Equals in the wild: How mathematical equality is talked about in lessons

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Abstract

Equality is a nuanced concept in mathematics, and the equals sign is used and interpreted in different ways. During professional development in language-responsive teaching, a group of teachers discussed the use of ‘equals’ in algebra lessons. This article considers how three of these teachers went on to invoke equality in their classroom interactions when teaching other topics. The cases observed suggest constructive resonances between conceptions of ‘equals’ and geometric relationships, and include examples of teachers balancing precision with fluency. We argue that attention to language in one mathematical topic may extend helpfully to other topics, but this is not automatic.

Keywords: angles, equals sign, notion of equality, language-responsive mathematics teaching

Introduction

The concept of equality is fundamental throughout school mathematics. Nevertheless, it has been well established that students come to understand the term ‘equals’ and the equality sign in a variety of ways (Kieran, 1981). Early mathematical activity can instil and reinforce a recognition of = as a “stimulus to do something” (Behr et al., 1980, p. 13), and although later work brings in the possibility of seeing this symbol as a signifier of a relationship of sameness or equivalence (Rittle-Johnson et al., 2011), this reading is not always fully accepted by students (Sumpter & Löwenhielm, 2022), leading to a range of interpretations which can impact later student performance in arithmetic and algebra (e.g., Knuth et al., 2006).

The authors were involved in facilitating a professional development project based around language-responsive mathematics teaching (Erath et al., 2021) across the secondary-school curriculum. At an early point in the programme, the participating teachers trialled activities and had discussions about teaching algebra, starting from the context of linear equations. This included discussion of the challenges surrounding ‘equals’, as described above. For example, it was suggested that small choices – such as saying ‘is equal to’ in place of ‘equals’ when a relationship is being represented by the equals sign – might, over time, support students in moving away from a predominantly operational conception. The teachers then went back and tried a selection of activities and approaches out within their own classrooms.

After moving on to different topics in the programme, we were curious to investigate whether, how, and to what extent the renewed awareness of attention to language which the teachers had developed when working with equals in the context of algebra might transfer to other curriculum topics. How would the teachers treat the concept of equality when the focus of the lesson switched to geometry or probability? In other words, and borrowing a phrase from Hutchins (1995), what happens to equals “in the wild”?

Conceptions of ‘=’

In formal mathematics, the equals sign implies equality, the canonical example of an equivalence relation. However, research has shown that the equals sign is used and understood in different ways within the teaching and learning of mathematics (e.g., Kieran, 1981; Matthews et al., 2012), and that it can invoke diverse notions of “sameness” and equivalence. This research distinguishes between two main ways in which the equals sign can be understood: operationally and relationally.

Many students first meet the equals sign in contexts where an operational view is most useful, where the equals sign comes before the answer and can be read as an instruction to calculate (Behr et al., 1980). As they progress, this meaning broadens to include the equals sign as representing a relationship (Sumpter & Löwenhielm, 2022), where the equal sign can be read as the association between two elements that are comparable. Within the operational view, Rittle-Johnson et al. (2011) make a further distinction between rigid operational and flexible operational, where a rigid conception is restricted to calculating from left to right.

By way of contrast, the relational view of the equals sign focuses on the relationship between, or the “sameness” of, two expressions or quantities. The task “find the missing number in 10 + 7 = ☐ + 1” makes more sense if the equals sign is read as “is the same as” rather than an instruction to calculate; a relational view also allows students to appraise statements such as “2 + 5 = 5 + 2” (Matthews & Fuchs, 2020). However, there are mathematical contexts where the operational view remains necessary (e.g., 3 + 4 = ?), and this is not mutually exclusive of recognising how the result of an operation relates to the original expression (3 + 4 is the same as, or has the same value as 7). Stephens et al. (2013) make a distinction between a relational-computational view, where establishing the sameness of two expressions requires computation and a relational-structural view, where equivalence is recognised without the need for computation. Kieran and Martínez-Hernández (2022) further distinguish between visible sameness and invisible sameness. This distinction is illustrated through transforming equations, in order to solve them or identify an identity where the equals sign within the equation is visible, but the equivalence of rearrangements of these equations (top-down) is invisible.

