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Variational problems with singular perturbation

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Abstract

In this paper, we construct the local minimum of a certain variational problem which we take in the form

$$\inf \int_{\Omega} \left\{ \frac{\epsilon}{2} k g^2 |\nabla w|^2 + \frac{1}{4\epsilon} f^2 g^4 (1 - w^2)^2 \right\} \mathrm{d}x,$$

where ϵ is a small positive parameter and $\Omega \subset \mathbb{R}^n$ is a convex bounded domain with smooth boundary. Here $f, g, k \in C^3(\Omega)$ are strictly positive functions in the closure of the domain $\bar{\Omega}$. If we take the inf over all functions $H^1(\Omega)$, we obtain the (unique) positive solution of the partial differential equation with Neumann boundary conditions (respectively Dirichlet boundary conditions).

We wish to restrict the inf to the local (not global) minimum so that we consider solutions of this Neumann problem which take both signs in Ω and which vanish on $(n-1)$ dimensional hypersurfaces $\Gamma_\epsilon \subset \Omega$. By using a Γ-convergence method, we find the structure of the limit solutions as $\epsilon \to 0$ in terms of the weighted geodesics of the domain Ω.

MSC: 35K60; 35B25

Keywords: Singular perturbation; Interface shape and motion

1. Introduction

We consider the (semilinear) equation for $u(x, \epsilon)$ defined for $x \in \bar{\Omega} \subset \mathbb{R}^n$, and $\epsilon > 0$,

$$\epsilon \nabla \cdot (k(x) \nabla u) + \epsilon^{-1} V_u(x, u) = 0, \qquad x \in \Omega, \tag{1.1}$$

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where ϵ is a small parameter and V is a bistable potential for u; here V depends on $x \in \Omega$ and V is even in u. We assume that u satisfies the boundary conditions

$$\partial_n u = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where ∂_n denotes the outward normal derivative on $\partial\Omega$ (assumed smooth).

We consider the symmetric in u bistable case for $V(x,u)$, when V has two local minimisers $\pm g(x)$. Thus our reaction term is a function $V_u(x,u)$ odd in u with first strictly positive zero $g(x) > 0$ for $x \in \bar{\Omega}$ such that $V_u(x, g(x)) \equiv 0$. Suppose that $V_{uuu} < 0$ and $V_u > 0$ for $0 < u < g(x)$ for all $x \in \bar{\Omega}$; then we may write $V_u(x,u) \equiv f^2(x,u)(g^2(x) - u^2)u$, for $f(x,u) > 0$ for $x \in \bar{\Omega}$, $0 < u < g(x)$. Note that $f(x,u)$ is even in u. For simplicity, we consider the case $f(x,u) \equiv f(x)$; henceforth, we rewrite $V(x,u) = \frac{1}{4} f^2(x)(g^2 - u^2)^2$.

Let $u = gw$, then we rewrite Eq. (1.1) as, for $M \equiv \frac{\nabla\cdot(k\nabla g)}{f^2 g^3}$,

$$\epsilon\nabla\cdot(kg^2\nabla w) + \epsilon^{-1} f^2 g^4 (1 + \epsilon^2 M - w^2) w = 0. \tag{1.3}$$

For $\epsilon > 0$ small we see that $w_0 = \pm 1 + \mathrm{O}(\epsilon^2)$ is a simple solution of (1.3) which is a global minimiser of $\hat{G}_\epsilon(w)$ defined in Eq. (1.4) in $H^1(\Omega)$. We are interested in more complicated solutions which consist of pieces of w_0 in different parts of the domain Ω joined by rapidly varying solutions defined in narrow $\mathrm{O}(\epsilon)$ regions based on hypersurfaces $\Gamma_\epsilon \subset \Omega$. We call these interface solutions.

We use the Γ-convergence method to solve our problem. The basic idea of Γ-convergence is due to De Giorgi [2]; the application to local minimisers is studied by Kohn and Sternberg [9,11]. We simplify the application to our partial differential equation related energy functional significantly by using hypersurfaces $\Gamma_\epsilon \in C^3$ rather than C^2. Finally, we relax this assumption ($\Gamma_\epsilon \in C^3$) to give a more general result, which characterises the interfaces of local minimisers which are not global minimisers, as weighted geodesics of the domain Ω.

The fundamental idea of Γ-convergence is to identify the first nontrivial term in an asymptotic expansion as $\epsilon \to 0$ for the energy functional in (1.5) of the perturbed problem (rather than to expand the solution and the differential equation given by (1.2)). The energy functional of our problem, for $M \equiv \frac{\nabla\cdot(k\nabla g)}{f^2 g^3}$, is

$$\hat{G}_\epsilon(w) = \int_\Omega \left\{ \frac{\epsilon}{2} k g^2 |\nabla w|^2 + \frac{1}{4\epsilon} f^2 g^4 (1 + \epsilon^2 M - w^2)^2 \right\} \mathrm{d}x \tag{1.4}$$

which for $w \in H^1(\Omega)$ has for $\epsilon \to 0$ the same limit as

$$G_\epsilon(w) = \int_\Omega \left\{ \frac{\epsilon}{2} k g^2 |\nabla w|^2 + \frac{1}{4\epsilon} f^2 g^4 (1 - w^2)^2 \right\} \mathrm{d}x. \tag{1.5}$$

We wish to show such solutions that locally minimise $G_\epsilon(w)$ in $H^1(\Omega)$ exist as they act as local long-time attractors of the corresponding time-dependent problem [5,6]. For $w \in L^1(\Omega)$ and $w \notin H^1(\Omega)$ we have $\hat{G}_\epsilon(w) = \infty = G_\epsilon(w)$; since we will be interested in taking the infimum of G_ϵ. Thus when it is finite, we will consider $w \in L^1(\Omega)$ but usually not consider the $w \in L^1(\Omega) \setminus H^1(\Omega)$ case. Our aim is to show that $\hat{G}_\epsilon$ has such minimisers by showing that G_ϵ has minimisers, and by characterising these minimisers for ϵ small.

