

Analysis of preconditioned Picard iterations for the Navier–Stokes equations

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Mixed finite element formulations of fluid flow problems lead to large systems of equations of saddle-point type for which iterative solution methods are mandatory for reasons of efficiency. A successful approach in the design of solution methods takes into account the structure of the problem; in particular, it is well-known that an efficient solution can be obtained if the associated Schur complement problem can be solved efficiently and robustly. In this work we analyze a preconditioner for the Schur complement for the Oseen problem which was introduced in [18]. We show that the spectrum of the preconditioned system is independent of the mesh parameter; moreover, we demonstrate that the number of GMRES iterations grows like the square-root of the Reynolds number for steady-state problems, while for time-dependent problems this dependence becomes negligible. In both the steady-state and time-dependent case the performance is mesh-independent.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary Γ . Consider the Navier–Stokes equations in primitive variables with the following boundary conditions

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \times (0, T), \quad (1.1a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T), \quad (1.1b)$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^*(\mathbf{x}, t) \quad \text{on } \Gamma_D \times (0, T), \quad (1.1c)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N \times (0, T), \quad (1.1d)$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ in Ω . Here \mathbf{n} is the outward normal to Γ and $\boldsymbol{\sigma} = -p\mathbf{I} + 2\nu\varepsilon(\mathbf{u})$ is the Cauchy stress tensor, with $\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)$, the symmetric part of the velocity gradient.

Fully-implicit and semi-implicit time discretization schemes [25, pp. 438–441] coupled with a Picard linearization of the non-linear part lead to the following Oseen-type problem

$$-\nu \Delta \mathbf{u} + (\mathbf{b} \cdot \nabla) \mathbf{u} + \theta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1.2a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.2b)$$

$$\mathbf{u} = \mathbf{u}^* \quad \text{on } \Gamma_D, \quad (1.2c)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N, \quad (1.2d)$$

where θ , typically of order $O(1/\Delta t)$, is due to the time discretization and \mathbf{b} is a divergence-free vector field which arises from the Picard linearization.

A mixed stabilized finite element formulation of (1.2) leads to the following system of equations (see Section 3)

$$K \mathbf{x} = \begin{pmatrix} F & B_1^t \\ B_2 & -C \end{pmatrix} \begin{pmatrix} \mathbf{x}_u \\ \mathbf{x}_p \end{pmatrix} = \begin{pmatrix} \mathbf{f}_u \\ \mathbf{f}_p \end{pmatrix}. \quad (1.3)$$

where F is a vector advection–diffusion operator, B_1^t, B_2 are discrete gradient and divergence operators including stabilization terms and C is a stabilization matrix.

Since the size of K is usually large, we restrict our attention to iterative solution methods for the system (1.3). In particular, we note here that two major classes of iterative solution methods are (i) multigrid methods; (ii) Krylov subspace methods. Both have been shown to be successful solvers for various choices of discretizations of (1.1) as well as for various ranges of the viscosity parameter ν ; we refer the reader to [8],[37] for a comparison of some of these methods.

In this work we analyze the performance of a preconditioning technique employed in conjunction with a Krylov subspace method. However, the resulting preconditioner could also be used in certain multigrid iterations, e.g., [33], which employ a pressure solution method (see below).

In general, the solution of (1.3) is usually sought in two different ways:

- a global approach, where the solution \mathbf{x} is computed iteratively.

A Krylov subspace approach employs an iterative solver such as GMRES or BiCGStab [27] with right or left preconditioners; some popular choices are given respectively by

$$P_R = \begin{pmatrix} F & B_1^t \\ O & \widehat{S} \end{pmatrix}, \quad P_L = \begin{pmatrix} F & O \\ B_2 & \widehat{S} \end{pmatrix}, \quad (1.4)$$

where \widehat{S} is an approximation to the *pressure Schur complement* $S = C + B_2 F^{-1} B_1^t$. We note here that if $\widehat{S} = S$ convergence is guaranteed in at most 3 iterations [24].

- a pressure solution method, where \mathbf{x}_u is eliminated from (1.3) and then the solution \mathbf{x}_p is computed iteratively; \mathbf{x}_u is then found in terms of \mathbf{x}_p .

This approach leads to a system for \mathbf{x}_p of the form

$$S\mathbf{x}_p = B_2 F^{-1} \mathbf{f}_u - \mathbf{f}_p \quad (1.5)$$

which, when solved iteratively, also needs an approximation to S .

It is clear that both approaches need to approximate (i) the Schur complement S and (ii) the vector ‘advection–diffusion–reaction’ operator F . Assuming that the latter task can be effectively achieved ([1]) we concentrate on approximations of the Schur complement.

We first note that for the Stokes problem a useful approximation of the Schur complement is due to Cahouet and Chabard [4]

$$\widehat{S} = (\nu M_p^{-1} + \theta A_p^{-1})^{-1} \quad (1.6)$$

where M_p and A_p are the projections of the identity and the Neumann Laplacian operators onto the pressure space. We note also that in the case of the steady-state Stokes equations (1.6) becomes

$$\widehat{S} = M_p / \nu, \quad (1.7)$$

the analysis of which proves its effectivity for this problem [35], [29], [19].

Naturally, the above choices of \widehat{S} were considered for the Navier–Stokes equations and in particular for the Oseen problem (1.2). Results are reported in [9], [19], [20] for steady-state problems and for *stable* formulations. We also note here the multigrid approach in [33] which requires an approximation of the Schur complement; the approach in [33] uses among other choices the preconditioner (1.6) to solve the pressure Schur complement problem (1.5). Analytic and numerical results in the above references show that the preconditioned system has a spectrum independent of the mesh parameter and conclude that convergence is mesh-independent. However, the viscosity parameter ν is also an important parameter and it is desirable that convergence be independent of or mildly dependent on ν as $\nu \rightarrow 0$. For example, the choice (1.7) yields a number of iterations which was shown to increase linearly with $1/\nu$ ([19]); moreover, numerical results in [19], [20] seem to indicate that the preconditioner is useful for a limited range of ν . That the convergence rate deteriorates as $\nu \rightarrow 0$ is a somewhat expected result since the non-normality of S cannot be matched by the symmetric preconditioner (1.6).

Alternative preconditioners which tried to deal with the non-symmetry inherent in the Schur complement were proposed in [9] and [18]. Elman suggested the approximation

$$(\widehat{S})^{-1} = (BB^t)^{-1} BFB^t (BB^t)^{-1}$$

for stable formulations for which $B_1 = B_2 = B$. This choice reduces the dependence on ν to $\nu^{-1/2}$ but introduces an h^{-1} dependence for the number of iterations. Moreover, though efficiently applied to the MAC finite difference scheme, the implementation for finite elements requires a further efficient approximation for $(BB^t)^{-1}$ which does not always have a straightforward implementation.

The choice of preconditioner we analyze in this paper was introduced in [18] and is given by

$$(\widehat{S})^{-1} = M_p^{-1} F_p A_p^{-1}, \quad (1.8)$$

with M_p, A_p defined as above and F_p the projection onto the pressure finite element space of the velocity operator of equation (1.2a), $-\nu\Delta + \mathbf{b} \cdot \nabla + \theta$. Note that when $\mathbf{b} = 0$ we recover the preconditioner (1.6). The numerical results presented in [18] together with those in [10] show no mesh dependence and a dependence on ν of order $\nu^{-1/2}$ or less. Our analysis confirms and refines these results. We provide mesh-independent bounds on the eigenvalues of the preconditioned system for the general case of time-dependent, stabilized problems. Moreover, our bounds are shown numerically to be descriptive with respect to the parameters in the problem. We show in particular that the modulus of the eigenvalues of the preconditioned system grows like $R = \|\mathbf{b}\|_{L^2(\Omega)}/\nu$, which for problems with length scale $O(1)$ is the Reynolds number, and demonstrate that this leads to the number of iterations growing linearly with $R^{1/2}$. As a byproduct of our analysis, we confirm the bounds in [9] for the steady-state case with $\widehat{S} = M_p/\nu$ and extend them to the case of stabilized problems.

The paper is structured as follows. In Section 2 we derive the preconditioner for the time-dependent problem (1.1) with the general set of boundary conditions (1.1c, 1.1d). In Section 3 we present the weak formulation of problem (1.2) and consider the stabilized formulation and some useful related bilinear forms. Moreover, we derive results for the discrete operators arising from our mixed stabilized finite element formulation. Section 4 presents the main results, Theorem 8 and Theorem 9 which establish eigenvalue bounds for preconditioners (1.7), (1.8). Finally Section 5 investigates the issue of spectral description of convergence for iterative solvers with special emphasis on pseudo-spectral bounds.

2 Derivation of preconditioner

The preconditioner (1.8) was originally derived for the steady-state Navier–Stokes equations in [18]. We include here an alternative derivation which includes more general boundary conditions and takes into account time-dependent schemes.

Consider the following modified Oseen equations in $\Omega \subset \mathbb{R}^d$

$$-\nu\Delta \mathbf{u} + (\mathbf{b} \cdot \nabla) \mathbf{u} + \theta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (2.1a)$$

$$\operatorname{div} \mathbf{u} = g \quad \text{in } \Omega, \quad (2.1b)$$

$$\mathbf{u} = \mathbf{u}^* \quad \text{on } \Gamma_D, \quad (2.1c)$$

$$\nu \mathbf{n} \cdot \nabla \mathbf{u} = \mathbf{n} p \quad \text{on } \Gamma_N, \quad (2.1d)$$

We note here the presence of g on the right side of equation (2.1b). This will allow us to derive the *continuous pressure Schur complement* and subsequently find a useful approximation to

it. Galdi [12] considered problem (2.1) for the case $\mathbf{b} = (1, 0)$, $c = 0$ in order to calculate the fundamental solution tensor for the resulting Oseen operator, including the inverse of the corresponding Schur complement in \mathbb{R}^2 .

Let $\mathbf{G}(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y})\mathbf{I}_d$ be the Green's tensor for a system of d advection–diffusion–reaction equations with boundary conditions (2.1c, 2.1d). The solution \mathbf{u} of equation (2.1a) for a given p is then given by (see [23, p. 66])

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \int_{\Omega} \mathbf{G}(\mathbf{x}, \mathbf{y})(\mathbf{f}(\mathbf{y}) - \nabla_{\mathbf{y}}p(\mathbf{y}))d\mathbf{y} \\ &\quad + \int_{\Gamma_N} \mathbf{G}(\mathbf{x}, \mathbf{y})\mathbf{n}(\mathbf{y})p(\mathbf{y})d\Gamma(\mathbf{y}) - \int_{\Gamma_D} \mathbf{u}^*(\mathbf{y})\nu\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\mathbf{G}(\mathbf{x}, \mathbf{y})d\Gamma(\mathbf{y}), \end{aligned}$$

which simplifies to

$$\mathbf{u}(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y})\mathbf{f}(\mathbf{y})d\mathbf{y} + \int_{\Omega} \text{grad}_{\mathbf{y}}G(\mathbf{x}, \mathbf{y})p(\mathbf{y})d\mathbf{y} - \int_{\Gamma_D} \mathbf{u}^*(\mathbf{y})\nu\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}G(\mathbf{x}, \mathbf{y})d\Gamma(\mathbf{y}).$$

Using (2.1b) we get

$$\text{div}_{\mathbf{x}} \int_{\Omega} \text{grad}_{\mathbf{y}}G(\mathbf{x}, \mathbf{y})p(\mathbf{y})d\mathbf{y} = g - \text{div}_{\mathbf{x}} \int_{\Omega} G(\mathbf{x}, \mathbf{y})\mathbf{f}(\mathbf{y})d\mathbf{y} + \text{div}_{\mathbf{x}} \int_{\Gamma_D} \mathbf{u}^*(\mathbf{y})\nu\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}G(\mathbf{x}, \mathbf{y})d\Gamma(\mathbf{y}).$$

This suggests the following definition.

