

# Accepted Manuscript

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PII: S0021-8693(18)30190-X

DOI: <https://doi.org/10.1016/j.jalgebra.2018.02.033>

Reference: YJABR 16628

To appear in: *Journal of Algebra*

Received date: 12 August 2016

Please cite this article in press as: K. Erdmann, A. Skowroński, Weighted surface algebras, *J. Algebra* (2018), <https://doi.org/10.1016/j.jalgebra.2018.02.033>

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Journal of Algebra 00 (2018) 1–49

Journal of  
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# Weighted surface algebras<sup>☆</sup>

Karin Erdmann<sup>a</sup>, Andrzej Skowroński<sup>b,\*</sup>

<sup>a</sup>Mathematical Institute, Oxford University, ROQ, Oxford OX2 6GG, United Kingdom

<sup>b</sup>Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

*Dedicated to Idun Reiten on the occasion of her 75th birthday*

## Abstract

A finite-dimensional algebra  $A$  over an algebraically closed field  $K$  is called *periodic* if it is periodic under the action of the syzygy operator in the category of  $A$ - $A$ -bimodules. The periodic algebras are self-injective and occurred naturally in the study of tame blocks of group algebras, actions of finite groups on spheres, hypersurface singularities of finite Cohen-Macaulay type, and Jacobian algebras of quivers with potentials. Recently, the tame periodic algebras of polynomial growth have been classified and it is natural to attempt to classify all tame periodic algebras. We introduce the weighted surface algebras of triangulated surfaces with arbitrarily oriented triangles and describe their basic properties. In particular, we prove that all these algebras, except the singular tetrahedral algebras, are symmetric tame periodic algebras of period 4. Moreover, we describe the socle deformations of the weighted surface algebras and prove that all these algebras are also symmetric tame periodic algebras of period 4. The main results of this paper form an important step towards a classification of all periodic symmetric tame algebras of non-polynomial growth, and lead to a complete description of all algebras of generalized quaternion type with 2-regular Gabriel quivers [36].

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**Keywords:** Syzygy, Periodic algebra, Self-injective algebra, Surface algebra, Tame algebra

**2010 MSC:** 16D50, 16E30, 16G20, 16G60, 16G70

## 1. Introduction and the main results

Throughout this paper,  $K$  will denote a fixed algebraically closed field. By an algebra we mean an associative finite-dimensional  $K$ -algebra with an identity. For an algebra  $A$ , we denote by  $\text{mod } A$  the category of finite-dimensional right  $A$ -modules and by  $D$  the standard duality  $\text{Hom}_K(-, K)$  on  $\text{mod } A$ . An algebra  $A$  is called *self-injective* if  $A_A$  is injective in  $\text{mod } A$ , or equivalently, the projective modules in  $\text{mod } A$  are injective. A prominent class of self-injective algebras is formed by the *symmetric algebras*  $A$  for which there exists an associative, non-degenerate symmetric  $K$ -bilinear form  $(-, -) : A \times A \rightarrow K$ . Classical examples of symmetric algebras are provided by the blocks of group algebras of finite groups and the Hecke algebras of finite Coxeter groups. In fact, any algebra  $A$  is the quotient algebra of its trivial extension algebra  $T(A) = A \ltimes D(A)$ , which is a symmetric algebra. Two self-injective algebras  $A$  and  $\Lambda$  are said to be *socle equivalent* if the quotient algebras  $A/\text{soc}(A)$  and  $\Lambda/\text{soc}(\Lambda)$  are isomorphic.

From the remarkable Tame and Wild Theorem of Drozd (see [17, 23]) the class of algebras over  $K$  may be divided into two disjoint classes. The first class consists of the *tame algebras* for which the indecomposable modules occur in each dimension  $d$  in a finite number of discrete and a finite number of one-parameter families. The second class is

<sup>☆</sup>The research was supported by the research grant DEC-2011/02/A/ST1/00216 of the National Science Center Poland.

\*Corresponding author

Email addresses: [erdmann@maths.ox.ac.uk](mailto:erdmann@maths.ox.ac.uk) (Karin Erdmann), [skowron@mat.uni.torun.pl](mailto:skowron@mat.uni.torun.pl) (Andrzej Skowroński)

formed by the *wild algebras* whose representation theory comprises the representation theories of all algebras over  $K$ . Accordingly, we may realistically hope to classify the indecomposable finite-dimensional modules only for the tame algebras. Among the tame algebras we may distinguish the *representation-finite algebras*, having only finitely many isomorphism classes of indecomposable modules, for which the representation theory is rather well understood. On the other hand, the representation theory of arbitrary tame algebras is still only emerging. The most accessible ones amongst the tame algebras are algebras of polynomial growth [70] for which the number of one-parameter families of indecomposable modules in each dimension  $d$  is bounded by  $d^m$ , for some positive integer  $m$  (depending only on the algebra).

Let  $A$  be an algebra. Given a module  $M$  in  $\text{mod } A$ , its *syzygy* is defined to be the kernel  $\Omega_A(M)$  of a minimal projective cover of  $M$  in  $\text{mod } A$ . The syzygy operator  $\Omega_A$  is a very important tool to construct modules in  $\text{mod } A$  and relate them. For  $A$  self-injective, it induces an equivalence of the stable module category  $\underline{\text{mod}} A$ , and its inverse is the shift of a triangulated structure on  $\underline{\text{mod}} A$  [46]. A module  $M$  in  $\text{mod } A$  is said to be *periodic* if  $\Omega_A^n(M) \cong M$  for some  $n \geq 1$ , and if so the minimal such  $n$  is called the *period* of  $M$ . The action of  $\Omega_A$  on  $\text{mod } A$  can effect the algebra structure of  $A$ . For example, if all simple modules in  $\text{mod } A$  are periodic, then  $A$  is a self-injective algebra. Sometimes one can even recover the algebra  $A$  and its module category from the action of  $\Omega_A$ . For example, the self-injective Nakayama algebras are precisely the algebras  $A$  for which  $\Omega_A^2$  permutes the isomorphism classes of simple modules in  $\text{mod } A$ . An algebra  $A$  is defined to be *periodic* if it is periodic viewed as a module over the enveloping algebra  $A^e = A^{\text{op}} \otimes_K A$ , or equivalently, as an  $A$ - $A$ -bimodule. It is known that if  $A$  is a periodic algebra of period  $n$  then for any indecomposable non-projective module  $M$  in  $\text{mod } A$  the syzygy  $\Omega_A^n(M)$  is isomorphic to  $M$ .

Finding or possibly classifying periodic algebras is an important problem. It is very interesting because of connections with group theory, topology, singularity theory and cluster algebras. Periodicity of an algebra, and its period, are invariant under derived equivalences [67] (see also [34]). Therefore, to study periodic algebras we may assume that the algebras are basic and indecomposable.

Preprojective algebras of Dynkin type are periodic and their periods divide 6 (see [4, 39]). They belong to a larger class of periodic algebras, the deformed preprojective algebras of generalized Dynkin type (see [5, 34]). With the exception of few small cases, all these algebras are wild (see [33]). Preprojective algebras of Dynkin type occur in other contexts, in particular they are the stable Auslander algebras of the categories of maximal Cohen-Macaulay of the Kleinian 2-dimensional hypersurface singularities (see [2, 3]). We refer to [4, 13, 25] for periodicity results on the stable Auslander algebras of arbitrary hypersurface singularities of finite Cohen-Macaulay type. It would be interesting to understand connections between the stable Auslander algebras of hypersurface singularities of finite Cohen-Macaulay type and the deformed mesh algebras of generalized Dynkin type introduced in [34]. For the simple plane curve singularities of Dynkin type  $\mathbb{A}_n$  this was clarified in [6, 7].

In [24] Dugas proved that every representation-finite self-injective algebra, without simple blocks, is a periodic algebra, this extended partial results from [12, 30, 31, 34] to the general case. We note that, by general theory (see [72, Section 3]), a basic, indecomposable, non-simple, symmetric algebra  $A$  is representation-finite if and only if  $A$  is socle equivalent to an algebra  $T(B)^G$  of invariants of the trivial extension algebra  $T(B)$  of a tilted algebra  $B$  of Dynkin type with respect to free action of a finite cyclic group  $G$ . Moreover, there are representation-finite indecomposable symmetric algebras of arbitrary large period (see [12]). Recently, the representation-infinite, indecomposable, periodic algebras of polynomial growth were classified by Białkowski, Erdmann and Skowroński in [8] (see also [71, 72]). In particular, it follows from [8] (see also [9, 10, 11, 71] and [72, Section 5]) that every basic, indecomposable, representation-infinite periodic tame symmetric algebra of polynomial growth is socle equivalent to an algebra  $T(B)^G$  of invariants of the trivial extension algebra  $T(B)$  of a tubular algebra  $B$  of tubular type  $(2, 2, 2, 2)$ ,  $(3, 3, 3)$ ,  $(2, 4, 4)$ ,  $(2, 3, 6)$  (introduced by Ringel [68]) with respect to free action of a finite cyclic group  $G$ . Then one knows that there is a common bound of the periods of all representation-infinite indecomposable symmetric algebras of polynomial growth (see [8]).

It would be interesting to classify all indecomposable periodic symmetric tame algebras of non-polynomial growth. We ask whether the following might hold.

**Problem.** *Let  $A$  be an indecomposable symmetric tame algebra of non-polynomial growth for which all simple modules in  $\text{mod } A$  are periodic. Is it true that  $A$  is a periodic algebra of period 4?*

Motivated by known properties of blocks with generalized quaternion defect groups in the group algebras of finite groups, Erdmann introduced and investigated in [27, 28, 29] the *algebras of quaternion type*, being the inde-

composable, representation-infinite tame symmetric algebras  $A$  with non-singular Cartan matrix  $C_A$  for which every indecomposable non-projective module in  $\text{mod } A$  is periodic of period dividing 4. In particular, Erdmann proved that every algebra  $A$  of quaternion type has at most 3 non-isomorphic simple modules and its basic algebra is isomorphic to an algebra belonging to 12 families of symmetric representation-infinite algebras defined by quivers and relations. Subsequently it has been proved in [50] (see also [62] for the polynomial growth cases) that all these algebras are tame, and are in fact periodic of period 4 (see [8, 33]). In particular it shows that a finite group  $G$  is periodic with respect to the group cohomology  $H^*(G, \mathbb{Z})$  if and only if all blocks with non-trivial defect groups of its group algebras  $KG$  over an arbitrary algebraically closed field  $K$  are periodic algebras. By the famous result of Swan [75] periodic groups can be characterized as the finite groups acting freely on finite CW-complexes homotopically equivalent to spheres (see [34, Section 4] for more details). Some of the algebras of quaternion type occur as endomorphism algebras of cluster tilting objects in the stable categories of maximal Cohen-Macaulay modules over odd-dimensional isolated hypersurface singularities (see [14, Section 7]).

New interesting families of tame symmetric algebras with all indecomposable non-projective finite-dimensional modules periodic of period dividing 4 appeared surprisingly in the theory of cluster algebras. In [19, 20], Derksen, Weyman and Zelevinsky introduced quivers with potentials and the associated Jacobian algebras, and established links between the theory of cluster algebras (invented by Fomin and Zelevinsky [42]) and the representation theory of algebras. On the other hand, in the beautiful paper [41], Fomin, Shapiro and Thurston associated to each bordered surface with marked points a cluster algebra, each of whose exchange matrices is defined in terms of the signed adjacencies between the arcs of an ideal triangulation of the surface, and such that the flips of triangulations correspond to the mutations of the associated skew-symmetric matrices (equivalently, mutations of the associated quivers). In particular, a wide class of 2-acyclic quivers of finite mutation type has been exhibited in [41]. Moreover, Felikson, Shapiro and Tumarkin proved in [40] that there are only 11 mutation equivalence classes of 2-acyclic quivers of finite mutation type not coming from triangulations of marked surfaces. Further, in [56] Labardini-Fragoso associated a quiver with potential to any ideal triangulation of a surface with marked points in such a way that flips of triangulations correspond to mutations of the associated quivers with potentials. Finally, Ladkani proved in [57] that the Jacobian algebras associated to ideal triangulations of surfaces with empty boundary and punctures and Labardini-Fragoso potentials are finite-dimensional tame symmetric algebras with singular Cartan matrices. Moreover, Valdivieso-Díaz proved in [76] that the stable Auslander-Reiten quivers of these Jacobian algebras consist of stable tubes of ranks 1 and 2. In particular, this showed that the list of tame symmetric algebras with periodic module categories announced in Theorem 6.2 of our article [34] (and hence also in [72, Theorem 8.7]) is not complete. In fact, this omission was pointed to us first by S. Ladkani.

The aim of this paper is to introduce a more general class of algebras, called weighted surface algebras, and describe their basic properties. In this paper, by a surface we mean a connected, compact, 2-dimensional real manifold  $S$ , orientable or non-orientable, with or without boundary. Then  $S$  admits a structure of a finite 2-dimensional triangular cell complex, and hence a triangulation. We say that  $(S, \vec{T})$  is a directed triangulated surface if  $S$  is a surface,  $T$  is a triangulation of  $S$  with at least 3 pairwise different edges, and  $\vec{T}$  is an arbitrary choice of orientations of the triangles in  $T$ . To such  $(S, \vec{T})$  we associate a triangulation quiver  $(Q(S, \vec{T}), f)$ , where  $Q(S, \vec{T})$  is a 2-regular quiver, that is every vertex is a source and target of exactly two arrows. The vertices of this quiver are the edges of  $T$ , and  $f$  is a permutation of the arrows in  $Q(S, \vec{T})$  reflecting the orientation  $\vec{T}$  of triangles in  $T$ . Since  $Q(S, \vec{T})$  is 2-regular there is a second permutation, denoted by  $g$ , of the arrows of  $Q(S, \vec{T})$ . If  $O(g)$  is the set of  $g$ -orbits of arrows in  $Q(S, \vec{T})$ , we will define two functions  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  and  $c_\bullet : O(g) \rightarrow K^*$ , called weight and parameter functions. Then the weighted surface algebra  $\Lambda(S, \vec{T}, m_\bullet, c_\bullet)$  will be defined as a quotient algebra  $KQ(S, \vec{T})/I(S, \vec{T}, m_\bullet, c_\bullet)$  of the path algebra  $KQ(S, \vec{T})$  of  $Q(S, \vec{T})$  over  $K$  by an admissible ideal  $I(S, \vec{T}, m_\bullet, c_\bullet)$  of  $KQ(S, \vec{T})$ . Certain algebras of this form which are defined via the tetrahedral triangulation of the sphere, play a special role, we call these tetrahedral algebras.

The following two theorems describe basic properties of the weighted surface algebras.

**Theorem 1.1.** *Let  $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$  be a weighted surface algebra over an algebraically closed field  $K$ . Then the following statements hold:*

- (i)  $\Lambda$  is a representation-infinite tame symmetric algebra.
- (ii)  $\Lambda$  is of polynomial growth if and only if  $\Lambda$  is a non-singular tetrahedral algebra.

**Theorem 1.2.** *Let  $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$  be a weighted surface algebra over an algebraically closed field  $K$ . Then the following statements are equivalent:*

- (i) All simple modules in  $\text{mod } \Lambda$  are periodic of period 4.
- (ii)  $\Lambda$  is a periodic algebra of period 4.
- (iii)  $\Lambda$  is not a singular tetrahedral algebra.

We would like to mention that the periodicity of module categories  $\text{mod } \Lambda$  of weighted surface (triangulation) algebras different from a singular tetrahedral algebra, was established independently using cluster theory methods (see [60, 76]). But we want stress that the periodicity of algebras established in the above theorem is a much stronger property than the periodicity of its module category. For example, the periodicity of an algebra implies the periodicity of its Hochschild cohomology. Moreover, we provide a self-contained proof of the above theorem, presenting explicit constructions of periodic bimodule resolutions of the considered algebras (see Section 7).

We obtain the following direct consequence of the above theorems and the main result of [26] (see also Theorem 2.5).

**Corollary 1.3.** *Let  $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$  be a weighted surface algebra over an algebraically closed field  $K$ , with the Grothendieck group  $K_0(\Lambda)$  of rank at least 4. Then the Cartan matrix  $C_\Lambda$  of  $\Lambda$  is singular.*

Let  $(S, \vec{T})$  be a directed triangulated surface, and  $m_\bullet, c_\bullet$  weight and parameter functions of  $(Q(S, \vec{T}), f)$ . Assume that the boundary  $\partial S$  of  $S$  is not empty. Then we may consider a border function  $b_\bullet : \partial(Q(S, \vec{T}), f) \rightarrow K$  on the set  $\partial(Q(S, \vec{T}), f)$  of vertices of  $Q(S, \vec{T})$  corresponding to the boundary edges of the triangulation  $T$  of  $S$ , and the associated socle deformed weighted surface algebra  $\Lambda(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet) = KQ(S, \vec{T})/I(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet)$ , where  $I(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet)$  is an admissible ideal of  $KQ(S, \vec{T})$  such that  $\Lambda(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet)$  is socle equivalent to  $\Lambda(S, \vec{T}, m_\bullet, c_\bullet)$ .

The following theorem is the third main result of the paper.

**Theorem 1.4.** *Let  $A$  be a basic, indecomposable, symmetric algebra over an algebraically closed field  $K$ . Assume that  $A$  is socle equivalent but not isomorphic to a weighted surface algebra  $\Lambda(S, \vec{T}, m_\bullet, c_\bullet)$ . Then the following statements hold:*

- (i) The surface  $S$  has non-empty boundary.
- (ii)  $K$  is of characteristic 2.
- (iii)  $A$  is isomorphic to a socle deformed weighted surface algebra  $\Lambda(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet)$ .
- (iv) The Cartan matrix  $C_A$  of  $A$  is singular.
- (v)  $A$  is a tame algebra of non-polynomial growth.
- (vi)  $A$  is a periodic algebra of period 4.

In Section 8 we will provide explicit constructions of periodic bimodule resolutions of the socle deformed weighted surface algebras.

The above theorems are the key new results towards classifications of distinguished classes of tame symmetric algebras. As the continuation [36] of this paper we classify basic, indecomposable, representation-infinite, tame symmetric algebras  $A$  with 2-regular Gabriel quiver having at least 3 vertices and where all simple modules are periodic of period 4 (called algebras of generalized quaternion type). These are the algebras socle equivalent to the weighted surface algebras  $\Lambda(S, \vec{T}, m_\bullet, c_\bullet)$ , different from the singular tetrahedral algebra, and the higher tetrahedral algebras investigated in [35].

Further, the orbit closures of the weighted surface algebras (and their socle deformations) in the affine varieties of associative  $K$ -algebra structures contain new wide classes of tame symmetric algebras related to algebras of dihedral and semidihedral types, which occurred in the study of blocks of group algebras with dihedral and semidihedral defect groups. We refer to [37, 38] for a classification of algebras of generalized dihedral type and a characterization of Brauer graph algebras, using biserial weighted surface algebras.

This paper is organized as follows. Section 2 contains some known preliminary results on algebras and modules. In Section 3 we describe our general approach and results for constructing a minimal projective bimodule resolution of an algebra with periodic simple modules. Section 4 introduces triangulation quivers and shows that they arise naturally from orientations of triangles of triangulated surfaces. In Section 5 we define weighted surface algebras of directed triangulated surfaces and prove that they are tame symmetric algebras. Section 6 is devoted to distinguished properties of a family of algebras given by the tetrahedral triangulation of the sphere. In Section 7 we discuss the periodicity of arbitrary weighted surface algebras. Section 8 deals with socle deformations of weighted surface algebras of directed



triangulated surfaces with boundary and their properties. In Section 9 we prove that all these algebras are periodic algebras of period 4. In Section 10 we discuss the representation type of the weighted surface algebras and their socle deformations.

For general background on the relevant representation theory we refer to the books [1, 29, 69, 74].

## 2. Preliminary results

A *quiver* is a quadruple  $Q = (Q_0, Q_1, s, t)$  consisting of a finite set  $Q_0$  of vertices, a finite set  $Q_1$  of arrows, and two maps  $s, t : Q_1 \rightarrow Q_0$  which associate to each arrow  $\alpha \in Q_1$  its source  $s(\alpha) \in Q_0$  and its target  $t(\alpha) \in Q_0$ . We denote by  $KQ$  the path algebra of  $Q$  over  $K$  whose underlying  $K$ -vector space has as its basis the set of all paths in  $Q$  of length  $\geq 0$ , and by  $R_Q$  the arrow ideal of  $KQ$  generated by all paths  $Q$  of length  $\geq 1$ . An ideal  $I$  in  $KQ$  is said to be *admissible* if there exists  $m \geq 2$  such that  $R_Q^m \subseteq I \subseteq R_Q^2$ . If  $I$  is an admissible ideal in  $KQ$ , then the quotient algebra  $KQ/I$  is called a *bound quiver algebra*, and is a finite-dimensional basic  $K$ -algebra. Moreover,  $KQ/I$  is indecomposable if and only if  $Q$  is connected. Every basic, indecomposable, finite-dimensional  $K$ -algebra  $A$  has a bound quiver presentation  $A \cong KQ/I$ , where  $Q = Q_A$  is the *Gabriel quiver* of  $A$  and  $I$  is an admissible ideal in  $KQ$ . For a bound quiver algebra  $A = KQ/I$ , we denote by  $e_i$ ,  $i \in Q_0$ , the associated complete set of pairwise orthogonal primitive idempotents of  $A$ , and by  $S_i = e_i A / e_i \text{rad } A$  (respectively,  $P_i = e_i A$ ),  $i \in Q_0$ , the associated complete family of pairwise non-isomorphic simple modules (respectively, indecomposable projective modules) in  $\text{mod } A$ .

Following [73], an algebra  $A$  is said to be *special biserial* if  $A$  is isomorphic to a bound quiver algebra  $KQ/I$ , where the bound quiver  $(Q, I)$  satisfies the following conditions:

- (a) each vertex of  $Q$  is a source and target of at most two arrows,
- (b) for any arrow  $\alpha$  in  $Q$  there are at most one arrow  $\beta$  and at most one arrow  $\gamma$  with  $\alpha\beta \notin I$  and  $\gamma\alpha \notin I$ .

Moreover, if in addition  $I$  is generated by paths of  $Q$ , then  $A = KQ/I$  is said to be a *string algebra* [15]. It was proved in [64] that the class of special biserial algebras coincides with the class of biserial algebras (indecomposable projective modules have biserial structure) which admit simply connected Galois coverings. Furthermore, by [77, Theorem 1.4] we know that every special biserial algebra is a quotient algebra of a symmetric special biserial algebra. We also mention that, if  $A$  is a self-injective special biserial algebra, then  $A / \text{soc}(A)$  is a string algebra.

The following has been proved by Wald and Waschbüsch in [77] (see also [15, 22] for alternative proofs).

**Proposition 2.1.** *Every special biserial algebra is tame.*

For a positive integer  $d$ , we denote by  $\text{alg}_d(K)$  the affine variety of associative  $K$ -algebra structures with identity on the affine space  $K^d$ . Then the general linear group  $\text{GL}_d(K)$  acts on  $\text{alg}_d(K)$  by transport of the structures, and the  $\text{GL}_d(K)$ -orbits in  $\text{alg}_d(K)$  correspond to the isomorphism classes of  $d$ -dimensional algebras (see [53] for details). We identify a  $d$ -dimensional algebra  $A$  with the point of  $\text{alg}_d(K)$  corresponding to it. For two  $d$ -dimensional algebras  $A$  and  $B$ , we say that  $B$  is a *degeneration* of  $A$  ( $A$  is a *deformation* of  $B$ ) if  $B$  belongs to the closure of the  $\text{GL}_d(K)$ -orbit of  $A$  in the Zariski topology of  $\text{alg}_d(K)$ .

Geiss' Theorem [44] shows that if  $A$  and  $B$  are two  $d$ -dimensional algebras,  $A$  degenerates to  $B$  and  $B$  is a tame algebra, then  $A$  is also a tame algebra (see also [18]). We will apply this theorem in the following special situation.

**Proposition 2.2.** *Let  $d$  be a positive integer, and  $A(t)$ ,  $t \in K$ , be an algebraic family in  $\text{alg}_d(K)$  such that  $A(t) \cong A(1)$  for all  $t \in K \setminus \{0\}$ . Then  $A(1)$  degenerates to  $A(0)$ . In particular, if  $A(0)$  is tame, then  $A(1)$  is tame.*

A family of algebras  $A(t)$ ,  $t \in K$ , in  $\text{alg}_d(K)$  is said to be *algebraic* if the induced map  $A(-) : K \rightarrow \text{alg}_d(K)$  is a regular map of affine varieties.

An important combinatorial and homological invariant of the module category  $\text{mod } A$  of an algebra  $A$  is its Auslander-Reiten quiver  $\Gamma_A$ . Recall that  $\Gamma_A$  is the translation quiver whose vertices are the isomorphism classes of indecomposable modules in  $\text{mod } A$ , the arrows correspond to irreducible homomorphisms, and the translation is the Auslander-Reiten translation  $\tau_A = D \text{Tr}$ . For  $A$  self-injective, we denote by  $\Gamma_A^s$  the stable Auslander-Reiten quiver of  $A$ , obtained from  $\Gamma_A$  by removing the isomorphism classes of projective modules and the arrows attached to them. By a *stable tube* we mean a translation quiver  $\Gamma$  of the form  $\mathbb{Z}\mathbb{A}_\infty / (\tau^r)$ , for some  $r \geq 1$ , and we call  $r$  the *rank* of  $\Gamma$ . We note that, for a symmetric algebra  $A$ , we have  $\tau_A = \Omega_A^2$  (see [74, Corollary IV.8.6]). In particular, we have the following equivalence.

**Proposition 2.3.** *Let  $A$  be an indecomposable, representation-infinite symmetric algebra. The following statements are equivalent:*

- (i)  $\Gamma_A^s$  consists of stable tubes.
- (ii) All indecomposable non-projective modules in  $\text{mod } A$  are periodic.

Therefore, we conclude that, if  $A$  is an indecomposable, representation-infinite, symmetric, periodic algebra (of period 4) then  $\Gamma_A^s$  consists of stable tubes (of ranks 1 and 2). We also note that, if  $A$  is a representation-infinite special biserial symmetric algebra, then  $\Gamma_A^s$  admits an acyclic component (see [32]), and consequently  $A$  is not a periodic algebra.

Let  $A$  be an algebra over  $K$  and  $\sigma$  a  $K$ -algebra automorphism of  $A$ . Then for any  $A$ - $A$ -bimodule  $M$  we denote by  ${}_1M_\sigma$  the  $A$ - $A$ -bimodule with the underlying  $K$ -vector space  $M$  and action defined as  $amb = am\sigma(b)$  for all  $a, b \in A$  and  $m \in M$ .

The following has been proved in [45, Theorem 1.4].

**Theorem 2.4.** *Let  $A$  be an algebra over  $K$  and  $d$  a positive integer. Then the following statements are equivalent:*

- (i)  $\Omega_A^d(S) \cong S$  in  $\text{mod } A$  for every simple module  $S$  in  $\text{mod } A$ .
- (ii)  $\Omega_{A^\sigma}^d(S) \cong {}_1A_\sigma$  in  $\text{mod } A^\sigma$  for some  $K$ -algebra automorphism  $\sigma$  of  $A$  such that  $\sigma(e)A \cong eA$  for any primitive idempotent  $e$  of  $A$ .

Moreover, if  $A$  satisfies these conditions, then  $A$  is self-injective.

The Cartan matrix  $C_A$  of an algebra  $A$  is the matrix  $(\dim_K \text{Hom}_A(P_i, P_j))_{1 \leq i, j \leq n}$  for a complete family  $P_1, \dots, P_n$  of a pairwise non-isomorphic indecomposable projective modules in  $\text{mod } A$ . The following main result from [26] shows why the original class of algebras of quaternion type is very restricted compared with the algebras which we will study in this paper.

**Theorem 2.5.** *Let  $A$  be an indecomposable, representation-infinite tame symmetric algebra with non-singular Cartan matrix such that every non-projective indecomposable module in  $\text{mod } A$  is periodic of period dividing 4. Then  $\text{mod } A$  has at most three pairwise non-isomorphic simple modules.*

### 3. Bimodule resolutions of self-injective algebras

In this section we describe a general approach for proving that an algebra  $A$  with periodic simple modules is a periodic algebra.

Let  $A = KQ/I$  be a bound quiver algebra, and  $e_i, i \in Q$ , be the primitive idempotents of  $A$  associated to the vertices of  $Q$ . Then  $e_i \otimes e_j, i, j \in Q_0$ , form a set of pairwise orthogonal primitive idempotents of the enveloping algebra  $A^e = A^{\text{op}} \otimes_K A$  whose sum is the identity of  $A^e$ . Hence,  $P(i, j) = (e_i \otimes e_j)A^e = Ae_i \otimes e_jA$ , for  $i, j \in Q_0$ , form a complete set of pairwise non-isomorphic indecomposable projective modules in  $\text{mod } A^e$  (see [74, Proposition IV.11.3]).

The following result by Happel [47, Lemma 1.5] describes the terms of a minimal projective resolution of  $A$  in  $\text{mod } A^e$ .

**Proposition 3.1.** *Let  $A = KQ/I$  be a bound quiver algebra. Then there is in  $\text{mod } A^e$  a minimal projective resolution of  $A$  of the form*

$$\cdots \rightarrow \mathbb{P}_n \xrightarrow{d_n} \mathbb{P}_{n-1} \rightarrow \cdots \rightarrow \mathbb{P}_1 \xrightarrow{d_1} \mathbb{P}_0 \xrightarrow{d_0} A \rightarrow 0,$$

where

$$\mathbb{P}_n = \bigoplus_{i, j \in Q_0} P(i, j)^{\dim_K \text{Ext}_A^n(S_i, S_j)}$$

for any  $n \in \mathbb{N}$ .

The syzygy modules have an important property, a proof for the next Lemma may be found in [74, Lemma IV.11.16].

**Lemma 3.2.** *Let  $A$  be an algebra. For any positive integer  $n$ , the module  $\Omega_{A^e}^n(A)$  is projective as a left  $A$ -module and also as a right  $A$ -module.*

There is no general recipe for the differentials  $d_n$  in Proposition 3.1, except for the first three which we will now describe. We have

$$\mathbb{P}_0 = \bigoplus_{i \in Q_0} P(i, i) = \bigoplus_{i \in Q_0} Ae_i \otimes e_i A.$$

The homomorphism  $d_0 : \mathbb{P}_0 \rightarrow A$  in  $\text{mod } A^e$  defined by  $d_0(e_i \otimes e_i) = e_i$  for all  $i \in Q_0$  is a minimal projective cover of  $A$  in  $\text{mod } A^e$ . Recall that, for two vertices  $i$  and  $j$  in  $Q$ , the number of arrows from  $i$  to  $j$  in  $Q$  is equal to  $\dim_K \text{Ext}_A^1(S_i, S_j)$  (see [1, Lemma III.2.12]). Hence we have

$$\mathbb{P}_1 = \bigoplus_{\alpha \in Q_1} P(s(\alpha), t(\alpha)) = \bigoplus_{\alpha \in Q_1} Ae_{s(\alpha)} \otimes e_{t(\alpha)} A.$$

Then we have the following known fact (see [8, Lemma 3.3] for a proof).

