Utility-based pricing of stochastic income



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Abstract

In this thesis we investigate an optimal investment problem for an investor who also receives an unhedegable stochastic income stream. We extend the log-Brownian model of Henderson [18] to a stochastic volatility model where the underlying market is incomplete and the income is a function of the stock volatility. With exponential preferences, explicit solutions to the primal and dual problems are possible using extensions of the distortion power solution of Zariphopoulou [48] and Henderson [18]. We derive expressions for the utility indifference price and the associated marginal utility-based price of the stochastic income stream.

Even in the absence of stochastic income the market is incomplete and the optimal dual measure is the minimal entropy measure $\mathbb{Q}^E$. We show how this horizon dependent measure serves as a pricing measure in the utility indifference valuation scheme. This is in contrast to Henderson [18] where the underlying market is a Black-Scholes model and the optimal dual measure, without the income, is the minimal martingale measure $\mathbb{Q}^M$. The horizon dependence of the pricing kernel in the stochastic volatility model can potentially lead to inconsistencies in the pricing of claims with multiple payment dates. We carefully construct a forward utility function which restores the minimal martingale measure as the dual minimiser for the model without the claim, thus removing the horizon dependence.

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Chapter 1

Introduction

This thesis investigates a utility-based method for valuing a claim which pays stochastic income at a continuous rate in a stochastic volatility diffusion model. This is an extension of the classical utility-based pricing paradigm, which has almost exclusively been addressed to the pricing of claims paying at a single future date, to a claim paying at multiple dates. Our goal is to examine the dual side of the problem, and to investigate the dependence of the resulting pricing measure on the times at which the claim makes payments.

Utility maximisation has remained a cornerstone in portfolio investment theory for over 40 years. Merton [37], one of the earliest and most widely cited papers in this field, derived a solution to a continuous optimal investment problem with intermediate consumption in a Black-Scholes market. Because of his many significant contributions this problem is now often referred to as the Merton problem. Given a fixed, finite time horizon T, and initial wealth x, an agent seeks to maximise expected utility of inter-temporal consumption and terminal bequest with dynamic trading in a financial market. With asset prices following continuous Markovian diffusions in a complete market, Merton used dynamic programming methods to derive a Hamilton-Jacobi-Bellman (HJB) equation for the value function representing the maximal expected utility. The HJB equation, a non-linear second-order partial differential equation, can be solved explicitly for power and logarithmic utility functions, a relative rarity in stochastic optimal control.

Subsequent research has extended Merton's problem in different directions, and with varying degrees of generality. The log-Brownian market in [37] was a complete market with the underlying asset price diffusion having constant drift and volatility.

Pliska [45], Cox and Huang [6, 7], and Karatzas et al. [30] developed a *dual* approach which exploits the martingale property of the wealth process to characterise the optimal strategy. This method did not require the Markovian assumption on asset price dynamics needed in the primal approach and so [30] worked with general Itô processes. The crucial result in the dual approach is the characterisation of the marginal utility of optimal terminal wealth (a non-negative random variable) as being proportional to the Radon-Nikodym density of the unique equivalent martingale measure (EMM) of the complete market.

Karatzas et al. [28] extended the duality technique to incomplete Itô process models. The multiplicity of EMMs in an incomplete market causes a modification of the characterisation of the optimal terminal wealth. The marginal utility of optimal terminal wealth is now proportional to the density of an *optimal* EMM, called the dual minimiser. This makes the dual methodology towards optimal investment and consumption more involved than in a complete market, but also much richer from a mathematical viewpoint. This avenue of research spawned a huge literature on the dual side of the problem. The theory when asset prices follow Itô processes is given an exhaustive treatment in the monograph by Karatzas and Shreve [31].

When asset prices follow general semi-martingales, the problem of optimal investment, without inter-temporal consumption, was developed by a number of authors, and culminated in the seminal work of Kramkov and Schachermayer [34]. They showed that a so-called reasonable asymptotic elasticity condition on the utility function was a necessary and sufficient condition for the crucial assertions of the theory (such as the existence and uniqueness of an optimal investment strategy) to hold.

For problems with intermediate consumption, the utility maximisation problem in semi-martingale markets has proven much more difficult, and is still an active area of research. Notable advances here include Karatzas and Žitkovic [32] and Žitkovic [49].

Another direction in which Merton's problem was generalised was the incorporation of transaction costs in asset trading. This was first accomplished in an infinite horizon consumption framework by Davis and Norman [10] and has spawned an enormous body of work. Notable in this regard is Cvitanić and Karatzas [8], who developed the dual approach to this problem, and this has recently been exploited

heavily by Muhle-Karbe and co-authors [25, 16] to develop asymptotic results for small levels of transaction costs.

Other directions in which Merton's problem has been extended include problems with stochastic volatility (for example Zariphopoulou [48]), real options (Henderson[19]), and infinite horizon consumption (Huang and Pagés [23]), possibly with bankruptcy (Karatzas et al. [29]), to name but a few.

In incomplete markets claims cannot be perfectly replicated and the seller is subject to unhedgeable risk. The problem of valuing derivatives in incomplete markets can therefore be viewed as an optimal risk management problem and hence as an extension of the optimal investment problem. The development of utility maximisation techniques for incomplete markets, applied to the economic principle of certainty equivalence, led to the idea of utility-based valuation of contingent claims, beginning with Hodges and Neuberger [22]. Utility maximisation acts as a selection criterion for a pricing measure from the infinitely many arbitrage-free pricing measures in an incomplete market. The whole area is reviewed in Carmona [5].

For exponential utility, the problem of a claim payoff with random endowment has been studied by many authors (see for example [11],[24],[43]), and the link to the dual side of the problem is well established. In virtually all this work though, the contingent claim is one which makes a single payment at a predetermined future date. One of the goals of this thesis is to consider the utility-based valuation of a claim which pays out at *multiple* dates. This issue is interesting because, in general, the dual minimiser, which serves as a pricing measure in a utility indifference valuation scheme, is horizon dependent. An issue arises as to whether income streams or American claims can be consistently valued within this framework, in the dual sense of selecting a pricing measure.

This has led to the development of alternative objective investment criteria, called horizon-unbiased utility functions by Henderson and Hobson [19, 21], or forward performance measures by Musiela and Zariphopolou [44, 42]. Berrier et al. [3] investigate the properties of these forward utilities.

In this thesis we first examine the horizon dependence of the pricing measure that emerges when classical utility-based methods are applied to the valuation of an income stream. We then apply the forward utility approach to the same problem, to eliminate the horizon dependence.

This thesis is organised as follows. Chapter 2 provides an overview of the primal and dual methods of utility maximisation with consumption on a finite horizon and provides an introduction to incomplete market portfolio optimisation. This sets the stage for subsequent chapters.

In Chapter 3 we review the infinite horizon optimal consumption problem in a complete market. The time-independence of the infinite horizon problems makes it easier to solve in some instances. We discuss in particular the infinite horizon Merton consumption problem with exponential utility. More than one solution to this problem has appeared in the literature, and indeed some of the work arising here appears to be incorrect. We discuss these issues in detail, and this is the first contribution of the thesis.

Chapter 4 contains the main contribution. We analyse a utility maximisation problem for an investor who receives an unhedgeable stochastic income stream in a stochastic volatility model. This work is contained in the paper [1], which is to be submitted for publication.

This extends a model of Henderson [18], who solved an investment problem in a Black-Scholes market in which stochastic income was also received. In our case, the underlying market is a stochastic volatility market, and the income is dependent on the stochastic volatility. This is a tractable model for which explicit solutions are available under exponential preferences using extensions to the distortion power solution found in [18], [40] and [48]. We derive explicit expressions for the utility indifference price of the income stream, and for the associated marginal utility-based price.

At first sight our model looks very similar to the one in [18]. However the extension of the underlying market to a stochastic volatility model introduces some significant differences when one examines the pricing measure that arises in valuing the income stream.

The presence of stochastic volatility changes the dual minimiser for the problem without stochastic income from the minimal martingale measure $\mathbb{Q}^M$ (uniquely defined in terms of the stock's market price of risk) to the minimal entropy measure $\mathbb{Q}^E$. The latter is the solution of a dual optimisation problem with a fixed horizon, and so is inherently *horizon dependent*, as we explicitly show. The consequence of this is that the marginal price of the income stream is the expectation of aggregated income with a pricing kernel that is horizon dependent, which is not the case in

[18]. One of our original goals was to show that, with underlying incompleteness, a classical applied to the valuing claims making payments at multiple dates.

We then examine ways of removing this horizon dependence using a carefully defined forward performance process. The adjustment from a classical utility to a forward utility turns out to be exactly the adjustment needed to restore $\mathbb{Q}^M$ as the dual minimiser in the zero income case. The horizon dependence in the marginal utility-based pricing kernel is thus removed. This is reminiscent of the early papers on horizon-unbiased utility, where American option problems were rendered insensitive to any preferred horizon. In particular Henderson [19] analysed a real option investment problem (a perpetual American option) which was given a meaningful solution by removing any preferred horizon from the associated investment problem without the real option. This was achieved by switching from a classical to a forward utility.

More recently, the same procedure has been carried out in Leung et al [35], who considered both forward and classical utility-based pricing of American options in a stochastic factor model, which is similar to ours. In all of these works, the adjustment to forward utility changes the optimal dual measure in the underlying market to $\mathbb{Q}^M$. To some extent, this is not a satisfactory answer to the problem in that $\mathbb{Q}^M$ essentially ignores unhedgeable risk. We finish with some concluding remarks.

Chapter 2

Utility maximisation on a finite horizon

In a classical investment-consumption problem an agent decides how to distribute and dynamically adjust their wealth amongst a collection of unpredictable financial assets. Additionally they decide on a proportion of wealth to be consumed at each moment in time. Both modern and classical theories of economic behaviour use utility functions to measure the satisfaction gained from wealth accumulated through investment and consumption. The objective of the investor is to maximise their expected utility of consumption and terminal wealth.

This chapter is split into two sections. The first section reviews the methodologies of primal and dual solutions to utility maximisation for complete markets and the second section provides an introduction to the dual approach for incomplete markets.

2.1 The complete market case

Merton [36, 37] was the first to provide a solution to the continuous-time portfolio selection problem. Using the Bellman principle of dynamic programming and techniques of stochastic control, Merton derived a non-linear PDE for the value function in a Markovian model. Later, a dual approach was developed by, among others, Pliska [45] and Karatzas et al. [30]. This approach is based on the martingale properties of the agents' wealth process and convex duality. In this section we compare both the primal and dual approaches for a fixed, finite, investment horizon.

2.1.1 Model formulation

Let $W = (W_t^1, \ldots, W_t^m)_{0 \le t \le T}$ be an m-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$ is the completion of the natural filtration generated by W;

$$\mathcal{F}_t = \sigma(W_s; 0 \le s \le t).$$

The multi-dimensional complete market M^0 consists of m risky assets and one riskless asset. The m risky assets, $S := (S_t^1, \ldots S_t^m)_{0 \le t \le T}$, are modelled as an Itô process satisfying

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^m \sigma_t^{ij} dW_t^j \right), \quad i = 1, \ldots, m,$$

and the riskless asset, S^0, has dynamics

$$dS_t^0 = r_t S_t^0 dt,$$

where r_t satisfies $r_t \ge 0$ for all $t \in [0, T]$. We assume throughout that the coefficients $\mu_t = (\mu_t^1, \ldots, \mu_t^m)^T$, $\sigma_t = (\sigma_t^{ij})_{1 \le i,j \le m}$ and r_t are progressively measurable with respect to the filtration $\mathbb{F}$. Further, we assume that the volatility matrix σ_t is invertible almost everywhere on $[0, T] \times \Omega$. Additional conditions on the model coefficients are given below.

We denote by $\pi := (\pi_t^1, \ldots, \pi_t^m)_{0 \le t \le T}$ the wealth invested in the risky assets and by $c := (c_t)_{0 \le t \le T}$ the rate of consumption of the investor. Consumption must be non-negative and satisfy $\int_0^T c_s ds < \infty$ almost surely. An investor trades a self-financing strategy (π, c) generating wealth $X := (X_t^{\pi,c})_{0 \le t \le T}$ following

$$\begin{aligned} dX_t^{\pi,c} &= \sum_{i=1}^m \frac{\pi_t^i}{S_t^i} dS_t^i + r_t \left(X_t^{\pi,c} - \sum_{i=1}^m \pi_t^i \right) dt - c_t dt \\ &= (r_t X_t^{\pi,c} - c_t) dt + \pi_t \big((\mu_t - r_t \mathbb{I}_m) dt + \sigma_t dW_t \big), \end{aligned} \tag{2.1.1}$$

where $\mathbb{I}_m$ denotes the m-dimensional column vector of ones. To ensure wealth does not become infinitely large we require that $\int_0^T \pi_t \big(\mu_t - r_t \mathbb{I}_m \big) dt < \infty$.

The market price of risk, or relative risk process, is the process $\lambda : [0, T] \times \Omega \to \mathbb{R}^m$ which solves

$$\sigma_t \lambda_t = \mu_t - r_t \mathbb{I}_m, \quad 0 \le t \le T. \tag{2.1.2}$$

The existence of this process is closely related to the absence of arbitrage, as given in Theorem 2.1.4 below.

We denote by $\widetilde{X}_t$ the discounted wealth process, $\widetilde{X}_t = X_t/S_t^0$, and similarly for the processes π, c and others defined later. This can be expressed in integral form as

$$\widetilde{X}_t^{\pi,c} + \int_0^t \widetilde{c}_s ds = x + \int_0^t \widetilde{\pi}_s \sigma_s (\lambda_s ds + dW_s). \tag{2.1.3}$$

In order for the Itô integral to be well defined we require $\int_0^T \parallel \widetilde{\pi}_t \sigma_t \parallel^2 \, dt < \infty$ almost surely. Consequently we adopt the following formal definition of a portfolio process provided by Karatzas [26].

Definition 2.1.1. *A progressively measurable, $\mathbb{R}^m$-valued, process $\pi := (\pi_t^1, \ldots, \pi_t^m)_{0 \le t \le T}$ satisfying*

$$\int_0^T \parallel \widetilde{\pi}_t \sigma_t \parallel^2 \, dt + \int_0^T \widetilde{\pi}_t \big(\mu_t - r_t \mathbb{I}_m\big) dt < \infty \ \ a.s.$$

is called a portfolio process.

We will also require the following definitions of the gains process and an arbitrage opportunity.

Definition 2.1.2. *For a portfolio process π the discounted gains process $\widetilde{G} := (\widetilde{G}_t^{\pi})_{t \ge 0}$ is defined as*

$$\widetilde{G}_t^{\pi} := \int_0^t \widetilde{\pi}_s \Big[(\mu_s - r_s \mathbb{I}_m) ds + \sigma_s dW_s \Big], \quad 0 \le t \le T.$$

Definition 2.1.3. *A portfolio π that satisfies*

$$\mathbb{P}(\widetilde{G}_T^{\pi} \ge 0) = 1, \quad \mathbb{P}(\widetilde{G}_T^{\pi} > 0) > 0,$$

for any $T > 0$ is called an arbitrage opportunity. We say that a market M^0 is arbitrage-free if no such opportunity exists.

Theorem 2.1.4. *(a) Suppose there exists a process $\lambda : [0,T] \times \Omega \to \mathbb{R}^m$ satisfying (2.1.2) and*

$$\mathbb{E}\left[\exp\left\{ \int_0^T \lambda_t^2 dt \right\} \right] < \infty. \tag{2.1.4}$$

Then the market M^0 is arbitrage-free.

(b) If the market M^0 is arbitrage-free then there exists an $\mathbb{F}$-adapted process $\lambda : [0,T] \times \Omega \to \mathbb{R}^m$ satisfying (2.1.2).

Proof: (a) Define the process $Z := (Z_t)_{0\le t\le T}$ given by

$$Z_t := \mathcal{E}(-\lambda \cdot W)_t, \quad 0 \le t \le T, \tag{2.1.5}$$

where $\mathcal{E}(\cdot)$ is the Doléans exponential. The Novikov condition (2.1.4) is sufficient to ensure Z is a martingale so, by Girsanov's theorem, we can define an equivalent measure $\mathbb{Q}$ with the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t, \quad 0 \le t \le T. \tag{2.1.6}$$

Then $W^{\mathbb{Q}}$ given by $dW_t^{\mathbb{Q}} := dW_t + \lambda_t dt$ is a $\mathbb{Q}$-Brownian motion and $\widetilde{S}$ is a $\mathbb{Q}$-local martingale. The First Fundamental Theorem of Asset Pricing [12] proves that if there is an equivalent measure under which the discounted asset price process is a local martingale then the market is arbitrage-free.