Definition 1 *The continuous pressure Schur operator \mathcal{S} is given by*

$$\mathcal{S}p(\mathbf{x}) = \text{div}_{\mathbf{x}} \int_{\Omega} \text{grad}_{\mathbf{y}}G(\mathbf{x}, \mathbf{y})p(\mathbf{y})d\mathbf{y}. \quad (2.2)$$

The pressure Schur operator is not available in closed form in general. One way to circumvent this problem is to replace the Green's function for the advection–diffusion–reaction operator $G(\mathbf{x}, \mathbf{y})$ with the corresponding fundamental solution which we denote by $\mathcal{G}(\mathbf{x})$. This function is known to always exist [15] and it is even available in closed form for special cases of \mathbf{b} [21]. However, we are interested in the following identity

$$\text{div}_{\mathbf{x}}\text{grad}_{\mathbf{y}}\mathcal{G}(\mathbf{y} - \mathbf{x}) = -\Delta_{\mathbf{x}}\mathcal{G}(\mathbf{y} - \mathbf{x}),$$

which leads to our continuous approximation $\widehat{\mathcal{S}}$ of the Schur operator \mathcal{S}

$$\widehat{\mathcal{S}}p(\mathbf{x}) = -\Delta_{\mathbf{x}} \int_{\Omega} \mathcal{G}(\mathbf{y} - \mathbf{x})p(\mathbf{y})d\mathbf{y}.$$

Remark 1 *Our approximation does not depend on the boundary conditions for the Oseen problem (2.1).*

We also note that the approximation $\widehat{\mathcal{S}}$ with \mathcal{G} restricted to Ω can be defined via

$$(-\nu\Delta + \mathbf{b} \cdot \nabla + \theta)\phi(\mathbf{x}) = p(\mathbf{x}) \quad \text{in } \Omega, \quad (2.3a)$$

$$\widehat{\mathcal{S}}p = -\Delta\phi(\mathbf{x}) \quad \text{in } \Omega. \quad (2.3b)$$

However, the above equations do not uniquely define $\widehat{\mathcal{S}}$. Numerical investigations in [18] suggest the following boundary conditions

$$\mathbf{n} \cdot \nabla\phi = 0 \quad \text{on } \Gamma. \quad (2.4)$$

together with the pressure constraint $\langle p, 1 \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the usual $L^2(\Omega)$ -inner product.

3 Problem formulation

Consider the following Oseen problem arising from the linearization of the time-dependent Navier–Stokes equations (1.1) in \mathbb{R}^2

$$-\nu\Delta\mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{u} + \theta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (3.1a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (3.1b)$$

$$\mathbf{u} = \mathbf{u}^* \quad \text{on } \Gamma_D, \quad (3.1c)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N, \quad (3.1d)$$

where θ is a non-negative constant and $\mathbf{b} \in [H^1(\Omega)]^2 \cap [L^\infty(\Omega)]^2$, with $\operatorname{div} \mathbf{b} = 0$. We also assume that $\mathbf{n} \cdot \mathbf{b} \geq 0$ on Γ_N which is a standard, mild restriction (see [26]).

A mixed formulation of problem (3.1) involves choosing appropriate spaces

$$\mathbf{V} \subset [H_E^1(\Omega)]^2 = \{\boldsymbol{\phi} \in [H^1(\Omega)]^2 : \boldsymbol{\phi}|_{\Gamma_D} = \mathbf{0}\}, \quad P \subset L_0^2(\Omega) = \{p \in L^2(\Omega) : \langle p, 1 \rangle = 0\}$$

for the velocity and pressure respectively and results in the following weak formulation

Given $\mathbf{f} \in [L^2(\Omega)]^2$, find $(\mathbf{u}, p) \in H = \mathbf{V} \times P$ such that

$$B(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in H, \quad (3.2)$$

where

$$B(\mathbf{w}, r; \mathbf{v}, q) = \nu \langle \nabla \mathbf{w}, \nabla \mathbf{v} \rangle + \langle \mathbf{b} \cdot \nabla \mathbf{w} + \theta \mathbf{w}, \mathbf{v} \rangle - \langle r, \operatorname{div} \mathbf{v} \rangle - \langle q, \operatorname{div} \mathbf{w} \rangle \quad (3.3)$$

and F is a linear functional on H given by

$$F(\mathbf{v}, q) = \langle \mathbf{f}, \mathbf{v} \rangle.$$

Existence and uniqueness of solutions to problems of type (3.2) is guaranteed provided the following conditions hold for all $(\mathbf{w}, r), (\mathbf{v}, q) \in H$ ([1])

$$|B(\mathbf{w}, r; \mathbf{v}, q)| \leq C_1 \|\mathbf{w}, r\|_H \|\mathbf{v}, q\|_H \quad (3.4a)$$

$$\sup_{(\mathbf{w}, r) \in H \setminus \{0\}} \frac{B(\mathbf{w}, r; \mathbf{v}, q)}{\|\mathbf{w}, r\|_H} \geq C_2 \|\mathbf{v}, q\|_H \quad (3.4b)$$

$$\sup_{(\mathbf{v}, q) \in H \setminus \{0\}} \frac{B(\mathbf{w}, r; \mathbf{v}, q)}{\|\mathbf{v}, q\|_H} \geq C_3 \|\mathbf{w}, r\|_H \quad (3.4c)$$

where $\|\cdot\|_H$ is a suitable norm on $H = \mathbf{V} \times P$.

Let $H^h = (\mathbf{V}^h, P^h) \subset (\mathbf{V}, P) = H$ be a pair of finite element spaces of functions defined on the corresponding subdivisions \mathcal{T}^h of Ω into simplices T with diameter h_T . Let \mathcal{E}^h denote the set of inter-element boundaries and let $h = \max_T h_T$. A stable finite element discretization of (3.2) requires that the discrete versions of equations (3.4) obtained by replacing H with H^h are satisfied with constants independent of the discretization parameters. While condition (3.4a)

can be shown to be satisfied for example with respect to the norm $\|\mathbf{w}, r\|_H = (\|\mathbf{w}\|_1^2 + \|r\|_0^2)^{1/2}$, conditions (3.4b, 3.4c) lead to the following well-known inf-sup stability condition [2]

$$\sup_{\mathbf{w} \in \mathbf{V}^h \setminus \{0\}} \frac{\langle q, \operatorname{div} \mathbf{w} \rangle}{\|\mathbf{w}\|_{\mathbf{V}^h}} \geq C_2 \|q\|_0 \quad \forall q \in P^h \quad (3.5)$$

which is not automatically satisfied for any choice of finite element spaces. One way of circumventing this problem is to choose a stabilized discrete formulation of (3.2)

Find $(\mathbf{u}, p) \in \mathbf{V}^h \times P^h$ such that for all $(\mathbf{v}, q) \in \mathbf{V}^h \times P^h$

$$B_\delta(\mathbf{u}, p; \mathbf{v}, q) = F_\delta(\mathbf{v}, q), \quad (3.6)$$

where

$$\begin{aligned} B_\delta(\mathbf{w}, r; \mathbf{v}, q) &= B(\mathbf{w}, r; \mathbf{v}, q) + \beta \langle \operatorname{div} \mathbf{w}, \operatorname{div} \mathbf{v} \rangle - \sigma \sum_{E \in \mathcal{E}^h} h_E \langle [r]_E, [q]_E \rangle_E \\ &\quad + \sum_{T \in \mathcal{T}^h} \delta_T \langle -\nu \Delta \mathbf{w} + \mathbf{b} \cdot \nabla \mathbf{w} + \theta \mathbf{w} + \nabla r, \rho \nu \Delta \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{v} - \nabla q \rangle_T \\ F_\delta(\mathbf{v}, q) &= (\mathbf{f}, \mathbf{v}) + \sum_{T \in \mathcal{T}^h} \delta_T \langle \mathbf{f}, \rho \nu \Delta \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{v} - \nabla q \rangle_T \end{aligned}$$

and δ_T is a non-negative mesh function with generic parameter $\delta = \max_T \delta_T$ which tends to zero as $h \rightarrow 0$.

The above formulation is equivalent to the saddle-point problem

Find $(\mathbf{u}, p) \in \mathbf{V}^h \times P^h$ such that for all $(\mathbf{v}, q) \in \mathbf{V}^h \times P^h$

$$a(\mathbf{u}, \mathbf{v}) + b_1(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle + \sum_{T \in \mathcal{T}^h} \delta_T \langle \mathbf{f}, \rho \nu \Delta \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{v} \rangle_T \quad (3.7a)$$

$$b_2(\mathbf{u}, q) + c(p, q) = - \sum_{T \in \mathcal{T}^h} \delta_T \langle \mathbf{f}, \nabla q \rangle_T \quad (3.7b)$$

where

$$\begin{aligned} a(\mathbf{w}, \mathbf{v}) &= \nu \langle \nabla \mathbf{w}, \nabla \mathbf{v} \rangle + \langle \mathbf{b} \cdot \nabla \mathbf{w}, \mathbf{v} \rangle + \theta \langle \mathbf{w}, \mathbf{v} \rangle + \beta \langle \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v} \rangle \\ &\quad + \sum_{T \in \mathcal{T}^h} \delta_T \langle -\nu \Delta \mathbf{w} + \mathbf{b} \cdot \nabla \mathbf{w} + \theta \mathbf{w}, \nu \Delta \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{v} \rangle_T \end{aligned} \quad (3.8)$$

$$b_1(\mathbf{v}, r) = - \langle \operatorname{div} \mathbf{v}, r \rangle - \sum_{T \in \mathcal{T}^h} \delta_T \langle -\nu \Delta \mathbf{v} - \mathbf{b} \cdot \nabla \mathbf{v}, \nabla r \rangle_T \quad (3.9)$$

$$b_2(\mathbf{w}, q) = - \langle \operatorname{div} \mathbf{w}, q \rangle - \sum_{T \in \mathcal{T}^h} \delta_T \langle -\nu \Delta \mathbf{w} + \mathbf{b} \cdot \nabla \mathbf{w} + \theta \mathbf{w}, \nabla q \rangle_T \quad (3.10)$$

$$c(r, q) = - \sum_{T \in \mathcal{T}^h} \delta_T \langle \nabla r, \nabla q \rangle_T - \sigma \sum_{E \in \mathcal{E}^h} h_E \langle [r]_E, [q]_E \rangle_E. \quad (3.11)$$

We note that for the case of piecewise constant basis elements for the pressure space P^h the first term in (3.11) vanishes, while for continuous elements the term involving jumps across inter-element boundaries is zero.

For various choices of parameters one can recover the various stabilization schemes previously proposed:

- $\rho = 0, \beta \neq 0, \theta = 0$: the SUPG formulation of Brooks and Hughes [3]
- $\rho = 0, \beta \neq 0, \theta \neq 0$: the SUPG formulation of Hansbo and Szepessy[14]
- $\rho = \pm 1, \beta \neq 0, \theta = 0$: the formulation in Franca and Frey [11]
- $\rho = 0, \beta = 0, \theta = 0$: the penalty method of Tobiska and Lube [31] – we note that this method had also the penalty term, $\sum_{T \in \mathcal{T}^h} \tau \langle \nabla r, \nabla q \rangle_T$ included.
- $\rho = -1, \beta = 0, \theta = 0$: the least-squares formulation of Zhou and Feng [38]

Note that in all the above references $\theta = \sigma = 0$. The case $\sigma \neq 0$, which allows for discontinuous pressure across inter-element boundaries is considered for example in [26]. Although we are interested in solving the time-dependent problem with both low-order methods ($\sigma \neq 0$) and high-order methods ($\sigma = 0$) we consider the above formulation only for continuous elements ($\sigma = 0$), but otherwise in its full generality with $\theta, \beta, \rho \neq 0$; for simplicity we set $\rho = 1$.

