Children's Understanding of the Commutativity and Complement Principles: A Latent Profile Analysis

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Abstract

This study examined patterns of individual differences in the acquisition of the knowledge of the commutativity and complement principles in 115 five- to six-year-old children and explored the role of concrete materials in helping children understand the principles. On the basis of latent profile analysis, four groups of children were identified: The first group succeeded in commutativity tasks with concrete materials but in no other tasks; the second succeeded in commutativity tasks in both concrete and abstract conditions, but not in complement tasks; the third group succeeded in all commutativity tasks and in complement tasks with concrete materials, and the final group succeeded in all the tasks. The four groups of children suggest a developmental trend – (1) Knowledge of the commutativity and of the complement principles seems to develop from thinking in the context of specific quantities to thinking about more abstract symbols; (2) There may be an order of understanding of the principles – from the commutativity to the complement principle; (3) Children may acquire the knowledge of the commutativity principle in the more abstract tasks before they start to acquire the knowledge of the complement principle. This study contributes to the literature by showing that assessing additive reasoning in different ways and identifying profiles with classification analyses may be useful for educators to understand more about the developmental stage where each child is placed. It appears that a more fine-grained assessment of additive reasoning can be achieved by incorporating both concrete materials and relatively abstract symbols in the assessment.

Keywords: Additive reasoning, commutativity principle, complement principle, latent profile analysis

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Children's understanding of logical principles in mathematics has received increasing empirical attention in recent years because of its importance in mathematical problem solving and computation (Baroody, Torbeyns, & Verschaffel, 2009; Ching & Nunes, 2016; Nunes, Bryant, Barros, & Sylva, 2012; Nunes, Bryant, Evans, Bell, Garnder, Garnder, & Carraher, 2007; Verschaffel, Bryant, & Torbeyns, 2012). It has been argued that children may initially understand mathematical principles in the context of concrete referents (Bruner, 1960; Bryant, Christie, & Rendu, 1999; Gilmore & Papadatou-Pastou, 2009; Hughes, 1981; Piaget & Inhelder, 1971; Resnick, 1992; Vygotsky, 1962). Mathematics models various aspects of the world effectively by creating abstract structures that have properties shared with its real-world counterpart. We can manipulate and use the mathematical model to predict and make conclusions about events if the model acts in ways that truly corresponds to it. Some researchers have proposed that concrete materials can be used as an intermediary between the symbolic-mathematical world and the real world (Bruner, 1966; Piaget, 1952; Resnick, 1992). The concrete model is often considered more abstract than the actual situation, but less abstract than the mathematical model represented by numerical symbols. Thus, they may act as a vehicle through which children model the quantitative aspects of the real world. However, the notion that children's thinking is inherently concrete in nature is not universally accepted (Gelman, 2000, 2003; Gelman & Wellman, 1991). Some evidence suggests that concrete materials may facilitate the understanding of certain principles only (Canobi, Reeve, & Pattison, 2003). In the present study, we examined patterns of individual differences in children's knowledge of two essential principles in additive reasoning, namely the commutativity principle and the complement principle. Using a 'person-centered' approach (Bergman & Magnusson, 1997; Bisanz, Watchorn, Piatt, & Sherman, 2009; Laursen & Hoff, 2006), we aimed to explore patterns of individual differences in children's performance on different reasoning tasks (commutativity and complement principles) in different testing contexts (with and without the support of concrete materials).

1.1 The Importance of the Understanding the Commutativity and Complement Principles

Additive reasoning is based on quantities connected by part-whole relations. Two essential properties of part-whole relations are (1) the commutativity principle and (2) the complement principle (Kilpatrick, Swafoord, & Findell, 2001). Commutativity refers to the irrelevance of addend order to the sum, i.e. ‘a + b = c’ implies ‘b + a = c’, whereas the complement principle refers to the inverse relation between addition and subtraction, i.e. ‘a + b = c’ implies ‘c - a = b’. These two principles are important for children to learn mathematics because they contribute to (1) the understanding of the nature of number, (2) computational fluency, and (3) the ability to solve problems in a variety of situations.

Piaget (1952) argues that numbers are not simply a set of words in a fixed order, but they also reflect the part-whole logic of the number system. For example, the mastery of additive reasoning involves the integration of the commutativity and complement principles. One should understand that three quantities e.g., 3 + 4 = 7 can be expressed in four mathematical relations, e.g., 7 – 3 = 4, 4 + 3 = 7, 7 – 4 = 3, and 3 + 4 = 7, and that these four expressions can be deduced from each other. A thorough understanding of the part-whole relations of quantities involves the recognition that these expressions are essentially describing the same relation.

The conceptual understanding of these two principles may contribute to children’s computational fluency (Baroody, Torbeyns, & Verschaffel, 2009; Canobi, 2004; Canobi, Reeve, & Pattison, 2003; Nunes, Bryant, Hallett, Bell, & Evans, 2009). It has been suggested that this understanding may form the basis for children to develop more advanced computational strategies that help them modify complex problems to make them easier to solve (Bryant & Nunes, 2009; Canobi, 2004; Canobi, Reeve, & Pattison, 2003; Fuson, 1990; Nunes & Bryant, 1996, 2015). For example, some efficient strategies (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013; Shrager & Siegler, 1998), such as counting-all starting with the larger addend (CAL) and counting-on from the larger addend (COL), require the knowledge that the order of numbers does not affect the outcome in addition (i.e. the commutativity principle). The understanding of the commutativity principle may also foster the development of other strategies, such as the ‘ten-strategy’ and ‘addends-compare strategy’. For example, children who grasp the commutativity principle can transform the problem ‘3 + 6 + 7’ into ‘(3 + 7) + 6’ that is easier to solve (the ten-strategy). For some arithmetic problems, children do not need to calculate if they recognize that the identical addends that had been shown (though in different order e.g., ‘2 + 7 + 8’) in a previous problem had already been solved e.g., ‘8 + 7 + 2’. This addends-compare strategy may also require the understanding of the commutativity principle (Gaschler et al., 2013). An understanding of the inversion principle may also facilitate the use of ‘indirect addition’ in which children can use additions to solve subtraction problems effectively if the numbers are close to each other. For example, to solve ‘21 – 18’, it is less likely to make mistakes if they count up from 18 to 21. Some researchers have suggested that the complement principle contributes to the mastery of basic subtraction combinations (Baroody, 1983, 1984, 1985, 1999; Baroody & Ginsburg, 1986; Baroody et al., 1983; Fuson, 1988, 1992; Putnam, deBettencourt, & Leinhardt, 1990).

Understanding the commutativity and complement principles may also help children solve problems in a variety of situations. The solution to many story problems relies on the knowledge of the underlying relations between the quantities in the problem. Sometimes the relations are not obvious to problem solvers, especially when those problems whose solutions rest on the understanding of the inverse relation between addition and subtraction. For instance, children may not find a Change problem difficult when the missing information is the result of the change (e.g., 'David had 8 books. Then Peter gave him 3 more books. How many books does David have now?'). It is because the action in the story and the arithmetic operation required to solve the problem are consistent – A problem that involves a change that increases the quantity can be solved by addition, whereas one that decreases the quantity can be solved by subtraction. In contrast, when the starting situation is not known (e.g., 'Alex had some cookies. He gave 3 cookies to his mother and had 8 cookies left. How many cookies did he have before?'), problem solvers have to decide which arithmetic operation to use based on the information about the change and its end result. These start-unknown problems are more difficult (e.g., Carpenter, Hiebert, & Moser, 1981; De Corte & Verschaffel, 1987; Ginsburg, 1982) because the relation between the action described in the story and the operation is inverse, i.e., A problem that involves a change that decreases the quantity has to be solved by addition. Students must understand that the operation 'addition' can be conceived as the inverse of 'subtraction' and analyze the quantitative relations underlying the problem situation.

Knowledge of the commutativity principle may also relate to children's solving some missing addend problems (Nunes & Bryant, 2015). Consider this example 'Jane had 3 cookies, got some more and now has 7. How many more cookies did she get?' Children can easily solve this problem by representing the first addend with 3 fingers, counting up to the final state i.e. 7 fingers, and evaluated how many fingers they had to add in the process. However, if the problem has the first rather than the second addend missing e.g., 'Jane had some cookies; her mother gave her 4 more and now she has 7; how many did she have to start with?' the children have to understand that the order does not affect the total. Those who understand the commutativity principle can start from the second addend i.e. 4, add up to 7, and count how many were added. Children who do not understand commutativity may find this problem difficult to solve because they do not know how many cookies Jane to start with.