Jones and Pratt (2012) make the case that a full relational conception of the equals sign should move beyond an appreciation of accepting sameness to include a substitutive interpretation, that is understanding that one expression can be substituted by an equivalent one. Kieran and Martínez-Hernández (2022) have gone on to argue that a substitutive interpretation depends on the support of a sameness interpretation, but also that substituting demonstrates sameness.

The relationship between these different interpretations of the equals sign is not unproblematic, in part because they draw on complex notions of equality and equivalence. Rittle-Johnson, Matthews and colleagues argue, for example, that knowledge of mathematical equivalence should be seen as a continuum rather than levels or stages in development (Matthews et al., 2012; Rittle-Johnson et al., 2011) and this is supported by other research showing that students’ understanding can draw on each of these interpretations in different ways at the same time (Sumpter & Löwenhielm, 2022). Further, while students could have different conceptions of the equals sign and the underlying concepts of equality and equivalence, they often express these in similar ways e.g. “the same as” (Lee & Pang, 2021). Furthermore, when asked to give or rate definitions of equivalence, these definitions are not necessarily representative of students’ larger understanding (Sumpter & Löwenhielm, 2022). Thus, the equals sign and the language used to accompany this symbol can mask the underlying relationship or interpretation needed to make sense of how it is being used. The meaning of sameness can likewise vary across mathematical contexts (Kieran & Martínez-Hernández, 2022; Melhuish & Czocher, 2020), primarily because “we often mean that the two objects are the same *in some (meaningful) respect* but are not the same in *every respect*" (Melhuish & Czocher, 2020, p. 39, italics in original).

Nevertheless, using the equals sign to represent an equivalence relation is widely considered to be fundamental to algebraic thinking (e.g., Donovan et al., 2022; Kieran, 2022; Stephens et al., 2013). Developing this understanding of the equals sign has been shown to be related to students' early algebraic competence, including their ability to solve equations (Hornburg et al., 2022; Matthews & Fuchs, 2020). This view of the equals sign is needed to work with equations where the operations appear on the right-hand side or both sides of the equals sign, rather than solely on the left-hand side (e.g., solving $25=5(x+3)$ or $5(x+3)=6(x+2)$ for x). Importantly, a relational perspective requires students to go beyond recognising the equivalence relation to acting upon it (Mason et al., 2009).

Most of this research into student performance has largely used assessments or specific tasks focused on students' conceptions of the equals sign. These tasks draw on the different ways in which the equals sign is used and the different structures being communicated within school mathematics. In the classroom, students have to navigate across these different uses and structures which are not necessarily made explicit. It is not necessarily obvious which meaning of the equals sign is most relevant. This is particularly the case in lessons involving a variety of algebraic manipulations such as those described by Kieran and Martínez-Hernández (2022) where students might need to simplify expressions, apply operations to both sides of the equals sign in order to reach a specific value for an unknown, or apply operations to demonstrate an identity, as well as assign specific values to variables (for example). Students also have to manage the use of the equals signs in different domains (Cook et al., 2022).

While much of the research within mathematics education has focused on the use of the equals sign in numerical or algebraic expressions and equations, the concepts of equality and equivalence pervade mathematics and school mathematics (Cook et al., 2021). There has been extensive research examining the teaching and learning of equivalent fractions where recognising that two fractions "have the same value" involves understanding their equivalence, rather than a more operational perspective where equivalent fractions are calculated using specific methods. Yet the connection between the meaning of the equals sign in these contexts is rarely made. Cook et al. (2022) have developed a framework focusing on students' reasoning about equivalence that is designed to cross these different domains. In this framework, they identify three interpretations of equivalence: common characteristic, descriptive, and transformational. They also offer illustrations of students' reasoning about equivalence in fractions, modular congruence, abstract algebra and combinatorics. Importantly they emphasise the importance of students recognising the equivalence of forms generated through transformations.

Polysemy, ambiguity and language in mathematics

The summary above has detailed how students may come to interpret and use the equals sign in diverse ways. A related challenge in teaching is then to help students recognise this polysemous nature of the equals sign, and of equalities such as $3+2=5$ or $3+x=5$, alongside the specific meaning involved in every situated use. Although teachers may expect students to infer correctly how the equals sign is being used across types of mathematical tasks including arithmetic and algebraic equalities, these distinctions may not be easy or immediate for students.

