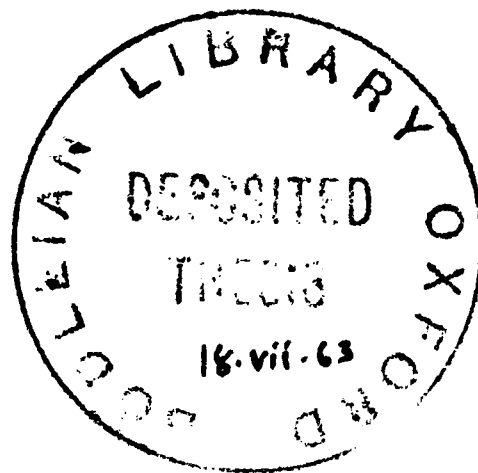


IDENTICAL RELATIONS
IN SIMPLE GROUPS

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CHAPTER 1

INTRODUCTION

An identical relation is an equation of the form

$$w(x_1, \dots, x_n) = 1$$

where w is an element of a free group F , freely generated by a countable set $x_1, \dots, x_n, \dots$ which are regarded as variables.

$w = 1$ is said to hold on a group G if $w(g_1, \dots, g_n) = 1$ for every substitution of elements $g_1, \dots, g_n$ of G for the variables $x_1, \dots, x_n$. If $w = 1$ holds on G , then so does $w^{-1} = 1$, and $w(f_1, \dots, f_n) = 1$, where $f_1, \dots, f_n$ are any elements of F .

If $v(x_1, \dots, x_m) = 1$ also holds on G , then so does the product $w(x_1, \dots, x_n)v(x_{n+1}, \dots, x_{n+m})$. By repeating these processes we

obtain the consequences of a set of identical relations (e.g.,

$[x_1, x_2] = 1$ is a consequence of $x_1^2 = 1$), and a set of identical relations is said to be closed if it contains all its consequences.

Since an identical relation which holds on a given set of groups X will also hold on subgroups, homomorphic images and direct products of groups of X , there is clearly a connection between sets of identical relations and varieties (which are classes of groups closed under the operations of taking subgroups, homomorphic images and

unrestricted direct products). In fact, we have as a special case of a theorem contained in G. Birkhoff's paper, "On the Structure of Abstract Algebras", (1), that there is a one-one correspondence between varieties and closed sets of identical relations. Thus the identical relations holding on a given group G hold on every group in the variety generated by G , and every group satisfying all the identical relations which hold on G belongs to the variety generated by G .

Now, a set of identical relations is said to be finitely based if it is the set of consequences of a finite subset (which may be taken to consist of just one relation; the product of those occurring in the basis). In his paper "Identical Relations in Groups, I" (8), B. H. Neumann considers the question of the existence of a finite basis for the identical relations of a group (and, in particular, a finite group). He shows that a variety of abelian groups has a finite basis for its identical relations (in fact, it is $x_1^e = [x_1, x_2] = 1$, where e is the exponent). There is also a finite basis for the identical relations of any variety of nilpotent groups; this is proved by R. Lyndon, in a paper entitled "Two Notes on Nilpotent Groups", (7). If $\mathcal{O}$ and $\mathcal{L}$ are varieties, then the variety $\mathcal{O} \circ \mathcal{L}$ consists of all those groups with a normal subgroup in $\mathcal{O}$ whose factor group belongs to $\mathcal{L}$. In his paper "Some Remarks on Varieties of Groups" (5), G. Higman extends Lyndon's results to

prove that if $\mathcal{A}$ is nilpotent, and the identical relations of $\mathcal{L}$ have a finite basis, then so do those of $\mathcal{A} \circ \mathcal{L}$.

None of these results requires that the varieties concerned should be generated by finite groups, but, in general, this should prove a more manageable problem, since a theorem in (8) tells us that those identical relations of a finite group which involve a bounded number of variables have a finite basis.

The case in which the generating group is soluble is considered by D. C. Cross in his D. Phil. dissertation (2) (a report on this work is contained in G. Higman's paper "Identical Relations in Finite Groups", (6)). He introduces the concept of a "finite" variety, which we shall call a Cross variety (2.2.10). Since one property of a Cross variety is that any group belonging to it has a finite basis for its identical relations (3.3.4), Neumann's problem for a given finite group G will be solved if we can find a Cross variety containing G . Cross proves that a finite soluble group G generates a Cross variety if it satisfies one of the following conditions:

- (1) The derived group of G is nilpotent;
- (2) All Sylow subgroups of G are elementary abelian;
- (3) G has a normal, elementary abelian, Sylow subgroup, S , and G/S generates a Cross variety. (In this case, G need not be soluble.)

This work has been continued by M. B. Powell (9), who uses similar methods, together with the concept of absolute rank, introduced by Professor Higman (unpublished), to show that any finite soluble group generates a Cross variety.

An important concept in this work is that of a critical group (i.e., one that does not belong to the variety generated by its proper factors (2.2.8)) and, to show that any finite group belongs to a Cross variety, and thus has a finite basis for its identical relations, it would be sufficient to prove:

Hypothesis A. Let B be a critical group. If $\mathcal{O}$ is a Cross variety containing all proper factors of B , then the variety generated by B and $\mathcal{O}$ is Cross.

For, let G be any finite group, and consider the critical groups occurring as factors of G . Those of minimal order will be abelian and will generate a Cross variety, $\mathcal{U}_0$, say. Now consider a critical group not contained in $\mathcal{U}_0$ and minimal with respect to this property. This will satisfy the conditions of hypothesis A, and so, together with $\mathcal{U}_0$, will generate a Cross variety, $\mathcal{U}_1$. Proceeding in this manner, we eventually obtain a Cross variety, $\mathcal{U}_r$, containing all proper factors of G . Now, either G already belongs to this variety, or G is critical, and so, together with

$\mathcal{U}_r$, generates a Cross variety. Thus, in either case, G belongs to a Cross variety.

Now, a critical group possesses a unique minimal normal subgroup (3.3.1) which must be either elementary abelian, or a direct product of isomorphic (non-abelian) simple groups. This thesis is concerned with the latter type of critical group, and we prove:

Theorem I. Let B be a critical group whose minimal normal subgroup is non-abelian. If $\mathcal{O}$ is a Cross variety which contains every proper factor of B , then the variety generated by B and $\mathcal{O}$ is Cross.

Although this gives only a partial solution to Hypothesis A, it does provide an affirmative answer to Neumann's problem for quite a number of groups. In fact, combining it with Powell's result on soluble groups we obtain:

Theorem II. The variety generated by a finite group, none of whose factors is a non-soluble critical group with abelian minimal normal subgroup, is Cross.

(We first note that the variety generated by all soluble factors of G , by Powell's result, is Cross, and then adjoin the remaining

critical groups one at a time by making use of Theorem I.)

Examples of groups satisfying the conditions of Theorem II are the linear fractional groups $L.F.(2, p^n)$, for, as is shown in L. E. Dickson's book, "Linear Groups" (3), the subgroups of such groups are either soluble, or linear fractional groups of smaller order.

The group of smallest order which does not satisfy the hypotheses of Theorem II is the binary icosahedral group, which is a central extension of a cyclic group of order two by the icosahedral group. This is a critical group, and its minimal normal subgroup is the cyclic group of order two.

The method used in the proof of Theorem I is to select certain properties of groups in the given variety, $\mathcal{U}$, which depend on n -generator subgroups, for n not greater than some fixed n_0 . These properties then hold in $\mathcal{U}^{(n_0)}$ (the variety defined by the identical relations of $\mathcal{U}$ involving at most n_0 variables) since n_0 -generator subgroups of groups in $\mathcal{U}^{(n_0)}$ belong to $\mathcal{U}$, and they are used to show that $\mathcal{U}^{(n_0)}$ is Cross. But then, by (3.3.3), $\mathcal{U}$, being a subvariety of a Cross variety, is itself Cross.

Chapter 2 contains notation and definitions, and in Chapter 3

preliminary results are proved. (Any others used without reference can be found in M. Hall's book on group theory (4)). Chapter 4 contains the proof of Theorem I.

I should like to thank Professor Higman for suggesting this problem, and for his helpful advice and criticism, which have been of great assistance to me.

CHAPTER 2

NOTATION AND DEFINITIONS

2.1 Notation

Groups are denoted by upper case Roman letters, and varieties by upper case Gothic letters. Group elements are denoted by lower case Roman letters.