1.2. Using Concrete Materials to Facilitate Understanding

Given the importance of the understanding of the commutativity and complement principles in mathematics learning, we should identify ways to help children learn these principles. Some theorists and evidence suggest that young children can obtain cognitive benefits from exploring mathematical concepts with concrete materials (Bruner, 1960; Bryant, Christie, & Rendu, 1999; Gilmore & Papadatou-Pastou, 2009; Hughes, 1981; Piaget & Inhelder, 1971; Resnick, 1992; Vygotsky, 1962). Classic developmental theories contend that the acquisition of symbolic competence proceeds through a concrete-to-abstract shift: The progression from thinking that is based on concrete reality to thinking that is less constrained by context. For example, Piaget (1952) postulates that the development of the ability to reason with abstract hypothetical propositions without the help of more concrete information was the final stage of cognitive development. Piaget observed that children at the concrete operational stage had difficulty in reasoning about false propositions that included relations that could not happen in the real world.

Other popular theories have also seen development in terms of a transition from concrete to abstract. For example, in research of early categorization, Bruner (1966) argues that conceptual development is a perceptual-to-conceptual shift. At first, children can only think of objects in terms of the features that are directly available to their senses. After that, they start to consider abstract properties of objects. For instance, children may regard that bats and birds belong to the same category because they can fly. With development, children become realize that objects and living things can be categorized based on abstract and non-observable information. So eventually children understand that bats and birds should be in different categories because bats are mammals, and that even an animal that does not fly, such as penguins, can be in the bird category. Thus, the developmental shift is from a reliance on concrete properties to more abstract ones. Vygotsky (1962) also suggests that young children classify things in terms of themes that are highly concrete and salient properties. He argues for the important role of concreteness in symbolic play. When young children play, they often use concrete objects to substitute things in the real world (e.g., a stick for a horse). He thinks that the use of concrete objects in this manner is an initial form of symbolization because the children are less constrained to the properties of the objects in the game. Thus, pretend play is important for children's cognitive development in a way that it helps children to recognize that physical objects can be considered as a representation of something else.

Specifically to mathematical development, Resnick (1992) proposes that children progress through four qualitatively different kinds of mathematical thinking. At the first level, children reason about physical objects without references to specific numerosities. At the second level, children can reason about numbers that are connected with a particular and meaningful context. For instance, they can understand that adding three more apples to a collection of four apples makes seven apples. At the third level, children can reason about specific numbers that do not correspond to concrete referents (e.g., 3 + 4 = 4 + 3). Finally, children construct abstract understanding of the general nature of mathematical principles (e.g., a + b = b + a).

Because children may understand a concept in the context of concrete referents, some researchers posit that manipulatives can be used to enhance children's understanding of abstract principles. For example, Bruner argues "what is most important for teaching basic concepts is that the child be helped to pass progressively from concrete thinking to the utilization of more conceptually adequate modes of thought" (Bruner, 1960, p. 38). The conception of Bruner's 'enactive-iconic-symbolic' modes of representation forms the basis for a spectrum of teaching practices in mathematics education. One such adaptation of his model is the Concrete-Representation-Abstract (CRA) approach to mathematics instruction. This teaching approach recommends teachers to start with modes of instruction that are more concrete to students and then gradually replace the representations into forms that involve formal mathematical symbols. Ketterlin-Geller, Chard, and Fien (2008) argue that "a gradated instructional sequence that proceeds from concrete to representational to abstract (CRA) benefits struggling students" (p. 35). More recently, Fyfe, McNeil, Son, and Goldstone (2014) identify several potential benefits to using concrete materials. For instance, concrete materials are advantageous because they ground the learning of a concept in familiar and meaningful contexts (Schliemann & Carraher, 2002). They may increase understanding and memory by inducing physical or imagined action (Glenberg, Gutierrez, Levin, Japuntich, & Kaschak, 2004). Concrete materials may also allow learners to create their own knowledge of abstract ideas (Brown, McNeil, & Glenberg, 2009).

1.2.1 Evidence supporting the concrete-to-abstract shift for mathematical reasoning

In the domain of additive reasoning, do children at first succeed in problems that are set in the presence of concrete materials, and succeed only later when the problems refer to more abstract symbols, in the absence of objects? In one study, Hughes (1981) examined this question by giving three kinds of tasks to children at the age of 3 to 5. One task was called the ‘closed box task’ in which the children were shown a box containing a certain number of objects, and then the experimenter added or took away a certain number of objects. The second task was called a ‘hypothetical’ task in which the children were told about additions and subtractions in a hypothetical situation (either a box or a shop). In the first and second tasks, the children were asked to solve a problem that refers to quantities, such as, ‘I put 2 bricks in this box and then I put 1 brick in this box; how many bricks are in the box now?’ The third kind of task was called the ‘formal code’ task in which the children were given numerical problems without reference to any concrete materials or situations. For example, the children were asked, ‘What is 2 plus 1?’ Hughes found that the children performed well in the first two types of task that consisted of real or imagined concrete materials. In contrast, the children performed poorly in the formal code task in which no concrete referents were used or mentioned. This study suggests that some children may need concrete materials in order to make sense of the problem that they are going to solve.

Bryant, Christie, and Rendu (1999) showed that the accuracy for kindergartners to solve three-term inverse problems ($a + b – b = ?$) was significantly higher when the problems were presented with concrete objects. Studying an older age group (6-9 years old), Gilmore and Bryant (2006) also found that children were more accurate at solving inversion problems when they were presented in pictures. Sherman and Bisanz (2009) found that children (7-9 year-old) who solved equivalence problems (e.g., $5 + 4 = 7 + ?$) in a concrete condition had significantly higher accuracy rates than children who solved the same problems in a numerical context. In a study on using the CRA approach in teaching subtraction with regrouping, Flores (2010) showed that Grade 3 students improved significantly on fluency and confidence in calculations that involved subtractions. Other studies demonstrated evidence for the positive influence of using CRA to teach different types of mathematical knowledge, including fractions (Butler, Miller, Crehan, Babbit, & Pierce, 2003), word problems (Maccini & Hughes, 2000), simple linear functions (Witzel, 2003), and advanced linear functions (Witzel, Mercer, & Miller, 2003).The CRA approach has also been shown to be effective in facilitating the learning of students who had learning difficulties in mathematics (e.g., Butler et al., 2003; Maccinni & Ruhl, 2000; Morin & Miller, 1998; Peterson, Mercer, & O’Shea, 1998; Witzel, Mercer, & Miller, 2003).

Sowell (1989) conducted the first meta-analysis related to the effectiveness of using concrete materials to teach mathematics. The findings showed a small effect size in favor of the use of concrete materials. The meta-analysis of Gilmore and Papadatou-Pastou (2009) also demonstrated that children obtained benefits in identifying the inverse relation between addition and subtraction when problems were presented with more context, such as pictures that described the action in a story problem. A more recent meta-analysis by Carbonneau, Marley, and Selig (2013) also showed positive evidence for the use of concrete materials in teaching mathematics. They found that the effect sizes of using manipulatives to teach mathematics were small to moderate, compared with instruction that only involved abstract numerical symbols. However, the relation between teaching mathematics with concrete materials and students' learning was complicated by factors such as, instructional strategies and methodological limitations.

1.2.2 Concrete materials are not always beneficial

The notion that young children's thinking is inherently concrete in nature is not universally accepted. Some studies have shown that preschoolers use abstract concepts as a foundation for generating inferences and problem solving. Gelman and colleagues (Gelman, 2000, 2003; Gelman & Wellman, 1991) have shown that children understand that there is an internal 'essence' in certain things that is distinct from their external appearance. For example, children as young as 3 years of age could understand that oranges and lemons belong to the same category even though they are of different colors, whereas oranges and orange balloons only look alike in terms of their color. Thus, it has been suggested that young children's understanding of objects is not inevitably bound to external appearances (Deloache, 1987, 2000; Deloache, Miller, & Rosengrem, 1997; Uttal, Liu, & Deloache, 2006).

Consistent with this view, Cowan and Renton (1996) did not find any difference between children's performance on recognizing commutativity ($a + b = b + a$) in concrete and abstract contexts. Some research has demonstrated little to no benefits of using concrete materials in teaching (e.g., Ball, 1992; Baranes, Perry, & Stigler, 1989; Resnick & Omanson, 1987; Thompson, 1992; Uttal et al., 1997). Some research has even shown negative influence of concrete materials (e.g., Goldstone & Sakamoto, 2003; Kaminski, Sloutsky, & Heckler, 2008; McNeil, Uttal, Jarvin, & Sternberg, 2009; Sloutsky, Kaminski, & Heckler, 2005; Son, Smith, & Goldstone, 2008). For example, Kaminiski and colleagues (2008) showed that participants who learned through abstract symbols performed better than those who learned through concrete materials (e.g., measuring cups, pizza slices etc.). They also found that learning from a single abstract representation promoted transfer better than learning from multiple concrete representations. Some researchers have cautioned the use of concrete materials during learning because these materials contain distracting details that draw attention to themselves rather than their referents (e.g., Belenky & Schalk 2014; DeLoache, 2000; DeLoache & Bruns, 1994; DeLoache, Miller, & Rosengren, 1997; Kaminski et al. 2008; Uttal, Scudder, & DeLoache, 1997). Thus, the evidence remains inconclusive regarding whether concrete materials help children learn abstract concepts.

