UNIVERSITY OF OXFORD

Robust & Stochastic Model Predictive Control

by

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Abstract

In the thesis, two different model predictive control (MPC) strategies are investigated for linear systems with uncertainty in the presence of constraints: namely robust MPC and stochastic MPC. Firstly, a Youla Parameter is integrated into an efficient robust MPC algorithm. It is demonstrated that even in the *constrained* cases, the use of the Youla Parameter can desensitize the costs to the effect of uncertainty while not affecting the nominal performance, and hence it strengthens the robustness of the MPC strategy. Since the controller $u = K_{LQ}x + c$ can offer many advantages and is used across the thesis, the work provides two solutions to the problem when the unconstrained nominal LQ-optimal feedback K_{LQ} cannot stabilise the whole class of system models.

The work develops two stochastic tube approaches to account for probabilistic constraints. By using a semi closed-loop paradigm, the nominal and the error dynamics are analyzed separately, and this makes it possible to compute the tube scalings offline. First, ellipsoidal tubes are considered. The evolution for the tube scalings is simplified to be affine and using Markov Chain model, the probabilistic tube scalings can be calculated to tighten the constraints on the nominal. The online algorithm can be formulated into a quadratic programming (QP) problem and the MPC strategy is closed-loop stable. Following that, a direct way to compute the tube scalings is studied. It makes use of the information on the distribution of the uncertainty explicitly. The tubes do not take a particular shape but are defined implicitly by tightened constraints. This stochastic MPC strategy leads to a non-conservative performance in the sense that the probability of constraint violation can be as large as is allowed. It also ensures the recursive feasibility and closed-loop stability, and is extended to the output feedback case.

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Chapter 1

Introduction

Model Predictive Control (MPC), as an advanced control strategy, does not specify a special strategy but rather includes a wide range of control methods which utilize an explicit model of a system to obtain the control input sequences by minimising an objective cost function. As its name implies, a model (linear or nonlinear) which describes the system dynamics is needed when applying an MPC strategy. The objective is usually set to be an index that reflects system behaviour from the current instant to some time in the future, which is the reason why it is called *predictive* control. MPC is always implemented in a *receding horizon* manner, thus it is also referred to as receding horizon control. The process of receding horizon control (for discrete time models) is described as follows:

(1). At time instant k, with the known state information $x^{(k)}$, a control input sequence $\{u_0^{(k)}, u_1^{(k)}, \ldots, u_{T-1}^{(k)}\}$ is calculated when minimising the objective function J_T^k over a horizon T, and only the first element of the sequence $u_0^{(k)}$ is applied to the system;

(2). At time $k+1$, the state information $x^{(k+1)}$ becomes available. Then another control input sequence $\{u_0^{(k+1)}, u_1^{(k+1)}, \ldots, u_{T-1}^{(k+1)}\}$ is optimised with a same horizon T. Similarly, only the first element $u_0^{(k+1)}$ is applied to the system;

(3). At time $k+2$, $k+3$, $\ldots$, repeat the procedure in time k or $k+1$.

The development of Model Predictive Control was underpinned by its success in industrial applications. Early researchers were mostly pioneers of industrial practitioners and their results were introduced to address issues of practical importance such as *Model Predictive Heuristic Control* [Richalet et al., 1978], *Dynamic Matrix Control* [Cutler and Ramaker, 1981], *Generalized Predictive Control* [Clarke et al., 1987], etc. The different design methods of MPC have been widely used in chemical engineering, process engineering, aero engineering, financial engineering and even policy assessment study [Qin and Badgwell, 2003, Kouvaritakis et al., 2006], In general, MPC has three distinct features, which justify its wide applications:

1. Real time optimization with an appropriate intrinsic model;
2. Receding horizon control strategy using the latest system information;
3. Applicable to multi-variable systems subject to diverse constraints.

As described above, mathematical models in MPC are developed to approximate the real system dynamics, and inevitably in practice there will exist modelling inaccuracies or uncertainty. How to apply MPC strategy in the presence of model uncertainty has been a focus of the research on MPC. During the recent years, two different MPC strategies have attracted a lot of attentions. One is to deal with the worst case scenario of model uncertainty, which means the MPC strategy can be valid for each realization of possible model parameters. This is called *robust* MPC. The other makes use of the distribution information of the uncertainty to ensure that constraints are satisfied with a certain probability, which is referred to as *stochastic* MPC. This thesis investigates and compares both MPC strategies for discrete-time linear systems with uncertainty.

In robust MPC, an *open-loop* strategy assumes that only the current information of the system is available throughout the predictions and a sequence of control input actions is optimised. Since the future uncertainty realizations will be available up to the prediction instants, this information can also be utilized. However, for open-loop MPC no consideration is given to the information of the future measured states. Although it is computationally attractive, open-loop MPC can be very conservative. Thus closed-loop MPC has been proposed, in which the control inputs in the predictions are functions of the future measured states. In general, online optimisation over arbitrary functions is not implementable. Therefore, many semi closed-loop controllers have been proposed and applied. Typical developments of those controllers are presented below. A linear state feedback law was proposed in Kothare et al. [1996], which ensured the recursive feasibility and closed-loop stability. But the constant feedback assumption over the prediction horizon brought in a degree of conservativeness, in the sense of both the region of attraction and the cost bound. On the basis of linear state feedback, a dual mode strategy with only one step in mode 1 and linear feedback law in mode 2 was developed by Park et al. [1999]. Later on, Schuurmans and Rossiter [2000] studied a dual mode strategy with multi-step input sequence in mode 1 and linear feedback law in mode 2. This leads to a larger region of attraction and a smaller cost. However, the number of LMI's optimised online grows exponentially, which limits the method of Schuurmans and Rossiter [2000] to short horizons and small number of vertices. One design that is used across the thesis makes use of control input which are made up of the sum of linear state feedback Kx_i and a perturbation signal c_i.

$$u_i = Kx_i + c_i. \tag{1.1}$$

Based on (1.1), Kouvaritakis et al. [2000] proposed an augmented autonomous state space description (see (2.51), (2.52)), which avoids the need to propagate the effects of the uncertainty on the predicted states. The perturbation signal c_k provides extra degrees of freedom which can enlarge the region of attraction and accommodate the constraints. The algorithm proposed in Kouvaritakis et al. [2000] aims at the minimisation of a quadratic cost (*nominal* performance) subject to an ellipsoidal invariant set (quadratic form), which can be solved through a Newton Raphson method. The implementation of this algorithm is very efficient when compared with previous formulations listed above.

Although the algorithm in Kouvaritakis et al. [2000] has many advantages, it employs the nominal performance as the online objective. This will not be so justified if the worst case performance differs greatly from the nominal. To address this issue, this thesis considers the use of a Youla Parameter method to desensitize the cost to the effect of uncertainty and thus improves the robustness of the MPC strategy. The idea of introducing Youla Parameter into MPC which can strengthen the robustness of the system while preserving the nominal performance dates back to Kouvaritakis et al. [1992], which has been studied further by Yoon and Clarke [1995], Wams and van den Boom [1997], Stoica et al. [2008]. Surprisingly, all these work does not address the constrained cases, which seems to defeat the justification for the popularity of using MPC. In Rossiter and Kouvaritakis [1998], it was mentioned the Youla parameter may achieve the superior robustness even with constraint present, yet it did not provide the analysis of recursive feasibility and stability. The research presented here then extends the Youla Parameter method to handle various constraints on input, state or output with feasibility guaranteed.

As is mentioned earlier, the controller (1.1) is employed across the whole thesis for beneficial effects with respect to uncertainty handling. For most of the cases the unconstrained nominal LQ-optimal feedback law K_{LQ} is chosen in the interest of nominal optimality. However, there exist uncertain plants where the K_{LQ} cannot stabilise the whole class of system models. This problem is addressed here firstly by replacing $K_{LQ}x$ with another linear feedback Hx, which can be obtained offline by optimising the worst case cost for the unconstrained system. If a feasible solution exists, this approach provides an effective answer with respect to stability but it may not converge to the LQ-optimal feedback. This drawback is circumvented through the design of a dynamic controller whenever that is feasible.

Robust MPC takes only account of the worst case of the uncertainty, while stochastic MPC makes use of the distribution information of the uncertainty. Correspondingly, the constraints are deterministic in robust MPC, and in stochastic MPC there usually exist probabilistic constraints (constraints to be met with a certain probability p). In such cases, by ignoring the probabilistic nature of the problem, robust MPC restricts unnecessarily the available control authority and therefore leads to conservative results. How to deal with probabilistic constraints forms a key challenge in the study of stochastic MPC. The most common mechanism is to transform the probabilistic constraints into deterministic ones. A large number of approaches have been developed. Schwarm and Nikolaou [1999] assumed the uncertainty to be Gaussian distributed and translated the online algorithm into a second order cone program through the use of the inverse cumulative distribution function. Probabilistic inscribed ellipsoids were presented in Van Hessem et al. [2002] to provide sufficient conditions for constraint satisfaction. In Primbs and Sung [2009], all the probabilistic constraints were reduced to constraints in expectation only. None of the above stochastic MPC strategies ensured recursive feasibility and the closed-loop stability.

Tube methods have been both applied to robust MPC [Langson et al., 2004, Mayne et al., 2005] and stochastic MPC [Cannon et al., 2009a]. Robust tubes define the set of all possible states at a certain time instant, accordingly probabilistic tubes represent the set in which states will lie with some probability. Cannon et al. [2009a] developed polytopic tubes and utilized the concept of Markov chains to address the probabilistic constraints, which ensured the recursive feasibility and closed-loop stability. However, it invoked one-step-ahead transitional probability constraints and one-step-ahead violation probability constraints among different layers of tubes. This results in a large number of constraints and thus limits the method to a small number of tubes and short prediction horizons.

The work presented here deals with systems which are subject to additive uncertainty, analyzes the dynamics of the nominal system and the error separately using the controller (1.1), and then makes use of the tube methodology to account for the probabilistic constraints. Firstly, an ellipsoidal tube similar to the form in Van Hessem et al. [2002] is proposed, but the point highlighted in this study is that the evolution for the tube scalings is simplified as an affine function offline. Then it is easy to compute the transitional probabilities for Markov Chain model with a process of discretization. Hence, one can use a large number of tubes and the probabilistic tube scalings can be calculated to transform the probabilistic constraints into deterministic ones. The online algorithm turns into a quadratic programming (QP) problem, which can be solved efficiently. Unlike Van Hessem et al. [2002], this algorithm guarantees recursive feasibility with the concept of Recursively Feasible Stochastic Tubes (RFST). This design is effective but deploys ellipsoidal tube cross-sections and this depends on some assumption which can lead to a degree of conservatism. Further improvements can be achieved through the exact computation of probabilistic tube scalings using the explicit distribution function of the uncertainty. No assumption on the shape of tube is made, but the cross-sections of the tubes are defined implicitly through a process of constraint tightening. This constraint tightening is non-conservative in the sense that, within the control structure adopted, the probability of constraint violations can be as large as is allowed. The main contribution here is that a *non-conservative* stochastic MPC strategy is formulated, which can run in a *computationally efficient* manner and provide the necessary *guarantees of feasibility* and *closed-loop stability*. Moreover, the method is extended to the case when states are not measurable and observers are needed. This state estimation approach can also be applied in robust MPC. The effects of robust MPC and stochastic MPC are then compared by numerical examples.

The outline of the thesis is as follows:

- In Chapter 2, earlier work on robust MPC and stochastic MPC is reviewed. Firstly open-loop and closed-loop MPC are described, then typical design-s of semi closed-loop controllers are discussed and tube methods in robust MPC are illustrated. Following that, different formulations to handle probabilistic constraints in stochastic MPC are surveyed and a stochastic tube method which was developed recently is introduced. In the last part, robust MPC and stochastic MPC are compared, and the main work of the thesis is discussed further.

- In Chapter 3, a Youla Parameter method to improve the robustness of an existing efficient MPC algorithm is proposed. This method preserves the nominal optimality but also handles constraints, which to date has not been addressed by many other researchers. Numerical evidence is presented to demonstrate that the introduction of Youla Parameter can effectively desensitize the cost to the effect of uncertainty.

- In Chapter 4, the problem when K_{LQ} in the controller $u_i = K_{LQ}x_i + c_i$ (K_{LQ} denotes the unconstrained nominal LQ-optimal feedback law) cannot stabilise the whole class of system models is remedied through two methods. One is to replace K_{LQ} with another linear feedback H, and the other is through the design of a dynamic controller which can render the nominal unconstrained LQ-optimal feedback law implementable.

- In Chapter 5, a stochastic ellipsoidal tube methodology is developed. The uncertainty and the errors are assumed to lie in ellipsoids (described by quadratic inequalities). A simple affine dynamics for evolution of the tube scaling is obtained first, then the Markov Chain model is employed to determine the probabilistic tubes at each time step. Recursively feasible stochastic tubes (RFST) are developed and this algorithm is proved to be closed-loop stable.

- In Chapter 6, a tube, whose scaling is computed directly from the distribution function of the uncertainty, is defined. The way to define recursively feasible probabilistic tubes is then described. This method leads to tight satisfaction of probabilistic constraints. It is also extended to output feedback case, where the state observer is used. Finally the method is compared with the ellipsoidal tube method of Chapter 5.

- In Chapter 7, main conclusions of this thesis are summarized and possible future research directions are suggested.

Chapter 2

Review on Robust and Stochastic MPC

In this chapter, the open-loop and closed-loop strategies used in MPC are first introduced and some common concepts, such as stability, dual mode strategy and terminal sets, are presented in detail. The review then considers robust MPC. We begin by introducing the robust MPC method and then select several typical formulations from the large number available in the existing literatures. An efficient robust MPC algorithm, which is based on an augmented autonomous formulation, is described, and afterwards the tube method in robust MPC to handle additive uncertainty is discussed. Following that, the review reports the progress made in stochastic MPC. Different ways of handling probabilistic constraints are considered and compared. Stochastic tube methods are also investigated. The review concludes with a discussion of the connections between the research on robust MPC and stochastic MPC and reveals several gaps which will be addressed in the thesis.

2.1 Open-loop and Closed-loop MPC Strategy

As discussed in Chapter 1, the basic strategy of MPC is as follows: it predicts future behaviour under various constraints on the input, state and/or output, and then performs an optimization in order to determine the future inputs; this is implemented in a receding horizon manner. For the purposes of illustration, use will be made of the discrete time state space model:

$$x_{k+1} = Ax_k + Bu_k, \tag{2.1a}$$

$$y_k = Cx_k, \tag{2.1b}$$

where $x_k \in \mathbb{R}^n,\ u_k \in \mathbb{R}^m,\ y_k \in \mathbb{R}^p$, A is the state matrix, B is the input matrix and C is the output matrix. The control and input sequences must satisfy

$$x_k \in \mathbb{X}, \quad u_k \in \mathbb{U}, \tag{2.2}$$

where ordinarily $\mathbb{U}$ is a convex, compact set and $\mathbb{X}$ is a convex and closed set. The control objective is to minimize a cost function $J(k)$

$$J(k) = \sum_{i=0}^{N-1} [x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k)], \tag{2.3}$$

where Q can be chosen as $Q = C^T Q_0 C$ with $Q_0 > 0$ and $R \geq 0$. Here for simplicity $J(k)$ is taken to be a finite horizon cost function and given an initial condition, a control input sequence is obtained with a receding horizon implementation

$$\mathbf{u} := \{u_{0|0},\ u_{0|1},\ \ldots,\ u_{0|N-1}\}, \tag{2.4}$$

where $u_{0|k}$ denotes the first element of control *actions* optimised at time instant k, i.e. $\arg\min_u J(k) = \{u_{0|k},\ u_{1|k},\ \ldots,\ u_{N-1|k}\}$. Use of this optimal input trajectory at future times would constitute an open-loop strategy in that it consists of a sequence of control *actions*. To deal with system uncertainties better, MPC performs the optimisation to determine a policy π which is composed of a sequence of control *laws*[1]

$$\pi := \arg\min_{u} J(k) = \{u_{0|0}(\cdot),\ u_{0|1}(\cdot),\ \ldots,\ u_{0|N-1}(\cdot)\}, \tag{2.5}$$

where $u_{0|k}(\cdot)$ represents the first element of control *laws* optimised at time k, i.e. $\arg\min_u J(k) = \{u_{0|k}(\cdot),\ u_{1|k}(\cdot),\ \ldots,\ u_{N-1|k}(\cdot)$. This results in what is called *closed-loop* MPC or *feedback* MPC [Langson et al., 2004]. In practice, closed-loop MPC is often difficult to implement on account of its prohibitive computational burden. Therefore, consideration will be given to open-loop control sequence $\mathbf{u}$ without special specifications in the remainder of this chapter.

2.2 Stability

MPC usually steers the state to the origin or to an equilibrium state x_r, also called the set point state in the tracking problem (see 2.21). This requires that the closed-loop system should be *stable*. The concept of stability is based on the notion of equilibrium [Maciejowski, 2002]. An equilibrium point $(0,0)$ is stable (in the sense of Lyapunov) if given any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $\|[x_0^T, u_0^T]\|_p < \epsilon$, [2]then $\left\|[x_k^T, u_k^T]\right\|_p < \delta$ for all $k > 0$. It is asymptotically stable if in addition $\left\|[x_k^T, u_k^T]\right\|_p \to 0$ as $k \to \infty$.

Theorem 2.1 (*Lyapunov's Theorem*)**.** If there exists a function $V(x,u)$ which is positive-definite, namely such that $V(x,u) \geq 0$ and $V(x,u) = 0$ only when $(x,u) = (0,0)$, and has the ('decrescent') property that

$$\text{If} \quad \left\|[x_1^T, u_1^T]\right\|_p \leq \left\|[x_2^T, u_2^T]\right\|_p, \quad \text{then} \quad V(x_1,u_1) \leq V(x_2,u_2), \tag{2.6}$$

[1] A control law is regarded as a function of current and/or previous states.

[2] $\|x\|_p$ denotes the p norm, namely $\|x\|_p \triangleq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, where $p \geq 1$ can be any real integer.

and if, along any trajectory of a system, in some neighborhood of $(0,0)$ the property

$$V(x_{k+1}, u_{k+1}) \leq V(x_k, u_k) \tag{2.7}$$

holds, then $(0,0)$ is a stable equilibrium point. If in addition, $V(x_k, u_k) \to 0$ as $k \to \infty$ then it is asymptotically stable. Such a function V is called a Lyapunov function.

Clearly, according to Lyapunov's theorem, if V is *monotonously decreasing*, then $(0,0)$ is a stable equilibrium point. Lyapunov's theorem can be applied to both linear or nonlinear systems.

MPC based on the optimisation of a finite horizon cost function $J(k)$, such as the form of (2.3), does not carry the guarantee of closed-loop stability. This difficulty can be avoided if $J(k)$ is defined to be the infinite horizon cost:

$$\begin{aligned} J(k) &= \sum_{i=0}^{\infty}[x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k)] \\ &= \sum_{i=0}^{N-1}[x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k)] + F\big(x(k+N|k)\big) \end{aligned} \tag{2.8}$$

where the additional term $F\big(x(k+N|k)\big)$ is the terminal cost representing the cost-to-go with a given terminal control law. In addition, to account for constraints from the N^{th} prediction time onwards, use is made of a terminal set X_f ($X_f \subseteq \mathbb{X}$) and a terminal constraint: $x(k+N|k) \in X_f$. The above handling of infinite horizon makes the variable (control sequence $\mathbf{u}$) finite dimensional and thus the online optimisation of the cost is implementable, the assumption usually made being that (as explained below) beyond the prediction horizon N, the predicted inputs are those stipulated by the LQ unconstrained optimal state feedback law.

2.2.1 Dual Mode Strategy

A convenient way to obtain the terminal cost $F\big(x(k+N|k)\big)$ is that adopted by the dual mode infinite horizon MPC.

Mode 1: for $i = 0, 1, \ldots, N-1$, control sequence $\mathbf{u}$ can be computed by (2.4);

Mode 2: for $i = N, N+1, \ldots, \infty$, the inputs obey the unconstrained optimal control law $u = Kx$, thus stabilizing the system. Then the cost function $J(k)$ in (2.8) can be explicitly written as

$$J(k) = \sum_{i=0}^{N-1} [x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k)] + x^T(k+N|k)\tilde{Q}x(k+N|k) \tag{2.9}$$

where $\tilde{Q}$ is the terminal weighting matrix. It can be obtained from the matrix Lyapunov equation (2.10) [Maciejowski, 2002].

$$\begin{aligned} F(x(k+N|k)) &= \sum_{i=N}^{\infty} [x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k)] \\ &= \sum_{i=N}^{\infty} x^T(k+i|k)(Q+K^TRK)x(k+i|k) \\ &= x^T(k+N|k)\{\sum_{i=0}^{\infty}(A+BK)^{i^T}(Q+K^TRK)(A+BK)^i\}x(k+N|k) \end{aligned}$$

Therefore $\tilde{Q} = \sum_{i=0}^{\infty}(A+BK)^{i^T}(Q+K^TRK)(A+BK)^i$ and this leads to

$$\tilde{Q} - (A+BK)^T\tilde{Q}(A+BK) = Q + K^TRK. \tag{2.10}$$

The conditions ensuring that the above system is stable are listed in Maciejowski [2002]:

1). A positive semi-definite solution ($\tilde{Q} \geq 0$) of (2.10) exists;

2). The optimal predicted input and state sequences satisfy the constraints over an infinite horizon.

Condition 1 makes sure that $J(k)$ is positive definite and can be seen as a candidate Lyapunov function. Condition 2 ensures that the optimal predicted cost $J^o(k)$ is non-increasing over k and satisfies

$$J^o(k+1) - J^o(k) \leq -[x^T(k)Qx(k) + u^T(k)Ru(k)]. \tag{2.11}$$

At time k an optimal input sequence of the first steps N is given by

$$\mathbf{u}^o(k) = \{u^o(k|k), u^o(k+1|k), \ldots, u^o(k+N-1|k)\}$$

Then at the next step $k+1$, a feasible input sequence is

$$\tilde{\mathbf{u}}(k+1) = \{u^o(k+1|k), u^o(k+2|k), \ldots, u^o(k+N-1|k), Kx^o(k+N|k)\},$$

which consists of two parts: the first $N-1$ element is the 'tail' of $\mathbf{u}^o(k)$, the last element is the first control move of mode 2 at time k. Then the cost $\tilde{J}(k+1)$ corresponding to this input is

$$\begin{aligned}
\tilde{J}(k+1) = &\sum_{i=1}^{N-1}[x^T(k+i|k)Qx(k+i|k) + u^{oT}(k+i|k)Ru^o(k+i|k)] + x^T(k+N|k)\tilde{Q}x(k+N|k)\\
=&\sum_{i=0}^{\infty}[x^T(k+i|k)Qx(k+i|k) + u^{oT}(k+i|k)Ru^o(k+i|k)] - [x^T(k)Qx(k) + u^T(k)Ru(k)]\\
=&J^o(k) - [x^T(k)Qx(k) + u^T(k)Ru(k)]
\end{aligned}$$

Therefore, it is clear that the optimal cost $J^o(k+1)$ at time $k+1$ satisfies

$$J^o(k+1) \leq \tilde{J}(k+1) \leq J^o(k) - [x^T(k)Qx(k) + u^T(k)Ru(k)].$$

Thus, if condition 1 and condition 2 are both satisfied, the stability of the infinite horizon control with dual mode strategy is guaranteed.

A more general analysis of stability appears in Mayne et al. [2000]. It employs a terminal set X_f associated with the terminal control law, $u_f(\cdot)$. The four conditions for ensuring the closed loop stability are listed below:

1. $X_f \subseteq \mathbb{X}$, X_f closed, $0 \in X_f$ (state constraint satisfied in X_f);

2. $u_f(x_k) \in \mathbb{U}, \forall x_k \in X_f$ (control constraint satisfied in X_f);

3. $x_{k+1}\big(x_k, u_f(x_k)\big) \in X_f$, $\forall x_k \in X_f$ (i.e. X_f is positively invariant under $u_f(\cdot)$);

4. $\hat{F}(x_k, u_f(x_k)) + [x^T(k)Qx(k) + u_f^T(k)Ru_f(k)] \leq 0$, $\forall x_k \in X_f$ (where $\hat{F}(x_k, u_f(x_k)) = F(x_{k+1}, u_f(x_{k+1})) - F(x_k, u_f(x_k))$, and $F(\cdot)$ is a local Lyapunov function).

2.2.2 Terminal Sets and the Tracking Problem

In terms of the terminal set X_f, two forms of sets are commonly used in MPC, namely ellipsoidal and polyhedral. In the sequel, derivation of X_f (both ellipsoidal and polyhedral) for system (2.1) will be given. For systems with uncertainties or terminal sets in other forms, the details of how to construct X_f can be referred to in Lee and Kouvaritakis [2000], Gilbert and Tan [1991], etc.

Inside X_f the terminal control law should satisfy the linear constraints which will be taken to be at the form

$$\left|g^T x\right| \leq u_{lim}, \tag{2.12}$$

where g is a known vector[3]. Moreover, X_f should be a positively invariant set (under the terminal law).

Ellipsoidal sets are defined as

$$E = \{x : x^T P x \leq 1\}, \quad P > 0. \tag{2.13}$$

[3]For convenience of illustration, (2.12) is one form of (2.2) for a single-input, single-output case. And since the terminal sets defined in this section are symmetric, the use of *symmetric* constraint (2.12) will bring no conservativeness compared with one-sided constraint $g^T x \leq u_{lim}$.

For E to be an invariant set, the following must be satisfied:

$$x^T(A+BK)^T P(A+BK)x \le x^T P x \le 1, \quad \forall x \in E$$

$$x^T(P-(A+BK)^T P(A+BK))x \ge 0 \quad \Leftrightarrow \quad P-(A+BK)^T P(A+BK) \ge 0$$

$$\Leftrightarrow \quad P^{-1}-P^{-1}(A+BK)^T P(A+BK)P^{-1} \ge 0$$

By Schur complementation, this can be further written as a linear matrix inequality (LMI)[4]:

$$\begin{bmatrix} P^{-1} & P^{-1}(A+BK)^T \\ (A+BK)P^{-1} & P^{-1} \end{bmatrix} \ge 0. \tag{2.14}$$

Constraint (2.12) can be handled as follows:

$$\left|g^T x\right|^2 \le \left|g^T P^{-1/2} P^{1/2} x\right|^2 \le \left\|g^T P^{-1/2}\right\|^2 \left\|P^{1/2} x\right\|^2 = (g^T P^{-1} g)(x^T P x) \le g^T P^{-1} g \le u_{lim}^2$$

This can be written as an LMI using Schur complement:

$$\begin{bmatrix} P^{-1} & P^{-1} g \\ g^T P^{-1} & u_{lim}^2 \end{bmatrix} \ge 0. \tag{2.15}$$

It is straightforward to observe that (2.14) and (2.15) are both sufficient and necessary conditions of invariance and feasibility, respectively. Our aim is to maximize the terminal set so as to enlarge the region of attraction of the feasible initial states. Therefore, P in (2.13) can be calculated using the objective function

$$P = \arg\min_{P} \det(P^{-1})^{-1} \quad \text{subject to } (2.14), (2.15). \tag{2.16}$$

[4]An LMI has the form of $F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0$, where $x \in \mathbb{R}^m$ is the design variable, $F_i = F_i^T \in \mathbb{R}^{n\times n}$ are known, and $F(x) > 0$ means that $u^T F(x) u > 0$ for all nonzero $u \in \mathbb{R}^n$ [Boyd et al., 1994].

Low complexity polyhedral sets can be defined as

$$\Delta = \{x : |Vx| \leq \alpha\}, \quad \alpha > 0 \tag{2.17}$$

where V is a square and invertible matrix with inverse $W = V^{-1}$ and α is a vector with each element positive. Δ is an invariant set if the following necessary and sufficient condition is satisfied

$$|V(A + BK)x| = |V(A + BK)WVx|| \leq |V\Phi W| |Vx| \leq |V\Phi W| \alpha \leq \alpha \tag{2.18}$$

where $\Phi = A + BK$. Constraints need satisfying at the same time and this leads to the necessary and sufficient condition:

$$\left|g^T x\right| = \left|g^T WVx\right| \leq \left|g^T W\right| |Vx| \leq \left|g^T W\right| \alpha \leq u_{lim}. \tag{2.19}$$

It is convenient to choose the rows of V to be the left eigenvectors of Φ in order to make $V\Phi W$ diagonal; this is possible if Φ has real eigenvalues and a simple Jordan canonical form[5]. Therefore, if $u = Kx$ is a stabilizing terminal law so that the modulus of the eigenvalues of Φ are less than 1, then (2.18) will hold true for any α. So the only constraint for α is (2.19) and in order to enlarge the volume of the terminal set it is possible to select α as:

$$\alpha = \arg\max_{\alpha} \det(diag(\alpha)) \quad \text{subject to (2.19)}. \tag{2.20}$$

Now the whole dual mode control strategy to implement MPC on system (2.1) is as follows.

Algorithm 2.1. *Offline:* Optimize the volume of the terminal set by solving (2.16) or (2.20);

Online: At each time instant $k = 0, 1, 2, \ldots, N - 1$,

[5]The case of complex eigenvalues can be treated in a similar manner but requires these eigenvalues to lie inside the square with vertices $(\pm 1, 0)$ and $(0, \pm j)$.

1) solve the optimization

$$\mathbf{u}(x) = \arg\min_{u} J(k) = \{u_k(k;x), u_k(k+1;x), \dots, u_k(k+N-1;x)\}$$
$$\text{s.t.} \quad x(k+i) \in \mathbb{X}, \; u(k+i) \in \mathbb{U}, \; \text{for } i = 0, 1, \dots, N-1; \; \text{and } x(k+N) \in X_f$$

2) Implement the first element of $\mathbf{u}(x)$ and measure the value of the state vector of the next step (or compute an appropriate estimate).

It is always straightforward to extend MPC to solve simple tracking problems: the controller will drive the state to a constant steady state point. For system (2.1), assume x_{ss}, u_{ss} and y_{ss} to be the state, input and output corresponding to the equilibrium point, so

$$x_{ss} = Ax_{ss} + Bu_{ss}, \quad y_{ss} = Cx_{ss}. \tag{2.21}$$

Then $x_{k+1} - x_{ss} = A(x_k - x_{ss}) + B(u_k - u_{ss}), y_k - y_{ss} = C(x_k - x_{ss})$, which corresponds to a same dynamical system

$$\delta x_{k+1} = A(\delta x_k) + B(\delta u_k), \; \delta y_k = C(\delta x_k), \; \text{where } \delta(\cdot)_k = (\cdot)_k - (\cdot)_{ss}. \tag{2.22}$$

Therefore, using MPC for system (2.22) will actually regulate the state, input and output to the steady state set point conditions if the closed-loop system is stable. MPC can also be used in reference trajectory tracking problem, the details of which will be omitted here but otherwise can be seen in Kwakernaak and Sivan [1972].

2.3 Robust MPC

Mathematical models like (2.1) used in control engineering, can only describe the dynamics of the real system in an approximate way, since there are always modelling inaccuracies or uncertainty. So model (2.1) gives an idealized representation, usually referred to as the *nominal* model. Real system models always contain disturbances and/or uncertainty, and the objective of the thesis is to study the application of MPC strategy to these perturbed system models. One way to design controllers for these problems is to preserve stability and performance for *each possible realization* of bounded disturbances or uncertainty, which is considered as *robust control*. Below, the open-loop and closed-loop methods in Section 2.1 are briefly analyzed in the study of robust MPC.

Open-loop MPC, where a sequence of control actions in (2.4) is optimised, proves to be very computationally attractive (e.g. [Mayne et al., 2000, Zheng and Morari, 1993]), yet it cannot avoid the 'propagation' of the predicted trajectories due to the effects of uncertainty, because it does not utilize the future state information. Thus, closed-loop (feedback) MPC in the form of (2.5) has become a focus of research. While the computational demand for closed-loop MPC usually gets overly high when determining a control policy π online, attempts to resolve this dilemma have resulted in the development of the simplifications of closed-loop MPC, which have been reported in a number of publications such as Kothare et al. [1996], Scokaert and Mayne [1998], Bemporad [1998], Park et al. [1999], Kouvaritakis et al. [2000], Schuurmans and Rossiter [2000], Löfberg [2003], Mayne et al. [2005], Goulart et al. [2006], Skaf and Boyd [2010]. Due to space limitation, we are not going to describe all such results; instead consideration is given to several typical designs of robust MPC controllers.

2.3.1 Linear State Feedback Law

In Kothare et al. [1996], a linear state feedback law is proposed as the simplification of (2.5):

$$u(k+i|k) = H(k)x(k+i|k),\ \ i \geq 0, \tag{2.23}$$

where $H(k)$ is assumed to be constant over the prediction horizon but is allowed to vary from one time step k to the next. Although paradigms both for polytopic uncertainty and structured feedback uncertainty have been considered by Kothare et al. [1996], robust MPC design for polytopic uncertainty is reviewed here. Accordingly use will be made of a linear time-varying (LTV) model

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \\ y(k) &= Cx(k), \\ [A(k)\ B(k)] &\in \Omega, \end{aligned} \tag{2.24}$$

where the set Ω represents a polytope: $\Omega \triangleq Co\{[A_1\ B_1],\ [A_2\ B_2],\ \ldots,\ [A_L\ B_L]\}$[6]. The constraints are in the form of either Euclidean norm or ∞-norm

$$\|u(k+i|k)\|_2 \leq u_{max}, \quad k, i \geq 0, \tag{2.25}$$

$$\|u(k+i|k)\|_\infty \leq u_{max}, \quad k, i \geq 0, \tag{2.26}$$

$$\|y(k+i|k)\|_2 \leq y_{max}, \quad k \geq 0, i \geq 1, \tag{2.27}$$

$$\|y(k+i|k)\|_\infty \leq y_{max}, \quad k \geq 0, i \geq 1, \tag{2.28}$$

[6] Co denotes the convex hull, which means if $[A\ B] \in \Omega$, then $[A\ B] = \sum_{i=1}^{L} \lambda_i [A_i\ B_i]$, $\sum_{i=1}^{L} \lambda_i = 1,\ \lambda_i \geq 0$.

where (2.25) or (2.26) defines input constraints, and (2.27) or (2.28) defines output constraints. A robust performance objective is minimised

$$\min_{u(k+i|k),\ i=0,1,\ldots,m}\ \max_{[A(k+i)\ \ B(k+i)]\in\Omega} J_\infty(k),$$

where

$$J_\infty(k) = \sum_{i=0}^{\infty}[x^T(k+i|k)Q_1x(k+i|k) + u^T(k+i|k)Ru(k+i|k)]. \tag{2.29}$$

An upper bound cost $V(x(k))$ of (2.29) is developed, which is a quadratic function $V(x) = x^TPx,\ P > 0$ and satisfies the following decrecent property

$$\begin{aligned} V(x(k+i+1|k)) - V(x(k+i|k)) \leq -\,[x^T(k+i|k)Q_1x(k+i|k)+ \\ u^T(k+i|k)Ru(k+i|k)],\ \forall k, i \geq 0. \end{aligned} \tag{2.30}$$

Sufficient condition for (2.30) is

$$[A(k+i) + B(k+i)H]^TP[A(k+i) + B(k+i)H] - P + H^TRH + Q_1 \leq 0, \tag{2.31}$$

where for notational convenience, H has been used instead of $H(k)$. For (2.29) to be finite, $x(\infty|k) = 0$ is required. So summing (2.30) from $i = 0$ to $i = \infty$, one can get $-V(x(k|k)) \leq -J_\infty(k)$. Thus

$$\max_{[A(k+i)\ \ B(k+i)]\in\Omega} J_\infty(k) \leq V(x(k|k)). \tag{2.32}$$

To handle the constraints, an invariant ellipsoid $x(k|k)^TPx(k|k) \leq \gamma$ is proposed. Considering (2.30), it is easy to get $\max_{[A(k+i)\ \ B(k+i)]\in\Omega} x(k+i|k)^TPx(k+i|k) \leq \gamma,\ i \geq 0$. With $P = \gamma Q^{-1}$, the ellipsoid is equivalent to $x(k|k)^TQ^{-1}x(k|k) \leq 1$ which can be rewritten as

$$\begin{bmatrix} 1 & x(k|k)^T \\ x(k|k) & Q \end{bmatrix} \geq 0, \quad Q \succ 0. \tag{2.33}$$

Thus, $\mathcal{E} = \{z|z^T Q^{-1} z \leq 1\} = \{z|z^T P z \leq \gamma\}$ is the invariant set for system (2.24) under (2.23). Let $Y = HQ$, with Schur complements, (2.31) can be transformed into the LMI

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R^{1/2} \\ * & Q & 0 & 0 \\ * & * & \gamma I & 0 \\ * & * & * & \gamma I \end{bmatrix} \geq 0, \quad j = 1, 2, \ldots, L, \tag{2.34}$$

where $*$ denotes an off-diagonal block of a symmetric matrix. Input constraints in Euclidean norm as (2.25) can be handled as follows [Boyd et al., 1994]:

$$\begin{aligned} \max_{i \geq 0} \|u(k+i|k)\|_2^2 &= \max_{i \geq 0} \left\|YQ^{-1}x(k+i|k)\right\|_2^2 \leq \max_{z \in \mathcal{E}} \left\|YQ^{-1}z\right\|_2^2 \\ &= \lambda_{max}(Q^{-1/2}Y^T Y Q^{-1/2}) = \lambda_{max}(YQ^{-1}Y^T), \end{aligned}$$

where $\lambda_{max}(\cdot)$ refers to the largest eigenvalue of a matrix. For $\|u(k+i|k)\|_2^2 \leq u_{max}^2$ to be met, one necessary and sufficient condition is obtained by simple manipulations:

$$\begin{bmatrix} u_{max}^2 I & Y \\ Y^T & Q \end{bmatrix} \geq 0. \tag{2.35}$$

When input constraint is in ∞-norm as (2.26), it can also be translated into an LMI constraint:

$$\max_{i\geq 0}\|u(k+i|k)\|_\infty^2 = \max_{i\geq 0}\left\|YQ^{-1}x(k+i|k)\right\|_\infty^2 \leq \max_{z\in\mathcal{E}}\left\|YQ^{-1}z\right\|_\infty^2 \leq \max_j [YQ^{-1}Y^T]_{jj} \leq u_{max}^2$$

$$\Leftrightarrow \begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \geq 0 \text{ with } \max_i X_{ii} \leq u_{max}^2. \tag{2.36}$$

With similar arguments, output constraints (2.27) or (2.28) can be expressed in LMI form:

$$\begin{bmatrix} Q & (A_jQ+B_jY)^TC^T \\ * & y_{max}^2 \end{bmatrix} \geq 0, \quad j = 1,2,\ldots,L, \tag{2.37}$$

$$\begin{bmatrix} X & C(A_jQ+B_jY) \\ * & Q \end{bmatrix} \geq 0, \quad j = 1,2,\ldots,L, \quad \text{with } \max_i X_{ii} \leq y_{max}^2. \tag{2.38}$$

Algorithm 2.2. At each sampling time $k = 0,1,2,\ldots,$

1) solve the online optimization

$$(\gamma, Q, Y) = \arg\min_{\gamma,Q,Y} \gamma$$

subject to (2.33), (2.34), and either (2.35) or(2.36) and either (2.37) or(2.38), depending on which forms of constraints are imposed.

2) Compute $H = YQ^{-1}$ and implement the current input $u(k) = Hx(k|k)$.

It has been proved that Algorithm 2.2 possesses the property of recursive feasibility[7] and is robustly asymptotically stable [Kothare et al., 1996]. While the drawback for this study is that it is computationally demanding (solving semi-definite programs online) and it incurs a degree of conservativeness through the

[7]Recursive Feasibility means that if the algorithm has a feasible solution at time k, then it is feasible for all time instants $t > k$.

use of a linear feedback law which is assumed constant over the prediction horizon. Thus, the region of attraction (the set of initial states which are feasible for the algorithm) can be small and $V(x(k))$ may not offer a tight bound on performance.

2.3.2 Robust One-step Receding Horizon Control

Rather than using a one mode strategy (linear feedback law) as in Kothare et al. [1996], a dual mode strategy with only one step in mode 1 and linear feedback law in mode 2 is studied by Park et al. [1999]:

$$u_{k+i|k} = \begin{cases} u_k, & i = 0 \\ H_k x_{k+i|k}, & i = 1, \dots, \infty \end{cases} \tag{2.39}$$

The system considered is the same as in (2.24) and the input and state constraints take the following form

$$\begin{aligned} -u_{lim} \leq u_k \leq u_{lim}, \qquad & k = 0, 1, \dots, \infty \\ -g_{lim} \leq Gx_k \leq g_{lim}, \qquad & k = 0, 1, \dots, \infty \end{aligned} \tag{2.40}$$

The cost function for this one-step design of receding horizon control is defined as

$$J(x_k, u_k, \Psi_k, k) = J_1(x_k, u_k, k) + J_2(x_k, \Psi_k, k), \tag{2.41}$$

where $J_1(x_k, u_k, k) = x_k^T Q x_k + u_k^T R u_k$ $(Q \geq 0,\ R > 0)$ and $J_2(x_k, \Psi_k, k) = x_{k+1|k}^T \Psi_k x_{k+1|k}$ $(\infty > \Psi_k > 0)$. The objective is then to minimise J_1 by choosing an optimal u_k and optimise J_2 through the design of Ψ_k, H_k at each sampling time k. $J_1 \leq \gamma_1$ can be easily recast into the following LMI:

$$\begin{bmatrix} \gamma_1 I & x_k^T & u_k^T \\ x_k & Q^{-1} & 0 \\ u_k & 0 & R^{-1} \end{bmatrix} \geq 0. \tag{2.42}$$

Starting form the $x_{k+1|k} = A_k x_k + B_k u_k$, it is easy to perform the optimization of γ_2 ($J_2 \le \gamma_2$) with the same method as in section 2.3.1: transforming all the conditions/constraints into LMI form. Therefore, the minimisation of $J(x_k, u_k, \Psi_k, k)$ in (2.41) is formulated as a semi-definite programming (SDP) problem: min $(\gamma_1 + \gamma_2)$.