$H \leq G$, $H < G$, $H \triangleleft G$ (or $G \geq H$, $G > H$, $G \triangleright H$) mean, respectively, that H is a subgroup, a proper subgroup, a normal subgroup of G .

$g \in G$ means that g is an element of G .

If $g_1, \dots, g_r \in G$ and $H_1, \dots, H_s \leq G$, then $\{g_1, \dots, g_r, H_1, \dots, H_s\}$ is the subgroup of G generated by $g_1, \dots, g_r$ together with the elements of $H_1, \dots, H_s$.

$[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$.

$Z(G)$ is the centre of G , and if $H \leq G$, then $C_G(H)$ is the centraliser of H in G .

$G = H_1 \times \dots \times H_r$ means that G is the direct product of the groups $H_1, \dots, H_r$. (To avoid having to make trivial exceptions, we make the convention that if $r = 0$, then $G = 1$ (the trivial group)).

$H \cong G$ means that H is isomorphic to G .

$G \in \mathcal{O}_f$ means that G belongs to the variety $\mathcal{O}_f$.

$\mathcal{H} \in \mathcal{O}_f$, $\mathcal{H} \subset \mathcal{O}_f$ (or $\mathcal{O}_f \supset \mathcal{H}$, $\mathcal{O}_f \supsetneq \mathcal{H}$) mean respectively that $\mathcal{H}$ is a subvariety, a proper subvariety, of $\mathcal{O}_f$.

$\mathcal{O}_f^{(n)}$ denotes the variety consisting of those groups which satisfy the identical relations of the variety $\mathcal{O}_f$ involving at most n variables (it is also the variety of those groups all of whose n -generator subgroups belong to $\mathcal{O}_f$).

Throughout we take "simple" to mean "non-abelian simple".

2.2 Definitions

2.2.1 Definition. If $G \leq H \times K$, then, for each $g \in G$, $g = hk$, where

$h \in H$, $k \in K$, and the projections ϕ, ψ from G to H, K are defined by $g\phi = h$, $g\psi = k$. (The images $G\phi$, $G\psi$ are called the projections of G on H, K .)

2.2.2 Definition. If $w(x_1, \dots, x_n)$ is a group word of finite length in a set of variables $x_1, \dots, x_n$, then the equation

$$w(x_1, \dots, x_n) = 1$$

is an identical relation. It is said to hold on a group G (or G is said to satisfy the identical relation) if

$$w(g_1, \dots, g_n) = 1$$

for any substitution of elements $g_1, \dots, g_n$ of G for the variables

$x_1, \dots, x_n$. Since identical relations are always written in the form $w = 1$, and not $w_1 = w_2$, we may unambiguously refer to the "identical relation w ".

2.2.3 Definition. Since, in an identical relation, it is immaterial what the variables are called, we take them as the generators of one particular countably generated free group F . Then the consequences of a set W of identical relations are obtained from the identical relations of W by multiplying them together, taking inverses, and substituting any element of F for any variable in a relation.

2.2.4 Definition. A set of identical relations W is said to be a basis for another set W' if every relation of W' is a consequence of W .

2.2.5 Definition. A set of identical relations is said to be closed if it contains all its consequences.

2.2.6 Definition. A variety is a set of groups which is closed under the operations of taking subgroups, homomorphic images, and unrestricted direct products; e.g., the set of all abelian groups forms a variety as does the set of all groups of exponent n , but the set of all p -groups does not form a variety, for, if it did, it would include the unrestricted direct product of cyclic groups of order p^r for all r . The product of the generators of these would be an element of infinite

order, and so would generate an infinite cyclic group. But the factor groups of such a group can have any order.

2.2.7 Definition. The variety generated by a set of groups X is the intersection of all those varieties which contain all the groups of X . It is also the closure of X under the operations of taking subgroups, homomorphic images, and unrestricted direct products (for this closure is clearly a variety containing X which is itself contained in any variety which contains X).

2.2.8 Definition. If G is a finite group, a factor of G is a quotient group H/K , where $G \geq H \triangleright K \geq 1$, and H/K is a proper factor unless $G = H$, and $K = 1$. (This meaning of factor is not to be confused with that of a factor of a direct product. Both meanings will be used, the context making it clear which is intended.)

2.2.9 Definition. A finite group G is said to be critical if it does not belong to the variety generated by its proper factors.

2.2.10 Definition. A variety $\mathcal{V}$ is said to be Cross if:

- (1) Finitely generated groups in $\mathcal{V}$ are finite;
- (2) $\mathcal{V}$ contains only a finite number of critical groups;
- (3) $\mathcal{V}$ has a finite basis for its identical relations.

2.2.11 Definition. Let $w(x_1, \dots, x_n)$ be a group word in n variables; then the subgroup of G generated by all elements of the form $w(g_1, \dots, g_n)$, where $g_1, \dots, g_n \in G$, is called a verbal subgroup of G , and is denoted by $w(G)$. Clearly, $w(G) \triangleleft G$ (in fact it is fully invariant) and $G/w(G)$ is a group satisfying the identical relation w .

CHAPTER 3

PRELIMINARY RESULTS

Throughout this chapter references are given as far as possible to the places where the results were originally stated. The proofs given are not necessarily the original ones.

3.1 Direct Products

3.1.1 Proposition (Remak (10)). Let $G \leq H \times K$, where the projections from G to H and K are ϕ, ψ respectively. If $G \cap H \cong G\phi$ then $G = (G \cap H) \times (G \cap K)$.

Proof: If $g \in G$, then $g = g\phi g\psi$, but, by hypothesis, $g\phi \in G$, and thus $g\psi \in G$, and so $g\psi \in G \cap K$. Hence $G \cap H$ and $G \cap K$ span G . But these are normal subgroups with trivial intersection, and so generate their direct product. Hence $G = (G \cap H) \times (G \cap K)$.

3.1.2 Proposition (Remak (10)). Let $G \leq H \times K$, where the projections from G to H and K are ϕ, ψ respectively; then $G \cap H \triangleleft G\phi$, $G \cap K \triangleleft G\psi$, $G/(G \cap H) \cong G\psi$, $G/(G \cap K) \cong G\phi$, and $G\phi/(G \cap H) \cong G\psi/(G \cap K)$.

Proof: Let $h \in G \cap H$, and $g\phi \in G\phi$. Then $g^{-1}hg \in G$, i.e., $(g\phi)^{-1}(g\psi)^{-1}h g\phi g\psi = (g\phi)^{-1}hg\phi \in G$. But $(g\phi)^{-1}hg\phi \in H$, and thus to $G \cap H$. Hence $G \cap H \triangleleft G\phi$. Similarly $G \cap K \triangleleft G\psi$.

Now ϕ and ψ are clearly homomorphisms of G , and their kernels consist precisely of those elements of G which have projection 1 on H and K respectively, i.e., they are $G \cap K$ and $G \cap H$. Thus $G\phi \simeq G/(G \cap K)$, and $G\psi \simeq G/(G \cap H)$.

Let $G \cap H = G_1$, $G \cap K = G_2$. Since these are normal subgroups of G with trivial intersection they generate their direct product $G_1 \times G_2$, which is also normal in G . But

$$G/(G_1 \times G_2) \simeq (G/G_2)/((G_1 \times G_2)/G_2) \simeq G\phi/G_1 = G\phi/(G \cap H) \quad \text{and}$$

$$G/(G_1 \times G_2) \simeq (G/G_1)/((G_1 \times G_2)/G_1) \simeq G\psi/G_2 = G\psi/(G \cap K).$$

Hence $G\phi/(G \cap H) \simeq G\psi/(G \cap K)$.

3.1.3 Proposition (Remak (10)). Let $G \triangleleft H \times K$, where the projections from G to H , K are ϕ , ψ ; then $G\phi/(G \cap H) \leq Z(H/(G \cap H))$.

Proof: Clearly $G \cap H \triangleleft H$. Let $g\phi \in G\phi$, and $h \in H$, then $h^{-1}g\phi g\psi h \in G$, and so $(g\phi g\psi)^{-1}h^{-1}g\phi g\psi h = (g\phi)^{-1}h^{-1}g\phi h \in G$. But it also belongs to H , and so to $G \cap H$. Hence $[H, G\phi] \leq G \cap H$, and so $G\phi/(G \cap H) \leq Z(H/(G \cap H))$.

