\right.\\&\left.\left.-\left(\sum_{l=1}^n a_lV^\varepsilon*\rho_l^\varepsilon\left(s,Y_k^{i,\varepsilon}(s)-z\right)\right)^{m-1}\right]\mathrm{d}z\right)^2\mathrm{d}s.\end{aligned}$$

The following equality holds by the scaling of $V^\varepsilon(\cdot) = \varepsilon^{-d}V(\cdot/\varepsilon)$,

$$\int_{\mathbb{R}^d} |\nabla V^\varepsilon(z)|\,\mathrm{d}z = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} |\nabla V(z)|\,\mathrm{d}z < \frac{C_V}{\varepsilon},$$

where C_V is independent with ε, then

$$\begin{aligned}
\left|X_k^{i,\varepsilon}(t) - Y_k^{i,\varepsilon}(t)\right|^2 \leqslant{}& t b_k^2 \int_0^t \left| \int_{\mathbb{R}^d} \nabla V^\varepsilon(z)\,\mathrm{d}z \times \sup_{y\in\mathbb{R}^d} \left[\left(\sum_{l=1}^n \frac{a_l}{N} \sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - X_l^{j,\varepsilon}(s) - y\right)\right)^{m-1} \right.\right. \\
&\left.\left. - \left(\sum_{l=1}^n a_l V^\varepsilon * \rho_l^\varepsilon\left(s, Y_k^{i,\varepsilon}(s) - y\right)\right)^{m-1}\right]\right|^2 \mathrm{d}s \\
\leqslant{}& \frac{C_V^2 b_k^2 t}{\varepsilon^2} \int_0^t \sup_{y\in\mathbb{R}^d} \left| \left(\sum_{l=1}^n \frac{a_l}{N} \sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - X_l^{j,\varepsilon}(s) - y\right)\right)^{m-1} \right. \\
&\left. - \left(\sum_{l=1}^n a_l V^\varepsilon * \rho_l^\varepsilon\left(s, Y_k^{i,\varepsilon}(s) - y\right)\right)^{m-1}\right|^2 \mathrm{d}s.
\end{aligned} \tag{2.1}$$

When $m \geqslant 2$, it holds that

$$\begin{aligned}
&\left| \left(\sum_{l=1}^n \frac{a_l}{N} \sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - X_l^{j,\varepsilon}(s) - y\right)\right)^{m-1} - \left(\sum_{l=1}^n a_l V^\varepsilon * \rho_l^\varepsilon\left(s, Y_k^{i,\varepsilon}(s) - y\right)\right)^{m-1}\right|^2 \\
&\leqslant C\|V^\varepsilon\|_{L^\infty}^{2m-4} \left| \sum_{l=1}^n \frac{a_l}{N} \sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - X_l^{j,\varepsilon}(s) - y\right) - \sum_{l=1}^n a_l V^\varepsilon * \rho_l^\varepsilon\left(s, Y_k^{i,\varepsilon}(s) - y\right)\right|^2.
\end{aligned}$$

And the quadratic term can be estimated as follows

$$\begin{aligned}
&\left| \sum_{l=1}^n \frac{a_l}{N} \sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - X_l^{j,\varepsilon}(s) - y\right) - \sum_{l=1}^n a_l V^\varepsilon * \rho_l^\varepsilon\left(s, Y_k^{i,\varepsilon}(s) - y\right)\right|^2 \\
&\leqslant n\sum_{l=1}^n a_l^2 \left| \frac{1}{N}\sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - X_l^{j,\varepsilon}(s) - y\right) - V^\varepsilon * \rho_l^\varepsilon\left(s, Y_k^{i,\varepsilon}(s) - y\right)\right|^2 \\
&\leqslant 3n\sum_{l=1}^n a_l^2 \left| \frac{1}{N}\sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - X_l^{j,\varepsilon}(s) - y\right) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - Y_l^{j,\varepsilon}(s) - y\right)\right|^2 \\
&\quad + 3n\sum_{l=1}^n a_l^2 \left| \frac{1}{N}\sum_{j=1}^N V^\varepsilon\left(X_k^{i,\varepsilon}(s) - Y_l^{j,\varepsilon}(s) - y\right) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon\left(Y_k^{i,\varepsilon}(s) - Y_l^{j,\varepsilon}(s) - y\right)\right|^2 \\
&\quad + 3n\sum_{l=1}^n a_l^2 \left| \frac{1}{N}\sum_{j=1}^N V^\varepsilon\left(Y_k^{i,\varepsilon}(s) - Y_l^{j,\varepsilon}(s\right) - y) - V^\varepsilon * \rho_l^\varepsilon(s, Y_k^{i,\varepsilon}(s) - y)\right|^2 \\
&=: J_1^{i,k} + J_2^{i,k} + J_3^{i,k}.
\end{aligned}$$

For any $t\in[0,T]$, we take the expectation of (2.1) to obtain

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)|^2\right] \leqslant C_0(\varepsilon)\, t \int_0^t \sup_{y\in\mathbb{R}^d}\left(\mathbb{E}\left[J_1^{i,k}\right] + \mathbb{E}\left[J_2^{i,k}\right] + \mathbb{E}\left[J_3^{i,k}\right]\right)\mathrm{d}s,$$

where the constant

$$C_0(\varepsilon) \sim \frac{\|V^\varepsilon\|_{L^\infty}^{2m-4}}{\varepsilon^2} \sim \frac{1}{\varepsilon^{2md-4d+2}}.$$

The first term $J_1^{i,k}$ and the second term $J_2^{i,k}$ can be estimated thanks to the Lipschitz continuity of V^ε fixed ε as

$$J_1^{i,k} \leqslant 3n\|\nabla V^\varepsilon\|_{L^\infty}^2 \sum_{l=1}^n \frac{a_l^2}{N}\sum_{j=1}^N \left|X_l^{j,\varepsilon}(s) - Y_l^{j,\varepsilon}(s)\right|^2,$$

and

$$J_2^{i,k} \leqslant 3n\|\nabla V^\varepsilon\|_{L^\infty}^2 \left|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)\right|^2 \sum_{l=1}^n a_l^2.$$

For the third term $J_3^{i,k}$, we let $Z^i := Y_k^{i,\varepsilon} - y$ to simplify the computation

$$\begin{aligned}
\mathbb{E}\left[J_3^{i,k}\right] &= 3n\sum_{l=1}^n a_l^2 \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N V^\varepsilon\left(Z^i - Y_l^{j,\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right|^2\right] \\
&\leqslant 3n\sum_{l=1}^n \frac{a_l^2}{N^2}\sum_{j,j'}\mathbb{E}\left[\left(V^\varepsilon\left(Z^i - Y_l^{j,\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\left(V^\varepsilon\left(Z^i - Y_l^{j',\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\right],
\end{aligned}$$

Recalling the definition of nonlinear process (1.7), we can see that the randomness of $Y_l^{j,\varepsilon}$ for different index j comes from i.i.d Brownian motions B_l^j and initial data $Y_l^{j,\varepsilon}(0)$. When $i \neq j \neq j'$, the sum vanishes because

$$\begin{aligned}
&\mathbb{E}\left[\left(V^\varepsilon\left(Z^i - Y_l^{j,\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\left(V^\varepsilon\left(Z^i - Y_l^{j',\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\right] \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[\left(V^\varepsilon\left(Z^i - Y_l^{j,\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\left(V^\varepsilon\left(Z^i - Y_l^{j',\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\Big|Z^i\right]\right] \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[V^\varepsilon\left(Z^i - Y_l^{j,\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\Big|Z^i\right]\mathbb{E}\left[V^\varepsilon\left(Z^i - Y_l^{j',\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\Big|Z^i\right]\right] = 0,
\end{aligned}$$

where the last line is due to that $Y_l^{j,\varepsilon}(s)$ and $Y_l^{j',\varepsilon}(s)$ have the same distribution $\rho_l^\varepsilon(s)$ for any $s \in [0,T]$. Fixed the index i, number of elements in the set

$$S = \{j, j' | \text{At least two of indexes } i, j, j' \text{ are equal}\}$$

is $3N - 2$. Thus we can bound $\mathbb{E}[J_3^{i,k}]$ as

$$\begin{aligned}
\mathbb{E}\left[J_3^{i,k}\right] =& 3n\sum_{l=1}^n \frac{a_l^2}{N^2}\sum_{i\neq j\neq j'}\mathbb{E}\left[\left(V^\varepsilon\left(Z^i - Y_l^{j,\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\left(V^\varepsilon\left(Z^i - Y_l^{j',\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\right] \\
&+ 3n\sum_{l=1}^n \frac{a_l^2}{N^2}\sum_{S}\mathbb{E}\left[\left(V^\varepsilon\left(Z^i - Y_l^{j,\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\left(V^\varepsilon\left(Z^i - Y_l^{j',\varepsilon}\right) - V^\varepsilon * \rho_l^\varepsilon\left(Z^i\right)\right)\right]\cdot \\
\leqslant& \frac{12(3N-2)n^2\|V^\varepsilon\|_{L^\infty}^2 \sum_{l=1}^n a_l^2}{N^2}.
\end{aligned}$$