**Lemma 3.3.** *Let  $A = KQ/I$  be a bound quiver algebra, and  $d_1 : \mathbb{P}_1 \rightarrow \mathbb{P}_0$  the homomorphism in  $\text{mod } A^e$  defined by*

$$d_1(e_{s(\alpha)} \otimes e_{t(\alpha)}) = \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha$$

*for any arrow  $\alpha$  in  $Q$ . Then  $d_1$  induces a minimal projective cover  $d_1 : \mathbb{P}_1 \rightarrow \Omega_{A^e}^1(A)$  of  $\Omega_{A^e}^1(A) = \text{Ker } d_0$  in  $\text{mod } A^e$ . In particular, we have  $\Omega_{A^e}^2(A) \cong \text{Ker } d_1$  in  $\text{mod } A^e$ .*

We will denote the homomorphism  $d_1 : \mathbb{P}_1 \rightarrow \mathbb{P}_0$  by  $d$ . For the algebras  $A$  we will consider, the kernel  $\Omega_{A^e}^2(A)$  of  $d$  will be generated, as an  $A$ - $A$ -bimodule, by some elements of  $\mathbb{P}_1$  associated to a set of relations generating the admissible ideal  $I$ . Recall that a relation in the path algebra  $KQ$  is an element of the form

$$\mu = \sum_{r=1}^n c_r \mu_r,$$

where  $c_1, \dots, c_r$  are non-zero elements of  $K$  and  $\mu_r = \alpha_1^{(r)} \alpha_2^{(r)} \dots \alpha_{m_r}^{(r)}$  are paths in  $Q$  of length  $m_r \geq 2$ ,  $r \in \{1, \dots, n\}$ , having a common source and a common target. The admissible ideal  $I$  can be generated by a finite set of relations in  $KQ$  (see [1, Corollary II.2.9]). In particular, the bound quiver algebra  $A = KQ/I$  is given by the path algebra  $KQ$  and a finite number of identities  $\sum_{r=1}^n c_r \mu_r = 0$  given by a finite set of generators of the ideal  $I$ . Consider the  $K$ -linear homomorphism  $\varrho : KQ \rightarrow \mathbb{P}_1$  which assigns to a path  $\alpha_1 \alpha_2 \dots \alpha_m$  in  $Q$  the element

$$\varrho(\alpha_1 \alpha_2 \dots \alpha_m) = \sum_{k=1}^m \alpha_1 \alpha_2 \dots \alpha_{k-1} \otimes \alpha_{k+1} \dots \alpha_m$$

in  $\mathbb{P}_1$ , where  $\alpha_0 = e_{s(\alpha_1)}$  and  $\alpha_{m+1} = e_{t(\alpha_m)}$ . Observe that  $\varrho(\alpha_1 \alpha_2 \dots \alpha_m) \in e_{s(\alpha_1)} \mathbb{P}_1 e_{t(\alpha_m)}$ . Then, for a relation  $\mu = \sum_{r=1}^n c_r \mu_r$  in  $KQ$  lying in  $I$ , we have an element

$$\varrho(\mu) = \sum_{r=1}^n c_r \varrho(\mu_r) \in e_i \mathbb{P}_1 e_j,$$

where  $i$  is the common source and  $j$  is the common target of the paths  $\mu_1, \dots, \mu_r$ . The following lemma shows that relations always produce elements in the kernel of  $d_1$ ; the proof is straightforward.

**Lemma 3.4.** *Let  $A = KQ/I$  be a bound quiver algebra and  $d_1 : \mathbb{P}_1 \rightarrow \mathbb{P}_0$  the homomorphism in  $\text{mod } A^e$  defined in Lemma 3.3. Then for any relation  $\mu$  in  $KQ$  lying in  $I$ , we have  $d_1(\varrho(\mu)) = 0$ .*

For an algebra  $A = KQ/I$  in our context, we will see that there exists a family of relations  $\mu^{(1)}, \dots, \mu^{(q)}$  generating the ideal  $I$  such that the associated elements  $\varrho(\mu^{(1)}), \dots, \varrho(\mu^{(q)})$  generate the  $A$ - $A$ -bimodule  $\Omega_{A^e}^2(A) = \text{Ker } d_1$ . In fact, using Lemma 3.2, we will be able to show that

$$\mathbb{P}_2 = \bigoplus_{j=1}^q P(s(\mu^{(j)}), t(\mu^{(j)})) = \bigoplus_{j=1}^q Ae_{s(\mu^{(j)})} \otimes e_{t(\mu^{(j)})} A,$$



and the homomorphism  $d_2 : \mathbb{P}_2 \rightarrow \mathbb{P}_1$  in  $\text{mod } A^e$  such that

$$d_2(e_{s(\mu^{(j)})} \otimes e_{t(\mu^{(j)})}) = \varrho(\mu^{(j)}),$$

for  $j \in \{1, \dots, q\}$ , defines a projective cover of  $\Omega_{A^e}^2(A)$  in  $\text{mod } A^e$ . In particular, we have  $\Omega_{A^e}^3(A) \cong \text{Ker } d_2$  in  $\text{mod } A^e$ . We will denote this homomorphism  $d_2$  by  $R$ .

For the next map  $d_3 : \mathbb{P}_3 \rightarrow \mathbb{P}_2$ , which we will call  $S := d_3$  later, we do not have a general recipe. To define it, we need a set of minimal generators for  $\Omega_{A^e}^3(A)$ , and Proposition 3.1 tells us where we should look for them.

#### 4. Triangulation quivers of surfaces

The aim of this section is to introduce triangulation quivers of directed triangulated surfaces and present several examples illustrating possible shapes of such quivers.

In this paper, by a *surface* we mean a connected, compact, 2-dimensional real manifold  $S$ , orientable or non-orientable, with or without boundary. It is well known that every surface  $S$  admits an additional structure of a finite 2-dimensional triangular cell complex, and hence a triangulation by the deep Triangulation Theorem (see for example [16, Section 2.3]).

For a natural number  $n$ , we denote by  $D^n$  the unit disk in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , which consists of all points of distance  $\leq 1$  from the origin. Then the boundary  $\partial D^n$  of  $D^n$  is the unit sphere  $S^{n-1}$  in  $\mathbb{R}^n$ , formed by all points of distance 1 from the origin. Further, by an  $n$ -cell we mean a topological space homeomorphic to the open disk  $\text{int } D^n = D^n \setminus \partial D^n$ . In particular,  $D^0$  and  $e^0$  consist of a single point, and  $S^0 = \partial D^1$  consists of two points. A finite  $m$ -dimensional cell complex is a topological space  $X = X^m$  constructed by the following procedure (see [49]):

- (1) Start with a finite discrete set  $X^0$ , whose points are regarded as 0-cells.
- (2) Inductively, for  $n \in \{1, \dots, m\}$ , form the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching a finite number of  $n$ -cells  $e_i^n$  via maps  $\varphi_i^n : S^{n-1} \rightarrow X^{n-1}$ . This means that  $X^n$  is the quotient space of the disjoint union  $X^{n-1} \coprod_i D_i^n$  of  $X^{n-1}$  and a finite collection of  $n$ -disks  $D_i^n$  under the identification  $x \sim \varphi_i^n(x)$  for  $x \in \partial D_i^n$ . The cell  $e_i^n$  is the homeomorphic image of  $\text{int } D_i^n = D_i^n \setminus \partial D_i^n$  under the quotient map. Hence, as a set  $X^n$  is a disjoint union of  $X^{n-1}$  and all attached  $n$ -cells  $e_i^n$ .

For each  $n$ -cell  $e_i^n$  of  $X$ , the composition of continuous maps  $D_i^n \hookrightarrow X^{n-1} \coprod_i D_i^n \rightarrow X^n \rightarrow X$  is denoted by  $\phi_i^n$  and called the characteristic map of  $e_i^n$ . We also note that a subset  $A \subset X$  is open (or closed) if and only if  $A \cap X^n$  is open (or closed) for any  $n \in \{0, \dots, m\}$ .

The following consequence of [49, Proposition A2] provides a convenient description of finite  $m$ -dimensional cell complexes.

**Proposition 4.1.** *Let  $m$  be a positive integer and  $X$  a Hausdorff space. Then a finite family of continuous maps  $\varphi_i^n : D_i^n \rightarrow X$ , with  $n \in \{0, \dots, m\}$  and  $D_i^n = D^n$ , is the family of characteristic maps of a finite  $m$ -dimensional cell complex structure on  $X$  if and only if the following conditions are satisfied:*

- (i) *Each  $\varphi_i^n$  restricts to a homeomorphism from  $\text{int } D_i^n$  into its image, a cell  $e_i^n \subset X$ , and these cells are all disjoint and their union is  $X$ .*
- (ii) *For each cell  $e_i^n$ ,  $\varphi_i^n(\partial D_i^n)$  is contained in the union of a finite number of cells of smaller dimension than  $n$ .*

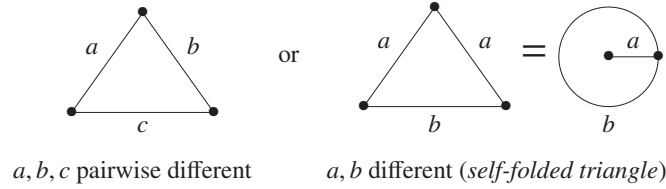
We refer to [49, Appendix] for some basic topological facts about cell complexes.

Let  $S$  be a surface. In this paper, by a *finite 2-dimensional triangular cell complex structure* on  $S$  we mean a finite family of continuous maps  $\varphi_i^n : D_i^n \rightarrow S$ , with  $n \in \{0, 1, 2\}$  and  $D_i^n = D^n$ , satisfying the following conditions:

- (1) Each  $\varphi_i^n$  restricts to a homeomorphism from  $\text{int } D_i^n$  to the  $n$ -cell  $e_i^n = \varphi_i^n(\text{int } D_i^n)$ , and these cells are disjoint and their union is  $S$ .
- (2) For each 2-cell  $e_i^2$ ,  $\varphi_i^2(\partial D_i^2)$  is contained in the union of  $k$  1-cells and  $k$  0-cells, with  $k \in \{2, 3\}$ .

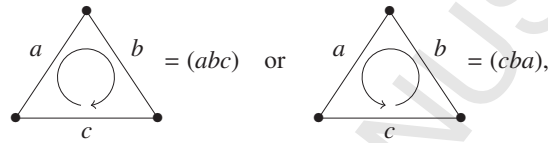
Then the closures  $\varphi_i^2(D_i^2)$  of all 2-cells  $e_i^2$  are called *triangles* of  $S$ , and the closures  $\varphi_i^1(D_i^1)$  of all 1-cells  $e_i^1$  are called *edges* of  $S$ . The collection  $T$  of all triangles  $\varphi_i^2(D_i^2)$  is said to be a *triangulation* of  $S$ . We assume that such a triangulation  $T$  of  $S$  has at least three pairwise different edges, or equivalently, there are at least three pairwise

different 1-cells in the considered cell complex structure on  $S$ . Then  $T$  is a finite collection  $T_1, \dots, T_n$  of triangles of the form

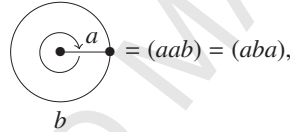


such that every edge of such a triangle in  $T$  is either the edge of exactly two triangles, or is the self-folded edge, or lies on the boundary. We note that a given surface  $S$  admits many finite 2-dimensional cell structures, and hence triangulations. We refer to [16, 51, 52] for general background on surfaces and constructions of surfaces from plane models.

Let  $S$  be a surface and  $T$  a triangulation  $S$ . To each triangle  $\Delta$  in  $T$  we may associate an orientation

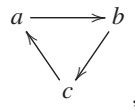


if  $\Delta$  has pairwise different edges  $a, b, c$ , and



if  $\Delta$  is self-folded, with the self-folded edge  $a$ , and the other edge  $b$ . Fix an orientation of each triangle  $\Delta$  of  $T$ , and denote this choice by  $\vec{T}$ . Then the pair  $(S, \vec{T})$  is said to be a *directed triangulated surface*. To each directed triangulated surface  $(S, \vec{T})$  we associate the quiver  $Q(S, \vec{T})$  whose vertices are the edges of  $T$  and the arrows are defined as follows:

- (1) for any oriented triangle  $\Delta = (abc)$  in  $\vec{T}$  with pairwise different edges  $a, b, c$ , we have the cycle



- (2) for any self-folded triangle  $\Delta = (aab)$  in  $\vec{T}$ , we have the quiver



- (3) for any boundary edge  $a$  in  $T$ , we have the loop



Then  $Q = Q(S, \vec{T})$  is a triangulation quiver in the following sense (introduced independently by Ladkani in [59, Definition 2.4] (see also [60, Definition 3.12])).

**Definition 4.2.** A triangulation quiver is a pair  $(Q, f)$ , where  $Q = (Q_0, Q_1, s, t)$  is a finite connected quiver and  $f : Q_1 \rightarrow Q_1$  is a permutation on the set  $Q_1$  of arrows of  $Q$  satisfying the following conditions:

- (a) every vertex  $i \in Q_0$  is the source and target of exactly two arrows in  $Q_1$ ,  
 (b) for each arrow  $\alpha \in Q_1$ , we have  $s(f(\alpha)) = t(\alpha)$ ,

(c)  $f^3$  is the identity on  $Q_1$ .

Let  $Q = Q(S, \vec{T})$  be the quiver associated to the directed triangulated surface  $(S, \vec{T})$ . The permutation  $f$  on its set of arrows is defined as follows:

$$(1) \quad \begin{array}{c} a \xrightarrow{\alpha} b \\ \gamma \swarrow \searrow \beta \\ c \end{array} \quad f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha,$$

for an oriented triangle  $\Delta = (abc)$  in  $\vec{T}$ , with pairwise different edges  $a, b, c$ ,

$$(2) \quad \begin{array}{c} \beta \\ \curvearrowright a \xrightarrow{\alpha} b \\ \gamma \end{array} \quad f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha,$$

for a self-folded triangle  $\Delta = (aab)$  in  $\vec{T}$ , and

$$(3) \quad \begin{array}{c} \alpha \\ \curvearrowright a \end{array} \quad f(\alpha) = \alpha,$$

for a boundary edge  $a$  of  $T$ .

We note that for such  $(Q, f)$ ,  $Q$  is 2-regular. We will consider only triangulation quivers with at least three vertices.

We will see below that different directed triangulated surfaces (even of different genus) may lead to the same triangulation quiver (see Example 4.4). We also mention that a similar construction of the triangulation quiver associated to an orientable surface was given in [59, Proposition 2.5] (see also [60, Definition 4.1]).

Let  $(Q, f)$  be a triangulation quiver. Then we have the involution  $\bar{\cdot} : Q_1 \rightarrow Q_1$  which assigns to an arrow  $\alpha \in Q_1$  the arrow  $\bar{\alpha}$  with  $s(\alpha) = s(\bar{\alpha})$  and  $\alpha \neq \bar{\alpha}$ . With this, we obtain another permutation  $g : Q_1 \rightarrow Q_1$  of the set  $Q_1$  of arrows of  $Q$  such that  $g(\alpha) = \bar{f(\bar{\alpha})}$  for any  $\alpha \in Q_1$ . We write  $O(g)$  for the set of  $g$ -orbits in  $Q_1$ .

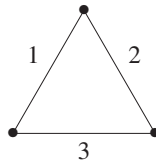
We will present now several examples of triangulation quivers. We will denote by  $\mathbb{S}$  the sphere  $S^2$ , by  $\mathbb{T}$  the torus and by  $\mathbb{P}$  the projective plane. For two surfaces  $X$  and  $Y$  we denote by  $X \# Y$  the connected sum of  $X$  and  $Y$ .

Recall that  $X \# Y$  is the surface constructed by the following steps:

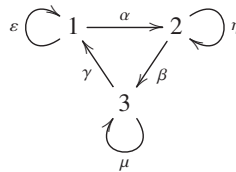
- Remove a small open 2-disk from each of the spaces  $X$  and  $Y$ , leaving the boundary 1-disks on each of the surfaces.
- Glue together the boundary 1-disks to form the connected sum.

In the first three examples we describe all possible triangulation quivers with exactly three vertices, and related directed triangulated surfaces.

**Example 4.3.** Let  $S = T$  be the triangle

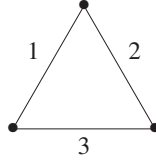


with the three pairwise different edges, forming the boundary of  $S$ , and consider the clockwise orientation  $\vec{T}$  of  $T$ . Then the triangulation quiver  $Q(S, \vec{T})$  is the quiver

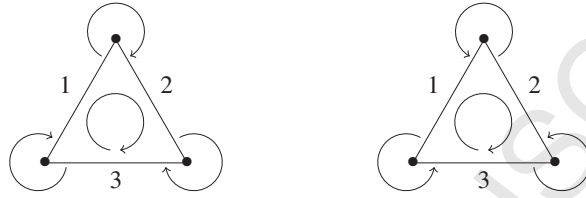


with  $f$ -orbits  $(\alpha \beta \gamma)$ ,  $(\epsilon)$ ,  $(\eta)$ ,  $(\mu)$ . Observe that we have only one  $g$ -orbit  $(\alpha \eta \beta \mu \gamma \epsilon)$  of arrows in  $Q(S, \vec{T})$ .

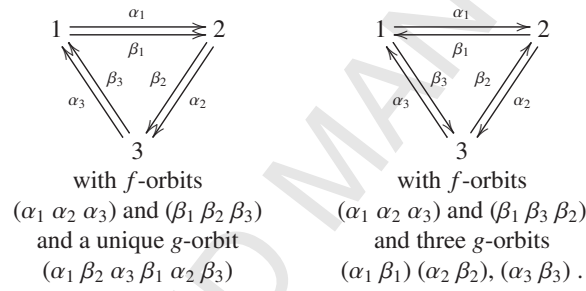
**Example 4.4.** Let  $S$  be the sphere  $\mathbb{S}$  with triangulation  $T$



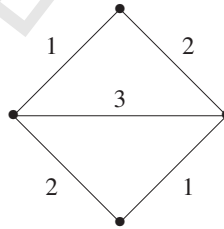
given by two unfolded triangles. There are two possible orientations  $\vec{T}$  of the triangles of  $T$  (up to duality)



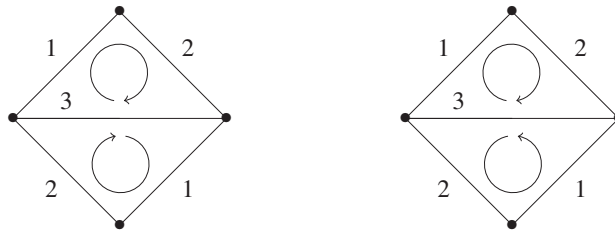
of  $T$ . The associated triangulation quivers  $Q(S, \vec{T})$  are



Consider also the torus  $\mathbb{T}$  with the triangulation  $T^*$

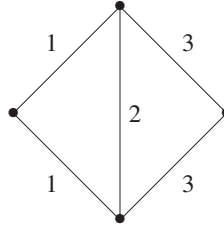


and the two possible orientations  $\vec{T}^*$  of the triangles of  $T^*$  (up to duality)



The associated triangulation quivers  $Q(\mathbb{T}, \vec{T}^*)$  are exactly the same as the triangulation quivers  $Q(S, \vec{T})$  above.

**Example 4.5.** Let  $S = \mathbb{P} \# \mathbb{P}$  be the connected sum of two copies of the projective plane  $\mathbb{P}$ . Then  $S$  admits the triangulation  $T$  of the form



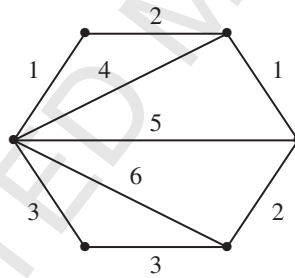
given by two self-folded triangles sharing a common edge. Then we have a unique orientation  $\vec{T}$  of these two triangles, and the associated triangulation quiver  $Q(S, \vec{T})$  is of the form

$$\alpha \curvearrowright 1 \xrightleftharpoons[\gamma]{\beta} 2 \xrightleftharpoons[\sigma]{\delta} 3 \curvearrowleft \varrho,$$

with the  $f$ -orbits  $(\alpha \beta \gamma)$  and  $(\varrho \sigma \delta)$ . Moreover,  $\mathcal{O}(g)$  consists of the three  $g$ -orbits  $(\alpha)$ ,  $(\varrho)$ ,  $(\beta \delta \sigma \gamma)$ . We also mention that  $\mathbb{P} \# \mathbb{P}$  is homeomorphic to the Klein bottle  $\mathbb{K}$  (see [16, Example 3.8]), and consequently the above triangulation quiver is also the quiver  $Q(\mathbb{K}, \vec{T})$ , for the induced directed triangulated structure on  $\mathbb{K}$ .

In the next examples, the shaded subquivers of a quiver  $Q(S, \vec{T})$  define the  $f$ -orbits of arrows in  $Q(S, \vec{T})$ . We note that for a loop, it is always clear from the context whether or not it is part of an  $f$ -orbit of length 3.

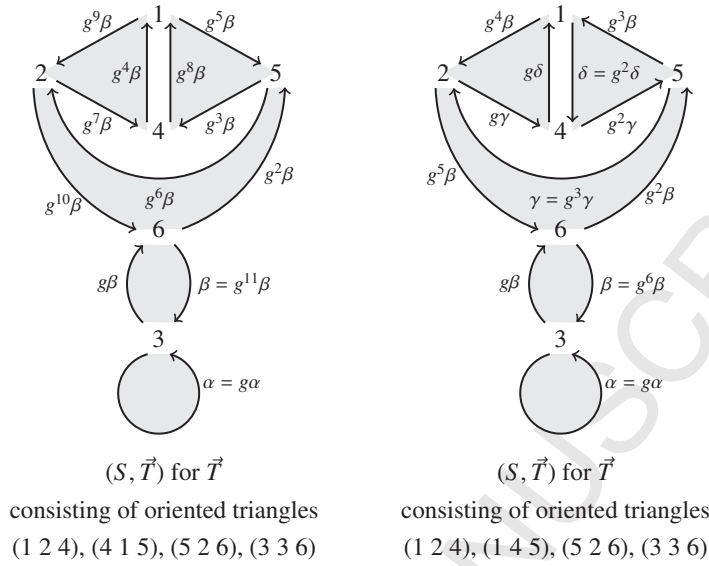
**Example 4.6.** Let  $S = \mathbb{T} \# \mathbb{P}$ , and let  $T$  be the following triangulation of  $S$



where the edges 1, 2 correspond to  $\mathbb{T}$ , and the edge 3 corresponds to  $\mathbb{P}$ . Observe that  $S$  has empty boundary. We

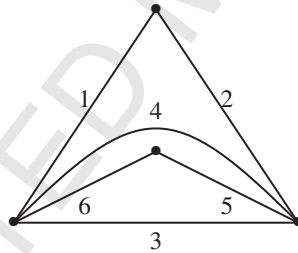


consider two orientations of the triangles of  $T$  and the associated quivers

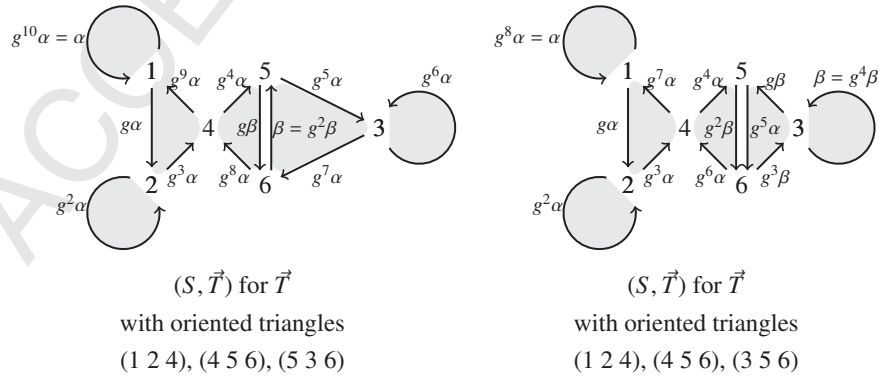


Observe that the first orientation gives two  $g$ -orbits of arrows in  $Q(S, \vec{T})$  (of lengths 1 and 11), while for the second orientation there are four  $g$ -orbits of arrows in  $Q(S, \vec{T})$  (of lengths 1, 2, 3, 6).

**Example 4.7.** Let  $S$  be a once punctured triangle, and let  $T$  be the triangulation of  $S$

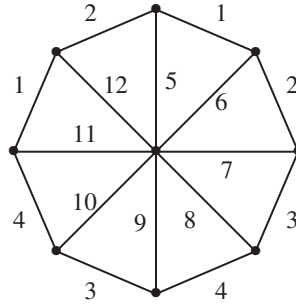


such that the edges 1, 2, 3 are on the boundary. We consider two orientations  $\vec{T}$  of the triangles of  $T$  and the associated quivers



In both orientations  $\vec{T}$ , there are two  $g$ -orbits of arrows in  $Q(S, \vec{T})$  but they have different length.

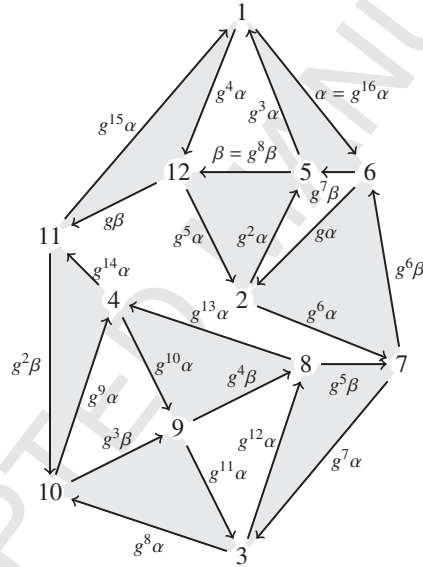
**Example 4.8.** Let  $S = \mathbb{T}\#\mathbb{T}$ , and  $T$  be the following triangulation of  $S$



Consider the orientation  $\vec{T}$  of triangles in  $T$

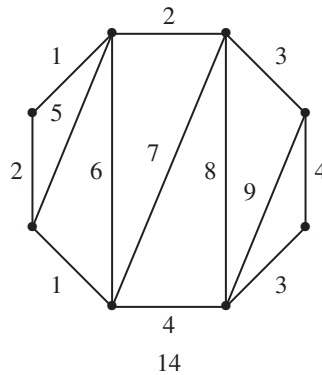
$$(1\ 6\ 5), (2\ 7\ 6), (7\ 3\ 8), (8\ 4\ 9), (9\ 3\ 10), (10\ 4\ 11), (11\ 1\ 12), (2\ 5\ 12).$$

Then the quiver  $Q(S, \vec{T})$  is of the form



and  $g$  has two orbits (of lengths 8 and 16).

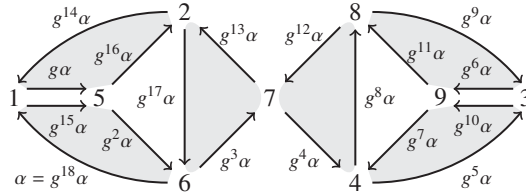
**Example 4.9.** Let  $S = \mathbb{T}\#\mathbb{T}$ , and  $T$  be the following triangulation of  $S$



We note that  $S$  has empty boundary. We consider the following orientation  $\vec{T}$  of triangles in  $T$

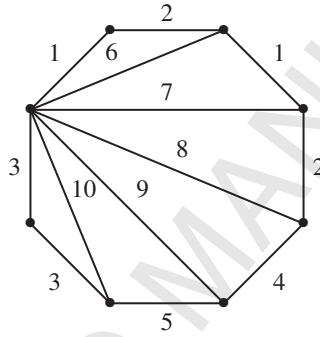
$$(1 \ 5 \ 2), (5 \ 6 \ 1), (2 \ 6 \ 7), (8 \ 7 \ 4), (3 \ 9 \ 8), (4 \ 3 \ 9).$$

Then the quiver  $Q(S, \vec{T})$  is of the form



There is only one  $g$ -orbit of arrows in  $Q(S, \vec{T})$  (of length 18).

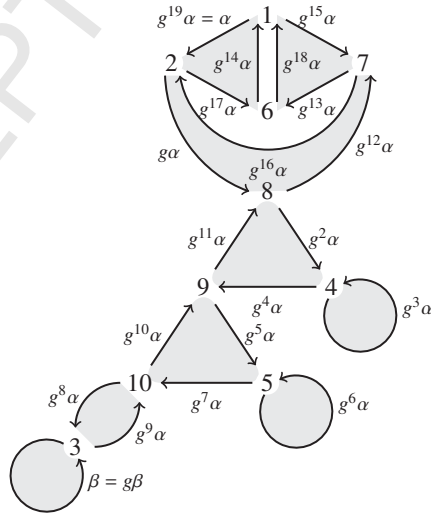
**Example 4.10.** Let  $S$  be obtained from  $\mathbb{T}\#\mathbb{P}$  by creating one boundary component, and  $T$  the following triangulation of  $S$



with two edges 4 and 5 on the boundary. Consider the following orientation  $\vec{T}$  of triangles in  $T$

$$(1 \ 2 \ 6), (6 \ 1 \ 7), (7 \ 2 \ 8), (8 \ 4 \ 9), (9 \ 5 \ 10), (3 \ 3 \ 10).$$

Then the quiver  $Q(S, \vec{T})$  is of the form



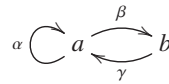
We note that there are two  $g$ -orbits of arrows in  $Q(S, \vec{T})$ , of lengths 1 and 19.

We will now show that every triangulation quiver comes from a directed triangulated surface.

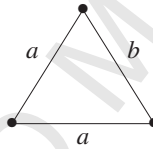
**Theorem 4.11.** *Let  $(Q, f)$  be a triangulation quiver with at least three vertices. Then there exists a directed triangulated surface  $(S, \vec{T})$  such that  $(Q, f) = Q(S, \vec{T})$ .*

*Proof.* Let  $Q = (Q_0, Q_1, s, t)$ . We denote by  $n(Q, f)$  the number of  $f$ -orbits in  $Q_1$  of length 3. We will prove the theorem by induction on  $n(Q, f)$ . Observe that if  $n(Q, f) = 1$  then  $(Q, f)$  is the triangulation quiver described in Example 4.3, because  $Q$  is a connected 2-regular quiver with  $|Q_0| \geq 3$ . Further, all possible triangulation quivers with three vertices are described in Examples 4.3, 4.4, 4.5. Therefore, we may assume that  $|Q_0| \geq 4$  and  $n(Q, f) \geq 2$ . We shall consider two cases.

