Constructive Mathematics -

Its Set Theory and Practice

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ABSTRACT

The thesis falls naturally into two parts, in the first of which (comprising Chapter 1) there is laid down a set-theoretic foundation for constructive mathematics as understood by Errett Bishop and his followers. The work of this part closely follows the lines of the corresponding classical development of set theory by Anthony Morse, highlights several classical definitions and results which are inadequate for a proper description of constructive mathematics, and develops constructive replacements for these where possible; of particular importance is the constructive proof of a general recursion theorem, from which the familiar theorems of simple and primitive recursion readily follow.

The second part of the thesis (Chapters 2 - 5) is concerned with various problems of constructive analysis, the link between these problems being their involvement with compactness or local compactness at some stage. Chapter 2 serves as an introduction to this analysis, and includes the definition of metric injectiveness and the proof of a constructive substitute for the classical result that a continuous injection of a compact Hausdorff space onto a Hausdorff space has continuous inverse.

In Chapter 3 we give an improved definition of one-point compactification of a locally compact space, and then develop the theory of existence and essential uniqueness of such compactifications of a given space. In turn, this is applied in Chapter 4, which deals in full with the space of continuous, complex-valued functions which vanish at infinity on a locally compact space, and with star homomorphisms between such spaces; interpolated within the main body of this chapter is the vital Backward Uniform Continuity Theorem, which leads to a discussion of possible constructive substitutes for the classical Uniform Continuity Theorem.

The final chapter deals with constructive substitutes for various topologies associated with spaces of bounded linear mappings between normed linear spaces. The main results of this chapter concern the weak operator topology on the space $\mathrm{Hom}(H,H)$ of bounded linear operators on a Hilbert space H, and include a constructive proof of the weak operator precompactness of the unit ball of $\mathrm{Hom}(H,H)$, and a proof that the compactness of this ball is an essentially non-constructive proposition. The chapter ends with a discussion of linear functionals and the weak operator topology on $\mathrm{Hom}(H,H)$, and a partial substitute for the classical characterisa-

of ultraweakly continuous linear functionals on a linear subset of Hom(H,H).

In addition, there are five appendices, three of which develop material arising from that in the main body of the thesis. In the first of these three, we describe an axiomatic theory of proofs within the formal system of Chapter 1, and derive (amongst other results) a very satisfactory characterisation of proofs of '$p \to q$'; the second deals with connectedness, and builds up to a constructive proof that a closed ball in finite dimensional Banach space is connected; finally, the last makes a remark on metric injectiveness in the light of a conjecture in Chapter 2.

ACKNOWLEDGEMENTS

Even in connection with such a meagre work as this, it is inevitable that there will be many to whom thanks are due. Of these, I should mention in particular: Michael Dummett and Robin Gandy, without whose willingness to supervise work in a field different from their own this dissertation would not have been possible; Mollie Phipps, who (assisted latterly by Julia Bayman) has patiently maintained her standards of typographical excellence throughout much difficult material; the Science Research Council, whose finances have supported much of my life here in Oxford; and, finally, Errett Bishop, to whose stimulating book I owe my timeous rescue from despair at so much of current mathematics.

PROLOGUE

The following dissertation is submitted under the regulations of the University of Oxford for the degree of Doctor of Philosophy in Pure Mathematics.

The dissertation falls naturally into two distinct parts. The first of these, comprising Chapter 1, lays down a set-theoretic foundation for constructive mathematics as understood by Bishop [1]; the work of this part closely follows the lines of the corresponding classical development of set theory by Morse [14]. With some relaxation of the formality of Chapter 1, the second part of the dissertation (Chapters 2 - 5) is concerned with various analytic problems in which the concepts of compactness and local compactness play important roles. In addition, there are five appendices, three of which develop material arising from, or closely related to, that found in the main body of the dissertation.

Many of the results and remarks in Chapters 1 and 5 are comparatively trivial, and have been included for the sake of completeness and clarity of exposition. The same motivation lies behind our decision to include proofs of certain results which are mentioned without proof in [1] or [14]. Unless statement is made to the contrary, the proofs of these and all other results in this dissertation are essentially due to the undersigned.

No part of this dissertation has been, or is being, submitted for a degree, diploma, or any other qualification at any other university.

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CONTENTS

CHAPTER 1.

SET THEORY FOR THE CONSTRUCTIVE MATHEMATICIAN.

The appearance of constructive mathematics as a serious contender for the attention and affections of practising mathematicians may be traced to that of L. E. J. Brouwer's Amsterdam doctoral dissertation, 'Over de grondslagen der Wiskunde' [7],in 1907. Although it is true to say that a few individuals - for example, Kronecker - had earlier expressed disapproval of the 'idealistic' methods of some of their nineteenth century contemporaries, it is in Brouwer's polemical writings, beginning with the above and continuing throughout the next forty-seven years, that the foundations of a precise and practical approach to constructive mathematics were laid.

Unfortunately - and perhaps inevitably, in the face of opposition from men of such stature as Hilbert - Brouwer's 'intuitionist' school became more and more involved in quasi-mystical speculation about the nature of constructive thought, to the detriment of the practice of constructive mathematics proper. Thus it remained for Errett Bishop, in his seminal book 'Foundations of Constructive Analysis' [1], to resurrect constructive mathematics in practice and produce some outstanding constructive proofs of important theorems already known in their classical form - in particular, many of the fundamental results in the theories of Banach spaces, measure, and locally compact groups. It is in the spirit of Bishop's book, freed from the shackles of Brouwer's intuitionism, that this present dissertation is written.

But what *is* constructive mathematics? For a concise, but definitive, answer we cannot do better than quote what we shall here and hereafter recognize as *Bishop's Thesis*:

'The primary concern of mathematics is number...'

- or, to expand this within its original context ([1], page 2),

'The primary concern of mathematics is number, and this means the positive integers... every mathematical statement ultimately expresses the fact that if we perform certain computations within the set of positive integers, we shall get certain results.'

However, in searching for a deeper understanding of the nature of mathematics, we must ask, and try to answer, the questions:

What is a number?

What do we mean by 'constructive'?

To take first things first, do we feel, with Bishop, that 'the development of the theory of the positive integers from the primitive concept of the unit, the concept of adjoining a unit, and the process of mathematical induction carries complete conviction'; or would we seek surer ground on which to build our mathematics? In our opinion, the latter alternative is to be preferred and, indeed, can be realised in the conflation of mathematics and logic, the logic in question being of a manifestly constructive nature (that is, eschewing such idealistic principles as the notorious one of 'excluded middle').

The details of a formal system in which this conflation is carried out will be described in the main body of this chapter. For the moment, we therefore content ourselves with a few preliminary comments.

For us, 'logic' will mean 'propositional calculus, predicate calculus and set theory'. Thus we interpret '$x \in P$' in the Fregean manner as 'the concept P applies to x', and draw no distinction between, for example,

'for all x, $P(x)$'

and

'the intersection, as x runs, of the sets $P(x)$'.

More generally, no distinction is made between 'terms' and 'well-formed formulae': *every mathematical/logical object may be regarded as either a set or a logical proposition.*

We do not entirely equate mathematics and logic: as well as the principles of logic itself, our mathematics requires certain 'construction rules', by which we can be sure that part of our mathematics has constructive content and meaning, and an axiom of choice. Of the construction rules, perhaps the most important is that which allows us to construct the set of all natural numbers, and therefore the important class of recursive functions. (Note that in our formal system we shall focus our attention on the natural numbers, rather than the positive integers. This is a purely technical matter, and in no way affects the import of Bishop's Thesis.)

It will be remarked that we have '*part* of our mathematics' with constructive content and meaning: within our formal system we can talk about a very large class of mathematical/logical objects, but not all of these will necessarily be constructively defined. This exactly parallels the situation in informal mathematics, where we can, for example, talk about

'the least upper bound of the set S of real numbers'

without necessarily recognising this as an object which has been, or ever can be, constructed.

In this context, it is worth noting that a mathematical proposition may have constructive significance without itself being, as a term, a constructive object. Thus, as we shall later see,

'every natural number is constructively well-defined'

is certainly a proposition with constructive significance, but as a term, it is not a constructive object (it actually equals the universe in our formal system!)

Bearing these remarks in mind, we may now begin the description of our formal theory. This theory is based on the classical system developed in [14]. The rules of inference and the powerful notation that underlie our mathematics proper are developed in the first part of this chapter (sections 1.1 - 1.5); to dispose of these preliminaries as quickly as is consistent with clarity, we shall draw heavily on quotations from [14], whose description can hardly be bettered. In section 1.6 we state the axioms of propositional and predicate logic, and sketch briefly a few ideas in the development of 'pure' logic from these axioms; before finally passing on to the most important part of the chapter, that concerned with set theory (sections 1.8 - 1.29). Perhaps the most significant sections of this last part are those containing remarks on constructivity (for example, 1.12 and 1.15), and 1.22, in which a very general recursion theorem is proved and then applied to produce the familiar recursion theorems of everyday analysis.

1.1. Variables and Schemators.

The symbols of our formal theory are precisely those inscriptions ('marks') which are not quotation marks; an *expression* is a linear array of such symbols. A distinction is made between a linear array of contiguous italic letters - such as "*xyzt*" - and one like "inf" of contiguous nonitalic letters: the former example is recognised as comprising four symbols - "x", "y", "z" and "t" - while the latter is considered as one symbol. This distinction affects what happens when we (systematically) replace symbols by symbols; for example, if C is the expression obtained from "αβ" by replacing "α" by A and "β" by B, then: if A is "w" and B is "h", then C is "w h"; if A is "wh" and B is "at", then C is the two-symbol expression "wh at"; if A is "*wh*" and B is "*at*", then C is the four-symbol expression "*what*".

Of particular interest are those symbols known as *schemators*, examples of which are u,u′,w″,... etc; intuitively, ux is to be read as either

x has the property u

or

the set corresponding to x under u.

(In the latter interpretation, we imagine x as running through some index set A, and ux as the corresponding element of some family of sets indexed by A.) Again we note the lack of distinction between 'terms' and 'well-formed formulae' in our theory.

The constants of our theory are: the *definor* "≡", the schemators, and all symbols fixed by some definition. (Roughly, a symbol c is *fixed* by a definition D if D has the appearance (A ≡ B), in which c occurs in A and does not occur in B.) A symbol which is not a constant is called a *variable*.

By a *variant* of an expression A we mean an expression B such that each of A and B can be obtained from the other by systematic replacement of variables by variables. (Note that this means that if in A we replace "x" by "y", we must do so at each occurrence of "x" in A.) A variant of "$(x \equiv y)$" or of some expression introduced by a definition is called a *form*.

Following the example of Morse ([14],0.0), we fix our most important constants by the following (otherwise meaningless) orienting definitions

.0 $((\wedge\vee \rightarrow x) \equiv x)$

.1 $(\cap\cup ; x \equiv x)$

(Note that, for reasons of typographical convenience, we use "$\cap$" and "$\cup$" where Morse uses "$\bigwedge$" and "$\bigvee$".) We also introduce certain basic forms with the definitions

.2 $((x \rightarrow x') \equiv (x \rightarrow x'))$

.3 $(\cap x\ \underline{u}x \equiv \cap x\ \underline{u}x)$

.4 $(\cup x\ \underline{u}x \equiv \cup x\ \underline{u}x)$

.5 $((x \wedge x') \equiv (x \wedge x'))$

.6 $((x \vee x') \equiv (x \vee x'))$

.7 $(x \equiv x)$

.8 $(\underline{u}x \equiv \underline{u}x)$

.9 $(x' \equiv x')$

.10 $(\underline{v}'xx' \equiv \underline{v}'xx')$

etc.

An expression is *schematic* if it can be obtained by replacing variables by variables in some expression like $\underline{u}x$, $\underline{u}'xx'$,... etc. Practical handling of schematic expressions requires the following agreement on schematic replacement:

B is obtained from A by *schematically replacing* S by R if and only if:

S is a schematic expression, and there is an expression Q in which the first symbol in S does not occur and a symbol q such that A is obtained from Q by replacing q by S and B is obtained from Q by replacing q by R.

Thus, for example, taking

S as "$\underline{\mathrm{u}}x$"

Q as "$(\cap y q \rightarrow q)$"

A as "$(\cap y\ \underline{\mathrm{u}}x \rightarrow \underline{\mathrm{u}}x)$"

R as "x"

and

B as "$(\cap y x \rightarrow x)$",

we see that

"$(\cap y x \rightarrow x)$"

is obtained from

"$(\cap y\ \underline{\mathrm{u}}x \rightarrow \underline{\mathrm{u}}x)$"

by schematically replacing "$\underline{\mathrm{u}}x$" by "x".

1.2. Free, indicial and accepted variables.

In order to formulate wide-ranging and unambiguous rules of inference for our system we need to distinguish three kinds of variables which may appear in expressions. The idea of 'free variable' may surely be introduced without further comment:

If α is free in A then α is a variable and A is an expression. A variable is *free* in a form if and only if it occurs therein less than twice. A is a *formula* if and only if some variable is free in A. If A is a formula, C is a formula, B is different from A and is obtained from A either by replacing some free variable of A by C or by schematically replacing some schematic expression by C, then a variable is *free* in B if and only if it is free in both A and C.

On the other hand, it is not immediately clear why we require the notions of 'indicial' and 'accepted' variables which we now describe:

If α is indicial in A, then α is a variable and A is a formula; if α is accepted in A, then α is a variable and A is a formula. A variable is *indicial* in a form if and only if it occurs therein more than once. A variable is *accepted* in a form if and only if it occurs therein less than twice.

If A, B and C are formulas with A different from B and B different from C, and if B can be obtained from A by replacing a free and accepted variable of A by C, then: α is *accepted* in B if and only if α is accepted in A, and α is *indicial* in B if and only if α is indicial in A and does not appear in C.

If A, B and C are formulas with A different from B and B different from C, S is a schematic expression, some variable in S is indicial in A, and if B is obtained from A by schematically replacing S by C, then: α is *indicial* in B if and only if α is indicial in A, and α is *accepted* in B if and only if α is accepted in A and does not appear in C.

The reason for our making these definitions will be explained in the remark at the end of the next section.

1.3. Rules of inference

The following are the rules of inference of our theory.

.0 *Initiation*: Every formula asserted to be a definition or an axiom is a theorem.

.1 *Detachment* (Modus Ponens): If a theorem is obtained from "$(p \to q)$" by replacing "p" by a theorem and "q" by a formula T, then T is a theorem.

.2 *Substitution*: If T is a theorem in which b is free and A is such a formula that each variable in it is free in T, then the expression obtained from T by replacing b by A is also a theorem.

.3 *Schematic substitution*: If T is a theorem, S is a schematic expression, and A is such a formula that each variable in it is either free in T or occurs explicitly in S, and T′ is a formula obtained from T by schematically replacing S by A, then T′ is a theorem.

.4 *Indicial substitution*: If q is free in Q, T is a theorem obtained from Q by replacing q by a formula A in which α is indicial, B is obtained from A by replacing α by a variable which is accepted in A, and finally T′ is obtained from Q by replacing q by B, then T′ is a theorem.

.5 *Universalization*: If T is a formula obtained from "$\cap xy$" by replacing "x" by a variable and "y" by a theorem, then T is a theorem.

(These rules are quoted from [14],0.24 - 0.29.)

As an example, we verify that if

"$(\cap x\ \underline{u}x \to \underline{u}x)$"

is a theorem, then so is

"$(\cap y\ \underline{u}y \to \underline{u}x)$".

To do so we note that "$(q \rightarrow p)$" is a form, that "p" and "q" are free in "$(q \rightarrow p)$", and that "$\underline{u}x$" is a (form and) formula; whence "q" is free in the expression "$(q \rightarrow \underline{u}x)$". Moreover, "$x$" is clearly indicial in the formula "$\cap x\ \underline{u}x$", in which also "y" is free and therefore accepted. Our demonstration is therefore completed by applying the rule of Indicial substitution, taking

Q as "$(q \rightarrow \underline{u}x)$"

T as "$(\cap x\ \underline{u}x \rightarrow \underline{u}x)$"

A as "$\cap x\ \underline{u}x$"

B as "$\cap y\ \underline{u}y$"

α as "x"

and

T′ as "$(\cap y\ \underline{u}y \rightarrow \underline{u}x)$".

Remark: Had we adopted the perfectly natural suggestion that the words 'is accepted in A' in the rule of Indicial Substitution be replaced by 'does not appear in A' we would have been unable to interpret such expressions as "$\Sigma\ x \in x\ x$" (where $\Sigma\ x \in A\ \underline{u}x$ is the sum of the terms $\underline{u}x$ as x runs through the index set A). However, with the wording as it stands, we can apply the rule of Indicial Substitution to show from the theorem

$$(\Sigma\ y \in x\ y = \Sigma\ y \in x\ y)$$

that

$$(\Sigma\ x \in x\ x = \Sigma\ y \in x\ y).$$

It is precisely in order that we can make this sort of interpretation if necessary that we introduce the notions of indicial and accepted variables into our discussion. ®†

† Throughout this dissertation we shall indicate the end of a section entitled 'Remark(s)' by the symbol ®.

1.4. Theory of notation.

One of the most appealing and beautiful aspects of Morse's development of logic and set theory is his theory of notation, which 'permits useful simplification of a vast number of complicated expressions and justifies many of the informal conventions of present day mathematics'. We do no more than give a very rough sketch of the outlines of Morse's notation theory; for a more thorough description we could not improve on that given in [14], pages 15 - 27, to which we refer the reader for full details.

We take over unchanged the first part of Morse's theory - in which, by means of a separation of symbols into types, and several agreements, definitions and definitional schemas, there is established a unique interpretation for a large number of quantifier-free expressions. The essence of this interpretation is illustrated by the following table:

Expression	Interpretation
"$(xx'x'')$"	"$(x \wedge x' \wedge x'')$"
"$(x \rightarrow x' \rightarrow x'')$"	"$((x \rightarrow x') \wedge (x \rightarrow x''))$"
"$(x \leftrightarrow x' \leftrightarrow x'')$"	"$((x \leftrightarrow x') \wedge (x' \leftrightarrow x''))$"
"$(x \in x' = x'' \subset x''')$"	"$((x \in x') \wedge (x'=x'') \wedge (x'' \subset x'''))$"
"$(x \in x' \cap x'')$"	"$(x \in (x' \cap x''))$"
"$(x \in x' \wedge x'')$"	"$((x \in x') \wedge x'')$"
"$(x \wedge x' \in x'')$"	"$(x \wedge (x' \in x''))$"

Of these, the last three in particular should be noted as illustrations of the dictum: if one can contrive one valid reading (that is, translation into the English language) of an expression in which "$\rightarrow$" and "$\leftrightarrow$" and primed symbols derived therefrom do not appear, then that reading will be correct. (The qualifying words are necessary in this dictum, as we believe that, without Morse's rules for reading formulae, one would be able to give several different English translations of such expressions as

"$(x \rightarrow x' \rightarrow x'' \leftrightarrow x''')$".)

For convenience, we remark that a *nexus* is a symbol of some type in accordance with Morse's separation into numbered types.

1.5. Further notation theory.

When we turn to expressions in which quantifiers appear, we have first the *definitional schema for negation:*

.0 We accept as a definition each expression which can be obtained from

"$((x \backsim \in y) \equiv (x \in y \rightarrow \cap zz))$"

by replacing "$\in$" by a nexus different from "$\backsim$".

We now separate some of our expressions into numbered classes, as shown in the following table:

Class	Expression
0	"E", "One", "The"
1	"$\cap$"
2	"$\cup$"
3	"sup", "inf", "λ"
4	all expressions of class 0, 1, 2 or 3

(Morse includes more expressions in his separation, and has more classes; we have included only those we need, but have kept to his numbering and separation in all cases except that of "λ": whereas Morse's "λ" is of class 1, our "λ" is of class 3. The importance of this change from the classical to the constructive classification of "λ" will be emphasised in section 1.18.)

.1 *Definitional schema:* We accept as a definition each expression which can be obtained by replacing "E" by an expression of class 0, "∩" by an expression of class 1, "∪" by an expression of class 2, and "sup" by an expression of class 3 in any one of the following expressions:

"$(\mathrm{E}x;\underline{\mathrm{u}}x\ \underline{\mathrm{v}}x \equiv \mathrm{E}x(\underline{\mathrm{u}}x \wedge \underline{\mathrm{v}}x))$"

"$(\cap x;\underline{\mathrm{u}}x\ \underline{\mathrm{v}}x \equiv \cap x(0 \in \underline{\mathrm{u}}x \rightarrow \underline{\mathrm{v}}x))$"

"$(\cup x;\underline{\mathrm{u}}x\ \underline{\mathrm{v}}x \equiv \cup x(0 \in \underline{\mathrm{u}}x \wedge \underline{\mathrm{v}}x))$"

"$(\mathrm{sup}\ x\ \underline{\mathrm{v}}x \equiv \mathrm{sup}\ x;(x = x)\ \underline{\mathrm{v}}x)$".

By a *march* we mean an expression, such as

"$x \subset x' \in \cap\ x'' \subset \subset x''' \rightarrow x''''$",

which is obtained from one of

"x", "xx'", "$xx'x''$",...

by inserting symbol(s) of some type(s) between each pair of adjacent symbols of the form x,x',x'',... For our purposes, an expression A is said to be *verbal* if it has one or more of

"$\rightarrow$", "$\leftrightarrow$", "$\in$", "$\ni$", "$\subset$", "$\supset$",

"$=$", "$\neq$", "$\nmid$", "$<$", "$\leq$", "$>$", "$\geq$"

among its symbols; otherwise, A is *verbless*.

We say that s is a *subject* of A if and only if s is such a verbless expression, whose terminal symbol is one of x,x',... etc., that either A is s or A can be obtained from "xyz" by replacing "x" by s, "y" by a verbal nexus, and "z" by an expression.

It is in the next few definitions, agreements and definitional schemas that the heart of much of the application of our theory of notation lies.

.2 $(\mathrm{st}\ zx\ \underline{\mathrm{u}}x \equiv \cup x(z = x\ \wedge \underline{\mathrm{u}}x))$

.3 (substitute z for x in $\underline{\mathrm{u}}x \equiv$ st zx $\underline{\mathrm{u}}x$)

.4.0 We accept as a definition each expression which can be obtained by replacing "p" by a march whose terminal symbol is "x" in

"(st zp $\underline{\mathrm{u}}'xx'$ $\equiv$ $\cup x \cup x'(z = (p) \wedge \underline{\mathrm{u}}'xx')$)".

. 1 We accept as a definition each expression which can be obtained by replacing "p" by a march whose terminal symbol is "x''" in

"(st zp $\underline{\mathrm{u}}''xx'x''$ $\equiv$ $\cup x \cup x' \cup x''(z = (p) \wedge \underline{\mathrm{u}}''xx'x'')$)"

etc.

We now agree that

.5.0 A is a *1 stencil* if and only if A can be obtained by replacing "E" by an expression of class 0, "$\cup$" by an expression of class 1, 2 or 3, and "$\cap$" by an expression of class 4 in one of the expressions:

"(Ep;qr $\equiv$ Ez st $zp(q \wedge r)$)"

"($\cup p$;qr $\equiv$ $\cup z$;st ztq st ztr)"

"($\cap pr$ $\equiv$ $\cap p$;$(x = x)$ r)".

.1 A is a *2 stencil* if and only if A can be obtained by replacing "$\cap$" by an expression of class 4 in any one of the expressions:

"($\cap p$; qr $\equiv$ $\cap s$;$((p) \wedge q)$ r)"

"($\cap pr$ $\equiv$ $\cap s$; (p) r)".

With these agreements in mind, we now state the definitional schemas

.6.0 We accept as a definition each expression which can be obtained from a 1 stencil by replacing "p" by a verbless march of order 2, "t" by "x,x'", "q" by "$\underline{\mathrm{u}}'xx'$", and "$r$" by "$\underline{\mathrm{v}}'xx'$".

.1 We accept as a definition each expression which can be obtained from a 1 stencil by replacing "p" by a verbless march of order 3, "t" by "x,x',x''", "q" by "$\underline{\mathrm{u}}''xx'x''$",

and "r" by "$\underline{\mathrm{v}}''xx'x''$".
etc.

.7.0 We accept as a definition each expression which can be obtained from a 2 stencil by replacing "p" by a verbal march M of order 1, "s" by a subject of M, "q" by "$\underline{\mathrm{u}}x$", and "r" by "$\underline{\mathrm{v}}x$".

.1 We accept as a definition each expression which can be obtained from a 2 stencil by replacing "p" by a verbal march M of order 2, "s" by a subject of M, "q" by "$\underline{\mathrm{u}}'xx'$", and "$r$" by "$\underline{\mathrm{v}}'xx'$".

etc.

Thus, for example; we have as theorems

"$(\cap x,y;\ \underline{\mathrm{u}}'xy\ \underline{\mathrm{v}}'xy \equiv \cap z;\ \mathrm{st}\ z\ x,y\ \underline{\mathrm{u}}'xy\ \mathrm{st}\ z\ x,y\ \underline{\mathrm{v}}'xy)$"

"$(\mathrm{E}\ x\ \ddot{\cup}\ y \in A\ \underline{\mathrm{u}}'xy = \mathrm{E}\ z\ \mathrm{st}\ z\ x \cup y\ (z \in A \wedge \underline{\mathrm{u}}'xy))$"

"$(\lambda x\ \underline{\mathrm{u}}x = \lambda x;\ (x=x)\ \underline{\mathrm{u}}x)$"

and

"$(\lambda x \in A\ \underline{\mathrm{u}}x = \lambda x;\ x \in A\ \underline{\mathrm{u}}x)$"

(cf. section 1.15).

Finally, we introduce the definitional schema:

.8 We accept as a definition each expression which can be obtained from

"$((A \cap\cap B) \equiv \mathrm{E}x \cap y(x \in A \wedge y \in B))$"

by replacing "$\cap$" by a verbless binarian.

From this we obtain, for example, the theorem

"$(A,,B \equiv \mathrm{E}x,y(x \in A \wedge y \in B))$".

(For all material in this section, see [14], 0.49 - 0.62.)

1.6. Logic.

As we are primarily interested in the constructive version of Morse's set theory, rather than in his theory of notation - which is, in any case, unchanged from that in [14] except for the class number of the expression "λ" - we shall go no deeper into the description of this notational theory; nor do we intend to say anything about Morse's remarks on demonstrations or on the construction of definitions ([14], pages 27-29 and appendix A).

However, before going on to discuss set theory, we must first deal with the axiomatisation of propositional and predicate logic in our constructive framework. Our logic is based on the three primitive connectives "$\rightarrow$", "$\wedge$", "$\vee$" and the two quantifiers "$\cap$", "$\cup$" (all of which were fixed by appropriate definitions in section 1.1); we shall also require the preliminary definitions

.0 (If x then $y \equiv (x \rightarrow y)$)

.1 ((x implies y) $\equiv (x \rightarrow y)$)

.2 $((x \leftrightarrow y) \equiv ((x \rightarrow y) \wedge (y \rightarrow x)))$

.3 (x if and only if $y \equiv (x \leftrightarrow y)$)

.4 (x is equivalent to $y \equiv (x \leftrightarrow y)$)

.5 (For each $x, \underline{u}x \equiv \cap x\ \underline{u}x$)

.6 (For some $x, \underline{u}x \equiv \cup x\ \underline{u}x$)

.7 ((x and y) $\equiv (x \wedge y)$)

.8 ((x or y) $\equiv (x \vee y)$)

.9 $(\mathrm{U} \equiv \cup xx)$,

the definitional axioms

*.10 $(0 \equiv \cap xx)$

*.11 $(\sim p \equiv (p \rightarrow 0))$

and the axioms of definition

.12 $((x \equiv y) \rightarrow (x \rightarrow y))$

.13 $((x \equiv y) \rightarrow (y \rightarrow x))$.

Remark: We recall that all definitions are to be considered as theorems. As definitions are characterised amongst the theorems by the appearance of the definor "$\equiv$" as their princial connective, we do not think it necessary to indicate definitions in general by any word or sign. There is, however, one exception to this: our *definitional axioms* (introduced to simplify the statement of our main axioms) will be singled out from other definitions and theorems by the mark of an asterisk on the left.

The fundamental axioms of our logic are the *propositional axioms*

.14 $(p \rightarrow (p \wedge p))$

.15 $((p \wedge q) \rightarrow (q \wedge p))$

.16 $((p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow (q \wedge r)))$

.17 $((p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)))$

.18 $(q \rightarrow (p \rightarrow q))$

.19 $((p \wedge (p \rightarrow q)) \rightarrow q)$

.20 $(p \rightarrow (p \vee q))$

.21 $((p \vee q) \rightarrow (q \vee p))$

.22 $(((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r))$

and the *predicate axioms*

.23 $((y \rightarrow \underline{\mathrm{u}}x) \rightarrow (y \rightarrow \cap x\ \underline{\mathrm{u}}x))$

.24 $((\underline{\mathrm{u}}x \rightarrow y) \rightarrow (\cup x\ \underline{\mathrm{u}}x \rightarrow y))$

.25 $(\cap x\ \underline{\mathrm{u}}x \rightarrow \underline{\mathrm{u}}x)$

.26 $(\underline{\mathrm{u}}x \rightarrow \cup x\ \underline{\mathrm{u}}x)$

.27 $(y \rightarrow \cap xy)$

.28 $(\cup xy \rightarrow y)$

Our propositional axioms are simply the first eight of those due to Heyting ([12], pages 105-106). The main difference between our approach and Heyting's is that, where he takes negation as a primitive concept, for us it is defined by (1.6.10), (1.6.11) and the explanatory definition

.29 (Not $p \equiv \sim p$).

As will be shown below, this enables us to prove Heyting's extra two propositional axioms

.30 $(((p \to q) \wedge (p \to \sim q)) \to \sim p)$

.31 $(\sim p \to (p \to q))$.

Our predicate axioms are essentially those described by Troelstra ([20], page 11). Bearing in mind the explanatory definitions

.32 (The principle of excluded middle $\equiv \cap x(x \vee \sim x)$)

.33 (omniscience $\equiv$ The principle of excluded middle)

we have been particularly anxious when choosing axioms both here and elsewhere to avoid the appearance of such terms as

"(omniscience)"

"$(\sim\sim p \to p)$"

and

"$(\sim \cap x\ \underline{u}x \leftrightarrow \cup x \sim \underline{u}x)$"

among our theorems. (Note, in passing, that our use of the word 'omniscience' is not exactly that of Bishop on page 9 of [1].)

As the development of propositional and predicate logic is much more straightforward, and less interesting, than that of set theory, we shall content ourselves with the following list of theorems.

.34 $(p \wedge q \to p)$

.35 $(p \to (q \to (p \wedge q))$

.36 $((p \to (p \to q)) \to (p \to q))$

.37 $((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$

.38 $((p \to (q \to r)) \to (q \to (p \to r)))$

.39 $(p \to ((p \to q) \to q))$

.40 $((p \to (q \to r)) \to ((p \wedge q) \to r)))$

.41 $(((p \to q) \wedge (q \to r)) \to (p \to r))$

.42 $(((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r)))$

.43 $(p \to p)$

.44 ~ 0

.45 U

and a sketch of the proof of (1.6.30). Noting (1.6.10), (1.6.11), (1.6.42), (1.6.19) and (1.6.41), we have

$$
\begin{aligned}
&(((p \to q) \wedge (p \to \sim q)) \\
&\to (p \to q) \wedge (p \to (q \to \cap xx)) \\
&\to (p \to (q \wedge (q \to \cap xx))) \\
&\to (p \to (q \wedge (q \to \cap xx))) \wedge ((q \wedge (q \to \cap xx)) \to \cap xx) \\
&\to (p \to \cap xx) \\
&\to \sim p). \qquad \square
\end{aligned}
$$

A proof of (1.6.31) is even simpler than this, and will be omitted.

<u>Remark</u>: Throughout this chapter, all our proofs (when given explicitly) will be of the abbreviated form just used. We take it for granted that a full proof, in accordance with the strictest meaning of the word 'proof', can always be reconstructed from the outlines that we give. We shall also indicate the end of a proof by the 'Halmos tombstone' $\square$. ®

1.7. The fundamental definitions of set theory.

For the remainder of this chapter we shall concern ourselves entirely with constructive set theory in the spirit of Morse.

Bearing in mind the orienting definitions

.0 $((\to \in)x \equiv x)$

.1 $((x \in x') \equiv (x \in x'))$,

we see that the following preliminary definitions introduce most of the familiar objects of elementary set theory:

.2 $((x \text{ is a set}) \equiv (x \rightarrow x))$

.3 $(\text{The empty set} \equiv 0)$

.4 $(\text{The universe} \equiv U)$

.5 $(\text{complement } x \equiv \sim x)$

.6 $((x \text{ is a member of } y) \equiv (x \in y))$

.7 $((x \text{ is a point}) \equiv \cup y(x \in y))$

.8 $((x \text{ is true}) \equiv (0 \in x))$

.9 $((x \text{ is false}) \equiv \sim(0 \in x))$

.10 $((x \ni y) \equiv (y \in x))$

.11 $((x \text{ holds } y) \equiv (x \ni y))$

*.12 $((x \subset y) \equiv \cap t((t \in x) \rightarrow (t \in y)))$

.13 $((x \text{ is a subset of } y) \equiv (x \subset y))$

.14 $((x \supset y) \equiv (y \subset x))$

.15 $((x \text{ is a superset of } y) \equiv (x \supset y))$

*.16 $((x = y) \equiv ((x \subset y) \wedge (y \subset x)))$

.17 $((x \text{ equals } y) \equiv (x = y))$

.18 $((x \neq y) \equiv \sim(x = y))$

.19 $((x \subset\cdot\, y) \equiv (x \subset y \neq x))$

.20 $((x \text{ is a proper subset of } y) \equiv (x \subset\cdot\, y))$

.21 $((x \cdot\!\supset y) \equiv (y \subset\cdot\, x))$

.22 $((x \text{ is a proper superset of } y) \equiv (x \cdot\!\supset y))$

.23 $((\text{The intersection as } x \text{ runs, of } \underline{u}x) \equiv \cap x\ \underline{u}x)$

.24 $((\text{The union as } x \text{ runs, of } \underline{u}x) \equiv \cup x\ \underline{u}x)$

*.25 $(\Pi A \equiv \cap y(y \in A \rightarrow y))$

.26 $(\text{The intersection of } A \equiv \Pi A)$

*.27 $(\nabla A \equiv \cup y(y \in A \wedge y))$

.28 $(\text{The union of } A \equiv \nabla A)$

.29 $((x \cap y) \equiv (x \wedge y))$

.30 $((x \text{ intersect } y) \equiv (x \cap y))$

.31 $((x \cup y) \equiv (x \vee y))$

.32 $((x \text{ union } y) \equiv (x \cup y))$

Before dealing with the set theoretic axioms of set theory, we also need two logical definitional axioms

*.33 $(\mathrm{U} \equiv \cup x x)$

*.34 $((x \leftrightarrow y) \equiv ((x \rightarrow y) \wedge (y \rightarrow x)))$

and the *Axiom of Definition* for set theory

.35 $((x \equiv y) = (x = y))$.

1.8. Axioms of set theory - first group.

Our first group of axioms is concerned with truth and certain rules for manipulation of sets:

.0 $(x \leftrightarrow (0 \in x))$

.1 $((t \in \mathrm{U}) \rightarrow ((t \in (x \in y)) \leftrightarrow (x \in y)))$

.2 $((t \in a) \rightarrow ((t \in (x \rightarrow y)) \leftrightarrow ((t \in x) \rightarrow (t \in y))))$

.3 $((t \in \cap x\ \underline{u}x) \leftrightarrow \cap x(t \in \underline{u}x))$

.4 $((t \in \cup x\ \underline{u}x) \leftrightarrow \cup x(t \in \underline{u}x))$

.5 $((t \in (x \wedge y)) \leftrightarrow ((t \in x) \wedge (t \in y)))$

.6 $((t \in (x \vee y)) \leftrightarrow ((t \in x) \vee (t \in y)))$

<u>Remark</u>: In view of Bishop's Thesis, it is interesting that our criterion of truth (axiom (1.8.0) intimately involves the natural number 0 (that 0 is a natural number will be shown in section 1.20). ®

Among the elementary deductions we can make from these axioms and the definitions and axioms in section 1.7, we have

.7 $(x \text{ is a set})$

.8 $(0 \in \mathrm{U})$

Proof. Apply axiom (1.8.0) to (1.6.45). □

.9 $(x \subset y \rightarrow (x \rightarrow y))$

.10 $(x = y \rightarrow (x \leftrightarrow y))$

.11 $((x \equiv y) \leftrightarrow (x = y))$

.12 $((a = b) \leftrightarrow \cap x(x \in a \leftrightarrow x \in b))$

.13 $(x \in a \to x \in \mathrm{U})$

Proof. Noting axiom (1.8.4) we have

$$(x \in a \to \cup y(x \in y) \to x \in \cup yy \to x \in \mathrm{U}) \quad \square$$

.14 $(x \in a \to x \in \sim 0)$

Proof. From (1.7.35) and (1.6.11),

$$(\sim 0 = (0 \to 0)).$$

On the other hand, axiom (1.8.2) and propositional logic give

$$(x \in a \to ((x \in 0 \to x \in 0) \to x \in (0 \to 0))),$$

$$((x \in 0 \to x \in 0) \to (x \in a \to x \in (0 \to 0))),$$

and therefore, by (1.6.43), detachment, the above and (1.8.12),

$$(x \in a \to x \in (0 \to 0) \to x \in \sim 0) \quad \square$$

.15 $(x \in 0 \to x \in a)$

Proof. Similar to that of (1.8.13). $\square$

.16 $(0 \subset a \subset \mathrm{U})$

.17 $(t \in (x \in y) \leftrightarrow x \in y \wedge t \in \mathrm{U})$

.18 $(t \in (x \to y) \leftrightarrow (t \in x \to t \in y) \wedge t \in \mathrm{U})$

.19 $(x \in 0 \to 0)$

Proof. By (1.8.15), (1.8.17) and (1.8.0)

$$(x \in 0 \to x \in (0 \in 0) \to (0 \in 0) \to 0) \quad \square$$

.20 $(x = 0 \leftrightarrow \cap y \sim (y \in x))$

Of great importance is

.21 $(t \in \mathrm{U} \to t \in \sim a \leftrightarrow t \mathbin{\backsim\!\in} a)$

Proof. Noting (1.5.0) and (1.8.19), we have

$$\begin{aligned}(t \in \mathrm{U} \to t \in \sim a &\leftrightarrow t \in (a \to 0)\\ &\leftrightarrow (t \in a \to t \in 0)\\ &\leftrightarrow (t \in a \to 0)\\ &\leftrightarrow t \mathbin{\backsim\!\in} a) \quad \square\end{aligned}$$

Finally,

.22 $(U = \sim\sim U)$

.23 $(0 = \sim\sim 0)$

Remarks: Our choice of definition for U was dictated by the desire to obtain (1.8.13): had we made the definition $(U \equiv \cup x \sim x)$ - perhaps nearer to that of Morse ([14], 1.0.6) - the best we could have obtained along these lines would have been

$$((x \in \sim a \to x \in U) \wedge (x \in \sim\sim a \to x \in U)).$$

Note also that, in spite of (1.8.22) and (1.8.23), theorem (1.8.10) leads us to trust that $(x = \sim\sim x)$ is not a theorem of our system. ®..

1.9. Equality.

Our axioms of equality are

.0 $((x \in U) \to ((x = y) \leftrightarrow \cap t((x \in t) \to (y \in t))))$

.1 $((x = y) \to (\underline{u}x = \underline{u}y))$

From these we readily obtain

.2 $(x = y \to (x \in t \to y \in t))$

.3 $(x = y \to (x \in t \leftrightarrow y \in t))$

.4 $(x = y \to \underline{v}x = \underline{v}y)$

.5 $(x = y \to \underline{u}'xz = \underline{u}'yz)$

.6 $(x = y \to \underline{u}'zx = \underline{u}'zy)$

.7 $(x = y \wedge s = t \to \underline{u}'xt = \underline{u}'yt)$

1.10. Singletons.

Singletons, which play a most important role in the subsequent development of our set theory, can be approached in two different ways: from the first of these

*.0 $(\text{sng } x \equiv \cap_y (y \to (x \in y)))$

.1 $(\text{singleton } x \equiv \text{sng } x)$

we derive the fundamental theorems

.2 $(y \in U \to y \in \text{sng } x \leftrightarrow y = x)$

.3 $(y \in \text{sng } x \leftrightarrow y = x \wedge y \in U)$.

While the second approach

*.4 $(\text{sngl } x \equiv \cap y((x \in y) \to y))$

.5 $(\text{single } x \equiv \text{sngl } x)$

produces the corresponding criterion

.6 $(x \in U \to y \in \text{sngl } x \leftrightarrow y = x)$

and is linked to the first by

.7 $(x \in U \to \text{sng } x = \text{sngl } x)$.

However, for a more general characterisation of singletons we must turn to

*.8 $(\text{singleton is } a \equiv (\Pi a = \nabla a \in a))$.

We then have

.9 $(\text{singleton is } a \to y \in a \leftrightarrow y = \Pi a = \nabla a)$

and, not unexpectedly,

.10 $(x \in U \to \text{singleton is sng } x \wedge \text{singleton is sngl } x)$

.11 $(\text{singleton is } a \to a = \text{sng } \nabla a = \text{sngl } \Pi a)$.

Remark: Morse ([14], 2.50.1) makes the definition

$(\text{singleton is } a \equiv (\Pi a = \nabla a))$,

from which he is able to derive as a theorem

$(\text{singleton is } a \leftrightarrow \Pi a = \nabla a \in a)$.

This derivation breaks down constructively where it uses omniscience to prove that $(a = 0 \wedge \sim (a = 0))$. ®

1.11. Classification

*.0 $(\text{E}x\ \underline{u}x \equiv \cup x(0 \in \underline{u}x \wedge \text{sng } x))$

.1 (The set of points x such that $\underline{u}x \equiv \text{E}x\ \underline{u}x$)

.2 $(\{x : \underline{u}x\} \equiv \text{E}x\ \underline{u}x)$

The *Classification Theorems* are

.3 $(x \in \mathrm{E}x\ \underline{\mathrm{u}}x \leftrightarrow \underline{\mathrm{u}}x \wedge x \in \mathrm{U})$

.4 $(x \in \mathrm{U} \rightarrow x \in \mathrm{E}x\ \underline{\mathrm{u}}x \leftrightarrow \underline{\mathrm{u}}x)$

The second of these follows from the first, whose proof is based on (1.8.0), (1.10.2), (1.9.1) and (1.8.10) (we omit the details).

1.12. Axioms of construction

The heading of this section derives from the apparent correspondence between membership of the Universe, and our intuitive notion of a mathematical object being 'constructively well-defined'. So much do we believe in the exactness of this correspondence that we are prepared to adopt as our *Fundamental Thesis on Constructivity:*

A mathematical object (set) is constructively well-defined if and only if it belongs to the Universe.

Viewed from another angle, this may also be taken as our definition of the expression 'constructively well-defined'.

As our first axioms of construction we have

.0 $(((A \in \mathrm{U}) \wedge \cap x((x \in A) \rightarrow (\underline{\mathrm{u}}x \in \mathrm{U}))) \rightarrow (\cup x((x \in A) \wedge \underline{\mathrm{u}}x) \in \mathrm{U}))$

.1 $(((x \in \mathrm{U}) \wedge (y \in \mathrm{U})) \leftrightarrow ((x \vee y) \in \mathrm{U}))$

.2 $((x \in \mathrm{U}) \leftrightarrow (\mathrm{sngl}\ x \in \mathrm{U}))$.

Remark: Morse is able to derive (1.12.0), (1.12.1) and several of our later theorems of construction from the one powerful *axiom of replacement*

$$(\cup x((A \in \mathrm{U}) \wedge (x \subset A) \wedge ((\underline{\mathrm{u}}x \in \mathrm{U}) \wedge \underline{\mathrm{u}}x)) \in \mathrm{U}),$$

via the consequent 'theorem of replacement'

$$(\cup x((A \in \mathrm{U}) \wedge (x \in A) \wedge (\underline{\mathrm{u}}x \in \mathrm{U}) \wedge \underline{\mathrm{u}}x) \in \mathrm{U}).$$

Unfortunately, his derivation of this theorem from his axiom of replacement rests heavily on the use of omniscience. We shall discuss this matter more fully later, in section 1.24. (cf. [14], 2.39 - 2.49). ®

The first two of our axioms of construction give rise to several elementary *theorems of construction:*

.3 $(A \in \mathrm{U} \to B \cap A \in \mathrm{U})$

Proof. By (1.12.1)

$$(A \in \mathrm{U} \to (B \cap A) \cup A = A \in \mathrm{U} \to B \cap A \in \mathrm{U}) \qquad \square$$

.4 $(B \subset A \in \mathrm{U} \to B \in \mathrm{U})$

.5 $((\cup x(x \in A \wedge \underline{u}x)) \in \mathrm{U} \to \cap x(x \in A \to \underline{u}x \in \mathrm{U}))$

.6 $(A \in \mathrm{U} \to \nabla A \in \mathrm{U})$

Of great importance also are

.7 $(x \in \mathrm{U} \to \mathrm{sng}\ x \in \mathrm{U})$

.8 $(\sim (x \in \mathrm{U}) \to \mathrm{sng}\ x = 0 \wedge \mathrm{sngl}\ x = \mathrm{U})$

.9 (singleton is $a \to a \in \mathrm{U}$)

Proof. Use (1.10.11), (1.10.9) and (1.12.7). $\square$

Remark: Classically we can infer from (1.12.8) and (1.10.7) that

$$(x \in \mathrm{U} \leftrightarrow \mathrm{sng}\ x = \mathrm{sngl}\ x)$$

and from (1.12.8) and (1.12.7) that

$$(\mathrm{sng}\ x \in \mathrm{U}).$$

We do not know of any constructive proof of these statements; indeed, the second of these seems highly undesirable as a constructive theorem: we would not expect the singleton of x to be constructively well-defined unless this was known to be the case for x itself! ®

1.13. Basicorderedpairs and basicrelations

*.0 $(\{x\} \equiv \mathrm{sngl}\ x)$

*.1 $(\{xx'\} \equiv (\mathrm{sngl}\ x \vee \mathrm{sngl}\ x'))$

*.2 $((x \,\lrcorner\, y) \equiv \{\{x\}\{xy\}\})$

.3 (basicorderedpair $xy \equiv (x \,\lrcorner\, y)$)

.4 (basicorderedpair is $p \equiv \cup x \cup y(p = x \,\lrcorner\, y \in \mathrm{U})$)

.5 $(\text{basicrelation is } R \equiv \cap p(p \in R \rightarrow \text{basicorderedpair is } p))$

.6 $(\text{bsvs } Rx \equiv \mathrm{E}y(x \mathbin{\lrcorner} y \in R))$

.7 $(\text{The basic vertical section of } R \text{ at } x \equiv \text{bsvs } Rx)$

*.8 $(A \mathbin{\lrcorner\lrcorner} B \equiv \mathrm{E}x \mathbin{\lrcorner} y(x \in A \wedge y \in B))$

.9 $(\text{basicartesianproduct } AB \equiv A \mathbin{\lrcorner\lrcorner} B)$

A familiar, and rather tedious 'argument by cases' enables us to prove

.10 $(x \mathbin{\lrcorner} y = s \mathbin{\lrcorner} t \in \mathrm{U} \leftrightarrow x = s \in \mathrm{U} \wedge y = t \in \mathrm{U})$

We omit the details. It is, however, worth noting that our theory would be unaffected if we were to replace (1.13.2) by the two axioms

$$(x \in \mathrm{U} \rightarrow \text{sng } x \in \mathrm{U})$$

$$(x \mathbin{\lrcorner} y = s \mathbin{\lrcorner} t \in \mathrm{U} \leftrightarrow x = s \in \mathrm{U} \wedge y = t \in \mathrm{U}).$$

For (1.10.7) would then give

$$(x \in \mathrm{U} \rightarrow \text{sngl } x \in \mathrm{U}),$$

and therefore, via (1.12.2) and (1.12.1),

$$\begin{aligned}(\text{sngl } x \in \mathrm{U} &\rightarrow \text{sngl sngl } x \in \mathrm{U} \\ &\rightarrow (\text{sngl sngl } x \vee \text{sngl}(\text{sngl } x \vee \text{sngl } x)) \in \mathrm{U} \\ &\rightarrow x \mathbin{\lrcorner} x \in \mathrm{U} \\ &\rightarrow x \in \mathrm{U}).\end{aligned}$$

Of course, it is better to have the one axiom (1.12.2) than the alternative two.