Now we possess all ingredients to estimate $\mathbb{E}\left[\sup_{0\leqslant s\leqslant t}|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)|^2\right]$ as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leqslant s\leqslant t}|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)|^2\right] \\
&\quad\leqslant C_0(\varepsilon)\, t\int_0^t \sup_{y\in\mathbb{R}^d}\left(\mathbb{E}\left[J_1^{i,k}\right] + \mathbb{E}\left[J_2^{i,k}\right] + \mathbb{E}\left[J_3^{i,k}\right]\right)\mathrm{d}s \\
&\quad\leqslant C_0(\varepsilon)\, t\int_0^t \Bigg(3n\|\nabla V^\varepsilon\|_{L^\infty}^2 \sum_{l=1}^n a_l^2 \mathbb{E}\left|X_l^{i,\varepsilon}(s) - Y_l^{i,\varepsilon}(s)\right|^2 \\
&\qquad + 3n\|\nabla V^\varepsilon\|_{L^\infty}^2 \mathbb{E}\left|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)\right|^2 \sum_{l=1}^n a_l^2 + \frac{12(3N-2)n^2\|V^\varepsilon\|_{L^\infty}^2 \sum_{l=1}^n a_l^2}{N^2}\Bigg)\mathrm{d}s.
\end{aligned}$$

We sum up species index k from 1 to n, then we can see that

$$\sum_{k=1}^n \mathbb{E}\left[\sup_{0\leqslant s\leqslant t}|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)|^2\right] \leqslant \frac{C_1(\varepsilon)t^2}{N} + C_2(\varepsilon, T)\int_0^t \sum_{k=1}^n \mathbb{E}\left|X_k^{i,\varepsilon}(s) - Y_k^{i,\varepsilon}(s)\right|^2 \mathrm{d}s,$$

where

$$C_1(\varepsilon) \sim C_0(\varepsilon)\|V^\varepsilon\|_{L^\infty}^2 \sim \frac{1}{\varepsilon^{2md-2d+2}}, \quad C_2(\varepsilon, T) \sim C_0(\varepsilon)\|\nabla V^\varepsilon\|_{L^\infty}^2 \sim \frac{1}{\varepsilon^{2md-2d+4}}.$$

Gronwall's inequality implies the estimate as

$$\sum_{k=1}^{n}\mathbb{E}\left[\sup_{0\leqslant s\leqslant t}|X_k^{i,\varepsilon}(s)-Y_k^{i,\varepsilon}(s)|^2\right]\leqslant\frac{2C_1(\varepsilon)}{N}\int_0^t se^{-C_2(\varepsilon,T)s}\mathrm{d}s\leqslant\frac{C(\varepsilon,t)}{N},$$

where

$$C(\varepsilon,t)=\frac{2C_1(\varepsilon)}{(C_2(\varepsilon,T))^2}e^{C_2(\varepsilon,T)t}\sim\varepsilon^{6+2d(m-1)}\exp\left(t/\varepsilon^{4+2d(m-1)}\right). \tag{2.2}$$

3. Proof of existence for the cross-diffusion systems

In this section, we will prove the nonlocal to local convergence, i.e. for any species k, the measure-valued solution ρ_k^ε of equation (1.5) converges to a weak solution ρ_k of the cross-diffusion system (1.1) when ε goes to 0 (up to a subsequence).

Let us define the nonnegative functions $g^\varepsilon:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ as

$$g^\varepsilon(t,x)=V^\varepsilon*\left(\sum_{l=1}^{n}a_lV^\varepsilon*\rho_l^\varepsilon(t)\right)^{m-1}(x),$$

further define the regularised solution of nonlocal equation (1.5) as

$$u_k^\varepsilon=V^\varepsilon*\rho_k^\varepsilon,$$

which is also a nonnegative probability measure, then $g^\varepsilon=V^\varepsilon*\left(\sum_{l=1}^{n}a_lu_l^\varepsilon\right)^{m-1}$. Then we convolve both sides of (1.5) with V^ε to obtain the equality

$$\partial_tu_k^\varepsilon=\nabla\cdot(b_k\rho_k^\varepsilon\nabla g^\varepsilon)*V^\varepsilon+\sigma\Delta u_k^\varepsilon=(b_k\rho_k^\varepsilon\nabla g^\varepsilon)*\nabla V^\varepsilon+\sigma\Delta u_k^\varepsilon,$$

which leads to

$$\partial_t(a_ku_k^\varepsilon)=(a_kb_k\rho_k^\varepsilon\nabla g^\varepsilon)*\nabla V^\varepsilon+a_k\sigma\Delta u_k^\varepsilon.$$

Summing up species index k from 1 to n, testing against it by $(\sum_k a_ku_k^\varepsilon)^{m-1}$ and integrating in time, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\sum_k a_ku_k^\varepsilon(t)\right)^m\mathrm{d}x-\int_{\mathbb{R}^d}\left(\sum_k a_ku_k^\varepsilon(0)\right)^m\mathrm{d}x\\
&=\int_0^t\int_{\mathbb{R}^d}\left(\sum_k a_ku_k^\varepsilon\right)^{m-1}\left(\sum_k a_kb_k\rho_k^\varepsilon\nabla g^\varepsilon\right)*\nabla V^\varepsilon\mathrm{d}x\mathrm{d}s\\
&\quad+\sigma\int_0^t\int_{\mathbb{R}^d}\left(\sum_k a_ku_k^\varepsilon\right)^{m-1}\Delta\left(\sum_k a_ku_k^\varepsilon\right)\mathrm{d}x\mathrm{d}s\\
&=-\int_0^t\int_{\mathbb{R}^d}\nabla V^\varepsilon*\left(\sum_k a_ku_k^\varepsilon\right)^{m-1}\cdot\left(\sum_k a_kb_k\rho_k^\varepsilon\nabla g^\varepsilon\right)\mathrm{d}x\mathrm{d}s\\
&\quad-(m-1)\sigma\int_0^t\int_{\mathbb{R}^d}\left(\sum_k a_ku_k^\varepsilon\right)^{m-2}\left|\nabla\left(\sum_k a_ku_k^\varepsilon\right)\right|^2\mathrm{d}x\mathrm{d}s\\
&=-\sum_k\int_0^t\int_{\mathbb{R}^d}a_kb_k\left|\nabla g^\varepsilon\right|^2\rho_k^\varepsilon(\mathrm{d}x)\mathrm{d}s-(m-1)\sigma\int_0^t\int_{\mathbb{R}^d}\left(\sum_k a_ku_k^\varepsilon\right)^{m-2}\left|\nabla\left(\sum_k a_ku_k^\varepsilon\right)\right|^2\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

where we applied the following fact in the second equality, for some integrable f and h,

$$\int_{\mathbb{R}^d}f(x)(h*\nabla V^\varepsilon)(x)\mathrm{d}x=-\int_{\mathbb{R}^d}(\nabla V^\varepsilon*f)(x)h(x)\mathrm{d}x.$$

By the assumption $\rho_{k,0}\in L^1\cap L^\infty\subset L^m$ and $u_k^\varepsilon(0)=\rho_{k,0}*V^\varepsilon$, which implies

$$\|u_k^\varepsilon(0)\|_{L^m(\mathbb{R}^d)}\leqslant\|\rho_{k,0}\|_{L^m(\mathbb{R}^d)}<\infty.$$

Then we get the uniform in ε estimate as follows.

Lemma 3.1. *For each species k and $t \geqslant 0$, the following estimate holds*

$$\left\| \sum_k a_k u_k^\varepsilon (t) \right\|_{L^m}^m + \sum_k \int_0^t \int_{\mathbb{R}^d} a_k b_k \left| \nabla g^\varepsilon \right|^2 \rho_k^\varepsilon (s, \mathrm{d}x) \\ + (m-1)\sigma \int_0^t \int_{\mathbb{R}^d} \left(\sum_k a_k u_k^\varepsilon \right)^{m-2} \left| \nabla \left(\sum_k a_k u_k^\varepsilon \right) \right|^2 \leqslant \left\| \sum_k a_k \rho_{k,0} \right\|_{L^m}^m < \infty.$$

Remark 3.2. From above lemma, we can deduce that for each k the nonnegative sequence $(u_k^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty([0,T], L^m(\mathbb{R}^d))$. And the equality

$$\int_0^t \int_{\mathbb{R}^d} \left(\sum_k a_k u_k^\varepsilon \right)^{m-2} \left| \nabla \left(\sum_k a_k u_k^\varepsilon \right) \right|^2 \mathrm{d}x\mathrm{d}s = \frac{4}{m^2} \int_0^t \int_{\mathbb{R}^d} \left| \nabla \left(\sum_k a_k u_k^\varepsilon \right)^{\frac{m}{2}} \right|^2 \mathrm{d}x\mathrm{d}s,$$

implies the sequence $\left((\sum_k a_k u_k^\varepsilon)^{m/2} \right)_{\varepsilon>0}$ is bounded in $L^2([0,T], H^1(\mathbb{R}^d))$; and for each k,

$$\int_0^t \int_{\mathbb{R}^d} \left| \nabla g^\varepsilon (s,x) \right|^2 \rho_k^\varepsilon (s, \mathrm{d}x) \, \mathrm{d}s \text{ is uniformly bounded in } \varepsilon.$$

Notice that we are not able to get higher regularity for u_k^ε from the estimate above, but only for the sum $\sum_k a_k u_k^\varepsilon$.