3.1.4 Proposition (Remak (10)). A normal subgroup of a direct product of simple groups must be a (possibly empty) direct product of a number of the factors.

Proof: Let $N \triangleleft H_1 \times \dots \times H_r$, where each H_i is simple, and let ϕ be the projection from N to H_1 . We proceed by induction on r (the proposition is clearly true for $r = 1$, since the only normal subgroups of a simple group are the group itself and 1). Assume true for $r - 1$. Since $N \triangleleft H_1 \times \dots \times H_r$, $N\phi \triangleleft H_1$, and so is H_1 or 1. But, from (3.1.3), $N\phi / (N \cap H_1) \leq Z(H_1 / (N \cap H_1)) = 1$ (since H_1 is simple). Thus $N\phi = N \cap H_1$, and so by (3.1.1) $N = (N \cap H_1) \times (N \cap (H_2 \times \dots \times H_r))$, where the first factor is H_1 or 1, and the second factor, by the induction hypothesis, consists of the direct product of a number of the H_i ($i = 2, \dots, r$). Thus N has the required form.

3.1.5 Corollary. The decomposition of a direct product of simple groups is unique, and any normal subgroup of such a direct product has a unique normal complement.

3.1.6 Proposition (Remak (10)). Let $H_i \triangleleft G$ ($i = 1, \dots, r$) and let $D = \bigcap_{i=1}^r H_i$; then $G/D \simeq \bar{G} \leq (G/H_1) \times \dots \times (G/H_r)$.

Proof: Let the natural map from G to G/H_i be α_i . Consider the mapping γ of G into $(G/H_1) \times \dots \times (G/H_r)$ defined by $g\gamma = g\alpha_1 \dots g\alpha_r$. This is clearly a homomorphism, and its kernel consists precisely of those elements of G which are mapped on to the identity by each α_i , i.e., it is D . Thus $G/D \simeq \bar{G} \leq (G/H_1) \times \dots \times (G/H_r)$.

3.1.7 Proposition. If $N \triangleleft G \leq H \times K$, and $C_{G/\phi}(N\phi) = 1$, where ϕ, ψ are the projections from G to H, K , then $N \cap H = 1$ if and only if $G \cap H = 1$.

Proof: If is trivial, since $N \cap H \leq G \cap H$.

Conversely, suppose $g\phi \in G \cap H$, $g\phi \neq 1$, and let $n = n\phi n\psi \in N$. Since $N \triangleleft G$, $(g\phi)^{-1}n\phi n\psi g\phi \in N$, and so $(n\phi n\psi)^{-1}(g\phi)^{-1}n\phi n\psi g\phi = [n\phi, g\phi] \in N$. But this is an element of H also, and so belongs to $N \cap H$. It cannot be 1 for all $n\phi$ since $C_{G/\phi}(N\phi) = 1$, and $g\phi \neq 1$.

3.1.8 Proposition. If $G = \{D_1, \dots, D_r\}$ where the D_i are direct products of simple groups, and $D_i \triangleleft G$, $D_i \cap D_j = 1$, then $G = D_1 \times \dots \times D_r$.

Proof: By induction on r . The lemma is certainly true for $r = 1$. Assume true for $r - 1$, then $\{D_2, \dots, D_r\} = D_2 \times \dots \times D_r \triangleleft G$. Now $D_1 \cap (D_2 \times \dots \times D_r)$ is a normal subgroup of $D_2 \times \dots \times D_r$, which is also a direct product of simple groups, and thus, by (3.1.4) this intersection is a direct product of a number of the factors. But $D_1 \cap D_i = 1$ ($i = 2, \dots, r$) and so $D_1 \cap (D_2 \times \dots \times D_r) = 1$. It follows that these groups generate their direct product, i.e., $G = D_1 \times D_2 \times \dots \times D_r$, as required.

3.1.9 Proposition. Let $G/N \cong H_1 \times \dots \times H_r$, where N is abelian, and each H_i is a direct product of simple groups. Let F_i be the subgroup of G such that $F_i/N \cong H_i$. If $F_i = D_i \times N$ ($D_i \cong H_i$) for $i = 1, \dots, r$, then $G = D_1 \times \dots \times D_r \times N$.

Proof: Certainly $G = \{ D_1, \dots, D_r, N \}$.

Now, $F_i \triangleleft G$, being the inverse image of a normal subgroup of G/N .

Thus, for any g in G , we have $g^{-1}D_i g \leq F = D_i \times N$. But the projection of $g^{-1}D_i g$ on N is an abelian homomorphic image of $g^{-1}D_i g$, which is a direct product of simple groups, and so this projection must be 1.

Thus $g^{-1}D_i g = D_i$, and $D_i \triangleleft G$. Also $F_i \cap F_j = N$ (since $(F_i/N) \cap (F_j/N) = 1$), and so $D_i \cap D_j = 1$. It follows from (3.1.8) that $D_1 \times \dots \times D_r$ generate their direct product, which will also be normal in G . But $N \cap (D_1 \times \dots \times D_r)$ is an abelian normal subgroup of a direct product of simple groups, and so is 1. Hence $G = D_1 \times \dots \times D_r \times N$, as required.

3.2 Varieties and Identical Relations

3.2.1 Proposition (Neumann (8)). Any variety is generated by its finitely generated groups.

Proof: Let $\mathcal{U}$ be a variety and F_n the free group on n generators $x_1, \dots, x_n$. The identical relations of $\mathcal{U}$ involving at most n variables, when regarded as elements of F_n , form a fully invariant subgroup R_n . They clearly form a group, and if $w(x_1, \dots, x_n) \in R$ and e is an endomorphism of F_n , then

$$we = w(x_1 e, \dots, x_n e).$$

But this is a consequence of w , and so belongs to R_n , which is thus fully invariant.

Let $V_n \cong F_n / R_n$. Then all such V_n 's belong to $\mathcal{U}$, and the identical

relations of the variety generated by $V_1, V_2, \dots$, are precisely those of $\mathcal{U}$, and so the two varieties coincide. But all the V_n 's are finitely generated, and so $\mathcal{U}$ is generated by its finitely generated groups, as required.

3.2.2 Proposition (Neumann (8)). The identical relations of a finite group, G , which involve at most n variables possess a finite basis.

Proof: Let R_n be defined as above; then R_n is contained in the kernel of every homomorphism of F_n into G . Conversely, any element of F_n which is contained in the kernel of every such homomorphism must be an identical relation of G involving at most n variables, and so R_n is the intersection of all these kernels. But, since G is finite, there are only a finite number of such homomorphisms. Thus R_n is the intersection of a finite number of normal subgroups of finite index, and so is itself of finite index. It follows from Schreier's theorem (3.5.1), since F_n is finitely generated, that R_n is finitely generated, and these generators provide the finite basis for the identical relations of G involving at most n variables.

3.2.3 Proposition (Higman (5)). A group which belongs to the variety generated by a set of groups X is a homomorphic image of a subgroup of a direct product of groups isomorphic to groups in X .

Proof. The class of groups described certainly contains X , and is contained in every variety which contains X . But it is also closed

under the operations of taking subgroups, homomorphic images, and direct products, and so is itself a variety.

3.2.4 Proposition (Higman (5)). Suppose that either:-

(i) X is a finite set of finite groups, and G is finitely generated;

or

(ii) G is finite;

Then G is isomorphic to a subgroup of a direct product of groups isomorphic to groups in X only if it is isomorphic to a subgroup of a direct product of a finite number of such groups.

Proof: Consider the intersections of G with the subgroups of the direct product obtained by omitting one direct factor. These give a family of normal subgroups of G, whose factor groups (3.1.2) are isomorphic to subgroups of groups in X, and which intersect in the identity. But, under either hypothesis, such a family is finite, and, as in (3.1.6), G is isomorphic to a subgroup of the direct product of the factor groups, which is a group of the required form.

3.2.5 Proposition (Higman (5)). Suppose that either:-

(i) X is a finite set of finite groups;

or

(ii) all groups in X belong to a fixed variety in which finitely generated groups are finite;

then a finitely generated group belonging to the variety generated

by λ is a homomorphic image of a subgroup of a direct product of a finite number of groups isomorphic to groups in λ .

Proof: By (3.2.3) G is a homomorphic image of a subgroup of a direct product of groups isomorphic to groups in λ . This subgroup can clearly be taken to be finitely generated, and the result follows from the previous proposition.