For completeness, we mention also the theorems

.11 $(x \mathbin{\lrcorner} y \in \mathrm{E}x \mathbin{\lrcorner} y\ \underline{\mathrm{u}}'xy \leftrightarrow \underline{\mathrm{u}}'xy \wedge x \mathbin{\lrcorner} y \in \mathrm{U})$

.12 $(R = \mathrm{E}x \mathbin{\lrcorner} y\ \underline{\mathrm{u}}'xy \rightarrow \text{basicrelation is } R)$

.13 $(A \mathbin{\lrcorner\lrcorner} B = \cup x \cup y(x \in A \wedge y \in B \wedge \text{sng}(x \mathbin{\lrcorner} y)))$

.14 $(\text{basicrelation is } A \mathbin{\lrcorner\lrcorner} B)$

.15 $(x \in A \rightarrow \text{bsvs}(A \mathbin{\lrcorner\lrcorner} B)x = B)$

.16 $(\sim (x \in A) \rightarrow \text{bsvs}(A \mathbin{\lrcorner\lrcorner} B)x = 0)$

.17 $(\sim (x \mathbin{\lrcorner} y = 0))$

.18 $(\sim (0 \in A \nearrow\nearrow B))$

1.14 Orderedpairs

Although basicorderedpairs have their uses (cf. section 1.27), we follow Morse's lead, and define a more powerful concept of ordered pair; one reason for doing so is to arrive at (1.14.15), which fits in very well with our theory of notation.

*.0 $(\mathrm{ss}\ a \equiv (\mathrm{sng}\ 0 \vee \cup x(x \in a \wedge \mathrm{sng}\ \mathrm{sng}\ x)))$

*.1 $((a,b) \equiv ((\mathrm{sng}\ 0 \nearrow\nearrow \mathrm{ss}\ a) \vee (\mathrm{sng}\ \mathrm{sng}\ 0 \nearrow\nearrow \mathrm{ss}\ b)))$

.2 $(\mathrm{orderedpair}\ ab \equiv (a,b))$

*.3 $(\mathrm{orderedpair\ is}\ p \equiv \cup a \cup b(p = a,b))$

.4 $(\mathrm{crd}'p \equiv \nabla \mathrm{bsvs}\ p0)$

.5 (The first coordinate of $p \equiv \mathrm{crd}'p$)

.6 $(\mathrm{crd}''p \equiv \nabla \mathrm{bsvs}\ p\ \mathrm{sng}\ 0)$

.7 (The second coordinate of $p \equiv \mathrm{crd}''p$)

.8 $(\mathrm{cartesianproduct}\ AB \equiv A,,B))$

Noting the lemmas

.9 $(\mathrm{bsvs}(a,b)0 = \mathrm{ss}\ a \wedge \mathrm{bsvs}(a,b)\ \mathrm{sng}\ 0 = \mathrm{ss}\ b)$

.10 $(\nabla \mathrm{ss}\ a = a)$

.11 $(a \in \mathrm{U} \leftrightarrow \mathrm{ss}\ a \in \mathrm{U})$

.12 $(x \in A \wedge A \nearrow\nearrow B \in \mathrm{U} \to B \in \mathrm{U})$,

we have

.13 $(a,b \in \mathrm{U} \leftrightarrow a \in \mathrm{U} \wedge b \in \mathrm{U})$

Proof. By axiom (1.12.1),

$(a,b \in \mathrm{U} \to (\mathrm{sng}\ 0 \nearrow\nearrow \mathrm{ss}\ a \in \mathrm{U}) \wedge (\mathrm{sng}\ \mathrm{sng}\ 0 \nearrow\nearrow \mathrm{ss}\ b \in \mathrm{U}))$

As we have

$(0 \in \mathrm{sng}\ 0 \wedge \mathrm{sng}\ 0 \in \mathrm{U})$

and

$(\mathrm{sng}\ 0 \in \mathrm{sng}\ \mathrm{sng}\ 0 \wedge \mathrm{sng}\ \mathrm{sng}\ 0 \in \mathrm{U})$,

it follows from (1.14.12) and (1.14.11) that

$(a,b \in \mathrm{U} \rightarrow \mathrm{ss}\ a \in \mathrm{U} \wedge \mathrm{ss}\ b \in \mathrm{U}$

$\rightarrow a \in \mathrm{U} \wedge b \in \mathrm{U}).$

On the other hand, (1.14.11), (1.12.7) and (1.12.0) show that

$(a \in \mathrm{U} \rightarrow \mathrm{sng}\ 0 \frown \mathrm{ss}\ a = \cup x(x \in \mathrm{ss}\ a \wedge \mathrm{sng}(0,x)) \in \mathrm{U})$

and

$(b \in \mathrm{U} \rightarrow \mathrm{sng}\ \mathrm{sng}\ 0 \frown \mathrm{ss}\ b = \cup x(x \in \mathrm{ss}\, b \wedge \mathrm{sng}(\mathrm{sng}\ 0,\ x)) \in \mathrm{U})$

Reference to (1.14.1) and (1.12.1) completes the proof. □

.14 $(\mathrm{crd}'(a,b) = a \wedge \mathrm{crd}''(a,b) = b)$

.15 $(a,b = c,d \leftrightarrow a = c \wedge b = d)$

Of theorems relating to cartesianproducts, we mention only

.16 $(x,y \in \mathrm{E}x,y\ \underline{\mathrm{u}}'xy \leftrightarrow \underline{\mathrm{u}}'xy \wedge x,y \in \mathrm{U}).$

1.15. Substitution

The theorems of this section - and in particular (1.15.7) and (1.15.8) - show very clearly some of the richness of our theory of notation.

.0 $(\mathrm{st}\ yx\ \underline{\mathrm{u}}x = \underline{\mathrm{u}}y)$

.1 $(\mathrm{st}\ ty\ \underline{\mathrm{u}}'xy = \underline{\mathrm{u}}'xt)$

.2 $(\mathrm{st}\ (s,t)x,y\ \underline{\mathrm{u}}'xy = \underline{\mathrm{u}}'st)$

.3 $(\sim \mathrm{orderedpair\ is}\ z \rightarrow \mathrm{st}\ z\ x,y\ \underline{\mathrm{u}}'xy = 0)$

.4 $(\cap x \cap y \cap z(\underline{\mathrm{w}}'xy = (0 \in \underline{\mathrm{u}}'xy \rightarrow \underline{\mathrm{v}}'xy) \wedge$

$\underline{\mathrm{u}}z = (0 \in \mathrm{st}\ z(x,y)\ \underline{\mathrm{u}}'xy \rightarrow \mathrm{st}\ z(x,y)\ \underline{\mathrm{v}}'xy)$

$\rightarrow \cap x \cap y\ \underline{\mathrm{w}}'xy = \cap z\ \underline{\mathrm{u}}z))$

Proof. $(c = a,b \rightarrow \cap z\ \underline{\mathrm{u}}z \subset \underline{\mathrm{u}}c = \underline{\mathrm{w}}'ab$

$\rightarrow \cap z\ \underline{\mathrm{u}}z \subset \underline{\mathrm{w}}'ab),$

whence

$(\cap a \cap b(\cap z\ \underline{\mathrm{u}}z \subset \underline{\mathrm{w}}'ab))$

and therefore

$(\cap z\ \underline{\mathrm{u}}z \subset \cap x \cap y\ \underline{\mathrm{w}}'xy).$

On the other hand,

$(t \in \cap x \cap y\ \underline{w}\ 'xy$

$\rightarrow \cap x \cap y(0 \in \underline{u}'xy \rightarrow t \in \underline{v}'xy)$

$\rightarrow (0 \in \text{st}\ z\ x,y\ \underline{u}'xy \rightarrow \text{st}\ z\ x,y\ \underline{u}'xy$

$\rightarrow \cup x \cup y(z = x,y \wedge \underline{u}'xy)$

$\rightarrow \cup x \cup y(z = x,y \wedge 0 \in \underline{u}'xy \wedge$

$(0 \in \underline{u}'xy \rightarrow t \in \underline{v}'xy))$

$\rightarrow \cup x \cup y(z = x,y \wedge t \in \underline{v}'xy))$

$\rightarrow (0 \in \text{st}\ z\ x,y\ \underline{u}'xy \rightarrow t \in \text{st}\ z\ x,y\ \underline{v}'xy)$

$\rightarrow t \in (0 \in \text{st}\ z\ x,y\ \underline{u}'xy \rightarrow \text{st}\ z\ x,y\ \underline{v}'xy)$

$\rightarrow t \in \underline{u}z),$

whence

$(\cap x \cap y\ \underline{w}'xy \subset \underline{u}z).$

Thus

$(\cap x \cap y\ \underline{w}'xy \subset \cap z\ \underline{u}z \subset \cap x \cap y\ \underline{w}'xy),$

from which the result follows. □

.5 $(\cap x \cap y \cap z(\underline{w}'xy = (0 \in \underline{u}'xy \rightarrow \underline{v}'xy) \wedge$

$\underline{u}z = (0 \in \text{st}\ z(x,y)\ \underline{u}'xy \rightarrow \text{st}\ z(x,y)\ \underline{v}'xy)$

$\rightarrow \cup x \cup y\ \underline{w}'xy = \cup z\ \underline{u}z)$

The last two lemmas are vital for the proof of

.6 $(\cap x,y;\ \underline{u}'xy\ \underline{v}'xy = \cap x \cap y;\ \underline{u}'xy\ \underline{v}'xy)$

Proof. From our theory of notation and (1.15.4) above we have

$(\cap x,y;\ \underline{u}'xy\ \underline{v}'xy$

$= \cap z;\ \text{st}\ z(x,y)\ \underline{u}'xy\ \text{st}\ z(x,y)\ \underline{v}'xy$

$= \cap z(0 \in \text{st}\ z(x,y)\ \underline{u}'xy \rightarrow \text{st}\ z(x,y)\ \underline{v}'xy)$

$= \cap x \cap y(0 \in \underline{u}'xy \rightarrow \underline{v}'xy)$

$= \cap x \cap y\ ;\ \underline{u}'xy\ \underline{v}'xy)$ □

.7 $(\cup x,y;\ \underline{u}'xy\ \underline{v}'xy = \cup x \cup y;\ \underline{u}'xy\ \underline{v}'xy)$

<u>Remark</u>: Morse's classical proofs of (1.15.6) and (1.15.7) depend

on the theorem

(orderedpair is p $\vee$ ~ orderedpair is p).

Now, it is tempting to think that this last term could be added as an axiom to our present system without destroying its constructivity: for, is it not reasonable to say that we can recognise whether or not an object is an orderedpair? However, this is unfortunately not the case, as the following argument shows.

Let

$$(z = (\text{sng } 0 \,\text{,,}\, \text{ss } x) \cup (\text{sng sng } 0 \,\text{,,}\, ((x = y) \wedge \text{ss } y)).$$

Then

$$\begin{aligned}(z = a,b &\rightarrow \text{sng sng } 0 \,\text{,,}\, \text{ss } b = \text{sng sng } 0 \,\text{,,}\, ((x{=}y) \wedge \text{ss } y) \\ &\rightarrow \text{ss } b = ((x{=}y) \wedge \text{ss } y) \\ &\rightarrow 0 \in ((x{=}y) \wedge \text{ss } y) \subset (x{=}y) \\ &\rightarrow 0 \in (x{=}y) \\ &\rightarrow x = y\ ),\end{aligned}$$

the third last line holding because

$$(0 \in \text{sng } 0 \in \text{ss } b).$$

On the other hand,

$$\begin{aligned}(x = y &\rightarrow (x = y) = \text{U} \\ &\rightarrow z = (\text{sng } 0 \,\text{,,}\, \text{ss } x) \cup (\text{sng sng } 0 \,\text{,,}\, \text{ss } y) \\ &\rightarrow \text{orderedpair is } z),\end{aligned}$$

so that

$$(\sim \text{orderedpair is } z \rightarrow \sim (x = y)).$$

It is now clear that

$$\begin{aligned}&(\cap p(\text{orderedpair is } p \vee \sim \text{orderedpair is } p) \\ &\rightarrow \cap x \cap y((x = y) \vee \sim (x = y))).\end{aligned}$$

But

$$(\cap x \cap y((x = y) \vee \sim (x = y))$$
$$\to ((t \in x) = \mathrm{U}) \vee \sim ((t \in x) = \mathrm{U})$$
$$\to (t \in x \vee \sim (t \in x))$$
$$\to \cap t(t \in (x \vee \sim x) \leftrightarrow t \in \mathrm{U})$$
$$\to x \vee \sim x = \mathrm{U}$$
$$\to x \vee \sim x),$$

whence, clearly,

($\cap p$(orderedpair is p $\vee \sim$ orderedpair is p) $\to$ omniscience).

It is for this reason that we do not add

(orderedpair is p $\vee \sim$ orderedpair is p)

as an axiom of our system.

Finally, we note that of the miscellaneous notational theorems in section 2.70 of [14], all but (2.70.10) and (2.70.18) go through constructively. Moreover, it is easy to see that the two which fail in a constructive setting entail

$$(\sim \cap x\ \underline{\mathrm{u}}x \leftrightarrow \cup x \sim \underline{\mathrm{u}}x)$$

and

$$(\cap x(y \vee \underline{\mathrm{u}}x) \leftrightarrow y \vee \cap x\ \underline{\mathrm{u}}x)$$

respectively, and so are essentially non-constructive. ®

1.16. Relations

*.0 (relation is $R \equiv \cap x \in R$ orderedpair is x)

.1 (relation $RS \equiv$ relation is $R \wedge$ relation is S)

*.2 (dmn $R \equiv \mathrm{E}x \cup y(x,y \in R)$)

.3 (The domain of $R \equiv$ dmn R)

*.4 (rng $R \equiv \mathrm{E}y \cup x(x,y \in R)$)

.5 (The range of $R \equiv$ rng R)

*.6 (vs $Rx \equiv \mathrm{E}y(x,y \in R)$)

.7 (The vertical section of R at $x \equiv$ vs Rx)

.8 (hs $Ry \equiv \mathrm{E}x(x,y \in R)$)

.9 (The horizontal section of R at $y \equiv$ hs Ry)

.10 (inv $R \equiv$ E$x,y(y,x \in R)$)

.11 (The inverse of $R \equiv$ inv R)

.12 ($(R:S) \equiv$ Ex,y∪$z(x,z \in S \wedge z,y \in R)$)

.13 (R composed with $S \equiv (R:S)$)

.14 ($(R\vdots S) \equiv (S:R)$)

.15 (strc $RA \equiv (R \cap (A,,\mathrm{U}))$)

.16 (The restriction of R to $A \equiv$ strc RA)

.17 (strn $RB \equiv (R \cap (\mathrm{U},,B))$)

.18 (The restriction in range of R to $B \equiv$ strn RB)

.19 ($_{*}RA \equiv$ ∪$x \in A$ vs Rx)

.20 (The image of A under $R \equiv {}_{*}RA$)

.21 ($^{*}RB \equiv$ ∪$y \in B$ hs Ry)

.22 (The inverse image of B under $R \equiv {}^{*}RB$)

Of the many elementary deductions made from these definitions, we mention only the more important

.23 (relation is Ex,y $\underset{\sim}{\mathrm{u}}'xy$)

.24 (relation is $R \rightarrow R =$ E$x,y(x,y \in R)$)

.25 (relation is $S \wedge R \subset S \rightarrow$ relation is R)

.26 (dmn inv R = rng R ∧ rng inv R = dmn R)

.27 (relation is $R \rightarrow$ inv inv $R = R$)

.28 ($R:(S:T) = (R:S):T$)

.29 (inv $(R:S)$ = (inv S):(inv R))

the lemma

.30 (relation is R

→ dmn R = ∪$p \in R$ sng crd$'p$ ∧ rng R = ∪$p \in R$ sng crd$''p$)

and the very important theorem of construction

.31 $(R \in \mathrm{U} \to \mathrm{dmn}\ R \in \mathrm{U} \wedge \mathrm{rng}\ R \in \mathrm{U})$

Proof. By (1.12.4),

$(R \in \mathrm{U} \to \mathrm{E}p \in R(\text{orderedpair is } p) \in \mathrm{U})$.

Moreover

$(\mathrm{dmn}\ R = \mathrm{dmn}\ \mathrm{E}p \in R(\text{orderedpair is } p) \wedge$

$\mathrm{rng}\ R = \mathrm{rng}\ \mathrm{E}p \in R(\text{orderedpair is } p))$,

so that we may without loss of generality suppose that

(relation is R).

We now have

$(p \in R \to \text{orderedpair is } p \wedge p \in \mathrm{U}$

$\to \mathrm{crd}'p \in \mathrm{U} \wedge \mathrm{crd}''p \in \mathrm{U}$

$\to \mathrm{sng}\ \mathrm{crd}'p \in \mathrm{U} \wedge \mathrm{sng}\ \mathrm{crd}''p \in \mathrm{U})$.

The result follows from this, (1.16.30) and (1.12.0). □

1.17. Functions

*.0 (function is $f \equiv$ (relation is $f \wedge \cap x \in \mathrm{dmn}\ f$ singleton is vs fx))

.1 (function $fg \equiv$ (function is $f \wedge$ function is g))

.2 (univalent is $f \equiv$ function f inv f)

*.3 $(.fx \equiv \Pi\mathrm{vs}\ fx)$

.4 (The value of f at $x \equiv .fx$)

.5 $(\_xf \equiv .fx)$

.6 (upon A is $f \equiv$ (function is $f \wedge \mathrm{dmn}\ f \subset A$))

.7 (on A is $f \equiv$ (function is $f \wedge \mathrm{dmn}\ f = A$))

.8 (upon A to B is $f \equiv$ (function is $f \wedge \mathrm{dmn}\ f \subset A \wedge \mathrm{rng}\ f \subset B$))

*.9 (on A to B is $f \equiv$ (function is $f \wedge \mathrm{dmn}\ f = A \wedge \mathrm{rng}\ f \subset B$))

.10 (upon A onto B is $f \equiv$ (upon A is $f \wedge \mathrm{rng}\ f = B$))

.11 (on A onto B is $f \equiv$ (on A is $f \wedge \mathrm{rng}\ f = B$))

.12 (map $AB \equiv \mathrm{E}f$(on A to B is f))

Again, as in earlier sections, we content ourselves with a judicious selection of theorems:

.13 $(\text{function is } f \to x,y \in f \leftrightarrow x \in \text{dmn } f \wedge y = .fx)$

.14 $(\text{function is } f \wedge x \in \text{dmn } f \to .fx \in \mathrm{U})$

.15 $(\text{function is } f \wedge \sim (x \in \text{dmn } f) \to .fx = \mathrm{U})$

.16 $(\text{function is } f \to y \in \text{rng } f \leftrightarrow \cup x \in \text{dmn } f\ (y = .fx))$

.17 $(\text{function is } f \to f = \cup x \in \text{dmn } f\ \text{sng}(x,.fx))$

From (1.17.14), (1.14.13), (1.12.7), (1.12.0) and (1.17.17) we obtain the very satisfactory theorem of construction

.18 $(\text{function is } f \wedge \text{dmn } f \in \mathrm{U} \to f \in \mathrm{U})$

In turn, this enables us to prove

.19 $(A \in \mathrm{U} \wedge B \in \mathrm{U} \to A,,B \in \mathrm{U})$

Proof. Clearly

$(b \in B \wedge f = \mathrm{E}x,y(x \in A \wedge y = x,b)$

$\to \text{on } A \text{ is } f \wedge \text{rng } f = A,,\{b\}),$

whence, by (1.17.18) and (1.16.31),

$(A \in \mathrm{U} \to \cap b \in B(A,,\{b\} \in \mathrm{U})).$

Noting that

$(A,,B = \cup b \in B(A,,\{b\}))$

we now deduce the result from axiom (1.12.0) $\square$

1.18. λ-calculus

.0 $(\lambda x;\underline{v}x\ \underline{u}x \equiv \mathrm{E}x,y(\underline{v}x \wedge y = \underline{u}x))$

.1 $(\text{lambda } x \text{ with } \underline{v}x,\ \underline{u}x \equiv \lambda x;\underline{v}x\ \underline{u}x)$

As we are taking "λ" as an 'expression of class 3', it follows from our theory of notation (Section 1.5) that

.2 $(\lambda x\ \underline{u}x \equiv \lambda x;(x = x)\ \underline{u}x)$

.3 $(\lambda x\ \underline{u}x = \mathrm{E}x,y(y = \underline{u}x))$

The main theorems of our λ-calculus are

.4 $(\text{function is } \lambda x;\underline{v}x\ \underline{u}x)$

.5 $(\text{function is } f \to f = \lambda x \in \text{dmn } f.fx$

.6 $(f = \lambda x;\underline{v}x\ \underline{u}x$

$\rightarrow \text{dmn } f = \text{E}x(\underline{v}x \wedge \underline{u}x \in \text{U}) \wedge (x \in \text{dmn } f \rightarrow .fx = \underline{u}x))$

.7 $(f = \lambda x\ \underline{u}x \rightarrow \text{dmn } f = \text{E}x(\underline{u}x \in \text{U}) \wedge (x \in \text{dmn } f \rightarrow .fx = \underline{u}x))$

.8 $(f = \lambda x \in A\ \underline{u}x \rightarrow \text{dmn } f = \text{E}x \in A(\underline{u}x \in \text{U}) \wedge (x \in \text{dmn } f \rightarrow .fx = \underline{u}x))$

.9 $(\lambda x\ \underline{u}x = \lambda x \in \text{U}\ \underline{u}x)$

Remark: Morse takes 'λ' as an expression of class 1, adopts (1.18.3) as a definition, and is able to derive (1.18.6) from the fact that, under these circumstances,

$$(\lambda x;\underline{v}x\ \underline{u}x = \lambda x(0 \in \underline{v}x \rightarrow \underline{u}x))$$

is a theorem. However, this procedure is essentially non-constructive: to see this, let us suppose that (1.18.3) and

$$(\text{dmn}\ \ \lambda x(0 \in \underline{v}x \rightarrow \underline{u}x) = \text{E}x(\underline{v}x \wedge \underline{u}x \in \text{U}))$$

are both theorems of our system. Then

$$\begin{aligned}(\text{dmn } \lambda x(0 \in (x \vee \sim x) \rightarrow x \neq x) &= \text{E}x((x \vee \sim x) \wedge (x \neq x) \in \text{U})\\ &= \text{E}x((x \vee \sim x) \wedge 0 \in \text{U})\\ &= \text{E}x(x \vee \sim x)).\end{aligned}$$

On the other hand, as

$$(\underline{u}x = (0 \in (x = x) \rightarrow \underline{u}x)),$$

we also have

$$\begin{aligned}(\text{dmn } \lambda x\ \underline{u}x &= \text{dmn } \lambda x(0 \in (x = x) \rightarrow \underline{u}x)\\ &= \text{E}x(x = x \wedge \underline{u}x \in \text{U})\\ &= \text{E}x(\underline{u}x \in \text{U})).\end{aligned}$$

But

$$(\sim\sim(x \vee \sim x)),$$

so that

$$\begin{aligned}(t \in (0 \in (x \vee \sim x) \rightarrow x \neq x) &\rightarrow (0 \in (x \vee \sim x) \rightarrow t \in (x \neq x))\\ &\rightarrow (x \vee \sim x \rightarrow t \in 0)\\ &\rightarrow \sim(x \vee \sim x)\\ &\rightarrow t \in 0)\end{aligned}$$

and therefore

$((0 \in (x \vee \sim x) \to x \neq x) = 0 \in \mathrm{U})$.

Hence

$(\mathrm{dmn}\ \lambda x(0 \in (x \vee \sim x) \to x \neq x) = \mathrm{E}x((0 \in (x \vee \sim x) \to x \neq x) \in \mathrm{U}) = \mathrm{U})$

It now follows that

$(\cap x \in \mathrm{U}(x \vee \sim x))$

- which clearly demonstrates the non-constructive nature of Morse's approach to λ-calculus. ®

1.19. Unicity and unique choice

Before dealing with natural numbers and recursion theory, we need to know how to express 'the unique x with property P' as a term of our formal system. To do this, we take over exactly Morse's definitions

.0 $(\text{One } x\ \underline{\mathrm{u}}x \equiv \cup y \cap x(\underline{\mathrm{u}}x \leftrightarrow x = y))$

.1 $(\text{There is just one } x \text{ such that } \underline{\mathrm{u}}x \equiv \text{One } x\ \underline{\mathrm{u}}x)$

.2 $(\text{The } x\ \underline{\mathrm{u}}x \equiv \cap x;(\text{One } x\ \underline{\mathrm{u}}x \wedge \underline{\mathrm{u}}x)\ x)$

and the consequent theorems

.3 $(\text{One } x\ \underline{\mathrm{u}}x \to \underline{\mathrm{u}}x \leftrightarrow x = \text{The } x\ \underline{\mathrm{u}}x)$

.4 $(\sim \text{One } x\ \underline{\mathrm{u}}x \to \text{The } x\ \underline{\mathrm{u}}x = \mathrm{U})$.

We are now also able to state and prove the *Theorem of Unique Choice*

.5 $(\cap x \in A \text{ One } y\ \underline{\mathrm{u}}'xy \to \text{One } f\ (\text{on } A \text{ is } f \wedge \cap x \in A\ \underline{\mathrm{u}}'x.fx)$

Proof. It is easily seen that

$(f = \lambda x \in A \text{ The } y\ \underline{\mathrm{u}}'xy \to \text{on } A \text{ is } f \wedge \cap x \in A\ \underline{\mathrm{u}}'x.fx)$.

The uniqueness is trivial. □

1.20. The natural numbers

*.0 $(\mathrm{N} \equiv \cap A;(0 \in A \wedge \cap x \in A((x \vee \mathrm{sng}\ x) \in A))\ A)$

.1 $(\text{The set of naturalnumbers} \equiv \mathrm{N})$

.2 $(\mathrm{scsr}\ x \equiv (x \vee \mathrm{sng}\ x))$

.3 (The successor of $x \equiv$ scsr x)

.4 (naturalnumberclass is $A \equiv (0 \in A \wedge \cap x \in A(\text{scsr } x \in A)))$

.5 $(1 \equiv \text{scsr } 0)$

.6 $(2 \equiv \text{scsr } 1)$

.7 $(3 \equiv \text{scsr } 2)$

.8 $(4 \equiv \text{scsr } 3)$

.9 $(5 \equiv \text{scsr } 4)$

.10 $(6 \equiv \text{scsr } 5)$

.11 $(7 \equiv \text{scsr } 6)$

.12 $(8 \equiv \text{scsr } 7)$

.13 $(9 \equiv \text{scsr } 8)$

From these definitions we immediately deduce

.14 (naturalnumberclass is N)

.15 $(\mathrm{N} = \cap A;$ (naturalnumberclass is A) $A)$,

the first three *Peano properties*

.16 $(0 \in \mathrm{N})$

.17 $(n \in \mathrm{N} \rightarrow \text{scsr } n \in \mathrm{N})$

.18 $(n \in \mathrm{N} \rightarrow \text{scsr } n \neq 0)$

and the *Theorem of Induction*

.19 $(0 \in S \subset \mathrm{N} \wedge \cap x \in S(\text{scsr } x \in S) \rightarrow S = \mathrm{N})$.

Amongst the innumerable consequences of this last theorem are the lemmas

.20 $(m \in \mathrm{N} \wedge n \in \mathrm{N} \wedge m \in n \rightarrow m \subset n)$

.21 $(m \in \mathrm{N} \rightarrow \sim(m \in m))$

.22 $(m \in \mathrm{N} \wedge n \in \mathrm{N} \rightarrow \sim(m \in n \wedge n \in m))$,

the last two of which enable us to prove the final Peano property

.23 $(m \in \mathrm{N} \wedge n \in \mathrm{N} \rightarrow \text{scsr } m = \text{scsr } n \leftrightarrow m = n)$.

Other useful applications of (1.20.19) are

.24 $(m \in \mathrm{N} \rightarrow m = 0 \vee \text{One } n \in \mathrm{N}\ (m = \text{scsr } n))$

.25 $(m \in \mathrm{N} \wedge n \in \mathrm{N} \rightarrow (m = n \vee m \neq n))$

.26 $(m \in \mathrm{N} \wedge n \in \mathrm{N} \to m \in n \leftrightarrow \mathrm{scsr}\ m \in \mathrm{scsr}\ n)$

.27 $(m \in \mathrm{N} \wedge n \in \mathrm{N} \to m \in n \leftrightarrow m \subset\!\cdot\ n)$

.28 $(\mathrm{N} = \nabla \mathrm{N})$

and the more general induction theorem

.29 $(m \in S \subset \mathrm{N} \sim m \wedge \cap x \in S(\mathrm{scsr}\ x \in S) \to S = \mathrm{N} \sim m)$

- from which we easily obtain the *Theorem of Precedence*

.30 $(n \in \mathrm{N} \sim 1 \to \mathrm{scsr}\ \nabla n = n)$

With the help of the above results we can now develop the theory of orderings on N from the definitions

.32 $(m \leq n \equiv m \in \mathrm{N} \wedge n \in \mathrm{N} \wedge m \in \mathrm{scsr}\ n)$

.33 $(m < n \equiv m \in \mathrm{N} \wedge n \in \mathrm{N} \wedge m \in n)$.

We omit the details.

1.21. Wellfounded sets

Our concept of 'wellfounded set' has its origins in Richman's definition of 'constructive ordinal' [16].

.0 $(\text{wellfounded is } A \equiv \cap S \subset A(\cap x \in A(x \cap A \subset S \to x \in S) \to S = A))$

.1 $(\text{transitive is } A \equiv \cap x \in A\ \cap y \in A\ \cap z \in A(x \in y \wedge y \in z \to x \in z))$

.2 $(\text{inducive is } A \equiv \cap x \in \mathrm{scsr}\ A(\text{wellfounded is } x \wedge \nabla x \subset x))$

Before discussing further the none-too-transparent definition (1.21.0), we mention

.3 $(\text{wellfounded is } A$

$\to \sim \cup f(\text{on N to } A \text{ is } f \wedge \cap n \in \mathrm{N}(.f\ \mathrm{scsr}\ n \in .fn))$

Proof. Let

$(S = \mathrm{E}x \in A \sim \cup f(\text{on N to } A \text{ is } f \wedge .f0 = x \wedge \cap n \in \mathrm{N}(.f\ \mathrm{scsr}\ n \in .fn)))$

Then

$(x \in A \wedge x \cap A \subset S$

$\to (\text{on N to } A \text{ is } f \wedge .f0 = x \wedge \cap n \in \mathrm{N}(.f\ \mathrm{scsr}\ n \in .fn) \wedge$

$g = \lambda n \in \mathrm{N}.f\ \mathrm{scsr}\ n$

$\to \text{on N to } A \text{ is } g \wedge .g0 \in x \cap A \subset S \wedge \cap n \in \mathrm{N}(.g\ \mathrm{scsr} n \in .gn)$

$\to 0)$

$\rightarrow \sim \bigcup f(\text{on N to } A \text{ is } f \wedge .f0 = x \wedge \bigcap n \in \text{N}(.f \text{ scsr } n \in .fn))$

$\rightarrow x \in S)$,

whence

$(S = A)$. □

.4 (wellfounded is $\text{U} \rightarrow \sim \bigcup f(\text{on N is } f \wedge \bigcap n \in \text{N}(.f \text{ scsr } n \in .fn)))$

Proof. This follows from (1.21.3) and the fact that

(on N is $f \rightarrow$ on N to U is f). □

.5 (wellfounded is N)

Proof. Let

$(S \subset \text{N} \wedge \bigcap x \in \text{N}(x \cap \text{N} \subset S \rightarrow x \in S))$

and apply (1.20.19) to show that

$(S = \text{N})$. □

.6 ($n \in \text{N} \rightarrow$ wellfounded is n)

Proof. It is trivial that

(wellfounded is 0).

Suppose that

($n \in \text{N} \wedge$ wellfounded is n)

and let

$(S \subset \text{scsr } n \wedge \bigcap x \in \text{scsr } n(x \cap \text{scsr } n \subset S \rightarrow x \in S))$.

We first note that, by (1.20.28), (1.21.3) and (1.21.5),

$(x \in \text{scsr } n \rightarrow x \cap \text{scsr } n = x \cap n)$.

Hence

$(x \in n \wedge x \cap n \subset S \cap n$

$\rightarrow x \in \text{scsr } n \wedge x \cap \text{scsr } n \subset S$

$\rightarrow x \in S)$,

so that

$(S \cap n = n)$.

Moreover,

$(n \in \text{scsr } n \wedge n \cap \text{scsr } n = n = S \cap n \subset S)$,

so that

$(n \in S)$.

Thus, clearly,

$(S = \text{scsr } n)$.

Reference to (1.20.19) completes the proof. □

This last result may also be obtained as a consequence of the interesting general theorem.

.7 (wellfounded is $A \wedge$ transitive is $A \wedge B \subset A \rightarrow$ wellfounded is B)

Proof. Let

$(T \subset B \wedge \bigcap x \in B(x \cap B \subset T \rightarrow x \in T))$

and define

$(S = \mathrm{E}x \in A(B \cap \text{scsr } x \subset T))$.

Then

$(x \in A \wedge x \cap A \subset S \wedge y \in B \wedge y \in \text{scsr } x$

$\rightarrow (t \in y \cap B \rightarrow t \in B \subset A \wedge (t \in y \in x \vee t \in y = x)$

$\rightarrow t \in x \cap A$

$\rightarrow t \in S$

$\rightarrow t \in B \cap \text{scsr } t \subset T)$

$\rightarrow y \cap B \subset T$

$\rightarrow y \in T)$,

whence

$(x \in A \wedge x \cap A \subset S$

$\rightarrow \bigcap y(y \in B \cap \text{scsr } x \rightarrow y \in T)$

$\rightarrow B \cap \text{scsr } x \subset T$

$\rightarrow x \in S)$.

It follows that

$(S = A)$,

and therefore that

$(x \in B \rightarrow x \in A = S$

$\rightarrow x \in B \cap \text{scsr } x \subset T$

$\rightarrow x \in T)$.

Hence

$(B = T)$

and the proof is complete. □

.8 $(S \subset \mathrm{N} \rightarrow \text{wellfounded is } S)$

Proof. Use (1.21.5), (1.20.20), (1.20.27) and (1.21.7). □

.9 (inducive is N)

Proof. Use (1.21.5), (1.21.6), (1.20.20) and (1.20.28). □

We now look a little more closely at our choice of definition (1.21.0). To begin with, we note that
$(\text{wellfounded is } A \leftrightarrow \sim \cup f(\text{on N to } A \text{ is } f \wedge \cap n \in \mathrm{N}(.f \text{ scsr } n \in .fn))$
is a classical theorem. For, supposing that

$$(S \subset A \wedge \cap x \in A(x \cap A \subset S \rightarrow x \in S) \wedge a \in A \sim S),$$

and arguing classically, we have

$$(a \cap A \subset S \rightarrow a \in A \sim S \wedge a \in S \rightarrow 0),$$

whence

$$(\cup b(b \in a \wedge b \in A \sim S)).$$

From this and the classical axiom of choice ([14], 2.5.8) it follows that

$$(\cup f(\text{on N to } A \text{ is } f \wedge .f0 = a \wedge \cap n \in \mathrm{N}(.f \text{ scsr } n \in .fn)),$$

which, together with (1.21.3), establishes the above proposition.

It is precisely because of this, and the fact that the classical idea of a set A being wellfounded is that

$$(\sim \cup f(\text{on N to } A \text{ is } f \wedge \cap n \in \mathrm{N}(.f \text{ scsr } n \in .fn)),$$

that we make our definition (1.21.0).

In view of these remarks and theorems (1.21.3) - (1.21.9), we consider it reasonable to postulate as our constructive

Axiom of Foundation

.10 $(((S \subset A) \wedge \cap x(((x \in A) \wedge ((x \wedge A) \subset S)) \rightarrow (x \in S))) \rightarrow (S = A))$.

The foregoing now immediately yield

.11 (wellfounded is A)

.12 $(\sim \cup f(\text{on N is } f \wedge \cap n \in \mathrm{N}(.f\text{'scsr } n \in .fn)))$

and

.13 (inducive is $A \leftrightarrow \cap x \in \mathrm{scsr}\ A(\nabla x \subset x)$).

The main reason for our introduction of (1.21.10) as an axiom is to simplify the statements and proofs of the theorems in section 1.22. However, should any doubt be cast on the validity of (1.21.10) as a constructive principle, we can recover the results of section 1.22 in a form sufficiently strong for all practical purposes by building the necessary wellfoundedness into the hypotheses; in particular, because the proofs of (1.21.5) - (1.21.9) do not depend in any way on (1.21.10), we can certainly obtain the everyday recursion theorems from such a modification of 1.22 (cf. section 1.23).

It may well be asked why we do not simply adopt the axiom of foundation in its familiar classical form

$$(x \in A \rightarrow \cup y(y \in A \wedge y \cap A = 0))$$

(equivalent to Morse's axiom ([1], 2.5.9)). One reason for our preferring (1.21.10) as an axiom is that it applies so neatly to the proofs in section 1.22; however, a far more cogent consideration in our preference is that the above classical axiom of foundation is essentially non-constructive! To see this, let

$$(A = (\{1\} \cup (x \cup \sim x)))$$

and suppose that the above classical axiom obtains. Then as

$$(1 \in A)$$

we have

$$(\cup y(y \in A \wedge y \cap A = 0)).$$

But

$$(\sim A = \sim\{1\} \cap \sim x \cap \sim\sim x = 0),$$

so that

$$(y \cap A = 0 \rightarrow y \subset \sim A = 0 \rightarrow y = 0)$$

and therefore

$$(0 \in A).$$

As

$(\sim(0 \in \{1\}))$

we conclude that

$(0 \in (x \cup \sim x))$,

from which (using (1.8.0) we obtain the unwanted theorem

(omniscience).

1.22. Recursion

It is in this section more than any other that the full power and beauty of our approach to set theory is revealed. We here develop a very general recursion theorem from which - in section 1.23 - we shall obtain the familiar recursion theorems as special cases. All this is only made possible by the definitions

.0 (Induced Rxy $\underline{u}'xy$ on A

$\equiv$ (relation is $R \wedge$ dmn $R \subset A \wedge \cap x \in A$(vs Rx = st strc Rx y $\underline{u}'xy$))

.1 (R is induced on A by $\underline{u}'xy$ in x and y $\equiv$ Induced Rxy $\underline{u}'xy$ on A)

.2 (Ndc Axy $\underline{u}'xy$ $\equiv$ The R(Induced Rxy $\underline{u}'xy$ on A))

We begin with two lemmas:

.3 (Induced Rxy $\underline{u}'xy$ on A $\wedge$ Induced Sxy $\underline{u}'xy$ on $A \rightarrow R = S$)

Proof. Let

$(T = \text{E}x \in A(\text{vs } Rx = \text{vs } Sx))$.

Then

$(x \in A \wedge x \cap A \subset T$

$\rightarrow (y \in \text{strc } Rx \rightarrow \cup p \cup q(y = p,q \in R \wedge p \in x \cap \text{dmn } R \subset x \cap A \subset T)$

$\rightarrow \cup p \cup q(y = p,q \in R \wedge \text{vs } Rp = \text{vs } Sp \wedge p \in x)$

$\rightarrow \cup p \cup q(y = p,q \in S \wedge p \in x)$

$\rightarrow y \in \text{strc } Sx)$

$\rightarrow \text{strc } Rx \subset \text{strc } Sx)$.

Similarly

$(x \in A \wedge x \cap A \subset T \rightarrow \text{strc } Sx \subset \text{strc } Rx)$,

so that

$(x \in A \wedge x \cap A \subset T$

$\rightarrow \text{strc } Rx = \text{strc } Sx$

$\rightarrow \text{vs } Rx = \text{st strc } Rx\ y\ \underline{u}'xy = \text{st strc } Sx\ y\ \underline{u}'xy = \text{vs } Sx$

$\rightarrow x \in T)$.

Thus (1.21.10)

$(T = A)$

and

$(x \in A \rightarrow \text{vs } Rx = \text{vs } Sx)$,

from which the result is almost immediate. □

.4 (Induced $Rxy\ \underline{u}'xy$ on $A \rightarrow R = \text{Ndc } Axy\ \underline{u}'xy)$

Proof. Use (1.22.3) and (1.19.3). □

With these behind us we can now prove

.5 *The General Recursion Theorem*

(inducive is $A \rightarrow$ Induced $Rxy\ \underline{u}'xy$ on $A \leftrightarrow R = \text{Ndc } Axy\ \underline{u}'xy)$

Proof. Let

$((R = \text{Ndc } Axy\ \underline{u}'xy) \wedge \cap a(\underline{u}a = \text{Ndc } axy\ \underline{u}'xy) \wedge$

$(S = \text{E}x,y(x \in A \wedge y \in \underline{u}'x\underline{u}x)))$.

Then

1° (relation is $S \wedge \text{dmn } S \subset A \wedge \cap x \in A(\text{vs } Sx = \underline{u}'x\ \underline{u}x))$

Next we have

$(a \in A \wedge x \in a \rightarrow x \subset a$

$\rightarrow \text{strc}(\text{strc } Sa)x = \text{strc } S(a \cap x) = \text{strc } Sx)$.

In view of this and the (easily verified) statement

$(x \in a \rightarrow \text{vs}(\text{strc } Sa)x = \text{vs } Sx)$,

it follows that

$(a \in A \wedge \cap x \in a(\text{vs } Sx = \underline{u}'x \text{ strc } Sx)$

$\rightarrow$ relation is strc $Sa \wedge \text{dmn strc } Sa \subset a \wedge$

$\cap x \in a(\text{vs}(\text{strc } Sa)x = \underline{u}'x \text{ strc}(\text{strc } Sa)x)$

$\rightarrow$ Induced $(\text{strc } Sa)xy\ \underline{u}'xy$ on $a))$.

Thus (1.22.4)

2° $(a \in A \wedge \cap x \in a(\mathrm{vs}\ Sx = \underline{\mathrm{u}}'x\ \mathrm{strc}\ Sx) \rightarrow \mathrm{strc}\ Sa = \underline{\mathrm{u}}a)$.

We now let

$(T = \mathrm{E}a \in A \cap x \in a(\mathrm{vs}\ Sx = \underline{\mathrm{u}}'x\ \mathrm{strc}\ Sx))$.

Then, as

$(a \in A \rightarrow a \subset A)$,

we have (by 2° and 1°)

$(a \in A \wedge a \cap A \subset T \wedge x \in a$

$\rightarrow x \in a \subset T$

$\rightarrow x \in A \wedge \cap y \in x(\mathrm{vs}\ Sy = \underline{\mathrm{u}}'y\ \mathrm{strc}\ Sy)$

$\rightarrow x \in A \wedge \mathrm{strc}\ Sx = \underline{\mathrm{u}}x$

$\rightarrow \mathrm{vs}\ Sx = \underline{\mathrm{u}}'x\ \underline{\mathrm{u}}x = \underline{\mathrm{u}}'x\ \mathrm{strc}\ Sx)$

Hence

$(a \in A \wedge a \cap A \subset T$

$\rightarrow \cap x \in a(\mathrm{vs}\ Sx = \underline{\mathrm{u}}'x\ \mathrm{strc}\ Sx)$

$\rightarrow a \in T)$,

from which - via (1.21.10) - we deduce that

$(T = A)$.

From this, 1° and 2° we obtain

$(a \in A \rightarrow \mathrm{strc}\ Sa = \underline{\mathrm{u}}a$

$\rightarrow \mathrm{vs}\ Sa = \underline{\mathrm{u}}'a\ \underline{\mathrm{u}}a = \underline{\mathrm{u}}'a\ \mathrm{strc}\ Sa)$,

whence (again with the help of 1°) we have

(Induced Sxy $\underline{\mathrm{u}}'xy$ on A).

Reference to (1.22.4) completes the proof. □

1.23. The familiar recursion theorems

.0 (ndc HA $\equiv$ Ndc Axy sng.Hy)

.1 (on A,f is induced by H $\equiv$

(function is H^ inducive is A^ on A is f^$\cap x \in A$(.fx = .Hstrcfx))

.2 (ndc'ha $\equiv$ ndc $\lambda g(g=0 \wedge a \vee g \neq 0 \wedge .h.g\nabla$ dmn $g)$N)

.3 (sequence is S ≡ on N is S)

.4 (sqnc A ≡ ES(on N to A is S))

.5 (sequence A ≡ sqnc A)

.6 (ndc″Sa ≡ ndc λg(g=0 ∧ a ∨ g≠0 ∧ ..S∇ dmn g .g∇ dmn g)N)

.7 *The Simple Recursion Theorem*

(on A to A is h ∧ a ∈ A

→ on N to A is f∧.f0 = a∧∩n ∈ N(.f scsr n = .h.fn)↔f=ndc′ha))

Proof. For convenience, we let

(H = λg(g=0 ∧ a ∨ g≠0 ∧ .h.g∇ dmn g)),

so that, in particular,

(ndc′ha = ndc HN).

Suppose to begin with that

(on N to A is f∧.f0 = a ∧ ∩n ∈ N(.f scsr n = .h.fn))

Then

(.f0 = a = .H0 = .H strc f0)

and

(0 ∈ n ∧ n ∈ N

→ (0,a) ∈ strc fn ∧ dmn strc fn = n

→ .H strc fn = .h.strc fn ∇n = .h.f ∇n = .fn).

With reference to (1.18.4) and (1.21.9) it is now clear that

(on N,f is induced by H),

and therefore that

(Induced fxy sng.Hy on N).

Hence (1.22.5)

(f = ndc′ha).

On the other hand, with

(f = ndc′ha),

we have (1.22.5)

(relation is f ∧ dmn f ⊂ N ∧ ∩n ∈ N(vs fn = sng.H strc fn))

Then, as

$(t \in \mathrm{vs}\ f0$

$\leftrightarrow t \in \mathrm{sng}.H\ \mathrm{strc}\ f0$

$\leftrightarrow t = .H\ \mathrm{strc}\ f0 = .H0 = a \in \mathrm{U}$

$\leftrightarrow t = a)$,

we see that

$(\mathrm{dmn\ strc}\ f1 = 1 \wedge \cap x \in 1(\mathrm{singleton\ is\ vs}\ fx \wedge .fx = a \in A))$.

Now suppose that

$(0 \in n \wedge n \in \mathrm{N} \wedge \mathrm{dmn\ strc}\ fn = n \wedge$

$\cap x \in n(\mathrm{singleton\ is\ vs}\ fx \wedge .fx \in A))$.

Then

$(.H\ \mathrm{strc}\ fn = .h.\mathrm{strc}\ fn\ \nabla \mathrm{dmn\ strc}\ fn$

$= .h.\mathrm{strc}\ fn\ \nabla n$

$= .h.f\ \nabla n)$,

whence

$(.H\ \mathrm{strc}\ fn = .h.f\ \nabla n \in A)$.

It follows that

$(n \in \mathrm{dmn\ strc}\ f\ \mathrm{scsr}\ n \wedge \mathrm{singleton\ is\ vs}\ fn \wedge .fn = .h.f\nabla n \in A)$,

from which - via our induction hypothesis - we deduce that

$(\mathrm{dmn\ strc}\ f\ \mathrm{scsr}\ n = \mathrm{scsr}\ n \wedge$

$\cap x \in \mathrm{scsr}\ n(\mathrm{singleton\ is\ vs}\ fx \wedge .fx \in A))$.

Thus (1.20.29)

$(\cap n \in \mathrm{N}\sim 1(\mathrm{dmn\ strc}\ fn = n \wedge \cap x \in n(\mathrm{singleton\ is\ vs}\ fx \wedge .fx \in A)))$

so that

$(\mathrm{on\ N\ to}\ A\ \mathrm{is}\ f)$.

Moreover, it is clear from the above argument that

$(.f0 = a \wedge \cap n \in \mathrm{N}(.f\ \mathrm{scsr}\ n = .h.f\nabla \mathrm{scsr}\ n = .h.fn))$.

The proof is therefore complete. □

.8 $(S \in \mathrm{sqnc\ map}\ AA \wedge a \in A$

$\rightarrow \mathrm{on\ N\ to}\ A\ \mathrm{is}\ f \wedge .f0 = a \wedge \cap n \in \mathrm{N}(.f\ \mathrm{scsr}\ n = ..Sn.fn)$

$\leftrightarrow f = \mathrm{ndc}''Sa)$

.9 *The Primitive Recursion Theorem*

$(\text{on } N,,A \text{ to } A \text{ is } h \wedge a \in A$

$\rightarrow \text{on } N \text{ to } A \text{ is } f \wedge .f0 = a \wedge \cap n \in N(.f \text{ scsr } n = .h(n,.fn))$

$\leftrightarrow f = \text{ndc}'' \lambda n \in N\ \lambda x \in A.h(n,x)a)$

Proof. Use (1.23.8) □

<u>Remark</u>: With these theorems, the arithmetic of natural numbers can be built up in the usual way; we shall not give the familiar details in this work. ®

It will be noticed that there is, as yet, no guarantee that our 'functions defined by recursion' in (1.23.7), (1.23.8) and (1.23.9) are constructively well-defined: indeed, it is clear from (1.16.31) and (1.17.18) that these functions are constructively well-defined if and only if this is true of the set N. Accordingly, we adopt as an axiom, and as a further expression of the leading role played by the natural numbers in constructive mathematics, the *Axiom of Content*:

.10 $(N \in U)$.