We now state the following lemma.

Lemma 3.3. *For each k, the sequence $(\rho_k^\varepsilon)_{\varepsilon>0}$ is relatively compact in $C\big([0,T], \mathcal{M}(\mathbb{R}^d)\big)$.*

Proof. To apply the Arzelà–Ascoli theorem, we need to verify the following two claims, for each kth species,

(1) there is a relatively compact subset $\mathcal{K}_k \subset \mathcal{M}(\mathbb{R}^d)$ which is independent of ε and t, that for any $t \in [0,T]$ and $\varepsilon > 0$, $\rho_k^\varepsilon(t) \in \mathcal{K}_k$;
(2) the sequence $(\rho_k^\varepsilon)_{\varepsilon>0}$ is equicontinuous, i.e. for every $\eta > 0$ there exists δ such that, for all $\varepsilon > 0$ and $t, s \in [0,T]$ such that $|t-s| < \delta$, then it implies $d(\rho_k^\varepsilon(s), \rho_k^\varepsilon(t)) < \eta$.

We start with proving the first statement. A subset of $\mathcal{M}(\mathbb{R}^d)$ is relatively compact if and only if it is tight, then it is equivalent to show for any $t \in [0,T]$ and $\eta > 0$, there exists a compact set $K_k \subset \mathbb{R}^d$ with $\rho_k^\varepsilon(K_k) \geqslant 1-\eta$ for all $\varepsilon > 0$. Recall the nonlinear process $Y_k^\varepsilon(t)$ defined by (1.7) with $\mathrm{Law}(Y_k^\varepsilon(t)) = \rho_k^\varepsilon(t)$ satisfies the SDE

$$\mathrm{d}Y_k^\varepsilon (t) = -b_k \nabla g^\varepsilon (t, Y_k^\varepsilon (t)) \, \mathrm{d}t + \sqrt{2\sigma} \mathrm{d}B_k (t) \,.$$

Then $\rho_k^\varepsilon(K_k) \geqslant 1-\eta$ is equivalent to $\mathbb{P}\big[Y_k^\varepsilon(t) \in K_k^c\big] \leqslant \eta$. We can take the compact set as a closed ball with radius $R > 0$, then the probability of Y_k^ε being outside the closed ball can be estimated as

$$\mathbb{P}\left[|Y_k^\varepsilon (t)| > R \right] = \mathbb{P}\left[\left| Y_k^\varepsilon (0) - b_k \int_0^t \nabla g^\varepsilon (s, Y_k^\varepsilon (s)) \, \mathrm{d}s + \sqrt{2\sigma} B_k (t) \right| > R \right] \\ \leqslant \mathbb{P}\left[|Y_k^\varepsilon (0)| > \frac{R}{3} \right] + \mathbb{P}\left[\left| b_k \int_0^t \nabla g^\varepsilon (s, Y_k^\varepsilon (s)) \, \mathrm{d}s \right| > \frac{R}{3} \right] + \mathbb{P}\left[\left| \sqrt{2\sigma} B_k (t) \right| > \frac{R}{3} \right],$$

where the first term and the third term goes to 0 as $R \to \infty$. For the second term, we deduce

$$\mathbb{P}\left[\left| b_k \int_0^t \nabla g^\varepsilon (s, Y_k^\varepsilon (s)) \, \mathrm{d}s \right| > \frac{R}{3} \right] \leqslant \frac{9}{R^2} \mathbb{E}\left[\left| b_k \int_0^t \nabla g^\varepsilon (s, Y_k^\varepsilon (s)) \, \mathrm{d}s \right|^2 \right] \\ \leqslant \frac{9tb_k^2}{R^2} \mathbb{E}\left[\int_0^t |\nabla g^\varepsilon (s, Y_k^\varepsilon (s))|^2 \, \mathrm{d}s \right] \\ = \frac{9tb_k^2}{R^2} \int_0^t \int_{\mathbb{R}^d} \left| \nabla g^\varepsilon (s,x) \right|^2 \rho_k^\varepsilon (s, \mathrm{d}x) \, \mathrm{d}s,$$

which goes to 0 by sending R to ∞ by remark 3.2.

Now we prove the second claim. For $s,t\in[0,T]$, the distance between $\rho_k^\varepsilon(s)$ and $\rho_k^\varepsilon(t)$ has the following estimate

$$\begin{aligned} d\left(\rho_k^\varepsilon(t),\rho_k^\varepsilon(s)\right) &= \sup_{f\in BL}\left|\int_{\mathbb{R}^d} f(x)\,\rho_k^\varepsilon(t,\mathrm{d}x)-\int_{\mathbb{R}^d} f(x)\,\rho_k^\varepsilon(s,\mathrm{d}x)\right| \\ &= \sup_{f\in BL}\left|\mathbb{E}\left[f(Y_k^\varepsilon(t))\right]-\mathbb{E}\left[f(Y_k^\varepsilon(s))\right]\right| \\ &\leqslant \left(\mathbb{E}\left[\left|Y_k^\varepsilon(t)-Y_k^\varepsilon(s)\right|^2\right]\right)^{1/2}, \end{aligned}$$

and by Minkowski's inequality

$$\begin{aligned} \left(\mathbb{E}\left[\left|Y_k^\varepsilon(t)-Y_k^\varepsilon(s)\right|^2\right]\right)^{1/2} &= \left(\mathbb{E}\left[\left|b_k\int_s^t \nabla g^\varepsilon\left(r,Y_k^\varepsilon(r)\right)\mathrm{d}r+\sqrt{2\sigma}B_k(t)-\sqrt{2\sigma}B_k(s)\right|^2\right]\right)^{1/2} \\ &\leqslant b_k\left(\mathbb{E}\left[\left|\int_s^t \nabla g^\varepsilon\left(r,Y_k^\varepsilon(r)\right)\mathrm{d}r\right|^2\right]\right)^{1/2}+\sqrt{2\sigma}\left(\mathbb{E}\left[\left|B_k(t)-B_k(s)\right|^2\right]\right)^{1/2} \\ &\leqslant b_k\left(\mathbb{E}\left[|t-s|\int_s^t\left|\nabla g^\varepsilon\left(r,Y_k^\varepsilon(r)\right)\right|^2\mathrm{d}r\right]\right)^{1/2}+\sqrt{2\sigma}|t-s|^{1/2} \\ &= |t-s|^{1/2}\left[b_k\left[\int_s^t\int_{\mathbb{R}^d}\left|\nabla g^\varepsilon(r,x)\right|^2\rho_k^\varepsilon(r,\mathrm{d}x)\mathrm{d}r\right)^{1/2}+\sqrt{2\sigma}\right] \\ &\leqslant C|t-s|^{1/2}, \end{aligned}$$

where the constant C is independent with ε by remark 3.2 again. In conclusion, lemma 3.3 is proved. □

We have shown that for each species k the sequence $(\rho_k^\varepsilon)_{\varepsilon>0}$ has a convergent subsequence. We now fix such a convergent subsequence, which is still denoted by $(\rho_k^\varepsilon)_{\varepsilon>0}$. Let $\rho_k\in C\left([0,T],\mathcal{M}\left(\mathbb{R}^d\right)\right)$ be its limit, i.e.

$$\rho_k^\varepsilon\to\rho_k \quad \text{in} \quad C\left([0,T],\mathcal{M}\left(\mathbb{R}^d\right)\right)\text{ as }\varepsilon\to 0. \tag{3.1}$$

Lemma 3.4. *For each species k, the sequence $(u_k^\varepsilon)_{\varepsilon>0}$ converges to ρ_k in $C\left([0,T],\mathcal{M}\left(\mathbb{R}^d\right)\right)$ up to a subsequence.*

Proof. The lemma can be implied by

$$\sup_{0\leqslant t\leqslant T} d\left(\rho_k^\varepsilon(t),u_k^\varepsilon(t)\right)\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