3.2.6 Corollary. A finitely generated group in a variety generated by a finite set of finite groups is finite.

3.3 Critical Groups and Cross Varieties

3.3.1 Proposition (Cross (2), Higman (6)). If G is a critical group, then it possesses a unique minimal normal subgroup.

Proof: If G is simple, then it is its own minimal normal subgroup.

If not, suppose G has two minimal normal subgroups, H, K . Then $H \cap K = 1$ (for it is a normal subgroup of G strictly contained in a minimal normal subgroup). But by (3.1.6), (taking $r = 2$),

$$G = G/(H \cap K) \simeq \bar{G} \leq (G/H) \times (G/K),$$

and G/H and G/K are proper factors of G , contradicting the hypothesis that G is critical.

3.3.2 Proposition (Cross (2), Higman (6)). If the variety $\mathcal{V}$ is generated by its finite groups, then it is generated by its critical groups.

Proof: Let $\mathcal{V}_0$ be the variety generated by the critical groups of $\mathcal{U}$; then $\mathcal{V}_0 \subseteq \mathcal{U}$, and to prove that $\mathcal{V}_0 = \mathcal{U}$ it is sufficient to show that any finite group G belonging to $\mathcal{U}$ necessarily belongs to $\mathcal{V}_0$. We proceed by induction on the order of G , the basis of the induction being the (non-trivial) group of smallest, and thus prime, order. This is certainly critical, and so belongs to $\mathcal{V}_0$. If G is critical, then it belongs to $\mathcal{V}_0$. If not, consider its proper factors. Each of these belongs to $\mathcal{U}$, and so, by the induction hypothesis, to $\mathcal{V}_0$. Thus the variety they generate is contained in $\mathcal{V}_0$. But G belongs to this variety, and thus to $\mathcal{V}_0$.

3.3.3 Proposition (Cross (2), Higman (6)). A subvariety of a Cross variety is itself Cross.

Proof: If $\mathcal{U}$ is Cross then (2.2.10) $\mathcal{U}$ satisfies:

- (1) finitely generated groups in $\mathcal{U}$ are finite;
- (2) $\mathcal{U}$ contains a finite number of critical groups;
- (3) $\mathcal{U}$ has a finite basis for its identical relations.

Let $\mathcal{V}$ be any subvariety of $\mathcal{U}$; then, trivially, $\mathcal{V}$ will satisfy (1) and (2), so we need show only that it satisfies (3).

From (1) and (3.2.1), $\mathcal{V}$ is generated by its finite groups, and so (3.3.2) by its critical groups. Consider a chain

$$\mathcal{U} = \mathcal{V}_0 \supset \mathcal{V}_1 \supset \dots \supset \mathcal{V}_i \supset \dots \supset \mathcal{V}_n = \mathcal{V}$$

of varieties between $\mathcal{U}$ and $\mathcal{V}$, where the inclusions are proper.

Since every subvariety of $\mathcal{V}$ is generated by its critical groups, there is for $i = 0, \dots, n-1$, a critical group in $\mathcal{V}_i$ not in $\mathcal{V}_{i+1}$. By (2) this implies a bound for n , and so we can suppose that the chain has no proper refinements. Suppose that the variety $\mathcal{V}_i$ has a finite basis for its identical relations (we know this to be true for $i = 0$). Because $\mathcal{V}_{i+1}$ is properly contained in $\mathcal{V}_i$, there is an identical relation $w = 1$ which holds in $\mathcal{V}_{i+1}$ but not in $\mathcal{V}_i$. The variety $\mathcal{H}$, whose identical relations are those of $\mathcal{V}_i$ together with $w = 1$, satisfies

$$\mathcal{V}_i \supset \mathcal{H} \supseteq \mathcal{V}_{i+1}$$

where the first inclusion is proper. Since our chain has no proper refinements, $\mathcal{H} = \mathcal{V}_{i+1}$, i.e., the identical relations of $\mathcal{V}_{i+1}$ have a finite basis. By induction, so have those of $\mathcal{V}$.

3.3.4 Corollary. If G is a finite group belonging to $\mathcal{V}$ the answer to Neumann's problem for G is affirmative, i.e., there is a finite basis for the identical relations holding on G .

3.4 Verbal Subgroups

3.4.1 Proposition. $w(G)$ is a fully invariant subgroup of G .

Proof: Let $w(g_1, \dots, g_n)$ be a generator of $w(G)$, and e an endomorphism of G ; then

$$w(g_1, \dots, g_n)e = w(g_1e, \dots, g_ne) \in w(G).$$

Thus the image of any generator of $w(G)$ is contained in $w(G)$, and it follows that this is a fully invariant subgroup of G .

3.4.2 Proposition. If $N \triangleleft G$, and $w(G/N) = 1$, then $w(G) \leq N$.

Proof: Let $g_1N, \dots, g_nN$ be any n elements of G/N ; then

$$w(g_1N, \dots, g_nN) = N;$$

i.e., since $N \triangleleft G$, $w(g_1, \dots, g_n)N = N$, and so $w(g_1, \dots, g_n) \in N$.

Hence $w(G) \leq N$, as required.

3.4.3 Proposition. If $N \triangleleft G$, then $w(G/N) = w(G)N/N$.

Proof: Let $w(g_1N, \dots, g_nN)$ be a generator of $w(G/N)$; then, as above,

$$w(g_1N, \dots, g_nN) = w(g_1, \dots, g_n)N. \text{ Hence } w(G/N) = w(G)N/N.$$

3.4.4 Proposition. Let $G \leq H \times K$; then $w(G)\phi = w(G\phi)$, and $w(G)\psi = w(G\psi)$, where ϕ, ψ are the projections from G to H, K .

Proof: Let $w(g_1, \dots, g_n)$ be a generator of $w(G)$; then $g_i = g_i\phi g_i\psi$,

$$\text{and so } w(g_1, \dots, g_n) = w(g_1\phi, \dots, g_n\phi) w(g_1\psi, \dots, g_n\psi) = w\phi w\psi,$$

and the truth of the proposition follows.

3.5 Schreier's Theorem

3.5.1 Proposition (Schreier (1)). A subgroup of finite index in a finitely generated group is itself finitely generated.

Proof: Let $H \leq G = \{g_1, \dots, g_r\}$ and let $s_1, \dots, s_n$ be a system of right coset representatives of H in G , chosen so that the representative of H is 1. Let γ be the mapping which sends each element of G on to the representative of the coset which contains it. Then the elements $s_i g_j (\gamma(s_i g_j))^{-1}$ generate H . For let $g_1 g_2 \dots g_k$ be any word in the generators of G which belongs to H ; then we may rewrite this in the form

$$\begin{aligned} g_1 g_2 \dots g_r &= g_1 (\gamma(g_1))^{-1} \gamma(g_1) g_2 g_3 \dots g_k \\ &\dots \dots \dots \\ &= g_1 (\gamma(g_1))^{-1} \gamma(g_1) g_2 (\gamma(\gamma(g_1) g_2))^{-1} \dots \gamma(\gamma \dots (\gamma(g_1) g_2) \dots g_k). \end{aligned}$$

But $\gamma(\gamma(g_1) g_2) = \gamma(g_1 g_2)$, etc., for, if g_1 belongs to the coset Hs_i , then $g_1 = hs_i$ and $s_i g_2$ clearly belongs to the same coset as $hs_i g_2$. Thus the last term in the above expression is $\gamma(g_1 g_2 \dots g_k) = 1$, since $g_1 g_2 \dots g_k \in H$, and the representative of H is 1. Thus we can express any element of H in terms of the elements $s_i g_j (\gamma(s_i g_j))^{-1}$. But there are at most rn of these, and so H is finitely generated, as required.

CHAPTER 4

CRITICAL GROUPS WITH NON-ABELIAN MINIMAL NORMAL SUBGROUPS

In this chapter we prove:

Theorem. Let B be a critical group whose minimal normal subgroup is non-abelian. If $\mathcal{O}$ is a Cross variety which contains every proper factor of B , then the variety, $\mathcal{U}$, generated by $\mathcal{O}$ and B is Cross.
(We may assume that B does not belong to $\mathcal{O}$, since the theorem is trivial if it does.)

4.1 Special notation and preliminary results

Let A be a finite group which generates $\mathcal{O}$ (e.g., the direct product of the critical groups of $\mathcal{O}$) and let $w(x_1, \dots, x_a) = 1$ be the basis of the identical relations of $\mathcal{O}$.