(1) Assume that there is an  $f$ -orbit of length 3 in  $(Q, f)$  containing a loop. Then  $Q$  contains a subquiver

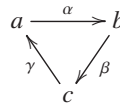


with  $f(\alpha) = \beta$ ,  $f(\beta) = \gamma$ ,  $f(\gamma) = \alpha$ . Consider the quiver  $Q' = (Q'_0, Q'_1, s', t')$  obtained from  $Q$  by removing the vertex  $a$ , the arrows  $\alpha, \beta, \gamma$ , and adding a loop  $\varepsilon$  at vertex  $b$ . Then we have the permutation  $f' : Q'_1 \rightarrow Q'_1$  such that  $f'(\sigma) = f(\sigma)$  for any arrow  $\sigma \in Q_1 \setminus \{\alpha, \beta, \gamma\}$  and  $f'(\varepsilon) := \varepsilon$ . Hence  $(Q', f')$  is a triangulation quiver with  $|Q'_0| = |Q_0| - 1 \geq 3$ . By the inductive assumption, there is a directed triangulated surface  $(S', \vec{T}')$ , with  $T'$  given by a finite 2-dimensional cell complex structure on  $S'$ , such that  $(Q', f') = Q(S', \vec{T}')$ . Moreover, the loop  $\varepsilon$  of  $Q'$  is created by a bordered edge  $b$  of the triangulation  $T'$  of  $S'$ . Consider the surface  $S''$  obtained from the projective plane  $\mathbb{P}$  by creating one boundary component, and its triangulation  $T''$



with boundary edge  $b$  and self-folded edge  $a$ . Moreover, let  $\vec{T}''$  be the orientation  $(aab)$  of  $T''$ . Let  $\phi'_b : D^1 \rightarrow S'$  be the characteristic map of the cell complex structure defining  $(S', T')$  whose image is the edge  $b$ , and  $\phi''_b : D^1 \rightarrow S''$  the characteristic map of the cell complex structure defining  $(S'', T'')$  whose image is the edge  $b$ . Denote by  $S$  the quotient space of the disjoint union  $S' \sqcup S''$  under the identification  $\phi'_b(x) \sim \phi''_b(x)$  for all  $x \in D^1$ . Then we have on  $S$  the cell complex structure induced by the cell complex structures of  $S'$  and  $S''$  and the characteristic map  $\phi_b : D^1 \rightarrow S$ , whose image is the edge  $b$ , obtained by gluing the two edges  $b$  in  $S'$  and  $S''$ , and replacing the characteristic maps  $\phi'_b$  and  $\phi''_b$ . In particular, applying Proposition 4.1, we infer that  $S$  is a surface with the triangulation  $T = T' \sqcup T''$ , and the orientation  $\vec{T}$  of triangles in  $T$  given by the orientations  $\vec{T}'$  and  $\vec{T}''$  of triangles in  $T'$  and  $T''$ . Moreover, we have  $(Q, f) = Q(S, \vec{T})$ .

(2) Assume that there is no loop in any  $f$ -orbit of length 3 in  $Q_1$ . Then  $Q$  contains a subquiver



with  $f(\alpha) = \beta$ ,  $f(\beta) = \gamma$ ,  $f(\gamma) = \alpha$ , and  $a, b, c$  are pairwise different vertices. Consider the quiver  $Q' = (Q'_0, Q'_1, s', t')$  obtained from  $Q$  by removing the arrows  $\alpha, \beta, \gamma$ , and adding the loops



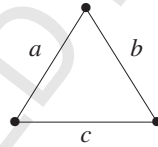
at the vertices  $a, b, c$ . Then  $Q'$  is a finite 2-regular quiver with  $Q'_0 = Q_0$  and it has at most three connected components. Moreover, there is the permutation  $f' : Q'_1 \rightarrow Q'_1$  such that  $f'(\sigma) = f(\sigma)$  for any arrow  $\sigma \in Q_1 \setminus \{\alpha, \beta, \gamma\}$  and

$f'(\varepsilon_a) = \varepsilon_a$ ,  $f'(\varepsilon_b) = \varepsilon_b$ ,  $f'(\varepsilon_c) = \varepsilon_c$ . For each  $i \in \{a, b, c\}$ , denote by  $Q(i) = (Q(i)_0, Q(i)_1, s(i), t(i))$  the connected component of  $Q'$  containing the vertex  $i$ , and by  $f_i : Q(i)_1 \rightarrow Q(i)_1$  the restriction of  $f'$  to  $Q(i)_1$ . Observe that each  $(Q(i), f_i)$  is a triangulation quiver with  $n(Q(i), f_i) \leq n(Q, f) - 1$ . Moreover, by the assumption imposed on the  $f$ -orbits in  $Q_1$ , we conclude that either  $|Q(i)_0| \geq 3$  or  $|Q(i)_0| = 1$ . Clearly, if  $|Q(i)_0| = 1$  then  $Q(i)$  is the loop  $\varepsilon_i$  at  $i$ . Since  $|Q_0| \geq 4$ , we conclude that  $|Q(i)_0| \geq 3$  for some  $i \in \{a, b, c\}$ . We may assume that  $|Q(a)_0| \geq 3$ , and  $|Q(c)_0| = 1$ , if  $|Q(i)_0| = 1$  for some  $i \in \{a, b, c\}$ . For each  $i \in \{a, b, c\}$  with  $|Q(i)_0| \geq 3$ , it follows from the inductive assumption that  $(Q(i), f_i) = Q(S(i), T(i))$  for a directed triangulated surface  $(S(i), T(i))$ . Observe also that, if  $Q(i) = Q(j)$  for some  $i \neq j$  in  $\{a, b, c\}$ , then  $|Q(i)_0| = |Q(j)_0| \geq 3$ . In such a case, we assume that  $(S(i), T(i)) = (S(j), T(j))$ . We may assume (without loss of generality) that, if  $Q'$  has at most two connected components, then  $Q(a) = Q(b)$ . We define the topological space  $S'$  as follows:

- $S' = S(a) \sqcup S(b) \sqcup S(c)$ , if  $Q(a)$ ,  $Q(b)$ ,  $Q(c)$  are pairwise different with  $|Q(a)_0| \geq 3$ ,  $|Q(b)_0| \geq 3$ ,  $|Q(c)_0| \geq 3$ ;
- $S' = S(a) \sqcup S(b)$ , if  $Q(a)$ ,  $Q(b)$ ,  $Q(c)$  are pairwise different with  $|Q(a)_0| \geq 3$ ,  $|Q(b)_0| \geq 3$ ,  $|Q(c)_0| = 1$ ;
- $S' = S(a)$ , if  $Q(a)$ ,  $Q(b)$ ,  $Q(c)$  are pairwise different with  $|Q(a)_0| \geq 3$ ,  $|Q(b)_0| = 1$ ,  $|Q(c)_0| = 1$ ;
- $S' = S(a) \sqcup S(c)$ , if  $Q(a) = Q(b)$ , different from  $Q(c)$ , and  $|Q(c)_0| \geq 3$ ;
- $S' = S(a)$ , if  $Q(a) = Q(b)$ , different from  $Q(c)$ , and  $|Q(c)_0| = 1$ ;
- $S' = S(a)$ , if  $Q(a) = Q(b) = Q(c)$ .

Observe that there is the finite 2-dimensional cell complex structure on  $S'$ , given by the finite 2-dimensional cell complex structures on the surfaces  $S(i)$ , defining the triangulations  $T(i)$ , for  $i \in \{a, b, c\}$  with  $|Q(i)_0| \geq 3$ , and consequently the induced triangulation  $T'$  of  $S'$ . We denote by  $\vec{T}'$  the orientation of triangles in  $T'$  given by the orientations  $\vec{T}(i)$  of triangles of  $T(i)$  in  $S(i)$ , for all  $i \in \{a, b, c\}$  with  $|Q(i)_0| \geq 3$ . Moreover, for any  $i \in \{a, b, c\}$  with  $|Q(i)_0| \geq 3$ , we denote by  $\phi'_i : D^1 \rightarrow S'$  the characteristic map of the defined cell complex structure on  $S'$  whose image is the edge  $i$ .

Consider now the triangle  $S'' = T''$



with the three pairwise different edges, forming the boundary of  $S''$ , and the orientation  $\vec{T}'' = (abc)$  (see Example 4.3). Let  $\phi''_a : D^1 \rightarrow S''$ ,  $\phi''_b : D^1 \rightarrow S''$ ,  $\phi''_c : D^1 \rightarrow S''$  be the characteristic maps of the 2-dimensional cell complex structure on  $S'' = T''$  whose images are respectively the edges  $a$ ,  $b$ ,  $c$ .

Let  $S$  be the quotient space of  $S' \sqcup S''$  under the identification  $\phi'_i(x) \sim \phi''_i(x)$  for all  $x \in D^1$  and  $i \in \{a, b, c\}$  with  $|Q(i)_0| \geq 3$ . Then, applying Proposition 4.1 again, we conclude that  $S$  is a surface with a 2-dimensional cell complex structure defining the triangulation  $T = T' \sqcup T''$ , and the orientation  $\vec{T}$  of triangles in  $T$  given by the orientations  $\vec{T}'$  and  $\vec{T}''$  of triangles in  $T'$  and  $T''$ . It follows from the above construction that  $(Q, f) = Q(S, \vec{T})$ .  $\square$

**Corollary 4.12.** *Let  $(Q, f)$  be a triangulation quiver with at least three vertices. Then  $Q$  contains a loop  $\alpha$  with  $f(\alpha) = \alpha$  if and only if  $(Q, f) = Q(S, \vec{T})$  for a directed triangulated surface  $(S, \vec{T})$  where  $S$  has non-empty boundary.*

We end this section with the comment that the setting of directed triangulated surfaces proposed in this paper is natural for the purposes of a self-contained representation theory of symmetric tame algebras of non-polynomial growth which we are currently developing. In particular, the realization Theorem 4.11 gives the option of changing orientation of any triangle independently.



## 5. Weighted surface algebras

In this section we define weighted surface algebras of directed triangulated surfaces and describe their basic properties.

Let  $(Q, f)$  be a triangulation quiver. Then we have two permutations  $f : Q_1 \rightarrow Q_1$  and  $g : Q_1 \rightarrow Q_1$  on the set  $Q_1$  of arrows of  $Q$  such that  $f^3$  is the identity on  $Q_1$  and  $g = \bar{f}$ , where  $\bar{\cdot} : Q_1 \rightarrow Q_1$  is the involution which assigns to an arrow  $\alpha \in Q_1$  the arrow  $\bar{\alpha}$  with  $s(\alpha) = s(\bar{\alpha})$  and  $\alpha \neq \bar{\alpha}$ . For each arrow  $\alpha \in Q_1$ , we denote by  $O(\alpha)$  the  $g$ -orbit of  $\alpha$  in  $Q_1$ , and set  $n_\alpha = n_{O(\alpha)} = |O(\alpha)|$ . Recall that  $O(g)$  is the set of all  $g$ -orbits in  $Q_1$ . A function

$$m_\bullet : O(g) \rightarrow \mathbb{N}^* = \mathbb{N} \setminus \{0\}$$

is said to be a *weight function* of  $(Q, f)$ , and a function

$$c_\bullet : O(g) \rightarrow K^* = K \setminus \{0\}$$

is said to be a *parameter function* of  $(Q, f)$ . We write briefly  $m_\alpha = m_{O(\alpha)}$  and  $c_\alpha = c_{O(\alpha)}$  for  $\alpha \in Q_1$ . In this paper, we will assume that  $m_\alpha n_\alpha \geq 3$  for any arrow  $\alpha \in Q_1$ .

For any arrow  $\alpha \in Q_1$ , we consider the path

$$A_\alpha = \begin{cases} (\alpha g(\alpha) \dots g^{n_\alpha-1}(\alpha))^{m_\alpha-1} \alpha g(\alpha) \dots g^{n_\alpha-2}(\alpha), & \text{if } n_\alpha \geq 2, \\ \alpha^{m_\alpha-1}, & \text{if } n_\alpha = 1, \end{cases}$$

in  $Q$  of length  $m_\alpha n_\alpha - 1$  from  $s(\alpha)$  to  $t(g^{n_\alpha-2}(\alpha))$ . Moreover, for any arrow  $\alpha \in Q_1$ , we have the oriented cycle

$$B_\alpha = (\alpha g(\alpha) \dots g^{n_\alpha-1}(\alpha))^{m_\alpha}$$

of length  $m_\alpha n_\alpha$ .

**Definition 5.1.** Let  $(Q, f)$  be a triangulation quiver with weight and parameter functions  $m_\bullet$  and  $c_\bullet$ . We define the bound quiver algebra

$$\Lambda(Q, f, m_\bullet, c_\bullet) = KQ/I(Q, f, m_\bullet, c_\bullet),$$

where  $I(Q, f, m_\bullet, c_\bullet)$  is the admissible ideal in the path algebra  $KQ$  of  $Q$  over  $K$  generated by:

- (1)  $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}$ , for all arrows  $\alpha \in Q_1$ ,
- (2)  $\beta f(\beta) g(f(\beta))$ , for all arrows  $\beta \in Q_1$ .

Then  $\Lambda(Q, f, m_\bullet, c_\bullet)$  is called a *weighted triangulation algebra* of  $(Q, f)$ .

We note that  $Q$  is the quiver of the algebra  $\Lambda(Q, f, m_\bullet, c_\bullet)$ , and the ideal  $I(Q, f, m_\bullet, c_\bullet)$  is an admissible ideal of  $KQ$ , by the assumption that  $m_\alpha n_\alpha \geq 3$  for all arrows  $\alpha \in Q_1$ . We also note that a weighted triangulation algebra defined above is a triangulation algebra defined in [59, 60] as a quotient of complete path algebra of the quiver by a closed ideal (see [59, Definition 2.10 and Theorem 1.1(a)] or [60, Definition 5.16 and Proposition 7.4]).

**Definition 5.2.** Consider the bound quiver algebra

$$B(Q, f, m_\bullet, c_\bullet) = KQ/J(Q, f, m_\bullet, c_\bullet),$$

where  $J(Q, f, m_\bullet, c_\bullet)$  is the admissible ideal in the path algebra  $KQ$  of  $Q$  over  $K$  generated by:

- (1)  $c_\alpha B_\alpha - c_{\bar{\alpha}} B_{\bar{\alpha}}$ , for all arrows  $\alpha \in Q_1$ ,
- (2)  $\beta f(\beta)$ , for all arrows  $\beta \in Q_1$ .

We call this algebra a *biserial weighted triangulation algebra*.

We note that a biserial weighted triangulation algebra is a Brauer graph algebra. In fact, it is shown in [38] that the class of Brauer graph algebras coincides with the class of indecomposable idempotent algebras of biserial weighted triangulation algebras (we refer to [38] for related references and results).

Let  $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$  be a weighted triangulation algebra. In order to study modules in  $\text{mod } \Lambda$  and properties of  $\Lambda$ , we specify a suitable basis of the algebra  $\Lambda$ , defined in terms of the permutations  $f$  and  $g$ . We will identify an element of  $KQ$  with its residue class in  $\Lambda = KQ/I(Q, f, m_\bullet, c_\bullet)$ . We will need also an extra notation. For each arrow  $\alpha$  in  $Q$ , we denote by  $A'_\alpha$  the subpath of  $A_\alpha$  from  $t(\alpha)$  to  $t(g^{n_\alpha-2}(\alpha))$  of length  $m_\alpha n_\alpha - 2$  such that  $\alpha A'_\alpha = A_\alpha$ . We note that  $A'_\alpha$  is a path of length  $\geq 1$  since we assume that  $m_\alpha n_\alpha \geq 3$ .

**Lemma 5.3.** *Let  $\alpha$  be an arrow in  $Q$ . We have in  $\Lambda$  the equalities:*

- (i)  $f^2(\alpha) = g^{n_\alpha-1}(\bar{\alpha})$ .
- (ii)  $A_{\bar{\alpha}} f^2(\alpha) = B_{\bar{\alpha}}$ .
- (iii)  $\alpha A_{g(\alpha)} = B_\alpha$ .
- (iv)  $c_\alpha B_\alpha = \alpha f(\alpha) f^2(\alpha) = \bar{\alpha} f(\bar{\alpha}) f^2(\bar{\alpha}) = c_{\bar{\alpha}} B_{\bar{\alpha}}$ .
- (v)  $A'_\alpha f^2(\bar{\alpha}) = A_{g(\alpha)}$ .

*Proof.* (i) The arrow  $g(f^2(\alpha))$  starts at  $t(f^2(\alpha)) = s(\alpha)$  and we have  $g(f^2(\alpha)) \neq f(f^2(\alpha)) = \alpha$ . Hence we have  $g(f^2(\alpha)) = \bar{\alpha} = g^{n_\alpha}(\bar{\alpha})$  and therefore  $f^2(\alpha) = g^{n_\alpha-1}(\bar{\alpha})$ .

Part (ii) follows from (i), and part (iii) holds by definition.

(iv) From the relations in  $\Lambda$ , (iii), and since  $c_\bullet$  is constant on  $g$ -orbits, we obtain

$$\alpha f(\alpha) f^2(\alpha) = \alpha (f(\alpha) f^2(\alpha)) = \alpha c_{\bar{f}(\alpha)} A_{\bar{f}(\alpha)} = c_{g(\alpha)} \alpha A_{g(\alpha)} = c_\alpha B_\alpha.$$

Similarly, we have  $\bar{\alpha} f(\bar{\alpha}) f^2(\bar{\alpha}) = c_{\bar{\alpha}} B_{\bar{\alpha}}$ . Then, by (ii), we obtain

$$\alpha f(\alpha) f^2(\alpha) = (\alpha f(\alpha)) f^2(\alpha) = c_{\bar{\alpha}} A_{\bar{\alpha}} f^2(\alpha) = c_{\bar{\alpha}} B_{\bar{\alpha}} = \bar{\alpha} f(\bar{\alpha}) f^2(\bar{\alpha}).$$

(v) By (i) we have that  $f^2(\bar{\alpha}) = g^{n_\alpha-1}(\alpha)$ , and hence the required equality holds.  $\square$

**Lemma 5.4.** *Let  $\alpha$  be an arrow in  $Q$ . Then the following hold:*

- (i)  $B_\alpha \text{rad } \Lambda = 0$ .
- (ii)  $B_\alpha$  is non-zero.

*Proof.* (i) We must show that  $B_\alpha \alpha = 0$  and  $B_\alpha \bar{\alpha} = 0$  in  $\Lambda$ . It follows from (i) and (iv) of Lemma 5.3 and the relations in  $\Lambda$  that

$$\begin{aligned} c_\alpha B_\alpha \alpha &= \bar{\alpha} f(\bar{\alpha}) f^2(\bar{\alpha}) \alpha = \bar{\alpha} (f(\bar{\alpha}) f^2(\bar{\alpha}) g(f^2(\bar{\alpha}))) = 0, \\ c_\alpha B_\alpha \bar{\alpha} &= \alpha f(\alpha) f^2(\alpha) \bar{\alpha} = \alpha (f(\alpha) f^2(\alpha) g(f^2(\alpha))) = 0, \end{aligned}$$

and hence  $B_\alpha \alpha = 0$  and  $B_\alpha \bar{\alpha} = 0$ , because  $c_\alpha \in K^*$ .

(ii) This follows from the relations defining  $\Lambda$ .  $\square$

It follows from Lemmas 5.3 and 5.4 that, for a vertex  $i$  of  $Q$  and the arrows  $\alpha$  and  $\bar{\alpha}$  starting at  $i$ , the element  $c_\alpha B_\alpha = c_{\bar{\alpha}} B_{\bar{\alpha}}$  generates the socle of the projective module  $e_i \Lambda$ . The next lemma shows that, for any vertex  $i$  of  $Q$ , the quotient  $e_i(\text{rad } \Lambda)^2 / \text{soc}(e_i \Lambda)$  is a direct sum of uniserial right  $\Lambda$ -modules, as well as gives most of a basis for the indecomposable projective module  $e_i \Lambda$ .

**Lemma 5.5.** *Let  $\alpha$  be an arrow of  $Q$ . Then the following hold:*

- (i)  $\alpha g(\alpha) f(g(\alpha)) = 0$  in  $\Lambda$ .
- (ii)  $\alpha g(\alpha) \Lambda$  is a uniserial right  $\Lambda$ -module, with basis given by all initial subwords of  $B_\alpha$  of length  $\geq 2$ . In particular,  $\dim_K \alpha g(\alpha) \Lambda = m_\alpha n_\alpha - 1$ .

*Proof.* (i) Since  $\overline{g(\alpha)} = f(\alpha)$  we obtain the equalities

$$\alpha g(\alpha) f(g(\alpha)) = \alpha c_{\overline{g(\alpha)}} A_{\overline{g(\alpha)}} = c_{f(\alpha)} \alpha A_{f(\alpha)} = c_{f(\alpha)} \alpha f(\alpha) g(f(\alpha)) \cdots = 0,$$

by the relations for the algebra  $\Lambda$ .

(ii) It follows from (i) that the right  $\Lambda$ -module  $\alpha g(\alpha) \text{rad } \Lambda$  is generated by  $\alpha g(\alpha) g^2(\alpha)$ . Then using (i) repeatedly we conclude that  $\alpha g(\alpha) \Lambda$  is a uniserial right  $\Lambda$ -module with basis formed by all initial subwords of  $B_\alpha$  of length  $\geq 2$ . Clearly, then  $\dim_K \alpha g(\alpha) \Lambda = m_\alpha n_\alpha - 1$ .  $\square$

**Corollary 5.6.** *Let  $i$  be a vertex of  $Q$  and  $\alpha, \bar{\alpha}$  the two arrows in  $Q$  with source  $i$ . Then  $\dim_K e_i \Lambda = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$ .*

*Proof.* It follows from the previous lemma, that a basis of  $\alpha g(\alpha) \Lambda$  is given by the set of initial subwords of  $B_\alpha$  of length  $\geq 2$ . Then we also see that  $\text{rad } e_i \Lambda$  has basis consisting of all initial subwords of  $A_\alpha$  and  $A_{\bar{\alpha}}$  together with  $B_\alpha$ . This shows that  $\dim_K e_i \Lambda = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$ .  $\square$

We present now basic properties of the algebras  $B(Q, f, m_\bullet, c_\bullet)$  and  $\Lambda(Q, f, m_\bullet, c_\bullet)$ .

**Proposition 5.7.** *Let  $(Q, f)$  be a triangulation quiver,  $m_\bullet$  and  $c_\bullet$  weight and parameter functions of  $(Q, f)$ , and  $B = B(Q, f, m_\bullet, c_\bullet)$ . Then the following statements hold:*

- (i)  *$B$  is a finite-dimensional special biserial algebra with  $\dim_K B = \sum_{O \in O(g)} m_O n_O^2$ .*
- (ii)  *$B$  is a symmetric algebra.*
- (iii)  *$B$  is a tame algebra.*

*Proof.* We write  $J = J(Q, f, m_\bullet, c_\bullet)$ .

(i) Let  $i$  be a vertex of  $Q$  and let  $\alpha, \bar{\alpha}$  be the two arrows in  $Q$  with source  $i$ . Then the indecomposable projective right  $B$ -module  $P_i = e_i B$  has dimension equal to  $\dim_K P_i = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$ . Indeed,  $P_i$  has a basis given by  $e_i$ , all initial subwords of  $A_\alpha$  and  $A_{\bar{\alpha}}$ , and  $B_\alpha$ . Then we deduce that

$$\dim_K B = \sum_{O \in O(g)} m_O n_O^2.$$

(ii) It is well known (see for example [74, Theorem IV.2.2]) that  $B$  is a symmetric algebra if and only if it has a symmetrizing form. That is, there exists a  $K$ -linear form  $\varphi : B \rightarrow K$  such that  $\varphi(ab) = \varphi(ba)$  for all  $a, b \in B$  and  $\text{Ker } \varphi$  does not contain non-zero one-sided ideal of  $B$ . Let  $i$  be a vertex of the quiver  $Q$  and  $\alpha, \bar{\alpha}$  be the arrows with source  $i$ . Then the element  $c_\alpha B_\alpha + J = c_{\bar{\alpha}} B_{\bar{\alpha}} + J$  generates the one-dimensional socle of the indecomposable projective right  $B$ -module  $P_i$  at the vertex  $i$ . Clearly, we have also that  $\text{top}(P_i) = S_i = \text{soc}(P_i)$ . We define a required  $K$ -linear form  $\varphi : B \rightarrow K$  by assigning to the coset  $u + J$  of a path  $u$  in  $Q$  the following element in  $K$

$$\varphi(u + J) = \begin{cases} c_\alpha^{-1} & \text{if } u = B_\alpha \text{ for an arrow } \alpha \in Q_1, \\ 0 & \text{otherwise,} \end{cases}$$

and extending to a  $K$ -linear form.

(iii) Since  $B$  is special biserial, it is tame, by Proposition 2.1.  $\square$

We refer to Section 6 for the tetrahedral algebras and their properties.

**Proposition 5.8.** *Let  $(Q, f)$  be a triangulation quiver,  $m_\bullet$  and  $c_\bullet$  weight and parameter functions of  $(Q, f)$ , and  $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ . Then the following statements hold:*

- (i)  *$\Lambda$  is a finite-dimensional algebra with  $\dim_K \Lambda = \sum_{O \in O(g)} m_O n_O^2$ .*
- (ii)  *$\Lambda$  is a symmetric algebra.*
- (iii)  *$\Lambda$  degenerates to the algebra  $B(Q, f, m_\bullet, c_\bullet)$ , provided  $\Lambda$  is not a tetrahedral algebra.*
- (iv)  *$\Lambda$  is a tame algebra.*

*Proof.* We abbreviate  $I = I(Q, f, m_\bullet, c_\bullet)$ .

(i) It follows from Corollary 5.6 that, for each vertex  $i$  of  $Q$ , the indecomposable projective right  $\Lambda$ -module  $P_i$  at the vertex  $i$  has the dimension  $\dim_K P_i = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$ , where  $\alpha, \bar{\alpha}$  are the two arrows in  $Q$  with source  $i$ . Then we get

$$\dim_K \Lambda = \sum_{O \in O(g)} m_O n_O^2.$$

(ii) Similarly, as in the above Proposition, we define a symmetrizing  $K$ -linear form  $\varphi : \Lambda \rightarrow K$  by assigning to the coset  $u + I$  of a path  $u$  in  $Q$  the following element in  $K$

$$\varphi(u + I) = \begin{cases} c_\alpha^{-1} & \text{if } u = B_\alpha \text{ for an arrow } \alpha \in Q_1, \\ 0 & \text{otherwise,} \end{cases}$$

and extending to a  $K$ -linear form.

(iii) Assume that  $\Lambda$  is not a tetrahedral algebra. For each  $t \in K$ , consider the bound quiver algebra  $\Lambda(t) = KQ/I^{(t)}$ , where  $I^{(t)}$  is the admissible ideal in the path algebra  $KQ$  of  $Q$  over  $K$  generated by the elements:

- (i)  $\alpha f(\alpha) - tc_{\bar{\alpha}} A_{\bar{\alpha}}$ , for all arrows  $\alpha \in Q_1$ ,
- (ii)  $\beta f(\beta)g(f(\beta))$ , for all arrows  $\beta \in Q_1$ .

Then a simple checking shows that  $\Lambda(t)$ ,  $t \in K$ , is an algebraic family in the variety  $\text{alg}_d(K)$ , with  $d = \dim_K \Lambda$ , such that  $\Lambda(t) \cong \Lambda(1) = \Lambda$  for all  $t \in K \setminus \{0\}$  and  $\Lambda(0) = B = B(Q, f, m_\bullet, c_\bullet)$ . Then it follows from Proposition 2.2 that  $\Lambda$  degenerates to  $B$ . We refer also to [60, Proposition 7.13] for a different algebraic family of intermediate algebras degenerating  $\Lambda$  to  $B$ .

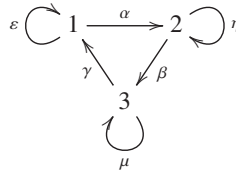
(iv) If  $\Lambda$  is not a tetrahedral algebra then it follows from Propositions 2.2 and 5.7 that  $\Lambda$  is tame. Assume  $\Lambda$  is a tetrahedral algebra. If  $\Lambda$  is non-singular then the tameness (even polynomial growth) of  $\Lambda$  follows from the old article [62] where the representation theory of the trivial extensions of arbitrary tubular algebras has been established. If  $\Lambda$  is singular, then the tameness of  $\Lambda$  follows from [21, Theorem] and [61, Theorem A].  $\square$

**Definition 5.9.** Let  $(S, \vec{T})$  be a directed triangulated surface,  $(Q(S, \vec{T}), f)$  the associated triangulation quiver, and let  $m_\bullet$  and  $c_\bullet$  be weight and parameter functions of  $(Q(S, \vec{T}), f)$ . Then the triangulation algebra  $\Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet)$  will be called a *weighted surface algebra*.

For further purposes, we would like to have two notions: a weighted surface algebra and a weighted triangulation algebra on the grounds, one of topological origin and the other purely algebraic.

We give now examples of weighted surface algebras, using the triangulation quivers from Examples 4.3, 4.4, 4.5.

**Example 5.10.** Let  $(Q(S, \vec{T}), f)$  be the triangulation quiver



with  $f$ -orbits  $(\alpha \beta \gamma)$ ,  $(\varepsilon)$ ,  $(\eta)$ ,  $(\mu)$ , considered in Example 4.3. Then  $g$  has only one orbit,  $(\alpha \eta \beta \mu \gamma \varepsilon)$ , and hence a weight function  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  and a parameter function  $c_\bullet : O(g) \rightarrow K^*$  are given by a positive integer  $m$  and a parameter  $c \in K^*$ . The associated weighted surface algebra  $\Lambda = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet)$  is given by the above quiver and the relations

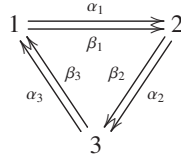
$$\begin{aligned} \alpha\beta &= c(\varepsilon\alpha\eta\beta\mu\gamma)^{m-1}\varepsilon\alpha\eta\beta\mu, & \varepsilon^2 &= c(\alpha\eta\beta\mu\gamma\varepsilon)^{m-1}\alpha\eta\beta\mu\gamma, & \alpha\beta\mu &= 0, & \varepsilon^2\alpha &= 0, \\ \beta\gamma &= c(\eta\beta\mu\gamma\varepsilon\alpha)^{m-1}\eta\beta\mu\gamma\varepsilon, & \eta^2 &= c(\beta\mu\gamma\varepsilon\alpha\eta)^{m-1}\beta\mu\gamma\varepsilon\alpha, & \beta\gamma\varepsilon &= 0, & \eta^2\beta &= 0, \\ \gamma\alpha &= c(\mu\gamma\varepsilon\alpha\eta\beta)^{m-1}\mu\gamma\varepsilon\alpha\eta, & \mu^2 &= c(\gamma\varepsilon\alpha\eta\beta\mu)^{m-1}\gamma\varepsilon\alpha\eta\beta, & \gamma\alpha\eta &= 0, & \mu^2\gamma &= 0. \end{aligned}$$

Moreover, the Cartan matrix  $C_\Lambda$  of  $\Lambda$  is of the form

$$\begin{bmatrix} 4m & 4m & 4m \\ 4m & 4m & 4m \\ 4m & 4m & 4m \end{bmatrix},$$

and hence is singular.