(b) Suppose the market is arbitrage-free. For $t \in [0,T]$ define

$$\begin{aligned} F_t : &= \{\omega \in \Omega : (2.1.2) \text{ has no solution}\} \\ &= \{\omega \in \Omega : \mu \notin \text{linear span of the columns of } \sigma\} \\ &= \{\omega \in \Omega : \exists v(t,\omega) \text{ with } \sigma^T(t,\omega)v(t,\omega) = 0 \text{ and } v(t,\omega)\mu(t,\omega) \neq 0\}. \end{aligned}$$

Define

$$\pi(t,\omega) = \begin{cases} v(t,\omega)\text{sgn}(v(t,\omega)(\mu(t,\omega) - r(t,\omega)\mathbb{I}_m)) & \omega \in F_t, \\ 0 & \omega \notin F_t. \end{cases}$$

The discounted gains process satisfies

$$\widetilde{G}_s^{\pi} = \int_0^s \frac{1}{S^0(t,\omega)} |v(t,\omega)(\mu(t,\omega) - r(t,\omega)\mathbb{I}_m)| \mathbf{1}_{\omega \in F_t} dt \ge 0, \quad 0 \le s \le T.$$

Since the market is arbitrage-free and the discounted gains process is non-negative for all $\omega \in \Omega$, it follows from Definition 2.1.3 that $\widetilde{G}_T^{\pi} = 0$ for all $\omega \in \Omega$. This implies F_t is empty and (2.1.2) must have a solution for almost all $(t,\omega) \in [0,T] \times \Omega$. $\square$

We assume that the market M^0 is arbitrage-free and that the market price of risk satisfies $\mathbb{E}\left[\int_0^T \parallel \lambda_t \parallel^2 dt\right] < \infty$. Let Z_t and $\mathbb{Q}$ be given by (2.1.5) and (2.1.6). Then $W^{\mathbb{Q}}$ given by

$$dW_t^{\mathbb{Q}} := dW_t + \lambda_t dt$$

is a $\mathbb{Q}$-Brownian motion and it follows from (2.1.3) that $\left(\widetilde{X}_t^{\pi,c} + \int_0^t \widetilde{c}_s ds\right)_{0\leq t\leq T}$ is a $\mathbb{Q}$-local martingale. If the stronger condition $\mathbb{E}\left[\int_0^T \parallel \widetilde{\pi}_s\sigma_s \parallel^2 ds\right] < \infty$ a.s. holds then $\left(\widetilde{X}_t^{\pi,c} + \int_0^t \widetilde{c}_s ds\right)_{0\leq t\leq T}$ is a $\mathbb{Q}$-martingale.

Utility functions measure the satisfaction gained from wealth and consumption and will be used to compare trading strategies.

Definition 2.1.5. *A function $U : (0,\infty) \to \mathbb{R}$ is called a utility function if it is continuously differentiable, strictly increasing, strictly concave and satisfies the Inada conditions*

$$U'(0) := \lim_{x\downarrow 0} U'(x) = +\infty, \quad U'(\infty) := \lim_{x\uparrow\infty} U'(x) = 0.$$

We can extend this definition to allow for negative wealth by changing the derivative limit at $x = 0$ to a limit at $x = -\infty$.

The requirements of this definition are very natural. Economic intuition tells us an investor will prefer a higher level of wealth to a lower one, but the utility gained from additional units decreases as their wealth increases. The concavity and the limits of the derivative imply the marginal utility, $U'(\cdot)$, is strictly decreasing, which coincides with our economic understanding.

Common utility functions that have been extensively used in classical utility maximisation literature include

- logarithmic utility: $U(x) = \log(x), \quad x \in \mathbb{R}^+,$
- power utility: $U(x) = \frac{x^\gamma}{\gamma}, \quad \gamma < 1, \gamma \neq 0, \quad x \in \mathbb{R}^+,$
- exponential utility $U(x) = -\exp(-\gamma x), \quad \gamma > 0, \quad x \in \mathbb{R}.$

Investors' preferences change through time so we define inter-temporal utility functions. Furthermore, we allow for the possibility that such preferences may be stochastic.

Definition 2.1.6. *A function $U(t,x,\omega) : [0,T]\times\mathbb{R}^+\times\Omega \to \mathbb{R}$, is an inter-temporal utility function if, for every fixed $(t,\omega) \in [0,T]\times\Omega$, $U(t,\cdot,\omega)$ is a utility function according to Definition 2.1.5 and $U(t,x,\omega)$ is a continuous function in the first variable.*

Unless indicated otherwise, we will use time-dependent utility functions of the form $U(t,x) = \exp(-\alpha t)U(x)$ where $U(x)$ is a utility function and α is a non-negative constant.

The expected utility from consumption and terminal wealth, given some capital $x > 0$ at time $t \in [0,T]$, for a strategy (π, c) is denoted by

$$J(t,x;\pi,c) := \mathbb{E}\left[e^{-\beta T}U_1(X_T^{\pi,c}) + \int_t^T e^{-\alpha s}U_2(c_s)ds \Big| X_t = x\right], \tag{2.1.7}$$

where U_1 and U_2 are utility functions and α and β are non-negative constants. The agent wishes to maximise $J(t,x;\pi,c)$ over the set of admissible trading strategies $\mathcal{A}_{t,T}(x)$ defined below. Denote $u(t,x)$ to be the value function

$$u(t,x) := \sup_{(\pi,c)\in\mathcal{A}_{t,T}(x)} J(t,x;\pi,c). \tag{2.1.8}$$

We assume throughout that $u(t,x)$ is finite.

Initially we define the set of admissible trading and consumption pairs as

$$\mathcal{A}_{t,T}(x) := \{(\pi,c) : \pi \text{ is a portfolio process and } (\widetilde{G}_s^{\pi})_{t\le s\le T} \text{ is bounded below}\}.$$

The restriction of admissible strategies to strategies with a bounded gains process is sufficient to prevent doubling strategies. A doubling strategy can generate arbitrarily large wealth from zero initial capital with probability one. This would result in infinite utility.

For logarithmic and power utility negative terminal wealth is disallowed. In the case of constant coefficients the optimal strategy, as we shall see later, is to invest and consume a fixed proportion of current wealth and the optimal wealth process remains positive at all times. For an investor with exponential utility negative wealth is allowed and in principle the wealth process can become unbounded below. This leads to subtleties in defining the class of admissible strategies, a topic investigated by Schachermayer [46]. Consider an investor with exponential utility who maximises utility of terminal wealth. The value function is given by

$$u(t,x) = \sup_{\pi\in\mathcal{A}_{t,T}(x)} \mathbb{E}[-e^{-\gamma X_T^{\pi}}|\mathcal{F}_t].$$

The optimal Merton strategy is $\pi_t^* = e^{-r(T-t)}\frac{\lambda}{\sigma\gamma}$, $0 \le t \le T$, and the optimal wealth process is given by

$$X_t^* = x + e^{-r(T-t)}\left(\frac{\lambda^2}{\gamma}t + \frac{\lambda}{\gamma}W_t\right), \quad 0 \le t \le T.$$

This wealth process can reach arbitrarily negative values.

One approach to deal with this issue is to restrict the set of admissible strategies to those which exhibit a wealth process that is bounded below. This lower bound is a liquidity constraint which prevents the investor from accumulating too much debt. Such a restriction can lead to difficulties in finding optimal controls.

Some alternatives to this formulation have been presented for the zero consumption case. Schachermayer [46] expands the set of admissible trading strategies to include those with unbounded wealth processes provided that the terminal wealth of such a strategy is dominated by the terminal wealth of a strategy with a bounded wealth process. Becherer [2] considers admissibility from the dual perspective which involves the selection of an equivalent martingale measure. This equivalent martingale measure is chosen to minimise the relative entropy with respect to the physical measure $\mathbb{P}$. Let $\mathcal{M}_f$ denote the set of equivalent martingale measures, $\mathbb{Q}$, with finite relative entropy with respect to $\mathbb{P}$. Becherer [2] defines the admissible set as any strategy π such that the the wealth process, $(\widetilde{X}_t^{\pi})_{0\leq t\leq T}$, is a $(\mathbb{Q},\mathbb{F})$-martingale for all $\mathbb{Q}\in\mathcal{M}_f$. Delbaen et al. [11] showed that the solution to the exponential utility maximisation problem is the same under a variety of choices of admissible sets, including Becherer's selection, and was, in some sense, robust.

With the exception of a small number of papers, the case of negative wealth in a consumption framework appears not to have been addressed. Karatzas and Žitković [32] consider the problem of maximising the expected utility from consumption in a semi-martingale market with a random endowment process. They define their admissible set of strategies as strategies, (π,c), which have a gains process bounded below and give a non-negative terminal wealth almost surely, though they allow for negative wealth prior to maturity. The infinite horizon problem with exponential utility has been studied by a number of authors and we discuss this in detail in the next chapter.

2.1.2 The dynamic programming method

In this section we review the original methods of Merton [36, 37]. For notational ease we restrict ourselves to a two-asset model with constant coefficients; $\lambda_t=\lambda\in\mathbb{R}$, $\sigma_t=\sigma\in\mathbb{R}^+$ and $r_t=r\in\mathbb{R}^+$, though all concepts readily extend to a multi-dimensional model in which the asset price process is a Markovian diffusion. The

asset price dynamics are

$$\begin{aligned} dS_t &= S_t\Big(\mu dt + \sigma dW_t\Big), \\ dS_t^0 &= rS_t^0 dt. \end{aligned}$$

If the investor selects an optimal strategy, (π^*, c^*), for the investment period $[t, T]$ their expected utility is given by $u(t, x)$. Compare this to using an arbitrary strategy, (π, c), over the period $[t, t + h)$ before adopting an optimal strategy for $[t + h, T]$. We observe that, for all $(\pi, c) \in \mathcal{A}_{t,T}(x)$,

$$u(t,x) \geq \mathbb{E}\left[u\left(t+h, X_{t+h}^{\pi,c}\right) + \int_t^{t+h} e^{-\alpha s} U_2(c_s) ds \,\middle|\, X_t = x\right]. \tag{2.1.9}$$

Equality is achieved if and only if the investor follows an optimal strategy on $[t, t+h)$. On the other hand, assuming sufficient regularity of $u(t, x)$, a simple application of Itô's Lemma to $u(t, X_t^{\pi,c})$ gives

$$\mathbb{E}\Big[u(t+h, X_{t+h}^{\pi,c}) - u(t, X_t^{\pi,c})\Big|\mathcal{F}_t\Big] = \mathbb{E}\left[\int_t^{t+h} \Big(u_t(s, X_s) + \mathcal{L}^{\pi,c} u(s, X_s)\Big) ds \Big| \mathcal{F}_t\right], \tag{2.1.10}$$

where $\mathcal{L}^{\pi,c}$ is the infinitesimal generator defined by

$$\mathcal{L}^{\pi,c} f(t,x) := (rx + \pi_t \sigma\lambda - c_t) f_x(t,x) + \frac{1}{2}\sigma^2 \pi_t^2 f_{xx}(t,x).$$

The asset prices are Markovian so we can exchange the expectation conditioned on $\mathcal{F}_t$ with one conditioned on $X_t = x$. Then, combining (2.1.10) and (2.1.9), we have

$$\mathbb{E}\left[\int_t^{t+h} \Big(u_t(s, X_s^{\pi,c}) + \mathcal{L}^{\pi,c} u(s, X_s^{\pi,c}) + e^{-\alpha s} U_2(c_s)\Big) ds \,\middle|\, X_t = x\right] \leq 0.$$

Dividing by $h > 0$ and taking the limit $h \to 0$, assuming sufficient regularity that we can take the limit inside the expectation, we have that $u(t, x)$ satisfies the Hamilton-Jacobi-Bellman equation

$$\max_{\pi,c}\left[u_t(t,x) + \mathcal{L}^{\pi,c} u(t,x) + e^{-\alpha t} U_2(c)\right] = 0, \quad u(T,x) = e^{-\beta T} U_1(x). \tag{2.1.11}$$

The HJB equation provides a necessary condition on the solution to the utility maximisation problem. If u is indeed the value function then it must satisfy (2.1.11), and if the pair (π^*, c^*) is the optimal strategy, then it attains the supremum in (2.1.11). The following verification theorem provides sufficient conditions for a candidate value function u.

Theorem 2.1.7. *(Verification theorem) Suppose the functions $h(t,x)$ and $(\hat{\pi},\hat{c})$ are such that*

1. *$h(t,x)$ solves the HJB equation (2.1.11),*
2. *$(\hat{\pi},\hat{c})$ is an admissible strategy and for each fixed (t,x), the supremum*

$$\max_{\pi,c}\Big(h_t(t,x)+\mathcal{L}^{\pi,c}h(t,x)+e^{-\alpha t}U_2(c)\Big) \tag{2.1.12}$$

   is attained by $(\hat{\pi},\hat{c})$,

3. *for all admissible strategies (π,c)*

$$\mathbb{E}\left[\int_0^T \sigma^2\pi_s^2 h_x^2(s,X_s^{\pi,c})ds\right]<\infty. \tag{2.1.13}$$

Then $h(t,x)=u(t,x)$, the optimal value function for the utility maximisation problem (2.1.8), and $(\hat{\pi},\hat{c})=(\pi^*,c^*)$, the optimal control law.

Proof: Let (π,c) be an arbitrary admissible strategy. Itô's formula applied to $h(t,X_t^{\pi,c})$ gives

$$\begin{aligned}h(T,X_T^{\pi,c})-h(t,X_t^{\pi,c}) &= \int_t^T\Big(h_t(s,X_s^{\pi,c})+\mathcal{L}^{\pi,c}h(s,X_s^{\pi,c})\Big)ds\\ &\quad+\int_t^T \pi_t\sigma h_x(s,X_s^{\pi,c})dW_s.\end{aligned} \tag{2.1.14}$$

Furthermore $h(t,x)$ solves the HJB equation (2.1.11), so

$$h_t(t,x)+\mathcal{L}^{\pi,c}h(t,x)+e^{-\alpha t}U_2(c)\le 0,\quad h(T,x)=e^{-\beta T}U_1(x). \tag{2.1.15}$$

Combining (2.1.14) and (2.1.15), we have that

$$e^{-\beta T}U_1(X_T^{\pi,c})\le h(t,X_t^{\pi,c})-\int_t^T e^{-\alpha s}U_2(c_s)ds+\int_t^T\pi_s\sigma h_x(s,X_s^{\pi,c})dW_s.$$

The integrability condition (2.1.13) means the stochastic integral vanishes when we take expectations conditioned on X_t giving

$$h(t,x)\ge\mathbb{E}\left[e^{-\beta T}U_1\big(X_T^{\pi,c}\big)+\int_t^T e^{-\alpha s}U_2(c_s)ds\,\middle|\,X_t=x\right]=J(t,x;\pi,c).$$

Since (π, c) is an arbitrary admissible control it follows that

$$h(t,x) \geq \sup_{\pi,c \in \mathcal{A}_{t,T}(x)} J(t,x;\pi,c) = u(t,x). \tag{2.1.16}$$

On the other hand, since $(\hat{\pi}, \hat{c})$ is admissible and attains the supremum in (2.1.12), we have

$$h_t(t,x) + \mathcal{L}^{\hat{\pi},\hat{c}} h(t,x) + e^{-\alpha t} U_2(c) = 0,$$

and

$$h(t,x) = J(t,x;\hat{\pi},\hat{c}) \leq u(t,x). \tag{2.1.17}$$

It follows from (2.1.16) and (2.1.17) that $h(t,x) = u(t,x)$ and $(\hat{\pi}, \hat{c}) = (\pi^*, c^*)$. $\square$

This verification theorem provides a strategy for solving the Merton problem. First we identify the optimal control, (π^*, c^*), as a function of the unknown $u(t,x)$, then substitute into the HJB equation and solve the resulting PDE. Finally we verify that the proposed controls are admissible.

Performing the maximisation of the HJB equation (2.1.11) over (π, c) gives the optimal Markov feedback controls $\pi^*(t,x)$ and $c^*(t,x)$ as

$$\pi^*(t,x) = -\frac{\lambda u_x(t,x)}{\sigma u_{xx}(t,x)}, \quad c^*(t,x) = I^{U_2}(e^{\alpha s} u_x(t,x)), \tag{2.1.18}$$

where $I^{U_2}(\cdot)$ is the inverse of the marginal utility $I^{U_2}(\cdot) = (U_2'(\cdot))^{-1}$. The optimal trading and consumption strategy is given by $(\pi_s^*, c_s^*) = (\pi^*(s, X_s^*), c^*(s, X_s^*))$, where X^* is the wealth process (2.1.1) with $\pi_t = \pi_t^*$ and $c_t = c_t^*$.