Note that if $u_k = H_k x_k$, then this one-step design transforms to the scheme of Section 2.3.1. Therefore, it is more general than the linear feedback law of Kothare et al. [1996]. It is also shown in Park et al. [1999] that the method can lead to a larger region of attraction (admissible set of initial states) and a smaller cost. Moreover, this proposed scheme maintains recursively feasible and closed-loop stable. Despite the fact that the one-step method has yielded certain amount of benefits, it is not surprising that further improvements were sought through the use of longer horizons.

2.3.3 Robust Multi-step Receding Horizon Control

A multi-step prediction horizon n_c in mode 1 of a dual mode strategy is investigated for the system (2.24) in Schuurmans and Rossiter [2000]. The controller dynamics is defined by

$$u(k+i|k) = \begin{cases} K(k+i|k)x(k+i|k) + c(k+i|k), & i = 0, 1, \dots, n_c - 1 \\ K(k+n_c)x(k+i|k), & i = n_c, \dots, \infty \end{cases} \tag{2.43}$$

where for the first n_c steps the input is a time-varying feedback control law plus a feedforward term[8] and from time $k \ge n_c$ a fixed control law is exploited. Let P be as defined in Section 2.3.1 and satisfy (2.30), and introduce variables γ_i that

[8] To reduce the online computational complexity, $K(k+i|k)$, $i = 0, 1, \dots, n_c - 1$ is assumed known (chosen offline) as in Schuurmans and Rossiter [2000].

satisfy

$$\gamma_i \geq x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k), \quad i = 0,1,\ldots,n_c-1,$$
$$\gamma_{n_c} \geq x^T(k+n_c|k)Px(k+n_c|k),$$

for all $x(k+i|k),\ u(k+i|k),\ i \geq 0$ satisfying system (2.24). Then an upper bound $\overline{V}(k)$ of the cost function $J(k)$ is obtained

$$J(k) = \sum_{i=0}^{n_c-1} J_i(k) + \sum_{i=n_c}^{\infty} J_i(k) \leq \sum_{i=0}^{n_c} \gamma_i = \overline{V}(k), \tag{2.44}$$

with $J_i(k) \triangleq x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k)$[9]. To render γ_i as tight as possible, the polytopic sets in which the future states will lie in have been computed according to the following lemma.

Lemma 2.2. Given $x(k+i|k) \in S(k+i|k)$ and the set $S(k+i|k)$ is a convex hull made up of L^i vertices:

$$S(k+i|k) = Co\{\nu_1(k+i|k), \nu_2(k+i|k), \ldots, \nu_{L^i}(k+i|k)\},\ \forall i > 0$$

then $S(k+i+1|k)$ is the tightest set which consists of the possible future states $x(k+i+1|k)$ if its vertices $\nu_j(k+i+1|k),\ j = 1,2,\ldots,L^{i+1}$ are obtained from

$$\nu_j(k+i+1|k) = A(l)\nu_m(k+i|k) + B(l)u(k+i|k),\ \forall l \in \{1\ldots L\},\ m = 1,2,\ldots,L^i. \tag{2.45}$$

The proof of Lemma 2.2 is given in Schuurmans and Rossiter [2000]. Considering the convexity of $S(k+i|k)$, it is reasonable to evaluate the value of γ_i by computing the maximum of $J_i(k)$ only on all of the vertices $\nu_j(k+i|k),\ j = 1,2,\ldots,L^i$, which leads to a tight bound. Combining the controller dynamics (2.43), (2.45) is

[9] Since P satisfies (2.30), according to the analysis in Section 2.3.1, we get $\sum_{i=n_c}^{\infty} J_i(k) \leq x^T(k+n_c|k)Px(k+n_c|k) \leq \gamma_{n_c}$.

converted into

$$\nu_j(k+i+1|k) = [A(l) + B(l)K(k+i)]\nu_m(k+i|k) + B(l)c(k+i|k). \tag{2.46}$$

Then in mode 1, the bound γ_i for step $i = 0, 1, \ldots, n_c - 1$ can be rewritten as

$$\begin{bmatrix} I & Q^{1/2}\nu_j(k+i|k) & 0 \\ \nu_j^T(k+i|k)Q^{1/2} & \gamma_i & w_j^T(k+i|k)R^{1/2} \\ 0 & R^{1/2}w_j(k+i|k) & I \end{bmatrix} \geq 0, \quad \forall \nu_j(k+i|k) \in \text{vert}\{S(k+i|k)\} \tag{2.47}$$

with $w_j(k+i|k) = K(k+i)\nu_j(k+i|k) + c(k+i|k)$. And the input constraint $\underline{u}_t \leq u_t(k) \leq \overline{u}_t,\ t = 1, 2, \ldots, r$ becomes

$$\underline{u}_t \leq K_t(k+i)\nu_j(k+i|k) + c_t(k+i|k) \leq \overline{u}_t,\ \forall \nu_j(k+i|k) \in \text{vert}\{S(k+i|k)\},\ j = 1, 2, \ldots, L^i, \tag{2.48}$$

for $i = 0, 1, \ldots, n_c - 1$. In mode 2, it is straightforward to implement the optimisation using the similar method as in Section 2.3.1 with the starting state $\nu_j(k+n_c) \in \text{vert}\{S(k+n_c)\}$. Hence, minimising $\overline{V}(k)$ is a semi-definite programming (SDP) problem. This multi-step formulation is also proved to be recursively feasible and robustly asymptotically stable. Since $\gamma_i,\ i = 0, 1, \ldots, n_c - 1$ provide exact bounds, the SDP optimisation results in a smaller cost than the method in Section 2.3.1 and 2.3.2 and allows for a larger region of attraction for the associated MPC strategy. However the number of LMI's grow quickly with prediction steps n_c and the numbers L of model vertices, and so this method suffers from 'curse of dimensionality', which may become intractable for large n_c and L.

2.3.4 Efficient Robust Predictive Control

Here the proposed controller [Rossiter et al., 1998] is similar to the form of (2.43), but the feedback control law is fixed at all time instants:

$$u(k+i|k) = Kx(k+i|k) + c(k+i|k); \quad c(i+n_c+k|k) = 0, \ i \geq 0 \tag{2.49}$$

with $c(k+i|k), \ i = 0, 1, \ldots, n_c - 1$ providing the degrees of freedom. The diagram of the structure is shown in Figure 2.1.

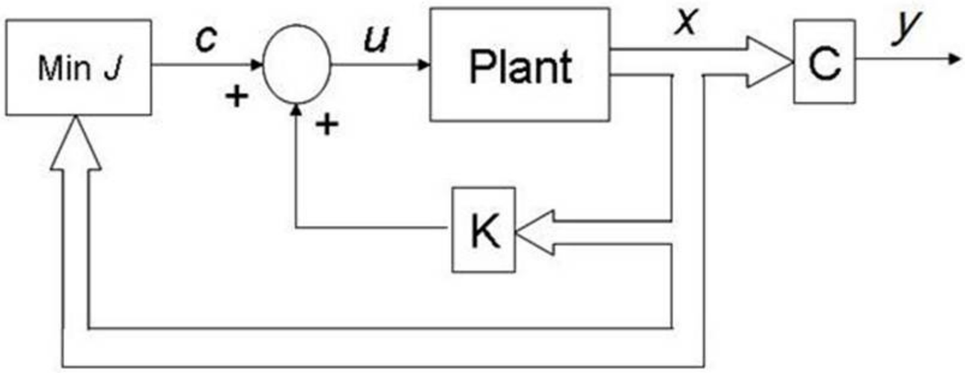

FIGURE 2.1: Perturbation from a linear feedback MPC

The use of the perturbation $c(k+i|k)$ provides a mechanism for ensuring the constraints are respected whenever $c(k+i|k) = 0$ is infeasible. However, if $c(k+i|k)$ is the deviation from the unconstrained LQ-optimal law, it should be kept as small as possible. When K in (2.49) is the unconstrained LQ-optimal feedback law for the nominal system $x_{k+1} = A_0 x_k + B_0 u_k$, the cost function $J(k)$ [Kouvaritakis et al., 2002] evaluated for nominal system can be simplified as

$$J(k) = \mathbf{c}^T W \mathbf{c}, \quad W = diag([B_0^T \Sigma B_0 + R, B_0^T \Sigma B_0 + R, \ldots, B_0^T \Sigma B_0 + R]) \tag{2.50}$$

where $\mathbf{c} = [c_k^T, c_{k+1}^T, \ldots, c_{k+n_c-1}^T]^T$ and Σ is the solution to the matrix Lyapunov equation $\Sigma - (A_0 + B_0 K)^T \Sigma (A_0 + B_0 K) = Q + K^T R K$. For the single input single output (SISO) case, $W = \mu I$, where μ is a scalar constant.

With the paradigm of (2.49), an efficient robust MPC scheme is developed in Kouvaritakis et al. [2000]. The closed-loop system of (2.24) becomes

$$x(k+1) = \Phi(k)x(k) + B(k)c(k), \quad x \in \mathbb{R}^n,\ c \in \mathbb{R}^m \tag{2.51}$$

where $\Phi(k)$ lies within the polytope whose vertices are $\Phi_l = A_l + B_l K,\ l = 1, 2, \ldots, L$. Then the prediction dynamics can be described equivalently by the augmented autonomous state space model

$$\begin{aligned}
z(k+1) =& \Psi(k)z(k), \quad z \in \mathbb{R}^{n+m\cdot n_c} \\
\Psi(k) = \begin{bmatrix} \Phi(k) & B(k)E \\ 0 & M \end{bmatrix}&, \quad z(k) = \begin{bmatrix} x(k) \\ f(k) \end{bmatrix}, \quad E = [I_m\ 0_m \cdots 0_m] \\
f(k) = \begin{bmatrix} c(k) \\ c(k+1) \\ \vdots \\ c(k+n_c-1) \end{bmatrix}&, \quad f(k+1) = \begin{bmatrix} c(k+1) \\ \vdots \\ c(k+n_c-1) \\ 0_{m\times 1} \end{bmatrix}, \\
f(k+2) = \begin{bmatrix} c(k+2) \\ \vdots \\ c(k+n_c-1) \\ 0_{m\times 1} \\ 0_{m\times 1} \end{bmatrix}&, \cdots, \quad M = \begin{bmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \cdots & 0_m & I_m \\ 0_m & 0_m & \cdots & \cdots & 0_m \end{bmatrix},
\end{aligned} \tag{2.52}$$

where 0_m is an $m \times m$ zero matrix. For this augmented model, an ellipsoidal invariant set $E_z = \{z | z^T P_z z \leq 1\}$ $(P_z > 0)$ is defined. To ensure its invariance and feasibility, using an analysis similar to that used for (2.14) and (2.15), P_z must satisfy

$$\begin{bmatrix} P_z^{-1} & P_z^{-1}\Psi_i^T \\ \Psi_i P_z^{-1} & P_z^{-1} \end{bmatrix} \geq 0, \quad i = 1, 2, \ldots, L \tag{2.53}$$

$$\begin{bmatrix} P_z^{-1} & P_z^{-1}[K_j \ e_j]^T \\ [K_j \ e_j]P_z^{-1} & d_j^2 \end{bmatrix} \geq 0, \quad j = 1, 2, \ldots, m \tag{2.54}$$

where (2.54) is a sufficient condition of the input constraint $|(u_k)_j| = |[K_j \ e_j]z_k| \leq d_j, \ j = 1, 2, \ldots, m$ and $K_j, \ e_j$ denotes the jth row of K and identity matrix respectively. The projection of E_z onto the x domain is [Kouvaritakis et al., 2000]:

$$E_{xz} = \{x | x^T P_{xz} x \leq 1\}, \quad P_{xz} = (T P_z^{-1} T^T)^{-1} \tag{2.55}$$

where $T = [I, \ 0]$ is the transformation matrix in $x = Tz$.

Algorithm 2.3 (Efficient Robust MPC)**.** *Offline:* Optimise the region of attraction defined in (2.55) by minimising $\log\det(TP_z^{-1}T^T)^{-1}$ subject to (2.53), (2.54); *Online:* At each time step $k = 0, 1, \ldots, n_c - 1$,

$$f_k = \arg\min_u J(k) = \arg\min_{f_k} f_k^T W f_k, \quad \text{subject to } z_k^T P_z z_k \leq 1. \tag{2.56}$$

Then use (2.49) to implement the first element of f_k and repeat the online step at the next time instant.

The first benefit of Algorithm 2.3 is that it obviates the need to propagate the effects of uncertainty over the prediction horizon and only requires that the current augmented state z_k lies in E_z. Secondly, (2.56) can be solved very efficiently through the use of a Newton Raphson iteration that is used to determine the unique negative root of a well behaved polynomial [Kouvaritakis et al., 2002], thus Algorithm 2.3 is much more efficient than the algorithms described in the previous sections which include solving a number of LMI's online. This also allows n_c to be large enough to have a big region of attraction. In addition, stability is ensured by requiring the initial state to lie inside the ellipsoid of (2.55).

2.3.5 Robust MPC with Tubes

The concept of a 'tube' (at a certain time instant) in robust MPC is used to describe the set which contains all possible states arising from uncertainty. To determine a tube, one needs to clarify three important parameters: i.e. position, shape (e.g. ellipsoidal or polyhedral cross-section) and volume. Therefore, the computation of these three parameters and the determination of their evolution in prediction time form the key components of tube MPC. From this point of view, it is not difficult to note that the linear feedback law approach of Section 2.3.1 optimises an invariant ellipsoidal tube online that will contain all the predicted states, and the method in Section 2.3.4 in fact defines an invariant ellipsoidal tube offline. There also exist studies on tubes of different shapes, e.g. polyhedral cross-section [Lee and Kouvaritakis, 2000].

One noteworthy formulation for robust tube MPC is investigated in Mayne et al. [2005]. In contrast to the *nominal* model (2.1), the system now is considered to be subject to bounded additive disturbances:

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad w_k \in W \tag{2.57}$$

where W is a compact set which includes the origin. The state and input constraints are in the same form of (2.2): $x_k \in \mathbb{X}$, $u_k \in \mathbb{U}$. State feedback $u = Kx$ is designed such that the nominal model $x_{k+1} = (A + BK)x_k = A_c x_k$ is stable and Z is a disturbance invariance set for the uncertain system $x_{k+1} = A_c x_k + w_k$ which thus satisfies

$$A_c Z \oplus W \subseteq Z, \tag{2.58}$$

where $\oplus$ denotes Minkowski addition[10]. It is shown [Kolmanovsky and Gilbert, 1998] that the minimal disturbance invariant set is $Z_{min} = \sum_{i=0}^{\infty} A_c^i W$, and a polytopic outer approximation of Z_{min}, Z can be computed according to the method in Raković et al. [2005]. The following controller is then introduced to describe the system state evolution with the concept of tubes.

Lemma 2.3. For $x_{k+1} = A_c x_k + w_k$, assume there is a disturbance invariant set Z. If $x_k \in \bar{x}_k \oplus Z$ and $u_k = \bar{v}_k + K(x_k - \bar{x}_k)$, then $x_{k+1} \in \bar{x}_{k+1} \oplus Z$ for all $w_k \in W$, where $x_{k+1} = Ax_k + Bu_k + w_k$ and $\bar{x}_{k+1} = A\bar{x}_k + B\bar{v}_k$.

Lemma 2.3 reveals that under the feedback policy $u_k = \bar{v}_k + K(x_k - \bar{x}_k)$, the trajectory of x_k in system (2.57) will lie in a set of tubes of cross-section Z centred at $\bar{x}_k$ satisfying the nominal dynamics $\bar{x}_{k+1} = A\bar{x}_k + B\bar{v}_k$. Thus, constraints can be tightened on the state $\bar{x}_k$ and the input $\bar{v}_k$ of the nominal model:

$$\begin{aligned} \bar{x}_k &\in \bar{\mathbb{X}} \triangleq \mathbb{X} \ominus Z, \quad k = 0, 1, \ldots, N-1 \\ \bar{v}_k &\in \bar{\mathbb{U}} \triangleq \mathbb{U} \ominus KZ, \quad k = 0, 1, \ldots, N-1 \\ \bar{x}_N &\in \bar{X}_f \subset \mathbb{X} \ominus Z. \end{aligned} \tag{2.59}$$

The cost function to be minimised is defined as:

$$\overline{V}_N(x_0, \mathbf{v}) \triangleq \sum_{i=0}^{N-1} \ell(\bar{x}_i, \bar{v}_i) + V_f(\bar{x}_N), \tag{2.60}$$

where $\ell(\bar{x}, \bar{v}) = \bar{x}^T Q \bar{x} + \bar{v}^T R \bar{v}$, $V_f(\bar{x}) = \bar{x}^T P \bar{x}$ $(Q, R, P \succ 0)$ and $\mathbf{v} \triangleq \{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{N-1}\}$. And the terminal set $\bar{X}_f$ and the terminal cost $V_f(\cdot)$ satisfy the four stability conditions [Mayne et al., 2000] listed in Section 2.2.1.

Let $\bar{\mathcal{U}}(\mathbf{v})$ represents the set of nominal inputs $\mathbf{v}$ which respect the tighter constraints (2.59)

$$\bar{\mathcal{U}}(\mathbf{v}) = \{\mathbf{v} | \bar{v}_i \in \bar{\mathbb{U}},\ \bar{x}_i \in \bar{\mathbb{X}},\ i = 0, 1, \ldots, N-1,\ \bar{x}_N \in \bar{X}_f\}. \tag{2.61}$$

[10]The Minkowski addition (sum) is: $\mathbb{A} \oplus \mathbb{B} \triangleq \{a + b,\ \forall a \in \mathbb{A},\ \forall b \in \mathbb{B}\}$

One typical way is then to assume $\bar{x}_0$ equal to known x_0, hence the conventional nominal control problem [Langson et al., 2004] is

$$\bar{P}_N(x_0): \quad \overline{V}_N^c(x_0) = \min_{\mathbf{v}} \{\overline{V}_N(x_0, \mathbf{v}) | \mathbf{v} \in \bar{\mathcal{U}}(\mathbf{v}),\ x_0 = \bar{x}_0\}. \tag{2.62}$$

The corresponding domain of the value function $\overline{V}_N^c(x_0)$ is $\bar{X}_N \triangleq \{x_0 | \bar{\mathcal{U}}(\mathbf{v}) \neq \varnothing,\ x_0 = \bar{x}_0\}$.

Another design in which the initial nominal state $\bar{x}_0$ is also dealt with as a decision variable is explored in Mayne et al. [2005]. The control problem is then modified by

$$P_N^*(x_0): \quad V_N^*(x_0) = \min_{\bar{x}_0, \mathbf{v}} \{\overline{V}_N(x_0, \mathbf{v}) | \mathbf{v} \in \bar{\mathcal{U}}(\mathbf{v}), x_0 \in \bar{x}_0 \oplus Z\}, \tag{2.63}$$

$$(\bar{x}_0^*, \mathbf{v}) = \arg\min_{\bar{x}_0, \mathbf{v}} \{\overline{V}_N(x_0, \mathbf{v}) | \mathbf{v} \in \bar{\mathcal{U}}(\mathbf{v}), x_0 \in \bar{x}_0 \oplus Z\}. \tag{2.64}$$

Both $P_N^*(x_0)$ and $\bar{P}_N(x_0)$ are quadratic programs. By solving $P_N^*(x_0)$, the implicit control law is

$$\kappa_N^*(x_0) \triangleq \bar{v}_0^* + K(x_0 - \bar{x}_0^*), \tag{2.65}$$

and the domain of the value function $V_N^*(x_0)$ is $X_N^* \triangleq \{x_0 | \exists \bar{x}_0 \text{ such that } x_0 \in \bar{x}_0 \oplus Z,\ \mathcal{U}(\mathbf{v}) \neq \varnothing\}$. It is worth noting that the value of $\overline{V}_N^c(x_0)$ will be no smaller than $\overline{V}_N^*(x_0)$ for a same system and initial condition x_0. In addition, one important solution of the problem $P_N^*(x_0)$ is that when $x_0 \in Z$, then $V_N^*(x_0) = 0,\ \bar{x}_0^* = 0,\ \mathbf{v} = \{0, 0, \ldots, 0\}$, which renders the system (2.57) robustly exponentially stable over the region of attraction X_N^*. Here the set Z for perturbed system (2.57) is analogous to the original point $\{0\}$ for nominal system (2.1).

In the above analysis, a simple form of tube is presented: *fixed geometry* and *fixed volume* with the centre varying. More general tube MPC methods have been addressed in Langson et al. [2004], Raković et al. [b,a], which interested readers can refer to.

2.4 Stochastic MPC

As described in the previous section, research on robust MPC has achieved a great number of encouraging results in tackling the worst case of the system uncertainty, however, it does not consider the information about how the uncertainty is distributed, and this is the main justification for the study of stochastic MPC. In addition to hard constraints that cannot be transgressed in any cases[11], stochastic MPC usually considers probabilistic constraints, also sometimes referred to as chance constraints. Probabilistic constraints imply that the constraints may be violated with a given probability. Stochastic MPC has found applications in a great variety of fields, such as sustainable development policy making [Kouvaritakis et al., 2004, 2006], building climate control [Oldewurtel et al., 2008, 2010], traffic control [Zegeye et al., 2010] and financial engineering [Pasik-Duncan et al., 2004], etc. A capability for handling probabilistic constraints is particularly important in problems with large unknown disturbances, but more generally it can help avoid the trade-off between computation and optimality of robust MPC. This is achieved by replacing the requirement in robust MPC for constraint satisfaction for all uncertainty realizations (which can lead to excessive computation or conservative performance [Scokaert and Mayne, 1998]) with a guarantee that constraints are satisfied with specified probabilities. In stochastic MPC, the mechanism for handling probabilistic constraints is to convert them to deterministic ones. As such, it is possible to utilize the controllers developed for robust MPC. Several typical formulations of stochastic MPC are introduced and analyzed below.

[11]The constraints incorporated in robust MPC are all hard constraints.

2.4.1 Normal Distributed Chance-constrained Problem

An early study of stochastic MPC appears in [Åström, 1970] aiming at the minimization of the variance. It was based on transfer function models and optimized the basic stochastic parameters. A typical formulation to deal with the probabilistic constraints is introduced in Schwarm and Nikolaou [1999]. This paper proposes a chance-constrained stochastic programming problem

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad g_1(\mathbf{x}) \leq 0, \; Pr\{g_2(\mathbf{p},\mathbf{x}) \leq 0\} \geq \alpha. \tag{2.66}$$

The chance constraint $Pr\{g_2(\mathbf{p},\mathbf{x}) \leq 0\} \geq \alpha$ is handled by substituting it with a deterministic constraint form $g_3(\mathbf{x}) \leq 0$ provided that $\mathbf{p}$ is known. If $g_2(\mathbf{p},\mathbf{x}) = A(\mathbf{x})\mathbf{p}+b(\mathbf{x})$, then the stochastic parameters $\mathbf{p}$ can be separated from the decision variable $\mathbf{x}$ in one constraint, such as

$$Pr\{\hat{g}_2(\mathbf{x}) \leq \mathbf{p}\} \geq \alpha. \tag{2.67}$$

In consequence, the deterministic equivalent form can be obtained easily through the use of the probability density function (pdf) of $\mathbf{p}$, i.e.

$$\hat{g}_2(\mathbf{x}) \leq \beta, \;\; \text{where } \beta \text{ is defined by } \alpha = \int \cdots \int_{-\infty}^{\beta} pdf(\mathbf{p})d\mathbf{p}. \tag{2.68}$$

Schwarm and Nikolaou [1999] consider a linear constraint form with normally distributed parameters

$$Pr\left\{\boldsymbol{a}^T(M\boldsymbol{x}+c) \leq b\right\} \geq \alpha, \tag{2.69}$$

where $\boldsymbol{a}$ is a normally distributed vector with mean $\bar{\boldsymbol{a}}$ and variance $\boldsymbol{P}_{\boldsymbol{a}}$. Due to linear transformation, it is known that $\boldsymbol{a}^T(M\boldsymbol{x}+c)$ will itself be normally distributed with mean $\bar{\boldsymbol{a}}^T(M\boldsymbol{x}+c)$ and variance $Var\left[\boldsymbol{a}^T(M\boldsymbol{x}+c)\right] = (M\boldsymbol{x} +$

$c)^T \boldsymbol{P_a} M\boldsymbol{x} + c)$. Then constraint (2.69) can be rearranged to the standard normal form

$$Pr\left\{\frac{\boldsymbol{a}^T(M\boldsymbol{x}+c) - \bar{\boldsymbol{a}}^T(M\boldsymbol{x}+c)}{\sqrt{(M\boldsymbol{x}+c)^T\boldsymbol{P_a}(M\boldsymbol{x}+c)}} \le \frac{b - \bar{\boldsymbol{a}}^T(M\boldsymbol{x}+c)}{\sqrt{(M\boldsymbol{x}+c)^T\boldsymbol{P_a}(M\boldsymbol{x}+c)}}\right\} \ge \alpha. \tag{2.70}$$

By calculating the confidence level K_α according to the inverse cumulative distribution of the standard normal distribution which corresponds to α, it is easy to convert (2.70) to $\frac{b-\bar{\boldsymbol{a}}^T(M\boldsymbol{x}+c)}{\sqrt{(M\boldsymbol{x}+c)^T\boldsymbol{P_a}(M\boldsymbol{x}+c)}} \ge K_\alpha$, which can be rewritten as

$$\bar{\boldsymbol{a}}^T(M\boldsymbol{x}+c) \le b - K_\alpha\sqrt{(M\boldsymbol{x}+c)^T\boldsymbol{P_a}(M\boldsymbol{x}+c)}. \tag{2.71}$$

Thus the original chance constraint (2.69) leads to a second order cone program and is tractable. However, it places the restriction that $g_2(\mathbf{p},\mathbf{x})$ should be affine or linear with stochastic parameters.

Similar methods to handle the probabilistic constraint are investigated by Warren and Marlin [2003, 2004], Yan and Bitmead [2005]. The system considered in Yan and Bitmead [2005] includes gaussian process noise $\omega_n \sim N(0,\Gamma_n)$ and gaussian measurement noise $\nu_n \sim N(0,\Lambda_n)$:

$$x_{n+i+1} = Ax_{n+i} + Bu_{n+i} + G\omega_{n+i}, \tag{2.72}$$

$$y_{n+i} = Cx_{n+i} + D\nu_{n+i}, \tag{2.73}$$

$$Pr(x_{n+i} \le \beta_i) \ge p_i \ (i = 1,\dots,N). \tag{2.74}$$

Let $\hat{x}_{n+i|n}$, $\tilde{x}_{n+i|n}$ denote the state estimate and estimate error respectively, generated by a finite dimensional Kalman filter:

$$\hat{x}_{n+i|n} = A\hat{x}_{n+i-1|n} + Bu_{n+i-1}, \tag{2.75}$$

$$\Sigma_{n+i|n} = A\Sigma_{n+i-1|n}A^T + G\Gamma_n G^T \ (i = 1,\dots,N), \tag{2.76}$$

where $\Sigma_{n+i|n}$ is the covariance of $\tilde{x}_{n+i|n}$. Since the covariances above do not depend on the current state or state estimate and since $\tilde{x}_{n+i|n} \sim N(0, \Sigma_{n+i|n})$, it is possible to translate (2.74) into

$$\hat{x}_{n+i|n} \leq \beta_i - \Sigma_{n+i|n}^{1/2} \beta_i^*. \tag{2.77}$$

(2.77) is a sequence of linear constraints, which transforms the stochastic optimisation into a computationally standard MPC. However the noises in Warren and Marlin [2003, 2004], Yan and Bitmead [2005] need to be Gaussian to keep the random variables (e.g., state and error) normally distributed and all the methods described thus far provide no guarantees for feasibility or stability in closed loop.

2.4.2 Ellipsoidal Relaxation for Probabilistic Constraint

Another way to handle probabilistic constraint was proposed in Van Hessem et al. [2002]. It is assumed that the system is described by

$$x_{k+1} = Ax_k + Bu_k + Gw_k, \tag{2.78a}$$

$$z_k = G_z x_k + E_z u_k, \tag{2.78b}$$

$$y_k = Cx_k + Fw_k, \tag{2.78c}$$

where y represents the measured output vector, w denotes the exogenous stochastic inputs including both measurement errors and process disturbances (normally distributed), z is the performance variable on which the objective as well as the constraints are imposed. The probabilistic constraint is in the form of

$$Pr\{z_k \in \mathcal{P}\} \geq \alpha \quad \text{where} \quad \mathcal{P} = \{\zeta : H^T \zeta \leq g\}, \tag{2.79}$$

which implies that z_k should lie inside some polytope with probability no smaller than some predefined value α. Given a Gaussian process, z_k can be fully characterized by its mean $\bar{z}$ and auto-covariance matrix Z, it follows that $Pr\{z_k \in \mathcal{P}\} \geq \alpha$ is equivalent to

$$\frac{1}{\sqrt{(2\pi)^n det(Z)}} \int_{\mathcal{P}} e^{-\frac{1}{2}(\zeta-\bar{z})^T Z^{-1}(\zeta-\bar{z})} d\zeta \geq \alpha, \tag{2.80}$$

which seems to be rather difficult to render an MPC optimization practicable. Van Hessem et al. [2002] exploit an ellipsoidal relaxation method shown in Figure 2.2. Define an ellipsoid $\varepsilon_r = \{\zeta : \zeta^T Z^{-1} \zeta \leq r^2\}$ such that

$$Pr\{z_k - \bar{z} \in \varepsilon_r\} \geq \alpha, \tag{2.81a}$$

$$\bar{z} + \varepsilon_r \subset \mathcal{P}. \tag{2.81b}$$

Then (2.81) provides a sufficient condition for the satisfaction of (2.79).

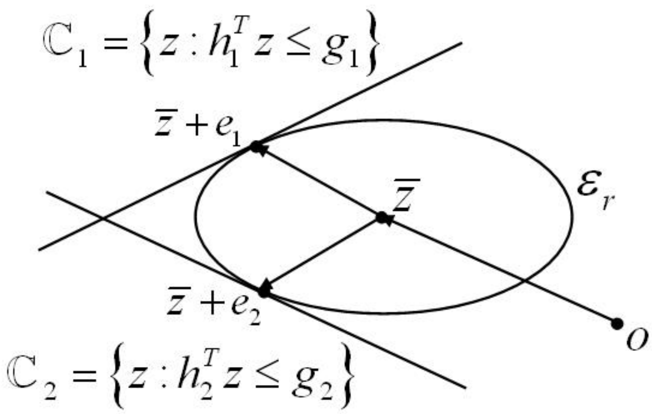

Figure 2.2: Ellipsoidal relaxation of probabilistic constraints

The first condition of (2.81) can be converted into a one-dimensional integration and the second one is easily proved to be equivalent to

$$h_j^T \bar{z} + r\sqrt{h_j^T Z h_j} \leq g_j \quad \forall j, \tag{2.82}$$

where h_j and g_j denote the jth row of matrix H and the jth element of vector g in (2.79) respectively. With the use of techniques such as a Youla parameterization [van Hessem and Bosgra, 2002] or "innovations feedback" [van Hessem and Bosgra, 2003], (2.82) can be translated into a second-order cone program [Lobo et al., 1998]. Thus it is possible to solve the problem using general SDP solvers. However for large dimensional systems with long prediction horizons, the computation burden for this optimization will become prohibitive and hence not applicable.

2.4.3 Constraint Satisfaction in Expectation Form

A stochastic receding horizon control strategy for linear systems with control and state multiplicative noise is developed in Primbs and Sung [2009], which includes constraints. The stochastic dynamical system (discrete time) is as follows

$$x(k+1) = Ax(k) + Bu(k) + \sum_{j=1}^{q} [C_j x(k) + D_j u(k)]\, w_j(k) \tag{2.83}$$

where $w_j(k),\ j = 1, \cdots, q$ are independent and identically distributed random variables with $\mathbb{E}[w_j(k)] = 0$, $\mathbb{E}[w_j(k)w_l(k)] = 0$ for $j \neq l$, and $\mathbb{E}[w_j(k)w_j(k)] = 1$. The constraints take the form of

$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T H_l \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + f_l^T \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leq \beta_l, \quad l = 1 \ldots L_1 \tag{2.84a}$$

$$x^T(k) H_{lx} x(k) + (f_{lx})^T x(k) \leq \beta_{lx}, \quad lx = 1 \ldots L_2 \tag{2.84b}$$

with $H_l \geq 0$ and $H_{lx} \geq 0$. Obviously, when $H_l = 0$ and $H_{lx} = 0$, (2.84) reduce into linear constraints. The finite horizon control optimization, denoted by $\mathcal{P}(x(k), N)$, exploits an open-loop plus linear feedback strategy (under the assumption that an appropriate terminal cost function and terminal constraints exist)

$$V(x(k)) = \min_{\bar{u}(\cdot|k),K(\cdot|k)} \mathbb{E}_{x(k)}\left(\sum_{i=0}^{N-1}\begin{bmatrix} x(i|k) \\ u(i|k)\end{bmatrix}^T M \begin{bmatrix} x(i|k) \\ u(i|k)\end{bmatrix} + x^T(N|k)\Phi x(N|k)\right)$$

$$\text{s.t.} x(0|k) = x(k), u(0|k) = \bar{u}(0|k),$$

$$u(i|k) = \bar{u}(i|k) + K(i|k)(x(i|k) - E_{x(k)}[x(i|k)]),\ i = 1 \dots N-1$$

$$\mathbb{E}_{x(k)}(x^T(N|k)\Phi x(N|k)) \le \alpha,\ (u(\cdot|k), x(\cdot|k) \in \mathcal{C}(x(k), N))$$

and aims to satisfy (2.84) in expectation. Thus the constraint set, $\mathcal{C}(x(k), N)$, is transformed into

$$\mathcal{C}(x(k), N) = \begin{cases} \mathbb{E}\left\{\begin{bmatrix} x(i) \\ u(i)\end{bmatrix}^T H_l \begin{bmatrix} x(i) \\ u(i)\end{bmatrix} + f_l^T \begin{bmatrix} x(i) \\ u(i)\end{bmatrix}\right\} \le \beta_l, & i = 0 \dots N-1, l = 1 \dots L_1 \\ \mathbb{E}\left\{x^T(i) H_{lx} x(i) + (f_{lx})^T x(i)\right\} \le \beta_{lx}, & i = 1 \dots N, l = 1 \dots L_2 \end{cases}$$

Matrix variables $P(i)$ for $i = 1 \dots N-1$ have been introduced which satisfy

$$P(i) \ge \mathbb{E}\left\{\begin{bmatrix} x(i) \\ u(i)\end{bmatrix}\begin{bmatrix} x(i) \\ u(i)\end{bmatrix}^T\right\} = \begin{bmatrix} \bar{x}(i) \\ \bar{u}(i)\end{bmatrix}\begin{bmatrix} \bar{x}(i) \\ \bar{u}(i)\end{bmatrix}^T + \begin{bmatrix} \Sigma(i) \\ U(i)\end{bmatrix}\Sigma^{-1}(i)\begin{bmatrix} \Sigma(i) \\ U(i)\end{bmatrix}^T, \tag{2.85}$$

with $\bar{x}(i) = \mathbb{E}_{x(k)}[x(i)]$, $\Sigma(i) = \mathbb{E}\{(x(i) - \bar{x}(i))(x(i) - \bar{x}(i))^T\}$ and $U(i) = K(i)\Sigma(i)$. Using Schur complements (2.85) can be rewritten as an LMI

$$\begin{bmatrix} P(i) & \begin{bmatrix} \Sigma(i) \\ U(i)\end{bmatrix} & \begin{bmatrix} \bar{x}(i) \\ \bar{u}(i)\end{bmatrix} \\ \begin{bmatrix} \Sigma(i) & U^T(i)\end{bmatrix} & \Sigma(i) & 0 \\ \begin{bmatrix} \bar{x}^T(i) & \bar{u}^T(i)\end{bmatrix} & 0 & 1 \end{bmatrix} \ge 0. \tag{2.86}$$

With the introduction of $P(i)$, the objective function can be written as

$$V = \min \sum_{i=0}^{N-1} Tr\left(MP(i)\right) + Tr\left(\Phi P^X(N)\right), \tag{2.87}$$

where $P^X(N) = \mathbb{E}_{x(k)}(x(N|k)x^T(N|k))$. Correspondingly, $\mathcal{C}(x(k), N)$ and the terminal constraint can be translated into

$$\mathcal{C}(x(k),N) = \begin{cases} Tr\left(H_l P(i)\right) + f_l^T \begin{bmatrix} \bar{x}(i) \\ \bar{u}(i) \end{bmatrix} \leq \beta_l, & i = 0 \dots N-1,\ l = 1 \dots L_1 \\ Tr\left(H_{lx} P^X(i)\right) + (f_{lx})^T \bar{x}(i) \leq \beta_{lx}, & i = 1 \dots N, l = 1 \dots L_2 \end{cases}$$

$$Tr\left(\Phi P(N)\right) \leq \alpha, \tag{2.88}$$

where $P^X(i) = \mathbb{E}(x(i)x^T(i))$, while the covariance matrices $\Sigma(i+1)$ and $\Sigma(i)$, for $i = 1 \dots N-1$, can be expressed in the form of LMI's as well. Then, $\mathcal{P}(x(k), N)$ becomes a semi-definite program such that the optimization can be implemented in a receding horizon manner. However, the constraint relaxation obtained by employing the expectation form is simplistic. Moreover, this method is not computationally efficient because of the constraints, which are in LMI form, and the large number of matrix variables that need to be determined online. In addition, no guarantee is given regarding the issue of feasibility.

2.4.4 Affine Disturbance Feedback Controller

In Oldewurtel et al. [2008, 2010], an affine disturbance feedback controller [Löfberg, 2003, Goulart et al., 2006] is exploited in analyzing a stochastic model predictive control problem which considers the following discrete-time LTI system and chance constraints

$$x^+ = Ax + Bu + Ew, \tag{2.89}$$

$$Pr\left[F_j x_k \leq f_j\right] \geq 1 - \alpha_{j,k},\ \ j = 1 \dots r, k = 1 \dots N-1 \tag{2.90}$$

$$Pr\left[S_l u_k \leq s_l\right] \geq 1 - \beta_{l,k},\ \ l = 1 \dots m, k = 0 \dots N-1 \tag{2.91}$$

where w is an independent and identically normally distributed disturbance with $w \sim \mathcal{N}(0, I)$. The controller moves are designed as

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + h_i, \tag{2.92}$$

and can be rewritten in the matrix form

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{h}, \quad \text{and} \tag{2.93}$$

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \quad \mathbf{h} := \begin{bmatrix} h_0 \\ \vdots \\ \vdots \\ h_{N-1} \end{bmatrix}.$$

Together with the lifted form of the system dynamics, chance constraints (2.90) and (2.91) are converted into

$$Pr\left[\mathbf{F}_j(\mathbf{A}x_0 + \mathbf{BMw} + \mathbf{Bh} + \mathbf{Ew}) \leq \mathbf{g}_i\right] \geq 1 - \alpha_j, \; j = 1 \ldots r, \tag{2.94}$$

$$Pr\left[\mathbf{S}_l(\mathbf{Mw} + \mathbf{h}) \leq \mathbf{s}_l\right] \geq 1 - \beta_l, \; l = 1 \ldots r \tag{2.95}$$

For brevity, the detailed form of the matrices will not be listed here but can be found in the literature. Since (2.94) and (2.95) are bi-affine in the decision variables and the disturbances, it is possible to transform them into tractable formulations: such as a linear program in [Oldewurtel et al., 2008] or a second-order cone program in [Oldewurtel et al., 2010]. The authors have shown that the stochastic MPC strategy significantly outperforms previous MPC practices in energy efficient building climate control. However, the deficiency of the study is that it provides no guarantees of closed-loop properties, such as the feasibility or stability issues.