To verify this, we notice that for any $t\in[0,T]$ and $f\in BL$, the following equality holds

$$\begin{aligned} \left|\int_{\mathbb{R}^d} f(x)\,u_k^\varepsilon(t,x)\,\mathrm{d}x-\int_{\mathbb{R}^d} f(x)\,\rho_k^\varepsilon(t,\mathrm{d}x)\right| &= \left|\int_{\mathbb{R}^d} f(x)\left(\rho_k^\varepsilon(t)*V^\varepsilon\right)(x)\,\mathrm{d}x-\int_{\mathbb{R}^d} f(x)\,\rho_k^\varepsilon(t,\mathrm{d}x)\right| \\ &= \left|\int_{\mathbb{R}^d}\left(\left(f*V^\varepsilon\right)(x)-f(x)\right)\rho_k^\varepsilon(t,\mathrm{d}x)\right|, \end{aligned}$$

where V^ε is even. And then it holds

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} f(x)\,u_k^\varepsilon(t,x)\,\mathrm{d}x-\int_{\mathbb{R}^d} f(x)\,\rho_k^\varepsilon(t,\mathrm{d}x)\right| \\ &\quad\leqslant \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|f(x+y)-f(y)\right|V^\varepsilon(x)\,\mathrm{d}x\right)\rho_k^\varepsilon(t,\mathrm{d}y) \\ &\quad\leqslant \int_{\mathbb{R}^d}|x|V^\varepsilon(x)\,\mathrm{d}x\int_{\mathbb{R}^d}\rho_k^\varepsilon(t,\mathrm{d}y)=C\varepsilon, \end{aligned}$$

which implies our lemma. □

By *a priori* estimate lemma 3.1, for any species k, the sequence $(u_k^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty([0,T],L^m(\mathbb{R}^d))$. Banach–Alaoglu theorem implies that, up to a subsequence, it weakly* converges in $L^\infty([0,T],L^m(\mathbb{R}^d))$. Thus by lemma 3.4, we get $\rho_k\in L^\infty([0,T],L^m(\mathbb{R}^d))$. Next, we are going to prove the convergence also holds in $L^m([0,T]\times\mathbb{R}^d)$.

Lemma 3.5. *For each species k, up to a subsequence,* $(u_k^\varepsilon)_{\varepsilon>0}$ *strongly converges in* $L^m([0,T]\times\mathbb{R}^d)$ *to* ρ_k.

We claim that it suffices to prove the convergence result in $L^1([0,T]\times B_R)$ for any fixed $R>0$, i.e.

$$u_k^\varepsilon \to \rho_k \text{ strongly in } L^1\left([0,T]\times B_R\right) \text{ as } \varepsilon\to 0. \tag{3.2}$$

It is true because of the following remark.

Remark 3.6. By *Vitali convergence theorem*, the sequence $(u_k^\varepsilon)_{\varepsilon>0}$ converges in $L^m([0,T]\times\mathbb{R}^d)$ to ρ_k if and only if

(i) the sequence $(u_k^\varepsilon)_{\varepsilon>0}$ converges in the Lebesgue measure on $[0,T]\times\mathbb{R}^d$ to ρ_k;
(ii) the functions $(u_k^\varepsilon)^m$ are uniformly integrable;
(iii) for every $\eta>0$, there exists a set $E_\eta\in[0,T]\times\mathbb{R}^d$ of finite measure, such that $\iint_{E_\eta^c}|u_k^\varepsilon|^m<\eta$ for all ε.

Actually, by lemma 3.4 and Prokhorov's theorem, for any k, $(u_k^\varepsilon)_{\varepsilon\geqslant0}$ are uniformly tight. Thus for any $\eta_k>0$ there exists $R_{\eta_k}>0$ such that for any $\varepsilon>0$, it holds

$$\int_0^T\int_{B^c_{R_{\eta_k}}} u_k^\varepsilon(t,x)\,\mathrm{d}x\mathrm{d}t\leqslant\eta_k.$$

Then for any $\delta>0$, there exists a ball B_{R_δ} such that for any $\varepsilon>0$

$$\int_0^T\int_{B^c_{R_\delta}} u_k^\varepsilon(t,x)\,\mathrm{d}x\mathrm{d}t\leqslant\frac{\delta}{4}\quad\text{and}\quad\int_0^T\int_{B^c_{R_\delta}}\rho_k(t,x)\,\mathrm{d}x\mathrm{d}t\leqslant\frac{\delta}{4}.$$

And we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}|u_k^\varepsilon-\rho_k|\mathrm{d}x\mathrm{d}t&=\int_0^T\int_{B^c_{R_\delta}}|u_k^\varepsilon-\rho_k|\mathrm{d}x\mathrm{d}t+\int_0^T\int_{B_{R_\delta}}|u_k^\varepsilon-\rho_k|\mathrm{d}x\mathrm{d}t\\&\leqslant\frac{\delta}{2}+\int_0^T\int_{B_{R_\delta}}|u_k^\varepsilon-\rho_k|\mathrm{d}x\mathrm{d}t.\end{aligned}$$

That is to say if for any B_R the sequence $(u_k^\varepsilon)_{\varepsilon>0}$ converges to ρ_k in $L^1([0,T]\times B_R)$, then it implies the convergence also holds in $L^1([0,T]\times\mathbb{R}^d)$, which further implies the convergence holds in Lebesgue measure. Statement (i) of the remark is satisfied for sure.

By remark 3.2 the sequence $\left(\sum_k a_k u_k^\varepsilon\right)^{m/2}$ is bounded in $L^2\left([0,T],H^1\left(\mathbb{R}^d\right)\right)$. If $\omega\in H^1(\mathbb{R}^d)$, then we have $\omega^2\in W^{1,1}(\mathbb{R}^d)\subset L^{d/(d-1)}(\mathbb{R}^d)$ for $d\geqslant2$. So $\left(\sum_k a_k u_k^\varepsilon\right)^m$ is bounded in $L^2\left([0,T],L^{d/(d-1)}(\mathbb{R}^d)\right)$. By the positivity of each a_k and u_k^ε, for any k, it holds

$$(u_k^\varepsilon)^m\in L^2\left([0,T],L^{d/(d-1)}\left(\mathbb{R}^d\right)\right)\subset L^{d/(d-1)}\left([0,T]\times\mathbb{R}^d\right),$$

which deduces that the following uniform in ε bound holds

$$\int_0^T\int_{\mathbb{R}^d}\left(u_k^\varepsilon(t,x)\right)^{\frac{md}{d-1}}\mathrm{d}x\mathrm{d}t\leqslant C_k.$$

Since for any set $A\subset[0,T]\times\mathbb{R}^d$ with the characteristic function χ_A and volume $|A|$, we have

$$\begin{aligned}\lim_{|A|\to0}\sup_{\varepsilon>0}\iint_A\left(u_k^\varepsilon(t,x)\right)^m\mathrm{d}x\mathrm{d}t&\leqslant\lim_{|A|\to0}\sup_{\varepsilon>0}\left(\int_0^T\int_{\mathbb{R}^d}(\chi_A)^d\,\mathrm{d}x\right)^{\frac{1}{d}}\left(\int_0^T\int_{\mathbb{R}^d}\left(u_k^\varepsilon(t,x)\right)^{\frac{md}{d-1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{d-1}{d}}\\&\leqslant\lim_{|A|\to0}|A|^{\frac{1}{d}}C_k^{\frac{d-1}{d}}=0,\end{aligned}$$

which verifies (ii). And we get

$$\begin{aligned}\int_0^T\int_{B^c_{R_{\eta_k}}}\left(u_k^\varepsilon(t,x)\right)^m\mathrm{d}x\mathrm{d}t&\leqslant\left(\int_0^T\int_{B^c_{R_{\eta_k}}}u_k^\varepsilon(t,x)\,\mathrm{d}x\mathrm{d}t\right)^{\frac{m}{1-d+md}}\left(\int_0^T\int_{B^c_{R_{\eta_k}}}u_k^\varepsilon(t,x)^{\frac{md}{d-1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{(m-1)(d-1)}{1-d+md}}\\&\leqslant C_k'\eta_k^{\frac{m}{1-d+md}},\end{aligned}$$

i.e. (iii) has been verified, then our claim follows.

The following discussion is analogous to that in [21]. To prove our strong convergence result (3.2), we need to prove for each k, the sequence $(u_k^\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $L^1([0,T]\times B_R)$. From lemma 3.1 and remark 3.2, we can only deduce higher regularity about the sum $\sum_k a_k u_k^\varepsilon$. For single species u_k^ε, we will take advantage of the mild form of the solution. We firstly introduce a fractional-type Sobolev space X_α, for $0<\alpha<1$:

$$X_\alpha := \left\{ w\in L^1\left(\mathbb{R}^d\right) \Big| \sup_{0<|h|\leqslant 1} \frac{\|w(\cdot+h)-w(\cdot)\|_{L^1(\mathbb{R}^d)}}{|h|^\alpha} < +\infty \right\}.$$

One can check that this is a Banach space endowed with the norm

$$\|w\|_{X_\alpha} := \|w\|_{L^1(\mathbb{R}^d)} + \sup_{0<|h|\leqslant 1} \frac{\|w(\cdot+h)-w(\cdot)\|_{L^1(\mathbb{R}^d)}}{|h|^\alpha}.$$

By the Riesz–Fréchet–Kolmogorov theorem [3, theorem 4.26], any bounded subset of X_α is compact in $L^1(\Omega)$ for any bounded domain $\Omega\subset\mathbb{R}^d$.