As just noted, we now have

.11 $(\text{on } A \text{ to } A \text{ is } h \wedge a \in A \rightarrow \text{ndc}'ha \in U)$

.12 $(S \in \text{sqnc map } AA \wedge a \in A \rightarrow \text{ndc}''Sa \in U)$

.13 $(\text{on } N,,A \text{ to } A \text{ is } h \wedge a \in A \rightarrow \text{ndc}'' \lambda n \in N\ \lambda x \in A.h(n,x)a \in \mathbf{U})$

We are now in a position to derive Morse's *axiom of infinity* ([14], 2.5.6) - namely

.14 $(U = \cup c \cap x(c \wedge (c \in U) \wedge ((x \in c) \rightarrow \text{sng } x \in c)))$

Proof. By (1.12.7), (1.18.7), (1.23.7) and (1.23.11),

$(f = \text{ndc}'\lambda x \text{ sng } x\ 0$

$\rightarrow \text{on } N \text{ is } f \wedge .f0 = 0 \wedge \cap n \in N(.f \text{ scsr } n = \text{sng}.fn) \wedge f \in U)$

whence (1.16.31)

$(c = \text{rng ndc}'\lambda x \text{ sng } x\ 0$

$\rightarrow 0 \in c \wedge c \in U \wedge (x \in c \rightarrow \text{sng } x \in c)$

$\to \cap x(0 \in c \wedge c \in \mathrm{U} \wedge (x \in c \to \operatorname{sng} x \in c)))$.

Thus

$(\cup c\ \cap x(0 \in c \wedge c \in \mathrm{U} \wedge (x \in c \to \operatorname{sng} x \in c))$,

from which the result readily follows. □

Remark: The purpose of (1.23.14) in Morse's classical system is to prove the theorems

$(x \in \mathrm{U} \to \operatorname{sng} x \in \mathrm{U})$

and

$(\mathrm{N} \in \mathrm{U})$.

We prefer to treat these as two axioms, replacing Morse's one - to do so seems more in keeping with the spirit of [1]. ®

1.24. The mapping set axiom

Our next axiom - the *Mapping Set Axiom*,

.0 $(A \in \mathrm{U} \wedge B \in \mathrm{U} \to \mathrm{E}f(\text{on } A \text{ to } B \text{ is } f) \in \mathrm{U})$

- is our constructive substitute for the classical *Power Set Axiom*

$(A \in \mathrm{U} \to \mathrm{E}x(x \subset A) \in \mathrm{U})$,

to the constructive application of which several authors (for example, Myhill [15]) have raised serious objections.

To make precise our own criticism of the Power Set Axiom, let us first note that

'A set is not an entity which has an ideal existence. A set exists only when it has been defined. To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set....
A similar remark applies to the definition of a function: in order to define a function from a set A to a set B, we prescribe a finite routine which leads from an element of A to [a unique] element of B...' ([1], page 2).

(Actually, Bishop requires more of us than this: in the case of a set, we must also prescribe 'what we must do in order to show that two elements of the set are equal'. As equality is an absolute, defined notion in our formal theory, and not prescribed for each set - as in Bishop's approach - we omit this extra requirement of his in the definition of a set.)

Now Church's Thesis suggests - although we must stress that it does not prove - that we have a clear idea of how to construct a mapping of N into N, and therefore that, according to the above criterion of Bishop, the set map NN is constructively well-defined. (No element of map NN is known which is generally accepted as 'constructive' and yet, not being general recursive, contradicts Church's Thesis.)

On the other hand, no such tentative general characterisation of constructively defined subsets of N is known or, we believe, likely to be found. For this reason, and until such time as a characterisation of constructive subsets of N be found, we are led to exclude $Ex(x \subset N)$ from our canon of constructively well-defined sets.

It may be argued that, although our criticism of the Power Set Axiom certainly provides good grounds for its rejection, nevertheless we have not, by one favourable example, produced a strong enough case for the acceptance of the Mapping Set Axiom. One way of meeting this objection is to restrict ourselves still further, to the adoption of the axiom

$$(\text{map NN} \in U).$$

It is interesting to note that, were we to take this approach, we could deduce the theorems

$$(N \in U),$$

$$(\cap n \in \mathrm{N}(\mathrm{map}\ n\mathrm{N} \in \mathrm{U}))$$

and

$$(\cap m \in \mathrm{N}\ \cap n \in \mathrm{N}(\mathrm{map}\ mn \in \mathrm{U})).$$

To derive the first of these, we simply remark that

$$(\mathrm{N} = \mathrm{rng}\ \lambda f \in \mathrm{E}f\ \cup n \in \mathrm{N}(f = \lambda x \in \mathrm{N}\ n).f0)$$

and refer to (1.17.18) and (1.16.31). To prove the second, we have

$$(n \in \mathrm{N} \wedge h \in \mathrm{map}\ n\mathrm{N} \wedge g = \lambda x \in \mathrm{N}(x \in n \wedge .hx \vee \sim(x \in n) \wedge 0)$$
$$\rightarrow g \in \mathrm{map}\ \mathrm{NN} \wedge \mathrm{strc}\ gn = h),$$

whence, clearly,

$(\mathrm{on}\ \mathrm{map}\ \mathrm{NN}\ \mathrm{onto}\ \mathrm{map}\ n\mathrm{N}\ \mathrm{is}\ \lambda g \in \mathrm{map}\ \mathrm{NN}\ \mathrm{strc}\ gn)$

- from which, again by (1.17.18) and (1.16.31), we obtain our result. Finally, the third of the above statements follows from the second, the fact that

$$(m \in \mathrm{N} \wedge n \in \mathrm{N} \rightarrow \mathrm{map}\ mn = \mathrm{E}f \in \mathrm{map}\ m\mathrm{N}(\mathrm{rng}\ f \subset n)),$$

and (1.12.4).

Unfortunately, the axiom

$$(\mathrm{map}\ \mathrm{NN} \in \mathrm{U})$$

is too restrictive, in that it does not allow us to construct as elements of our universe such objects as 'the set of all finite subsets of A' (where A is constructively well-defined): to do this, we appear to need at least the axiom

$$(A \in \mathrm{U} \rightarrow \mathrm{map}\ \mathrm{N}A \in \mathrm{U}),$$

coupled with (1.23.10). But even this is not strong enough to enable us to construct, for example, the set of all continuous mappings of R into R as an element of the universe, where R is the set of all real numbers. It is precisely such considerations as these which serve to confirm our original resolve to adopt the Mapping Set Axiom (1.24.0) as our constructive substitute for the Power Set Axiom of classical mathematics.

Having said all this, let us now make the convenient definition

.1 $(\text{sb } A = \text{E}x(x \subset A))$,

and comment further on the relation between the Power Set Axiom and (1.24.0). To begin with, we note that with the Power Set Axiom we can deduce the converse to (1.12.6) - that is

$$(\nabla A \in \text{U} \rightarrow A \in \text{U}).$$

For we then have

$$(\nabla A \in \text{U} \rightarrow A \subset \text{sb } \nabla A \in \text{U} \rightarrow A \in \text{U}).$$

In fact, as an axiom or theorem,

$$(\nabla A \in \text{U} \rightarrow A \in \text{U})$$

is equivalent to the Power Set Axiom: for the former gives

$$(A \in \text{U} \rightarrow \nabla \text{ sb } A = A \in \text{U} \rightarrow \text{sb } A \in \text{U}).$$

Note also that, in any event, the converse of the Power Set Axiom is true within our present system:

.2 $(\text{sb } A \in \text{U} \rightarrow A = \nabla \text{sb } A \in \text{U})$

Next, we remark that (1.24.0) can be deduced as a theorem with the aid of the Power Set Axiom: for, with the latter,

$$(A \in \text{U} \wedge B \in \text{U} \rightarrow \nabla \text{ map } AB = A,,B \in \text{U} \rightarrow \text{map } AB \in \text{U}).$$

Moreover, (1.24.0) is classically equivalent to the Power Set Axiom: for, given that

$$(\text{omniscience} \wedge A \in \text{U})$$

is a theorem, and setting

$$(S = \text{E}f \cup X \in \text{sb } A\ (f =_{\!-} \lambda x \in A(x \in X \wedge 1 \vee \sim(x \in X) \wedge 0))),$$

we have

$$(S \subset \text{E}f(\text{on } A \text{ to } \{0\ 1\} \text{ is } f) \in \text{U}),$$

whence (1.24.0)

$$(S \in \text{U}).$$

As

$$(g = \lambda f \in S\ \text{E}x \in A(.fx = 1) \rightarrow \text{on } S \text{ onto sb } A \text{ is } g)$$

we conclude from (1.17.18) and (1.16.31) that

$$(\text{sb } A \in \text{U}).$$

To end this section, we show how Morse's axiom of replacement ([14], 2.5.7) - from which he is able to derive

$$(A \in \mathrm{U} \to \mathrm{sb}\ A \in \mathrm{U})$$

as a theorem - fits into our system:

.3 $(\text{omniscience} \to \cup x((A \in \mathrm{U} \wedge x \subset A) \wedge (\underline{\mathrm{u}}x \in \mathrm{U}) \wedge \underline{\mathrm{u}}x)) \in \mathrm{U})$

Proof. Let

$$(S = \cup x((A \in \mathrm{U} \wedge x \subset A) \wedge (\underline{\mathrm{u}}x \in \mathrm{U}) \wedge \underline{\mathrm{u}}x)).$$

Then

$$\begin{aligned}(\sim(A \in \mathrm{U}) &\to (A \in \mathrm{U}) = 0\\ &\to (A \in \mathrm{U} \wedge x \subset A \wedge \underline{\mathrm{u}}x \in \mathrm{U} \wedge \underline{\mathrm{u}}x) \subset (A \in \mathrm{U}) = 0\\ &\to S = \cup x\ 0 = 0\\ &\to S \in \mathrm{U}).\end{aligned}$$

On the other hand, it is readily shown that

$$(A \in \mathrm{U} \to S = \cup x(x \in \mathrm{sb}\ A \wedge \underline{\mathrm{u}}x \in \mathrm{U} \wedge \underline{\mathrm{u}}x)).$$

Moreover,

$$\begin{aligned}(\underline{\mathrm{u}}x \in \mathrm{U} &\to (\underline{\mathrm{u}}x \in \mathrm{U}) = \mathrm{U}\\ &\to (\underline{\mathrm{u}}x \in \mathrm{U} \wedge \underline{\mathrm{u}}x) \in \mathrm{U})\end{aligned}$$

and

$$\begin{aligned}(\sim(\underline{\mathrm{u}}x \in \mathrm{U}) &\to (\underline{\mathrm{u}}x \in \mathrm{U}) = 0\\ &\to (\underline{\mathrm{u}}x \in \mathrm{U} \wedge \underline{\mathrm{u}}x) = 0 \in \mathrm{U}).\end{aligned}$$

Thus

$$(\text{omniscience} \to \cap x \in \mathrm{sb}\ A((\underline{\mathrm{u}}x \in \mathrm{U} \wedge \underline{\mathrm{u}}x) \in \mathrm{U})).$$

It follows from the remarks preceding this theorem, and (1.12.0), that

$$(\text{omniscience} \wedge A \in \mathrm{U} \to S \in \mathrm{U}).$$

The proof is completed by one further application of the Principle of Omniscience. □

1.25. Families of sets

.0 $(\text{finite is } A \equiv \cup f(\text{univalent is } f \wedge \text{dmn } f \in \mathrm{N} \wedge \text{rng } f = A))$

.1 $(\text{fnt} \equiv \mathrm{E}A \text{ finite is } A)$

.2 $(\text{subfinite is } A \equiv \cup f(\text{function is } f \wedge \text{dmn } f \in \mathrm{N} \wedge \text{rng } f = A))$

.3 $(\text{subfnt} \equiv \mathrm{E}A \text{ subfinite is } A)$

.4 $(\text{denumerable is } A \equiv \cup f(\text{on N onto } A \text{ is } f \wedge \text{univalent is } f))$

.5 $(\text{denmbl} \equiv \mathrm{E}A \text{ denumerable is } A)$

.6 $(\text{countable is } A \equiv \cup f(\text{upon N onto } A \text{ is } f))$

.7 $(\text{cbl} \equiv \mathrm{E}A \text{ countable is } A)$

Note that - in contrast to Bishop's usage - the empty set is both subfinite and countable.

.8 $(\text{fnt} \subset \text{subfnt} \subset \text{cbl})$

.9 $(\text{denmbl} \subset \text{cbl})$

.10 $(A \in \mathrm{U} \rightarrow \text{subfnt} \cap \text{sb } A \in \mathrm{U})$

Proof. As (1.24.0)

$(\cap n \in \mathrm{N}\ (\text{map } nA \in \mathrm{U}))$

we have (1.12.0)

$(\cup n(n \in \mathrm{N} \wedge \text{map } nA) \in \mathrm{U})$.

The result now follows from (1.18.4), (1.17.18), (1.16.31) and the fact that

$(\text{subfnt} \cap \text{sb } A = \text{rng } \lambda f \in \cup n(n \in \mathrm{N} \wedge \text{map } nA)\text{rng } f)$. □

.11 $(A \in \mathrm{U} \rightarrow \text{fnt} \cap \text{sb } A \in \mathrm{U})$

.12 $(A \in \mathrm{U} \rightarrow \text{denmbl} \cap \text{sb } A \in \mathrm{U})$

Note that

$(A \in \mathrm{U} \rightarrow \text{cbl} \cap \text{sb } A \in \mathrm{U})$

as a theorem or axiom is equivalent to

$(\text{sb N} \in \mathrm{U})$.

This is a consequence of the theorems

$(\mathrm{N} \in \mathrm{U} \wedge (\text{sb N} = \text{cbl} \cap \text{sb N}))$,

$(\text{cbl} \cap \text{sb } A = \text{rng } \lambda f \in \cup S \in \text{sbN map } SA \text{ rng } f)$

More generally, if we were to accept the power set axiom as a theorem, then we would be able to derive (1.25.10), (1.25.11), (1.25.12) and

$(A \in \mathrm{U} \to \mathrm{cbl} \cap \mathrm{sb}\ A \in \mathrm{U})$

quite simply, using (1.12.4).

The remainder of this section is intended as preparation for the next, on Borel sets.

.13 $(\text{inhabited is } A \equiv \cup x(x \in A))$

.14 $(\mathrm{htd} \equiv \mathrm{E}\, A \text{ inhabited is } A)$

.15 $(\nabla' F \equiv \cup G \in \mathrm{subfnt} \cap \mathrm{sb}\ F\ \mathrm{sng}\ \nabla G)$

.16 $(\Pi' F \equiv \cup G \in \mathrm{htd} \cap \mathrm{subfnt} \cap \mathrm{sb}\ F\ \mathrm{sng}\ \Pi G)$

.17 $(\nabla'' F \equiv \cup G \in \mathrm{cbl} \cap \mathrm{sb}\ F\ \mathrm{sng}\ \nabla G)$

.18 $(\Pi'' F \equiv \cup G \in \mathrm{htd} \cap \mathrm{cbl} \cap \mathrm{sb}\ F\ \mathrm{sng}\ \Pi G)$

.19 $(\nabla^- F \equiv \cup G \in \mathrm{sb}\ F\ \mathrm{sng}\ \nabla G)$

.20 $(\Pi^- F \equiv \cup G \in \mathrm{htd} \cap \mathrm{sb}\ F\ \mathrm{sng}\ \Pi G)$

The extra condition of being inhabited is necessary in (1.25.16), (1.25.18), and (1.25.20) to avoid the appearance of $\Pi 0$ (equal to U).

.21 $(F \subset \nabla' F \subset \nabla'' F \subset \nabla^- F)$

.22 $(F \subset \Pi' F \subset \Pi'' F \subset \Pi^- F)$

.23 $(0 \in \nabla' F)$

.24 $(F \subset G \to \nabla'' F \subset \nabla'' G)$

.25 $(F \subset G \to \Pi'' F \subset \Pi'' G)$

1.26. Borel sets

We believe that, barring the fact that Bishop's definition applies to 'complemented sets' ([1], chapter 3, section 2), the following captures the spirit of his inductive approach to Borel sets.

.0 $(\mathrm{Borel}\ F \equiv \cap B;(F \subset B = \nabla'' B = \Pi'' B)B)$

Of the following theorems, perhaps the most important in practice are (1.26.3) and (2.26.7), each of which illustrates the inductive nature of definition (1.26.0).

.1 $(F \subset \text{Borel } F)$

.2 $(F \subset B = \nabla''B = \Pi''B \to \text{Borel } F \subset B)$

.3 $(F \subset S = \nabla''S = \Pi''S \subset \text{Borel } F \to S = \text{Borel } F)$

.4 $(\text{Borel } F = \nabla''\text{Borel } F = \Pi''\text{Borel } F)$

Proof. By (1.25.24), (1.25.25) and (1.26.2),

$(F \subset B = \nabla''B = \Pi''B$

$\to \text{Borel } F \subset B$

$\to \nabla''\text{Borel } F \subset \nabla''B = B \wedge \Pi''\text{Borel } F \subset \Pi''B = B)$,

whence

$(\nabla''\text{Borel } F \subset \cap B(F \subset B = \nabla''B = \Pi''B \to B) \wedge$

$\Pi''\text{Borel } F \subset \cap B(F \subset B = \nabla''B = \Pi''B \to B))$,

and the result follows by (1.26.0), (1.25.21) and (1.25.22).□

.5 $(S \subset \text{Borel } F \to \nabla''S \subset \text{Borel } F \wedge \Pi''S \subset \text{Borel } F)$

.6 $(\text{Borel } F = \hat{\cap} B;(F \subset B \wedge \cap S \subset B(\nabla''S \subset B \wedge \Pi''S \subset B))B)$

Proof. By (1.25.21), (1.25.22), (1.25.24) and (1.25.25) it is clear that

$(\cap S \subset B(\nabla''S \subset B \wedge \Pi''S \subset B) \leftrightarrow B = \nabla''B = \Pi''B)$.

The result follows almost immediately. □

.7 $(F \subset X \subset \text{Borel } F \wedge \cap S \subset X(\nabla''S \subset X \wedge \Pi''S \subset X) \to X = \text{Borel } F)$

.8 $(F \subset \text{sb } X \to \text{Borel } F \subset \text{sb } X)$

Proof. Using (1.25.24), (1.25.25), (1.25.21), (1.25.22) and (1.26.4), we readily see that

$(S = \text{Borel } F \cap \text{sb } X$

$\to \nabla''S \subset \nabla''\text{Borel } F \cap \nabla''\text{sb } X \subset S \wedge$

$\Pi''S \subset \Pi''\text{Borel } F \cap \Pi''\text{sb } X \subset S$

$\to S = \nabla''S = \Pi''S)$.

The proof is completed with reference to (1.26.1) and (1.26.6). □

.9 (Borel $F \subset$ sb ∇F)

In many ways it would be satisfactory to have

($F \in$ U $\rightarrow$ Borel $F \in$ U)

as a theorem. In view of (1.26.9), this will certainly be the case if we accept the power set axiom. On the other hand, despite Bishop's one mention of the set of borel subsets ([1], page 183, definition 1), it appears that, in practice, we only need individual borel sets, or countable families of borel sets, and not the set Borel F of all such sets. Moreover, Bishop's 'improved' constructive measure theory [4] does not seem to bother with borel sets at all. For these reasons, we remain content without

($F \in$ U $\rightarrow$ Borel $F \in$ U)

as a theorem.

1.27. Tuples

The constructive theory of tuples follows the classical development almost exactly. Of particular notational interest are the definitional schemas (1.27.6) and (1.27.7) and theorem (1.27.18)

.0 (bsdmn $x \equiv$ Et(bsvs $xt \in$ htd))

.1 (tuple is $x \equiv$

(basicrelation is $x \wedge \cap t \in$ bsdmn x (bsvs xt = ss ∇bsvs xt)))

.2 (tuple a is $x \equiv$ (tuple is $x \wedge$ bsdmn x = a))

.3 (crd $tx \equiv \nabla$bsvs xt)

.4 (The t coordinate of $x \equiv$ crd tx)

.5 (bstrc $xa \equiv x \cap$ (a ⤏ U))

.6 *Definitional schema*

We accept as a definition each expression obtained from

'$((x, \in y) \equiv (\text{tuple is } x \wedge \cap t \in \text{bsdmn } x\ (\text{crd}tx \in y)))$'

by replacing '$\in$' by a nexus which is not a comma.

.7 *Definitional schema*

We accept as a definition each expression obtained from

'(Each coordinate of $x \in y \equiv (x, \in y)$)'

by replacing '$\in$' by a nexus which is not a comma.

.8 $((x,x',x'') \equiv ((x,x') \cup (\text{sng } 2 \nearrow\!\!\nearrow \text{ss } x'')))$

.9 $((x,x',x'',x''') \equiv ((x,x',x'') \cup (\text{sng } 3 \nearrow\!\!\nearrow \text{ss } x''')))$

These last two definitions clearly generalise.

.10 $(\text{tuple } a \text{ is } x \wedge \text{tuple } a \text{ is } y \rightarrow x = y \leftrightarrow \cap t \in a(\text{crd}tx = \text{crd}ty))$

.11 $(x = \cup t \in a(\text{sng}t \nearrow\!\!\nearrow \text{ss } \underline{u}t) \rightarrow \text{tuple } a \text{ is } x \wedge \cap t \in a(\text{crd}tx = \underline{u}t))$

.12 $(\text{crd}'x = \text{crd } 0x \wedge \text{crd}''x = \text{crd } 1x)$

.13 $(p = 0 \leftrightarrow \text{tuple } 0 \text{ is } p \wedge \cap t(\text{crd } tp = 0))$

.14 $(p = \text{sng } 0 \nearrow\!\!\nearrow \text{ss } x \rightarrow \text{tuple } 1 \text{ is } p \wedge \text{crd } 0p = x)$

.15 $(p = x,x' \rightarrow \text{tuple } 2 \text{ is } p \wedge \text{crd } 0p = x \wedge \text{crd } 1p = x')$

.16 $(p = x,x',x'' \rightarrow \text{tuple } 3 \text{ is } p \wedge \text{crd } 0p = x \wedge \text{crd } 1p = x' \wedge \text{crd } 2p = x'')$

.17 $(m \in \mathrm{N} \wedge n \in \mathrm{N} \wedge m \leq n \wedge \text{tuple } n \text{ is } x$
$\rightarrow \text{tuple } m \text{ is bstrc } xm \wedge \cap t \in m(\text{crd } t \text{ bstrc } xm = \text{crd } tx))$

What does an N-tuple look like? The answer is given by the next two theorems, which also reveal the pardonable confusion between sequences and N-tuples in informal mathematics.

.18 $(X = \mathrm{E}x(\text{tuple N is } x \wedge \cap n \in \mathrm{N}(\text{crd } nx \in A)) \wedge f = \lambda x \in X \lambda n \in \mathrm{N}\,\text{crd}nx$
$\rightarrow \text{univalent is } f \wedge \text{on } X \text{ onto sqnc } A \text{ is } f)$

.19 $(\text{sequence is } S \wedge X = \mathrm{E}x(\text{tuple N is } x \wedge \cap n \in \mathrm{N}(\text{crd } nx \in .Sn)) \wedge$
$f = \lambda x \in X\ \lambda n \in \mathrm{N}\ \text{crd } nx$
$\rightarrow \text{univalent is } f \wedge \text{on } X \text{ onto } \mathrm{E}g(\text{sequence is } g \wedge \cap n \in \mathrm{N}(.gn \in .Sn) \text{ is } f)$

Finally, we have two illustrations of the usefulness of (1.27.6) and (1.27.7).

.20 $\quad (x,y, \in A \leftrightarrow x \in A \wedge y \in A)$

.21 $\quad (\cap x,y,z, \in A\ \underline{u}''xyz \leftrightarrow \cap x \in A\ \cap y \in A\ \cap z \in A\ \underline{u}''xyz)$

1.28. Axioms of Choice

'When a classical mathematician claims he is a constructivist, he probably means he avoids the axiom of choice. This axiom is unique in its ability to trouble the conscience of the classical mathematician, but in fact it is not a real source of the unconstructivities of classical mathematics. A choice function exists in constructive mathematics, because a choice is implied by the very meaning of existence.' ([1], page 9).

Perhaps it is the vagueness of these remarks of Bishop that led Myhill to write: 'there is not a single use of the axiom of choice in [1]. This is truly extraordinary; in the introduction to his book, a mathematician defends the use of a certain axiom, and then doesn't use it in hundreds of pages. What _is_ used over and over again is the principle that if for every element x of a certain set A there is determined a _unique_ y such that $\phi(x,y)$ then there is a function f defined on A such that $\phi(x,f(x))$ for all x in A.' [15]. (This last principle is just our Theorem of Unique Choice (1.19.5). Note, incidentally, that Myhill uses the standard notation $f(x)$ where we would write $.fx$)

In fact, as Bishop is vague, so Myhill is wrong! Over and over again Bishop's informal analysis in [1] uses not only unique choice but also dependent choice; we shall see examples of this later (cf. section 2.1). We should note, however, that Myhill has shown in [15] that a general axiom of choice contradicts Church's Thesis; we therefore trust that such a general principle will not be required in our constructive analysis.

To completely cover the situations discussed in [1], we are therefore forced to postulate the *Axiom of Dependent Choice:*

.0 $(a \in A \wedge \cap x(x \in A \rightarrow \cup y(y \in A \wedge \underline{\mathrm{u}}'xy))$

$\rightarrow \cup f(\text{on N to } A \text{ is } f \wedge .f0 = a \wedge \cap n(n \in \mathrm{N} \rightarrow \underline{\mathrm{u}}'.fn.f \text{ scsr } n)))$

This leads to

.1 $(a \in A \wedge \cap n \in \mathrm{N} \sim 1 \cap x \in \text{map } nA \cup y \in A\ \underline{\mathrm{u}}'xy$

$\rightarrow \cup f \in \text{sqnc } A(.f0 = a \wedge \cap n \in \mathrm{N} \sim 1\ \underline{\mathrm{u}}'\text{strc } fn.fn))$

Proof. Let

$(B = \mathrm{E}n, f(n \in \mathrm{N} \sim 1 \wedge f \in \text{map } nA))$

and

$(\cap p \cap q(\underline{\mathrm{v}}'pq =$

$(\text{st}(\text{crd}''p,.\text{crd}''q\ \text{crd}'p)x,y\ \underline{\mathrm{u}}'xy \wedge \text{crd}'q = \text{scsr crd}'p \wedge$

$\text{crd}''p = \text{strc crd}''q\ \text{crd}'p)).$

Then

$(n \in \mathrm{N} \sim 1 \wedge x \in \text{map } nA \wedge y \in A \wedge \underline{\mathrm{u}}'xy \wedge$

$q = (\text{scsr } n, \lambda t \in \text{scsr } n(t \in n \wedge .xt \vee t = n \wedge y))$

$\rightarrow q \in B \wedge \underline{\mathrm{v}}'(n,x)q),$

so that

$(\cap p \in B \cup q \in B\ \underline{\mathrm{v}}'pq).$

With the help of (1.28.0) we can clearly construct a term g so that

$(\text{on N} \sim 1 \text{ to } B \text{ is } g \wedge .g1 = (1,\{(0,a)\}) \wedge$

$\cap n \in \mathrm{N} \sim 1\ \underline{\mathrm{v}}'.gn.g \text{ scsr } n).$

A simple application of (1.20.29) then proves that

$(\cap n \in \mathrm{N} \sim 1(\text{dmn crd}''.gn{=}n \wedge \text{crd}''.gn{=}\text{strc crd}''.g\text{scsr}n\ n))$

from which - again via (1.20.29) - we readily obtain

$(f = \lambda n \in \mathrm{N}.\mathrm{crd}''.g \text{ scsr } n\ n$

$\to \text{on N to } A \text{ is } f \wedge .f0 = a \wedge \cap n \in \mathrm{N} \sim 1\ \underline{\mathrm{u}}'\mathrm{strc}\ fn.fn)$. □

Remark: In fact (1.28.1) is equivalent to (1.28.0): for if (1.28.1) obtains it is easy to show that

$(a \in A \wedge \cap x \in A \cup y \in A\ \underline{\mathrm{u}}'xy$

$\to \cup f \in \mathrm{sqnc}\ A(.f0 = a \wedge \cap n \in \mathrm{N} \sim 1\ \underline{\mathrm{u}}'.(\mathrm{strc}\ fn)\nabla\ \mathrm{dmn\ strc}\ fn.fn))$,

from which (1.28.0) follows almost immediately. ®

.2 *The Principle of Countable Choice*

$(\cap n \in \mathrm{N} \cup x \in A\ \underline{\mathrm{u}}'nx \to \cup f \in \mathrm{sqnc}\ A \cap n \in \mathrm{N}\ \underline{\mathrm{u}}'n.fn))$,

Proof. Let

$(B = \mathrm{E}n,x(n \in \mathrm{N} \wedge x \in A \wedge \underline{\mathrm{u}}'nx))$.

Then it is clear that

$(\cup a \in A(0,a \in B) \wedge \cap p \in B \cup q \in B(\mathrm{crd}'q = \mathrm{scsr\ crd}'p))$,

whence (1.28.0) there exists a term g such that

$(g \in \mathrm{sqnc}\ B \wedge .g0 = (0,a) \wedge$

$\cap n \in \mathrm{N}(\mathrm{crd}'.g \text{ scsr } n = \mathrm{scsr\ crd}'.gn))$.

To complete the proof we now need only set

$(f = \lambda n \in \mathrm{N}\ \mathrm{crd}''.gn)$. □

Remark: (1.28.0) can be derived from (1.28.2) with the help of the extra *Axiom of Internal Choice*

$(\cap x \in A \cup y \in A\ \underline{\mathrm{u}}'xy \to \cup f \in \mathrm{map}\ AA \cap x \in A\ \underline{\mathrm{u}}'x.fx)$.

I am grateful to Peter Hancock for pointing out the essence of the following proof (and that of (1.28.3) below).

Let

$(a \in A \wedge \cap x \in A \cup y \in A\ \underline{\mathrm{u}}'xy)$.

Then the Axiom of Internal Choice ensures that

$(\cup h \in \mathrm{map}\ AA \cap x \in A\ \underline{\mathrm{u}}'x.hx)$,

whence (1.23.7) there exists a term f such that

$(f \in \mathrm{sqnc}\ A \wedge .f0 = a \wedge \cap n \in \mathrm{N}(.f \text{ scsr } n = .h.fn))$.

For such f, it is clear that

$(\cap n \in \mathrm{N}\ \underline{\mathrm{u}}'.fn.f$ scsr $n)$,

as we required.

However, although 'internal choice' is certainly valid within our system in the special case $A = \mathrm{N}$, we think it unwise to allow it for general sets A: indeed, even in the case $A = \mathrm{R}$ (the set of real numbers), 'internal choice' does not appear to us as a satisfactory constructive principle. ®

Finally, we mention the more general choice principle

.3 (sequence is $T \wedge a \in .T0 \wedge \cap n \in \mathrm{N}\ \cap x \in .Tn\ \cup y \in .T$ scsr $n\ \underline{\mathrm{u}}'xy$

$\rightarrow \cup f$(sequence is $f \wedge .f0 = a \wedge \cap n \in \mathrm{N}(.fn \in .Tn \wedge \underline{\mathrm{u}}'.fn.f$ scsr $n))$

Proof. Let

$(A = \mathrm{E}n,x(n \in \mathrm{N} \wedge x \in .Tn))$.

Then

$(y \in A \rightarrow \cup n \in \mathrm{N}\ \cup x \in .Tn(y = n,x)$

$\rightarrow \cup n \in \mathrm{N}\ \cup x \in .Tn\ \cup x' \in .T$ scsr $n(y = n,x \wedge \underline{\mathrm{u}}'xx')$

$\rightarrow \cup z \in A(\mathrm{crd}'z =$ scsr $\mathrm{crd}'y \wedge \underline{\mathrm{u}}'\ \mathrm{crd}''y\ \mathrm{crd}''z))$,

whence (1.28.0) there exists a term h such that

$(h \in \mathrm{sqnc}\ A \wedge .h0 = 0,a \wedge$

$\cap n \in \mathrm{N}(\mathrm{crd}'.h$ scsr $n =$ scsr $\mathrm{crd}'.hn \wedge \underline{\mathrm{u}}'\mathrm{crd}''.hn\mathrm{crd}''.h$ scsr $n)$

A straightforward application of (1.20.29) now shows that

$(\cap n \in \mathrm{N}(\mathrm{crd}'.hn = n \wedge \mathrm{crd}''.hn \in .Tn \wedge \underline{\mathrm{u}}'\mathrm{crd}''.hn\ \mathrm{crd}''.h$ scsr$n)$

whence

$(f = \lambda n \in \mathrm{N}\ \mathrm{crd}''.hn$

$\rightarrow$ sequence is $f \wedge .f0 = a \wedge$

$\cap n \in \mathrm{N}(.fn \in .Tn \wedge \underline{\mathrm{u}}'.fn.f$ scsr $n))$. □

1.29. Concluding remarks on set theory

We do not intend to pursue further the detailed formalisation of constructive analysis within the system described above. Suffice it to say that our axioms and theorems of construction ensure that the set R of real numbers is constructively well-defined (of particular importance in this context are axioms (1.23.10) and (1.24.0), and that certain concepts in [1] become much easier to handle within the formal system: for example, a formal proof that the empty set in a metric space is not located is quite simple, whereas an informal one is rather difficult to describe precisely. Another situation which is clarified by our formalisation appears at the start of the next chapter of this work (section 2.1).

We should also mention the 'counterexamples in the style of Brouwer' so beloved of constructive mathematicians. These are best handled by means of the definitions

.0 $(\text{limniscience} \equiv \cap S \in \text{sqnc } N\ (\cap n \in N(.Sn = 0) \vee \cup n \in N(.Sn \neq 0)))$

.1 (The limited principle of omniscience $\equiv$ limniscience)

A 'Brouwer counterexample' to the proposition p then becomes simply a proof that

$(p \rightarrow \text{limniscience})$

- a much more precise description than that usually made in such cases (cf. [1], page 26).

Finally, we refer the reader to Appendices 1 and 2, in which we gather together the axioms of set theory for convenient reference, and sketch briefly an axiomatic approach to proof theory within the above formal system.

CHAPTER 2

PROLOGUE TO ANALYSIS

From this point on (when we turn to the consideration of certain problems in constructive analysis, as distinct from set theory), we shall relax the notational rigidity of Chapter 1, and allow those notations which have become standard in mathematical practice: thus, for example, when we are dealing with functions, we shall use either $f(x)$ or $.fx$, where the spirit of Chapter 1 would restrict us to the latter; moreover, we shall write "$\Rightarrow$" and "$\Leftrightarrow$" instead of the "$\rightarrow$" and "$\leftrightarrow$" of chapter 1, and allow "$\rightarrow$" both its common employments, as a sign for convergence (as in '$\lim_{n\rightarrow\infty} x_n$'), and as one for mappings (as in 'the mapping $T \rightarrow \langle Tx|y\rangle$' where, in such cases, the domain of the mapping is known or understood). We shall also write Q,R,R^{0+},R^{+} for the sets of rational numbers, real numbers, non-negative real numbers and positive real numbers respectively; C for the set of complex numbers; and z^* for the complex conjugate of an element z of C.

Before we pass on to the main body of this chapter, we had better gather together a few important preliminary results on metric spaces. Accordingly, let (E,d) be a metric space, ξ a point of E, and r a positive number. When there is no likelihood of confusion over the metric in question - a caveat which applies to all definitions and notations which make no explicit mention of a metric, norm, scalar product or underlying algebraic structure on which terms under discussion are intimately dependent† - we write $B(\xi,r)$, $\overline{B}(\xi,r)$ respectively for the open ball $\{x \in E: d(\xi,x) < r\}$ and the closed ball $\{x \in E: d(\xi,x) \leq r\}$ in

†If we want to emphasise the dependence on a particular metric d, we can speak of d-completeness, d-compactness, d-continuity,etc.

E with centre ξ and radius r; and $A^{\circ}, \overline{A}$ respectively for the interior and closure of a subset A of E.

If A and B are subsets of E, we write dist(A,B) for the term inf $x,y \in A,,B\ .d(x,y)$ whenever this is constructively well-defined; in the case where $A = \{\xi\}$, we write dist(ξ,B) rather than dist$(\{\xi\},B)$. The subset B of E is said to be *located* (in E) if dist(x,B) is well-defined for each x in E; in which case we write E-B - or, when no confusion is likely, simply $-B$ - for the *metric complement* $\{x \in E\colon 0 < \mathrm{dist}(x,B)\}$ of B in E. On the other hand, the *diameter* of a subset A of E is the term sup $x,y, \in A\ .d(x,y)$ - otherwise written diam A - when this is well-defined.

A mapping f of E into a metric space (E',d') is *uniformly continuous* if there exists a relation ω, with dmn $\omega = \mathrm{R}^{+}$ and rng $\omega \subset \mathrm{R}^{+}$, such that

$$(\cap \varepsilon,\delta \in \omega\ \cap x,y, \in E(d(x,y) \leq \delta \Rightarrow d'(.fx,.fy) \leq \varepsilon)).$$

ω is then called a *modulus of uniform continuity* for f (on E). We commonly abuse notation by writing $\omega(\varepsilon)$ for any δ such that $(\varepsilon,\delta) \in \omega$; this amounts to a notationally convenient - but dispensable - application of 'internal choice'. If f is an injective, uniformly continuous mapping of E onto E' with uniformly continuous inverse, we say that E and E' are *metrically equivalent* and that f is a *metric equivalence* of E with E' (or between E and E'); in particular, if this obtains for $E = E'$ and $f = \lambda x \in Ex$, then d and d' are said to be *equivalent metrics* on E.

Given $\varepsilon > 0$, by an *ε-approximation* to our metric space E we mean an inhabited, subfinite subset S of E such that

$$(\cap x \in E\ \cup y \in S\ (d(x,y) < \varepsilon)).$$

If E has an ε-approximation for each $\varepsilon > 0$, then E is *precompact*; if, in addition, E is complete, we say that it is *compact*. We note that *if f is a uniformly continuous mapping of a precompact metric space E into a metric space E', then ${}_*fE$ is precompact; moreover, if $E' = R$, then* $\sup x \in E\ .fx$ *and* $\inf x \in E\ .fx$ *are well-defined real numbers;*

that

a precompact subset of a metric space is located, and a located subset of a precompact metric space is itself precompact;

and that

a subset of a finite dimensional Banach space is compact if and only if it is closed, located and bounded.

By a *cover* of a set X we mean a family S of subsets of X whose union is X. Of great importance is the proposition:

if E is a compact metric space and $\varepsilon > 0$, then there exists a finite cover of E in which each set is compact, and of diameter less than ε

and the consequent theorem

if E is a compact metric space, and f a uniformly continuous mapping of E into R, then, for all but countably many real numbers $\alpha > \inf x \in E\ .fx$, the set $\{x \in E: .fx \leq \alpha\}$ is compact.

For further results on compactness, we refer the reader to Chapter 4 of [1], from which these last two were extracted.

In the remaining chapters of this thesis we shall be concerned with various problems associated with compactness and local compactness in metric spaces (cf. Chapter 3 for the definition of local compactness). We begin this present chapter with an important result on locatedness (2.1.0), a variant of which - namely, our theorem (2.1.1) - appears, and is incorrectly proved, in Bishop's book ([1], Chapter 6, Lemma 7). From this we are led to two conjectures of considerable importance for later work in Chapters 3 and 4. The chapter ends with some important definitions and a constructive analogue of the classical result that an injective, uniformly continuous mapping of a compact space into a Hausdorff space is a homeomorphism of its domain onto its range.

2.1 Some important results on locatedness.

.0 *Let A be a complete, located subset of a metric space (E,d), and ξ a point of E. Then there exists y in A such that, for each n in N, $2^{-n+2} < d(\xi,y)$ entails $2^{-n-2} <$ dist(ξ,A).*

Proof. For each n in N we have either a proof that $2^{-n-2} < \mathrm{dist}(\xi,A)$ or a proof that $\mathrm{dist}(\xi,A) < 2^{-n-1}$. We may therefore define recursively a mapping δ of N into $\{0,1\}$ such that, for each n,

$\delta(n+1) \leq \delta(n)$,

$\delta(n) = 0 \Rightarrow 2^{-n-2} < \mathrm{dist}(\xi,A)$

and

$\delta(n) = 1 \Rightarrow \mathrm{dist}(\xi,A) < 2^{-n-1}$.

With ζ any point of A, we now construct a sequence $(y_n)_{n\in\mathrm{N}}$ in A so that: if $\delta(0) = 0$, then $y_n = \zeta$ for each n; if $\delta(0) = 1 = \delta(n)$, then y_n is chosen in A so that $d(\xi,y_n) < 2^{-n-1}$; while if $\delta(0) = 1$ and $\delta(n) = 0$, then $y_n = y_m$, where m is that unique integer such that $\delta(m) = 1$ and $\delta(m+1) = 0$. Let p,q be natural numbers with $q \leq p$. Then

$d(y_p,y_q) \leq 2^{-p+1} + 2^{-q+1}$.

For if $\delta(p) = 1$, then $\delta(q) = 1$ and so

$$\begin{aligned} d(y_p,y_q) &\leq d(\xi,y_p) + d(\xi,y_q) \\ &< 2^{-p-1} + 2^{-q-1} \\ &< 2^{-p+1} + 2^{-q+1} \quad ; \end{aligned}$$

if $\delta(p) = 0 = \delta(q)$, then $y_p = y_q$, $d(y_p, y_q) = 0$; while if $\delta(p) = 0$ and $\delta(q) = 1$, then there exists (unique) m such that $q \leq m < p$, $\delta(m) = 1$, $\delta(m+1) = 0$, and

$$d(y_p, y_q) \leq \Sigma_{k=q+1}^{m} d(y_k, y_{k-1})$$

$$< \Sigma_{k=q+1}^{m} (2^{-k-1} + 2^{-k})$$

$$< \Sigma_{k=q}^{\infty} 2^{-k}$$

$$< 2^{-p+1} + 2^{-q+1}$$

It follows that (y_n) is a Cauchy sequence in A, and therefore converges to a point y of A, where $\cap n \in \mathrm{N}(d(y, y_n) \leq 2^{-n+1})$.

Now let n belong to N, and suppose that $2^{-n+2} < d(\xi, y)$.

Then

$$d(\xi, y_n) \geq d(\xi, y) - d(y, y_n)$$

$$> 2^{-n+2} - 2^{-n+1}$$

$$> 2^{-n-1}$$

whence $\delta(n) = 0$, and therefore $2^{-n-2} < \mathrm{dist}(\xi, A)$. □

An immediate corollary of this is

.1 *Let A be a complete, located subset of a metric space (E, d), and ξ a point of E such that $0 < d(\xi, x)$ for each x in A. Then $0 < \mathrm{dist}(\xi, A)$.* □

Remarks: (i) Before finally taking our leave of the formal mathematics of Chapter 1, it is well worth our while to sketch the formal layout of the first part of the proof of (2.1.0) (the construction of the sequence - or, strictly, the N-tuple - $(y_n)_{n \in \mathrm{N}}$). One good reason for doing this is to show up an apparently unavoidable need of the Axiom of Dependent Choice (1.28.0); another is that the formal proof makes the construction of the function δ very much clearer than does the above argument (even in a less terse form). For the duration of this present remark only, we shall revert to the notation of Chapter 1, with the assumption that all the various operations of addition, exponentiation, etc, on real numbers, and such terms as dist(ξ,A), have been defined in the appropriate manner, and have the expected properties.

Let

$$(\phi = \lambda m,n\in\{12\},,\mathrm{N}(m\wedge((m = 1 \wedge \mathrm{dist}(\xi,A) < 2^{-n-1}) \vee (m = 2 \wedge 2^{-n-2} < \mathrm{dist}(\xi,A))))).$$

Then (1.18.6)

$$(\text{on } \{12\},,\mathrm{N} \text{ is } \phi).$$

Moreover, as

$$(\cap n \in \mathrm{N}(2^{-n-2} < \mathrm{dist}(\xi,A) \vee \mathrm{dist}(\xi,A) < 2^{-n-1}))$$

and

$$(\cap x,y, \in \mathrm{R}((x < y \rightarrow (x < y) = \mathrm{U})\wedge(\sim(x < y) \rightarrow (x < y) = 0))),$$

it is clear that

$$(\cap n \in \mathrm{N} \cup m\in\{12\}(.\phi(m,n) = m)).$$

Thus (1.28.2)

$$(\text{on N to } \{12\} \text{ is } g \wedge \cap n \in \mathrm{N}(.\phi(.gn,n) =.gn))$$

for some term g. We now set

$$(h = \lambda m,n \in \mathrm{N},,\mathrm{N} \min(2 - .g \text{ scsr } m,n))$$

$$(\delta = \mathrm{ndc}'' \lambda n \in \mathrm{N}\ \lambda x \in \mathrm{N}.h(n,x)(2 - .g0))$$

$$(\alpha = \text{The } m \in \mathrm{N}(.\delta m = 1 \wedge .\delta \text{ scsr } m = 0)).$$

Then, (1.23.9)

$$(\text{on N to N is } \delta\wedge.\delta 0=2-.g0\wedge\cap n\in\mathrm{N}(.\delta\,\mathrm{scsr} n=\min(2-.g \text{ scsr } n,.\delta n))).$$

Noting that

$$\begin{aligned}&(n \in \mathrm{N} \wedge .\phi(1,n) = 1\\ &\rightarrow 1 = 1 \cap (\mathrm{U} \cap (\mathrm{dist}(\xi,A) < 2^{-n-1}))\cap(0 \cup (2^{-n-2} < \mathrm{dist}(\xi,A)))\\ &\quad = 1 \cap (\mathrm{dist}(\xi,A) < 2^{-n-1})\\ &\rightarrow 0 \in (\mathrm{dist}(\xi,A) < 2^{-n-1})\\ &\rightarrow \mathrm{dist}(\xi,A) < 2^{-n-1})\end{aligned}$$

and that (likewise)

$$(n \in \mathrm{N} \wedge .\phi(2,n) = 2 \rightarrow 2^{-n-2} < \mathrm{dist}(\xi,A)),$$

it is now a straightforward matter to apply the Theorem of Induction (1.20.19), to obtain

$$(\mathrm{rng}\ \delta \subset \{01\} \wedge \cap n \in \mathrm{N}(.\delta\ \mathrm{scsr}\ n \le .\delta n \wedge (.\delta n = 0 \rightarrow 2^{-n-2} < \mathrm{dist}(\xi,A)) \wedge (.\delta n = 1 \rightarrow \mathrm{dist}(\xi,A) < 2^{-n-1})))$$

We also have

$$(.\delta 0 = 1 \rightarrow \cap n \in \mathrm{N}{\sim}1(.\delta n = 0 \rightarrow \alpha \in \mathrm{N} \wedge .\delta\alpha = 1 \wedge .\delta\ \mathrm{scsr}\alpha = 0))$$

and

$$(\cap n \in \mathrm{N} \cap x \in A \cup y \in A((.\delta\ \mathrm{scsr}n = 0 \wedge x = y) \vee (.\delta\ \mathrm{scsr}n = 1 \wedge .d(\xi,y) < 2^{-n-2}))).$$

Now, a simple corollary of (1.28.3) is

$$(a \in \mathrm{A} \wedge \cap n \in \mathrm{N} \cap x \in A \cup y \in A\ \underline{\mathrm{u}}''nxy \rightarrow \cup f \in \mathrm{sqnc}\ A(.f0 = a \wedge \cap n \in \mathrm{N}\ \underline{\mathrm{u}}''n.fn.f\ \mathrm{scsr}n)).$$

(To prove this, we simply set

$$(T = \lambda n \in \mathrm{N}\{n\},,A),$$

apply (1.28.3) to produce a term F such that

$$(\mathrm{sequence\ is}\ F \wedge .F0 = (0,a) \wedge \cap n \in \mathrm{N}(.Fn \in .Tn \wedge \underline{\mathrm{u}}''n\ \mathrm{crd}''.Fn\ \mathrm{crd}''.F\mathrm{scsr}n))$$

and then set

$$(f = \lambda n \in \mathrm{N}\ \mathrm{crd}''.Fn))$$

It should be clear that, with ζ any point of A, there exists a term y such that

$$(y \in \mathrm{sqnc}A \wedge ((.\delta 0 = 0 \wedge .y0 = \zeta) \vee (.\delta 0 = 1 \wedge .d(\xi,.y0) < 2^{-1})) \wedge \cap n \in \mathrm{N}((.\delta\mathrm{scsr}n = 0 \wedge .y\mathrm{scsr}n = .yn) \vee (.\delta\mathrm{scsr}n = 1 \wedge .d(\xi,.y\mathrm{scsr}n) < 2^{-n-2})))$$

To show that such a term y fulfils our requirements for the proof of (2.1.0), it only remains to prove the statements

$$(.\delta 0 = 0 \rightarrow y = \lambda n \in \mathrm{N}\zeta)$$

$$(n \in \mathrm{N} \wedge .\delta n = 1 \rightarrow .d(\xi,.yn) < 2^{-n-1}$$

and

$$(n \in \mathrm{N} \wedge .\delta 0 = 1 \wedge .\delta n = 0 \rightarrow .yn = .y\alpha \in A)$$

- the simple details of whose proofs we shall omit.