2.4.5 Invariance with Probability p

An MPC strategy which ensures the closed-loop stability has been proposed in Cannon et al. [2009b]. It considers the system with both multiplicative and additive stochastic noise in the presence of probabilistic constraints:

$$x_{k+1} = A_k x_k + B_k u_k + d_k, \tag{2.96}$$

$$\psi_k = C_k x_k + D_k u_k + \eta_k, \tag{2.97}$$

$$[A_k \ B_k \ d_k \ C_k \ D_k \ \eta_k] = [\bar{A} \ \bar{B} \ 0 \ \bar{C} \ \bar{D} \ 0] + \sum_{j=1}^{m} [\tilde{A}_j \ \tilde{B}_j \ \tilde{g}_j \ \tilde{C}_j \ \tilde{D}_j \ \tilde{\eta}_j] q_{k,j}$$

with independently and (temporally) identically distributed (i.i.d.) variable $q_k = [q_{k,1} \cdots q_{k,m}]^T \sim \mathcal{N}(0, I)$, and ψ_k being the system output, which is required to respect the *joint chance constraint*[12][Chen et al., 2010]:

$$\frac{1}{N_c} \sum_{i=0}^{N_c-1} Pr\{\psi_{k+i} \notin I_\psi\} \leq \frac{N_{max}}{N_c}, \tag{2.98}$$

where I_ψ is the constrained set. (2.98) means that within the future horizon N_c the expected times of constraint violation (i.e. ψ_k lies outside I_ψ) should not exceed N_{max}. This study employs the dual mode strategy and autonomous augmented

[12] *Joint chance constraint* means the mean of the N_c constraint satisfaction has to be no smaller than the given probability, as opposed to *individual chance constraint* which implies each chance constraint needs to be fulfilled with the pre-defined probability.

description of the prediction dynamics at time k:

$$
\begin{gathered}
z_{i+1|k} = \Psi_{k+i} z_{i|k} + \delta_{k+i}, \quad i = 0, 1, \ldots \\
z_{0|k} = \begin{bmatrix} x_k \\ f_k \end{bmatrix}, \quad f_k = \begin{bmatrix} c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}, \quad \begin{bmatrix} \Psi_k & \delta_k \end{bmatrix} = \begin{bmatrix} \bar{\Psi} & 0 \end{bmatrix} + \sum_{j=1}^{m} [\tilde{\Psi}_j \ \tilde{\gamma}_j] q_{k,j}, \\
\bar{\Psi} = \begin{bmatrix} \bar{\Phi} & \bar{B}E \\ 0 & M \end{bmatrix}, \quad \tilde{\Psi}_j = \begin{bmatrix} \tilde{\Phi}_j & \tilde{B}_j E \\ 0 & M \end{bmatrix}, \quad \tilde{\gamma}_j = \begin{bmatrix} \tilde{g}_j \\ 0 \end{bmatrix}, \\
\bar{\Phi} = \bar{A} + \bar{B}K, \quad \tilde{\Phi}_j = \tilde{A}_j + \tilde{B}_j K, \quad E = [I_{n_u} \ 0 \cdots 0] \\
M = \begin{bmatrix} 0_{n_u} & I_{n_u} & 0_{n_u} & \cdots & 0_{n_u} \\ 0_{n_u} & 0_{n_u} & I_{n_u} & \cdots & 0_{n_u} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0_{n_u} & \cdots & \cdots & 0_{n_u} & I_{n_u} \\ 0_{n_u} & 0_{n_u} & \cdots & \cdots & 0_{n_u} \end{bmatrix}.
\end{gathered}
\tag{2.99}
$$

Then the cost function for system (2.96) is developed. Due to the existence of persistent non-zero additive uncertainty, the stage cost will tend to a limit asymptotically and the control objective is defined to minimize the cumulative mean deviation away from that limit:

$$
J_k = \sum_{i=0}^{\infty} \mathbb{E}_k(L_i), \ L_i = z_i^T \tilde{Q} z_i - \lim_{i \to \infty} \mathbb{E}_k(z_i^T \tilde{Q} z_i) = z_i^T \tilde{Q} z_i - tr(\Theta \tilde{Q}), \tag{2.100}
$$

where Θ is the solution of the Lyapunov equation $\Theta - \bar{\Psi}\Theta\bar{\Psi}^T - \sum_{j=1}^m \tilde{\Psi}_j \Theta \tilde{\Psi}_j^T = \sum_{j=1}^m \tilde{\gamma}_j \tilde{\gamma}_j^T$ and $\tilde{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix}$. In this setting, it has been proved [Cannon et al., 2009b] that the cost function can be written as a quadratic form of the current augmented state z_k:

$$J_k = \begin{bmatrix} z_k \\ 1 \end{bmatrix}^T \begin{bmatrix} P_z & P_{1z}^T \\ P_{1z} & P_1 \end{bmatrix} \begin{bmatrix} z_k \\ 1 \end{bmatrix}, \tag{2.101}$$

where $P_z,\ P_{1z}$ and P_1 are defined by

$$P_z - \bar{\Psi}^T P_z \bar{\Psi} - \sum_{j=1}^{m} \tilde{\Psi}_j^T P_z \tilde{\Psi}_j = \tilde{Q},\ \ P_{1z} = \sum_{j=1}^{m} \tilde{\gamma}_j^T P_z \tilde{\Psi}_j (I - \bar{\Psi})^T,\ \ P_1 = -tr(\Theta P_z).$$

The probabilistic constraint (2.98) is handled by the introduction of the Markov Chain. It is assumed that there are two nested sets $\mathcal{E}_1, \mathcal{E}_2\,(\mathcal{E}_1 \subset \mathcal{E}_2)$ such that x_k can either lie in $S_1 = \mathcal{E}_1$ or $S_2 = \mathcal{E}_2 - \mathcal{E}_1$ (assuming that $\mathcal{E}_2$ is constructed so as to make the probability of $x_k \notin \mathcal{E}_2$ negligible). The conditional violation probabilities are defined as

$$Pr\{\psi_{k+i} \notin I_\psi | x_k \in S_j\} = p_j, \quad j = 1, 2, \tag{2.102}$$

and the matrix of transitional probabilities is defined as

$$\Pi = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \tag{2.103}$$

where p_{ij} is the probability that x_k is steered from the region S_j to S_i in one step under the online algorithm. Then on account of the properties of Markov Chains, after i steps one can get

$$\begin{bmatrix} Pr\{x_{k+i} \in S_1\} \\ Pr\{x_{k+i} \in S_2\} \end{bmatrix} = \Pi^i \begin{bmatrix} Pr\{x_k \in S_1\} \\ Pr\{x_k \in S_2\} \end{bmatrix}. \tag{2.104}$$

In consequence, the probability of constraint violation at instant $k+i$ is

$$Pr\{\psi_{k+i} \notin I_\psi\} = [p_1 \;\; p_2]\Pi^i \begin{bmatrix} Pr\{x_k \in S_1\} \\ Pr\{x_k \in S_2\} \end{bmatrix}. \tag{2.105}$$

Since the eigenvalue/eigenvector decomposition of Π [Kushner, 1971] is given by

$$\Pi = [w_1 \;\; w_2] \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}, \quad 0 \le \lambda_2 < 1 \tag{2.106}$$

then, as $i \to \infty$, the average rate of constraint violation given $x_k \in S_j$ is

$$R_j = [p_1 \;\; p_2] w_1 v_1^T e_j, \quad j = 1, 2 \tag{2.107}$$

where e_j denotes the jth column of the 2×2 identity matrix. If $R_j (j = 1, 2)$ are no greater than N_{max}/N_c , then there must exsit a finite number i^* such that, for all $i \ge i^*$, the total expected number of constraint violations will be no greater than $i^* N_{max}/N_c$. Hence, if $i^* \le N_c$, this formulation successfully ensures satisfaction of constraint (2.98).

The inner set $S_1 = E_x : \{x | x^T P_x x \le 1\}$ is constructed in Cannon et al. [2009b] to have the property of Invariance with Probability p (IWPp), i.e. each state $x_k \in E_x$ will be steered into $x_{k+1} \in E_x$ with probability p under a given control law. The p in this construction in fact means transitional probability p_{11} in (2.103). Considering the constraint violation probability p_1, then the offline optimisation computes S_1, $\mathcal{E}_z = \{z : z^T \hat{P} z \le 1\}$, and its projection onto x-subspace $S_1 = E_x : \{x | x^T P_x x \le 1\}$ (i.e. whenever $x \in S_1$, there exists f such that $z = [x^T \;\; f^T]^T \in \mathcal{E}_z$).

Algorithm 2.4. At time steps $k = 0, 1, \ldots$:

(1). If $x_k \in S_1$, compute

$$f_k^* = \arg\min_{f_k} \begin{bmatrix} z_{0|k} \\ 1 \end{bmatrix}^T \begin{bmatrix} P_z & P_{1z}^T \\ P_{1z} & P_1 \end{bmatrix} \begin{bmatrix} z_{0|k} \\ 1 \end{bmatrix} \text{ subject to } z_{0|k}^T \hat{P} z_{0|k} \le 1; \quad (2.108)$$

(2). If $x_k \notin S_1$, compute $f_k^* = \arg\min_{f_k} z_{0|k}^T \bar{\Psi} \Gamma P_x \Gamma^T \bar{\Psi} z_{0|k}$ subject to

$$\begin{bmatrix} z_{0|k} \\ 1 \end{bmatrix}^T \begin{bmatrix} P_z & P_{1z}^T \\ P_{1z} & P_1 \end{bmatrix} \begin{bmatrix} z_{0|k} \\ 1 \end{bmatrix} \le \begin{bmatrix} x_k \\ M f_{k-1}^* \\ 1 \end{bmatrix}^T \begin{bmatrix} P_z & P_{1z}^T \\ P_{1z} & P_1 \end{bmatrix} \begin{bmatrix} x_k \\ M f_{k-1}^* \\ 1 \end{bmatrix}; \quad (2.109)$$

(3). Implement $u_k = K x_k + E f_k^*$.

In Algorithm 2.4, (2.108) minimizes the cost function and one does not need to consider the constraint explicitly when $z = [x^T \ f^T]^T \in \mathcal{E}_z$; and (2.109) aims to drive the state back into S_1 since $\Gamma^T \bar{\Psi} z_{0|k}$ ($\Gamma = [I \ 0]^T, x = \Gamma^T z$) denotes the nominal state $\bar{x}_{k+1}$ at the next time step, which is an indirect way to increase the transitional probability p_{12}. In conclusion, the main contribution of this study [Cannon et al., 2009b] is to use the Markov chain model to address the issue of the constraint satisfaction and to transform the objective into a quadratic form. However, the method employs only two layers of sets to reduce computational complexity and the selection of transitional probabilities is ad hoc rather than optimal; these factors both have inevitably lead to conservativeness.

2.4.6 Probabilistic Polytopic Tubes for Stochastic MPC

A systematic approach for linear systems subject to both multiplicative and additive noise is proposed in Cannon et al. [2009a]. The concept of discrete Markov chains is employed once again to address the probabilistic constraint, yet this study allows the tubes parameters to be optimized online[13]. The system considered here is of the form:

$$x_{k+1} = \hat{A}_k x_k + \hat{B}_k u_k + \hat{d}_k, \tag{2.110}$$

$$[\hat{A}_k\ \hat{B}_k\ \hat{d}_k] = [\hat{A}^0\ \hat{B}^0\ \hat{d}^0] + \sum_{j=1}^{\rho} [\hat{A}^{(j)}\ \hat{B}^{(j)}\ \hat{d}^{(j)}] q_{jk} \tag{2.111}$$

with q_{jk} an i.i.d. random variable with finite support for all k, and dual mode strategy is adopted. The joint chance constraints take the same form as (2.98):

$$\frac{1}{N_c}\left\{\sum_{i=1}^{N_c} Pr\{\psi_{k+i} \notin I_\psi :\ F_s x_{k+i} + G_s c_{k+i} > h_S\}\right\} \leq \frac{N_{max}}{N_c}, \tag{2.112}$$

and the hard constraints are in the form of

$$\psi_k^H = F_H x_k + G_H c_k \leq h_H. \tag{2.113}$$

First, a sequence of μ tubes with polytopic cross-sections $\mathbb{X}_{i|k}^{(l)}$, $l = 1, \ldots, \mu$, centred on a nominal trajectory $x_{k+i|k}^c$ (which can be easily obtained from the recursion of the nominal dynamics $x_{k+i|k}^c = [\Phi^0]^i x_{k|k} + \sum_{j=1}^{i} [\Phi^0]^{i-j} B^0 c_{j|k}$) is constructed in mode 1, whereas in mode 2 the polytopic cross-sections, $\mathbb{X}_T^{(l)}$, are centred around the origin:

$$\mathbb{X}_{i|k}^{(l)} = \{x : |x - x_{k+i|k}^c| \leq \overline{x}_{i|k}^{(l)}\}, \quad l = 1, \ldots, \mu,\ i = 1, \ldots, N \tag{2.114a}$$

$$\mathbb{X}_T^{(l)} = \{x : |x| \leq \overline{x}_T^{(l)}\}, \quad l = 1, \ldots, \mu \tag{2.114b}$$

[13]Tube methods have been studied in different MPC formulations [Lee and Kouvaritakis, 2000, Mayne et al., 2005, 2009], which are introduced in Section 2.3.5.

To ensure the tubes are nested, the following constraints need to be imposed for $i = 1, \ldots, N$:

$$\overline{x}_{i|k}^{(\mu)} \geq \overline{x}_{i|k}^{(\mu-1)} \geq \ldots \geq \overline{x}_{i|k}^{(1)} > 0 \tag{2.115a}$$

$$\overline{x}_{T}^{(\mu)} \geq \overline{x}_{T}^{(\mu-1)} \geq \ldots \geq \overline{x}_{T}^{(1)} > 0. \tag{2.115b}$$

In addition, the initial tube cross-section is set to $\mathbb{X}_{0|k}^{(l)} = x_k,\ l = 1, \ldots, \mu$ and the terminal constraint is imposed as $\mathbb{X}_{N|k}^{(l)} \subset \mathbb{X}_{T}^{(l)},\ l = 1, \ldots, \mu$.

Then satisfaction of both hard and probabilistic constraints can be guaranteed through one-step-ahead predictions, which suggest the following conditions.

(1) Transitional probability constraints:

$$Pr\{x_{k+i+1|k} \in \mathbb{X}_{i+1|k}^{(l)} | x_{k+i|k} \in \mathbb{X}_{i|k}^{(m)}\} \geq p_{lm} \quad \textit{in mode } 1 \tag{2.116a}$$

$$Pr\{x_{k+i+1|k} \in \mathbb{X}_{T}^{(l)} | x_{k+i|k} \in \mathbb{X}_{T}^{(m)}\} \geq p_{lm} \quad \textit{in mode } 2. \tag{2.116b}$$

(2) Robust universality constraint on outer tube:

$$Pr\{x_{k+i+1|k} \in \mathbb{X}_{i+1|k}^{(\mu)} | x_{k+i|k} \in \mathbb{X}_{i|k}^{(l)}\} = 1 \quad \textit{in mode } 1 \tag{2.117a}$$

$$Pr\{x_{k+i+1|k} \in \mathbb{X}_{T}^{(\mu)} | x_{k+i|k} \in \mathbb{X}_{T}^{(l)}\} = 1 \quad \textit{in mode } 2. \tag{2.117b}$$

(3) One-step-ahead constraint violation probability:

$$Pr\{\psi_{k+i+1|k}^{S} > h_S | x_{k+i|k} \in \mathbb{X}_{i|k}^{(l)}\} \leq p_l \quad \textit{in mode } 1 \tag{2.118a}$$

$$Pr\{\psi_{k+i+1|k}^{S} > h_S | x_{k+i|k} \in \mathbb{X}_{T}^{(l)}\} \leq p_l \quad \textit{in mode } 2. \tag{2.118b}$$

(4) Feasibility of outer tube with regard to hard constraints:

$$Pr\{\psi^H_{k+i|k} \leq h_H | x_{k+i|k} \in \mathbb{X}^{(\mu)}_{i|k}\} = 1 \quad \textit{in mode } 1 \tag{2.119a}$$

$$Pr\{\psi^H_{k+i|k} \leq h_H | x_{k+i|k} \in \mathbb{X}^{(\mu)}_{T}\} = 1 \quad \textit{in Mode } 2. \tag{2.119b}$$

Analogous to the analysis of (2.105), the probability of constraint violation at the ith-step-ahead can be demonstrated to have an upper bound as

$$Pr\{\psi^S_{k+i+1|k} > h_S\} \leq [p_1 \ \ldots \ p_\mu]^T (T^{-1}\Pi)^i e_1, \tag{2.120}$$

where Π is the transition probability matrix, and T is a lower-triangular matrix:

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & -1 & 1 \end{bmatrix} \tag{2.121}$$

and $e_1 = [1 \ 0 \cdots 0]^T$. The bound of (2.120) provides a mechanism for choosing the probabilities p_{lm}, p_l for $l, m = 1, \cdots, \mu$ so as to make sure they satisfy

$$\frac{1}{N_c} \sum_{i=0}^{N_c-1} [p_1 \cdots p_\mu](T^{-1}\Pi)^i e_1 \leq \frac{N_{max}}{N_c}. \tag{2.122}$$

It is shown in [Cannon et al., 2009a] that by invoking the constraints (2.116)-(2.119) at the vertices, $q^s(p), s = 1, \cdots, \nu$, of a polytopic confidence region $\mathcal{Q}(q(p))$, (2.116)-(2.119) can be transformed into a series of linear constraints in the degrees of freedom, namely c_{k+i} and the parameters $\overline{x}^{(l)}_{i|k}$, $\overline{x}^{(l)}_T$ which define the tubes. The aim of the offline algorithm is to maximize the volume of the terminal set so as to enlarge the region of attraction and the online algorithm consists of the minimization of an objective function in the quadratic form of decision variables (see (2.101)) subject to a set of linear constraints. To conclude, this study accounts for both hard and probabilistic constraints and guarantees the closed-loop feasibility for stochastic systems. However, the need to invoke the transitional and violation constraints at each vertex of the confidence polytope $\mathcal{Q}(q(p))$ and the optimisation of the tube parameters online can result in a high computational complexity, which limits the applicability of this algorithm to low dimensional problems with short horizons. Using a primal barrier interior method can make it more computational efficient [Ng, 2011], yet the fixed tube geometry (polytopes) and the finite number of tubes both cause a degree of conservativeness. Furthermore, the choice of the confidence region of q is not unique, it can nevertheless affect the degree of conservativeness in closed-loop performance.

2.5 Comparison and Discussion

The review of the research on robust and stochastic MPC undertaken in this chapter shows that both MPC strategies handle uncertainty which is present in most real systems, namely model uncertainty (multiplicative noises) and/or additive disturbances. The main difference between the two strategies is that robust MPC takes account of only the worst case scenario of the uncertainty, while stochastic MPC makes use of the information on the distribution of the uncertainty. In other words, when applying the robust MPC strategy, the uncertainty is regarded as lying inside some 'black container' with boundaries known, no consideration will be given to the internal information of that black container. Accordingly, the constraints that are incorporated into robust MPC are deterministic constraints, unlike the stochastic MPC which deal with both deterministic constraints and probabilistic (chance) constraints. If probabilistic constraints

$$Pr\{x_k \in \mathbb{X},\ \text{and/or}\ \ u_k \in \mathbb{U}\} \geq \alpha, \tag{2.123}$$

such as (2.69) and (2.79), need to be fulfilled with probability $\alpha = 1$, then constraints (2.123) are in essence deterministic. In this situation, the stochastic MPC is in fact robust MPC. Based on this understanding, robust MPC can be regarded as a special form of stochastic MPC, where all the constraints have to be met with probability 1. As a result, there exist many similarities between the two approaches which will be summarised below. In general, for a same system, stochastic MPC often outperforms robust MPC since the former exploits more information about the system (uncertainty). For example, stochastic MPC will often lead to a smaller cost (better performance) and have a larger region of attraction, as will be demonstrated by numerical evidences to be presented later in the thesis.

One key challenge, probably the most difficult point in stochastic MPC, arises from the probabilistic constraints (2.123). The basic idea is to transform (2.123) into deterministic constraints, which will provide *sufficient* conditions for satisfying (2.123). Diverse methods have been developed to implement the transformation. In Schwarm and Nikolaou [1999] and Warren and Marlin [2004], the uncertainty is assumed to be normally distributed so that the calculation of the inverse cumulative distribution function of the stochastic variables becomes easily implementable. The stochastic MPC problem is then translated into a second-order cone programming problem [Lobo et al., 1998]. In Primbs and Sung [2009], all the constraints are simplified by being applied in expected value only. In Van Hessem et al. [2002], an ellipsoidal relaxation method shown in Figure 2.2 is deployed to ensure constraint satisfaction. The Markov chain model has also been introduced to handle the joint chance constraint [Cannon et al., 2009b,a], transforming probabilistic constraints into deterministic ones. Although they may take different forms (e.g. linear constraints or conic constraints), stochastic MPC is accordingly converted into a standard conventional MPC or robust MPC problem. Then it follows that most controller designs can be effective for both robust and stochastic MPC. Table 2.1 lists several controller designs that have been deployed in stochastic MPC.

The perturbation plus a fixed linear state feedback law is proposed in Rossiter et al. [1998] for the robust MPC problem, and it has been exploited in Cannon et al. [2009b,a] for stochastic MPC. Oldewurtel et al. [2008, 2010] employ the affine disturbance feedback law to address stochastic MPC, which has been studied for robust MPC in Löfberg [2003], Goulart et al. [2006]. The 'innovations feedback' law utilized for stochastic MPC in van Hessem and Bosgra [2003], Primbs and Sung [2009] also appeared in the analysis for robust MPC in Mayne et al. [2005].

Literature	Controller	Online Computation	Guarantee of Closed-loop Properties
van Hessem and Bosgra [2003]	$u_i = \bar{u}_i + K_i(x_i - \mathbb{E}(x_i))$	Second-order cone programm (SOCP)	No
Oldewurtel et al. [2008, 2010]	$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + h_i$	Linear program or SOCP	No
Primbs and Sung [2009]	$u_i = \bar{u}_i + K_i(x_i - \mathbb{E}(x_i))$	Semidefinite program	No guarantee of feasibility, but closed-loop stable under a modified control policy
Cannon et al. [2009b]	$u_i = Kx_i + c_i$	Newton Raphson method	Yes
Cannon et al. [2009a]	$u_i = Kx_i + c_i$	Quadratic program (tube parameters as decision variables)	Yes

Table 2.1: Comparison of several stochastic MPC formulations

Before offering any further discussion of stochastic MPC, let us summarise some interesting observations on robust MPC. In this chapter, it is argued that Algorithm 2.3 is very efficient in handling robust MPC problem compared with the methods of other sections. The augmented autonomous formulation (2.52) in use obviates the need to consider the effects of the propagation of the uncertainty on the predicted states and thus the online computation can be realized efficiently by the Newton Raphson method. However, Algorithm 2.3 takes the nominal cost to be the objective function, which seems to be very optimistic, especially if the worst case cost differs significantly from the nominal cost. To tackle this problem, this thesis introduces a Youla parameter into Algorithm 2.3 to improve the robustness. It has been demonstrated that the Youla parameter can be used to enhance the robustness of the closed-loop system without degrading the nominal performance [Kouvaritakis et al., 1992], and this appealing idea has been studied further by [Yoon and Clarke, 1995, Wams and van den Boom, 1997, Stoica et al., 2008]. Surprisingly, all these work considered the Youla parameter in the context of the unconstrained case, which is surprising given that the popularity of MPC is mainly because of its strong ability to handle constraints. In Rossiter and Kouvaritakis [1998], it was mentioned that the Youla parameter might achieve the superior robustness even with constraint present, however it did not have a rigorous analysis of recursive feasibility and stability. The study in this thesis advances the research by showing that the Youla parameter can be imbedded into an efficient robust MPC algorithm in presence of constraints with feasibility guaranteed.

The controller design $u_i = Kx_i + c_i$ is widely used in robust MPC (e.g., Alg. 2.3) and stochastic MPC (see Table 2.1), since it is convenient to construct the augmented autonomous formulation (2.52). Additionally, the perturbation sequence c_i can provide additional degrees of freedom to accommodate the constraints and enlarge the region of attraction. This design will also be exploited in the stochastic MPC study of this thesis. In the interest of nominal behaviour, the unconstrained nominal LQ-optimal feedback law K_{LQ} is mostly employed. However, there are some uncertain plants where K_{LQ} cannot stabilise the whole class of system models. Two approaches are presented in the thesis to address this problem. Firstly, it is straightforward to replace K_{LQ} with another linear state feedback law H, in which the optimisation process is somewhat similar to the method of Kothare et al. [1996]. This approach is easily understood and implemented, yet it cannot explicitly recover the unconstrained nominal LQ-optimal trajectory whenever applicable. Therefore, a novel dynamical controller is developed, which can not only stabilise the whole class of models, but also keeps the capability of generating (asymptotically) the nominal LQ-optimal trajectory.

It has been stated that tube method can be applied in robust MPC as well as stochastic MPC. The set containing all possible states at a certain time instant is called a robust tube, while the set in which states will lie with some probability is defined as a probabilistic tube. The tube method provides a mechanism to analyze the constraints, i.e. to tighten the constraints on the nominal model, as in (2.59). Different shapes of tube cross-sections are explored, such as ellipsoidal in Van Hessem et al. [2002] and polyhedral in Cannon et al. [2009a], where the tube parameters are both incorporated into the online optimization. This carries with it the disadvantage that there are too many variables and the computational price is overly high. In addition, it is seen from Table 2.1 that the stochastic MPC approaches in van Hessem and Bosgra [2003], Oldewurtel et al. [2008, 2010], Primbs and Sung [2009] do not ensure the closed-loop stability. The main contribution made in this thesis is that two kinds of stochastic tube methods have been developed which carry the necessary guarantees of recursive feasibility and closed-loop stability. Moreover, the stochastic tubes can be determined offline so that the online algorithm is reduced to the level of quadratic programming (QP) problem and thus is computationally attractive.

The first stochastic tube is ellipsoidal, based on the assumption that the uncertainty and the error are described by quadratic inequalities. A simple affine dynamics for tube scaling evolution is derived, which can be used to calculate the transitional probability matrix for a Markov chain model. Thus the probabilistic tube scalings can be computed offline, which are then used to tighten the constraints on the nominal predictions. The second stochastic tube may not take a common shape, but deploys cross-sections which are defined implicitly by the tightened constraints (through direct calculation of probabilistic distributions of the uncertainty). In consequence, the transformed deterministic constraints give *not only sufficient but* *also necessary* conditions (within the adopted controller structure) for the satisfaction of probabilistic constraints (2.123). Thus probabilistic constraints can be tightened to the minimum level, i.e. the probability of constraint violation can be equal to the largest limit allowed. In this sense, the second stochastic tube approach with no particular shape is less conservative than the ellipsoidal tube approach. The thesis compares the two approaches through a numerical example. Importantly, the second stochastic tube method is successfully extended to handle the case where states are not available but are instead estimated through the use of observers.

Chapter 3

Youla Parameter in Robust Constrained Linear MPC

[1]Previous work [Kouvaritakis et al., 1992] proposed the introduction of a Youla parameter into the receding horizon control strategy to enhance the degree of robustness of the closed-loop system without affecting the optimality of nominal performance. The idea is clearly appealing and has been investigated further by a number of other researchers. Yet to date the application of the idea has only been realized for the case of unconstrained systems, which is surprising given that the presence of constraints is the main justification for the use of MPC for linear systems. It is the purpose of the research reported in this chapter to combine recent results on invariance, feasibility and an augmented and autonomous formulation of the predicted dynamics [Kouvaritakis et al., 2000] to derive an effective robust constrained MPC algorithm that takes full advantage of the introduction of a suitable Youla parameter. The efficacy of the proposed algorithm is illustrated by means of a numerical example.

[1]The work in this chapter has been published in Cheng et al. [2009].

3.1 The Need for the Youla Parameter

A worst case objective function is employed by Kothare et al. [1996] (see Section 2.3.1), while Algorithm 2.3 proposed by Kouvaritakis et al. [2000] minimises a nominal objective. The worst case strategy can be overly pessimistic (conservative) with respect to nominal behaviour, while optimising the nominal cost can be overly optimistic with respect to worst case behaviour, especially so if the worst case and nominal performance differ significantly. To reduce the difference, a Youla Parameter will be introduced here to provide extra degrees of freedom, which is an idea proposed in Kouvaritakis et al. [1992]. The interesting observation made there was that this Youla Parameter can be used to improve the robustness of MPC feedback scheme without affecting the optimality of unconstrained nominal performance. Thus the minimization of the RH_∞-norm of an appropriate sensitivity function would lead to superior robustness with respect to model uncertainty to the system transfer function; robustness to additive disturbances could be optimized instead by minimizing the RH_∞-norm of a different transfer function. This idea of achieving optimal nominal performance first and then optimizing robust performance was subsequently adopted by Yoon and Clarke [1995], Wams and van den Boom [1997], Stoica et al. [2008]. In all these work, surprisingly, consideration was only given to the unconstrained case, which somehow seems to defeat the purpose behind the use of MPC. In Rossiter and Kouvaritakis [1998], it was mentioned the Youla parameter may achieve the superior robustness even with constraint present, yet it did not provide the analysis of recursive feasibility and stability. The aim here is to close this gap through the combined use of the method of the augmented autonomous formulation in (2.52), with its concomitant effect of robust invariance and feasibility, and the use of the Youla Parameter as recommended in Kouvaritakis et al. [1992].

Despite its significant computational advantages, Algorithm 2.3 optimizes the performance of the nominal system, which could differ considerably from that obtained for other models within the uncertainty class of (2.51). The nominal performance objective of Algorithm 2.3 would be better justified if it were possible to desensitize performance over the model uncertainty. From the perspective of frequency response desensitization, the Q parameter provides an effective solution and is introduced in the context of MPC in the following way.

In the absence of constraints, the MPC control law can be implemented using the fixed term feedback scheme as Figure 3.1, where r denotes a reference set point and the polynomials $M(z)$, $N(z)$ are solutions of

$$a_0(z)M(z) + b_0(z)N(z) = p_{opt}(z), \tag{3.1}$$

$$p_{opt}(z) = \det[zI - (A_0 + B_0 K)]. \tag{3.2}$$

where a_0, b_0 represent nominal values of a, b in Figure 3.1, A_0, B_0 denote the nominal state space description of the system, and K is the unconstrained LQ-optimal feedback for nominal system.

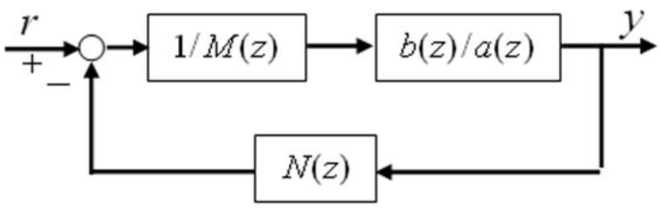

FIGURE 3.1: The MPC feedback loop

However, equation (3.1) admits a whole family of solutions:

$$M(z) = Y(z) - b_0(z)Q(z), \quad N(z) = X(z) + a_0(z)Q(z) \tag{3.3}$$

where for convenience X, Y, and Q are selected to be polynomials. The transfer functions from r to y and u are $b(z)/p_{opt}(z)$, $a(z)/p_{opt}(z)$, respectively and do not depend on the free variable Q. Thus Q can be used to minimise the RH_∞-norm of an appropriate sensitivity function in order to optimize robustness to model uncertainty without affecting the nominal optimality of the system. Remarkably, useful as this observation may be, it does not address the issue of constraint handling, which is the main justification for using MPC.

3.2 Sensitivity Reduction and the Youla Parameter

The research here redresses this omission using the augmented autonomous MPC formulation. We first reformulate the DOF (degrees of freedom) as perturbations on the optimal feedback control law depicted in Figure 3.1. This leads to the feedback loop of Figure 3.2 [Kouvaritakis et al., 1992], where the perturbation signal c is introduced to force the input u to respect the constraints of (2.25), but otherwise should be kept as small as possible in the interest of optimality. This strategy is optimal for the nominal model only, while its approximate validity can be extended to a wider class of models (centred on the nominal) provided that the system of Figure 3.2 is desensitized to the effects of model uncertainty. An indirect, but convenient way to achieve this desensitization is to minimize the RH_∞-norm of the sensitivity transfer function

$$S_1 = \frac{\Delta H}{H} / \frac{\Delta G}{G} = \frac{a_0(Y(z) - b_0 Q(z))}{p_{opt}(z)}. \tag{3.4}$$

where H, G denotes the closed-loop and open-loop transfer function respectively, and ΔH, ΔG their respective additive uncertainty. Since this function, S_1, is affine in Q, it is amenable to the usual RH_∞ treatment (including that of Kouvaritakis et al. [1992] which is based on Lawson's weighted squares algorithm). Furthermore, S_1 is known to give a measure of reduction (due to feedback) of sensitivity to uncertainty, but that reduction concerns the frequency response sensitivity. Thus, choosing Q in this manner will optimise the worst case (over all frequencies) sensitivity reduction. To be more precise, S_1 should be scaled by a suitable weighting factor that allows information of the frequency response of the open loop sensitivity to be taken into account. Hence S_1 can be rewritten as

$$S = \frac{\Delta H}{H} = \frac{\Delta G}{G} S_1 = \frac{\Delta G}{G} \cdot \frac{a_0(Y - b_0 Q)}{p_{opt}} = \frac{(b/a - b_0/a_0)}{b_0/a_0} \cdot \frac{a_0(Y - b_0 Q)}{p_{opt}} = F_2 - F_1 Q, \tag{3.5}$$

where $F_2 = \frac{(\delta b \cdot a_0 - \delta a \cdot b_0) a_0 Y}{(a_0 + \delta a) b_0 p_{opt}}$ and $F_1 = \frac{(\delta b \cdot a_0 - \delta a \cdot b_0) a_0}{(a_0 + \delta a) p_{opt}}$.

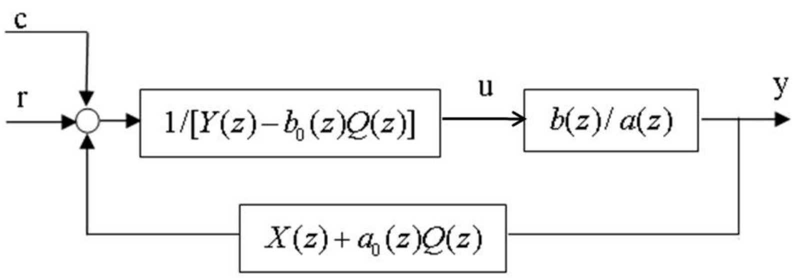

FIGURE 3.2: The MPC feedback loop with the Q parameter

To ensure the stability of the system as well as keep the model simple, in reality $Q(z)$ is first chosen as a FIR model (polynomial), and then a low order transfer function $N_Q(z)/D_Q(z)$ is used to approximate the desensitizing benefit of the polynomial $Q(z)$. This procedure results in a Q which desensitizes the closed-loop transfer function to additive uncertainty, but may well not be optimal in desensitizing the conventional cost function of (2.8). However, simulation results (see Section 3.4) show that the solutions for Q based on the well known RH_∞ optimal solutions produce results which are close to optimal, even in the context of the cost of (2.8).

The analysis above applies to the unconstrained case only. However when constraints are present, as described in Section 2.3.4, to avoid infeasibility it is necessary to perturb the optimal control law, and this was achieved in (2.49) through the use of the closed-loop configuration of Figure 3.2. Then, provided the MPC strategy is able to *ensure the feasibility of predictions at all time instants*, the feedback loop of Figure 3.2 will always *operate within its linear region* and therefore the desensitization achieved through Q will remain valid, throughout the receding horizon of the MPC strategy. The following section shows how the Youla Parameter Q will be embedded in the context of the augmented autonomous state space formulation.

3.3 The Youla Parameter in Robust Constrained MPC

The first task here is to derive a state space model for the system of Figure 3.2. As is described above, we have substituted $N_Q(z)/D_Q(z)$ for the polynomial $Q(z)$. Then the system dynamics in transfer function form become

$$u(z) = \frac{D_Q(z)a(z)}{\hat{M}(z)a(z) + \hat{N}(z)b(z)}v(z), \;\; y(z) = \frac{D_Q(z)b(z)}{\hat{M}(z)a(z) + \hat{N}(z)b(z)}v(z), \tag{3.6}$$

where $v(z) = r(z) + c(z)$ and

$$\hat{M}(z) = Y(z)D_Q(z) - b_0(z)N_Q(z), \tag{3.7}$$

$$\hat{N}(z) = X(z)D_Q(z) + a_0(z)N_Q(z). \tag{3.8}$$

The controllable canonical form for (3.6) can be written as

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}v_k, \;\; y_k = \tilde{C}_y\tilde{x}_k, \;\; u_k = \tilde{C}\tilde{x}_k + \tilde{D}v_k, \tag{3.9}$$

$$\tilde{A} = \begin{bmatrix} 0 & I \\ & -d^T \end{bmatrix}, \;\; \tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \;\; d = [C_{\hat{M}} \;\; C_{\hat{N}}]\theta, \;\; \tilde{D} = 1,$$

$$\tilde{C}_y^T = [0^T \quad C_D]\theta, \quad \tilde{C}^T = [C_D' \quad 0^T]\theta - d,$$

where θ is the column vector of coefficients (in order of ascending powers of z) of $a(z)$, $b(z)$ (with the coefficients of $a(z)$ appearing first), $\tilde{C}_y$, $\tilde{C}$ are the row vectors of the coefficients of $D_Q(z)b(z)$, $D_Q(z)a(z) - [\hat{M}(z)a(z) + \hat{N}(z)b(z)]$ respectively taken in ascending order, $C_{\hat{M}}$, $C_{\hat{N}}$, C_D, C'_D are the convolution matrix that relate θ to the coefficients of $\hat{M}(z)a(z)$, $\hat{N}(z)b(z)$, $D_Q(z)b(z)$, $D_Q(z)a(z)$, respectively. In this formulation,

$$\tilde{x}_k = [x_{k-\tau+1}^T \; x_{k-\tau+2}^T \cdots x_{k-1}^T \; x_k^T]^T, \quad \tau = n_a + n_b + n_Q, \tag{3.10}$$

where n_a, n_b, n_Q denote the degrees of $a(z)$, $b(z)$, and $D_Q(z)$. Model (3.9) describes a tracking problem with a constant set point. The more general case of time varying set points requires knowledge of the dynamics that generate any particular reference signal $r(t)$. With r a constant, we can compute the steady state value for $\tilde{x}$ and re-write the model in terms of a regulation problem by considering deviation from the steady state:

$$\tilde{x}_{ss} = \tilde{A}\tilde{x}_{ss} + \tilde{B}r_{ss}, \;\; y_{ss} = \tilde{C}_y\tilde{x}_{ss}, \;\; u_{ss} = \tilde{C}\tilde{x}_{ss} + \tilde{D}r_{ss}, \tag{3.11}$$

Subtracting (3.9) from (3.11), one can get

$$\Delta\tilde{x}_{k+1} = \tilde{A}(\Delta\tilde{x}_k) + \tilde{B}(\upsilon_k - r_{ss}), \;\; \Delta y_k = \tilde{C}_y(\Delta\tilde{x}_k), \;\; \Delta u_k = \tilde{C}\Delta\tilde{x}_k + \tilde{D}(\upsilon_k - r_{ss}), \tag{3.12}$$

where $\Delta\tilde{x}_k = \tilde{x}_k - \tilde{x}_{ss}$. Correspondingly, we take the input $\upsilon_k - r_{ss}$ to be c_k (the deviation of υ_k from the steady state). On account of the linear dependence on θ in (3.12), it is possible to generate the polytopic uncertainty model for the system of Figure 3.2 given the vertices θ^i of a corresponding polytopic uncertainty class of the open-loop input-output transfer function description. The problem then becomes equivalent to that defined in Section 2.3.4, after the application of the closed-loop paradigm law of (2.49). This leads to the augmented autonomous model:

$$\tilde{z}_{k+1} = \tilde{\Psi}^{(i)}(k)\tilde{z}_k,\ \ \tilde{z}_k = \begin{bmatrix} \Delta\tilde{x}_k \\ f_k \end{bmatrix},\ \tilde{\Psi}^{(i)}(k) = \begin{bmatrix} \tilde{A}^{(i)} & [\tilde{B}\ 0 \cdots 0] \\ 0 & M \end{bmatrix} \tag{3.13}$$

The perturbation c_k can be optimized using an approach analogous to Algorithm 2.3, but modified for the new augmented autonomous dynamics of (3.13). As is the case for the predictions of (2.52), the minimization of the nominal predicted cost of (2.50) can be likewise proved to be equivalent to the minimization of $f_k^T f_k$ (for SISO systems). The procedure differs however in one aspect: unlike the model of Section 2.3.4 where $\tilde{u}_k = \tilde{K}\tilde{z}_k$ ($\tilde{K}$ is fixed), here $\tilde{C}\Delta\tilde{x}_k + \tilde{D}c_k$, where $\tilde{C}$ is subject to polytopic uncertainty with vertices $\tilde{C}^{(i)} = ([C_D' \quad 0^T]\theta^{(i)} - d^{(i)})^T$. This means that the conditions to be satisfied by the parameter, $\tilde{P}_z$, defining the augmented ellipsoid $\tilde{z}_k^T\tilde{P}_z\tilde{z}_k \le 1$, should be modified to the analogous invariance and feasibility conditions:

$$\begin{bmatrix} \tilde{P}_z^{-1} & \tilde{P}_z^{-1}(\tilde{\Psi}^{(i)})^T \\ \tilde{\Psi}^{(i)}\tilde{P}_z^{-1} & \tilde{P}_z^{-1} \end{bmatrix} \ge 0, \quad i = 1, 2, \dots, L, \tag{3.14}$$

$$\begin{bmatrix} \tilde{P}_z^{-1} & \tilde{P}_z^{-1}(\tilde{K}_j^{(i)})^T \\ \tilde{K}_j^{(i)}\tilde{P}_z^{-1} & d_j^2 \end{bmatrix} \ge 0, \quad i = 1, 2, \dots, L,\ j = 1, 2, \dots, m. \tag{3.15}$$

Then the overall MPC control strategy is summarized below.

Algorithm 3.1 (Robust MPC with Youla Parameter)**.**

Offline: 1). Compute the $Q(z)$ which minimizes the RH_∞-norm of the weighted sensitivity function S of (3.5);

2). Compute $\tilde{P}_z = \arg\min_{\tilde{P}_z} \det(T\tilde{P}_z^{-1}T^T)^{-1}$ (T is the transformation matrix in $\tilde{x} = T\tilde{z}$) subject to (3.14) and (3.15).

Online: At each time instant $k = 0, 1, 2, \dots, N_c - 1$, perform

$$f_k = \arg\min_f f_k^T f_k, \quad \text{subject to } \tilde{z}_k^T \tilde{P}_z \tilde{z}_k \leq 1. \tag{3.16}$$

Then implement (3.9) with the first element of f_k and repeat the online step at the next time instant.

Remark 3.1. Note that the offline optimization of $\tilde{P}_z$ can be formulated as a convex LMI problem (e.g. see [Boyd et al., 1994]). Furthermore, the online optimization can be performed using the efficient Newton Raphson method of Kouvaritakis et al. [2000].

In common with Algorithm 2.3, Algorithm 3.1 enjoys the benefit of an extremely efficient online optimization and has the guarantee of closed-loop stability and feasibility (provided that the online optimization is feasible at the initial time). However, in addition, the desirable property of robustness of performance is enhanced on account of the use of the Q parameter.

