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Monotone normality in products

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Abstract

Monotone normality in finite and infinite topological products is investigated. As shown in (Heath et al., 1973), the countable (Tychonoff) power of a space is monotonically normal if and only if the space is stratifiable. It is shown that if the square of a space is monotonically normal, then all finite powers are monotonically normal and hereditarily paracompact. For certain special cases, it is observed that a space has all finite powers monotonically normal if and only if it linearly stratifiable. Nonetheless, a monotonically normal topological group is constructed, all of whose finite powers are monotonically normal, but which is not linearly stratifiable. The group is constructed using special filters and nonstandard topologies on infinite products. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The product theory of separation and covering properties has been a central theme in general topology ever since the discovery of the ‘pathological’ Sorgenfrey line (a hereditarily Lindelöf space whose square is not normal, see [15]) and Michael line (a hereditarily paracompact space whose product with the irrationals is not normal, see [12]). The aim of this note is to investigate the product theory of monotonically normal spaces.

Recall that a space is said to be monotonically normal [10] if to every pair of disjoint closed sets, A and B say, there is assigned an open set $H(A, B)$ so that

- (1) $A \subseteq H(A, B)$ and $\overline{H(A, B)} \cap B = \emptyset$,
- (2) $H(A, B) \subseteq H(A', B')$, whenever $A \subseteq A'$ and $B' \subseteq B$.

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Every metrizable space, and every linearly ordered space is monotonically normal. Arbitrary subspaces, and closed images, of monotonically normal spaces are again monotonically normal. Thus, many of the basic spaces of topology are monotonically normal. In particular, both the Sorgenfrey and Michael lines are monotonically normal.

The class of monotonically normal spaces has been extensively studied because it is fairly broad, but nonetheless has a rich structure in which deep theorems prevail over counter-examples. (See Gruenhage's survey articles [7,8].) This is very much in contrast to the class of all normal spaces where 'pathology' seems to be the rule rather than the exception. From this perspective, the 'pathological' behaviour in products of such monotonically spaces as the Sorgenfrey and Michael lines appears anomalous. We will see, however, that the behaviour of monotone normality in products is a mixture of theorems and counter-examples. Indeed we will see that two natural conjectures, arrived at by analogy, are both false, leading in one case to a theorem, and in the other to a counter-example.

The first (false) conjecture concerns finite products. Going beyond the Sorgenfrey line (a Lindelöf space with nonnormal square), there are related examples, X_n for each n in \mathbb{N} , so that X_n^n is Lindelöf but X_n^{n+1} is not normal [14]. This makes it reasonable to ask the following.

Question 1. Are there spaces X_n for each n in \mathbb{N} , so that X_n^n is monotonically normal but X_n^{n+1} is not (monotonically) normal?

The answer is 'no', as is made clear by the following theorem.

Theorem 1. *If a space has a monotonically normal square, then all its finite powers are monotonically normal and hereditarily paracompact.*

Results in which from " X^2 has property \mathcal{P} " one can deduce " X^n has \mathcal{P} , for every n in \mathbb{N} ", appear to be scarce in topology. In fact the most plausible scenario where this might happen is when from " X^2 has \mathcal{P} " one can prove that X has a stronger property \mathcal{Q} , which is known to be finitely productive. In the case where our property \mathcal{P} is monotone normality, there is a natural candidate for the stronger property \mathcal{Q} . This stronger property is (linear) stratifiability.

Let X be a topological space, and denote its open sets by \mathcal{T} , and its closed sets by \mathcal{C} . Then X is said to be κ -semistratifiable [16], for some cardinal κ , if there is an operator $G: \kappa \times \mathcal{C} \rightarrow \mathcal{T}$ such that

- (1) $G(\alpha, H) \subseteq G(\alpha, K)$, if H and K are closed, and $H \subseteq K$,
- (2) $G(\beta, H) \subseteq G(\alpha, K)$, if $\alpha < \beta$,
- (3) $\bigcap_{\alpha \in \kappa} G(\alpha, H) = H$.

And X is κ -stratifiable if in addition we have

- (4) $\bigcap_{\alpha \in \kappa} \overline{G(\alpha, H)} = H$.