Lemma 3.7. *The sequence $(u_k^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1\left([0,T],X_\alpha\right)$ for any $0<\alpha<1$.*

Proof. We recall that

$$\partial_t u_k^\varepsilon = \nabla\cdot(b_k\rho_k^\varepsilon\nabla g^\varepsilon)*V^\varepsilon + \sigma\Delta u_k^\varepsilon, \quad u_k^\varepsilon(0) = \rho_{k,0}*V^\varepsilon.$$

Let $f_k^\varepsilon = \left(b_k\rho_k^\varepsilon\nabla g^\varepsilon\right)*V^\varepsilon$, then we can write above equation into mild form as:

$$\begin{aligned} u_k^\varepsilon(t) &= \Gamma(t)*_x u_k^\varepsilon(0) + \int_0^t \left(\Gamma(t-s)*_x \operatorname{div} f_k^\varepsilon(s)\right)\mathrm{d}s \\ &= \Gamma(t)*_x u_k^\varepsilon(0) + \int_0^t \left(\nabla\Gamma(t-s)*_x f_k^\varepsilon(s)\right)\mathrm{d}s, \end{aligned}$$

where $\Gamma(t,x)$ is the heat kernel given by

$$\Gamma(t,x) := \begin{cases} \frac{1}{(4\sigma\pi t)^{d/2}} e^{-\frac{|x|^2}{4\sigma t}} & \text{for } t>0 \\ \delta_x & \text{for } t=0. \end{cases}$$

In [21], for any $0<\alpha<1$ we know $\Gamma,\nabla\Gamma\in L^1\left([0,T],X_\alpha\right)$. Also, by the fact

$$\left\|\int_0^t\int_{\mathbb{R}^d}\nabla_x\Gamma(t-s,x-z)f_k^\varepsilon(s,z)\,\mathrm{d}z\mathrm{d}s\right\|_{L^1([0,T],X_\alpha)} \leqslant \|\nabla\Gamma\|_{L^1([0,T],X_\alpha)}\|f\|_{L^1([0,T]\times\mathbb{R}^d)},$$

one need the L^1-estimate of f_k^ε that

$$\begin{aligned} \|f_k^\varepsilon\|_{L^1([0,T]\times\mathbb{R}^d)} &\leqslant \int_0^T\int_{\mathbb{R}^d} b_k|\nabla g^\varepsilon(t,x)|\,\rho_k^\varepsilon(t,\mathrm{d}x)\,\mathrm{d}t \\ &\leqslant \left(T\int_0^T\int_{\mathbb{R}^d} b_k^2|\nabla g^\varepsilon(t,x)|^2\rho_k^\varepsilon(t,\mathrm{d}x)\,\mathrm{d}t\right)^{1/2}. \end{aligned}$$

Therefore, the sequence $(u_k^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1\left([0,T],X_\alpha\right)$. *A priori* estimate lemma 3.1, for any species k tells us f_k^ε is bounded in $L^1\left([0,T]\times\mathbb{R}^d\right)$ uniformly in ε. □

Fix $0<\alpha<1$, and take some big $s>0$ such that $\mathcal{M}(B_R)\hookrightarrow H^{-s}(B_R)$ continuously. And for any $\delta>0$, there exists a constant C_δ such that for any smooth function f on $\mathbb{R}^d$, the following inequality holds (see also [21])

$$\|f\|_{L^1(B_R)} \leqslant \delta\|f\|_{X_\alpha} + C_\delta\|f\|_{H^{-s}(B_R)}. \tag{3.3}$$

Taking ε and ε', applying (3.3) to $u_k^{\varepsilon}(t)-u_k^{\varepsilon'}(t)$ and integrating in time, we obtain

$$\begin{aligned}\left\|u_k^{\varepsilon}-u_k^{\varepsilon'}\right\|_{L^1([0,T]\times B_R)} &\leqslant \delta\left\|u_k^{\varepsilon}-u_k^{\varepsilon'}\right\|_{L^1([0,T],X_\alpha)}+C_\delta\left\|u_k^{\varepsilon}-u_k^{\varepsilon'}\right\|_{L^1([0,T],H^{-s}(B_R))}\\ &\leqslant \delta\left(\left\|u_k^{\varepsilon}\right\|_{L^1([0,T],X_\alpha)}+\left\|u_k^{\varepsilon'}\right\|_{L^1([0,T],X_\alpha)}\right)+C_\delta\left\|u_k^{\varepsilon}-u_k^{\varepsilon'}\right\|_{L^1([0,T],H^{-s}(B_R))}\\ &\leqslant C_k\left(\delta+C_\delta\int_0^T d\left(u_k^{\varepsilon}(t),u_k^{\varepsilon'}(t)\right)\mathrm{d}t\right).\end{aligned}$$

By the convergence of $(u_k^{\varepsilon})_{\varepsilon>0}$ in $C([0,T],\mathcal{M}(\mathbb{R}^d))$, we have

$$\limsup_{\varepsilon,\varepsilon'\to0}\left\|u_k^{\varepsilon}-u_k^{\varepsilon'}\right\|_{L^1([0,T]\times B_R)}\leqslant C_k\delta,$$

which implies that $(u_k^{\varepsilon})_{\varepsilon>0}$ is a Cauchy sequence in $L^1([0,T]\times B_R)$ by the arbitrariness of δ. Together with claim (3.2), lemma 3.5 has been proved.

We finally show that for any k, the limit ρ_k is a weak solution of cross-diffusion system (1.1) with initial data $\rho_{k,0}$ as in definition 1.5.

Proposition 3.8. *For each k species and any test function $f\in C^1([0,T],C_b^2(\mathbb{R}^d))$, the limit ρ_k satisfies the following equation:*

$$\begin{aligned}&\int_{\mathbb{R}^d}f(t,x)\,\rho_k(t,x)\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}^d}\partial_s f(s,x)\,\rho_k(s,x)\,\mathrm{d}x\mathrm{d}s=\int_{\mathbb{R}^d}f(0,x)\,\rho_{k,0}(x)\,\mathrm{d}x\\ &\quad+\sigma\int_0^t\int_{\mathbb{R}^d}\Delta f(s,x)\,\rho_k(s,x)\,\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{R}^d}b_k\rho_k(s,x)\,\nabla f(s,x)\cdot\nabla\left(\sum_{l=1}^n a_l\rho_l(s,x)\right)^{m-1}\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{3.4}$$

Proof. In terms of (1.5), for any $f\in C^1([0,T],C_b^2(\mathbb{R}^d))$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}f(t,x)\,\rho_k^{\varepsilon}(t,\mathrm{d}x)+\int_0^t\int_{\mathbb{R}^d}\partial_s f(s,x)\,\rho_k^{\varepsilon}(s,\mathrm{d}x)\,\mathrm{d}s&=\int_{\mathbb{R}^d}f(0,x)\,\rho_{k,0}(x)\,\mathrm{d}x\\ &\quad+\sigma\int_0^t\int_{\mathbb{R}^d}\Delta f(s,x)\,\rho_k^{\varepsilon}(s,\mathrm{d}x)\,\mathrm{d}s\\ &\quad-\int_0^t\int_{\mathbb{R}^d}b_k\nabla f(s,x)\cdot\nabla g^{\varepsilon}(s,x)\,\rho_k^{\varepsilon}(s,\mathrm{d}x)\,\mathrm{d}s.\end{aligned}\tag{3.5}$$

Given that ρ_k^{ε} converges to ρ_k in $C([0,T],\mathcal{M}(\mathbb{R}^d))$ as ε approaches 0, we are able to pass to the limit for the first four terms of (3.5). For the last term, it has

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla g^{\varepsilon}(s,x)\,\rho_k^{\varepsilon}(s,\mathrm{d}x)\,\mathrm{d}s-\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla\left(\sum_l a_l\rho_l(s,x)\right)^{m-1}\rho_k(s,x)\,\mathrm{d}x\mathrm{d}s\right|\\ &\leqslant\left|\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla g^{\varepsilon}(s,x)\,\rho_k^{\varepsilon}(s,\mathrm{d}x)\,\mathrm{d}s-\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla\left(\sum_l a_l u_l^{\varepsilon}(s,x)\right)^{m-1}u_k^{\varepsilon}(s,x)\,\mathrm{d}x\mathrm{d}s\right|\\ &\quad+\left|\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla\left(\sum_l a_l u_l^{\varepsilon}(s,x)\right)^{m-1}u_k^{\varepsilon}(s,x)\,\mathrm{d}x\mathrm{d}s\right.\\ &\quad\left.-\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla\left(\sum_l a_l\rho_l(s,x)\right)^{m-1}\rho_k(s,x)\,\mathrm{d}x\mathrm{d}s\right|\\ &=:I_1^{\varepsilon}+I_2^{\varepsilon}.\end{aligned}$$