From the variational method, we seek a local minimiser (distinct from the global minimiser), so we consider the problem, for certain bounded sets $\hat{B} \subset H^1(\Omega)$, and fixed $\epsilon \in (0, \epsilon_0)$,

$$\inf_{w \in \hat{B}} G_\epsilon(w). \tag{1.6}$$

For a fixed choice of a ball B in $\mathrm{BV}(\Omega)$, the space of functions of bounded variation in Ω, for each $\epsilon > 0$, let w_ϵ denote a (local and interior) minimiser of problem (1.6) for $\hat{B} = B \cap H^1(\Omega)$. Here we choose B to avoid global minimisers and to allow us to prove that local minimisers exists.

We define a weighted parameter $h(x)$ by

$$h(x) = \frac{2\sqrt{2}}{3}\sqrt{k}\, f g^3. \tag{1.7}$$

The goal is to characterise $w_0 = \lim_{\epsilon_j \to 0} w_{\epsilon_j}$ for any L^1-convergent subsequence of $\{w_\epsilon\} \subset \hat{B}$. We will show that $\lim_{\epsilon_j \to 0} G_{\epsilon_j}(w_{\epsilon_j}) = G_0(w_0) = \int_\Gamma h(x(\sigma, 0))\mathrm{d}\sigma$, where G_ϵ, G_0 are defined in Eqs. (2.2) and (2.3), and Γ is a hypersurface is defined after Eq. (2.1). Thus Γ is a weighted geodesics hypersurface in the domain Ω.

2. Local minimisers of the energy functional

First, we introduce a function v_0 in the following. Let $A \subset \Omega$ have finite perimeter and suppose $\partial A \cap \Omega = \Gamma$ is smoothly parameterised by (arc length coordinates) σ. Further, let $v_0 = 2\chi_A - 1 \in \mathrm{BV}(\Omega)$ such that

$$v_0(x) = \begin{cases} -1, & \text{if } x \in \Omega \setminus \bar{A} \\ 0, & \text{if } x \in \Omega \cap \partial A \\ 1, & \text{if } x \in A. \end{cases} \tag{2.1}$$

First, we consider the $(n-1)$-dimensional hypersurface $\Gamma \in C^3(\sigma)$, where σ defines the position on Γ by $x(\sigma) \in \Gamma$, so that Γ has continuous and bounded curvature. Let ρ be the (signed) distance along the normal $\vec{n}(\sigma)$ to Γ in Ω. Then we attach labels (σ, ρ) to points in a neighbourhood $\Omega^* = \{x(\sigma, \rho) \in \Omega : |\rho| < \rho_*\}$ of Γ where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) \in \Gamma$ is an $(n-1)$-dimensional hypersurface coordinate σ which identifies Γ. The fact that Γ has bounded curvature guarantees that there exists $\rho_* > 0$ such that $x(\sigma, \rho)$ is a 1–1 mapping onto Ω^*.

Suppose that $G_\epsilon, G_0 : L^1(\Omega) \to \mathbb{R}$ are given by, for $\epsilon > 0$,

$$G_\epsilon(v) = \begin{cases} \displaystyle\int_\Omega \left\{ \frac{1}{4\epsilon} f^2 g^4 (1 - v(x)^2)^2 + \frac{\epsilon}{2} k g^2 |\nabla v|^2 \right\} \mathrm{d}x & \text{for } v \in B \subset H^1(\Omega), \\ \infty, & \text{otherwise,} \end{cases} \tag{2.2}$$

and

$$G_0(v_0) = \begin{cases} \displaystyle\int_\Omega h(x) |D\chi_{\{v_0 = 1\}}| \mathrm{d}x, & \text{for } v_0 \in B \subset \mathrm{BV}(\Omega), \text{ and } v_0 \in \{-1, 1\} \text{ a.e.} \\ \infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

We note that $\inf_B G_\epsilon(v) = \inf_{\hat{B}} G_\epsilon(v)$ because the values of $G_\epsilon(v)$ for $v \in B$ and $v \notin \hat{B}$ must be infinity.

We now state the main result of this paper, which characterises the structure of the limit solutions as $\epsilon \to 0$ in terms of the weighted geodesics of Ω with the weight $h(x)$. Note that a definition of Γ-convergence is given by De Giorgi [2] (see also [10]). Then the Γ limit of $G_\epsilon(u)$ is $G_0(v_0)$ as $u \to v_0$ in $L^1(\Omega)$ and $\epsilon \to 0$, that is,

$$\Gamma(L^1(\Omega)^-) \lim_{\substack{u \to v_0 \\ \epsilon \to 0}} G_\epsilon(u) = G_0(v_0), \tag{2.4}$$

if and only if the following Properties 2.1 and 2.2 have been satisfied.

Property 2.1. *For each $v \in L^1(\Omega)$ and for any $\{v_\epsilon\}$ in $L^1(\Omega)$ such that $v_\epsilon \to v$ in $L^1(\Omega)$ as $\epsilon \to 0$, then we have*

$$\liminf_{\epsilon\to 0} G_\epsilon(v_\epsilon) \geq G_0(v). \tag{2.5}$$

Property 2.2. *For each $v_0 \in \mathrm{BV}(\Omega)$ given by (2.1), there exists a sequence $\{v_{\epsilon_j}\}$ in $L^1(\Omega)$ satisfying*

$$v_{\epsilon_j} \to v_0 \qquad \textit{as } \epsilon_j \to 0 \qquad \textit{in } L^1(\Omega), \tag{2.6}$$

$$\textit{and} \qquad \lim_{\epsilon_j\to 0} G_{\epsilon_j}(v_{\epsilon_j}) = G_0(v_0). \tag{2.7}$$

From Properties 2.1 and 2.2, we have the following theorem. (The details of the proof are in Section 3.)

Theorem 2.3. *For a fixed suitable choice of a ball B in $\mathrm{BV}(\Omega)$, for each $\epsilon > 0$ let w_ϵ be a local and interior minimiser of problem (1.6) for $\hat{B} = B \cap H^1(\Omega)$. Let $w_\epsilon \to w_0$ in $L^1(\Omega)$ for some sequence $\epsilon \to 0$. Then w_0 is a solution of*

$$\inf_{\substack{v_0 \in B\subset \mathrm{BV}(\Omega)\\ v_0\in\{-1,1\}\text{ a.e.}}} \int_\Omega h(x)|\mathrm{D}\chi_{\{v_0=1\}}| = \inf_\Gamma \int_\Gamma h(x(\sigma,0))\mathrm{d}\sigma, \tag{2.8}$$

if Γ is sufficiently smooth, where $\Gamma = \{w_0 = 0\} \subset \Omega$ and σ is the arc length along Γ.