Inserting (2.1.18) into the HJB equation gives

$$\begin{aligned} 0 &= u_t + u_x(rx - I^{U_2}(e^{\alpha s} u_x)) - \frac{1}{2}\lambda^2 \frac{u_x^2}{u_{xx}} + e^{-\alpha t} U_2(I^{U_2}(e^{\alpha s} u_x)), \\ u(T,x) &= e^{-\beta T} U_1(x). \end{aligned} \tag{2.1.19}$$

For some utility functions, it is possible to solve this PDE and obtain explicit solutions for $u(t,x)$.

2.1.3 The dual problem

In this section we examine the work of Karatzas et al. [30] on portfolio optimisation in a complete market. We revert back to using general $\mathbb{F}$-adapted parameters in the market model M^0.

For a utility function $U(\cdot)$, we denote by $I^U(\cdot)$ the inverse of the marginal utility, $I^U(\cdot) = (U'(\cdot))^{-1}$. By the continuity and concavity of $U(\cdot)$ it is easily deduced that $U'(\cdot)$ is continuous and strictly decreasing, so $I^U(\cdot)$ exists and is also continuous and strictly decreasing.

Definition 2.1.8. *The convex conjugate of a utility function $U(\cdot)$ is $V : \mathbb{R}^+ \to \mathbb{R}$ defined as*

$$V(\eta) := \sup_{x \in \mathrm{dom(U)}} \big[U(x) - x\eta\big], \quad \eta > 0. \tag{2.1.20}$$

Let $V(\eta)$ be the convex dual of a utility function $U(x)$. The supremum in (2.1.20) is attained by $x = I^U(\eta)$, so $V(\eta)$ maybe written as

$$V(\eta) = U(I^U(\eta)) - \eta I^U(\eta), \quad \eta > 0. \tag{2.1.21}$$

Differentiating (2.1.21) gives $V'(\eta) = -I^U(\eta)$. Consider $\inf_{\eta>0} \big[V(\eta) + x\eta\big]$. The infimum is achieved by $\eta = U'(x)$ and using (2.1.21) we have the bi-dual relation

$$\inf_{\eta>0} \big[V(\eta) + x\eta\big] = U(x), \quad x \in \mathrm{dom}(U).$$

In particular, we will use the inequality

$$U(x) \leq V(\eta) + x\eta, \quad x \in \mathrm{dom}(U), \tag{2.1.22}$$

which holds with equality if and only if $\eta = U'(x)$.

We denote by $\xi = (\xi_t)_{0 \leq t \leq T}$ the state price density, $\xi_t = \widetilde{Z}_t = Z_t/S^0_t$. Further we use the notation

$$\xi_{s,t} = \frac{\xi_t}{\xi_s}, \quad 0 \leq s \leq t \leq T.$$

Recall from (2.1.3) that $(\widetilde{X}^{\pi,c}_t + \int_0^t \widetilde{c}_s ds)_{0 \leq t \leq T}$ is a $\mathbb{Q}$-local martingale. In the case of non-negative wealth this process is a non-negative local martingale and therefore a supermartingale, hence

$$\mathbb{E}\left[X^{\pi,c}_T \xi_{t,T} + \int_t^T c_u \xi_{t,u} du \Big| \mathcal{F}_t\right] \leq x. \tag{2.1.23}$$

When we allow for negative wealth, we assume that $\mathbb{E}\left[\int_0^T \parallel \widetilde{\pi}_s \sigma_s \parallel^2 ds\right] < \infty$ so that $(\widetilde{X}^{\pi,c}_t + \int_0^t \widetilde{c}_s ds)_{0 \leq t \leq T}$ is a $\mathbb{Q}$-martingale and (2.1.23) holds with equality. The inequality (2.1.23) is called the budget constraint and is used in a Lagrange maximisation problem specified below

The Lagrangian is given by

$$\begin{aligned}\mathcal{L}(X_T, c, \eta): \quad & \mathbb{E}\left[e^{-\beta T}U_1\big(X_T^{\pi,c}\big) + \int_t^T e^{-\alpha u}U_2(c_u)du\Big|\mathcal{F}_t\right] \\ & \qquad +\eta\left(x - \mathbb{E}\left[X_T^{\pi,c}\xi_{t,T} + \int_t^T c_u\xi_{t,u}du\Big|\mathcal{F}_t\right]\right) \\ & \leq x\eta + \mathbb{E}\left[e^{-\beta T}V_1\left(e^{\beta T}\eta\xi_{t,T}\right) + \int_t^T e^{-\alpha u}V_2\left(e^{\alpha u}\eta\xi_{t,u}\right)du\Big|\mathcal{F}_t\right],\end{aligned} \tag{2.1.24}$$

where $\eta > 0$ is a Lagrange multiplier and V_1 and V_2 are the convex conjugates of the utility functions U_1 and U_2. The last line holds with equality if and only if $X_T^{\pi,c} = X_T^*$ and $c_u = c_u^*$, where X_T^* and c_u^* are given by

$$X_T^* = I^{U_1}\Big(e^{\beta T}\eta\xi_{t,T}\Big), \quad c_u^* = I^{U_2}(e^{\alpha u}\eta\xi_{t,u}), \quad t \leq u \leq T. \tag{2.1.25}$$

The Lagrange multiplier, $\eta > 0$, is fixed using the budget constraint (2.1.23).

Define the dual value function by

$$v(t,\eta) := \mathbb{E}\left[e^{-\beta T}V_1\left(e^{\beta T}\eta\xi_{t,T}\right) + \int_t^T e^{-\alpha u}V_2\left(e^{\alpha u}\eta\xi_{t,u}\right)du\Big|\mathcal{F}_t\right]. \tag{2.1.26}$$

Differentiating v with respect to η we have

$$-v_\eta(t,\eta) = \mathbb{E}\left[\xi_{t,T}X_T^* + \int_t^T \xi_{t,u}c_u^*du\Big|\mathcal{F}_t\right] = x.$$

We assume $v'(t,\eta) < \infty$, for all $\eta > 0$, so that $-v'(t,\cdot)$ maps $(0,\infty)$ onto itself and is continuous and strictly increasing, thus its inverse exists. Then $\eta = I^v(t,-x)$, where $I^v(t,\cdot)$ denotes the inverse of the derivative $v(t,\cdot)$, $I^v(t,\cdot) = (v'(t,\cdot))^{-1}$. This specification of η via the budget constraint implies the optimal strategy given in (2.1.25) is unique.

Next we prove that $v(t,\eta)$ is the convex conjugate of the value function $u(t,x)$.

$$\begin{aligned}
v(t,\eta) &= \mathbb{E}\left[e^{-\beta T}V_1\big(e^{\beta T}\eta\xi_{t,T}\big) + \int_t^T e^{-\alpha u}V_2\big(e^{\alpha u}\eta\xi_{t,u}\big)du\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}\Big[e^{-\beta T}\Big(U_1\big(I^{U_1}(e^{\beta T}\eta\xi_{t,T})\big) - e^{\beta T}\eta\xi_{t,T}I^{U_1}\big(e^{\beta T}\eta\xi_{t,T}\big)\Big) \\
&\qquad + \int_t^T e^{-\alpha u}\Big(U_2\big(I^{U_2}(e^{\alpha u}\eta\xi_{t,u})\big) - e^{\alpha u}\eta\xi_{t,u}I^{U_2}\big(e^{\alpha u}\eta\xi_{t,u}\big)\Big)du\Big|\mathcal{F}_t\Big] \\
&= \mathbb{E}\left[e^{-\beta T}U_1(X_T^*) + \int_t^T e^{-\alpha u}U_2(c_u^*)du\Big|\mathcal{F}_t\right] - \eta\mathbb{E}\left[\xi_{t,T}X_T^* + \int_t^T \xi_{t,u}c_u^*du\Big|\mathcal{F}_t\right] \\
&\leq u(t,x) - \eta x \\
&\leq \sup_{x\in\mathrm{dom(u)}}[u(t,x) - \eta x]. \qquad (2.1.27)
\end{aligned}$$

On the other hand, (2.1.24) holds for arbitrary admissible strategies (π, c), with $x \in \mathrm{dom}(u)$ and $\eta > 0$, so using the optimal controls (π^*, c^*) in (2.1.24) we have that

$$u(t,x) \leq x\eta + v(t,\eta), \quad \forall x \in \mathrm{dom}(u),$$

which implies

$$\sup_{x\in\mathrm{dom(u)}}[u(t,x) - x\eta] \leq v(t,\eta). \qquad (2.1.28)$$

Combining (2.1.27) and (2.1.28) we conclude u and v are convex conjugates.

We summarise these results in the following theorem.

Theorem 2.1.9. *Suppose U_1 and U_2 are utility functions with convex conjugates V_1 and V_2. For $x, \eta > 0$ define the primal and dual value functions by*

$$u(t,x) := \sup_{(\pi,c)\in\mathcal{A}_{t,T}(x)} \mathbb{E}\left[e^{-\beta T}U_1\big(X_T^{\pi,c}\big) + \int_t^T e^{-\alpha u}U_2(c_u)du\Big|\mathcal{F}_t\right], \quad 0 \leq t \leq T, \qquad (2.1.29)$$

and

$$v(t,\eta) := \mathbb{E}\left[e^{-\beta T}V_1(e^{\beta T}\eta\xi_{t,T}) + \int_t^T e^{-\alpha u}V_2(e^{\alpha u}\eta\xi_{t,u})du\Big|\mathcal{F}_t\right], \quad 0 \leq t \leq T.$$

Then u and v are convex conjugates and the optimal terminal wealth, X_T^*, and consumption strategy, $(c_u^*)_{t\leq u\leq T}$, of problem (2.1.29) are unique and given by

$$X_T^* = I^{U_1}\big(e^{\beta T}\eta\xi_{t,T}\big), \quad c_u^* = I^{U_2}\big(e^{\alpha u}\eta\xi_{t,u}\big), \quad t \leq u \leq T. \qquad (2.1.30)$$

2.1.4 Example: power utility with constant coefficients

In this section, we present the explicit solution to the finite horizon Merton problem with consumption for power utility functions. We restrict ourselves to the case of constant parameters: $\lambda_t = \lambda \in \mathbb{R}$, $\sigma_t = \sigma \in \mathbb{R}^+$, and $r_t = r \in \mathbb{R}^+$. The value function is given by

$$u(t,x) := \mathbb{E}\left[e^{-\beta T} U(X_T) + \int_t^T e^{-\alpha s} U(c_s) ds \middle| X_t = x \right],$$

where α and β are positive constants and $U(x) = x^\gamma/\gamma$, $0 < \gamma < 1$.

Solution via dynamic programming

Using $I^U(y) = y^{\frac{1}{\gamma-1}}$ in (2.1.19) it follows that $u(t,x)$ satisfies

$$\begin{aligned} u_t + u_x\Big(rx - \left(e^{\alpha t}u_x\right)^{\frac{1}{\gamma-1}}\Big) - \frac{1}{2}\lambda^2 \frac{u_x^2}{u_{xx}} + e^{-\alpha t}\frac{\left(e^{\alpha t}u_x\right)^{\frac{\gamma}{\gamma-1}}}{\gamma} &= 0, \\ u(T,x) &= e^{-\beta T}\frac{x^\gamma}{\gamma}. \end{aligned}$$

Seeking a solution of the form $u(t,x) = \frac{x^\gamma}{\gamma} e^{-\alpha t} f(t)$ it follows that $f(t)$ satisfies

$$\begin{aligned} 0 &= \frac{x^\gamma}{\gamma} e^{-\alpha t}\left[f'(t) + \left(r\gamma - \alpha - \frac{1}{2}\lambda^2 q \right) f(t) + (1-\gamma) f(t)^q \right], \\ F(T) &= e^{(\alpha-\beta)T}, \end{aligned}$$

where $q = \frac{\gamma}{\gamma-1}$. Let $f(t) = g(t)^{1-\gamma}$, then $g(t)$ satisfies

$$g'(t) - \nu(\alpha) g(t) + 1 = 0, \quad g(T) = e^{\frac{\alpha-\beta}{1-\gamma}T},$$

where $\nu(\alpha)$ is the constant

$$\nu(\alpha) = \frac{\alpha - r\gamma - \frac{\lambda^2}{2}\frac{\gamma}{1-\gamma}}{1-\gamma}. \tag{2.1.31}$$

It is easily verified that $g(t)$ is given by

$$g(t) = \frac{1}{\nu(\alpha)} + \left(e^{\frac{\alpha-\beta}{1-\gamma}T} - \frac{1}{\nu(\alpha)} \right) e^{-\nu(\alpha)(T-t)}, \tag{2.1.32}$$

and $u(t,x)$ is given by

$$u(t,x) = \frac{x^\gamma}{\gamma} e^{-\alpha t}\left[\frac{1 - e^{-\nu(\alpha)(T-t)}}{\nu(\alpha)} + e^{-\nu(\beta)T + \nu(\alpha)t} \right]^{1-\gamma}. \tag{2.1.33}$$

We require that $\nu(\alpha) \neq 0$ to ensure the value function is finite. If $(X_s^*)_{t\leq s\leq T}$ represents the optimal wealth process then the optimal feedback controls are $(\pi^*(s, X_s^*), c^*(s, X_s^*))$, where (π^*, c^*) are given by

$$\pi^*(s,x) = \frac{\lambda}{\sigma(1-\gamma)}x, \quad c^*(s,x) = \frac{x}{g(s)}, \quad t \leq s \leq T. \tag{2.1.34}$$

The optimal investment strategy involves investing a fixed proportion of current wealth in the stock S at all times.

Solution using the dual approach

Recall the martingale property of the wealth process

$$\mathbb{E}\left[X_T^*\xi_{t,T} + \int_t^T c_s^*\xi_{t,s}ds \middle| X_t = x\right] = x. \tag{2.1.35}$$

From (2.1.30) it follows that the optimal terminal wealth and optimal consumption strategy satisfy

$$X_T^* = \left(\eta e^{\beta T}\xi_{t,T}\right)^{\frac{1}{\gamma-1}}, \quad c_s^* = \left(\eta e^{\alpha s}\xi_{t,s}\right)^{\frac{1}{\gamma-1}}, \quad t \leq s \leq T.$$

Inserting these into the budget constraint, (2.1.35), it follows that

$$\begin{aligned} x &= \eta^{\frac{1}{\gamma-1}}\mathbb{E}\left[e^{\frac{\beta T}{\gamma-1}}\xi_{t,T}^q + \int_t^T e^{\frac{\alpha s}{\gamma-1}}\xi_{t,s}^q ds \middle| X_t = x\right] \\ &= \eta^{\frac{1}{\gamma-1}}e^{-\frac{\alpha t}{1-\gamma}}g(t), \end{aligned}$$

where $g(t)$ is defined in (2.1.32) and $\nu(\alpha)$ is defined in (2.1.31). This identifies the Lagrange multiplier η. Substituting the optimal terminal wealth and optimal consumption strategy into the value function gives

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[e^{-\beta T}\frac{\left(\eta e^{\beta T}\xi_{t,T}\right)^q}{\gamma} + \int_t^T e^{-\alpha s}\frac{\left(\eta e^{\alpha s}\xi_{t,s}\right)^q}{\gamma}ds \middle| X_t = x\right] \\ &= \frac{\eta^q}{\gamma}\mathbb{E}\left[e^{\frac{\beta T}{\gamma-1}}\xi_{t,T}^q + \int_t^T e^{\frac{\alpha s}{\gamma-1}}\xi_{t,s}^q ds \middle| X_t = x\right] \\ &= \frac{x^\gamma}{\gamma}e^{-\alpha t}g(t)^{1-\gamma}. \end{aligned}$$

This reconciles with (2.1.33), the solution obtained using the primal approach. The consumption strategy in feedback form is $c^*(s, X_s^*), s \in [t, T]$, where $c^*(s, x)$ is given by

$$c^*(s,x) = I^U(\eta e^{\alpha s}) = \frac{x}{g(s)}, \quad t \leq s \leq T.$$

We can use the martingale property of the wealth process to recover the optimal trading strategy. For $t \leq s \leq T$ the optimal wealth process satisfies

$$\begin{aligned} X_s^* &= \mathbb{E}\left[X_T^* \xi_{s,T} + \int_s^T c_u^* \xi_{s,u} du \middle| \mathcal{F}_s\right] \\ &= \mathbb{E}\left[(\eta e^{\beta T} \xi_{t,T})^{\frac{1}{\gamma-1}} \xi_{s,T} + \int_s^T (\eta e^{\alpha u} \xi_{t,u})^{\frac{1}{\gamma-1}} \xi_{s,u} du \middle| \mathcal{F}_s\right] \\ &= e^{-\frac{\alpha s}{1-\gamma}} (\eta \xi_{t,s})^{-\frac{1}{1-\gamma}} g(s). \end{aligned}$$

Itô's lemma applied to $\widetilde{X}_s^*$ gives

$$d\widetilde{X}_s^* = \widetilde{X}_s^* \frac{\lambda}{1-\gamma}\big(\lambda ds + dW_s\big) - \frac{\widetilde{X}_s^*}{g(s)} ds = \widetilde{X}_s^* \frac{\lambda}{1-\gamma} dW_s^{\mathbb{Q}} - \widetilde{c}_s^* ds.$$

Comparing this to (2.1.3) we see that $\pi^*(s,x) = \frac{\lambda}{\sigma(1-\gamma)} x$ for $t \leq s \leq T$ which matches (2.1.34).