(ii) In general, we shall find more use for (2.1.1) than for (2.1.0). For applications of the latter to the constructive theory of connectedness in metric spaces, we refer the reader to Appendix 3. ®

2.2 Two important conjectures.

Being familiar with classical mathematics, and bearing in mind (2.1.0), we are naturally led to

Conjecture 1: If K is a compact subset, and A a complete, located subset, of a metric space E, then there exists y in K such that $0 < \mathrm{dist}(y,A)$ entails $0 < \inf x \in K \; \mathrm{dist}(x,A)$

- the constructive validity of which would certainly be of some practical value (cf. remark following (A3.3.2) in Appendix 3). Unfortunately, neither constructive proof nor 'Brouwer counter-example' is known for this conjecture, even in the weaker form in which A is also compact. We should note, however, that the proposition

if f is a uniformly continuous mapping of a compact metric space K into R^{0+}, then there exists y in K such that $0 < .fy$ entails $0 < \inf x \in K.fx$

- a classically valid generalisation of Conjecture 1 - is essentially non-constructive: for, applied to the function $\lambda y \in K(.fy - \inf x \in K.fx)$, with f any uniformly continuous mapping of K into R^{0+}, it entails that f attains its infimum; which, in turn, entails the limited principle of omniscience (cf. [20], (8.3.2)).

Closely related to Conjecture 1, and clearly a generalisation of (2.1.1), is

Conjecture 2: If f is a uniformly continuous mapping of a compact metric space K into R, and $0 < .fx$ for each x in K, then $0 < \inf x \in K.fx$.

Again, we are unfortunate in having neither proof nor counter-example for this statement; indeed, Bishop (in a private communication) has expressed the opinion that Conjecture 2 'will never be either proved or disproved' within the framework of constructive mathematics.

In spite of this, we can prove Conjecture 2 within the wider system of Brouwer's intuitionistic mathematics. Before doing so, however, we require some of the basic definitions of this system, which we now state in the form most suited to our present purpose (cf. [12], Chapter 3).

By a *spread law* we mean a subset S of $\{f:\cap n\in N\sim 1(\text{on } n \text{ to } N \text{ is } f)\}$ with the property

$$(\cap n\in N((\{0,n\}\in S\vee\{0,n\}\notin S)\wedge\cap f\in S(f\cup\{\text{dmn}f,n\}\in S\vee f\cup\{\text{dmn}f,n\}\notin S))\wedge$$
$$\cap f\in S\cup n\in N(f\cup\{\text{dmn}f,n\}\in S)\wedge$$
$$\cap f\in S\cap n\in N\sim 1(\text{strc } fn\in S)).$$

A *spread* is a pair (S,C) comprising a spread law S and a mapping C with domain S. A sequence f is *admissible* for the spread (S,C) if strc fn belongs to S for each n in $N\sim 1$; in which case $\lambda n\in N\sim 1.C$ strc fn is called an *element of the spread* in question.

A spread (S,C) is *finitary* if

$$((\text{finite is } \{n\in N:\{0,n\}\in S\})\wedge\cap f\in S(\text{finite is } \{n\in N:f\cup\{\text{dmn}f,n\}\in S\})).$$

A finitary spread is also called a *fan*. We shall shortly have need of *Brouwer's Fan Theorem* : if ϕ is a (constructively defined) mapping of the set σ of elements of a finitary spread into N, then there exists a positive integer ν such that

$$(\cap f,g,\in\sigma(\text{strc } f\nu = \text{strc } g\nu \Rightarrow .\phi f = .\phi g)).$$

By a *finitary point representation* of a metric space E, we mean a pair $(\alpha,(S,C))$, where α is a sequence in E and (S,C) a finitary spread such that $.Cf = \alpha:f$ for each f in S; every element of (S,C) converges in E; and every element of E is the limit of some element of (S,C). For our present purpose, it suffices to note that a compact metric space has a finitary point representation ([21], (3.11)). We are now in a position to give an intuitionistic proof of Conjecture 2.

Let $(\alpha,(S,C))$ be a finitary point representation of the compact metric space K, and f a uniformly continuous mapping of K into R with $0 < .fx$ for each x in K. A straightforward application of the Axiom of Choice in the form (acceptable to an intuitionist)

$$(\cap g\in\tau\cup n\in N\ \underline{u}'gn \Rightarrow \cup\phi(\text{on } \tau \text{ to N is } \phi\wedge\cap g\in\tau\ \underline{u}'g.\phi g)),$$

where τ is the set of elements of a spread, enables us to construct a mapping ϕ of the set σ of elements of (S,C) into N~1, such that

$$(\cap g\in\sigma((.\phi g)^{-1} \le .f\ \lim_{n\to\infty}.gn))).$$

Using the Fan Theorem, we now obtain ν in N~1 with the property

$$(\cap g,h,\in\sigma(\text{strc } g\nu = \text{strc } h\nu \Rightarrow .\phi g = .\phi h)).$$

As K is inhabited and (S,C) is finitary, the set

$$\{g\in\sigma:\cup s\in S(g = \text{strc } s\nu)\}$$

is both inhabited and finite, as is therefore rng ϕ. With

$$\delta \equiv \inf g\in\sigma(.\phi g)^{-1},$$

it is now clear that $0 < \delta \le \inf x\in K.fx$, as we required.

Remarks: The uniform continuity of f is superfluous to the needs of the intuitionist, as his mathematics admit a 'proof' that every mapping of a compact metric space K into a separable metric space is uniformly continuous ([21], (3.12)).

It must be stressed that the above intuitionistic proof does not count as a constructive proof as understood by Bishop or ourselves: we do not accept such ideas as the Fan Theorem as principles on which constructive mathematics may be based, any more than we accept the law of excluded middle or the limited principle of omniscience. Nevertheless, *we do not expect to find a constructive situation in which Brouwer's intuitionistic principles fail to hold in retrospect:* thus, for example, if we have a solution (in our sense) of a constructive mathematical problem to which the intuitionist would have applied the Fan Theorem, we expect that what is asserted in the Fan Theorem will be seen to hold good from the vantage point of one who has actually solved the problem - but not necessarily from that of one to whom the solution of the problem is still a mystery. It is for this reason that we do not expect to produce a constructive proof that Conjecture 2 entails the limited principle of omniscience. On the other hand, it is presumably the similarity between the above intuitionistic proof of Conjecture 2 and that of the uniform continuity of real-valued functions defined on a compact metric space that has led Bishop to the conclusion that Conjecture 2 'will never be either proved or disproved' in constructive mathematics. ®

2.3. Metric injectiveness.

We conclude this chapter by discussing a situation in which a proof of Conjecture 2 would be valuable. But first we need several definitions.

Let (E,d) and (E',d') be metric spaces. A subset A of E is called a *compact image* in E if there exists a compact metric space K and a uniformly continuous mapping of K onto A. A mapping f of E into E' is *continuous* if it is uniformly continuous 'near' each compact image A in E, in the following precise sense:

$$(\cap\varepsilon\in R^{+}\cup\delta\in R^{+}\cap x\in A\cap y\in E(d(x,y)\leq\delta\Rightarrow d'(.fx,.fy)\leq\varepsilon)).$$

This definition is made to coincide with the definitions given in [1] for continuity on compact or locally compact metric spaces, and to ensure both that the composition of two continuous functions is continuous, and that a continuous function is 'pointwise' continuous in the usual sense. The definition first appears in [2]

If there exists an injective, continuous mapping f of E onto E' with continuous inverse, then we say that E and E' are *homeomorphic* metric spaces, and that f is a *homeomorphism* of E on E' (or 'between E and E' '). In particular, if $E = E'$ and the mapping $\lambda x \in E\ x$ is a homeomorphism of E on itself, then d and d' are said to be *homeomorphic metrics* on E.

A mapping f of E into E' is *precontinuous* if it is uniformly continuous on each compact image in E. Clearly, a continuous mapping is precontinuous, and every precontinuous mapping on a compact metric space is continuous.

A precontinuous mapping f of E into E' is *metrically injective* if, for any compact subsets A,B of E with $0 < \mathrm{dist}(A,B)$, we have $0 < \mathrm{dist}({}_*fA, {}_*fB)$. (Note that we need precontinuity of f in this definition to ensure that $\mathrm{dist}({}_*fA, {}_*fB)$ is well-defined.) Finally, a mapping (precontinuous or otherwise) f of E into E' is *metrically weak-injective* if

$$(\cap x \in E\ \cap y \in E(0 < d(x,y) \Rightarrow 0 < d'(.fx,.fy)))$$

It is comparatively trivial to show that, were Conjecture 2 valid, the concepts of metric injectiveness and metric weak-injectiveness would coincide for precontinuous functions; moreover, these concepts are equivalent for the intuitionist.

The fundamental result on metric injectiveness is

.0 *Let (E,d) be a compact metric space and f a uniformly continuous, metrically injective mapping of E into a metric space (E',d'). Then* inv f *is uniformly continuous on ${}_*fE$, and ${}_*fE$ is compact.*

Proof. Given $\varepsilon > 0$, we construct a finite cover $\{K_0,\ldots,K_\nu\}$ of E in which each set is compact, and has diameter less than $\varepsilon/3$. There exists r in R such that $0 < r < \varepsilon/3$, and

$S_j \equiv \{x \in E : r \leq \text{dist}(x,K_j)\}$

is compact or empty for each j in N~scsr ν.

Without loss of generality, we suppose that each S_j is compact, and then set

$\delta_j \equiv \text{dist}({}_*fS_j, {}_*fK_j)$,

$\delta \equiv \min\{\delta_0,\ldots,\delta_\nu\}$.

Then, as $0 < r \leq \text{dist}(S_j,K_j)$ for each j in N~scsrν and f is metrically injective, we see that $0 < \delta$. Choosing x and y in E such that $d'(.fx,.fy) \leq \delta/2$, and then j in N~scsrν with y in K_j, we now clearly have $\text{dist}(.fx, {}_*fK_j) < \delta_j$, $\text{dist}(x,K_j) \leq r$, and therefore $d(x,z) < 2r$ for some z in K_j. It follows that

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(y,z) \\ &< 2r + \text{diam } K_j \\ &< 2\varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

This proves uniform continuity of inv f on ${}_*fE$. It is now comparatively trivial to prove ${}_*fE$ complete, and therefore compact. □

Remark: It might be feared that (2.3.0) would lead to a contradiction of the well-known 'Brouwer counterexamples' to the classical theorem that every uniformly continuous mapping of a compact metric space into R attains its supremum and infimum. To remove this fear, we point out that the propositions

a. for each ζ in R^{0+} it is decidable whether or not $\lambda x \in [0,1]\ \zeta x$ is metrically injective.

b. every uniformly continuous mapping of [0,1] into R attains its supremum and infimum

and

c. for each x in R^{0+}, either $x > 0$ or $x = 0$

are equivalent to each other, and therefore to the limited principle of omniscience (cf. [1], page 26). ®

CHAPTER 3

LOCALLY COMPACT SPACES AND THEIR COMPACTIFICATIONS

By a *locally compact space* we mean an inhabited metric space E with the property: for each bounded subset B of E, there exists a compact subset K of E with $B \subset K$. Such a space is both separable and complete. Moreover, a mapping of a locally compact space E into a metric space E' is continuous if and only if it is uniformly continuous on each compact subset of E - or, equivalently, on each bounded subset of E; in which case, it carries bounded subsets of E into bounded subsets of E'.

The main concern of this chapter is to demonstrate the existence and uniqueness (up to metric equivalence) of a useful embedding of a given locally compact space as a subset of a compact metric space - the constructive analogue of the Alexandrov compactification of classical topology (Sections 3.3 - 3.5). Some of our theorems are mentioned - and, in the case of (3.3.7), proved in detail - by Bishop ([1], Chapter 4); we have included these results for one or more of the following reasons: for the sake of completeness (as with the results of Section 3.1); as motivation for later work (3.3.0); and (as with (3.3.7)) because our proof has substantial improvements on that given by Bishop.

Section 3.2 appears as an interpolation in the main body of the chapter; however, its results - and especially (3.2.2) - will be of considerableuse in Chapter 5, and should not be overlooked.

We conclude this introduction with a remark on notation: if A,B are subsets of a metric space (E,d), we shall write $d(A,B)$ - rather than $\text{dist}(A,B)$ - when we wish to make no mistake about the metric under consideration.

3.1. Compactifiers

By a *compactifier* for the metric space E we mean a continuous mapping f of E into R such that ${}^*f(]-\infty,r])$ is bounded for each real number r. In the constructive theory of locally compact spaces, compactifiers play a particularly important role, the performance of which depends on

.0 *If f is a compactifier for the locally compact space E, then* $\inf x \in E.fx$ *is well-defined; and, for all but countably many real numbers* $r > \inf x \in E.fx$, *the set* ${}^*f(]-\infty,r])$ *is compact.*

Proof. With a any point of E, K a compact subset of E such that ${}^*f(]-\infty,.fa+1]) \subset K$, $m \equiv \inf x \in K.fx$, and x any point of E, we have either $.fx < .fa + 1$ - in which case $x \in K$, and $m \leq .fx$ - or $m \leq .fa < .fx$. Thus $\inf x \in E.fx$ exists, and equals m. Without loss of generality, we suppose that $m < 1$. With $(K_n)_{1 \leq n}$ a sequence of compact subsets of E such that

$\cap n \in N{\sim}1({}^*f(]-\infty,n]) \subset K_n)$,

we see that, for each positive integer n,

$\inf x \in K_n.fx = m$,

and there exists a sequence $(c_{nk})_{k\in N}$ in $]m,\infty[$ with the property: $K_n \cap {}^*f(]-\infty,r])$ is compact whenever $r \in R$, $m < r \leq n$, and $r \neq c_{nk}$ for each k. As

${}^*f(]-\infty,r]) = K_n \cap {}^*f(]-\infty,r])$

for each r in $]-\infty,n]$, we conclude that ${}^*f(]-\infty,r])$ is compact whenever $r \in R$, $m < r$, and $r \neq c_{nk}$ for each n and k. □

Using this result, we can prove ([1], Chapter 4, Proposition 13):

.1 *A locally compact subset of a metric space is closed and located; and a closed, located subset of a locally compact space is itself locally compact.* □

Remark: It might be thought that we require the subset under consideration in the second part of (3.1.1) to be also inhabited. However, this requirement is superfluous, as a located subset of an inhabited metric space (E,d) must itself be inhabited: for, given any point a of E, we can find x in S with $d(a,x) < \mathrm{dist}(a,S)+1$. ®

3.2. Applications to dimensionality

An important consequence of (3.1.1) is ([1], Chapter 9, Proposition 5):

.0 *A closed ball in a finite dimensional Banach space is compact.* □

The converse of this is proved along the familiar classical lines (cf.[8], (5.9.4)):

.1 *If the closed unit ball of a normed linear space E is compact - or, equivalently, if E is locally compact - then E is finite dimensional.* □

As an application of (3.2.0), we shall derive simple, but interesting, alternative criteria of finite and infinite dimensionality for Hilbert spaces, in terms of orthonormal bases. But before doing so we had better make quite clear exactly what we mean by the terms 'infinite dimensional', and 'orthonormal basis'.

A normed linear space $(E,\|\ \|)$ is *infinite dimensional* if, whenever V is a finite dimensional subspace of E, there exists x in E such that

$$0 < \inf y \in V \|x-y\| .$$

On the other hand, a sequence $(a_n)_{1\leq n}$ in the Hilbert space $(H,\langle\ |\rangle)$ is an *orthonormal basis* for H if it is *orthonormal*- in the sense that $\langle a_m|a_n\rangle = 0$ whenever $m \neq n$; and, for each n, either $a_n = 0$ or $\|a_n\| = 1$ - and each x in H has a unique representation of the form

$$x = \Sigma_{n=1}^{\infty} x_n a_n$$

with each x_n in C, and $x_n = 0$ when $a_n = 0$. It can be shown ([1], Chapter 9, Theorem 7) that every (separable)† Hilbert space $(H, \langle\,|\,\rangle)$ has an orthonormal basis, and that, for each orthonormal basis $(a_n)_{1\leq n}$ for H and each x, y in H,

$$x = \Sigma_{n=1}^{\infty} \langle x | a_n \rangle a_n$$

and

$$\langle x|y\rangle = \Sigma_{n=1}^{\infty} \langle x|a_n\rangle \langle a_n|y\rangle .$$

With these definitions behind us, we can now prove

.2 *Let* $(a_n)_{1\leq n}$ *be an orthonormal basis in the Hilbert space* $(H, \langle\,|\,\rangle)$. *Then*

(a) *in order that* H *be infinite dimensional, it is necessary and sufficient that there exist a strictly increasing sequence* $(n_k)_{k\in\mathrm{N}}$ *of positive integers such that* $\|a_{n_k}\| = 1$ *for each* k.

(b) *in order that* H *be finite dimensional, it is necessary and sufficient that there exist* ν *in* N *such that* $a_n = 0$ *for* $n \geq \nu$.

Proof. We first prove that: if V is a finite dimensional subspace of H, then there exists ν in N~1 such that $0 < \inf x \in V\|x - a_n\|$ whenever $n \geq \nu$ and $\|a_n\| = 1$. To do this, we let $\{x_0, \ldots, x_m\}$ be a 1/2-approximation to the unit ball $V_1 \equiv \{x \in V : \|x\| \leq 1\}$ of V (3.2.0). For each k in scsr m, as

$$\|x_k\|^2 = \Sigma_{j=1}^{\infty} |\langle x_k|a_j\rangle|^2,$$

we may choose ν_k in N~1 so that $|\langle x_k|a_n\rangle| \leq 3/8$ whenever $n \geq \nu_k$. With P the projection of H on V,

† It should be remembered that, in constructive analysis, all Banach and Hilbert spaces are separable by definition; indeed, Bishop 'knows of no constructively defined nonvoid metric space that can be proved to be nonseparable' ([1], Appendix A).

$\nu \equiv \max(\nu_0,\ldots,\nu_m)$, and $n \geq \nu$, we now suppose that $\|a_n\| = 1$. Then $.Pa_n \in V_1$, and so there exists k in scsr m with $\|Pa_n - x_k\| < 1/2$. For this same k, we have

$$\begin{aligned} \|x_k - a_n\|^2 &= \|x_k\|^2 - 2\mathrm{Re}\langle x_k|a_n\rangle + \|a_n\|^2 \\ &\geq 1 - 2\mathrm{Re}\langle x_k|a_n\rangle \\ &> 1/4 \end{aligned}$$

(where 'Re' denotes 'real part of'), whence

$$\begin{aligned} \inf x \in V\|a_n - x\| &= \|a_n - .Pa_n\| \\ &\geq \|a_n - x_k\| - \|x_k - .Pa_n\| \\ &> 0 \end{aligned}$$

as we required.

There is now no difficulty in proving 'sufficiency' in (a), and the non-trivial part of (b). The proof of 'necessity' in (a) follows the lines of the Gram-Schmidt construction in Theorem 8, Chapter 9 of [1], and is omitted here. □

3.3. Compactifications

We now turn to the main subject of this chapter, the embedding of locally compact spaces in compact spaces. Our first theorem - which appears without proof on page 103 of [1] - is important both in its own right, and as motivation for certain later parts of this section.

.0 *Let E be an inhabited set, the metric complement of a located subset Y of a locally compact space (X,d). Let h be a compactifier for (X,d), and g the mapping $\lambda x \in E(.hx + d(x,Y)^{-1})$ of E into* R. *Then E is locally compact with respect to the metric*

$$d_0 \equiv \lambda x,y \in E,,E(d(x,y) + |.gx - .gy|),$$

and g is a compactifier for (E,d_0).

Proof. To begin with, we note that, for real numbers r,s with $s \geq r$, ${}^*g(]-\infty,r]) \subset {}^*h(]-\infty,s])$; that

${}^{*}h(]-\infty,r])$ is a compact subset of X for all but countably many $r > c$, where $c \equiv \inf x \in X.hx$ (3.1.0); and that ${}^{*}g(]-\infty,r])$ is nonempty for all large enough r. Choosing a in E, and then r in R so that

$\max(c, .ha + d(a,Y)^{-1}) < r$,

we can therefore find $s > r$ with ${}^{*}h(]-\infty,s])$ compact. As

$\sup x \in {}^{*}h(]-\infty,s]).d(x,Y) \geq d(a,Y) \geq (s - c)^{-1}$,

there exists α such that $0 < \alpha < (s - c)^{-1}$ and

$A \equiv {}^{*}h(]-\infty,s]) \cap \{x : d(x,Y) \geq \alpha\}$

is compact (cf. introduction to chapter 2). Then $A \subset E = X - Y$. Moreover, for $r \leq t \leq s$, ${}^{*}g(]-\infty,t]) \subset A$, so that

${}^{*}g(]-\infty,t]) = \{x \in A : .gx \leq t\} = {}^{*}\text{strc } gA(]-\infty,t])$.

Now,

$\inf x \in A\ .gx \leq .ga < s$,

so that we may choose real t with $r < t < s$ and ${}^{*}\text{strc } gA\ (]-\infty,t])$ (nonempty and) compact - that is, ${}^{*}g(]-\infty,t])$ compact. Moreover, for such t, it is readily seen that g is uniformly continuous on ${}^{*}g(]-\infty,t])$ with respect to the metric d.

It is now clear that we can construct a strictly increasing sequence $(r_k)_{k\in N}$ of positive integers greater than c such that, for each k, ${}^{*}g(]-\infty,r_k])$ is compact and g is uniformly continuous on ${}^{*}g(]-\infty,r_k])$ with respect to d. This last condition also ensures that ${}^{*}g(]-\infty,r_k])$ is d_0-precompact for each k; moreover, as $d(x,y) \leq d_0(x,y)$ for x,y in E, a d_0-Cauchy sequence in ${}^{*}g(]-\infty,r_k])$ is d-Cauchy, hence d-convergent, and so (again by continuity of g on $({}^{*}g(]-\infty,r_k]),d)$) ${}^{*}g(]-\infty,r_k])$ is (complete and) compact with respect to d_0.

If S is a d_0-bounded subset of E, $\delta > 0$ is chosen so that

$d_0(x,a) \leq \delta$ - and therefore $d(x,a) \leq \delta$ and $|.gx - .ga| \leq \delta$ - for each x in S, and then k is chosen so that $|.ga| + \delta \leq r_k$, we see that S is contained in the d_0-compact subset ${}^*g(]-\infty,r_k])$ of E. Hence (E,d_0) is locally compact.

Now, it is clear that g is uniformly continuous on E with respect to d_0. This, together with the d_0-boundedness of ${}^*g(]-\infty,r_k])$ for each k, and the inclusion ${}^*g(]-\infty,r]) \subset {}^*g(]-\infty,r_k])$ - valid for $r \leq r_k$ - shows that g is a compactifier for (E,d_0). □

.1 *Under the conditions of* (3.3.0), *if* $Y = \{\omega\}$ *is a singleton subset of* X, *then a subset* K *of* E *is* d-*compact if and only if it is* d_0-*compact, in which case* g *is uniformly continuous on* (K,d).

Proof. Let $K \subset E$ be d-compact. Then (2.1.1) $d(\omega,K) > 0$, so that $\lambda x \in K.d(x,\omega)^{-1}$ is uniformly continuous on (K,d), $\lambda x \in K\ x$ is a uniformly continuous mapping of (K,d) onto (K,d_0), and therefore K is d_0-compact. On the other hand, if $K \subset E$ is d_0-compact, then - in the notation of the proof of (3.3.0) - $K \subset {}^*g(]-\infty,r_k])$ for some k, so that g is uniformly continuous on (K,d), and (as a simple argument shows) K is d-compact. □

As a corollary, we have

.2 *Under the conditions of* (3.3.1), *if* E *is locally compact with respect to the metric* d, *then* d *and* d_0 *are homeomorphic metrics on* E. □

<u>Remark</u>: The generalisation of (3.3.1) to the case where Y is complete and located in X, but not necessarily a singleton, appears to depend on Conjecture 1 of Section 2.2. ®

.3 *Let* (E,d) *and* (X,d') *be locally compact spaces,* e *a continuous, injective mapping of* E *onto the metric complement of a single point* ω *in* X, *and* d_0 *the metric*

$\lambda x,y \in E,,E(d'(.ex\ .ey) + |d'(.ex,\omega)^{-1} - d'(.ey,\omega)^{-1}|)$ *on E. Then the following two conditions are equivalent:*

(a) *e is metrically injective, and* inv *e is precontinuous*

(b) *d,d_0 are equivalent metrics on E.*

Moreover, if either of these conditions obtains, then inv *e is metrically injective; a subset K of E is d-compact if and only if ${}_*eK$ is d'-compact; and, if X is also compact, e is uniformly continuous on (E,d).*

Proof. By (3.3.0) ${}_*eE$ is locally compact with respect to the metric

$d_0' \equiv \lambda x,y \in {}_*eE,,{}_*eE.d_0(.\text{inv } e\ x,\ .\text{inv } e\ y)$.

As e is an isometry of (E,d_0) onto $({}_*eE,d_0')$, it follows that (E,d_0) is locally compact, and (3.3.1) that $K \subset E$ is d_0-compact if and only if ${}_*eK$ is d'-compact; in which case $0 < d'(\omega,{}_*eK)$ (2.1.1), $\lambda z \in {}_*eK.d'(z,\omega)^{-1}$ is uniformly continuous on $({}_*eK,d')$, and therefore strc inv e_*eK is uniformly continuous as a mapping of $({}_*eK,d')$ onto (K,d_0). That (a) entails (b) follows from this, the (obvious) uniform continuity of e as a mapping of (E,d_0) onto $({}_*eE,d')$, the definition of 'precontinuous function', and (2.3.0).

Conversely, let us suppose that condition (b) obtains. Then it is clear that e is continuous as a mapping of (E,d) onto $({}_*eE,d')$, that $K \subset E$ is d-compact if and only if ${}_*eK$ is d'-compact, and that inv e is precontinuous as a mapping of $({}_*eE,d')$ onto (E,d). Let A and B be d-compact subsets of E with $c \equiv d(A,B) > 0$. Then, choosing d-compact $K \subset E$ with $A \cup B \subset K$, and a modulus δ of uniform continuity for strc inv $e\ {}_*eK$ as a mapping of $({}_*eK,d')$ onto (K,d), we see that

$d'({}_*eA,{}_*eB) \geq \delta(c/2) > 0$

Thus e is metrically injective. This completes the proof of the equivalence of conditions (a) and (b).

We now suppose that either, and therefore both, of conditions (a) and (b) obtains, and note first that, by an argument similar to one used above, inv e is metrically injective as a mapping of $({}_*eE,d')$ onto (E,d). Supposing, in addition, that X is compact, it remains to prove that e is uniformly continuous as a mapping of (E,d) onto $({}_*eE,d')$. Given $\varepsilon > 0$, we choose real r so that $0 < r < \varepsilon/2$ and

$$S \equiv \{x \in X : d'(x,\omega) \geq r\}$$

is d'-compact (this is possible because X is compact and $X - \{\omega\}$ is inhabited - cf. introduction to Chapter 2). Then $S \subset {}_*eE$, ${}^*eS$ is compact, and we can find $c > 0$ so that

$$T \equiv \{x \in E : d(x, {}^*eS) \leq 3c\}$$

is d-compact (3.1.0). With δ a modulus of uniform continuity for e on T, and x and y points of E such that $d(x,y) \leq \min(\delta(\varepsilon),c)$, we now have: either $d(x, {}^*eS) < 2c$ - in which case x and y both belong to T, and therefore $d'(.ex,.ey) \leq \varepsilon$ - or $d(x, {}^*eS) > c$. In the latter case, $d(y, {}^*eS) > 0$, and the metric injectiveness of e ensures that $0 < d'(.ex,S)$ and $0 < d'(.ey,S)$, whence

$$d'(.ex,.ey) \leq d'(.ex,\omega) + d'(.ey,\omega) \leq 2r < \varepsilon.$$

Thus indeed, e is uniformly continuous on (E,d). □

Under the conditions of (3.3.3), if X is compact and either (and therefore both) of conditions (a) and (b) obtains, we say that $((X,d'),e,\omega)$ - or, when no confusion is likely, (X,d') - is a *one-point compactification* of (E,d) with *point at infinity* ω and *canonical injection* e.

An immediate consequence of (3.3.0) and (3.3.3) is

.4 *Let* (X,d) *be a compact space,* E *the metric complement of the single point* ω *in* X, *and* d_0 *the metric*

$$\lambda x,y \in E,,E(d(x,y) + |d(x,\omega)^{-1} - d(y,\omega)^{-1}|)$$

on E. *Then* (E,d_0) *is locally compact, and* $((X,d), \lambda x \in E\ x,\omega)$ *is a one-point compactification of* (E,d_0). □

As a partial converse of this, we have

.5 *Let* (X,d) *be a complete, bounded metric space,* E *the metric complement of a single point* ω *in* X, *and suppose that* E *is locally compact with respect to the metric*

$$d_0 \equiv \lambda x,y \in E,,E(d(x,y) + |d(x,\omega)^{-1} - d(y,\omega)^{-1}|).$$

Then X *is compact.*

Proof. Let ε be a positive number, and M a positive number such that $d(x,y) \leq M$ for each x and y in X. Then

$A \equiv \{x \in X : d(x,\omega) > \varepsilon/2\}$

is a subset of E, and $d_0(x,y) < M + 4\varepsilon^{-1}$ for each x and y in A. Thus, there exists a d_0-compact subset K of E with $A \subset K$. Let $\{x_0,\ldots,x_\nu\}$ be an ε-approximation to (K,d_0). Then, given x in X, we haveeither $d(x,\omega) < \varepsilon$ or $d(x,\omega) > \varepsilon/2$; in the latter case, x belongs to A, and so

$d(x,x_j) \leq d_0(x,x_j) < \varepsilon$

for some j, $0 \leq j \leq \nu$. Thus, $\{\omega,x_0,\ldots,x_\nu\}$ is an ε-approximation to (X,d), which is therefore precompact. As X is complete, this proves our theorem. □

Our next result covers the special case when our locally compact space is in fact compact.

.6 *Let* $((X,d'),e,\omega)$ *be a one-point compactification of the locally compact space* (E,d). *Then a subset* A *of* E *is bounded if and only if there exists* $c > 0$ *such that* $d'(.ex,\omega) \geq c$ *for each* x *in* A, *in which case* inv e *is uniformly continuous on* $_*eA$. *In particular,* E *is compact if and only if there exists* $c > 0$ *such that* $d'(.ex,\omega) \geq c$ *for each* x *in* E.

Proof. If $A \subset E$ and there exists $c > 0$ such that $d'(.ex,\omega) \geq c$ for each x in A, we choose t so that $0 < t < c$ and

$K' \equiv \{x \in X : d'(x,\omega) \geq t\}$

is compact. As ${}_*eA \subset K'$, it follows that A is bounded (3.3.3) and inv e is uniformly continuous on ${}_*eA$. The converse follows simply from the definition of local compactness, (3.3.3) and (2.1.1). The last part of the theorem is an immediate consequence of the first. □

We conclude this section with the fundamental theorem on the existence of one-point compactifications; our proof is based on that of Theorem 9 of Chapter 4 of [1], the main difference being the simplifications we introduce by the use of the concept of metric injectiveness.

.7 *Every locally compact space has a one-point compactification.*

Proof. Let (E,d) be a locally compact space, $(a_n)_{1\leq n}$ a dense sequence in E, and define

$f \equiv \lambda n \in \mathrm{N}{\sim}1\ \lambda x \in E\ \min(1,d(x,a_n))$.

Let Y be the set map $\mathrm{N}{\sim}1\ [0,1]$; d' the metric

$\lambda s,s' \in Y,,Y\ \Sigma_{n=1}^{\infty} 2^{-n}|.sn - .s'n|$

(with respect to which Y is compact); and e the mapping

$\lambda x \in E\ \lambda n \in \mathrm{N}{\sim}1\ ..fn\ x$.

Then, for each x and y in E,

$$\begin{aligned} d'(.ex,.ey) &= \Sigma_{n=1}^{\infty} 2^{-n}|\min(1,d(x,a_n)) - \min(1,d(y,a_n))| \\ &\leq \Sigma_{n=1}^{\infty} 2^{-n}|d(x,a_n) - d(y,a_n)| \\ &\leq \Sigma_{n=1}^{\infty} 2^{-n}d(x,y) \\ &= d(x,y), \end{aligned}$$

so that e is uniformly continuous on E.

Now let A and B be compact subsets of E such that $0 < c \equiv d(A,B)$. With α chosen so that $0 < \alpha < \min(2^{-1},c^{-1})$, let $n_1,\ldots,n_\nu$ be positive integers such that

$n_1 \leq n_2 \leq \cdots \leq n_\nu$ and

$A \subset \cup_{k=1}^{\nu}\{x \in E : d(x,a_{n_k}) < \alpha\ c\}$.

Let a belong to A, b to B, and choose k so that

$1 \leq k \leq \nu$ and $d(a,a_{n_k}) < \alpha\, c$. Then

$d(b,a_{n_k}) \geq d(a,b) - d(a,a_{n_k}) > c(1 - \alpha)$,

$d(a,a_{n_k}) < \alpha\, c < c/2 < (1 - \alpha)c < d(b,a_{n_k})$

and so

$$\begin{aligned} d'(.ea,.eb) &\geq 2^{-n_k}|\min(1,d(a,a_{n_k})) - \min(1,d(b,a_{n_k}))| \\ &= 2^{-n_k}(\min(1,d(b,a_{n_k})) - \min(1,d(a,a_{n_k}))) \\ &\geq 2^{-n_\nu}(\min(1,(1 - \alpha)c) - \alpha\, c) \\ &= 2^{-n_\nu} \min(1-\alpha c,(1 - 2\alpha)c) \end{aligned}$$

Thus

$d'({}_*eA,{}_*eB) \geq 2^{-n_\nu}\min(1-\alpha c,(1-2\alpha)c) > 0$,

and e is metrically injective.

We next show that ${}_*eE$ is d'-precompact. To this end, for each positive integer n we let

$Y_n \equiv \{x \in E : d(x,a_1) \geq n-1\}$.

Given $\varepsilon > 0$, we choose n' so that $\Sigma_{n=n'+1}^{\infty}2^{-n} < \varepsilon$, set

$k \equiv 2 + \max n \in \mathrm{N} \wedge 1 \leq n \leq n'.d(a_1,a_n)$,

and consider an arbitrary point x in Y_k: for $1 \leq n \leq n'$,

$d(x,a_n) \geq d(x,a_1) - d(a_1,a_n) \geq (k - 1) - (k - 2) = 1$

so that $..fn\ x = 1$. Thus, for all x,y in Y_k,

$d'(.ex,.ey) = \Sigma_{n=n'+1}^{\infty} 2^{-n}|..fn\ x - ..fn\ y| \leq \Sigma_{n=n'+1}^{\infty} 2^{-n} < \varepsilon$.

On the other hand, choosing r in $]k-1,k[$ so that

$E_r \equiv \{x \in E : d(x,a_1) \leq r\}$

is compact, we see that ${}_*eE_r$ is compact (2.3.0). With $\{y_0,\ldots,y_q\}$ an ε-approximation to ${}_*eE_r$, and y_{q+1} any point of ${}_*eY_k$, it is now easy to show that $\{y_0,\ldots,y_{q+1}\}$ is an ε-approximation to ${}_*eE$. Thus ${}_*eE$ is precompact.

With

$\omega \equiv \lambda n \in \mathrm{N}\sim 1\ 1$

it now follows that the closure X of ${}_*e(E) \cup \{\omega\}$ in Y is compact in the metric d' . Moreover, ${}_*eE \subset X - \{\omega\}$: for, given x in E and choosing n so that $d(x,a_n) < 1/2$, we have

$$d'(.ex,\omega) \geq 2^{-n}|\min(1,d(x,a_n)) - 1| > 2^{-n-1}.$$

We now prove that: if $S \subset {}_*eE$ and there exists $\beta > 0$ such that $d'(x,\omega) \geq \beta$ for each x in S, then inv e is uniformly continuous on S. Indeed, choosing a positive integer p so that $\Sigma_{n=p+1}^{\infty} 2^{-n} < \beta$, and x in ${}^*eS$, we have

$$\Sigma_{n=1}^{p} 2^{-n}|..fn\ x - 1| = d'(.ex,\omega) - \Sigma_{n=p+1}^{\infty} 2^{-n}|..fn\ x - 1| > \beta - \beta = 0,$$

whence $0 < |..fj\ x - 1|$, and therefore $d(x,a_j) < 1$, for some integer j with $1 \leq j \leq p$. Thus

$${}^*eS \subset \{x \in E : \min j \in \mathrm{N} \wedge 1 \leq j \leq p . d(x,a_j) < 1\},$$

so that there exists compact $K \subset E$ with ${}^*eS \subset K$. That inv e is uniformly continuous on S now follows from (2.3.0). In particular, we see from this and (2.1.1) that inv e is uniformly continuous on compact subsets of ${}_*eE$. Now let z belong to $X - \{\omega\}$, set $\beta \equiv 2^{-1}d'(z,\omega)$, and choose a sequence $(z_k)_{1\leq k}$ in ${}_*e(E) \cup \{\omega\}$ with $d'(z_k,z) < \min(k^{-1},\beta)$ for each k. Then, for each k, $d'(z_k,\omega) > \beta$, so that z_k belongs to the subset

$$S \equiv {}_*e\{x \in E : d'(.ex,\omega) \geq \beta\}$$

of ${}_*e(E)$. By the foregoing, inv e is uniformly continuous on S, so that $(.\mathrm{inv}e\ z_k)_{1\leq k}$ is Cauchy in E, and therefore converges to a point x of E. Finally, as e is uniformly continuous on E, we have

$$.ex = \lim_{k\to\infty} .e.\mathrm{inv}e\ z_k = z,$$

whence z belongs to ${}_*e(E)$, $X - \{\omega\} \subset {}_*e(E)$, and our proof is complete. □

Remark: Were Conjecture 2 of Section 2.2. valid, we could prove that the mapping e defined in the above proof was actually a homeomorphism of E onto ${}_*eE$. For, under that condition, and given a uniformly continuous mapping ϕ of a compact metric space K into ${}_*eE$, we have

$$0 < c \equiv \inf x \in K.d'(\omega,.\phi\, x),$$

whence

$$L \equiv \{x \in X : d'(x,{}_*\phi K) \leq c/2\} \subset {}_*eE,$$

$d'(x,\omega) \geq c/2 > 0$ for each x in L, and there exists a modulus δ of uniform continuity for strc inv $e\, L$ on (L,d') (cf. proof of (3.3.7)); it is then clear that

$$\cap\varepsilon\in\mathbb{R}^+\cap x\in{}_*\phi K\cap y\in{}_*eE(d'(x,y) \leq \min(c/2,\delta(\varepsilon)) \Rightarrow d(.\mathrm{inv}\ ex,.\mathrm{inv}\ ey) \leq \varepsilon)$$

and therefore that inv e is continuous on ${}_*eE$.

We could then simplify our definition of 'one-point compactification' by replacing the conditions that the canonical injection be continuous, metrically injective, and have pre-continuous inverse, with the single condition that it be a homeomorphism of E onto ${}_*eE$. ®

3.4. Uniqueness of one-point compactifications

The essential uniqueness of the one-point compactification of a given locally compact space should come as no surprise; we prove this first in the very general form

.0 *For* $k = 1,2$, *let* (E_k,d_k) *be a locally compact space with one-point compactification* $((X_k,d_k'),e_k,\omega_k)$. *Then a mapping* f *is a homeomorphism of* E_1 *onto* E_2 *if and only if there exists a metric equivalence* ϕ *between* X_1 *and* X_2 *such that* $f = \mathrm{inv}\ e_2 : \phi : e_1$ *and* $.\phi\omega_1 = \omega_2$.

Proof. Suppose first that f is a homeomorphism of E_1 onto E_2. Given $\varepsilon > 0$, we choose real r so that $0 < r < \varepsilon$ and the set $\{x \in X_2 : d_2'(x,\omega_2) \geq r\}$ is compact. Then

$$K \equiv \{x \in E_2 : d_2'(.e_2x,\omega_2) \geq r\}$$

is compact, as are therefore the sets ${}_*\mathrm{inv}fK$ and ${}_*(e_1:\mathrm{inv}f)K$ (3.3.3). It follows that

$$c \equiv 3^{-1}d_1'(\omega_1, {}_*(e_1:\mathrm{inv}f)K)$$

exists, and is positive (2.1.1). Moreover, by (3.3.6),

$e_2 : f : \mathrm{inv} e_1$ is uniformly continuous on the set

$_*(e_1 : \mathrm{inv} f)(K)_{2c} \equiv \{x \in {}_*e_1 E_1 : d_1'(x, {}_*(e_1 : \mathrm{inv} f)K) \leq 2c\}$.

On the other hand, $0 < r \leq d_2'(\omega_2, {}_*e_2 K)$, so that there exists real s with

$$d_2'(\omega_2, {}_*e_2 K) - r < s < d_2'(\omega_2, {}_*e_2 K)$$

and

$$\{x \in X_2 : d_2'(x, \omega_2) \geq d_2'(\omega_2, {}_*e_2 K) - s\}$$

compact (cf. introduction to Chapter 2). It follows that

$$K' \equiv \{x \in E_2 : d_2'(.e_2 x, \omega_2) \geq d_2'(\omega_2, {}_*e_2 K) - s\}$$

is compact (3.3.3), and (3.3.6) that $e_1 : \mathrm{inv} f : \mathrm{inv} e_2$ is uniformly continuous on ${}_*e_2 K'$. Moreover, given x in E_2 with $d_2'(.e_2 x, {}_*e_2 K) \leq s$, we have

$$\begin{aligned} d_2'(\omega_2, {}_*e_2 K) - s &\leq d_2'(\omega_2, {}_*e_2 K) - d_2'(.e_2 x, {}_*e_2 K) \\ &\leq d_2'(\omega_2, .e_2 x), \end{aligned}$$

whence x belongs to K'; in particular, $K \subset K'$.

With δ a modulus of uniform continuity for $e_1 : \mathrm{inv} f : \mathrm{inv} e_2$ on ${}_*e_2 K'$ we now see that, for any x in E_1 with $0 < d_1'(.e_1 x, {}_*(e_1 : \mathrm{inv} f)K)$, either $0 < d_2'(.e_2 . fx, {}_*e_2 K)$, or $d_2'(.e_2 . fx, {}_*e_2 K) < s$; in the latter case, $.e_2 . fx$ belongs to ${}_*e_2 K'$, and so

$$d_2'(.e_2 . fx, {}_*e_2 K) \geq \delta(2^{-1} d_1'(.e_1 x, {}_*(e_1 : \mathrm{inv} f)K)) > 0.$$

Thus, in both cases $.e_2 . fx$ belongs to $X_2 - {}_*e_2 K$, and therefore

$$d_2'(.e_2 . fx, \omega_2) \leq r < \varepsilon.$$

In particular, we note that, for x in E_1 and $d_1'(.e_1 x, \omega_1) \leq c$, we have

$$0 < 2c \leq d_1'(.e_1 x, {}_*(e_1 : \mathrm{inv} f)K)),$$

and so $d_2'(.e_2 . fx, \omega_2) < \varepsilon$.

We are now in a position to prove uniform continuity of the mapping

$\phi \equiv \lambda x \in {}_*e_1(E_1) \cup \{\omega_1\}(x \in {}_*e_1E_1 \wedge .e_2.f.\mathrm{inv}e_1 x \vee x = \omega_1 \wedge \omega_2)$.
With δ_1 a modulus of uniform continuity for $e_2:f:\mathrm{inv}e_1$ on ${}_*(e_1:\mathrm{inv}f)(K)_{2c}$, we choose x and y in ${}_*e_1(E_1) \cup \{\omega_1\}$ so that $d_1'(x,y) \leq \min(c,\delta_1(\varepsilon))$. If one of x,y - say y - equals ω_1, then either $x = \omega_1$ and $d_2'(.\phi x,.\phi y) = 0$, or x belongs to ${}_*e_1E_1, d_1'(x,\omega_1) \leq c$, and therefore
$d_2'(.\phi x,.\phi y) = d_2'(.e_2.f.\mathrm{inv}\ e_1\ x,\omega_2) \leq \varepsilon$.
If x and y both belong to ${}_*e_1E_1$, then: either
$0 < \min(d_1'(x,{}_*(e_1:\mathrm{inv}f)K),\ d_1'(y,{}_*(e_1:\mathrm{inv}f)K))$,
in which case
$d_2'(.\phi x,.\phi y) \leq d_2'(.e_2.f.\mathrm{inv}e_1 x,\omega_2) + d_2'(.e_2.f.\mathrm{inv}e_1 y,\omega_2) \leq 2\varepsilon$;
or
$\min(d_1'(x,{}_*(e_1:\mathrm{inv}f)K), d_1'(y,{}_*(e_1:\mathrm{inv}f)K)) < c$,
when x and y both belong to ${}_*(e_1:\mathrm{inv}f)(K)_{2c}$, and therefore
$d_2'(.\phi x,.\phi y) = d_2'(.e_2.f.\mathrm{inv}e_1 x,.e_2.f.\mathrm{inv}e_1 y) \leq \varepsilon$.
Thus, in all cases, $d_2'(.\phi x,.\phi y) \leq 2\varepsilon$, and uniform continuity of ϕ on ${}_*e_1(E_1) \cup \{\omega_1\}$ is established.
As $e_1(E_1) \cup \{\omega_1\}$ is dense in X_1, ϕ extends by continuity to a uniformly continuous mapping of X_1 into X_2. In the same way, we can show that $\mathrm{inv}\phi$ equals
$\lambda x \in {}_*e_2(E_2) \cup \{\omega_2\}(x \in {}_*e_2E_2 \wedge .e_1.\mathrm{inv}f.\mathrm{inv}e_2 x \vee x = \omega_2 \wedge \omega_1)$
and extends to a uniformly continuous mapping of X_2 into X_1. It is now comparatively trivial to show that the extension of ϕ is our desired metric equivalence between X_1 and X_2, the inverse equivalence being the extension of $\mathrm{inv}\phi$. This completes the first part of our proof.
The second part is much easier to deal with: we suppose that ϕ is a metric equivalence between X_1 and X_2 such that $.\phi\omega_1 = \omega_2$, and let f be the mapping $\mathrm{inv}e_2:\phi:e_1$.
Then, with δ a common modulus of uniform continuity

for ϕ and invϕ on their respective domains, and $K \subset E_1$ compact, we have ${}_*e_1K$ compact (3.3.3), so that $0 < d_1'(\omega_1, {}_*e_1K)$ (2.1.1) and ${}_*(\phi:e_1)K \subset \{x \in X_2 : d_2'(\omega_2, x) \geq \delta(2^{-1}d_1'(\omega_1, {}_*e_1K))\} \subset {}_*e_2E_2$. Thus f maps E_1 into E_2; moreover, ${}_*(\phi:e_1)K$ is compact, so that strc fK is uniformly continuous. In the same way, we show that inve_1:invϕ:e_2 maps E_2 into E_1 and is uniformly continuous on compact subsets of E_2. Finally, it is clear that f is a bijection of E_1 onto E_2 with inverse inve_1:invϕ:e_2, and therefore that f is a homeomorphism of E_1 onto E_2. □

An immediate corollary of this is

.1 *Any two one-point compactifications of a locally compact space are metrically equivalent.* □

3.5. Subspaces and compactifications

.0 *Let* (E,d) *be a locally compact space,* $((X,d'),e,\omega)$ *a one-point compactification of* E, F *a closed subset of* E, *and* Y *the closure in* X *of the subset* ${}_*e(F) \cup \{\omega\}$. *Then* F *is locally compact if and only if* Y *is compact, in which case* $((Y, \text{strc}d'Y,,Y)$, strc $eY, \omega)$ *is a one-point compactification of* F.

Proof. We first show that ${}_*eF = Y-\{\omega\}$. Certainly, ${}_*eF \subset Y-\{\omega\}$. On the other hand, if z belongs to $Y-\{\omega\}$, then $z = .ex$ for some x in E. As ${}_*e(F) \cup \{\omega\}$ is dense in Y, there exists a sequence (x_n) in F such that $(.ex_n)$ converges to z, and $d'(.ex_n, \omega) \geq c > 0$ for each n, where $c \equiv 2^{-1}d'(z,\omega)$. By (3.3.6), inv$e$ is uniformly continuous on $\{y \in Y : d'(y,\omega) \geq c\}$, so that (x_n) converges to x, x is in the closed subset F of E, and z belongs to ${}_*eF$. Thus ${}_*eF = Y-\{\omega\}$.