1.3 Different Principles in Additive Reasoning

The benefits of concrete materials in learning may depend on the type of concept that is involved. Canobi, Reeve, and Pattison (2003) assessed additive reasoning in different contexts (e.g., concrete and numerical) and revealed some interesting interactions between the use of concrete materials and the understanding of the logical relations in additive reasoning. They found that there was no difference in children's understanding of the commutativity principle in concrete and abstract contexts, whereas concrete materials facilitated children to understand the associativity principle. At present, it is not clear why concrete materials enhanced children's understanding of one principle but not the other even though they belong to the same conceptual category 'additive reasoning'. One possibility is that knowledge of these principles does not develop at the same rate and concrete materials may only benefit children to learn a concept that is not yet well developed.

Some evidence suggests that the understanding of the commutativity principle may develop earlier before that of other related addition concepts. For example, Canobi (2005) presented Combine problems with concrete objects (putting two sets of blocks together) and questions that only included numbers (e.g., 23 + 38) and asked children to determine whether knowledge of one addition result (a + b = c) is helpful for solving another addition question (b + a = ?). All the 7-year-olds in her study understood that the knowledge a + b helped to solve a question about b + a. Canobi also examined the understanding of the complement principle in the same group of children. She presented children with an operation and its result (e.g., 23 + 38 = 61) and asked them whether this information is helpful for solving another question (e.g., 61 – 23). She demonstrated that only 31% of the children recognized that they could use the inverse relation between addition and subtraction to solve the problems. This study suggests that there may be an order of acquisition of the understanding of the commutativity and complement principles because the same children solved both tasks. Thus, it is possible that concrete materials may not make a difference in children's performance for tasks that assessed commutativity knowledge because most of them may have already mastered the principle.

There is other evidence that suggests that the understanding of the complement principle may be developmentally more advanced than that of the commutativity principle. Baroody (1999) found that only 17% of the 4- to 7-year-old children showed understanding of the complement principle on at least 4 out of 6 problems. Canobi (2009) found that only about 30% of the 8-year-old children could use the complement principle to solve problems and 16% of those could explain the use of this principle correctly. Recently, Torbeyns, Peters, de Smelt, Ghesquière, and Verschaffel (2016) showed a wide range of individual differences in children's knowledge of the complement principle even at the ages of 9 to 10 years. They argued that the difficulties in children's understanding of the complement principle are in contrast with children's understanding of other related knowledge in additive reasoning, such as the commutativity principle. Thus, children's knowledge of the commutativity principle may develop before that of the complement principle. Because most of the previous research did not assess both principles together in the same study, more evidence is needed to evaluate whether there is an order of understanding of the principles (from the commutativity to the complement principle).

1.4 The Present Study

In summary, it remains to be verified (1) whether there is a progress in children's understanding of mathematical principles from concrete to abstract situations, and (2) whether concrete materials are beneficial for the understanding of certain principles only. According to Resnick's (1992) model and other classic developmental theories (e.g., Bruner, 1966; Piaget, 1952), mathematical knowledge evolves from a relatively concrete context to a relatively abstract one. Thus, in the present study, we hypothesized that knowledge of the commutativity and the complement principles develops over time rather than emerges in an all-or-nothing fashion – At first, children succeed in tasks that assess their knowledge of each of these principles when the problems are set in the presence of concrete materials, and succeed only later when the problems refer to more abstract materials, in the absence of objects.

We also hypothesized that there is an order of acquisition for these two principles. Existing evidence [e.g., Baroody, Wilkins, & Tiilikainen, 2003; Canobi, 2004, 2005; De Corte & Verschaffel, 1987; Sophian & McCorgray, 1994; Torbeyns et al., 2016; Wilkins, Baroody & Tiilikainen, 2001; Wright (cited in Nunes & Bryant, 1996)] suggests that schemas of young children are not entirely integrated. Although both principles rest on children's understanding of part-whole relations, commutativity refers to the simplest logical aspect of part-whole, which is that the order in which you add the parts does not affect the whole (a + b = b + a). Knowledge of the complement principle appears to be logically related to, but to require more than commutativity. In order to understand that if a + b = c, then c – b = a, and c – a = b, children need to think of (1) a and b as interchangeable parts (i.e. they need to understand commutativity) and that (2) if you take the first part away from the whole, you are left with the second, and if you take the second part away from the whole, you are left with the first.

Thus, two basic hypotheses are being tested in this study:

H1: Knowledge of the commutativity and of the complement principles develops from thinking in the context of specific quantities to thinking about more abstract symbols.

H2: There is an order of understanding of the principles – from the commutativity to the complement principle.

In this study, we used a 'person-centered' approach (Bergman & Magnusson, 1997; Laursen & Hoff, 2006) to examine these two hypotheses. The person-centered approach considers each individual holistically and aims to identify consistent and meaningful patterns of performance across variables within individuals. This approach is particularly relevant to studying children's development of conceptual knowledge of mathematics because research suggests that children do not develop an understanding of a mathematical concept in an all-or-none manner [e.g., Baroody, Wilkins, & Tiilikainen, 2003; Bryant, Christie, & Rendu, 1999; Canobi, 2004, 2005; Canobi, Reeve, & Pattison, 2003; De Corte & Verschaffel, 1987; Sophia & McCorgray, 1994; Torbeyns et al., 2016; Wilkins, Baroody & Tiilikainen, 2001; Wright (cited in Nunes & Bryant, 1996)]. Some children, for example, may demonstrate their knowledge in the presence of concrete materials only but not in a relatively abstract context. The existence of this group of children will be obscured if conceptual knowledge is assessed merely in one single context. Bisanz, Watchorn, Piatt, and Sherman (2009) also suggest that identifying profiles of children's understanding is a good way to highlight the diverse manners in which children can show their understanding. Therefore, through latent profile analysis, we aimed to explore patterns of individual differences in children's performance on different reasoning tasks (commutativity and complement principles) in different testing contexts (with and without the support of concrete materials).

The hypotheses lead to the following predictions in group identification. First, if children initially understand a principle in the context of concrete referents, all children who perform well for the principle in the abstract condition should also do well in the concrete condition. It is not expected to find a group of children who perform well in the abstract condition, but not in the concrete condition as well. Identification of the latter group of children would lead to falsifying the first hypothesis.

Second, if children develop the understanding of the commutativity principle before the complement principle, then all children who obtain high scores in the complement tasks should also perform well in the commutativity tasks, and one should not find children who perform well in the complement tasks but not in the commutativity tasks. If a sizeable group of children with the latter profile of performance were to be identified, this could not be attributed simply to error of measurement, but would be evidence against the hypothesis of an order of acquisition of these two principles.

One aspect in this developmental sequence remains unclear: Do children need to master the commutativity principle in the more abstract condition before they start to succeed in the complement principle tasks? Or is it sufficient to succeed in the commutativity tasks with concrete materials to make progress in the complement tasks with concrete materials? If the first alternative is true, one would expect to find four groups of children who display some knowledge of the principles: the first would succeed in commutativity tasks with concrete materials but in no other tasks; the second could succeed in commutativity tasks in both the concrete and abstract conditions, but not in complement tasks; the third would succeed in all commutativity tasks and in complement tasks with concrete materials; the final group would succeed in all the tasks. However, if the second alternative is true, and it is sufficient to succeed in commutativity tasks with concrete materials in order to make some progress in the complement principle, one should identify a fifth group, that succeeds in commutativity and complement tasks with concrete materials but in neither task in the abstract conditions only. It is, of course, possible to identify yet a sixth group, which has no success at all in any of the tasks. However, its identification is not relevant to the test of these alternatives.

2. Method

2.1 Participants

One hundred and fifteen children (61 boys, 54 girls) studying in three primary schools in Hong Kong participated in both waves of assessments in this longitudinal study. All of these children spoke Cantonese and attended the first year of primary school, with a mean age of 76.32 months (*SD* = 2.81 months, ranging from 67.8 to 82.1 months), during the first wave of assessment. The highest education levels attained by the mothers of the children in the sample were as follows: No schooling/pre-primary school level – 5.2%, primary school graduates –

20.8%, secondary school graduates – 57.4%, and university graduates – 16.5%. The relative distribution of education levels was similar to that of the overall Hong Kong population (Hong Kong Population Census, 2011), in which the majority of the population was secondary school graduates whereas a small proportion received no schooling or had pre-primary school educational level. School teachers of the participating children confirmed that all had intelligence within the range accepted as normal for their ages, and did not have learning difficulties or emotional/behavioral problems, such as, dyslexia, specific language impairments, attention deficits and hyperactivity disorders, or any neurological disorders.