Next we would like to relax the assumption that Γ is smooth ($\Gamma = \partial A \cap \Omega \in C^3$). We consider $v_0 \in \mathrm{BV}(\Omega)$ defined by (2.3). Let $A \subset \Omega$ be a set of finite perimeter in Ω with $0 < |A| < |\Omega|$. From [11], we know that there exists a sequence of open sets $\{A_k\}$ in Ω and a sequence $\{v_k\} \in \mathrm{BV}(\Omega)$ where

$$v_k(x) = \begin{cases} -1, & \text{if } x \in \Omega \setminus \bar{A}_k,\\ 0, & \text{if } x \in \Omega \cap \partial A_k,\\ 1, & \text{if } x \in A_k,\end{cases} \tag{2.9}$$

such that v_k and A_k satisfy the following properties:

(i) $\partial A_k \cap \Omega \in C^3$;
(ii) $|((A_k \cap \Omega) \setminus A) \cup (A \setminus (A_k \cap \Omega))| \to 0$ as $k \to \infty$;
(iii) $\int_\Omega h(x)|\mathrm{D}\chi_{\{v_k=1\}}| \to \int_\Omega h(x)|\mathrm{D}\chi_{\{v_0=1\}}|$ as $k \to \infty$.

Idea of proof for these properties: First extend χ_A to a function $\tilde{u} \in \mathrm{BV}(\mathbb{R}^n)$ such that $\tilde{u}(x) = \chi_A(x)$ for $x \in \Omega$, and $\int_{\partial\Omega} |\mathrm{D}\tilde{u}| = 0$ (see [3]).

From [3], we have a sequence of C^∞ functions $\{u_\epsilon\}$ satisfying $u_\epsilon \to \tilde{u}$ in $L^1(\Omega)$,

$$\text{and} \qquad \lim_{\epsilon\to 0} \int_\Omega |\mathrm{D}u_\epsilon| = \int_\Omega |\mathrm{D}\tilde{u}|.$$

Then define sets $C_{\epsilon,m} = \{u_\epsilon(x) > m\}$. In [11], it was shown that there exists a value of $m \in (0, 1)$ and a sequence $\epsilon_k \to 0$ such that $\partial C_{\epsilon_k,m} \in C^\infty$, $\chi_{C_{\epsilon_k,m}} \to \chi_A$ in $L^1(\Omega)$, and $\int_\Omega h(x)|\mathrm{D}\chi_{\{v_k=1\}}| \to \int_\Omega h(x)|\mathrm{D}\chi_{\{v_0=1\}}|$ as $k \to \infty$. Here $|\cdot|$ is the n-dimensional Lebesgue measure. □

From property (iii) above, for a choice of A_k and associated v_k we have

$$\lim_{k\to\infty} G_0(v_k) = \lim_{k\to\infty} \int_\Omega h(x)|D\chi_{\{v_k=1\}}| = \int_\Omega h(x)|D\chi_{\{v_0=1\}}| = G_0(v_0).$$

Since $v_k - v_0 \equiv 0$ in each $(A_k \cap \Omega) \cap A$ domain, we have

$$\begin{aligned}\int_\Omega |v_k - v_0|\,dx &= \int_{(A_k\cap\Omega)\setminus A} |v_k - v_0|\,dx + \int_{(A_k\cap\Omega)\cap A} |v_k - v_0|\,dx \\ &\quad + \int_{A\setminus(A_k\cap\Omega)} |v_k - v_0|\,dx, \\ &= \int_{(A_k\cap\Omega)\setminus A} |v_k - v_0|\,dx + \int_{A\setminus(A_k\cap\Omega)} |v_k - v_0|\,dx, \\ &= \int_{((A_k\cap\Omega)\setminus A)\cup(A\setminus(A_k\cap\Omega))} |v_k - v_0|\,dx. \qquad (2.10)\end{aligned}$$

From property (ii) above, and Eq. (2.10) show that $\lim_{k\to\infty} \int_\Omega |v_k - v_0| dx = 0$, that is, there exists $v_k \to v$ as $k \to \infty$ in $L^1(\Omega)$ for each $v_0 \in \mathrm{BV}(\Omega)$ associated with a finite perimeter set $A \subset \Omega$.

From Property 2.2, for each $v_k \in \mathrm{BV}(\Omega)$ there exists a sequence $\{v_{\epsilon_k}\} \in \mathrm{BV}(\Omega)$ such that $v_{\epsilon_k} \to v_k$ as $\epsilon_k \to 0$ in $L^1(\Omega)$ and $\lim_{\epsilon_k\to 0} G_{\epsilon_k}(v_{\epsilon_k}) = G_0(v_k)$. From a diagonalisation argument, there exists a subsequence $\{v_{\epsilon_{kj}}\}$ satisfying Eqs. (2.6) and (2.7) for each choice of a finite perimeter set $A \subset \Omega$ with associated $v_0 \in \mathrm{BV}(\Omega)$.

3. Proof of theorem

In the following, we use the definition of a bounded variation function and some properties in [3] to prove the main Theorem 2.3. First, we give a basic and useful Property 3.1.