We now suppose that F is locally compact. Then (3.3.3)

$_*eF$ is locally compact with respect to the metric $\lambda x,y\in {}_*e(F),,{}_*e(F)(d'(x,y) + |d'(x,\omega)^{-1} - d'(y,\omega)^{-1}|)$. As Y is closed in X, and therefore complete and bounded, it follows from (3.3.5) that Y is compact. Conversely, if Y is compact, it is clear from (3.3.4) that F is locally compact, and that $((Y,\text{strc } d'Y,,Y), \text{ strc } eY,\omega)$ is a one-point compactification of F. □

.2 *Let h be a homeomorphism of a locally compact space $(E_1.d_1)$ onto a locally compact subspace F of a locally compact space (E_2,d_2). Let $((X_2,d_2'),e_2,\omega_2)$ be a one-point compactification of E_2, and Y the closure in X of the subset $_*e_2(F)\cup\{\omega_2\}$. Then $((Y,\text{strc}d_2'Y,,Y),e_2:h,\omega_2)$ is a one-point compactification of E_1.*

Proof. In view of (3.5.0), we lose no generality in taking $F = E_2$, $Y = X_2$. It is clear from (3.3.3) that $e_2:h$ is a continuous mapping of E_1 onto $_*e_2E_2$ with precontinuous inverse. On the other hand, with $((X_1,d_1'),e_1,\omega_1)$ a one-point compactification of (E_1,d_1) (3.3.7), and A and B compact subsets of E_1 such that $0 < d_1(A,B)$, we have $_*e_1A$ and $_*e_1B$ compact, and $0 < d_1'(_*e_1A,{}_*e_1B)$ (3.3.3). As the mapping $e_1:\text{inv}h:\text{inv}e_2$ is uniformly continuous on $_*e_2E_2$ (3.4.0), it follows that

$$0 < d_2'(_*(e_2:h:\text{inv}e_1)_*e_1A,{}_*(e_2:h:\text{inv}e_1)_*e_1B)=d_2'(_*(e_2:h)A,{}_*(e_2:h)B)$$

Thus $e_2:h$ is metrically injective, and the proof is complete. □

CHAPTER 4.

ALGEBRAS OF FUNCTIONS ON LOCALLY COMPACT SPACES.

By a *star algebra*, we mean a complex linear algebra A taken with an *algebra involution* on A - that is, a mapping $x \to x^*$ of A into itself such that, for each x,y in A and ζ in $\mathbb{C}$,

$$(x + y)^* = x^* + y^*, \quad (xy)^* = y^*x^*, \quad (\zeta x)^* = \zeta^* x^*, \quad (x^*)^* = x.$$

If, in addition, $\| \ \|$ is a norm on A with the property

$$\cap x,y, \in A(\|xy\| \leq \|x\| \, \|y\|)$$

then $(A,\| \ \|,x \to x^*)$ - or, commonly, A itself - is called a *normed star algebra* (or, in the case where $(A,\| \ \|)$ is complete, a *Banach star algebra*).

If $(A,x \to x^*)$ and $(B,x \to x^*)$ are star algebras, we say that a homomorphism ϕ of A into B is a *star homomorphism* (of A into B) if $.\phi(x^*) = (.\phi x)^*$ for each x in A.

If X,A are sets, n a natural number, and τ an n-ary operation on A (that is, a mapping of map nA - or, less strictly, of $\{x$: tuple n is $x \wedge x, \in A\}$ - into A), then

$$\underset{\sim}{\tau} = \lambda\phi \in \text{map } n(\text{map } XA)\ \lambda x \in X.\tau(\lambda t \in n..\phi tx)$$

is an n-ary operation on map XA - the *pointwise operation* of τ; we shall commonly make no notational distinction between τ and $\underset{\sim}{\tau}$. In particular, if A is a (real or complex) linear algebra, then the same is true of map XA under the corresponding pointwise operations of addition, multiplication by scalars and multiplication. (Note that we regard 'multiplication by the scalar ζ' as a 1-ary operation for each ζ in $\mathbb{C}$.) Moreover, if A is a star algebra, then so is map XA when taken with the corresponding pointwise operations and involution. On the other hand, if

$$\|f\|_X \equiv \sup x \in X \ |.fx|$$

is well-defined for each f in a complex (resp. real) linear subalgebra $\mathcal{R}$ of map $X\mathbb{C}$ (resp. map $X\mathbb{R}$) under pointwise operations, then $f \to \|f\|_X$ defines a norm on $\mathcal{R}$ with respect to which $\mathcal{R}$ is a normed algebra.

In this chapter, we discuss a particular star algebra of the above type - namely, the algebra of continuous functions which vanish at infinity on a locally compact space. The first section describes the algebraic and norm structure on these algebras, and proves their norm completeness. Following this, there are two results necessary for the third and final section (which deals in detail with the characterisation and properties of star homomorphisms between the algebras described in Section 4.1); the second of these results (4.2.1) - a constructive substitute for the classical theorem that there exists a unique uniform structure compatible with the given topology on a compact Hausdorff space (namely, that generated by all uniformly continuous mappings of the space into R) - leads us to make some remarks on another, as yet only partially answered, question relating to continuity of functions in constructive analysis.

4.1. The spaces $C^{0}(E)$, $C(E)$

A mapping f of a metric space E into C is said to *vanish at infinity* if, for each $\varepsilon > 0$, there exists a compact subset K of E such that $|.fx| \leq \varepsilon$ whenever x belongs to $E-\dot{K}$. The set of all continuous mappings of E into C (resp. R) which vanish at infinity is written $C^{0}(E)$ (resp. $C^{0}_{R}(E)$), and is a complex (resp. real) linear algebra under pointwise operations of addition, multiplication and multiplication by scalars.

In this first section, we show how to define on $C^{0}(E)$ a natural Banach star algebra structure. (Note that, although we shall be concerned almost entirely with $C^{0}(E)$, simple modifications of our results will enable them to apply to $C^{0}_{R}(E)$ as well.)

Throughout the section, (E,d) will be a locally compact space.

.0 *Every element* f *of* $C^0(E)$ *is uniformly continuous on* E.

Proof. Given $\varepsilon > 0$, we choose compact $K \subset E$ so that $|.fx| \leq \varepsilon/2$ for each x in $E - K$, and then $c > 0$ with $\{x \in E : d(x,K) \leq 3c\}$ compact (3.1.0). With δ a modulus of uniform continuity for f on $\{x \in E : d(x,K) \leq 3c\}$, x and y points of E, and $d(x,y) \leq \min(c,\delta(\varepsilon))$, we then have: either $c < d(x,K)$ - in which case $0 < d(y,K)$, and $|.fx - .fy| \leq |.fx| + |.fy| \leq \varepsilon$ - or $d(x,K) < 2c$, when $d(y,K) < 3c$, and again $|.fx - .fy| \leq \varepsilon$. □

Our next result is stated without proof on page 249 of [1].

.1 *Let* f *belong to* $C^0(E)$. *Then* $\|f\|_E$ *is a well-defined real number.*

Proof. We first prove that, given $\varepsilon > 0$ and a compact subset K of E, there exists compact $L \subset E$ such that $K \subset L$ and

$\cap x \in E(|.fx| \leq \|f\|_L + \varepsilon)$.

To this end, we choose compact $K' \subset E$ such that $|.fx| \leq \|f\|_K + \varepsilon$ for each x in $-K'$, and then compact $L \subset E$ with $K \cup K' \subset L$. Then $|.fx| \leq \|f\|_L + \varepsilon$ for each x in $L \cup -L$. But

$\cap\alpha \in R^+\cap x \in E(0 < d(x,L) \vee d(x,L) < \alpha)$,

so that $L \cup -L$ is dense in E. It is now clear that L fulfils our requirements.

With the help of 'dependent choice', it is now a simple matter to construct a sequence $(K_n)_{n\in N}$ of compact subsets of E such that, for each n in N, $K_n \subset K_{scsrn}$ and

$\cap x \in E(|.fx| \leq \|f\|_{K_n} + 2^{-n})$.

It should be clear from these properties that

$\cap m,n, \in N(m \leq n \Rightarrow 0 < \|f\|_{K_n} - \|f\|_{K_m} \leq 2^{-m})$, so that $(\|f\|_{K_n})_{n\in N}$ is a Cauchy sequence in R, with limit s, say. Moreover, for each x in E and each n in N,

$|\cdot fx| \leq \|f\|_{K_n} + 2^{-n} \leq s + 2^{-n}$,

so that $|\cdot fx| \leq s$. On the other hand, with ε a positive number, ν chosen in N so that

$(4\varepsilon^{-1} \leq \nu \wedge |s - \|f\|_{K_\nu}| \leq \varepsilon/2)$,

and x an element of K_ν with

$||\cdot fx| - \|f\|_{K_\nu}| \leq 2\nu^{-1}$,

we have

$|s - |\cdot fx|| \leq \varepsilon/2 + 2\nu^{-1} \leq \varepsilon$.

Thus indeed, $\|f\|_E$ is well-defined, and equals s. □

From now on, it will be understood that when we speak of $C^o(E)$ as a normed star algebra, it will always be to the pointwise operations and involution, and the norm $f \to \|f\|_E$, that we refer.

A mapping f of E into C is said to *have compact support* if there exists a compact subset K of E such that $_*f(E-K) = \{0\}$; such a set K is then called a *support* of f (in E).† The set $C(E)$ of continuous mappings of E into C with compact support is a complex linear star subalgebra of $C^o(E)$; moreover, when E is compact, the algebras $C(E)$ and $C^o(E)$ coincide, and comprise all uniformly continuous mappings of E into C. In general, we have

.2 *$C(E)$ is dense in $C^o(E)$.*

Proof. Given $\varepsilon > 0$ and f in $C^o(E)$, we choose compact $K \subset E$ so that $|\cdot fx| \leq \varepsilon/2$ for each x in $-K$, and then

† These definitions apply equally well to a general metric space E; however, this is not true of the succeeding theorem.

r, s with $0 < s < r$ and $\{x \in E : d(x,K) \leq s\}$ and $\{x \in E : d(x,K) \leq r\}$ both compact (3.1.0). With ϕ the mapping
$\lambda y \in E \max(0, 1 - (r-s)^{-1} d(y, \{x \in E : d(x,K) \leq s\}),$
we see that ϕ is continuous and has compact support $\{x \in E : d(x,K) \leq r\}$; that $0 \leq .\phi x \leq 1$ for each x in E, and $.\phi x = 1$ throughout $\{x \in E : d(x,K) \leq s\}$; and that ϕf is continuous and has compact support $\{x \in E : d(x,K) \leq r\}$in E. For each x in E, we now have: either $d(x,K) < s$ - in which case $|(.\phi x)(.fx) - .fx| = 0$ - or $0 < d(x,K)$, when $|(.\phi x)(.fx) - .fx| \leq 2|.fx| \leq \varepsilon$.
Thus $\|\phi f - f\|_E \leq \varepsilon$, and $C(E)$ is dense in $C^0(E)$. □

<u>Remark:</u> The set $C_R(E)$ of real-valued elements of $C(E)$ is clearly a real linear subalgebra of $C_R^0(E)$, and is dense in $C_R^0(E)$. ®

Our next theorem is vital for the later work of this chapter.

.3 *Let* $((X,d'),e,\omega)$ *be a one-point compactification of the locally compact space* (E,d). *Then there exists a unique isometric, algebraic star isomorphism* u *of* $C^0(E)$ *onto the ideal* $C(X,\omega) \equiv \{f \in C(X) : .f\omega = 0\}$ *of* $C(X)$ *such that* $\mathrm{strc}.uf\ {}_*eE = f{:}\mathrm{inve}$ *for each* f *in* $C^0(E)$.

Proof. Given $\varepsilon > 0$, we let g belong to $C(X,\omega)$ and δ be a modulus of uniform continuity for g on X, and then choose real r so that $0 < r < \delta(\varepsilon)$ and $\{x \in E : d'(x,\omega) \geq r\}$ is oompact. With K the compact subset ${}^*e\{x \in X : d'(x,\omega) \geq r\}$ of E (3.3.3), and x an element of $E - K$, we have $d'(.ex,\omega) \leq r < \delta(\varepsilon)$, whence $|.g.ex| = |.g.ex - .g\omega| \leq \varepsilon$.

Thus $g{:}e$ belongs to $C^0(E)$.

On the other hand, given f in $C^0(E)$, we let g be the mapping

$\lambda x \in {}_*e(E) \cup \{\omega\}(x \in {}_*eE \wedge .f.\mathrm{inv}\ e\ x \vee x = \omega \wedge 0)$,

choose compact $K \subset E$ such that $|.fx| \leq \varepsilon/2$ for each x in $E - K$, and set $c \equiv 3^{-1}d'(\omega, {}_*eK)$ (which exists, by (3.3.3)). Then $c > 0$ (2.1.1) and

$({}_*eK)_{2c} \subset \{x \in X : d'(x,\omega) \geq c\}$,

where

$({}_*eK)_{2c} \equiv \{x \in X : d'(x, {}_*eK) \leq 2c\}$.

Hence (3.3.6) inv e - and therefore g (4.1.0) - is uniformly continuous on $({}_*eK)_{2c}$. Let δ_ε be a modulus of uniform continuity for g on $({}_*eK)_{2c}$, and x,y elements of E such that $d'(.ex,.ey) \leq \min(c, \delta_\varepsilon(\varepsilon))$.

Then either

$\min(d'(.ex, {}_*eK), d'(.ey, {}_*eK)) < c$

- in which case both $.ex, .ey$ belong to $({}_*eK)_{2c}$ and $|.g.ex - .g.ey| \leq \varepsilon$ - or

$0 < \min(d'(.ex, {}_*eK), d'(.ey, {}_*eK))$.

In this latter case, with δ' a modulus of uniform continuity for e, we have

$0 < \delta'(2^{-1}d'(.ex, {}_*eK)) \leq d(x,K)$,

$0 < \delta'(2^{-1}d'(.ey, {}_*eK)) \leq d(y,K)$,

and therefore

$|.g.ex - .g.ey| = |.fx - .fy| \leq |.fx| + |.fy| \leq \varepsilon$.

To complete the proof that g is uniformly continuous on ${}_*e(E) \cup \{\omega\}$, it suffices to note that if x belongs to E and $d'(.ex,\omega) \leq \min(c, \delta_\varepsilon(\varepsilon))$, then

$d'(.ex, {}_*eK) \geq d'(\omega, {}_*eK) - d'(\omega, .ex) \geq 2c$,

so that $0 < \delta'(c) \leq d(x,K)$, and therefore

$|.g.ex - .g\omega| = |.fx| \leq \varepsilon/2 < \varepsilon.$

As ${}_{*}e(E)\cup\{\omega\}$ is dense in X, it follows from all the foregoing that there eixsts a unique bijection u of $C^{o}(E)$ onto $C(X,\omega)$ such that

strc.uf ${}_{*}eE = f$:inv e

for each f in $C^{o}(E)$. Moreover, this last property, the fact that $..uf\ \omega = 0$ for each f in $C^{o}(E)$, and the denseness of ${}_{*}e(E)\cup\{\omega\}$ in X, ensure that u is an isometry. The proof that u is an algebraic star homomorphism - and therefore a star isomorphism - is straightforward. □

Remark: It is an immediate consequence of (4.1.3) that the mapping $\lambda x \in E.d'(.ex,\omega)$ - the *distance from infinity* (relative to the one-point compactification $((X,d'),e,\omega)$) - is an element of $C^{o}(E)$. ®

.4 $C^{o}(E)$ *is a Banach star algebra.*

Proof. Let $((X,d'),e,\omega)$ be a one-point compactification of (E,d) (3.3.7). As $\{g \in C(X) : .g\omega = 0\}$ is a closed, and therefore complete, subset of the complete space $C(X)$, it follows from (4.1.3) that $C^{o}(E)$ is complete. (We omit the simple proof of completeness of $C(X)$.) On the other hand, in view of the well-known consequence of the constructive Stone-Weierstrass Theorem that the set of all polynomials in the functions $\lambda x \in X.d'(x,y)$, with y in X, is dense in $C_{\mathbb{R}}(X)$ ([1], Chapter 4, Theorem 7, Corollary 2), it is a straightforward, if rather tedious, matter to prove that the set Γ of all polynomials in the functions $\lambda x \in X.d'(x,x_n)$ with complex rational coefficients is dense in $C(X)$, where (x_n) is a dense sequence in X. As Γ is clearly countable, we conclude that there is a dense sequence $(g_n)_{n\in\mathrm{N}}$ in $\mathrm{C}(X)$. It now follows from (4.1.3) that the sequence

$(\lambda x \in E(.g_n.ex - .g_n\omega))_{n\in\mathrm{N}}$

is dense in $\mathrm{C}^{\mathrm{O}}(E)$. □

Our second application of (4.1.3) requires knowledge of the Tietze Extension Theorem ([1], Chapter 4, Theorem 10):

Let Y be a compact subset of a metric space X, I a compact proper interval in R, *and f a continuous mapping of Y into I. Then there exists a mapping h of X into I which is uniformly continuous on bounded subsets of X, and such that* strc $hY = f$.

Bearing this in mind, we prove

.5 *Let F be a locally compact subspace of E, and f an element of* $\mathrm{C}^{\mathrm{O}}(F)$. *Then there exists g in* $\mathrm{C}^{\mathrm{O}}(E)$ *such that* strc $gF = f$.

Proof. Without loss of generality, we may suppose that f belongs to $\mathrm{C}^{\mathrm{O}}_{\mathrm{R}}(F)$. Let $((X,d'),e,\omega)$ be a one-point compactification of (E,d) (3.3.7), and Y the closure of ${}_{*}e(F)\cup\{\omega\}$ in X. Then (3.5.0) $((Y,\text{strc } d'\ Y,.Y)$, strc $eF,\omega)$ is a one-point compactification of $(F,\text{strc } d\ F,,F)$, so that there exists f_1 in $\mathrm{C}(Y)$ with

strc $f_1\ {}_{*}eF = f$:strc inv $e\ {}_{*}eF$

and $.f_1\omega = 0$ (4.1.3). By the Tietze Extension Theorem, there exists f_2 in $\mathrm{C}(X)$ such that strc $f_2 Y = f_1$. One more reference to (4.1.3) suffices to show that we may take $f_2{:}e$ as our desired function g. □

4.2. Two important preliminary results

Let A be a commutative Banach algebra. A *character* of A is a homomorphism χ of A into C which is nonzero, in the sense that

$0 < |.\chi a|$ for some a in A. The classical proofs that every character χ of A is bounded, and that if A has an identity e then $.\chi e = 1$ and χ has norm equal to 1, carry over unchanged into our constructive framework (cf. [5],§16, proposition 3).

Bishop ([1], Chapter 9, Proposition 10) has shown that the characters of $C(X)$, where X is compact, are precisely the *evaluation mappings* $\lambda f \in C(X).fx$, with x in X. Our work in Section 4.3. requires the corresponding result for $C^{0}(E)$.

.0 *Let E be a locally compact space. Then a mapping χ of $C^{0}(E)$ into C is a character of $C^{0}(E)$ if and only if there exists x in E such that $\chi = \lambda f \in C^{0}(E).fx$.*

Proof. It is clear that every mapping $\lambda f \in C^{0}(E).fx$ with x in E is a character of $C^{0}(E)$. Conversely, given a character χ of $C^{0}(E)$, (4.1.2) and the continuity of χ enable us to find ϕ in $C(E)$ with $0 < |.\chi\phi|$. Let K be a compact support of ϕ in E. Then $.\chi f = 0$ for each f in $C^{0}(E)$ which vanishes throughout K: for, given such f, we see that $f\phi$ vanishes throughout $K \cup -K$, and therefore throughout E; were $0 < |.\chi f|$, we would therefore have the contradiction

$$0 < |.\chi f|\,|.\chi\phi| = |.\chi(f\phi)| = 0.$$

It follows from this and (4.1.5) that there is a unique mapping χ_K of $C(K)$ into C such that $.\chi_K(\mathrm{strc}\, fK) = .\chi f$ for each f in $C^{0}(E)$. Moreover, χ_K is clearly a character of $C(K)$, so that there exists x in K with $\chi_K = \lambda g \in C(K).gx$, and therefore $\chi = \lambda f \in C^{0}(E).fx$. □

Our next result is, perhaps, the key to the remainder of this chapter.

.1 *The Backward Uniform Continuity Theorem.*

Let (E,d) be a metric space, and h a mapping of E into a compact space (K,d') such that $f{:}h$ is uniformly continuous on E for each f in $C_R(K)$. Then h is uniformly continuous on E.

Proof. Given $\varepsilon > 0$, we construct a finite cover $\{K_0,\ldots,K_\nu\}$ of K in which each set is compact, and has diameter less than $\varepsilon/2$, and then choose a common modulus of uniform continuity δ_ε for the functions $p_j{:}h$, where, for each j in N~scsrν,

$p_j \equiv \lambda x \in K \max(0,\ 1 - 2\varepsilon^{-1}d'(x,K_j))$.

Then, with x and y in $E, d(x,y) \leq \delta_\varepsilon(1/2)$, and j chosen in N~scsrν so that $.hx$ belongs to K_j, we have

$.p_j.hy \geq .p_j.hx - 2^{-1} = 1 - 2^{-1} > 0$

whence

$d'(.hy,K_j) < \varepsilon/2$,

and therefore $d'(.hy,z) < \varepsilon/2$ for some z in K_j. It follows that

$d'(.hx,.hy) \leq \operatorname{diam} K_j + d'(.hy,z) < \varepsilon$,

whence $\varepsilon \to \delta_\varepsilon(1/2)$ is a modulus of uniform continuity for h on E. □

<u>Remarks</u>: (i) Classically, this theorem would appear in a much more general setting - that of uniform spaces (cf. [6], Chapitre 2, §4, No.1). A natural elementary classical proof of (4.2.1) as it stands takes the following form: suppose that h is not uniformly continuous on E, so that there exist $\alpha > 0$, and sequences (x_n), (y_n) in E, with $(d(x_n,y_n))_{n\in N}$ convergent to 0, but $\alpha \leq d'(.hx_n,.hy_n)$ for each n in N. As K is compact, there exists (sic) a subsequence $(x_{n_k})_{k\in N}$ of (x_n) such that $(.hx_{n_k})_{k\in N}$ converges to a point z of K. The uniform continuity of $\lambda x \in E.d'(z,.hx)$ on E (ex hyp.) now ensures that $(.hy_{n_k})_{k\in N}$ also converges to z, whence, for large enough k, $d'(.hx_{n_k},.hy_{n_k}) \leq \alpha/2$.

This contradiction shows that we must actually have h uniformly continuous on E.

It is interesting to contrast these constructive and classical proofs of (4.2.1), and to observe how much more information is yielded by the former!

(ii) Consider the situation ('dual' to that in (4.2.1)) where (K,d) is compact, h is a mapping of K into a metric space (E,d'), and $f{:}h$ is uniformly continuous on K for each uniformly continuous mapping f of E into R. If we can prove that h is also uniformly continuous on K, then we will have established the *Forward Uniform Continuity Theorem*, a good constructive substitute for the classical Uniform Continuity Theorem (cf. [8], (3.16.5) and [6], Chapitre 2, §4, No.2).

As $\lambda x \in K.d'(.hx,.ha)$ is uniformly continuous for each a in K, we see that h is continuous at each point of K, and that ${}_*hK$ is a bounded subset of E. Taken with (4.2.1), this last fact enables us to prove the Forward Uniform Continuity Theorem in the special case where E is locally compact. For then, choosing compact $K' \subset E$ with ${}_*hK \subset K'$, and a uniformly continuous mapping f of K' into R, we apply (4.1.5) to construct an element g of $C^0(E)$ such that strc $gK' = f$. We then see from (4.1.0) and our hypotheses that $f{:}h = g{:}h$ is uniformly continuous on K. An application of (4.2.1) now completes the proof that h itself is uniformly continuous on K.

In view of this particular case of the Forward Uniform Continuity Theorem we feel that it is unlikely that the general form of the theorem will prove to be an essentially non-constructive proposition; this feeling is reinforced by the observation that, if E is separable, then h is always uniformly continuous from the standpoint of intuitionistic mathematics ([21],3.12). However, a direct, constructive proof of the uniform continuity of h in the general case has so far eluded us, the best we have achieved being such partial results as the equivalence of the three conditions:

(a) h is uniformly continuous on K

(b) ${}_*hK$ is precompact

(c) for each $\varepsilon > 0$, there exists a finite subset $\{x_0,\ldots,x_\nu\}$ of K such that the sets

$$\{x \in K : \cap y \in K(\ |d'(.hx,.hy) - d'(.hx_j,.hy)| \leq \varepsilon)\},$$

with j in N~scsrν, form a cover of K.

Indeed, it is clear that condition (a) entails each of the other two. On the other hand, let $\varepsilon > 0$ be given, and suppose first that ${}_*hK$ is precompact. Let $\{\xi_0,\ldots,\xi_m\}$ be points of K such that $\{.h\xi_0,\ldots,.h\xi_m\}$ is an ε-approximation to ${}_*hK$, δ a common modulus of uniform continuity for the functions $\lambda x \in K.d'(.hx,.h\xi_j)$ (where j belongs to N~scsr ν) and x,y points of K with $d(x,y) \leq \delta(\varepsilon)$. Then, choosing j in N~scsrν so that $d'(.hx,.h\xi_j) \leq \varepsilon$, we have

$$\begin{aligned} d'(.hx,.hy) &\leq d'(.hx,.h\xi_j) + d'(.hy,.h\xi_j) \\ &\leq d'(.hx,.h\xi_j) + |d'(.hx,.h\xi_j) - d'(.hy,.h\xi_j)| + d'(.hx,.h\xi_j \\ &\leq 3\varepsilon \end{aligned}$$

Thus (b) entails (a).

To complete the proof of the equivalence of conditions (a), (b) and (c), it now clearly suffices to prove that: if (X,ρ) is a metric space and, for each $\varepsilon > 0$, there exists a finite subset $\{x_0,\ldots,x_\nu\}$ of X such that the sets

$$\{x \in X : \cap z \in X(|\rho(x,z) - \rho(x_j,z)| \leq \varepsilon)\},$$

(with j in N~scsrν) cover X, then X is precompact. To prove this, we let x and y be points of X such that

$$\cap z \in X(|\rho(x,z) - \rho(y,z)| \leq \varepsilon),$$

and suppose that $\rho(x,y) > 4\varepsilon$. Then, for each z in X,

$$(\rho(x,z) - 2\varepsilon) + (\rho(y,z) - 2\varepsilon) > 0,$$

so that

$$\max(\rho(x,z) - 2\varepsilon, \rho(y,z) - 2\varepsilon) > 0.$$

Hence $\min(\rho(x,z),\rho(y,z)) > \varepsilon$ for each z in X - which is plainly absurd. We conclude that $\rho(x,y) \leq 4\varepsilon$. The given condition on X now shows that X is precompact. ®

4.3. Star homomorphisms between algebras $C^{\circ}(E)$.

Throughout this section, (E_k,d_k) will be a locally compact space, $((X_k,d_k'),e_k,\omega_k)$ a one-point compactification of E_k, and ϕ_k the corresponding distance from infinity, $\lambda x \in E_k . d_k'(.e_k x,\omega_k)$.

It is clear that, if h is a continuous mapping of E_2 into E_1, then

$$H \equiv \lambda f \in C^{\circ}(E_1)\ f{:}h$$

is a star homomorphism of $C^{\circ}(E_1)$ into the algebra of continuous mappings of E_2 into C (under pointwise operations), and that H is nonzero in the sense that $0 < |..Hfx|$ for some f in $C^{\circ}(E_1)$ and x in E_2; moreover, if E_2 is compact, then H maps $C^{\circ}(E_1)$ into $C^{\circ}(E_2)$. In this section, we shall be concerned with various problems associated with the converse question: if H is a star homomorphism of $C^{\circ}(E_1)$ into $C^{\circ}(E_2)$, under what conditions is there a mapping (continuous mapping?) h of E_2 into E_1 such that $H = \lambda f \in C^{\circ}(E_1)\ f{:}h$?

To begin with, we dispose of the case where E_1,E_2 are both compact:

.0 *Let* E_1,E_2 *be compact metric spaces, and* H *a star homomorphism of* $C(E_1)$ *into* $C(E_2)$ *such that* $.H(\lambda x \in E_1\ 1) = \lambda x \in E_2\ 1$. *Then there exists a uniformly continuous mapping* h *of* E_2 *into* E_1 *such that* $H = \lambda f \in C(E_1)\ f{:}h$.

Proof. Given y in E_2, we note that $\lambda f \in C(E_1)..Hfy$ is a character of $C(E_1)$, so that (4.2.0) there clearly exists a mapping h of E_2 into E_1 with $H = \lambda f \in C(E_1)\ f{:}h$. The uniform continuity of h follows immediately from (4.2.1). □

In general we have

.1 *If H is a star homomorphism of $C^{\circ}(E_1)$ into $C^{\circ}(E_2)$, then a necessary and sufficient condition for the existence of a mapping h of E_2 into E_1 such that $H = \lambda f \in C^{\circ}(E_1)\ f{:}h$ is that $..H\phi_1 y$ be positive for each y in E_2; in which case the following conditions on h are equivalent:*

(i) *h is continuous*

(ii) *h maps bounded subsets of E_2 onto bounded subsets of E_1*

(iii) *for each compact $K \subset E_2$, $0 < \inf y \in K\ ..H\phi_1 y$*

Proof. If h exists, then

$\cap y \in E_2(..H\phi_1 y = d_1{}'(.e_1.hy, \omega_1) > 0)$.

On the other hand, if y belongs to E_2 and $..H\phi_1 y > 0$, then $\lambda f \in C^{\circ}(E_1)..Hfy$ is a character of $C^{\circ}(E_1)$; reference to (4.2.0) now completes the first part of the proof.

Suppose then that $H = \lambda f \in C^{\circ}(E_1)\ f{:}h$ with h a mapping of E_2 into E_1. It is clear that (i) entails (ii); moreover, that (ii) and (iii) are equivalent follows from (3.3.6) and the fact that, for each compact $K \subset E_2$, $\inf y \in K..H\phi_1 y = d_1{}'(\omega_1, {}_*(e_1{:}h)K)$.

It therefore remains to prove that (iii) entails (i). To this end, we suppose that $K \subset E_2$ is compact and that $0 < \inf y \in K..H\phi_1 y$. Then, choosing r so that

$0 < r < \inf y \in K..H\phi_1 y$

and

$L \equiv \{x \in E_1 : d_1{}'(.e_1 x, \omega_1) \geq r\}$

is compact, and any element f of $C(L)$, we may apply (4.1.5) to obtain f_1 in $C^0(E_1)$ with strc $f_1 L = f$. As $_*hK \subset L$, we then have

strc $f{:}h\ K$ = strc $f_1{:}h\ K$ = strc $.Hf_1\ K$,

so that $f{:}h$ is uniformly continuous on K. It follows from (4.2.1) that h is uniformly continuous on K. That (iii) entails (i) is now immediate. □

Remark: Were Conjecture 2 of Section 2.2 valid, we could prove that if there exists a mapping h of E_2 into E_1 with $H = \lambda f \in C^0(E_1)\ f{:}h$, then condition (iii) of (4.3.1) obtains, and h is necessarily continuous. ®

We recall that a mapping f between metric spaces E, E' is *proper* if, for each bounded subset B of E', $^*fB$ is bounded in E.

.2 *Let h be a continuous mapping of E_2 into E_1 . In order that $f{:}h$ belong to $C^0(E_2)$ for each f in $C^0(E_1)$, it is necessary and sufficient that h be proper; in which case the closure of $_*hE_2$ in E_1 is locally compact. Moreover, if h is proper and metrically injective, then $_*hE_2$ is locally compact and h is a homeomorphism of E_2 onto $_*hE_2$.*

Proof. Suppose first that

rng $\lambda f \in C^0(E_1)\ f{:}h \subset C^0(E_2)$.

If B is a bounded subset of $_*hE_2$ and K a compact subset of E_1 with $B \subset K$, we apply (4.1.5) to construct ϕ in $C^0(E_1)$ such that $.\phi x = 1$ for each x in K. Then, choosing compact $L \subset E_2$ so that $|.\phi.hy| < 1$ for each y in $E_2 - L$, we see that $^*hB \subset L$ - whence h is proper.

To prove $_*hE_2$ located, we now let a, b belong to E_1, E_2 respectively, choose real $c > 2d_1(a, .hb)$, and take the particular case where

$B \equiv \{x \in {}_*hE_2 : d_1(x, .hb) \le c\}$.

Then

$\cap y \in E_2(d_1(a, .hy) \le d_1(a, .hb) \Rightarrow .hy \in B \subset {}_*hL)$

from which it follows that $d_1(a, {}_*hE_2)$ exists, and equals $d_1(a, {}_*hL)$. Thus ${}_*hE_2$ is located in E_1, and its closure in E_1 is locally compact (3.1.1).

On the other hand, supposing h to be proper, and given f in $C^0(E_1)$ and ε in R^+, we choose in turn a compact subset K' of E_1 such that $|.fx| \le \varepsilon$ for each x in $E_1 - K'$, a positive number r so that

$K'_r \equiv \{x \in E_1 : d_1(x, K') \le r\}$

is compact, and compact $L' \subset E_2$ with ${}^*h(K'_r) \subset L'$. It is then clear that, for each y in $E_2 - L'$, $d_1(.hy, K') \ge r$, and therefore $|.f.hy| \le \varepsilon$. Thus $f{:}h$ belongs to $C^0(E_2)$. Reference to (2.3.0) and the first part of the proof now suffices to establish the remainder of the theorem. □

Remark: If h is proper, then ${}_*hE_2$ is classically locally compact: for, if s is a Cauchy sequence in ${}_*hE_2$, then ${}^*h$ rng s is a bounded subset of E_2, and is therefore contained in a compact set $K \subset E_2$; classically, ${}_*hK$ is (compact and) complete, so that s converges in ${}_*hK$, ${}_*hE_2$ is (complete and) closed, and therefore (4.3.2) ${}_*hE_2$ is locally compact. However, it is clear that if (4.3.2) holds with the words 'the closure of' deleted, then a uniformly continuous mapping of a compact metric space into R has compact range. That this last proposition is essentially non-constructive is well-known (cf. [1], Chapter 2, Exercise 9). ®

This brings us to the first really important theorem in this section:

.3 *Let H be a star homomorphism of $C^0(E_1)$ into $C^0(E_2)$. In order that H map $C^0(E_1)$ onto $C^0(E_2)$, it is necessary and sufficient that there exist a metrically injective, continuous* *mapping h of E_2 into E_1 such that $H = \lambda f \in C^O(E_1)\ f:h$.*

Proof. Suppose that H maps $C^O(E_1)$ onto $C^O(E_2)$, and choose ϕ in $C^O(E_1)$ with $.H\phi = \phi_2$. Then, given y in E_2, we see that - as $0 < .\phi_2 y$ - $\lambda f \in C^O(E_1)\ ..Hfy$ is a character of $C^O(E_1)$; whence (4.2.0) there exists a mapping h of E_2 into E_1 with $H = \lambda f \in C^O(E_1)\ f:h$. With $K \subset E_2$ compact, we have ((3.3.3) and (2.1.1)).

$0 < m \equiv \inf y \in K.d_2'(.e_2 y, \omega_2)$

whence there exists compact $L \subset E_1$ such that $|.\phi x| < m$ for each x in $E_1 - L$. As

$|.\phi.hy| = |..H\phi y| = d_2'(.e_2 y, \omega_2) \geq m$

for each y in K, we have ${}_*hK \subset L$. It is now clear that h maps bounded subsets of E_2 onto bounded subsets of E_1, whence (4.3.1) h is continuous.

To prove h metrically injective, we let A,B be compact subsets of E_2 such that $0 < c \equiv d_2(A,B)$, apply (4.1.5) to construct g in $C^O(E_2)$ with

$\text{strc}\ g\ A \cup B = \lambda x \in A \cup B.d_2(x,B)$,

and then choose f in $C^O(E_1)$ with $.Hf = g$. With x in A and y in B, we have

$|.f.hx - .f.hy| = |.gx - .gy| = d_2(x,B)$

whence

$\text{dist}({}_*(f:h)A, {}_*(f:h)B) \geq c > 0.$

As both ${}_*hA$ and ${}_*hB$ are precompact subsets of E_1, there exists compact $K_1 \subset E_1$ with ${}_*h(A) \cup {}_*h(B) \subset K_1$. With δ a modulus of uniform continuity for f on K_1, it is now clear that

$0 < \delta(c/2) \leq d_1({}_*hA, {}_*hB)$,

whence h is metrically injective.

Conversely, suppose there exists a metrically injective, continuous mapping h of E_2 into E_1 such that

$H = \lambda f \in C^{O}(E_1)\ f{:}h$. Then (4.3.2) ${}_*hE_2$ is a locally compact subset of E_1 and h is a homomorphism of E_2 on ${}_*hE_2$. With $\varepsilon > 0$, and g any element of $C^{O}(E_2)$, we choose compact $S \subset E_2$ such that $|.gx| \leq \varepsilon$ for each x in $E_2 - S$, and then a modulus δ of uniform continuity for h on the bounded set

$S_1 \;\equiv\; \{x \in E_2 : d_2(x,S) \leq 1\}$.

With y in E_2 and $d_1(.hy, {}_*hS) > 1/2$, we must have

$0 < \min(1,\delta(1/2)) \leq d_2(y,S)$,

and therefore $|.gy| \leq \varepsilon$. Choosing r so that $r > 1/2$ and the set

$T \;\equiv\; \{x \in {}_*hE_2 : d_1(x, {}_*hS) \leq r\}$

is compact (3.1.0), we now see that

$\cap x \in {}_*h(E_2) - T(|.g.\ \mathrm{inv}\, h\ x| < \varepsilon)$.

Thus $g{:}\mathrm{inv}\ h$ - which is clearly continuous on ${}_*hE_2$ - belongs to $C^{O}({}_*hE_2)$. By (4.1.5) there exists f in $C^{O}(E_1)$ with strc $f\ {}_*hE_2 = g{:}\mathrm{inv}\ h$, whence

$.Hf = f{:}h = (\mathrm{strc}\ f\ {}_*hE_2){:}h = g$.

Thus H maps $C^{O}(E_1)$ onto $C^{O}(E_2)$. □

Remarks: (i) Although there is no reason why, in the case H maps $C^{O}(E_1)$ onto $C^{O}(E_2)$, the satisfactory condition $.H\phi_1 = \phi_2$ should obtain, we can choose a one-point compactification Y of E_2 relative to which $.H\phi_1$ is the distance from infinity: for, by (4.3.2) (4.3.3) and (3.5.1), the closure Y of ${}_*(e_1{:}h)(E_2) \cup \{\omega_1\}$ in X_1 is a one-point compactification of E_2 with point at infinity ω_1 and canonical injection $e_1{:}h$.

(ii) If Conjecture 2 of section 2.2 obtains, then we can replace the words 'metrically injective' by 'metrically weak-injective' throuthout (4.3.3) and its proof. In view of this, it is interesting to note the equivalence of the statements:

(a) a uniformly continuous, metrically weak-injective mapping of a compact space into a metric space is metrically injective.

(b) if X, Y are compact spaces, and h a uniformly continuous metrically weak-injective mapping of Y into X, then the star homomorphism $\lambda f \in C(X)\ f{:}h$ maps $C(X)$ onto $C(Y)$.

Indeed, that (a) entails (b) is a consequence of (4.3.3). On the other hand, if (b) holds and h is a uniformly continuous, metrically weak-injective mapping of the compact space Y into the metric space X, we lose no generality in taking X as the completion of ${}_*hY$, so that X is compact and ${}_*hY$ is dense in X. It now follows from (b) and (4.3.3) that h is metrically injective. ®

.4 *A star homomorphism H of $C^0(E_1)$ into $C^0(E_2)$ is injective if and only if it is an isometry. Moreover, if there exists a continuous mapping h of E_2 into E_1 such that $H = \lambda f \in C^0(E_1)\ f{:}h$ then a necessary and sufficient condition for H to be injective is that ${}_*hE_2$ be dense in E_1.*

Proof. To begin with, let $H = \lambda f \in C^0(E_1)\ f{:}h$ with h a continuous mapping of E_2 into E_1. Then ${}_*hE_2$ is located in E_1 ((4.3.2) and (3.1.1)). If H is injective, z belongs to E_1, and we suppose that $0 < d_1(z, {}_*hE_2)$, then there exists $r > 0$ such that

$\{x \in E_1 : d_1(x,z) < r\} \subset E_1 - {}_*hE_2$

With f the mapping

$\lambda x \in E_1\ \max(0, 1 - r^{-1}d_1(x,z))$,

we have $f \in C^0(E_1)$, $.fx = 0$ for each x in ${}_*hE_2$, and therefore

$.Hf = f{:}h = \lambda x \in E_2\ 0$.

On the other hand, $.fz = 1$, so that $f \neq \lambda x \in E_1\ 0$. This contradicts the injective nature of H, whence $d_1(z, {}_*hE_2) = 0$,

z belongs to $({}_*hE_2)^-$, and ${}_*hE_2$ is dense in E_1.

Conversely, if ${}_*hE_2$ is dense in E_1, f and g are elements of $C^O(E_1)$, and $.Hf = .Hg$, then

strc f ${}_*hE_2$ = strc g ${}_*hE_2$,

whence (by continuity) $f = g$.

To handle the general case (where the mapping h need not exist), for $k = 1,2$ we let u_k be an isometric star isomorphism of $C^O(E_k)$ onto $C(X_k,\omega_k) = \{g \in C(X_k):$ $.g\omega_k = 0\}$ (4.1.3) It is then straightforward to verify that

$H\# \equiv \lambda g{\in}C(X_1)(.u_2.H.\mathrm{inv}\ u_1\ \lambda x{\in}X_1(.gx - .g\omega_1){+}\lambda x{\in}X_2.g\omega_1)$

is a star homomorphism of $C(X_1)$ into $C(X_2)$ such that

$.H\#(\lambda x \in X_1\ 1) = \lambda x \in X_2\ 1$

$\cap f \in C^O(E_1)(.H\#.u_1 f = .u_2.Hf)$

and

$\cap g \in C(X_1)(..H\#g\omega_2 = .g\omega_1)$.

Given that H is injective, we now prove the same of $H\#$. To this end, with $g \in C(X_1)$ and $.H\#g = \lambda x \in X_2\ 0$, and supposing that $0 < |.g\omega_1|$, we have

$-1 = (.g\omega_1)^{-1}.(.u_2.H.\mathrm{inv}\ u_1\ \lambda x \in X_1(.gx - .g\omega_1))\omega_2 = 0$

(Remember that rng $u_2 = C(X_2,\omega_2)$!). This contradiction ensures that $.g\omega_1 = 0$, so that g belongs to $C(X_1,\omega_1)$, and

$.u_2.H.\mathrm{inv}\ u_1 g = \lambda x \in X_2\ 0$.

As u_1, H and u_2 are all injective, it now follows that $g = 0$. Hence $H\#$ is injective.

Now, by (4.3.0) and the first part of this proof, there exists a uniformly continuous mapping γ of X_2 into X_1 such that

$H\# = \lambda g \in C(X_1)\ g{:}\gamma$

and ${}_*\gamma X_2$ is dense in X_1. For each f in $C^o(E_1)$ we now have

$$\begin{aligned}
\| .Hf\|_{E_2} &= \| .(\text{inv } u_2).H\#.u_1 f\|_{E_2} \\
&= \| .H\#.u_1 f\| \\
&= \sup y \in X_2 | ..u_1 f.\gamma y| \\
&= \sup x \in {}_*\gamma X_2 | ..u_1 fx| \\
&= \| .u_1 f\|_{X_1} \\
&= \| f\|_{E_1}.
\end{aligned}$$

The proof is completed by noting that if H is isometric, then it is trivially injective. □

Remarks: (i) The propositions

(a) If E_1, E_2 are compact, and h is a continuous mapping of E_2 into E_1 such that $\lambda f \in C(E_1)\ f{:}h$ is injective, then ${}_*hE_2 = E_1$.

(b) If h is a continuous mapping of a compact metric space into R, then rng h is compact.

are equivalent. Indeed, that (b) entails (a) is a straightforward consequence of (4.3.4). On the other hand, that (a) entails (b) follows from the fact that, if h is a continuous mapping of a compact metric space E into R, then the closure E' of ${}_*hE$ in R is compact, ${}_*hE$ is dense in E', and (4.3.4) $\lambda f \in C(E')\ f{:}h$ is an injective mapping of $C(E')$ into $C(E)$.

It follows immediately that (a) is classically true, and essentially non-constructive.

(ii) Even when H is injective, there may not exist a mapping h of E_2 into E_1 with $H = \lambda f \in C^o(E_1)\ f{:}h$. To see this, take $X_1 \equiv [0,1]$, $E_2 \equiv [0,1] \cup \{2\}$, with metrics d_1', d_2 the corresponding restrictions of the usual metric on R; $E_1 \equiv]0,1]$, with metric

$$d_1 \equiv \lambda x,y \in E_1,,E_1(|x - y| + |x^{-1} - y^{-1}|);$$

and e_1 the mapping $\lambda x \in E_1\, x$ of E_1 into X_1. Then (E_2,d_2) is compact, and (3.3.4) (E_1,d_1) is locally compact, with one-point compactification $((X_1,d_1{}'),e_1,0)$. Defining

$$\gamma \equiv \lambda x \in E_2 \min(x,2 - x),$$

we see that γ is a uniformly continuous mapping of E_2 onto X_1, so that (4.3.4)

$$H\# \equiv \lambda g \in C(X_1)\; g{:}\gamma$$

is an injective star homomorphism of $C(X_1)$ into $C^O(E_2)$. With u_1 the unique isometric star isomorphism of $C^O(E_1)$ onto $\{g \in C(X_1) : .g0 = 0\}$ such that

$$\cap f \in C^O(E_1)(\text{strc}.u_1 f\; {}_*e_1 E_1 = f{:}\text{inv } e_1),$$

(4.1.3), it follows that

$$H \equiv \lambda f \in C^O(E_1)\; .H\#.u_1 f$$

is an injective star homomorphism of $C^O(E_1)$ into $C^O(E_2)$.

However, as

$$\begin{aligned} .H(\lambda x \in E_1.d_1{}'(.e_1 x,0)) &= .H\#(\lambda x \in X_1.d_1{}'(x,0)) \\ &= .H\#(\lambda x \in X_1 x) \\ &= \gamma \end{aligned}$$

and $.\gamma 2 = 0$, we conclude from (4.3.1) that there exists no mapping h of E_2 into E_1 with $H = \lambda f \in C^O(E_1)\; f{:}h$. ®

Putting together (4.3.3) and (4.3.4) we obtain

.5 *The Banach-Stone Theorem.*

In order that star homomorphism H of $C^O(E_1)$ into $C^O(E_2)$ be an (isometric) isomorphism, it is necessary and sufficient that there exist a homeomorphism h of E_2 on E_1 such that $H = \lambda f \in C^O(E_1)\; f{:}h$. □

For the discussion of this theorem in a classical setting, we refer the reader to ([10], IV.6.26 - 27).

CHAPTER 5

OPERATOR TOPOLOGIES IN CONSTRUCTIVE ANALYSIS

We recall that, in classical mathematics, the *weak operator topology* on the space of bounded linear mappings of a Hilbert space H into itself is the weak topology generated by the mappings $T \to \langle .Tx|y\rangle$, where x,y belong to H([19],§Q). The discussion of this topology naturally leads to that of the more general situation in which E,F and G are normed linear spaces, and we are interested in the weak topology generated on the set $\mathrm{Hom}(E,F)$ of bounded linear mappings of E into F by mappings $T \to .\phi.Tx$, where x and ϕ belong to certain given subsets of E and $\mathrm{Hom}(F,G)$ respectively. In this final chapter we shall consider the natural constructive substitutes for such topologies, paying particular attention to that for the weak operator topology (cf. Sections 5.4 - 5.6). Of especial interest to us (in keeping with the focus of our attention in Chapters 2 - 4) will be the question of precompactness of the unit ball

$$\mathrm{Hom}_1(E,F) \equiv \{T \in \mathrm{Hom}(E,F) : \cap x \in E(\|.Tx\| \leq \|x\|)\}$$

with respect to the various pseudometrics under discussion. †

5.1. Weak seminorms on Hom(E,F)

Apart from a small adaptation in the remark following (5.1.3), we shall adopt the following notation throughout the first four sections of this chapter: E,F and G will be normed linear spaces over C (the same symbol $\|\ \|$ being used for the norm in each case); $(a_n)_{1\leq n}, (a'_n)_{1\leq n}$ sequences in the unit ball of E; $(\phi_n)_{1\leq n}$, $(\phi'_n)_{1\leq n}$ sequences in $\mathrm{Hom}_1(F,G)$; and $[\![\ ]\!]$, $[\![\ ]\!]'$ the

† Note that the definitions of properties associated with pseudometrics are the obvious analogues of those associated with metrics.

seminorms defined on Hom(E,F) by

$$[\![T]\!] \equiv \Sigma_{j,k=1}^{\infty} 2^{-j-k} \|.\phi_j.Ta_k\|,$$

$$[\![T]\!]' \equiv \Sigma_{j,k=1}^{\infty} 2^{-j-k} \|.\phi_j'.Ta_k'\|$$

respectively.