2.2 Measures

2.2.1 Additive reasoning (the commutativity and complement principles).

The commutativity principle refers to the irrelevance of addend order to the sum, i.e. ‘a + b = c’ implies ‘b + a = c’, whereas the complement principle refers to the inverse relation between addition and subtraction, i.e. ‘a + b = c’ implies ‘c - a = b’. This study adapted a similar conceptual task used by Canobi, Reeve, and Pattison (2003) in which children were tested whether they could recognize conceptual relations between pairs of addition/subtraction problems. In general, children were shown a puppet that was going to solve two problems, namely base and target problems. The puppet ‘solved’ the base problem by counting very quickly and told the answer to the researcher, who then told the children that the answer was correct. After that, the children were shown a target problem and were asked to determine whether the puppet needed to count again to solve the problem or whether the puppet could find out the answer by ‘looking back’ at the base problem.

All of the problems were presented as story problems that involved a change in quantity (e.g., Mary has 3 fish and her mother gave her 5 more). We used Change problems instead of Combine problems because the performance of the children in the pilot study reached ceiling for the Combine problems that assessed the commutativity principle. This result was consistent with previous studies [e.g., Wilkins, Baroody & Tiilikainen, 2001; Wright (cited in Nunes & Bryant, 1996)] that showed that children performed better on commutativity tasks that were presented in the format of Combine problems.

The number of words in each problem did not vary considerably. The experimenter presented each child with a written version of the problem as it was read and kept it in front of the child until the problem was solved. For example, after showing the base and target problems that were printed on two separate cards, the researcher asked, ‘Now look at these two problems. If we gave Pika (the puppet) this problem next (pointing to the target problem), do you think Pika would need to count to work out the answer or could Pika look back at the problem he has already done (pointing to the base problem)?'

The conceptual judgment task involved two parts: a 'testing session' immediately after a 'warm-up session'. In the 'warm-up session', the children were given six practice problems to familiarize with the procedure. Half of the practice problems were identical (e.g., base: 4 + 4 and target: 4 + 4) and half of them were different (e.g., base: 4 + 3 and target: 6 + 7). Children were given feedback on whether they were correct in judging the same/different relation between the target and base problems. The answers were 100% correct for all participants in this session, indicating that they understood the task instructions.

In the 'testing session', the researcher showed six target problems in random order after asking the puppet to solve the base problem (e.g., Mary has 3 fish and her mother gave her 5 more). The target problems included (1) an identity problem, which was identical with the target problem (e.g., Mary has 3 fish and her mother gave her 5 more); (2) a different problem, which was completely unrelated to the target problem (e.g., Mary has 7 fish and her mother gave her 2 more). The identity and different problems were designed to detect possible responses biases, which may involve inattention, difficulty in understanding the procedure, and random responses. The accuracy rates for all the identity and different problems were 100%.

To assess children's knowledge in each of the additive reasoning principles, two types of items were used: test items and control items. Examples of these items are presented in Table 1.

Insert Table 1 about here

The test items were designed to assess children's understanding of a particular principle. They included (1) commutativity test items, which were related to the corresponding base problems on the basis of the commutativity principle (e.g., 5 + 3); (2) complement test items, which were related to the corresponding base problems according to the complement principle (e.g., 8 – 5).

Control items were included to detect whether children answer the question correctly because of biases. For example, children may answer that '3 + 5 = 8' is helpful for solving '5 + 3' correctly just because they realize that two numbers in the base problem (i.e. 3 and 5) are present in the target problem (5 + 3). These children may not understand the commutativity principle but simply have a response bias to say 'yes' when the numbers are the same. Children with such a response bias would also answer that '3 + 5 = 8' helps to solve the question '5 – 3'.

These control items included (1) commutativity controls: subtraction items that evaluate whether the children did not simply ignore the operation to make a judgment (e.g., 5 – 3); and (2) complement controls, which involved addition problems that comprised the sum and one term of the base problem added together (e.g., 8 + 5).

Thus, the control items did not serve to measure the constructs, but they were there to allow for a correction for response biases. A child was only credited one point if they answered both the test and the control items correctly. There were 6 commutativity items and 6 control items for commutativity; if the child passed one commutativity item and its control, the child was awarded one point; otherwise, no points were awarded. Similarly, there were 6 complement items and 6 control items for the complement principle; if the child passed one complement item and its control, the child was awarded one point; otherwise, no points were awarded.

The problems were presented to children in two conditions: one with bricks (concrete condition) versus another without bricks (abstract condition). The use of concrete materials was a within-subject factor. In the concrete condition, children were presented with the same problems as in the abstract condition but with bricks that represented the addends of the problems. In the warm-up session of the concrete condition, the experimenter modeled the problems by attaching or pulling apart groups of bricks. The bricks were attached to each other so that they could not be counted easily. Prior to the judgment task, the experimenter allowed the children to play with the bricks to ensure that the children felt comfortable with the bricks and recognized that the bricks were attached to each other. All of the bricks were of the same size and color.

In the concrete condition, the experimenter moved the bricks from a base problem card to a target problem card and at the same time attached and separated bricks in front of the children in order to help children identify the part-whole relations of the quantities. For example, the commutativity problems involved swapping the order of addends by moving the bricks representing these addends so that they were arranged in a different order. The complement problems involved adding or taking away the bricks representing the addends or subtrahends, respectively. The concrete and abstract conditions were conducted on two separate days. The possible range of scores for each principle (commutativity and complement) in each condition (concrete versus abstract) was 0 to 6. In all trials, feedback was not given. The internal consistencies of the additive reasoning measures were satisfactory (Commutativity-concrete:

Cronbach's α = 0.73; Commutativity-abstract: Cronbach's α = 0.82; Complement-concrete: Cronbach's α = 0.83; Complement-abstract: Cronbach's α = 0.86).

In all conditions, half of the problems had sums less than 10 (small number) and half of them had sums between 15 and 25 (large number). If the children solve the problems with their understanding of the commutativity and complement principles, they should be able to provide correct answers regardless of the size of the numbers involved in the problems. If the children solve the problems by calculating, they should make significantly more mistakes for large-number problems than small-number problems. Another indicator of the use calculation to solve problems is how long it takes for the children to respond. If they rely on calculation, exceptionally long response latency can be observed. In this study, it is likely that the children solved the additive reasoning tasks on the basis of conceptual knowledge rather than calculation because all of them responded within a short period of time (within 10 seconds) and the number size did not make a significant difference in accuracy rates for all conditions (*ps* > .05). However, the response latencies were not used for testing the hypotheses because it is not clear what they represent. A child who has long response latency may indicate that she or he is not good at the concept or simply slow to respond in general.

2.2.2 Mathematical achievement – Calculation.

We included mathematical achievement measures at Time 1 and Time 2 to evaluate the concurrent and predictive validity of the profiles identified by the classification analysis. It was measured by children's performance on sixteen simple calculation tasks and thirty-two story problems, all of which were designed with reference to the curriculum guide developed by the Hong Kong Education Bureau. Thirty items (15 addition and 15 subtraction) were constructed and tested in the pilot. On the basis of pilot findings, sixteen items (8 addition and 8 subtraction) were selected for each wave of data collection in the main study. Of these sixteen items, four are considered as "easy" (average correct rate: 70-100%), six are "moderate" in difficulty (average correct rate: 40-70%), and six are "difficult" items (average correct rate: 0-40%). At Time 1, children were orally presented with addition and subtraction combinations that involved eight addition of numbers up to 25 and eight subtractions from numbers less than 25 (i.e. 6 + 7; 3 + 8; 2 + 6; 9 + 16; 7 + 4; 2 + 16; 14 + 4; 11 + 7; 7 – 5; 9 – 6; 6 – 4; 12 – 3; 21 – 16; 22 – 18; 25 – 6; 18 – 5). At time 2, children were orally presented with ten addition and subtraction problems with large numbers (24 + 4; 8 + 19; 7 + 23; 21 + 5; 9 + 19; 28 – 9; 31 – 8; 27 – 5; 28 – 19; 26 – 8);

three 3-addend single digit problems (3 + 9 + 2, 7 + 2 + 4; 8 + 5 + 2), three 3-subtrahend single digit problems (8 – 4 – 3; 13 – 3 – 8; 15 – 7 – 5). A printed version of each calculation problem was presented as each problem was read and kept in full view of the child during problem solving. Feedback was not provided and no time limit was set. The maximum possible score for calculation was 16. The measures appeared to have good internal consistency (T1: α = .87; T2: α = .92).