**Example 5.11.** Let  $(Q(S, \vec{T}), f)$  be the triangulation quiver



with  $f$ -orbits  $(\alpha_1 \alpha_2 \alpha_3)$  and  $(\beta_1 \beta_2 \beta_3)$ , considered in Example 4.4. Then  $g$  has only one orbit, which is  $(\alpha_1 \beta_2 \alpha_3 \beta_1 \alpha_2 \beta_3)$ , and hence a weight function  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  and a parameter function  $c_\bullet : O(g) \rightarrow K^*$  are given by a positive integer  $m$  and a parameter  $c \in K^*$ . The associated weighted surface algebra  $\Lambda = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet)$  is given by the above quiver and the relations

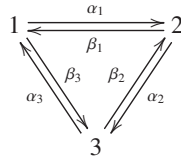
$$\begin{aligned} \alpha_1 \alpha_2 &= c(\beta_1 \alpha_2 \beta_3 \alpha_1 \beta_2 \alpha_3)^{m-1} \beta_1 \alpha_2 \beta_3 \alpha_1 \beta_2, & \alpha_2 \alpha_3 &= c(\beta_2 \alpha_3 \beta_1 \alpha_2 \beta_3 \alpha_1)^{m-1} \beta_2 \alpha_3 \beta_1 \alpha_2 \beta_3, \\ \alpha_3 \alpha_1 &= c(\beta_3 \alpha_1 \beta_2 \alpha_3 \beta_1 \alpha_2)^{m-1} \beta_3 \alpha_1 \beta_2 \alpha_3 \beta_1, & \beta_1 \beta_2 &= c(\alpha_1 \beta_2 \alpha_3 \beta_1 \alpha_2 \beta_3)^{m-1} \alpha_1 \beta_2 \alpha_3 \beta_1 \alpha_2, \\ \beta_2 \beta_3 &= c(\alpha_2 \beta_3 \alpha_1 \beta_2 \alpha_3 \beta_1)^{m-1} \alpha_2 \beta_3 \alpha_1 \beta_2 \alpha_3, & \beta_3 \beta_1 &= c(\alpha_3 \beta_1 \alpha_2 \beta_3 \alpha_1 \beta_2)^{m-1} \alpha_3 \beta_1 \alpha_2 \beta_3 \alpha_1, \\ \alpha_1 \alpha_2 \beta_3 &= 0, & \alpha_2 \alpha_3 \beta_1 &= 0, & \alpha_3 \alpha_1 \beta_2 &= 0, \\ \beta_1 \beta_2 \alpha_3 &= 0, & \beta_2 \beta_3 \alpha_1 &= 0, & \beta_3 \beta_1 \alpha_2 &= 0. \end{aligned}$$

Moreover, the Cartan matrix  $C_\Lambda$  of  $\Lambda$  is of the form

$$\begin{bmatrix} 4m & 4m & 4m \\ 4m & 4m & 4m \\ 4m & 4m & 4m \end{bmatrix},$$

and hence is singular.

**Example 5.12.** Let  $(Q(S, \vec{T}), f)$  be the triangulation quiver



with  $f$ -orbits  $(\alpha_1 \alpha_2 \alpha_3)$  and  $(\beta_1 \beta_3 \beta_2)$ , considered in Example 4.4. Then  $O(g)$  consists of the three  $g$ -orbits  $(\alpha_1 \beta_1)$   $(\alpha_2 \beta_2)$ ,  $(\alpha_3 \beta_3)$  of length 2. Let  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  be a weight function and  $m_1 = m_{\alpha_1}$ ,  $m_2 = m_{\alpha_2}$ ,  $m_3 = m_{\alpha_3}$ . By our assumption, we must take  $m_1 \geq 2$ ,  $m_2 \geq 2$ ,  $m_3 \geq 2$ , because  $|O(\alpha_1)| = 2$ ,  $|O(\alpha_2)| = 2$ ,  $|O(\alpha_3)| = 2$ . Let  $c_\bullet : O(g) \rightarrow K^*$  be a parameter function and  $c_1 = c_{\alpha_1}$ ,  $c_2 = c_{\alpha_2}$ ,  $c_3 = c_{\alpha_3}$ . Then the associated weighted surface algebra  $\Lambda = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet)$  is given by the above quiver and the relations

$$\begin{aligned} \alpha_1 \alpha_2 &= c_3(\beta_3 \alpha_3)^{m_3-1} \beta_3, & \alpha_2 \alpha_3 &= c_1(\beta_1 \alpha_1)^{m_1-1} \beta_1, & \alpha_3 \alpha_1 &= c_2(\beta_2 \alpha_2)^{m_2-1} \beta_2, \\ \beta_1 \beta_3 &= c_2(\alpha_2 \beta_2)^{m_2-1} \alpha_2, & \beta_3 \beta_2 &= c_1(\alpha_1 \beta_1)^{m_1-1} \alpha_1, & \beta_2 \beta_1 &= c_3(\alpha_3 \beta_3)^{m_3-1} \alpha_3, \\ \alpha_1 \alpha_2 \beta_2 &= 0, & \alpha_2 \alpha_3 \beta_3 &= 0, & \alpha_3 \alpha_1 \beta_1 &= 0, \\ \beta_1 \beta_3 \alpha_3 &= 0, & \beta_3 \beta_2 \alpha_2 &= 0, & \beta_2 \beta_1 \alpha_1 &= 0. \end{aligned}$$



Moreover, the Cartan matrix  $C_\Lambda$  of  $\Lambda$  is of the form

$$\begin{bmatrix} m_1 + m_3 & m_1 & m_3 \\ m_1 & m_1 + m_2 & m_2 \\ m_3 & m_2 & m_2 + m_3 \end{bmatrix},$$

and  $\det C_\Lambda = 4m_1m_2m_3$ . Hence  $C_\Lambda$  is non-singular.

**Example 5.13.** Let  $(Q(S, \vec{T}), f)$  be the triangulation quiver

$$\alpha \curvearrowright 1 \xrightleftharpoons[\gamma]{\beta} 2 \xrightleftharpoons[\sigma]{\delta} 3 \curvearrowright e,$$

with  $f$ -orbits  $(\alpha \beta \gamma)$  and  $(\varrho \sigma \delta)$ , considered in Example 4.5. Then  $O(g)$  consists of the  $g$ -orbits  $(\alpha)$ ,  $(\varrho)$ ,  $(\beta \delta \sigma \gamma)$ . Let  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  be a weight function and  $m_\alpha = p$ ,  $m_\varrho = q$ ,  $m_\beta = r$ . By our assumption, we have  $p \geq 3$  and  $q \geq 3$ , because  $|O(\alpha)| = 1$  and  $|O(\varrho)| = 1$ . Moreover, let  $c_\bullet : O(g) \rightarrow K^*$  be a parameter function and  $c_\alpha = a$ ,  $c_\varrho = b$ ,  $c_\beta = c$ . Then the associated weighted surface algebra  $\Lambda = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet)$  is given by the above quiver and the relations

$$\begin{aligned} \alpha\beta &= c(\beta\delta\sigma\gamma)^{r-1}\beta\delta\sigma, & \beta\gamma &= a\alpha^{p-1}, & \gamma\alpha &= c(\delta\sigma\gamma\beta)^{r-1}\delta\sigma\gamma, \\ \varrho\sigma &= c(\sigma\gamma\beta\delta)^{r-1}\sigma\gamma\beta, & \sigma\delta &= b\varrho^{q-1}, & \delta\varrho &= c(\gamma\beta\delta\sigma)^{r-1}\gamma\beta\delta, \\ \alpha\beta\delta &= 0, \quad \beta\gamma\beta = 0, & \gamma\alpha^2 &= 0, \quad \varrho\sigma\gamma = 0, & \sigma\delta\sigma &= 0, \quad \delta\varrho^2 = 0. \end{aligned}$$

Moreover, the Cartan matrix  $C_\Lambda$  of  $\Lambda$  is of the form

$$\begin{bmatrix} p+r & 2r & r \\ 2r & 4r & 2r \\ r & 2r & q+r \end{bmatrix},$$

and  $\det C_\Lambda = 4pqr$ . Hence  $C_\Lambda$  is non-singular.

The class of weighted surface algebras contains as a very special subclass the class of Jacobian algebras of surfaces with punctures. Recall that a *surface with punctures* is a pair  $(S, P)$ , where  $S$  is an orientable surface with empty boundary, and  $P$  is a finite set of points in  $S$ , called punctures. Then an ideal triangulation (briefly, triangulation) of  $(S, P)$  is any maximal collection  $T$  of pairwise compatible arcs with the ends in  $P$  whose relative interiors do not intersect each other (see [41, Section 2]), and the triple  $(S, P, T)$  is called a *triangulated surface with punctures*. Moreover, it is always assumed that a triangulated surface with punctures  $(S, P, T)$  satisfies the following conditions:

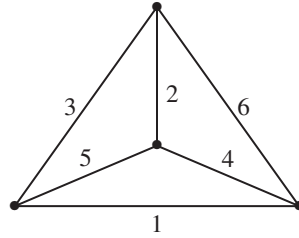
- if  $S$  is a sphere then  $|P| \geq 4$ ;
- there is no arc in  $P$  starting and ending at the same puncture;
- for each puncture  $p \in P$ , there are at least 3 arcs in  $T$  incident to  $p$ .

A triangulated surface with punctures  $(S, P, T)$  may be viewed as directed triangulated surface  $(S, \vec{T})$ , where  $\vec{T}$  is one of the two possible choices of coherent orientations of triangles in  $T$ , using the fact that  $S$  is orientable. Then the quiver  $Q(S, \vec{T})$  of  $(S, \vec{T})$  is the adjacency quiver  $Q(S, P, T)$  of  $(S, P, T)$  defined by Fomin, Shapiro and Thurston [41]. Moreover, the quiver  $Q(S, \vec{T}) = Q(S, P, T)$  has no loops nor 2-cycles  $\bullet \rightleftarrows \bullet$ . Finally, the Jacobian algebra of  $(S, P, T)$  with respects to the Labardini-Fragoso potential [56] is the surface algebra  $\Lambda(Q, f, m_\bullet, c_\bullet)$  of the directed triangulated surface  $(S, \vec{T})$  given by  $(S, P, T)$ , and the weight function  $m_\bullet$  taking only value 1 (see [57]). For an arbitrary weight function  $m_\bullet$  of  $(S, \vec{T})$  we obtain a *weighted Jacobian algebra* of  $(S, P, T)$ , as investigated by Ladkani [58, 59].

## 6. Tetrahedral algebras

In this section we present a family of algebras given by the tetrahedral triangulation of the sphere, which has exceptional properties among all weighted surface algebras considered in this paper.

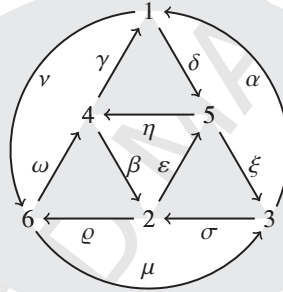
**Example 6.1.** Let  $\mathbb{S} = S^2$  be the sphere in  $\mathbb{R}^3$ . Consider the tetrahedral triangulation  $T$  of  $\mathbb{S}$



and its coherent orientation  $\vec{T}$

$$(1\ 5\ 4), (2\ 5\ 3), (2\ 6\ 4), (1\ 6\ 3).$$

Then the associated quiver  $Q(\mathbb{S}, \vec{T})$  is of the form



where the shaded subquivers denote the  $f$ -orbits.

In  $Q(\mathbb{S}, \vec{T})$  we have the four  $g$ -orbits which are, written in cycle notation,

$$(\beta\ \varepsilon\ \eta), (\varrho\ \mu\ \sigma), (\gamma\ \nu\ \omega), (\alpha\ \delta\ \xi).$$

Let  $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^* = \mathbb{N} \setminus \{0\}$  be the weight function taking the value 1 on each  $g$ -orbit. Consider a parameter function  $c_\bullet : \mathcal{O}(g) \rightarrow K^* = K \setminus \{0\}$ , and let  $c_{\mathcal{O}(\beta)} = a$ ,  $c_{\mathcal{O}(\varrho)} = b$ ,  $c_{\mathcal{O}(\gamma)} = c$  and  $c_{\mathcal{O}(\alpha)} = d$ , for elements  $a, b, c, d \in K^*$ . Then the algebra  $\Lambda(\mathbb{S}, a, b, c, d) = \Lambda(\mathbb{S}, \vec{T}, m_\bullet, c_\bullet)$  is given by the above quiver  $Q(\mathbb{S}, \vec{T})$  and the relations

$$\begin{array}{llllll} \delta\eta = c\nu\omega, & \eta\gamma = d\xi\alpha, & \gamma\delta = a\beta\varepsilon, & \delta\eta\beta = 0, & \eta\gamma\nu = 0, & \gamma\delta\xi = 0, \\ \varrho\omega = a\varepsilon\eta, & \omega\beta = b\mu\sigma, & \beta\varrho = c\gamma\nu, & \varrho\omega\gamma = 0, & \omega\beta\varepsilon = 0, & \beta\varrho\mu = 0, \\ \sigma\varepsilon = d\alpha\delta, & \varepsilon\xi = b\varrho\mu, & \xi\sigma = a\eta\beta, & \sigma\varepsilon\eta = 0, & \varepsilon\xi\alpha = 0, & \xi\sigma\varrho = 0, \\ \alpha\nu = b\sigma\varrho, & \nu\mu = d\delta\xi, & \mu\alpha = c\omega\gamma, & \alpha\nu\omega = 0, & \nu\mu\sigma = 0, & \mu\alpha\delta = 0, \end{array}$$

corresponding to the four  $f$ -orbits in  $Q(\mathbb{S}, \vec{T})$  where an orbit is given by the arrows around a shaded triangle. Moreover, a minimal set of relations defining  $\Lambda(\mathbb{S}, a, b, c, d)$  is given by the above twelve commutativity relations and the six zero relations

$$\delta\eta\beta = 0, \quad \varrho\omega\gamma = 0, \quad \sigma\varepsilon\eta = 0, \quad \beta\varrho\mu = 0, \quad \eta\gamma\nu = 0, \quad \omega\beta\varepsilon = 0,$$

so the remaining six of the above zero relations are superfluous.

We note now that the algebra  $\Lambda(\mathbb{S}, a, b, c, d)$  is isomorphic to the algebra  $\Lambda(\mathbb{S}, abcd, 1, 1, 1)$ . Indeed, there is an isomorphism of algebras  $\varphi : \Lambda(\mathbb{S}, abcd, 1, 1, 1) \rightarrow \Lambda(\mathbb{S}, a, b, c, d)$  given by

$$\begin{aligned} \varphi(\alpha) &= a\alpha, & \varphi(\mu) &= b\mu, & \varphi(\nu) &= c\nu, \\ \varphi(\delta) &= bcd\delta, & \varphi(\omega) &= bcd\omega, & \varphi(\sigma) &= bcd\sigma, \\ \varphi(\xi) &= \xi, & \varphi(\varrho) &= \varrho, & \varphi(\gamma) &= \gamma, & \varphi(\eta) &= \eta, & \varphi(\varepsilon) &= \varepsilon, & \varphi(\beta) &= \beta. \end{aligned}$$

An algebra  $\Lambda(\mathbb{S}, a, b, c, d)$ , with  $a, b, c, d \in K^*$ , is said to be a *tetrahedral algebra*. Moreover, the triangulation quiver  $Q(\mathbb{S}, \vec{T})$  of  $\Lambda(\mathbb{S}, a, b, c, d)$  is said to be the *tetrahedral triangulation quiver*.

For each  $\lambda \in K^* = K \setminus \{0\}$ , we abbreviate  $\Lambda(\mathbb{S}, \lambda) = \Lambda(\mathbb{S}, \lambda, 1, 1, 1)$ . We shall discuss now distinguished properties of the tetrahedral algebras.

Recall that the trivial extension algebra  $T(B) = B \ltimes D(B)$  of an algebra  $B$  by the injective cogenerator  $D(B) = \text{Hom}_K(B, K)$  has underlying  $K$ -vector space  $T(B) = B \oplus D(B)$ , and the multiplication in  $T(B)$  is given by

$$(b_1, f_1)(b_2, f_2) = (b_1b_2, b_1f_2 + f_1b_2)$$

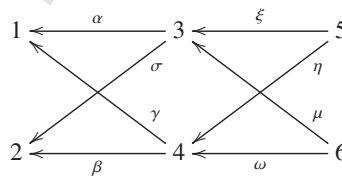
for  $b_1, b_2 \in B$  and  $f_1, f_2 \in D(B)$ . Then there is a canonical associative, non-degenerate, symmetric  $K$ -bilinear form  $(-, -) : T(B) \times T(B) \rightarrow K$  defined by

$$((b_1, f_1), (b_2, f_2)) = f_1(b_2) + f_2(b_1)$$

for  $b_1, b_2 \in B$  and  $f_1, f_2 \in D(B)$ .

A prominent role in the representation theory of tame symmetric algebras of polynomial growth is played by the trivial extensions of the tubular algebras (in the sense of Ringel [68]), whose representation theory was described by Nehring and Skowroński in [62]. Moreover, the derived equivalence classification of these algebras follows from results established in [48, 67]. We refer also to the article [65] for the invariance of the trivial extensions of tubular algebras under stable equivalences. It follows also from [71, Example 3.3] that there are exactly four families of the trivial extensions of tubular algebras of tubular type  $(2, 2, 2, 2)$ , and the tetrahedral algebras  $\Lambda(\mathbb{S}, \lambda)$  with  $\lambda \in K \setminus \{0, 1\}$  form one of these families. For the purposes of this section, we will describe now the identification of a tetrahedral algebra  $\Lambda(\mathbb{S}, \lambda)$  with the trivial extension algebra  $T(B(\lambda))$  of an algebra  $B(\lambda)$  of global dimension 2, being for  $K \setminus \{0, 1\}$  a tubular algebra of tubular type  $(2, 2, 2, 2)$ .

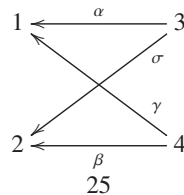
For each  $\lambda \in K^*$ , we denote by  $B(\lambda)$  the  $K$ -algebra given by the quiver



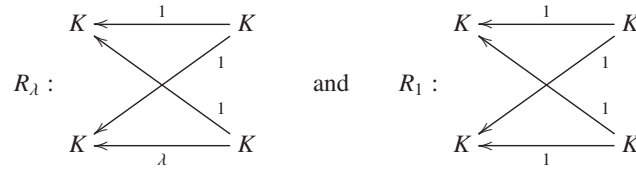
and the relations

$$\eta\gamma = \xi\alpha, \quad \xi\sigma = \lambda\eta\beta, \quad \mu\alpha = \omega\gamma, \quad \omega\beta = \mu\sigma.$$

We note that  $B(\lambda)$  is the double one-point extension algebra of the path algebra  $H = K\Delta$  of the quiver  $\Delta$



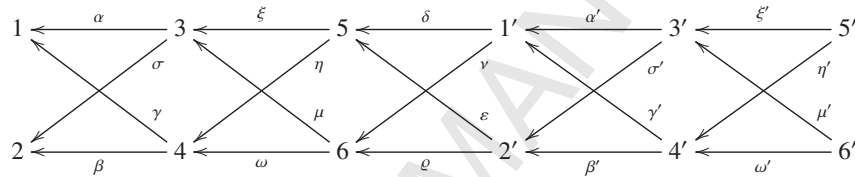
of Euclidean type  $\widetilde{\mathbb{A}}_3$  by two indecomposable modules



lying on the mouth of stable tubes of rank 1 in  $\Gamma_H$ . For  $\lambda \in K \setminus \{0, 1\}$ , the modules  $R_\lambda$  and  $R_1$  are not isomorphic, and then  $B(\lambda)$  is a tubular algebra of type  $(2, 2, 2, 2)$  in the sense of [68], and consequently it is an algebra of polynomial growth. On the other hand,  $B(1)$  is the tame minimal non-polynomial growth algebra (30) from [63]. We also mention that all algebras  $B(\lambda)$ ,  $\lambda \in K^*$ , are simply connected and of global dimension 2.

**Lemma 6.2.** *For any  $\lambda \in K^*$ , the algebras  $\Lambda(\mathbb{S}, \lambda)$  and  $T(B(\lambda))$  are isomorphic.*

*Proof.* By general theory (see [72]), the trivial extension algebra  $T(B(\lambda))$  is isomorphic to the orbit algebra  $\widehat{B(\lambda)}/(\nu_{\widehat{B(\lambda)}})$  of the repetitive category  $\widehat{B(\lambda)}$  of  $B(\lambda)$  with respect to the infinite cyclic group  $(\nu_{\widehat{B(\lambda)}})$  generated by the Nakayama automorphism  $\nu_{\widehat{B(\lambda)}}$  of  $\widehat{B(\lambda)}$ . One checks directly that  $\widehat{B(\lambda)}$  contains the full convex subcategory  $B(\lambda)^{(2)}$  given by the quiver



and the relations

$$\begin{aligned} \eta\gamma &= \xi\alpha, & \xi\sigma &= \lambda\eta\beta, & \mu\alpha &= \omega\gamma, & \omega\beta &= \mu\sigma, \\ \nu\mu &= \delta\xi, & \delta\eta &= \nu\omega, & \varepsilon\xi &= \varrho\mu, & \varrho\omega &= \lambda\varepsilon\eta, \\ \sigma'\varepsilon &= \alpha'\delta, & \alpha'\nu &= \sigma'\varrho, & \gamma'\delta &= \lambda\beta'\varepsilon, & \beta'\varrho &= \gamma'\nu, \\ \eta'\gamma' &= \xi'\alpha', & \xi'\sigma' &= \lambda\eta'\beta', & \mu'\alpha' &= \omega'\gamma', & \omega'\beta' &= \mu'\sigma', \\ \delta\eta\beta &= 0, & \varrho\omega\gamma &= 0, & \sigma'\varepsilon\eta &= 0, & \beta'\varrho\mu &= 0, & \eta'\gamma'\nu &= 0, & \omega'\beta'\varepsilon &= 0, \end{aligned}$$

where  $\nu_{\widehat{B(\lambda)}}(i) = i'$  for any vertex  $i \in \{1, 2, 3, 4, 5, 6\}$  and  $\nu_{\widehat{B(\lambda)}}(\theta) = \theta'$  for any arrow  $\theta \in \{\alpha, \beta, \gamma, \sigma, \xi, \omega, \eta, \mu\}$ .

We conclude that  $T(B(\lambda))$  is isomorphic to the algebra  $\Lambda(\mathbb{S}, \lambda) = \Lambda(\mathbb{S}, \lambda, 1, 1, 1)$ .  $\square$

We note that the algebra (category)  $B(\lambda)^{(2)}$  is isomorphic to the duplicated algebra

$$\begin{bmatrix} B(\lambda) & 0 \\ D(B(\lambda)) & B(\lambda) \end{bmatrix} = \left\{ \begin{bmatrix} b_1 & 0 \\ f & b_2 \end{bmatrix} \mid b_1, b_2 \in B(\lambda), f \in D(B(\lambda)) \right\}$$

of  $B(\lambda)$ .

The next two propositions describe some distinguished properties of the tetrahedral algebras  $\Lambda(\mathbb{S}, \lambda)$ ,  $\lambda \in K^*$ .

**Proposition 6.3.** *For any  $\lambda \in K \setminus \{0, 1\}$  the following statements hold:*

- (i)  $\Lambda(\mathbb{S}, \lambda)$  is an algebra of polynomial growth.
- (ii)  $\Lambda(\mathbb{S}, \lambda)$  is a periodic algebra of period 4.
- (iii) The simple modules in  $\text{mod } \Lambda(\mathbb{S}, \lambda)$  are periodic of period 4.
- (iv) The simple modules in  $\text{mod } \Lambda(\mathbb{S}, \lambda)$  lie in six pairwise different stable tubes of rank 2 of  $\Gamma_{\Lambda(\mathbb{S}(\lambda))}$ .

*Proof.* It follows from Lemma 6.2 that  $\Lambda(\mathbb{S}, \lambda)$  is isomorphic to the trivial extension algebra  $T(B(\lambda))$ . We identify  $\Lambda(\mathbb{S}, \lambda)$  and  $T(B(\lambda))$ . Since  $\lambda \in K \setminus \{0, 1\}$ , the algebra  $B(\lambda)$  is a tubular algebra of tubular type  $(2, 2, 2, 2)$ , and  $T(B(\lambda))$  is the orbit algebra  $\widehat{B(\lambda)}/(\nu_{\widehat{B(\lambda)}})$ . Then, applying the results of [62, Section 3], we conclude that  $T(B(\lambda))$  is an algebra of polynomial growth and the six pairwise nonisomorphic indecomposable projective-injective  $T(B(\lambda))$ -modules  $P_1, P_2, P_3, P_4, P_5, P_6$ , at the vertices  $1, 2, 3, 4, 5, 6$ , lie in six pairwise different components  $C_1, C_2, C_3, C_4, C_5, C_6$  of  $\Gamma_{T(B(\lambda))}$  such that their stable parts  $C_1^s, C_2^s, C_3^s, C_4^s, C_5^s, C_6^s$  are stable tubes of rank 2 and do not contain simple modules. Further, since  $T(B(\lambda))$  is a symmetric algebra, the six pairwise nonisomorphic simple  $T(B(\lambda))$ -modules  $S_1, S_2, S_3, S_4, S_5, S_6$ , at the vertices  $1, 2, 3, 4, 5, 6$ , are the socles of the modules  $P_1, P_2, P_3, P_4, P_5, P_6$ , respectively. Observe also that  $P_i/S_i$  belongs to  $C_i$ , for any  $i \in \{1, 2, 3, 4, 5, 6\}$ . Then  $S_i = \Omega_{T(B(\lambda))}(P_i/S_i)$  belongs to a component  $\mathcal{T}_i$  such that  $\mathcal{T}_i^s = \Omega_{T(B(\lambda))}(C_i^s)$ , for any  $i \in \{1, 2, 3, 4, 5, 6\}$ . Hence, we obtain that  $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6$  are pairwise different stable tubes of rank 2 containing the simple modules  $S_1, S_2, S_3, S_4, S_5, S_6$ , respectively. We also note that  $\tau_{T(B(\lambda))} = \Omega_{T(B(\lambda))}^2$ , because  $T(B(\lambda))$  is a symmetric algebra. Therefore, the simple modules  $S_1, S_2, S_3, S_4, S_5, S_6$  are periodic modules of period 4.

We will prove now that  $T(B(\lambda))$  is periodic as an algebra, of period 4. Consider the cyclic group  $H$  of automorphisms of the algebra  $T(B(\lambda)) = \Lambda(\mathbb{S}, \lambda)$  generated by the automorphism  $h$  given by the following cyclic rotations of the vertices and arrows of the quiver  $Q(\mathbb{S}, \vec{T})$  from Example 6.1

$$(1 \ 6 \ 3), \quad (4 \ 2 \ 5), \quad (\nu \ \mu \ \alpha), \quad (\beta \ \varepsilon \ \eta), \quad (\gamma \ \varrho \ \xi), \quad (\delta \ \omega \ \sigma).$$

Then  $H$  is of order 3 and acts freely on the set of primitive idempotents of  $\Lambda(\mathbb{S}, \lambda)$  corresponding to the vertices of  $Q(\mathbb{S}, \vec{T})$ . Further, the orbit algebra  $\Lambda(\mathbb{S}, \lambda)/H = T(B(\lambda))/H$  is isomorphic to the algebra  $\Lambda'_3(\lambda)$  from [8, Section 6], given by the quiver

$$\alpha \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 1 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\gamma} \end{array} 2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \beta$$

and the relations

$$\alpha^2 = \sigma\gamma, \quad \gamma\sigma = \lambda\beta^2, \quad \gamma\alpha = \beta\gamma, \quad \alpha\sigma = \sigma\beta.$$

We note that the above relations imply the zero relations

$$\gamma\alpha^2 = 0, \quad \alpha^2\sigma = 0, \quad \sigma\beta^2 = 0, \quad \beta^2\gamma = 0, \quad \alpha\sigma\beta = 0, \quad \beta\gamma\alpha = 0, \quad \sigma\gamma\sigma = 0, \quad \gamma\sigma\gamma = 0,$$

because  $\lambda \in K \setminus \{0, 1\}$ . It has been proved in [8, Proposition 7.1] that  $\Lambda'_3(\lambda)$  is a periodic algebra of period 4. Since the order of  $H$  is coprime to 4, it follows from [24, Theorem 3.7] that  $\Lambda(\mathbb{S}, \lambda) = T(B(\lambda))$  is also a periodic algebra of period 4.  $\square$

**Proposition 6.4.** *The algebra  $\Lambda(\mathbb{S}, 1)$  is a tame algebra of non-polynomial growth and there exist three pairwise different components  $C_1, C_3, C_5$  in  $\Gamma_{\Lambda(\mathbb{S}, 1)}$  having the following properties:*

- (i) *For each  $r \in \{1, 3, 5\}$ ,  $C_r$  is isomorphic to the stable translation quiver  $\mathbb{Z}\mathbb{D}_\infty$ .*
- (ii) *For each  $r \in \{1, 3, 5\}$ , the component  $C_r$  contains a full translation subquiver of the form*

$$\begin{array}{ccccc} \tau_{\Lambda(\mathbb{S}, 1)} S_r & & & & S_r \\ & \searrow & & \nearrow & \\ & M_r & & & \\ & \nearrow & & \searrow & \\ \tau_{\Lambda(\mathbb{S}, 1)} S_{r+1} & & & & S_{r+1} \end{array}$$

where  $S_r$  and  $S_{r+1}$  are the simple  $\Lambda(\mathbb{S}, 1)$ -modules at the vertices  $r$  and  $r + 1$ .

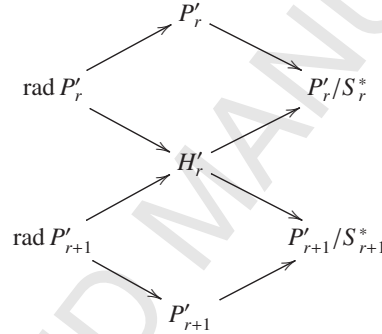
In particular,  $\Lambda(\mathbb{S}, 1)$  does not have a simple periodic module, and hence  $\Lambda(\mathbb{S}, 1)$  is not a periodic algebra.



*Proof.* We identify  $\Lambda(\mathbb{S}, 1) = T(B(1)) = \widehat{B(1)}/(\nu_{\widehat{B(1)}})$  using Lemma 6.2. Consider the Galois covering  $F : \widehat{B(1)} \rightarrow \widehat{B(1)}/(\nu_{\widehat{B(1)}}) = T(B(1))$  and the push-down functor  $F_\lambda : \text{mod } \widehat{B(1)} \rightarrow \text{mod } T(B(1))$  induced by  $F$ . It follows from [43, Theorem 3.6] that  $F_\lambda$  preserves the projective modules and almost split sequences. Recall that  $B(1)$  is the pg-critical algebra (30) from [63, Theorem 3.2], and hence is a tame algebra of non-polynomial growth, by [63, Proposition 3.1]. Then the trivial extension algebra  $T(B(1))$  is of non-polynomial growth, because  $B(1)$  is a quotient algebra of  $T(B(1))$ . Then, applying Proposition 5.8, we conclude that  $\Lambda(\mathbb{S}, 1) = T(B(1))$  is a tame algebra of non-polynomial growth.