Property 3.1. *From the definition of generalised derivatives in* BV(Ω) *(see* [3]*), we have, for any* $v_0 \in \mathrm{BV}(\Omega)$ *satisfying Eq.* (2.1) *for* $v_0 = 0$ *only on a set* $\Gamma = \partial A \cap \Omega$ *for some* $A \subset \Omega$ *of finite perimeter,*

$$\begin{aligned}G_0(v_0) &= \int_\Omega h(x)|D\chi_{\{v_0=1\}}| = \sup_{\overline{\boldsymbol\phi}} \left\{ \int_A \nabla\cdot\overline{\boldsymbol\phi}\,dx : \overline{\boldsymbol\phi} \in C_0^1(\Omega,\mathbb{R}^n), |\overline{\boldsymbol\phi}| \le h(x) \right\} \\ &= \sup_{\boldsymbol\phi} \left\{ \int_A \nabla\cdot(h\boldsymbol\phi)dx : \boldsymbol\phi \in C_0^1(\Omega,\mathbb{R}^n), |\boldsymbol\phi| \le 1 \right\} \\ &= \sup_{\boldsymbol\phi} \left\{ \int_{\partial A\cap\Omega} (h\boldsymbol\phi)\cdot dS : \boldsymbol\phi \in C_0^1(\Omega,\mathbb{R}^n), |\boldsymbol\phi| \le 1 \right\} = \int_\Gamma h(x(\sigma,0))d\sigma. \qquad (3.11)\end{aligned}$$

Remark. If A does not have bounded perimeter then in the result (2.5) or the result (2.7), we have $\infty = \infty$, so we do not consider this case: henceforth we take $A \subset \Omega$ to have finite perimeter.

Proof of Theorem 2.3. We prove the theorem by using Properties 2.1 and 2.2. First, for any $v_0 \in \mathrm{BV}(\Omega)$ satisfying the assumptions in Property 3.1, using Property 2.2, there exists a sequence $\{v_{\epsilon_j}\}$ which converges to v_0 in $L^1(\Omega)$ and satisfies Eq. (2.7). Since the minimisers $\{w_{\epsilon_j}\}$ of (2.2) occupy a bounded set in BV(Ω), compactness theory implies there exists a $w_0 \in L^1(\Omega)$

such that a subsequence, $\{w_{\epsilon_j}\}$, converges to the limit w_0 in $L^1(\Omega)$ (using Theorem 1.19 in [3]). From Property 2.1 Eq. (2.5), we have

$$\liminf_{\epsilon_j \to 0} G_{\epsilon_j}(w_{\epsilon_j}) \geq G_0(w_0).$$

Since $\{w_{\epsilon_j}\}$ is a sequence of minimisers for each ϵ_j, $G_{\epsilon_j}(w_{\epsilon_j}) \leq G_{\epsilon_j}(v_{\epsilon_j})$ for the above sequence $\{v_{\epsilon_j}\}$. Hence, for any v_0 satisfying the assumptions of Property 3.1, we have

$$G_0(w_0) \leq \liminf_{\epsilon_j \to 0} G_{\epsilon_j}(w_{\epsilon_j}) \leq \liminf_{\epsilon_j \to 0} G_{\epsilon_j}(v_{\epsilon_j}) = \lim_{\epsilon_j \to 0} G_{\epsilon_j}(v_{\epsilon_j}) = G_0(v_0).$$

For all other $v \in B$ we have $G_0(v) = \infty$. Therefore w_0 must be a minimiser of G_0 in B and Theorem 2.3 follows. □

Proof of Property 2.1. We consider $\{v_\epsilon\} \in H^1(\Omega)$, otherwise $G_\epsilon(v_\epsilon) = \infty$. Suppose $v \in L^1(\Omega)$ such that $v_\epsilon \to v$ in $L^1(\Omega)$; note that we can take $v(x) \in \{1, -1\}$ a.e. and $v \in \mathrm{BV}(\Omega)$ otherwise $G_0(v) = \infty$, and the inequality (2.5) is trivial ($\infty = \infty$). Using the Cauchy–Schwartz inequality, we have

$$\begin{aligned} G_\epsilon(v_\epsilon) &= \int_\Omega \left\{ \frac{1}{4\epsilon} f^2 g^4 (1 - v_\epsilon^2)^2 + \frac{\epsilon}{2} k g^2 |\nabla v_\epsilon|^2 \right\} \mathrm{d}x, \\ &\geq \frac{1}{\sqrt{2}} \int_\Omega f g^2 (1 - v_\epsilon^2) \sqrt{k} g |\nabla v_\epsilon| \mathrm{d}x = \frac{1}{\sqrt{2}} \int_\Omega \sqrt{k} f g^3 (1 - v_\epsilon^2) |\nabla v_\epsilon| \mathrm{d}x, \\ &:= \frac{1}{\sqrt{2}} \sup_{\substack{\boldsymbol{\phi} \in C_0^1(\Omega, \mathbb{R}^n) \\ |\boldsymbol{\phi}| \leq 1}} \int_\Omega \sqrt{k} f g^3 (1 - v_\epsilon^2)(\nabla v_\epsilon \cdot \boldsymbol{\phi}) \mathrm{d}x. \end{aligned} \tag{3.12}$$

Let $\psi_\epsilon : \Omega \to \mathbb{R}$ be defined by (so that if $v_\epsilon = -1$, $\psi_\epsilon(x) = 0$ which we need to get Eq. (3.16))

$$\psi_\epsilon(x) = \frac{1}{\sqrt{2}} \sqrt{k} f g^3 \left(v_\epsilon - \frac{1}{3} v_\epsilon^3 + \frac{2}{3} \right). \tag{3.13}$$

For a fixed $\boldsymbol{\phi} = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x)) \in C_0^1(\Omega, \mathbb{R}^n)$, $|\boldsymbol{\phi}| \leq 1$, we have

$$\begin{aligned} (\nabla_x \psi_\epsilon(x) \cdot \boldsymbol{\phi}) &= \left(\frac{\partial \psi_\epsilon(x)}{\partial x_1}, \ldots, \frac{\partial \psi_\epsilon(x)}{\partial x_n} \right) \cdot (\phi_1, \phi_2, \ldots, \phi_n) = \sum_{i=1}^n \frac{\partial \psi_\epsilon(x)}{\partial x_i} \phi_i, \\ &= \sum_{i=1}^n \left\{ \frac{(\sqrt{k} f g^3)_{x_i}}{\sqrt{2}} \left(v_\epsilon - \frac{1}{3} v_\epsilon^3 + \frac{2}{3} \right) + \frac{\sqrt{k} f g^3}{\sqrt{2}} (1 - v_\epsilon^2) \frac{\partial v_\epsilon}{\partial x_i} \right\} \phi_i \\ &= \frac{1}{\sqrt{2}} \left(v_\epsilon - \frac{1}{3} v_\epsilon^3 + \frac{2}{3} \right) \sum_{i=1}^n (\sqrt{k} f g^3)_{x_i} \phi_i \\ &\quad + \frac{\sqrt{k} f g^3}{\sqrt{2}} (1 - v_\epsilon^2)(\nabla v_\epsilon \cdot \boldsymbol{\phi}). \end{aligned} \tag{3.14}$$