2.2.3 Mathematical achievement – Story problem solving.

Similarly, eight types of word problems were tested from the same pilot study as in calculation. Thirty-two problems were chosen in the main study – On the basis of Riley, Greeno, and Heller's (1983) classification of story problems, Time 1 assessment included four result unknown Change problems, four start unknown Change problems, four change unknown Change problems, four unknown difference set Compare problems, four unknown compare set Compare problems, four unknown reference set Compare problems, four different unknown Combine problems, and four Equalize problems. At Time 2, the number of each type of problems was the same, except that two Combine problems were replaced by two more difficult 'de-combine transformations problems' (e.g., John played two games of marbles. In the second game he lost seven marbles. His final result, with the two games together, was that he had won three marbles. What happened in the first game?). For each type of problems, half of them (i.e. two for each type) involved small numbers (sum < 10), whereas half of them involved larger numbers (10 < sum > 20). To reduce the working memory demands of the task, the experimenter presented the each child with a written version of the story problem as it was read and kept it in front of the child until the problem was solved. In this way, children were easier to keep track of the contents and to make relevant judgments accordingly. The maximum possible score for story problem solving was 32. The measures appeared to have good internal consistency (T1: α = .92; T2: α = .91).

2.2.4 Control variables.

We included several control variables in this study (1) general intelligence, (2) demographic characteristics, (3) working memory, and (4) procedural counting. General intelligence was measured with Raven's Standard Progressive Matrices (Raven, Raven, & Court, 2003). Demographic information reported by parents in a questionnaire regarding children's sex and mothers' highest education level. Mothers' highest educational level has been regarded as a proxy variable for socioeconomic status. Among various indicators of socioeconomic status, Nunes, Bryant, Sylva, and Barros (2009) showed that mothers' highest educational level was the best predictor of mathematical achievement. Therefore, it was selected as the indicator of socioeconomic status in the present study.

Working memory was assessed with four tasks, including (1) digit span forward (phonological loop), (2) digit span backward (central executive), (3) counting recall (central executive), and (4) Corsi blocks (visuospatial) [Working Memory Test Battery for Children (WMTB-C); Pickering & Gathercole, 2001]. In the digit span forward task, children listened to a series of single-digit numbers and were asked to repeat the numbers in the correct order. All digits were presented at a rate of one per second. The series of numbers initially consisted of two numbers, and increased by one number after every other presentation, to a maximum of nine. Children were given one point for each sequence correctly recalled. The maximum possible score for this task was 16. The internal consistency of this task was satisfactory (Cronbach's $\alpha = 0.89$).

The digit span backward was similar to the digit span forward, except that the children were asked to recite the numbers backward. In counting recall, children were asked to count the triangles in a series of shape arrays and then to recall the total number of triangles in each series. The number of arrays started from two and increased by one array after every other presentation to a maximum of nine. The total number of correct trials was used as an indicator of participants' performance on these tasks. The maximum possible score for this task was 16. The internal consistency of this task was satisfactory (Cronbach's $\alpha = 0.81$).

The Corsi block task involved nine blocks and the experimenter tapped a sequence of blocks at a rate of one per second. Then, children were asked to replicate the sequence. The sequence involved two blocks initially and increased by one block every other presentation, to a maximum of nine. The maximum possible score for this task was 16. The internal consistency of this task was satisfactory (Cronbach's $\alpha = 0.83$). For each of the above tasks, there were two trials for each span length and testing was terminated when a child failed two trials of the same length. In each task, two practice items were given to the children and no feedback was given to the children in any of the testing trials.

Procedural counting was assessed with two tasks: oral rote counting and object counting. In oral rote counting, children counted some numerical sequences verbally in ascending and descending orders. They were first asked to count from 5 to 16 as a practice trial. There were then eight testing trials in which children were asked to count a set of numbers in ascending orders (e.g., 25 to 32; 56 to 63; 76 to 81; 118 to 123) and in descending orders (e.g., 46 to 38; 73

to 65; 34 to 27; 121 to 115). Testing within a set was discontinued when a child had committed errors on two sequences in a set. Children received one point for each sequence completed correctly. Another task, object counting, was also included as one of the measures of procedural counting to test whether the children could count correctly using one-to-one correspondence between words and objects. In object counting, they were required to count two trials of geometric shapes (e.g., circles, squares) and two trials of recognizable objects (e.g., pens, rubber). The numbers of objects were 6, 9, 13, and 15 for rubber, pens, squares, and circles, respectively. On any given trials, the objects were identical in appearance. Children received one point for each correct counting. The total scores for procedural knowledge of counting for each child was the sum of his/her performance on the oral rote and object counting tasks. The maximum possible score was 12. The internal consistency of this procedural counting task was satisfactory (Cronbach's $\alpha = 0.71$).

2.3 Procedure

This study was approved by a research ethics committee of the university. Participating children were recruited through local schools and non-profit child-related community centers in Hong Kong. Parents were informed of the study via letters sent home by teachers or/and administrators. Upon receipt of parental consent, the children were asked for verbal assent and participated individually with the author in a quiet location, which was separate from other children in the primary school or centre At Time 1 (first grade), the children were tested in two 30-40 min sessions separated by approximately 1 week. For all children, order of task presentation was the same. The first session included Raven's Standard Progressive Matrices, the central executive, phonological loop, and visuospatial sketchpad tasks, as well as the tasks that assessed children's knowledge of the commutativity and complement principles in the abstract condition. The second session involved tasks that assessed procedural counting, additive reasoning (concrete condition), calculation and story problem solving. At Time 2 (second grade), the children were tested in one session that lasted for approximately 10 to 20 minutes in which calculation and story problem solving tasks were administered. For each child, the interval between the first and second wave of assessments was between 9 and 11 months, with 10 months being the commonest interval (83%). For all children, testing was conducted by a researcher in Cantonese during the day.

2.4 Overview and Explanation of the Analysis

We aimed to uncover groups of children who show similarities and differences in their performance on the reasoning tasks in the different testing conditions (with or without concrete materials). With latent profile analysis, the profiles are inferred from the patterns of observed responses. Children are classified into latent classes on the basis of similar patterns of observed data (Magidson & Vermunt, 2002; Pastor & Gagné, 2013). On the basis of the hypotheses, we expected to find five groups of children with latent profile analysis (Table 2):

Insert Table 2 about here

The first group of children would obtain high scores in all the tasks. The second group would have high scores in all commutativity tasks and in complement tasks with concrete materials only. The third group would get high scores in commutativity tasks in both concrete and abstract conditions, but not in complement tasks. The fourth group would have high scores in commutativity tasks with concrete materials but not in other tasks. The final group would get low scores in all of the tasks. This group is not relevant to the test of the hypothesis, but it is reasonable to expect that some children will not succeed in any of the tasks.

There are classification techniques other than latent profile analysis that serve similar analytical purposes, such as cluster analysis, but latent profile analysis is a better technique to address the hypotheses of this study. First, cluster analysis is a non-inferential procedure, which means that we cannot assume the identified clusters are applicable to the population. The groups are arbitrarily based on the similarities among individuals in a particular sample. In contrast, it is assumed in latent profile analysis that there is an unobserved latent variable in the population that explains responses on the indicator variables. Thus, it takes a confirmatory approach, which is more appropriate than cluster analysis for the present study. It is because the hypotheses and predictions were set up according to developmental theories and previous research. Second, no statistical tests are available to evaluate the clustering solution in cluster analysis. Thus, the decisions for the groupings are rather subjective (Hair, Anderson, Tatham, & Black, 1998; Magidson & Vermunt, 2002; Whiteman & Loken, 2006). In contrast, latent profile analysis employs rigorous statistical measures of fit to facilitate group identifications in a given population (Pastor, Barron, Miller, & Davis, 2007). The application of statistical tests of model-data fit confers a major advantage of mixture modeling over cluster analysis by allowing researchers to identify groups in a more objective manner. Comparative and simulation research has demonstrated that latent profile analysis offers more accurate grouping solutions than cluster analysis (DiStefano & Kamphaus, 2006; Magidson & Vermunt, 2002; Whiteman & Loken 2006).