We will find it useful to introduce the following modified bilinear form

$$\tilde{B}_\delta(\mathbf{w}, r; \mathbf{v}, q) = \tilde{a}(\mathbf{w}, \mathbf{v}) + b_1(\mathbf{v}, r) + b_2(\mathbf{w}, q) + \tilde{c}(r, q) \quad (3.12)$$

where

$$\begin{aligned} \tilde{a}(\mathbf{w}, \mathbf{v}) &= \nu \langle \nabla \mathbf{w}, \nabla \mathbf{v} \rangle + \theta \langle \mathbf{w}, \mathbf{v} \rangle + \beta \langle \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v} \rangle + \frac{1}{2} \int_{\Gamma_N} (\mathbf{b} \cdot \mathbf{n})(\mathbf{w} \cdot \mathbf{v}) d\Gamma \\ &\quad + \sum_{T \in \mathcal{T}^h} \delta_T \langle -\nu \Delta \mathbf{w} + \mathbf{b} \cdot \nabla \mathbf{w} + \theta \mathbf{w}, \nu \Delta \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{v} \rangle_T \\ \tilde{c}(r, q) &= -(\Gamma + 1) \sum_{T \in \mathcal{T}} \delta_T \langle \nabla r, \nabla q \rangle_T \end{aligned}$$

with Γ to be defined later. Note that $\tilde{a}(\cdot, \cdot)$ was chosen so that $\tilde{a}(\mathbf{v}, \mathbf{v}) = a(\mathbf{v}, \mathbf{v})$ for $\mathbf{v} \in \mathbf{V}^h$. We also need to introduce suitable norms on \mathbf{V}^h and H^h . We first note that after integration by parts

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}) &= \nu |\mathbf{v}|_1^2 + \theta \|\mathbf{v}\|_0^2 + \beta \|\operatorname{div} \mathbf{v}\|_0^2 + \frac{1}{2} \int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{v}|^2 d\Gamma - \sum_{T \in \mathcal{T}^h} \delta_T \|\nu \Delta \mathbf{v}\|_{0,T}^2 + \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 \\ &\quad + \theta \sum_{T \in \mathcal{T}^h} \delta_T \nu \int_{\Gamma_T} \frac{1}{2} (\mathbf{n} \cdot \nabla) |\mathbf{v}|^2 d\Gamma_T - \theta \sum_{T \in \mathcal{T}^h} \delta_T \nu |\mathbf{v}|_1^2 + \theta \sum_{T \in \mathcal{T}^h} \delta_T \int_{\Gamma_T} \mathbf{b} \cdot \mathbf{n} |\mathbf{v}|^2 d\Gamma_T \end{aligned}$$

defines a norm on \mathbf{V}^h if it is positive for all $\mathbf{v} \in \mathbf{V}^h \setminus \{\mathbf{0}\}$. This property holds for a particular choice of δ_T . Let $\|\cdot\|_h : \mathbf{V}^h \rightarrow \mathbb{R}$, $\|\cdot, \cdot\|_h : H^h \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned}\|\mathbf{v}\|_h^2 &:= a(\mathbf{v}, \mathbf{v}) \\ \|\mathbf{v}, q\|_h^2 &:= \|\mathbf{v}\|_h^2 + \frac{1}{\nu}\|q\|_0^2.\end{aligned}$$

We have

Lemma 1 *Let $\mathbf{v}, \mathbf{w} \in \mathbf{V}^h$ and assume the following inverse inequality holds for all $\mathbf{v} \in \mathbf{V}^h$*

$$\|\Delta \mathbf{v}\|_{0,T} \leq \mu h_T^{-1} \|\nabla \mathbf{v}\|_{0,T}, \quad \mu > 0.$$

Let $\delta_T = \alpha h_T^2 / \nu$ with $\alpha < \min\{2/(3\mu^2), \nu/(h^2\theta)\}$. Then $\|\cdot\|_h$ is a norm on \mathbf{V}^h with

$$a(\mathbf{v}, \mathbf{v}) = \tilde{a}(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_h^2$$

and there exist constants C_0, \tilde{C}_0 such that

$$a(\mathbf{w}, \mathbf{v}) \leq C_0 \frac{b}{[\nu(\nu + \theta)]^{1/2}} \|\mathbf{w}\|_h \|\mathbf{v}\|_h, \quad \tilde{a}(\mathbf{w}, \mathbf{v}) \leq \tilde{C}_0 \|\mathbf{w}\|_h \|\mathbf{v}\|_h$$

where $b = \|\mathbf{b}\|_0$.

Proof See Appendix. ■

Lemma 2 *Let $H^h = (\mathbf{V}^h, P^h)$ be a pair of finite element spaces and let $\delta_T = \alpha h_T^2 / \nu$ with α satisfying the restrictions in Lemma 1. Let also Γ in (3.12) satisfy $\Gamma = \max\{0, \tilde{C}_0^{-1} - 1\}$. Then there exist constants C_1, C_2 such that*

$$\left| \tilde{B}_\delta(\mathbf{w}, r; \mathbf{v}, q) \right| \leq C_1 \|\mathbf{w}, r\|_h \|\mathbf{v}, q\|_h \quad \forall (\mathbf{w}, r), (\mathbf{v}, q) \in H^h \quad (3.13a)$$

$$\sup_{(\mathbf{w}, r) \in H^h \setminus \{0\}} \frac{\tilde{B}_\delta(\mathbf{w}, r; \mathbf{v}, q)}{\|\mathbf{w}, r\|_h} \geq C_2 \|\mathbf{v}, q\|_h \quad \forall (\mathbf{v}, q) \in H^h \quad (3.13b)$$

$$\sup_{(\mathbf{v}, q) \in H^h \setminus \{0\}} \frac{\tilde{B}_\delta(\mathbf{w}, r; \mathbf{v}, q)}{\|\mathbf{v}, q\|_h} \geq C_2 \|\mathbf{w}, r\|_h \quad \forall (\mathbf{w}, r) \in H^h \quad (3.13c)$$

Proof See Appendix. ■

3.1 Discrete operator results

In this section we derive the results necessary in the analysis of preconditioned iterations for the discrete problem (3.6). First we introduce some notation and conventions.

Given a matrix $M \in \mathbb{R}^{n \times n}$, we denote by $\sigma_i(M), \lambda_i(M)$ the i th largest singular value and eigenvalue of M respectively. Moreover, the following are standard results [17]

$$\sigma_n(M) \leq |\lambda_i(M)| \leq \sigma_1(M). \quad (3.14a)$$

$$\lambda_n(H(M)) \leq \sigma_n(M) \leq \lambda_1(H(M)), \quad (3.14b)$$

where $H(M) = (M + M^t)/2$. Finally, the set

$$\mathcal{F}(M) = \left\{ \frac{\mathbf{x}^* M \mathbf{x}}{\mathbf{x}^* \mathbf{x}}, \mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\} \right\}$$

is called the *field of values* of M .

Let \mathcal{T}^h be a subdivision of Ω into simplices T ; let h_T (resp. τ_T) be the diameter of the smallest disc containing T (resp. largest disc contained in T) and let

$$h = \max_T h_T, \quad \tau = \min_T \frac{\tau_T}{h}.$$

In the following we consider *quasi-uniform* families of subdivisions which satisfy $\tau \geq \tau_0 > 0$ for all h .

Let $(\mathbf{w}, r), (\mathbf{v}, q) \in H^h$. We define the following discrete operators

$$\langle F \mathbf{w}, \mathbf{v} \rangle = a(\mathbf{w}, \mathbf{v}) \tag{3.15a}$$

$$\langle A \mathbf{w}, \mathbf{v} \rangle = \frac{1}{2} (a(\mathbf{w}, \mathbf{v}) + a(\mathbf{v}, \mathbf{w})) \tag{3.15b}$$

$$\langle N \mathbf{w}, \mathbf{v} \rangle = \frac{1}{2} (a(\mathbf{w}, \mathbf{v}) - a(\mathbf{v}, \mathbf{w})) \tag{3.15c}$$

$$\langle B_1 \mathbf{v}, r \rangle = b_1(\mathbf{v}, r) \tag{3.15d}$$

$$\langle B_2 \mathbf{w}, q \rangle = b_2(\mathbf{w}, q) \tag{3.15e}$$

$$\langle C r, q \rangle = -c(r, q) \tag{3.15f}$$

$$\langle M_p r, q \rangle = \langle r, q \rangle. \tag{3.15g}$$

With a standard abuse of notation, we let the above operators denote also their respective representations in our choice of bases for H^h . We note here that under the assumption of mesh uniformity one can show that there exist constants γ, Γ such that

$$\gamma h^2 \leq \frac{\mathbf{x}^t M_p \mathbf{x}}{\mathbf{x}^t \mathbf{x}} \leq \Gamma h^2 \quad \forall \mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}. \tag{3.16}$$

Moreover, we assume the following inverse inequality holds for functions $q \in P^h \setminus \{0\}$

$$h \|\nabla q\|_0 \leq \eta \|q\|_0 \tag{3.17}$$

which yields the following bound

$$\frac{\mathbf{x}^t C \mathbf{x}}{\mathbf{x}^t M_p \mathbf{x}} \leq \frac{\alpha \eta^2}{\nu} \quad \forall \mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}. \tag{3.18}$$

The above choice of finite element spaces produces the following block matrix problem

$$\begin{pmatrix} F & B_1^t \\ B_2 & -C \end{pmatrix} \begin{pmatrix} \mathbf{x}_u \\ \mathbf{x}_p \end{pmatrix} = \begin{pmatrix} \mathbf{f}_u \\ \mathbf{f}_p \end{pmatrix} \tag{3.19}$$

where $F = A + N$ and we denoted by \mathbf{x}_u the vector of coefficients of the velocities u_i , $i = 1, 2$ expanded in the basis of \mathbf{V}^h and by \mathbf{x}_p the vector of coefficients of the pressure p^h expanded in the basis of P^h .

With this choice of finite element spaces, Lemma 1 yields the following discrete operator results

$$1 = \frac{\mathbf{x}^t F \mathbf{x}}{\mathbf{x}^t A \mathbf{x}}, \quad \frac{\mathbf{x}^t F \mathbf{y}}{(\mathbf{x}^t A \mathbf{x})^{1/2} (\mathbf{y}^t A \mathbf{y})^{1/2}} \leq C_0 \frac{b}{\nu} \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Similar relations can be derived for the Rayleigh quotient involving the inverses of F and A . We have

$$\frac{\mathbf{x}^t F^{-1} \mathbf{x}}{\mathbf{x}^t A^{-1} \mathbf{x}} = \frac{\mathbf{y}^t \tilde{F}^{-1} \mathbf{y}}{\mathbf{y}^t \mathbf{y}}$$

where $\tilde{F} = A^{-1/2} F A^{-1/2} = I + A^{-1/2} N A^{-1/2} = I + \tilde{N}$ is a normal matrix since N is a skew-symmetric matrix. Under this assumption, the field of values of \tilde{F}^{-1} satisfies [16, p. 11]

$$\mathcal{F}(\tilde{F}^{-1}) = Co(\Lambda(\tilde{F}^{-1}))$$

where $Co(\chi)$ denotes the convex hull of a set χ . Thus,

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^t F^{-1} \mathbf{x}}{\mathbf{x}^t A^{-1} \mathbf{x}} &= \min \operatorname{Re}(\mathcal{F}(\tilde{F}^{-1})) \\ &= \min_k \operatorname{Re} \lambda_k(\tilde{\mathbf{F}}^{-1}) \\ &= \min_k \operatorname{Re} \frac{1}{\lambda_k(\tilde{F})} \\ &= \min_k \operatorname{Re} \frac{1}{1 + \lambda_k(\tilde{N})} \\ &= \frac{1}{\max_k |\lambda_k(\tilde{F})|^2} \\ &\geq C_0^{-1} \frac{\nu(\nu + \theta)}{b^2}. \end{aligned}$$

We also have

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^t F^{-1} \mathbf{x}}{\mathbf{x}^t A^{-1} \mathbf{x}} &= \max \operatorname{Re}(\mathcal{F}(\tilde{F}^{-1})) \\ &\leq \max_k \sigma_k(\tilde{F}^{-1}) \\ &= \frac{1}{\min_k \sigma_k(\tilde{F})} \\ &\leq 1. \end{aligned}$$

We summarise these results in the following

Lemma 3 *Let $A, F = A + N$ be defined as in (3.15). Then*

$$\frac{\mathbf{x}^t F \mathbf{x}}{\mathbf{x}^t A \mathbf{x}} = 1, \quad C_0^{-1} \frac{\nu(\nu + \theta)}{b^2} \leq \frac{\mathbf{x}^t F^{-1} \mathbf{x}}{\mathbf{x}^t A^{-1} \mathbf{x}} \leq 1.$$

In order to derive similar results involving the matrices B_1, B_2 we employ Lemma 2. The inf-sup condition (3.13b) with $\mathbf{v}^h = 0$ leads to the following discrete inequality

$$C_2 \left(\mathbf{x}_q^t (M_p/\nu) \mathbf{x}_q \right)^{1/2} \leq \max_{(\mathbf{x}_w, \mathbf{x}_r)} \frac{\begin{pmatrix} \mathbf{x}_w \\ \mathbf{x}_r \end{pmatrix}^t \begin{pmatrix} B_1^t \\ -\tilde{C} \end{pmatrix} \mathbf{x}_q}{\left(\begin{pmatrix} \mathbf{x}_w \\ \mathbf{x}_r \end{pmatrix}^t \begin{pmatrix} A & O \\ O & M_p/\nu \end{pmatrix} \begin{pmatrix} \mathbf{x}_w \\ \mathbf{x}_r \end{pmatrix} \right)^{1/2}} \quad \forall \mathbf{x}_q \in \mathbb{R}^m,$$

where $(\mathbf{x}_w^t \ \mathbf{x}_r^t)^t \in \mathbb{R}^{n+m} \setminus \{\mathbf{0}\}$ is the vector of coefficients of $(\mathbf{w}^h, \mathbf{r}^h)$ in the basis for H^h and $\tilde{C} = (\Gamma + 1)C$.