2.2 The incomplete market case

Although complete markets are easy to work with they are a rarity. Market frictions, such as transaction costs or portfolio constraints, possibly on short-selling or non-traded assets, give rise to incomplete markets. Mathematically, the Second Fundamental Theorem of Asset Pricing [12] tells us there is more than one equivalent probability measure under which the traded assets are local martingales.

If the asset dynamics are Markovian it is still possible to apply techniques of dynamic programming and derive a HJB equation for the value function. The absence of a unique equivalent martingale measure makes the dual method more involved than in a complete market. In this section we review the approach developed by Karatzas, Lehoczky, Shreve and Xu [28].

In the multi-dimensional model M^0 of the previous section, the number of risky assets is equal to the dimension of the Brownian motion driving each stock and the market is complete. We now define a convex cone, K, which will act as a constraint on our admissible strategies and will create an incomplete market. Let $(\pi_1, \ldots, \pi_m)$ represent the amount invested in each asset and fix a positive integer, d, strictly less than m. Denote by K the cone

$$K := \{\pi \in \mathbb{R}^m : \pi_{d+1}, \ldots, \pi_m = 0\}.$$

The restricted admissible trading set is denoted $\mathcal{A}_{t,T}(x;K)$ and given by

$$\mathcal{A}_{t,T}(x;K) := \{(\pi,c) \in \mathcal{A}_{t,T}(x) : \pi \in K, (\lambda \otimes \mathbb{P})\text{-a.e on } [t,T] \times \Omega\}. \tag{2.2.1}$$

The investor is prohibited from trading the assets $S^{d+1}, \ldots, S^m$, though in reality these assets may not even exist. In addition to the previous assumptions made on the parameters we require that σ_t and σ_t^{-1} are uniformly bounded. We denote this constrained market by M^K. The value function $u^K(t,x)$ of the portfolio optimisation problem in this constrained market is given by

$$u^K(t,x) := \sup_{(\pi,c)\in\mathcal{A}_{t,T}(x;K)} \mathbb{E}\left[e^{-\beta T}U_1(X_T^{\pi,c}) + \int_t^T e^{-\alpha s}U_2(c_s)ds\Big|\mathcal{F}_t\right]. \tag{2.2.2}$$

The maximum expected utility of the constrained problem is dominated by the maximum expected utility of the unconstrained problem since $\mathcal{A}_{t,T}(x;K) \subseteq \mathcal{A}_{t,T}(x)$, so

$$u^K(t,x) \leq u^0(t,x), \quad 0 \leq t \leq T, \quad x > 0. \tag{2.2.3}$$

Heuristically the aim is to find a larger complete model, by adding fictitious assets to the traded assets, $\{S^1, \ldots, S^d\}$, in which the optimal strategy is also admissible for the constrained market. Then the dominance of the maximum utility attainable in M^K by the maximum utility attainable in a larger complete market, (2.2.3), implies this strategy must also be optimal for the constrained, incomplete model. To do this we embed the incomplete market M^K into a family of auxiliary models $\{M^\nu\}_{\nu\in\mathcal{D}}$, for a suitably constructed set $\mathcal{D}$.

Let $\mathcal{H}$ represent the Hilbert space of progressively measurable processes, $\nu : [0,T] \times \Omega \to \mathbb{R}^d$, satisfying $\mathbb{E}\left[\int_0^T \parallel \nu_s \parallel^2 ds\right] < \infty$ and denote by

$$\bar{K} := \{x \in \mathbb{R}^m : \pi x \geq 0,\ \forall \pi \in K\} = \{x \in \mathbb{R}^m : x_1 = \cdots = x_d = 0\}$$

the polar cone of $-K$. Denote by $\mathcal{D}$ the set

$$\mathcal{D} := \{\nu \in \mathcal{H} : \nu(t,\omega) \in \bar{K}, (\lambda \otimes \mathbb{P})\text{-a.e } (t,\omega) \in [0,T]\}. \tag{2.2.4}$$

For each $\nu \in \mathcal{D}$, the market M^ν consists of the usual risk-free asset and m risky assets, $(S_t^{\nu,1}, \ldots, S_t^{\nu,m})$, with the dynamics

$$dS_t^{\nu,i} \;=\; S_t^{\nu,i}\left[\left(\mu_t^i + \nu_t^i\right)dt + \sum_{j=1}^m \sigma_t^{ij}dW_t^j\right], \quad S_t^{\nu,i} = s^{\nu,i} \in (0,\infty).$$

Since $\nu^1 = \cdots = \nu^d = 0$, the assets $\{S_t^{\nu,1}, \ldots, S_t^{\nu,d}\}$ in the complete market M^ν are identical to the traded assets in the constrained market. For $t \in [0,T]$ define the processes

$$\lambda_t^\nu := \sigma_t^{-1}(\mu_t + \nu_t - r_t \mathbb{I}_m),$$
$$W_t^\nu := W_t + \int_0^t \lambda_s^\nu ds = W_t^{\mathbb{Q}} + \int_0^t \sigma_s^{-1}\nu_s ds,$$

where $dW_t^{\mathbb{Q}} = dW_t + \lambda_t^0 dt$, and

$$Z_\nu(t) := \mathcal{E}(-\lambda_\nu \cdot W)_t.$$

We shall assume Z^ν is a martingale so, by Girsanov's theorem, we can define a new measure $\mathbb{Q}^\nu$. Then $\widetilde{S}^\nu$ is a $\mathbb{Q}^\nu$-local martingale and $\mathbb{Q}^\nu$ is the unique equivalent martingale measure for the market M^ν.

Since $(S^1, \ldots, S^d) = (S_t^{\nu,1}, \ldots, S_t^{\nu,d})$ each measure $\mathbb{Q}^\nu$ is also an equivalent martingale measure for the constrained market M^K. So each fictitious completion of the constrained market, parameterised by ν, corresponds to an equivalent martingale measure change in the constrained model.

We can solve the utility maximisation problem for each complete market, M^ν, using the techniques of Section 2.1. Fix $\nu \in \mathcal{D}$. For any admissible strategy $(\pi, c) \in \mathcal{A}_{t,T}^\nu(x)$ the discounted wealth process satisfies

$$d\widetilde{X}_t^{\nu,\pi,c} = -\widetilde{c}_t dt + \widetilde{\pi}_t \sigma_t dW_t^{\mathbb{Q}} + \widetilde{\pi}_t \nu_t dt.$$

If $\pi \in K$ then, by the definition of $\bar{K}$, we have that $\pi\nu \geq 0$ for all $\nu \in \mathcal{D}$ and

$$X_t^{\nu,\pi,c} \geq X^{\pi,c}(t), \quad \forall\, 0 \leq t \leq T \text{ a.s.} \tag{2.2.5}$$

with equality if and only if $\pi_t \nu_t = 0$ almost everywhere. The aim is to identify an auxiliary market M^ϕ, for some $\phi \in \mathcal{D}$, with an optimal pair (π^ϕ, c^ϕ) which satisfies the complimentary slackness condition $\pi^\phi \phi = 0$. Then (2.2.5) holds with equality and (π^ϕ, c^ϕ) must be admissible for the constrained model M^K. Furthermore,

$$u^K(t,x) \geq \mathbb{E}_{t,x}\left[e^{-\beta T} U_1(X_T^{\pi^\phi, c^\phi}) + \int_t^T e^{-\alpha s} U_2(c^\phi(s)) ds\right] = u^\phi(t,x). \tag{2.2.6}$$

It follows from (2.2.3) that $u^K(t,x) \leq u^\phi(t,x)$, so we have equality in (2.2.6) and (π^ϕ, c^ϕ) is the optimal strategy in the constrained market M^K. All that remains is to find the appropriate $\phi \in \mathcal{D}$. Karatzas et al. [28] provide some equivalent conditions on the optimality of ϕ which we present here.

Theorem 2.2.1. *Let $\mathcal{D}$ be given by (2.2.4) and denote the dual value function of $u^\nu(t,x)$ by*

$$v^\nu(t,\eta) := \mathbb{E}\left[e^{-\beta T}V_1\left(e^{\beta T}\eta\xi^\nu_{t,T}\right) + \int_t^T e^{-\alpha s}V_2\left(e^{\alpha s}\eta\xi^\nu_{t,s}\right)ds\Big|\mathcal{F}_t\right], \quad \eta > 0,$$

for all $\nu \in \mathcal{D}$. Suppose $\phi \in \mathcal{D}$, then the following are equivalent:

1. *Financeability: There exists $(\pi^\phi, c^\phi) \in \mathcal{A}(x;K)$ such that $\pi^\phi\phi = 0$, and (π^ϕ, c^ϕ) is the optimal strategy for the market M^ϕ a.e on $[0,T]\times\Omega$.*
2. *Minimality of ϕ: $u^\phi(t,x) \le u^\nu(t,x)$ for all $\nu \in \mathcal{D}$.*
3. *Dual optimality: With $\eta = I^{v^\phi}(t,-x)$, $v^\phi(t,\eta) \le v^\nu(t,\eta)$ for all $\nu \in \mathcal{D}$.*

Proof: (1) $\Rightarrow$ (2): Suppose $(\pi^\phi, c^\phi) \in \mathcal{A}(x;K)$ is the optimal control for the market M^ϕ and satisfies $\pi^\phi\phi = 0$. Then, for all $\nu \in \mathcal{D}$, equations (2.2.6) and (2.2.3) imply

$$u^\phi(t,x) \le u^K(t,x) \le u^\nu(t,x).$$

(2) $\Rightarrow$ (3): Suppose that $u^\phi(t,x) \le u^\nu(t,x)$ for all $\nu \in \mathcal{D}$ and set $\eta = I^{v^\phi}(t,-x)$. Using the conjugacy of u^ν and v^ν proved in the complete market case, we have that

$$v^\phi(t,\eta) = u^\phi(t,x) - x\eta \le u^\nu(t,x) - x\eta \le \sup_{x>0}[u^\nu(t,x) - x\eta] = v^\nu(t,\eta).$$

We refer the reader to [26] for the technical proof of (3) $\Rightarrow$ (1). □

The last condition leads us to the dual optimisation problem. Define

$$v^K(t,\eta) := \inf_{\nu\in\mathcal{D}} v^\nu(t,\eta), \quad 0 < \eta < \infty, \tag{2.2.7}$$

where $v^\nu(t,\eta)$ is the dual value function of $u^\nu(t,x)$.

Karatzas and co-authors showed that, if the utility functions are sufficiently smooth, the infimum in (2.2.7) is attained and there exists $\phi_\eta \in \mathcal{D}$ such that $v^K(\eta) = v^{\phi_\eta}(\eta)$ for every $\eta \in (0,\infty)$. Furthermore, they prove that $v^K(\eta)$ is the convex conjugate of the constrained optimisation value function, $u^K(x)$. Finally, the correspondence between each $\nu \in \mathcal{D}$ and the equivalent martingale measure $\mathbb{Q}^\nu$ means we can replace the infimum over $\nu \in \mathcal{D}$ with an infimum over the set of equivalent martingale measures for $(S^1, \dots S^d)$.

We summarise the results in the following theorem.

Theorem 2.2.2. *Denote by $u^K(t,x)$ the value function*

$$u^K(t,x) := \sup_{(\pi,c)\in\mathcal{A}_{t,T}(x;K)} \mathbb{E}\left[e^{-\beta T}U_T(X_T^{\pi,c}) + \int_t^T e^{-\alpha s}U(c_s)ds\Big|\mathcal{F}_t\right].$$

Let $\mathcal{M}$ denote the set of equivalent martingale measures in the incomplete market M^K and define the dual value function

$$v^K(t,\eta) := \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[e^{-\beta T}V_1(e^{\beta T}\eta\widetilde{Z}_T^{\mathbb{Q}}) + \int_t^T e^{-\alpha s}V_2(e^{\alpha s}\eta\widetilde{Z}_s^{\mathbb{Q}})ds\Big|\mathcal{F}_t\right], \quad \eta>0, \tag{2.2.8}$$

where $Z_t^{\mathbb{Q}} = \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ and $\widetilde{Z}_t^{\mathbb{Q}} = Z_t^{\mathbb{Q}}/S_t^0$. Denote by $\mathbb{Q}^*$ the dual minimiser which attains the infimum in (2.2.8) with density $\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t^{\mathbb{Q}^*}$. Then $u^K(t,x)$ and $v^K(t,\eta)$ are convex conjugates, and with $\eta = I^{v^K}(t,-x)$, the optimal terminal wealth, X_T^*, and optimal consumption process, c_s^* are given by

$$X_T^* = I^{U_1}(e^{\beta T}\eta\widetilde{Z}_T^{\mathbb{Q}^*}), \quad c_s^* = I^{U_2}(e^{\alpha s}\eta\widetilde{Z}_s^{\mathbb{Q}^*}), \quad t\le s\le T. \tag{2.2.9}$$

Finally, to conclude this section, a comment on existence and uniqueness.

Definition 2.2.3. *The asymptotic elasticity of a utility function $U(x)$, as defined by Kramkov and Schachermayer [34], is*

$$AE(U) := \limsup_{x\to\infty}\frac{xU'(x)}{U(x)}. \tag{2.2.10}$$

A utility function is said to have reasonable asymptotic elasticity if its asymptotic elasticity is strictly less than 1. Many common utility functions, including the logarithmic and power utility, satisfy this condition. Kramkov and Schachermayer showed in their seminal contribution [34] that the condition of reasonable asymptotic elasticity of the utility function was a necessary and sufficient condition for the existence of a unique solution to the portfolio optimisation problem without consumption in their generalised incomplete market driven by semi-martingale price processes.

Chapter 3

Complete market utility maximisation on an infinite horizon

In some problems, the specification of fixed horizon T is too restrictive or not appropriate and an infinite trading horizon is more natural. The time-independence of the infinite horizon problem can make it easier to solve in some cases, however duality theory for infinite horizon consumption problems in incomplete markets is not yet fully understood and this remains an active area of research.

3.1 The infinite horizon Merton problem

We adopt a two asset Black-Scholes model as presented at the beginning of Section 2.1.1. The agent seeks to maximise their discounted utility of consumption over an infinite horizon. The value function is

$$u(x) := \sup_{(\pi,c)\in\mathcal{A}_\infty} \int_0^\infty e^{-\alpha t} U(c_t) dt, \tag{3.1.1}$$

where α is a positive discount factor, U is a utility function and $\mathcal{A}_\infty$ is an appropriately defined set of admissible controls. We only allow consumption strategies $(c_t)_{t\geq 0}$ that are non non-negative and satisfy

$$\int_0^T c_s ds < \infty \text{ a.s. } \quad \forall\, T \in [0,\infty).$$

We further stipulate that the wealth process $X = (X_t^{\pi,c})_{t\geq 0}$ must be non-negative for all $0 \leq t < \infty$.

On a finite horizon we saw that wealth may become negative under exponential preferences. It turns out, however, that the optimal wealth process for exponential utility on an infinite horizon is non-negative. A closer examination of the optimal controls in this case follows this section. A transversality condition, defined in (3.2.2) below, is needed and may imply further feasibility conditions on the model parameters.