Pimm (1981) observed that in the English language different words and phrases are polysemous in the mathematics register, and that this polysemy can be extended to symbols. The same symbol in mathematics can be associated with different concepts and meanings, and these differences can be communicated by using different words and phrases in talking, writing or thinking. Mamolo (2010) similarly referred to polysemous symbols, such as '+¿' and '1', and their nature as signs of ambiguity. This ambiguity is not a disadvantage *per se*. It is part of mathematics, and it can be addressed in teaching as supportive of students' mathematical thinking and learning. Zazkis and Kontorovich (2016) discussed the polysemous symbol '-1' in the mathematics register, and how it can indicate both an inverse function and the reciprocal of a rational number.

When writing or speaking in English, we can use the same expressions and the equals sign '=' to create several different statements. Some of these more clearly entail an operational conception of '=' (e.g., "three plus two gives five", "three plus two makes five") than others (e.g., "three plus two is five", "three plus two is also five", "five is also three plus two"). The words and phrases we chose, therefore, support some meanings over others for the symbol and the mathematical expressions and statements.

The distinction between meanings for polysemous symbols (and words and phrases) of the mathematics register often remains implicit in classroom teaching (Pimm, 2021). When such distinctions are not explicitly communicated or introduced, the ambiguity at the root of the variety of polysemous elements of the mathematics register is not then exploited for the benefit of mathematical thinking and learning. Language-responsive mathematics teaching (e.g., Avalos & Secada, 2019; Erath et al., 2021), however, might attend to this distinction and support the goal of supporting students' understanding of the equivalence concept across mathematical topics.

Far from simplified views and discourses of communication as requiring one linguistic form and one content meaning (Cowie, 2002), we can see polysemy and ambiguity as beneficial and intrinsic to communication, thinking and learning in mathematics. In mathematics teaching, disambiguation of polysemous words, phrases and symbols, such as the equals sign, is a powerful strategy for enacting students' mathematical understanding, and a responsive use of language is key for this. When teachers notice polysemy and ambiguity in mathematics they are better placed to help students notice themselves this polysemy and ambiguity and build on them. Polysemy needs to be noticed when it appears, and it needs to be created deliberately by showing tasks, expressions or topics in which the same word, phrase or symbol takes different meanings.

Methods

The presented study is part of the larger project "Developing language-responsive mathematics classrooms" (DLRMC) that aims at adapting and developing classroom resources and professional development (PD) materials for supporting teachers to address the linguistic challenges students meet when learning mathematics. DLRMC is a mixed-methods study that combines a design research approach for PD and classroom resources with some assessments of students' achievement in mathematics.

Relevant to the research presented in this article is the design research part (Bakker & van Eerde, 2015; The Design-Based Research Collective, 2003) of the larger project. Here, classroom and PD resources were developed collectively by members of the research team and seven teachers from English state schools. At the heart of the design processes was the intertwinement of principles for language-responsive teaching (Erath et al., 2021) with the learning and teaching of key mathematical ideas around linear equations, angles in parallel lines, and probability. Teachers from the project implemented the collectively designed resources in their classrooms and video-recorded their lessons with one camera focusing on the front of the classroom and showing the students from the back. Episodes from these videos were then used in follow-up PD sessions to enhance teachers' work following the Discipline of Noticing

(Mason, 2002) as well as to revise the materials as part of the design cycles.

The data for this article is drawn from three recorded lessons: two on angles and one on probability. The lessons are from classrooms in three different schools. The larger project follows a four-step-data-analysis method. For this article, we build on steps one and two. In the first step, all video data was analysed with a focus on lesson structure which involved coding lessons into episodes based on types of activity around tasks. In the second step, the quality of interaction was coded by two trained raters according to four of the domains of the Global Teaching Insights observation framework (Bell et al., 2020). For this article, we focus on episodes from the three lessons that were coded as including detailed student contributions.