Hence

$$C_2 \min_{\mathbf{x}_q \in \mathbb{R}^m \setminus \{\mathbf{0}\}} \left(\frac{\mathbf{x}_q^t (M_p/\nu) \mathbf{x}_q}{\mathbf{x}_q^t \mathbf{x}_q} \right)^{1/2} \leq \min_{\mathbf{x}_q \in \mathbb{R}^m \setminus \{\mathbf{0}\}} \max_{(\mathbf{y}_w, \mathbf{y}_r)} \frac{\begin{pmatrix} \mathbf{y}_w \\ \mathbf{y}_r \end{pmatrix}^t \begin{pmatrix} A^{-1/2} B_1^t \\ -(M_p/\nu)^{-1/2} \tilde{C} \end{pmatrix} \mathbf{x}_q}{\|(\mathbf{y}_w^t \ \mathbf{y}_r^t)^t\| \|\mathbf{x}_q\|},$$

after setting $\mathbf{y}_w = A^{1/2} \mathbf{x}_w, \mathbf{y}_r = (M_p/\nu)^{1/2} \mathbf{x}_r$. Thus

$$C_2 \lambda_m^{1/2} (M_p/\nu) \leq \sigma_m \left(\begin{pmatrix} A^{-1/2} B_1^t \\ -(M_p/\nu)^{-1/2} \tilde{C} \end{pmatrix} \right)$$

and standard singular value inequalities [16, p. 149] yield the following bound

$$C_2 \lambda_m^{1/2} (M_p/\nu) \leq \sigma_m (B_1 A^{-1/2}). \quad (3.20)$$

Similarly, using the continuity relation (3.13a) with $\mathbf{v}^h = 0, \mathbf{r}^h = 0$ we get

$$\sup_{\mathbf{x}_w, \mathbf{x}_q} \mathbf{x}_w^t B_1^t \mathbf{x}_q \leq C_1 \left(\mathbf{x}_w^t A \mathbf{x}_w \right)^{1/2} \left(\mathbf{x}_q^t (M_p/\nu) \mathbf{x}_q \right)^{1/2} \quad (3.21)$$

Similar results can be derived for B_2 which together with (3.21) give

Lemma 4 *Let the assumptions of Lemma 2 hold and let A, B, M_p be defined as above. Then there exist constants C_3, C_4 such that*

$$\max_{\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^t B_i A^{-1} B_i^t \mathbf{x}}{\mathbf{x}^t M_p \mathbf{x}} \leq C_3 \frac{1}{\nu} \quad i = 1, 2, \quad (3.22)$$

$$\max_{\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^t B_i^t B_i \mathbf{x}}{\mathbf{x}^t A \mathbf{x}} \leq C_4 \frac{h^2}{\nu} \quad i = 1, 2. \quad (3.23)$$

Proof Note first that (3.21) leads to the following bound on the singular values of $B_i A^{-1/2}$

$$\sigma_k(B_i A^{-1/2}) \leq C_1 \lambda_1^{1/2}(M_p/\nu)$$

Thus,

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^m \setminus \{0\}} \frac{\mathbf{x}^t B_i A^{-1} B_i^t \mathbf{x}}{\mathbf{x}^t \mathbf{x}} &\leq \sigma_1(B_i A^{-1} B_i^t) \\ &\leq \sigma_1^2(B_i A^{-1/2}) \\ &\leq C_1^2 \lambda_1(M_p/\nu) \end{aligned}$$

and the first part follows immediately with $C_3 = C_1^2$. On the other hand, since for any rectangular matrix R we have $\max_k \sigma_k(R)^2 = \max_k \lambda_k(R^t R)$, we can divide (3.21) by $(\mathbf{x}_w^t A \mathbf{x}_w)^{1/2}$ and then square the result to get

$$\begin{aligned} C_1^2 \Gamma h^2 \frac{1}{\nu} &\geq \max_k \lambda_k(A^{-1/2} B_i^t B_i A^{-1/2}) \\ &= \max_{\mathbf{x} \in \mathbb{R}^m \setminus \{0\}} \frac{\mathbf{x}^t A^{-1/2} B_i^t B_i A^{-1/2} \mathbf{x}}{\mathbf{x}^t \mathbf{x}} \\ &= \max_{\mathbf{y} \in \mathbb{R}^m \setminus \{0\}} \frac{\mathbf{y}^t B_i^t B_i \mathbf{y}}{\mathbf{y}^t A \mathbf{y}}. \end{aligned}$$

■

3.2 Discrete representation of the preconditioner

We recall here the approximation (2.3) to the continuous Schur complement (2.2) derived in Section 2

$$(-\nu \Delta + \mathbf{b} \cdot \nabla + \theta) \phi(\mathbf{x}) = p(\mathbf{x}) \quad \text{in } \Omega, \quad (3.24a)$$

$$\widehat{S} p = -\Delta \phi(\mathbf{x}) \quad \text{in } \Omega, \quad (3.24b)$$

$$\mathbf{n} \cdot \nabla \phi = 0 \quad \text{on } \Gamma. \quad (3.24c)$$

In order to derive the discrete form of our preconditioner, we project (3.24) onto $L_0^2(\Omega)$. In particular, we choose to project equation (3.24a) in the same way as we projected the velocity operator in equation (3.1a). Let $p, \phi \in P^h$ and let $a(\cdot, \cdot)$ be defined as in (3.8). We seek $(\widehat{S} p)^h$ such that

$$a(\phi, \psi) = \langle p, \psi \rangle \quad (3.25a)$$

$$\left\langle (\widehat{S} p)^h, \psi \right\rangle = \langle \nabla \phi, \nabla \psi \rangle \quad (3.25b)$$

for all $\psi \in P^h$. Let $q, r \in P^h$. Defining the discrete operators

$$\begin{aligned} \langle F_p q, r \rangle &= a(q, r), \\ \langle M_p q, r \rangle &= \langle q, r \rangle, \\ \langle A_p q, r \rangle &= \langle \nabla q, \nabla r \rangle, \\ \langle S_p q, r \rangle &= \left\langle (\widehat{S} q)^h, r \right\rangle, \end{aligned}$$

equations (3.25) become

$$\begin{aligned} F_p \Phi &= M_p \mathbf{p}, \\ S_p \mathbf{p} &= A_p \Phi, \end{aligned}$$

where \mathbf{p}, Φ are the vectors of coefficients of p, ϕ in the basis of P^h .

Remark 2 Since $P^h \subset P \subset L_0^2(\Omega)$ the operators F_p, A_p are non-singular. This is due to the constraint $\langle p, 1 \rangle = 0$ imposed on functions of P^h . Therefore the inverse of F_p exists and we can write the projection of \widehat{S} onto P^h as

$$S_p = A_p F_p^{-1} M_p. \quad (3.26)$$

We note here that the Poincaré inequality [5, p. 12]

$$\|q\|_0 \leq C(\Omega) \|\nabla q\|_0 \quad \forall q \in P^h \setminus \{0\} \quad (3.27)$$

together with (3.16) and (3.17) yield the bound

$$\frac{\gamma h^2}{C(\Omega)} \leq \frac{\mathbf{x}^t A_p \mathbf{x}}{\mathbf{x}^t \mathbf{x}} \leq \eta \Gamma \quad \forall \mathbf{x} \in \mathbb{R}^m \setminus \{0\}. \quad (3.28)$$

We end this section with the following two results.

Theorem 5 Let F_p, A_p be defined as above. Then

$$\|F_p A_p^{-1}\| \leq C_6 b + C_5 \theta.$$

Proof Let H_p, N_p be defined by

$$\begin{aligned} \langle H_p q, r \rangle &= \frac{1}{2} (a(q, r) + a(r, q)) \\ \langle N_p q, r \rangle &= \frac{1}{2} (a(q, r) - a(r, q)) \\ &= \frac{1}{2} (s(q, r) - s(r, q)) \\ \langle Q_p q, r \rangle &= s(q, r), \end{aligned}$$

where

$$s(q, r) = \langle \mathbf{b} \cdot \nabla q, r \rangle - \theta \sum_T \delta_T \langle \mathbf{b} \cdot q, r \rangle_T + \theta \nu \sum_T \delta_T \int_{\Gamma_N} \mathbf{n} \cdot \nabla q \, r \, d\Gamma.$$

Then $F_p = H_p + N_p$, $N_p = (Q_p - Q_p^t)/2$ and

$$\|F_p A_p^{-1}\| \leq \|H_p A_p^{-1}\| + \frac{1}{2} (\|Q_p A_p^{-1}\| + \|Q_p^t A_p^{-1}\|).$$

Let $q = \sum(\mathbf{q})_i \psi_i$, $r = \sum(\mathbf{r})_i \psi_i$ be arbitrary (pressure) grid functions defined on Ω . Then it is easy to see that

$$\langle H_p q, q \rangle \leq \gamma_1 \langle A_p q, q \rangle \quad (3.29)$$

$$|s(q, r)| \leq \gamma_2 b |q|_1 \|r\|_0, \quad (3.30)$$

for some constants γ_1, γ_2 . In particular

$$\gamma_1 = \max \{ \nu, \delta b^2, 2\beta, C(\Omega)\theta, C(\Omega)\theta\delta/2, C(\Omega)\theta b\delta \} = C(\Omega)\theta,$$

under the assumption that ν is small and δ, β decrease with the mesh parameter. Inequality (3.29) gives

$$\frac{\mathbf{q}^t H_p \mathbf{q}}{\mathbf{q}^t A_p \mathbf{q}} = \frac{\mathbf{q}^t \tilde{H}_p \mathbf{q}}{\mathbf{q}^t \mathbf{q}} \leq C(\Omega)\theta$$

where $\tilde{H}_p = A_p^{-1/2} H_p A_p^{-1/2}$. We get

$$\begin{aligned} \|H_p A_p^{-1}\|^2 &= \max_{\mathbf{q} \neq \mathbf{0}} \frac{\mathbf{q}^t A_p^{-1} H_p^2 A_p^{-1} \mathbf{q}}{\mathbf{q}^t \mathbf{q}} \\ &= \max_{\mathbf{q} \neq \mathbf{0}} \frac{\mathbf{q}^t \tilde{H}_p A_p^2 \tilde{H}_p \mathbf{q}}{\mathbf{q}^t \mathbf{q}} \\ &\leq \eta^2 \Gamma^2 \max_{\mathbf{q} \neq \mathbf{0}} \frac{\mathbf{q}^t \tilde{H}_p^2 \mathbf{q}}{\mathbf{q}^t \mathbf{q}} \\ &\leq \eta^2 \Gamma^2 C(\Omega)^2 \theta^2. \end{aligned}$$

On the other hand, inequality (3.30) gives

$$\frac{|\mathbf{q}^t Q_p^t \mathbf{r}|}{(\mathbf{r}^t M_p \mathbf{r})^{1/2}} \leq \gamma_2 b (\mathbf{q}^t A_p \mathbf{q})^{1/2} \quad \forall \mathbf{q}, \mathbf{r} \in \mathbb{R}^m \setminus \{\mathbf{0}\}.$$

Setting $\mathbf{r} = Q_p \mathbf{q}$ we get

$$\frac{|\mathbf{q}^t Q_p^t Q_p \mathbf{q}|}{(\mathbf{q}^t Q_p^t M_p Q_p \mathbf{q})^{1/2}} \leq \gamma_2 b (\mathbf{q}^t A_p \mathbf{q})^{1/2} \quad \forall \mathbf{q} \in \mathbb{R}^m \setminus \{\mathbf{0}\}.$$

By (3.16), $|\mathbf{q}^t Q_p^t M_p Q_p \mathbf{q}| \leq \Gamma h^2 |\mathbf{q}^t Q_p^t Q_p \mathbf{q}|$. Thus

$$|\mathbf{q}^t Q_p^t Q_p \mathbf{q}|^{1/2} \leq \gamma_2 b \Gamma^{1/2} h (\mathbf{q}^t A_p \mathbf{q})^{1/2} \quad \forall \mathbf{q} \in \mathbb{R}^m \setminus \{\mathbf{0}\}.$$

and hence

$$\begin{aligned} \|Q_p A_p^{-1}\|^2 &= \max_{\mathbf{q} \in \mathbb{R}^m \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^t Q_p^t Q_p \mathbf{q}}{\mathbf{q}^t A_p^2 \mathbf{q}} \\ &\leq \gamma_2^2 b^2 \Gamma h^2 \max_{\mathbf{q} \in \mathbb{R}^m \setminus \{\mathbf{0}\}} \frac{\mathbf{q}^t A_p \mathbf{q}}{\mathbf{q}^t A_p^2 \mathbf{q}} \\ &\leq \gamma_2^2 b^2 \Gamma h^2 \frac{C(\Omega)}{\gamma} h^{-2} \\ &= \frac{\gamma_2^2 \Gamma C(\Omega)}{\gamma} b^2. \end{aligned}$$

Similarly we get $\|Q_p^t A_p^{-1}\|^2 \leq b^2 \gamma_2^2 \Gamma C(\Omega)/\gamma$ and therefore

$$\|F_p A_p^{-1}\| \leq \eta \Gamma C(\Omega)\theta + \gamma_2 \left(\frac{\Gamma C(\Omega)}{\gamma} \right)^{1/2} b,$$

and the result follows with $C_5 = \eta \Gamma C(\Omega)$, $C_6 = \gamma_2 \left(\frac{\Gamma C(\Omega)}{\gamma} \right)^{1/2}$. ■

One can also show that the operator $A_p F_p^{-1}$ is bounded. In fact, this follows from the following result which can be found in [22].