In this first section, we smooth the path of our later discussion of particular examples of such *weak seminorms* by obtaining general answers to the following important questions:

Under what conditions do $[\![\ ]\!]$, $[\![\ ]\!]'$ *induce equivalent pseudometrics on* $\mathrm{Hom}_1(E,F)$?

Under what conditions is $\mathrm{Hom}_1(E,F)$ *precompact with respect to* $[\![\ ]\!]$?

The answer to the first of these questions is a consequence of

.0 *In order that a mapping f of a pseudometric space (X,d) into $(\mathrm{Hom}_1(E,F),[\![\ ]\!])$ be continuous (resp. uniformly continuous), it is necessary and sufficient that $(\lambda T \in \mathrm{Hom}_1(E,F).\phi_j.Ta_k):f$ be continuous (resp. uniformly continuous) for all positive integers j,k.*

Proof. The proof of necessity is comparatively trivial.

To prove sufficiency, given $\varepsilon > 0$ and compact $K \subset X$, and supposing that

$(\lambda T \in \mathrm{Hom}_1(E,F).\phi_j.Ta_k):f$

is continuous for all positive integers j,k, we choose a positive integer ν such that $\Sigma_{j=\nu+1}^{\infty} 2^{-j} < \varepsilon/8$, and then $\delta > 0$ with the property

$\cap x \in K \cap y \in X(d(x,y) \leq \delta \Rightarrow$

$\cap j,k, \in \mathrm{N}\text{~scsr}\nu(\|.\phi_j..fxa_k - .\phi_j..fya_k\| \leq \varepsilon/2).$

Noting that, for all S,T in $\mathrm{Hom}_1(E,F)$,

$$[\![S-T]\!] = (\Sigma_{j,k=1}^{\nu} + \Sigma_{j=1}^{\nu}\Sigma_{k=\nu+1}^{\infty} + \Sigma_{j=\nu+1}^{\infty}\Sigma_{k=1}^{\infty})2^{-j-k}\|.\phi_j.(S-T)a_k\|$$

$$\leq \Sigma_{j,k=1}^{\nu} 2^{-j-k}\|.\phi_j.(S-T)a_k\| + \Sigma_{j=1}^{\infty}2^{-j}\Sigma_{k=\nu+1}^{\infty} 2^{-k+1} + \Sigma_{k=1}^{\infty}2^{-k}\Sigma_{j=\nu+1}^{\infty}2^{-j+1}$$

$$\leq \Sigma_{j,k=1}^{\nu}2^{-j-k}\|.\phi_j.(S-T)a_k\| + \varepsilon/2$$

we now see that

$$\cap x \in K \cap y \in X(d(x,y) \leq \delta \Rightarrow [\![.fx - .fy]\!] \leq \varepsilon).$$

Thus f is continuous on X. The case of uniform continuity is similar, but simpler, and we shall omit the details. □

Immediate corollaries of (5.1.0) are

.1 *For all positive integers* j,k, $\lambda T \in \mathrm{Hom}_1(E,F).\phi_j.Ta_k$ *is uniformly continuous with respect to the seminorm* $[\![\,]\!]$. □

.2 *In order that the seminorms* $[\![\,]\!]$, $[\![\,]\!]'$ *induce equivalent pseudo-metrics on* $\mathrm{Hom}_1(E,F)$, *it is necessary and sufficient that for all positive integers* j,k *the functions* $\lambda T \in \mathrm{Hom}_1(E,F).\phi_j.Ta_k$ *and* $\lambda T \in \mathrm{Hom}_1(E,F).\phi_j'.Ta_k'$ *be uniformly continuous with respect to both* $[\![\,]\!]$ *and* $[\![\,]\!]'$. □

Remark: Even if $[\![\,]\!]$ and $[\![\,]\!]'$ do induce equivalent pseudometrics on $\mathrm{Hom}_1(E,F)$ - and therefore on all uniformly bounded subsets of $\mathrm{Hom}(E,F)$ - they need not be equivalent seminorms on $\mathrm{Hom}(E,F)$ (cf. remark (i) following (5.3.1)). ®

A general answer to the second question asked above is given by

.3 *A necessary and sufficient condition that* $\mathrm{Hom}_1(E,F)$ *be pre-compact in the pseudometric induced by the seminorm* $[\![\,]\!]$ *is that, for each positive integer* n, *the set*

$$M_n \equiv \{(.\phi_j.Ta_k)_{j,k,\in \text{ scsr } n \sim 1} : T \in \mathrm{Hom}_1(E,F)\}$$

be precompact in the metric induced by the norm $|||\ |||$, *where*

$$|||(.\phi_j.Ta_k)_{j,k,\in \text{ scsr } n \sim 1}||| \equiv \max\{\|.\phi_j.Ta_k\| : j,k,\in \text{scsr} n \sim 1\}$$

for each T in $\mathrm{Hom}_1(E,F)$.

Proof. The necessity of the condition is a simple consequence of (5.1.1). Conversely, supposing that M_n is $|||\ |||$ - precompact for each positive integer n, and given $\varepsilon > 0$, we choose a positive integer ν so that

$[\![S-T]\!] \leq \Sigma_{j,k=1}^{\nu} 2^{-j-k} \|.\phi_j.(S-T)a_k\| + \varepsilon/2$

for all S,T in $\mathrm{Hom}_1(E,F)$. With $T_0,\ldots,T_\alpha$ elements of $\mathrm{Hom}_1(E,F)$ such that

$\{(.\phi_j.T_r a_k)_{j,k,} \in \mathrm{scsr}\ n \sim 1 : r \in \mathrm{scsr}\ \alpha\}$

is an $\varepsilon/2$-approximation to $(M_\nu, |||\ |||)$, we let T be any element of $\mathrm{Hom}_1(E,F)$ and choose r in scsr α with

$$|||(.\phi_j.Ta_k)_{j,k,} \in \mathrm{scsr}\ n \sim 1 - (.\phi_j.T_r a_k)_{j,k,} \in \mathrm{scsr}\ n \sim 1||| < \varepsilon/2$$

Then

$[\![T - T_r]\!] \leq \Sigma_{j,k=1}^{\nu} 2^{-j-k}\varepsilon/2 + \varepsilon/2 < \varepsilon$

so that $\{T_0,\ldots,T_\alpha\}$ is an ε-approximation to $(\mathrm{Hom}_1(E,F),[\![\ ]\!])$, and the last-named is, indeed, precompact. □

Remark: Let X,Y,F and G be normed linear spaces over C, and E the set $\mathrm{Hom}(X,Y)$. We recall that an element S of E is *normable* if the *operator norm*

$$\|S\| \equiv \sup x \in X \wedge \|x\| \leq 1\ \|.Sx\|$$

of S is constructively well-defined. Now, it is by no means assured that this norm will be well-defined for each element of E; so that E need not be a constructive normed linear space with respect to the operator norm. However, theorems (5.1.0) - (5.1.3) remain valid if $\mathrm{Hom}(E,F)$ is interpreted as the set of linear mappings T of E into F which are bounded, in the sense that there exists $\delta > 0$ - called a *bound* of T - with the property

$$\cap S \in \mathrm{Hom}_1(X,Y)(\|.TS\| \leq \delta);$$

and $\mathrm{Hom}_1(E,F)$ is interpreted as the set of those T in $\mathrm{Hom}(E,F)$ for which δ may be chosen in $]0,1]$. We then extend the definitions at the beginning of this remark, and say that an element T of $\mathrm{Hom}(E,F)$ is *normable* if its *operator norm*

$$\|T\| \equiv \sup S \in \mathrm{Hom}_1(X,Y)\ \|.TS\|$$

is constructively well-defined. Where this situation arises in future, it should always be understood that it is these interpretations and definitions that we have in mind. ®

5.2. The weak* norm

Throughout this section, we shall consider the situation where E is separable, $(a_n)_{1\le n}$ is a dense sequence in the unit ball of E, and

$$[\![T]\!] \equiv \Sigma_{n=1}^{\infty} 2^{-n}\|Ta_n\|$$

for each T in $\mathrm{Hom}(E,F)$: this is just the situation of Section 5.1, with $F = G$ and $\phi_k = \lambda x \in F\ x$ for each k. Clearly, $[\![\ ]\!]$ is now a norm - the *weak* norm* defined by the sequence $(a_n)_{1\le n}$.

.0 *For each x in E, $\lambda T \in \mathrm{Hom}_1(E,F).Tx$ is uniformly continuous with respect to the weak* norm defined by the sequence $(a_n)_{1\le n}$ dense in the unit ball of E.*

Proof. Given x in the unit ball of E, and $\varepsilon > 0$, we choose k in N~1 so that $\|x-a_k\| \le \varepsilon/4$, and then a modulus δ of uniform continuity for $\lambda T \in \mathrm{Hom}_1(E,F).Ta_k$ with respect to the weak* norm $[\![\ ]\!]$ defined by $(a_n)_{1\le n}$ (5.1.1). Then, with S,T in $\mathrm{Hom}_1(E,F)$ and $[\![S-T]\!] \le \delta(\varepsilon/2)$, we have

$$\|.Sx-.Tx\| \le \|.(S-T)(x-a_k)\| + \|.(S-T)a_k\| \le \varepsilon.$$

The result follows almost immediately. □

From this and (5.1.2) we obtain.

.1 *All weak* norms defined on* $\mathrm{Hom}(E,F)$ *induce equivalent metrics on* $\mathrm{Hom}_1(E,F)$. □

Remarks: (i) (5.2.1) is our justification for referring in future to 'the weak* norm' on $\mathrm{Hom}(E,F)$, when we really mean 'any weak* norm'. We shall adopt similar, and similarly justified, abuses of language without further mention.

(ii) An argument similar to that of (5.2.1) proves that the metric induced on $\mathrm{Hom}_1(E,F)$ by the weak* norm is equivalent to that induced by the *double norm* :

$$\lambda T \in \mathrm{Hom}(E,F)\Sigma_{n=1}^{\infty} 2^{-n}(1 + \|x_n\|)^{-1}\|.Tx_n\|,$$

where$(x_n)_{1\leq n}$ is dense in E (cf. [1], Chapter 9, Section 4). ®

.2 *If F is complete, then $\mathrm{Hom}_1(E,F)$ is complete in the metric induced by the weak* norm.*

Proof. Let (T_n) be a sequence in $\mathrm{Hom}_1(E,F)$ that is Cauchy in the weak* norm $[\![\ ]\!]$. Then, for each x in E, $(.T_n x)$ is a Cauchy sequence in F (5.1.1), and so converges to an element of F. It is now easily seen that the mapping $x \to \lim_{n\to\infty}.T_n x$ belongs to $\mathrm{Hom}_1(E,F)$, and is the $[\![\ ]\!]$ - limit of the sequence (T_n). □

Our next two theorems answer the question of compactness of $\mathrm{Hom}_1(E,F)$ in the weak* norm.

.3 *If the separable normed linear space E is nontrivial - that is, contains an element of positive norm - and $\mathrm{Hom}_1(E,F)$ is pre-compact in the weak* norm, then F is finite dimensional, and $\mathrm{Hom}_1(E,F)$ is actually weak* norm compact.*

Proof. Choosing a in E with $\|a\| = 1$, we let X be the set $\{ta : t\in \mathbf{C}\}$, and u the linear functional which maps ta in X to t in $\mathbf{C}$. Then, by the Hahn-Banach Theorem (cf. Appendix 5), for each $c > 0$ there exists a normable linear functional u_c on E with strc $u_c X = u$ and $\|u_c\| \leq \|u\| + c = 1 + c$. For each y in the unit ball of F, the mapping $\lambda x \in E(.u_c x)y$ clearly belongs to

$\mathrm{Hom}_{1+c}(E,F)$, and maps a to y. Thus

$\cap c\in\mathrm{R}^+ \cap y\in F(\|y\|\leq 1 \Rightarrow \cup T(T\in\mathrm{Hom}_{1+c}(E,F) \wedge .Ta = y))$

With δ a modulus of uniform continuity for the mapping $\lambda T \in \mathrm{Hom}_2(E,F).Ta$ with respect to the weak* norm $[\![\ ]\!]$ (5.2.0) and ε an arbitrary positive number, we construct a $\delta(\varepsilon/3)$-approximation $\{T_0,\ldots,T_\nu\}$ to $\mathrm{Hom}_{1+\min(1,\varepsilon/3)}(E,F)$ in the norm $[\![\ ]\!]$. For each k in N~scsrν,$.T_k a$ then belongs to $\{x\in F : \|x\| \leq 1 + \varepsilon/3\}$, so that we may choose y_k in F with $\|y_k\| \leq 1$ and $\|y_k - .T_k a\| < 2\varepsilon/3$. Given y in the unit ball of F, we now choose T in $\mathrm{Hom}_{1+\min(1,\varepsilon/3)}(E,F)$ with $.Ta = y$, and then k in N~scsrν so that $[\![T-T_k]\!] < \delta(\varepsilon/3)$. Then

$\|y-y_k\| \leq \|.Ta-.T_k a\| + \|.T_k a - y_k\| < \varepsilon.$

Thus $\{y_0,\ldots,y_\nu\}$ is an ε-approximation to the unit ball of F, which ball is therefore precompact. It now follows that F is finite dimensional. The proof is completed with reference to (5.2.2). □

.4 *If E is separable and F is finite dimensional, then* $\mathrm{Hom}_1(E,F)$ *is compact in the weak* norm.*

Proof. When F has dimension 0, this is trivial. On the other hand, when F has finite dimension $n \geq 1$, the result readily follows from the special case where $n = 1$, which case has been fully dealt with by Bishop ([1], Chapter 9, Theorem 9 - note that Bishop uses the double norm in his proof, and cf. remark (ii) following (5.2.1)).□

5.3. The strong operator norm

Throughout this brief section we shall consider the situation where $(E,\langle\ |\rangle)$ is a complex Hilbert space, $(a_n)_{1\leq n}$ an orthonormal basis for E, and

$$[\![T]\!] \equiv \Sigma_{n=1}^{\infty} 2^{-n}\|.Ta_n\|$$

for each T in Hom(E,F). It should be clear that $\llbracket\ \rrbracket$ is a norm on Hom(E,F) - the *strong operator norm* defined by the orthonormal basis $(a_n)_{1\leq n}$ - and that, in the case where $E = F$, the metric induced on Hom$_1(E,E)$ by this norm is the constructive analogue of the strong operator topology on Hom$_1(E,E)$ (cf. [19], §9).

A by-now-familiar type of argument (which we omit) produces

.0 *For each x in the Hilbert space E, $\lambda T \in$ Hom$_1(E,F).Tx$ is uniformly continuous with respect to any strong operator norm.* □

This, with (5.2.0) and (5.1.2), yields

.1 *All strong operator norms on* Hom$_1(E,F)$ *induce metrics on* Hom$_1(E,F)$ *that are equivalent to those induced by the weak* norm.* □

Remarks: (i) If the Hilbert space E is infinite dimensional, then the strong operator norms defined by different orthonormal bases need not be equivalent norms on Hom(E,E). To see this, let $\llbracket\ \rrbracket$ be the strong operator norm defined by the orthonormal basis $(a_n)_{1\leq n}$ construct a strictly increasing sequence $(n_k)_{1\leq k}$ of positive integers such that $\|a_{n_k}\| = 1$ for each k (3.2.2), and define a sequence $(T_k)_{1\leq k}$ in Hom(E,E) by

$$T_k \equiv \lambda x \in E\ 2^{n_k/2} \langle x|a_{n_k}\rangle \quad a_{n_k}$$

Then $(\|.T_k a_{n_k}\|)_{1\leq k}$ diverges to ∞, whence - by the Uniform Boundedness Theorem (Appendix 5) - there exists a in E with $(\|.T_k a\|)_{1\leq k}$ unbounded, and therefore $(.T_k a)_{1\leq k}$ not convergent. We lose no generality in taking $\|a\| = 1$, so that we can construct an orthonormal basis $(a_n')_{1\leq n}$ in E with $a_1' = a$. Clearly, the sequence $(T_k)_{1\leq k}$ cannot converge in the strong operator norm defined by $(a_n')_{1\leq n}$. However, as

$$\Sigma_{j=1}^{\infty}\ 2^{-j} \|.T_k a_j\| = 2^{-n_k/2}$$

for each positive integer k, $(T_k)_{1\leq k}$ converges to 0 in the strong operator norm $\llbracket\ \rrbracket$. We therefore conclude that the two strong operator norms in question are not equivalent on $\mathrm{Hom}(E,F)$.

(ii) The following is an explicit proof that, if E is infinite dimensional, then $\mathrm{Hom}_1(E,E)$ is not precompact in the strong operator norm (cf. (5.3.1), (5.2.3) and (5.2.4)). Given ε such that $0 < \varepsilon < 2^{-n_1-1}\sqrt{2}$, and supposing that $\{T_0,\ldots,T_\nu\}$ is an ε-approximation to $\mathrm{Hom}_1(E,E)$ in the strong operator norm $[\![\ ]\!]$ defined by the orthonormal basis $(a_n)_{1\leq n}$, we set

$$S_k \equiv \lambda x \in E\langle x|a_{n_1}\rangle a_{n_k}$$

for each positive integer k. Then each S_k is in $\mathrm{Hom}_1(E,E)$; moreover, given positive integers j,k with $j\neq k$, and choosing r_j,r_k in scsrv so that $[\![S_j - T_{r_j}]\!] < \varepsilon$ and $[\![S_k - T_{r_k}]\!] < \varepsilon$, we have

$$\begin{aligned} 2^{-n_1}\sqrt{2} &= 2^{-n_1}\|.S_j a_{n_1} - .S_k a_{n_1}\| \\ &\leq [\![S_j - S_k]\!] \\ &< 2\varepsilon + [\![T_{r_j} - T_{r_k}]\!] \end{aligned}$$

and therefore

$$2^{-n_1-1}\sqrt{2} < [\![T_{r_j} - T_{r_k}]\!].$$

It readily follows from this that the set $\{T_0,\ldots,T_\nu\}$ must be infinite. This contradiction, in turn, establishes that $\mathrm{Hom}_1(E,E)$ is not $[\![\ ]\!]$-precompact. ®

5.4. The weak operator norm

Perhaps the most interesting of the situations considered in this chapter is that of this section, in which $(E,\langle|\rangle)$ is again a complex Hilbert space with orthonormal basis $(a_n)_{1\leq n}$, and

$$T \equiv \Sigma_{j,k=1}^{\infty} 2^{-j-k}|\langle .Ta_j | a_k\rangle|$$

for each T in $\mathrm{Hom}(E,E)$. In this case - which corresponds to the situations of Section 5.1 with $E = F$, $G = \mathbf{C}$ and $\phi = \lambda x \in E\langle x|a_k\rangle$ for each k - our seminorm $[\![\ ]\!]$ is again a norm, the *weak operator* *norm* defined by the orthonormal basis $(a_n)_{1\leq n}$; as far as the unit ball of Hom(E,E) is concerned, the metric induced by the weak operator norm is the constructive analogue of the weak operator topology (cf. [10], §9).

.0 *For all x,y in E, the mapping $\lambda T \in \mathrm{Hom}_1(E,E)\langle .Tx|y\rangle$ is uniformly continuous with respect to the weak operator norm defined by the orthonormal basis $(a_n)_{1\leq n}$.*

Proof. Given $\varepsilon > 0$ and points x,y of E, we choose a positive integer r so that

$$|\Sigma_{j=r+1}^{\infty}\langle x|a_j\rangle a_j| \leq (6(1+\|y\|))^{-1}\varepsilon$$

and then a positive integer s so that

$$|\Sigma_{k=s+1}^{\infty}\langle y|a_k\rangle a_k| \leq (6r(1+\|x\|))^{-1}\varepsilon.$$

With S,T in $\mathrm{Hom}_1(E,E)$ and

$$[\![S-T]\!] < 2^{-r-s}(3rs(1+\|x\|)(1+\|y\|))^{-1}\varepsilon,$$

we then have

$$|\langle .Sx|y\rangle - \langle .Tx|y\rangle|$$

$$\leq \Sigma_{j=1}^{r}|\langle x|a_j\rangle|\,|\langle .(S-T)a_j|y\rangle| + |\langle .(S-T)(\Sigma_{j=r+1}^{\infty}\langle x|a_j\rangle a_j)|y\rangle|$$

$$\leq \|x\|\,\Sigma_{j=1}^{r}\Sigma_{k=1}^{s}|\langle y|a_k\rangle|\,|\langle .(S-T)a_j|a_k\rangle|$$

$$+ \|x\|\,\Sigma_{j=1}^{r}|\langle .(S-T)a_j|\Sigma_{k=s+1}^{\infty}\langle y|a_k\rangle a_k\rangle|$$

$$+ 2\|\Sigma_{j=r+1}^{\infty}\langle x|a_j\rangle a_j\|\,\|y\|$$

$$\leq \|x\|\,\|y\|\,\Sigma_{j=1}^{r}\Sigma_{k=1}^{s}|\langle .(S-T)a_j|a_k\rangle| + 2\|x\|\,\Sigma_{j=1}^{r}\|\Sigma_{k=s+1}^{\infty}\langle y|a_k\rangle a_k\|$$

$$+ \varepsilon/3$$

$$< \varepsilon \quad \square$$

It follows from this and (5.1.2) that

.1 *All weak operator norms on* Hom(E,E) *induce equivalent metrics on* $\mathrm{Hom}_1(E,E)$. □

<u>Remarks</u>: (i) Similar arguments show that the metrics induced on $\mathrm{Hom}_1(E,E)$ by weak operator norms, norms of the form

$$\lambda T \in \mathrm{Hom}(E,E)\ \Sigma_{j,k=1}^{\infty}\ 2^{-j-k}|\langle .Tx_j|x_k\rangle|.$$

with $(x_n)_{1\leq n}$ dense in the unit ball of E, and norms of the form

$$\lambda T \in \mathrm{Hom}(E,E)\ \Sigma_{j,k=1}^{\infty} 2^{-j-k}(1 + \|x_j\|)^{-1}(1 + \|x_k\|)^{-1}|\langle .Tx_j|x_k\rangle|$$

with $(x_n)_{1\leq n}$ dense in E, are all equivalent.

(ii) If E is infinite dimensional, then the weak operator norms defined by different orthonormal bases of E need not be equivalent on $\mathrm{Hom}(E,E)$. This is seen by constructing a strictly increasing sequence $(n_k)_{1\leq k}$ of positive integers such that $\|a_{n_k}\| = 1$ for each k and then arguing (as in remark (i) following (5.3.1)) with the sequence $(\langle .T_k a_{n_k}|\xi\rangle)_{1\leq k}$, where

$$\xi \equiv \Sigma_{k=1}^{\infty}\ 2^{-n_k}\ a_{n_k}$$

and

$$T_k \equiv \lambda x \in E\ 2^{3n_k/2}\ \langle x|a_{n_k}\rangle\ a_{n_k}$$

for each positive integer k.

(iii) If $[\![\ ]\!]_s$, $[\![\ ]\!]_w$ are respectively the strong and weak operator norms defined on $\mathrm{Hom}(E,E)$ by the orthonormal basis $(a_n)_{1\leq n}$, then $[\![\ ]\!]_w$ is 'weaker' than $[\![\ ]\!]_s$, in the sense that the identity mapping $T \to T$ of $(\mathrm{Hom}_1(E,E),[\![\ ]\!]_s)$ onto $(\mathrm{Hom}_1(E,E),[\![\ ]\!]_w)$ is uniformly continuous. Moreover, if E is finite dimensional, then all strong and weak operator norms on $\mathrm{Hom}(E,E)$ are equivalent to the usual operator norm

$$\lambda T \in \mathrm{Hom}(E,E)\ \sup x \in E \wedge \|x\| \leq 1\ \|.Tx\|$$

However, if E is infinite dimensional, $(n_k)_{1\leq k}$ is a strictly increasing sequence of positive integers such that $\|a_{n_k}\| = 1$ for each k, and

$$T_k \equiv \lambda x \in E\langle x|a_{n_1}\rangle a_{n_k},$$

then T_k belongs to $\mathrm{Hom}_1(E,E)$, $([\![T_k]\!]_w)_{1\leq k}$ converges to 0, but

$$\|\cdot T_j a_{n_1} - \cdot T_k a_{n_1}\| = \sqrt{2}$$

whenever $j \neq k$; so that $(⟦T_k⟧_s)_{1\leq k}$ is not convergent. Thus $⟦\ ⟧_s$ and $⟦\ ⟧_w$ are not equivalent when E is infinite dimensional.

In fact, we can readily show that if E is nontrivial and $⟦\ ⟧_s, ⟦\ ⟧_w$ induce equivalent metrics on $\mathrm{Hom}_1(E,E)$, then E is finite dimensional. For, choosing r so that $\| a_r \| = 1$, and setting

$$S_k \equiv \lambda x \in E \ \langle x | a_r \rangle a_k$$

for each positive integer k, we see that $S \in \mathrm{Hom}_1(E,E)$ and - as $⟦S_k⟧_w \leq 2^{-r-k}$ for each k in $\mathrm{N}{\sim}1$ - $(⟦S_k⟧_s)_{1\leq k}$ converges to 0. Thus there exists ν in $\mathrm{N}{\sim}1$ such that

$$2^{-r}\| a_k \| = ⟦S_k⟧_s < 2^{-r}$$

for each k in $\mathrm{N}{\sim}\nu$. For such k, we then have $\| a_k \| < 1$, $a_k = 0$. Thus (3.2.2) E is finite dimensional. ®

It is a well-known theorem of classical mathematics that $\mathrm{Hom}_1(E,E)$ is compact in the weak operator topology ([19], §8, Theorem 13). As we shall see shortly, things are not quite so straightforward in constructive mathematics; our first step, however, is in accord with the classical situation.

.2 $\mathrm{Hom}_1(E,E)$ *is precompact in the weak operator norm.*

Proof. Given $\varepsilon > 0$ and a positive integer ν, we let E_0 be the finite dimensional subspace of E spanned by $\{a_1,\ldots,a_\nu\}$, and P the projection of E on E_0. Then every element of $\mathrm{Hom}(E_0,E_0)$ is normable, and $\mathrm{Hom}(E_0,E_0)$ is a finite dimensional Banach space under the usual operator norm, with compact unit ball $\mathrm{Hom}_1(E_0,E_0)$ (3.2.0). Let $\{T_0^\circ,\ldots,T_r^\circ\}$ be an ε-approximation in norm to $\mathrm{Hom}_1(E_0,E_0)$. For each k in scsr r, define $T_k \equiv T_k^\circ : P$. Then, given T in $\mathrm{Hom}_1(E,E)$, we have strc $P:T\ E_0$ in $\mathrm{Hom}_1(E_0,E_0)$, so that $\|\mathrm{strc}\ P:T\ E_0 - T_k^\circ\| < \varepsilon$ for some k in scsr r. For this same k, and integers m,n with $1 \leq m,n \leq \nu$, we have

$$\begin{aligned}
|\langle .Ta_m|a_n\rangle - \langle .T_k a_m|a_n\rangle| &= |\langle .Ta_m|.Pa_n\rangle - \langle .T_k^{\circ}.Pa_m|a_n\rangle| \\
&= |\langle .P.Ta_m|a_n\rangle - \langle .T_k^{\circ}a_m|a_n\rangle| \\
&= |\langle .\mathrm{strc}\ P{:}T\ E_0\ a_m|a_n\rangle - \langle .T_k^{\circ}a_m|a_n\rangle \\
&\le \|\mathrm{strc}\ P{:}T\ E_0 - T_k^{\circ}\| \\
&< \varepsilon.
\end{aligned}$$

Reference to (5.1.3) completes the proof. □

Remarks: (i) We recall that an operator T on E is *positive* if it is selfadjoint and $\langle .Tx|x\rangle \geq 0$ for each x in E- or, equivalently, if there exists an element S of Hom(E,E) with adjoint S^*,† such that $T = S^*S$. Now, using the notation of the proof of (5.4.2), we readily see that if T and T_k° are positive so are strc $P{:}T\ E_0$ and T_k. In order to prove that

the set $\mathrm{Hom}_1^+(E,E)$ of positive elements of Hom(E,E) is precompact in the weak operator norm,

it is therefore sufficient to prove that, when E is finite dimensional, $\mathrm{Hom}_1^+(E,E)$ is precompact in the usual operator norm. Accordingly, let us suppose that E is finite dimensional, so that each element of Hom(E,E) is normable and has an adjoint and

$$\phi \equiv \lambda T \in \mathrm{Hom}_1(E,E)\ T^*T$$

is a mapping of $\mathrm{Hom}_1(E,E)$ onto $\mathrm{Hom}_1^+(E,E)$. For each x in E, and S,T in $\mathrm{Hom}_1(E,E)$, we have

$$\begin{aligned}
|\langle ..\phi Sx|x\rangle - \langle ..\phi Tx|x\rangle| &= |\|.Sx\|^2 - \|.Tx\|^2| \\
&= (\|.Sx\| + \|.Tx\|)|\|.Sx\| - \|.Tx\|| \\
&\le 2\|x\|\ \|.Sx - .Tx\| \\
&< 2\|x\|^2\|S - T\|,
\end{aligned}$$

† We shall use the standard notation T^* for the adjoint of our operator T on H, when this adjoint is well-defined.

so that

$$\lambda T \in \mathrm{Hom}_1(E,E)\langle\, ..\phi Tx|x\rangle$$

is uniformly continuous on $\mathrm{Hom}_1(E,E)$ with respect to the operator norm. It follows from this, (5.1.0) and the equality

$$4\langle\, .Tx|y\rangle \;=\; \langle\, .T(x+y)|(x+y)\rangle \;-\; \langle\, .T(x-y)|(x-y)\rangle$$
$$+\; i\langle\, .T(x+iy)|(x+iy)\rangle \;-\; i\langle\, .T(x-iy)|(x-iy)\rangle$$

(valid for x,y in E and T in $\mathrm{Hom}(E,E)$) that ϕ is uniformly continuous as a mapping of $\mathrm{Hom}_1(E,E)$ (under the operator norm) into $(\mathrm{Hom}_1^+(E,E),[\![\;]\!])$, where $[\![\;]\!]$ is the weak operator norm. As the former set is compact, $\mathrm{Hom}_1^+(E,E)$ is precompact in the weak operator norm, and therefore in the usual operator norm (cf. remark (iii) following (5.4.1)).

(ii) In the proof of (5.4.2), note that T_k is normable, with $\|T_k\| = \|T_k^{\circ}\|$. Thus the normable elements are dense in $\mathrm{Hom}_1(E,E)$ in the weak operator norm. In view of remark (i) above, the same applies to $\mathrm{Hom}_1^+(E,E)$. ®

We omit the simple proof of

.3 $\mathrm{Hom}_1(E,E)$ *is closed in* $\mathrm{Hom}(E,E)$ *in the weak operator norm.* □

In discussing the possible compactness of $\mathrm{Hom}_1(E,E)$ in the weak operator norm, we are now left with the question of its completeness. Let us therefore consider a sequence (T_k) in $\mathrm{Hom}_1(E,E)$ that is Cauchy in the weak operator norm $[\![\;]\!]$. By (5.4.0)

$$\phi \equiv \lambda x,y \in E,,E\ \lim_{n\to\infty}\langle\, .T_n x|y\rangle$$

is a well-defined (and clearly sesquilinear) mapping of $E,,E$ into $\mathbf{C}$; moreover it is clear that (T_k) is $[\![\;]\!]$-convergent to an element T of $\mathrm{Hom}_1(E,E)$ if and only if

$$\cup T \in \mathrm{Hom}_1(E,E)\cap x,y, \in E(.\phi(x,y) = \langle\, .Tx|y\rangle\,)$$

By the Riesz Representation Theorem (Appendix 5), this last is equivalent to the condition that, for each x in E, the linear functional $\lambda y \in E.\phi(x,y)^*$ be normable.

Unfortunately, there does not seem to be any constructive means of proving normability of these functionals.

Another condition equivalent to the $[\![\ ]\!]$-convergence of (T_k) in $\mathrm{Hom}_1(E,E)$ is that, for each x in E, the series

$$\Sigma_{n=1}^{\infty}|\lim_{k\to\infty}\langle .T_k x|a_n\rangle|^2$$

be convergent. It is the examination of this condition that provides the somewhat unexpected solution to the problem of completeness of $\mathrm{Hom}_1(E,E)$ in the weak operator norm.

.4 *If E is an infinite dimensional Hilbert space, then the completeness of* $\mathrm{Hom}_1(E,E)$ *in the weak operator norm entails the limited principle of omniscience.*

Proof. In view of (3.2.2), we may assume without loss of generality that $\|a_n\| = 1$ for each n. Let $(n_k)_{1\le k}$ be an increasing sequence in $\{0,1\}$, set

$$\phi \equiv \lambda r \in \mathrm{N}{\sim}1(\Sigma_{j=1}^{r} j^{-2})^{-1},$$

and for each positive integer r, define a linear mapping T_r of E into E by

$$T_r \equiv \lambda x \in E(.\phi r)\Sigma_{j=1}^{r} j^{-1}\langle x|a_j\rangle\,\Sigma_{k=1}^{r}(n_{k+1}-n_k)^{1/2}a_k.$$

Then, for x in E and r in $\mathrm{N}{\sim}1$,

$$\begin{aligned}\|.T_r x\|^2 &= (.\phi r)^2|\Sigma_{j=1}^{r} j^{-1}\langle x|a_j\rangle|^2\Sigma_{k=1}^{r}(n_{k+1} - n_k)\\ &\le (.\phi r)^2(\Sigma_{m=1}^{r}|\langle x|a_m\rangle|^2)(\Sigma_{j=1}^{r} j^{-2})(n_{r+1} - n_1)\\ &\le (.\phi r)(n_{r+1} - n_1)\|x\|^2\\ &\le \|x\|^2\end{aligned}$$

Thus, T_r belongs to $\mathrm{Hom}_1(E,E)$. Now, with r,s positive integers such that $r > s$, we have

$$\begin{aligned}[\![T_r - T_s]\!] = {}&\Sigma_{j,k=1}^{s} 2^{-j-k}(n_{k+1} - n_k)^{1/2} j^{-1}|.\phi r - .\phi s| +\\ &(.\phi r)\Sigma_{j=1}^{r}\Sigma_{k=s+1}^{r} 2^{-j-k}(n_{k+1} - n_k)^{1/2} j^{-1} +\\ &(.\phi r)\Sigma_{j=s+1}^{r}\Sigma_{k=1}^{s} 2^{-j-k}(n_{k+1} - n_k)^{1/2} j^{-1}.\end{aligned}$$

Given $\varepsilon > 0$ and a modulus δ of uniform continuity for the mapping $\lambda t \in [1,2]t^{-1}$, we now choose a positive integer s_ε so that

$$\Sigma_{j=s_\varepsilon+1}^{\infty} 2^{-j} \leq \varepsilon/4$$

and, for all r,s in $\mathrm{N}{\sim}s_\varepsilon$,

$$|\Sigma_{j=1}^{r} j^{-2} - \Sigma_{j=1}^{s} j^{-2}| \leq \delta(\varepsilon/2).$$

Then, with r,s in $\mathrm{N}{\sim}s_\varepsilon$ and $r > s$, we have

$$\begin{aligned}[\![T_r - T_s]\!] &\leq |.\phi r - .\phi s|\Sigma_{j,k=1}^{\infty} 2^{-j-k} + \\ &\quad (.\phi r)\Sigma_{j=1}^{r}\Sigma_{k=s+1}^{r} 2^{-j-k} + (.\phi r)\Sigma_{j=s+1}^{r}\Sigma_{k=1}^{s} 2^{-j-k} \\ &\leq |.\phi r - .\phi s| + 2\Sigma_{j=s+1}^{\infty} 2^{-j} \\ &\leq \varepsilon.\end{aligned}$$

Thus $(T_r)_{1\leq r}$ is a $[\![\;]\!]$ -Cauchy sequence in $\mathrm{Hom}_1(E,E)$. We now let $\xi \equiv \Sigma_{j=1}^{\infty} j^{-1}a_j$. Then, for positive integers r,k with $r \geq k$, we have

$$\langle .T_r\xi|a_k\rangle = (.\phi r)\Sigma_{j=1}^{r}j^{-2}(n_{k+1} - n_k)^{1/2} = (n_{k+1} - n_k)^{1/2}$$

so that

$$\lim_{r\to\infty}\langle .T_r\xi|a_k\rangle = (n_{k+1} - n_k)^{1/2}.$$

From this it follows that if (T_r) is $[\![\;]\!]$ -convergent to an element T of $\mathrm{Hom}_1(E,E)$ then

$$\langle .T\xi|a_k\rangle = (n_{k+1} - n_k)^{1/2}$$

for each k; so that the series $\Sigma_{k=1}^{\infty}(n_{k+1} - n_k)$ must converge, its sum being $\|.T\xi\|^2$. As this last series is convergent if and only if

$$\sup k \in \mathrm{N}{\sim}1\; n_k = \lim_{k\to\infty}n_k = n_1 + \Sigma_{k=1}^{\infty}(n_{k+1} - n_k)$$

is well-defined, we conclude that the $[\![\;]\!]$-completeness of $\mathrm{Hom}_1(E,E)$ entails the proposition: every increasing sequence in $\{0,1\}$ has a well-defined least upper bound. That this, in turn, entails the limited principle of omniscience is well-known (cf. [1], pages 4-5). □

Remarks: (i) This theorem provides the only example known to the author of a classically compact subset of a metric space X, which is constructively both precompact and closed in X, but whose completeness is an essentially non-constructive property.

(ii) The remarks preceding (5.4.4), and (5.4.4) itself, combine to show that the constructive version of the Riesz Representation Theorem (as stated in Appendix 5) is the best we can hope for; more explicitly,

> if E is an infinite dimensional Hilbert space on which each bounded linear functional has the form $\lambda x \in E\langle x|a\rangle$ with a in E, then the limited principle of omniscience is constructively valid. ®

5.5 Linear functionals and the weak operator norm

It seems reasonable to expect that, as in the classical, so in the constructive theory of algebras of bounded linear operators on a Hilbert space H, an important role will be played by those linear functionals on such an algebra $\mathcal{R}$ which are uniformly continuous on $\mathcal{R} \cap \mathrm{Hom}_1(H,H)$ with respect to the weak operator norm; classically, these are precisely the ultraweakly continuous linear functionals on $\mathcal{R}$ ([9], Chapter 1, §3). In this section we shall look at such functionals in the case $\mathcal{R} = \mathrm{Hom}(H,H)$.

Throughout this and the next section, $(H,\langle\,|\,\rangle)$ will be a complex Hilbert space; $(a_n)_{1\leq n}$, $(a_n')_{1\leq n}$ orthonormal bases of H, defining weak operator norms $[\![\;]\!]_w$, $[\![\;]\!]_w'$ respectively; and, for each j,k in $\mathbb{N}\sim 1$, S_{jk}, S_{jk}' the mappings $x \to \langle x|a_j\rangle a_k$ and $x \to \langle x|a_j'\rangle a_k'$ respectively.

.0 *Let f be a linear functional on* $\mathrm{Hom}(H,H)$. *In order that f be uniformly continuous on* $\mathrm{Hom}_1(H,H)$ *with respect to the weak operator norm, it is necessary and sufficient that there exist* *a double sequence* $(c_{jk})_{j,k,} \in \mathrm{N}\sim 1$ *in* C *such that the series* $\Sigma_{j,k=1}^{\infty} c_{jk}\langle .Ta_j|a_k\rangle$ *is uniformly convergent to* $.fT$ *on* $\mathrm{Hom}_1(H,H)$; *in which case* $c_{jk} = .fS_{jk}$ *for each* j,k *in* N~1.

Proof. Given $\varepsilon > 0$, we choose a positive integer ν so that $\Sigma_{n=\nu+1}^{\infty} 2^{-n} < \varepsilon/2$. Then, with p,q elements of $\mathrm{N}\sim\nu$, and T any element of $\mathrm{Hom}_1(H,H)$, we have

$$\llbracket T - \Sigma_{j=1}^{p}\Sigma_{k=1}^{q}\langle .Ta_j|a_k\rangle S_{jk}\rrbracket_w$$

$$= \Sigma_{m,n=1}^{\infty}2^{-m-n}|\langle .Ta_m|a_n\rangle - \Sigma_{j=1}^{p}\Sigma_{k=1}^{q}\langle .Ta_j|a_k\rangle\langle .S_{jk}a_m|a_n\rangle|$$

$$= \Sigma_{m,n=1}^{\infty}2^{-m-n}|\langle .Ta_m|a_n\rangle - \Sigma_{j=1}^{p}\Sigma_{k=1}^{q}\langle .Ta_j|a_k\rangle\delta_{jm}\delta_{kn}|$$

$$= \Sigma_{m=1}^{p}\Sigma_{n=q+1}^{\infty}2^{-m-n}\langle .Ta_m|a_n\rangle + \Sigma_{m=p+1}^{\infty}\Sigma_{n=1}^{\infty}2^{-m-n}\langle .Ta_m|a_n\rangle$$

$$\leq \Sigma_{m=1}^{\infty}2^{-m}\varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

Thus the series

$$\Sigma_{j,k=1}^{\infty}\langle .Ta_j|a_k\rangle S_{jk}$$

is $\llbracket\,\rrbracket_w$ - uniformly convergent on $\mathrm{Hom}_1(H,H)$, its sum being T. If the linear functional f is $\llbracket\,\rrbracket_w$-uniformly continuous on $\mathrm{Hom}_1(H,H)$, it is now clear that

$$.fT = \Sigma_{j,k=1}^{\infty}(.fS_{jk})\langle .Ta_j|a_k\rangle,$$

the series being uniformly convergent on $\mathrm{Hom}_1(H,H)$.

Conversely, supposing that

$$f = \lambda T \in \mathrm{Hom}(H,H)\Sigma_{j,k=1}^{\infty}c_{jk}\langle .Ta_j|a_k\rangle$$

with each c_{jk} in C and the series uniformly convergent on $\mathrm{Hom}_1(H,H)$, we note first that

$$\cap j,k, \in \mathrm{N}\sim 1(c_{jk} = .fS_{jk}).$$

Moreover, choosing μ in N~1 so that

$$\cap S,T, \in \mathrm{Hom}_1(H,H)(|.f(S-T) - \Sigma_{j,k=1}^{\mu}c_{jk}\langle .(S-T)a_j|a_k\rangle| < \varepsilon/2)$$

and then a common modulus δ of uniform continuity for the functions $T \to \langle .Ta_j|a_k\rangle$, $1 \leq j,k \leq \mu$, on $(\mathrm{Hom}_1(H,H),\llbracket\,\rrbracket_w)$, and setting

$M \equiv \max\{|c_{jk}| : j,k, \in \mathrm{N} \wedge 1 \leq j,k \leq \mu\}$,

we have

$$|.fS - .fT| \leq \Sigma_{j,k=1}^{\mu} |c_{jk}| |\langle .(S-T)a_j | a_k\rangle| + \varepsilon/2$$
$$\leq \varepsilon$$

whenever S,T belong to $\mathrm{Hom}_1(H,H)$ and satisfy

$[\![S-T]\!]_w \leq \delta(2^{-1}M^{-1}\mu^{-2}\varepsilon)$.

This proves uniform continuity of f on $(\mathrm{Hom}_1(H,H),[\![\,]\!]_w)$. □

Remarks: (i) Classically, a linear functional f on $\mathrm{Hom}(H,H)$ is uniformly continuous on $(\mathrm{Hom}_1(H,H),[\![\,]\!]_w)$ if and only if there exist sequences $(x_n)_{1 \leq n}$, $(y_n)_{1 \leq n}$ in H such that $\Sigma\|x_n\|^2, \Sigma\|y_n\|^2$ both converge, and

$$f = \lambda T \in \mathrm{Hom}(H,H)\Sigma_{n=1}^{\infty}\langle .Tx_n | y_n\rangle$$

([9], Chapter 1, §3). No constructive proof is known as yet for this extremely elegant classical criterion.

(ii) Bearingin mind (5.4.2) and (5.5.0), we can easily show that the set of linear functionals on $\mathrm{Hom}(H,H)$ that are uniformly continuous on $(\mathrm{Hom}_1(H,H),[\![\,]\!]_w)$ is a Banach space under pointwise operations of addition and multiplication by scalars, and the operator norm (cf. remark following 5.1.3)). ®

In view of (5.2.4), we might hope that the set of linear functionals on $\mathrm{Hom}(H,H)$ that are uniformly continuous on $(\mathrm{Hom}_1(H,H),[\![\,]\!]_w)$ and have operator norm at most 1 would be precompact in some appropriate adaptation of the weak* norm. Whether or not this is the case, we can describe very natural weak seminorms with respect to which this set is precompact.

.1 *Let* $[\![\,]\!], [\![\,]\!]'$ *be the seminorms defined on the set* $V \equiv \mathrm{Hom}(\mathrm{Hom}(H,H),\mathrm{C}$ *by*

$$[\![f]\!] \equiv \Sigma_{j,k=1}^{\infty} 2^{-j-k}|.fS_{jk}|$$
$$[\![f]\!]' \equiv \Sigma_{j,k=1}^{\infty} 2^{-j-k}|.fS'_{jk}|.$$

Then $[\![\,]\!]$ *and* $[\![\,]\!]'$ *induce equivalent pseudometrics on the unit ball*

$V_1 \equiv \mathrm{Hom}_1(\mathrm{Hom}(H,H),\mathrm{C})$ of V. *Moreover, the restriction of either of these seminorms to the set* V^w *of elements of* V *which are uniformly continuous on* $(\mathrm{Hom}_1(H,H),[\![\ ]\!]_w)$ *is a norm.*

Proof. To begin with, let p,q be positive integers, and ε a real number such that $0 < \varepsilon < 1$. Choose a positive integer ν so that

$$\Sigma_{k=\nu+1}^{\infty}|\langle a_p'|a_k\rangle|^2 < \varepsilon^2/4$$

and

$$\Sigma_{k=\nu+1}^{\infty}|\langle a_q'|a_k\rangle|^2 < \varepsilon^2/4,$$

let P be the projection of H on the linear subspace spanned by $\{a_1,\ldots,a_\nu\}$, and let x be any point of H. Then

$$\begin{aligned}
&\|.S_{pq}'x - \Sigma_{j,k=1}^{\nu}\langle .S_{pq}'a_j|a_k\rangle .S_{jk}x\| \\
= &\|\langle x|a_p'\rangle a_q' - \Sigma_{k=1}^{\nu}\langle\langle\Sigma_{j=1}^{\nu}\langle x|a_j\rangle a_j|a_p'\rangle a_q'|a_k\rangle a_k\| \\
= &\|\langle x|a_p'\rangle a_q' - \Sigma_{k=1}^{\nu}\langle\langle .Px|a_p'\rangle a_q'|a_k\rangle a_k\| \\
= &\|\langle x|a_p'\rangle a_q' - .P(\langle .Px|a_p'\rangle a_q')\| \\
= &\|\langle x|a_p'\rangle a_q' - \langle x|.Pa_p'\rangle .Pa_q'\| \\
\leq &\|\langle x|a_p' - .Pa_p'\rangle a_q'\| + \|\langle x|.Pa_p'\rangle(a_q' - .Pa_q')\| \\
\leq &\|x\|\ \|a_p' - .Pa_p'\| + \|x\|\ \|a_q' - .Pa_q'\| \\
\leq &\varepsilon\|x\|.
\end{aligned}$$

Thus ε is a bound for the operator

$$S_{pq}' - \Sigma_{j,k=1}^{\nu}\langle .S_{pq}'a_j|a_k\rangle S_{jk}.$$

Now let δ be a common modulus of uniform continuity for the functions $f \to .fS_{jk}$, $1 \leq j,k \leq \nu$, on $(V_1,[\![\ ]\!])$ (5.1.1). Then, with f,g in V_1 and $[\![f - g]\!] \leq \delta(\nu^{-2}\varepsilon)$, we have

$$|.fS'_{pq} - .gS'_{pq}|$$
$$\leq |.f(S'_{pq} - \Sigma^{\nu}_{j,k=1}\langle .S'_{pq}a_j|a_k\rangle S_{jk})| +$$
$$|\Sigma^{\nu}_{j,k=1}\langle .S'_{pq}a_j|a_k\rangle (.fS_{jk} - .gS_{jk})| +$$
$$|.g(S'_{pq} - \Sigma^{\nu}_{j,k=1}\langle .S'_{pq}a_j|a_k\rangle S_{jk})|$$
$$\leq \varepsilon + \Sigma^{\nu}_{j,k=1}|.fS_{jk} - .gS_{jk}| + \varepsilon$$
$$\leq 3\varepsilon$$

Thus $f \to .fS'_{pq}$ is uniformly continuous on $(V_1,[\![\,]\!])$. In view of (5.1.2) and the remark following (5.1.3), it is now clear that $[\![\,]\!]$ and $[\![\,]\!]'$ induce equivalent pseudometrics on V_1.