Consider now the full convex subcategory  $\widehat{B(1)}^{(2)}$  of  $\widehat{B(1)}$  presented in the proof of Lemma 6.2. For each  $i \in \{1, 2, 3, 4, 5, 6\}$ , we denote by  $S_i^*$  the simple  $\widehat{B(1)}$ -module at the vertex  $i$ , and by  $P_i'$  the indecomposable projective  $\widehat{B(1)}$ -module at the vertex  $i'$ . Then, for each  $i \in \{1, 2, 3, 4, 5, 6\}$ ,  $S_i = F_\lambda(S_i^*)$  is the simple  $T(B(1))$ -module and  $P_i = F_\lambda(P_i')$  the indecomposable projective  $T(B(1))$ -module the vertex  $i$ . Applying [63, Theorem 6.1], we conclude that the Auslander-Reiten quiver  $\Gamma_{\widehat{B(1)}}$  of  $\widehat{B(1)}$  admits three pairwise different components  $\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6$  having the following properties:

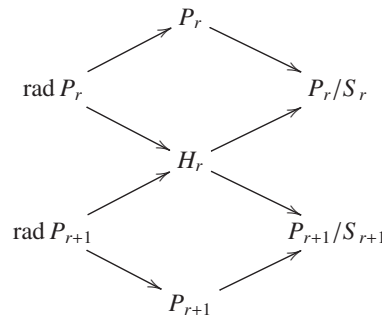
- For each  $r \in \{1, 3, 5\}$ , the stable part  $\mathcal{D}_{r+1}^s$  of  $\mathcal{D}_{r+1}$  is isomorphic to the translation quiver  $\mathbb{Z}\mathbb{D}_\infty$ .
- For each  $r \in \{1, 3, 5\}$ , the component  $\mathcal{D}_{r+1}$  does not contain a simple module.
- For each  $r \in \{1, 3, 5\}$ , the component  $\mathcal{D}_{r+1}$  contains a full translation subquiver of the form



where  $\text{rad } P'_r/S_r^* = H_r = \text{rad } P'_{r+1}/S_{r+1}^*$ .

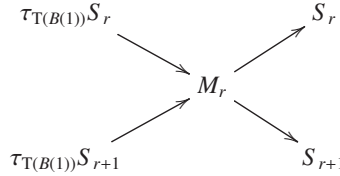
Then, applying the push-down functor  $F_\lambda : \text{mod } \widehat{B(1)} \rightarrow \text{mod } T(B(1))$ , we conclude that the Auslander-Reiten quiver  $\Gamma_{T(B(1))}$  of  $T(B(1))$  admits three pairwise different components  $\mathcal{C}_2 = F_\lambda(\mathcal{D}_2)$ ,  $\mathcal{C}_4 = F_\lambda(\mathcal{D}_4)$ ,  $\mathcal{C}_6 = F_\lambda(\mathcal{D}_6)$ , having the following properties:

- For each  $r \in \{1, 3, 5\}$ , the stable part  $\mathcal{C}_{r+1}^s$  of  $\mathcal{C}_{r+1}$  is isomorphic to the translation quiver  $\mathbb{Z}\mathbb{D}_\infty$ .
- For each  $r \in \{1, 3, 5\}$ , the component  $\mathcal{C}_{r+1}$  does not contain a simple module.
- For each  $r \in \{1, 3, 5\}$ , the component  $\mathcal{C}_{r+1}$  contains a full translation subquiver of the form



where  $\text{rad } P_r/S_r = H_r = \text{rad } P_{r+1}/S_{r+1}$ .

Observe that  $C_2, C_4, C_6$  are all components of  $\Gamma_{T(B(1))}$  containing projective modules, and do not contain simple modules. For each  $r \in \{1, 3, 5\}$ , let  $C_r$  be the component of  $\Gamma_{T(B(1))}$  such that  $C_r^s = \Omega_{T(B(1))}(C_{r+1}^s)$ . Then, for each  $r \in \{1, 3, 5\}$ ,  $C_r^s$  is isomorphic to the translation quiver  $\mathbb{Z}\mathbb{D}_\infty$ , and  $C_r$  contains a full translation subquiver of the form



where  $M_r = \Omega_{T(B(1))}(H_r)$ . Clearly,  $C_1, C_3, C_5$  are pairwise different components of  $\Gamma_{T(B(1))}$ , and different from the components  $C_2, C_4, C_6$ . In particular, we conclude that  $C_1 = C_1^s, C_3 = C_3^s, C_5 = C_5^s$ .  $\square$

We note also the following common property of all algebras  $\Lambda(\mathbb{S}, \lambda)$ ,  $\lambda \in K^*$ .

**Proposition 6.5.** *Let  $\lambda \in K^*$ . Then all uniserial modules of length 2 in  $\text{mod } \Lambda(\mathbb{S}, \lambda)$  are periodic of period 4 and form the mouth of six pairwise different stable tubes of rank 2 in  $\Gamma_{\Lambda(\mathbb{S}, \lambda)}$ .*

*Proof.* For each arrow  $\theta$  in the quiver  $Q(\mathbb{S}, \vec{T})$  of  $\Lambda(\mathbb{S}, \lambda)$  we denote by  $U_\theta$  the uniserial module of length 2 in  $\text{mod } \Lambda(\mathbb{S}, \lambda)$  whose top is the simple module  $S_{s(\theta)}$  at  $s(\theta)$  and the socle is the simple module  $S_{t(\theta)}$  at  $t(\theta)$ . One checks directly that  $\Omega_{\Lambda(\mathbb{S}, \lambda)}^2(U_\theta) = U_{fgf\theta}$ . Moreover, we have  $fgf\theta \neq \theta$ , and  $fgf^2gf\theta = \theta$ . Hence, the uniserial modules  $U_\theta$  and  $U_{fgf\theta}$  are periodic of period 4 and form the mouth of a stable tube of rank 2 in  $\Gamma_{\Lambda(\mathbb{S}, \lambda)}$  (recall that  $\tau_{\Lambda(\mathbb{S}, \lambda)} = \Omega_{\Lambda(\mathbb{S}, \lambda)}^2$ ). Observe also that every uniserial module of length 2 in  $\text{mod } \Lambda(\mathbb{S}, \lambda)$  is of the form  $U_\theta$  for some arrow  $\theta$  in  $Q(\mathbb{S}, \vec{T})$ . In fact, for each arrows  $\theta$  in  $Q(\mathbb{S}, \vec{T})$  the uniserial module  $U_\theta$  is isomorphic to the module  $\pi g(\pi)\Lambda(\mathbb{S}, \lambda)$  with  $\pi = g^{-2}(\theta)$ , described in Lemma 5.5.  $\square$

The Gabriel quiver of a tetrahedral algebra has the following characterization:

**Lemma 6.6.** *Let  $(Q, f)$  be a triangulation quiver with at least three vertices. The following statements are equivalent:*

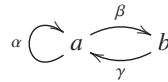
- (i)  $(Q, f)$  is the tetrahedral triangulation quiver.
- (ii) For any arrow  $\alpha$  in  $Q_1$ , we have  $n_\alpha = 3$ .
- (iii)  $g^3$  is the identity on  $Q_1$ .
- (iv) There is an arrow  $\beta$  in  $Q_1$  such that  $n_\beta = 3, n_{f\beta} = 3, n_{f^2\beta} = 3, n_{f^3\beta} = 3$ .

*Proof.* The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) and (ii)  $\Rightarrow$  (iv) are obvious. We will prove first that (iii) implies (ii).

Assume that  $g^3$  is the identity on  $Q_1$ . Suppose that  $Q_1$  contains a loop



Since  $Q$  is a 2-regular connected quiver with at least three vertices, and since  $\alpha$  belongs to a 3-cycle of either  $f$  or  $g$ , it contains a subquiver



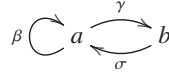
and one of the two cases hold:

- (1)  $f(\alpha) = \alpha, g(\alpha) = \beta, g(\beta) = \gamma, g(\gamma) = \alpha;$
- (2)  $f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha, g(\alpha) = \alpha.$

In case (1), we obtain  $f(\gamma) = \beta$ , and hence  $f(\beta) = f^2(\gamma)$ , so this is a loop at  $b$  since  $f^3(\gamma) = \gamma$ . In case (2), we obtain  $g(\gamma) = \beta$ , and hence  $g(\beta) = g^2(\gamma)$  which is again a loop at  $b$  since  $g^3(\gamma) = \gamma$ . Thus, in the both cases,  $Q$  is a quiver with two vertices, a contradiction. Hence,  $Q$  has no loops, and the statement (ii) holds.

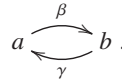
It remains to show that (iv) implies (i). Assume that  $\beta$  is an arrow in  $Q_1$  such that  $n_\beta = 3$ ,  $n_{\bar{\beta}} = 3$ ,  $n_{f(\beta)} = 3$ ,  $n_{f(\bar{\beta})} = 3$ . We prove statement (i) in several steps.

We first claim that  $\beta, \bar{\beta}, f(\beta), f(\bar{\beta})$  are not loops. Suppose that  $\beta$  is a loop. Then  $Q$  contains a subquiver of the form



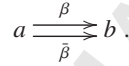
with  $a \neq b$ ,  $g(\beta) = \gamma$ ,  $g(\gamma) = \sigma$ ,  $g(\sigma) = \beta$ . Then  $f(\beta) = \beta$  and  $f(\sigma) = \gamma$ . Since  $f^3(\sigma) = \sigma$  we have  $f(\gamma) = f^2(\sigma)$  and this is a loop at  $b$ , and consequently  $Q$  has only two vertices, a contradiction. Similarly, we conclude that  $\bar{\beta}, f(\beta), f(\bar{\beta})$  are not loops.

We claim now that  $\beta, \bar{\beta}, f(\beta), f(\bar{\beta})$  do not belong to 2-cycles. Suppose that  $\beta$  belongs to a 2-cycle



Then  $\gamma = f(\beta)$  or  $\gamma = g(\beta)$ . Since  $f^3(\beta) = \beta$  and  $g^3(\beta) = \beta$  we infer that  $f(\gamma) = f^2(\beta)$  or  $g(\gamma) = g^2(\beta)$  and hence is a loop. This is a contradiction because such a loop is equal to  $\bar{\beta}$ . We note also that  $\bar{f}(\bar{\beta}) = g(\beta)$  and  $\bar{f}(\bar{\beta}) = g(\bar{\beta})$ , and hence  $n_{\bar{f}(\bar{\beta})} = n_\beta = 3$ ,  $n_{\bar{f}(\bar{\beta})} = n_{\bar{\beta}} = 3$ . Then we conclude that  $\bar{\beta}, f(\beta), f(\bar{\beta})$  do not belong to 2-cycles.

In the next step, we prove that  $\beta, \bar{\beta}, f(\beta), f(\bar{\beta})$  are not part of double arrows. Suppose that  $Q$  has double arrow



Note that  $f(\beta) \neq f(\bar{\beta})$  and therefore they are the arrows starting at  $b$ , and similarly  $f^2(\beta)$  and  $f^2(\bar{\beta})$  are the arrows ending at  $a$ .

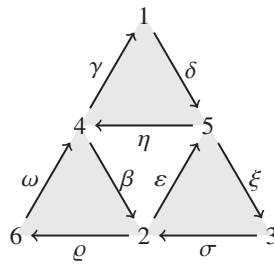
Now  $g(\beta) \neq g(\bar{\beta})$ , and these also start at  $b$ . Since  $g(\beta) \neq f(\beta)$ , we must have  $f(\beta) = g(\bar{\beta})$  and  $f(\bar{\beta}) = g(\beta)$ .

Since  $n_\beta = 3$ , the arrow  $g^2(\beta)$  ends at  $a$  and therefore it must be one of  $f^2(\beta)$  or  $f^2(\bar{\beta})$ . Similarly  $g^2(\bar{\beta})$  ends at  $a$ . Now  $g^2(\beta) \neq g^2(\bar{\beta})$  and therefore  $\{f^2(\beta), f^2(\bar{\beta})\} = \{g^2(\beta), g^2(\bar{\beta})\}$ . If  $g^2(\beta) = f^2(\beta)$  then  $f(g^2(\beta)) = f^3(\beta) = \beta$ . But then

$$g(g^2(\beta)) = \overline{f(g^2(\beta))} = \bar{\beta}$$

and  $n_\beta > 3$ , a contradiction. So we can only have  $g^2(\beta) = f^2(\bar{\beta})$ . This means that if  $\gamma := f(\bar{\beta}) = g(\beta)$ , then we have  $g(\gamma) = f(\gamma)$  which also is a contradiction. Similarly one shows that  $f(\beta)$  and  $f(\bar{\beta})$  are not double arrows.

Summing up, we conclude that  $Q$  contains a subquiver of the form

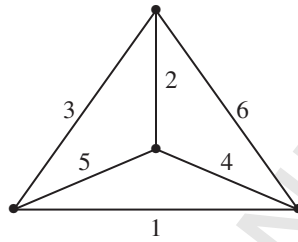


where  $\varepsilon = g(\beta)$ ,  $\eta = g(\varepsilon)$ ,  $\beta = g(\eta)$ , and the shaded triangles denote the  $f$ -orbits of the arrows  $\beta, \varepsilon, \eta$ . Observe that  $\xi = g(\delta)$ ,  $\gamma = g(\omega)$ ,  $\varrho = g(\sigma)$ . Moreover, we have  $\gamma = \bar{\beta}$ ,  $\varrho = f(\beta)$ ,  $\delta = f(\bar{\beta})$ . Hence, by the imposed assumption, there exist arrows  $\alpha, \nu, \mu$  in  $Q_1$  with  $t(\alpha) = 1 = s(\nu)$ ,  $t(\nu) = 6 = s(\mu)$ ,  $t(\mu) = 3 = s(\alpha)$  such that  $g(\alpha) = \delta$ ,  $g(\nu) = \omega$ ,  $g(\mu) = \sigma$ . Obviously, then  $f(\alpha) = \nu$ ,  $f(\nu) = \mu$ ,  $f(\mu) = \alpha$ . Therefore,  $(Q, f)$  is the required tetrahedral triangulation quiver.  $\square$

An algebra  $\Lambda(\mathbb{S}, a, b, c, d)$  for  $a, b, c, d \in K^*$  with  $abcd = 1$  is said to be a *singular tetrahedral algebra*. It follows from Lemma 6.2 and Proposition 6.4 that the singular tetrahedral algebras do not have periodic simple modules, and hence are not periodic algebras. We will prove in the next section that all other weighted surface algebras are periodic algebras. We also mention that the tetrahedral algebras  $\Lambda(\mathbb{S}, a, b, c, d)$  with  $abcd \neq 1$  are all weighted surface algebras of polynomial growth.

We would like to stress that, starting from the triangulation quiver  $Q(\mathbb{S}, \vec{T})$  defined in Example 6.1 and taking weight functions with value different from 1 on some  $g$ -orbits, we may create infinitely many weighted surface algebras which are not isomorphic to the tetrahedral algebras, discussed above. Similarly, we may create infinitely many new weighted surface algebras by changing the orientation of triangles in the tetrahedral triangulation of the sphere. The following example shows that we obtain new algebras even if the weight function takes value 1 on all  $g$ -orbits.

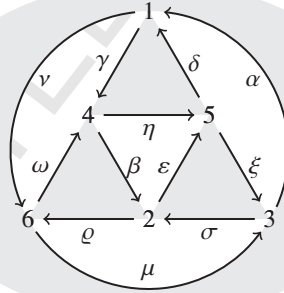
**Example 6.7.** Let  $T$  be the tetrahedral triangulation of the sphere  $\mathbb{S}$



and  $\vec{T}$  the orientation

$$(1\ 4\ 5), (2\ 5\ 3), (2\ 6\ 4), (1\ 6\ 3)$$

of triangles in  $T$ , obtained from the coherent orientation of triangles in  $T$  considered in Example 6.1 by changing the orientation of one triangle on the opposite orientation, and keeping the orientations of all other triangles unchanged. Then the associated triangulation quiver  $Q(\mathbb{S}, \vec{T})$  is of the form



Then we have only two  $g$ -orbits of arrows in  $Q(\mathbb{S}, \vec{T})$

$$O(\beta) = \{\beta, \varepsilon = g\beta, \delta = g^2\beta, \nu = g^3\beta, \omega = g^4\beta, \eta = g^5\beta, \xi = g^6\beta, \alpha = g^7\beta, \gamma = g^8\beta\},$$

$$O(\varrho) = \{\varrho, \mu = g\varrho, \sigma = g^2\varrho\}.$$

Moreover, let  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  be the weight function taking the value 1 on each  $g$ -orbit in  $O(g)$ ,  $c_\bullet : O(g) \rightarrow K^*$  a parameter function, and  $a = c_{O(\beta)}$ ,  $b = c_{O(\varrho)}$ . Then the associated algebra  $\Lambda(\mathbb{S}, \vec{T}, m_\bullet, c_\bullet)$  is given by the above quiver

$Q(\mathbb{S}, \vec{T})$  and the relations

$$\begin{array}{lll} \eta\delta = a\beta\epsilon\delta\nu\omega\xi\alpha, & \delta\gamma = a\xi\alpha\gamma\beta\epsilon\delta\nu\omega, & \gamma\eta = a\nu\omega\eta\xi\alpha\gamma\beta\epsilon, \\ \varrho\omega = a\epsilon\delta\nu\omega\xi\alpha\gamma, & \omega\beta = b\mu\sigma, & \beta\varrho = a\eta\xi\alpha\gamma\beta\epsilon\delta\nu, \\ \sigma\epsilon = a\alpha\gamma\beta\epsilon\delta\nu\omega\eta, & \epsilon\xi = b\varrho\mu, & \xi\sigma = a\delta\nu\omega\eta\xi\alpha\gamma\beta, \\ \alpha\nu = b\sigma\varrho, & \nu\mu = a\gamma\beta\epsilon\delta\nu\omega\xi, & \mu\alpha = a\omega\eta\xi\alpha\gamma\beta\epsilon\delta, \\ \eta\delta\nu = 0, & \delta\gamma\beta = 0, & \gamma\eta\xi = 0, \\ \varrho\omega\eta = 0, & \omega\beta\epsilon = 0, & \beta\varrho\mu = 0, \\ \sigma\epsilon\delta = 0, & \epsilon\xi\alpha = 0, & \xi\sigma\varrho = 0, \\ \alpha\nu\omega = 0, & \nu\mu\sigma = 0, & \mu\alpha\gamma = 0. \end{array}$$

Observe that this algebra  $\Lambda(\mathbb{S}, \vec{T}, m_\bullet, c_\bullet)$  is not isomorphic to a tetrahedral algebra. We will prove in Section 10 that  $\Lambda(\mathbb{S}, \vec{T}, m_\bullet, c_\bullet)$  is a tame algebra of non-polynomial growth. It is also known that derived equivalence of self-injective algebras preserves the representation type (see [54, 55, 66]). Hence it follows from Proposition 6.3 that  $\Lambda(\mathbb{S}, \vec{T}, m_\bullet, c_\bullet)$  is not derived equivalent to a non-singular tetrahedral algebra. We will show in Section 7 that  $\Lambda(\mathbb{S}, \vec{T}, m_\bullet, c_\bullet)$  is a periodic algebra. Then, applying Proposition 6.4, we conclude that  $\Lambda(\mathbb{S}, \vec{T}, m_\bullet, c_\bullet)$  is not derived equivalent to a singular tetrahedral algebra, because periodicity of algebras is invariant under derived equivalence (see [34, 67]). This shows that changing orientation of one triangle in a directed triangulated surface may lead to a non-derived equivalent weighted surface algebra.

## 7. Periodicity of weighted surface algebras

In this section we will prove that every weighted surface algebra with at least three simple modules, not isomorphic to a tetrahedral algebra, is a periodic algebra of period 4. We note that, by Propositions 6.3 and 6.4, a tetrahedral algebra  $\Lambda(\mathbb{S}, \lambda)$ ,  $\lambda \in K^*$ , is a periodic algebra if and only if  $\Lambda(\mathbb{S}, \lambda)$  is nonsingular ( $\lambda \neq 1$ ). Moreover, for  $\lambda \in K \setminus \{0, 1\}$  the algebra has period 4.

Throughout this section, we fix  $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$  for a triangulation quiver  $(Q, f)$  with at least three vertices, a weight function  $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$  and a parameter function  $c_\bullet : \mathcal{O}(g) \rightarrow K^*$ . Moreover, we assume that  $\Lambda$  is not a tetrahedral algebra.

We start by describing minimal projective resolutions of simple modules in  $\text{mod } \Lambda$ .

**Proposition 7.1.** *Let  $i$  be a vertex of  $Q$  and  $\alpha, \bar{\alpha}$  the arrows of  $Q$  starting at  $i$ . Then there is an exact sequence in  $\text{mod } \Lambda$*

$$0 \rightarrow S_i \rightarrow P_i \xrightarrow{\pi_3} P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))} \xrightarrow{\pi_2} P_{t(\alpha)} \oplus P_{t(\bar{\alpha})} \xrightarrow{\pi_1} P_i \rightarrow S_i \rightarrow 0,$$

which give rise to a minimal projective resolution of  $S_i$  in  $\text{mod } \Lambda$ . In particular,  $S_i$  is a periodic module of period 4.

*Proof.* We take for  $S_i$  the simple quotient of  $P_i = e_i\Lambda$ , and then  $\Omega_\Lambda(S_i)$  can be identified with  $\text{rad } P_i = \alpha\Lambda + \bar{\alpha}\Lambda$ . We define the homomorphism of right  $\Lambda$ -modules

$$\pi_1 : P_{t(\alpha)} \oplus P_{t(\bar{\alpha})} \rightarrow P_i$$

by  $\pi_1(x, y) = \alpha x + \bar{\alpha}y$  for  $x \in P_{t(\alpha)}$  and  $y \in P_{t(\bar{\alpha})}$ . Clearly,  $\pi_1$  induces a projective cover of  $\text{rad } P_i = \Omega_\Lambda(S_i)$  and its kernel is isomorphic to  $\Omega_\Lambda^2(S_i)$ . We know the dimension of  $\Omega_\Lambda^2(S_i)$ . Namely, using the projective cover  $\pi_1$  and Corollary 5.6, we obtain the equalities

$$\begin{aligned} \dim_K \Omega_\Lambda^2(S_i) &= \dim_K P_{t(\alpha)} + \dim_K P_{t(\bar{\alpha})} - (\dim_K P_i - 1) \\ &= m_{f(\alpha)}n_{f(\alpha)} + m_{g(\alpha)}n_{g(\alpha)} + m_{f(\bar{\alpha})}n_{f(\bar{\alpha})} + m_{g(\bar{\alpha})}n_{g(\bar{\alpha})} - m_\alpha n_\alpha - m_{\bar{\alpha}} n_{\bar{\alpha}} + 1 \\ &= m_{f(\alpha)}n_{f(\alpha)} + m_{f(\bar{\alpha})}n_{f(\bar{\alpha})} + 1, \end{aligned}$$

because  $m_{g(\alpha)} = m_\alpha$ ,  $n_{g(\alpha)} = n_\alpha$ ,  $m_{g(\bar{\alpha})} = m_{\bar{\alpha}}$ ,  $n_{g(\bar{\alpha})} = n_{\bar{\alpha}}$ .

Consider the elements in  $P_{t(\alpha)} \oplus P_{t(\bar{\alpha})}$

$$\varphi = (f(\alpha), -c_{\bar{\alpha}}A'_{\bar{\alpha}}) \quad \text{and} \quad \psi = (-c_{\alpha}A'_{\alpha}, f(\bar{\alpha})).$$

Observe that

$$\begin{aligned} \pi_1(\varphi) &= \alpha f(\alpha) - c_{\bar{\alpha}}\bar{\alpha}A'_{\bar{\alpha}} = \alpha f(\alpha) - c_{\bar{\alpha}}A_{\bar{\alpha}} = 0, \\ \pi_1(\psi) &= -c_{\alpha}\alpha A'_{\alpha} + \bar{\alpha}f(\bar{\alpha}) = -c_{\alpha}A_{\alpha} + \bar{\alpha}f(\bar{\alpha}) = 0, \end{aligned}$$

and hence  $\varphi, \psi$  belong to  $\text{Ker } \pi_1 = \Omega_{\Lambda}^2(S_i)$ . We note that  $\varphi$  and  $\psi$  are independent modulo the radical, even in the case when  $A'_{\bar{\alpha}}$  or  $A'_{\alpha}$  is an arrow. Indeed, if  $A'_{\bar{\alpha}}$  (respectively,  $A'_{\alpha}$ ) is an arrow then  $A'_{\bar{\alpha}} = g(\bar{\alpha})$  (respectively,  $A'_{\alpha} = g(\alpha)$ ), and is linearly independent from  $f(\bar{\alpha})$  (respectively,  $f(\alpha)$ ). We find the intersection of  $\varphi\Lambda$  and  $\psi\Lambda$ . Note that

$$\begin{aligned} \varphi f^2(\alpha) &= (f(\alpha)f^2(\alpha), -c_{\bar{\alpha}}A'_{\bar{\alpha}}f^2(\alpha)) = (f(\alpha)f^2(\alpha), -c_{\bar{\alpha}}A_{g(\bar{\alpha})}), \\ \psi f^2(\bar{\alpha}) &= (-c_{\alpha}A'_{\alpha}f^2(\bar{\alpha}), f(\bar{\alpha})f^2(\bar{\alpha})) = (-c_{\alpha}A_{g(\alpha)}, f(\bar{\alpha})f^2(\bar{\alpha})), \end{aligned}$$

by Lemma 5.3 (v). Moreover, we have  $g(\alpha) = \overline{f(\bar{\alpha})}$ ,  $g(\bar{\alpha}) = \overline{f(\alpha)}$ ,  $c_{\alpha} = c_{g(\alpha)}$ ,  $c_{\bar{\alpha}} = c_{g(\bar{\alpha})}$ . Hence we conclude that  $\varphi f^2(\alpha) = -\psi f^2(\bar{\alpha})$ . It follows from Lemmas 5.3 and 5.4 that  $f(\alpha)f^2(\alpha)f^3(\alpha) = c_{f(\alpha)}B_{f(\alpha)}$  is a non-zero element of the socle of  $P_{t(\alpha)} = P_{s(f(\alpha))}$  and  $f(\bar{\alpha})f^2(\bar{\alpha})f^3(\bar{\alpha}) = c_{f(\bar{\alpha})}B_{f(\bar{\alpha})}$  is a non-zero element of the socle of  $P_{t(\bar{\alpha})} = P_{s(f(\bar{\alpha}))}$ . On the other hand, we have  $-c_{\alpha}A_{g(\alpha)}g(f^2(\alpha)) = f(\alpha)f^2(\alpha)g(f^2(\alpha)) = 0$ ,  $-c_{\bar{\alpha}}A_{g(\bar{\alpha})}g(f^2(\bar{\alpha})) = f(\bar{\alpha})f^2(\bar{\alpha})g(f^2(\bar{\alpha})) = 0$ , and  $g(f^2(\alpha)) = f^3(\bar{\alpha})$ ,  $g(f^2(\bar{\alpha})) = f^3(\alpha)$ . Hence, the socle of  $P_{t(\alpha)} \oplus P_{t(\bar{\alpha})}$  is contained in  $\varphi\Lambda \cap \psi\Lambda$ . In particular, we have that  $\dim_K(\varphi\Lambda \cap \psi\Lambda) \geq 3$ , because  $\varphi f^2(\alpha) = -\psi f^2(\bar{\alpha})$  is not in the socle of  $P_{t(\alpha)} \oplus P_{t(\bar{\alpha})}$ . We claim that  $\dim_K(\varphi\Lambda \cap \psi\Lambda) = 3$ . Suppose that  $\dim_K(\varphi\Lambda \cap \psi\Lambda) \geq 4$ . Observe that if  $A'_{\alpha}$  (respectively,  $A'_{\bar{\alpha}}$ ) is not an arrow, then it follows from Lemma 5.5 (i) that  $A'_{\alpha}g(f(\bar{\alpha})) = 0$ , (respectively,  $A'_{\bar{\alpha}}g(f(\alpha)) = 0$ ), and consequently  $\dim_K(\varphi\Lambda \cap \psi\Lambda) = 3$ . Suppose that  $\dim_K(\varphi\Lambda \cap \psi\Lambda) \geq 4$ . Then  $A'_{\alpha}$  and  $A'_{\bar{\alpha}}$  are arrows, and hence  $A'_{\alpha} = g(\alpha)$  and  $A'_{\bar{\alpha}} = g(\bar{\alpha})$ . Observe that then  $n_{\alpha} = 3$ ,  $n_{\bar{\alpha}} = 3$ ,  $f(g(\alpha)) = g(f(\bar{\alpha}))$ ,  $f(g(\bar{\alpha})) = g(f(\alpha))$ . Moreover, there exists an element  $a \in K^*$  such that  $\varphi g(f(\alpha)) = a\psi g(f(\bar{\alpha}))$ . Then we obtain the equalities

$$\begin{aligned} f(\alpha)g(f(\alpha)) &= -ac_{\alpha}A'_{\alpha}g(f(\bar{\alpha})) = -ac_{\alpha}g(\alpha)g(f(\bar{\alpha})) = -ac_{\alpha}g(\alpha)f(g(\alpha)), \\ a f(\bar{\alpha})g(f(\bar{\alpha})) &= -c_{\bar{\alpha}}A'_{\bar{\alpha}}g(f(\alpha)) = -c_{\bar{\alpha}}g(\bar{\alpha})g(f(\alpha)) = -c_{\bar{\alpha}}g(\bar{\alpha})f(g(\bar{\alpha})). \end{aligned}$$

In particular, we conclude  $t(f(g(\alpha))) = t(g(f(\bar{\alpha})))$  and  $t(f(g(\bar{\alpha}))) = t(g(f(\alpha)))$ , and so  $n_{f(\alpha)} = 3$  and  $n_{f(\bar{\alpha})} = 3$ . Then we conclude that  $n_{\alpha} = 3$ ,  $n_{\bar{\alpha}} = 3$ ,  $n_{f(\alpha)} = 3$ ,  $n_{f(\bar{\alpha})} = 3$ . Hence, applying Lemma 6.6, we conclude that  $(Q, f)$  is the tetrahedral triangulation quiver, a contradiction. Therefore, indeed  $\dim_K(\varphi\Lambda \cap \psi\Lambda) = 3$ . Further, we have the equalities

$$\begin{aligned} \dim_K \varphi\Lambda &= \dim_K f(\alpha)\Lambda + \dim_K \text{soc}(P_{t(\bar{\alpha})}) = m_{f(\alpha)}n_{f(\alpha)} + 2, \\ \dim_K \psi\Lambda &= \dim_K f(\bar{\alpha})\Lambda + \dim_K \text{soc}(P_{t(\alpha)}) = m_{f(\bar{\alpha})}n_{f(\bar{\alpha})} + 2. \end{aligned}$$