Let A be a uniformly elliptic operator and let $A_h u^h = f^h$ denote the discretization of $Au = f$ with given boundary conditions. Let B be another elliptic operator with the same boundary conditions and let B_h denote its discretization.

Theorem 6 (*Manteuffel and Parter, 1989*) *Let A, B be defined as above and let A_h, B_h be the corresponding discretizations given a finite element space S^h . Moreover assume that the following conditions hold for $f \in L^2(\Omega)$, $u \in H^2(\Omega)$, $u^h \in S^h$*

$$\|u\|_{H^2(\Omega)} \leq K_1(A) \|f\|_{L^2(\Omega)}, \quad (3.31a)$$

$$\|Au\|_{L^2(\Omega)} \leq K_2(A) \|u\|_{H^2(\Omega)}, \quad (3.31b)$$

$$\|A_h^{-1} f - A^{-1} f\|_{L^2(\Omega)} \leq h^2 M_1(A) \|f\|_{L^2(\Omega)}, \quad (3.31c)$$

$$\|A_h u^h\|_{L^2(\Omega)} \leq h^{-2} M_2(A) \|u^h\|_{L^2(\Omega)}. \quad (3.31d)$$

and similarly for B, B_h . Then there exists a constant $M(A : B)$ such that

$$\|A_h B_h^{-1}\| \leq M(A : B),$$

with $M(A : B) = K_2(A)K_1(B)[1 + M_1(A)M_2(A)] + M_1(A)M_2(B)$.

Proof See [22], Theorem 5.2. ■

Theorem 7 *Let A_p, F_p be defined as above. Then*

$$\|A_p F_p^{-1}\| \leq C_7 \frac{1}{\nu}.$$

Proof Follows from Theorem 6 with $A_p = A_h, F_p = B_h$ by noting that the corresponding constants satisfy $M_1(A), M_2(A), K_2(A) = O(1)$, $K_1(B), M_2(B) = O(\nu^{-1})$. ■

Remark 3 *The above result requires H^2 regularity of our solution which may exist for special domains such as convex polygons. We note here that a result which does not need such a restriction can be found in [13]. However, although the authors prove independence of the mesh parameter, their result does not yield the right ν dependence. In practice the above bound seems to hold without any regularity restrictions.*

4 Singular value and eigenvalue bounds

Theorem 8 *Let $S = C + B_2 F^{-1} B_1^t$ denote the Schur complement associated with the Oseen problem. Then there exist constants C_8, C_9 such that*

$$\|S M_p^{-1}\| \leq C_8 \frac{1}{\nu}, \quad \|M_p S^{-1}\| \leq C_9 \frac{b^2}{\nu + \theta}.$$

Corollary 8A *The modulus of the eigenvalues of SM_p^{-1} is bounded independently of the mesh parameter h .*

$$C_9^{-1} \frac{\nu + \theta}{b^2} \leq |\lambda_i(SM_p^{-1})| \leq C_8 \frac{1}{\nu}.$$

Theorem 9 *Let $S_p^{-1} = M_p^{-1}F_pA_p^{-1}$ and let $S = C + B_2F^{-1}B_1^t$ be the Schur complement for the Oseen problem. Then*

$$\begin{aligned} \|SS_p^{-1}\| &\leq C_{10} \frac{\theta + b}{\nu}, \\ \|S_pS^{-1}\| &\leq C_{11} \frac{b^2}{\nu(\nu + \theta)}. \end{aligned}$$

Proof Follows immediately from the following inequalities together with the bounds of Theorem 8 and Theorem 5

$$\|SS_p^{-1}\| \leq \|SM_p^{-1}\| \|F_pA_p^{-1}\|, \quad \|S_pS^{-1}\| \leq \|A_pF_p^{-1}\| \|M_pS^{-1}\|.$$

■

Corollary 9A *The modulus of the eigenvalues of SS_p^{-1} is bounded independently of the mesh parameter h .*

$$C_{11}^{-1} \frac{\nu(\nu + \theta)}{b^2} \leq |\lambda_i(SS_p^{-1})| \leq C_{10} \frac{\theta + b}{\nu}.$$

Before we prove Theorem 8 some remarks are in place. First, the bounds derived above are descriptive. Thus, the bounds on the eigenvalues of SM_p^{-1} appear to be more promising from a preconditioning point of view. On the other hand, the Schur complement is a non-normal matrix and preconditioning by the symmetric matrix M_p is bound to deteriorate as the non-normality increases, which is the case when $\nu \rightarrow 0$. As we will see in Section 5, the preconditioner S_p manages to capture the non-normality of S and the result is better convergence of iterative solvers, as demonstrated in [18]. Finally, we note that both preconditioners exhibit mesh independence. However, while the number of iterations for preconditioner M_p grows initially with h to settle for a constant value, the number of iterations for preconditioner S_p decreases with h to settle for a considerably lower value (see [18] for details).

Proof of Theorem 8 Since $\|SM_p^{-1}\| \leq \|B_2F^{-1}B_1^tM_p^{-1}\| + \|CM_p^{-1}\|$, we derive bounds for each of the terms on the right hand side. We have

$$\begin{aligned} \|B_2F^{-1}B_1^tM_p^{-1}\mathbf{x}\|^2 &= \mathbf{x}^t M_p^{-1} B_1 F^{-t} B_2^t B_2 F^{-1} B_1^t M_p^{-1} \mathbf{x} \\ &\leq C_4 h^2 \frac{1}{\nu} \mathbf{x}^t M_p^{-1} B_1 F^{-t} A F^{-1} B_1^t M_p^{-1} \mathbf{x} \\ &= C_4 h^2 \frac{1}{\nu} \mathbf{x}^t M_p^{-1} B_1 F^{-t} B_1^t M_p^{-1} \mathbf{x} \\ &\leq C_4 h^2 \frac{1}{\nu} \mathbf{x}^t M_p^{-1} B_1 A^{-t} B_1^t M_p^{-1} \mathbf{x} \\ &\leq C_3 C_4 h^2 \frac{1}{\nu^2} \mathbf{x}^t M_p^{-1} \mathbf{x} \\ &\leq \gamma C_3 C_4 \frac{1}{\nu^2} \|\mathbf{x}\|^2. \end{aligned}$$

Since

$$\begin{aligned}
\|CM_p^{-1}\| &\leq \|M_p^{-1/2}CM_p^{-1/2}\|\kappa_2^{1/2}(M_p) \\
&\leq \left(\frac{\Gamma}{\gamma}\right)^{1/2} \lambda_1(M_p^{-1/2}CM_p^{-1/2}) \\
&= \left(\frac{\Gamma}{\gamma}\right)^{1/2} \max_{\mathbf{x}} \frac{\mathbf{x}^t C \mathbf{x}}{\mathbf{x}^t M_p \mathbf{x}} \\
&\leq \left(\frac{\Gamma}{\gamma}\right)^{1/2} \frac{\alpha\gamma^2}{\nu}
\end{aligned}$$

the first bound follows with $C_7 = (\gamma C_3 C_4)^{1/2} + (\Gamma/\gamma)^{1/2} \alpha \Gamma^2$. To obtain the second bound note that

$$\begin{aligned}
\|M_p S^{-1}\|^{-1} &= \sigma_m(SM_p^{-1}) \\
&\geq \sigma_m(S)/\sigma_1(M_p) \\
&\geq \sigma_m(B_2 F^{-1} B_1^t)/\sigma_1(M_p) \quad (\text{since } C \text{ is positive semi-definite}) \\
&\geq \frac{1}{\Gamma h^2} \min_{\mathbf{x}} \max_{\mathbf{y}} \frac{\mathbf{x}^t B_2 F^{-1} B_1^t \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}
\end{aligned}$$

Now, since B_1, B_2 have full rank, we can introduce $\tilde{\mathbf{x}} = A^{-1/2} B_2^t \mathbf{x}, \tilde{\mathbf{y}} = A^{-1/2} B_1^t \mathbf{y}$ such that

$$\begin{aligned}
\min_{\mathbf{x}} \max_{\mathbf{y}} \frac{\mathbf{x}^t B_2 F^{-1} B_1^t \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} &= \min_{\tilde{\mathbf{x}}} \max_{\tilde{\mathbf{y}}} \frac{\tilde{\mathbf{x}}^t A^{1/2} F^{-1} A^{1/2} \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{y}}\|} \\
&= \min_{\tilde{\mathbf{x}}} \max_{\tilde{\mathbf{y}}} \frac{\tilde{\mathbf{x}}^t A^{1/2} F^{-1} A^{1/2} \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{y}}\|} \frac{\|A^{-1/2} B_2^t \mathbf{x}\|}{\|\mathbf{x}\|} \frac{\|A^{-1/2} B_1^t \mathbf{y}\|}{\|\mathbf{y}\|} \\
&\geq \sigma_m(\tilde{F}^{-1}) \sigma_m(B_1 A^{-1/2}) \sigma_m(B_2 A^{-1/2}) \\
&\geq C_0^{-1} \frac{\nu(\nu + \theta)}{b^2} C_2^2 \gamma \frac{h^2}{\nu}
\end{aligned}$$

and the result follows with $C_8 = \frac{\Gamma}{\gamma} C_0 / C_2^2$. ■

5 Convergence analysis

5.1 Bounds for the GMRES algorithm

In this section we consider the convergence behaviour of GMRES applied to the global system (1.3) with coefficient matrix K and preconditioned with right preconditioner P_R defined in (1.4), (1.8). In particular, we consider bounds on the norm of the residuals \mathbf{r}^k generated after k iterations of the GMRES algorithm applied to a system with coefficient matrix $\hat{K} = K P_R^{-1}$ of the form

$$\frac{\|\mathbf{r}^k\|}{\|\mathbf{r}^0\|} \leq \min_{p_k(0)=1} \|p_k(\hat{K})\|, \tag{5.1}$$

where p_k denotes a polynomial of degree k .

Various bounds for the right-hand side of (5.1) have been proposed in the literature, although it is not clear which is sufficiently descriptive given any particular problem. For our problem, many of these bounds are not applicable. For example, the field of values bound of Eiermann [6] together with the bounds of Starke [30] and Elman [7] require that the origin is not contained in the field of values of the system matrix. Numerical experiments indicate that this is not the case for \widehat{K} . Moreover, the classical eigenvalue-eigenvector bound of Saad [27]

$$\frac{\|\mathbf{r}^k\|}{\|\mathbf{r}^0\|} \leq \kappa_2(V) \min_{p_k(0)=1} \max_{\lambda \in \Lambda} |p_k(\lambda)|. \quad (5.2)$$

assumes an eigenvector decomposition $\widehat{K} = V\Lambda V^{-1}$ and involves the 2-norm condition number $\kappa_2(V)$ of the eigenvector matrix V which in our case turns out to grow with decreasing mesh-size.

Thus we turn to the following pseudo-spectral bound of Trefethen [32] which seems to be appropriate for our preconditioned system. Let

$$\Lambda_\varepsilon(\widehat{K}) := \left\{ z \in \mathbb{C} : \|(zI - \widehat{K})^{-1}\| > \varepsilon^{-1} \right\}$$

denote the ε -pseudo-spectrum with contour length $\mathcal{L}(\Gamma_\varepsilon)$ of our preconditioned system matrix. Then the residuals \mathbf{r}^k in the GMRES iteration satisfy

$$\frac{\|\mathbf{r}^k\|}{\|\mathbf{r}^0\|} \leq \frac{\mathcal{L}(\Gamma_\varepsilon)}{2\pi\varepsilon} \min_{p_k(0)=1} \max_{z \in \Lambda_\varepsilon(\widehat{K})} |p_k(z)|, \quad (5.3)$$

Given the independence of the mesh parameter of the spectrum of \widehat{K} proved in Theorem 5 it is not unreasonable to assume that the ε -pseudo-spectrum also is mesh-independent (although this is not true in general) to obtain the following result.