The notion of "in the wild" is taken from a range of interactional focused research that attends to the use of different aspects of cognition in contexts where the focus of the interaction is not that aspect of cognition. It originates from Hutchins' seminal work studying problem solving on a US Navy ship (Hutchins, 1995). While this research often takes place in real-life or out-of-school contexts, we found the idea useful for thinking about how students experience the use of the equals sign and the accompanying concepts of equality and equivalence when lessons do not focus on explicitly teaching about the meaning of "=".

We adopted an ethnomethodological stance, looking specifically at how the equals sign arises in naturally occurring classroom interactions where the lesson is not focused on the equals sign specifically. The meanings for the equals sign are co-constructed through the interactions and are indicated through the words spoken, the symbols written, as well as the descriptions or explanations surrounding its use. The interactional approach focuses not only on how teachers or students use the equals sign, but also how this use is responded to by others in the interaction (Ingram, 2018, 2020).

Words and phrases such as "makes" or "gives" to describe equals might be associated with an operational view, whilst "the same as" might suggest a relational view (Knuth et al., 2006). These distinctions could be clarified within an ethnomethodological stance by bringing in additional interactional features from the lesson, such as gestures that could evoke particular conceptions which focus on the relationship between the two sides of the equals sign (Stylianou et al., 2024). However, some words, gestures and symbols such as "is" are more ambiguous. We therefore used some elements of Systemic Functional Linguistics (SFL) to provide a complementary perspective when examining the different ways in which the meanings of equality arise in interaction.

Within SFL (e.g., Halliday & Matthiessen, 2004; Morgan, 1998), verbs, or *processes*, encode activity in the clauses in which they occur. In the case of *is*, this can be the activity of identifying with something, of having an attribute, or of existing. These different meanings are bound up with different portrayals of the world, with the first two forms occurring widely in mathematics, construing mathematics as a world of relations between objects rather than, for example, a world of material actions (Halliday & Martin, 1993). For the same verb to make available this range of meanings presents long-recognised challenges (famously noted by Bertrand Russell (Davidse, 1992)). Rather than viewing this as a deficiency of language, SFL analysis involves unpacking the meanings made available by looking beyond the verb to the whole clause and the wider text and context and examining these linguistic choices in relation to alternative choices (Halliday & Matthiessen, 2004).

The different uses of *is* to identify or attribute a property differ in that only identifying is reversible (Halliday & Matthiessen, 2004). That is, *x is y* is reversible, but attributive clauses such as *x is 75°* or *ABCD is a parallelogram* are not usually reversed. It would therefore seem that unpacking these meanings is, unsurprisingly, bound up in the meanings of the objects and attributes as well as the processes used to relate them, but this raises questions about how students learn to distinguish between attributes and identification when these clauses both involve the same process.

Alternative phrasings of *A is B* include *A equals B* (which identifies A and B through process), and *A is equal to B* or *A is the same as B* (which attributes to A an unspecified property that B also has). These examples also illustrate an ambiguity related to the brevity of mathematical language in not specifying the sense in which A and B are equal or the same (e.g. value, size). In this sense, the mathematical use of *equal* and *same* may be less precise than everyday usage. One interpretation of this omission is that it foregrounds the shared-ness of an attribute and avoids complex nominal groups often employed for precision such as *[the size of angle x] is [the same as [the size of angle y]]*. From a mathematical perspective, making this aspect explicit could be viewed as a form of redundancy, because it is taken as shared by experts (Morgan, 2001), and yet omitting this detail requires students to draw meanings from the context for themselves.

Results

We offer here three findings from our consideration of the data.

Resonances between different conceptions of 'equals', geometrical relationships and probability

First, choices and variations within the language used in the two angles lessons suggested a resonance between certain geometric relationships and different conceptions of equality.

When working with pairs of alternate, corresponding, or vertically opposite angles, teachers and students regularly used the term 'equal' in reference to the two considered angles having the same size, thus invoking a relational conception of equals. For instance, whilst sharing with the students a diagram similar to Figure 1 below, Teacher 1 asked *"is f equal to any angle here?"* and Teacher 2 voiced *"e is equal to a".* Equals was often invoked verbally, and sometimes links were made to written mathematics: *"Do we agree, a is equal to c?* [Writes $a=c$ on the board.] *So that's the kind of equation I'm looking at."* (Teacher 1).