A space which is \aleph_0 -stratifiable is usually just called a stratifiable space, while a space which is κ -stratifiable for some κ is said to be linearly stratifiable. Every linearly

stratifiable space is monotonically normal and hereditarily paracompact. This is most easily seen in the case of stratifiable spaces, for then conditions (3) and (4) in the definition are equivalent to saying that the space is perfectly normal, while conditions (1) and (2) are ‘monotonicity’ restrictions. In fact a κ -semitratifiable space is κ -stratifiable if and only if it is monotonically normal. It can be shown that a countable product of stratifiable spaces is again stratifiable, and a finite product of κ -stratifiable spaces (κ fixed) is κ -stratifiable. (See Vaughan [16], for proofs.)

As is well known, if a space X has a nontrivial convergent sequence and if X^2 is hereditarily normal, then X is perfectly normal. Similarly, it is known that, if X^2 is monotonically normal and X has a convergent sequence, then X is stratifiable. The analogous result for κ -stratifiable spaces is given by Lemma 3 below. Thus, if a space, X say, is such that X^ω is monotonically normal, then either X is trivial, or X^ω contains a copy of 2^ω , and since $X^\omega = X \times X^\omega$, it is clear that X is stratifiable (this is proved in [10]). All this makes it very natural to pose our second question.

Question 2. If a space has all finite powers monotonically normal, then is it true that the space is linearly stratifiable?

As with our first question, the answer is in the negative.

Example 2. There is a topological group, each of whose finite powers is monotonically normal, but which is not linearly stratifiable.

The fact that our example is a topological group is significant. Unlike the situation in topological spaces, the situation regarding preservation of covering and separation properties in products and powers of groups remains unclear. On the one hand, there is the famous Comfort–Ross theorem [4] which states that an arbitrary product of pseudo-compact topological groups is pseudo-compact, on the other there is (for example) a *consistent* example showing that the square of a Lindelöf topological group need not be normal (see [11]). Heath raised in [9] the question as to whether the square of a monotonically normal topological group is always monotonically normal. This remains unanswered. However, every separable monotonically normal group is stratifiable, [6], and so certainly has monotonically normal square. A natural approach to a positive solution, would have been to show that every monotonically normal topological group is linearly stratifiable. Example 2 eliminates any such possibility.

In this paper, only T_3 spaces will be considered, so that Lindelöf spaces are paracompact, and paracompact spaces are normal. Rather than the definition of monotone normality given above, it will be convenient to use an alternative characterization. A space is monotonically normal if and only if there is an operator $V(\cdot, \cdot)$ assigning to each point x and open neighbourhood U , an open set $V(x, U)$ containing x so that

(MN) if $V(x, U)$ meets $V(x', U')$, then either $x \in U'$ or $x' \in U$.

So as to avoid uninteresting special cases, we say that a space is nontrivial provided it contains at least two elements.

2. Finite powers

If p is a free filter on set S , then $X(p)$ is defined to be the space with underlying set $S \cup \{p\}$, and the topology in which points of S are isolated and neighbourhoods of p are of the form $F \cup \{p\}$, for F in p . Given an infinite cardinal κ , we write D_κ^* for the space $X(p)$, where S is κ , and p consists of all co- $< \kappa$ subsets of S . Observe that D_ω^* is just a convergent sequence.

Lemma 3. *Let X be a space. If $X \times D_\kappa^*$ is monotonically normal, then X is κ -stratifiable.*

Proof. It is notationally convenient to identify the point p with κ , so that D_κ^* has underlying set $\kappa + 1$, all $\alpha < \kappa$ are isolated, and the basic neighbourhoods of κ are of the form $(\alpha, \kappa]$, for $\alpha < \kappa$. Now suppose X is a space such that $X \times D_\kappa^*$ is monotonically normal. Then X is monotonically normal, so it is sufficient to prove that X is κ -semitractable.

For each $\alpha < \kappa$, and closed H in X , define

$$G(\alpha, H) = \pi_X \left(\bigcup \{ V(\langle x, \beta \rangle, X \times (\alpha, \kappa)) : x \in H \text{ and } \alpha < \beta < \kappa \} \right).$$

Here π_X is the projection from $X \times D_\kappa^*$ onto X , and so the $G(\alpha, H)$'s are open.

Observe that,

- (1) $G(\alpha, H) \subseteq G(\alpha, K)$, if $H \subseteq K$, and
- (2) $G(\beta, H) \subseteq G(\alpha, H)$, if $\alpha \leq \beta$.