Let the minimal normal subgroup of B be M . Let m be the minimum number of generators of M , and let b be the minimum number of generators in a generating set for B which includes a minimum generating set for M .

Definition. We call a subgroup D of a group G an M-subgroup if it satisfies:

- (1) $D \simeq M$;
- (2) There exists a subgroup H of G such that $D \triangleleft H$, and $H/C_H(D) \simeq B$
(H is called an M-normaliser of D).

If a group G contains a subgroup K which has as a direct factor an M-subgroup D of G , we call D an M-factor of K (in G).

4.1.1 Lemma. If D is an M-subgroup of G , and $N \triangleleft G$, then $D \cap N = D$ or 1 , and, in the latter case, DN/N is an M-subgroup of G/N .

Proof: Let H be an M-normaliser of D , so that $D \triangleleft H$, and $H/C_H(D) \simeq B$. Under this homomorphism the image of D is a normal subgroup of B , isomorphic to D (since $D \cap C_H(D) = 1$), i.e., isomorphic to M . Thus it must be M itself. Since M is the minimal normal subgroup of B , it follows that no proper non-trivial subgroup of D can be normal in H . But $D \cap N = D \cap (H \cap N)$ is a normal subgroup of H , and so must be D or 1 . If $D \cap N = 1$, then $H \cap N$ and D commute elementwise (both being normal in H) and so $H \cap N \leq C_H(D)$. Consider $DN/N \triangleleft HN/N \leq G/N$.

$$DN/N \simeq D/(D \cap N) \simeq D \simeq M.$$

Let $\bar{c}$, an element of $C_{HN/N}(DN/N)$, be the image under the homomorphism from G to G/N of c , an element of H , so that $dc = cdn$, where $n \in N$, for any element d of D . Thus $d^{-1}c^{-1}dc = n$. But $d^{-1}c^{-1}dc \in D$ (since $c \in H$, and $D \triangleleft H$), and thus $d^{-1}c^{-1}dc \in D \cap N = 1$. Hence $c \in C_H(D)$. Conversely, if $c \in C_H(D)$ then certainly its image in the

homomorphism belongs to $C_{HN/N}(DN/N)$; i.e., $C_{HN/N}(DN/N) = C_H(D)N/N$. Thus, since $H \cap N \leq C_H(D)$,

$$\begin{aligned} (HN/N)/C_{HN/N}(DN/N) &= (HN/N)/(C_H(D)N/N) \simeq (H/H \cap N)/(C_H(D)/C_H(D) \cap N) \\ &\simeq H/C_H(D) \simeq B. \end{aligned}$$

It follows that DN/N is an M -subgroup of G/N .

4.1.2 Lemma. If $G > K = D_1 \times \dots \times D_r$, where the D_i 's are M -subgroups of G , and $N \triangleleft G$, then KN/N is a direct product of M -subgroups of G/N .

Proof: From the previous lemma, we have that $D_i \cap N = D_i$ or 1 , and that, in the latter case, $D_i N/N$ is an M -subgroup of G/N . Since each D_i , being isomorphic to M , is a direct product of simple groups, it follows that K is a direct product of simple groups, and so (3.1.4) any normal subgroup of K must be the direct product of a number of these. Thus $K \cap N$ must consist precisely of the direct product of those D_i s with which N has intersection D_i . But

$$KN/N \simeq K/(K \cap N) = D_1 \times \dots \times D_s,$$

where these are the D_i 's with which N has intersection 1 . It follows that KN/N is the direct product of the corresponding $D_i N/N$, and thus has the required form.

4.1.3 Corollary. If in the above lemma each D_i is normal in G , then

$$\underline{KN = D_1 \times \dots \times D_s \times N, \text{ where these are the } D_i\text{'s such that } D_i \cap N = 1.}$$

Proof: This follows immediately from the fact that $D_1 \times \dots \times D_s$ and N are normal subgroups of G with trivial intersection.

4.1.4 Lemma. Let X, Y be normal in G and let D/X be an M -subgroup of G/X , where $D \leq XY$. Then $(D \cap Y)/(X \cap Y)$ is an M -subgroup of $G/(X \cap Y)$.

Proof: Since $X \leq D \leq XY$, $D = X(D \cap Y)$, and so

$$M \simeq D/X = X(D \cap Y)/X \simeq \text{[REDACTED]} \simeq (D \cap Y)/((D \cap Y) \cap X) = (D \cap Y)/(X \cap Y).$$

Let H/X be an M -normaliser of D/X . Then $D \cap Y \triangleleft H$, since $D \triangleleft H$, $Y \triangleleft G$.

Suppose $C_{H/X}(D/X) = U/X$, and $C_{H/X \cap Y}((D \cap Y)/(X \cap Y)) = V/(X \cap Y)$.

Let $u \in U$, $d \in D \cap Y$; then, since uX centralises D/X , $u^{-1}du = dx$,

$x \in X$. But $D \cap Y \triangleleft H$, and so $x \in D \cap Y$. Hence $x \in X \cap D \cap Y = X \cap Y$.

But this means that $u(X \cap Y)$ centralises $(D \cap Y)/(X \cap Y)$, and so $u \in V$.

Hence $U \leq V$. Now, let $v \in V$, $d \in D$. Then $d = d_1x$, where $d_1 \in D \cap Y$,

and $x \in X$. Then $v^{-1}dv = v^{-1}d_1vv^{-1}xv = d_1x_1v^{-1}xv$, where $x_1 \in X \cap Y$,

since $v(X \cap Y)$ centralises $D \cap Y$. Thus $v^{-1}dv = dx^*$, where $x^* \in X$ (since

$X \triangleleft G$). Hence vX centralises D/X , and so $v \in U$, i.e., $V \leq U$. It

follows that $U = V$, and

$$(H/(X \cap Y))/C_{H/X \cap Y}((D \cap Y)/(X \cap Y)) \simeq H/V = H/U \simeq (H/X)/C_{H/X}(D/X) \simeq B.$$

Thus $(D \cap Y)/(X \cap Y)$ is an M -subgroup of $G/(X \cap Y)$, with M -normaliser $H/(X \cap Y)$.

4.2 The Variety $\mathcal{V}$

Throughout this section G is a finitely generated (and thus finite (3.2.6)) group in $\mathcal{V}$.

4.2.1 Lemma. In order to show that a property P holds for G it is sufficient to show that 1 has P, and that, if all factors of a group U in $\mathcal{U}$ have P, then so have all factors of both $U \times A$ and $U \times B$.

Proof: From (3.2.5), G is a homomorphic image of a subgroup of a direct product of a finite number of groups isomorphic to A and B. This is sufficient to show that all factors of a direct product

$$A_1 \times \dots \times A_r \times B_1 \times \dots \times B_r$$

have P. (There is no loss of generality in assuming that the same number of A's and B's occur.) We proceed by induction on r, the basis of the induction being $r = 0$, since we are assuming that 1 has P.

Assume all factors of $A_1 \times \dots \times A_{r-1} \times B_1 \times \dots \times B_{r-1} = U$ have P; then so have all factors of $U \times A$ and thus of $U \times A \times B = A_1 \times \dots \times A_r \times B_1 \times \dots \times B_r$.

4.2.2 Lemma. In applying the previous lemma, in order to show that all factors of $U \times A$ or $U \times B$ have P, it is sufficient to consider those factors K/N such that $K \cap A > 1$, or $K \cap B > 1$.

Proof: From (3.1.2), $K \cap A$ or $K \cap B = 1$ implies that $K \simeq K\phi$ (its projection on U). Thus K/N is isomorphic to a factor of U, and so, by assumption, has P.

4.2.3 Lemma. $w(G)$ is a (finite) direct product of M-subgroups of G, each of which is normal in G (and, by (3.1.5), this expression is unique).

Proof: This certainly is true for 1, so we must show that if it is true for all factors of U , where $U \in \mathcal{U}$, then it is true for all factors of $U \times V$, where $V \simeq A$ or $V \simeq B$.

Let $G \simeq K/N$, where $K \leq U \times V$. Let ϕ, ψ be the projections from K to U, V ; then (3.4.4)

$$w(K)\phi = w(K\phi) \qquad w(K)\psi = w(K\psi)$$

Now, unless $K\psi = V \simeq B$, $K\psi \in \mathcal{U}$, and so $w(K)\psi = w(K\psi) = 1$, and $w(K) = w(K\phi) \times 1$.