For I_1^ε, noticing $V^\varepsilon(x-y)=V^\varepsilon(y-x)$, we have

$$\begin{aligned}
I_1^\varepsilon = &\Big|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla\Big(\sum_l a_l u_l^\varepsilon(s,y)\Big)^{m-1} V^\varepsilon(x-y)\,\rho_k^\varepsilon(s,\mathrm{d}x)\,\mathrm{d}y\mathrm{d}s \\
&-\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla f(y)\cdot\nabla\Big(\sum_l a_l u_l^\varepsilon(s,y)\Big)^{m-1} V^\varepsilon(y-x)\,\rho_k^\varepsilon(s,\mathrm{d}x)\,\mathrm{d}y\mathrm{d}s\Big| \\
\leqslant &\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big|\nabla f(x)-\nabla f(y)\big|\Big|\nabla\Big(\sum_l a_l u_l^\varepsilon(s,y)\Big)^{m-1}\Big|V^\varepsilon(x-y)\,\rho_k^\varepsilon(s,\mathrm{d}x)\,\mathrm{d}y\mathrm{d}s \\
\leqslant &\big\|\nabla^2 f\big\|_{L^\infty}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big|\nabla\Big(\sum_l a_l u_l^\varepsilon(s,y)\Big)^{m-1}\Big|\,|x-y|\,V^\varepsilon(x-y)\,\rho_k^\varepsilon(s,\mathrm{d}x)\,\mathrm{d}y\mathrm{d}s;
\end{aligned}$$

recall that $V\in C_c^\infty(\mathbb{R}^d)$, then

$$\sup_{x,y\in\operatorname{supp}V^\varepsilon}|x-y|<C\varepsilon,$$

which implies that

$$I_1^\varepsilon\leqslant\varepsilon C\big\|\nabla^2 f\big\|_{L^\infty}\int_0^t\int_{\mathbb{R}^d}\Big|\nabla\Big(\sum_l a_l u_l^\varepsilon(s,y)\Big)^{m-1}\Big|u_k^\varepsilon(s,y)\,\mathrm{d}y\mathrm{d}s.$$

And we have

$$\begin{aligned}
&\Big\|u_k^\varepsilon\nabla\Big(\sum_l a_l u_l^\varepsilon\Big)^{m-1}\Big\|_{L^1([0,T]\times\mathbb{R}^d)} \\
&=(m-1)\Big\|u_k^\varepsilon\Big(\sum_l a_l u_l^\varepsilon\Big)^{m-2}\nabla\Big(\sum_l a_l u_l^\varepsilon\Big)\Big\|_{L^1([0,T]\times\mathbb{R}^d)} \\
&\leqslant(m-1)\Big\|\Big(\sum_l a_l u_l^\varepsilon\Big)^{m/2-1}\nabla\Big(\sum_l a_l u_l^\varepsilon\Big)\Big\|_{L^2([0,T]\times\mathbb{R}^d)}\Big\|u_k^\varepsilon\Big(\sum_l a_l u_l^\varepsilon\Big)^{m/2-1}\Big\|_{L^2([0,T]\times\mathbb{R}^d)} \\
&\leqslant\frac{m-1}{\min_k a_k}\Big\|\Big(\sum_l a_l u_l^\varepsilon\Big)^{m-2}\Big|\nabla\Big(\sum_l a_l u_l^\varepsilon\Big)\Big|^2\Big\|_{L^1([0,T]\times\mathbb{R}^d)}^{1/2}\Big\|\sum_l a_l u_l^\varepsilon\Big\|_{L^m([0,T]\times\mathbb{R}^d)}^{m/2},
\end{aligned}$$

which is bounded according to *a priori* estimate lemma 3.1. Thus we obtain I_1^ε goes to 0 as $\varepsilon\to0$.

For I_2^ε, we have

$$I_2^\varepsilon\leqslant\|\nabla f\|_{L^\infty}\int_0^t\int_{\mathbb{R}^d}\Big|\nabla\Big(\sum_l a_l u_l^\varepsilon(s,x)\Big)^{m-1}u_k^\varepsilon(s,x)-\nabla\Big(\sum_l a_l\rho_l(s,x)\Big)^{m-1}\rho_k(s,x)\Big|\mathrm{d}x\mathrm{d}s.$$

To prove $I_2^\varepsilon\to0$ as $\varepsilon\to0$, we need to prove

$$\nabla\Big(\sum_l a_l u_l^\varepsilon(s,x)\Big)^{m-1}u_k^\varepsilon(s,x)\to\nabla\Big(\sum_l a_l\rho_l(s,x)\Big)^{m-1}\rho_k(s,x)\quad\text{strongly in}\quad L^1\big([0,T]\times\mathbb{R}^d\big).$$

We possess that for any k

$$u_k^\varepsilon\to\rho_k\quad\text{strongly in}\quad L^m\big([0,T]\times\mathbb{R}^d\big)\text{ as }\varepsilon\to0.\tag{3.6}$$

So we only need to verify that

$$\nabla\Big(\sum_l a_l u_l^\varepsilon\Big)^{m-1}\rightharpoonup\nabla\Big(\sum_l a_l\rho_l\Big)^{m-1}\quad\text{weakly in}\quad L^{\frac{m}{m-1}}\big([0,T]\times\mathbb{R}^d\big)\text{ as }\varepsilon\to0.\tag{3.7}$$

Lemma 3.9. *Assume $m \geqslant 2$, for any function h such that $h \in L^m([0,T]\times\mathbb{R}^d)$ and $h^{m/2} \in L^2([0,T],H^1(\mathbb{R}^d))$, then $h^{m-1} \in L^{\frac{m}{m-1}}([0,T],W^{1,\frac{m}{m-1}}(\mathbb{R}^d))$.*

Proof. For $m=2$, the lemma holds trivially; for $m>2$, $h^{m-1} \in L^{\frac{m}{m-1}}([0,T]\times\mathbb{R}^d)$ is apparent, we need to prove $\nabla h^{m-1} \in L^{\frac{m}{m-1}}([0,T]\times\mathbb{R}^d)$ as follows

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\left|\nabla h^{m-1}\right|^{\frac{m}{m-1}}\mathrm{d}x\mathrm{d}s &= C_m\int_0^T\int_{\mathbb{R}^d}\left|h^{\frac{m}{2}-1}\nabla h^{\frac{m}{2}}\right|^{\frac{m}{m-1}}\mathrm{d}x\mathrm{d}s\\ &\leqslant C_m\left(\int_0^T\int_{\mathbb{R}^d}h^{\left(\frac{m}{2}-1\right)\frac{m}{m-1}\frac{2m-2}{m-2}}\mathrm{d}x\mathrm{d}s\right)^{\frac{m-2}{2m-2}}\left(\int_0^T\int_{\mathbb{R}^d}\left|\nabla h^{\frac{m}{2}}\right|^{\frac{m}{m-1}\frac{2m-2}{m}}\mathrm{d}x\mathrm{d}s\right)^{\frac{m}{2m-2}}\\ &= C_m\left(\int_0^T\int_{\mathbb{R}^d}h^m\mathrm{d}x\mathrm{d}s\right)^{\frac{m-2}{2m-2}}\left(\int_0^T\int_{\mathbb{R}^d}\left|\nabla h^{\frac{m}{2}}\right|^2\mathrm{d}x\mathrm{d}s\right)^{\frac{m}{2m-2}} < \infty.\end{aligned}$$

□

By remark 3.2, we have $u_k^\varepsilon \in L^m([0,T]\times\mathbb{R}^d)$ and $(\sum_k a_k u_k^\varepsilon)^{m/2} \in L^2([0,T],H^1(\mathbb{R}^d))$, which implies the limit $\rho_k \in L^m([0,T]\times\mathbb{R}^d)$ and $(\sum_k a_k\rho_k)^{m/2} \in L^2([0,T],H^1(\mathbb{R}^d))$. Applying the lemma above, we have

$$\left(\sum_l a_l u_l^\varepsilon\right)^{m-1} \text{ and } \left(\sum_l a_l\rho_l\right)^{m-1} \in L^{\frac{m}{m-1}}\left([0,T],W^{1,\frac{m}{m-1}}\left(\mathbb{R}^d\right)\right).$$