The latent profile analysis was conducted using MPlus7 (Muthén & Muthén, 2007). In order to determine the optimal number of classes, we evaluated various models against a number of criteria. First, models with different numbers of classes were compared using information criteria (IC)-based fit statistics. These include the Bayesian Information Criteria (BIC; Schwartz, 1978), Akaike Information Criteria (AIC; Akaike, 1987), and Adjusted BIC (Sclove, 1987). Lower values on these fit statistics represent better model fit. Second, the accuracy with which models group children into their most likely class was examined. Entropy is a statistic that evaluates this accuracy, and can range from 0 to 1, with higher scores indicating higher classification accuracy. On the basis of the hypotheses, we expected to obtain the lowest values on the IC statistics and the highest value on Entropy in the five-class model.

Third, a statistical model comparison likelihood ratio test was used – the Lo–Mendell–Rubin test (LMR; Lo, Mendell, & Rubin, 2001). This test compares the fit of a target model (e.g., four-class model) to a comparison model that specifies one less class (e.g., three-class model). The *p*-value generated for the LMR test indicates whether the solution with more classes ($p < .05$) or less classes ($p > .05$) fits better. In the subsequent analysis, we expected that the five-class model would fit better than the four-class model, whereas the inclusion of more classes would not show improvement, suggesting that the five-class model is the most parsimonious model.

An additional consideration is the size of each class because a small class could lead to unstable parameter estimates if the sample size is not large. The final and the most important consideration of model selection is whether the classes make theoretical sense. Thus, we would examine the patterns of variables within each class to see whether a particular classification provides any meaningful information (McLachlan & Peel, 2000; Pastor & Gagné, 2013). In summary, deciding on the number of classes involve the consideration of theories, statistical indices, parsimony, and the substantive meaning of each solution (Bauer & Curran, 2003).

3. Results

3.1 Profile Identification

Latent profile models containing 3, 4, 5, and 6 classes were fit to the data. The model fit indices for each LPA are available in Table 3.

Insert Table 3 about here

To determine the optimal number of classes, we began by reviewing the IC indices (AIC, BIC, and SSABIC) and the entropy values. The indices of all profile models were close to each other. Thus, it is difficult to make a clear-cut decision merely based on the IC indices and the entropy values. However, the three-class model was not selected as the optimal model because Class 2 contains all children except the high-performing and low-performing children. This model is not meaningful in theoretical sense because this classification made no distinctions in the middle group. The LMR test also showed that the inclusion of one more class (i.e. the four-class model) fit better than the three-class model ($p< 0.05$).

The six-class model was also not chosen as the optimal model because it yielded a class size that was too small to be of substantive value (9 children only). The LMR test also indicated that it did not fit better than the five-class model ($p> 0.05$). Comparing the four- and five-class models, the LMR test showed that the five-class model was not better than the four-class model ($p > 0.05$). The four-class model also revealed lower AIC, BIC, and SSABIC values than the five-class model.

In summary, the four-class model gave a solution that fits the model better than the three-class model and the other comparisons did not show an improvement. Thus, the latent profile analysis led us to choose the four-class model as the best fit to the data in this study. This finding did not support our hypothesis because we expected to find five distinct groups of children. Having decided on the four-class model, the next step is to examine whether these classes show a relation to the groups identified in our hypotheses. It is thus necessary to examine the characteristics of the children within each class to identify which hypothetical group of children is missing.

3.2 Description of Classes

The means and standard deviations of each class in the four-class model are presented in Table 4 and in Figure 1 graphically.

Insert Table 4 about here

Insert Figure 1 about here

Children in Class 1 had good performance of the commutativity and complement principles in both the concrete and abstract conditions. Within-group comparisons showed that the children in this Class did not differ significantly in their performance across the four conditions (all comparisons showed $p > .05$).

Children in Class 2 demonstrated good performance on the commutativity tasks in both conditions, but their performance on the complement tasks was high only in the presence of concrete referents. It appears that they could not think about the inverse relation between addition and subtraction without the support of concrete materials. Within-group comparisons showed that the only significant differences happened between the 'complement-abstract' condition and the other three conditions (all comparisons showed $p < .05$).

In Class 3, children demonstrated good performance on the commutativity tasks, but they did not perform well on the complement tasks. Within group comparisons indicated that the children in this Class performed significantly better in the two commutativity conditions than the two complement conditions (all comparisons showed $p < .05$).

Finally, children in Class 4 showed good performance on the commutativity tasks in the presence of concrete materials only, but they did not do well on all of the complement tasks. Although it may appear that the children in this Class performed poorly across all four conditions, a closer look at the performance difference between the 'commutativity-concrete' and the 'commutativity-abstract' conditions showed that they were significantly better at understanding the commutativity principle with objects than in the abstract condition ($p < .05$).

A comparison of the present results and my hypothetical predictions of children's profiles is summarized in Table 5.

Insert Table 5 about here

It shows that while the hypothetical Class 5 (children who demonstrate low competence in all tasks) was absent in the data, the existence of the first four Classes were supported by the latent profile analysis. Although Class 4 did not have a truly 'high' score (only 2.21) on the commutativity tasks in the concrete condition, the score was significantly higher than that on the same tasks in the abstract condition (1.65), which supports our prediction.

3.3 Validation Analyses – Mean Differences in Time 1 Mathematical Achievement

In order to provide additional evidence for the meaningfulness of the classes, we examined whether the profiles predict children's mathematical achievement. In this study, it was hypothesized that children with a higher additive reasoning ability perform better in mathematics. Thus, if we find that a group of children that is characterized by high levels of additive reasoning shows significantly higher scores on calculation and story problem solving than a group of children that exhibits a weaker profile, both concurrently and longitudinally, then we can demonstrate some validity for the four-class model.

We entered the two variables – calculation scores and scores on solving story problem at Time 1 – in the latent profile analysis as 'auxiliary variables' using MPlus7 to assess the equality of means. Comparisons of the means of validity variables across classes are summarized in Table 6. The results showed that all four classes were statistically differed from each other on calculation scores and story problem solving scores at Time 1 (Class 1 > Class 2 > Class 3 > Class 4). This finding provides some evidence for the validity of the four-class model.

Insert Table 6 about here

3.4 Validation Analyses – Using Profiles to Predict Time 2 Mathematical Achievement

The test of validity can be obtained from analyses that assess whether the profiles that were obtained at an earlier time predict children's performance in calculation and story problem solving at a later time beyond the effects of age, IQ, working memory, and counting ability. Because the latent classes are categorical, they were dummy-coded and entered into a series of multiple regression analyses.

Class 4 served as the comparison group (i.e. the group coded 0) in the first set of regression analyses on T2 calculation and story problem solving as criterion variables. Table 7 shows that the means of Classes 1 to 3 were significantly higher than the mean of Class 4 [all standardized beta-values were significant and positive (*ps* < .001)]. Using Class 3 as the comparison group, the model showed that the means of Classes 1 and 2 were significantly higher than the mean of Class 3 (*ps* < .001). When the comparison group was Class 2, the standardized beta-value indicated that the mean of Class 1 was significantly higher than that of Class 2 (*ps* < .001).

Insert Table 7 about here

4. Discussion

This study examined children's patterns of addition competencies in different tasks. Three main findings are noteworthy. First, knowledge of the commutativity and of the complement principles seems to develop from thinking in the context of specific quantities to thinking about more abstract symbols. Second, there may be an order of understanding of the principles – from the commutativity to the complement principle. Third, children may acquire the knowledge of the commutativity principle in the more abstract tasks before they start to acquire the knowledge of the complement principle. In this discussion, we elaborate more specifically on the key findings.

4.1 The Role of Concrete Materials

On the basis of theories and research that suggest that children's thinking becomes increasingly more abstract as they grow older (e.g., Bruner, 1966; Bryant, Christie, & Rendu, 1999; Hughes, 1981; Piaget & Inhelder, 1971; Resnick, 1992; Sherman & Bisanz, 2009; Sowell, 1989), a concrete-to-abstract ordering in children's responses to both commutativity and complement problems is expected. We hypothesized that at first, children succeed in tasks that assess their additive reasoning when the problems are set in the presence of concrete materials, and succeed only later when the problems refer to symbols, in the absence of objects. This hypothesis is supported by the findings that all children who performed well for the principles in the abstract condition also did well in the concrete condition. There was not a group of children who performed well in the abstract condition only but not in the concrete condition as well. Thus, the conceptual profiles suggest that some children may initially understand additive reasoning in the context of concrete referents.

This finding is not consistent with what Kaminiski and colleagues (2008) showed that abstract symbols, compared with concrete materials, facilitated mathematical performance. However, it has to be noted that the participants in their study were undergraduate students, whereas those in the present study were children at the age of 6 years. Young children may derive greater cognitive benefits from concrete materials relative to adults. One reason for this may be that children are assumed to have a greater reliance on interaction with their physical environment to construct meaning and to gain proficiency with higher-level representations (Bruner, 1966; Piaget, 1952).