If $K\psi = B$, then $w(K\psi) = w(B) > 1$, since $B \notin \mathcal{U}$. Now $w(B) \triangleleft B$, and so $w(B) \geq M$. But $w(B/M) = 1$, and so (3.4.2) $w(B) \leq M$, hence $w(B) = M$. By (4.2.2) we may assume $K \cap B > 1$. But $C_{K\psi}(w(K)\psi) = C_B(M) = 1$, and so by (3.1.7), $w(K) \cap B > 1$. But $w(K) \triangleleft K$, and so $w(K) \cap B \triangleleft K\psi = B$. Thus $w(K) \cap B = M = w(K)\psi$, and so by (3.1.1) M is a direct factor of $w(K)$. But $C_K(M) = K \cap U$ and so $K/C_K(M) = K/(K \cap U) \simeq K\psi = B$ (3.1.2). Hence M is an M -subgroup of K . Also $M \triangleleft K\psi$, and so $M \triangleleft K$. Thus $w(K) = w(K\phi) \times M$, where M satisfies the given conditions.

Now consider $w(K\phi)$, which we have seen to be a direct factor of $w(K)$. $K\phi$ is a factor of U , and so, by hypothesis, $w(K\phi)$ is a direct product of M -subgroups of $K\phi$ each of which is normal in $K\phi$. Let D be one of these factors, and $H\phi$ an M -normaliser of D in $K\phi$. Let H be the inverse image of D in K . Then $D \triangleleft H$, and $h \in C_H(D)$ if and only if $h\phi \in C_{H\phi}(D)$. Thus

$$H/C_H(D) \simeq H\phi / C_{H\phi}(D) \simeq B$$

Also $D \triangleleft K$, since $D \triangleleft K\phi$. Thus $w(K\phi)$ is a direct product of M -subgroups of K , and it follows that in both the above cases, $w(K)$ has the required form.

Now let $G = K/N$ be a factor group of K , then, if $w(K) = D_1 \times \dots \times D_r$ by (4.1.3) we have $w(K)N = D_1 \times \dots \times D_s \times N$, where $D_1, \dots, D_s$ are the M -factors of $w(K)$ which have intersection 1 with N . But $w(G) = w(K)N/N$, and, by (4.1.2), this is a direct product of M -subgroups of G . Clearly, each of these, being a homomorphic image of a normal subgroup of K , will be normal in G , and the truth of the lemma follows.

4.2.4 Lemma. If D is an M -subgroup of G , then $D \leq w(G)$.

Proof. $D \triangleleft H \leq G$, and $H/C_H(D) \simeq B$. Thus $DC_H(D)/C_H(D) = w(H/C_H(D)) = w(H)C_H(D)/C_H(D)$, and so $DC_H(D) = D_1 \times \dots \times D_s \times C_H(D)$ where $D_1, \dots, D_s$ are the factors of $w(H)$ which have intersection 1 with $C_H(D)$. But $DC_H(D) = D \times C_H(D)$, since D has no centre, and thus $s = 1$, and $D \times C_H(D) = D_1 \times C_H(D)$. Now, if D had non-trivial component in $C_H(D)$, this would necessarily be abelian, but it is a homomorphic image of $D (\simeq M)$, so this is impossible. Thus $D = D_1 \leq w(H) \leq w(G)$.

4.2.5 Lemma. If $N \triangleleft G$, and L/N is an M -subgroup of G/N , with M -normaliser H/N , then $L = D \times N$, where D is an M -subgroup of H (and thus of G).

Proof: As in the above lemma, L/N is one of the direct factors of $w(H/N)$, and so is of the form DN/N , where D is an M -factor of $w(G)$. But $N \triangleleft H$, $D \triangleleft H$, and $D \cap N = 1$; thus $L = D \times N$, and has the required form.

4.2.6 Lemma. If D is an M -subgroup of G , then the projection of D on any D_i which is an M -factor of $w(G)$ is either D_i or 1.

Proof: The lemma is certainly true for 1, and so we must show that if it is true for all factors of U ($\in \mathcal{U}$), then it is true for $G = K/N$, where $K \leq U \times V$, and $V \simeq A$ or B .

Let the inverse image of D in K be L ; then, from the previous lemma, $L = D' \times N$, where D' is an M -subgroup of K . If D' has projection D_i or 1 on each M -factor D_i of $w(K)$, then, since $w(G) = (D_1 \times \dots \times D_s \times N)/N$, where $D_1, \dots, D_s$ are factors of $w(K)$, and $D = (D' \times N)/N$, D will also have this property. Hence it is sufficient to prove the lemma for K . Let ϕ, ψ be the projections from K to U, V . As in (4.2.3), $w(K) = w(K\phi) \times 1$, or $w(K) = w(K\phi) \times M$, the latter holding if $K\psi = V \simeq B$. Let H be an M -normaliser of D in K , then $D \cap V \triangleleft H$, and so is D or 1.

- (i) If $D \cap V = D$, then, since $D \in w(K)$, we must have $V \simeq B$, and $D = M$, so that D is itself an M -factor of $w(K)$.
- (ii) If $D \cap V = 1$, then (3.1.2) $D \simeq D\phi$, and (as in (4.1.1)) $D\phi$ is an M -subgroup of $K\phi$ (with M -normaliser $H\phi$). Thus, by hypothesis, $D\phi$ has projection D_i or 1 on each factor of $w(K\phi)$, and thus so has D .

It remains to consider $D\psi$. Since $D \leq w(H)$, this will be 1 unless $H\psi = V \simeq B$ (in which case $w(K) = w(K\phi) \times M$). But $D\psi \triangleleft H\psi$, and $D\psi \leq M$. Thus $D\psi = 1$ or M . Hence, in all cases, D has the required properties.

4.2.7 Lemma. If D is an M -subgroup of G , then $C_G(D) \triangleleft G$.

Proof: From (4.2.6), $D \leq w(G) = D_1 \times \dots \times D_r$, and has projection D_i on $D_1, \dots, D_s$, say, and 1 on the other M -factors. Let $d = d_1 \dots d_s$ be an element of D , and let $c \in C_G(D)$, so that $c^{-1}dc = d$. Since $D_i \triangleleft G$, we have $c_i^{-1}d_i c = d'_i \in D_i$, and so $d'_1 \dots d'_s = d_1 \dots d_s$. Hence $d'_i = d_i$. But this will hold for any element d_i of D_i since D has projection D_i on D_i , and so c centralises each of $D_1, \dots, D_s$ also. Conversely, any element which centralises each of these groups will certainly centralise D , so we have $C_G(D) = \bigcap_{i=1}^s C_G(D_i)$. But $C_G(D_i) \triangleleft G$, since $D_i \triangleleft G$, and so $C_G(D) \triangleleft G$, as required.

4.2.8 Lemma. If $G \geq S \triangleright D \times N$, where D is an M -subgroup of G , then $D \triangleleft S$.

Proof: As above, $D \leq w(G) = D_1 \times \dots \times D_r$, having projection D_i on $D_1, \dots, D_s$, say, and 1 on the other M -factors of $w(G)$. Now, $N \leq C_G(D)$, and so $N \leq C_G(D_i)$ ($i = 1, \dots, s$). Thus $N \cap (D_1 \times \dots \times D_s) = 1$. Let $d \in D$, $s \in S$. Since $D \times N \triangleleft S$, $s^{-1}ds = d'n$, where $d' \in D$, and $n \in N$, i.e., $n = s^{-1}dsd'^{-1}$. But, since each D_i is normal in G , the right hand side is an element of $D_1 \times \dots \times D_s$, which has intersection 1 with N . Hence $n = 1$, and so $D \triangleleft S$, as required.

4.2.9 Lemma. If D is an M -subgroup of $L \leq G$, then any element of G which centralises D also centralises every M -factor of $w(L)$ on which D has non-trivial projection.

Proof:

(i) If $L = G$, then the result has already been proved in the course of proving (4.2.7).