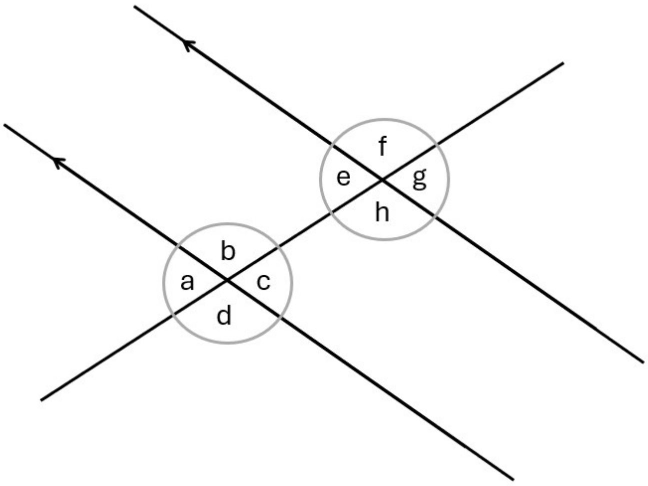

Figure 1: A diagram similar to one used by Teachers 1 and 2

Versions of "X is equal to Y" were by far the most common type of case observed in the angles dataset. There were also some related variations which continued to suggest a relational conception of equals. On occasion, teachers and students opted to use the word 'same' in place of equal or equals: *"They are the same, why? What type of an angle are they?"* (Teacher 2). Here, the teacher adds to the idea of a relationship by prompting students to name that relationship. Elsewhere, Teacher 2 was observed extending the common "equal" phrase by adding "to each other", using this, as well as repetition and an increase in specificity, to direct attention:

> *[Student name] has told us that some of the angles are equal to each other, that opp- some of the opposite angles are equal to each other. [Student name] which two angles could be equal to each other? (Teacher 2)*

By way of contrast, in cases when the focus was on relationships such as angles around a point or co-interior angles, the use of equals more closely evoked an operational conception, at least at first:

Teacher 2:	(whilst writing $a+b+c+d=360$) a plus b plus c plus d equals three hundred and sixty degrees. Can someone tell me why is that the case? Why – what is the reason for this answer? What's the reason for this equation? Err, [student name], what's the reason for this?
Student:	'cause it's a complete circle?
Teacher 2:	It's a complete circle. So angles around a what equal three hundred and sixty?
Student:	A point?
Teacher 2:	Yeah, good. Angles around a point equal three hundred and sixty.

There is some ambiguity in this interaction. Although the original case of $a+b+c+d=360$ could be argued to have involved an operational conception of the equals sign, since the letters referred to a specific example within a diagram on the board (see Figure 1), both the student and the teacher begin to refer to a general rule, which raises the possibility of a relational view.

A similar caveat could be applied to a related instance, where the teacher challenged the use of the word 'equal' to describe co-interior angles, leading a student to suggest "*co-interior angles add up to a hundred and eighty degrees*"*.* At another point, the same teacher, when working with an individual, asked about some angles "*what would they make?*"

It was explicit in one of the angles lessons that the two conceptions of equals were beginning to be brought alongside each other as the students worked on a task which involved multiple angle relationships:

> *So, think about using some numbers if you haven't done so already. I'm seeing lots of you being able to compare things that are the same, so you're saying 'this is equal to this, this is equal to this', but think about it – if you added two of the angles together, what could they make?* (Teacher 2)

Here the teacher is encouraging their students to diversify their responses, moving from only offering statements which are fully predicated on a relational use of equals to including further statements where equals is working (at least in part) operationally as a lead into an angle sum. At a different point in the same lesson, when three rules – for alternate, corresponding, and co-interior angles – were summarised on the board by Teacher 2 with a set of three diagrams, equations, and statements, 'equals' might likewise be considered to be functioning in both a relational and operational sense.

Compared with the angles lessons, there were only a small number of cases involving equals or equality in the probability lesson. Nonetheless, one spoken connection arose when the class was discussing whether a game was fair. Referencing the options for winning or losing the game, Teacher 3 commented, "*So being fair means equal chances, that is what you understand".* Equal here is functioning as an adjective in a way which is distinct from how it was observed being used in the angle lessons.