In contrast to the findings by Canobi, Reeve, and Pattison (2003) that children performed equally well on commutativity tasks in concrete and abstract conditions, the present study showed that children were more accurate at recognizing the commutativity principle when they were given some concrete materials to think about quantities. One factor that may explain the divergent results is that a more stringent test for commutativity was adopted in this study. In both studies, children were asked to judge whether a puppet could use a solved problem to figure out the answer to another. In Canobi et al.'s study, children were asked to judge if 'a + b' was helpful for a puppet to solve 'b + a'. For these tasks, there are two possibilities for why children can give a correct answer: they may truly understand the commutativity principle, but it is also likely that they produce a correct response without paying attention to the order of the addends. That is, they may say 'a + b' is helpful for a puppet to solve 'b + a' simply because both numbers 'a' and 'b' are present. If the second possibility is true, then the commutativity task used in Canobi et al.'s study may overestimate children's understanding of the principle, which may also reduce the variations observed in their study.

By contrast, the present study used a more stringent measure that required children to solve two types of items: test items and control items. The test items were designed to assess children's understanding of a particular principle (e.g., whether the puppet can solve '5 + 3' without counting if it knows that '3 + 5 = 8'), whereas the control items (e.g., whether the puppet can solve '5 – 3' without counting if it knows that '3 + 5 = 8') were used to allow for a correction for response biases. Children were considered to understand the principles only if they answered both test and control items correctly. This more stringent test may represent a more valid measure of commutativity understanding and may capture more variations in children's responses.

Helping the children to represent reasoning problems with concrete materials may lead children to pay more attention to the relations between quantities. From an educational perspective, some children may need support from concrete materials to reason about how quantities can be physically combined and separated in order to learn about addition and subtraction concepts. However, for both the commutativity and complement knowledge, there are children who performed well in the concrete condition, but not in the abstract condition, even though the additive relations assessed in the conditions are the same. This finding is consistent with previous work that some children were not able to see the connection between solving mathematical problems with physical objects and solving the same or similar problems with abstract symbols (DeLoache, 1991, 1995, 2000; DeLoache & Bruns, 1994; DeLoache, Miller, & Rosengren, 1997; Uttal, Scudder, & DeLoache, 1997). These children may not consider the solutions to the additive reasoning problems that they derive from concrete materials are also relevant to similar but more abstract versions of the same problems. According to the dual representation hypothesis (DeLoache, 1991, 1995, 2000; DeLoache & Bruns, 1994; DeLoache, Miller, & Rosengren, 1997), children must represent the concrete model itself as an object and, at the same time as a symbol for what it represents. More research is needed to examine the conditions under which concrete materials are effective in facilitating the transfer of children's understanding of mathematical principles from concrete to abstract situations (Fyfe, McNeil, Son, & Goldstone, 2014; McNeil & Fyfe, 2012).

4.2 The Order of the Understanding of the Principles

We also hypothesized that there is an order of understanding of the principles – from the commutativity to the complement principle. The finding that all children who obtained high scores in the complement tasks also performed well in the commutativity tasks supported this hypothesis. There was not a group of children who performed well in the complement tasks only but not in the commutativity tasks as well. There has been a discussion in the literature concerning the sequences in which children learn about the commutativity and complement principles. For example, Resnick (1992) suggests that the two principles are not distinct but governed by a single part-whole schema. By contrast, some evidence challenges the generality and coherence of the part-whole schema in children [e.g., Baroody, Wilkins, & Tiilikainen, 2003; Canobi, 2004, 2005; De Corte & Verschaffel, 1987; Sophia & McCorgray, 1994; Torbeyns et al., 2016; Wilkins, Baroody & Tiilikainen, 2001; Wright (cited in Nunes & Bryant, 1996)] by showing that children's early knowledge in mathematics is compartmentalized. These studies suggest that various mathematical schemas in children are weak and loosely connected.

The present study supports the latter view and Canobi's (2005) finding that whereas all 7-year-old children mastered the commutativity principle, whereas only one-third of the children demonstrated understanding of the complement principle. Our results suggest that some children may develop commutativity knowledge prior to their complement knowledge. Although the present study showed evidence that is consistent with the concrete-to-abstract shift hypothesis in Resnick's (1992) model, her model regarding the development of different mathematical principles might be more complete if it also considers the incompleteness of children's emerging concept of addition. For these children, the knowledge of the commutativity principle may represent part of the conceptual basis for understanding the complement principles that involve the appreciation of the consequences of subtracting a part from the whole. They may find it easier to master how two parts are added together to form the whole before understanding concepts that involve subtraction.

Do children need to grasp the commutativity principle in more abstract tasks before they start to succeed in the complement principle tasks? Or is it sufficient to succeed in the commutativity tasks with concrete materials to make progress in the complement tasks with concrete materials? The present findings showed that the first alternative is more likely to be true. On the basis of additive reasoning performance, four groups of children were identified: The first group succeeded in commutativity tasks with concrete materials but in no other tasks; the second succeeded in commutativity tasks in both concrete and abstract conditions, but not in complement tasks; the third group succeeded in all commutativity tasks and in complement tasks with concrete materials, and the final group succeeded in all the tasks. The patterns of individual differences in additive reasoning may reflect a developmental trend. It is likely that children acquire the knowledge of the commutativity principle in more abstract tasks before they start to acquire the knowledge of the complement principle. If the second alternative were true, that it is sufficient to succeed in commutativity tasks with concrete materials in order to make some progress in the complement principle, one would have identified a fifth group, that succeeded in commutativity and complement tasks with concrete materials but in neither task in the abstract conditions only. In contrast to this prediction, this group of children was not found in the present study.

However, additive reasoning was measured at Time 1 only. A solid conclusion on the development of additive reasoning cannot be made on the basis of cross-sectional data because there can be different interpretations of the findings regarding the developmental trajectory of the two types of knowledge. The first possibility is that the children in different subgroups develop along the same developmental path but in different rates. That is, all children go through the same developmental process (from commutativity to complement, from concrete to abstract for each principle) in which some children are faster in completing their understanding of the part-whole relations in addition and subtraction. The second possibility is that the profiles may indicate different paths of development. Whereas some children develop the concepts in a piecemeal fashion, for other children both types of understanding develop together. For example, those children who succeeded in all tasks may have acquired the understanding of commutativity and complement principles at the same time. Thus, longitudinal studies that keep track of the conceptual development of the same group of children are needed to examine whether the distinct profiles found in the present study indicate a single path of development or differential paths of development.

4.3 Educational Implications

One educational implication of the present study concerns the assessment of additive reasoning in young children. The latent profile analysis suggests that patterns of individual differences are present in the development of different aspects of additive reasoning. It would not be easy for teachers to devise strategies that are tailored to the needs of different children if they do not know anything about the particular strengths and weaknesses of each child. This study shows that the needs of different children may relate to the developmental order of the commutativity and complement principles, and the role of concrete materials in this development. Thus, in order to provide more appropriate teaching support, teachers may need to understand more about the developmental stage where each child is placed. The present study supports the view of Bisanz and colleagues (2009) by showing that testing different additive concepts in different contexts and identifying profiles with classification analyses may be useful for the assessment and teaching of addition reasoning. Using symbolic tasks alone, for example, may underestimate children's understanding of a mathematical principle at the quantity level. It seems that a more fine-grained assessment of additive reasoning can be achieved by incorporating both concrete materials and abstract symbols in the assessment. Subsequent intervention efforts can target the specific areas of deficit associated with a given profile.

This study may also have implications in teaching mathematics to children. The findings showed that some children's additive reasoning develops from thinking in the context of concrete quantities to thinking about more abstract symbols, but none of the participating children showed evidence that suggests the reverse order. In other words, learning about numbers and arithmetic may start from situations in which children are invited to connect it meaningfully to the reality, rather than the other way round. One implication is that learning about how quantities are logically connected to each other may be assigned a higher priority than learning formal operations of addition and subtraction in early mathematics education. Concrete materials are important because they enhance children's ability to represent quantities and relations between quantities. Thus, in order to search for meaningful mathematics teaching, educators should find ways to keep teaching connected to quantities in the world. Children may be given simple representational tools, such as blocks and diagrams that represent information about relations to solve problems, before they are taught about formalizations.