(ii) In the general case, let $D_1, \dots, D_r$ be the M -factors of $w(G)$, and $E_1, \dots, E_s$ the M -factors of $w(L)$. E_1 is an M -subgroup of G , so we may suppose it to have projection D_i on D_i for $i = 1, \dots, t$, and 1 otherwise. Then, for $j > 1$, E_j centralises E_1 , and so, by (i), centralises $D_1, \dots, D_t$, so that its projection on D_i must be 1 for $i = 1, \dots, t$. It follows that we can calculate the projection of an element of $w(L)$ on D_i ($i = 1, \dots, t$) by first projecting on E_1 and then projecting the result, i.e., if ϕ_i is the projection of $w(G)$ on D_i , and η_j is the projection of $w(L)$ on E_j , then

$$x\phi_i = x\eta_1\phi_i \quad (x \in w(L), i = 1, \dots, t)$$

Thus, if D has projection E_1 on E_1 , it has projection D_i on D_i , $i = 1, \dots, t$. By (i), an element which centralises D also centralises $D_1 \times \dots \times D_t$, and so centralises E_1 , which is contained in $D_1 \times \dots \times D_t$.

4.3 The variety $\mathcal{V}^{(n)}$

In this section, G is a finitely generated group in $\mathcal{V}^{(n)}$, where

$$n \geq n_0 = \max(a, 2b + 1).$$

4.3.1 Lemma. $w(G)$ is generated by a finite number of M -subgroups of G .

Proof: $G/w(G) \in \mathcal{O}$, and is finitely generated, and so (3.2.6) is finite. It follows from (3.5.1) that $w(G)$ is finitely generated, and so is generated by a finite number of elements of the form $w(g_1, \dots, g_a)$. Let $L = \{g_1, \dots, g_a\}$, then L has at most n_0 generators, and so belongs to $\mathcal{V}$. From (4.2.3) we have that $w(L)$ is a (finite) direct product of M -subgroups of L . But these will also be M -subgroups of G , and so each of the finite number of generators of $w(G)$ is contained in the direct product of a finite number of M -subgroups of G , which is itself contained in $w(G)$. It follows that the totality of such M -subgroups generates $w(G)$.

4.3.2 Lemma. Let D be an M -subgroup of G ; then there is an M -normaliser H of D in G having only b generators, and $D \leq w(H) \leq w(G)$.

Proof: Let H' be an M -normaliser of D in G , and let $a_1, \dots, a_m$ be generators of D . Consider $H'/C_{H'}(D)$. We can choose a set of generators $a_1 C_{H'}(D), \dots, a_b C_{H'}(D)$ for this, of which the first m correspond to the generators of D (because $D \cap C_{H'}(D) = 1$). Let

$H = \{a_1, \dots, a_b\}$. Then $D \triangleleft H$, $H' = HC_{H'}(D)$, and

$$B \simeq H'/C_{H'}(D) = HC_{H'}(D)/C_{H'}(D) \simeq H/(H \cap C_{H'}(D)) = H/C_H(D).$$

Thus H is also an M -normaliser of D in G , and has only b generators as required. But, from our choice of n_0 , this means that $H \in \mathcal{U}$, and so (4.2.4) $D \leq w(H)$. But $w(H) \leq w(G)$.

4.3.3 Lemma. If D is an M -subgroup of G , then $C_G(D) \triangleleft G$.

Proof: Let H be an M -normaliser of D with b generators, and let $g \in G$, $c \in C_G(D)$. If $L = \{H, c, g\}$, then, since $b + 2 \leq 2b + 1$, $L \in \mathcal{U}$. But D is an M -subgroup of L , and so (4.2.7) $C_L(D) \triangleleft L$. But $c \in C_L(D)$, and so $g^{-1}cg \in C_L(D) \leq C_G(D)$. But g was any element of G , and c any element of $C_G(D)$, and so $C_G(D) \triangleleft G$, as required.

4.3.4 Lemma. If $G \geq S \triangleright D \times N$, where D is an M -subgroup of G , then $D \triangleleft S$.

Proof: Let H be an M -normaliser of D with b generators, and let $s \in S$. If $L = \{H, s\}$, then, as before, $L \in \mathcal{U}$. But D is an M -subgroup of L , and $L \cap S \triangleright L \cap (D \times N) = D \times (L \cap N)$. Thus the conditions of (4.2.8) are satisfied, and we have $D \triangleleft L \cap S$, i.e., $s^{-1}Ds = D$. Thus $D \triangleleft S$.

4.3.5 Lemma. If $G \triangleright N$, where N is abelian, and K/N is a direct product of M -subgroups of G/N , then $K = E \times N$, where E is a direct product of M -subgroups of G .

Proof: Let J/N be one of the M -factors of K/N and H'/N an

M -normaliser of J/N having b generators, $a_1N, \dots, a_bN$. Let $H = \{a_1, \dots, a_b\}$. Then $H/(H \cap N) \simeq HN/N = H'/N$. Hence H must contain a subgroup L such that $L/(H \cap N)$ is an M -subgroup of $H/(H \cap N)$, corresponding to J/N under this isomorphism. But H , having only b generators, belongs to $\mathcal{UL}$, and so, from (4.2.5), we have that $L = D \times (H \cap N)$, where D is an M -subgroup of H (and thus of H' and G). Also, $LN = DN$, and so $DN/N = J/N$.

Now suppose H' contains two M -subgroups, D_1, D_2 such that $D_1N/N = D_2N/N = J/N$. Let H_1, H_2 be M -normalisers of D_1, D_2 , each having b generators; then, if $P = \{H_1, H_2\}$, $P \in \mathcal{UL}$, since it has at most $2b$ generators, and so $\{D_1, D_2\} \leq w(P)$ (4.2.4). Now, $w(P)$, as a direct product of M -subgroups of P , has no non-trivial abelian normal subgroup, so that $w(P) \cap N = 1$. Thus $w(P)$ is isomorphic to its image $w(P)N/N$ in the natural map of H' on H'/N . But $w(P)N/N \geq J/N$, which is the image of both D_1 and D_2 . It follows that $D_1 = D_2$. Now, for any n in N , $n^{-1}Dn$ will be an M -subgroup of H' such that its image in the map on to H'/N is J/N . Hence we have $n^{-1}Dn = D$, i.e., $D \triangleleft DN$. But $D \cap N = 1$, and so $DN = D \times N$. This will hold for each M -factor of K/N , and so K satisfies the hypotheses of (3.1.9). Thus $K = D_1 \times \dots \times D_r \times N$, where the D are M -subgroups of G .

4.3.6 Lemma. Let D be an M -subgroup of G , and put $X = C_G(D)$, $Y = C_G(X)$; then XY contains every M -subgroup of G .

Proof: XY certainly contains D itself. Let D' be any other

M-subgroup of G , and let H, H' be M -normalisers of D and D' respectively, each having only b generators. Let $L = \{H, H'\}$; then L has at most $2b$ generators, and so $L \in \mathcal{U}$, and $\{D, D'\} \leq w(L)$. By (4.2.6) D has projection D_i or 1 on each M -factor, D_i , of $w(L)$. If D has projection 1 on D_i then $D_i \leq X$. If it has projection D_i , then $D_i \leq Y$; for let $x \in X$, and consider the group $P = \{L, x\}$. P has at most $2b + 1$ generators, and so $P \in \mathcal{U}$, and D, L and P satisfy the conditions of (4.2.9). But $x \in C_P(D)$, and so $x \in C_P(D_i)$. But this will hold for every x in X , and so $D_i \leq C_G(X) = Y$. Hence $w(L) \leq \{X, Y\} = XY$ (since, by (4.3.3), $X \triangleleft G$). Thus $D' \leq XY$.

4.3.7 Lemma. If I is an M -subgroup of G , and J is a finite direct product of M -subgroups of G , then the subgroup $\{I, J\}$ which they generate is contained in a finite direct product of M -subgroups of G .

Proof: Let J be a direct product of s M -subgroups. We proceed by induction on s . If $s = 1$, let H and H' be M -normalisers of I and J respectively, with b generators. Then $L = \{H, H'\} \in \mathcal{U}$, and $\{I, J\} \leq w(L)$, a (finite) direct product of M -subgroups of L , and thus of G .