3.2 The solution to the primal problem

Merton [36] extended the primal PDE-based approach of the finite horizon problem to the infinite horizon problem. If one adopts an arbitrary admissible strategy, (π, c), on the time interval $[0, h)$ and the optimal strategy, (π^*, c^*), on $[h, \infty)$, the value function should satisfy the dynamic programming principle

$$u(x) \geq \mathbb{E}\left[\int_0^h e^{-\alpha s} U(c_s) ds + e^{-\alpha h} u(X_h^{\pi,c})\right].$$

From this we can derive the infinite horizon HJB equation

$$\max_{\pi,c}\left[U(c) - \alpha u(x) + u_x(x)(rx - c + \lambda\sigma\pi) + \frac{1}{2}u_{xx}(x)\pi^2\sigma^2\right] = 0. \tag{3.2.1}$$

There is no maturity date to which we can assign a boundary condition so instead we require a transversality condition. For all admissible strategies $(\pi, c) \in \mathcal{A}_\infty(x)$, $u(X_t^{\pi,c})$ must satisfy

$$\lim_{t\to\infty} \mathbb{E}\big[e^{-\alpha t} u(X_t^{\pi,c})\big] = 0. \tag{3.2.2}$$

The economic interpretation of the transversality condition is that there is no utility gained from having non-zero wealth at infinity and that all wealth must be consumed in the long-run. Mathematically, we require the transversality condition for the verification theorem which follows.

Theorem 3.2.1. *Suppose the functions $h(x)$ and (π^*, c^*) are such that*

1. *$h(x)$ satisfies the infinite horizon HJB equation (3.2.1) and the transversality condition (3.2.2),*
2. *$(\hat{\pi}, \hat{c})$ is an admissible strategy and for each fixed $x > 0$, the supremum*

$$\max_{\pi,c}\left[U(c) - \alpha h(x) + h_x(x)\big(rx + \pi\sigma\lambda - c\big) + \frac{1}{2}h_{xx}(x)\pi^2\sigma^2\right] = 0 \tag{3.2.3}$$

is attained by $(\hat{\pi}, \hat{c})$,

3. $\mathbb{E}\left[\int_0^\infty \sigma^2\pi_s^2 h_x^2(X_s^{\pi,c})ds\right] < \infty$ *for all admissible strategies* $(\pi, c) \in \mathcal{A}_\infty(x)$.

Then $h(x) = u(x)$*, the optimal value function for problem (3.1.1), and* $(\hat{\pi}, \hat{c}) = (\pi^*, c^*)$*, the optimal control law.*

Proof: Let (π, c) be an arbitrary admissible strategy. Using Itô's formula, $e^{-\alpha t}h(X_t^{\pi,c})$ satisfies

$$d\big(e^{-\alpha t}h(X_t^{\pi,c})\big) = e^{-\alpha t}\Big[\Big(-\alpha h(X_t^{\pi,c}) + h_x(X_t^{\pi,c})(rX_t^{\pi,c} + \pi_t\sigma\lambda - c_t) + \frac{1}{2}h_{xx}(X_t^{\pi,c})\pi_t^2\sigma^2\Big)dt + h_x(X_t^{\pi,c})\pi_t\sigma dW_t\Big]. \quad (3.2.4)$$

$h(x)$ satisfies the HJB equation (3.2.1), so

$$-\alpha h(X_t^{\pi,c}) + h_x(X_t^{\pi,c})\big(rX_t^{\pi,c} + \pi_t\sigma\lambda - c_t\big) + \frac{1}{2}h_{xx}(X_t)\pi_t^2\sigma^2 \le -U(c_t). \quad (3.2.5)$$

Substituting (3.2.5) into (3.2.4) and integrating gives

$$\int_0^\infty d\big(e^{-\alpha t}h(X_t^{\pi,c})\big) \le -\int_0^\infty e^{\alpha t}U(c_t)dt + \int_0^\infty e^{-\alpha t}h_x(X_t^{\pi,c})\pi_t\sigma dW_t. \quad (3.2.6)$$

Now take expectations conditional on $X_0 = x$. Since $h(X_t^{\pi,c})$ satisfies the transversality condition (3.2.2) it follows that

$$\mathbb{E}\left[\int_0^\infty d\big(e^{-\alpha t}h(X_t^{\pi,c})\big)\right] = -h(x).$$

The integrability condition (3) of the theorem ensures the stochastic integral in (3.2.6) is a martingale, so

$$h(x) \ge \mathbb{E}\left[\int_0^\infty e^{-\alpha t}U(c_t)dt\right].$$

Furthermore, since (π, c) is an arbitrary admissible strategy, we have

$$h(x) \ge \sup_{(\pi,c)\in\mathcal{A}_\infty(x)} \mathbb{E}\left[\int_0^\infty e^{-\alpha t}U(c_t)dt\right] = u^\infty(x). \quad (3.2.7)$$

By assumption $(\hat{\pi}, \hat{c})$ is admissible and attains the supremum in (3.2.3), so using this control we have equality in (3.2.5) and

$$h(x) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t}U(\hat{c}_t)dt\right] \le u^\infty(x). \quad (3.2.8)$$

Together (3.2.7) and (3.2.8) imply $h(x) = u(x)$ and $(\hat{\pi}, \hat{c}) = (\pi^*, c^*)$. □

Without the transversality condition a function satisfying the infinite horizon HJB equation (3.2.1) may not be the value function to the infinite horizon consumption problem. In practice we first seek a solution to the HJB equation and then verify that the optimal strategy satisfies the transversality condition.

Optimising (3.2.1) over strategies (π, c), we have the optimal feedback controls $\pi_t^* = \pi(X_t^*)$ and $c_t^* = c(X_t^*)$, where π^* and c^* are satisfy

$$\pi^*(x) = -\frac{\lambda}{\sigma}\frac{u_x(x)}{u_{xx}(x)}, \quad c^*(x) = I^U(u_x(x)). \tag{3.2.9}$$

Substituting these controls in the HJB equation the value functions solves the PDE

$$V(u_x(x)) - \alpha u(x) + u_x r x - \frac{1}{2}\lambda^2 \frac{u_x^2(x)}{u_{xx}(x)} = 0, \tag{3.2.10}$$

where $V(\cdot)$ is the convex conjugate of the utility function $U(\cdot)$, as before.

3.3 The solution to the dual problem

The infinite horizon problem can also be tackled using the dual approach. Recall from (2.1.3) that $\left(\widetilde{X}_t + \int_0^t \widetilde{c}_s ds\right)_{t\geq 0}$ is a $\mathbb{Q}$-local martingale. Since $X_t^{\pi,c} \geq 0$ for all $t \in [0, \infty)$ it follows that

$$\mathbb{E}\left[\int_0^t \xi_s c_s ds\right] \leq \mathbb{E}\left[\xi_t X_t + \int_0^t \xi_s c_s ds+\right] \leq x. \tag{3.3.1}$$

Letting $t \to \infty$ and using the Monotone Convergence Theorem we get the infinite horizon budget constraint as

$$\mathbb{E}\left[\int_0^\infty \xi_s c_s ds\right] \leq x. \tag{3.3.2}$$

We follow the same solution method as the finite horizon problem. The Lagrangian is given by

$$\begin{aligned}
\mathcal{L}(c,\eta): \quad & \mathbb{E}\left[\int_0^\infty e^{-\alpha s} U(c_s) ds\right] + \eta\left(x - \mathbb{E}\left[\int_0^\infty \xi_s c_s ds\right]\right) \\
&= x\eta + \mathbb{E}\left[\int_0^\infty \left(e^{-\alpha s} U(c_s) - \eta \xi_s c_s\right) ds\right] \\
&\leq x\eta + \mathbb{E}\left[\int_0^\infty e^{-\alpha s} V(e^{\alpha s}\eta \xi_s) ds\right].
\end{aligned} \tag{3.3.3}$$

The last line holds with equality if $c_s = c_s^*$, where c_s^* is given by

$$c_s^* = I^U(e^{\alpha s}\eta\xi_s), \quad s \geq 0. \tag{3.3.4}$$

Denote by $v(\eta)$ the dual value function

$$v(\eta) := \mathbb{E}\left[\int_0^\infty e^{-\alpha s} V(e^{\alpha s}\eta\xi_s) ds\right], \quad \eta > 0. \tag{3.3.5}$$

Using the same arguments as in the finite horizon problem it can be shown that $u(x)$ and $v(\eta)$ are convex conjugates.

3.4 Example: power utility with constant coefficients

In this section we calculate explicit solutions for the infinite horizon problem with power utility. As in the finite horizon example we restrict ourselves to constant parameters, $\lambda_t = \lambda \in \mathbb{R}$, $\sigma_t = \sigma \in \mathbb{R}^+$, and $r_t = r \in \mathbb{R}^+$, satisfying the feasibility condition

$$\alpha > \gamma\left(r + \frac{\lambda^2}{2(1-\gamma)}\right). \tag{3.4.1}$$

This is necessary for the transversality conditions to hold. If this condition is violated the discounted utility can grow arbitrarily large if we adopt a strategy of prolonged investment followed by massive consumption. Set

$$u(x) := \sup_{(\pi,c)\in\mathcal{A}_\infty(x)} \mathbb{E}\left[\int_0^\infty e^{-\alpha s}\frac{c_s^\gamma}{\gamma} ds\right].$$

The value function $u(x)$ satisfies the infinite HJB equation

$$\max_{\pi,c}\left[u_x(x)\big(rx - c + \pi\sigma\lambda\big) + \frac{1}{2}\sigma^2\pi^2 u_{xx}(x) + U(c) - \alpha u(x)\right] = 0, \tag{3.4.2}$$

and the optimal control law $(\pi^*(X_t^*), c^*(X_t^*))$ satisfies

$$\pi^*(x) = -\frac{\lambda u_x(x)}{\sigma u_{xx}(x)}, \quad c^*(x) = I^U\left(u_x(x)\right) = u_x(x)^{\frac{1}{\gamma-1}}. \tag{3.4.3}$$

Substituting (3.4.3) into (3.4.2) we obtain the following PDE for the value function:

$$u_x(x)\left(rx - u_x(x)^{\frac{1}{\gamma-1}}\right) - \frac{1}{2}\lambda^2\frac{u_x(x)^2}{u_{xx}(x)} + \frac{\left(u_x(x)^{\frac{1}{\gamma-1}}\right)^\gamma}{\gamma} - \alpha u(x) = 0. \tag{3.4.4}$$

It is easily verified that the solution to (3.4.4) is

$$u(x) = \frac{x^\gamma}{\gamma}\nu(\alpha)^{\gamma-1}, \tag{3.4.5}$$

where $\nu(\alpha)$ is previously defined in (2.1.31). The optimal trading strategy is $\pi_t^* = \pi^*(X_t^*)$ and the optimal consumption strategy is $c_t^* = c^*(X_t^*)$, where π^* and c^* are given in feedback form by

$$\pi^*(x) = \frac{\lambda}{\sigma(1-\gamma)}x, \quad c^*(x) = x\nu(\alpha). \tag{3.4.6}$$

The optimal portfolio process has the dynamics

$$\begin{aligned} dX_t^* &= \left(rX_t^* + \pi_t^*\sigma\lambda - c_t^*\right)dt + \pi_t^*\sigma dW_t \\ &= X_t^*\left[\left(r + \frac{\lambda^2}{1-\gamma} - \nu(\alpha)\right)dt + \frac{\lambda}{1-\gamma}dW_t\right]. \end{aligned}$$

The optimal strategy is to invest and consume a fixed proportion of current wealth at all times and the wealth process has the dynamics of a geometric Brownian motion. Finally we check the transversality condition. Let (π, c) be an arbitrary admissible strategy. Then

$$\begin{aligned} \mathbb{E}[e^{-\alpha t}u(X_t^{\pi,c})|X_0 = x] &= e^{-\alpha t}\nu(\alpha)^{\gamma-1}\mathbb{E}\left[\frac{(X_t^{\pi,c})^\gamma}{\gamma}\Big|X_0 = x\right] \\ &\leq e^{-\alpha t}\nu(\alpha)^{\gamma-1}\sup_{(\pi,c)\in\mathcal{A}_{0,t}(x)}\mathbb{E}\left[U(X_t^{\pi,c})|X_0 = x\right]. \end{aligned} \tag{3.4.7}$$

The last term on the right hand side is simply the finite horizon Merton problem with zero consumption, which has the solution

$$u(0,x;t) := \sup_{\pi\in\mathcal{A}_{0,t}(x)}\mathbb{E}\left[U(X_t^{\pi,c})|X_0 = x\right] = \frac{x^\gamma}{\gamma}\exp\left\{\gamma\left(r + \frac{\lambda^2}{2(1-\gamma)}\right)t\right\}. \tag{3.4.8}$$

Substituting (3.4.8) into (3.4.7) gives

$$0 \leq \mathbb{E}[e^{-\alpha t}u(X_t^{\pi,c})|X_0 = x] \leq \nu(\alpha)^{\gamma-1}\frac{x^\gamma}{\gamma}\exp\left\{\left(-\alpha + \gamma\left(r + \frac{\lambda^2}{2(1-\gamma)}\right)\right)t\right\}.$$

It is easy to see that the transversality condition is satisfied if and only if the feasibility condition (3.4.1) is satisfied. Without this condition, infinite consumption would be possible.

3.5 Exponential utility: enforcement of the non-negative consumption constraint

We now examine the solution in the case of exponential utility more closely. Setting $U(c) = -e^{-\gamma c}$ a simple calculation verifies that the solution of (3.2.10) is

$$u(x) = -Ae^{-\gamma r x}, \tag{3.5.1}$$

where A is the positive constant

$$A = \frac{1}{r} \exp\left\{ \frac{r - \alpha - \lambda^2/2}{r} \right\}. \tag{3.5.2}$$

From this we can recover the optimal controls and optimal wealth process in feedback form

$$\pi^*(x) = \frac{\lambda}{\gamma r \sigma}, \quad c^*(x) = rx - \frac{r - \alpha - \lambda^2/2}{\gamma r}. \tag{3.5.3}$$

The optimal wealth process is a drifting Brownian motion with the dynamics

$$dX_t^* = \frac{r - \alpha - \lambda^2/2}{\gamma r} dt + \frac{\lambda}{\gamma r} dW_t.$$

We can then deduce that the optimal consumption process is also an arithmetic Brownian motion and hence, in principle, unbounded below and potentially negative. This violates the non-negative consumption constraint and thus the controls are not admissible.

The inadmissibility of the consumption strategy has been overlooked by several authors who have used this solution. In both [20], [39] the authors do not identify or comment on the admissible set of controls or examine the behaviour of the optimal strategies of this solution

This inadmissible optimal consumption policy stems from finite marginal utility at zero wealth. Negative consumption is not infinitely penalised. Several attempts have been made to resolve this problem in which the non-negative consumption constraint is binding. Implicitly they adopt the modified utility function

$$U(c) = \begin{cases} -e^{-\gamma c}, & c \geq 0, \\ -\infty, & c < 0. \end{cases}$$

In his monograph [38], Merton recognised the consumption constraint was binding in the case of finite marginal utility and attempted to tackle the issue by specifying a free-boundary problem. He considers a general utility function of the form

$U(t,c) = e^{-\alpha t}U(c)$ with finite marginal utility at $c = 0$, $U'(0) < \infty$. We outline here his method in the specific case of exponential utility.

Since the problem is time-independent, there must be a level of wealth, $\widehat{X}$, below which consumption is zero. Under the restriction that wealth must always be non-negative, Merton claims that finite marginal utility, $U'(c) \leq M < \infty$ for all $c \geq 0$, implies that $\widehat{X}$ is strictly positive, $\widehat{X} > 0$. While we have not managed to verify this claim, we later establish that his method leads to a suboptimal solution, and so this need not concern us.

Define $u^1(x) \equiv u(x)$ for $0 \leq x \leq \widehat{X}$ and $u^2(x) \equiv u(x)$ for $x > \widehat{X}$. On the boundary Merton specifies value matching and smooth pasting conditions,

$$u^1(\widehat{X}) = u^2(\widehat{X}) \tag{3.5.4}$$

and

$$u^1_x(\widehat{X}) = u^2_x(\widehat{X}) \tag{3.5.5}$$

$$u^1_{xx}(\widehat{X}) = u^2_{xx}(\widehat{X}). \tag{3.5.6}$$

For $x > \widehat{X}$ consumption is non-negative, $u^2(x)$ satisfies the PDE (3.2.10) which has the solution (3.5.1)

$$u^2(x) = -A\exp\{-\gamma r x\}, \tag{3.5.7}$$

where A is the constant given in (3.5.2), and the optimal controls in feedback form, $\pi^*(X^*_t), c^*(X^*_t)$, satisfy (3.5.3).