Then we conclude that

$$\begin{aligned} \dim_K(\varphi\Lambda + \psi\Lambda) &= \dim_K \varphi\Lambda + \dim_K \psi\Lambda - \dim_K(\varphi\Lambda \cap \psi\Lambda) \\ &= m_{f(\alpha)}n_{f(\alpha)} + m_{f(\bar{\alpha})}n_{f(\bar{\alpha})} + 1. \end{aligned}$$

Since  $\varphi\Lambda + \psi\Lambda$  is contained in  $\text{Ker } \pi_1 = \Omega_{\Lambda}^2(S_i)$ , comparing the dimensions, we conclude that  $\Omega_{\Lambda}^2(S_i) = \varphi\Lambda + \psi\Lambda$ . Hence we have found generators of  $\Omega_{\Lambda}^2(S_i)$ . In particular, we conclude that a projective cover of  $\Omega_{\Lambda}^2(S_i)$  in  $\text{mod } \Lambda$  is induced by the homomorphism of right  $\Lambda$ -modules

$$\pi_2 : P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))} \rightarrow P_{t(\alpha)} \oplus P_{t(\bar{\alpha})}$$

given by  $\pi_2(u, v) = \varphi u + \psi v$  for  $u \in P_{t(f(\alpha))}$  and  $v \in P_{t(f(\bar{\alpha}))}$ . We have seen that  $\varphi f^2(\alpha) = -\psi f^2(\bar{\alpha})$ . This shows that the element in  $P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))} = P_{s(f^2(\alpha))} \oplus P_{s(f^2(\bar{\alpha}))}$

$$\theta = (f^2(\alpha), f^2(\bar{\alpha}))$$

lies in  $\text{Ker } \pi_2 = \Omega_\Lambda^3(S_i)$ . We may calculate the dimension of  $\Omega_\Lambda^3(S_i)$  as follows

$$\begin{aligned} \dim_K \Omega_\Lambda^3(S_i) &= \dim_K P_{s(f^2(\alpha))} + \dim_K P_{s(f^2(\bar{\alpha}))} - \dim_K \Omega_\Lambda^2(S_i) \\ &= m_{f^2(\alpha)} n_{f^2(\alpha)} + m_{g(f(\alpha))} n_{g(f(\alpha))} + m_{f^2(\bar{\alpha})} n_{f^2(\bar{\alpha})} + m_{g(f(\bar{\alpha}))} n_{g(f(\bar{\alpha}))} \\ &\quad - m_{f(\alpha)} n_{f(\alpha)} - m_{f(\bar{\alpha})} n_{f(\bar{\alpha})} - 1 \\ &= m_{f^2(\alpha)} n_{f^2(\alpha)} + m_{f^2(\bar{\alpha})} n_{f^2(\bar{\alpha})} - 1, \end{aligned}$$

because  $m_{g(f(\alpha))} = m_{f(\alpha)}$ ,  $n_{g(f(\alpha))} = n_{f(\alpha)}$ ,  $m_{g(f(\bar{\alpha}))} = m_{f(\bar{\alpha})}$ ,  $n_{g(f(\bar{\alpha}))} = n_{f(\bar{\alpha})}$ . Applying Corollary 5.6 to the opposite algebra  $\Lambda^{\text{op}}$  we conclude that  $\dim_K \Lambda e_i = m_{f^2(\alpha)} n_{f^2(\alpha)} + m_{f^2(\bar{\alpha})} n_{f^2(\bar{\alpha})}$ . Since  $\Lambda$  is a symmetric algebra, we have  $P_i \cong D(\Lambda e_i)$  in  $\text{mod } \Lambda$ , and hence  $\dim_K P_i = m_{f^2(\alpha)} n_{f^2(\alpha)} + m_{f^2(\bar{\alpha})} n_{f^2(\bar{\alpha})}$ . Hence we obtain that  $\dim_K \Omega_\Lambda^3(S_i) = \dim_K P_i - 1$ . Consider now the homomorphism of right  $\Lambda$ -modules

$$\pi_3 : P_i \rightarrow P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))}$$

given by  $\pi_3(z) = \theta z$  for any  $z \in P_i$ . Clearly,  $\pi_3$  induces a projective cover of  $\Omega_\Lambda^3(S_i)$  in  $\text{mod } \Lambda$ . Moreover,  $\text{Ker } \pi_3 = S_i = \text{soc}(P_i)$ , because  $\dim_K \Omega_\Lambda^3(S_i) = \dim_K(P_i/S_i)$ . In particular, we have  $\Omega_\Lambda^4(S_i) \cong S_i$  and  $\Omega_\Lambda^j(S_i) \not\cong S_i$  for any  $j \in \{1, 2, 3\}$ . This finishes the proof.  $\square$

We would like to mention that Proposition 7.1 holds also for any non-singular tetrahedral algebra  $\Lambda(\mathbb{S}, a, b, c, d)$ , which can be checked directly. On the other hand, for a singular tetrahedral algebra  $\Lambda = \Lambda(\mathbb{S}, a, b, c, d)$ , the proof given above is incorrect because we have  $\dim_K(\varphi\Lambda \cap \psi\Lambda) = 4$  (instead of 3). Clearly, it is also impossible by Proposition 6.4.

The next aim is to construct the first steps of a minimal projective bimodule resolution of  $\Lambda$ . Then we will show that  $\Omega_\Lambda^4(\Lambda) \cong \Lambda$  in  $\text{mod } \Lambda^e$ . We shall use the notation introduced in Section 3. Recall the first few steps of a minimal projective resolution of  $\Lambda$  in  $\text{mod } \Lambda^e$ ,

$$\mathbb{P}_3 \xrightarrow{S} \mathbb{P}_2 \xrightarrow{R} \mathbb{P}_1 \xrightarrow{d} \mathbb{P}_0 \xrightarrow{d_0} \Lambda \rightarrow 0$$

where

$$\begin{aligned} \mathbb{P}_0 &= \bigoplus_{i \in Q_0} P(i, i) = \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda, \\ \mathbb{P}_1 &= \bigoplus_{\alpha \in Q_1} P(s(\alpha), t(\alpha)) = \bigoplus_{\alpha \in Q_1} \Lambda e_{s(\alpha)} \otimes e_{t(\alpha)} \Lambda, \end{aligned}$$

the homomorphism  $d_0$  is defined by  $d_0(e_i \otimes e_i) = e_i$  for all  $i \in Q_0$ , and the homomorphism  $d : \mathbb{P}_1 \rightarrow \mathbb{P}_0$  is defined by

$$d(e_{s(\alpha)} \otimes e_{t(\alpha)}) = \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha$$

for any arrow  $\alpha$  in  $Q_1$  (see Lemma 3.3). In particular, we have  $\Omega_\Lambda^1(\Lambda) = \text{Ker } d_0$  and  $\Omega_\Lambda^2(\Lambda) = \text{Ker } d$ . We define now the homomorphism  $R : \mathbb{P}_2 \rightarrow \mathbb{P}_1$ . For each arrow  $\alpha$ , consider the element in  $KQ$

$$\mu_\alpha := \alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}.$$

Note that  $\mu_\alpha = e_{s(\alpha)} \mu_\alpha e_{t(f(\alpha))}$ . It follows from Propositions 3.1 and 7.1 that  $\mathbb{P}_2$  is of the form

$$\mathbb{P}_2 = \bigoplus_{\alpha \in Q_1} P(s(\alpha), t(f(\alpha))) = \bigoplus_{\alpha \in Q_1} \Lambda e_{s(\alpha)} \otimes e_{t(f(\alpha))} \Lambda.$$

We define the homomorphism  $R : \mathbb{P}_2 \rightarrow \mathbb{P}_1$  in  $\text{mod } \Lambda^e$  by

$$R(e_{s(\alpha)} \otimes e_{t(f(\alpha))}) = \varrho(\mu_\alpha)$$

for any arrow  $\alpha$  in  $Q_1$ , where  $\varrho : KQ \rightarrow \mathbb{P}_1$  is the  $K$ -linear homomorphism defined in Section 3. It follows from Lemma 3.4 that  $\text{Im } R \subseteq \text{Ker } d$ .



**Lemma 7.2.** *The homomorphism  $R : \mathbb{P}_2 \rightarrow \mathbb{P}_1$  induces a projective cover  $\Omega_{\Lambda^e}^2(\Lambda)$  in  $\text{mod } \Lambda^e$ . In particular, we have  $\Omega_{\Lambda^e}^3(\Lambda) = \text{Ker } R$ .*

*Proof.* We know that  $\text{rad } \Lambda^e = \text{rad } \Lambda^{\text{op}} \otimes \Lambda + \Lambda^{\text{op}} \otimes \text{rad } \Lambda$  (see [74, Corollary IV.11.4]). It follows from the definition that the generators  $\varrho(\mu_\alpha)$ ,  $\alpha \in Q_1$ , of the image  $R$  are elements of  $\text{rad } \mathbb{P}_1$  which are linearly independent in  $\text{rad } \mathbb{P}_1 / \text{rad}^2 \mathbb{P}_1$ . Moreover, the form of  $\mathbb{P}_2$  tells us where the generators of  $\Omega_{\Lambda^e}^2(\Lambda) = \text{Ker } d$  must be. Then we conclude that  $\varrho(\mu_\alpha)$ ,  $\alpha \in Q_1$ , form a minimal set of generators of the right  $\Lambda^e$ -module  $\Omega_{\Lambda^e}^2(\Lambda)$ . Summing up, we obtain that  $R : \mathbb{P}_2 \rightarrow \Omega_{\Lambda^e}^2(\Lambda)$  is a projective cover of  $\Omega_{\Lambda^e}^2(\Lambda)$  in  $\text{mod } \Lambda^e$ .  $\square$

By Propositions 3.1 and 7.1 we have that  $\mathbb{P}_3$  is of the form

$$\mathbb{P}_3 = \bigoplus_{i \in Q_0} P(i, i) = \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda.$$

For each vertex  $i \in Q_0$ , consider the following element of  $\mathbb{P}_2$

$$\begin{aligned} \psi_i &= (e_i \otimes e_{\ell(f(\alpha))})f^2(\alpha) + (e_i \otimes e_{\ell(f(\bar{\alpha}))})f^2(\bar{\alpha}) - \alpha(e_{\ell(\alpha)} \otimes e_i) - \bar{\alpha}(e_{\ell(\bar{\alpha})} \otimes e_i) \\ &= (e_{s(\alpha)} \otimes e_{\ell(f(\alpha))})f^2(\alpha) + (e_{s(\bar{\alpha})} \otimes e_{\ell(f(\bar{\alpha}))})f^2(\bar{\alpha}) - \alpha(e_{s(f(\alpha))} \otimes e_{\ell(f^2(\alpha))}) \\ &\quad - \bar{\alpha}(e_{s(f(\bar{\alpha}))} \otimes e_{\ell(f^2(\bar{\alpha}))}), \end{aligned}$$

where  $\alpha$  and  $\bar{\alpha}$  are the arrows starting at vertex  $i$ . Then we define the homomorphism  $S : \mathbb{P}_3 \rightarrow \mathbb{P}_2$  in  $\text{mod } \Lambda^e$  by

$$S(e_i \otimes e_i) = \psi_i$$

for any vertex  $i \in Q_0$ .

**Lemma 7.3.** *The homomorphism  $S : \mathbb{P}_3 \rightarrow \mathbb{P}_2$  induces a projective cover of  $\Omega_{\Lambda^e}^3(\Lambda)$  in  $\text{mod } \Lambda^e$ . In particular, we have  $\Omega_{\Lambda^e}^4(\Lambda) = \text{Ker } S$ .*

*Proof.* We will prove first that  $R(\psi_i) = 0$  for any  $i \in Q_0$ . Fix a vertex  $i \in Q_0$ . Then we have the equalities in  $\mathbb{P}_1$

$$\begin{aligned} R(\psi_i) &= \varrho(\mu_\alpha)f^2(\alpha) + \varrho(\mu_{\bar{\alpha}})f^2(\bar{\alpha}) - \alpha\varrho(\mu_{f(\alpha)}) - \bar{\alpha}\varrho(\mu_{f(\bar{\alpha})}) \\ &= (\varrho(\alpha f(\alpha)) - c_{\bar{\alpha}}\varrho(A_{\bar{\alpha}}))f^2(\alpha) + (\varrho(\bar{\alpha} f(\bar{\alpha})) - c_{\alpha}\varrho(A_{\alpha}))f^2(\bar{\alpha}) \\ &\quad - \alpha(\varrho(f(\alpha)f^2(\alpha)) - c_{\bar{f}(\alpha)}\varrho(A_{\bar{f}(\alpha)})) - \bar{\alpha}(\varrho(f(\bar{\alpha})f^2(\bar{\alpha})) - c_{f(\bar{\alpha})}\varrho(A_{f(\bar{\alpha})})) \\ &= \varrho(\alpha f(\alpha))f^2(\alpha) + \varrho(\bar{\alpha} f(\bar{\alpha}))f^2(\bar{\alpha}) - \alpha\varrho(f(\alpha)f^2(\alpha)) - \bar{\alpha}\varrho(f(\bar{\alpha})f^2(\bar{\alpha})) \\ &\quad - c_{\bar{\alpha}}\varrho(A_{\bar{\alpha}})f^2(\alpha) - c_{\alpha}\varrho(A_{\alpha})f^2(\bar{\alpha}) + c_{\alpha}\alpha\varrho(A_{g(\alpha)}) + c_{\bar{\alpha}}\bar{\alpha}\varrho(A_{g(\bar{\alpha})}) \\ &= e_i \otimes f(\alpha)f^2(\alpha) + \alpha \otimes f^2(\alpha) + e_i \otimes f(\bar{\alpha})f^2(\bar{\alpha}) + \bar{\alpha} \otimes f^2(\bar{\alpha}) \\ &\quad - \alpha \otimes f^2(\alpha) - \alpha f(\alpha) \otimes e_i - \bar{\alpha} \otimes f^2(\bar{\alpha}) - \bar{\alpha} f(\bar{\alpha}) \otimes e_i \\ &\quad - c_{\bar{\alpha}}\varrho(A_{\bar{\alpha}})f^2(\alpha) - c_{\alpha}\varrho(A_{\alpha})f^2(\bar{\alpha}) + c_{\alpha}\alpha\varrho(A_{g(\alpha)}) + c_{\bar{\alpha}}\bar{\alpha}\varrho(A_{g(\bar{\alpha})}) \\ &= c_{\alpha}(e_i \otimes A_{g(\alpha)} + \alpha\varrho(A_{g(\alpha)}) - \varrho(A_{\alpha})f^2(\bar{\alpha}) - A_{\alpha} \otimes e_i) \\ &\quad + c_{\bar{\alpha}}(e_i \otimes A_{g(\bar{\alpha})} + \bar{\alpha}\varrho(A_{g(\bar{\alpha})}) - \varrho(A_{\bar{\alpha}})f^2(\alpha) - A_{\bar{\alpha}} \otimes e_i) \\ &= 0, \end{aligned}$$

because  $f^2(\bar{\alpha}) = g^{n_{\alpha}-1}(\alpha)$  and  $f^2(\alpha) = g^{n_{\bar{\alpha}}-1}(\bar{\alpha})$ . Hence  $\text{Im } S \subseteq \text{Ker } R$ . Further, it follows from the definition that the generators  $\psi_i$ ,  $i \in Q_0$ , of the image of  $S$  are elements of  $\text{rad } \mathbb{P}_2$  which are linearly independent in  $\text{rad } \mathbb{P}_2 / \text{rad}^2 \mathbb{P}_2$ . Then we conclude from the form of  $\mathbb{P}_2$  that these elements form a minimal set of generators of  $\text{Ker } R = \Omega_{\Lambda^e}^3(\Lambda)$ . Hence  $S : \mathbb{P}_3 \rightarrow \Omega_{\Lambda^e}^3(\Lambda)$  is a projective cover of  $\Omega_{\Lambda^e}^3(\Lambda)$  in  $\text{mod } \Lambda^e$ .  $\square$

**Theorem 7.4.** *There is an isomorphism  $\Omega_{\Lambda^e}^4(\Lambda) \cong \Lambda$  in  $\text{mod } \Lambda^e$ . In particular,  $\Lambda$  is a periodic algebra of period 4.*

*Proof.* This is very similar to the proof of [33, Theorem 5.9]. For each vertex  $i \in Q_0$ , we denote by  $\mathcal{B}_i$  the basis of  $e_i\Lambda$  consisting of  $e_i$ , all initial subwords of  $A_\alpha$  and  $A_{\bar{\alpha}}$ , and  $\omega_i = c_\alpha \mathcal{B}_\alpha = c_{\bar{\alpha}} \mathcal{B}_{\bar{\alpha}}$  (see Lemma 5.3 and Corollary 5.6). We note that  $\omega_i$  generates the socle of  $e_i\Lambda$ . Then  $\mathcal{B} = \bigcup_{i \in Q_0} \mathcal{B}_i$  is a  $K$ -linear basis of  $\Lambda$ . In the proof of Proposition 5.8, we have defined the symmetrizing  $K$ -linear form  $\varphi : \Lambda \rightarrow K$  which assigns to the coset  $u + I$  of a path  $u$  in  $Q$  the element in  $K$

$$\varphi(u + I) = \begin{cases} c_\alpha^{-1} & \text{if } u = \mathcal{B}_\alpha \text{ for an arrow } \alpha \in Q_1, \\ 0 & \text{otherwise,} \end{cases}$$

where  $I = I(Q, f, m_\bullet, c_\bullet)$ . Then, by general theory, we have the symmetrizing form  $(-, -) : \Lambda \times \Lambda \rightarrow K$  such that  $(x, y) = \varphi(xy)$  for any  $x, y \in \Lambda$ . Observe that, for any elements  $x \in \mathcal{B}_i$  and  $y \in \mathcal{B}$ , we have

$$(x, y) = \text{the coefficient of } \omega_i \text{ in } xy,$$

when  $xy$  is expressed as a linear combination of the elements of  $e_i\mathcal{B}$  over  $K$ . Consider also the dual basis  $\mathcal{B}^* = \{b^* \mid b \in \mathcal{B}\}$  of  $\Lambda$  such that  $(b, c^*) = \delta_{bc}$  for  $b, c \in \mathcal{B}$ . Observe that, for  $x \in e_i\mathcal{B}$  and  $y \in \mathcal{B}$ , the element  $(x, y)$  can only be non-zero if  $y = ye_i$ . In particular, if  $b \in e_i\mathcal{B}e_j$  then  $b^* \in e_j\mathcal{B}e_i$ .

For each vertex  $i \in Q_0$ , we define the element of  $\mathbb{P}_3$

$$\xi_i = \sum_{b \in \mathcal{B}_i} b \otimes b^*.$$

We note that  $\xi_i$  is independent of the basis of  $\Lambda$  (see [33, part (2a) on the page 119]). It follows from [33, part (2b) on the page 119] that, for any element  $a \in e_i(\text{rad } \Lambda)e_j \setminus e_i(\text{rad } \Lambda)^2e_j$ , we have

$$a\xi_i = \xi_j a.$$

Consider now the homomorphism

$$\theta : \Lambda \rightarrow \mathbb{P}_3$$

in  $\text{mod } \Lambda^e$  such that  $\theta(e_i) = \xi_i$  for any  $i \in Q_0$ . Then  $\theta(1_\Lambda) = \sum_{i \in Q_0} \xi_i$ , and consequently we have

$$a \left( \sum_{i \in Q_0} \xi_i \right) = \theta(a) = \left( \sum_{i \in Q_0} \xi_i \right) a$$

for any element  $a \in \Lambda$ . We claim that  $\theta$  is a monomorphism. It is enough to show that  $\theta$  is a monomorphism of right  $\Lambda$ -modules. We know that  $\Lambda = \bigoplus_{i \in Q_0} e_i\Lambda$  and each  $e_i\Lambda$  has simple socle generated by  $\omega_i$ . For each  $i \in Q_0$ , we have

$$\theta(\omega_i) = \left( \sum_{j \in Q_0} \xi_j \right) \omega_i = \xi_i \omega_i = \sum_{b \in \mathcal{B}_i} (b \otimes b^*) \omega_i = \sum_{b \in \mathcal{B}_i} b \otimes b^* \omega_i = \omega_i \otimes \omega_i \neq 0.$$

Hence the claim follows. Our next aim is to show that  $S(\xi_i) = 0$  for any  $i \in Q_0$ , or equivalently, that  $\text{Im } \theta \subseteq \text{Ker } S = \Omega_{\Lambda^e}^4(\Lambda)$ . Applying arguments from [33, part (3) on the pages 119 and 120], we obtain that

$$\sum_{b \in \mathcal{B}} b(a^r \otimes a^s)b^* = \sum_{b \in \mathcal{B}} b \otimes a^{r+s}b^*$$

for all integers  $r, s \geq 0$  and any element  $a = e_p a e_q$  in  $\text{rad } \Lambda$ , with  $p, q \in Q_0$ . In particular, for each arrow  $\alpha$  in  $Q_1$ , we have

$$\sum_{b \in \mathcal{B}} b\alpha \otimes b^* = \sum_{b \in \mathcal{B}} b \otimes \alpha b^*,$$

and hence

$$\sum_{b \in \mathcal{B}_i} b\alpha \otimes b^* = \sum_{b \in \mathcal{B}_i} b \otimes \alpha b^*$$

for any  $i \in Q_0$ . We note that every arrow  $\beta$  in  $Q$  occurs once as a left factor of some  $\psi_j$  (with negative sign) and once as a right factor of some  $\psi_k$  (with positive sign), because  $\beta = f^2(\alpha)$  for a unique arrow  $\alpha$ . Then, for any  $i \in Q_0$ , the following equalities hold

$$\begin{aligned} S(\xi_i) &= \sum_{b \in \mathcal{B}_i} S(b \otimes b^*) = \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} S(b e_j \otimes e_j b^*) = \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} b S(e_j \otimes e_j) b^* \\ &= \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} b \psi_j b^* = \sum_{\alpha \in Q_1} \left[ \sum_{b \in \mathcal{B}_i} -(b \alpha \otimes b^*) + \sum_{b \in \mathcal{B}_i} b \otimes \alpha b^* \right] = 0. \end{aligned}$$

Hence, indeed  $\text{Im } \theta \subseteq \text{Ker } S = \Omega_{\Lambda^e}^4(\Lambda)$ , and we obtain a monomorphism  $\theta : \Lambda \rightarrow \Omega_{\Lambda^e}^4(\Lambda)$  in  $\text{mod } \Lambda^e$ .

Finally, it follows from Theorem 2.4 and Proposition 3.1 that  $\Omega_{\Lambda^e}^4(\Lambda) \cong {}_1\Lambda_\sigma$  in  $\text{mod } \Lambda^e$  for some  $K$ -algebra automorphism  $\sigma$  of  $\Lambda$ . Then  $\dim_K \Lambda = \dim_K \Omega_{\Lambda^e}^4(\Lambda)$ , and consequently  $\theta$  is an isomorphism. Therefore, we have  $\Omega_{\Lambda^e}^4(\Lambda) \cong \Lambda$  in  $\text{mod } \Lambda^e$ . Clearly, then  $\Lambda$  is a periodic algebra of period 4.  $\square$

**Corollary 7.5.** *Let  $(Q, f)$  be a triangulation quiver with at least four vertices, let  $m_\bullet$  and  $c_\bullet$  be weight and parameter functions of  $(Q, f)$ , and let  $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$  be the associated weighted triangulation algebra. Then the Cartan matrix  $C_\Lambda$  of  $\Lambda$  is singular.*

*Proof.* This follows from Theorems 2.5 and 7.4.  $\square$

## 8. Socle deformed weighted surface algebras

In this section we introduce socle deformations of weighted surface algebras of surfaces with boundary, and describe their basic properties. We will show in the next section that these algebras are periodic algebras of period 4.

Let  $(Q, f)$  be a triangulation quiver with at least three vertices. A vertex  $i \in Q_0$  is said to be a *border vertex* of  $(Q, f)$  if there is a loop  $\alpha$  at  $i$  with  $f(\alpha) = \alpha$ . If so, then  $\bar{\alpha} = g(\alpha)$ ,  $\alpha = f^2(\alpha) = g^{n_\alpha-1}(\bar{\alpha})$ , and  $f^2(\bar{\alpha}) = g^{-1}(\alpha)$ . In particular, we have  $n_\alpha = n_{\bar{\alpha}} \geq 3$ , because  $|Q_0| \geq 3$ . Hence the loop  $\alpha$  is uniquely determined by the vertex  $i$ , and we call it a *border loop* of  $(Q, f)$ . We also note that the following equalities hold (see before Definition 5.1):  $\alpha A_{\bar{\alpha}} = B_\alpha = A_\alpha f^2(\bar{\alpha})$  and  $\bar{\alpha} A_{g(\bar{\alpha})} = B_{\bar{\alpha}} = A_{\bar{\alpha}} \alpha$ . We denote by  $\partial(Q, f)$  the set of all border vertices of  $(Q, f)$ , and call it the *border* of  $(Q, f)$ . Observe that, if  $(S, \vec{T})$  is a directed triangulated surface with  $(Q(S, \vec{T}), f) = (Q, f)$ , then the border vertices of  $(Q, f)$  correspond bijectively to the boundary edges of the triangulation  $T$  of  $S$ . Hence, the border  $\partial(Q, f)$  of  $(Q, f)$  is non-empty if and only if the boundary  $\partial S$  of  $S$  is not empty. A function

$$b_\bullet : \partial(Q, f) \rightarrow K$$

is said to be a *border function* of  $(Q, f)$ . Assume that  $\partial(Q, f)$  is not empty. Then, for a weight function  $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ , a parameter function  $c_\bullet : \mathcal{O}(g) \rightarrow K^*$ , and a border function  $b_\bullet : \partial(Q, f) \rightarrow K$ , we may consider the bound quiver algebra

$$\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet) = KQ/I(Q, f, m_\bullet, c_\bullet, b_\bullet),$$

where  $I(Q, f, m_\bullet, c_\bullet, b_\bullet)$  is the admissible ideal in the path algebra  $KQ$  of  $Q$  over  $K$  generated by the elements:

- (1)  $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}$ , for all arrows  $\alpha \in Q_1$  which are not border loops,
- (2)  $\alpha^2 - c_{\bar{\alpha}} A_{\bar{\alpha}} - b_{s(\alpha)} B_{\bar{\alpha}}$ , for all border loops  $\alpha \in Q_1$ ,
- (3)  $\beta f(\beta) g(f(\beta))$ , for all arrows  $\beta \in Q_1$ .

Then  $\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$  is said to be a *socle deformed weighted triangulation algebra*. We note that if  $b_\bullet$  is a zero border function ( $b_i = 0$  for all  $i \in \partial(Q, f)$ ) then  $\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet) = \Lambda(Q, f, m_\bullet, c_\bullet)$ . Moreover, if  $(Q, f) = Q(S, \vec{T})$  for a directed triangulated surface  $(S, \vec{T})$  with non-empty boundary, then  $\Lambda(Q(S, \vec{T}), m_\bullet, c_\bullet, b_\bullet)$  is said to be a *socle deformed weighted surface algebra*.

**Proposition 8.1.** *Let  $(Q, f)$  be a triangulation quiver with at least three vertices and  $\partial(Q, f)$  not empty,  $m_\bullet$ ,  $c_\bullet$ ,  $b_\bullet$  weight, parameter, border functions of  $(Q, f)$ ,  $\bar{\Lambda} = \Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$ , and  $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ . Then the following hold:*

- (i)  $\bar{\Lambda}$  is a finite-dimensional algebra with  $\dim_K \bar{\Lambda} = \sum_{O \in O(g)} m_O n_O^2$ .
- (ii)  $\bar{\Lambda}$  is socle equivalent to  $\Lambda$ .
- (iii)  $\bar{\Lambda}$  degenerates to  $\Lambda$ .
- (iv)  $\bar{\Lambda}$  is a tame algebra.
- (v)  $\bar{\Lambda}$  is a symmetric algebra.
- (vi) The Cartan matrix  $C_{\bar{\Lambda}}$  of  $\bar{\Lambda}$  is singular.

*Proof.* We abbreviate  $\bar{I} = I(Q, f, m_\bullet, c_\bullet, b_\bullet)$ .

(i) Let  $i$  be a vertex of  $Q$ , and let  $\alpha, \bar{\alpha}$  be the arrows in  $Q$  with source  $i$ . Then the indecomposable projective right  $\bar{\Lambda}$ -module  $P_i = e_i \bar{\Lambda}$  has basis given by  $e_i$ , all initial subwords of  $A_\alpha$  and  $A_{\bar{\alpha}}$ , and  $c_\alpha B_\alpha = c_{\bar{\alpha}} B_{\bar{\alpha}}$ , and hence  $\dim_K P_i = m_\alpha n_\alpha + m_{\bar{\alpha}} n_{\bar{\alpha}}$ . Then we obtain

$$\dim_K \bar{\Lambda} = \sum_{O \in O(g)} m_O n_O^2.$$

(ii) We note that  $\text{soc}(\bar{\Lambda})$  and  $\text{soc}(\Lambda)$  are generated by the elements  $c_\alpha B_\alpha = c_{\bar{\alpha}} B_{\bar{\alpha}}$  for all arrows  $\alpha$  in  $Q_1$ , and  $B_\alpha = B_{\bar{\alpha}}$ ,  $c_\alpha = c_{\bar{\alpha}} = c_{g(\alpha)}$  for all loops  $\alpha$  in  $Q_1$  with  $s(\alpha) \in \partial(Q, f)$ . Therefore the algebras  $\bar{\Lambda}/\text{soc}(\bar{\Lambda})$  and  $\Lambda/\text{soc}(\Lambda)$  are isomorphic. Hence  $\bar{\Lambda}$  is socle equivalent to  $\Lambda$ .

(iii) For each  $t \in K$ , consider the bound quiver algebra  $\bar{\Lambda}(t) = KQ/\bar{I}^{(t)}$ , where  $\bar{I}^{(t)}$  is the admissible ideal in the path algebra  $KQ$  of  $Q$  over  $K$  generated by the elements:

- (1)  $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}$ , for all arrows  $\alpha \in Q_1$  which are not border loops,
- (2)  $\alpha^2 - c_{\bar{\alpha}} A_{\bar{\alpha}} - t b_{s(\alpha)} B_{\bar{\alpha}}$ , for all border loops  $\alpha \in Q_1$ ,
- (3)  $\beta f(\beta) g(f(\beta))$ , for all arrows  $\beta \in Q_1$ .

Then  $\bar{\Lambda}(t)$ ,  $t \in K$ , is an algebraic family in the variety  $\text{alg}_d(K)$ , with  $d = \dim_K \bar{\Lambda}$ , such that  $\bar{\Lambda}(t) \cong \bar{\Lambda}(1) = \bar{\Lambda}$  for all  $t \in K^*$  and  $\bar{\Lambda}(0) \cong \Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ . It follows from Proposition 3.1 that  $\bar{\Lambda}$  degenerates to  $\Lambda$ .