Hence, integrating terms in Eq. (3.14), we have

$$\begin{aligned} &\frac{1}{\sqrt{2}} \int_\Omega \sqrt{k} f g^3 (1 - v_\epsilon^2)(\nabla v_\epsilon \cdot \boldsymbol{\phi}) \mathrm{d}x \\ &\quad = \int_\Omega (\nabla \psi_\epsilon(x) \cdot \boldsymbol{\phi}) \mathrm{d}x - \int_\Omega \frac{1}{\sqrt{2}} \left(v_\epsilon - \frac{1}{3} v_\epsilon^3 + \frac{2}{3} \right) \sum_{i=1}^n (\sqrt{k} f g^3)_{x_i} \phi_i(x) \, \mathrm{d}x, \end{aligned}$$

$$
\begin{aligned}
&= -\int_{\Omega} \psi_\epsilon(x)\nabla\cdot\boldsymbol{\phi}\,\mathrm{d}x - \int_{\Omega}\frac{1}{\sqrt{2}}\left(v_\epsilon - \frac{1}{3}v_\epsilon^3 + \frac{2}{3}\right)\sum_{i=1}^{n}(\sqrt{k}\,fg^3)_{x_i}\phi_i(x)\,\mathrm{d}x,\\
&= -\int_{\Omega}\left\{\frac{1}{\sqrt{2}}\left(v_\epsilon - \frac{1}{3}v_\epsilon^3 + \frac{2}{3}\right)\sum_{i=1}^{n}\left(\sqrt{k}\,fg^3\frac{\partial\phi_i}{\partial x_i} + (\sqrt{k}\,fg^3)_{x_i}\phi_i\right)\right\}\mathrm{d}x,\\
&= -\int_{\Omega}\frac{1}{\sqrt{2}}\left(v_\epsilon - \frac{1}{3}v_\epsilon^3 + \frac{2}{3}\right)\nabla\cdot(\sqrt{k}\,fg^3\boldsymbol{\phi}(x))\,\mathrm{d}x. \qquad (3.15)
\end{aligned}
$$

Therefore, from Eqs. (3.12) and (3.15) we have, for all $\boldsymbol{\phi}\in C_0^1(\Omega)$,

$$
\begin{aligned}
\liminf_{\epsilon\to 0} G_\epsilon(v_\epsilon) &\geq -\lim_{\epsilon\to 0}\int_{\Omega}\frac{1}{\sqrt{2}}\left(v_\epsilon - \frac{1}{3}v_\epsilon^3 + \frac{2}{3}\right)\nabla\cdot(\sqrt{k}\,fg^3\boldsymbol{\phi}(x))\,\mathrm{d}x,\\
&= -\int_{\Omega}\chi_{\{v=1\}}\frac{2\sqrt{2}}{3}\nabla\cdot(\sqrt{k}\,fg^3\boldsymbol{\phi}(x))\,\mathrm{d}x,\\
&= -\int_{\Omega}\chi_{\{v=1\}}\nabla\cdot(h(x)\boldsymbol{\phi}(x))\,\mathrm{d}x. \qquad (3.16)
\end{aligned}
$$

Finally, taking the supremum over all such appropriate $\boldsymbol{\phi}$ with $|\boldsymbol{\phi}|\leq 1$, from the definition of D in $\mathrm{BV}(\Omega)$ (see [3]), we have

$$
\liminf_{\epsilon\to 0} G_\epsilon(v_\epsilon) \geq \sup_{\boldsymbol{\phi}}\int_{\Omega}\chi_{\{v=1\}}\nabla\cdot(h\boldsymbol{\phi})\mathrm{d}x = \int_{\Omega}h(x)|\mathrm{D}\chi_{\{v=1\}}| = G_0(v).
$$

Proof of Property 2.2. For each $v_0\in\mathrm{BV}(\Omega)$ given by (2.1) which satisfies the assumptions in Property 3.1, we need to construct a sequence $v_{\epsilon_j}\to v_0$ in $L^1(\Omega)$ as $\epsilon_j\to 0$, so that

$$
\lim_{\epsilon_j\to 0} G_{\epsilon_j}(v_{\epsilon_j}) = G_0(v_0). \qquad (3.17)
$$

First, we consider $v_0\in\mathrm{BV}(\Omega)$ given by Eq. (2.1) for bounded perimeter A and $\Gamma = \partial A\cap\Omega$ sufficiently smooth. We use $\Gamma\in C^3$ to construct local coordinates (σ,ρ) near Γ where (σ,ρ) have been defined following Eq. (2.1). Let Ω_1 be neighbourhoods of Γ defined by, for $\epsilon\in(0,\epsilon_*)$ with $2\sqrt{\epsilon_*}\leq\rho_*$,

$$
\Omega_1 = \{x(\sigma,\rho)\in\Omega : |\rho| < \sqrt{\epsilon}\}. \qquad (3.18)
$$

Next we introduce an inner variable to describe the inner layer behaviour near Γ by defining $\xi(\sigma,\rho)$ such that

$$
\epsilon\xi(\sigma,\rho) = \frac{1}{\sqrt{2}}\int_0^{\rho}\frac{f(\sigma,\bar{\rho})g(\sigma,\bar{\rho})}{\sqrt{k(\sigma,\bar{\rho})}}\,\mathrm{d}\bar{\rho}. \qquad (3.19)
$$

Hence we can construct a transition layer sequence v_ϵ given by

$$
v_\epsilon(x) = \begin{cases}\tanh\xi_1(\sigma), & \text{if } x\in A\cap(\Omega\setminus\overline{\Omega_1}),\\ \tanh\xi(\sigma,\rho), & \text{if } x(\sigma,\rho)\in\Omega_1,\\ -\tanh\xi_1(\sigma), & \text{if } x\in(\Omega\setminus A)\cap(\Omega\setminus\overline{\Omega_1}),\end{cases} \qquad (3.20)
$$

where ξ is given by Eq. (3.19) and $\xi_1(\sigma) = \frac{1}{\sqrt{2}\epsilon}\int_0^{\sqrt{\epsilon}}\frac{f(\sigma,\bar{\rho})g(\sigma,\bar{\rho})}{k(\sigma,\bar{\rho})}\mathrm{d}\bar{\rho} = \frac{\overline{M}(\sigma)}{\sqrt{\epsilon}}$ for some $\overline{M}(\sigma)>0$.