3.4 Numerical Example

Consider a second order system with an input-output description given by

$$a(z) = [1 \;\; z \;\; z^2 \;\; 0 \;\; 0]\theta, \quad b(z) = [0 \;\; 0 \;\; 0 \;\; 1 \;\; z]\theta \tag{3.17}$$

where θ lies in the convex hull of vertices $\theta^{(i)}$, $i = 1, 2, 3, 4$:

$$\theta^{(0)} = [0.3 \;\; 0.8 \;\; 1 \;\; 0.6 \;\; 1]^T,$$

$$\theta^{(i)} = \begin{cases} \theta^{(0)} \pm [0.06 \;\; 0.16 \;\; 0 \;\; 0.012 \;\; 0.02]^T, & i = 1, 2 \\ \theta^{(0)} \pm [0.06 \;\; 0.16 \;\; 0 \;\; -0.012 \;\; -0.02]^T, & i = 3, 4 \end{cases}$$

The cost function is $J = \sum_0^\infty (y^T y + \Delta u^T R \Delta u)$ and for this example $R = 10^{-6}$. This results in the closed loop optimal pole polynomial

$$p_{opt}(z) = 0.6z + z^2 \tag{3.18}$$

and a pair of particular solutions to (3.1)

$$Y(z) = 1 + 0.6z^{-1},\ X(z) = -0.8 - 0.3z^{-1}. \tag{3.19}$$

The Q parameter is chosen to minimise the worst case of the sensitivity function S in (3.5) over all frequencies for all the vertices $i = 1, 2, 3, 4$:

$$\min_Q \max_i \{\|S^{(i)}\|_\infty\} = \min_Q \max_i \{\|F_2^{(i)} - F_1^{(i)} Q\|_\infty\}.$$

Take the Q for the minimization of the RH_∞-norm to be a 17th order FIR model first (labeled as Q_p) and then approximate the (weighted) polynomial with a 2nd order transfer function $Q_t = N_Q(z)/D_Q(z)$, since

$$\|F_2 - F_1 Q_t\|_\infty = \|F_2 - F_1 Q_p + F_1 Q_p - F_1 Q_t\|_\infty \leq \|F_2 - F_1 Q_p\|_\infty + \|F_1 Q_p - F_1 Q_t\|_\infty.$$

Then the 17th order FIR model for Q is

$$\begin{aligned} Q(z) = 0.1(&3.976 - 0.422z^{-1} - 0.531z^{-2} + 0.213z^{-3} + 0.253z^{-4} - 0.356z^{-5} + 0.162z^{-6} + \\ &0.046z^{-7} - 0.098z^{-8} + 0.031z^{-9} + 0.036z^{-10} - 0.037z^{-11} + 0.023z^{-13} - \\ &0.014z^{-14} - 0.002z^{-15} + 0.007z^{-16} - 0.003z^{-17}) \end{aligned}$$

and the 2nd order transfer function Q_t is

$$\frac{N_Q(z)}{D_Q(z)} = \frac{0.3958 + 0.0529z^{-1}}{1 + 0.2403z^{-1} + 0.1835z^{-2}} \tag{3.20}$$

Figures 3.3 to 3.6 show the effects on the reduction of the sensitivity function for the use of Q, which plot the gain of the frequency response of S for the Q in FIR and transfer function form (solid line) and for $Q = 0$ (dashed line), against values of ωT, $0 \leq \omega T \leq \pi$. Although the maximum sensitivity function value for vertex 1 after using Q seems to be larger, the worst case sensitivity function value for all vertices have been reduced from 0.291 ($Q = 0$) to 0.200 (Q as an FIR) or 0.204 (Q as a 2nd order transfer function).

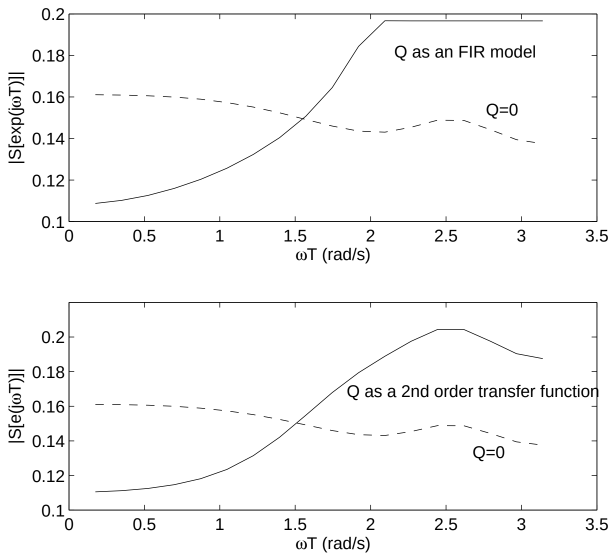

FIGURE 3.3: Sensitivity reductions due to Q - vertex 1

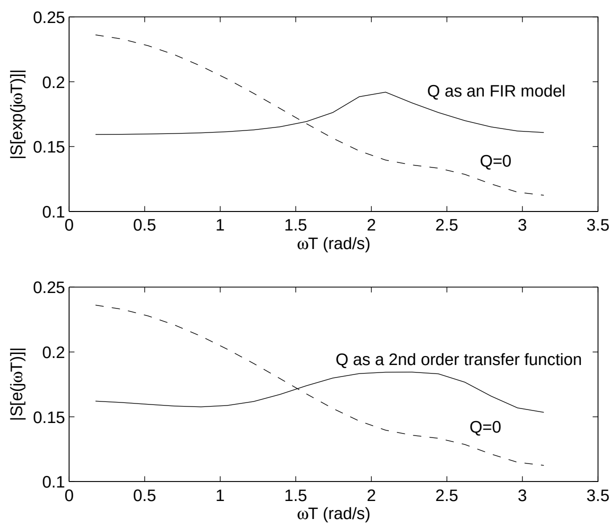

FIGURE 3.4: Sensitivity reductions due to Q - vertex 2

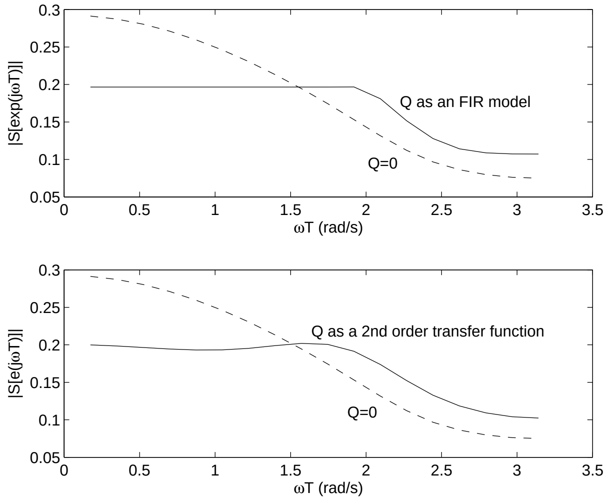

FIGURE 3.5: Sensitivity reductions due to Q - vertex 3

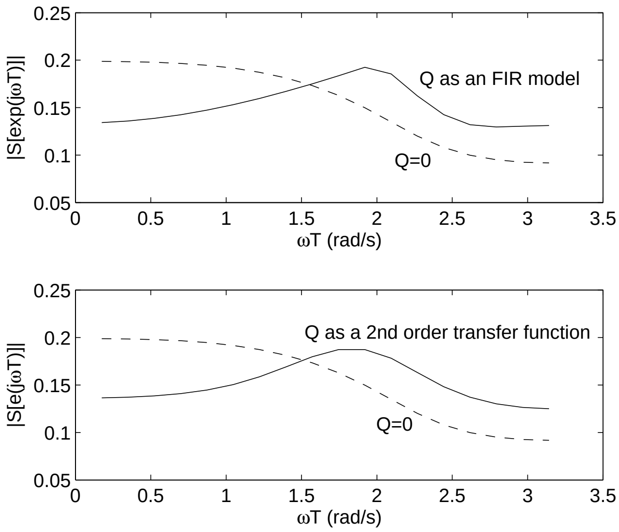

FIGURE 3.6: Sensitivity reductions due to Q - vertex 4

Finally the offline algorithm is implemented and a maximum volume ellipsoidal invariant set is calculated, for a control horizon of $N_c = 10$ and an input constraint of $-0.3 \leq \Delta u \leq 0.3$. To illustrate the efficacy of Algorithm 3.1, Table 3.1 lists the closed-loop costs obtained by Algorithm 2.3[2] and 3.1 for zero initial condition and a reference set point $r = 0.36$. The costs are computed for fixed values of θ, namely the nominal value $\theta^{(0)}$ as well as the four vertices $\theta^{(i)}, i = 1, 2, 3, 4$. The assumption

[2]In fact, here Algorithm 2.3 is also seen as Algorithm 3.1 without using the Youla Parameter, or $Q = 0$ in (3.6).

being made here is that the model is linear time invariant (LTI) when updating the system states. In fact, a truly worst case cost calculation would entail the permutation of different vertices at each time instant [Schuurmans and Rossiter, 2000] and hence the online computation becomes intractably computationally intensive for a long prediction horizon such as 10. Therefore, we approximate the worst case cost with that calculated for the LTI system assumption.

	θ^0	θ^1	θ^2	θ^3	θ^4	Max cost sensitivity
Algorithm 2.3	1.296	1.402	1.319	1.445	1.303	11.5%
Algorithm 3.1	1.296	1.377	1.278	1.383	1.270	6.25%

TABLE 3.1: Values of closed-loop cost times 10

As expected, the desensitization of the closed-loop system to additive uncertainty has the significant effect of reducing (almost halving) the cost sensitivity (measured as a percentage of nominal cost) with the maximum value for Algorithm 3.1 being more than 5.25% smaller than that of Algorithm 2.3. It is noteworthy that Algorithm 3.1 achieves this with a comparable online computation load to that of Algorithm 2.3. Clearly there is a certain amount of extra offline computation with Algorithm 3.1 but that involves two simple optimization problems which are known to converge and are not particularly demanding.

3.5 Conclusions

The use of the Youla parameter in the context of MPC has been advocated as far back as 1992 and has been endorsed by various researchers since [Yoon and Clarke, 1995, Wams and van den Boom, 1997, Rossiter and Kouvaritakis, 1998, Stoica et al., 2008]. Yet to date, there has been no rigorous analysis to apply this in the presence of constraints (considering the feasibility issues), which is surprising when considering that handling constraints offers the main justification for the use of MPC. The results presented in this section propose a strategy that redresses this situation and the efficacy of this strategy is demonstrated by means of a numerical example. The use of Youla parameter in this context is shown to have improved the robustness of the original problem by desensitizing the closed-loop system to additive uncertainty without influencing the nominal behaviour. The difference between the worst case cost and nominal cost has been reduced while the nominal cost stays the same, and thus more justification is provided for employing the nominal cost as the objective function online.

There remains one issue and that concerns the tuning of the Youla parameter which, as is sensible, aims at the desensitization of the closed-loop system to model uncertainty. However this desensitization is performed for the sake of convenience (given the body of available work) in the frequency response. As a result, there is no guarantee that the resulting Q parameter is optimal for the purposes of the robust MPC framework proposed here. This particular point was tested in the numerical example, and it was found that although some improvement to the results of the proposed algorithm could be brought about by the further tuning of the Youla parameter, such improvements were very modest indeed. It is nevertheless left as a topic for further research to devise more effective algorithms for the choice of Q which address directly the needs of the robust constrained MPC algorithm proposed here.

Chapter 4

A Dynamic Controller Design in Robust MPC

In Algorithm 2.3 and 3.1, the design of controllers makes use of the nominal LQ-optimal feedback plus a perturbation signal $u = K_{LQ}x + c$, which constructs efficient MPC strategies with the use of the augmented autonomous state space formulation (2.52). However, there exist uncertain plants such that the LQ-optimal feedback K_{LQ} cannot stabilize the whole class of systems due to uncertainty (see (2.24)), thus $u = K_{LQ}x + c$ becomes invalid. The work in this chapter discusses two approaches that can address the difficulty: one is to replace K_{LQ} with another linear feedback H, which can be obtained through an offline minimization of the unconstrained worst case cost; the other method optimises a dynamic feedback law to robustly stabilize the uncertain plants while retaining the ability to implement the nominal LQ-optimal control whenever applicable. The two approaches are validated and compared by a numerical example.

4.1 Optimising Linear Feedback Law

Recall the augmented autonomous state space model (2.52) for system $x(k+1) = A(k)x(k) + B(k)u(k)$:

$$z(k+1) = \Psi(k)z(k), \quad \Psi(k) = \begin{bmatrix} \Phi(k) & B(k)E \\ 0 & M \end{bmatrix}, \quad z(k) = \begin{bmatrix} x(k) \\ f(k) \end{bmatrix}, \tag{4.1}$$

where $\Phi(k)$ lies within the polytope whose vertices are $\Phi_l = A_l + B_l K,\ l = 1, 2, \ldots, L$. Using the semi closed-loop controller $u_{k+i|k} = Kx_{k+i|k} + c_{k+i|k}$ with $c_{k+i|k} = 0$ for all $i \geq N$, the stage cost is $x^T Q_1 x + u^T R u = z^T \tilde{Q} z$ with $\tilde{Q} = \begin{bmatrix} Q_1 + K^T RK & K^T RE \\ E^T RK & E^T RE \end{bmatrix}$, where E is the transformation matrix defined in (2.52).

In general, the unconstrained nominal LQ-optimal feedback K_{LQ} is employed to provide optimal nominal performance. Yet it is based on the knowledge that K_{LQ} can stabilise the whole class of systems, namely $\Phi = A + BK_{LQ}$ is strictly stable for each realization of $\{A, B\}$. In reality, this may not always be possible: Φ may be unstable for some $\{A, B\}$. This section extends the approach in Algorithm 2.3 to problems in which the uncertain plant under nominal LQ-optimal control is either not quadratically stabilizable, or has unacceptably poor worst-case performance. The idea is straightforward: since the LQ-optimal control cannot stabilise the whole class of systems, another linear feedback law H is developed to replace K_{LQ}.

Define S_x such that it satisfies

$$S_x^{-1} - (A_l + B_l H)^T S_x^{-1} (A_l + B_l H) > Q_1 + H^T RH, \quad l = 1, 2, \ldots, L \tag{4.2}$$

then, analogous to the analysis in Section 2.3.1, $x^T S_x^{-1} x$ describes the unconstrained worst case cost for system $x(k+1) = A(k)x(k) + B(k)u(k)$ when using linear state feedback $u(k) = Hx(k)$. Therefore, one sensible choice of H is to minimise the largest eigenvalue of S_x^{-1} subject to (4.2).

Pre- and post-multiplying by S_x on both sides of (4.2), it is converted into

$$S_x - S_x(A_l + B_l H)^T S_x^{-1}(A_l + B_l H)S_x \succ S_x(Q_1 + H^T RH)S_x, \quad l = 1, 2, \ldots, L$$

With Schur complements, the above inequality can be further transformed into[1]

$$\begin{bmatrix} S_x & (A_l S_x + B_l H S_x)^T & S_x & (HS_x)^T \\ \star & S_x & 0 & 0 \\ \star & \star & Q_1^{-1} & 0 \\ \star & \star & \star & R^{-1} \end{bmatrix} \succ 0. \tag{4.3}$$

Assuming $Y_H = HS_x$, (4.3) becomes an LMI in S_x and Y_H. Consequently, the approach to designing H is given below.

$$H = Y_H S_x^{-1}, \quad \{S_x, Y_H\} = \arg \max_{S_x, Y_H} \lambda_S \tag{4.4}$$
$$\text{subject to} \quad S_x \geq \lambda_S I \text{ and } (4.3).$$

After H is obtained, with the formulation of $u_{k+i|k} = Hx_{k+i|k} + c_{k+i|k}$ with $c_{k+i|k} = 0$ for all $i \geq N$, the augmented autonomous model (2.52) is still valid. The perturbation sequence $c_{k+i|k}$, $i = 0, 1, \ldots, n_c - 1$ provides n_c degrees of freedom to enlarge the region of attraction as well as to accommodate the constraints. To optimise the nominal behaviour, the objective function should be defined as:

$$J_H(k) = \begin{bmatrix} x_k \\ f_k \end{bmatrix}^T W_H \begin{bmatrix} x_k \\ f_k \end{bmatrix}, \tag{4.5}$$

[1] $\star$ denotes an off-diagonal block of a symmetric matrix in this chapter.

where W_H is the solution of the Lyapunov equation:

$$W_H - \begin{bmatrix} A_0 + B_0 H & B_0 E \\ 0 & M \end{bmatrix}^T W_H \begin{bmatrix} A_0 + B_0 H & B_0 E \\ 0 & M \end{bmatrix} = \begin{bmatrix} Q_1 + H^T R H & H^T R E \\ E^T R H & E^T R E \end{bmatrix}.$$

In order to ensure the quadratic stability of the uncertain system, one approach is to make the ellipsoidal invariant set $E_z = \{z | z^T P_z z \leq 1\}$ $(P_z > 0)$ contractive [Boyd et al., 1994]:

$$\rho P_z - \Psi_l^T P_z \Psi_l \geq 0, \ \ l = 1, 2, \ldots, L, \quad 0 < \rho < 1, \tag{4.6}$$

This can lead to

$$z_k^T \Psi_k^T P_z \Psi_k z_k \leq \rho z_k^T P_z z_k,$$

such that $V(z_k) = z_k^T P_z z_k$ is a Lyapunov function (see Theorem 2.1). The value of ρ will influence the volume of the region of attraction, and a comprise needs to be made between the volume and the rate of contraction.

Algorithm 4.1. *Offline:*

(i). Compute H using (4.4) and define the nominal cost function $J_H(k)$ in (4.5);

(ii). Choose a contractive parameter ρ, then optimise the region of attraction defined in (2.55) by minimising $\log\det(TP_z^{-1}T^T)^{-1}$ subject to (4.6) and (2.54);[2]

Online: At each time step $k = 0,\ 1,\ \cdots$, solve

$$f_k = \arg\min J_H(k), \quad \text{subject to} \ \ z_k^T P_z z_k \leq \rho^k. \tag{4.7}$$

Then implement $u_k = Hx_k + c_k$ using the first element of f_k and repeat the online step at the next time instant.

[2] The offline optimisation (ii) can be applied iteratively in order to minimise ρ subject to the constraint that the region of attraction contains a specified operating region in state space. The details of this are straightforward and are omitted here.

4.2 Designing the Controller Dynamics

The previous section describes a method to compute H in the controller $u = Hx + c$ when K_{LQ} is unable to robustly stablilise the whole class of models. The H is optimised in the sense of minimising the worst case cost, hence $u = Hx + c$ may not provide good performance for the case of the nominal system dynamics. One problem is that the predictions generated by $u = Hx + c$ cannot explicitly generate the unconstrained nominal LQ-optimal feedback law when this is feasible with respect to the constraints in the online MPC optimisation. To circumvent this problem, a dynamic feedback controller is proposed in this section to pre-stabilise the prediction system, and it is shown that its dynamics can be optimised offline using semi-definite programming (SDP) so as to impose a similar upper bound on worst-case performance in the absence of constraints as in last section. The dynamic controller retains the capacity to generate the predictions of the nominal LQ-optimal control law when the unconstrained system is given by the nominal model dynamics.

On the basis of the augmented autonomous construction (2.52), several pieces of research [Drageset et al., 2003, Imsland et al., 2005, Cannon and Kouvaritakis, 2005] were conducted to further enlarge the region of attraction of Algorithm 2.3 and hence reduce the conservativeness. The aim was achieved by optimising the prediction dynamics of the perturbation signal c_k. As in last section, denote the augmented variable $z_k = [x_k^T \ f_k^T]^T$. In Imsland et al. [2005], the perturbation vector f_k was assumed to evolve as

$$f_{k+1} = Fx_k + Gf_k, \tag{4.8}$$

with $c_k = Df_k$. Compared with (2.52), extra degrees of freedom F, G, D are introduced, which can be incorporated in a non-convex optimisation for a larger region of attraction offline [Imsland et al., 2005].

Similar to the form of (4.8), another prediction dynamics was proposed by Cannon and Kouvaritakis [2005]:

$$f_{k+1} = A_{c,k} f_k, \tag{4.9}$$

with $c_k = C_c f_k$. Here $A_{c,k}$ was assumed to lie in a convex hull as A, B in (2.52): $A_c \in Co\{A_{c,1}, A_{c,2}, \dots, A_{c,L}\}$. Using the congruence transformation, the offline optimisation was reformulated as a convex problem. In addition, it is proved that the ellipsoidal invariant set under (4.9) is equal to the invariant set under a certain linear feedback law $u = \tilde{H}x$ [Cannon and Kouvaritakis, 2005].

Both (4.8) and (4.9) are designed for optimising the region of attraction, in this section a dynamical controller is proposed to stabilise the whole class of system models, which cannot be realized with the use of the nominal LQ-optimal feedback law K_{LQ}.

In order to determine a suitable dynamic controller, we define the predicted inputs[3]

$$u_{k+i|k} = K x_{k+i|k} + \upsilon_{k+i|k}, \quad i = 0, 1, \dots \tag{4.10}$$

where the predictions $\upsilon_{k+i+1|k} \in Co\{F_l x_{k+i|k} + N_l \upsilon_{k+i|k}, l = 1, \dots, L\}$ for all $i \geq 0$, and $\upsilon_{k|k} \in \mathbb{R}^{n_u}$ is determined through the *online* optimization at time k. The set of quadratically stabilizing $\{F_l, N_l\}$ can be obtained from the set $(W_1, \{F_l, N_l\})$ satisfying $W_1 \succ 0$ and

$$W_1 - \begin{bmatrix} A_l + B_l K & B_l \\ F_l & N_l \end{bmatrix}^T W_1 \begin{bmatrix} A_l + B_l K & B_l \\ F_l & N_l \end{bmatrix} \succ \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K^T \\ I_{n_u} \end{bmatrix} R \begin{bmatrix} K^T \\ I_{n_u} \end{bmatrix}^T, \quad l = 1, \dots, L \tag{4.11}$$

The following theorem shows that $\{F_l, N_l\}$ can be chosen so that the minimum of the worst case cost $\bar{J}_k$ (over $\upsilon_{k|k}$) along predicted trajectories of system (2.24),

[3]For simplicity of notation, K_{LQ} will be written as K in the following contexts of this chapter.

(4.10) for given x_k is equal to the minimum $\bar{J}_k$ achievable using linear state feedback.

Theorem 4.1. There exist $\{F_l, N_l\}$, S_{xf} and S_f such that $\left(W_1 = \left[\begin{smallmatrix} S_x & S_{xf} \\ * & S_f \end{smallmatrix}\right]^{-1}, \{F_l, N_l\}\right)$ is feasible for (4.11) if and only if there exists a feedback gain H such that S_x satisfies (4.2), namely $S_x^{-1} - (A_l + B_l H)^T S_x^{-1}(A_l + B_l H) \succ Q_1 + H^T R H$.

Proof. Considering $W_1^{-1} = \left[\begin{smallmatrix} S_x & S_{xf} \\ * & S_f \end{smallmatrix}\right]$, define $\{\Gamma_l^x, \Gamma_l^{xf}, Y_l^{fx}, Y_l^f\}$ by

$$\begin{bmatrix} \Gamma_l^x & \Gamma_l^{xf} \\ Y_l^{fx} & Y_l^f \end{bmatrix} = \begin{bmatrix} A_l + B_l K & B_l \\ F_l & N_l \end{bmatrix} W_1^{-1} = \begin{bmatrix} (A_l{+}B_l K)S_x {+} B_l S_{xf}^T & (A_l{+}B_l K)S_{xf} {+} B_l S_f \\ F_l S_x + N_l S_{xf}^T & F_l S_{xf} + N_l S_f \end{bmatrix}.$$

Also let $[\Sigma_x \ \Sigma_f] = [D_x \ D_f]W_1^{-1} = [D_x S_x {+} D_f S_{xf}^T \ \ D_x S_{xf} + D_f S_f]$,

where $[D_x \ D_f]^T[D_x \ D_f] = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K^T \\ I_{n_u} \end{bmatrix} R \, [K \ I_{n_u}]$.

Let (4.11) times W_1^{-1} from both left and right, then the conditions of (4.11) can be written for $l = 1, \ldots, L$ as

$$\begin{bmatrix} \begin{bmatrix} S_x & S_{xf} \\ S_{xf}^T & S_f \end{bmatrix} & \begin{bmatrix} \Gamma_l^x & \Gamma_l^{xf} \\ Y_l^{fx} & Y_l^f \end{bmatrix}^T & \begin{bmatrix} \Sigma_x^T \\ \Sigma_f^T \end{bmatrix} \\ \begin{bmatrix} \Gamma_l^x & \Gamma_l^{xf} \\ Y_l^{fx} & Y_l^f \end{bmatrix} & \begin{bmatrix} S_x & S_{xf} \\ S_{xf}^T & S_f \end{bmatrix} & 0 \\ \begin{bmatrix} \Sigma_x & \Sigma_f \end{bmatrix} & 0 & I \end{bmatrix} \succ 0. \tag{4.12}$$

By Schur complements, (4.12) is equivalent to $S_x \succ 0$ and

$$\begin{bmatrix} S_x & S_{xf} & {Y_l^{fx}}^T & \Sigma_x^T \\ S_{xf}^T & S_f & {Y_l^f}^T & \Sigma_f^T \\ Y_l^{fx} & Y_l^f & S_f & 0 \\ \Sigma_x & \Sigma_f & 0 & I \end{bmatrix} - \begin{bmatrix} {\Gamma_l^x}^T \\ {\Gamma_l^{xf}}^T \\ S_{xf}^T \\ 0 \end{bmatrix} S_x^{-1} \begin{bmatrix} \Gamma_l^x & \Gamma_l^{xf} & S_{xf} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} S_x - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^x & S_{xf} - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^{xf} & {Y_l^{fx}}^T - {\Gamma_l^x}^T S_x^{-1} S_{xf} & \Sigma_x^T \\ * & S_f - {\Gamma_l^{xf}}^T S_x^{-1} \Gamma_l^{xf} & {Y_l^f}^T - {\Gamma_l^{xf}}^T S_x^{-1} S_{xf} & \Sigma_f^T \\ * & * & S_f - S_{xf}^T S_x^{-1} S_{xf} & 0 \\ * & * & * & I \end{bmatrix} \succ 0. \tag{4.13}$$

Choosing $\{F_l, N_l\}$ so that ${Y_l^{fx}}^T = {\Gamma_l^x}^T S_x^{-1} S_{xf}$, ${Y_l^f}^T = {\Gamma_l^{xf}}^T S_x^{-1} S_{xf}$, (4.13) is transformed into

$$\begin{bmatrix} S_x - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^x & S_{xf} - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^{xf} & 0 & \Sigma_x^T \\ * & S_f - {\Gamma_l^{xf}}^T S_x^{-1} \Gamma_l^{xf} & 0 & \Sigma_f^T \\ * & * & S_f - S_{xf}^T S_x^{-1} S_{xf} & 0 \\ * & * & * & I \end{bmatrix} \succ 0.$$

For $S_f - S_{xf}^T S_x^{-1} S_{xf} \succ 0$ and $S_x \succ 0$ are ensured by $W_1^{-1} \succ 0$, equivalent condition is given by

$$\begin{bmatrix} S_x - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^x & S_{xf} - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^{xf} & \Sigma_x^T \\ * & S_f - {\Gamma_l^{xf}}^T S_x^{-1} \Gamma_l^{xf} & \Sigma_f^T \\ * & * & I \end{bmatrix} \succ 0.$$

It is straightforward to get the following condition considering Schur complements again:

$$\begin{bmatrix} S_x - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^x - \Sigma_x^T \Sigma_x & S_{xf} - {\Gamma_l^x}^T S_x^{-1} \Gamma_l^{xf} - \Sigma_x^T \Sigma_f \\ * & S_f - {\Gamma_l^{xf}}^T S_x^{-1} \Gamma_l^{xf} - \Sigma_f^T \Sigma_f \end{bmatrix} \succ 0. \tag{4.14}$$

A necessary condition of (4.14) is the first block should be positive definite:

$$S_x - \Gamma_l^{xT} S_x^{-1} \Gamma_l^x - \Sigma_x^T \Sigma_x \succ 0. \tag{4.15}$$

Pre- and post-multiplication by S_x^{-1} inequality (4.15) converts it into (4.2) for $H = K + S_{xf}^T S_x^{-1}$, which demonstrates that (4.2) is *necessary* for (4.14). Next we show (4.15) is also the *sufficient* condition for (4.14). (4.15) can be expressed as

$$S_x - [S_x,\, S_{xf}][A_l + B_l K,\, B_l]^T S_x^{-1} [A_l + B_l K,\, B_l] \begin{bmatrix} S_x \\ S_{xf}^T \end{bmatrix} - [S_x,\, S_{xf}][D_x,\, D_f]^T [D_x,\, D_f] \begin{bmatrix} S_x \\ S_{xf}^T \end{bmatrix} \succ 0.$$

Correspondingly, $S_{xf} - \Gamma_l^{xT} S_x^{-1} \Gamma_l^{xf} - \Sigma_x^T \Sigma_f$ and $S_f - \Gamma_l^{xf^T} S_x^{-1} \Gamma_l^{xf} - \Sigma_f^T \Sigma_f$ can be rewritten as

$$S_{xf} - [S_x,\, S_{xf}][A_l + B_l K,\, B_l]^T S_x^{-1} [A_l + B_l K,\, B_l] \begin{bmatrix} S_{xf} \\ S_f \end{bmatrix} - [S_x,\, S_{xf}][D_x,\, D_f]^T [D_x,\, D_f] \begin{bmatrix} S_{xf} \\ S_f \end{bmatrix},$$

$$S_f - [S_{xf}^T,\, S_f][A_l + B_l K,\, B_l]^T S_x^{-1} [A_l + B_l K,\, B_l] \begin{bmatrix} S_{xf} \\ S_f \end{bmatrix} - [S_{xf}^T,\, S_f][D_x,\, D_f]^T [D_x,\, D_f] \begin{bmatrix} S_{xf} \\ S_f \end{bmatrix}.$$

If (4.15) is satisfied, assuming $S_{xf} = S_x \mathbb{K}^T$ and $S_f = \mathbb{K} S_x \mathbb{K}^T + \epsilon I_{n_u}$ ($\mathbb{K}$ is a $n_u \times n_x$ full rank matrix, then simple manipulations suggest that (4.14) is guaranteed for sufficiently small positive ϵ and for $H = K + \mathbb{K}$. ■

Remark 4.2. The preceding formulation does not ensure explicitly that the case when $\upsilon_{k+i|k} = 0$ for all $i \geq 0$ can be executed in (4.10), and hence it does not guarantee that the set of predicted trajectories includes the nominal LQ-optimal feedback law. To remedy this, we define $F_0 = 0$, $N_0 = 0$, and with $\{A_0, B_0\}$ denoting as before the nominal value of $\{A, B\}$, we introduce an additional constraint on W_1 by invoking (4.11) for $l = 1, \ldots, L$ as well as $l = 0$. This will render the nominal unconstrained LQ-optimal feedback law implementable.

It is noted that $\min[x^T\, \upsilon^T]W_1[x^T\, \upsilon^T]^T = x^T S_x^{-1} x$ in the unconstrained case , which then leads to a way of computing $(W_1, \{F_l, N_l\})$ $(l = 1, \dots, L)$ via the minimisation of the largest eigenvalue of S_x^{-1}:

$$\{F_l,\, N_l\} = \arg \max_{W_1, F_l, N_l} \lambda_S \tag{4.16}$$

$$\text{subject to} \quad S_x \geq \lambda_S I \text{ and (4.11) for } l = 0, 1, \dots, L.$$

Remark 4.3. (4.16) is a convex optimisation, and can be solved by semi-definite programming techniques.

After pre-stabilisation by means of (4.10), the controller can be further formulated into

$$u_{k+i|k} = Kx_{k+i|k} + \upsilon_{k+i|k} + c_{k+i|k}, \quad c_{k+N+i|k} = 0. \tag{4.17}$$

The perturbation input $c_{k+i|k}$, $i = 0, 1, \dots, N-1$ is employed to satisfy the constraints and robustify the MPC strategy, e.g. to enlarge the region of attraction. Thus the matrices in the augmented model (2.52) turn into

$$\Psi_l = \begin{bmatrix} A_l + B_l K & B_l & B_l E \\ F_l & N_l & 0 \\ 0 & 0 & M \end{bmatrix}, \; l = 0, 1, \dots, L,$$

$$z_k = \begin{bmatrix} x_k \\ \upsilon_k \\ f_k \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_1 + K^T R K & K^T R & K^T R E \\ RK & R & RE \\ E^T R K & E^T R & E^T R E \end{bmatrix}$$

Then the nominal cost function for this controller design becomes

$$J_0(k) = \begin{bmatrix} x_k \\ \upsilon_k \\ f_k \end{bmatrix}^T W_0 \begin{bmatrix} x_k \\ \upsilon_k \\ f_k \end{bmatrix}, \tag{4.18}$$

where W_0 is the solution of the Lyapunov equation $W_0 - \Psi_0^T W_0 \Psi_0 = \tilde{Q}$.

To guarantee the closed-loop stability, the contractive invariant set is computed in the same way as in Section 4.1.

Algorithm 4.2. *Offline:*

(i). Compute $\{F_l, N_l\}$ using (4.16) and the nominal cost function $J_0(k)$ in (4.18);

(ii). Choose a contractive parameter ρ, then optimise the region of attraction defined in (2.55) by minimising $\log\det(TP_z^{-1}T^T)^{-1}$ subject to (4.6) (for $l = 0, 1, \ldots, L$) and (2.54);

Online: At each time step $k = 0,\ 1,\ \cdots$, compute

$$(\upsilon_k,\ f_k) = \arg\min J_0(k), \quad \text{subject to} \ \ z_k^T P_z z_k \leq \rho^k. \tag{4.19}$$

Then implement $u_k = Kx_k + \upsilon_k + c_k$ using υ_k and the first element of f_k, and repeat the online step at the next time step.

Remark 4.4. (4.19) is a convex optimisation problem on $(\upsilon_k,\ f_k)$. It can be solved efficiently through the Newton-Raphson method.

Remark 4.5. Algorithm 4.2 is asymptotically stable for system $x(k+1) = A(k)x(k) + B(k)u(k)$. This is ensured due to the contractive constraint with $\rho < 1$ imposed on the online optimisation (see (4.19)), where $V(z_k) = z_k^T P_z z_k$ is a Lyapunov function.

4.3 Numerical Example

Take a second order single-input, single-output (SISO) system model:

$$
\begin{gathered}
A_0=\begin{bmatrix}4.2025 & -1.2275\\ 1.9500 & -0.0325\end{bmatrix},\quad A_1=\begin{bmatrix}4.20 & -1.23\\ 2.00 & 0\end{bmatrix},\quad A_2=\begin{bmatrix}4.28 & -1.05\\ 1.93 & -0.19\end{bmatrix}\\
A_3=\begin{bmatrix}4.07 & -1.25\\ 2.00 & 0.06\end{bmatrix},\quad A_4=\begin{bmatrix}4.26 & -1.38\\ 1.87 & 0\end{bmatrix}\\
B_0=\begin{bmatrix}0.5600\\ -0.0025\end{bmatrix},\quad B_1=\begin{bmatrix}0.50\\ 0\end{bmatrix},\quad B_2=\begin{bmatrix}0.48\\ -0.05\end{bmatrix},\quad B_3=\begin{bmatrix}0.58\\ 0.10\end{bmatrix},\quad B_4=\begin{bmatrix}0.68\\ -0.06\end{bmatrix}
\end{gathered}
\tag{4.20}
$$

with $y = Cx,\ C = [-0.51\ 0.48]$ and cost weight $Q = C' \times C,\ R = 0.01$.

The nominal unconstrained LQ-optimal gain is $K_{LQ} = [-6.25\ 2.12]$. The K_{LQ} here cannot stabilize the whole class of systems (4.20), because the eigenvalues of $A_2 + B_2K_{LQ}$ are 1.23 and −0.25. Therefore, controller $u = K_{LQ}x + c$ cannot be applied directly, and hence it is replaced by $u = Hx+c$, from (4.4) $H = [-6.77\ 2.11]$. When using $u = Kx + \upsilon + c$, from (4.16) the matrices $\{F_l,\ N_l\}$ are obtained:

$$
\begin{cases}
F_1=\begin{bmatrix}-0.5805 & 0.0929\end{bmatrix}, & F_2=\begin{bmatrix}-0.6857 & 0.0171\end{bmatrix}\\
F_3=\begin{bmatrix}-0.2404 & 0.0131\end{bmatrix}, & F_4=\begin{bmatrix}0.0038 & -0.0326\end{bmatrix}\\
N_1=-0.2806,\quad N_2=-0.2608, & N_3=-0.3186,\quad N_4=-0.3681
\end{cases}
$$

As a result, the eigenvalues of $\begin{bmatrix}A_2 + B_2K_{LQ} & B_2\\ F_2 & N_2\end{bmatrix}$ become $\begin{pmatrix}0.9622\\ -0.0001\\ -0.2388\end{pmatrix}$, which indicates a stable matrix.

4.3.1 Comparison of Eigenvalues of S_x^{-1}

The offline optimisation objectives, to compute H in (4.4) and $\{F_l,\ N_l\}$ in (4.16), are identical: minimising the largest eigenvalue of S_x^{-1}. Table 4.1 compares the index of the worst case cost by giving the eigenvalues of S_x^{-1}, where $\bar{J} = \|x\|^2_{S_x^{-1}}$, for three controllers:

(i). Linear feedback $u = Hx$, where H minimises $\lambda_{\max}(S_x^{-1})$ subject to (4.2);

(ii). $\{\bar{F}_l,\ \bar{N}_l\}$ are optimized for $l = 1, \ldots, L$ in (4.11), this will not include nominal LQ-optimal feedback law explicitly;

(iii). $\{F_l,\ N_l\}$ are optimized for $l = 0, 1, \ldots, L$ in (4.16), which makes $\upsilon_{k+i|k} = 0$ for all $i \geq 0$ possible.

Optimization method	$\lambda(S_x^{-1})$
(i) H optimized subject to (4.2)	$(4.6536, 0.2925)$
(ii) $\{\bar{F}_l,\ \bar{N}_l\}$ optimized for $l = 1, \ldots, L$ in (4.11)	$(4.6536, 0.2925)$
(iii) $\{F_l,\ N_l\}$ optimized for $l = 0, 1, \ldots, L$ in (4.16)	$(4.9652, 0.3005)$

Table 4.1: Eigenvalues of S_x^{-1}, where $\bar{J}_k = \|x_k\|^2_{S_x^{-1}}$

Clearly, the cost bound is identical for (i) and (ii), which is a direct consequence of Theorem 4.1. The increase in (iii) arises because of the additional constraint $(l = 0)$ discussed in Remark 4.2 which allows for $F_0 = 0,\ N_0 = 0$.

4.3.2 Comparison of the Closed-loop Behaviour

In the previous two sections, it is discussed that both controller designs, namely $u = Hx + c$ and $u = Kx + \upsilon + c$, can be incorporated in the augmented autonomous formulation to construct robust MPC strategies, i.e. Algorithm 4.1 and 4.2. And they can both be solved efficiently through a Newton-Raphson method.

Assume the perturbation length $n_c = 10$, the input bound $\bar{u} = 3.2$, and that the model is linear time invariant when updating the system states, hence there are five realizations of system models (nominal plus four vertices, i.e. $l = 0, 1, \ldots, 4$). 20 transient steps are implemented under Algorithm 4.1 and 4.2.

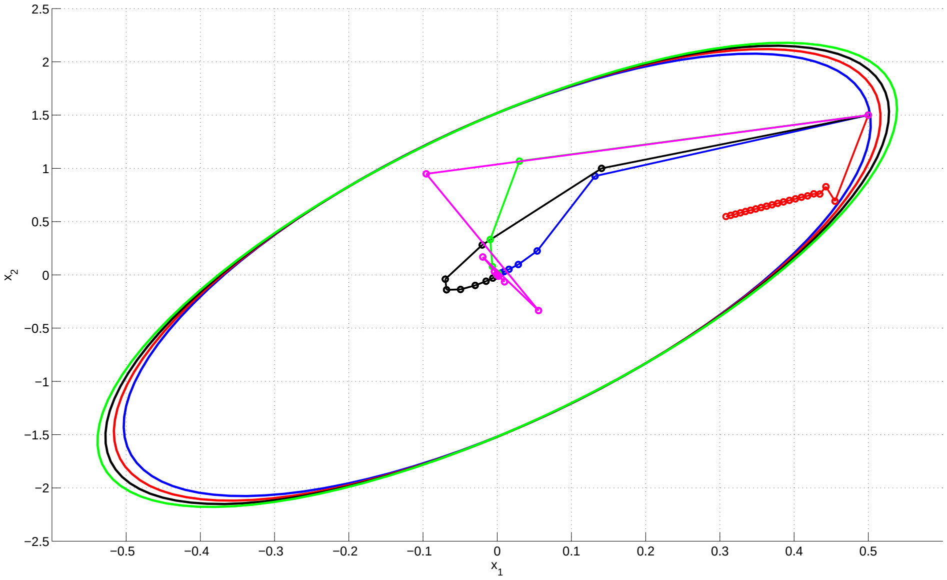

FIGURE 4.1: The invariant ellipsoids for $\rho = 0.94, 0.96, 0.98, 1$ and the closed-loop trajectories for $l = 0, 1, \ldots, 4$ under Algorithm 4.2

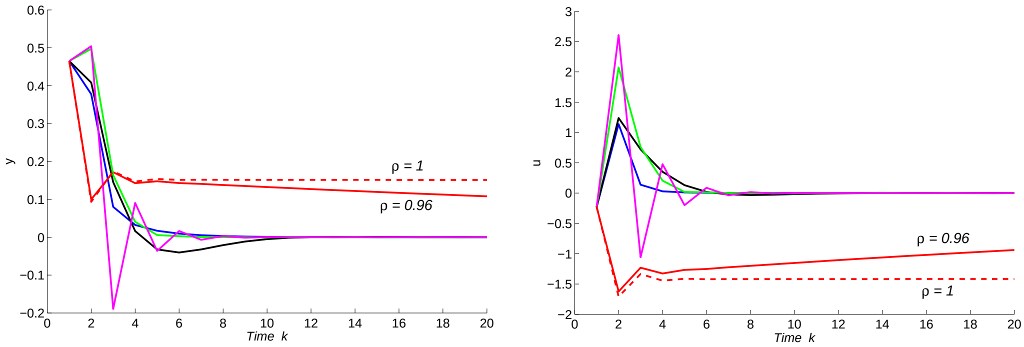

FIGURE 4.2: Output and input trajectories: solid line for $\rho = 0.96$, dashed line for $\rho = 1$ and $l = 2$

Figure 4.1 reveals the invariant ellipsoids for different contractive parameters $\rho = 0.94, 0.96, 0.98, 1$ in (4.6) under Algorithm 4.2, and the volume of these sets gets larger as ρ increases. However the volume is not particularly sensitive to the value of ρ within this interval, and $\rho = 0.96$ is chosen in the online optimisation. The closed-loop trajectories of x_k for different realizations of models ($l = 0, 1, \ldots, 4$) are also depicted in Figure 4.1, from an initial condition $x_0 = [0.5 \quad 1.5]^T$. It is found that the states will converge to the origin very fast except for $l = 2$ (the red line), which was forced to approach the origin by the contractive invariant set with $\rho = 0.96$. The output and input trajectories are shown in Figure 4.2. For non-contractive invariance, i.e. if $\rho = 1$ is chosen, it is noted that the output and input trajectories for $l = 2$, when $A_l + B_l K_{LQ}$ is not stable, will converge to another equilibrium point[4] other than the original point, thus the closed-loop cost will tend to infinity over an infinite costing horizon.