(iv)  $\bar{\Lambda}$  is a tame algebra because  $\bar{\Lambda}/\text{soc}(\bar{\Lambda}) \cong \Lambda/\text{soc}(\Lambda)$  and  $\Lambda$  is tame, by Proposition 5.8. This also follows from Propositions 2.2 and 5.8.

(v) We define a symmetrizing form  $\bar{\varphi} : \bar{\Lambda} \rightarrow K$  of  $\bar{\Lambda} = KQ/\bar{I}$  by assigning to the coset  $u + \bar{I}$  of a path  $u$  in  $Q$  the following element of  $K$

$$\bar{\varphi}(u + \bar{I}) = \begin{cases} c_\alpha^{-1} & \text{if } u = B_\alpha \text{ for an arrow } \alpha \in Q_1, \\ b_i c_\alpha^{-1} & \text{if } u = \alpha^2 \text{ for some border loop } \alpha \in Q_1, \\ 0 & \text{otherwise.} \end{cases}$$

We note that for a border loop  $\alpha$  of  $(Q, f)$  we have  $B_\alpha = B_{\bar{\alpha}}$  and  $c_\alpha = c_{\bar{\alpha}}$ . Moreover, for any arrow  $\beta$  in  $Q_1$ , we have  $\beta f(\beta) g(f(\beta)) = 0$  and  $\beta g(\beta) f(g(\beta)) = 0$  in  $\bar{\Lambda}$  (see Lemma 5.5). Hence, if  $\alpha$  is a border loop, then  $\alpha^2 \bar{\alpha} = \alpha^2 g(\alpha) = 0$  and  $f^2(\bar{\alpha}) \alpha^2 = g^{-1}(\alpha) \alpha^2 = g^{-1}(\alpha) g(g^{-1}(\alpha)) f(g(g^{-1}(\alpha))) = 0$ .

(vi) This follows from (ii), (v), Corollary 7.5, and the fact that all weighted triangulation algebras given by the triangulation quivers with three vertices and non-empty border have singular Cartan matrices (see Examples 4.3, 4.4, 4.5, and 5.10).  $\square$

We note that in general a selfinjective algebra which is socle equivalent to a tame symmetric algebra, need not be symmetric (see [8, Theorems 6.4, 6.7, and Proposition 6.8]).

**Proposition 8.2.** *Let  $(Q, f)$  be a triangulation quiver with at least three vertices and  $\partial(Q, f)$  not empty, and  $m_\bullet, c_\bullet, b_\bullet$  weight, parameter, border functions of  $(Q, f)$ . Assume that  $K$  has characteristic different from 2. Then the algebras  $\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$  and  $\Lambda(Q, f, m_\bullet, c_\bullet)$  are isomorphic.*

*Proof.* Since  $K$  has characteristic different from 2, for any vertex  $i \in \partial(Q, f)$  there exists a unique element  $a_i \in K$  such that  $b_i = 2a_i$ . Then we have an isomorphism of  $K$ -algebras  $h : \Lambda(Q, f, m_\bullet, c_\bullet) \rightarrow \Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$  such that

$$h(\alpha) = \begin{cases} \alpha & \text{for any arrow } \alpha \in Q_1 \text{ which is not a border loop,} \\ \alpha - a_{s(\alpha)} \alpha^2 & \text{for any border loop } \alpha \in Q_1. \end{cases}$$

We note that, if  $\alpha$  is a border loop in  $Q_1$ , then  $\alpha^2 g(\alpha) = 0$  and  $g^{-1}(\alpha) \alpha^2 = 0$ .  $\square$

**Proposition 8.3.** *Let  $A$  be a basic, indecomposable, symmetric algebra with the Grothendieck group  $K_0(A)$  of rank at least 3 which is socle equivalent to a weighted triangulated algebra  $\Lambda(Q, f, m_\bullet, c_\bullet)$ . Then  $A$  is isomorphic to an algebra  $\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$  for some border function  $b_\bullet$  of  $(Q, f)$ .*

*Proof.* Let  $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ ,  $I = I(Q, f, m_\bullet, c_\bullet)$ , and so  $\Lambda = KQ/I$ . Since  $A$  is socle equivalent to  $\Lambda$ , there is a  $K$ -algebra isomorphism  $\varphi : A/\text{soc}(A) \rightarrow \Lambda/\text{soc}(\Lambda)$ . Then  $A$  is isomorphic to a bound quiver algebra  $KQ/J$  for an admissible ideal  $J$  of  $KQ$ , because  $A$  is a basic algebra. Moreover, we may assume that  $\varphi(\alpha) = \alpha$  for any arrow  $\alpha$  in  $Q_1$ . Because  $A$  is a symmetric algebra, each indecomposable projective right  $A$ -module  $e_i A$  has one-dimensional socle generated by an element  $\omega_i \in e_i A e_i$  such that  $\omega_i \text{rad } A = 0$ . We have the following relations in  $A$ :

- (1)  $\alpha f(\alpha) + \text{soc}(A) = c_{\bar{\alpha}} A_{\bar{\alpha}} + \text{soc}(A)$ , for all arrows  $\alpha \in Q_1$ ,
- (2)  $\beta f(\beta) g(f(\beta)) \in \text{soc}(A)$ , for all arrows  $\beta \in Q_1$ .

Let  $\beta$  be an arrow in  $Q_1$  and  $i = s(\beta)$ . Then  $i = t(f^2(\beta)) \neq t(g(f(\beta)))$ . Since  $\beta f(\beta) g(f(\beta)) = e_i \beta f(\beta) g(f(\beta))$ , we conclude that  $\beta f(\beta) g(f(\beta)) \in \text{soc}(e_i A)$ , and hence  $\beta f(\beta) g(f(\beta)) = \lambda \omega_i$  for an element  $\lambda \in K$ . But then  $\beta f(\beta) g(f(\beta)) = 0$  because  $\omega_i \in e_i A e_i$ .

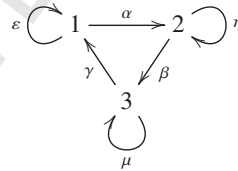
Take now an arrow  $\alpha \in Q_1$ , and let  $i = s(\alpha)$ . We know that  $\alpha f(\alpha)$  and  $A_{\bar{\alpha}}$  are paths in  $Q$  from  $i$  to  $t(f(\alpha)) = s(f^2(\alpha)) = g^{n_{\bar{\alpha}}-1}(\bar{\alpha})$ . Hence, we deduce that  $\alpha f(\alpha) + \text{soc}(e_i A) = c_{\bar{\alpha}} A_{\bar{\alpha}} + \text{soc}(e_i A)$ , and consequently  $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}} = b_i \omega_i$  for some element  $b_i \in K$ . We also note that if  $i \notin \partial(Q, f)$  then  $i \neq t(f(\alpha))$ , and then we conclude as above that  $\alpha f(\alpha) = c_{\bar{\alpha}} A_{\bar{\alpha}}$ . Clearly, for  $i \in \partial(Q, f)$ , we have  $\alpha f(\alpha) = c_{\bar{\alpha}} A_{\bar{\alpha}} + b_i \omega_i$ . Moreover, in this case have  $B_\alpha = B_{\bar{\alpha}}$  and  $c_\alpha = c_{\bar{\alpha}}$ , because  $\bar{\alpha} = g(\alpha)$ , so we may take  $\omega_i = B_\alpha$ . Hence, we have the border function  $b_\bullet : \partial(Q, f) \rightarrow K$  such that  $A$  is isomorphic to the algebra  $\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$ .  $\square$

The above results show that it is worthwhile to distinguish the weighted surface (triangulation) algebras from the socle deformed weighted surface (triangulation) algebras, occurring only in characteristic 2.

It may seem at the first sight that the notion of a socle deformed weighted surface (triangulation) algebra is a special case of the notion of a triangulation algebra defined in [60, Definition 5.16 and Proposition 7.4]. But it follows from Theorem 4.11, [36, Main Theorem], and [60, Theorem 7.1 and Proposition 7.4] that these two notions actually coincide.

We end this section with an example showing that there exist socle deformed weighted surface algebras which are not isomorphic to a weighted surface algebra.

**Example 8.4.** Let  $(Q(S, \vec{T}), f)$  be the triangulation quiver



with the  $f$ -orbits  $(\alpha \beta \gamma)$ ,  $(\varepsilon)$ ,  $(\eta)$ ,  $(\mu)$ , considered in Examples 4.3 and 5.10. Then  $O(g)$  consists of one  $g$ -orbit  $(\alpha \eta \beta \mu \gamma \varepsilon)$ . Let  $m_\bullet : O(g) \rightarrow \mathbb{N}$  be the weight function with  $m_{O(g)} = 1$  and  $c_\bullet : O(g) \rightarrow K^*$  the parameter function with  $c_{O(g)} = 1$ . Then the associated weighted surface algebra  $\Lambda = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet)$  is given by the above quiver and the relations

$$\begin{array}{llll} \alpha\beta = \varepsilon\alpha\eta\beta\mu, & \varepsilon^2 = \alpha\eta\beta\mu\gamma, & \alpha\beta\mu = 0, & \varepsilon^2\alpha = 0, \\ \beta\gamma = \eta\beta\mu\gamma\varepsilon, & \eta^2 = \beta\mu\gamma\varepsilon\alpha, & \beta\gamma\varepsilon = 0, & \eta^2\beta = 0, \\ \gamma\alpha = \mu\gamma\varepsilon\alpha\eta, & \mu^2 = \gamma\varepsilon\alpha\eta\beta, & \gamma\alpha\eta = 0, & \mu^2\gamma = 0. \end{array}$$

Observe that the border  $\partial(Q(S, \vec{T}), f)$  of  $(Q(S, \vec{T}), f)$  is the set  $Q_0 = \{1, 2, 3\}$  of vertices of  $Q$ , and  $\varepsilon, \eta, \mu$  are the border loops. Take now a border function  $b_\bullet : \partial(Q(S, \vec{T}), f) \rightarrow K$ . Then the associated socle deformed weighted surface

algebra  $\bar{\Lambda} = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet, b_\bullet)$  is given by the above quiver and the relations

$$\begin{aligned} \alpha\beta &= \varepsilon\alpha\eta\beta\mu, & \varepsilon^2 &= \alpha\eta\beta\mu\gamma + b_1\alpha\eta\beta\mu\gamma\varepsilon, & \alpha\beta\mu &= 0, & \varepsilon^2\alpha &= 0, \\ \beta\gamma &= \eta\beta\mu\gamma\varepsilon, & \eta^2 &= \beta\mu\gamma\varepsilon\alpha + b_2\beta\mu\gamma\varepsilon\alpha\eta, & \beta\gamma\varepsilon &= 0, & \eta^2\beta &= 0, \\ \gamma\alpha &= \mu\gamma\varepsilon\alpha\eta, & \mu^2 &= \gamma\varepsilon\alpha\eta\beta + b_3\gamma\varepsilon\alpha\eta\beta\mu, & \gamma\alpha\eta &= 0, & \mu^2\gamma &= 0. \end{aligned}$$

Assume that  $K$  has characteristic 2 and  $b_\bullet$  is non-zero, say  $b_1 \neq 0$ . We claim that the algebras  $\Lambda$  and  $\bar{\Lambda}$  are not isomorphic. Suppose that there is an isomorphism  $h : \Lambda \rightarrow \bar{\Lambda}$  of  $K$ -algebras. Then there exist elements  $r_1, s_1, t_1, u_1, v_1, w_1 \in K^*$  and  $r_i, s_i, t_i, u_j, v_j, w_j \in K$ ,  $i \in \{2, 3, 4\}$ ,  $j \in \{2, 3\}$ , such that

$$\begin{aligned} h(\alpha) &= r_1\alpha + r_2\varepsilon\alpha + r_3\alpha\eta + r_4\varepsilon\alpha\eta, & h(\varepsilon) &= u_1\varepsilon + u_2\varepsilon^2 + u_3\varepsilon^3, \\ h(\beta) &= s_1\beta + s_2\eta\beta + s_3\beta\mu + s_4\eta\beta\mu, & h(\eta) &= v_1\eta + v_2\eta^2 + v_3\eta^3, \\ h(\gamma) &= t_1\gamma + t_2\mu\gamma + t_3\gamma\varepsilon + t_4\mu\gamma\varepsilon, & h(\mu) &= w_1\mu + w_2\mu^2 + w_3\mu^3. \end{aligned}$$

Observe that we have in  $\bar{\Lambda}$  the equalities

$$\begin{aligned} \varepsilon^3 &= \varepsilon(\alpha\eta\beta\mu\gamma + b_1\alpha\eta\beta\mu\gamma\varepsilon) = \varepsilon\alpha\eta\beta\mu\gamma, \\ \varepsilon^3 &= (\alpha\eta\beta\mu\gamma + b_1\alpha\eta\beta\mu\gamma\varepsilon)\varepsilon = \alpha\eta\beta\mu\gamma\varepsilon. \end{aligned}$$

Since  $K$  has characteristic 2, we conclude that the following equalities hold in  $\bar{\Lambda}$

$$\begin{aligned} u_1^2\alpha\eta\beta\mu\gamma + u_1^2b_1\alpha\eta\beta\mu\gamma\varepsilon &= u_1^2\varepsilon^2 = h(\varepsilon)^2 \\ &= h(\varepsilon^2) = h(\alpha\eta\beta\mu\gamma) = h(\alpha)h(\eta)h(\beta)h(\mu)h(\gamma) \\ &= r_1v_1s_1w_1t_1\alpha\eta\beta\mu\gamma + r_2v_1s_1w_1t_1\varepsilon\alpha\eta\beta\mu\gamma + r_1v_1s_1w_1t_3\alpha\eta\beta\mu\gamma\varepsilon \\ &= r_1v_1s_1w_1t_1\alpha\eta\beta\mu\gamma + v_1s_1w_1(r_2t_1 + r_1t_3)\alpha\eta\beta\mu\gamma\varepsilon, \end{aligned}$$

and hence  $u_1^2 = r_1v_1s_1w_1t_1$  and  $u_1^2b_1 = v_1s_1w_1(r_2t_1 + r_1t_3)$ . In particular, we obtain that  $r_2t_1 + r_1t_3 \neq 0$ , because  $u_1, b_1, v_1, s_1, w_1 \in K^*$ . On the other hand, we have the following equalities in  $\bar{\Lambda}/(\text{rad } \bar{\Lambda})^4$

$$\begin{aligned} 0 + (\text{rad } \bar{\Lambda})^4 &= h(\mu\gamma\varepsilon\alpha\eta) + (\text{rad } \bar{\Lambda})^4 = h(\gamma\alpha) + (\text{rad } \bar{\Lambda})^4 \\ &= h(\gamma)h(\alpha) + (\text{rad } \bar{\Lambda})^4 = (r_2t_1 + r_1t_3)\gamma\varepsilon\alpha + (\text{rad } \bar{\Lambda})^4, \end{aligned}$$

and hence  $r_2t_1 + r_1t_3 = 0$ , a contradiction. This proves that the algebras  $\Lambda$  and  $\bar{\Lambda}$  are not isomorphic. We note that then, by Proposition 8.3, the algebra  $\bar{\Lambda}$  is not isomorphic to any weighted surface algebra.

It would be interesting to know when, for  $K$  of characteristic 2, a socle deformed weighted surface algebra is isomorphic to a weighted surface algebra.

## 9. Periodicity of socle deformed weighted surface algebras

In this section we prove that all socle deformed weighted surface algebras introduced in the previous section are periodic algebras of period 4.

Assume that  $K$  has characteristic 2. Let  $(Q, f)$  be a triangulation quiver with at least three vertices and non-empty border  $\partial(Q, f)$ . Moreover, let  $m_\bullet : \mathcal{O}(g) \rightarrow \mathbb{N}^*$  be a weight function,  $c_\bullet : \mathcal{O}(g) \rightarrow K^*$  a parameter function and  $b_\bullet : \partial(Q(S, \vec{T}), f) \rightarrow K$  a border function, which we assume to be non-zero. Moreover, let  $\Lambda = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet)$  be the associated weighted triangulation algebra and  $\bar{\Lambda} = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet, b_\bullet)$  the associated socle deformed weighted triangulation algebra. We note that  $(Q, f)$  is not the tetrahedral triangulation quiver, because  $\partial(Q, f)$  is not empty.

We have the following analogue of Proposition 7.1.



**Proposition 9.1.** *Let  $i$  be a vertex of  $Q$  and  $\alpha, \bar{\alpha}$  the arrows of  $Q$  starting at  $i$ . Then there is in  $\text{mod } \bar{\Lambda}$  a short exact sequence*

$$0 \rightarrow S_i \rightarrow P_i \xrightarrow{\pi_3} P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))} \xrightarrow{\pi_2} P_{t(\alpha)} \oplus P_{t(\bar{\alpha})} \xrightarrow{\pi_1} P_i \rightarrow S_i \rightarrow 0,$$

which give rise to a minimal projective resolution of  $S_i$  in  $\text{mod } \bar{\Lambda}$ . In particular,  $S_i$  is a periodic module of period 4.

*Proof.* If  $i$  is not a border vertex, the claim follows by arguments as in the proof of Proposition 7.1. Therefore, assume that  $i \in \partial(Q, f)$ . In this case, we have  $\alpha = f(\alpha)$ ,  $\bar{\alpha} = g(\alpha)$ , and hence  $c_\alpha = c_{\bar{\alpha}}$ ,  $B_\alpha = B_{\bar{\alpha}}$ . We take for  $S_i$  the simple quotient of  $P_i = e_i \bar{\Lambda}$ , and hence  $\Omega_{\bar{\Lambda}}(S_i)$  is identified with  $\text{rad } P_i = \alpha \bar{\Lambda} + \bar{\alpha} \bar{\Lambda}$ . We define, as in the proof of Proposition 7.1, the homomorphism of right  $\bar{\Lambda}$ -modules

$$\pi_1 : P_{t(\alpha)} \oplus P_{t(\bar{\alpha})} \rightarrow P_i$$

by  $\pi_1(x, y) = \alpha x + \bar{\alpha} y$  for  $x \in P_{t(\alpha)}$ ,  $y \in P_{t(\bar{\alpha})}$ , and show that it induces a projective cover of  $\text{rad } P_i = \Omega_{\bar{\Lambda}}(S_i)$  in  $\text{mod } \bar{\Lambda}$ . In particular, we obtain that  $\Omega_{\bar{\Lambda}}^2(S_i) = \text{Ker } \pi_1$ .

Consider now the following elements in  $P_{t(\alpha)} \oplus P_{t(\bar{\alpha})} = P_i \oplus P_{t(\bar{\alpha})}$

$$\bar{\varphi} = (f(\alpha), -c_{\bar{\alpha}} A'_{\bar{\alpha}} - b_i B'_{\bar{\alpha}}) \quad \text{and} \quad \bar{\psi} = (-c_{\alpha} A'_{\alpha} - b_i A_{\alpha}, f(\bar{\alpha})),$$

where  $B'_{\bar{\alpha}}$  is the subpath of  $B_{\bar{\alpha}}$  from  $t(\bar{\alpha})$  to  $i$  of length  $m_{\bar{\alpha}} n_{\bar{\alpha}} - 1 = m_{\alpha} n_{\alpha} - 1$  such that  $\bar{\alpha} B'_{\bar{\alpha}} = B_{\bar{\alpha}}$ . Then we have the equalities

$$\begin{aligned} \pi_1(\bar{\varphi}) &= \alpha^2 - c_{\bar{\alpha}} \bar{\alpha} A'_{\bar{\alpha}} - b_i \bar{\alpha} B'_{\bar{\alpha}} = \alpha^2 - c_{\bar{\alpha}} A_{\bar{\alpha}} - b_i B_{\bar{\alpha}} = 0, \\ \pi_1(\bar{\psi}) &= -c_{\alpha} \alpha A'_{\alpha} - b_i \alpha A_{\alpha} + \bar{\alpha} f(\bar{\alpha}) = -c_{\alpha} A_{\alpha} + \bar{\alpha} f(\bar{\alpha}) = 0, \end{aligned}$$

because  $\alpha A_{\alpha} = \alpha^2 A'_{\alpha} = 0$  due to  $\alpha^2 g(\alpha) = \alpha f(\alpha) g(f(\alpha)) = 0$ , and hence  $\bar{\varphi}, \bar{\psi}$  belong to  $\text{Ker } \pi_1 = \Omega_{\bar{\Lambda}}^2(S_i)$ . We have also the equalities

$$\begin{aligned} \bar{\varphi} f^2(\alpha) &= \bar{\varphi} \alpha = (\alpha^2, -c_{\bar{\alpha}} A'_{\bar{\alpha}} \alpha - b_i B'_{\bar{\alpha}} \alpha) = (\alpha^2, -c_{\bar{\alpha}} B'_{\bar{\alpha}}), \\ \bar{\psi} f^2(\bar{\alpha}) &= (-c_{\alpha} A'_{\alpha} f^2(\bar{\alpha}) - b_i A_{\alpha} f^2(\bar{\alpha}), f(\bar{\alpha}) f^2(\bar{\alpha})) = (-c_{\alpha} A_{g(\alpha)} - b_i B_{\alpha}, f(\bar{\alpha}) f^2(\bar{\alpha})), \end{aligned}$$

because  $B'_{\bar{\alpha}} \alpha = A_{\bar{f}(\bar{\alpha})} \alpha = c_{\bar{f}(\bar{\alpha})}^{-1} f(\bar{\alpha}) f^2(\bar{\alpha}) \alpha = 0$  due to the equality  $\alpha = g(f^2(\bar{\alpha}))$ . Moreover, we have the equalities  $c_{\alpha} = c_{\bar{\alpha}}$ ,  $A_{g(\alpha)} = A_{\bar{\alpha}}$ ,  $B_{\alpha} = B_{\bar{\alpha}}$ ,  $c_{\bar{\alpha}} = c_{g(\bar{\alpha})}$ ,  $B'_{\bar{\alpha}} = A_{g(\bar{\alpha})}$ , and  $g(\bar{\alpha}) = \bar{f}(\bar{\alpha})$ . Hence we conclude that  $\bar{\varphi} f^2(\alpha) = -\bar{\psi} f^2(\bar{\alpha})$ . Recall also that  $(Q, f)$  is not the tetrahedral triangulation quiver. Then, as in the proof of Proposition 7.1, we conclude that  $\dim_K(\bar{\varphi} \bar{\Lambda} \cap \bar{\psi} \bar{\Lambda}) = 3$ ,  $\bar{\varphi} \bar{\Lambda} + \bar{\psi} \bar{\Lambda} = \Omega_{\bar{\Lambda}}^2(S_i)$ , and the homomorphism of right  $\bar{\Lambda}$ -modules

$$\pi_2 : P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))} \rightarrow P_{t(\alpha)} \oplus P_{t(\bar{\alpha})}$$

given by  $\pi_2(u, v) = \bar{\varphi} u + \bar{\psi} v$  for  $u \in P_{t(f(\alpha))}$ , and  $v \in P_{t(f(\bar{\alpha}))}$  induces a projective cover  $\Omega_{\bar{\Lambda}}^2(S_i)$  in  $\text{mod } \bar{\Lambda}$ . In particular, we obtain that  $\Omega_{\bar{\Lambda}}^3(S_i) = \text{Ker } \pi_2$ . Further, since  $\bar{\varphi} f^2(\alpha) = -\bar{\psi} f^2(\bar{\alpha})$ , the element

$$\bar{\theta} = (f^2(\alpha), f^2(\bar{\alpha}))$$

of  $P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))} = P_{s(f^2(\alpha))} \oplus P_{s(f^2(\bar{\alpha}))}$  lies in  $\text{Ker } \pi_2 = \Omega_{\bar{\Lambda}}^3(S_i)$ . We may then consider the homomorphism of right  $\bar{\Lambda}$ -modules

$$\pi_3 : P_i \rightarrow P_{t(f(\alpha))} \oplus P_{t(f(\bar{\alpha}))}$$

given by  $\pi_3(z) = \bar{\theta} z$  for  $z \in P_i$ . Applying arguments as in the final part of the proof of Proposition 7.1, we conclude that  $\text{Ker } \pi_3 = S_i$  and that  $\pi_3$  induces a projective cover of  $\Omega_{\bar{\Lambda}}^3(S_i)$  in  $\text{mod } \bar{\Lambda}$ . Hence  $\Omega_{\bar{\Lambda}}^4(S_i) = \text{Ker } \pi_3 = S_i$ . Moreover, we have  $\Omega_{\bar{\Lambda}}^j(S_i) \not\cong S_i$  for any  $j \in \{1, 2, 3\}$ . This finishes the proof.  $\square$

We recall now the notation for the first few steps of a minimal projective resolution of  $\bar{\Lambda}$  in  $\text{mod } \bar{\Lambda}^e$

$$\mathbb{P}_3 \xrightarrow{S} \mathbb{P}_2 \xrightarrow{R} \mathbb{P}_1 \xrightarrow{d} \mathbb{P}_0 \xrightarrow{d_0} \bar{\Lambda} \rightarrow 0,$$



where

$$\begin{aligned}\mathbb{P}_0 &= \bigoplus_{i \in Q_0} P(i, i) = \bigoplus_{i \in Q_0} \bar{\Lambda} e_i \otimes e_i \bar{\Lambda}, \\ \mathbb{P}_1 &= \bigoplus_{\alpha \in Q_1} P(s(\alpha), t(\alpha)) = \bigoplus_{\alpha \in Q_1} \bar{\Lambda} e_{s(\alpha)} \otimes e_{t(\alpha)} \bar{\Lambda},\end{aligned}$$

the homomorphism  $d_0 : \mathbb{P}_0 \rightarrow \bar{\Lambda}$  in  $\text{mod } \bar{\Lambda}^e$  is defined by  $d_0(e_i \otimes e_i) = e_i$  for all  $i \in Q_0$ , and the homomorphism  $d : \mathbb{P}_1 \rightarrow \mathbb{P}_0$  in  $\text{mod } \bar{\Lambda}^e$  is defined by

$$d(e_{s(\alpha)} \otimes e_{t(\alpha)}) = \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha$$

for any arrow  $\alpha$  in  $Q_1$ . In particular, we have  $\Omega_{\bar{\Lambda}^e}^1(\bar{\Lambda}) = \text{Ker } d_0$  and  $\Omega_{\bar{\Lambda}^e}^2(\bar{\Lambda}) = \text{Ker } d$ . It follows from Propositions 3.1 and 9.1 that  $\mathbb{P}_2$  is of the form

$$\mathbb{P}_2 = \bigoplus_{\alpha \in Q_1} P(s(\alpha), t(f(\alpha))) = \bigoplus_{\alpha \in Q_1} \bar{\Lambda} e_{s(\alpha)} \otimes e_{t(f(\alpha))} \bar{\Lambda}.$$

For each arrow  $\alpha$  in  $Q_1$ , we define the element  $\bar{\mu}_\alpha = e_{s(\alpha)} \bar{\mu}_\alpha e_{t(f(\alpha))}$  as follows

$$\begin{aligned}\bar{\mu}_\alpha &= \alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}} && \text{if } \alpha \text{ is not a border loop,} \\ \bar{\mu}_\alpha &= \alpha^2 - c_{\bar{\alpha}} A_{\bar{\alpha}} - b_i B_{\bar{\alpha}} && \text{if } \alpha \text{ is a border loop.}\end{aligned}$$

Then we define the homomorphism  $R : \mathbb{P}_2 \rightarrow \mathbb{P}_1$  in  $\text{mod } \bar{\Lambda}^e$  by

$$R(e_{s(\alpha)} \otimes e_{t(f(\alpha))}) = \varrho(\bar{\mu}_\alpha)$$

for any arrow  $\alpha$  in  $Q_1$ , where  $\varrho : KQ \rightarrow \mathbb{P}_1$  is the  $K$ -linear homomorphism defined in Section 3. It follows from Lemma 3.4 that  $\text{Im } R \subseteq \text{Ker } d$ .

**Lemma 9.2.** *The homomorphism  $R : \mathbb{P}_2 \rightarrow \mathbb{P}_1$  induces a projective cover  $\Omega_{\bar{\Lambda}^e}^2(\bar{\Lambda})$  in  $\text{mod } \bar{\Lambda}^e$ . In particular, we have  $\Omega_{\bar{\Lambda}^e}^3(\bar{\Lambda}) = \text{Ker } R$ .*

*Proof.* This follows by the arguments in the proof of Lemma 7.2.  $\square$

By Propositions 3.1 and 9.1 the module  $\mathbb{P}_3$  is of the form

$$\mathbb{P}_3 = \bigoplus_{i \in Q_0} P(i, i) = \bigoplus_{i \in Q_0} \bar{\Lambda} e_i \otimes e_i \bar{\Lambda}.$$

For each vertex  $i \in Q_0$ , we consider the element in  $\mathbb{P}_2$

$$\psi_i = (e_i \otimes e_{t(f(\alpha))}) f^2(\alpha) + (e_i \otimes e_{t(f(\bar{\alpha}))}) f^2(\bar{\alpha}) - \alpha(e_{t(\alpha)} \otimes e_i) - \bar{\alpha}(e_{t(\bar{\alpha})} \otimes e_i).$$

Moreover, for each vertex  $i \in \partial(Q, f)$  and the border loop  $\alpha$  at  $i$ , we consider the elements in  $\mathbb{P}_2$

$$\begin{aligned}\psi_i^{(1)} &= (b_i c_\alpha^{-1})(\alpha \otimes \alpha + e_i \otimes \alpha^2), \\ \psi_i^{(2)} &= (b_i c_\alpha^{-1})^2(\alpha \otimes \alpha^2 + e_i \otimes \alpha^3), \\ \psi_i^{(3)} &= (b_i c_\alpha^{-1})^3(\alpha \otimes \alpha^3).\end{aligned}$$

Then, for each vertex  $i \in Q$ , we define the element  $\bar{\psi}_i$  in  $\mathbb{P}_2$  as follows

$$\begin{aligned}\bar{\psi}_i &= \psi_i && \text{if } i \notin \partial(Q, f), \\ \bar{\psi}_i &= \psi_i + \psi_i^{(1)} + \psi_i^{(2)} + \psi_i^{(3)} && \text{if } i \in \partial(Q, f).\end{aligned}$$

We define the homomorphism  $S : \mathbb{P}_3 \rightarrow \mathbb{P}_2$  in  $\text{mod } \bar{\Lambda}^e$  by

$$S(e_i \otimes e_i) = \bar{\psi}_i$$

for any vertex  $i \in Q_0$ . Then we have the following analogue of Lemma 7.3.