Fluency, precision and reasoning

The dialogue in the lessons also exhibited some features which pointed towards compromises being made between mathematical precision and fluency in communication. We have written elsewhere on how effective fluency in classroom talk is steered by what is important and what is sufficient (Ingram et al., 2024). Zazkis (2000) describes how teachers engage in code-switching between the formal mathematics register and informal speech, addressing the teacher's dilemma (after Adler, 1998) of modelling communication skills whilst encouraging participation and facilitating access.

The most common type of example in the cases from the angle lessons consisted of phrases similar to Teacher 1's "*a is equal to c*", which only holds from a strict mathematical perspective if "*a*" and "*c*" are mutually understood to indicate the sizes of the angles, and not only their names, or some other qualitative sameness such as both angles being acute or visually similar. (We do, however, recognise that this shorthand might be more or less salient for readers working within different geometry traditions.) This ellipsis was later highlighted in writing when Teacher 2 projected a preprepared question which contained both the full and abbreviated forms: "*Angle d is 47°. Calculate the size of angle x.*" Such phrasing could be inferred to be intended to balance fluency and precision.

More substantial contractions surrounded the language used to justify why two angles were in fact equal. Teacher 1 appeared satisfied with writing or saying a reference to the relevant fact, e.g. "*corresponding angle*", whilst Teacher 2 was observed promoting a longer form of justification: "*because* *corresponding angles are equal*", "*because vertically opposite angles are equal*."

These two types of statement direct the students' activity and attention differently, and both are abbreviated forms; although their approach is longer, Teacher 2 still implies the naming of the two angles under consideration, opening with 'because', and a further stage of precision could also be added by clarifying that the lines were parallel. Nonetheless, we note here that reducing the reasoning step as far as naming a relationship risks obscuring the place of equality within that relationship. If a student's attention is only ever centred on identifying the correct shape or association from a given set, they might have a diminished experience of recognising and working with a relational conception of equals.

The probability lesson included other cases where mathematical precision was limited in the service of fluency. Teacher 3 appeared to use the equals symbol as a notational shorthand for "is", writing *W = 12* on the board to signify that there were twelve outcomes which resulted in a win, and shortly afterwards $W=\frac{12}{20}$ to record the probability of winning. From an SFL perspective, these are attributive, non-reversible statements which appear to make limited appeal to mathematical equality.

Further connections between equality and geometry

Both of the angle lessons included cases where connections were made between the idea of equality and the vocabulary of geometry. In Teacher 1's lesson, a student introduced a new term in the middle of some reasoning, which was promptly questioned:

Teacher 1:	OK, again, why?
Student:	'cause a and e are similar, and …
Teacher 1:	Is it similar?
Student:	Equal.
Teacher 1:	Equal, good.

It is not possible, though, from the interaction to determine whether the student was using the word 'similar' in a strict mathematical sense or otherwise, for example to reference how the position and orientations of the named angles were visually alike in the diagram, drawing on the everyday use of similar in English.

A comparable situation arose in Teacher 2's lesson, this time having more of an impact on what followed. At the start of this case, the teacher is asking the class why two angles were equal. After asking for an answer and trying to build on one student's partial response, confirming that "*it is* *something to do with position"*, they offer a word search prompt for the term 'corresponding'. This is not the answer that is given; the teacher's tone indicates that they are surprised by the result, but they choose to take advantage of it:

Teacher 2: *Does anyone think, it begins with a c.*

Student: *Congruent.*

Teacher 2: *They are congruent! What makes these two angles congruent?*

The teacher continues in this manner, asking the students "*what makes two things congruent to each other?"*, thus echoing the *"equal to each other*" phrasing they had used earlier in the same lesson. The teacher even shares with the students, "*we're going off on a little tangent, but I think it's a good tangent*", before breaking off the class into pairs to talk about congruency, then bringing them together for further discussion.