Similarly, under Algorithm 4.1, the invariant sets for $\rho = 0.94$, 0.96, 0.98, 1 and the closed-loop trajectories for states x_k are shown in Figure 4.3, and the output and input trajectories in Figure 4.4. From simulations of more time steps, it is found that the output and input converge to zero when $\rho = 0.96$. However, both of them will tend to non-zero limits for $\rho = 1$, correspondingly the plant state remains at the boundary of the augmented invariant set.

	Ξ_0	Ξ_1	Ξ_2	Ξ_3	Ξ_4
Algorithm 4.1	3.822	4.409	7.427	5.456	6.019
Algorithm 4.2	3.804	4.311	7.884	5.417	5.985

TABLE 4.2: Values of closed-loop cost times 10 (for 20 sampling steps)

The closed-loop behaviour of Algorithm 4.1 and 4.2 is compared in Table 4.2[5] by analyzing the cumulative costs for 20 time steps. Both algorithms employ the contractive invariant set with $\rho = 0.96$ so that the states can converge to the original point.

It is found that the nominal cost for $u = Hx + c$ (Algorithm 4.1) is larger than that for $u = Kx + \upsilon + c$ (Algorithm 4.2). The reason for this is that the latter design keeps

[4]The equilibrium is restricted by the augmented invariant set $E_z = \{z | z^T P_z z \leq 1\}$, in fact on its boundary.

[5]Ξ represents $\{A,\ B\}$ in Table 4.2.

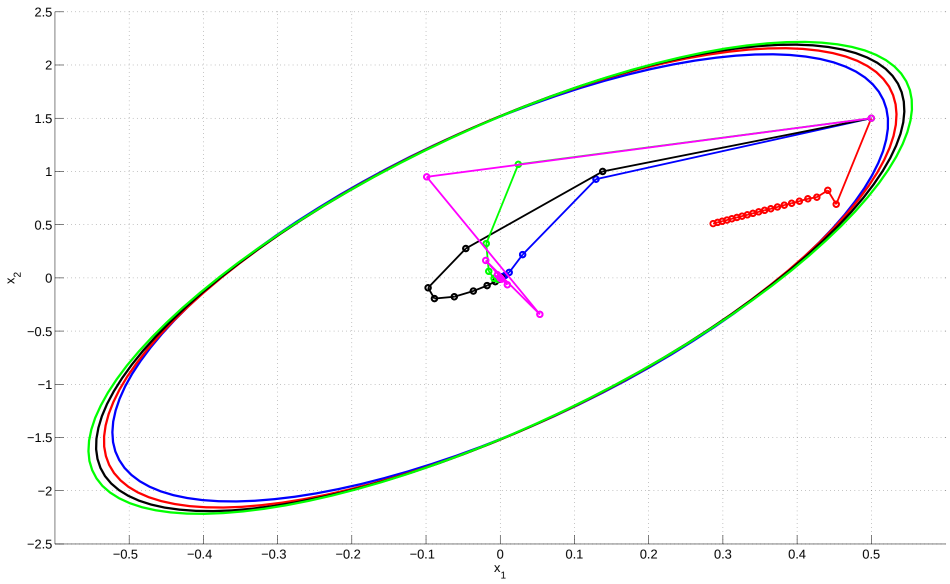

FIGURE 4.3: The invariant ellipsoids for $\rho = 0.94, 0.96, 0.98, 1$ and the closed-loop trajectories for $l = 0, 1, \ldots, 4$ under Algorithm 4.1

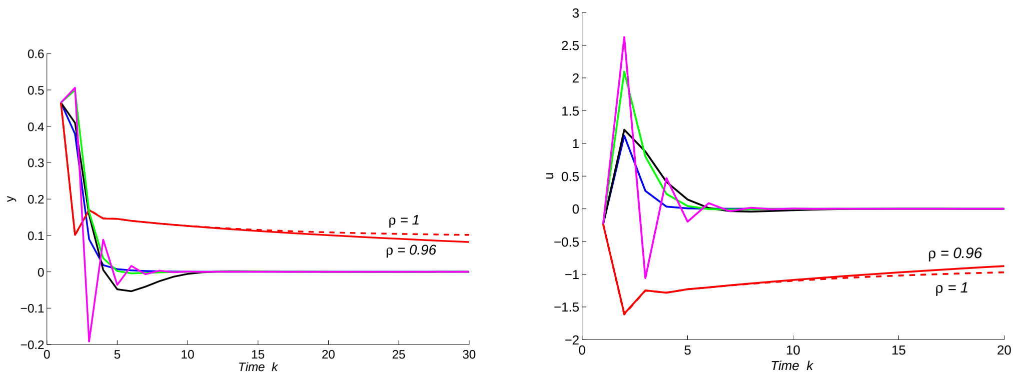

FIGURE 4.4: Output and input trajectories: solid line for $\rho = 0.96$, dashed line for $\rho = 1$ and $l = 2$

the *'distinct'* capacity to generate the unconstrained nominal LQ-optimal feedback law whenever applicable, which the former design does not possess. On the other hand, in order to gain that *'distinct'* capacity, the computation of $\{F_l, N_l\}$ has involved additional constraint of $F_0 = 0,\ N_0 = 0$, which has resulted in a larger worst case cost (0.7884) than the former (0.7427).

4.4 Conclusions

This chapter develops two methods to address the problem in using $u = Kx + c$ that the nominal LQ-optimal feedback law K_{LQ} may not stabilize the whole class of the uncertain system. One is to replace K_{LQ} with linear state feedback H; the other is to design a dynamic control law $u = K_{LQ}x + \upsilon + c$ with $\upsilon_{k+i+1|k} \in Co\{F_l x_{k+i|k} + N_l \upsilon_{k+i|k},\ l = 0, 1, \ldots, L\}$. The two approaches share three common properties: (i) Both are shown to be effective in stabilizing the system; (ii) Both can combine the augmented autonomous state space formulation to construct an efficient MPC strategy; (iii) H, $\{F_l,\ N_l\}$ are both computed through the offline minimisation of the unconstrained worst case cost using convex optimisation (semi-definite programming). The main difference is that $u = K_{LQ}x + \upsilon + c$ can generate the unconstrained LQ-optimal feedback trajectory. This is achieved by including an additional constraint in the offline optimisation of the worst case cost, and hence this property comes at the price of a higher bound on worst case performance.

Chapter 5

Stochastic Ellipsoidal Tube MPC with Probabilistic Constraints

[1]This chapter studies an approach of accounting for probabilistic constraints through the use of stochastic tubes and ellipsoidal relaxation methods which successfully ensures recursive feasibility and the closed-loop stability. Unlike the method investigated in Cannon, Kouvaritakis, and Ng [2009a] which optimizes the tube scalings online, here a large number layers of tubes (say 1000) can be constructed *offline* and all the online constraints are transformed into linear ones, thus the online computation becomes a quadratic programming (QP) problem and is computationally efficient even for long prediction horizons, which is a restriction for the tube approach of Cannon et al. [2009a] . Numerical examples are given to illustrate the efficacy of this stochastic ellipsoidal tube MPC.

[1]The work in this chapter has been published in Cannon et al. [2010].

5.1 Introduction

To address the gaps of the stochastic MPC study listed in Chapter 2, such as computational complexity, the study of recursive feasibility or closed-loop stability, a stochastic ellipsoidal tube MPC strategy with probabilistic constraints is developed. The constraint here is *individual chance constraint* which is used to imply that the probability of constraint violation should be less than p at any given time. This is a stronger condition than that used in (2.98) (joint chance constraint) but may be appropriate to a number of practical solutions. The effect of this is that it obviates the need for constraining the one-step-ahead violation probability.

As discussed in Section 2.3.5 and 2.5, a stochastic tube at time instant k in MPC denotes the set in which the states x_k will lie with some probability. To clarify the properties of tubes, three parameters need careful treatment: i.e. position, shape and volume (size). The position is always characterized by a reference point. For example, in this study nominal states of the uncertain system are regarded as the references. Rather than allowing for variable polytopic cross-sections for the tubes [Cannon et al., 2009a], the cross-sections to be used here will be fixed (for convenience ellipsoids are adopted) but the centres and sizes of these cross-sections are allowed to vary with time, which implies that there is a significant reduction in the number of degrees of freedom (DOF). The (time) evolution of the tubes is described by a simple dynamic system, while in this context the dynamics governing the tube scalings[2] are *stochastic*, and the computation of the predicted distributions of tube scalings is facilitated using *a process of discretization*. The distributions enable bounds to be imposed on the probability of violation of state constraints; these are computed offline and are invoked online by tightening the constraints on the predictions of a nominal model. The offline computation of distributions for the stochastic variables allows the use of many discretized levels and

[2] Tube scalings are used to depict the volume of the ellipsoidal tubes, like β_i in (5.8).

hence this, in turn, allows for the implicit use of many layered tubes without any concomitant increase in the online computation. An ellipsoidal relaxation method similar to the form of Van Hessem et al. [2002] (see Figure 2.2) is exploited here to handle the constraint, but the contribution of this study is that the closed-loop feasibility over an infinite horizon is ensured through the concept of Recursively Feasible Stochastic Tubes (RFST)[3] and the online algorithm is turned into a much more efficient quadratic programming (QP) problem with only a limited number of the perturbation sequence in the controller variables.

5.2 Problem Definition

Consider the linear time-invariant model with state $x_k \in \mathbb{R}^{n_x}$, control input $u_k \in \mathbb{R}^{n_u}$ and disturbance input $w_k \in \mathbb{R}^{n_w}$,

$$x_{k+1} = Ax_k + B_u u_k + B_w w_k, \tag{5.1}$$

Here w_k for $k = 0, 1, \ldots$, are zero-mean, independent and identically distributed (i.i.d.) random variables, and it is known that $w_k \in \mathcal{E}(W, \alpha_k)$ where $\mathcal{E}(W, \alpha_k)$ is an ellipsoidal set

$$w_k \in \mathcal{E}(W, \alpha_k); \quad \mathcal{E}(W, \alpha) = \{w : \; w^T W w \leq \alpha\}, \quad W = W^T > 0, \tag{5.2}$$

Here $\alpha_k \geq 0, \; k = 0, 1, \ldots$ are independently and (temporally) identically distributed (i.i.d.) random variables. The distribution function $F_\alpha(a) = \Pr(\alpha_k \leq a)$ is assumed to be known, either in closed form or by numerically integrating the density of w, and we make the following assumptions on α.

Assumption 5.1. F_α is continuously differentiable and α is bounded: $\alpha \in [0, \bar{\alpha}]$.

[3] Recursive Feasibility in stochastic MPC implies that if the probabilistic constraint such as (5.3) is satisfied at time k, then (5.3) will be met at all future time instants $t > k$.

The system is subject to probabilistic constraints of the form $\Pr(g_i^T x_k \leq h_i) \geq p_i$, $i = 1, \ldots n_c$, for given $g_i \in \mathbb{R}^{n_x}$, $h_i \in \mathbb{R}$ and probabilities $p_i \geq 0.5$. General linear constraints on states and inputs can be handled using this study's framework, and hard constraints can be included as a special case of probabilistic constraints invoked with probability 1 ($w.p.1$). To simplify presentation, the approach is developed for the case of a single probabilistic constraint ($n_r = 1$):

$$\Pr(g^T x_k \leq h) \geq p; \quad p \geq 0.5. \tag{5.3}$$

and the case of more than one soft constraint $n_r > 1$, can be treated easily by taking the intersection of n_r constraint sets. The control problem is to minimize, at each time k, the expected quadratic cost

$$J_k = \sum_{i=0}^{\infty} \mathbb{E}_k\Big(x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i}\Big) \tag{5.4}$$

subject to $\Pr(g^T x_{k+i} \leq h) \geq p$ for all $i > 0$, while ensuring closed loop stability (in a suitable sense) and convergence of x_k to a neighborhood of the origin as $k \to \infty$. We make use of the semi closed-loop paradigm and decompose the state and input of the original system (5.1) into:

$$x_k = z_k + e_k, \;\; u_k = K x_k + c_k \tag{5.5}$$

$$z_{k+1} = \Phi z_k + B_u c_k \tag{5.6}$$

$$e_{k+1} = \Phi e_k + B_w w_k \tag{5.7}$$

where $\Phi = A + B_u K$ is assumed to be strictly stable and c_{k+i}, $i = 0, 1, \ldots, N-1$ ($c_{k+i} = 0$ for $i \geq N$) are free variables in a receding horizon optimization at time k. This allows the effect of disturbances on the i-step-ahead predicted state, x_{k+i}, to be considered separately (via e_{k+i}) from the nominal prediction, z_{k+i}, and thus simplifies the handling of constraints.

5.3 Prediction Dynamics for Propagation of Uncertainty

The uncertainty in the i-step-ahead prediction e_{k+i} can be characterized using (5.2) and (5.7) in terms of the scalings β_i of ellipsoidal sets containing e_{k+i}:

$$e_{k+i} \in \mathcal{E}(V, \beta_i); \quad \mathcal{E}(V, \beta_i) = \{e : \ e^T V e \le \beta_i\}, \ V = V^T \succ 0. \tag{5.8}$$

This section constructs a dynamic system to define the random sequence $\{\beta_i, \ i = 1, 2, \ldots\}$ and proposes a method of approximating numerically the distribution functions of β_i, $i \ge 1$.

Given $e_{k+i} \in \mathcal{E}(V, \beta_i)$ and $w_{k+i} \in \mathcal{E}(W, \alpha)$ we have $e_{k+i+1} \in \mathcal{E}(V, \beta_{i+1})$ if and only if

$$\max_{\substack{e \in \mathcal{E}(V, \beta_i) \\ w \in \mathcal{E}(W, \alpha)}} (\Phi e + \ B_w w)^T V (\Phi e + \ B_w w) \le \beta_{i+1}. \tag{5.9}$$

The problem of calculating the minimum of β_{i+1} in (5.9) is NP-hard, but sufficient conditions for (5.9) can be stated as follows.

Lemma 5.2. If β_i , β_{i+1} and V satisfy

$$\beta_{i+1} = \lambda \beta_i + \alpha_i \tag{5.10}$$

$$V^{-1} - \frac{1}{\lambda} \Phi V^{-1} \Phi^T - B_w W^{-1} B_w^T \ge 0 \tag{5.11}$$

for some $\lambda > 0$, then $e_{k+i+1} \in \mathcal{E}(V, \beta_{i+1})$ whenever $e_{k+i} \in \mathcal{E}(V, \beta_i)$ and $w_{k+i} \in \mathcal{E}(W, \alpha)$.

Proof. Using the S-procedure [Boyd et al., 1994], a sufficient condition for (5.9) is given by

$$\begin{aligned}\beta_{i+1} - \begin{bmatrix} e \\ w \end{bmatrix}^T \begin{bmatrix} \Phi^T \\ B_w^T \end{bmatrix} V \begin{bmatrix} \Phi & B_w \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} \geq \lambda\beta_i - \lambda \begin{bmatrix} e \\ w \end{bmatrix}^T \begin{bmatrix} I \\ 0 \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} \\ + \mu\alpha_i - \mu \begin{bmatrix} e \\ w \end{bmatrix}^T \begin{bmatrix} 0 \\ I \end{bmatrix} W \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix},\end{aligned} \tag{5.12}$$

for some $\lambda > 0$ and $\mu > 0$. Then it is straightforward to obtain sufficient conditions for (5.12):

$$\beta_{i+1} = \lambda\beta_i + \mu\alpha_i, \tag{5.13}$$

$$\begin{bmatrix} \Phi^T \\ B_w^T \end{bmatrix} V \begin{bmatrix} \Phi & B_w \end{bmatrix} \leq \lambda \begin{bmatrix} I \\ 0 \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ I \end{bmatrix} W \begin{bmatrix} 0 & I \end{bmatrix}. \tag{5.14}$$

However, μ can be removed from (5.13), (5.14) and (5.8) by scaling β_i, β_{i+1} and V, so (5.10) is obtained and by Schur complementation (5.14) is transformed into

$$\begin{bmatrix} \lambda V & 0 & \Phi^T \\ 0 & W & B_w^T \\ \Phi & B_w & V^{-1} \end{bmatrix} \geq 0.$$

Then condition (5.11) is derived by Schur complementation again. ∎

The distribution of β_i for any $i > 0$ can be determined from the distributions of α and β_0 using (5.10). In the sequel λ, V in (5.10) (5.11) are taken to be constants (independent of α_i, β_i); a procedure for optimizing the values of λ and V is described in Section 5.4.3. We further assume that the distribution function, F_{β_0}, of β_0 is known and has the following properties.

Assumption 5.3. F_{β_0} is right-continuous with only a finite number of discontinuities and $\beta_0 \in [0, \bar{\beta}_0]$.

Note that Assumption 5.3 includes the case of fixed, deterministic β_0 (e.g. $\beta_0 = 0$ corresponds to $F_{\beta_0}(x) = 1$ for all $x \geq 0$). The following result shows that F_{β_i} is well-defined if $\lambda \in (0,1)$.

Theorem 5.4. If $0 < \lambda < 1$, then:

(i) β_i is bounded for all i: $\beta_i \in [0, \bar{\beta}_i]$, where $\bar{\beta}_{i+1} = \lambda\bar{\beta}_i + \bar{\alpha}$, and $\bar{\beta}_i$ is contractive with the unique, globally attractive, fixed point $\bar{\beta}_\infty = \big(1/(1-\lambda)\big)\bar{\alpha}$. We assume $\bar{\beta}_0 = 0$ which indicates current states are measurable, then $\bar{\beta}_i \leq \bar{\beta}_{i+1} \leq \bar{\beta}$ for all $i \geq 0$.

(ii) F_{β_i} is continuously differentiable for all $i \geq 1$.

(iii) β_i converges in distribution to the random variable $\beta_L = \sum_{k=0}^{\infty} \lambda^k \alpha_k$ as $i \to \infty$.

Proof. Assumption 5.1 and (5.10) together imply that $\beta_i \in [0, \bar{\beta}_i]$ with $\bar{\beta}_{i+1} = \lambda\bar{\beta}_i + \bar{\alpha}$. Hence if $\lambda \in (0,1)$, then it implies, $\bar{\beta}_i \to \frac{1}{1-\lambda}\bar{\alpha}$ as $i \to \infty$. Define two sets $\mathcal{B} := [0, \frac{\bar{\alpha}}{1-\lambda}]$ and $\mathcal{A} := [0, \bar{\alpha}]$, $\lambda \in (0,1)$ implies directly that the mapping $f(\mathcal{B}) := \lambda\mathcal{B} \oplus \mathcal{A}$ (where $\oplus$ denotes Minkowski set addition) is a contraction on the space of non-empty compact subsets of the real line. The uniqueness of the fixed point and its global attractivity follow then immediately by the Banach Contraction Principle which proves the first part of (i). When $\bar{\beta}_0 = 0$, $\bar{\beta}_i = \lambda^{i-1}\bar{\alpha} + \cdots + \bar{\alpha}$, thus, leads to the solution of the second part of (i).

The distribution of β_{i+1} is given for any $i \geq 0$ by the convolution integral (see e.g.[Feller, 1968]):

$$F_{\beta_{i+1}}(x) = \frac{1}{\lambda}\int_0^{\lambda\bar{\beta}} F_\alpha(x-y)\, f_{\beta_i}\!\left(\frac{y}{\lambda}\right) dy \tag{5.15}$$

where f_{β_i} denotes the density of β_i. It follows from Assumption 5.1 and the dominated convergence principle that $F_{\beta_{i+1}}$ is continuously differentiable, as asserted in (ii).

To prove (iii), let β_i' denote the random variable $\beta_i' = \gamma_i + \lambda^i \beta_0$ for $i = 0, 1, \ldots$ where

$$\gamma_{i+1} = \gamma_i + \lambda^i \alpha_i, \quad \gamma_0 = 0. \tag{5.16}$$

Then $\beta_i' = \sum_{k=0}^{i-1} \lambda^k \alpha_k + \lambda^i \beta_0$, and since (5.10) gives $\beta_i = \sum_{k=0}^{i-1} \lambda^k \alpha_{i-1-k} + \lambda^i \beta_0$ where $\{\alpha_i\}$ is i.i.d., the distributions of β_i and β_i' are identical for all $i \geq 0$. Furthermore the bounds on α_i and β_0 in Assumptions 5.1 and 5.3 imply that for every $\epsilon > 0$ there exists n such that

$$\Pr(|\beta_j' - \beta_i'| < \epsilon) = 1 \quad \forall\ i > n,\ j > i$$

and it follows that β_i' converges almost everywhere to a limit as $i \to \infty$ [Papoulis, 2002]. From the definition of β_L, we therefore have $\beta_i' \to \beta_L$ almost everywhere, and hence β_i converges in distribution to β_L [Papoulis, 2002]. ∎

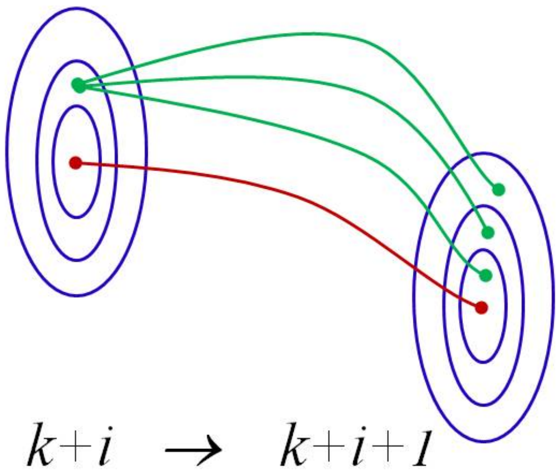

FIGURE 5.1: System state trajectories from time instant $k+i$ to $k+i+1$

A Markov Chain model can now be constructed to illustrate the process that system states transfer form one layer of tube at time $k+i$ to another layer of tube at $k+i+1$. As shown in Figure 5.1, the red line denotes the trajectories for the nominal states (the tube centres), the blue ellipsoids represent the probabilistic tubes[4] around the tube centre, and the green lines imply the possible trajectories for the

[4]For ease of clarity, only 3 layers are shown in the figure.

system actual states. Therefore, from (5.10) it is straightforward to calculate the transitional probabilities for the Markov Chain Model, the details of which are discussed in the following contexts.

The simple form of (5.10) makes it easy to introduce into the problem the stochastic nature of β_i provided that one is prepared to cover the set of $[0, \bar{\beta}]$ with discrete intervals,

$$b_i = \left[0,\ \widetilde{\beta}_i\right],\ i = 0,\ 1,\ \cdots,\ \rho \tag{5.17}$$

where

$$0 = \widetilde{\beta}_0 \leq \widetilde{\beta}_1 \leq \cdots \leq \widetilde{\beta}_\rho = \bar{\beta}_\infty \tag{5.18}$$

Then, by performing the necessary convolutional integrals (which can be done numerically if one partitions the range of values for α as well), it is possible to determine the matrix Π of probabilities π_{lj} that if $\widetilde{\beta}_{j-1} \leq \beta_{k+i|k} \leq \widetilde{\beta}_j$, then $\widetilde{\beta}_{l-1} \leq \beta_{k+i+1|k} \leq \widetilde{\beta}_l$. Accordingly, the vectors $\pi^{(k+i)}$ of the probabilities $\pi_j^{(k+i)}$ that $\widetilde{\beta}_{j-1} \leq \beta_{k+i|k} \leq \widetilde{\beta}_j$ and vectors $p^{(k+i)}$, of probabilities $p_j^{(k+i)}$ that $\beta_{k+i|k} \in b_j$, can be generated, recursively as

$$\pi^{(k+i+1)} = \Pi\pi^{(k+i)}, \tag{5.19}$$

$$p^{(k+i)} = T\pi^{(k+i)}, \tag{5.20}$$

where T is the lower-triangular matrix with each element 1 (see (2.121)).

Theorem 5.5. The predicted values of e_k will, with probability p, lie in the ellipsoids

$$e_{k+i|k} \in \mathcal{E}\left(V,\ \beta_{k+i}(p)\right), \tag{5.21}$$

where

$$\beta_{k+i}\left(p\right) = \widetilde{\beta}_j \tag{5.22}$$

with

$$j = \min_{l} \left(l : \; p_l^{(k+i)} \geq p \right) \tag{5.23}$$

and with $p_l^{(k+i)}$ for $l = 1, \ldots, \rho$ being the elements of the vector:

$$p^{(k+i)} = T \; \Pi^i \; \pi^{(k)} = \left(T \Pi T^{-1} \right)^i \; p^{(k)}. \tag{5.24}$$

Proof. Recursive use of (5.19) gives

$$\pi^{(k+i)} = \Pi^i \; \pi^{(k)} \tag{5.25}$$

which, taken together with (5.20) gives (5.24). Form the definition of j in (5.23) it follows that with probability equal to or greater than p, $e_{k+i|k} \in \mathcal{E}(V, \; \beta)$ for all $\beta \geq \widetilde{\beta}_j$, which therefore implies (5.21). ■

For convenience, for the probability p, the map of Theorem 5.5 from $p^{(k+i)}$ to $\beta_{k+i}(p)$ will be denoted by

$$\beta_{k+i} = \phi \left(p^{(k+i)}, \; p \right) \tag{5.26}$$

Theorem 5.5 implies that the predicted values of e_{k+i} lie inside tubes with ellipsoidal cross-sections (defined by V and $\beta_{k+i}(p)$) with probability p; hence we refer to these tubes as stochastic tubes. Although the stochastic tubes apply them to e_{k+i}, one can refer to x_{k+i} through the use of (5.5), which translates the tube centres to the nominal state predictions z_{k+i}.

Corollary 5.6. The parameter $\beta_{k+i}(p)$ of Theorem 5.5 is bounded from above by

$$\beta_{k+i}(p) \leq \beta_{k+i}(1) = \lambda^i \beta_k + \frac{1 - \lambda^i}{1 - \lambda} \overline{\alpha}, \;\; i = 0, \; 1, \; \ldots, \tag{5.27}$$

and for $\lambda \in (0,\ 1)$ the $\beta_{k+i}(1)$ converges to $\bar{\beta}_\infty$. Furthermore, the limit of $\beta_{k+i}(p)$ with i is given by

$$\lim_{i\to\infty} \beta_{k+i}(p) = \phi\left(p_{ss},\ p\right), \tag{5.28}$$

where p_{ss} is the eigenvector of $T\Pi T^{-1}$ that corresponds to the largest eigenvalue, scaled so that its last element is 1.

Proof. The inequality of (5.27) is self evident since $p \leq 1$ while the equality of (5.27) follows from $\beta_{k+i}(1) \to \bar{\beta}_\infty$. The limit of (5.28) is a consequence of the fact that the eigenvalues of $T\Pi T^{-1}$ are all positive and less than 1, except of the largest eigenvalue which is 1 [Norris, 1997]. Hence in the limit $p^{(k+i)}$ will be in the direction of the eigenvector that corresponds to the eigenvalue at 1, and by definition the last element of $p^{(k+i)}$ is always equal to 1. ∎

Given that there is an implicit map from β_{k+i-1} to $p^{(k+i-1)}$, and another two maps from $p^{(k+i-1)}$ to $p^{(k+i)}$ and from $p^{(k+i)}$ to β_{k+i}, respectively, it is convenient to describe the composite map from β_{k+i-1} to β_{k+i} as:

$$\beta_{k+i} = \psi\left(\beta_{k+i-1}\right). \tag{5.29}$$

5.4 The MPC Strategy

To avoid an infinite dimensional optimization problem, we adopt the usual dual mode MPC paradigm [Mayne et al., 2000]: at time k, mode 1 consists of the first N steps of the prediction horizon over which the control moves c_{k+i} in (5.5) and (5.6) are free, and mode 2 comprises the remainder of the horizon, with $c_{k+i} = 0$ for all $i \geq N$. On account of the additive uncertainty appearing in (5.1), the quadratic cost (5.4) is unbounded. Hence the modified cost like (2.100) will be used:

$$J_k = \sum_{i=0}^{\infty}\Big(\mathbb{E}_k(L_{k+i|k}) - L_{ss}\Big), \quad L_{ss} = \lim_{i\to\infty}\mathbb{E}_k(L_{k+i}) \tag{5.30}$$

where $L_{k+i|k} = x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}$. This cost is shown as (2.101) to be a quadratic function of the vector $\mathbf{c}_k$ of free variables $\{c_{k+i},\ i = 0,\ldots,N-1\}$ at time k

$$J_k = \begin{bmatrix} x_k^T & \mathbf{c}_k^T & 1\end{bmatrix} P \begin{bmatrix} x_k^T & \mathbf{c}_k^T & 1\end{bmatrix}^T \tag{5.31}$$

for a suitably defined positive definite matrix P. The satisfaction of constraints will be handled explicitly in mode 1 and implicitly, through the use of a terminal set, S, in mode 2.

5.4.1 Constraint Handling in Mode 1

The distribution of β_0 is dictated by the information available on the plant state at the beginning of the prediction horizon. Assuming that x_k is known at time k, we set $z_k = x_k$, $e_k = 0$ and $\beta_0 = 0$, so that $F_{\beta_0}(x) = 1$ for all $x \geq 0$. Then, using $v_{k+i|k}$ to denote the corresponding predictions of variable v at time k, we have from (5.7):

$$e_{k|k} = 0, \quad e_{k+i|k} = \Phi^{i-1}B_w w_k + \cdots + B_w w_{k+i-1}, \quad \forall i \geq 1.$$

These predictions are independent of time k, so the sets defining the stochastic tubes containing $e_{k+i|k}$ can be computed offline, and Theorem 5.5 gives $e_{k+i|k} \in \mathcal{E}(V,\ \beta_{k+i}(p))$ with probability p, where

$$p^{(k+i|k)} = \left(T\Pi T^{-1}\right)^i p^{(k|k)}, \ i = 1,2,\ldots, \quad p^{(k|k)} = [1 \ \cdots \ 1]^T.$$

In this setting the probabilistic constraint of (5.3) can be enforced as follows.

Lemma 5.7. The constraint $\Pr(g^T x_{k+i|k} \le h) \ge p$ for $p \ge 0.5$ is ensured by the condition

$$g^T z_{k+i|k} \le h - \left[\beta_{i|0}(q)\right]^{1/2} \sqrt{g^T V^{-1} g} \tag{5.32}$$

where $q = 2p - 1$ and $z_{k+i+1|k} = \Phi z_{k+i|k} + B_u c_{k+i|k}$ for $i = 0, 1, \dots$, with $z_{k|k} = x_k$.

Proof. Assume that the distribution of $g^T e_{k+i|k}$ is symmetric about 0 (note that there is no loss of generality in this assumption due to the symmetric bound employed in (5.8)). Then $g^T z_{k+i|k} + g^T e_{k+i|k} \le h$ holds with probability $p > 0.5$ iff $g^T z_{k+i|k} + |g^T e_{k+i|k}| \le h$ with probability $q = 2p - 1$. The sufficient condition (5.32) then follows from $e_{k+i|k} \in \mathcal{E}\big(V, \beta_{i|0}(q)\big)$ and $\max_{e \in \mathcal{E}(V, \beta_{i|0}(q))} |g^T e| = \left[\beta_{i|0}(q)\right]^{1/2} \sqrt{g^T V^{-1} g}$. ■

Note that the replacement of $|g^T e|$ by its maximum value of $\beta_{i|0}(q)^{1/2}\sqrt{g^T V^{-1} g}$ describes the constraint tightening process whereby the constraint $g^T z_{k+i|k} \le h$ on the nominal predicted state is replaced by the tighter condition given in (5.32). Although (5.32), when invoked for $i = 1, \dots, N$, ensures that at time k the predicted sequence $\{x_{k+i|k}\}$ satisfies (5.3), it does not ensure *recursive feasibility*, namely that there exists a predicted sequence at time $k + 1$ satisfying (5.3). This is because constraints of the form $\Pr(g^T x_{k+i|k} \le h) \ge p$ do not guarantee that $g^T x_{k+i|k+1} \le h$ holds with probability p. What is guaranteed is that the future state is contained at all times within the robust tube associated with upper bounds on β_i:

$$e_{k+i|k} \in \mathcal{E}\big(V, \beta_{i|0}(1)\big), \quad \beta_{i|0}(1) = \frac{1 - \lambda^i}{1 - \lambda}\bar{\alpha}, \ \forall i \ge 1, \quad \beta_{0|0}(1) = 0 \tag{5.33}$$

with probability 1. The stochastic tubes that define constraints such as (5.32) at times $k+1, k+2, \dots$ are therefore based on initial distributions for β_0 which should be conjectured from the worst case, the robust tube evolved from time k, and this enables the construction of constraints to ensure recursive feasibility. The simplest way is to invoke conditions that will ensure that all the extensions of $\mathbf{c}_k$, $M^i \mathbf{c}_k$, over the prediction horizon are feasible; the matrix M is the zero matrix with the identity matrix I on the super-diagonal above the principal-diagonal. Catering for the worst case requires taking the $\beta_{i|0}(1)$ of (5.33) as the starting $\beta_{i|i}$ at each future prediction time $k+i$, when defining the corresponding probabilistic tube cross-section sizes over the horizon $i+1, \cdots, N$:

$$\beta_{j|i} = \psi\left(\beta_{j-1|i}\right). \tag{5.34}$$

As a consequence, at the first prediction time instant, there will be one tube cross-section scaling, $\beta_{1|0}(q)$, at the second there will be two sizes, $\beta_{2|0}(q)$, $\beta_{2|1}(q)$, at the third there will be three sizes, $\beta_{3|0}(q)$, $\beta_{3|1}(q)$, $\beta_{3|2}(q)$, etc. Then, correspondingly for $i = 1$, (5.32) must be satisfied for $\beta_{1|0}(q)$, for $i = 2$ it must be satisfied for $\beta_{2|0}(q)$, $\beta_{2|1}(q)$, etc. A convenient way to summarize the situation is to define what can be termed Recursively Feasible Stochastic Tubes (RFST) as:

$$\mathcal{E}\left(V,\ \bar{\beta}_i(q)\right) = \left\{e:\quad e^T V e \le\ \bar{\beta}_i(q)\right\},\quad \bar{\beta}_i(q) = \max_{j\epsilon[0,\ i-1]}\beta_{i|j}(q) \tag{5.35}$$

and by replacing $\beta_{i|0}(q)$ with $\bar{\beta}_i(q)$ in (5.32)

$$g^T z_{k+i|k} \le h - \left[\bar{\beta}_i(q)\right]^{1/2}\sqrt{g^T V^{-1} g},\quad q = 2p-1. \tag{5.36}$$

The preceding argument is summarized as follows.

Theorem 5.8. If at $k = 0$ there exists $\mathbf{c}_0$ satisfying (5.36) for all $i > 0$, then for all $k > 0$ there exists $\mathbf{c}_k$ satisfying (5.36) for all $i > 0$, and hence the probabilistic constraint (5.3) is feasible for all $k > 0$.

Proof. This is by construction because at all future times it is possible to choose $\mathbf{c}_k = M^k\mathbf{c}_0$ which is guaranteed to satisfy (5.36) for the maxima of (5.34), and therefore *a-fortiori*, for $\beta_{i|0}(q)$. ∎

It is noteworthy that the computation of Recursively Feasible Stochastic Tubes (RFST) does not depend on the information available at time k, and hence can be performed offline. A further consequence of this is that one may use as large a number of tube layers as desired to render the numerical calculation error negligible without increasing the online computation.

5.4.2 Constraint Handling in Mode 2

The constraint handling framework of mode 1 can also be used to define a terminal constraint, $z_{k+N|k} \in S$, which ensures that (5.36) is satisfied for all $i > N$. Since (5.36) constitutes a set of linear constraints on the deterministic predicted trajectory $\{z_{k+i|k}\}$, the infinite horizon of mode 2 can be accounted for using techniques similar to those deployed in Gilbert and Tan [1991] for the computation of maximal invariant sets. Before describing the construction of S, we first derive some fundamental properties of the constraints (5.36).

Lemma 5.9. The scaling of the tube cross-section defined in (5.35) satisfies $\bar{\beta}_i(q) \le \bar{\beta}_{i+1}(q)$ for all i and all $q \in [0,1]$, and converges as $i \to \infty$ to a limit which is bounded by

$$\lim_{i\to\infty} \bar{\beta}_i(q) \le \bar{\beta}_\infty, \quad \bar{\beta}_\infty = \frac{\bar{\alpha}}{1-\lambda}. \tag{5.37}$$

Proof. The monotonic increase of the scaling of the robust tube (on account of $\beta_{0|0} = 0$) with j in (5.33) implies $\beta_{j|0}(1) < \beta_{j+1|0}(1)$, then it is straightforward to get

$$\beta_{j|j} < \beta_{j+1|j+1}. \tag{5.38}$$

Thus for any $i > j$, after the same $(i-j)$ steps (Markov Chain) the bigger initial scaling of error will result in a larger final one almost surely, namely $\beta_{i|j}(q) < \beta_{i+1|j+1}(q)$, $\forall j$, which implies $\bar{\beta}_i(q) \le \bar{\beta}_{i+1}(q)$ for all i. And since $\bar{\beta}_i(q) \le \bar{\beta}_i(1)$ for all i, $\bar{\beta}_i(q)$ must converge as $i \to \infty$ to a limit no greater than the asymptotic value of the robust tube scaling, $\bar{\beta}_\infty$. ∎

Lemma 5.10. Let S_∞ be the maximal set with the property that (5.36) is satisfied for all $i > N$ whenever $z_{k+N|k} \in S_\infty$, then a sufficient condition for S_∞ to be non-empty is:

$$h \geq \bar{\beta}_\infty^{1/2} \sqrt{g^T V^{-1} g}. \tag{5.39}$$

Proof. Satisfaction of constraints (5.36) over the infinite horizon of mode 2 requires that

$$g^T \Phi^i z_N \leq h - \left[\bar{\beta}_{N+i}(q)\right]^{1/2} \sqrt{g^T V^{-1} g}, \quad i = 1, 2, \ldots \tag{5.40}$$

where for convenience the initial state of mode 2 is labelled z_N. By Lemma 5.9, $\bar{\beta}_i(q)$ increases monotonically with i, and Φ is strictly stable by assumption, so (5.39) is obtained by taking the limit of (5.40) as $i \to \infty$ and using the upper bound in (5.37). Thus, if (5.39) holds, then S_∞ will at least contain the original point $\{0\}$. ∎

Lemma 5.9 implies that $\bar{\beta}_i(q)$ in (5.40) can be replaced by $\bar{\beta}_\infty$ for all $i > \hat{N}$, for some finite $\hat{N}$, in order to define the terminal set S. In this setting, the terminal set is defined by $S = S_{\hat{N}}$, where

$$\begin{aligned} S_{\hat{N}} = \{z_0 : g^T \Phi^i z_N \leq h - \left[\bar{\beta}_{N+i}(q)\right]^{1/2} \sqrt{g^T V^{-1} g}\,, \quad i = 1, \ldots, \hat{N} \\ g^T \Phi^{\hat{N}+j} z_0 \leq h - \bar{\beta}_\infty^{1/2} \sqrt{g^T V^{-1} g}\,, \quad j = 1, 2, \ldots \ \}. \end{aligned} \tag{5.41}$$

The corresponding conditions of (5.41) are conservative, but the implied constraint tightening can be made insignificant for sufficiently large $\hat{N}$. This follows from the assumption that Φ is strictly stable, which implies that $S_{\hat{N}} \to S_\infty$ as $\hat{N} \to \infty$.

Although (5.41) involves an infinite number of inequalities, $S_{\hat{N}}$ has the form of a maximal admissible invariant set. The procedure of Gilbert and Tan [1991] can therefore be used to determine an equivalent representation of $S_{\hat{N}}$ in terms of a finite number of inequalities. This involves solving a sequence of linear programs to find the smallest integer n^* such that $g^T\Phi^{\hat{N}+n^*+1}z_N \le h - \bar{\beta}_\infty^{1/2}\sqrt{g^TV^{-1}g}$ for all z_N satisfying $g^T\Phi^{\hat{N}+j}z_N \le h - \bar{\beta}_\infty^{1/2}\sqrt{g^TV^{-1}g}$ for $j = 1,\dots,n^*$, then $S_{\hat{N}}$ in (5.41) is given by

$$\begin{aligned} S_{\hat{N}} = \{z_N : g^T\Phi^i z_N \le h - \left[\bar{\beta}_{N+i}(q)\right]^{1/2}\sqrt{g^TV^{-1}g}, \quad i = 1,\dots,\hat{N} \\ g^T\Phi^{\hat{N}+j}z_N \le h - \bar{\beta}_\infty^{1/2}\sqrt{g^TV^{-1}g}, \quad j = 1,\dots,n^* \ \}. \end{aligned} \tag{5.42}$$

Remark 5.11. The scalings of the tube cross-sections in (5.41) are computed on the basis of (5.10); this is done for convenience, given the simple form of (5.10). However tighter tubes would be obtained if (5.9) were used in place of (5.10) to generate the robust bounds $\bar{\beta}_{i|0}(1)$ of (5.33) and the probabilistic bounds $\bar{\beta}_i(q)$ of (5.35). The implied computation is practicable even though (5.9) involves an implicit optimization because of the discretization of the range for β in (5.18).