For any test function $h \in L^m([0,T],W^{1,m}(\mathbb{R}^d))$, it holds

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^d}h(x,s)\left(\nabla\left(\sum_l a_l u_l^\varepsilon(s,x)\right)^{m-1}-\nabla\left(\sum_l a_l\rho_l(s,x)\right)^{m-1}\right)\mathrm{d}x\mathrm{d}s\right|\\ &=\left|\int_0^t\int_{\mathbb{R}^d}\nabla h(x,s)\left(\left(\sum_l a_l u_l^\varepsilon(s,x)\right)^{m-1}-\left(\sum_l a_l\rho_l(s,x)\right)^{m-1}\right)\mathrm{d}x\mathrm{d}s\right|\\ &\leqslant\|\nabla h\|_{L^m([0,T]\times\mathbb{R}^d)}\left(\int_0^t\int_{\mathbb{R}^d}\left|\left(\sum_l a_l u_l^\varepsilon(s,x)\right)^{m-1}-\left(\sum_l a_l\rho_l(s,x)\right)^{m-1}\right|^{\frac{m}{m-1}}\mathrm{d}x\mathrm{d}s\right)^{\frac{m-1}{m}},\end{aligned}$$

which goes to 0 as $\varepsilon\to 0$ thanks to

$$\left(\sum_l a_l u_l^\varepsilon\right)^{m-1}\to\left(\sum_l a_l\rho_l\right)^{m-1}\quad\text{strongly in}\quad L^{\frac{m}{m-1}}\left([0,T]\times\mathbb{R}^d\right).$$

Now for any $h \in L^m([0,T]\times\mathbb{R}^d)$, it can be approximated by a sequence $(h_n)_{n\geqslant 1} \in L^m([0,T],W^{1,m}(\mathbb{R}^d))$. Thus

$$\int_0^t\int_{\mathbb{R}^d}h\nabla\left(\sum_l a_l u_l^\varepsilon(s,x)\right)^{m-1}\mathrm{d}x\mathrm{d}s\to\int_0^t\int_{\mathbb{R}^d}h\nabla\left(\sum_l a_l\rho_l(s,x)\right)^{m-1}\mathrm{d}x\mathrm{d}s,\text{ as }\varepsilon\to 0,$$

which is due to both $\left(\sum_l u_l^\varepsilon\right)^{m-1}$ and $\left(\sum_l\rho_l\right)^{m-1}$ are bounded in $L^{\frac{m}{m-1}}([0,T],W^{1,\frac{m}{m-1}}(\mathbb{R}^d))$ and the dominated convergence theorem enables to let $h_n\to h$. Hence, the weak convergence as expressed in (3.7) is achieved as desired. Combining (3.6) and (3.7), we obtain $I_2^\varepsilon\to 0$.

Thus we finish the proof of proposition 3.8. □

In conclusion, $\rho=(\rho_1,\ldots,\rho_n)$ is a weak solution of the cross-diffusion system (1.1).

4. Proof of uniqueness for the cross-diffusion systems

In this section, we will prove theorem 1.8 under the assumption that mobilities $b_1,\ldots,b_k$ are the same. Namely, the cross-diffusion equation (1.1) reduces to the following system

$$\partial_t\rho_k=b\nabla\cdot\left(\rho_k\nabla\left(\sum_{l=1}^n a_l\rho_l\right)^{m-1}\right)+\sigma\Delta\rho_k,\quad k=1,2,\ldots,n, \tag{4.1}$$

If we sum up the cross-diffusion system with weights a_k, then it holds

$$\partial_t\left(\sum_l a_l\rho_l\right) = b\nabla\cdot\left(\sum_l a_l\rho_l\nabla P(\rho)\right) + \sigma\Delta\sum_l a_l\rho_l.$$

Let $u = \sum_l a_l\rho_l$ with $u_0 = \sum_l a_l\rho_{l,0}$, which satisfies formally that

$$\partial_t u = b\nabla\cdot\left(u\nabla u^{m-1}\right) + \sigma\Delta u, \tag{4.2}$$

where $u_0 \in L^1\cap L^\infty(\mathbb{R}^d)$. Weak solutions of (4.2) are understood in sense of the following definition.

Definition 4.1. A weak solution of (4.2) on the time interval $[0,T]$ satisfies $u\in L^m([0,T]\times\mathbb{R}^d)$ and $u^{m-1}\in L^{\frac{m}{m-1}}(0,T;W^{1,\frac{m}{m-1}}(\mathbb{R}^d))$. And for any $f\in C^1([0,T],C_b^2(\mathbb{R}^d))$ and almost any $t\in[0,T]$, it holds

$$\begin{aligned}\int_{\mathbb{R}^d} f(t,x)\,u(t,x)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^d} u(s,x)\,\partial_s f(s,x)\,\mathrm{d}x\mathrm{d}s = \int_{\mathbb{R}^d} f(0,x)\,u_0(x)\,\mathrm{d}x \\ +\sigma\int_0^t\int_{\mathbb{R}^d}\Delta f(s,x)\,u(s,x)\,\mathrm{d}x\mathrm{d}s - b\int_0^t\int_{\mathbb{R}^d} u(s,x)\,\nabla f(s,x)\cdot\nabla u^{m-1}(s,x)\,\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{4.3}$$

Given $\rho_{k,0}\in L^1\cap L^\infty(\mathbb{R}^d)$ for $k=1,\ldots,n$, if the weak solution of (4.2) defined as definition 4.1 is unique with initial data $u_0=\sum_k a_k\rho_{k,0}$, then the pressure of the cross-diffusion (4.1) satisfied by $P(\rho)=u^{m-1}$ is uniquely determined with certain regularity. Thus, each ρ_k satisfies the linear Fokker–Planck equation

$$\partial_t\rho_k = b\nabla\cdot\left(\rho_k\nabla u^{m-1}\right) + \sigma\Delta\rho_k,$$

which has a unique weak solution $\rho_k\in L^m([0,T]\times\mathbb{R}^d)$ in sense of definition 1.5 [2, theorem 9.3.6]. Formally, we test (4.2) with u^{m-1} and integrate in space and time to obtain the *a priori* estimate

$$\frac{\|u(T)\|_{L^m}^m}{m} + b\int_0^T\int_{\mathbb{R}^d} u|\nabla u^{m-1}|^2\mathrm{d}x\mathrm{d}t + \sigma(m-1)\int_0^T\int_{\mathbb{R}^d} u^{m-2}|\nabla u|^2\mathrm{d}x\mathrm{d}t \leqslant \frac{\|u(0)\|_{L^m}^m}{m}.$$

The above estimate can be derived by regularising u as $u^\varepsilon = V^\varepsilon * u$, showing the analogous estimate to lemma 3.1 for u^ε, and then passing to the limit as $u^\varepsilon\to u$. It implies that, for any k, the vector field $\nabla u^{m-1}\in L^2([0,T]\times\mathbb{R}^d,\rho_k\mathrm{d}x\mathrm{d}t)$, i.e.

$$\int_0^T\int_{\mathbb{R}^d}\rho_k|\nabla u^{m-1}|^2\mathrm{d}x\mathrm{d}t \leqslant \int_0^T\int_{\mathbb{R}^d} u|\nabla u^{m-1}|^2\mathrm{d}x\mathrm{d}t < \infty.$$

It also implies $\nabla u^{m-1}\in L^1([0,T]\times\mathbb{R}^d,\rho_k\mathrm{d}x\mathrm{d}t)$ as follows

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\rho_k|\nabla u^{m-1}|\mathrm{d}x\mathrm{d}t &\leqslant \int_0^T\int_{\mathbb{R}^d} u|\nabla u^{m-1}|\mathrm{d}x\mathrm{d}t \\ &\leqslant \|u\|_{L^1([0,T]\times\mathbb{R}^d)}^{\frac12}\left(\int_0^T\int_{\mathbb{R}^d} u|\nabla u^{m-1}|^2\mathrm{d}x\mathrm{d}t\right)^{\frac12} < \infty,\end{aligned}$$

which verifies the condition needed in [2, theorem 9.3.6].

Therefore, to show theorem 1.8, it is sufficient to prove the following proposition, which is well-known [21, 40] but we adapt it here to our regularity setting. Note that we perform integration by parts only once in the last term of (4.3). This differs from the standard definition of weak solutions to equations of the form $\partial_t u = \Delta F(u)$, where integration by parts is applied twice to transfer the Laplacian onto the test function.