Suppose, for a contradiction, $p \in \bigcap_{\alpha < \kappa} G(\alpha, H)$ but $p \notin H$. For each $\alpha < \kappa$, there is a β_α where $\alpha < \beta_\alpha < \kappa$, and an x_α in H , such that $\langle p, \beta_\alpha \rangle \in V(\langle x_\alpha, \beta_\alpha \rangle, X \times (\alpha, \kappa))$. Take any basic neighbourhood $U \times (\gamma, \kappa]$ of $\langle p, \kappa \rangle$. Then

$$\langle p, \beta_\gamma \rangle \in (U \times (\gamma, \kappa]) \cap V(\langle x_\gamma, \beta_\gamma \rangle, X \times (\gamma, \kappa))$$

and hence,

$$\langle p, \kappa \rangle \in \overline{\bigcup_{\alpha < \kappa} V(\langle x_\alpha, \beta_\alpha \rangle, X \times (\alpha, \kappa))}.$$

So, for some $\delta < \kappa$,

$$V(\langle p, \kappa \rangle, (X \setminus H) \times (\kappa + 1)) \cap V(\langle x_\delta, \beta_\delta \rangle, X \times (\delta, \kappa)) \neq \emptyset.$$

Now we see that, either $\langle p, \kappa \rangle \in X \times (\delta, \kappa)$, or $\langle x_\delta, \beta_\delta \rangle \in (X \setminus H) \times (\kappa + 1)$. But both options are impossible. \square

Thus, for classes of spaces which always contain a homeomorphic copy of some D_κ^* , Question 2 has a positive answer. For example, this is the case for any protometrisable space (see [13] for the definition) and any subspace of a linearly ordered space. (The author has been informed by the referee that M.J. Harris independently proved Lemma 3 in her doctoral thesis *Linearly Stratifiable Spaces*, University of Pittsburg, 1991). Another useful consequence of Lemma 3 concerns paracompactness.

Corollary 4. *If $X \times Y$ is monotonically normal, then either X and Y are hereditarily paracompact, or one of X and Y is linearly stratifiable. Hence, if X^{n+1} is monotonically normal, then X^n is hereditarily paracompact.*

Proof. Suppose $X \times Y$ is monotonically normal. If both X and Y are hereditarily paracompact then we are done. So suppose (relabelling if necessary) that Y is not hereditarily paracompact. Then by a deep theorem due to Balogh and Rudin [2], Y contains a subspace, S say, homeomorphic to a stationary subset of an uncountable regular cardinal. The space S must contain a subspace homeomorphic to some D_κ^* . Therefore, we conclude from Lemma 3 that X is linearly stratifiable.

For the second part, just note that linearly stratifiable spaces are hereditarily paracompact, so if $X^n \times X$ is monotonically normal, then in either of the above cases, X^n , is indeed hereditarily paracompact. \square

Now we are in a position to prove Theorem 1, and so answer Question 1. Theorem 1*, below, in fact proves rather more.

Theorem 1*. *Suppose the spaces X_1, X_2, \dots, X_n are such that $X_i \times X_j$ is monotonically normal if $i \neq j$. Then $\prod_{i=1}^n X_i$ is monotonically normal.*

In particular, if a space has monotonically normal square then all its finite powers are monotonically normal and hereditarily paracompact.

Proof. We proceed by induction on n . If $n \leq 2$, then there is nothing to prove. So, inductively, suppose $n > 2$ and that the claim holds for all $m < n$. Set $Y_1 = \prod_{i=1}^{n-2} X_i$, $Y_2 = X_{n-1}$ and $Y_3 = X_n$. By our inductive hypothesis, if $i < j$, then $Y_i \times Y_j$ has a monotone normality operator $V_{(i,j)}(\cdot, \cdot)$. We may assume $V_{(i,j)}(\langle y_i, y_j \rangle, U_i \times U_j)$ is a basic neighbourhood of $\langle y_i, y_j \rangle$ in $Y_i \times Y_j$, and may write

$$V_{(i,j)}(\langle y_i, y_j \rangle, U_i \times U_j) = V_{(i,j)}^A(\langle y_i, y_j \rangle, U_i \times U_j) \times V_{(i,j)}^B(\langle y_i, y_j \rangle, U_i \times U_j).$$