5.4.3 Stochastic MPC Algorithm

The offline computation of the tube scalings requires knowledge of V and λ. A possible choice is given by the pair V, λ that minimizes the steady state effect on the constraints in (5.36). This effect is revealed by the steady state value of $\left[\bar{\beta}_i(q)\right]^{1/2}\sqrt{g^TV^{-1}g}$, which by (5.37) is $\left[\bar{\alpha}/(1-\lambda)\right]^{1/2}\sqrt{g^TV^{-1}g}$, and this suggests defining V, λ via the following optimization

$$\begin{aligned} &\arg\min_{V^{-1},\lambda} \ \frac{1}{(1-\lambda)}\ g^TV^{-1}g \\ &\text{subject to} \quad V^{-1} - \frac{1}{\lambda}\Phi V^{-1}\Phi^T - B_wW^{-1}B_w^T \ge 0. \end{aligned} \tag{5.43}$$

For a fixed value of $\lambda > 0$, (5.43) is a semidefinite program (SDP) in the variable V^{-1}, and since λ is univariate and restricted to the interval $(0, 1)$, (5.43) can be solved by solving successive SDPs for V^{-1} with alternate iterations of a univariate optimization (such as bisection) for λ.

Given the definition of the cost (5.31), the constraints (5.36), and the terminal set (5.41), the MPC algorithm can be stated as follows.

Algorithm 5.1 (Stochastic MPC)**.**

Offline: Compute V, λ via (5.43). Determine Π in (5.19), and hence for $i = 1, \ldots, N + \hat{N}$, compute the scalings $\bar{\beta}_i(q)$ in (5.35). Compute the smallest value of n^* such that $S_{\hat{N}}$ in (5.41) can be expressed in terms of a finite number of inequalities in (5.42).

Online: At each time step $k = 0, 1, \ldots$

1) solve the quadratic programming (QP) optimization

$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} J_k \quad \text{subject to (5.36) for } i = 1, \ldots, N, \text{ and } z_{k+N|k} \in S_{\hat{N}} \tag{5.44}$$

2) implement $u_k = Kx_k + c_k^*$ (c_k^* is the first element of $\mathbf{c}_k^*$).

Theorem 5.12. Given feasibility at $k = 0$, Algorithm 5.1 is feasible for all $k \geq 1$. The closed loop system satisfies the probabilistic constraint (5.3) and the quadratic stability criterion

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{E}_0(L_k) \leq L_{ss}. \tag{5.45}$$

Proof. The recurrent feasibility guarantee follows from Theorem 5.8, which implies that the "tail": $\mathbf{c}_{k+1|k} = M\mathbf{c}_{k|k} = \{c_{k+1}, \ldots, c_{k+N-1}, 0\}$ of the minimizing sequence: $\mathbf{c}_k^* = \{c_k, \ldots, c_{k+N-1}\}$ of (5.44) at time k is feasible for (5.44) at time $k+1$, and so is the $\mathbf{c}_{k+i|k} = M^i\mathbf{c}_{k|k}$ for $i \geq 1$. The feasibility of the tail also implies that the optimal cost satisfies the bound:

$$J_k^* - \mathbb{E}_k(J_{k+1}^*) \geq L_k - L_{ss}, \tag{5.46}$$

Summing up (5.46) from k to ∞ implies (5.45). ∎

5.5 Illustrative Examples

5.5.1 Single-input-single-out Systems

Example 1. Consider a second order system with model and constraint

$$A = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w = I, \quad W = I$$
$$\Pr(g^T x_k \leq h) \geq p, \quad g = [1 \ \ 0.1364]^T, \quad h = 0.5, \quad p = 0.8 \tag{5.47}$$

Each element of the disturbance w_k is truncated[5] from a normal distribution with zero mean and variance $\sigma^2 = 1/144$ and the bounds are $-0.2236 \leq (w_k)_{1,2} \leq 0.2236$. The distribution of α is then a modified χ-squared distribution with $\bar{\alpha} = 0.1$, as shown in Figure 5.2.

The cost weights are $Q = I$, $R = 1$, and K in (5.5) is chosen as the LQG optimal feedback for the unconstrained case: $K = [1.0410 \ \ 1.0474]$. The offline optimization of V, λ in (5.43) gives

$$\lambda = 0.4220, \quad V = \begin{bmatrix} 0.7863 & 0.1497 \\ 0.1497 & 0.0724 \end{bmatrix}$$

The robust tube has asymptotic scaling $\bar{\beta}_\infty = 0.1730$, and the interval $[0, \ 0.1730]$ is divided into 1000 equal subintervals to define $\widetilde{\beta}_j$ in (5.18) and hence compute the matrix of transition probabilities, Π. From (5.42), horizons $N = 6$, $\hat{N} = 5$, give $n^* = 2$ and the scalings of the recursively feasible stochastic tubes $\bar{\beta}_i(q)$ ($q = 2p - 1 = 0.6$) are

$$\{\bar{\beta}_i(q)\} = \{1.28, \ 5.49, \ 7.27, \ 8.02, \ 8.34, \ 8.48, 8.53, \ 8.56, \ 8.56, \ 8.57, \ldots, \ 8.57\} \times 10^{-2}.$$

[5] Although the elements of w_k are independent in this example, they are allowed to be correlated in the formulation of Algorithm 5.1.

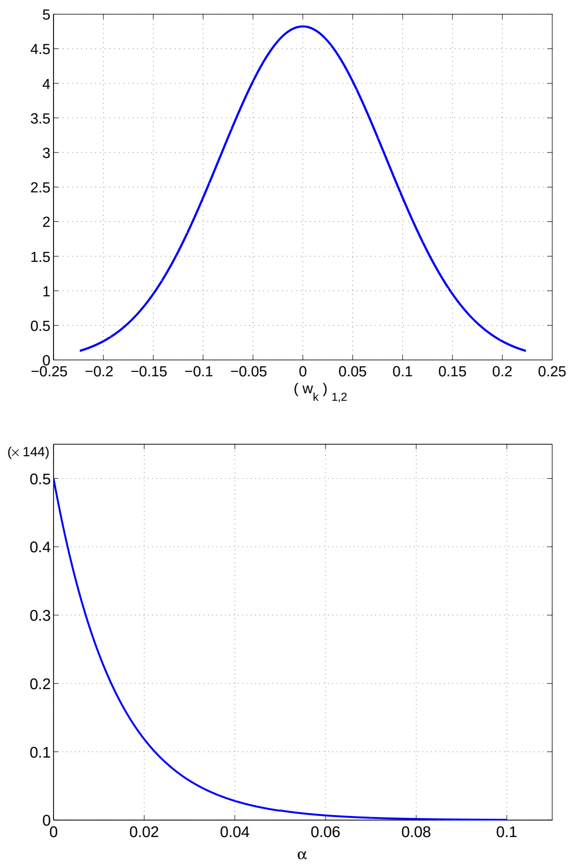

FIGURE 5.2: Probability density function of w_k and α

Figure 5.3 shows two trajectory bands with the same initial condition, $[-5\ 60]^T$, resulting from 200 realizations of the disturbance sequence: band 1 is the stochastic MPC law of Algorithm 5.1; band 2 is unconstrained LQ-optimal control law. The ellipsoids on band 1 show the tube cross-sections with scalings $\bar{\beta}_1(q)$ and $\bar{\beta}_1(1)$ corresponding to the recurrently feasible stochastic tube with $q = 0.6$ (inner ellipsoid) and the robust tube (outer ellipsoid). Algorithm 5.1 departs from the unconstrained optimal in order to meet constraints with probability greater than 0.8, while allowing some constraint violations (5.9% at $k = 1$). As expected, LQ-optimal control achieves a lower value of average cost:

$$\sum_{k=0}\Big(\mathbb{E}_0(L_k) - L_{ss}\Big) = \begin{cases} 7657 & \text{(LQG)} \\ 7728 & \text{(Algorithm 5.1)} \end{cases}$$

but achieves this at the expense of violating the probabilistic constraint (LQ-optimal leads to 100% violations at $k = 1$).

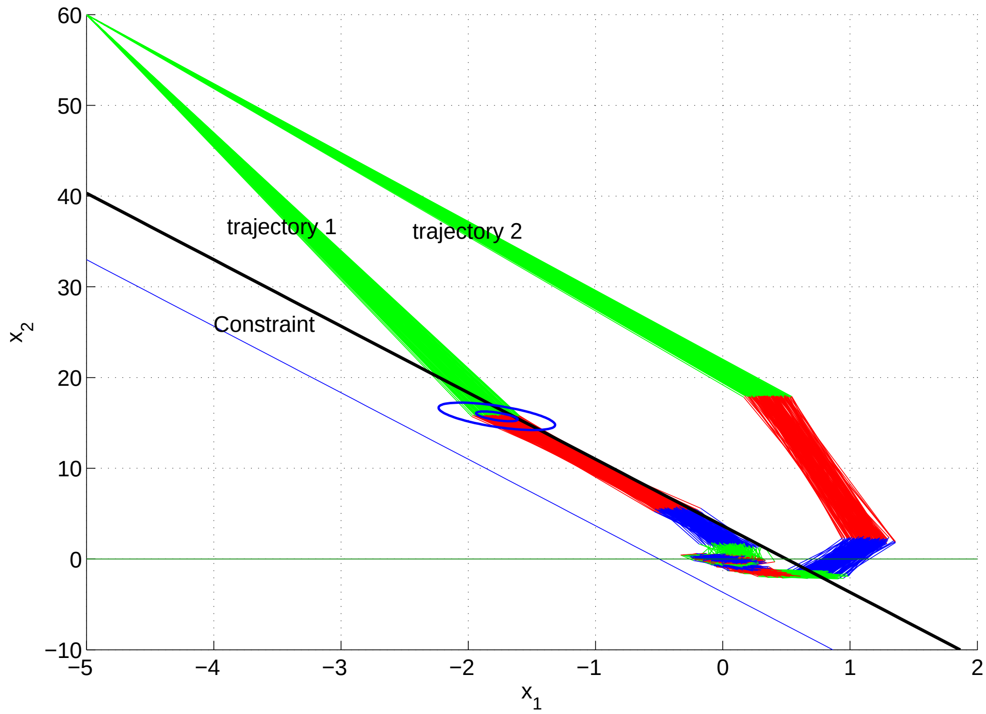

FIGURE 5.3: Closed-loop response of the system (5.47) under Algorithm 5.1 (trajectory 1) and unconstrained LQ-optimal control (trajectory 2). The ellipsoidal tube cross-sections corresponding to $q = 1$ and $q = 0.6$ predicted at time $k = 0$ are also shown.

Example 2. This example is based on the 2nd order system with model data

$$A = \begin{bmatrix} 1.8 & 0.5 \\ 1.5 & 1.2 \end{bmatrix}, \quad B_u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad B_w = I, \quad W = I$$
$$\Pr(g^T x_k \le h) \ge p, \quad g = [-1.75 \ 1]^T, \quad h = 1.5, \quad p = 0.6 \tag{5.48}$$

The cost weights are $Q = I$, $R = 5$, and K in (5.5) is chosen as the LQ-optimal feedback for the unconstrained case: $K = [1.5184 \quad -0.6447]$. The disturbance model is as defined in Example 1.

Closed-loop trajectories are shown in Figure 5.4. The two ellipsoids of the left plot at $k = 1$ denote stochastic tube with $q = 2p - 1 = 0.2$ and the robust tube, while the ellipse of the right plot represents the robust tube. The constraint violation rate is 100% (for the unconstrained LQ-optimal control), 17% (for Algorithm 5.1) and 0 (for robust MPC) at $k = 1$ respectively (all evaluated using 200 disturbance realizations). However, the average closed loop costs are compared as follows: 303 for LQG, 328 for Algorithm 5.1 with $p = 0.6$ in (5.3) and 357 for Algorithm 5.1 with $p = 1$ (robust MPC). For this example, stochastic MPC gives a cost improvement of around 8% over the cost of robust MPC.

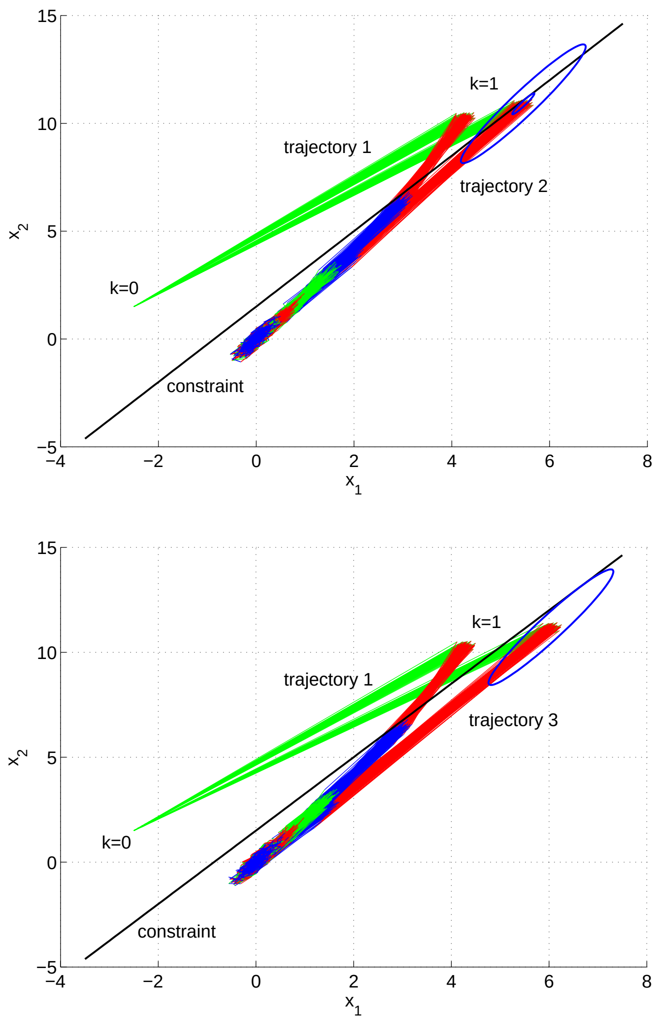

FIGURE 5.4: Closed-loop response under unconstrained LQ-optimal control (trajectory 1), Algorithm 5.1 (trajectory 2) and robust MPC (trajectory 3) for the same initial state $[-2.5\ 1.5]^T$

5.5.2 Multi-input-multi-out Case

Example 3. This is a fifth order system with two inputs and two outputs:

$$A = \begin{bmatrix} 0.213 & -0.120 & 0.321 & -0.099 & -0.426 \\ -0.308 & 0.407 & -0.045 & 0.002 & -0.234 \\ -0.253 & -0.333 & 0.343 & 0.070 & 0.223 \\ -0.170 & -0.160 & -0.315 & 0.084 & 0.160 \\ 0.347 & 0.029 & -0.158 & -0.406 & 0.210 \end{bmatrix}, \quad B_u = \begin{bmatrix} -2.171 & 1.692 \\ -0.059 & 0 \\ -1.011 & 0 \\ 0.614 & 0.380 \\ 0.508 & -1.009 \end{bmatrix},$$

$$y_k = Cx_k, \text{ with } C = \begin{bmatrix} -0.020 & 0 & 0 & 0.428 & 0.731 \\ 0 & 0 & -1.874 & 0 & 0.578 \end{bmatrix},$$

$$B_w = I, \; W = I, \quad \Pr(y_{k,1} \le h) \ge p, \quad h = 0.8, \quad p = 0.75. \tag{5.49}$$

Here $y_{k,1}$ means the first element of output and the cost weights are $Q = C^T C$ and $R = I$. Each element of the disturbance w_k is truncated from a normal distribution with zero mean and variance $\sigma^2 = 1/144$ and the bounds are $-0.1732 \le (w_k)_j \le 0.1732$ $(j = 1, \ldots, 5)$. The distribution of α is then a modified χ-squared distribution with $\bar{\alpha} = 0.15$, as shown in Figure 5.5.

K in (5.5) is chosen as the LQ-optimal feedback for the unconstrained case:

$$K = \begin{bmatrix} -0.2178 & -0.2269 & 0.2634 & 0.1199 & 0.1002 \\ 0.0086 & -0.1764 & -0.0118 & -0.0549 & 0.2467 \end{bmatrix},$$

and the offline optimization of λ in (5.43) gives $\lambda = 0.6690$. The robust tube has asymptotic scaling $\bar{\beta}_\infty = 0.4532$, and the interval $[0,\ 0.4532]$ is divided into 1000 equal subintervals to define $\widetilde{\beta}_j$ in (5.18) and hence compute the matrix of transition probabilities, Π. From (5.42), horizons $N = 9$, $\hat{N} = 8$, give $n^* = 6$ and the scalings of the recursively feasible stochastic tubes $\bar{\beta}_i(q)$ $(q = 2p - 1 = 0.5)$ are

$$\{\bar{\beta}_i(q)\} = \{0.304,\ 1.304,\ 1.977,\ 2.426,\ 2.726,\ 2.926,\ 3.063,\ 3.154,\ 3.211,\ 3.254,\ 3.281,\ 3.296,\ 3.308,\ 3.317,\ 3.323,\ 3.326,\ 3.329,\ 3.329,\ 3.332\} \times 10^{-1}.$$

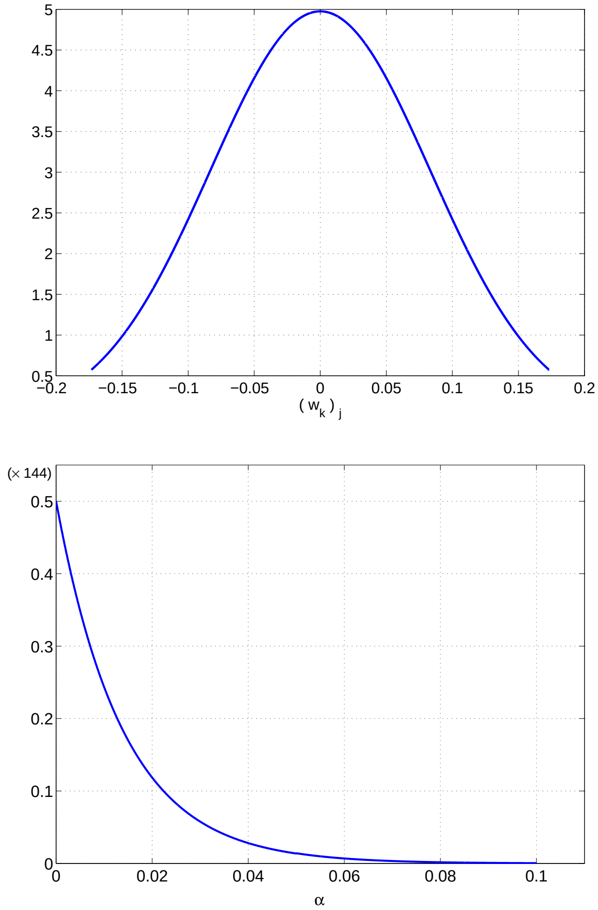

FIGURE 5.5: Probability density function of w_k and α for a 5th order system

The closed-loop trajectories of y_1 with Monte-Carlo simulations (for 200 realizations) are depicted in Figure 5.7, from a same initial state $x_0 = [-1\ 0\ 0\ -3.8\ 3.2]^T$. For the unconstrained LQ-optimal control law, band 3 suggests that the violation rate is 100% at time $k = 1$. And there are as expected no violations for robust MPC (band 2)[6], while under Algorithm 5.1 with $p = 0.75$ it is shown in band 1 that there is no violation as well at $k = 1$. However, it is noted that trajectories

[6]Robust MPC (band 2) is implemented under Algorithm 5.1 with $p = 1$ in (5.3).

under stochastic MPC with $p = 0.75$ are closer to the constraint line than those under robust MPC. The average cumulative closed-loop costs (for 18 transient steps) are:

$$\sum_{k=0}^{18}\left(\mathbb{E}_0(L_k) - L_{ss}\right) = \begin{cases} 6.65 & \text{(LQG)} \\ 7.67 & \text{(Algorithm 5.1)} \\ 9.17 & \text{(Robust MPC)} \end{cases} \tag{5.50}$$

The LQ-optimal control leads to the lowest cost at the expense of violating the constraint with probability 100% and the robust MPC results in the highest cost with no violations. The stochastic MPC guarantees the satisfaction of the probabilistic constraint with the cost 19.6% smaller than that of the robust MPC. The stochastic tube scalings ($\left[\bar{\beta}_{N+i}(q)\right]^{1/2}\sqrt{g^T V^{-1} g}$ in (5.36) with $q = 0.5$) and robust tube scalings (with $q = 1$) that are used to tighten the probabilistic constraints are compared in Figure 5.6. Stochastic tube scalings (solid line) at each predicted time are smaller than robust tube scalings (dashed line), thus the constraints (5.36) in stochastic MPC are less stringent than in robust MPC. That is the reason why stochastic MPC can achieve a better performance (smaller cost).

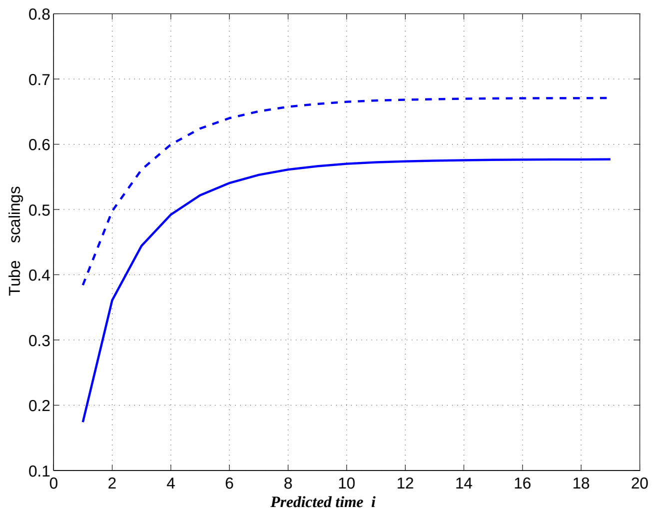

FIGURE 5.6: Stochastic and robust tube scalings $\left[\bar{\beta}_{N+i}(q)\right]^{1/2}\sqrt{g^T V^{-1} g}$

5.6 Discussion and Conclusion

The proposed stochastic MPC strategy handles probabilistic constraints with an online computational load similar to that of nominal MPC. This is achieved by fixing the shape of the cross-section of a tube containing predicted trajectories, but allowing the centres and scalings of the cross-sections to vary. Since the predicted tube scalings are univariate, their probability distributions can be computed efficiently offline. The online optimization is performed on the nominal disturbance-free model with tightened constraints. The resulting MPC law has guarantees of recursive feasibility and convergence in closed loop operation.

From the simulation results, it is noted the violation rate p_v under the proposed MPC strategy is much smaller than that is allowed (denoted by p_m): in Example 1, $p_v = 5.9\%$, $p_m = 20\%$; in Example 2, $p_v = 17\%$, $p_m = 40\%$ and in Example 3, $p_v = 0$, $p_m = 25\%$. This leads to the possibility of tightening the constraints further and obtaining a better performance. The degree of conservatism is brought in here due to the use of *fixed* shape of tubes (ellipsoids). Under the same controller structure (5.5), another stochastic MPC strategy with a less conservative result is presented in the following chapter.

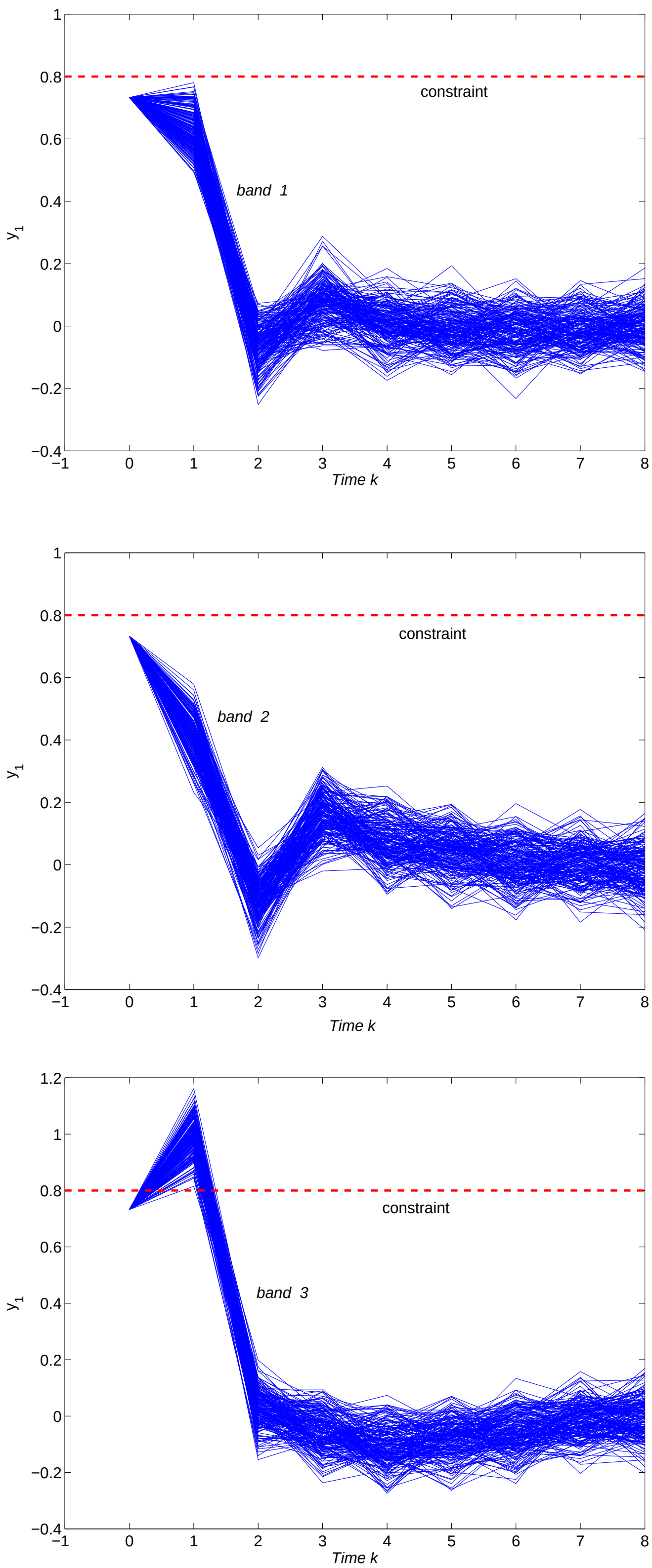

FIGURE 5.7: Closed-loop trajectories under Algorithm 5.1 (band 1), robust MPC (band 2) and unconstrained LQ-optimal control (band 3)

Chapter 6

Explicit Use of Probabilistic Distributions in Linear MPC

[1]The guarantee of feasibility given feasibility at initial time is an issue that has been overlooked by many of the recent research on stochastic model predictive control. Effective solutions have recently been proposed [Cannon et al., 2009a, 2010], but these carry considerable online computational load and/or a degree of conservativism. For the case that the elements of the random additive disturbance vector are independent, the current study in this chapter ensures that probabilistic constraints are met and that a quadratic stability condition is satisfied. This study is also extended to the cases when not all states are measurable and observers are needed. Numerical examples illustrate the efficacy of the proposed algorithm, which achieves tight satisfaction of constraints and thereby attains near-optimal performance.

[1]The work in this chapter has been published in Kouvaritakis et al. [2010b] and Cannon et al. [2012].

6.1 Introduction

Stochastic model predictive control (SMPC) is emerging as a research area of both practical and theoretical interest. The former derives from the fact that most real life applications are subject to stochastic uncertainty and have to obey constraints which may be probabilistic in nature (allowing violations to occur with a frequency below a given threshold). The latter stems from the fact that predictive control solves an open loop optimization problem but then implements the result in a receding horizon fashion, so guarantees of stability and feasibility of the open loop optimal response do not extend to closed loop behaviour; these control theoretic issues were not explored by the early contributions in the field (e.g. van Hessem and Bosgra [2002], Yan and Bitmead [2005], Oldewurtel et al. [2008]).

As described in Chapter 2, recent work [Cannon et al., 2009a,c] considered systems which are subject to additive stochastic disturbances, $w = Dq$, where the elements of the vector q are taken to be independently and (temporally) identically distributed (i.i.d.) with finite support. The approach of Cannon et al. [2009a,c] deployed various layers of probabilistic tubes with variable polytopic cross-sections and used these in conjunction with constraints controlling the probability of: (i) one-step-ahead transitions from one tube layer to another; (ii) one-step-ahead constraint violations given that the predicted state lies in a particular layer at any given prediction time. However, the use of variable polytopic cross-sections together with the use of many tube layers results in a large number of degrees of freedom and a large number of inequalities, which make the online implementation impractical for any situation other than for systems with low dimension and short prediction horizons.

To overcome this difficulty, the research in last chapter replaced the probabilistic assumptions on the additive disturbance w with a finite i.i.d. assumption on a quadratic function of w (see (5.2)), and was thus able to construct (offline) probabilistic tubes for the state which guaranteed satisfaction of probabilistic constraints and also ensured recursive feasibility, thus enabling the assertion of feasibility over all times, given feasibility at initial time. Ellipsoidal cross-section is employed for the tubes with variable centres and scalings, but their computation offline implies a significant reduction in the online computational burden. On account of this gain in efficiency, it is possible to use many layered tubes (say 1000) and therefore reduce considerably the degree of conservativeness. However, the ellipsoidal nature of the tube cross-sections brings the consequence that some degree of conservatism may persist. It is seen from the simulation results given in the last chapter, in terms of the violation probability and the state trajectories, that the tube approach does not achieve tight bound and that further improvements should be possible.

The aim of this chapter is to circumvent the above difficulty while retaining the computational convenience of Chapter 5 and this is achieved by reverting to the assumption on the disturbance w of Cannon et al. [2009a,c] but retaining some of the features of Chapter 5, namely the definition and treatment of recursive feasibility. However, unlike Chapter 5 where the constraint conditions are derived through projections of the state tubes, the tubes of this chapter are constructed directly on the probabilistically constrained variables. This is achieved by making explicit use of the distributions defining the additive disturbance w. The approach is based on a semi closed-loop formulation in which the degrees of freedom are perturbations on a given feedback law, but otherwise both the online optimization and the offline computation of the constraints are in essence open loop. With this understanding, the results of the chapter are claimed to be as tight as possible, given the assumed class of additive uncertainty. This is illustrated by two numerical examples, which achieve near-optimal performance by allowing the probability of constraint violation to reach the specified limit. Further improvements may be possible with optimal closed loop strategies, but these require significantly higher computational complexity, and are not considered here.

The above approach is based on state feedback, which assumes that the states are measurable. In practice this is often not the case, and it is then necessary to estimate the state via an observer. The introduction of state estimation into Robust MPC is well understood (see e.g. Bemporad and Garulli [2000], Lee and Kouvaritakis [2001], Chisci and Zappa [2002], Wan and Kothare [2002], Lvaas et al. [2008], Mayne et al. [2009]) and uses lifting to describe the combined system and observer dynamics. This chapter extends these ideas to include probabilistic information both on measurement noise and the initial plant state. Extending the developed tube method, which employs the explicit distribution of bounded additive stochastic disturbances, this chapter also examines how the effects of bounded measurement noise and uncertain state estimates propagate along predicted input/state trajectories. This analysis provides necessary and sufficient conditions for the satisfaction of probabilistic constraints over an infinite prediction horizon as well as conditions that guarantee the existence of feasible solutions at all times, given at least one feasible solution at initial time. Thus, unlike earlier work (e.g. Yan and Bitmead [2005]) which did not consider feasibility and stability, the algorithm here guarantees recursive feasibility with respect to both hard and probabilistic constraints and ensures stability and convergence of the plant state in a mean square sense. A numerical example is provided which verifies that the rate of constraint violation can be as tight as the specified probability allows. Throughout this study, all stochastic uncertainty is assumed to have bounded support. Not only is this necessary for asserting feasibility and stability, but it matches the real world more closely than the mathematically convenient but practically unrealistic Gaussian assumption, since noise and disturbances derived from physical processes are finite.

This chapter is organised as follows. Section 6.2~6.5 describes the newly developed stochastic tube method and Section 6.6 gives one example to illustrate the efficacy of this approach. Section 6.7 investigates its application into the cases when states are not all measurable and observers are introduced. Section 6.8 compares this tube method with previous research on stochastic MPC and draws appropriate conclusions.

6.2 Problem Formulation

Consider the uncertain linear system with model:

$$x_{k+1} = A_k x_k + B u_k + v_k\,, \qquad v_k = D w_k \tag{6.1}$$

where $u_k \in \mathbb{R}^l$, $w_k \in \mathbb{R}^{n_w}$, and the state $x_k \in \mathbb{R}^n$ is known at time k. It is assumed that the uncertain sequence $\{w_0, w_1, \ldots\}$ is independent and identically distributed, that the elements of w_k have zero-mean and are independent, and that the distribution of the ith element of w_k has the form:

$$\Pr\{w_{k,i} \le \xi_i\} = F_i(\xi_i), \quad F_i(\xi) = \begin{cases} 1 & \text{for } \xi \ge \alpha_i \\ 0 & \text{for } \xi < -\alpha_i \end{cases} \tag{6.2}$$

where $\alpha_i > 0$ and the distribution functions F_i are right-continuous with finitely many discontinuities. A consequence of (6.2) is that w_k is assumed to lie in the orthotope:

$$w_k \in \Pi = \{w : |w| \le \alpha\}, \quad \alpha = \begin{bmatrix} \alpha_1 & \cdots & \alpha_m \end{bmatrix}^T$$

(here and in the sequel the inequality sign $\le$ and absolute value $|\cdot|$ apply elementwise to vectors and matrices). The matrix D can be computed from the distribution of v; thus if $v \sim \mathcal{N}(0, \Sigma)$, where $\mathcal{N}$ denotes the normal distribution, then an obvious choice is $D = R\Lambda^{\frac{1}{2}}$, with R and Λ denoting the eigenvector and eigenvalue matrices of Σ. It is common practice to take F_i to be the distribution function for the scalar normal distribution $\mathcal{N}(0,1)$. However, in this scenario the distribution of w has infinite support, which is not considered to be reasonable from a practical point of view, and this provides the motivation for the more general setting of (6.2), which allows a more general (but simply supported) class of distributions.

The system (6.1) is assumed to be subject to the probabilistic constraint

$$\Pr\{g^T x_k \leq h\} \geq p\,. \tag{6.3}$$

The case of several probabilistic constraints, each of which is to be satisfied with a given probability, can be treated similarly by generating the necessary online conditions for each of the constraints. This framework also includes hard constraints, which are obtained if $p = 1$. Note that although (6.3) involves x_k and not u_k, the predicted future states are functions of both the current state and predicted inputs, so the approach applies to general linear input/state constraints.

Like (5.6) and (5.7), using the semi closed-loop paradigm [Kouvaritakis et al., 2000], the state and input trajectories predicted at time k are expressed for $i = 0, 1, \ldots$ as

$$x_{k+i} = z_{k+i} + e_{k+i} \tag{6.4a}$$

$$u_{k+i} = Kx_{k+i} + c_{k+i} \tag{6.4b}$$

$$z_{k+i+1} = \Phi z_{k+i} + Bc_{k+i} \tag{6.4c}$$

$$e_{k+i+1} = \Phi e_{k+i} + Dw_{k+i} \tag{6.4d}$$

where $c_{k+i} \in \mathbb{R}^l$, $i = 0, \ldots, N-1$ are optimization variables, $c_{k+i} = 0$ for $i \geq N$, for some finite prediction horizon N, and $\Phi = A + BK$ is assumed to be strictly stable. The dynamics for the nominal state z_k and the error e_k are separated from each other, which makes it convenient to study the 'centre' evolution[2] and the scalings evolution of the tube separately.

The problem is to devise a receding horizon MPC strategy that minimizes the cost

$$J = \sum_{k=0}^{\infty} \mathbb{E}\left(x_k^T Q x_k + u_k^T R u_k\right) \tag{6.5}$$

(where $\mathbb{E}$ denotes expectation) and guarantees that the closed loop system is stable, while its state converges to a neighbourhood of the origin, subject to the constraint (6.3).

6.3 Predictions and Probabilistic Tubes

A first step towards guaranteeing that the constraint (6.3) is met in closed-loop operation is to ensure that the constraint is satisfied by the predicted dynamics at all time instants k. Necessary and sufficient conditions for this are given below.

Theorem 6.1. At time k, and for a given prediction horizon N and vector of perturbations $\mathbf{c}_k^T = \left[c_k^T \; c_{k+1}^T \cdots c_{k+N-1}^T\right]$, the predictions of (6.1) satisfy the probabilistic constraint of (6.3) if and only if:

$$g^T H_i \mathbf{c}_k + g^T \Phi^i z_k \le h - \gamma_i, \quad i = 1, 2, \ldots \tag{6.6}$$

where $H_i = \left[\Phi^{i-1}B \; \cdots \; B \quad 0 \; \cdots \; 0\right]$ and γ_i for each $i = 1, 2, \ldots$ is defined as the minimum value such that

$$\Pr\{g^T(\Phi^{i-1}Dw_k + \ldots + Dw_{k+i-1}) \le \gamma_i\} = p. \tag{6.7}$$

[2]If the tube is assumed to have a symmetric shape like ellipsoids in Chapter 5 and the disturbances are symmetric about the original point $\{0\}$, the nominal state z_k can be regarded as the 'centre' of the tube, otherwise strictly speaking, it can be seen as the reference point which is used to determine the position of the tube.

Proof. The predictions of (6.4a-d) with $e_k = 0$ give

$$z_{k+i|k} = \Phi^i z_k + H_i \mathbf{c}_k \tag{6.8a}$$

$$e_{k+i|k} = \Phi^{i-1} D w_k + \ldots + D w_{k+i-1} \tag{6.8b}$$

$$x_{k+i|k} = z_{k+i|k} + e_{k+i|k} \tag{6.8c}$$

Therefore (6.6) implies $\Pr\{g^T x_{k+i} \le h\} \ge p$, $i = 1, 2, \ldots$, and since γ_i is the minimum value that satisfies (6.7), it follows that (6.6) is equivalent to the constraint $\Pr\{g^T x_{k+i} \le h\} \ge p$, invoked for $i = 1, 2, \ldots$. ∎

In general there may be more than one soft or hard constraint, in which case each leads to a condition such as (6.6). In their totality, these conditions define a region in state space which may be unbounded, and which in essence constitutes the cross-sections of a constraint tube in which the predictions x_{k+i} must lie in order to satisfy the constraints. For this reason we use (6.6) to define the state probabilistic tubes.

Bounds on γ_i in (6.7) are implied by the following result.

Corollary 6.2. For each i, γ_i satisfies the bound

$$\gamma_i \le \kappa \sqrt{g^T P_i g}, \quad \kappa^2 = p/(1-p) \tag{6.9}$$

where $P_1 = D\mathbb{E}(ww^T)D^T$ and

$$P_{i+1} = \Phi P_i \Phi^T + D\mathbb{E}(ww^T)D^T, \; i = 1, 2, \ldots \tag{6.10}$$

Proof. The bound of (6.9) follows from a direct application of Chebyshev's one-sided inequality [Feller, 1968] since (6.4d) with $e_k = 0$ implies that $\mathbb{E}(g^T e_{k+i}) = 0$ and $\mathbb{E}[(g^T e_{k+i})^2] = g^T P_i g$. ∎

Since Chebychev's inequality applies to an arbitrary distribution, the inequality (6.9) is likely to give a conservative estimate of γ_i. However (6.9) is not used directly in the design of the MPC algorithm described in the sequel, which is based instead on numerically approximating the relevant probability distribution, as we now discuss.

Exact computation of γ_i involves the calculation of the distribution function of $g^T(\Phi^{i-1}Dw_k+\ldots+Dw_{k+i-1})$, which in turn requires the evaluation of a multivariate convolution integral, and which, in general, is not practicable (for systems of large dimensions and over a long prediction horizon). However, by discretizing the distributions of the elements of w, the values of γ_i may be approximated, at a reasonable computational cost, using univariate convolutions.

For this, a discrete convolution is proposed corresponding to rectangular integration, for which the global error of approximation (under the usual continuity assumptions) is of order $O(1/q)$. Computing the discrete density function of a sum of two variables, each of which has q discrete intervals, involves $O(q^2)$ multiplications. In addition, the computation of γ_i does not depend on the current state x_k, and can therefore be performed (through numerical integration) offline, and this permits the use of values for q that are large enough to make the approximation error as small as required.

6.4 Recursively Feasible Probabilistic Tubes

Consider the prediction of the state of (6.4d) at time $k+j+i$, made on the basis of information available at time $k+j$:

$$e_{k+j+i|k+j} = \Phi^i e_{k+j|k+j} + \Phi^{i-1}Dw_{k+j|k+j} + \ldots + Dw_{k+j+i-1|k+j}. \tag{6.11}$$

This provides the prediction of $g^T x_{k+j+i|k+j}$ based on information available at time k, under the assumption that the perturbations, c_{k+i}, in (6.4b) are given by the elements of $\mathbf{c}_k$ for $i = 1, \ldots, N-1$, and $c_{k+i} = 0$ for $i \geq N$:

$$\begin{aligned} g^T x_{k+j+i|k+j} &= g^T z_{k+j+i|k} + g^T e_{k+j+i|k+j} \\ &= g^T \Phi^{j+i} z_k + g^T H_{j+i} \mathbf{c}_k + g^T \Phi^i e_{k+j|k+j} + \psi_i, \end{aligned} \tag{6.12}$$

where $\psi_i = g^T \Phi^{i-1} D w_{k+j|k+j} + \ldots + g^T D w_{k+j+i-1|k+j}$.

The reason for expressing $e_{k+j+i|k+j}$ in (6.11) in terms of $e_{k+j|k+j}$ rather than e_k (with $e_k = 0$ as implied by (6.8b)) is that $e_{k+j|k+j}$ depends on $w_k, w_{k+1}, \ldots, w_{k+j-1}$, which, given that $e_{k+j+i|k+j}$ is conditioned on information available at time $k+j$, cannot be considered to be random variables. Instead it is necessary to consider the worst case value for $g^T \Phi^{j-1} e_{k+j|k+j}$ over the class of allowable additive uncertainty. As a result of this, the usual feasibility properties do not apply, namely that feasibility with respect to the constraints of Theorem 6.1 cannot be ensured recursively given feasibility of $\mathbf{c}_0$ at time $k = 0$, since the extension of $\mathbf{c}_0$ is not guaranteed to provide at least one feasible solution at each time $k > 0$. The reason for that and the conditions which enable recursive feasibility to be asserted are stated by the following theorem.