4.4 Limitations and Future Directions

The present study represents one of the few studies that examine patterns of individual differences in children's understanding of addition concepts with a person-centered approach in a group of non-Caucasian children. It also demonstrates that the resulting classes from latent profile analysis are meaningful, showing that the class differences predicted children's performance in calculation and story problem solving beyond the effects of age, IQ, working memory, and counting ability. However, this study is limited in some ways that merit cautious interpretations of the findings. First, future studies are needed to examine whether the classes are replicable in another sample. Second, additive reasoning was only measured once at Time 1. The findings from the latent profile analysis based on the cross-sectional data do not allow us to make credible conclusions about the developmental trajectory of additive reasoning. Longitudinal studies that keep track of the conceptual development of the same group of children are needed to examine whether the distinct profiles found in the present study indicate a single path of development or differential paths of development. It would be interesting to explore whether the number of classes remains the same and whether each child's class membership is stable or changes over time. Third, future research may also include other types of tasks to assess additive reasoning, such as justification tasks (Prather & Alibali, 2009). The use of multi-faceted assessments may contribute to our knowledge about the variety in developmental trajectories of additive reasoning. However, justifying the use of a logical principle is not easy for young and mathematically weaker children. For example, a couple of studies have shown only minor successes for children to adequately justify the application of the complement principle (Baroody, 1999; Dowker, 2014; Torbeyns et al., 2016). Thus, it appears that developing age-appropriate versions of this type of measures for young children remains a challenge.

4.5 Conclusion

In summary, this study examined patterns of individual differences in children's understanding of the commutativity and complement principles. Latent profile analyses suggest that concrete materials help children recognize addition concepts and the acquisition of the commutativity principle may precede that of the complement principle. The findings contribute to the understanding of the development of addition concepts in children by showing that the two concepts of addition may not be initially integrated. A stronger addition schema that coherently embodies all principles in additive reasoning evolves gradually and may be aided by teaching with concrete materials. Assessment with both symbolic and concrete materials as well as profile identification may be conducive to a more valid assessment of children's understanding of the commutativity and complement principles.

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Table 1. Types of target problems, examples, and their purpose

Base Problem	Types of its Corresponding Target Problems	Purpose of the Target Problems
Mary has 3 fish and her mother gave her 5 more. How many fish does Mary have now? (Answer: 3 + 5 = 8)	*Commutativity test item:* 'Mary has 5 fish and her mother gave her 3 more. How many fish does Mary have now?'	To test children's understanding of the commutativity principle. The base problem should be helpful to solve this item because the answer of '5 + 3' can be deduced by '3 + 5 = 8' according to the commutativity principle.
	Commutativity control item: 'Mary has 5 fish and her mother took away 3 from her. How many fish does Mary have now?'	To allow for a correction for response biases. The base problem should not be helpful to solve this control item because the answer of '5 - 3' cannot be deduced by '3 + 5 = 8' according to the commutativity principle.
	Complement test item: 'Mary has 8 fish and her mother took away 5 from her. How many fish does Mary have now?'	To test children's understanding of the complement principle. The base problem should be helpful to solve this item because the answer of '8 – 5' can be deduced by '3 + 5 = 8' according to the complement principle.
	Complement control item: 'Mary has 8 fish and her mother gave 5 more to her. How many fish does Mary have now?'	To allow for a correction for response biases. The base problem should not be helpful to solve this control item because the answer of '8 + 5' cannot be deduced by '3 + 5 = 8' according to the complement principle.

Table 2. Predictions of children's profiles on their understanding of mathematical principles

	Commutativity-concrete	Commutativity-abstract/numerical symbols	Complement-concrete	Complement-abstract/numerical symbols
Class 1	High score	High score	High score	High score
Class 2	High score	High score	High score	Low score
Class 3	High score	High score	Low score	Low score
Class 4	High score	Low score	Low score	Low score
Class 5	Low score	Low score	Low score	Low score

Table 3. Model fit indices for each class model (N = 115)

	3 classes	4 classes	5 classes	6 classes
AIC	1356.996	1341.798	1344.278	1340.597
BIC	1406.405	1407.412	1418.656	1431.179
SSABIC	1349.51	1330.154	1334.713	1326.873
Entropy	0.962	0.898	0.862	0.870
LMR test	2 vs 3	3 vs 4	4 vs 5	5 vs 6
	value = 69.70	value = 21.80	value = 11.98	value = 10.75
	$p < 0.01$	$p = 0.02$	$p = 0.16$	$p = 0.33$
Number of children	C1=21	C1=21	C1=21	C1=9
for each class (C)	C2=59	C2=21	C2=38	C2=12
	C3=35	C3=38	C3=21	C3=38
		C4=35	C4=17	C4=18
			C5=18	C5=21
				C6=17

Table 4. Means and standard deviations of each class in the 4-class solution

	Means (standard deviations)			
	Commutativity-concrete	Commutativity-abstract	Complement-concrete	Complement-abstract
Class 1	5.18 (0.65)	5.24 (0.69)	5.16 (0.73)	5.11 (0.63)
Class 2	5.16 (0.68)	5.26 (0.64)	5.14 (0.78)	1.56 (0.71)
Class 3	4.98 (0.72)	5.22 (0.55)	1.43 (0.62)	1.35 (0.72)
Class 4	2.21 (0.59)	1.62 (0.71)	1.30 (0.67)	1.26 (0.65)

Table 5. A comparison of the results with the hypothetical predictions of children's profiles

	Commutativity-concrete	Commutativity-abstract/numerical symbols	Complement-concrete	Complement-abstract	**Results**
Class 1	High score	High score	High score	High score	**Present**
Class 2	High score	High score	High score	Low score	**Present**
Class 3	High score	High score	Low score	Low score	**Present**
Class 4	High score	Low score	Low score	Low score	**Present**
Class 5	Low score	Low score	Low score	Low score	**Absent**

Table 6. Means and standard errors of each class on calculation and story problem solving at T1

	Means (standard errors)			
	Class 1	Class 2	Class 3	Class 4
Time 1 Calculation	13.35 (0.40)	11.62 (0.52)	10.21 (0.36)	8.05 (0.43)
Time 1 Story Problem Solving	26.14 (0.69)	23.34 (0.68)	21.03 (0.50)	17.57 (0.55)

Note. The maximum scores of T1 and T2 calculation are 16.

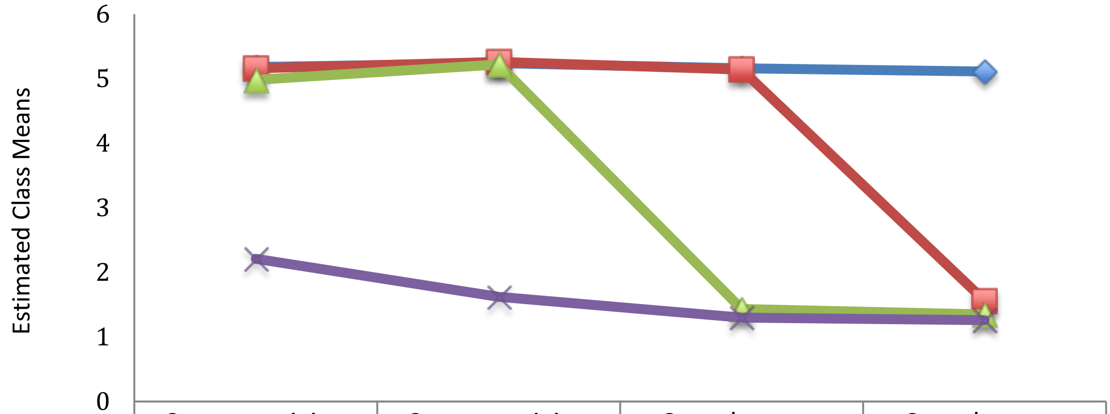
The maximum scores of T1 and T2 story problem solving are 32.

Table 7. Standardized beta-values of dummy-coded cluster variables in multiple regression analyses on children's performance on calculation and story problem solving at Time 2

	T2 Calculation		T2 Story problem solving	
	standardized betas	*t-values*	*standardized betas*	*t-values*
Class 1 vs Class 4	0.78***	9.03	0.85***	9.50
Class 2 vs Class 4	0.55***	6.05	0.66***	7.02
Class 3 vs Class 4	0.27***	3.28	0.36***	4.16
Class 1 vs Class 3	0.49***	6.17	0.47***	5.72
Class 2 vs Class 3	0.25**	2.98	0.26**	3.04
Class 4 vs Class 3	-0.25***	-3.28	-0.33***	-4.16
Class 1 vs Class 2	0.26***	3.55	0.22**	3.00
Class 3 vs Class 2	-0.23**	-2.98	-0.23**	-3.04
Class 4 vs Class 2	-0.45***	-9.06	-0.54***	-7.02

** *p-values* significant at the 0.01 level, *** *p-values* significant at the 0.001 level

Figure 1. Estimated class means of the four-class solution using latent profile analysis



	Commutativity-concrete	Commutativity-abstract	Complement-concrete	Complement-abstract
Class 1	5.18	5.24	5.16	5.11
Class 2	5.16	5.26	5.14	1.56
Class 3	4.98	5.22	1.43	1.35
Class 4	2.21	1.62	1.3	1.26