Theorem 10 *The GMRES algorithm converges in a number of iterations independent of the mesh-parameter.*

Proof Follows immediately since under the above assumption both $\Lambda_\varepsilon(\widehat{K})$ and $\mathcal{L}(\Gamma_\varepsilon)$ are independent of the mesh parameter. ■

In the following section we present numerical results which validate the above assumption and the results in Theorem 9.

5.2 Numerical results

In this section we present numerical results for a standard 2D test problem: the regularized driven-cavity flow. The domain is the unit square with boundary $\Gamma_N = \emptyset$ and zero boundary conditions except for $\mathbf{u}^*(x, y = 1) = 16x^2(1 - x)^2$. The problem was solved for a range of ν between 1/10 and 1/5000, for a range of time-steps $\Delta t = 1/\theta$ and for a range of mesh parameters.

5.2.1 Steady-state problems

We used the weak formulation (3.7) with $\beta = \theta = 0$ and δ_T as suggested in [11]. Two methods of discretization were employed: the so-called Q2-Q1 and Q1-Q1 discretizations. These methods correspond to a choice of finite element space $H^h = \mathbf{V}^h \times P^h$ of piecewise polynomials of degree two for velocities and degree one for pressure defined on quadrilateral subdivisions of Ω (Q2-Q1) and piecewise linear polynomials for both velocity and pressure (Q1-Q1). Other discretizations are tested in [18].

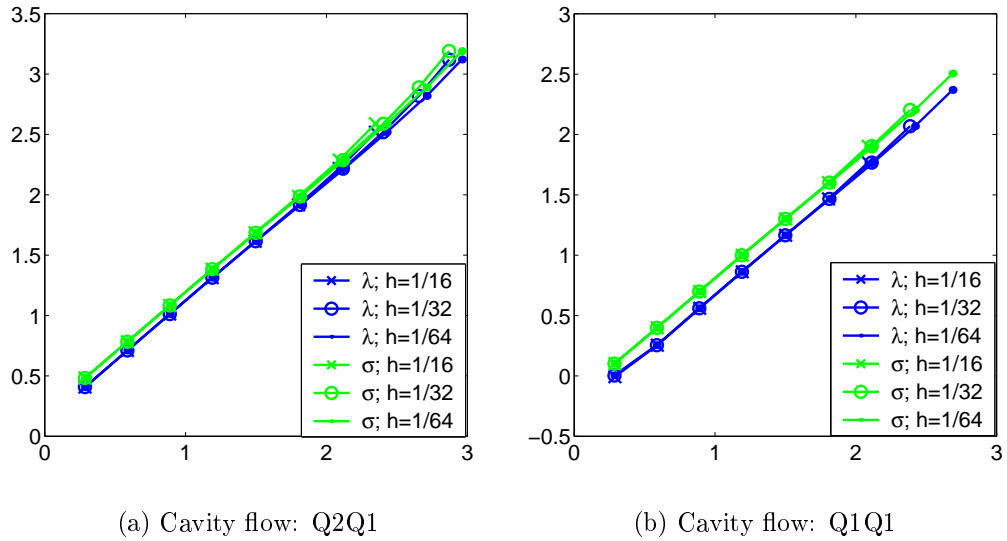


Figure 1: *Loglog plot of $\max_i |\lambda_i|$ (σ_i) versus $\|\mathbf{u}\|/\nu$.*

The eigenvalues of the preconditioned system together with the singular values are displayed in Figs 1, 2 for three mesh parameters and a range of ν . The figures show loglog plots of the moduli of the eigenvalues (singular values) versus the quantity $\|\mathbf{u}\|/\nu$ which appears in the result of Theorem 9. The GMRES performance for the same range of ν and for the same mesh parameters is presented in Fig. 3, which is also a loglog plot of the *average* number of GMRES iterations over the Picard steps versus $\|\mathbf{u}\|/\nu$. Here $\|\mathbf{u}\|$ represents the $L^2(\Omega)$ -norm of the solution of the last Picard step.

First, we note that the spectrum, the singular values and the GMRES performance are indeed mesh-independent as predicted by the theory. Moreover, we note that the singular value bound of Theorem 9 turned out to be remarkably descriptive for this problem for both the smallest and the largest eigenvalues and for any ν or h . Indeed, the dependence on $\|\mathbf{u}\|/\nu$ of the extreme singular values is mirrored by the extreme moduli of the eigenvalues.

On the other hand, the dependence on $\|\mathbf{u}\|/\nu$ of the singular values and of the spectrum is also correctly predicted by the theory. Fig. 1 displays a linear increase of the maximum singular value with $\|\mathbf{u}\|/\nu$ for all ranges of ν and for all mesh sizes. This growth seems to be directly related to the GMRES performance as seen in Fig. 3, although the bound (5.3) is not particularly descriptive with respect to this parameter. Thus, the number of GMRES iterations

seems to grow like $(\|\mathbf{u}\|/\nu)^{1/2}$, a result which was noted also in [21], [10]. We also note that in [10] it is also conjectured that the GMRES performance does not depend on the smallest modulus eigenvalues.

The smallest modulus eigenvalue decreases, as predicted, like $(\nu/\|\mathbf{u}\|)^2$, although this bound is attained for smaller values of ν . Fig. 2 shows a plateau which indicates that the smallest modulus eigenvalue seems to be independent of both h and ν for larger values of ν . However, for smaller values of ν the smallest modulus eigenvalue settles for a quadratic decrease with respect to ν . We conjecture that this quadratic decrease noticed for smaller values of ν may be related to the fact that a critical point is approached (around $\nu^{-1} = 10, 500$, see [28]).

Finally, we present in Figs 4, 5 ε -pseudo-spectra of the preconditioned system for different values of ν and h . More precisely, we chose to display the pseudo-spectra corresponding to the main cluster of eigenvalues around $z = 1$. The rest of the spectrum consists of isolated eigenvalues of modulus that grows linearly with $\|\mathbf{u}\|/\nu$ (including the largest modulus eigenvalue). The number of these isolated eigenvalues grows slowly with $\|\mathbf{u}\|/\nu$. It is argued in [10] that this eigenvalue migration from the cluster around $z = 1$ is responsible for the deterioration in GMRES convergence. However, this argument necessarily uses the bound (5.2) which we found unsuitable to describe convergence due to the large and mesh-dependent factor $\kappa_2(V)$.

As conjectured in Theorem 10, there is no dependence on the mesh parameter of the pseudo-spectrum as can be seen in Fig. 4 (the pseudospectrum actually shrinks as h is reduced!). This essentially indicates that the non-normality of the system matrix K due to this parameter was ‘captured’ by the preconditioner P_R . As expected, on the other hand, we note in Fig. 5 a clear dependence with respect to ν . In particular, the size of the ε -pseudospectrum seems to increase with decreasing ν with the immediate consequence that the bound (5.3) deteriorates. This is paralleled by a deterioration in the GMRES performances, which leads us to conjecture that the bound (5.3) is tight.

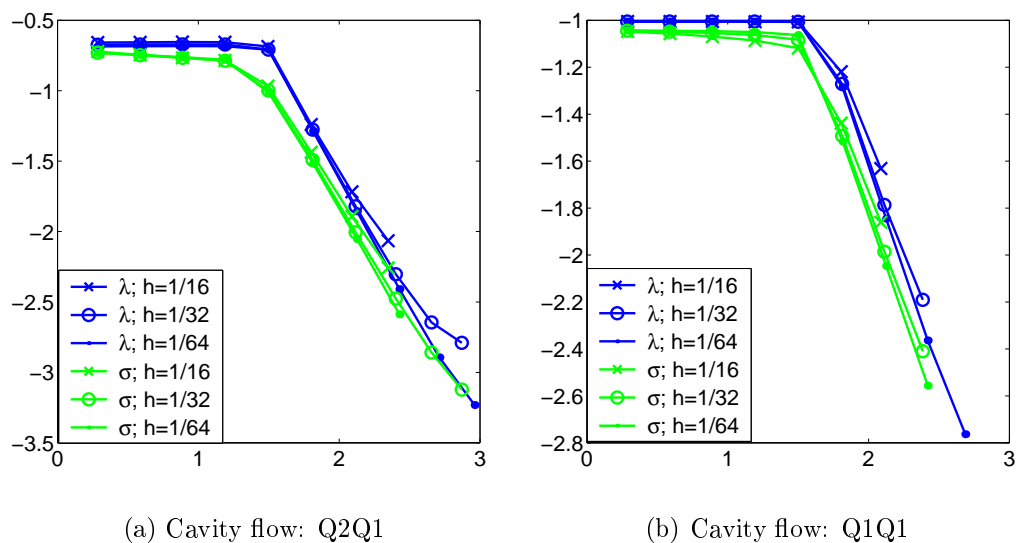


Figure 2: *Loglog plot of $\min_i |\lambda_i|$ (σ_i) versus $\|\mathbf{u}\|/\nu$.*

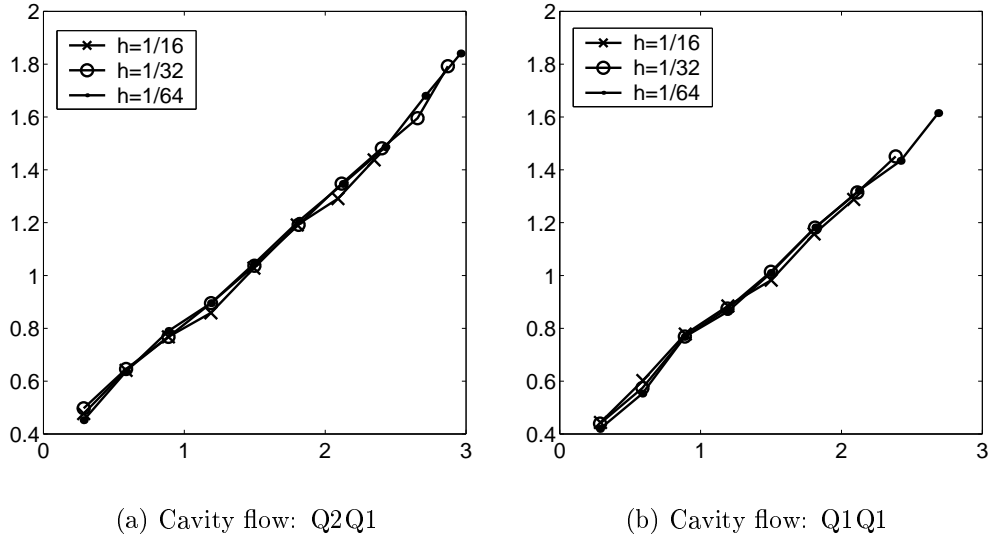


Figure 3: *Loglog plot of the number of GMRES iterations versus $\|\mathbf{u}\|/\nu$.*

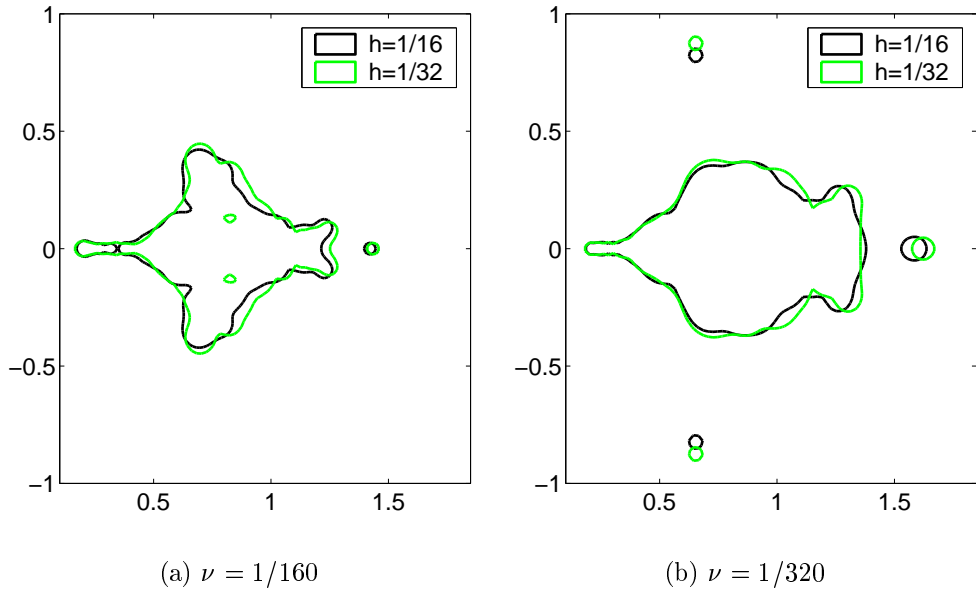


Figure 4: *Spectrum and ε -pseudo-spectrum set of \widehat{K} ; $\varepsilon = 10^{-2}$.*