Step 1. We would like to claim $v_\epsilon\to v_0$ as $\epsilon\to 0$ in $L^1(\Omega)$ (that is, $\lim_{\epsilon\to 0}\int_{\Omega}|v_\epsilon - v_0|\mathrm{d}x = 0$). In the $\Omega\setminus\Omega_1$ domain, there exist positive constants c_2, and c_3, such that $|v_\epsilon - v_0| =$

$|\tanh\xi_1 - 1| \le c_2 e^{-c_3/\sqrt{\epsilon}}$. Therefore, we have a (soft) upper bound of $\sqrt{\epsilon}$ for

$$\int_{\Omega\setminus\Omega_1} |v_\epsilon - v_0| \mathrm{d}x \le \int_{\Omega\setminus\Omega_1} c_2 e^{-c_3/\sqrt{\epsilon}} \mathrm{d}x = \mathrm{O}(\sqrt{\epsilon}). \tag{3.21}$$

In the Ω_1 domain, we have $|v_\epsilon - v_0| \le 2$, so that for some constant $M_1 > 0$, using the fact that Ω_1 has width $2\sqrt{\epsilon}$ in the ρ direction, we obtain

$$\int_\Gamma \int_{|\rho|<\sqrt{\epsilon}} |v_\epsilon - v_0| \mathrm{d}\bar{\rho}\, \mathrm{d}\sigma \le M_1\sqrt{\epsilon}. \tag{3.22}$$

Therefore, from Eqs. (3.21) and (3.22), we have, since $|\mathrm{d}x| = |J\mathrm{d}\rho \mathrm{d}\sigma| \le c|\mathrm{d}\rho \mathrm{d}\sigma|$,

$$\int_\Omega |v_\epsilon - v_0|\, \mathrm{d}x = \mathrm{O}(\sqrt{\epsilon}),$$

so $v_\epsilon \to v$ as $\epsilon \to 0$ in $L^1(\Omega)$, as claimed.

Step 2. We next show that, as $\epsilon \to 0$, $G_{\epsilon_j}(v_\epsilon)$ approaches $G_0(v_0)$. From Property 2.1, we only need to show that $\lim_{\epsilon\to 0} G_\epsilon(v_\epsilon) \le G_0(v)$. First, since there exist positive constants c_2 and c_3 such that $|v_\epsilon - v_0| \le c_2 e^{-c_3/\sqrt{\epsilon}}$ and $|\nabla v_\epsilon| = \mathrm{O}(e^{-1/\epsilon})$ in the $\Omega \setminus \Omega_1$ domain. Therefore, we note that, taking $\mathrm{O}(\epsilon)$ for simplicity,

$$\int_{\Omega\setminus\Omega_1} \left\{ \frac{1}{4\epsilon} f^2 g^4 (1 - v_\epsilon^2)^2 + \frac{\epsilon}{2} k g^2 |\nabla v_\epsilon|^2 \right\} \mathrm{d}x = \mathrm{O}(\epsilon), \tag{3.23}$$

so that, noting $\mathrm{d}x = |J|\mathrm{d}\rho\, \mathrm{d}\sigma = (1 + K\rho)\mathrm{d}\rho\, \mathrm{d}\sigma$, where K is the bounded curvature of Γ,

$$\begin{aligned} G_\epsilon(v_\epsilon) &= \int_\Omega \left\{ \frac{1}{4\epsilon} f^2 g^4 (1 - v_\epsilon^2)^2 + \frac{\epsilon}{2} k g^2 |\nabla v_\epsilon|^2 \right\} \mathrm{d}x, \\ &= \int_\Gamma \int_{|\rho|<\sqrt{\epsilon}} \left\{ \frac{1}{4\epsilon} f^2 g^4 (1 - v_\epsilon^2)^2 + \frac{\epsilon}{2} k g^2 |\nabla v_\epsilon|^2 \right\} (1 + K\bar{\rho})\, \mathrm{d}\bar{\rho}\, \mathrm{d}\sigma + \mathrm{O}(\epsilon). \end{aligned} \tag{3.24}$$

First, the first term on the right-hand side of Eq. (3.24) is, again using $f(x) = f(\sigma, \rho)$, $g(x) = g(\sigma, \rho)$ etc.,

$$\begin{aligned} &\int_\Gamma \int_{|\rho|<\sqrt{\epsilon}} \left\{ \frac{1}{4\epsilon} f^2 g^4 (1 - v_\epsilon^2)^2 + \frac{\epsilon}{2} k g^2 |\nabla v_\epsilon|^2 \right\} (1 + K\bar{\rho})\mathrm{d}\bar{\rho}\, \mathrm{d}\sigma, \\ &\quad = \int_\Gamma \int_{|\xi|<M_2/\sqrt{\epsilon}} \left\{ \frac{1}{4\epsilon} f^2 g^4 (1 - \tanh^2\xi)^2 + \frac{\epsilon}{2} k g^2 |\nabla_x \tanh\xi|^2 \right\} \\ &\qquad \times (1 + K\epsilon\xi M_1(\sigma)) \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\xi}\, \mathrm{d}\xi\, \mathrm{d}\sigma, \\ &\quad = \int_\Gamma \int_{|\xi|<M_2/\sqrt{\epsilon}} \left\{ \frac{1}{4\epsilon} f^2 g^4 \mathrm{sech}^4\xi + \frac{\epsilon}{2} k g^2 \left(\mathrm{sech}^2\xi \frac{fg}{\epsilon\sqrt{2k}} \right)^2 \right\} \\ &\qquad \times \left(\frac{\epsilon\sqrt{2k}}{fg} \right) \mathrm{d}\xi\, \mathrm{d}\sigma + \mathrm{O}(\sqrt{\epsilon}), \\ &\quad = \int_\Gamma \int_{|\xi|<M_2/\sqrt{\epsilon}} \frac{\sqrt{k}\, f g^3}{\sqrt{2}} \mathrm{sech}^4\xi\, \mathrm{d}\xi\, \mathrm{d}\sigma + \mathrm{O}(\sqrt{\epsilon}). \end{aligned} \tag{3.25}$$