Theorem 6.3. Recursive feasibility is ensured if (6.6) is replaced by

$$g^T H_i \mathbf{c}_k + g^T \Phi^i z_k \leq h - \beta_i, \quad i = 1, 2, \ldots \tag{6.13}$$

where β_i is the maximum element of the ith column of the matrix

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots \\ 0 & \gamma_1 + a_1 & \gamma_2 + a_2 & \gamma_3 + a_3 & \cdots \\ \vdots & 0 & \gamma_1 + a_1 + a_2 & \gamma_2 + a_2 + a_3 & \cdots \\ \vdots & \vdots & 0 & \gamma_1 + a_1 + a_2 + \alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6.14}$$

and

$$a_i = \max_{w \in \Pi} g^T \Phi^i D w = |g^T \Phi^i D| \, \alpha \tag{6.15}$$

Proof. The predictions of (6.12) are derived assuming the future vectors of perturbations $\mathbf{c}_{k+j}$, $j > 0$, from the extensions of the current perturbation vector $\mathbf{c}_k$, namely that $\mathbf{c}_{k+j} = M^j \mathbf{c}_k$, where M is the shift matrix with the identity matrix on the superdiagonal and zeros elsewhere given in (2.52). A sufficient condition for recursive feasibility requires (i) $\mathbf{c}_k$ is feasible, (ii) all the extensions of $\mathbf{c}_k$ are feasible. The former is achieved by the conditions of Theorem 6.1, while the latter requires a combination of probabilistic bounds (on the 'future' term ψ_i at the 'current' time $k+j$ in (6.12)) and worst case bounds (on the 'known' term $g^T \Phi^i e_{k+j|k+j}$ at the 'current' time $k+j$ in (6.12)). The term $g^T \Phi^i e_{k+j|k+j}$ depends on the disturbances w_{k+l}, $l = 0, \ldots j-1$, which can take any value in the orthotope Π, and for this reason it is necessary to consider the worst case values of (6.15).

The first row of the matrix (6.14) gives the probabilistic bounds for the perturbation vector $\mathbf{c}_k$ at time k given information at k, the second does the same for the extension $M\mathbf{c}_k$ at time $k+1$ given the information at $k+1$, etc. It follows therefore that the conditions of (6.13) guarantee recursive feasibility if $\beta_1, \beta_2, \ldots$ are selected as stated in the theorem. ∎

Corollary 6.4. The sequence $\beta_1, \beta_2, \ldots$ is monotonically non-decreasing and converges to a limit $\bar{\beta}$ bounded by

$$\bar{\beta} \leq \gamma_1 + a_1 + \cdots + a_{\nu-1} + \frac{\rho^\nu}{1-\rho}\|g\|_S \tag{6.16}$$

for any nonnegative integer ν, where $\|g\|_S = \sqrt{g^T S g}$ and ρ, S satisfy the conditions

$$\max_{w\in\Pi} \|Dw\|_{S^{-1}} \leq 1, \quad S \succ 0 \tag{6.17a}$$

$$\Phi^T S^{-1} \Phi \leq \rho^2 S^{-1}, \quad \rho \in (0,1)\,. \tag{6.17b}$$

Proof. The definitions of γ_i in (6.7) and a_i in (6.15) imply that $\gamma_{i+1} \leq \gamma_i + a_i$. From (6.14) it therefore follows that β_i appears in the diagonal and $\beta_i \leq \beta_{i+1}$ for all i and

$$\lim_{i\to\infty} \beta_i = \bar{\beta} \leq \gamma_1 + \sum_{i=1}^{\infty} a_i\,. \tag{6.18}$$

Furthermore, with the understanding that conditions (6.17a,b) are feasible since Φ is strictly stable, we have from (6.17a)

$$\max_{w\in\Pi} g^T\Phi^j Dw \leq \max_{\|v\|_{S^{-1}}\leq 1} g^T\Phi^j v = \max_{\|v\|_{S^{-1}}\leq 1} g^T\Phi^j S^{1/2}\big(S^{-1/2}v\big) \leq \|\Phi^{j^T} g\|_S\,.$$

Therefore (6.17b) implies $\max_{w\in\Pi} g^T\Phi^j Dw \leq \rho^j\|g\|_S$, which leads to (6.16) since $\rho < 1$. ∎

Remark 6.5. The bound on $\bar{\beta}$ in (6.16) can be made arbitrarily close to $\bar{\beta}$ by choosing ν sufficiently large. Since a_i converges to zero as $i \to \infty$, in practice it is possible to choose a suitable value for ν by computing $\gamma_1 + \sum_{i=1}^{\infty} a_i$ for a range of ν and selecting ν such that: (a) this sum is close enough to the limit of $\bar{\beta}$; and (b) the last term in (6.16) is sufficiently small. Note that the computation of a_i is a linear programming problem and thus easily obtained. It is worthwhile mentioning that $\bar{\beta}$ is connected with the probabilistic bound γ_1, so more precisely it should be denoted as $\bar{\beta}(p)$. In the following we still use $\bar{\beta}$ for this would not give rise to any misunderstanding.

6.5 SMPC Algorithm

To ensure that constraints (6.13) are satisfied over an infinite prediction horizon, we make use of the dual mode prediction paradigm [Mayne et al., 2000] implied by the parameterization of (6.4b): mode 1 comprises the first N steps over which the control moves are free and are defined by (6.4b), and mode 2 comprises the following infinite prediction horizon in which the control moves are prescribed by the predetermined feedback control law $u_{k+i} = Kx_{k+i}$. Constraints are invoked explicitly in mode 1 and implicitly in mode 2 through a terminal constraint. To derive the terminal constraint, note that in mode 2 (6.13) can be written as

$$g^T \Phi^j z_N \leq h - \beta_{N+j} \quad j = 1, 2, \ldots \tag{6.19}$$

and, following Gilbert and Tan [1991], define the maximal admissible set, $\mathcal{S}_\infty$, as the set of all z_N such that (6.19) holds for all j if $z_N \in \mathcal{S}_\infty$.

Lemma 6.6. The maximal admissible set $\mathcal{S}_\infty$ is non-empty if and only if

$$h \geq \bar{\beta}\,. \tag{6.20}$$

Proof. (6.20) is implied by (6.19) in the limit as $j \to \infty$ (since Φ is strictly stable by assumption), whereas (6.20) implies that $\mathcal{S}_\infty$ is non-empty (at least contains the original point $\{0\}$) due to the monotonicity of $\beta_{N+1}, \beta_{N+2}, \ldots$. ∎

Because the exact value of $\bar{\beta}$ is not known, (6.20) may be replaced by the sufficient condition implied by the bound (6.16):

$$h \geq \gamma_1 + a_1 + \cdots + a_{\nu-1} + \frac{\rho^\nu}{1-\rho}\|g\|_S\,. \tag{6.21}$$

Correspondingly, replacing β_{N+j} by the bound in (6.16) beyond a horizon $j = \hat{N}$ of mode 2, defines a set $S_{\hat{N}} \subset \mathcal{S}_\infty$:

$$\begin{aligned} S_{\hat{N}} = \big\{ z_N : g^T \Phi^l z_N \le h - \beta_{N+l}, \quad l = 1, \dots, \hat{N} \\ g^T \Phi^l z_N \le h - (\gamma_1 + a_1 + \dots + a_{\nu-1}) - \frac{\rho^\nu}{1-\rho} \|g\|_S, \quad l > \hat{N} \big\} \end{aligned} \tag{6.22}$$

Thus $S_{\hat{N}}$ provides an inner approximation of $\mathcal{S}_\infty$ which can be used to define the MPC terminal constraint set. There appears to be little advantage in using the inner approximation in that the definition of $S_{\hat{N}}$ involves an infinite number of inequalities. However the arguments of Gilbert and Tan [1991] imply that $S_{\hat{N}}$ is in fact defined by only the first $\hat{N}+n^*$ inequalities of (6.22), provided n^* is sufficiently large, i.e.

$$\begin{aligned} S_{\hat{N}} = \big\{ z_N : g^T \Phi^l z_N \le h - \beta_{N+l}, \quad l = 1, \dots, \hat{N} \\ g^T \Phi^{\hat{N}+j} z_N \le h - (\gamma_1 + a_1 + \dots + a_{\nu-1}) - \frac{\rho^\nu}{1-\rho} \|g\|_S, \quad j = 1, 2, \dots, n^* \big\} \end{aligned} \tag{6.23}$$

The determination of the smallest allowable value for n^* involves the offline solution of a sequence of linear programs [Gilbert and Tan, 1991] to determine the smallest integer n^* such that $g^T \Phi^{\hat{N}+n^*+1} z_N \le h - (\gamma_1 + a_1 + \dots + a_{\nu-1}) - \frac{\rho^\nu}{1-\rho}\|g\|_S$ for all z_N satisfying $g^T \Phi^{\hat{N}+j} z_N \le h - (\gamma_1 + a_1 + \dots + a_{\nu-1}) - \frac{\rho^\nu}{1-\rho}\|g\|_S$ for $j = 1, \dots, n^*$.

Before describing the overall SMPC algorithm, it is noted that for the case of additive stochastic uncertainty considered here, the stage cost, $\mathbb{E}\big(x_k^T Q x_k + u_k^T R u_k\big)$, of (6.5) converges to a finite, non-zero limit, L_{ss} [Cannon et al., 2009b]. In the interest therefore of obtaining a finite cost it is necessary to remove this limiting value from the stage cost. It is then possible to show [Cannon et al., 2009b] that the modified cost:

$$\tilde{J}_k = \sum_{i=0}^{\infty} \mathbb{E}\big(x_k^T Q x_k + u_k^T R u_k - L_{ss}\big)$$

is quadratic in the degrees of freedom contained in $\mathbf{c}_k$ (see (2.101)).

Algorithm 6.1 (SMPC)**.**

Offline: Compute the parameters β_i, $i = 1, \ldots, N$ of (6.13) that appear in mode 1 constraints, and also the parameters β_{N+l}, $l = 1, \ldots, \hat{N}$ that appear in the terminal constraint at the beginning of mode 2. Also compute parameters $a_i, i = 0, \ldots, \nu - 1$ and n^* to complete the definition of the terminal constraint.

Online: At each time $k = 0, 1, \ldots$ solve the quadratic programming optimization:

$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \tilde{J}_k \quad \text{subject to (6.13) for } i = 1, \ldots, N, \text{ and (6.23)}$$

and set $u_k = Kx_k + c_k^*$, where c_k^* is the first element of $\mathbf{c}_k^*$.

Theorem 6.7. Given feasibility at time $k = 0$, Algorithm 6.1 remains feasible at all future time instants, and causes the closed loop system to satisfy the probabilistic constraint (6.3) and the quadratic stability condition:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{E}\Big(x_k^T Q x_k + u_k^T R u_k\Big) \le L_{ss}\,. \tag{6.24}$$

Proof. The recurrence of feasibility is a direct consequence of the definition of the constraints in the online optimization of Algorithm 6.1. By Theorem 6.3 and the definition of $S_{\hat{N}}$ in (6.23), if $\mathbf{c}_0^*$ exists at $k = 0$, then the extension $M^k \mathbf{c}_0^*$ is feasible at all times $k = 1, 2, \ldots$. The feasibility of these extensions also implies that the optimal cost satisfies the bound:

$$\tilde{J}_k^* - \mathbb{E}(\tilde{J}_{k+1}^*) \ge x_k^T Q x_k + u_k^T R u_k - L_{ss}\,. \tag{6.25}$$

Summing up the expectation form for both sides of (6.25) from $k = 0$ to $k = \infty$ gives

$$\mathbb{E}(\tilde{J}_k^* - \tilde{J}_\infty^*) \ge \mathbb{E}\Big(x_k^T Q x_k + u_k^T R u_k - L_{ss}\Big),$$

which leads to (6.24) taking $\mathbb{E}(\tilde{J}_\infty^*) = 0$ into account. ∎

6.6 Examples

6.6.1 Single-input-single-out Case

The system and constraint (the same as Example 1 in Section 5.5.1) are defined by

$$A = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; D = I,$$
$$\Pr(g^T x_k \le h) \ge p, \quad g = [1 \;\; 0.1364]^T, \quad h = 0.5, \quad p = 0.8. \tag{6.26}$$

The disturbance sequence w_k is zero mean, i.i.d., and each element $w_{k,i}$, for $i = 1, 2$, has a truncated Gaussian distribution with variance $\sigma^2 = 1/144$ such that $|w_{k,i}| \le 0.2236$ for $i = 1, 2$. The feedback gain in (6.4) is $K = [1.041 \;\; 1.047]$, which is optimal for the LQ problem of minimizing $\tilde{J}_k$ in the absence of constraints with $Q = I$, $R = 1$.

The mode 1 prediction horizon N and the mode 2 horizon $\hat{N}$ over which constraints are handled explicitly were chosen (by considering the transient response in the absence of constraints) as $N = 6$ and $\hat{N} = 7$. The values of γ_i were computed for $i = 1, \ldots, N + \hat{N}$ by using a discrete approximation of the distribution of w based on $q = 10^4$ equi-spaced points on the interval $[-0.2236, \; 0.2236]$. The approximation errors in γ_i for $p = 0.8$ when q is chosen to be 10^3 and 10^4 are insignificant (0.3% and 0.2% respectively). In fact the computed value of γ_i is sufficiently close to the exact value that it would hardly make any difference to the simulation results whether $q = 10^3$ or $q = 10^4$ is used. For the chosen N and $\hat{N}$ the set $\mathcal{S}_{\hat{N}}$ is given by (6.23) with $n^* = 2$.

The two sets of trajectories in Figure 6.1 show the responses of the closed loop system for 200 realizations of the random disturbance sequence and initial condition $x_0 = [-5 \;\; 60]^T$. One set of trajectories (labelled trajectory 2) is obtained using the unconstrained LQ optimal feedback law. For 2×10^4 realizations of the

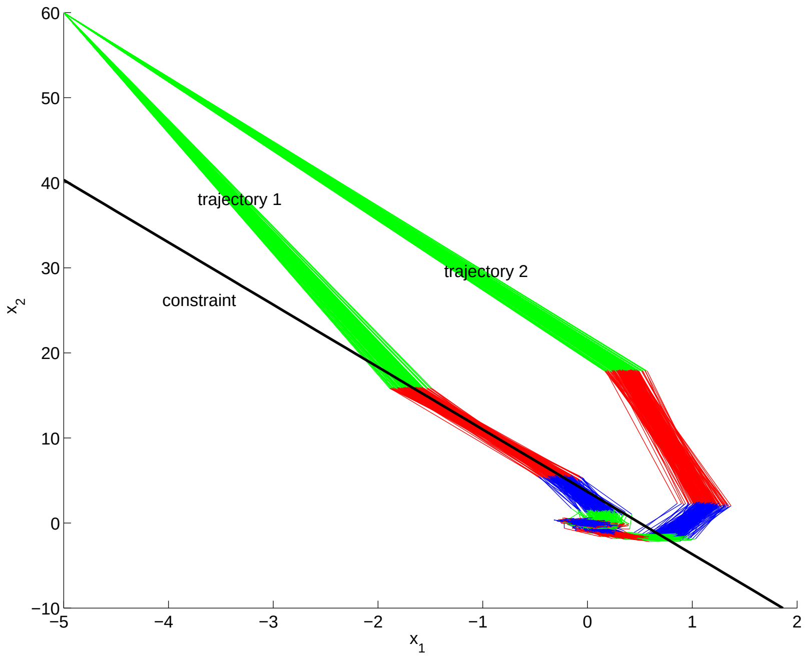

Figure 6.1: Closed loop responses of Algorithm 6.1 (labelled as trajectory 1) and unconstrained optimal control (trajectory 2) for 200 realizations of the uncertainty sequence $\{w_0, w_1, \ldots\}$.

disturbance sequence, this was found to violate the constraint at times $k = 1, 2$ for every realization of the disturbance sequence. The other set (labelled trajectory 1) is obtained using the control law of Algorithm 6.1, for which 19.84% and 19.88% of the same 2×10^4 disturbance realizations violate the constraint at time $k = 1, 2$. This is close to, but just within the probabilistic constraint, which requires that at least 80% of closed loop trajectories satisfy $g^T x_k \leq h$ at all times k. Note however that trials involving more disturbance realizations indicate that the rate of violations is asymptotic to the target of 20%. On the other hand, the performance cost, evaluated for the closed loop system over the duration of the transients (12 samples), is only 0.87% greater for Algorithm 6.1 than for the unconstrained LQ optimal control law.

6.6.2 Multi-input-multi-out Case

The system and constraint are the same as Example 3 in Section 5.5.2:

$$A = \begin{bmatrix} 0.213 & -0.120 & 0.321 & -0.099 & -0.426 \\ -0.308 & 0.407 & -0.045 & 0.002 & -0.234 \\ -0.253 & -0.333 & 0.343 & 0.070 & 0.223 \\ -0.170 & -0.160 & -0.315 & 0.084 & 0.160 \\ 0.347 & 0.029 & -0.158 & -0.406 & 0.210 \end{bmatrix}, \quad B = \begin{bmatrix} -2.171 & 1.692 \\ -0.059 & 0 \\ -1.011 & 0 \\ 0.614 & 0.380 \\ 0.508 & -1.009 \end{bmatrix},$$

$$y_k = Cx_k, \text{ with } C = \begin{bmatrix} -0.020 & 0 & 0 & 0.428 & 0.731 \\ 0 & 0 & -1.874 & 0 & 0.578 \end{bmatrix},$$

$$D = I, \quad \Pr(y_{k,1} \leq h) \geq p, \quad h = 0.8, \quad p = 0.75, \tag{6.27}$$

with the cost weights $Q = C^TC$ and $R = I$. Each element of the disturbance w_k is independent, truncated from a normal distribution with zero mean and variance $\sigma^2 = 1/144$ and the bounds are $-0.1732 \leq (w_k)_j \leq 0.1732$ $(j = 1, \ldots, 5)$.

The mode 1 prediction horizon N and the mode 2 horizon $\hat{N}$ over which constraints are handled explicitly were chosen (by considering the transient response in the absence of constraints) as $N = 9$ and $\hat{N} = 10$. The values of γ_i in (6.7) were computed for $i = 1, \ldots, N + \hat{N}$ by using a discrete approximation of the distribution of w based on $q = 10^4$ equi-spaced points on the interval $[-0.1732, 0.1732]$. For the chosen N and $\hat{N}$ the set $\mathcal{S}_{\hat{N}}$ is given by (6.23) with $n^* = 16$.

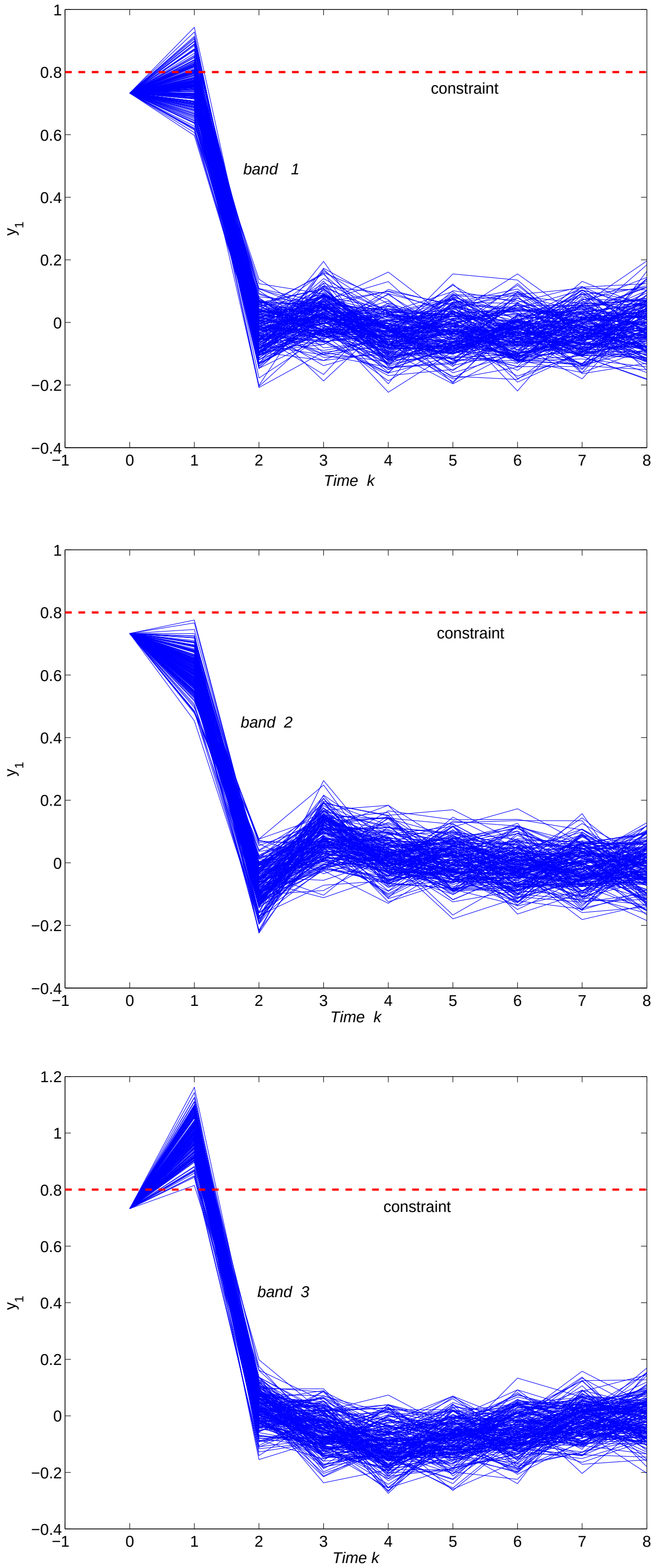

FIGURE 6.2: Closed-loop trajectories under Algorithm 6.1 (band 1), robust MPC (band 2) and unconstrained LQ-optimal control (band 3)

Closed-loop trajectories of y_1 with Monte-Carlo simulations (for 200 realizations) are depicted in Figure 6.2, from a same initial condition $x_0 = [-1\ 0\ 0\ \ -3.8\ 3.2]^T$, under the unconstrained LQ-optimal control law, Algorithm 6.1 with $p = 0.75$ and robust MPC. Note that:

The robust MPC strategies in Chapter 5 and this chapter in essence denote Algorithm 5.1 and Algorithm 6.1 respectively, with $p = 1$ in probabilistic constraints. They are different algorithms such that they lead to different system behaviour.

For the unconstrained LQ-optimal control law, band 3 suggests that the violation rate is 100% at time $k = 1$, and there are as expected no violations for robust MPC (band 2). Under Algorithm 6.1 with $p = 0.75$ it is shown in band 1 that there are 53 violations out of 200 realizations at $k = 1$, and the violation rate approaches 25% asymptotically when involving more disturbance realizations. The average cumulative closed-loop costs (for 18 transient steps) are:

$$\sum_{k=0}^{18}\left(\mathbb{E}_0(L_k) - L_{ss}\right) = \begin{cases} 6.65 & \text{(LQG)} \\ 7.10 & \text{(Algorithm 6.1)} \\ 7.74 & \text{(Robust MPC)} \end{cases} \tag{6.28}$$

The stochastic MPC produces a cost, which is 6.3% larger than the LQ-optimal control and 9.0% smaller than the corresponding robust MPC strategy. The stochastic tube scalings (β_i in (6.13) with $p = 0.75$) are compared with the robust tube scalings (with $p = 1$) in Figure 6.3. Stochastic tube scalings (solid line) at each predicted time are smaller than robust tube scalings (dash-dot line), thus the constraints (6.13) in stochastic MPC are less stringent than in robust MPC. Therefore, stochastic MPC can realize a better performance (smaller cost).

When comparing (6.28) with (5.50), one observation is that the cost under Algorithm 6.1 (7.10) is about 8.0% smaller than under Algorithm 5.1 (7.67). It implies that Algorithm 6.1 is less conservative than Algorithm 5.1. More details of this point will be analyzed in the last section of this chapter.

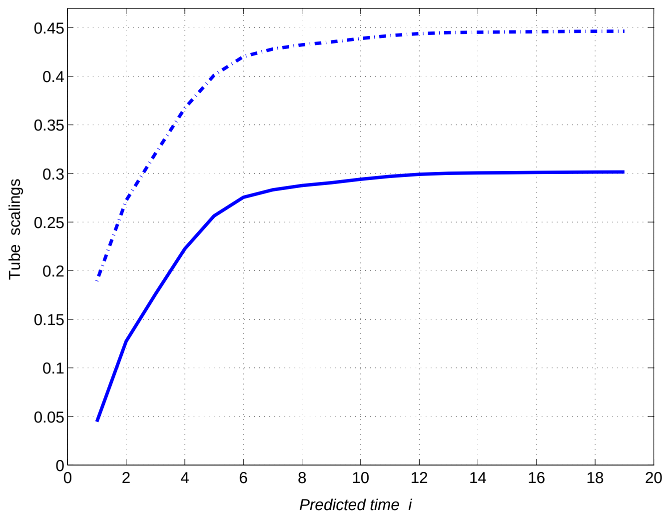

FIGURE 6.3: Stochastic and robust tube scalings β_i of (6.13)

6.7 Stochastic Tube MPC with State Estimation

6.7.1 Problem Formulation

Consider the same system as (6.1) with state $x_k \in \mathbb{R}^n$ not measurable, control input $u_k \in \mathbb{R}^{n_u}$ and measured output $y_k \in \mathbb{R}^{n_y}$:

$$x_{k+1} = Ax_k + Bu_k + Dw_k \tag{6.29a}$$

$$y_k = Cx_k + F\nu_k \tag{6.29b}$$

where the disturbance $w_k \in \mathbb{R}^{n_w}$ and measurement noise $\nu_k \in \mathbb{R}^{n_\nu}$ sequences are independent, identically distributed, with known, compactly supported distributions such that the elements of w_k and ν_k are zero-mean and independent. Let

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_{k+1} - C(A\hat{x}_k + Bu_k)) \tag{6.30a}$$

$$\hat{y}_k = C\hat{x}_k \tag{6.30b}$$

where $\hat{x}_k$ is the state estimate and $L \in \mathbb{R}^{n \times n_y}$ is chosen such that $A - LCA$ in (6.31) is strictly stable. The error dynamics are then[3]:

$$\varepsilon_{k+1} = (A - LCA)\varepsilon_k + (D - LCD)w_k - LF\nu_{k+1} \tag{6.31}$$

where $\varepsilon_k \triangleq x_k - \hat{x}_k$ is the estimation error; ε_0 is assumed to be a random variable with a known and compactly supported distribution, so that $\varepsilon_0 \in \Pi_0$ almost surely (a.s.) for some compact set $\Pi_0 \subset \mathbb{R}^n$. Let the state estimate and error sequences predicted at time k be $\{\hat{x}_{k+i|k}, \varepsilon_{k+i|k},\ i = 0, 1, \ldots\}$ and let $\xi_{k+i} = [\hat{x}_{k+i}^T\ \varepsilon_{k+i}^T]^T \in \mathbb{R}^{2n}$. Then using the closed loop paradigm [Kouvaritakis et al., 2000], the predictions evolve with

$$\xi_{k+i+1|k} = \Psi\xi_{k+i|k} + \tilde{B}c_{k+i|k} + \tilde{D}\delta_{k+i|k} \tag{6.32a}$$

$$u_{k+i|k} = K\hat{x}_{k+i|k} + c_{k+i|k} \tag{6.32b}$$

where: $\delta_{k+i|k} = [w_{k+i|k}^T \quad \nu_{k+i+1|k}^T]^T$ is the stochastic disturbance; $c_{k+i|k}$ for $i = 0, \ldots, N-1$ are optimization variables and $c_{k+i} = 0$ for $i \geq N$, for some chosen finite prediction horizon N; also Ψ (and hence also $A + BK$) is assumed to be strictly stable, with

$$\Psi = \begin{bmatrix} A + BK & LCA \\ 0 & A - LCA \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} LCD & LF \\ D - LCD & -LF \end{bmatrix}.$$

Given the assumptions on w_k and ν_k, the distribution of δ_k is compactly supported, so $\delta_k \in \Delta$ for all k almost surely where $\Delta \subset \mathbb{R}^{n_w + n_\nu}$.

[3]It is noted that (6.31) is base on a posterior state estimate, if this is not available, a prior estimation scheme may be exploited, i.e. replacing RHS of (6.31) by $\varepsilon_{k+1} = (A - LC)\varepsilon_k + Dw_k - LF\nu_k$.

The system is subject to probabilistic constraints:

$$\Pr\{\eta_x^T x_{k+1} + \eta_u^T u_k \le h\} \ge p, \quad k = 0, 1, \dots \tag{6.33}$$

for fixed scalar h, vectors η_x, η_u, and probability $p \in (0, 1]$. Applied to the predictions of (6.32), these are equivalent to

$$\Pr\Big\{g_0^T \xi_{k+i|k} + g_1^T \tilde{D}\delta_{k+i|k} + f^T c_{k+i|k} \le h\Big\} \ge p, \quad i = 0, 1, \dots \tag{6.34}$$

where $g_0^T = [\eta_x^T(A + BK) + \eta_u^T K^T \quad \eta_x^T A]$, $g_1^T = [\eta_x^T \quad \eta_x^T]$, and $f^T = \eta_x^T B + \eta_u^T$. The case of hard constraints is treated simply by setting $p = 1$ in (6.33) and (6.34). Like (6.4), we split $\xi_{k+i|k}$ into nominal ($z_{k+i|k}$) and uncertain ($e_{k+i|k}$):

$$\xi_{k+i|k} = z_{k+i|k} + e_{k+i|k} \tag{6.35a}$$

$$z_{k+i+1|k} = \Psi z_{k+i|k} + \tilde{B}c_{k+i|k} \tag{6.35b}$$

$$e_{k+i+1|k} = \Psi e_{k+i|k} + \tilde{D}\delta_{k+i|k} \tag{6.35c}$$

with initial conditions[4] pertaining to $i = 0$: $z_{k|k} = [\hat{x}_k^T \; 0]^T$, $e_{k|k} = [0 \; \varepsilon_k^T]^T$, $e_{k|k} \sim \mathfrak{D}_k$; and the distribution $\mathfrak{D}_k$ is calculated according to (6.31) given the distribution of the initial estimation error ε_0. This decomposition simplifies constraint handling because it separates the propagation of uncertainty in $e_{k+i|k}$ from the nominal state prediction, $z_{k+i|k}$.

Our problem is to devise a MPC strategy that minimizes, subject to (6.33) for all $k \ge 0$, the predicted cost:

$$J_k = \sum_{i=0}^{\infty} \mathbb{E}\left(x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}\right) \tag{6.36}$$

[4]Note that the initial conditions for predictions are updated using output measurements and observer states at each time k in a receding horizon way, while the predictions generated by (6.35c) do not retain the structure of the initial condition $e_{k|k}$ since the first block of $e_{k+i|k}$ contains a random variable for $i > 0$.

with $R \succ 0$, while ensuring that the closed loop system is stable and that x_k converges to a neighbourhood of the origin.

6.7.2 Recursively Feasible Probabilistic Tubes

To handle the constraints of (6.33), we first give necessary and sufficient conditions that guarantee that (6.34) is satisfied by predicted state and input sequences (like Theorem 6.1). In the sequel we let $\mathbf{c}_k = [c_{k|k}^T \ c_{k+1|k}^T \ \dots \ c_{k+N-1|k}^T]^T \in \mathbb{R}^{Nn_u}$.

Theorem 6.8. At time k, the constraints (6.34) are satisfied by predictions of (6.32) if and only if $\mathbf{c}_k$ satisfies

$$(g_0^T \tilde{H}_i + f^T E_i)\mathbf{c}_k + g_0^T \Psi^i z_{k|k} \leq h - \hat{\gamma}_{i|k}, \quad i = 0, 1, \dots \tag{6.37}$$

where $\tilde{H}_i = [\Psi^{i-1}\tilde{B} \cdots \tilde{B} \ 0 \cdots 0]$, $E_i \mathbf{c}_k = c_{k+i|k}$, and, for each k, $\hat{\gamma}_{i|k}$ is defined as the minimum value such that

$$\Pr\Big\{g_0^T e_{k|k} + g_1^T \tilde{D}\delta_{k|k} \leq \hat{\gamma}_{0|k}\Big\} = p, \quad \text{for } i = 0, \tag{6.38a}$$

$$\Pr\Big\{g_0^T(\Psi^i e_{k|k} + \Psi^{i-1}\tilde{D}\delta_{k|k} + \cdots + \tilde{D}\delta_{k+i-1|k}) + g_1^T \tilde{D}\delta_{k+i|k} \leq \hat{\gamma}_{i|k}\Big\} = p, \quad \text{for } i = 1, 2, \dots \tag{6.38b}$$

Proof. The predictions of (6.35) give

$$z_{k+i|k} = \Psi^i z_{k|k} + \tilde{H}_i \mathbf{c}_k \tag{6.39a}$$

$$e_{k+i|k} = \Psi^i e_{k|k} + \Psi^{i-1}\tilde{D}\delta_{k|k} + \cdots + \tilde{D}\delta_{k+i-1|k} \tag{6.39b}$$

and since $\hat{\gamma}_{i|k}$ is the minimum value that satisfies (6.38), it follows that (6.37) is equivalent to the constraint of (6.34). ∎

Many problems of practical interest have more than one constrained variable. For the case of r constrained scalar variables, we assume that each constraint can be expressed as a condition of the form of (6.33), and hence (6.34), i.e.

$$\Pr\left\{g_{0,j}^T \xi + g_{1,j}^T \tilde{D}\delta + f_j^T c \leq h_j\right\} \geq p_j$$

for $j = 1, \ldots, r$. Then Theorem 6.8 can be applied to each of the r individual constraints to derive a set of constraints like (6.37), which together define the probabilistic tubes for state predictions.

Due to linearity, (6.37) can be easily incorporated into an online optimization of $\mathbf{c}_k$. To determine $\hat{\gamma}_{i|k}$ one needs the distribution of $g_0^T(\Psi^i e_{k|k} + \Psi^{i-1}\tilde{D}\delta_k + \cdots + \tilde{D}\delta_{k+i-1}) + g_1^T \tilde{D}\delta_{k+i}$, and this requires a multivariate convolution integral. Instead, by discretizing the distributions of δ_k and ε_0 and performing discrete scalar convolutions, the values of $\hat{\gamma}_{i|k}$ can be approximated (to a specified degree of accuracy) at reasonable computational cost. Importantly, the calculation of $\hat{\gamma}_{i|k}$ does not require knowledge of x_k and can be performed offline. Moreover, the error, δp_i, due to discretization, can be accounted for by replacing p by $p + \delta p_i$ thereby incurring some conservativism, but as mentioned above, this can be made to be as small as desired.

Like the analysis in 6.4, although Theorem 6.8 ensures that (6.34) is satisfied over the entire prediction horizon at time k, the existence of $\mathbf{c}_k$ satisfying Theorem 6.8 does not ensure the existence of $\mathbf{c}_{k+1}$ that generate predictions at time $k+1$ satisfying (6.34). Consequently (6.37) does not guarantee the future feasibility of an online optimization problem incorporating (6.37) as the constraint. This is because $\hat{\gamma}_{i|k}$ in (6.38) is a probabilistic bound on the stochastic components of the predicted value of $g_0^T \xi_{k+i|k} + g_1^T \tilde{D}\delta_{k+i|k}$ at time k, whereas the same probabilistic bound on $g_0^T \xi_{k+i|k+1} + g_1^T \tilde{D}\delta_{k+i|k+1}$ could, on the basis of information available at time k, take its maximum value over *all possible realizations* of the estimation error e_k and disturbance δ_k at time k.

Define $M\mathbf{c}_k = [c_{k+1|k}^T \ \cdots \ c_{k+N-1|k}^T \ 0]^T \in \mathbb{R}^{Nn_u}$, then the preceding argument implies that $\mathbf{c}_{k+1} = M\mathbf{c}_k$ satisfies (6.37) at time $k+1$ if and only if $\hat{\gamma}_{i|k}$ is replaced in (6.37) by the bounds implied by the *worst-case* values for $e_{k|k}$ and $\delta_{k|k}$ in (6.38). The following theorem generalizes this approach by deriving the conditions for feasibility of $\mathbf{c}_{k+i} = M^i\mathbf{c}_k$ in (6.37) at time $k+i$ for $i \geq 1$. These conditions therefore ensure that the probabilistic constraints (6.34) are satisfied at time k and are also recursively feasible in the sense that they remain feasible at all times if feasible at time k. We define γ_i as the minimum value such that

$$\Pr\Big\{g_0^T(\Psi^{i-1}\tilde{D}\delta_k + \cdots + \tilde{D}\delta_{k+i-1}) + g_1^T\tilde{D}\delta_{k+i} \leq \gamma_i\Big\} = p, \tag{6.40}$$

and also define $\alpha_{i|k}$ and d_i for $i = 1, 2, \dots$ by

$$d_i = \max_{\delta\in\Delta}\ g_0^T\Psi^{i-1}\tilde{D}\delta \tag{6.41a}$$

$$\alpha_{i|k} = \operatorname{ess}\ \sup_{e_{k|k}\sim\mathfrak{D}_k}\ g_0^T\Psi^i e_{k|k} \tag{6.41b}$$

where $\operatorname{ess}\ \sup_{e\sim\mathfrak{D}} f(e)$ denotes the maximum of a function f over the support of the distribution $\mathfrak{D}$.

Theorem 6.9. At time k, the constraints (6.34) are satisfied by the predictions of (6.32) if $\mathbf{c}_k$ satisfies

$$(g_0^T\tilde{H}_i + f^TE_i)\mathbf{c}_k + g_0^T\Psi^i z_{k|k} \leq h - \beta_{i|k}, \quad i = 0, 1, \dots \tag{6.42}$$

where $\beta_{i|k}$ is the maximum element of the $(i+1)$th column of the matrix $\mathfrak{B}_k$ defined by

$$\mathfrak{B}_k = \begin{bmatrix} \hat{\gamma}_{0|k} & \hat{\gamma}_{1|k} & \hat{\gamma}_{2|k} & \cdots \\ 0 & \alpha_{1|k} + d_1 + \gamma_0 & \alpha_{2|k} + d_2 + \gamma_1 & \cdots \\ 0 & 0 & \alpha_{2|k} + d_2 + d_1 + \gamma_0 & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6.43}$$

Furthermore, if (6.42) is feasible at time k, then (6.42) will remain feasible at all times $k+1, k+2, \ldots$.

Proof. If $\beta_{i|k}$ is defined by the $(i+1)$th element of the first row of $\mathfrak{B}_k$, then (6.42) is equivalent to (6.37). The $(i+1)$th element of the second row of $\mathfrak{B}_k$ is obtained by replacing $e_{k|k}$ and $\delta_{k|k}$ in (6.38b) by their worst-case values, so that (6.42) also ensures that $M\mathbf{c}_k$ is feasible for (6.37) at time $k+1$. Similarly the jth row corresponds to the worst-case values of $e_{k|k}$ and $\delta_{k|k}, \ldots, \delta_{k+j-2|k}$ in (6.38b), thus ensuring that $M^j\mathbf{c}_k$ is feasible for (6.37) at time $k+j$. The proof is completed by noting that (6.42) is itself recursively feasible due to its definition in terms of the worst-case future uncertainty. ∎

Lemma 6.10. For all $i \geq 1$ we have $\beta_{i|k} = \alpha_{i|k} + \sum_{j=1}^{i} d_j + \gamma_0$ which appear on the diagonal.

Proof. From the definitions of d_i and $\alpha_{i|k}$ in (6.41b) we have $g_0^T \Psi^{i-1}\tilde{D}\delta_k \leq d_i$ and $g_0^T \Psi^i e_{k|k} \leq \alpha_{i|k}$ for all realizations of $e_{k|k}$ and δ_k. Hence $\hat{\gamma}_{1|k} \leq \alpha_{1|k} + d_1 + \gamma_0$, and from the definition of γ_{i-1} it also follows that

$$\Pr\left\{ g_0^T (\Psi^i e_{k|k} + \Psi^{i-1}\tilde{D}\delta_k + \cdots + \tilde{D}\delta_{k+i-1}) + \gamma_0 \leq \alpha_{i|k} + d_i + \gamma_{i-1} \right\} \geq p$$

so that $\hat{\gamma}_{i|k} \leq \alpha_{i|k} + d_i + \gamma_{i-1}$ for $i = 2, 3, \ldots$. Similarly, we obtain $\gamma_i \leq d_i + \gamma_{i-1}$ for $i = 1, 2, \ldots$, and therefore the maximum element in each column of $\mathfrak{B}_k$ lies on the diagonal. ∎

Remark 6.11. Although the bounds in (6.41) account for worst-case uncertainty, the bounds in (6.38a) and (6.40) hold with probability $p \leq 1$. The constraints of (6.42)-(6.43) are therefore less restrictive than the corresponding Robust MPC constraints which require $p = 1$ in (6.34), and hence compute the worst-case bounds on all future uncertainty variables.

Since $e_{k|k}$ and $\{\delta_k, \delta_{k+1}, \ldots\}$ are bounded, and since Ψ is strictly stable, the predicted sequence $\{e_{k|k}, e_{k+1|k}, \ldots\}$ generated by (6.35c) and hence $\{\beta_{i|k}, \beta_{i+1|k} \ldots\}$ are also bounded for any k. We next show how to compute bounds on $\beta_{i|k}$ that hold for all i, k exceeding given values. These are used in the next section to derive a recursively feasible SMPC algorithm that involves only a finite number of constraints. As we show below, these bounds also enable the constraints in the online SMPC optimization to be formed offline, at $k = 0$.