For $0 \leq x \leq \widehat{X}$ we have that $c^*(x) = 0$, $U(c^*(x)) = -1$, and $u^1(x)$ satisfies

$$0 = \max_{\pi}\left[-1 - \alpha u^1(x) + u^1_x(x)(rx + \pi\sigma\lambda) + \frac{1}{2}\pi^2\sigma^2 u^1_{xx}(x)\right],$$

with the boundary condition (3.5.4). The form of the optimal feedback investment strategy remains unchanged and $u^1(x)$ satisfies

$$0 = -1 - \alpha u^1(x) + u^1_x(x)rx - \frac{1}{2}\lambda^2\frac{(u^1_x(x))^2}{u^1_{xx}(x)}. \tag{3.5.8}$$

Seeking a solution of the form $u^1(x) = Bx^k - 1/\alpha$, for some constant B, k can be shown to be the solution of the quadratic equation $rk^2 - (a + r + \lambda^2/2)k + a = 0$, hence $k = k_+$ or $k = k_-$ given by

$$k_\pm = \frac{\left(\alpha + r + \frac{\lambda^2}{2}\right) \pm \sqrt{\left(\alpha + r + \frac{\lambda^2}{2}\right)^2 - 4\alpha r}}{2r}.$$

It is easily verified that $k_+ > 1 > k_- > 0$. It follows from (3.5.5) and (3.5.6) that

$$\frac{u^2_{xx}(\widehat{X})}{u^2_x(\widehat{X})} = \frac{u^1_{xx}(\widehat{X})}{u^1_x(\widehat{X})} = \frac{k-1}{\widehat{X}}. \tag{3.5.9}$$

On the other hand $u^2_{xx}(\widehat{X})/u^2_x(\widehat{X}) = -r\gamma < 0$ and, noting $\widehat{X} > 0$, it follows that $k < 1$, hence $k = k_-$. Furthermore, the boundary $\widehat{X}$ is given by

$$\widehat{X} = \frac{1-k_-}{r\gamma}, \tag{3.5.10}$$

and the constant B is given by

$$B = \frac{u^2(\widehat{X}) + \frac{1}{\alpha}}{\widehat{X}^k}.$$

In summary

$$u(x) = \begin{cases} -A\exp\{-\gamma r x\}, & x > \widehat{X}, \\ Bx^{k_-} - \frac{1}{\alpha}, & 0 \le x \le \widehat{X}. \end{cases} \tag{3.5.11}$$

$$\pi^*(x) = \begin{cases} \frac{\lambda}{\sigma\gamma r}, & x > \widehat{X}, \\ \frac{\lambda}{\sigma(1-k_-)}x, & 0 \le x \le \widehat{X}. \end{cases} \tag{3.5.12}$$

$$c^*(x) = \begin{cases} rx - \frac{r-\alpha-\lambda^2/2}{r\gamma}, & x > \widehat{X}, \\ 0, & 0 \le x \le \widehat{X}. \end{cases} \tag{3.5.13}$$

$$dX^*_t = \begin{cases} \frac{r-\alpha+\lambda^2/2}{r\gamma}dt + \frac{\lambda}{r\gamma}dW_t, & X^*_t > \widehat{X}, \\ rX^*_t + X^*_t\frac{\lambda}{1-k_-}(\lambda dt + dW_t), & 0 \le X^*_t \le \widehat{X}. \end{cases} \tag{3.5.14}$$

Further investigation shows, however, that Merton's solution is suboptimal. A simple example demonstrates this. Set $r = \alpha = 2$, $\lambda^2/2 = 1$ and $\gamma = 1/4$. Then $k = 1/2$ and $\hat{X} = 1$. Consider an investor with wealth $x = 0.8$. According to Merton's free-boundary solution, for this level of wealth, $c^*(x) = 0$ and $u(x) = Bx^k + 1/\alpha$. It is easy to verify that

$$\frac{\partial}{\partial c}\left(U(C) - cu_x(x)\right)\Big|_{c=0} > 0.$$

This implies that

$$\underset{c\ge 0}{\operatorname{argmax}}[U(c) - cu_x(x)] > 0 \tag{3.5.15}$$

and hence the HJB equation is violated.

While the concept of a boundary in wealth below which consumption is zero is well placed, the use of a smooth pasting condition to determine this value has not been successful.

Karatzas et al. [29] and Huang and Pagés [23] also identified the issue resulting from finite marginal utility at zero. They both tackle the problem using a careful extension to inverse of the marginal utility. For utility functions defined on $\mathbb{R}^+$ the inverse of the marginal utility $I(\eta)$ maps $(0, U'(0))$ to $[0, \infty]$. This definition is then extended by setting $I(\eta) = 0$ for $\eta \in [U'(0), \infty)$. Huang and Pagés define

$$I^+(\eta) := \inf\{c \in \mathbb{R}^+ : U'(c) \leq \eta\}.$$

For exponential utility $I(\eta)$ is given by

$$I^+(\eta) = \left[\frac{1}{\gamma}\left(\log\frac{y}{\gamma} - 1\right)\right]^+.$$

Using this definition Huang and Pagés use dual techniques to prove existence of solutions. In particular the optimal consumption policy is given by

$$c_s^* = I^+(e^{\alpha s}\eta\xi_s),$$

where η is the Lagrange multiplier specified via the budget constraint (3.3.2).

Some limited progress can be made on obtaining partial solutions to the infinite horizon optimal consumption problem with exponential utility. The optimal wealth process satisfies

$$\begin{aligned} X_t^* &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty e^{-rs}c_s^* ds \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty e^{-r(u-t)}I^+(e^{\alpha t}\eta\xi_{t,u})du \middle| \mathcal{F}_t\right] \end{aligned}$$

This is a function of time and Z_t, the Radon-Nikodym derivative of the equivalent martingale measure $\mathbb{Q}$. Define

$$F(t,z) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty e^{-r(u-t)}I^+\left(e^{\alpha t}\eta e^{-r(u-t)}\frac{Z_s}{z}\right)du \middle| \mathcal{F}_t\right].$$

Huang and Pagés prove smoothness properties of the function F allowing us to apply Itô's Lemma to $F(t, Z_t)$:

$$F(T,Z_T) - F(0,Z_0) = \int_0^T \left(F_t(t,Z_t) + \frac{1}{2}\lambda^2 Z_t^2 F_{zz}(t,Z_t)\right)dt - \int_0^T \lambda Z_t F_z(t,Z_t)dW_t. \tag{3.5.16}$$

On the other hand we know the wealth process satisfies

$$\widetilde{X}_t^* - x = \int_0^t \widetilde{\pi}_s^* \sigma(\lambda ds + dW_s) - \int_0^t \widetilde{c}_s^* ds. \tag{3.5.17}$$

Noting that $F(t, Z_t) = X_t^*$ and equating the integrands of stochastic integrals in (3.5.16) and (3.5.17) it follows that

$$\pi_t^* = -\frac{\lambda}{\sigma} Z_t F_Z(t, Z_t). \tag{3.5.18}$$

With a simple change of variables we have the relation $F(t, Z_t) = \phi(\eta e^{\alpha t})$, where $\phi(\eta)$ is given by

$$\phi(\eta) := \mathbb{E}^{\mathbb{Q}} \left[\int_0^\infty e^{-rs} I^+(e^{\alpha s} \eta \xi_s) ds \right].$$

Writing the expectation as an integral we have

$$\phi(\eta) = \int_{-\infty}^{\infty} \frac{I(e^x)}{\lambda\sqrt{2\pi}} e^{\frac{(x-\log\eta)(\alpha - r + \lambda^2/2)}{\lambda^2}} \int_0^\infty \frac{1}{\sqrt{s}} e^{-\frac{(x-\log\eta)^2 + (\alpha - r + \lambda^2/2)^2}{2\lambda^2 s}} ds dx$$

which can be reduced to

$$\phi(\eta) = \frac{1}{\lambda^2/2(\delta_1 + \delta_2)} \left[\int_0^\infty I^U(\eta e^{-u}) e^{-\delta_1 u} du + \int_0^\infty I^U(\eta e^u) e^{-\delta_2 u} du \right],$$

where δ_1 and δ_2 are the constants

$$\delta_1 = \frac{\left(\alpha - r + \frac{\lambda^2}{2}\right) + \sqrt{\left(\alpha - r + \frac{\lambda^2}{2}\right)^2 + 2r\lambda^2}}{\lambda^2},$$
$$\delta_2 = \frac{-\left(\alpha - r + \frac{\lambda^2}{2}\right) + \sqrt{\left(\alpha - r + \frac{\lambda^2}{2}\right)^2 + 2r\lambda^2}}{\lambda^2}.$$

Consumption is zero if and only if $U'(0) \leq e^{\alpha t} \eta \xi_t$ which occurs if wealth falls below the boundary $\overline{X} = \phi(U'(0))$, given by

$$\overline{X} = \frac{1}{\gamma \delta_1^2 \sqrt{(r - \alpha - \frac{\lambda^2}{2})^2 + 2r\lambda^2}}.$$

Furthermore in this region the wealth process is given by

$$\pi^*(x) = \frac{\lambda}{\sigma} \delta_1 x.$$

In the zero-consumption region, the optimal strategy is to invest a constant proportion of wealth into the risk asset and the optimal wealth process is a geometric

Brownian motion. The boundary $\overline{X}$ is positive so it follows that the investor's wealth is always strictly positive and the probability of bankruptcy is zero. This is a form of constant proportion portfolio insurance as presented by Black and Perold [4]. Merton notes that this optimal strategy minimises the expected time until optimal consumption becomes positive.

Chapter 4

Utility-based pricing of stochastic income in a stochastic volatility model

In this chapter we apply utility-based valuation techniques to the pricing of a stochastic income stream under classical utility and forward utility preferences. We focus on a stochastic volatility model with exponential preferences where explicit solutions are possible using extensions of the power distortion solution found in Henderson [18], Zariphopoulou [48] and Monoyios [40], among others.

There is a large literature on optimal investment problems with income addressing a variety of issues such as liquidity constraints on wealth (see for example [13], [17], [41]). We focus on the utility-based valuation of a stochastic income stream which is imperfectly correlated to traded assets. We extend the results of Henderson [18], in which the traded asset is modelled as a geometric Brownian motion, to a stochastic volatility model.

The continuous stochastic income stream may be thought of as the limiting case of a claim with multiple payment dates. It is well-known that when the underlying market is incomplete, the dual minimiser for the problem without the claim, $\mathbb{Q}_0^*$ is horizon dependent. Consequently the marginal utility-based price of a claim payable at a single fixed time is computed under a pricing kernel which depends on that maturity date. This can potentially lead to inconsistencies when valuing claims with multiple payment dates.

4.1 Utility indifference pricing

We first establish the notions of utility-based valuation. Consider a European claim with payoff Ψ_T at expiry T. In a complete market, by definition, there exists a self-financing portfolio, π, which replicates the claim payoff, $X_T^\pi = \Psi_T$, almost surely. No-arbitrage arguments then tell us the price of the claim at time 0 is X_0^π, the initial value of the replicating portfolio. A short position in the claim can be perfectly hedged by holding the portfolio π. Mathematically, there is a unique martingale measure, $\mathbb{Q}$, equivalent to the physical measure, $\mathbb{P}$.

In an incomplete market the opposite is true. Not all claims can be replicated and a short position in the claim is subject to a potential loss on maturity $X_T^\pi - \Psi_T$. The set of equivalent martingale measures, $\mathcal{M}$, is no longer a singleton and Kramkov [33] showed that arbitrage-free prices of Ψ_T span the interval

$$\left(\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\widetilde{\Psi}_T, \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\widetilde{\Psi}_T \right).$$

This interval is typically very wide and a natural framework for selecting a pricing measure is to introduce risk preferences via a utility function. The pricing and hedging of the residual risk may be viewed as an extension to the portfolio optimisation problem. The aim is to reconcile the maximum expected utility an investor may obtain when they buy and hold n units of the claim with their maximum expected utility if they do not invest in the security. Define the primal value function with random endowment, $u^{(n)}$, as the maximum expected utility when receiving a terminal payoff of n claims:

$$u^{(n)}(x) := \sup_{\pi\in\mathcal{A}_{0,T}(x)} \mathbb{E}\Big[U(X_T^\pi + n\Psi_T)\Big]. \tag{4.1.1}$$

Definition 4.1.1. *The utility indifference price per claim, $p^{(n)}$, for a holding of n claims, is defined as the solution to*

$$u^{(n)}(x - np^{(n)}) = u(x). \tag{4.1.2}$$

In other words, when trading the claim the agent experiences no loss or gain of utility compared with the alternative strategy of not trading it.

The marginal utility-based price of the claim, $\hat{p}$, may be defined in a number of ways. One way is to define it as the limit of the indifference price as the number of claims tends to zero:

Definition 4.1.2. *The marginal utility-based price of a claim is $\hat{p}$, defined by*

$$\hat{p} := \lim_{n \to 0} p^{(n)}. \tag{4.1.3}$$

Davis [9] provides an alternative, but equivalent, definition which postulates a zero rate of marginal substitution. Suppose an investor diverts an amount δ of initial capital into buying or selling claims at a price $\hat{p}$. The number of claims traded is $\delta/\hat{p}$ and the value function is given by

$$u^{(\delta/\hat{p})}(x-\delta) := \sup_{\pi \in \mathcal{A}(x-\delta)} \mathbb{E}\left[U\left(X_T^{\pi} + \frac{\delta}{\hat{p}}\Psi_T\right)\right]. \tag{4.1.4}$$

Definition 4.1.3. (Davis [9]) *The marginal price, $\hat{p}$, of the claim is the price which solves*

$$\frac{\partial}{\partial \delta} u^{(\delta/\hat{p})}(x-\delta)\bigg|_{\delta=0} = 0. \tag{4.1.5}$$

Davis showed that if the value function satisfies suitable regularity conditions we can formally differentiate (4.1.4) to obtain

$$\frac{\partial}{\partial \delta} u^{(\delta/\hat{p})}(x-\delta)\bigg|_{\delta=0} = -\hat{p}u_x^{(0)}(x) + \mathbb{E}[U_x(X_T^0)\Psi_T], \tag{4.1.6}$$

where X_T^0 is the optimal terminal wealth for investor who does not trade the claim. Equating to zero and rearranging the marginal utility price is given by

$$\hat{p} = \frac{\mathbb{E}[U_x(X_T^0)\Psi_T]}{u_x^{(0)}(x)}. \tag{4.1.7}$$

It can be shown from the results of the dual problem with zero consumption that $U_x(X_T^0)/u_x^{(0)}(x) = \xi_T^0$, where ξ^0 is the state price density of the optimal dual measure, $\mathbb{Q}_0^*$, for an investor not holding the claim.

It follows that the marginal price of the claim is the expected discounted value under the dual optimal measure,

$$\hat{p} = \mathbb{E}^{\mathbb{Q}_0^*}\left[\widetilde{\Psi}_T\right].$$

This approach relies on knowing the value function is sufficiently smooth, but in practice the differentiability of $u^{(\delta/p)}(x)$ with respect to δ is often difficult to check directly.

4.2 The market model

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space supporting a two-dimensional Brownian motion $\mathbf{W} = (W, \widetilde{W})_{0\leq t\leq T}$, where W and $\widetilde{W}$ have fixed correlation $\rho \in [-1, 1]$. The filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ is the completion of the natural filtration generated by $\mathbf{W}$. Denote by $\widetilde{\mathbb{F}} := (\widetilde{\mathcal{F}}_t)_{0\leq t\leq T}$ the filtration generated by $\widetilde{W}$. We write $\widetilde{W}_t = \rho W_t + \bar{\rho} W_t^{\perp}$, with $\bar{\rho}^2 = 1 - \rho^2$ and where $W^{\perp}$ is a Brownian motion independent of W.

We consider a stochastic volatility model with a single traded stock $S := (S_t)_{0\leq t\leq T}$ and a strictly positive stochastic volatility $Y := (Y_t)_{0\leq t\leq T}$, following

$$dS_t = S_t\left(\mu(Y_t)dt + Y_t dW_t\right), \tag{4.2.1}$$

$$dY_t = a(Y_t)dt + b(Y_t)d\widetilde{W}_t. \tag{4.2.2}$$

The parameter functions μ, a and b are $\widetilde{\mathbb{F}}$-adapted and such that unique, strong solutions to (4.2.1) and (4.2.2) exist. We shall often write $\beta_t \equiv \beta(Y_t)$ for a process depending on Y. Lipschitz conditions are sufficient for the existence of solutions to (4.2.1) and (4.2.2) but we do not impose them so as not to rule out some well-know stochastic volatility models.

The agent also has access to a riskless bond paying constant, non-negative interest rate r. Much literature sets the interest rate to be zero without loss of generality, but we do not do this as we wish to observe the role of the interest rate in discounting payments made at different times.