On the other hand, if f belongs to V_1 and $[\![f]\!] = 0$, then $.fS_{jk} = 0$ for all positive integers j,k. If in addition, f is uniformly continuous on $(\mathrm{Hom}_1(H,H),[\![\,]\!]_w)$, then (5.5.0) for each T in $\mathrm{Hom}(H,H)$,

$$0 = \Sigma^{\infty}_{j,k=1}\langle .Ta_j|a_k\rangle .fS_{jk} = .fT,$$

from which it follows that $[\![\,]\!]$ is a norm on V^w. □

.2 *In the notation of* (5.5.1), V_1 *is* $[\![\,]\!]$*-precompact, and* V_1^w *is* $[\![\,]\!]$*-dense in* V_1.

Proof. Let ε be given in $\mathbf{R}^+$, and ν in $\mathbf{N}{\sim}1$. In view of the remark following (5.1.3), and (5.1.3) itself, it will suffice to find a subfinite subset $\{f_0,\ldots,f_r\}$ of V_1^w with the property: for each f in V_1 there exists m in $\mathbf{N}{\sim}\mathrm{scsr}\ r$ such that

$$|.fS_{jk} - .f_mS_{jk}| < \varepsilon$$

for all integers j,k with $1 \leq j,k \leq \nu$. To do this, we let P be the projection of H on the finite dimensional subspace H_0 spanned by $\{a_1,\ldots,a_\nu\}$. Then every element of $\mathrm{Hom}(H_0,H_0)$ is normable, as is every element of

$V^0 \equiv \mathrm{Hom}(\mathrm{Hom}(H_0,H_0),\mathrm{C})$,

$\mathrm{Hom}(H_0,H_0)$ and V^0 are finite dimensional Banach spaces with respect to the corresponding pointwise operations and operator norms, and the unit ball

$V_1^0 \equiv \mathrm{Hom}_1(\mathrm{Hom}(H_0,H_0),\mathrm{C})$

of V^0 is compact in the operator norm. With $\{f_0^0,\ldots,f_r^0\}$ an ε-approximation to V_1^0 in the operator norm, we set

$f_k \equiv \lambda T \in \mathrm{Hom}(H,H).f_k^0\ (P{:}\mathrm{strc}TH_0)$

for each k in N~scsr r. Then f_k clearly belongs to $\mathrm{Hom}_1(\mathrm{Hom}(H,H),\mathrm{C})$. On the other hand, as $T \to \langle .Tx|.Py\rangle$ is uniformly continuous on $(\mathrm{Hom}_1(H,H),[\![\ ]\!]_w)$ for each x,y in H (5.4.0), given $\varepsilon' > 0$, there exists $\delta > 0$ such that

$\Sigma_{j,k=1}^{\nu}|\langle .P.(S - T)a_j|a_k\rangle|^2 \leq \varepsilon'$

whenever S,T belong to $\mathrm{Hom}_1(H,H)$ and $[\![S - T]\!]_w \leq \delta$. For such S,T, and any x in H_0, we then have

$$\begin{aligned}
&\|.P.(S - T)x\|^2\\
&= \|\Sigma_{j,k=1}^{\nu}\langle x|a_j\rangle\langle .P.(S - T)a_j|a_k\rangle a_k\|^2\\
&= \Sigma_{k=1}^{\nu}|\Sigma_{j=1}^{\nu}\langle x|a_j\rangle\langle .P.(S - T)a_j|a_k\rangle|^2\\
&\leq \Sigma_{k=1}^{\nu}(\Sigma_{m=1}^{\nu}|\langle x|a_m\rangle|^2)(\Sigma_{n=1}^{\nu}|\langle .P.(S - T)a_n|a_k\rangle|^2)\\
&= \|x\|^2\Sigma_{j,k=1}^{\nu}|\langle .P.(S - T)a_j|a_k\rangle|^2\\
&\leq \varepsilon'\|x\|^2
\end{aligned}$$

whence $T \to P{:}\mathrm{strc}\ TH_0$ is uniformly continuous as a mapping of $(\mathrm{Hom}_1(H,H),[\![\ ]\!]_w)$ into $\mathrm{Hom}_1(H_0,H_0)$ (under the operator norm). It follows that f_k belongs to V_1^w.

Now let f be any element of V_1, and j,k integers with $1 \leq j,k \leq \nu$. Then, defining

$f^0 \equiv \lambda T \in \mathrm{Hom}(H_0,H_0).f(T{:}P)$,

we see that f^0 belongs to V_1^0, so that there exists m in N~scsrr with $\|f^0 - f_m^0\| < \varepsilon$. Noting that

$P:(\mathrm{strc}\ S_{jk}H_0):P = S_{jk}$,

we now have

$$
\begin{aligned}
&|.fS_{jk} - .f_m S_{jk}| \\
= &|.f(P:(\mathrm{strc}\ S_{jk}H_0):P) - .f_m^0(P:\mathrm{strc}\ S_{jk}H_0)| \\
= &|.f^0(P:\mathrm{strc}\ S_{jk}H_0) - .f_m^0(P:\mathrm{strc}\ S_{jk}H_0)| \\
\le &\|f^0 - f_m^0\| \\
< &\ \varepsilon
\end{aligned}
$$

The subfinite subset $\{f_0,\ldots,f_r\}$ of V_1^w therefore has the desired properties. □

Remarks: (i) In order that the mapping $f \to .fT$ be uniformly continuous on $(V_1^w, [\![\ ]\!])$ for each T in $\mathrm{Hom}(H,H)$, it is necessary and sufficient that H be finite dimensional. Indeed, the sufficiency of this last condition is clear; while, conversely, if I denotes the mapping $\lambda x \in Hx$, and $f \to .fI$ is uniformly continuous on $(V_1^w, [\![\ ]\!])$, then there exists $\delta > 0$ and a positive integer ν with the property:

$$\cap f,g,\in V_1^w(\cap j,k,\in \mathrm{scsr}\ \nu \sim 1(|.fS_{jk} - .gS_{jk}| < \delta) \Rightarrow |.fI - .gI| < 1).$$

Choosing n in $\mathbf{N}\sim\mathrm{scsr}\nu$, supposing that $\|a_n\| = 1$, and setting

$$F \equiv \lambda T \in \mathrm{Hom}(H,H)\langle .Ta_n | a_n\rangle\ ,$$

we now see that $1 = .FI < 1$. This contradiction ensures that $a_n = 0$ for each integer $n \ge \nu$, and therefore that H is finite dimensional (3.2.2).

In particular, when H is infinite dimensional, $[\![\ ]\!]$ and the (analogue of the) weak* seminorm on V do not induce equivalent pseudometrics on V_1.

(ii) If H is infinite dimensional, then the completeness of V_1^w in the seminorm $[\![\ ]\!]$ entails the limited principle of omniscience. For, with $(n_k)_{1\le k}$ an increasing sequence in $\{0,1\}$, supposing that $\|a_n\| = 1$ for each n and setting

$$f_k \equiv \lambda T \in \mathrm{Hom}(H,H)\Sigma_{j=1}^{k}(n_{j+1} - n_j)\langle .Ta_j|a_j\rangle$$

for each positive integer k, we readily see that $(f_k)_{1\leq k}$ is a $[\![\,]\!]$-Cauchy sequence in V_1^w; however, the $[\![\,]\!]$-convergence of (f_k) to an element f of V_1^w entails that

$$.fS_{pq} = \lim_{k\to\infty} .f_k S_{pq}$$

for each p,q, whence (5.5.0)

$$.f(\lambda x \in Hx) = \Sigma_{j=1}^{\infty}(n_{j+1} - n_j)$$

and therefore

$$\sup k{\in}\mathrm{N}{\sim}1\ n_k = n_1 + \Sigma_{j=1}^{\infty}(n_{j+1} - n_j)$$

is well-defined. ®

5.6. Linear functionals on linear subsets of Hom(H, H).

Throughout this final section, $\mathfrak{R}$ will be a linear subset of Hom(H,H), and f a linear functional on $\mathfrak{R}$ that is uniformly continuous on $(\mathfrak{R} \cap \mathrm{Hom}_1(H,H),[\![\,]\!]_w)$, and nonzero (in the sense that $.fR > 0$ for some R in $\mathfrak{R}$).

A natural question - whose classical answer is in the affirmative - is:

> does there exist a linear functional $f\#$ on Hom(H,H) which is uniformly continuous on $(\mathrm{Hom}_1(H,H),[\![\,]\!]_w)$ and satisfies strc $f\#\mathfrak{R} = f$?

Using (5.5.0), it is not difficult to show that if such an extension $f\#$ of f exists, and ε belongs to R^+, then there exists a positive integer ν and a finite double sequence $(c_{jk})_{j,k}, \in$ scsrν ~1 in C such that

$$\Sigma_{j,k=1}^{\nu}|c_{jk}| = 1,$$

and

$$|\Sigma_{j,k=1}^{\nu} c_{jk}\langle .Ra_j|a_k\rangle| \leq \varepsilon$$

for each R in the *unit kernel* $*f(\{0\}) \cap \mathrm{Hom}_1(H,H)$ of f.

Although the question of the existence of $f\#$ remains open (within the framework of constructive analysis), we are able to give a partial answer, in the form of a converse to this last result (5.6.1). As a first step towards this answer, we require the lemma:

.0 *If* $\mathcal{R} \cap \mathrm{Hom}_1(H,H)$ *is precompact in the weak operator norm, then the same holds of the unit kernel of* f.

Proof. It is clear that

$$c \equiv \sup R \in \mathcal{R} \cap \mathrm{Hom}_1(H,H) \ |.fR|$$

is a well-defined positive number, and that there exists R_0 in $\mathcal{R} \cap \mathrm{Hom}_1(H,H)$ with $.fR_0 = c/2$. On the other hand, with ε given in R^+, the set

$$A_t \equiv \{R \in \mathcal{R} \cap \mathrm{Hom}_1(H,H) : |.fR| \leq t\}$$

is weak operator precompact for some t with $0 < t < (1+4c^{-1})^{-1}\varepsilon$. For such t, let $\{T_0,\ldots,T_\nu\}$ be a t-approximation to A_t in the weak operator norm $[\![\ ]\!]_w$, and set

$$T'_k \equiv (1+2c^{-1}t)^{-1}(T_k - 2c^{-1}(.fT_k)R_0)$$

for each k in N~scsrν. Then $.fT'_k = 0$, and

$$\|.T'_kx\| \leq (1+2c^{-1}t)^{-1}(\|.T_kx\| + 2c^{-1}t\|.R_0x\|) \leq \|x\|$$

for each x in H. Thus T'_k belongs to $*f(\{0\}) \cap \mathrm{Hom}_1(H,H)$. Let T be any element of this last set. Then T belongs to A_t, so that $[\![T - T_k]\!]_w < t$ for some k in N~scsrν. For this same k, we have

$$\begin{aligned}
[\![T - T'_k]\!]_w &< t + [\![2c^{-1}(1+2c^{-1}t)^{-1}(tT_k+(.fT_k)R_0)]\!]_w \\
&\leq t + 2c^{-1}(t[\![T_k]\!]_w + |.fT_k|[\![R_0]\!]_w) \\
&\leq (1+ 4c^{-1})t \\
&< \varepsilon
\end{aligned}$$

Thus $\{T'_0,\ldots,T'_\nu\}$ is an ε-approximation to $*f(\{0\}) \cap \mathrm{Hom}_1(H,H)$ in the weak operator norm. □

.1 *If* $\mathcal{R} \cap \mathrm{Hom}_1(H,H)$ *is precompact in the weak operator norm and* ε *belongs to* R^+, *then there exists a positive integer* ν *and a finite double sequence* $(c_{jk})_{j,k}, \in$ scsrν ~1 *in* C *such that*

$$\Sigma_{j,k=1}^{\nu}|c_{jk}| = 1,$$

and

$$|\Sigma_{j,k=1}^{\nu} c_{jk}\langle .Ra_j|a_k\rangle| \leq \varepsilon$$

for each R *in the unit kernel of* f.

Proof. Let δ be a modulus of uniform continuity for f on $(\mathcal{R} \cap \mathrm{Hom}_1(H,H), [\![\,]\!]_w)$, and ν a positive integer such that

$$[\![S - T]\!]_w \leq \Sigma_{j,k=1}^{\nu} 2^{-j-k}|\langle .(S - T)a_j|a_k\rangle| + 2^{-1}\delta(\varepsilon/2)$$

for all S,T in $\mathrm{Hom}_1(H,H)$. Let M be the set map(scsrν~1)C; for convenience, we shall write c_{jk} for $.c(j,k)$, whenever c belongs to M and j,k to scsrν ~1. For each c in M define

$$\|c\| \equiv \Sigma_{j,k=1}^{\nu}|c_{jk}|,$$

$$\|c\|_0 \equiv \sup R \in {}^*f(\{0\}) \cap \mathrm{Hom}_1(H,H)|\Sigma_{j,k=1}^{\nu} c_{jk}\langle .Ra_j|a_k\rangle|.$$

(The latter is well-defined in virtue of (5.6.0) and (5.4.0). Then $\|\,\|$ and $\|\,\|_0$ are seminorms on M, (taken with pointwise operations of addition and multiplication by scalars) $\|c\|_0 \leq \|c\|$ for each c in M, $(M,\|\,\|)$ is a finite dimensional Banach space, and

$$\beta \equiv \inf c \in M \wedge \|c\| = 1\ \|c\|_0$$

is well-defined. To complete the proof, it will suffice to show that $\beta < \varepsilon$. Moreover, as either $\beta > 0$ or $\beta < \varepsilon$, we lose no generality in supposing that $\beta > 0$. Thus $\|c\| \leq \beta^{-1}\|c\|_0$ for each c in M, $\|\,\|$ and $\|\,\|_0$ are equivalent norms on M, and $(M,\|\,\|_0)$ is a finite dimensional Banach space. Each linear functional ϕ on M is therefore $\|\,\|_0$-normable, in the sense that

$$\|\phi\|^0 \equiv \sup c \in M \wedge \|c\|_0 \leq 1\ |.\phi c|$$

is well-defined.

Let V be the set of (bounded) linear mappings of M into $\mathbf{C}$, S_0 the unit ball $\{\phi \in V : \|\phi\|^0 \leq 1\}$ of $(V, \| \|^0)$, and U the unit kernel $^*f(\{0\}) \cap \mathrm{Hom}_1(H,H)$ of f. Define a mapping

$\psi \equiv \lambda R \in U\ \lambda c \in M\ \Sigma_{j,k=1}^{\nu} c_{jk}\ \langle Ra_j | a_k \rangle$

of U into S_0. We shall show that rng ψ is $\| \|^0$-dense in S_0. To this end, we first prove that ψ is uniformly continuous as a mapping of $(U, [\![\ ]\!]_w)$ into $(S_0, \| \|^0)$. Given $\varepsilon' > 0$, we choose in turn an ε'-approximation $\{c^0, \ldots, c^r\}$ to the unit ball of $(M, \| \|_0)$, a common modulus $\delta_{\varepsilon'}$ of uniform continuity for the functions $T \to {..}\psi Tc^j$, $j \in$ scsr r, on $(U, [\![\ ]\!]_w)$ (5.4.0), and elements S, T of U with $[\![S - T]\!]_w \leq \delta_{\varepsilon'}(\varepsilon')$. Then, with c any element of the unit ball of $(M, \| \|_0)$, and t chosen in scsr r so that $\|c - c^t\|_0 < \varepsilon'$, we have

$|{..}\psi Sc - {..}\psi Tc|$

$\leq 2|\Sigma_{j,k=1}^{\nu}(c_{jk} - c_{jk}^t)\langle 2^{-1}(S - T)a_j|a_k\rangle| + |{..}\psi Sc^t - {..}\psi Tc^t|$

$\leq 2\|c - c^t\|_0 + \varepsilon'$

$\leq 3\varepsilon'$.

Thus

$\|.\psi S - .\psi T\|^0 \leq 3\varepsilon'$,

whence, clearly ψ is uniformly continuous as a mapping of $(U, [\![\ ]\!]_w)$ into $(S_0, \| \|^0)$, and (5.6.0) rngψ is $\| \|^0$-precompact. To complete the proof that rngψ is $\| \|^0$-dense in S_0, we let ϕ belong to S_0 and suppose that

$0 < \gamma \equiv \inf R \in U \|\phi - .\psi R\|^0$.

Then, with the help of the Separation Theorem (Appendix 5), we can construct a normable linear functional F on $(V, \| \|^0)$ such that

$.F\phi > |.F.\psi R| + \gamma/2$

for each R in U. As $(M, \|\|_0)$ is finite dimensional, the operator norm $\|\|^0$ and the double norm on V are equivalent so that (Appendix 5) there exists c' in M with

$F = \lambda g \in V.gc'$. It follows that

$.\phi c' \geq \sup R \in U|..\psi Rc'| + \gamma/2 > \|c'\|_0$

- a contradiction of our original choice of ϕ as an element of S_0. Thus

$\inf R \in U \|\phi - .\psi R\|^0 = 0,$

and rngψ is $\|\|^0$-dense in S_0.

As f is nonzero, we lose no generality in supposing that $.fR_0 = 1$ for some R_0 in $\mathcal{R} \cap \mathrm{Hom}_1(H,H)$. We then have

$\sup c \in M \wedge \|c\|_0 \leq 1 \ |\Sigma_{j,k=1}^{\nu} c_{jk} \langle .R_0 a_j | a_k \rangle|$

$\leq \sup c \in M \wedge \|c\| \leq \beta^{-1} \ |\Sigma_{j,k=1}^{\nu} c_{jk} \langle .R_0 a_j | a_k \rangle|$

$\leq \beta^{-1}.$

Thus

$\sigma \equiv \lambda c \in M \ \Sigma_{j,k=1}^{\nu} c_{jk} \langle \beta R_0 a_j | a_k \rangle$

belongs to S_0, and we can choose R_1 in U with

$\|\sigma - .\psi R_1\|^0 < 2^{-1}\delta(\varepsilon/2).$

Now, for each c in M with $\|c\| \leq 1$, we have $\|c\|_0 \leq 1$ and therefore

$|\Sigma_{j,k=1}^{\nu} c_{jk} \langle .(\beta R_0 - R_1) a_j | a_k \rangle| < 2^{-1}\delta(\varepsilon/2).$

In particular, choosing real numbers θ_{jk} so that

$\langle .(\beta R_0 - R_1) a_j | a_k \rangle = |\langle .(\beta R_0 - R_1) a_j | a_k \rangle| \exp(i\theta_{jk})$

and setting

$c_{jk} = 2^{-j-k}\exp(-i\theta_{jk})$

for each j,k in scsrν ~1, we have

$\Sigma_{j,k=1}^{\nu} |c_{jk}| = \Sigma_{j,k=1}^{\nu} 2^{-j-k}$

≤ 1

and

$$\Sigma_{j,k=1}^{\nu} 2^{-j-k} |\langle \cdot(\beta R_0 - R_1)a_j | a_k \rangle| = \Sigma_{j,k=1}^{\nu} c_{jk} \langle \cdot(\beta R_0 - R_1)a_j | a_k \rangle$$
$$< 2^{-1}\delta(\varepsilon/2).$$

As βR_0 and R_1 both belong to $\mathcal{R} \cap \mathrm{Hom}_1(H,H)$ (it is clear that $\beta \leq 1$), it follows that

$$[\![\beta R_0 - R_1]\!]_w \leq \delta(\varepsilon/2),$$

and therefore that

$$\beta = \cdot f(\beta R_0 - R_1) \leq \varepsilon/2 < \varepsilon.$$

This completes the proof. □

Comparison of the above proof and that of Theorem 10, page 287 of [1] (on which it is based) suggests that we might be able to adapt our argument to produce an affirmative answer to the question asked at the beginning of this section, under the extra condition that $\mathcal{R} \cap \mathrm{Hom}_1(H,H)$ be precompact in the weak operator norm. Unfortunately, all our attempts at such adaptation have met with frustration at some stage or another, and we have been forced to the conclusion that a full answer to our question may well require an approach altogether different from that used above and by Bishop.

EPILOGUE

Amongst the many questions outstanding after our discussion of constructive set theory and analysis, one more than any other cries out for an answer: *what is the value of constructive, as opposed to classical, mathematics?* Indeed, might it not be reasonable to argue, as did Hilbert in 1927, that

> 'Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists'?

Perhaps the perfect answer to this last question is to be found in the depth and scope of the analysis developed by Bishop [1]. However, this in itself does not provide the answer to our original question: to deal with that we contrast the remark of Bertrand Russell,

> '...mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true',

with what we have termed 'Bishop's Thesis':

> 'The primary concern of mathematics is number...'.

Here then lies our answer: prior to the work of Brouwer, Bishop and other constructive mathematicians, and in spite of its outward grandeur, mathematics was ultimately devoid of real meaning. In sharp contrast, the adoption of a constructivist philosophy enables us to view mathematics, not as an empty, abstract discipline

> 'full of sound and fury, signifying nothing',

but as a fulfilment of the apparently inborn concern of man with numeracy.

Whether or not there will come what Bishop sees as 'the inevitable day when constructive mathematics will be the accepted norm' remains the subject of another story; the one we wished to offer the reader is ended.

APPENDIX 1

AXIOMS OF SET THEORY.

The following is a complete list of the axioms of our set theory, as developed in Chapter 1, each axiom being indexed by the reference number under which it originally appeared in that chapter.

Definitional Axioms

(1.7.12) $((x \subset y) \equiv \cap t((t \in x) \rightarrow (t \in y)))$

(1.7.16) $((x = y) \equiv ((x \subset y) \wedge (y \subset x)))$

(1.7.25) $(\Pi A \equiv \cap y(y \in A \rightarrow y))$

(1.7.27) $(\cup A \equiv \cup y(y \in A \wedge y))$

(1.7.33) $(\mathrm{U} \equiv \cup xx)$

(1.7.34) $((x \leftrightarrow y) \equiv ((x \rightarrow y) \wedge (y \rightarrow x)))$

(1.10.0) $(\mathrm{sng}\ x \equiv \cap y(y \rightarrow (x \in y)))$

(1.10.4) $(\mathrm{sngl}\ x \equiv \cap y((x \in y) \rightarrow y))$

(1.10.8) $(\mathrm{singleton\ is}\ a \equiv (\Pi a = \nabla a \in a))$

(1.11.0) $(\mathrm{E}x\ \underline{u}x \equiv \cup x(0 \in \underline{u}x \wedge \mathrm{sng}\ x))$

(1.13.0) $(\{x\} \equiv \mathrm{sngl}\ x)$

(1.13.1) $(\{xx'\} \equiv (\mathrm{sngl}\ x \vee \mathrm{sngl}\ x'))$

(1.13.2) $((x{\scriptstyle\nearrow}y) \equiv \{\{x\}\{xy\}\})$

(1.13.8) $(A{\scriptstyle\nearrow\nearrow}B \equiv \mathrm{E}x{\scriptstyle\nearrow}y(x \in A \wedge y \in B))$

(1.14.0) $(\mathrm{ss}\ a \equiv (\mathrm{sng}\ 0 \vee \cup x(x \in a \wedge \mathrm{sng\ sng}\ x)))$

(1.14.1) $((a,b) \equiv ((\mathrm{sng}\ 0{\scriptstyle\nearrow\nearrow}\mathrm{ss}\ a) \vee (\mathrm{sng\ sng}\ 0{\scriptstyle\nearrow\nearrow}\mathrm{ss}\ b)))$

(1.14.3) $((\mathrm{orderedpair\ is}\ p \equiv \cup a \cup b(p = a,b))$

(1.16.0) $(\mathrm{relation\ is}\ R \equiv \cap x \in R\ \mathrm{orderedpair\ is}\ x)$

(1.16.2) $(\mathrm{dmn}\ R \equiv \mathrm{E}x\cup y(x,y \in R))$

(1.16.4) $(\mathrm{rng}\ R \equiv \mathrm{E}y\cup x(x,y \in R))$

(1.16.6) $(\mathrm{vs}\ Rx \equiv \mathrm{E}y(x,y \in R))$

(1.17.0) $(\mathrm{function\ is}\ f \equiv (\mathrm{relation\ is}\ f \wedge \cap x \in \mathrm{dmn}\ f\ \mathrm{singleton}$
$\mathrm{is\ vs}\ fx))$

(1.17.3) $(.fx \equiv \Pi\text{vs } fx)$

(1.17.9) $(\text{on } A \text{ to } B \text{ is } f \equiv (\text{function is } f \wedge \text{dmn } f = A \wedge \text{rng}f \subset B)$

(1.20.0) $(\mathrm{N} \equiv \cap A; (0 \in A \wedge \cap x \in A((x \vee \text{sng } x) \in A))A)$

Axiom of Definition

(1.7.35) $((x \equiv y) = (x = y)$

Set Theoretic Axioms of Set Theory

(1.8.0) $(x \leftrightarrow (0 \in x))$

(1.8.1) $((t \in \mathrm{U}) \rightarrow ((t \in (x \in y)) \leftrightarrow (x \in y)))$

(1.8.2) $((t \in a) \rightarrow ((t \in (x \rightarrow y)) \leftrightarrow ((t \in x) \rightarrow (t \in y))))$

(1.8.3) $((t \in \cap x\ \underline{u}x) \leftrightarrow \cap x(t \in \underline{u}x))$

(1.8.4) $((t \in \cup x\ \underline{u}x) \leftrightarrow \cup x(t \in \underline{u}x))$

(1.8.5) $((t \in (x \wedge y)) \leftrightarrow ((t \in x) \wedge (t \in y)))$

(1.8.6) $((t \in (x \vee y)) \leftrightarrow ((t \in x) \vee (t \in y)))$

(1.9.0) $((x \in \mathrm{U}) \rightarrow ((x = y) \leftrightarrow \cap t((x \in t) \rightarrow (y \in t))))$

(1.9.1) $((x = y) \rightarrow (\underline{u}x = \underline{u}y))$

(1.12.0) $(((A \in \mathrm{U}) \wedge \cap x((x \in A) \rightarrow (\underline{u}x \in \mathrm{U}))) \rightarrow (\cup x((x \in A) \wedge \underline{u}x) \in \mathrm{U})$

(1.12.1) $(((x \in \mathrm{U}) \wedge (y \in \mathrm{U})) \leftrightarrow ((x \vee y) \in \mathrm{U}))$

(1.12.2) $((x \in \mathrm{U}) \leftrightarrow (\text{sngl } x \in \mathrm{U}))$

(1.21.10) $(((S \subset A) \wedge \cap x(((x \in A) \wedge ((x \wedge A) \subset S)) \rightarrow (x \in S))) \rightarrow (S = A)$

(1.23.10) $(\mathrm{N} \in \mathrm{U})$

(1.24.0) $((A \in \mathrm{U}) \wedge (B \in \mathrm{U}) \rightarrow \mathrm{E}f(\text{on } A \text{ to } B \text{ is } f) \in \mathrm{U})$

(1.28.0) $(a \in A \wedge \cap x(x \in A \rightarrow \cup y(y \in A \wedge \underline{u}'xy))$

$\rightarrow \cup f(\text{on N to } A \text{ is } f \wedge .f0 = a \wedge \cap n(n \in \mathrm{N} \rightarrow \underline{u}'.fn.f\text{scsr}n))$

APPENDIX 2

AXIOMATIC PROOF THEORY

In keeping with our philosophy that 'every mathematical/logical object may be regarded as either a set or a proposition', we expect to be able to include proofs as terms of the set theory described in Chapter 1. In this appendix we describe an axiomatic approach to such a proof theory, and sketch briefly some consequences of our proof-theoretic axioms.

Although our proof theory has certain interesting features (notably, a derivable characterisation (A2.4.0) of proofs of the proposition (p $\rightarrow$ q)), we feel that its inclusion in an appendix, rather than in the main body of our dissertation, is justified by its one considerable defect: an inability to describe proofs of such fully general propositions as $\cap x \underline{u} x$ and $\cup x \underline{u} x$ (cf. remark (ii) in Section A2.3).

Note that we shall use the strictly formal notation of Chapter 1 throughout this appendix.

A2.1 Preliminary definitions

The description of our proof theory requires a primitive constant π , introduced by the orienting definition

.0 $(\pi(p) \equiv \pi(p))$

and explained by the definitions

.1 (The set of proofs of $p \equiv \pi(p)$)

.2 (x proves $p \equiv (x \in \pi(p))$)

We also require

.3 (basic is $x \equiv \cap t \in x(\pi(t \in x) = \text{sng } t)$)

This last definition - our formal expression of Bishop's notion of a *basic set* as 'a set for which no computations are necessary to check that an element belongs to the set' ([3], page 71) - is introduced with a view to the derivation of the Implication Theorem (A2.4.0) and a strong principle of choice from basic sets (the latter being necessary for the translation of the propositional axioms (1.6.16) and (1.6.22) into our proof-theoretic language - cf. proofs of (A2.7.2) and (A2.7.8)).

A2.2 Definitional axioms for proof theory

.0 $(\text{inhabited is } A \equiv \cup x(x \in A))$

.1 $(\text{htd} \equiv \mathrm{E}A(\text{inhabited is } A))$

.2 $(\text{basicorderedpair is } p \equiv \cup x \cup y(p = x,y \in \mathrm{U}))$

.3 $(\text{basicrelation is } R \equiv \cap p(p \in R \rightarrow \text{basicorderedpair is } p))$

.4 $(\text{bsvs } Rx \equiv \mathrm{E}y(x,y \in R))$

.5 $(\text{bsdmn } x \equiv \mathrm{E}t(\text{bsvs } xt \in \text{htd}))$

.6 $(\text{tuple is } x \equiv (\text{basicrelation is } x \wedge$

$\cap t \in \text{bsdmn } x(\text{bsvs } xt = \text{ss } \nabla\text{bsvs } xt)))$

7 $(\text{tuple } a \text{ is } x \equiv (\text{tuple is } x \wedge \text{bsdmn } x = a))$

.8 $(\text{crd } tx \equiv \nabla\text{bsvs } xt)$

.9 $(A,,B \equiv \mathrm{E}x,y(x \in A \wedge y \in B))$

A2.3 Axioms of proof theory

.0 $(\pi(p) = \pi(0 \in p))$

.1 $(\pi(p) \in \mathrm{U})$

.2 $(p \leftrightarrow \cup x(\text{ inhabited is } \pi(p))$

.3 $((A \in \mathrm{U}) \rightarrow \pi(\cap x \in A\ \underline{u}x) = \mathrm{E}f(\text{function is } f\ \wedge$

$\text{dmn } f = \mathrm{E}x,y(x \in A \wedge y \in \pi(x \in A)) \wedge \cap t \in \text{dmn } f(.ft \in \pi(\underline{u}\text{crd}0t))))$

.4 $((A \in \mathrm{U}) \rightarrow \pi(\cup x \in A\ \underline{u}x) = \mathrm{E}t((\text{tuple } 3 \text{ is } t) \wedge$

$(\text{crd}1t \in \pi(\text{crd}0t \in A)) \wedge (\text{crd}2t \in \pi(\underline{u}\ \text{crd}0t))))$

.5 $(\pi(p \wedge q) = ((\pi(p),,\pi(q)) \vee (\pi(q),,\pi(p))))$

.6 $(\pi(p \vee q) = (\pi(p) \vee \pi(q)))$

.7 $((x \in \pi(p)) \to (\pi(x \in \pi(p)) = \mathrm{sng}\ x))$

.8 $((n \in \mathrm{N}) \to (\pi(n \in \mathrm{N}) = \mathrm{sng}\ n))$

Remarks: (i) Axiom (A2.3.0) is our proof-theoretic analogue of the axiom of truth (1.8.0), and is necessary for the proof of the Implication Theorem (A2.4.0). Axiom (A2.3.1) is the expression of our belief that the set of proofs of p is always constructively well-defined, even if we do not know if p is true or false; in other words, we have a sound idea of what we mean by 'a proof of the proposition p', this idea being framed in terms of clearly described rules of inference, etc. The motivation behind axiom (A2.3.2) should be obvious.

Axioms (A2.3.3) - (A2.3.6) are expressions of the usual intuitionistic interpretations of the quantifiers and logical connectives (cf. [1], Chapter 1, Section 3). Note that the appearance of

$((\pi(p),,\pi(q)) \vee (\pi(q),,\pi(p)))$

rather than

$(\pi(p),,\pi(q))$

on the right hand side of (A2.3.5) is necessary to ensure that

$(\pi(p \wedge q) = \pi(q \wedge p))$

(which must obtain, by (1.8.5) and (1.9.1)).

Finally, (A2.3.7) and (A2.3.8) are the formal expressions of our belief that the set of proofs of p, and the set of natural numbers, are basic sets; in other words, it is (or should be !) immediately and unmistakeably clear from the construction of a proof x of p (resp. of a natural number n) that x is a proof of p (resp. a natural number).

(ii) It does not seem possible to produce criteria along the lines of (A2.3.3) and (A2.3.4) to describe the sets $\pi(\cap x\underline{u}x)$ and $\pi(\cup x\underline{u}x)$. For example, were we to add even the axiom

$$(\pi(\cap x \in U\ \underline{u}x) = Ef(\text{function is } f \wedge \text{dmn } f = Ex,y(x \in U \wedge y \in \pi(x \in U))$$
$$\cap t \in \text{dmn } f(.ft \in \pi(\underline{u}\ \text{crd}0t))))$$

then (A2.3.2) we would have

$$(\cup f(f \in Ef(\text{function is } f \wedge \text{dmn } f = Ex,y(x \in U \wedge y \in \pi(x \in U)) \wedge$$
$$\cap t \in \text{dmn } f(.ft \in \pi(\text{crd}0t = \text{crd}0t)))))$$

whence (again noting (A2.3.2))

$(\cup f(f \in U \wedge \text{dmn dmn } f = U))$.

From this and (1.16.31) we readily obtain the contradiction

$(U \in U)$.

On the other hand, the addition of

$(\pi(\cup x\underline{u}x) = Ep(\text{orderedpair is } p \wedge \text{crd}''p \in \pi(\underline{u}\ \text{crd}'p)))$

to our collection of axioms would entail ((A2.3.2) and (1.14.13))

$(\cup x\underline{u}x \rightarrow \cup x \in U\ \underline{u}x)$

whence,in particular,

$(\cup x \in U(x = U))$

and therefore

$(U \in U)$.

Finally, the (at first sight most reasonable) axiom

$$(\pi(\cup x \in U\ \underline{u}x) = Et(\text{tuple 3 is } t \wedge \text{crd}1t \in \pi(\text{crd}0t \in U) \wedge$$
$$\text{crd}2t \in \pi(\underline{u}\ \text{crd}0t)))$$

is incompatible with (A2.3.1) and (A2.3.2) taken together: for,by (A2.3.2), it entails

$(U = Ey\cup t \in \pi(\cup x \in U(x = x))(y = \text{crd}0t))$

whence ((A2.3.1) and an obvious analogue of (1.16.31))

$(U \in U)$.

As the exclusion of (A2.3.1) or (A2.3.2) from our list of proof-theoretic axioms would divorce our proof theory from reality, we are forced to reject the above - and,as far as we can see, all other - stronger forms of (A2.3.3) ands(A2.3.4) as candidates for election to our axiom system. In view of this apparent defect in our proof theory, it is worth observing that such fully general statements as $\cap x \underline{u}x$, $\cup x \underline{u}x$ have no place in the practical mathematics of Bishop's book [1]. Of course, this fits in well with the intuitive idea of constructive mathematics as a theory built up 'from below', starting with the natural numbers and discussing statements of the form $\cap x \in A\ \underline{u}x$, $\cup x \in A\ \underline{u}x$ where A is constructively well-defined, rather than overreaching itself in the discussion of idealistic statements like $\cap x \underline{u}x$ and $\cup x \underline{u}x$. Moreover, even in the formal system of Chapter 1 it appears that fully universal statements are normally of a trivial nature - such as, for example, $\cap x(x = x)$ - and could, if we so wished, be replaced by free variable statements (like $(x = x)$, in the case of $\cap x(x = x)$). Nevertheless, any attempt to recast the theory of Chapter 1 so that fully general statements are replaced by free variable ones, and the only universal and existential expressions permitted are those of the form $\cap x \in A\ \underline{u}x$ and $\cup x \in A\ \underline{u}x$ with $A \in \mathrm{U}$, runs into immediate difficulty with the definitions of 0 and U. One way out of this might be to take U as a primitive constant and define 0 to be $\cap x \in \mathrm{U}\ x$: but this would lead to the highly unsatisfactory situation in which our logic (of negation) depends on the definition and properties of the set theoretic connective '$\in$'! The inadequacy of the above proof theoretic axioms therefore remains, both irritating and seemingly unavoidable.

(iii) Noting that

$$(Ef(\text{function is } f \wedge \text{dmn } f = \mathrm{E}\, x,y(x \in A \wedge y \in \pi(x \in A)) \wedge$$
$$\cap t \in \text{dmn } f(.ft \in \pi(\underline{u}\mathrm{crd}0t)))$$
$$\subset \text{map } \mathrm{E}\, x,y(x \in A \wedge y \in \pi(x \in A)) \cup x \in A\pi(\underline{u}x))$$

that

$$(\mathrm{E}x,y(x \in A \wedge y \in \pi(x \in A)) \subset A,,\cup x \in A\ \pi(x \in A))$$

and that ((A2.3.1), (1.12.0) and (1.17.19))

$$(A \in \mathrm{U} \to A,,\cup x \in A\ \pi(x \in A) \in \mathrm{U} \wedge \cup x \in A\ \pi(\underset{\sim}{u}x) \in \mathrm{U})$$

we need only refer to (1.24.0) and (1.12.4) to complete an independent verification that

$$(A \in \mathrm{U} \to \mathrm{E}f(\text{function is } f \wedge \text{dmn } f = \mathrm{E}x,y(x \in A \wedge y \in \pi(x \in A)) \wedge$$
$$\cap t \in \text{dmn } f(.ft \in \pi(\underset{\sim}{u}\mathrm{crd}0t))) \in \mathrm{U})$$

(cf. (A2.3.3) and (A2.3.1)).

We omit the corresponding verification that

$$(A \in \mathrm{U} \to \mathrm{E}t(\text{tuple 3 is } t \wedge \mathrm{crd}1t \in \pi(\mathrm{crd}0t \in A) \wedge$$
$$\mathrm{crd}2t \in \ddot{\pi}(\underset{\sim}{u}\mathrm{crd}0t)) \in \mathrm{U}).$$ ®

A2.4 Implication

.0 *The Implication Theorem*

$$(\pi(p \to q) \doteq \pi(\cap x \in \pi(p)q) = \text{map } \mathrm{E}x,y(x = y \in \pi(p))\pi(q))$$

Proof To begin with, we note that ((1.8.0) and (A2.3.2))

$$(\cap x(x \in \pi(p) \to t \in \alpha) \leftrightarrow (0 \in p \to t \in \alpha)),$$

whence ((1.8.2) and (1.8.3))

$$(\cap x(x \in \pi(p) \to \alpha) = (0 \in p \to \alpha)).$$

On the other hand, it is clear from (A2.3.1), (A2.3.3) and (A2.3.0) that

$$(\pi(\cap x \in \pi(p)q) = \pi(\cap x \in \pi(p)(0 \in q)))$$

It now follows from all this, (A2. 3.0) and (1.8.2) that

$$\begin{aligned}(\pi(p\to q) &= \pi(0 \in (p \to q)) \\ &= \pi(0 \in p \to 0 \in q) \\ &= \pi(\cap x \in \pi(p)(0 \in q)) \\ &= \pi(\cap x \in \pi(p)q))\end{aligned}$$

It is now a simple matter to complete the proof, using (A2.3.3) and (A2.3.7). □

To connect this theorem with the standard intuitionistic interpretation of the connective '$\rightarrow$' we have

.1 $(\phi = \lambda f \in \pi(p \rightarrow q)\ \lambda x \in \pi(p).f(x,x)$

$\rightarrow$ univalent is $\phi \wedge$ on $\pi(p \rightarrow q)$ onto map $\pi(p)\pi(q)$ is $\phi)$

A2.5 Negation

.0 $(\sim p \leftrightarrow \pi(p) = 0)$

.1 $(\sim p \leftrightarrow \pi(\sim p) = 1)$

Proof By (A2.3.2), (1.6.11), (A2.4.0), (1.6.44) and (A2.5.0)

$(f \in \pi(\sim p)$

$\rightarrow \sim p \wedge$ function is $f \wedge \operatorname{dmn} f = \mathrm{E}x,y(x = y \in \pi(p)) \wedge$

$\cap t \in \operatorname{dmn} f(.ft \in \pi(0))$

$\rightarrow$ on 0 to 0 is f

$\rightarrow f = 0)$

The result follows from this and (A2.3.2). □

Remark: It would greatly simplify matters on occasion if we could have

$((x \in \pi(p)) \vee \sim(x \in \pi(p)))$

as a theorem or axiom. Unfortunately, such a theorem would entail

$(\sim p \vee \sim\sim p)$

and

$(\cap n \in \mathrm{N}\ \cap x(x = n \vee x \neq n))$

when coupled with (A2.5.1) and (A2.3.8) respectively.

A2.6 Choice

.0 (basic is $A \wedge A \in \mathrm{U} \wedge B \in \mathrm{U} \wedge \cap x \in A\ \cup y \in B\ \underline{\mathrm{u}}'xy$

$\rightarrow \cup f($on A to B is $f \wedge \cap x \in A\ \underline{\mathrm{u}}'x.fx))$

Proof By (A2.3.2) and (A2.3.3) there exists a term ϕ such that

$$(\text{function is } \phi \wedge \text{dmn } \phi = \text{E}x,y(x \in A \wedge y \in \pi(x \in A)) \wedge$$
$$\cap t \in \text{dmn } \phi(.\phi t \in \pi(\cup y \in B \ \underset{\sim}{\text{u}}'\text{crd}'ty)))$$

It is easily seen from (A2.1.3) that

$$(\text{dmn } \phi = \text{E}x,y(x = y \in A))$$

On the other hand, by (A2.3.4),

$$(\cap t \in \text{dmn } \phi \ (\text{tuple } 3 \text{ is } .\phi t \wedge \text{crd1}.\phi t \in \pi(\text{crd0}.\phi t \in B) \wedge$$
$$\text{crd2}.\phi t \in \pi(\underset{\sim}{\text{u}}'\text{crd}'t \ \text{crd0}.\phi t))).$$

Setting

$$(f = \lambda x \in A \ \text{crd0}.\phi(x,x)),$$

we now see that

$$(\text{on } A \text{ to } B \text{ is } f \wedge \cap x \in A \ \underset{\sim}{\text{u}}'x.fx)$$

as required. □

.1 $(A \in \text{U} \wedge \cap x \in \pi(p) \cup y \in A \ \underset{\sim}{\text{u}}'xy$

$\rightarrow \cup f(\text{on } \pi(p) \text{ to } A \text{ is } f \wedge \cap x \in \pi(p) \ \underset{\sim}{\text{u}}'x.fx))$

Proof Follows from (A2.3.7), (A2.1.3), (A2.3.1) and (A2.6.0). □

.2 $(A \in \text{U} \wedge \cap n \in \text{N} \cup y \in A \ \underset{\sim}{\text{u}}'ny$

$\rightarrow \cup f \in \text{sqnc } A \cap n \in \text{N} \ \underset{\sim}{\text{u}}'n.fn)$

Proof Follows from (A2.3.8), (A2.1.3), (1.23.10) and (A2.6.0). □

Although we feel that axiom (A2.3.8) is worthy of inclusion in our system, it is not necessary for the derivation (A2.6.2): indeed, the latter can be derived by an argument similar to that of (1.28.2), given a proof of the following restricted form of the Axiom of Dependent Choice (cf. (1.28.0)):

.3 $(A \in \text{U} \wedge a \in A \wedge \cap x \in A \cup y \in A \ \underset{\sim}{\text{u}}'xy$

$\rightarrow \cup f \in \text{sqnc } A(.f0 = a \wedge \cap n \in \text{N} \ \underset{\sim}{\text{u}}'.fn.f \text{ scsr } n))$

Proof Let

$$(B = \text{E}\,x,y(x \in A \wedge y \in \pi(x \in A))).$$

By (A2.3.2), (A2.3.3) and (A2.3.4) there exists a term ϕ such that

$(\text{function is } \phi \wedge \text{dmn } \phi = B \wedge \cap t \in B(\text{tuple } 3 \text{ is } .\phi t \wedge$

$\text{crd1}.\phi t \in \pi(\text{crd0}.\phi t \in A) \wedge \text{crd2}.\phi t \in \pi(\underline{\text{u}}'\text{crd}'t \ \text{crd0}.\phi t)))$

On the other hand, by (A2.3.2) there exists a term ζ with

$(\zeta \in \pi(a \in A)).$

Setting

$(h = \lambda t \in B(\text{crd0}.\phi t,\ \text{crd1}.\phi t))$

$(g = \text{ndc}'h(a,\ \zeta))$

and

$(f = \lambda n \in \text{N}\ \text{crd}'.gn),$

we now see that

$(\text{on } B \text{ to } B \text{ is } h \wedge (a,\zeta) \in B)$

whence (1.23.7)

$(\text{on N to } B \text{ is } g \wedge .g0 = (a,\zeta) \wedge$

$\cap n \in \text{N}(.g \text{ scsr } n = .h.gn = (\text{crd0}.\phi.gn,\ \text{crd1}.\phi.gn)))$

It readily follows from this and (A2.3.2) that

$(f \in \text{sqnc } A \wedge .f0 = a \wedge \cap n \in \text{N}\ \underline{\text{u}}'.fn.f \text{ scsr } n).$ □

<u>Remark</u>: It is not hard to show that, were we to adopt

$(A \in \text{U} \rightarrow \pi(\cap x \in A\ \underline{\text{u}}x) = \text{E}f(\text{on } A \text{ is } f \wedge \cap x \in A(.fx \in \pi(\underline{\text{u}}x))))$

as an axiom instead of (A2.3.3) we could derive the choice principle

\# $(A \in \text{U} \wedge B \in \text{U} \wedge \cap x \in A \cup y \in B\ \underline{\text{u}}'xy$

$\rightarrow \cup f(\text{on } A \text{ to } B \text{ is } f \wedge \cap x \in A\ \underline{\text{u}}'x.fx))$

- a principle which we find it hard to accept among our constructively valid theorems. In fact, however, we would be wrong in seeking to include # as an axiom or theorem of our system: for it is quite common in practice to find that a proof of the proposition

$(x \in A \rightarrow \underline{\text{u}}x)$

depends not only on the construction of the element x of A, but also on our proof that x actually does belong to A (cf. [15],

Appendix D). ®

A2.7 Propositional Calculus

The first nine theorems of this section translate the axioms of propositional calculus ((1.6.14) - (1.6.22)) into the language of our proof theory.