**Proposition 9.3.** *The homomorphism  $S : \mathbb{P}_3 \rightarrow \mathbb{P}_2$  induces a projective cover of  $\Omega_{\bar{\Lambda}^e}^3(\bar{\Lambda})$  in  $\text{mod } \bar{\Lambda}^e$ . In particular, we have  $\Omega_{\bar{\Lambda}^e}^4(\bar{\Lambda}) = \text{Ker } S$ .*

*Proof.* We will prove in several steps that  $R(\bar{\psi}_i) = 0$  for any vertex  $i \in Q_0$ . Fix a vertex  $i \in Q_0$ . If  $i \notin \partial(Q, f)$  then  $R(\bar{\psi}_i) = R(\psi_i) = 0$  by the identities as in the proof of Lemma 7.3. Assume that  $i \in \partial(Q, f)$ , and let  $\alpha \in Q_1$  be the border loop at  $i$ . Then we have  $\alpha = f(\alpha)$ ,  $\bar{\alpha} = g(\alpha)$ ,  $\alpha = f^2(\alpha) = g^{n_{\bar{\alpha}}-1}(\bar{\alpha})$ ,  $f^2(\bar{\alpha}) = g^{n_{\alpha}-1}(\alpha)$ ,  $c_{\alpha} = c_{\bar{\alpha}}$ ,  $B_{\alpha} = B_{\bar{\alpha}}$ . We abbreviate  $c = c_{\alpha} = c_{\bar{\alpha}}$  and  $b = b_i$ . We have in  $\mathbb{P}_1$  the following equalities describing  $R(\psi_i)$

$$\begin{aligned} R(\psi_i) &= \varrho(\bar{\mu}_{\alpha})f^2(\alpha) + \varrho(\bar{\mu}_{\bar{\alpha}})f^2(\bar{\alpha}) - \alpha\varrho(\bar{\mu}_{f(\alpha)}) - \bar{\alpha}\varrho(\bar{\mu}_{f(\bar{\alpha})}) \\ &= \varrho(\bar{\mu}_{\alpha})\alpha + \varrho(\bar{\mu}_{\bar{\alpha}})f^2(\bar{\alpha}) - \alpha\varrho(\mu_{\alpha}) - \bar{\alpha}\varrho(\bar{\mu}_{f(\bar{\alpha})}) \\ &= (\varrho(\alpha^2) - c\varrho(A_{\bar{\alpha}}) - b\varrho(B_{\bar{\alpha}}))\alpha + (\varrho(\bar{\alpha}f(\bar{\alpha})) - c\varrho(A_{\alpha}))f^2(\bar{\alpha}) \\ &\quad - \alpha(\varrho(\alpha^2) - c\varrho(A_{\bar{\alpha}}) - b\varrho(B_{\bar{\alpha}})) - \bar{\alpha}(\varrho(f(\bar{\alpha})f^2(\bar{\alpha})) - c\varrho(A_{g(\bar{\alpha})})) \\ &= e_i \otimes \alpha^2 + \alpha \otimes \alpha + e_i \otimes f(\bar{\alpha})f^2(\bar{\alpha}) + \bar{\alpha} \otimes f^2(\bar{\alpha}) \\ &\quad - \alpha \otimes \alpha - \alpha^2 \otimes e_i - \bar{\alpha} \otimes f^2(\bar{\alpha}) - \bar{\alpha}f(\bar{\alpha}) \otimes e_i \\ &\quad - c\varrho(A_{\bar{\alpha}})\alpha - b\varrho(B_{\bar{\alpha}})\alpha - c\varrho(A_{\alpha})f^2(\bar{\alpha}) \\ &\quad + c\alpha\varrho(A_{\bar{\alpha}}) + b\alpha\varrho(B_{\bar{\alpha}}) + c\bar{\alpha}\varrho(A_{g(\bar{\alpha})}) \\ &= e_i \otimes cA_{\bar{\alpha}} + e_i \otimes bB_{\bar{\alpha}} + e_i \otimes cA_{g(\bar{\alpha})} - cA_{\bar{\alpha}} \otimes e_i - bB_{\bar{\alpha}} \otimes e_i - cA_{\alpha} \otimes e_i \\ &\quad - c\varrho(A_{\bar{\alpha}})\alpha - b\varrho(B_{\bar{\alpha}})\alpha - c\varrho(A_{\alpha})f^2(\bar{\alpha}) \\ &\quad + c\alpha\varrho(A_{\bar{\alpha}}) + b\alpha\varrho(B_{\bar{\alpha}}) + c\bar{\alpha}\varrho(A_{g(\bar{\alpha})}) \\ &= c(e_i \otimes A_{g(\alpha)} + \alpha\varrho(A_{g(\alpha)}) - \varrho(A_{\alpha})f^2(\bar{\alpha}) - A_{\alpha} \otimes e_i) \\ &\quad + c(e_i \otimes A_{g(\bar{\alpha})} + \bar{\alpha}\varrho(A_{g(\bar{\alpha})}) - \varrho(A_{\bar{\alpha}})\alpha - A_{\bar{\alpha}} \otimes e_i) \\ &\quad + b(e_i \otimes B_{\bar{\alpha}} - B_{\bar{\alpha}} \otimes e_i + \alpha\varrho(B_{\bar{\alpha}}) - \varrho(B_{\bar{\alpha}})\alpha) \\ &= b(e_i \otimes B_{\bar{\alpha}} + B_{\bar{\alpha}} \otimes e_i + \alpha\varrho(B_{\bar{\alpha}}) + \varrho(B_{\bar{\alpha}})\alpha), \end{aligned}$$

since  $K$  has characteristic 2. We note that, if  $b = b_i = 0$ , then then  $\bar{\psi}_i = \psi_i$  and  $R(\bar{\psi}_i) = R(\psi_i) = 0$ . Hence we may assume that  $b \neq 0$ .

In order to calculate  $R(\psi_i^{(1)})$ ,  $R(\psi_i^{(2)})$ ,  $R(\psi_i^{(3)})$ , we use the following identities in  $\mathbb{P}_1$

- (1)  $\alpha\varrho(A_{\bar{\alpha}})\alpha = B_{\bar{\alpha}} \otimes e_i + \alpha\varrho(B_{\bar{\alpha}})$ ,
- (2)  $\varrho(B_{\bar{\alpha}})\alpha + \varrho(A_{\bar{\alpha}})\alpha^2 = A_{\bar{\alpha}} \otimes \alpha$ ,

which follow from the equalities  $\bar{\alpha} = g(\alpha)$  and  $g^{n_{\bar{\alpha}}-1}(\bar{\alpha}) = g^{n_{\alpha}}(\alpha) = \alpha$ .

We have the following equalities in  $\mathbb{P}_1$  describing  $R(\psi_i^{(1)})$

$$\begin{aligned} R(\psi_i^{(1)}) &= bc^{-1}(R(\alpha \otimes \alpha) + R(e_i \otimes \alpha^2)) \\ &= bc^{-1}(\alpha R(e_i \otimes e_i)\alpha + R(e_i \otimes e_i)\alpha^2) = bc^{-1}(\alpha\varrho(\mu_{\alpha})\alpha + \varrho(\mu_{\alpha})\alpha^2) \\ &= bc^{-1}(\alpha\varrho(\alpha^2 + cA_{\bar{\alpha}} + bB_{\bar{\alpha}})\alpha + \varrho(\alpha^2 + cA_{\bar{\alpha}} + bB_{\bar{\alpha}})\alpha^2) \\ &= bc^{-1}(\alpha \otimes \alpha^2 + \alpha^2 \otimes \alpha + c\alpha\varrho(A_{\bar{\alpha}})\alpha + b\alpha\varrho(B_{\bar{\alpha}})\alpha \\ &\quad + e_i \otimes \alpha^3 + \alpha \otimes \alpha^2 + c\varrho(A_{\bar{\alpha}})\alpha^2 + b\varrho(B_{\bar{\alpha}})\alpha^2) \\ &= b(c^{-1}(\alpha^2 \otimes \alpha) + B_{\bar{\alpha}} \otimes e_i + \alpha\varrho(B_{\bar{\alpha}}) + bc^{-1}\alpha\varrho(B_{\bar{\alpha}})\alpha \\ &\quad + e_i \otimes B_{\bar{\alpha}} + \varrho(A_{\bar{\alpha}})\alpha^2 + bc^{-1}\varrho(B_{\bar{\alpha}})\alpha^2). \end{aligned}$$

Then we obtain the equalities

$$\begin{aligned}
 R(\psi_i) + R(\psi_i^{(1)}) &= b\varrho(B_{\bar{\alpha}})\alpha + bc^{-1}(\alpha^2 \otimes \alpha) + b^2c^{-1}\alpha\varrho(B_{\bar{\alpha}})\alpha + b\varrho(A_{\bar{\alpha}})\alpha^2 + b^2c^{-1}\varrho(B_{\bar{\alpha}})\alpha^2 \\
 &= b(\varrho(B_{\bar{\alpha}})\alpha + \varrho(A_{\bar{\alpha}})\alpha^2) + bc^{-1}(cA_{\bar{\alpha}} \otimes \alpha + bB_{\bar{\alpha}} \otimes \alpha) \\
 &\quad + b^2c^{-1}(\alpha\varrho(B_{\bar{\alpha}})\alpha + \varrho(B_{\bar{\alpha}})\alpha^2) \\
 &= bA_{\bar{\alpha}} \otimes \alpha + bA_{\bar{\alpha}} \otimes \alpha + b^2c^{-1}(B_{\bar{\alpha}} \otimes \alpha + \alpha\varrho(B_{\bar{\alpha}})\alpha + \varrho(B_{\bar{\alpha}})\alpha^2) \\
 &= b^2c^{-1}(B_{\bar{\alpha}} \otimes \alpha + \alpha\varrho(B_{\bar{\alpha}})\alpha + \varrho(B_{\bar{\alpha}})\alpha^2) \\
 &= b^2c^{-1}(\alpha\varrho(A_{\bar{\alpha}})\alpha^2 + A_{\bar{\alpha}} \otimes \alpha^2 + \varrho(A_{\bar{\alpha}})\alpha^3) \\
 &= b^2c^{-1}(\alpha\varrho(A_{\bar{\alpha}})\alpha^2 + A_{\bar{\alpha}} \otimes \alpha^2 + A''_{\bar{\alpha}}\alpha^3),
 \end{aligned}$$

where  $A''_{\bar{\alpha}}$  is the subpath of  $A_{\bar{\alpha}}$  such that  $A''_{\bar{\alpha}}g^{n_{\bar{\alpha}}-2}(\bar{\alpha}) = A_{\bar{\alpha}}$ .

We have the following expressions of  $R(\psi_i^{(2)})$

$$\begin{aligned}
 R(\psi_i^{(2)}) &= (bc^{-1})^2(R(e_i \otimes \alpha^3) + R(\alpha \otimes \alpha^2)) \\
 &= (bc^{-1})^2(\varrho(\mu_{\alpha})\alpha^3 + \alpha\varrho(\mu_{\alpha})\alpha^2) \\
 &= (bc^{-1})^2((\varrho(\alpha^2) + c\varrho(A_{\bar{\alpha}}) + b\varrho(B_{\bar{\alpha}}))\alpha^3 + \alpha(\varrho(\alpha^2) + c\varrho(A_{\bar{\alpha}}) + b\varrho(B_{\bar{\alpha}}))\alpha^2) \\
 &= (bc^{-1})^2(\alpha \otimes \alpha^3 + c\varrho(A_{\bar{\alpha}})\alpha^3 + b\varrho(B_{\bar{\alpha}})\alpha^3 + \alpha \otimes \alpha^3 + \alpha^2 \otimes \alpha^2 \\
 &\quad + c\alpha\varrho(A_{\bar{\alpha}})\alpha^2 + b\alpha\varrho(B_{\bar{\alpha}})\alpha^2) \\
 &= (bc^{-1})^2(\alpha^2 \otimes \alpha^2 + c\varrho(A_{\bar{\alpha}})\alpha^3 + c\alpha\varrho(A_{\bar{\alpha}})\alpha^2 + b\varrho(B_{\bar{\alpha}})\alpha^3 + b\alpha\varrho(B_{\bar{\alpha}})\alpha^2).
 \end{aligned}$$

Moreover, we have  $\alpha^2 \otimes \alpha^2 = cA_{\bar{\alpha}} \otimes \alpha^2 + bB_{\bar{\alpha}} \otimes \alpha^2$ , and  $A''_{\bar{\alpha}} \otimes \alpha^3 = \varrho(A_{\bar{\alpha}})\alpha^3$ . Then we obtain the equalities

$$\begin{aligned}
 R(\psi_i) + R(\psi_i^{(1)}) + R(\psi_i^{(2)}) &= b^3c^{-2}(\varrho(B_{\bar{\alpha}})\alpha^3 + \alpha\varrho(B_{\bar{\alpha}})\alpha^2 + B_{\bar{\alpha}} \otimes \alpha^2) \\
 &= b^3c^{-2}(A_{\bar{\alpha}} \otimes \alpha^3 + \alpha\varrho(A_{\bar{\alpha}})\alpha^3 + \alpha A_{\bar{\alpha}} \otimes \alpha^2 + B_{\bar{\alpha}} \otimes \alpha^2) \\
 &= b^3c^{-2}(A_{\bar{\alpha}} \otimes \alpha^3 + A_{\alpha} \otimes \alpha^3 + B_{\alpha} \otimes \alpha^2 + B_{\bar{\alpha}} \otimes \alpha^2) \\
 &= b^3c^{-2}(A_{\bar{\alpha}} \otimes \alpha^3 + A_{\alpha} \otimes \alpha^3),
 \end{aligned}$$

because  $B_{\alpha} = B_{\bar{\alpha}}$ . We have also the following equalities

$$\begin{aligned}
 R(\psi_i^{(3)}) &= (bc^{-1})^3R(\alpha \otimes \alpha^3) = (bc^{-1})^3\alpha\varrho(\mu_{\alpha})\alpha^3 \\
 &= (bc^{-1})^3(\alpha(\varrho(\alpha^2) + c\varrho(A_{\bar{\alpha}}) + b\varrho(B_{\bar{\alpha}}))\alpha^3) \\
 &= (bc^{-1})^3(\alpha^2 \otimes \alpha^3 + cA_{\bar{\alpha}} \otimes \alpha^3 + bB_{\bar{\alpha}} \otimes \alpha^3) \\
 &= (bc^{-1})^3(cA_{\bar{\alpha}} \otimes \alpha^3 + bB_{\bar{\alpha}} \otimes \alpha^3 + cA_{\alpha} \otimes \alpha^3 + bB_{\alpha} \otimes \alpha^3) \\
 &= b^3c^{-2}(A_{\bar{\alpha}} \otimes \alpha^3 + A_{\alpha} \otimes \alpha^3).
 \end{aligned}$$

Summing up, we obtain the required vanishing equality

$$R(\bar{\psi}_i) = R(\psi_i) + R(\psi_i^{(1)}) + R(\psi_i^{(2)}) + R(\psi_i^{(3)}) = 0.$$

Therefore we have  $\text{Im } S \subseteq \text{Ker } R$ . Further, it follows from the definition that the generators  $\bar{\psi}_i$ ,  $i \in Q_0$ , of the image of  $S$  are elements of  $\text{rad } \mathbb{P}_2$  which are linearly independent in  $\text{rad } \mathbb{P}_2 / \text{rad}^2 \mathbb{P}_2$ . Then we conclude from the form of  $\mathbb{P}_2$  that these elements form a minimal set of generators of  $\text{Ker } R = \Omega_{\bar{\Lambda}^e}^3(\bar{\Lambda})$ . Hence  $S : \mathbb{P}_3 \rightarrow \Omega_{\bar{\Lambda}^e}^3(\bar{\Lambda})$  is a projective cover of  $\Omega_{\bar{\Lambda}^e}^3(\bar{\Lambda})$  in  $\text{mod } \bar{\Lambda}^e$ .  $\square$

**Theorem 9.4.** *There is an isomorphism  $\Omega_{\bar{\Lambda}^e}^4(\bar{\Lambda}) \cong \bar{\Lambda}$  in  $\text{mod } \bar{\Lambda}^e$ . In particular,  $\bar{\Lambda}$  is a periodic algebra of period 4.*

*Proof.* We proceed as in the proof of Theorem 7.4, and use [33, part (3) on the pages 119 and 120]. In particular, we fix some basis  $\mathcal{B} = \bigcup_{i \in Q_0} \mathcal{B}_i$  of  $\bar{\Lambda}$  over  $K$ , the socle elements  $\omega_i$  of  $e_i \bar{\Lambda}$ , and consider the symmetrizing form  $(-, -) : \bar{\Lambda} \times \bar{\Lambda} \rightarrow K$  such that, for any two elements  $x \in \mathcal{B}_i$  and  $y \in \mathcal{B}$ , we have

$$(x, y) = \text{the coefficient of } \omega_i \text{ in } xy,$$

when  $xy$  is expressed as a linear combination of the elements of  $e_i \mathcal{B} = \mathcal{B}_i$  over  $K$ . Moreover, we consider the dual basis  $\mathcal{B}^*$  of  $\mathcal{B}$  with respect to  $(-, -)$ . Then, for each vertex  $i \in Q_0$ , we define the element of  $\mathbb{P}_3$

$$\xi_i = \sum_{b \in \mathcal{B}_i} b \otimes b^*.$$

Then we conclude as in the proof of Theorem 7.4, that there is a monomorphism in  $\text{mod } \bar{\Lambda}^e$

$$\theta : \bar{\Lambda} \rightarrow \mathbb{P}_3$$

such that  $\theta(e_i) = \xi_i$  for any  $i \in Q_0$ . It follows also from Theorem 2.4 and Proposition 9.1 that  $\Omega_{\bar{\Lambda}^e}^4(\bar{\Lambda}) \cong {}_1 \bar{\Lambda}_\sigma$  in  $\text{mod } \bar{\Lambda}^e$  for some  $K$ -algebra automorphism  $\sigma$  of  $\bar{\Lambda}$ . Hence, we conclude that  $\dim_K \bar{\Lambda} = \dim_K \Omega_{\bar{\Lambda}^e}^4(\bar{\Lambda})$ . Moreover, by Proposition 9.3, we have  $\Omega_{\bar{\Lambda}^e}^4(\bar{\Lambda}) = \text{Ker } S$ . Therefore, in order to show that  $\theta$  induces an isomorphism  $\theta : \bar{\Lambda} \rightarrow \Omega_{\bar{\Lambda}^e}^4(\bar{\Lambda})$  in  $\text{mod } \bar{\Lambda}^e$ , it remains to prove that  $S(\xi_i) = 0$  for any  $i \in Q_0$ . Since  $K$  has characteristic 2, applying [33, part (3) on the pages 119 and 120], we conclude that for any vertex  $i \in \partial(Q, f)$  and the border loop  $\alpha$  at  $i$ , the following equalities hold in  $\mathbb{P}_2$

$$\begin{aligned} \sum_{b \in \mathcal{B}_i, e_i} b(\alpha \otimes \alpha + e_i \otimes \alpha^2) b^* &= 0, \\ \sum_{b \in \mathcal{B}_i, e_i} b(\alpha \otimes \alpha^2 + e_i \otimes \alpha^3) b^* &= 0, \\ \sum_{b \in \mathcal{B}_i, e_i} b(\alpha \otimes \alpha^3) b^* &= 0, \end{aligned}$$

because  $\alpha^4 = 0$ . Then, for any  $i \in Q_0$ , we obtain the equalities

$$\begin{aligned} S(\xi_i) &= \sum_{b \in \mathcal{B}_i} S(b \otimes b^*) = \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} S(b e_j \otimes e_j b^*) \\ &= \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} b S(e_j \otimes e_j) b^* = \sum_{b \in \mathcal{B}_i} \sum_{j \in Q_0} b \bar{\psi}_j b^* = 0. \end{aligned}$$

This completes the proof that  $\bar{\Lambda}$  is a periodic algebra of period 4. □

## 10. The representation type

In this section we discuss the representation type of weighted surface algebras and their socle deformations. In particular, we complete the proofs of Theorems 1.1 and 1.4.

Let  $A = KQ/I$  be a string algebra. For a given arrow  $\alpha \in Q_1$ , we denote by  $\alpha^{-1}$  the formal inverse of  $\alpha$  and set  $s(\alpha^{-1}) = t(\alpha)$  and  $t(\alpha^{-1}) = s(\alpha)$ . By a *walk* in  $(Q, I)$  we mean a sequence  $w = \alpha_1 \dots \alpha_n$ , where each  $\alpha_i$  is an arrow or the inverse of an arrow in  $Q$ , satisfying the following conditions:

- (i)  $t(\alpha_i) = s(\alpha_{i+1})$  for any  $i \in \{1, \dots, n-1\}$ ;
- (ii)  $\alpha_{i+1} \neq \alpha_i^{-1}$  for any  $i \in \{1, \dots, n-1\}$ ;
- (iii)  $w$  does not contain a subpath  $v$  such that  $v$  or  $v^{-1}$  belongs to  $I$ .

Moreover,  $w$  is said to be a *bipartite walk* if, for any  $i \in \{1, \dots, n-1\}$ , exactly one of  $\alpha_i$  and  $\alpha_{i+1}$  is an arrow. A walk  $w = \alpha_1 \dots \alpha_n$  in  $(Q, I)$  with  $s(\alpha_1) = t(\alpha_n)$  is called a *closed walk*. Following [73, 77], we say that a closed walk  $w$  in  $(Q, I)$  is a *primitive walk* if the following conditions are satisfied:

- (i)  $w^m$  is a walk in  $(Q, I)$  for any positive integer  $m$ ;
- (ii)  $w \neq v^r$  for any closed walk  $v$  in  $(Q, I)$  and positive integer  $r$ .

It is known that a string algebra  $A = KQ/I$  is representation-infinite if and only if  $(Q, I)$  admits a primitive walk (see [73, Theorem 1]). Moreover, if  $A = KQ/I$  is a representation-infinite string algebra then the primitive walks in  $(Q, I)$  create one-parameter families of stable tubes of rank 1 in the Auslander-Reiten quiver  $\Gamma_A$  (see [15, 77]).

We need the following combinatorial lemma.

**Lemma 10.1.** *Let  $A = KQ/I$  be a string algebra with  $Q$  a 2-regular quiver. Then, for any arrow  $\alpha \in Q_1$ , there is a bipartite primitive walk  $w(\alpha)$  containing the arrow  $\alpha$ .*

*Proof.* Since  $Q$  is a 2-regular quiver, we have two involutions  $\bar{\cdot} : Q_1 \rightarrow Q_1$  and  $^* : Q_1 \rightarrow Q_1$  of the set  $Q_1$  of arrows of  $Q$ . The first involution assigns to each arrow  $\alpha \in Q_1$  the arrow  $\bar{\alpha}$  with  $s(\alpha) = s(\bar{\alpha}^*)$  and  $\alpha \neq \bar{\alpha}$ . The second involution assigns to each arrow  $\alpha \in Q_1$  the arrow  $\alpha^*$  with  $t(\alpha) = t(\alpha^*)$  and  $\alpha \neq \alpha^*$ . Consider the automorphisms  $h : Q_1 \rightarrow Q_1$  such that  $h(\alpha) = \bar{\alpha}^*$  for any arrow  $\alpha \in Q_1$ . Clearly,  $h$  has finite order. In particular, for a given arrow  $\alpha \in Q_1$ , there exists a minimal positive integer  $r$  such that  $h^r(\alpha) = \alpha$ . Then the required bipartite primitive walk  $w(\alpha)$  is of the form

$$\alpha(\alpha^*)^{-1}h(\alpha)(h(\alpha)^*)^{-1} \dots h^{r-1}(\alpha)(h^{r-1}(\alpha)^*)^{-1}.$$

□

Let  $(Q, f)$  be a triangulation quiver,  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  a weight function, and  $c_\bullet : O(g) \rightarrow K^*$  a parameter function. We consider the bound quiver algebra

$$\Gamma(Q, f, m_\bullet, c_\bullet) = KQ/L(Q, f, m_\bullet, c_\bullet),$$

where  $L(Q, f, m_\bullet, c_\bullet)$  is the admissible ideal in the path algebra  $KQ$  of  $Q$  over  $K$  generated by the elements  $\alpha f(\alpha)$  and  $A_\alpha$ , for all arrows  $\alpha \in Q_1$ . Then  $\Gamma(Q, f, m_\bullet, c_\bullet)$  is a string algebra, called the string algebra of the weighted triangulation algebra  $\Lambda(Q, f, m_\bullet, c_\bullet)$ . We note that it is the largest string quotient algebra of  $\Lambda(Q, f, m_\bullet, c_\bullet)$ , with respect to dimension. Observe also that  $\Gamma(Q, f, m_\bullet, c_\bullet)$  is a quotient algebra of the special biserial degeneration algebra  $B(Q, f, m_\bullet, c_\bullet)$  of  $\Lambda(Q, f, m_\bullet, c_\bullet)$ . Moreover, if the border  $\partial(Q, f)$  of  $(Q, f)$  is not empty and  $b_\bullet : \partial(Q, f) \rightarrow K$  is a border function, then  $\Gamma(Q, f, m_\bullet, c_\bullet)$  is a quotient algebra of the socle deformed weighted triangulation algebra  $\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$ .

**Proposition 10.2.** *Let  $(Q, f)$  be a triangulation quiver,  $m_\bullet : O(g) \rightarrow \mathbb{N}^*$  a weight function, and  $c_\bullet : O(g) \rightarrow K^*$  a parameter function. Then the following statements hold:*

- (i)  $\Gamma(Q, f, m_\bullet, c_\bullet)$  is a representation-infinite tame algebra.
- (ii) If there is an arrow  $\alpha \in Q_1$  with  $n_\alpha \geq 4$  or  $m_\alpha \geq 2$ , then  $\Gamma(Q, f, m_\bullet, c_\bullet)$  is of non-polynomial growth.

*Proof.* We write  $\Gamma = \Gamma(Q, f, m_\bullet, c_\bullet)$  and  $L = L(Q, f, m_\bullet, c_\bullet)$ .

(i) Since a string algebra is special biserial, it is tame, by Proposition 2.1. Moreover, since  $Q$  is a 2-regular quiver, it follows from Lemma 10.1 that there is a (bipartite) primitive walk in  $(Q, L)$ , and consequently  $\Gamma$  is representation-infinite.

(ii) Assume that there is an arrow  $\alpha \in Q_1$  such that  $n_\alpha \geq 4$  or  $m_\alpha \geq 2$ . Recall that we assume  $m_\alpha n_\alpha \geq 3$ . Hence, if  $n_\alpha = 1$ , then  $m_\alpha \geq 3$ . Moreover,  $\bar{f}(\alpha) = g(\alpha)$ ,  $m_\alpha = m_{g(\alpha)}$ ,  $n_\alpha = n_{g(\alpha)}$ . Then  $u = g(\alpha)g^2(\alpha) \dots g^{n_\alpha-2}(\alpha)$  is a path of length  $\geq 2$  and is a proper subpath of  $A_{g(\alpha)}$ , and consequently  $u$  does not belong to  $L$ . Observe also that  $g(\bar{\alpha}) = \bar{f}(\bar{\alpha})$  and  $m_{\bar{\alpha}} n_{\bar{\alpha}} \geq 3$ , again by our general assumption. Hence,  $\bar{u} = g(\bar{\alpha}) \dots g^{n_{\bar{\alpha}}-2}(\bar{\alpha})$  is a path of length  $\geq 1$  and is a proper subpath of  $A_{g(\bar{\alpha})}$ , and then  $\bar{u}$  does not belong to  $L$ . Consider the following closed walk in  $(Q, f)$

$$v = uf(\bar{\alpha})^{-1}\bar{u}f(\alpha)^{-1}$$

and observe that it is a primitive walk. Applying Lemma 10.1, we may also consider the bipartite primitive walk  $w = w(g(\alpha))$ . We note that  $w$  is of the form  $w = g(\alpha) \dots f(\alpha)^{-1}$ . In particular, we conclude that, for any prime number  $q$  and positive integers  $r_1, s_1, \dots, r_t, s_t$  with  $\sum_{i=1}^t (r_i + s_i) = q$ , the closed walks in  $(Q, L)$  of the form

$$v_1^{r_1} w_1^{s_1} \dots v_t^{r_t} w_t^{s_t}$$

are primitive walks. Then it follows by the arguments applied in the proof of [70, Lemma 1] that the string algebra  $\Gamma$  is not of polynomial growth.  $\square$

Let  $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet)$  be a weighted surface algebra, and  $(Q, f) = (Q(S, \vec{T}), f)$  its triangulation quiver. It follows from Proposition 5.8 that  $\Lambda$  is a tame algebra. Further, the associated string algebra  $\Gamma = \Gamma(Q, f, m_\bullet, c_\bullet)$  is a quotient algebra of  $\Lambda$  and is representation-infinite, by Proposition 10.2 (i). Hence,  $\Lambda$  is representation-infinite. Applying Lemma 6.6, we conclude that  $\Lambda$  is a tetrahedral algebra if and only if  $n_\alpha = 3$  and  $m_\alpha = 1$  for any arrow  $\alpha \in Q_1$ . Moreover, if this is the case, then  $\Lambda$  is of polynomial growth if and only if  $\Lambda$  is a non-singular tetrahedral algebra, by Propositions 6.3 and 6.4. Assume now that  $\Lambda$  is not a tetrahedral algebra. Then it follows from Proposition 10.2 (ii) that the string quotient algebra  $\Gamma$  of  $\Lambda$  is not of polynomial growth. Hence,  $\Lambda$  is of non-polynomial growth.

Let  $A$  be a basic, indecomposable, symmetric algebra which is socle equivalent to  $\Lambda$ . We may assume that  $A$  is not isomorphic to  $\Lambda$ . Then it follows from Propositions 8.2 and 8.3 that the border  $\partial(Q, f)$  of  $(Q, f)$  is not empty,  $K$  has characteristic 2, and  $A$  is isomorphic to an algebra  $\tilde{\Lambda} = \Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$ , for some border function  $b_\bullet$  of  $(Q, f)$ . In particular, we know that  $\Lambda$  is not a tetrahedral algebra. Then the string algebra  $\Gamma$  of  $\Lambda$  is an algebra of non-polynomial growth and a quotient algebra of  $\tilde{\Lambda}$ . Therefore,  $A$  is a tame algebra of non-polynomial growth.

We mention that in the special case when  $\Lambda$  is the Jacobian algebra of an orientable surface with empty boundary and non-empty collection of punctures, different from a sphere with at most four punctures, the fact that  $\Lambda$  is of non-polynomial growth was proved in [76, Theorem].

## Acknowledgements

We are grateful to S. Ladkani for pointing out to us (September 2013) that there are tame symmetric algebras whose indecomposable non-projective modules are periodic of period dividing 4, coming from cluster theory, which we had not been aware of.

The research was done during the visits of the second named author at the Mathematical Institute in Oxford (March 2014) and the first named author at the Faculty of Mathematics and Computer Sciences in Toruń (October 2014). The results of the paper were partially presented at the conferences “Advances of Representation Theory of Algebras” (Montreal, June 2014), “Cluster Algebras and Geometry” (Münster, March 2016), “Workshop on Brauer Graph Algebras” (Stuttgart, March 2016).

There is an overlap of the arXiv update 1608.00321 of [59] from August 2016 with our paper, which we incidentally submitted the first time early 2016. We think it makes the paper harder to read if we were to cite the details of the overlap and we have therefore decided not to do this.

We thank the referees for the valuable comments and the editor for giving us a chance to prepare a more complete version of our paper.

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