Now suppose $s > 1$, and assume the lemma to be true when J is a direct product of fewer than s M -subgroups. Then $J = D_1 \times J_2$ say, and $\{I, D_1\} \leq K_1$, $\{I, J_2\} \leq K_2$, where K_1 and K_2 are (finite) direct products of M -subgroups. Let $X = C_G(D_1)$, $Y = C_G(K_2)$, so that $X \triangleleft G$ (4.3.3) and thus $Y \triangleleft G$. Also, $J_2 \leq X$, $D_1 \leq Y$, so that, if

$K = \{I, J\}$ then $K \leq K_1X$, and $K \leq K_2Y$. But, by (4.3.6), every M -subgroup of G is contained in XY , so that $K_1 \leq XY$, $K_2 \leq XY$, and $K \leq XY$. By (4.1.2), K_1X/X is a direct product of M -subgroups of G/X . If R/X is one of these factors, then it corresponds, in the natural isomorphism between K_1X/X and $(K_1X \cap Y)/(X \cap Y)$ to $(R \cap Y)/(X \cap Y)$, and by (4.1.4) this is an M -subgroup of $G/(X \cap Y)$. Thus $(K_1X \cap Y)/(X \cap Y)$, and, by symmetry, $(K_2Y \cap X)/(X \cap Y)$ are direct products of M -subgroups of $G/(X \cap Y)$. But these groups, being contained respectively in $Y/(X \cap Y)$ and $X/(X \cap Y)$, generate their direct product. That is,

$$((K_1X \cap Y)(K_2Y \cap X))/(X \cap Y)$$

is a direct product of M -subgroups of $G/(X \cap Y)$. By (4.3.5), since $X \cap Y$, being the intersection of a group with its centraliser, is abelian, $(K_1X \cap Y)(K_2Y \cap X) = E \times (X \cap Y)$, where E is a direct product of M -subgroups of G . Finally, from $K \leq XY$, $K \leq K_1X$, $K \leq K_2Y$, we have that $K \leq (K_1X \cap Y)(K_2Y \cap X)$. (For, if $k = xy = k_1x' = k_2y'$, where $k_1 \in K_1$, $k_2 \in K_2$, $x, x' \in X$, $y, y' \in Y$, we have that $y = x^{-1}k_1x' \in K_1X$, since X is normal, and so $y \in K_1X \cap Y$. Similarly $x \in K_2Y \cap X$.) Thus $K \leq E \times (X \cap Y)$. But K , being generated by M -subgroups, is its own derived group, so that its projection on $X \cap Y$ is 1. Hence $K \leq E$, a group of the required form.

4.3.8 Lemma. Let $G \triangleright K = \{D_1, \dots, D_s\}$ where the D_i are M -subgroups of G ; then K is a subgroup of a (finite) direct product of M -subgroups of G .

Proof: We proceed by induction on s . The lemma is certainly true

for $s = 1$. Assume it to be true for $s-1$; then $K = \{I, L\}$ where $I = D_1$, and $L = \{D_2, \dots, D_s\} \leq J$, a (finite) direct product of M -subgroups of G . From the previous lemma we have that $\{I, J\}$, and thus K , is contained in a (finite) direct product of M -subgroups of G .

4.3.9 Lemma. $w(G)$ is a (finite) direct product of M -subgroups of G , each of which is normal in G .

Proof: By (4.3.1), $w(G)$ is generated by a finite number of M -subgroups of G , and thus by (4.3.8) is contained in a (finite) direct product of M -subgroups of G . But (4.3.2) each such M -subgroup is itself contained in $w(G)$, and so $w(G)$ must actually be equal to this direct product. The normality of each M -factor follows from (4.3.4), since $w(G) \triangleleft G$.

4.4 Proof of Theorem

In order to show that $\mathcal{V}$ is Cross, we must show that it is contained in a Cross variety (3.3.3), and, for this variety, we choose $\mathcal{V}^{(n)}$, where $n \geq n_0 = \max(a, 2b + 1)$.

From (3.2.2) we know that this variety has a finite basis for its identical relations, so it remains to show that finitely generated groups in it are finite, and that it contains a finite number of

critical groups.

Now, from (4.3.9), we have that, if G is a finitely generated group in $\mathcal{U}^{(n)}$, then $w(G)$ is a finite direct product of M -subgroups of G , each of which is normal in G . Since $G/w(G)$ is finite, it follows that G is finite, and, if G is critical, then either $w(G) = 1$, in which case $G \in \mathcal{OL}$, which, by hypothesis, is Cross, and so possesses only a finite number of critical groups, or $w(G) = M$. In this case, since $w(G) \cap C_G(w(G)) = 1$, we must have $C_G(w(G)) = 1$, and so G is a subgroup of the full automorphism group of M . Only a finite number of such groups exist, and the truth of the theorem follows.

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IDENTICAL RELATIONS IN SIMPLE GROUPS

A B S T R A C T

An identical relation of a group G is an equation of the form $w(x_1, \dots, x_n) = 1$, where w is an element of the free group generated by $x_1, \dots, x_n$, which is satisfied by any substitution of elements of G for the variables $x_1, \dots, x_n$. A consequence of a given set of identical relations W is a relation which holds on every group on which each member of W holds. A set of identical relations is said to be closed if it contains all its consequences. As a special case of a theorem of G. Birkhoff, contained in his paper "On the Structure of Abstract Algebras", Proc. Cam. Phil. Soc. 31 (1935) 433-454, we have that there is a one-one correspondence between varieties of groups and closed sets of identical relations.

A set of identical relations is said to be finitely based if it is the set of consequences of a finite subset (which can be taken to consist of just one element, the direct product of the elements of this subset). In his paper "Identical Relations in Groups, I", Math. Ann. 14 (1937) 506-525, B. H. Neumann considers the question of

whether the identical relations of a given variety (and, in particular, the variety generated by a finite group) are finitely based. He shows this to be true for a variety of abelian groups, and R. C. Lyndon, in "Two Notes on Nilpotent Groups", Proc. Amer. Math. Soc. 3 (1952) 579-583, extends this to nilpotent groups. If $\mathcal{O}$ and $\mathcal{L}$ are varieties, then $\mathcal{O}.\mathcal{L}$ is the variety of all groups with a normal subgroup in $\mathcal{O}$ whose factor group belongs to $\mathcal{L}$. G. Higman in "Some Remarks on Varieties of Groups", Quart. J. of Math. Oxford (2) 10 (1959) 165-178, shows that, if $\mathcal{O}$ is nilpotent, then the identical relations of $\mathcal{O}.\mathcal{L}$ possess a finite basis if those of $\mathcal{L}$ do.

None of these results requires that the variety concerned should be generated by a finite group. This should prove a more manageable problem, since a theorem of Neumann's tells us that the set of those identical relations of a finite group which involve a bounded number of variables has a finite basis.

Now, a subvariety of a variety which possesses a finite basis for its identical relations does not necessarily have this property, otherwise we would have immediately that every variety, being a subvariety of the variety of all groups, which has no identical relations, possesses a finite basis. This difficulty is overcome by D. C. Cross in his D. Phil. Dissertation, Oxford, 1960. He introduces

a type of variety satisfying certain finiteness conditions (one of these being the possession of a finite basis for its identical relations) which we shall call a Cross variety. He proves that any subvariety of a Cross variety is itself Cross, so that, in order to prove that a given finite group G has a finite basis for its identical relations, it is sufficient to show that it is contained in a Cross variety.

An important concept in this work is that of a critical group, which is a group not contained in the variety generated by its proper factors (if $G \geq H \triangleright K \geq 1$, then H/K is a factor of G ; it is proper unless $G = H$, and $K = 1$). A critical group has a unique minimal normal subgroup, which must be either elementary abelian, or a direct product of isomorphic (non-abelian) simple groups. Here we are concerned with the latter type of critical group, and we prove:

Theorem. Let B be a critical group whose minimal normal subgroup is non-abelian. If $\mathcal{O}$ is a Cross variety which contains every proper factor of B , then the variety generated by B and $\mathcal{O}$ is Cross.

We make use of the fact that the minimal normal subgroup M of B is a verbal subgroup, in fact, it is $w(B)$, where $w(x_1, \dots, x_a) = 1$ is the basis of the identical relations of $\mathcal{O}$. We define a type of

subgroup called an M -subgroup, which has similar properties to those of M , and show that, if G is any finitely generated (and thus finite) group in $\mathcal{U}$, the variety generated by B and $\mathcal{O}$, then $w(G)$ is a direct product of M -subgroups of G , each of which is normal in G .

By making use of various properties of M -subgroups, we show that the w -group of any finitely generated group in $\mathcal{U}^{(n_0)}$, the variety of those groups satisfying the identical relations of $\mathcal{U}$ involving at most n_0 variables (for some fixed n_0), also has this form. From this we prove that $\mathcal{U}^{(n_0)}$ is a Cross variety. But $\mathcal{U} \subseteq \mathcal{U}^{(n_0)}$, and so $\mathcal{U}$ is Cross, as required.