.0 $(x \in \pi(p) \to (x,x) \in \pi(p \wedge p))$

.1 $(\pi(p \wedge q) = \pi(q \wedge p))$

.2 $(f \in \pi(p \to q) \to \cup g(g \in \pi(p \wedge r \to q \wedge r)))$

Proof Let

$$(\phi = \lambda m,x \in \{12\},,\pi(p \wedge r)(m \wedge ((m = 1 \wedge x \in \pi(p),,\pi(r)) \vee (m = 2 \wedge x \in \pi(r),,\pi(p)))))$$

Then as

$$((x \in A \to (x \in A) = \mathrm{U}) \wedge (\sim(x \in A) \to (x \in A) = 0))$$

it is clear from (1.18.8) and (A2.3.5) that

$$(\cap x \in \pi(p \wedge r)\cup m \in \{12\}(.\phi(m,x) = m))$$

whence (A2.6.1) there exists a term h such that

$$(\text{on } \pi(p \wedge r) \text{ to } \{12\} \text{ is } h \wedge \cap x \in \pi(p \wedge r)(.\phi(.hx,x) = .hx)).$$

With

$$(g = \lambda x,y \in \mathrm{E}x,y(x = y \in \pi(p \wedge r)). (.hx = 1 \wedge (.f(\mathrm{crd}'x,\mathrm{crd}'x),\mathrm{crd}''x) \vee .hx = 2 \wedge (\mathrm{crd}'x,.f(\mathrm{crd}''x,\mathrm{crd}''x))))$$

and noting (1.8.0) and (A2.4.0), we now see that

$$(x \in \pi(p \wedge r) \wedge .hx = 1$$

$$\to 1 = 1 \cap ((\mathrm{U} \cap (x \in \pi(p),,\pi(r))) \cup (0 \cap (x \in \pi(r),,\pi(p))))$$

$$= 1 \cap (x \in \pi(p),,\pi(r))$$

$$\to 0 \in (x \in \pi(p),,\pi(r))$$

$$\to x \in \pi(p),,\pi(r)$$

$\rightarrow (.hx = 1 \wedge (.f(\mathrm{crd}'x,\mathrm{crd}'x),\mathrm{crd}''x) \vee .hx = 2 \wedge$
$(\mathrm{crd}'x,.f(\mathrm{crd}''x,\mathrm{crd}''x)))$

$= (.f(\mathrm{crd}'x,\mathrm{crd}'x),\mathrm{crd}''x) \in \pi(q),,\pi(r))$

and that, similarly,

$(x \in \pi(p \wedge r) \wedge .hx = 2$

$\rightarrow x \in \pi(r),,\pi(p)$

$\rightarrow (.hx = 1 \wedge (.f(\mathrm{crd}'x,\mathrm{crd}'x),\mathrm{crd}''x) \vee .hx = 2 \wedge$
$(\mathrm{crd}'x,.f(\mathrm{crd}''x,\mathrm{crd}''x)))$

$= (\mathrm{crd}'x,.f(\mathrm{crd}''x,\mathrm{crd}''x)) \in \pi(r),,\pi(q))$

whence ((1.18.8) and (A2.3.5))

$(g \in \mathrm{map}\ \mathrm{E}x,y(x = y \in \pi(p \wedge r))\pi(q \wedge r)).$

To complete the proof, it now suffices to refer to (A2.4.0). □

.3 $(f \in \pi(p \rightarrow q)$

$\rightarrow \lambda g \in \pi(q \rightarrow r)\lambda x,y \in \mathrm{E}x,y(x = y \in \pi(p)).g(.f(x,y),.f(x,y))$

$\in \mathrm{map}\ \pi(q \rightarrow r)\ \pi(p \rightarrow r))$

.4 $(z \in \pi(q) \rightarrow \lambda x,y \in \mathrm{E}x,y(x = y \in \pi(p))z \in \pi(p \rightarrow q))$

.5 $(x \in \pi(p) \wedge f \in \pi(p \rightarrow q) \rightarrow .f(x,x) \in \pi(q))$

.6 $(\pi(p) \subset \pi(p \vee q))$

.7 $(\pi(p \vee q) = \pi(q \vee p))$

.8 $(f \in \pi(p \rightarrow r) \wedge g \in \pi(q \rightarrow r) \rightarrow \cup\psi(\psi \in \pi(p \vee q \rightarrow r)))$

Proof Define

$(\phi = \lambda m,x \in \{12\},,\pi(p \vee q)(m \wedge (m = 1 \wedge x \in \pi(p) \vee$
$m = 2 \wedge x \in \pi(q)))$

Then ((1.18.8) and (A2.3.6))

$(\cap x \in \pi(p \vee q)\cup m \in \{12\}(.\phi(m,x) = m))$

whence (A2.6.1) there exists a term h such that

$(\mathrm{on}\ \pi(p \vee q)\ \mathrm{to}\ \{12\}\ \mathrm{is}\ h \wedge \cap x \in \pi(p \vee q)(.\phi(.hx,x) = .hx)).$

With

$(\psi =$

$\lambda x,y \in \mathrm{E}x,y(x = y \in \pi(p \vee q))(.hx = 1 \wedge .f(x,x) \vee .hx = 2 \wedge$
$.g(x,x)))$

we now see (by an argument similar to that used in the proof of (A2.7.2)) that

$$(\psi \in \pi(p \vee q \rightarrow r)). \quad \Box$$

For the sake of completeness we add here the translations of the two extra propositional axioms of Heyting ((1.6.30) and (1.6.31)):

.9 $\quad (f \in \pi(p \rightarrow q) \wedge g \in \pi(p \rightarrow \sim q) \rightarrow \pi(p) = 0)$

Proof With reference to (A2.4.0) and (A2.3.2) we have

$$\begin{aligned}(t \in \pi(p) \rightarrow .f(t,t) \in \pi(q) \wedge .g(t,t) \in \pi(\sim q)\\ &\rightarrow q \wedge \sim q\\ &\rightarrow 0)\end{aligned}$$

whence result. $\Box$

.10 $\quad (\pi(\sim p) \subset \pi(p \rightarrow q))$

Proof By (A2.3.2) and (A2.5.1)

$$(x \in \pi(\sim p) \rightarrow x = 0).$$

On the other hand, by (A2.3.2), (A2.5.0) and (A2.4.0),

$$\begin{aligned}(x \in \pi(\sim p) \rightarrow \pi(p) = 0\\ &\rightarrow \text{on } Ex,y(x = y \in \pi(p)) \text{ to } \pi(q) \text{ is } 0\\ &\rightarrow 0 \in \pi(p \rightarrow q)).\end{aligned}$$

The result is now obvious. $\Box$

A2.8 Predicate Calculus

When we turn to consider the proof theoretic description of predicate calculus, we come up against difficulties of the same sort as we discussed in remark (ii) following the axioms in Section A2.3. It appears that we can do little or no better than produce the following translations of restricted forms of the predicate axioms ((1.6.23) - (1.6.28)) into our proof theory:

.0 $(A \in \mathrm{U} \wedge (y \to \underline{\mathrm{u}}x) \to (y \to \cap x \in A\ \underline{\mathrm{u}}x))$

Proof From propositional logic and our rules of inference we readily obtain

$(\cap x \in A(y \to \underline{\mathrm{u}}x))$,

whence ((A2.3.2) and (A2.3.3)) there exists a term f such that

$(\text{function is } f \wedge \text{dmn } f = \mathrm{E}x,y(x \in A \wedge y \in \pi(x \in A)) \wedge$

$\cap t \in \text{dmn } f(.ft \in \pi(y \to \underline{\mathrm{u}}\mathrm{crd}'t)))$.

Setting

$(\phi = \lambda s \in \pi(y)\ \lambda t \in \text{dmn } f..ft(s,s))$

and referring to (A2.4.0) and (A2.3.3) we now see that

$(\text{on } \pi(y) \text{ to } \pi(\cap x \in A\ \underline{\mathrm{u}}x) \text{ is } \phi)$.

The result follows from this, (A2.3.1), (1.17.18), (A2.4.1) and (A2.3.2). □

We omit the proof of the remaining translations of the predicate axioms.

.1 $(A \in \mathrm{U} \wedge (\underline{\mathrm{u}}x \to y) \to (\cup x \in A\ \underline{\mathrm{u}}x \to y))$

.2 $(A \in \mathrm{U} \wedge a \in A \wedge \cap x \in A\ \underline{\mathrm{u}}x \to \underline{\mathrm{u}}a)$

.3 $(A \in \mathrm{U} \wedge a \in A \wedge \underline{\mathrm{u}}a \to \cup x \in A\ \underline{\mathrm{u}}x)$

.4 $(A \in \mathrm{U} \wedge y \to \cap x \in Ay)$

Let us conclude this appendix with some comments on the possibility of identifying $\pi(p)$ with some simpler object in our theory.

The first suggestion that comes to mind in this context is to introduce as an axiom

$(p = \pi(p))$.

However, we cannot have both this and the (far more natural) axiom (A2.3.2) without contradicting the axioms of Chapter 1:

for, were (A2.3.2) and

$$(\text{sng } 1 = \pi(\text{sng } 1))$$

both theorems we would have

$$(1 \in \text{sng } 1 = \pi(\text{sng } 1))$$

whence

$$(\text{sng } 1)$$

and therefore

$$(0 \in \text{sng } 1)$$

-.that is

$$(0 = 1).$$

Another, and perhaps more appealing, possibility is the introduction of the axiom

#

$$(p \in \text{U} \rightarrow p = \pi(\text{U}x(x \in p)))$$

(note that the condition $(p \in \text{U})$ is necessary to avoid contradiction of the very desirable axiom (A2.3.1)). However, this also leads to a contradiction. To see this, we first note that, by (1.8.0), (1.8.1) and (A2.3.2),

$$((0 \in (p \wedge q)) = \text{U}z(z \in \text{E}z\text{U}x\text{U}y(x \in \pi(p) \wedge y \in \pi(q) \wedge z = \{xy\}))).$$

It is easy to show that

$$(\text{E}z\text{U}x\text{U}y(x \in \pi(p) \wedge y \in \pi(q) \wedge z = \{xy\})$$
$$= \text{U}t(t \in \pi(p),,\pi(q) \wedge \text{sng } \{\text{crd}'t \ \text{crd}''t\}))$$

whence (with reference to (A2.3.1)) we see that

$$(\text{E}z\text{U}x\text{U}y(x \in \pi(p) \wedge y \in \pi(q) \wedge z = \{xy\}) \in \text{U}).$$

Thus,supposing that # obtains, we have (A2.3.0)

$$(\pi(p \wedge q) = \pi(0 \in (p \wedge q))$$
$$= \pi(\text{U}z(z \in \text{E}z\text{U}x\text{U}y(x \in \pi(p) \wedge y \in \pi(q) \wedge z = \{xy\}))$$
$$= \text{E}z\text{U}x\text{U}y(x \in \pi(p) \wedge y \in \pi(q) \wedge z = \{xy\})).$$

From this, (A2.3.5), (1.6.44) and (A2.5.1), it follows that

$$(\text{sng } 0 = \{00\} = (0,0) = (\text{sng } 0 \text{,,sng } 0) \cup (\text{sng sng } 0 \text{,,sng } 0)),$$

from which we readily deduce a contradiction to (1.13.18).

Bearing in mind the work of these last two paragraphs, we feel inclined to doubt the existence of a consistent identification of a general proof set $\pi(p)$ with some simpler object of our set theory. Of course, such an identification does appear possible for special examples of $\pi(p)$ - this is precisely the content of our axioms (A2.3.3) - (A2.3.7).

APPENDIX 3

ON CONNECTEDNESS

A metric space E is said to be *connected* if $A = E$ for each inhabited, located subset A of E that is both open and closed in E. The main purpose of this appendix is to present an application of (2.1.0) in the proof that a compact, convex subset of a normed linear space is connected. However, we also take the opportunity to mention and prove certain additional results on connectedness, particularly in connection with subsets of R.

A3.1 Connected subsets of R

.0 *A bounded, inhabited, open interval in* R *is connected.*

Proof. Let a,b be points of R with $a < b$, I the open interval $]a,b[$ in R and A an inhabited subset of I that is open, closed and located in I. Let z be a point of I, and suppose that $r \equiv \mathrm{dist}(z,A) > 0$. To begin with, we prove that

$m \equiv \min(\mathrm{dist}(z - r,A), \mathrm{dist}(z + r,A)) = 0$.

Indeed, given x in A and supposing $m > 0$, we have either $z - r - m < x$ or $x < z - r$. In the latter case we immediately obtain $x \leq z - r - m$, $d(x,z) \geq r + m$. In the case $z - r - m < x$, the definitions of r and m ensure that, in turn, $z - r < z - r + m \leq x$, $z + r \leq x$, $z + r + m \leq x$, whence, again, $d(x,z) \geq r + m$. Thus we obtain the contradiction $\mathrm{dist}(z,A) \geq r + m > r$; from which we conclude that m must equal 0. We now also suppose that $0 < \mathrm{dist}(z - r,A)$, so that $\mathrm{dist}(z + r,A) = 0$, and $z + r \in \bar{A}$. Then, with x chosen in A so that $d(z + r,x) < r/2$, the definition of r ensures that $z < z + r \leq x < b$, so that $z + r \in I \cap \bar{A} = A$, and there exists $\delta > 0$ with $]z + r - \delta, z + r + \delta[\subset A$. Thus

$(z + r - \min(r/2,\delta/2)) \in A$.

and

$d(z, z + r - \min(r/2,\delta/2)) < r$.

This contradicts the definition of r, so that we must have $\mathrm{dist}(z - r,A) = 0$. But a similar argument to that just presented shows that this, too, contradicts the definition of r; whence, in fact, $r = 0$ and z belongs to the closure of A in I - that is, to A. Thus, indeed, $A = I$. □

A similar, but slightly simpler, argument proves

.1 *An unbounded, open interval in* R *is connected.* □

To deal with more general intervals, we need

.2 *Let F be a connected subset of a metric space E, and A a subset of E with $F \subset A \subset \overline{F}$. Then A is connected.*

Proof. Let S be an inhabited subset of A that is open, closed and located in A. Then $S \cap F$ is inhabited, and both open and closed in F. On the other hand, given x in F, $\varepsilon > 0$ and y in S such that

$d(x,y) < \mathrm{dist}(x,S) + \varepsilon/2$

and choosing $\delta > 0$ so that $A \cap \mathrm{B}(y,\delta) \subset S$, we see that

$d(y,z) < \min(\delta,\varepsilon/2)$.

for some z in F. For this z, we have

$z \in F \cap \mathrm{B}(y,\delta) = F \cap A \cap \mathrm{B}(y,\delta) \subset S \cap F$

and

$d(x,z) \leq d(x,y) + d(y,z) < \mathrm{dist}(x,S) + \varepsilon$.

It is now clear that $\mathrm{dist}(x,S \cap F)$ exists, and equals $\mathrm{dist}(x,S)$, whence $S \cap F$ is located in F. As F is connected it follows that $S \cap E = E$, and therefore that $F \subset S \subset A \subset \overline{F}$. Thus $\overline{S} \cap A = \overline{F} \cap A = A$. But S is closed in A, so that $\overline{S} \cap A = S$; hence $S = A$, and A is connected. □

From this, (A3.1.0) and (A3.1.1) it follows that

.3 *Any inhabited interval in* R *is connected.* □

A3.2. Partial converses of the results in section A 3.1.

.0 *Let S be a located, connected subset of* R, *and a, b points of S with $a \leq b$. Then $S \cap [a,b]$ is dense in $[a,b]$*

Proof. Let x belong to $[a,b]$ and suppose that

$r \equiv \mathrm{dist}(x,S) > 0$. Then

$a + r \leq x \leq b - r$

and

$\min(\mathrm{dist}(x - r,S), \mathrm{dist}(x + r,S)) = 0.$

Supposing also that $0 < \mathrm{dist}(x + r,S)$, we see immediately that $\mathrm{dist}(x - r,S) = 0$. We show that the inhabited set

$A \equiv S \cap]-\infty, x[= S \cap]-\infty,x]$

- which is both open and closed in S - is located in S. Indeed, given y in S, we have either $y < x$ - in which case $y \in A$ and $\mathrm{dist}(y,A) = 0$ - or $x - r<y$; in the latter case, for each z in A we have

$d(y,z) = y - z \geq d(y,x - r)$,

and so - as $x - r$ belongs to $\overline{S}$ - $\mathrm{dist}(y,A)$ exists and equals $d(y,x - r)$. Thus A is located in S, and so $A = S$. But this entails that b belongs to A, and so $b \leq x$ - contradicting the fact that $x < b$. Hence we must have $\mathrm{dist}(x + r,S) = 0$. But (as a simple modification of the foregoing argument shows) this entails $x \leq a$, contrary to the fact that $a < x$. It follows that we must have $r = 0$, $x \in \overline{S}$, and therefore $S \cap [a,b]$ dense in $[a,b]$. □

From this, we obtain

.1 *Let S be a closed, located, connected subset of* R, *and a,b points of S with $a \leq b$. Then $[a,b] \subset S$.* □

.2 *Let S be a bounded, located, connected subset of* R. *Then $\overline{S}$ is a compact interval.*

Proof. $\overline{S}$ is closed, located and bounded in R, and therefore compact. Let $p \equiv \inf_{x \in \overline{S}} x$ and $q \equiv \sup_{x \in \overline{S}} x$, and choose any point x of $[p,q]$. Then, given $\varepsilon > 0$, either $q - p < \varepsilon$ - in which case there clearly exists y in S with $|x - y| < \varepsilon$ - or $q - p > 0$. In this latter case, we may choose x' in $]p,q[$ with $|x - x'| < \varepsilon/2$, and then p',q' in S such that $p \leq p' < x' < q' \leq q$; applying (A3.2.0), we immediately obtain y in S with $|x' - y| < \varepsilon/2$, and therefore $|x - y| < \varepsilon$. It is now clear that S is dense in $[p,q]$, from which the result follows. □

We omit the details of the proofs of the corresponding results for unbounded intervals:

.3 *Let S be a located, connected subset of* R *that is bounded below but unbounded above (resp. bounded above but unbounded below). Then there exists p in* R *such that* $\overline{S} = [p,\infty[$ *(resp.* $\overline{S} =]-\infty\, p]$). □

.4 *Let S be a located, connected subset of* R *that is unbounded above and below. Then S is dense in* R. □

Remarks: (i) The proposition

every closed, located, connected subset of R is either bounded or unbounded

is essentially non-constructive. To see this, let $(n_k)_{0\leq k}$ be a sequencein $\{0,1\}$;if $n_0,\ldots,n_k$ are all 0, set $m_k = k$; otherwise let m_k be the smallest integer j with $n_j = 1$. We shall show that

$$S \equiv \cup_{k=0}^{\infty}[0,m_k]$$

is closed, located and connected in R. Tobegin with, let $(x_n)_{0\leq n}$ be a sequence in S converging to a point ξ of R. Then, choosing k so that $\xi \in [0,k]$, we have either

$$\cap j \in \mathrm{N}(j \leq k \Rightarrow n_j = 0)$$

or

$$\cup j \in \mathrm{N}(j \leq k \wedge n_j = 1).$$

In the former case, $[0,k] \subset S$ and so $\xi \in S$; in the latter, we let

$$\nu \equiv \min\{p : p \leq k \wedge n_p = 1\},$$

so that $S = [0,\nu]$, $x_n \leq \nu$ for each n, and therefore $\xi \leq \nu$,

$\xi \in [0,\nu] = S$. This shows that S is closed.

Now let x belong to R. If $0 < x$, we choose an integer $k > x$. As above, we have either $[0,k] \subset S$ or $S = [0,j]$ for some $j \leq k$; in both cases, $\mathrm{dist}(x,S)$ certainly exists. On the other hand, if $x < 1$, we have either $n_1 = 1$ - when $S = \{0\}$ or $S = [0,1]$, and S is certainly located in R - or $n_1 = 0, [0,1] \subset S$. In this latter case, we have $\mathrm{dist}(x,S) = \mathrm{dist}(x,[0,1])$: for, choosing β in R with

$$\max(0,x) < \beta < 1,$$

and any element s of S, we have either $s < 1$ - in which case $s \in [0,1]$ and $d(x,s) \geq \mathrm{dist}(x,[0,1])$ - or $\beta < s$, when

$$d(x,s) = s - x > \beta - x \geq \mathrm{dist}(x,[0,1]);$$

the conclusion follows because $[0,1] \subset S$.

The proof that S is connected is similar tothe proof of (A3.1.0), and will be omitted. Finally, it is clear that if S is bounded, then $n_j = 1$ for some j, while if S is unbounded, then $n_k = 0$ for each k. It should now be clear that the proposition in question entails the limited principle of omniscience.

(ii) Two open questions in the theory of connectedness are:

> if S is a located, connected subset of R and a,b are points of S, does S contain $[a,b]$?

and

> if f is a uniformly continuous mapping of an interval I in R into a metric space is ${}_*fI$ connected?

Note that we cannot expect to obtain affirmative constructive answers to both these questions: for we would then be able to prove the classical 'Intermediate Value Theorem', which is known to entail the limited principle of omniscience (cf. [1], page 5). On the other hand, a negative answer to the first question would entail a loss of elegance in the constructive theory of connectedness; while a negative answer to the second would entail a corresponding loss of power. ®

A3.3 Connectedness in normed linear spaces

A most satisfactory general result on connectedness is

.0 *A compact, convex subset of a normed linear space is connected.*

Proof. Let S be a compact, convex subset of the normed linear space $(E, \|\ \|)$, and A an inhabited subset of S that is open, closed and located in S. Let z be a point of S, suppose that $r \equiv \operatorname{dist}(z,A) > 0$, and let $K \equiv \overline{B}(z,r)$. We first prove that $K \cap S$ is located in S: given x in S and y in $K \cap S$, we certainly have

$\|x - y\| \geq \max(0, \|x - z\| - r)$.

On the other hand, either $\|x - z\| < r$ - in which case $\operatorname{dist}(x, K \cap S)$ exists and equals 0 - or $0 < \|x - z\|$; in the latter case,

$y \equiv z + \min(1, r\|x - z\|^{-1})(x - z)$ belongs to S, and satisfies $\|y - z\| \leq r$, $\|x - y\| = \max(0, \|x - z\| - r)$; so that $\operatorname{dist}(x, K \cap S)$ again exists, and equals

$\max(0, \|x - z\| - r)$.

It now follows that $K \cap S$ is precompact, so that

$\alpha \equiv \inf_{x \in K \cap S} \operatorname{dist}(x,A)$

is defined; we now show that $\alpha = 0$. Supposing $\alpha > 0$, we choose x in A with

$r = \operatorname{dist}(z,A) \leq \|x - z\| < r + \alpha$.

Then

$x' \equiv z + r\|x - z\|^{-1}(x - z)$

belongs to S (by convexity), $\|x' - z\| = r$ and

$\|x' - x\| = (1 - r\|x - z\|^{-1})\|x - z\| = \|x - z\| - r < \alpha$

- a contradiction to the definition of α. Hence $\alpha = 0$.

We may therefore choose y in A with $\operatorname{dist}(y, K \cap S) < r/2$.

Defining

$x_1 \equiv z + r\|y - z\|^{-1}(y - z)$,

we see that $x_1 \in S$, $\|x_1 - z\| = r$, and

$\mathrm{dist}(x_1,A) \leq \|x_1 - y\| = \|y - z\| - r = \mathrm{dist}(y,K \cap S) < r/2$.

We now use (2.1.0) to show that $\mathrm{dist}(x_1,A) > 0$. As A is closed in the compact set S, it is complete, whence (2.1.0) there exists y_1 in A with the property

$0 < \|x_1 - y_1\| \Rightarrow 0 < \mathrm{dist}(x_1,A)$.

As A is open in S, there exists $\delta > 0$ with $S \cap \mathrm{B}(y_1,\delta) \subset A$. Were $\|x_1 - y_1\| < \delta$, there would exist δ' with $0 < \delta' < r$ and $S \cap \mathrm{B}(x_1,2\delta') \subset A$; the point

$x_1 - \delta'\|x_1 - z\|^{-1}(x_1 - z)$

would then belong to $A \cap \mathrm{B}(z,r)$, contrary to the definition of r. It follows that we must have

$0 < \delta \leq \|x_1 - y_1\|$, and therefore $\mathrm{dist}(x_1,A) > 0$.

The Axiom of Dependent Choice now enables us to construct a sequence $(x_n)_{0 \leq n}$ in S such that $x_0 = z$ and, for each n,

(i) $\|x_{n+1} - x_n\| = \mathrm{dist}(x_n,A)$

and

(ii) $0 < \mathrm{dist}(x_{n+1},A) < 2^{-1}\mathrm{dist}(x_n,A)$.

Clearly, (x_n) is a Cauchy sequence, and so converges to a point ξ of S. Moreover,

$\mathrm{dist}(\xi,A) = \lim_{n\to\infty} \mathrm{dist}(x_n,A) = 0$,

so that ξ belongs to the closure of A in S - that is, to A. Choosing $t > 0$ so that $S \cap \mathrm{B}(\xi,t) \subset A$, and n large enough, we now have $x_n \in S \cap \mathrm{B}(\xi,t) \subset A$ and $\mathrm{dist}(x_n,A) = 0$ - contradicting (ii) above. This final contradiction ensures that $r = \mathrm{dist}(z,A) = 0$, and z belongs to the closure of A in S. Thus, z belongs to A, $A = S$, and S is connected. □

From this and the compactness of the closed unit ball in a finite dimensional Banach space (3.2.0), we obtain

.1 *The closed unit ball in a finite dimensional Banach space is connected.* □

In turn, this and (A3.2.2) yield

.2 *A compact subset of* R *is connected if and only if it is a compact interval.* □

Remark: We might well expect that connectedness would obtain also for the open unit ball in a finite dimensional Banach space. However, we do not know of any proof of this which does not depend on our first having a proof of Conjecture 1 of section 2.2. ®

APPENDIX 4

ON METRIC INJECTIVENESS

We have already remarked that if Conjecture 2 of Section 2.2 were valid then the concepts of metric injectiveness and metric weak-injectiveness would coincide for precontinuous functions (cf. Section 2.3). In view of this, it is interesting to note that the proposition

(a) a uniformly continuous, metrically weak-injective mapping of a compact metric space into a metric space is metrically injective

entails the following restricted form of Conjecture 2:

(b) if f is a uniformly continuous, metrically weak-injective mapping of a compact metric space K into R, and $0 < .fx$ for each x in K, then $0 < \inf x \in K\ .fx$

(*cf.* remark (ii) following (4.3.3)). To prove this, we let f be a uniformly continuous,metrically weak-injective mapping of a compact metric space (K,d) into R^{+}, and $((X,d'),e,\omega)$ a one-point compactification of K (3.3.7). Then (3.3.6) $0 < d'(\omega,{}_{*}eK)$ and $X = {}_{*}e(K) \cup \{\omega\}$. Defining

$$\phi \equiv \lambda x \in X\ (x \in {}_{*}eK \wedge .f.\mathrm{inv}\ e\ x \vee x = \omega \wedge 0)$$

and referring to (3.3.3),we see that ϕ is uniformly continuous on X and metrically weak-injective. Thus, if (a) obtains, we have

$$0 < \mathrm{dist}(.\phi\omega,{}_{*}\phi\,{}_{*}eK)$$

$$= \mathrm{dist}(0,{}_{*}fK)$$

$$= \inf x \in K\ .fx,$$

as required.

Taken with the remarks in Section 2.2, this suggests that it is unlikely that (a) will be either proved or disproved within the framework of constructive mathematics.

APPENDIX 5

FUNDAMENTAL PRINCIPLES OF CONSTRUCTIVE FUNCTIONAL ANALYSIS

Listed below for convenient reference are the constructive versions of several of the fundamental principles of linear analysis, together with the following (classically vacuous) criterion of normability of linear functionals:

.0 *A nonzero, bounded linear functional* ϕ *on a normed linear space* E *is normable if and only if* $^*\phi(\{0\})$ *is located in* E ([1], Chapter 9, Proposition 8). □

.1 <u>*The Separation Theorem*</u> ([1], Chapter 9, Theorem 3)

Let X *and* Y *be bounded, convex subsets of a separable normed linear space* E*, such that the set* $\{y - x: x \in X \wedge y \in Y\}$ *is located, and the real number*

$$d \equiv \inf_{x \in X \wedge y \in Y} \|x - y\|$$

is positive. Then, for each $\varepsilon > 0$*, there exists a normable linear functional* ϕ *on* E *such that* $\|\phi\| = 1$ *and*

$$\cap x \in X \ \cap y \in Y \ (\mathrm{Re}.\phi y > \mathrm{Re}.\phi x + d - \varepsilon). \quad □$$

.2 <u>*The Hahn-Banach Theorem*</u> ([1], Chapter 9, Theorem 4)

Let ϕ *be a nonzero linear functional on a linear subset* V *of a separable normed linear space* E*, such that* $^*\phi(\{0\})$ *is located in* E*. Then, for each* $\varepsilon > 0$*, there exists a normable linear functional* ψ *on* E *such that* $\mathrm{strc}\ \psi V = \phi$ *and*

$\|\psi\| \leq \|\phi\| + \varepsilon.$ □

The next two theorems give useful characterisations of linear functionals in certain special situations. The first of these (which is not mentioned by Bishop in [1]) is readily proved along the classical lines, with reference to (A5.0) above; the second appears as Theorem 10 of Chapter 9 of [1], and is vital for our proof of (5.6.1).

.3 *The Riesz Representation Theorem*

Let ϕ *be a nonzero, bounded linear functional on a Hilbert space* $(H,\langle\,|\,\rangle)$. *Then* ϕ *is normable if and only if there exists a (unique) element* ξ *of* H *such that* $\phi = \lambda x \in H\langle x|\xi\rangle$ *– in which case* $\|\phi\| = \|\xi\|$. □

.4 *Let* E *be a Banach space, and* ϕ *a linear functional on* $\mathrm{Hom}(E,\mathrm{C})$ *which is uniformly continuous on* $\mathrm{Hom}_1(E,\mathrm{C})$ *with respect to the double norm. Then there exists* ξ *in* E *such that* $\phi = \lambda u \in \mathrm{Hom}(E,\mathrm{C}).u\xi$. □

Finally, we have ([1], Chapter 9, Problem 6):

.5 *The Uniform Boundedness Principle*

Let E *be a Banach space,* F *a normed linear space,* $(T_n)_{n\in\mathrm{N}}$ *a sequence in* $\mathrm{Hom}(E,F)$, *and suppose that there exists a sequence* $(x_n)_{n\in\mathrm{N}}$ *in* E *such that* $\|x_n\| = 1$ *for each* n *in* N, *and* $(\|.T_nx_n\|)_{n\in\mathrm{N}}$ *diverges to* ∞. *Then there exists* ξ *in* E *such that the sequence* $(\|.T_n\xi\|)_{n\in\mathrm{N}}$ *is unbounded.* □

REFERENCES

[1] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill Co., New York, 1967.

[2] E. Bishop: *'The Neat Category of Stratified Spaces'* (To Appear?)

[3] E. Bishop: *'Mathematics as a Numerical Language'*, pp. 53-71 of Proceedings of the Conference on Intuitionism and Proof Theory at Buffalo, New York, in 1968, North-Holland Publishing Co., Amsterdam, 1970.

[4] E. Bishop and H. Cheng: *Constructive Measure Theory*, A.M.S. Memoir No. 116, 1972

[5] F. F. Bonsall and J. Duncan: *Complete Normed Algebras*, Springer Verlag, Berlin, 1973.

[6] N. Bourbaki: *Espaces Topologiques* (Premiere Partie, Livre III), Hermann, Paris, Troisieme Edn. 1961.

[7] L. E. J. Brouwer: *'Over de Grondslagen der Wiskunde'*, Thesis, University of Amsterdam, 1907.

[8] J. Dieudonne: *Foundations of Modern Analysis*, Academic Press Inc., New York, 1960.

[9] J. Dixmier: *Les Algebres d'Operateurs dans l'Espace Hilbertien (Algebres de von Neumann)*, Gauthier Villars, Paris, Deuxieme Edn. 1969.

[10] N. Dunford and J. T. Schwartz: *Linear Operators* (Volume I). Interscience Publishers, New York, 1958.

[11] G. Frege: *Die Grundlagen der Arithmetik* (Trans. J. Austin) Basil Blackwell, Oxford, 1950.

[12] A. Heyting: *Intuitionism - an Introduction*, North-Holland Publishing Co., Amsterdam, Third Edn, 1971.

[13] S. C. Kleene and R. E. Vesley: *The Foundations of Intuitionistic Mathematics*, North-Holland Publishing Co., Amsterdam, 1965.

[14] A. P. Morse: *A Theory of Sets* Academic Press Inc., New York, 1965.

[15] J. Myhill: *'Formal Systems of Intuitionistic Analysis III - Lawlike Analysis'* (To Appear?)

[16] F. Richman: *'The Constructive Theory of Countable Abelian p-groups'*,
Pacific J. Math.. 45 (1973), pp. 621-638.

[17] W. Rudin: *Functional Analysis*,
McGraw-Hill Co., New York, 1973.

[18] G. Stolzenberg: *Review of* [1]
Bull. A.M.S., March 1970, pp. 301-323.

[19] D. M. Topping: *Lectures on von Neumann Algebras*,
van Nostrand Rheinhold Co., New York, 1971.

[20] A. S. Troelstra: *Principles of Intuitionism*,
Lecture Notes in Mathematics No. 95,
Springer Verlag, Berlin, 1969.

[21] A. S. Troelstra: *'One-point Compactifications of Intuitionistic Locally Compact Spaces'*,
Fund. Math. LXII, 1968, pp. 75-93.

[22] A. S. Troelstra: *'Intuitionistic General Topology'*
Thesis, University of Amsterdam, 1966.

D.S. Bridges. D.Phil thesis.

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ADDENDA and CORRIGENDA

Each of the following remarks is indexed on the left by the page and line to which it refers. Thus, for example, the index (12,5) refers to 'page 12, line 5'.

(12,5) Replace by: 'For convenience, we remark that a *binarian* is a symbol of some type in accordance with Morse's separation into numbered types; and that a *nexus* is an expression in which each symbol is a binarian.'

(25,19) From (1.12.1), via (1.12.3) and (1.23.10), we obtain

$(\cap x(x \vee \sim x) \cap \mathrm{N} \in \mathrm{U})$

\- a result whose constructive validity has been questioned by Dana Scott. To answer his doubts, we note that

$(\cap x(x \vee \sim x) \cap \mathrm{N} = (\cap x(x \vee \sim x) \wedge \mathrm{N} \ \ \vee \ \cap x(x \vee \sim x) \wedge 0))$

\- so that '$\cap x(x \vee \sim x) \cap \mathrm{N}$' is the set equal to N if the principle of excluded middle is true, and 0 if it is false - and call to our aid the following remarks of Stoltzenberg ([18], pages 311 - 312):

'Constructive mathematics is completely general in its scope, and yet it is commonly claimed that the opposite is true. Usually this is caused by confusing the matter of defining a set with the problem of constructing elements of it. A typical instance of this reads, "the set consisting of 5 if Fermat's Last Theorem is true and 7 if it is false is not well-defined, according to Brouwer ". Not so. True, as it stands, this does not define an in teger. But it does define a subset of $\{5,7\}$, containing at most one integer.'

More formally, the substance of Stoltzenberg's remarks is the proposition

$((F \wedge \{5\} \vee \sim F \wedge \{7\}) \in U)$

where 'F' stands for the formal statement of Fermat's Last Theorem; that this proposition is true within our constructive set theory is, of course, a consequence of (1.12.1) (via (1.8.13), (1.12.2) and (1.12.3))!

In keeping with the above, we feel that any subset of an already well-constructed set is itself well-constructed (even if there may be considerable problems involved in the construction of individual elements of such a subset). Applying this to the example originally criticised by Scott, we see that it is the fact that the elements of '$\cap x(x \vee \sim x) \cap N$' (whatever they may be!) are drawn from the already well-constructed set N (1.23.10) that distinguishes the former as a well-constructed set (in contrast to the set '$\cap x(x \vee \sim x)$', about whose elements we know nothing, not even that they belong to some other, well-constructed, set).

(29,5) Replace 'sng(0,x)' by 'sng(0$\lrcorner x$)'

(29,7) Replace 'sng(sng 0,x)' by 'sng(sng 0$\lrcorner x$)'

(37,5) It is worth noting that

$\# \quad (\cap x \in U(x \vee \sim x))$

entails

(omniscience)

To see this, we note that

$((1 \wedge x) = (1 \cap x) \subset 1 \in \mathrm{U})$,

whence (1.12.4)

$((1 \wedge x) \in \mathrm{U})$.

Assuming that # holds, we therefore have

$((1 \wedge x) \vee \sim(1 \wedge x))$.

But

$$(\sim(1 \wedge x) \to (x \to (1 \wedge x) \wedge \sim(1 \wedge x))$$
$$\to (x \to 0)$$
$$\to \sim x),$$

so that we have

$((1 \wedge x) \vee \sim x)$.

This, in turn, yields

$(x \vee \sim x)$

and therefore

(omniscience).

(58,5) Both the definition of 'N' and that of 'Borel F' (1.26.0) are clearly impredicative. However, to accept the first as defining a well-constructed set (1.23.10) is merely to reflect within our formal system the intuitive conviction that the constructively well-defined set of natural numbers is the smallest set including 0 and the successors of all elements of the set. On the other hand, our definition of 'Borel F' accords equally well with our intuitive idea of the class of borel sets of F as the smallest class B which includes all elements of F and all countable unions and intersections of elements of B. But this time we have no good intuitive reason for supposing that our impredicative definition yields a constructively well-defined set of *all* borel sets of F; in other words, we would hesitate to accept the impredicative definition of 'Borel F' as defining an element of U.

(61,1) To describe in detail the incompatibility of the general axiom of choice

$$\#(\cap x \in A \cup y \in B\ \underline{u}'xy \rightarrow \cup f(\text{on } A \text{ to } B \text{ is } f \wedge \cap x \in A\ \underline{u}'x.fx))$$

and Church's Thesis, it is necessary to introduce Kleene's T_1-predicate. We recall the fundamental property of this predicate (cf. Kleene, *Introduction to Metamathematics*, §57): that, given any recursive binary relation $R(x,y)$ on N, there are natural numbers α,β such that

$$(\cup y\ R(x,y) \leftrightarrow \cup y\ T_1(\alpha,x,y))$$

and

$$(\cap y\ R(x,y) \leftrightarrow \cap y{\sim}T_1(\beta,x,y)).$$

Arguing classically, we then have

$$(\cup y\ R(\alpha,y) \leftrightarrow {\sim}\cap y{\sim}T_1(\alpha,\alpha,y))$$

and

$$(\cap y\ R(\beta,y) \leftrightarrow {\sim}\cup y{\sim}T_1(\beta,\beta,g)).$$

Now Kleene has shown that $T_1(x,y,z)$ is primitive recursive; so that the predicate ${\sim}T_1(x,x,y)$ is recursive. We shall show that $\cap t{\sim}T_1(x,x,t)$ is not recursive, and then apply this to demonstrate the incompatibility between # and Church's Thesis.

Indeed, supposing that $\cap t{\sim}T_1(x,x,t)$ is recursive, and noting that

$(\cap t \sim T_1(x,x,t) \leftrightarrow \cup y \cap t \sim T_1(x,x,t)),$

we see that there exists a natural number ν with the property

$(\cap t \sim T_1(x,x,t) \leftrightarrow \cup y \ T_1(\nu,x,y) \leftrightarrow \cup t \ T_1(\nu,x,t)).$

In particular,

$(\cap t \sim T_1(\nu,\nu,t) \leftrightarrow \cup t \ T_1(\nu,\nu,t))$

- which is absurd. Thus $\cap t \sim T_1(x,x,t)$ is not recursive.

We now let χ be the characteristic function of the recursive relation $\sim T_1(x,x,y)$, define

$(A = \mathrm{E}x \in \mathrm{N} \cap y \in \mathrm{N}(.\chi(x,y) = 0)),$

and assume the constructive validity of #. Then, in particular, there exists a function ϕ which carries a given bounded element g of map NN to a bound $.\phi g$ for g. With

$$(h = \lambda x \in \mathrm{N} \ \lambda y \in \mathrm{N}(.\chi(x,y) = 0 \wedge 0 \vee$$
$$.\chi(x,y) \neq 0 \wedge .\phi(\lambda t \in \mathrm{N} \ 0) + 1)),$$

we see that, for each x in N, $.hx$ is a bounded mapping of N into N, so that $.\phi(.hx)$ is defined. Now

$$(x \in A \rightarrow \cap y \in \mathrm{N}(.\chi(x,y) = 0)$$
$$\rightarrow .hx = \lambda t \in \mathrm{N} \ 0$$
$$\rightarrow .\phi.hx = .\phi(\lambda t \in \mathrm{N} \ 0))$$

On the other hand,

$$(x,y, \in \mathrm{N} \wedge .\chi(x,y) \neq 0 \rightarrow ..hx \ y = .\phi(\lambda t \in \mathrm{N} \ 0) + 1$$
$$\rightarrow .\phi.hx > .\phi(\lambda t \in \mathrm{N} \ 0))$$

Thus

$$(x \in \mathrm{N} \rightarrow .\phi.hx = .\phi(\lambda t \in \mathrm{N} \ 0) \vee .\phi.hx \neq .\phi(\lambda t \in \mathrm{N} \ 0)$$

$\rightarrow \cap y \in N\sim\sim(.\chi(x,y) = 0) \vee x \notin A$

$\rightarrow \cap y \in N(.\chi(x,y) = 0) \vee x \notin A$

$\rightarrow x \in A \vee x \notin A)$

- in other words,

$(\cap x \in N(\cap y \in N \sim T_1(x,x,y) \vee \sim\cap y \in N \sim T_1(x,x,y))$

It follows that the characteristic function χ of $\cap y\sim T_1(x,x,y)$ is effectively computable. According to Church's Thesis, this function is therefore recursive, so that $\cap y\sim T_1(x,x,y)$ is a recursive predicate. As this contradicts our working above, we are forced to the conclusion that Church's Thesis and the axiom of choice # cannot both be valid.

We should point out that all quantifiers applied to T_1 or $\sim T_1$ are to be taken as ranging over N, whether or not this is stated explicitly.

(70,28) This should read :

$(1 = 1 \cap (U\cap(\mathrm{dist}(\xi,A) < 2^{-n-1}))\cup(0\cap(2^{-n-2} < \mathrm{dist}(\xi,A)))$

(80,12) Let E be a locally compact normed linear space. Choose a compact subset K of E with $\bar{B}(0,1) \subset K$, a $\frac{1}{4}$-approximation $\{x_1,\dots,x_\nu\}$ to K, and a finite dimensional subspace V of E such that $\mathrm{dist}(x_k,V) < \frac{1}{4}$ for each k, $1 \le k \le \nu$. Let a be any point of E, suppose that $\delta \equiv \mathrm{dist}(a,V) > 0$, and choose v in V with $\delta < \|a - v\| < 3\delta/2$. Then

$z \equiv \|a - v\|^{-1}(a - v)$

belongs to K, so that there exists r with $1 \le r \le \nu$ and $\|z - x_r\| < \frac{1}{4}$. With v_r a point of V such that $\|x_r - v_r\| < \frac{1}{4}$, we now have

$$\|a - v\|\|z - x_r\| = \|a - v - \|a - v\|x_r\|$$

$$\geq \|a - v - \|a - v\|v_r\| - \|a - v\|\|x_r - v_r\|$$

$$\geq \operatorname{dist}(a,V) - \|a - v\|\|x_r - v_r\|$$

$$\geq \delta - \tfrac{1}{4}\|a - v\|.$$

Thus

$$\|a - v\|(\|z - x_r\| + \tfrac{1}{4}) \geq \delta$$

and so

$$\|a - v\| \geq \delta(\|z - x_r\| + \tfrac{1}{4})^{-1}$$
$$> \delta(\tfrac{1}{4} + \tfrac{1}{4})^{-1}$$
$$= 2\delta.$$

This contradicts the choice of v; so that, in fact, δ must equal 0. Hence a belongs to the closed subspace V of E, E and V coincide, and E is finite dimensional. □

(84,17) This should be expanded as follows:
'(K,d_o), and therefore K is d_o-precompact. Moreover, a d_o-Cauchy sequence in K is clearly d-Cauchy, hence d-convergent to a limit in K, and therefore d_o-convergent in K; so that (K,d_o) is (complete and) compact. On the other hand....'

(94,20) 'strc $eF,\omega)$ is a one-point compactification of F.'

(113,2) '...h is a homeomorphism of E_2....'

(123,14) Replace '(5.1.1)' by '(5.2.1)' .

(126,5) 'that $0 < \varepsilon < 2^{-n_1-2}\sqrt{2}$, and supposing....'

(127,4) '....(cf. [19],§9).'

(141,21) To prove this, let f be a nonzero linear functional on

$\mathrm{Hom}(H,H)$ that is uniformly continuous on $(\mathrm{Hom}_1(H,H),[\![\ ]\!]_w)$, and suppose without loss of generality that $\|f\| > \frac{1}{2}$, so that there exists R in $\mathrm{Hom}_1(H,H)$ with $|.fR| > \frac{1}{2}$. By (5.5.0), there exist complex numbers a_{jk} such that

$$.fT = \Sigma_{j,k=1}\ a_{jk}\langle .Ta_j | a_k\rangle$$

for each T in $\mathrm{Hom}(H,H)$, the series being uniformly convergent on $\mathrm{Hom}_1(H,H)$. Given $\varepsilon > 0$, we can therefore find a positive integer ν such that

$$|\Sigma^{\nu}_{j,k=1} a_{jk}\langle .Ra_j | a_k\rangle| > \tfrac{1}{2}$$

and

$$|.fT - \Sigma^{\nu}_{j,k=1} a_{jk}\langle .Ta_j | a_k\rangle| < \varepsilon/2$$

for all T in $\mathrm{Hom}_1(H,H)$. As

$$\tfrac{1}{2} < |\Sigma^{\nu}_{j,k=1}| a_{jk}||\langle .Ra_j | a_k\rangle| \leq \Sigma^{\nu}_{j,k=1}| a_{jk}|$$

we may define complex numbers c_{jk} by

$$c_{jk} \equiv (\Sigma^{\nu}_{p,q=1}| a_{pq}|)^{-1} a_{jk}.$$

Then $\Sigma^{\nu}_{j,k=1}| c_{jk}| = 1$. Moreover, for any T in the unit kernel of f,

$$\begin{aligned} |\Sigma^{\nu}_{j,k=1}\ c_{jk}\langle .Ta_j | a_k\rangle| &= (\Sigma^{\nu}_{p,q=1}| a_{pq}|)^{-1}|\Sigma^{\nu}_{j,k=1} a_{jk}\langle .Ta_j | a_k\rangle| \\ &< (\Sigma^{\nu}_{p,q=1}| a_{pq}|)^{-1}\varepsilon/2 \\ &< \varepsilon. \end{aligned}$$

This completes the proof.

(156,18) In view of the intuitionist's beliet that the notion 'a proof of p' is decidable, our assertion that

$$(x \in \pi(p) \vee x \notin \pi(p))$$

should not be a theorem of our system requires some further explanation.

To see how reasonable our assertion is, we consider a proof of the proposition p: the intuitive (metamathematical) idea of such a proof is a finite sequence

$s \equiv \{s_1,\ldots,s_n\}$

of statements, each of which is an axiom or is inferred from previous statements in the sequence by rules of inference, and the last of which is "p". If s' is the sequence

$\{s_1,\ldots,s_n \cap \cap x(x \vee \sim x)\}$,

then we do not know if

$(s = s' \vee s \neq s')$

is true within our system, and until we have such knowledge, we are unable to tell whether or not s' is a proof of p.

(163,13) The failure of # as an axiom or theorem of our system could have been demonstrated otherwise : for, if # obtains, then (A2.3.4)

$(p \in \mathrm{U} \rightarrow \cap x \in p(\text{tuple } 3 \text{ is } x))$,

whence

$(\text{tuple } 3 \text{ is } 0)$.

Thus ((1.27.2) and (1.27.0))

$(3 = \mathrm{E}t(\mathrm{bsvs0}t \in \mathrm{htd})$

$= \mathrm{E}t\cup y(y \in \mathrm{bsvs0}t)$

$= \mathrm{E}t\cup y(t_{\wp} y \in 0)$

$= 0)$

However, without any further postulation, we do have a most satisfactory theorem linking the concepts of 'constructively well-defined set' and 'proof set' :

$$(p \in \mathrm{U} \rightarrow p = \mathrm{E}x \cup y \cup z(x, y, z \in \pi(\cup x(x \in p))))$$

The proof of this is quite straightforward:

$$(p \in \mathrm{U} \rightarrow (x \in \mathrm{E}x \cup y \cup z(x, y, z \in \pi(\cup x(x \in p))$$

$$\leftrightarrow \cup y \cup z(x, y, z \in \pi(\cup x(x \in p)) = \pi(\cup x \in p(x \in p)))$$

$$\leftrightarrow \cup y \cup z(x \in p \wedge y \in \pi(x \in p) \wedge z \in \pi(x \in p))$$

$$\leftrightarrow x \in p)$$

$$\rightarrow p = \mathrm{E}x \cup y \cup z(x, y, z \in \pi(\cup x(x \in p))) \qquad \square$$

(166,27) 'it follows that $S \cap F = F$, and therefore that'

LIST OF SYMBOLS

The following is a by-no-means complete list of symbols used in the dissertation. Logical symbols (including those of our set and proof theories) are indexed by the reference number of their definition; the symbols of analysis by that of the Chapter and section in which they first appear.

Logical symbols

	1.5.0
$\mathrm{E}x;\underline{\mathrm{u}}x\ \underline{\mathrm{v}}x$	1.5.1
$\cap x;\underline{\mathrm{u}}x\ \underline{\mathrm{v}}x$	1.5.1
$\cup x;\underline{\mathrm{u}}x\ \underline{\mathrm{v}}x$	1.5.1
$\mathrm{sup}\ x\ \underline{\mathrm{v}}x$	1.5.1
$\mathrm{st}\ zx\ \underline{\mathrm{u}}x$	1.5.2
$A\cap\cap B$	1.5.8
$\mathbf{U}$	1.6.9
0	1.6.10
$\sim p$	1.6.11
omniscience	1.6.33
$x \subset y$	1.7.12
$x \supset y$	1.7.14
$x = y$	1.7.16
$x \neq y$	1.7.18
$x \subset\cdot\ y$	1.7.19
$x \cdot\supset y$	1.7.21
ΠA	1.7.25
∇A	1.7.27
$x \cap y$	1.7.29

Symbols of analysis

INDEX

The entries in this index are numbered by the Chapter and section of their first, or most important, appearance.