Henderson [18], analyses a model where S was a geometric Brownian motion with constant volatility. Our model is similar, but extends the constant volatility model to one with stochastic volatility.

Let $(\lambda_t)_{0\leq t\leq T}$ denote the stock's market price of risk, defined by $\lambda_t := (\mu_t - r)/Y_t$. The mean-variance trade-off process is the increasing process $K := (K_t)_{0\leq t\leq T}$ given by

$$K_t := \int_0^t \lambda_s^2 ds, \quad 0 \leq t \leq T.$$

It is assumed throughout that $K_t < \infty$ for $0 \leq t \leq T$, $\mathbb{P}$-almost surely.

The set $\mathcal{M}$ of local martingale measures $\mathbb{Q}$ is characterised by the density processes

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} \equiv Z_t := \mathcal{E}(-\lambda \cdot W - \psi \cdot W^{\perp})_t, \quad 0 \leq t \leq T.$$

where $\mathcal{E}$ is the Doléans exponential and $\psi = (\psi_t)_{0\leq t\leq T}$ is an $\mathbb{F}$-adapted process satisfying $\int_0^T \psi_t^2 dt \leq \infty$ $\mathbb{P}$-almost surely. If ψ is such that $\mathbb{E}[Z_T] = 1$, then Z is a true martingale and $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$. We assume that λ is such that the minimal martingale measure $\mathbb{Q}^M$ of Föllmer and Schweizer [14], defined by

$$\left.\frac{d\mathbb{Q}^M}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t^M := \mathcal{E}(-\lambda \cdot W)_t, \quad 0 \leq t \leq T, \tag{4.2.3}$$

is a probability measure. Denote by $\widetilde{\mathbb{Q}}^M$ the projection of the minimal martingale measure on to the filtration $\widetilde{\mathbb{F}}$ with density

$$\frac{d\widetilde{\mathbb{Q}}^M}{d\mathbb{P}} := \left.\frac{d\mathbb{Q}^M}{d\mathbb{P}}\right|_{\widetilde{\mathcal{F}}_t} = \mathcal{E}(-\rho\lambda \cdot \widetilde{W})_t, \quad 0 \leq t \leq T.$$

This measure, which we shall call the minimal distortion measure, appears frequently in the solutions to optimal investment problems with stochastic volatility under exponential preferences (see, for example, [40]). Denote by $\mathcal{M}_f$ the set of equivalent martingale measures with finite entropy relative to $\mathbb{P}$,

$$\mathcal{M}_f := \left\{ \mathbb{Q} \in \mathcal{M} : \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right] < \infty \right\}.$$

In addition to wealth generated from trading S, the investor receives a continuous stochastic income stream over $[0, T]$ at a rate $f(Y_t)$. We assume that, for all $\mathbb{Q} \in \mathcal{M}_f$, the stochastic income stream satisfies

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T f(Y_t)dt\right] < \infty. \tag{4.2.4}$$

This is tantamount to assuming that the income stream can be super-replicated. As is well-known, super-replicating bounds for derivatives are very wide, so in what follows we shall consider exponential utility-based pricing of the income stream. Condition (4.2.4) ensures that the associated value function, defined in (4.2.6) below, is finite and all the necessary expectations are well defined. Much literature on exponential indifference pricing (for example [2], [24]) stipulates exponential moment conditions on contingent claims of the form

$$\mathbb{E}\left[e^{(\gamma+\epsilon)\Psi}\right] < \infty, \quad \mathbb{E}\left[e^{-\epsilon\Psi}\right] < \infty,$$

where Ψ is a time-T payable endowment. Becherer [2] demonstrates the equivalence of these two conditions, with $\mathbb{E}^{\mathbb{Q}}[\Psi] < \infty$, using Fenchel's inequality.

The financial agent trades a self-financing portfolio involving the stock S and receives the stochastic income $f(Y)$, generating wealth process $X^\pi \equiv X =: (X_t)_{0\le t\le T}$ satisfying

$$dX_t = rX_t dt + \pi_t Y_t\big(\lambda_t dt + dW_t\big) + f(Y_t)dt, \tag{4.2.5}$$

where $\pi = (\pi_t)_{0\le t\le T}$ is the cash amount invested in the stock. Any such strategy π must be $\mathbb{F}$-adapted and satisfy $\int_0^T \pi_s^2\sigma_s^2 ds < \infty$. A further admissibility condition is given in (4.2.7) below.

The agent's risk preferences are modelled with an exponential utility function with risk aversion $\gamma > 0$. Starting with wealth $x \in \mathbb{R}^+$ at time $t \in [0,T]$, the objective is to maximise expected utility of terminal wealth, with value function

$$u(t,x,y) := \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[-e^{\gamma X_T^\pi}\,\middle|\, X_t = x, Y_t = y\right], \tag{4.2.6}$$

where $\mathcal{A}$ is the set of admissible trading strategies. We call this the primal problem.

Defining the admissible set $\mathcal{A}$ is subtle for exponential utility. Since the utility function is defined on the whole real line the wealth process can become negative. Schachermayer [46] notes that the optimal wealth process can in principle become unbounded below, and goes to some lengths to reconcile this fact with the absence of arbitrage and the necessity to rule out so-called doubling strategies. We adopt a definition formulated by Becherer [2] using the dual side of the optimal investment problem. We define the set of admissible strategies to be

$$\mathcal{A} := \left\{\pi : \int_0^t \frac{\pi_u}{S_u} dS_u,\ t\in[0,T],\ \text{is a } (\mathbb{Q},\mathbb{F}) - \text{martingale for all}\, \mathbb{Q}\in\mathcal{M}_f\right\}. \tag{4.2.7}$$

As shown by Delbaen et al. [11] and Kabanov and Stricker [24], there are a number of possibilities for defining the set of admissible strategies via the dual side of the problem and these all lead to the same maximal expected utility in the primal problem.

4.3 The solution to the primal problem

The perfect correlation between stochastic income and the stochastic volatility allows us to derive an explicit solution to the optimal investment problem (4.2.6). We extend the distortion power solution for optimal investment with stochastic volatility

studied by, among others, Zariphopoulou [48], Tehranchi [47], Monoyios [40], and in particular, in a log-Brownian model with non-traded stochastic income, Henderson [18].

Define the function $H \equiv H^{(T)} : [0,T] \times \mathbb{R} \to \mathbb{R}^+$ by

$$H^{(T)}(t,y) = \mathbb{E}^{\widetilde{\mathbb{Q}}^M}\left[\exp\left\{-\bar{\rho}^2 \int_t^T \left(\frac{1}{2}\lambda^2(Y_u) + \gamma e^{r(T-u)} f(Y_u)\right) du\right\} \middle| Y_t = y\right]. \tag{4.3.1}$$

We use $H^{(T)}$ to emphasise this function will in general be horizon-dependent. When no confusion occurs we shall occasionally drop the superscript T for brevity.

Theorem 4.3.1. *The value function (4.2.6) is given by*

$$u(t,x,y) = -e^{-\gamma e^{r(T-t)}x} H^{(T)}(t,y)^{\delta}, \quad 0 \le t \le T, \tag{4.3.2}$$

where

$$\delta = \frac{1}{1-\rho^2},$$

and with $H^{(T)}(t,y)$ given in (4.3.1).

Proof: The HJB equation, associated with the value function (4.2.6) is

$$\begin{aligned} 0 = \max_{\pi}\Big[& u_t + (rx + f(y))u_x + \mathcal{L}^Y u + \pi y \lambda(y) u_x \\ & + \frac{1}{2}\pi^2 y^2 u_{xx} + \pi\rho b(y) y u_{xy}\Big], \end{aligned} \tag{4.3.3}$$

where $\mathcal{L}^Y$ is the infinitesimal generator of the stochastic volatility Y,

$$\mathcal{L}^Y f(y) := a(y) f_y(y) + \frac{1}{2} b^2(y) f_{yy}(y).$$

Maximising over the π, the optimal investment strategy as a feedback control is $\pi^*(t, X_t^*, Y_t)$, where X^* is the optimal wealth process and π^* satisfies

$$\pi^*(t,x,y) = -\frac{\lambda(y)u_x(t,x,y) + b(y)\rho u_{xy}(t,x,y)}{y u_{xx}(t,x,y)}. \tag{4.3.4}$$

Substituting this optimal control into the HJB equation (4.3.3) gives

$$0 = u_t + (rx + f(y))u_x + \mathcal{L}^Y u - \frac{1}{2}\frac{(\lambda(y)u_x + b(y)\rho u_{xy})^2}{u_{xx}},$$

with $u(T,x,y) = -e^{-\gamma x}$. We seek a separable solution of the form (4.3.2). A simple calculation shows that $H(t,y)$ solves

$$0 = H_t + \big(a(y) - \lambda(y)b(y)\rho\big)H_y + \frac{1}{2}b^2(y)H_{yy} - (1-\rho^2)\left(\frac{1}{2}\lambda(y)^2 + \gamma e^{r(T-t)}f(y)\right)H, \tag{4.3.5}$$

with the boundary condition $H(T,y) = 1$. It follows from the Feynman-Kac theorem that $H(t,y)$ is given by (4.3.1). A verification argument then shows that $u(t,x,y)$ is indeed the value function. □

Setting the drift and volatility of the traded asset to be constant recovers the solution of Henderson [18], where the underlying stock is modelled as a geometric Brownian motion. In our stochastic volatility model, the expectation in definition of $H^{(T)}$ contains a term involving the mean-variance trade-off process which becomes deterministic in the constant coefficient model. This has a noticeable impact on the utility-based valuation of the income, which we demonstrate later.

For $f \equiv 0$ in (4.3.1), corresponding to the zero income case, the solution reduces to the well-known distortion power solution for exponential utility (see [48] for example).

Using the explicit solution (4.3.2), it follows from (4.3.4) that $\pi^*(t,x,y)$ is given by

$$\pi^*(t,x,y) = \frac{\lambda(y)}{y\gamma}e^{-r(T-t)} + \frac{b(y)\rho e^{-r(T-t)}}{y\gamma(1-\rho^2)}\frac{\partial}{\partial y}\log H^{(T)}(t,y), \quad 0 \le t \le T.$$

We note that this optimal control is independent of the investor's wealth X_t^*, a well-known property of exponential utility. The first term of π^* comes from the expression $-\lambda(y)u_x/yu_{xx}$, which is the form of the Merton optimal strategy where $\sigma = y$ and λ is a constant. The second term can be interpreted as a hedging component. When $b(Y)$ is zero, the volatility process Y is deterministic, the income $f(Y)$ is non-stochastic and the hedging component is zero. In the zero correlation case, $\rho = 0$, the stochastic income and stochastic volatility cannot be hedged and again the second hedging component of the optimal investment strategy is zero.

4.4 The solution to the dual problem

Let $(Z_t)_{0\le t\le T}$ be the density process of an equivalent martingale measure $\mathbb{Q} \in \mathcal{M}_f$. We denote by $\xi = (\xi_t)_{0\le t\le T}$ the state price density, $\xi_t := Z_t/S_t^0$. For $s \le t$ we use

the notation $\xi_{s,t} = \xi_t/\xi_s$. It follows from a simple application of Itô's Lemma that $\left(X_t\xi_t - \int_0^t \xi_s f(Y_s)ds\right)_{0\le t\le T}$ is a local $\mathbb{P}$-martingale, for all equivalent martingale measures $\mathbb{Q} \in \mathcal{M}_f$. This is used to derive the form of the dual problem:

$$v(t,\eta,y) := \inf_{\mathbb{Q}\in\mathcal{M}_f} \mathbb{E}\left[V\left(\eta\xi_{t,T}\right) + \eta\int_t^T \xi_{t,u}f(Y_u)du \,\middle|\, Y_t = y\right], \quad \eta > 0, \tag{4.4.1}$$

where $V(\eta)$ is the convex conjugate of the utility function U,

$$V(\eta) := \sup_{x\in\mathrm{dom}(U)} \Big[U(x) - \eta x\Big], \quad \eta > 0.$$

The convex conjugate of the exponential utility function is

$$V(\eta) = \frac{\eta}{\gamma}\left[\log\left(\frac{\eta}{\gamma}\right) - 1\right]. \tag{4.4.2}$$

For the zero income case, $f \equiv 0$, the dual problem is equivalent to minimising $\mathbb{E}[\xi_T \log \xi_T]$. The measure which achieves this infimum is, by definition, the minimal entropy measure which we denote by $\mathbb{Q}^E$. We write $\mathbb{Q}^E \equiv \mathbb{Q}^E(T)$ to emphasise the dependence on the horizon.

An eminent collection of literature, Karatzas et al.[29], Kramkov and Schachermayer [34], Delbaen et al.[11], and Frittelli[15], among others, has established the fundamental duality between the primal and dual value functions, showing the convex conjugate relationship between u and v and identifying conditions for existence of solutions.

Theorem 4.4.1. *The dual value function (4.4.1) is given by*

$$v(t,\eta,y) = H^{(T)}(t,y)^\delta V\left(e^{-r(T-t)}\frac{\eta}{H^{(T)}(t,y)^\delta}\right), \quad \eta > 0, \tag{4.4.3}$$

with V given in (4.4.2) and $H^{(T)}(t,y)$ given in (4.3.1). The optimal dual measure $\mathbb{Q}^*$, which achieves the infimum in (4.4.1), has density given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \mathcal{E}(-\lambda\cdot W - \psi^*\cdot W^\perp)_T,$$

with

$$\psi^*(t,Y_t;T) = -\frac{b(Y_t)}{\sqrt{1-\rho^2}}\frac{\partial}{\partial y}\log H^{(T)}(t,Y_t), \quad t\in[0,T]. \tag{4.4.4}$$

Proof: The dual value function (4.4.1) satisfies the Bellman equation

$$\inf_{\psi}\Big[v_t - r\eta v_\eta + \mathcal{L}^Y v + \frac{1}{2}\eta^2(\lambda^2(y)+\psi^2)v_{\eta\eta} \\ -\eta b(y)(\rho\lambda(y)+\bar{\rho}\psi)v_{\eta y} + \eta f(y)\Big] = 0, \tag{4.4.5}$$

with $v(T,\eta,y) = V(\eta)$. Minimising over ψ, it follows that the optimal value ψ^* is given by

$$\psi^*(t,\eta,y) = \frac{\bar{\rho}b(y)}{\eta}\frac{v_{\eta y}(t,\eta,y)}{v_{\eta\eta}(t,\eta,y)}. \tag{4.4.6}$$

Substituting this optimal control back into (4.4.5) gives

$$v_t - r\eta v_\eta + \mathcal{L}^Y v + \frac{1}{2}\eta^2\lambda^2(y)v_{\eta\eta} - \eta b(y)\rho\lambda(y)v_{\eta y} - \frac{1}{2}\bar{\rho}^2 b^2(y)\frac{v_{\eta y}^2}{v_{\eta\eta}} + \eta f(y) = 0. \tag{4.4.7}$$

We seek a solution to this PDE of the form (4.4.3). It follows that $H^{(T)}(t,y)$ solves the semi-linear PDE (4.3.5) with boundary condition $H^{(T)}(T,y) = 1$, as before, so $H^{(T)}(t,y)$ is indeed given by (4.3.1). It is also possible to achieve this result by directly calculating the convex conjugate of the explicit solution (4.3.2). Computation of the derivatives of v from (4.4.3) shows ψ^* is given by (4.4.4). □

One of the key features of this result is the dependence of dual minimiser ψ on the horizon T. In the zero income case the optimal dual measure is $\mathbb{Q}_0^* = \widetilde{\mathbb{Q}}^E(T)$.

Using the conjugate relationship between u and v we can characterise the primal value function as an optimisation problem over equivalent martingale measures. Combining (4.4.1) and (4.4.3) it follows that $H(t,Y_t)$ has the representation

$$-\log H^{(T)}(t,y)^\delta = \inf_{\mathbb{Q}\in\mathcal{M}_f}\mathbb{E}\left[Z_{t,T}\log Z_{t,T} + \gamma e^{r(T-t)}\int_t^T \xi_{t,u}f(Y_u)du\middle|\mathcal{F}_t\right],$$

where we have used the notation $Z_{t,T} = Z_T/Z_t$. Substituting this into the explicit solution (4.3.2), the primal value function can be written as

$$u(t,x,y) = -\exp\left\{-\gamma e^{r(T-t)}x - \inf_{\mathbb{Q}\in\mathcal{M}_f}\mathbb{E}\left[Z_{t,T}\log Z_{t,T} + e^{r(T-t)}\gamma\int_t^T \xi_{t,u}f(Y_u)du\middle|\mathcal{F}_t\right]\right\}.$$

4.5 Utility-based pricing

Explicit solutions to the primal problem allow us to calculate the utility indifference price and the marginal utility indifference price of the stochastic income. Throughout this section we set the initial time t to be zero.