5.2.2 Time-dependent problems

We used the Q2-Q1 discretization as in the steady-state case with the stabilization terms switched off ($\beta = \delta_T = 0$). The Oseen problem (1.2) arising from a backward Euler time-stepping routine together with Picard linearization was solved for various choices of θ . The results are presented in Figs 6, 7, 8. We first note that the bounds established in Theorem 9

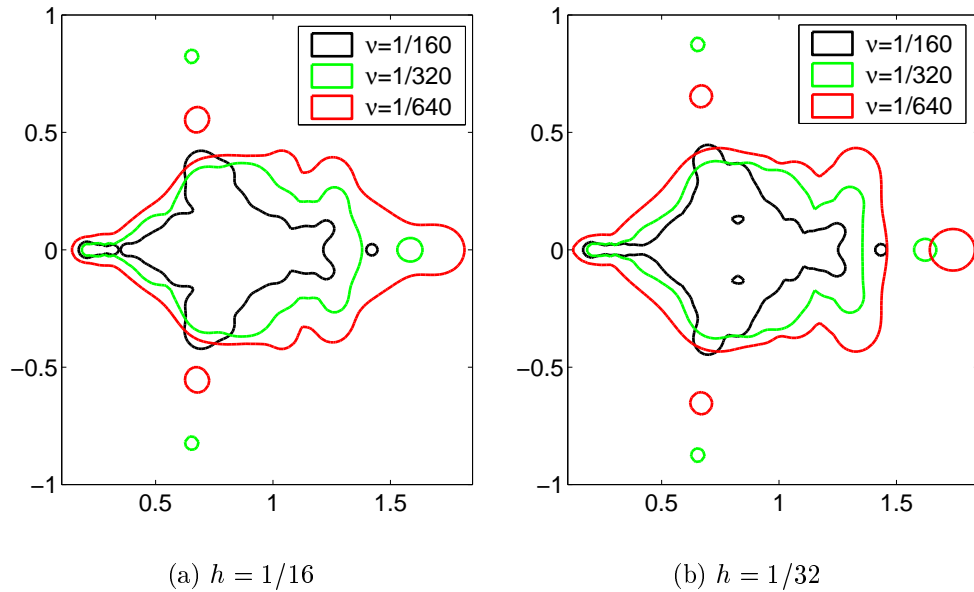


Figure 5: ε -pseudospectrum set of \widehat{K} for various values of ν ; $\varepsilon = 10^{-2}$.

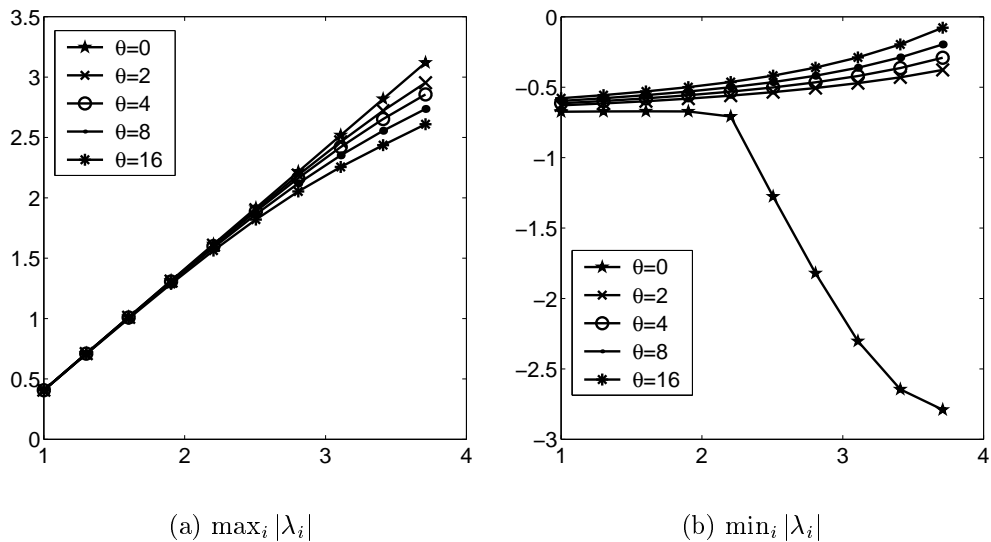


Figure 6: *Loglog plot of $\max_i |\lambda_i|$ ($\min_i |\lambda_i|$) versus $1/\nu$ for various values of θ .*

hold, although some of them are not as tight. The bound on the largest modulus eigenvalues is still descriptive as can be seen from Fig. 6(a). However, the smallest modulus eigenvalue seem to be bounded away from the origin independently of ν and whenever $\theta > 0$ (Fig. 6(b)). The quadratic decrease with respect to ν predicted analytically is therefore pessimistic.

Finally, the GMRES performance is presented in Fig. 7. We note that for $\theta > 0$ the average number of GMRES iterations becomes almost insensitive with respect to ν . This can

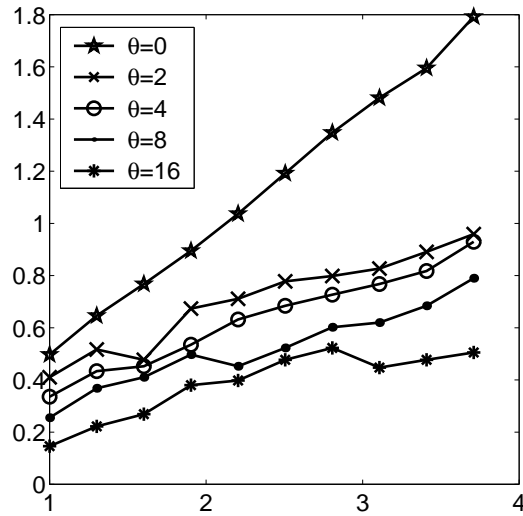


Figure 7: Loglog plot of the number of GMRES iterations versus $1/\nu$ for various values of θ .

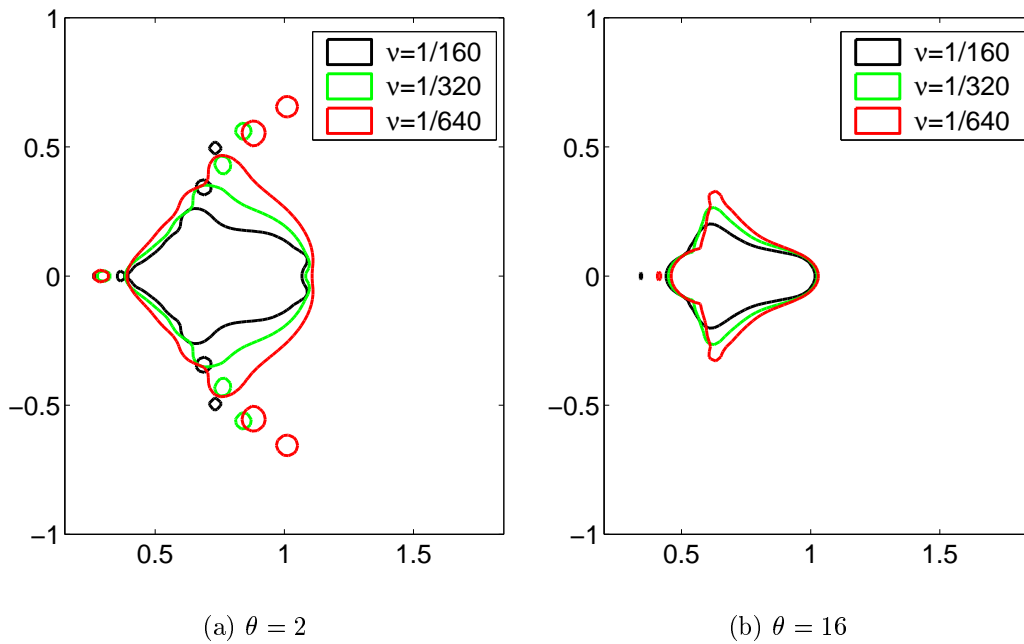


Figure 8: ε -pseudospectrum set of \widehat{K} for various values of ν ; $\varepsilon = 10^{-2}$.

be explained again via a pseudo-spectral argument. Fig. 8 displays the pseudo-spectra for various values of ν and for two values of θ . We note that the larger θ is the smaller the increase in the pseudo-spectral sets. Moreover, they seem to extend more slowly to the origin compared to the case $\theta = 0$, which makes approximation problem (5.3) easier and leads to the improvement in GMRES performance exhibited in Fig. 7.

6 Conclusion

The performance of the preconditioner for the Oseen problem introduced in [18] was analyzed with respect to the relevant parameters. Although the analysis led to descriptive bounds for the spectrum and the singular values, the performance of GMRES could only be considered in terms of the associated pseudo-spectra. The theoretical results also hold for the pseudo-spectra as demonstrated in our experiments. Thus, mesh-independent bounds for the performance of iterative solvers can be derived. Numerical experiments also indicate that although GMRES exhibits mesh-independent convergence, its performance deteriorates like the square-root of the Reynolds number. However, this is a result that remains to be established analytically.

Acknowledgements. The computation of the pseudo-spectral plots was greatly speeded up with the help of Tom Wright and his MATLAB pseudo-spectra GUI version 2.1 [36].

7 Appendix

Proof of Lemma 1. We first need to prove that $\|\cdot\|_h$ is a norm on \mathbf{V}^h . We have (cf. (3.8))

$$\begin{aligned}
a(\mathbf{v}, \mathbf{v}) &= \nu |\mathbf{v}|_1^2 + \frac{1}{2} \int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{v}|^2 d\Gamma + \theta \|\mathbf{v}\|_0^2 + \beta \|\operatorname{div} \mathbf{v}\|_0^2 + \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 \\
&\quad - \sum_{T \in \mathcal{T}^h} \delta_T \|\nu \Delta \mathbf{v}\|_{0,T}^2 + \sum_{T \in \mathcal{T}^h} \delta_T \langle \theta \mathbf{v}, \nu \Delta \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{v} \rangle_T \\
&\geq \nu |\mathbf{v}|_1^2 + \frac{1}{2} \int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{v}|^2 d\Gamma + \theta \|\mathbf{v}\|_0^2 + \beta \|\operatorname{div} \mathbf{v}\|_0^2 + \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 \\
&\quad - \alpha \mu^2 \sum_{T \in \mathcal{T}^h} \nu |\mathbf{v}|_{1,T}^2 - \frac{1}{2} \left(\sum_{T \in \mathcal{T}^h} \delta_T \|\theta \mathbf{v}\|_{0,T}^2 + \sum_{T \in \mathcal{T}^h} \delta_T \|\nu \Delta \mathbf{v}\|_{0,T}^2 \right) \\
&\quad - \frac{1}{2} \left(\sum_{T \in \mathcal{T}^h} \delta_T \|\theta \mathbf{v}\|_{0,T}^2 + \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 \right) \\
&\geq (1 - \alpha \mu^2) \nu |\mathbf{v}|_1^2 + \frac{1}{2} \int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{v}|^2 d\Gamma + (1 - \theta \delta) \theta \|\mathbf{v}\|_0^2 + \beta \|\operatorname{div} \mathbf{v}\|_0^2 \\
&\quad + \frac{1}{2} \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 - \frac{1}{2} \alpha \mu^2 \nu |\mathbf{v}|_1^2,
\end{aligned}$$

which is positive due to our choice of α .