Some opportunities for making connections were only partly leveraged. At one point in Teacher 1's lesson when the class was looking for pairs of equal angles, one student offered "*f and d*". These two angles did have the same size but did not match one of the established angle relationships; however, f and b were corresponding angles, and b and d were vertically opposite. The teacher lays out the chain of reasoning, posing the question "*If f is equal to b and b is equal to d … is he right then?*". However, instead of explicitly drawing attention to the transitivity property of the equality relation, the teacher appears to accept sounds of consent from much of the class, notes that they have "*made a leap there, that f is also equal to, err, to d"* and moves the discussion on.

Discussion and conclusion

The cases above provide evidence that equals can operate well "in the wild". The clearest instance of a productive resonance occurred in the angles data. From an interactionist perspective, we can offer that when moving between specific and general examples of angle relationships, and among different angle relationships, the students in the data had to act in ways that were consistent with both operational and relational conceptions of the equals sign. In this way, we might determine that students who can act flexibly and in line with both conceptions are better equipped for subsequent work with angle relationships, and not just for moving on with number and algebra.

It is also possible, though, to envision support working in the other direction. Angles offer a context for students to meet and consider the polysemy of the equals sign; the specific statement “in this diagram angle b equals 40 degrees” and the general one that “the sum of interior angles in a triangle is always equal to 180 degrees” form a pair of examples through which the polysemic aspect of equals can be explicitly highlighted by a teacher, or noted by a student. The diagrams of an angle lesson might likewise operate as visual cues for students as they negotiate this lexicosemantic variation, perhaps even more so when dynamic diagrams are used to emphasise generality. Would a relational conception of the equals sign in $a=b$, where a and b are alternate angles inside parallel lines, be reinforced by seeing how the statement remains true even as the lines – or even the diagram as a whole – is adjusted?

There are however limitations inherent in the data which have specific relevance to any argument for a (potentially) productive resonance as described above. All of the episodes of the data identified for analysis involving Teacher 1, and the majority of those involving Teacher 2, were based around the use of a diagram with no numerical values (see Figure 1 above). This would have impacted the balance of opportunities for students to offer responses which were clearly consistent with operational and relational views of the equals sign. The video data also only affords a view of the front of the classroom, rather than students’ individual work, favouring the teacher and students who were speaking. Nonetheless, the cases are suggestive of connections which might be meaningfully leveraged, and this is a potential area for further research – if this aspect of these tasks had been pre-empted as part of the professional development, would it have made a difference to practice or outcome? A related line of inquiry is whether and how a dynamic approach to angle diagrams (or an approach more in line with transformational geometry) might further develop such connections – would being able to ‘move’ angles on a diagram be sympathetic to a substitutive conception (Jones & Pratt, 2012) as the representation of one angle could be visually put in the place of another?

However, the data also began to suggest that equals might not automatically do well “in the wild”. The limited probability data offered a potential link to “equal chance”, but this was not developed in this case by the teacher. Likewise, there was some blurring of mathematical concepts as equality was brought alongside both congruence and similarity. The limited number of cases involved limits discussion here, but existing research (such as Cook et al., 2022) highlights both potential and risk; in the words of Melhuish and Czocher (2020), “[t]reating sameness as-if-well-defined in pedagogical contexts obscures the students’ thinking” (p. 38).

A second main finding in the data was the many examples of teachers balancing fluency and precision. In some cases, such as when the students were volunteering angle relationships grounded in a visual reference, different (possible) interpretations of equals occurred simultaneously, and this had to be navigated by both the teacher and the students. An additional complication arose when the students were invited to give reasons for their answers, as the expectations for this practice also demanded attention and co-construction.

In summary, the data supports the position that, in mathematics teaching, language matters – and not just the language which is immediate to the subject of a lesson. Whilst research has demonstrated the importance of paying attention to the language attendant to specific topics, this short study offers that diverse lessons may also give rise to opportunities for learners' conceptions of fundamental concepts, such as equals, to be reinforced, clarified or extended. However, such occasions and outcomes are not guaranteed; the specific cases outlined above also offer that a teacher's attention to language surrounding a concept cannot be assumed fully and automatically to transfer between curriculum topics. Equals can work well "in the wild" – to the benefit of both teachers and students – but it merits our attention and handling with care.

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