Proposition 4.2. *The weak solution of (4.2) defined as definition 4.1 is unique.*

Proof. Without loss of the generality, we assume $b=\sigma=1$ in the proof. The first step is to show $u\in L^{m+1}([0,T]\times\mathbb{R}^d)$. Recall the smooth mollifier V^ε defined as in (1.5) and let

$$u_\varepsilon(t,\cdot) = u(t,\cdot) * V^\varepsilon,$$

then it holds in sense of weak form (4.3) that

$$\partial_t u_\varepsilon = \nabla\cdot\left(\left(u\nabla u^{m-1}\right) * V^\varepsilon\right) + \Delta u * V^\varepsilon.$$

We take smooth test function

$$f(x,t)=\int_t^T\left(\frac{m-1}{m}u^m+u\right)*V^\varepsilon \mathrm{d}s$$

with

$$\nabla f(x,t)=\int_t^T\left(u\nabla u^{m-1}+\nabla u\right)*V^\varepsilon \mathrm{d}s.$$

We plug in f into (4.3). The right-hand side of the equality follows

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(0,x)\,u_\varepsilon(0,x)\,\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d}u_\varepsilon\Delta f\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{R}^d}\nabla f\cdot\left(\left(u\nabla u^{m-1}\right)*V^\varepsilon\right)\mathrm{d}x\mathrm{d}t\\
&=\int_{\mathbb{R}^d}u_\varepsilon(0)\int_0^T\left(\frac{m-1}{m}u^m+u\right)*V^\varepsilon\mathrm{d}t\\
&\quad-\int_0^T\int_{\mathbb{R}^d}\left(\left(u\nabla u^{m-1}+\nabla u\right)*V^\varepsilon\right)\left(\int_t^T\left(\left(u\nabla u^{m-1}+\nabla u\right)*V^\varepsilon\right)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t\\
&=\int_{\mathbb{R}^d}u_\varepsilon(0)\int_0^T\left(\frac{m-1}{m}u^m+u\right)*V^\varepsilon\mathrm{d}t-\frac{1}{2}\int_{\mathbb{R}^d}\left|\int_0^T\left(\left(u\nabla u^{m-1}+\nabla u\right)*V^\varepsilon\right)\mathrm{d}t\right|^2\mathrm{d}x,
\end{aligned}$$

where integration by parts is performed to transfer one derivative from f to u_ε, allowing us to handle the second-order term. The last step follows from Fubini's theorem. While the left-hand side of (4.3) yields

$$\begin{aligned}
&\int_{\mathbb{R}^d}f(T,x)\,u_\varepsilon(T,x)\,\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d}u_\varepsilon(t,x)\,\partial_t f(t,x)\,\mathrm{d}x\mathrm{d}t\\
&=-\int_0^T\int_{\mathbb{R}^d}u_\varepsilon\partial_t\left(\int_t^T\left(\frac{m-1}{m}u^m+u\right)*V^\varepsilon\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{R}^d}u_\varepsilon\left(\frac{m-1}{m}u^m+u\right)*V^\varepsilon\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Combining both identities above with the fact $u_\varepsilon^m\leqslant u^m*V^\varepsilon$ due to Jensen's inequality, we get that

$$\begin{aligned}
&\frac{m-1}{m}\int_0^T\int_{\mathbb{R}^d}(u_\varepsilon)^{m+1}\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^d}(u_\varepsilon)^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_{\mathbb{R}^d}\left|\int_0^T\left(\left(u\nabla u^{m-1}+\nabla u\right)*V^\varepsilon\right)\mathrm{d}t\right|^2\mathrm{d}x\\
&\leqslant\int_{\mathbb{R}^d}u_\varepsilon(0)\int_0^T\left(\frac{m-1}{m}u^m+u\right)*V^\varepsilon\mathrm{d}t\mathrm{d}x\\
&\leqslant\|u_\varepsilon(0)\|_{L^\infty}\int_{\mathbb{R}^d}\int_0^T\left(\frac{m-1}{m}u^m+u\right)*V^\varepsilon\mathrm{d}t\mathrm{d}x\\
&\leqslant\|u_\varepsilon(0)\|_{L^\infty}\left(\frac{m-1}{m}\|u\|_{L^m([0,T]\times\mathbb{R}^d)}+\|u\|_{L^1([0,T]\times\mathbb{R}^d)}\right).
\end{aligned}$$

Taking the limit as $\varepsilon\to 0$, we obtain $u\in L^{m+1}(\mathbb{R}^d\times[0,T])$.

Similar as the argument above, we assume that two solutions u and $\bar{u}$ of (4.2) have the same initial data i.e. $u(0)=\bar{u}(0)$. Their regularised versions $u_\varepsilon:=u*V^\varepsilon$ and $\bar{u}_\varepsilon:=\bar{u}*V^\varepsilon$ satisfy the following equation

$$\partial_t(u_\varepsilon-\bar{u}_\varepsilon)=\nabla\cdot\left(\left(u\nabla u^{m-1}-\bar{u}\nabla\bar{u}^{m-1}\right)*V^\varepsilon\right)+\Delta(u_\varepsilon-\bar{u}_\varepsilon).\tag{4.4}$$

We take test function

$$f=\int_t^T\left(\frac{m-1}{m}u^m+u-\frac{m-1}{m}\bar{u}^m-\bar{u}\right)*V^\varepsilon\mathrm{d}s,$$

and plug in it into the weak form of (4.4) as follows

$$\begin{aligned}&\int_{\mathbb{R}^d} f(T,x)\left(u_\varepsilon(T,x)-\bar{u}_\varepsilon(T,x)\right)\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d}\left(u_\varepsilon(t,x)-\bar{u}_\varepsilon(t,x)\right)\partial_t f(t,x)\,\mathrm{d}x\mathrm{d}t\\&\quad=\int_{\mathbb{R}^d} f(0,x)\left(u_\varepsilon(0)-\bar{u}_\varepsilon(0)\right)\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d}\left(u_\varepsilon-\bar{u}_\varepsilon\right)\Delta f\mathrm{d}x\mathrm{d}t\\&\qquad-\int_0^T\int_{\mathbb{R}^d}\nabla f\cdot\left(\left(u\nabla u^{m-1}\right)*V^\varepsilon-\left(\bar{u}\nabla\bar{u}^{m-1}\right)*V^\varepsilon\right)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.5}$$

The right-hand side reads as

$$\begin{aligned}&\int_{\mathbb{R}^d} f(0,x)(u_\varepsilon(0)-\bar{u}_\varepsilon(0))\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d}(u_\varepsilon-\bar{u}_\varepsilon)\Delta f\mathrm{d}x\mathrm{d}t\\&\quad-\int_0^T\int_{\mathbb{R}^d}\nabla f\cdot\left((u\nabla u^{m-1})*V^\varepsilon-(\bar{u}\nabla\bar{u}^{m-1})*V^\varepsilon\right)\mathrm{d}x\mathrm{d}t\\&=-\int_0^T\int_{\mathbb{R}^d}((u\nabla u^{m-1}+\nabla u-\bar{u}\nabla\bar{u}^{m-1}-\nabla\bar{u})*V^\varepsilon\\&\quad\times\left(\int_t^T(u\nabla u^{m-1}+\nabla u-\bar{u}\nabla\bar{u}^{m-1}-\nabla\bar{u})*V^\varepsilon)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t\\&=-\frac{1}{2}\int_{\mathbb{R}^d}\left|\int_0^T\left((u\nabla u^{m-1}+\nabla u-\bar{u}\nabla\bar{u}^{m-1}-\nabla\bar{u})*V^\varepsilon\right)\mathrm{d}t\right|^2\mathrm{d}x.\end{aligned}$$

And the left-hand side of the identity (4.5) yields

$$\begin{aligned}&\int_{\mathbb{R}^d} f(T,x)\left(u_\varepsilon(T,x)-\bar{u}_\varepsilon(T,x)\right)\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d}\left(u_\varepsilon(t,x)-\bar{u}_\varepsilon(t,x)\right)\partial_t f(t,x)\,\mathrm{d}x\mathrm{d}t\\&\quad=\int_0^T\int_{\mathbb{R}^d}\left(u_\varepsilon-\bar{u}_\varepsilon\right)\left(\frac{m-1}{m}u^m+u-\frac{m-1}{m}\bar{u}^m-\bar{u}\right)*V^\varepsilon\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Namely, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\left(u_\varepsilon-\bar{u}_\varepsilon\right)\left(\frac{m-1}{m}u^m+u-\frac{m-1}{m}\bar{u}^m-\bar{u}\right)*V^\varepsilon\mathrm{d}x\mathrm{d}t\\&\quad=-\frac{1}{2}\int_{\mathbb{R}^d}\left|\int_0^T\left((u\nabla u^{m-1}+\nabla u-\bar{u}\nabla\bar{u}^{m-1}-\nabla\bar{u})*V^\varepsilon\right)\mathrm{d}t\right|^2\mathrm{d}x\leqslant 0,\end{aligned}$$

where we take the limit $\varepsilon\to 0$ and infer that

$$\int_0^T\int_{\mathbb{R}^d}(u-\bar{u})\left(\frac{m-1}{m}u^m+u-\frac{m-1}{m}\bar{u}^m-\bar{u}\right)\mathrm{d}x\mathrm{d}t\leqslant 0.$$

But convexity of the function $\frac{m-1}{m}u^m+u$ implies

$$(u-\bar{u})\left(\frac{m-1}{m}u^m+u-\frac{m-1}{m}\bar{u}^m-\bar{u}\right)\geqslant 0$$

holds everywhere, which is also integrable because $u\in L^{m+1}([0,T]\times\mathbb{R}^d)$. Therefore, it follows $u=\bar{u}$ almost everywhere. $\square$

Data availability statement

No new data were created or analysed in this study.

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