Corollary 4.5.1. *The utility indifference price, p, for the stochastic income stream paying at a rate $f(Y_t)$ over the interval $[0,T]$, is given by*

$$p = -\frac{e^{-rT}}{\gamma(1-\rho^2)} \log\left(\mathbb{E}^{\widetilde{\mathbb{Q}}^E(T)}\left[e^{-(1-\rho^2)\gamma\int_0^T e^{r(T-s)}f(Y_s)ds}\right]\right). \tag{4.5.1}$$

Proof: It follows immediately from Definition 4.1.1 that p is given by

$$p = -\frac{e^{-rT}}{\gamma(1-\rho^2)} \log\left(\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}^M}\left[e^{-(1-\rho^2)\left(\frac{1}{2}K_T+\gamma\int_0^T e^{r(T-s)}f(Y_s)ds\right)}\right]}{\mathbb{E}^{\widetilde{\mathbb{Q}}^M}\left[e^{-(1-\rho^2)\frac{1}{2}K_T}\right]}\right). \tag{4.5.2}$$

The minimal entropy measure and the minimal martingale measure, projected on to $\widetilde{\mathcal{F}}_T$, are related by the Esscher transform

$$\frac{d\widetilde{\mathbb{Q}}^E}{d\widetilde{\mathbb{Q}}^M} = \frac{e^{-\frac{1}{2}(1-\rho^2)K_T}}{\mathbb{E}^{\widetilde{\mathbb{Q}}^M}\left[e^{-\frac{1}{2}(1-\rho^2)K_T}\right]}.$$

A proof of this result can be found in Theorem 3 of Monoyios [40]. The result then follows. □

The form of the utility indifference price is exactly the same as that of Henderson [18] except for the measure under which the expectation is taken. In our stochastic volatility model the expectation in (4.5.1) is taken under the measure $\widetilde{\mathbb{Q}}^E(T)$ while in the Black-Scholes market of Henderson this is replaced by an expectation taken under the minimal distortion measure $\widetilde{\mathbb{Q}}^M$.

Denote by F_T the integrated inflated income over the time period $[0,T]$, $F_T := \int_0^T e^{r(T-u)}f(Y_u)du$. Due to the scaling of the problem, the marginal utility-based price can be equivalently calculated as the limit of the indifference price as risk aversion tends to zero.

Corollary 4.5.2. *The marginal utility-based price of the stochastic income $(f(Y_s))_{0\leq s\leq T}$ is $\hat{p}$, given by*

$$\hat{p} = \mathbb{E}^{\widetilde{\mathbb{Q}}^E}\left[\int_0^T e^{-rs}f(Y_s)ds\right].$$

Proof: Expanding the exponential $\exp\{-\bar{\rho}^2\gamma F_T\}$ as a power series, the utility indifference price (4.5.1) can be written as

$$p = -\frac{e^{-rT}}{\gamma(1-\rho^2)} \log\left(1 - \bar{\rho}^2\gamma\mathbb{E}^{\widetilde{\mathbb{Q}}^E}[F_T] + \frac{1}{2}\bar{\rho}^4\gamma^2\mathbb{E}^{\widetilde{\mathbb{Q}}^E}[F_T^2] - \mathcal{O}(\gamma^3)\right).$$

Using the Taylor series for $\log(1+x)$ for small values of x it follows that

$$\hat{p} = \lim_{\gamma\to 0} p = \mathbb{E}^{\widetilde{\mathbb{Q}}^E}\left[\int_0^T e^{-rs} f(Y_s)ds\right].$$

□

The marginal utility-based price is the expected discounted value of the aggregated stochastic income under the measure $\widetilde{\mathbb{Q}}^E(T)$. Although Henderson [18] does not provide an expression, it is easy to see that, for a log-Brownian market, the marginal utility-based price will be the expected value of the aggregated income under the minimal distortion measure $\widetilde{\mathbb{Q}}^M$. The incompleteness in the underlying stochastic volatility model causes a change in the measure used to compute the marginal utility-based price compared to the case of constant volatility. The dual minimiser, $\widetilde{\mathbb{Q}}^E(T)$, depends on the the horizon, T, through the process, ψ^*, and consequently the pricing kernel is horizon-dependent. This is not the case in [18] when the price is calculated under $\widetilde{\mathbb{Q}}^M$.

If we use the projection of minimal martingale measure to value the income, we do not price the orthogonal risk associated with the Brownian motion $W^\perp$. For this reason, one would anticipate that the marginal utility-based price of the income stream in our stochastic volatility would be lower than in the Black-Scholes model of Henderson [18], but we have not verify this conjecture.

The horizon dependence of the pricing kernel in this stochastic volatility model can lead to a possible inconsistency when pricing a collection of claims payable at different dates. To demonstrate this let us briefly consider a discrete collection of payments $\{f(Y_{s_i})\}_{i=1,\ldots,n}$ made at times $0 < s_1 < s_2 < \ldots < s_n = T$. Valuing these payments collectively as a single claim it is an easy exercise to show that the marginal price will be given by

$$\hat{p} = \sum_{i=1}^{n} \mathbb{E}^{\widetilde{\mathbb{Q}}^E(T)}[f(Y_{s_i})]. \tag{4.5.3}$$

On the other hand if we conduct a series of individual utility maximisation problems for each payment and then sum the individual marginal utility prices we will have

$$\hat{p} = \sum_{i=1}^{n} \mathbb{E}^{\widetilde{\mathbb{Q}}^E(s_i)}[f(Y_{s_i})]. \tag{4.5.4}$$

The expectations in (4.5.3) and (4.5.4) are taken under different probability measures and in general they may not be equal. This can potentially lead to pricing inconsistencies for claims with multiple payment dates.

Motivated by the horizon dependence of this example we carefully construct a forward utility that removes this dependence in the next section.

4.6 Forward utility-based pricing

In recent years there has been much research in the area of maturity independent utility functions. Motivated by a real option problem Henderson and Hobson [21] proposed the notion of horizon-unbiased utility functions. A financial agent had the opportunity to sell an indivisible, non-traded asset Y, which was partially correlated to other assets in a financial market. This mixed optimal control and optimal stopping problem is an example of a perpetual American option. The agent has an infinite planning horizon and their value function is given by

$$u(t,x,y) = \sup_{\tau \in \mathcal{T}} \sup_{\pi \in \mathcal{A}_{t,\tau}(x)} \mathbb{E}\left[U\big(\tau, X_\tau^\pi + Y_\tau\big) | X_t = x, Y_t = y\right], \tag{4.6.1}$$

where $\mathcal{T}$ is the set of possible stopping times.

Henderson and Hobson used the following fundamental modelling principle to motivate their ideas: in the absence of the asset Y, the investor solving (4.6.1) should have no preference over the choice of $\tau \in \mathcal{T}$. In the presence of Y the optimisation over $\tau \in \mathcal{T}$ is not biased by the underlying trading problem.

Independently, Musiela and Zariphopoulou [42] established the concept of forward performance measures:

Definition 4.6.1. *An $\mathcal{F}_t$ adapted process $(U(t,x))_{t\geq 0}$ is a forward performance process if for $t \geq 0$ and $x \in \mathbb{R}$*

1. *the mapping $x \mapsto U(t,x)$ is strictly increasing and strictly concave with $U(0,x) = u_0(x)$,*

2. for each admissible strategy

$$\mathbb{E}\left[U(T, X_T^\pi)|\mathcal{F}_t\right] \leq U(t, X_t^\pi), \quad \forall\, T \geq t,$$

3. there exists an admissible strategy π^* for which

$$\mathbb{E}\left[U(T, X_T^{\pi^*})|\mathcal{F}_t\right] = U(t, X_t^{\pi^*}), \quad \forall\, T \geq t.$$

This construction maintains the martingale properties of the value function in the classical utility maximisation problem while defining a stochastic utility which evolves forward in time without reference to a specific horizon T. It can incorporate market factors to reflect investors' changing preferences with market conditions.

To specify the agents preference we use the exponential forward performance measure $U_F(t, x)$, given by

$$U_F(t, x) = -\exp\left\{-\gamma e^{-rt}x + \frac{1}{2}K_t\right\}.$$

This performance measure is similar to the one used by Leung et al. [35] in an American option model, but we have an extra factor to handle our non-zero interest rate r. Shortly, we will observe from our explicit solution that this is indeed a forward performance measure for an agent in our stochastic volatility model.

4.6.1 The primal solution

Denote by $u_F(t, x, y)$ the value function for an investor with this exponential forward preference,

$$u_F(t, x, y) := \sup_{\pi \in \mathcal{A}} \mathbb{E}\big[U_F(T, X_T^\pi)|X_t = x, Y_t = y\big], \tag{4.6.2}$$

where X has the dynamics (4.2.5).

Define the function $H_F : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ by

$$H_F(t, y) = \mathbb{E}^{\widetilde{\mathbb{Q}}^M}\left[e^{-(1-\rho^2)\int_t^T \gamma e^{-ru} f(Y_u)du}\Big|Y_t = y\right]. \tag{4.6.3}$$

Using exactly the same methods as in Theorem 4.3.1 we can prove the following result.

Theorem 4.6.2. *Let H_F be given by (4.6.3). Then the value function (4.6.2) is given by*

$$u_F(t, x, y) = U_F(t, x)H_F(t, y)^{\frac{1}{1-\rho^2}}. \tag{4.6.4}$$

Under this forward exponential performance measure the function H_F does not contain a term involving the mean-variance trade-off process K. Furthermore, in the absence of income, $f \equiv 0$, we have $H_F = 1$ and $u_F(t,x,y) = U_F(t,x)$. This demonstrates that U_F is indeed a forward performance measure for an investor in this market.

4.6.2 The dual solution

The dual to the primal value function (4.6.2) is $v_F(t,\eta,y)$, given by

$$v_F(t,\eta,y) = \inf_{\mathbb{Q}\in\mathcal{M}_f} \mathbb{E}\left[V_F(T,\eta\xi_{t,T}) + \int_t^T \eta\xi_{t,u} f(Y_u)du \middle| \mathcal{F}_t \right], \tag{4.6.5}$$

where V_F in the convex conjugate of U_F,

$$V_F(t,\eta) := \sup_{x\in \mathrm{dom(U)}} \left[U_F(t,x) - \eta x\right], \quad \eta > 0.$$

A straightforward calculation gives

$$\begin{aligned} V_F(t,\eta) &= e^{rt}\left[V(\eta) + \frac{\eta}{\gamma}\left(rt - \frac{1}{2}K_t\right)\right] \\ &= e^{\frac{1}{2}K_t} V\left(\eta e^{rt-\frac{1}{2}K_t}\right), \end{aligned}$$

where V is given by (4.4.2). Using identical arguments in the proof of Theorem 4.4.1 we have the following representation:

Theorem 4.6.3. *The dual value function (4.6.5) is given by*

$$v_F(t,\eta,y) = H_F(t,y) V_F\left(t, \frac{\eta}{H_F(t,y)^{\frac{1}{\rho^2}}}\right), \tag{4.6.6}$$

with $H_F(t,y)$ given by (4.6.3). The dual minimiser, $\mathbb{Q}_F^*$, achieving the infimum in (4.6.5) has density

$$\frac{d\mathbb{Q}_F^*}{d\mathbb{P}} = \mathcal{E}(-\lambda\cdot W - \psi_F^*\cdot W^\perp)_T,$$

with

$$\psi_F^*(t,Y_t;T) = -\frac{b(Y_t)}{\sqrt{1-\rho^2}}\frac{\partial}{\partial y}\log H_F(t,Y_t), \quad t\in[0,T]. \tag{4.6.7}$$

In the absence of income, $f \equiv 0$, it follows that $H_F(t, Y_t) = 1$ and thus the dual optimal control, $\psi_F^*(t, Y_t; T)$, is zero at all times. The careful specification of the forward performance measure results in the minimal martingale being the optimal dual measure in the absence of the stochastic income. This is in contrast to the classical exponential utility case, when the optimal dual measure is the minimal entropy measure. A similar pattern was observed in Leung et al. [35] and this seems to be a general result for certain forms of forward performance preferences as observed by Berrier et al. [3].

4.6.3 The forward utility pricing

The disparity in the optimal measures for classical and forward utility functions in the absence of income extends to the utility valuation of the stochastic income. The following forward utility indifference prices and marginal-utility based price follow from analogous arguments used in Section 4.5.

Corollary 4.6.4. *For a single unit of claim, the exponential forward indifference price p_F is given by*

$$p_F = -\frac{1}{\gamma e^{-rt}(1-\rho^2)} \log\left(\mathbb{E}^{\widetilde{\mathbb{Q}}^{\mathbb{M}}}\left[e^{-\vec{\rho}^2 \gamma F_T}\right]\right).$$

and the marginal forward utility-based price is given by

$$\hat{p}_F = \lim_{\gamma \to 0} p_F = e^{rT} \mathbb{E}^{\widetilde{\mathbb{Q}}^{\mathbb{M}}}\left[\int_0^T e^{-rs} f(Y_s) ds\right].$$

With exponential forward preferences the utility indifference price and the marginal utility-based prices are calculated as expectations under the minimal martingale measure $\widetilde{\mathbb{Q}}^M$ and not the minimal entropy measure $\widetilde{\mathbb{Q}}^E$. The adjustment from a classical to a forward utility is exactly the adjustment needed to restore $\mathbb{Q}^M$ as the dual minimiser in the stochastic volatility model without income; horizon dependence in the marginal utility-based pricing kernel is thus removed.

4.7 Concluding remarks

We have investigated an optimal investment problem in a stochastic volatility model in which the agent receives an additional unhedgeable stochastic income stream. The

incompleteness of the underlying market results in the dual optimiser being horizon dependent for the market without the income. This is in contrast to when the underlying market is complete and the dual optimiser is the minimal martingale measure.

We have obtained closed form solutions for the primal and dual problems with stochastic income, and computed the indifference price and the marginal utility-based price. These are given as expectations under the horizon-dependent dual optimal measure $\mathbb{Q}^E(T)$. The horizon dependence of $\mathbb{Q}^E(T)$ can potentially lead to inconsistencies when attempting to value claims with multiple payment dates, of which the stochastic income is an example.

We attempt to deal with the possible inconsistencies by specifying a forward utility function for which the optimal dual measure, in the zero income case, is the minimal martingale measure. Under this preference, the utility-based prices are taken as expectations under the minimal martingale measure. Consequently the orthogonal risk associated with the Brownian motion $W^{\perp}$ is unpriced. One might view this as a slightly artificial construction to resolve the issue of horizon dependence.

One suggested alternative method to remove the horizon dependence of the pricing measure might be to embed the problem within a consumption framework, possibly over an infinite horizon. Motivated by the construction of the finite horizon random endowment problem we suggest modifying the specification of utility maximisation with stochastic income, that we used above, as follows. Define the value function $u^{(\epsilon)}(x, y)$ by

$$u^{(\epsilon)}(x, y) = \sup_{(\pi,c)\in\mathcal{A}_\infty(x)} \mathbb{E}\left[\int_0^\infty e^{-\alpha t} U(c_t + \epsilon Y_t) dt\right], \tag{4.7.1}$$

where the wealth dynamics of the investor follow

$$dX_t^{\pi,c} = rX_t^{\pi,c}dt + \pi_t\sigma\lambda dt + \pi_t\sigma dW_t - c_t dt.$$

We suggest modifying the set of admissible consumption strategies to include strategies c which satisfy $c_t \geq -\epsilon Y_t$, for all $t \geq 0$. Consumption in the interval $[-\epsilon Y_t, 0]$ corresponds to consuming part of the income and reinvesting the remainder.

In private discussions between David Hobson and Michael Monoyios, they conjecture the following characterisation for the marginal price

$$\hat{p}(x) = \frac{\mathbb{E}\left[\int_0^\infty e^{-\alpha s} U_x(c_s^{0,x}) Y_s ds\right]}{u_x^{(0)}(x)}, \quad \forall x > 0,$$

where $c_s^{0,x}$ is the optimal consumption strategy and $u^{(0)}(x)$ is the optimal value function, for an investor with zero income. Davis' characterisation of the marginal price was proved assuming suitable regularity of the value function, but this may be improved using the notion of generalised derivatives presented by Karatzas and Kou [27].

Since explicit results for infinite horizon consumption in incomplete markets are not available, it is an open question as to whether this method would give meaningful results.

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