Define the symmetric positive definite matrix $S \in \mathbb{R}^{2n\times 2n}$ and scalar $\rho \in (0,1)$ as the solution of the semidefinite program:

$$(\rho, S) = \arg \min_{\substack{\rho>0 \\ S=S^T>0}} \rho$$

$$\text{subject to} \begin{bmatrix} \rho^2 & (\Psi S)^T \\ \Psi S & S \end{bmatrix} \succ 0 \tag{6.44a}$$

$$\begin{bmatrix} 1 & (\tilde{D}\delta)^T \\ \tilde{D}\delta & S \end{bmatrix} \succ 0 \quad \text{for all } \delta \in \Delta \tag{6.44b}$$

Note that the convexity of (6.44a,b) and the strict stability of Ψ imply the existence and uniqueness of the optimal solution to (6.44). Define $\mathcal{E}$ as the ellipsoidal set $\mathcal{E} = \{e : \|e\|_{S^{-1}} \le 1\}$, where $\|e\|_{S^{-1}} = \sqrt{e^T S^{-1} e}$, and let $\mathcal{E}_\varepsilon$ denote its projection onto the subspace $\{\varepsilon \in \mathbb{R}^n : \varepsilon = [0 \;\; I_n]e,\ e \in \mathcal{E}\}$, so that

$$\mathcal{E}_\varepsilon = \{\varepsilon : \varepsilon^T S_{22}^{-1} \varepsilon \le 1\},$$

where S_{22} is the relevant block of $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{smallmatrix}\right]$. Also define positive scalars τ and σ by

$$\tau = \operatorname*{ess\ sup}_{e_0 \sim \mathfrak{D}_0} \|e_0\|_{S^{-1}} = \max_{\varepsilon_0 \in \Pi_0} \left\| \begin{bmatrix} 0 \\ \varepsilon_0 \end{bmatrix} \right\|_{S^{-1}} \tag{6.45a}$$

$$\sigma = \max_{\varepsilon \in \mathcal{E}_\varepsilon} \left\| \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \right\|_{S^{-1}} = \left\{ \lambda_{\max}\left(S_{22}^{1/2} \begin{bmatrix} 0 & I \end{bmatrix} S^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} S_{22}^{1/2} \right) \right\}^{1/2}. \tag{6.45b}$$

(where $\lambda_{\max}(A)$ denotes the maximum eigenvalue of A).

Lemma 6.12. For any integer $m > 1$, $\beta_{i|k}$ satisfies:

$$\text{for} \quad 1 \le i < m, \quad \beta_{i|k} = \sum_{j=1}^{k} b_{ij} + a_{ik} + \sum_{j=1}^{i} d_j + \gamma_0, \quad \text{if } k < m; \tag{6.46a}$$

$$\text{for} \quad 1 \le i < m, \quad \beta_{i|k} \le \bar{\beta}_{i|m} = \sum_{j=1}^{m-i} b_{ij} + \bar{\alpha} + \sum_{j=1}^{i} d_j + \gamma_0, \quad \text{if } k \ge m; \tag{6.46b}$$

$$\text{for} \quad i \ge m, \quad \beta_{i|k} \le \bar{\beta} = \bar{\alpha} + \sum_{j=1}^{m} d_j + \gamma_0, \tag{6.46c}$$

where $b_{ij} = \max_{\delta \in \Delta} g_0^T \Psi^i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \Psi^{j-1} \tilde{D} \delta$, and

$$a_{ik} = \max_{e_0 \sim \mathfrak{D}_0} g_0^T \Psi^i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \Psi^k e_0, \quad \bar{\alpha} = \rho^m \sigma \left(\frac{1}{1-\rho} + \tau \right) \|g_0\|_S.$$

Proof. (i). Using $e_{k|k} = [0 \ \ \varepsilon_k^T]^T$, (6.31) and the structure of Ψ, one can get

$$e_{k|k} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \left[\Psi^k e_0 + \tilde{D}^{k-1} \delta_0 + \cdots + \tilde{D} \delta_{k-1} \right],$$

Thus the definition of $\alpha_{i|k}$ in (6.41) give $\alpha_{i|k} = \sum_{j=1}^{k} b_{ij} + a_{ik}$, which implies that the expression for $\beta_{i|k}$ in (6.46a) holds for all $i \ge 1$ and $k \ge 0$.

(ii). To demonstrate the bounds in (6.46b,c), note that (6.44b) implies $\tilde{D}\Delta \subseteq \mathcal{E}$ (i.e. $\tilde{D}\delta \in \mathcal{E}$ for all $\delta \in \Delta$), and by Schur complement, (6.44a) is equivalent to

$$S\Psi^T S^{-1}\Psi S < \rho^2 \quad \Leftrightarrow \Psi^T S^{-1}\Psi < \rho^2 S^{-1} \;\Rightarrow\; (\Psi e)^T S^{-1}\Psi e < \rho^2.$$

This leads to $\Psi\mathcal{E} \subseteq \rho\mathcal{E}$, therefore $\Psi^j \tilde{D}\Delta \subseteq \rho^j\mathcal{E}$. Since $\max_{e\in\mathcal{E}} g_0^T e = \|g_0\|_S$, it follows that

$$d_j = \max_{\delta\in\Delta}\; g_0^T\Psi^{j-1}\tilde{D}\delta \;\le\; \max_{e\in\mathcal{E}}\; g_0^T\Psi^{j-1}e \;\le\; \max_{e\in\mathcal{E}}\; \rho^{j-1}g_0^T e \;\le\; \rho^{j-1}\|g_0\|_S$$

Furthermore, (6.45b) implies

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\mathcal{E} \subseteq \sigma\mathcal{E}, \quad \sigma > 1,$$

so we have, similarly, $b_{ij} \le \rho^{i+j-1}\sigma\|g_0\|_S$, and from (6.45a) we have

$$a_{ik} \;\le\; \max_{e\in\mathcal{E}}\; \tau g_o^T\Psi^i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\Psi^k e \;\le\; \max_{e\in\mathcal{E}}\; \rho^{i+k}\sigma\tau g_o^T e \;\le\; \rho^{i+k}\sigma\tau\|g_0\|_S.$$

According to (6.46a), $\beta_{i|k} = \sum_{j=1}^{k} b_{ij} + a_{ik} + \sum_{j=1}^{i} d_j + \gamma_0 = \Big(\sum_{j=m-i+1}^{k} b_{ij} + a_{ik}\Big) + \sum_{j=1}^{m-i} b_{ij} + \sum_{j=1}^{i} d_j + \gamma_0$, and we can bound the first part of $\beta_{i|k}$ as follows:

$$\begin{aligned}
\sum_{j=m-i+1}^{k} b_{ij} + a_{ik} \;&\le\; \sum_{j=m-i+1}^{k} \rho^{i+j-1}\sigma\|g_0\|_S + \rho^{i+k}\sigma\tau\|g_0\|_S \\
&\le\; \frac{\rho^m - \rho^{i+k}}{1-\rho}\sigma\|g_0\|_S + \rho^{i+k}\sigma\tau\|g_0\|_S \\
&\le\; \rho^m\sigma\Big(\frac{1}{1-\rho} + \tau\Big)\|g_0\|_S \quad \big(\text{considering } k \ge m \text{ and } \rho < 1\big)
\end{aligned} \tag{6.47}$$

Thus (6.46b) is obtained.

(iii). $\beta_{i|k} = \sum_{j=1}^{k} b_{ij} + a_{ik} + \sum_{j=1}^{i} d_j + \gamma_0 = \big(\sum_{j=1}^{k} b_{ij} + a_{ik} + \sum_{j=m+1}^{i} d_j\big) + \big(\sum_{j=1}^{m} d_j + \gamma_0\big)$, and below we show $\sum_{j=1}^{k} b_{ij} + a_{ik} + \sum_{j=m+1}^{i} d_j \leq \bar{\alpha}$.

$$\begin{aligned}
\sum_{j=1}^{k} b_{ij} + a_{ik} + \sum_{j=m+1}^{i} d_j &\leq \frac{\rho^m - \rho^i}{1-\rho}\|g_0\|_S + \frac{\rho^i - \rho^{i+k}}{1-\rho}\sigma\|g_0\|_S + \rho^{i+k}\sigma\tau\|g_0\|_S \\
&< \frac{\rho^m - \rho^{i+k}}{1-\rho}\sigma\|g_0\|_S + \rho^{i+k}\sigma\tau\|g_0\|_S \quad \big(\text{considering } \sigma > 1\big) \\
&< \rho^m\sigma\Big(\frac{1}{1-\rho} + \tau\Big)\|g_0\|_S \quad \big(\text{considering } i \geq m \text{ and } \rho < 1\big) \qquad (6.48)
\end{aligned}$$

■

Remark 6.13. From the proof above (see (6.47) and (6.48)), it is obvious to argue that the bounds in (6.46b,c) can be made arbitrarily tight by using a sufficiently large value for m.

Remark 6.14. The values of $\beta_{0|k} = \hat{\gamma}_{0|k}$ for $k \geq m$ may be approximated for sufficiently large m (since Ψ is strictly stable) using the asymptotic distribution $\mathfrak{D}_\infty$ of $\lim_{k\to\infty} e_{k|k}$, i.e.

$$\beta_{0|k} \approx \hat{\gamma}_{0|\infty} = \gamma_\infty \;\; \forall k \geq m.$$

Combining this with the bounds on $\beta_{i|k}$ for $i \geq 1$ in Lemma 6.12 enables recursively feasible constraints to be computed offline for all $k \geq 0$ on the basis of a finite number of bounds:

$$\begin{aligned}
&\gamma_0,\; \{\hat{\gamma}_{0|k},\; k = 0,\dots,m-1\},\; \gamma_\infty,\; \{d_j,\; j = 1,\dots,m\}, \\
&\{b_{ij},\; i,j = 1,\dots m-1\},\; \{a_{ij},\; i,j = 1,\dots m-1\},\; \bar{\alpha}.
\end{aligned} \qquad (6.49)$$

This approach assumes that the transient response of the combined plant and observer system due to the initial estimation error becomes negligible after m time steps. For m less than the duration of these transients, the degree of conservativeness of bounds of Lemma 6.12 could be non-negligible.

6.7.3 Stochastic MPC with State Estimation

This section describes a receding horizon control strategy based on the predictions of (6.32), and establishes closed loop stability and convergence properties. The strategy deploys the online minimization of the predicted expected cost (6.36) over the decision variables $\mathbf{c}_k = [c_{k|k}^T, \ldots, c_{k+N-1|k}^T]^T$ subject to the constraints (6.34), which are invoked via the recursively feasible constraints of (6.42). As mentioned earlier, these are less restrictive than the *robust MPC* constraints and therefore allow for more control authority with which to improve performance and enlarge the region of attraction. The predictions of (6.32b), with $c_{k+i|k} = 0$ for $i \geq N$, imply a dual mode prediction strategy, with mode 1 comprising the first N prediction time steps during which the control inputs are defined by $\mathbf{c}_k$, and mode 2, the subsequent infinite horizon, over which the inputs are given by the fixed feedback law $u_{k+i|k} = Kx_{k+i|k}$. Constraints are handled explicitly in mode 1 and implicitly in mode 2 through the constraint that the predicted state at the end of mode 1 should lie in a terminal set. This set is derived by re-writing (6.42) in mode 2 at time k as

$$g_0^T \Psi^j z_{k+N|k} \leq h - \beta_{N+j|k}\,, \quad j = 0, 1, \ldots \tag{6.50}$$

The maximal admissible set at time k, denoted by $\mathcal{S}_{\infty|k}$, is the subset of $\mathbb{R}^n$ containing all $z_{k+N|k}$ such that (6.50) holds. The bounds (6.46) imply conditions for the existence of $S_{\infty|k}$.

Lemma 6.15. If the probability p is such that $h > \bar{\beta}$ and

$$h > \begin{cases} \max\limits_{i=N,\ldots,m-1} \beta_{i|k} & \text{if } k < m \\ \max\limits_{i=N,\ldots m-1} \bar{\beta}_{i|m} & \text{otherwise} \end{cases} \tag{6.51}$$

then $\mathcal{S}_{\infty|k}$ is non-empty.

Proof. Using (6.46b,c) to bound $\beta_{N+j|k}$ in (6.50) defining $\mathcal{S}_{\infty|k}$, it is easy to show that the conditions on h given in the lemma ensure that $\mathcal{S}_{\infty|k}$ contains the origin, and so is non-empty. ■

The approach of Gilbert and Tan [1991] provides a method of computing an inner approximation of $\mathcal{S}_{\infty|k}$. Consider first the case of $k < m$, and define a terminal set $\mathcal{S}_{m|k}$ as follows:

$$\begin{aligned} \mathcal{S}_{m|k} = \{z : g_0^T \Psi^{i-N} z \le h - \beta_{i|k}, \quad & i = N, \dots, m-1 \\ g_0^T \Psi^{i-N} z \le h - \bar{\beta}, \quad & i = m, m+1, \dots\} \end{aligned} \tag{6.52}$$

From (6.46c) it follows that $\mathcal{S}_{m|k} \subseteq \mathcal{S}_{\infty|k}$, and furthermore, it can be shown [Gilbert and Tan, 1991] that there exists N_k^* such that $\mathcal{S}_{m|k}$ is determined by the finite set of inequalities:

$$\begin{aligned} \mathcal{S}_{m|k} = \{z : g_0^T \Psi^{i-N} z \le h - \beta_{i|k}, \quad & i = N, \dots, m-1 \\ g_0^T \Psi^{i-N} z \le h - \bar{\beta}, \quad & i = m, \dots, m + N_k^*\} \end{aligned} \tag{6.53}$$

whenever $\mathcal{S}_{\infty|k}$ is bounded. For $k \ge m$, (6.46b) enables a time-invariant terminal set to be defined as $\mathcal{S}_{m|k} = \mathcal{S}_m$, where

$$\begin{aligned} \mathcal{S}_{m} = \{z : g_0^T \Psi^{i-N} z \le h - \bar{\beta}_{i|m}, \quad & i = N, \dots, m-1 \\ g_0^T \Psi^{i-N} z \le h - \bar{\beta}, \quad & i = m, \dots, m + N_m^*\} \end{aligned} \tag{6.54}$$

for some finite N_m^*. Thus a finite collection of terminal sets: $\mathcal{S}_{m|k}$, for $k = 0, \dots, m-1$ and $\mathcal{S}_m$ for all $k \ge m$ can be used to invoke the mode 2 constraints at any time k. Moreover, for each $k = 0, \dots, m$, the parameter N_k^* can be determined by solving a finite number of linear programming problems (see Section 6.5). Using the approach described above, it is then possible to express $\mathcal{S}_{m|k}$ and $\mathcal{S}_m$ in terms of a finite number of inequalities as in (6.53) and (6.54).

The cost defined by (6.36) is the summation of the expected value of each stage cost, which is infinite over the infinite prediction horizon since the asymptotic value:

$$L_{ss} = \lim_{i\to\infty} \mathbb{E}\big(x^T_{k+i|k} Q x_{k+i|k} + u^T_{k+i|k} R u_{k+i|k}\big)$$

is non-zero for the case of persistent additive uncertainty. Thus we redefine the cost function in terms of the deviation of expected stage cost from this asymptotic limit:

$$\breve{J}_k = \sum_{i=0}^{\infty}\Big[\mathbb{E}\big(x^T_{k+i|k} Q x_{k+i|k} + u^T_{k+i|k} R u_{k+i|k}\big) - L_{ss}\Big]. \tag{6.55}$$

This modification provides a finite cost that can be minimized online, yet clearly the optimal value of $\mathbf{c}_k$ is unchanged.

Remark 6.16. Given the distribution of δ, according to (2.101), the cost (6.55) can be expressed as a quadratic function of the degrees of freedom $\mathbf{c}_k$ in the predictions:

$$\breve{J}_k(\mathbf{c}_k) = \mathbf{c}^T_k P_{cc}\mathbf{c}_k + 2\hat{x}^T_k P^T_{xc}\mathbf{c}_k + p_k(\hat{x}_k) + f_k(\varepsilon_k), \tag{6.56}$$

where $P_{cc} \in \mathbb{R}^{Nn_u \times Nn_u}$, $P_{xc} \in \mathbb{R}^{n\times Nn_u}$ are constants that can be computed offline, and $p_k(\hat{x}_k)$, $f_k(\varepsilon_k)$[5] are independent of $\mathbf{c}_k$. Furthermore, if K is the optimal feedback gain for the case of no constraints, then $P_{xc} = 0$ by construction.

Algorithm 6.2 (Stochastic MPC with state estimation)**.**

Offline: Use the distributions of e_0 and δ_k to compute the parameters (6.49) that define the recursively feasible probabilistic constraints (6.34). Determine the constraint checking horizons N^*_k, $k = 0,\dots,m$ in the terminal constraint sets defined in (6.53) and (6.54).

Online: At each time $k = 0, 1, \dots$:

1. Compute the state estimate $\hat{x}_k$ and set $z_{k|k} = [\hat{x}^T_k\ 0]^T$.

[5]By (6.31), $f_k(\varepsilon_k) = \mathbb{E}\big(\varepsilon^T_k P_\varepsilon \varepsilon_k\big) = tr\big(\mathbb{E}(\varepsilon_k\varepsilon^T_k)P_\varepsilon\big)$ is determined by the recursive equation $\Sigma_{k+1} = (A - LCA)\Sigma_k(A - LCA)^T + [D \quad - LF]\mathbb{E}(\delta_k\delta^T_k)[D \quad - LF]^T$ and the known initial Σ_0, where $\Sigma_k \triangleq \mathbb{E}(\varepsilon_k\varepsilon^T_k)$.

2. Solve the quadratic program (QP):

$$
\begin{aligned}
\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \; & \breve{J}_k(\mathbf{c}_k) \\
\text{subject to } & (g_0^T \tilde{H}_i + f^T E_i)\mathbf{c}_k + g_0^T \Psi^i z_{k|k} \le h - \beta_i, \;\; i = 0, 1, \ldots, N-1 \\
& z_{k+N|k} \in \mathcal{S}_k
\end{aligned}
$$

where $\beta_i = \beta_{i|k}$ and $\mathcal{S}_k = \mathcal{S}_{m|k}$ if $k < m$

$\beta_i = \bar{\beta}_{i|m}$ and $\mathcal{S}_k = \mathcal{S}_m$ if $k \ge m$.

3. Set $u_k = K\hat{x}_{k+i|k} + c^*_{k|k}$, where $c^*_{k|k}$ is the first element of $\mathbf{c}_k^*$.

Theorem 6.17. Given feasibility at $k = 0$, Algorithm 6.2 remains feasible for all $k > 0$. The closed loop system under Algorithm 6.2 satisfies the probabilistic constraint (6.33) and the quadratic stability condition:

$$
\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r} \mathbb{E}\big(x_k^T Q x_k + u_k^T R u_k\big) \le L_{ss}. \tag{6.57}
$$

Proof. The recurrence of feasibility is due to the definition of the constraints in the online optimization of Algorithm 6.2, and this ensures that the probabilistic constraints (6.33) are satisfied in closed loop operation. In addition, Theorem 2 ensures that $M\mathbf{c}_k^*$ is feasible for the online optimization at $k+1$, so that

$$
\breve{J}_k(\mathbf{c}_k^*) = \mathbb{E}_k\big[\breve{J}_{k+1}(M\mathbf{c}_k^*)\big] + x_k^T Q x_k + u_k^T R u_k - L_{ss}
$$

which therefore implies that the optimal cost value satisfies the stochastic Lyapunov-like condition

$$
\breve{J}_k(\mathbf{c}_k^*) \ge \mathbb{E}_k\big[\breve{J}_{k+1}(\mathbf{c}_{k+1}^*)\big] + x_k^T Q x_k + u_k^T R u_k - L_{ss} \tag{6.58}
$$

Summing (6.58) over $k = 0, 1, \ldots, r$ gives

$$
\sum_{k=0}^{r} \Big[\mathbb{E}\big(x_k^T Q x_k + u_k^T R u_k\big) - L_{ss}\Big] \le \breve{J}_0(\mathbf{c}_0^*) - \mathbb{E}\big[\breve{J}_{r+1}(\mathbf{c}_{r+1}^*)\big]
$$

which implies (6.57) since $\breve{J}(\mathbf{c}_0^*)$ is by assumption finite and (6.58) ensures that $\mathbb{E}\big[\breve{J}_r(\mathbf{c}_r^*)\big]$ is finite for all r. ■

Corollary 6.18. If K is the optimal feedback gain for the case of no constraints, then $x_k \in \Omega$ in the limit as $k \to \infty$ under Algorithm 6.2, where Ω denotes the the minimal robust invariant set for the state of (6.29a,b) under $u_k = K\hat{x}_k$.

Proof. The quadratic form of $\breve{J}(\mathbf{c}_k)$ in (6.56) and the feasibility of $\mathbf{c}_{k+1} = M\mathbf{c}_k$ imply that

$$\mathbf{c}_k^T P_{cc}\mathbf{c}_k \geq \mathbf{c}_{k+1}^* P_{cc}\mathbf{c}_{k+1}^* + {c_{k|k}^*}^T (R + B^T P_x B) c_{k|k}^*$$

(where P_x is the steady state solution of the Riccati equation associated with (6.29) and (6.36)). From the sum of both sides of this inequality over $k \geq 0$ and the fact that $R + B^T P_x B > 0$, it follows that $c_{k|k}^* \to 0$ as $k \to \infty$, so that x_k converges to Ω. ■

6.7.4 Simulation Example

In this example, the model parameters in system (6.29) are:

$$A = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0.9 & 0.2 \end{bmatrix}, \quad D = I, \quad F = 1.$$

There are four probabilistic constraints of the form (6.33), with

$$\eta_x^T = \begin{bmatrix} \pm 1 & \pm 0.3 \end{bmatrix}, \quad \eta_u = 0, \quad h = 1.5, \quad p = 0.8.$$

and the cost weights $Q = C^T C$ and $R = 0.1$. The disturbance w_k, initial state estimate uncertainty ε_0, and measurement noise ν_k are bounded random variables, with truncated and scaled normal distributions. For example ν_k is derived from normally distributed $\nu_k' \sim \mathcal{N}(0, 1/100)$, which is truncated so that $|\nu_k| \leq 0.12$. Therefore the cumulative distribution function for ν_k, $F_v(V) = \Pr\{\nu_k \leq V\}$, is obtained from that of ν'_k by an affine transformation

$$F_v(V) = aF_{v'}(V) + b \quad \text{for all } V \in [-0.12, 0.12],$$

where a, b are such that $F_v(-0.12) = 0$ and $F_v(0.12) = 1$. As explained in the introduction, this is not an approximation which ignores large uncertainties but rather conforms with real-world uncertainties which have distributions with compact support. The distributions of w_k and ε_0 are defined analogously from truncated normal distributions:

$$w'_k \sim \mathcal{N}(0, I/100), \quad \|w_k\|_\infty \leq 0.12,$$
$$\varepsilon'_0 \sim \mathcal{N}(0, I/100), \quad \|\varepsilon_0\|_\infty \leq 0.12.$$

In accordance with the assumptions of Section 6.7.1, the sequences $\{w_k, k = 0, 1, \ldots\}$ and $\{\nu_k, k = 0, 1, \ldots\}$ are i.i.d. The feedback gain in (6.32b) was chosen as $K = -[1.0313 \;\; 1.0682]$, which is the unconstrained LQ-optimal gain for the given plant model and cost. The observer gain (based on the posterior estimation scheme) was chosen to be $L = -[0.8348 \;\; 1.2234]^T$, which gives the spectral radius of $A - LCA$ as 0.12. The mode 1 prediction horizon N and the horizon m over which constraints are handled explicitly are chosen (by considering the transient response when constraints are absent) as $N = 6$ and $m = 13$. The sequence $\{\hat{\gamma}_{0|k},\ k = 0, 1, \ldots, m-1\}$ was computed by discrete convolution based on an equi-spaced grid with spacing fixed at 10^{-4}. The $\beta_{i|k}$ were bounded using (6.46). This gives the condition of (6.51) as $h \geq \bar{\beta}$ where, respectively, for each of the constraints $\bar{\beta}$ was $1.31, 1.20, 1.31, 1.20$; as required, all these values are less than $h = 1.5$. For the chosen N and m, the terminal sets $\mathcal{S}_{m|k}$ and $\mathcal{S}_m$ are given by (6.53) and (6.54) with $N^*_k = 15$ for all k. For 10^4 realizations of the disturbance and noise sequences, the response of Algorithm 6.2 was simulated and compared with that of the optimal unconstrained feedback law. In each simulation, the initial state was defined as $x_0 = \hat{x}_0 + \varepsilon_0$, with the initial state estimate fixed: $\hat{x}_0 = (1.8, -4)$, and with ε_0 obtained by sampling $\mathfrak{D}_0$.

The evolution of the state sequence $\{x_k,\ k = 0, 1,\ \ldots\}$ for 200 realizations of the disturbance and noise sequences is shown in Figure 6.4, which also shows the four state constraints and shows that the chosen $\hat{x}_0$ lies on the boundary of the feasible region for Algorithm 6.2 and well outside that for robust MPC. For stochastic M-PC, the observed probability of violating the only one active constraint (constraint 1 in Figure 6.4) was 20.37% at $k = 1$ (as expected close to 20%), whereas violation rate was 100% at $k = 1$ under the unconstrained optimal control law. It is worth noting that although there are some violations of constraint 2 (the probability is 13.5%), constraint 2 is not active since the tube we construct is tight and therefore implies that the violation probability (for active constraints) should be about 20%. This argument is verified by the observation that the same simulation results will be obtained even if constraint 2 is excluded in the online optimisation.

The cumulative costs of (6.55), for closed loop operation averaged over 10^4 simulations, were 4.00 and 2.40 under Algorithm 6.2 and unconstrained optimal control respectively. Although the constrained stochastic MPC performance, as expected, is worse than the unconstrained LQR, it is significant that robust MPC is actually infeasible for the given initial condition. Interestingly for the initial condition $\hat{x}_0 = (1.35, -4)$ which lies on the boundary of the feasible regions for robust MPC the respective stochastic MPC and robust MPC costs were 1.72 and 2.42 indicating that stochastic MPC outperforms robust MPC.

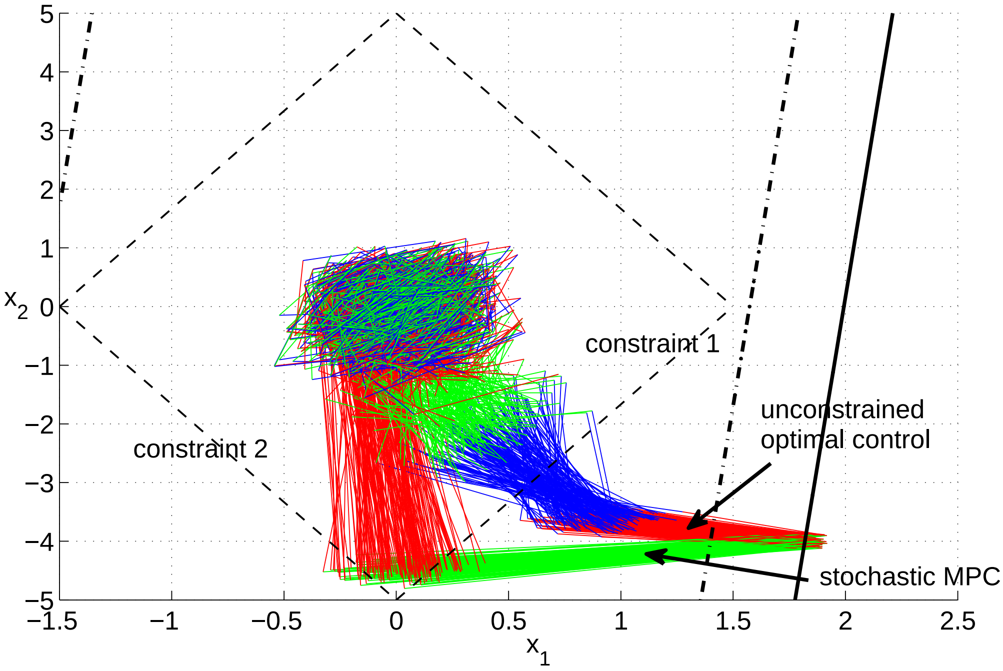

FIGURE 6.4: Evolution of x_k (two dimensions) for stochastic MPC and unconstrained optimal control; state constrains (dashed line); the boundaries of feasible initial conditions $\hat{x}_0$ for stochastic MPC (solid black line) and robust MPC (dash-dot line).

6.8 Discussion and Conclusion

6.8.1 Previous Tube Research versus Algorithm 5.1 and 6.1

First, Algorithm 5.1 and 6.1 are compared with robust tube methods [Langson et al., 2004, Mayne et al., 2005] described in Section 2.3.5. One similarity between these designs lies that the controllers are used to seperate the nominal dynamics and the error dynamics. In Langson et al. [2004], Mayne et al. [2005], the controller is $u_k = \bar{v}_k + K(x_k - \bar{x}_k)$. This is in fact equivalent to $u_k = Kx_k + c_k$ used in Algorithm 5.1 and 6.1 given $c_k = \bar{v}_k + K\bar{x}_k$. The key benefit provided by such design is that one does not necessarily consider the coupling of nominal states (considered as the reference point of the tube) and the errors (related to the tube scalings), which renders the calculation of tube scalings easily tractable.

A disturbance invariant set Z for the error dynamics $e_{k+1} = (A + BK)e_k + w_k$ is computed and used to tighten the constraints in the *whole prediction horizon* (see (2.59)) in Langson et al. [2004], Mayne et al. [2005], which is regarded as the term including $\bar{\beta}$ for Algorithm 5.1 and 6.1. And the tube scalings $\left[\bar{\beta}_{N+i}(q)\right]^{1/2}\sqrt{g^T V^{-1} g}$ in Algorithm 5.1 and β_i in Algorithm 6.1 are both monotonically increasing as i increase, which tend to the limit $\bar{\beta}_\infty^{1/2}\sqrt{g^T V^{-1} g}$ or $\bar{\beta}$. Clearly, when $\left[\bar{\beta}_{N+i}(q)\right]^{1/2}\sqrt{g^T V^{-1} g}$ and β_i are incorporated to tighten the (probabilistic) constraints on nominal states, it will be less stringent than that of Langson et al. [2004], Mayne et al. [2005]. So, the tube construction method in Algorithm 5.1 and 6.1 will generally bring in a larger region of attraction and less conservative results (smaller costs).

Secondly, Section 2.4.6 in the literature review introduces the probabilistic tubes developed earlier by Cannon et al. [2009a]. This contribution employed the concept of Markov Chains: assigning one-step-ahead transitional probability between tubes at successive time instants and one-step-ahead constraint violation probability. Applying these constraints on the confidence polytope $\mathcal{Q}(q(p))$ and optimising the polytopic tube parameters online give rise to too many variables thus this algorithm is restricted to application into low dimensional problems with short horizons. On the other hand, there is no direct way to choose the confidence region of q and only a small number of tubes can be used to reduce the computational burden, both of which have resulted in a degree of conservativeness. In contrast, Algorithm 5.1 calculates the transitional probability matrix with a simple tube scaling dynamics (5.10) and obtains the tube scalings to tighten the constraints. All these computations are performed offline, thus a large number layers of tubes (say 1000) can be deployed to reduce the conservativeness. Algorithm 6.1 computes the tight tube scalings directly, which is also completed offline.

6.8.2 Comparison between Algorithm 5.1 and 6.1

Both Algorithm 6.1 and Algorithm 5.1 are proposed for systems with random additive disturbances and probabilistic constraints. There are three similar properties between the two algorithms:

(a) Feasibility is guaranteed recursively, allowing for a closed loop quadratic stability property for both algorithms;

(b) The tube scalings used to relax the constraints, are both computed offline;

(c) The online optimisation reduces to a quadratic programming (QP) problem, which is the same as a conventional MPC problem.

The three points above make both algorithms fairly advantageous over many other algorithms proposed in Chapter 2: they are computationally competitive and provide a guarantee for closed loop feasibility and stability.

Furthermore, it is important to note the significant differences between Algorithm 6.1 and Algorithm 5.1. The first and most important one is the tube generating process. Algorithm 5.1 constructs the ellipsoidal tubes using a quadratic function of the error e_k with the assumption that w_k can be characterized by $w^T W w \leq \alpha$ (see (5.2)), which has caused a degree of conservativeness because of the use of the certain shape. Yet Algorithm 6.1 computes the tube parameters using the probabilistic distribution of the uncertainty, which is a direct way. The second is about the assumption with respect to the additive uncertainty. Apart from requiring the disturbance sequences w_k, $k = 1, 2, \ldots$ to be i.i.d, Algorithm 6.1 assume that each element of w_k should be independent so that it is possible to perform the numerical calculation of the probabilistic distribution for the sum of several random variables.
From the simulation results, it is demonstrated that Algorithm 6.1 is less conservative than Algorithm 5.1. Figure 6.5 compares the stochastic tube scalings of

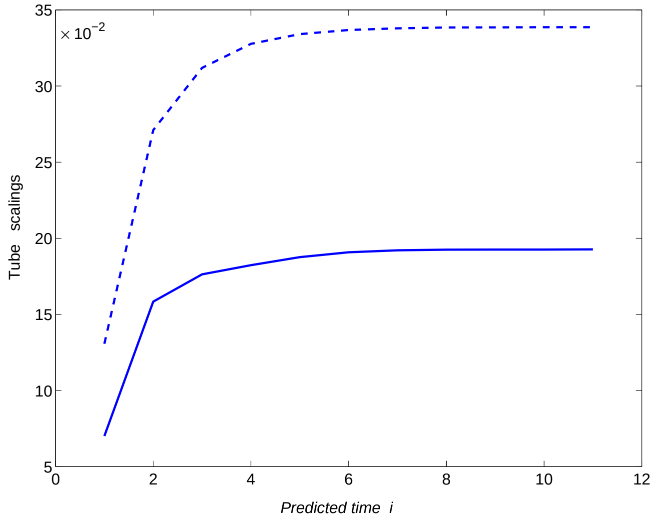

FIGURE 6.5: Stochastic tube scalings comparison: Algorithm 5.1 (dashed line) and Algorithm 6.1 (solid line);
dashed line: $[\bar{\beta}_i(q)]^{1/2}\sqrt{g^TV^{-1}g}$ in $g^Tz_{k+i|k} \leq h - [\bar{\beta}_i(q)]^{1/2}\sqrt{g^TV^{-1}g}$;
solid line: β_i in $g^Tz_{k+i|k} \leq h - \beta_i$.

the single-input-single-out case (Example 1 in Section 5.5.1): under Algorithm 5.1 (dashed line) and Algorithm 6.1 (solid line). It is easy to conclude that Algorithm 6.1 has relaxed the probabilistic constraints further than Algorithm 5.1, which therefore is expected to lead to a better performance (lower cost). This is also proved to be right for the multi-input-multi-out system: the costs are compared directly through (6.28) and (5.50). On the other hand, the violation probability for Algorithm 6.1 is just as specified, whereas the violation probability (5.9% of Example 1) for Algorithm 5.1 is much smaller than the maximum allowed (20%) by the algorithm. This also implies constraints cannot be relaxed further than that of Algorithm 6.1 with the current design of controller.

6.8.3 Conclusion

This chapter discussed and proposed a solution for a stochastic MPC problem for systems with bounded random additive disturbances and probabilistic constraints, and for both cases whether the states are available for direct measurement or not.

The proposed algorithm carries a guarantee of recursive feasibility and ensures a form of the quadratic stability and convergence of the state to a neighbourhood of the origin. It employs a direct way which explicitly calculates the tube parameters with the use of the probabilistic distribution of the uncertainty (additive disturbances and/or measurement noise, the initial state error), therefore, simulation results show that it achieves near-optimal performance while satisfying the probabilistic constraints non-conservatively, in the sense that the observed probability of constraint satisfaction in closed loop operation is the same as the specified value. This stochastic tube MPC methodology was compared with the tube MPC approaches discussed in previous chapters. In addition, the stochastic MPC was also compared with the corresponding robust MPC (assuming the constraints to be satisfied with a probability 1) by means of several numerical examples: the former has an advantage over the latter by possessing a larger initial region of attraction and producing a lower cost.

Chapter 7

Conclusions

7.1 Summary

Robust MPC and stochastic MPC strategies for the control of constrained linear systems with uncertainty are studied in the thesis.

Based on an augmented autonomous state space description (2.52), an efficient robust MPC algorithm, Algorithm 2.3, was constructed in Kouvaritakis et al. [2000]. This algorithm can be very efficient online through the use of the Newton Raphson method. The objective is to minimise the nominal costs, which forms a defensible strategy provided that the worst case cost does not differ significantly from the nominal. Research presented in this thesis embeds a Youla Parameter into Algorithm 2.3 to desensitize the cost to the effect of uncertainty without degrading the nominal performance and thus consolidates the robustness attributes of MPC. The research applies the Youla Parameter to the constrained cases with guaranteed feasibility, which to our knowledge has not been addressed before.

The controller design $u = K_{LQ}x + c$ is employed across the thesis for its convenience relating to an augmented autonomous formulation (2.52), which avoids the need to propagate the effects of uncertainty on the predicted trajectories. The reason for using unconstrained nominal LQ-optimal feedback K_{LQ} is to prioritize the interests of nominal performance, yet one possible problem arises that K_{LQ} may not stabilize the whole class of system models (2.24). Two approaches have been developed to tackle this problem. The first is to optimise another linear state feedback H in place of K_{LQ}. The H is designed so as to minimise the unconstrained worst case cost in mode 2. The second is to design a dynamic controller $u = K_{LQ}x + \upsilon + c$ and $\upsilon_{k+i+1|k} \in Co\{F_l x_{k+i|k} + N_l \upsilon_{k+i|k}\}$. The use of $\{F_l, N_l\}$ not only stabilizes the whole class of systems, but also allows for generating the nominal LQ-optimal trajectories whenever applicable, which cannot be realized explicitly by the first design $u = Hx + c$.

A stochastic ellipsoidal tube method is developed in the thesis. With the use of the semi closed-loop paradigm $u = K_{LQ}x + c$, the nominal dynamics and the error dynamics can be separated, for systems with only additive uncertainty. It is assumed that the disturbances and errors can both be characterized in terms of quadratic inequalities. The probabilistic constraint satisfaction is achieved through the probabilistic tubes (inscribed ellipses in Figure 5.3). Simple dynamics, an affine function (5.10), is obtained regarding the tube scalings . It is then easy to compute the transitional probabilities between different layers of tubes. Using the concept of Markov Chain, the tube probabilities and the corresponding tube scalings can be calculated for each prediction instant. To ensure the recursive feasibility, the recursively feasible probabilistic tubes are proposed. Moreover, an invariant terminal set is defined in mode 2 in order to stabilise the algorithm. All of the above calculations are performed offline. Therefore, the proposed stochastic MPC algorithm, Algorithm 5.1, is translated into a quadratic programming (QP) problem, which is similar to the conventional MPC strategy. This algorithm provides the necessary guarantees for closed-loop stability and feasibility.

A direct approach to calculate the tube scalings is investigated. The nominal dynamics and the error dynamics are once again analyzed separately. The assumption is made that each element of the uncertainty is independent. Hence the probabilistic tube scalings at particular predicted time instants can be obtained by computing the distribution of the sum of random variables. The constraints on nominal states are then tightened with the use of those tube scalings. The cross-sections of the tubes have no fixed shape, but are defined exactly by the tightened constraints. Recursively feasible probabilistic tubes are also put into use to guarantee the recursive feasibility. The proposed stochastic MPC strategy (Algorithm 6.1) achieves tight satisfaction of constraints in the sense that the probability of constraint violation can be as large as specified, which implies the constraints cannot be further tightened. This algorithm has also been extended to the cases when states are not measurable and observers are needed.

Instead of optimising tube parameters online [Cannon et al., 2009b,a], the two stochastic MPC algorithms developed in the thesis both compute the tube scalings offline, and thus the online algorithms are very efficient. Under the explicit assumption that the uncertainty is finitely supported, the algorithms can guarantee the recursive feasibility and closed-loop stability; this would not be possible if the uncertainty were allowed to assume arbitrary large values (albeit at very low probabilities). This assumption, apart from its theoretical convenience, corresponds better to the physical reality of most real life applications.

7.2 Future Research Implications

It is shown in the thesis how to introduce the Youla Parameter to the constrained system with a transfer function description, and the numerical evidence indicates that the Q Parameter is effective in strengthening the robustness for single-input, single-output (SISO) systems. It is then desirable to conduct further study on Youla Parameter so as to extend the results to the general case of multi-input, multi-output (MIMO) systems. This study will enlarge the applicability of the robust MPC strategy (Algorithm 2.3).

The stochastic MPC strategies in the thesis employ the semi closed-loop controller $u = K_{LQ}x + c$, which might be expected to improve the performance with a closed-loop controller. Closed-loop controllers will usually be computationally demanding. An approach of *towards closed-loop optimization* is studied in Kouvaritakis et al. [2010a] and has been shown to get a larger region of attraction.

As summarized in the previous section, the stochastic tube methods for systems with only additive disturbances developed in the thesis possess a number of desirable properties, such as the online computation which is comparable to conventional MPC strategy, guarantees of recursive feasibility and closed-loop stability. In future, the development of stochastic MPC algorithms with these attributes will focus on systems which are subject to multiplicative noise. Since such systems include the products of random variables, further work will be needed if the computation of tube scalings is to be practicable (preferably as direct as in Chapter 6). Moreover, it can be argued that those attributes will facilitate the application and theoretical research of stochastic MPC. For example, there is a number of recent studies on stochastic MPC applications in networked control systems (NCSs) like power generation plants and unmanned autonomous vehicles [Wu et al., 2009, Liu and Li, 2009, Bernardini et al., 2010]. NCSs usually have some common features: e.g. large scale, time varying delays and packet dropouts. It is thus preferred to apply control strategies with receding horizon fashion into NCSs to handle the network-induced uncertainties such as delays or packet dropouts. Since the uncertainties can always be defined by random processes, stochastic MPC is an appropriate choice for NCSs. Consequently, it is reasonable to anticipate that more research on stochastic MPC, with respect to both the theory and the applications for various complex systems like NCSs, will be ahead of us.

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