For a fixed σ with $x(\sigma, 0) \in \Gamma$ and $\hat{\xi} = \epsilon\xi$, from the Taylor expansion of $(\sqrt{k} f g^3)$ at $\rho = 0 = \epsilon\xi$ we have

$$(\sqrt{k} f g^3)(\sigma, \rho) = (\sqrt{k} f g^3)(\sigma, \rho(\hat{\xi})) = (\sqrt{k} f g^3)(\sigma, 0) + (\sqrt{k} f g^3)_\rho\Big|_{\rho=0}(\epsilon\xi) + \mathrm{O}(\epsilon^2\xi^2),$$

so

$$\begin{aligned}(\sqrt{k} f g^3)(\sigma, \rho)\operatorname{sech}^4\xi &= (\sqrt{k} f g^3)(\sigma, \rho(\hat{\xi}))\operatorname{sech}^4\xi \\ &= (\sqrt{k} f g^3)(\sigma, 0)\operatorname{sech}^4\xi + (\sqrt{k} f g^3)_\rho\Big|_{\rho=0}(\epsilon\xi)\operatorname{sech}^4\xi + \mathrm{O}(\epsilon^2\xi^2\operatorname{sech}^4\xi). \qquad (3.26)\end{aligned}$$

In the Ω_1 domain $|\xi| < \frac{M_2}{\sqrt{\epsilon}}$ for some constant $M_2 > 0$, so we have $\epsilon|\xi| < M_2\sqrt{\epsilon}$. Hence, from Eq. (3.26) we have, noting $\xi \operatorname{sech}\xi \to 0$ as $\xi \to \infty$,

$$(\sqrt{k} f g^3)(\sigma, \rho(\hat{\xi}))\operatorname{sech}^4\xi = (\sqrt{k} f g^3)(\sigma, 0)\operatorname{sech}^4\xi + \mathrm{O}(\sqrt{\epsilon}). \qquad (3.27)$$

Therefore, from using Eq. (3.27) in (3.25), and again using $\mathrm{d}x = (1 + \mathrm{O}(\sqrt{\epsilon}))\mathrm{d}\sigma\,\mathrm{d}\xi$ since the curvature K of Γ is bounded and $\epsilon|\xi| \le M_2\sqrt{\epsilon}$, we have

$$\begin{aligned}\lim_{\epsilon\to 0} G_\epsilon(v_\epsilon) &= \lim_{\epsilon\to 0}\int_\Omega \left\{\frac{1}{4\epsilon} f^2 g^4 (1 - v_\epsilon^2)^2 + \frac{\epsilon}{2} k g^2 |\nabla v_\epsilon|^2\right\}\mathrm{d}x, \\ &= \lim_{\epsilon\to 0}\int_\Gamma \int_{|\xi|<M_2/\sqrt{\epsilon}} \frac{\sqrt{k} f g^3}{\sqrt{2}}\operatorname{sech}^4\xi\,\mathrm{d}\xi\,\mathrm{d}\sigma + \lim_{\epsilon\to 0}\mathrm{O}(\sqrt{\epsilon}), \\ &= \lim_{\epsilon\to 0}\int_\Gamma \int_{|\xi|<M_2/\sqrt{\epsilon}} \frac{1}{\sqrt{2}}(\sqrt{k} f g^3)(\sigma, 0)\operatorname{sech}^4\xi\,\mathrm{d}\xi\mathrm{d}\sigma + \lim_{\epsilon\to 0}\mathrm{O}(\sqrt{\epsilon}), \\ &\le \frac{1}{\sqrt{2}}\int_\Gamma \sqrt{k} f g^3(\sigma, 0)\mathrm{d}\sigma \int_{-\infty}^{\infty}\operatorname{sech}^4\xi\,\mathrm{d}\xi = \frac{2\sqrt{2}}{3}\int_\Gamma \sqrt{k} f g^3(\sigma, 0)\mathrm{d}\sigma, \\ &= \int_\Gamma h(\sigma, 0)\mathrm{d}\sigma = G_0(v). \qquad (3.28)\end{aligned}$$

This completes the proof of Property 2.2. □

Remarks. (i) In the $\Omega \subset \mathbb{R}^2$ case, let $L(\Gamma) = \int_\Gamma \sqrt{k} f g^3(\sigma, 0)\mathrm{d}\sigma \equiv \int_\Gamma h(\sigma, 0)\mathrm{d}\sigma$, where σ is the arc length along Γ. From Theorem 2.3, we have w_0 is a solution of $\inf_\Gamma L(\Gamma)$. Since $\inf_\Gamma L(\Gamma)$ implies for the minimiser's curve Γ_0, $\delta L(\Gamma_0) = 0$ w.r.t. Γ. Let $\rho(\sigma)$ be the distance along $\vec{\mathbf{n}}(\sigma)$ from Γ_0 to $\Gamma \subset \Omega$. (This is well defined for curve Γ with bounded curvature in a strip $|\rho| \le \delta$ about Γ_0 for $\delta > 0$ sufficiently small.) For $\rho(\sigma) \in \mathbb{R}$, where $\vec{\mathbf{n}}$ is the normal direction, and $\frac{\partial}{\partial n}$ is the normal derivative to Γ_0 at the point σ given by

$$\begin{aligned}L(\Gamma_0 + \rho(\sigma)\vec{\mathbf{n}}) - L(\Gamma_0) &= \int_{\Gamma_0+\rho\vec{\mathbf{n}}} h(\hat{s}, 0)\mathrm{d}\hat{s} - \int_{\Gamma_0} h(\sigma, 0)\mathrm{d}\sigma, \\ &= \int_{\Gamma_0} h(\sigma, \rho(\sigma))(1 + \rho(\sigma)K)\mathrm{d}\sigma - \int_{\Gamma_0} h(\sigma, 0)\mathrm{d}\sigma, \\ &= \int_{\Gamma_0} \left(h(\sigma, 0) + \frac{\partial h}{\partial n}(\sigma, 0)\rho + \mathrm{O}(\rho^2)\right)(1 + \rho K)\mathrm{d}\sigma - \int_{\Gamma_0} h(\sigma, 0)\mathrm{d}\sigma, \\ &= \int_{\Gamma_0}\left\{\left(h(\sigma, 0)K + \frac{\partial h}{\partial n}(\sigma, 0)\right)\rho + \mathrm{O}(\rho^2)\right\}\mathrm{d}\sigma. \qquad (3.29)\end{aligned}$$