If we now prove the continuity bound for $\tilde{a}(\cdot, \cdot)$, the bound for $a(\cdot, \cdot)$ follows from

$$|a(\mathbf{w}, \mathbf{v})| \leq |\tilde{a}(\mathbf{w}, \mathbf{v})| + b |\mathbf{w}|_1 \|\mathbf{v}\|_0 + \frac{1}{2} \left(\int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{w}|^2 d\Gamma \right)^{1/2} \left(\int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{v}|^2 d\Gamma \right)^{1/2}$$

and the inequality

$$b |\mathbf{w}|_1 \|\mathbf{v}\|_0 \leq C \frac{b}{[\nu(\nu + \theta)]^{1/2}} \|\mathbf{w}\|_h \|\mathbf{v}\|_h,$$

where $C = \max\{1, C(\Omega)\}$ and $C(\Omega)$ the constant arising in the Poincaré inequality (3.27). Applying the Cauchy-Schwartz inequality $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$ to the terms comprising $\tilde{a}(\mathbf{w}, \mathbf{v})$ we get

$$\begin{aligned} |\tilde{a}(\mathbf{w}, \mathbf{v})|^2 \leq & 10 \left\{ \nu^2 \|\mathbf{w}\|_1^2 \|\mathbf{v}\|_1^2 + \theta^2 \|\mathbf{w}\|_0^2 \|\mathbf{v}\|_0^2 + \beta^2 \|\operatorname{div} \mathbf{w}\|_0^2 \|\operatorname{div} \mathbf{v}\|_0^2 \right. \\ & + \frac{1}{4} \left(\int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{w}|^2 d\Gamma \right) \left(\int_{\Gamma_N} \mathbf{b} \cdot \mathbf{n} |\mathbf{v}|^2 d\Gamma \right) \\ & + \sum_{T \in \mathcal{T}^h} \delta_T^2 \left(\nu^4 \|\Delta \mathbf{w}\|_0^2 \|\Delta \mathbf{v}\|_{0,T}^2 + \nu^2 \|\Delta \mathbf{w}\|_{0,T}^2 \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 + \nu \|\Delta \mathbf{v}\|_{0,T}^2 \|\mathbf{b} \cdot \nabla \mathbf{w}\|_{0,T}^2 \right) \\ & \left. + \sum_{T \in \mathcal{T}^h} \delta_T^2 \left(\|\mathbf{b} \cdot \nabla \mathbf{w}\|_{0,T}^2 \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 + \theta^2 \|\mathbf{w}\|_{0,T}^2 \nu^2 \|\Delta \mathbf{v}\|_{0,T}^2 + \theta^2 \|\mathbf{w}\|_{0,T}^2 \|\mathbf{b} \cdot \nabla \mathbf{v}\|_{0,T}^2 \right) \right\} \end{aligned}$$

and the result follows since every term on the right hand side can be bounded by a term in the product $\|\mathbf{w}^h\|_h^2 \|\mathbf{v}^h\|_h^2$ times a constant $\tilde{C}_0 = \tilde{C}_0(\mu, \eta, \Omega)$ for sufficiently small h . ■

To prove Lemma 2 we need the following lemma which can be found in [34]

Lemma 11 *Let $(\mathbf{v}^h, p^h) \in H^h$. Then*

$$\sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{\langle \operatorname{div} \mathbf{v}^h, p^h \rangle}{(\nu \|\mathbf{v}^h\|_1^2)^{1/2}} \geq \tilde{C}_1 \left(\frac{1}{\nu} \|p^h\|_0^2 \right)^{1/2} - \tilde{C}_2 \left(\sum_{T \in \mathcal{T}^h} \frac{h_T^2}{\nu} \|\nabla p^h\|_{0,T}^2 \right)^{1/2}, \quad (7.1)$$

for some constants \tilde{C}_1, \tilde{C}_2 .

Remark 4 *We note here that if $\tilde{\mathbf{z}}^h$ is a function for which the supremum is attained then the scaling $\mathbf{z}^h = \gamma \tilde{\mathbf{z}}^h$ will yield the same supremum. Given $p^h \in P^h$ we choose γ to be given by $\gamma = \nu^{-1/2} \|p^h\|_0 / \|\tilde{\mathbf{z}}^h\|_h$ so that*

$$\|\mathbf{z}^h\|_h = \left(\frac{1}{\nu} \|p^h\|_0^2 \right)^{1/2}.$$

Proof of Lemma 2. With the supremum in (7.1) scaled as above we have

$$\begin{aligned}
\tilde{B}(\mathbf{w}^h, r^h; -\mathbf{z}^h, 0) &= -\tilde{a}(\mathbf{w}^h, \mathbf{z}^h) - b_1(\mathbf{z}^h, r^h) \\
&\geq -\tilde{C}_0 \|\mathbf{w}^h\|_h \|\mathbf{z}^h\|_h + \langle \operatorname{div} \mathbf{z}^h, r^h \rangle - \sum_{T \in \mathcal{T}^h} \delta_T \langle \nu \Delta \mathbf{z}^h, \nabla r^h \rangle_T - \sum_{T \in \mathcal{T}^h} \delta_T \langle \mathbf{b} \cdot \nabla \mathbf{z}^h, \nabla r^h \rangle_T \\
&\geq -\tilde{C}_0 \|\mathbf{w}^h\|_h \|\mathbf{z}^h\|_h + \tilde{C}_1 \left(\frac{1}{\nu} \|r^h\|_0^2 \right)^{1/2} (\nu |\mathbf{z}^h|_1^2)^{1/2} - \tilde{C}_2 \left(\sum_{T \in \mathcal{T}^h} \frac{h_T^2}{\nu} \|\nabla r^h\|_{0,T}^2 \right)^{1/2} (\nu |\mathbf{z}^h|_1^2)^{1/2} \\
&\quad - \alpha \mu \left(\sum_{T \in \mathcal{T}^h} \nu \|\nabla \mathbf{z}^h\|_{0,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \right)^{1/2} - \left(\sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{z}^h\|_{0,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \right)^{1/2} \\
&\geq -\tilde{C}_0 \left(\frac{\epsilon}{2} \|\mathbf{w}^h\|_h^2 + \frac{1}{2\epsilon} \|\mathbf{z}^h\|_h^2 \right) + \tilde{C}_1 \left(\frac{1}{2\epsilon} \nu |\mathbf{z}^h|_1^2 + \frac{\epsilon}{2\nu} \|r^h\|_0^2 \right) \\
&\quad - (\alpha \mu + \tilde{C}_2/\alpha) \left(\frac{1}{2\epsilon} \nu |\mathbf{z}^h|_1^2 + \frac{\epsilon}{2} \sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \right) \\
&\quad - \left(\frac{1}{2\epsilon} \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{z}^h\|_{0,T}^2 + \frac{\epsilon}{2} \sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \right) \\
&\geq -\tilde{C}_0 \left(\frac{\epsilon}{2} \|\mathbf{w}^h\|_h^2 + \frac{1}{2\epsilon} \|\mathbf{z}^h\|_h^2 \right) - \frac{1}{2\epsilon} \left((\alpha \mu + \tilde{C}_2/\alpha - \tilde{C}_1) \nu |\mathbf{z}^h|_h^2 + \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{z}^h\|_{0,T}^2 \right) \\
&\quad + \tilde{C}_1 \frac{\epsilon}{2\nu} \|r^h\|_0^2 - \frac{\epsilon}{2} (\alpha \mu + \tilde{C}_2/\alpha + \alpha) \left(\sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \right) \\
&\geq -\tilde{C}_0 \frac{\epsilon}{2} \|\mathbf{w}^h\|_h^2 - \frac{1}{2\epsilon} (\tilde{C}_0 + \tilde{C}_3) \|\mathbf{z}^h\|_h^2 + \tilde{C}_1 \frac{\epsilon}{2\nu} \|r^h\|_0^2 - \frac{\epsilon}{2} (\alpha \mu + \tilde{C}_2/\alpha + \alpha) \sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \\
&\geq -\tilde{C}_4 \|\mathbf{w}^h\|_h^2 + \tilde{C}_5 \frac{1}{\nu} \|r^h\|_0^2 - \tilde{C}_6 \sum_{T \in \mathcal{T}^h} \frac{h_T^2}{\nu} \|\nabla r^h\|_{0,T}^2,
\end{aligned}$$

where we set $\tilde{C}_3 = \max \{1, \alpha \mu + \tilde{C}_2/\alpha + \alpha\}$, $\tilde{C}_5 = \tilde{C}_1 \epsilon/2 - (\tilde{C}_0 + \tilde{C}_3)/(2\epsilon)$ and we chose ϵ such that $\tilde{C}_5 > 0$.

On the other hand,

$$\begin{aligned}
\tilde{B}(\mathbf{w}^h, r^h; \mathbf{w}^h, -r^h) &= \tilde{a}(\mathbf{w}^h, \mathbf{w}^h) + b_1(\mathbf{w}^h, r^h) - b_2(\mathbf{w}^h, r^h) - \tilde{c}(r^h, r^h) \\
&\geq \tilde{C}_0 \|\mathbf{w}^h\|_h^2 + 2 \sum_{T \in \mathcal{T}^h} \delta_T \langle \mathbf{b} \cdot \nabla \mathbf{w}^h, \nabla r^h \rangle_T + (\Gamma + 1) \sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \\
&\geq \tilde{C}_0 \|\mathbf{w}^h\|_h^2 - \epsilon \sum_{T \in \mathcal{T}^h} \delta_T \|\mathbf{b} \cdot \nabla \mathbf{w}^h\|_{0,T}^2 - \frac{1}{\epsilon} \sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 + (\Gamma + 1) \sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \\
&\geq \tilde{C}_7 \left(\|\mathbf{w}^h\|_h^2 + \sum_{T \in \mathcal{T}^h} \delta_T \|\nabla r^h\|_{0,T}^2 \right),
\end{aligned}$$

where we chose $\epsilon < \tilde{C}_0$, $\Gamma = \max \{0, 1/\tilde{C}_0 - 1\}$ and we set $\tilde{C}_7 = \max \{\tilde{C}_0 - \epsilon, \Gamma + 1 - 1/\epsilon\}$. Hence, setting $(\mathbf{v}^h, q^h) = (\mathbf{w}^h - \rho \mathbf{z}^h, -r^h)$ we get

$$\begin{aligned} \tilde{B}(\mathbf{w}^h, r^h; \mathbf{v}^h, q^h) &= \tilde{B}(\mathbf{w}^h, r^h; -\mathbf{z}^h, 0) + \tilde{B}(\mathbf{w}^h, r^h; \mathbf{w}^h, -r^h) \\ &\geq (\tilde{C}_7 - \rho \tilde{C}_4) \|\mathbf{w}^h\|_h^2 + \rho \tilde{C}_5 \frac{1}{\nu} \|r^h\|_0^2 + (\tilde{C}_7 - \rho \tilde{C}_6) \sum_{T \in \mathcal{T}^h} \frac{h_T^2}{\nu} \|\nabla r^h\|_{0,T}^2 \\ &\geq \tilde{C}_8 \left(\|\mathbf{w}^h\|_h^2 + \frac{1}{\nu} \|r^h\|_0^2 \right) \end{aligned}$$

with $\rho = \min \{\tilde{C}_7 \tilde{C}_4^{-1}, \tilde{C}_7 \tilde{C}_6^{-1}\}$. Since

$$\begin{aligned} \|\mathbf{v}^h\|_h^2 + \frac{1}{\nu} \|q^h\|_0^2 &\leq 2(\|\mathbf{w}^h\|_h^2 + \rho^2 \|\mathbf{z}^h\|_h^2) + \frac{1}{\nu} \|q^h\|_0^2 \\ &= 2\|\mathbf{w}^h\|_h^2 + (2\rho^2 + 1) \frac{1}{\nu} \|r^h\|_0^2 \\ &\leq \tilde{C} \left(\|\mathbf{w}^h\|_h^2 + \frac{1}{\nu} \|r^h\|_0^2 \right), \end{aligned}$$

we get

$$\begin{aligned} \sup_{(\mathbf{w}^h, r^h) \in H^h \setminus \{0\}} \frac{\tilde{B}(\mathbf{w}^h, r^h; \mathbf{v}^h, q^h)}{\|\mathbf{w}^h, r^h\|_h \|\mathbf{v}^h, q^h\|_h} &\geq \frac{\tilde{B}(\mathbf{v}^h + \rho \mathbf{z}^h, -q^h; \mathbf{v}^h, q^h)}{\|\mathbf{v}^h + \rho \mathbf{z}^h, -q^h\|_h \|\mathbf{v}^h, q^h\|_h} \geq \tilde{C}_9 \quad \forall (\mathbf{v}^h, q^h) \in H^h, \\ \sup_{(\mathbf{v}^h, q^h) \in H^h \setminus \{0\}} \frac{\tilde{B}(\mathbf{w}^h, r^h; \mathbf{v}^h, q^h)}{\|\mathbf{w}^h, r^h\|_h \|\mathbf{v}^h, q^h\|_h} &\geq \frac{\tilde{B}(\mathbf{w}^h, r^h; \mathbf{w}^h - \rho \mathbf{z}^h, -r^h)}{\|\mathbf{w}^h, r^h\|_h \|\mathbf{w}^h - \rho \mathbf{z}^h, -r^h\|_h} \geq \tilde{C}_9 \quad \forall (\mathbf{w}^h, r^h) \in H^h. \end{aligned}$$

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