Here $K(\sigma)$ is the curvature of Γ_0 at the point. From the definition of a Gateaux (or even Frechet with the appropriate norm) derivative, we have the Gateaux derivative of $L(\Gamma)$ at Γ_0, for all small $\rho(\sigma)$ which define nearby curves Γ,

$$\delta L(\Gamma_0)\rho = \int_{\Gamma_0} \left\{ hK + \frac{\partial h}{\partial n} \right\} \rho \, \mathrm{d}\sigma. \tag{3.30}$$

Note that if we introduce a $\|\cdot\|$ for ρ in an appropriate space then we could show that (3.30) is a Frechet derivative.

At the minimiser (in fact, any critical point) Γ_0, $\delta L(\Gamma_0) \equiv 0$ means $\delta L(\Gamma_0)\rho = 0$, for all $\rho(\sigma)$. Thus we must have $h(\sigma, 0)K + \frac{\partial h}{\partial n}(\sigma, 0) \equiv 0$ for all $\sigma \in \Gamma_0$, where Γ_0 is a stationary value of $L(\Gamma)$. That is, the curvature of Γ_0 is $K = -\frac{1}{h}\frac{\partial h}{\partial n}$ at $\sigma \in \Gamma_0$, where $h = \sqrt{k}\, f g^3$.

For the radially symmetrical in $\mathbb{R}^n$ case, the curvature of Γ_0 is $K = \frac{(n-1)}{r}$, and so radially symmetric locally minimising solutions $w_\epsilon(r)$ will have nodal curves where $w_\epsilon = 0$ that are circles (sphere) of radius r_ϵ near

$$r_0 = (n-1)/K_0 = -\frac{(n-1)h(r_0)}{\mathrm{d}h/\mathrm{d}r\,(r_0)}.$$

(ii) From Eq. (3.30), we consider, consistently dropping terms that are $o(\rho_1, \rho_2)$,

$$\begin{aligned}
&\delta L(\Gamma_0 + \rho_2 \vec{\mathbf{n}})\rho_1 - \delta L(\Gamma_0)\rho_1 \\
&\quad = \int_{\Gamma_0 + \rho_2 \vec{\mathbf{n}}} \left\{ hK + \frac{\partial h}{\partial \mathbf{n}} \right\} (\bar{s}, 0)\rho_1 \mathrm{d}\bar{s} - \int_{\Gamma_0} \left\{ hK + \frac{\partial h}{\partial \mathbf{n}} \right\} (\sigma, 0)\rho_1 \mathrm{d}\sigma \\
&\quad = \int_{\Gamma_0} \left\{ hK + \frac{\partial h}{\partial \mathbf{n}} \right\} (\sigma, \rho_2(\sigma))(1 + \rho_2 K)\rho_1 \mathrm{d}\sigma - \int_{\Gamma_0} \left\{ hK + \frac{\partial h}{\partial \mathbf{n}} \right\} (\sigma, 0)\rho_1 \mathrm{d}\sigma \\
&\quad = \int_{\Gamma_0} \left\{ \left(hK + \frac{\partial h}{\partial \mathbf{n}} \right)(\sigma, 0) + \left(hK + \frac{\partial h}{\partial \mathbf{n}} \right)_{\mathbf{n}} (\sigma, 0)\rho_2 \right\} (1 + \rho_2 K)\rho_1 \mathrm{d}\sigma \\
&\qquad - \int_{\Gamma_0} \left\{ hK + \frac{\partial h}{\partial \mathbf{n}} \right\} (\sigma, 0)\rho_1 \mathrm{d}\sigma.
\end{aligned} \tag{3.31}$$

From the definition of Gateaux derivative, or Frechet derivative (with the appropriate norm), we have the second derivative of $L(\Gamma)$ at Γ_0 for all small ρ_1 and ρ_2 given by

$$\begin{aligned}
\delta^2 L(\Gamma_0)\rho_1\rho_2 &= \int_{\Gamma_0} \left(hK + \frac{\partial h}{\partial \mathbf{n}} \right)_{\mathbf{n}} (\sigma, 0)\rho_1\rho_2 \mathrm{d}\sigma \\
&\quad + \int_{\Gamma_0} \left\{ hK + \frac{\partial h}{\partial \mathbf{n}} \right\} (\sigma, 0)\rho_1\rho_2 K \mathrm{d}\sigma.
\end{aligned} \tag{3.32}$$

Since $K = -\frac{\partial h}{h\partial \mathbf{n}}$ on Γ_0, from Eq. (3.32) we have

$$\delta^2 L(\Gamma_0)\rho_1\rho_2 = \int_{\Gamma_0} \left(hK + \frac{\partial h}{\partial \mathbf{n}} \right)_{\mathbf{n}} (\sigma, 0)\rho_1\rho_2 \mathrm{d}\sigma.$$

Hence, if $\delta^2 L(\Gamma_0)\rho_1\rho_2 > 0$ for all ρ_1 and ρ_2, that is $\left(hK + \frac{\partial h}{\partial \mathbf{n}}\right)_{\mathbf{n}} (\sigma, 0) > 0$ for $x(\sigma, 0) \in \Gamma_0$, then Γ_0 is a local minimiser; also the steady-state solution of the corresponding time-dependent problem is stable.

Although we use a completely different approach in this paper, it is encouraging that the results stated in remarks (i) and (ii) are the consistent with the asymptotic and numerical results of the one-dimensional case in [7] and the multi-dimensional case in [8].

Uncited references

[1] and [4].

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