Define an operator $V(\cdot, \cdot)$ on pairs of points and basic neighborhoods of $Y_1 \times Y_2 \times Y_3$ by

$$\begin{aligned} V(\langle y_1, y_2, y_3 \rangle, U_1 \times U_2 \times U_3) \\ = \left[V_{(1,2)}^A(\langle y_1, y_2 \rangle, U_1 \times U_2) \times V_{(1,2)}^B(\langle y_1, y_2 \rangle, U_1 \times U_2) \times Y_3 \right] \\ \cap \left[Y_1 \times V_{(2,3)}^A(\langle y_2, y_3 \rangle, U_2 \times U_3) \times V_{(2,3)}^B(\langle y_2, y_3 \rangle, U_2 \times U_3) \right] \\ \cap \left[V_{(1,3)}^A(\langle y_1, y_3 \rangle, U_1 \times U_3) \times Y_2 \times V_{(1,3)}^B(\langle y_1, y_3 \rangle, U_1 \times U_3) \right]. \end{aligned}$$

Clearly $V(\langle y_1, y_2, y_3 \rangle, U_1 \times U_2 \times U_3)$ is an open neighbourhood of $\langle y_1, y_2, y_3 \rangle$. Suppose then that,

$$V(\langle y_1, y_2, y_3 \rangle, U_1 \times U_2 \times U_3) \cap (\langle \hat{y}_1, \hat{y}_2, \hat{y}_3 \rangle, \hat{U}_1 \times \hat{U}_2 \times \hat{U}_3) \neq \emptyset.$$

Considering pairs of co-ordinates, and recalling the definitions of $V(\cdot, \cdot)$, $V_{(i,j)}^A(\cdot, \cdot)$ and $V_{(i,j)}^B(\cdot, \cdot)$, we see that taking (i, j) equal to $(1, 2)$, $(2, 3)$ and $(1, 3)$, the following statements hold:

Assertion $_{(i,j)}$: $V_{(i,j)}(\langle y_i, y_j \rangle, U_i \times U_j) \cap V_{(i,j)}(\langle \hat{y}_i, \hat{y}_j \rangle, \hat{U}_i \times \hat{U}_j) \neq \emptyset$.

Hence, either $A_{(i,j)}$: $\langle y_i, y_j \rangle \in \hat{U}_i \times \hat{U}_j$.

or $B_{(i,j)}$: $\langle \hat{y}_i, \hat{y}_j \rangle \in U_i \times U_j$.

Now of the six statements $\{A_{(i,j)}, B_{(i,j)}: (i,j) = (1,2), (2,3) \text{ or } (1,3)\}$ at least three are thus; so from by the pigeon-hole principle, and relabelling if necessary, we suppose $A_{(1,2)}$ and $A_{(2,3)}$ are true. Thus $\langle y_1, y_2 \rangle \in \hat{U}_1 \times \hat{U}_2$ and $\langle y_2, y_3 \rangle \in \hat{U}_2 \times \hat{U}_3$, and therefore $\langle y_1, y_2, y_3 \rangle \in \hat{U}_1 \times \hat{U}_2 \times \hat{U}_3$.

In other words, $V(\cdot, \cdot)$ is a monotone normality operator for

$$\prod_{i=1}^n X_i = Y_1 \times Y_2 \times Y_3. \quad \square$$

3. p -products

Our example is a subspace of an infinite product. Let p be a free filter on a set S , and let $(X_s)_{s \in S}$ be an indexed family of spaces. Define the p -product of the X_s 's, denoted $p - \prod_{s \in S} X_s$, to be the topological space with underlying set $\prod_{s \in S} X_s$ and basic open sets

$$\bigcap_{s \notin F} \pi_s^{-1} U_s, \quad \text{where } F \in p, \text{ and } U_s \text{ is open in } X_s, \text{ for each } s \notin F.$$

Observe that if p is the co-finite filter on S , then $p - \prod_{s \in S} X_s$ is the normal Tychonoff product. While if $p = \mathbb{P}(S)$, then $p - \prod_{s \in S} X_s$ is the box product. These p -products are rarely monotonically normal, and we focus on some rather small subspaces. Fix $x \in \prod_{s \in S} X_s$, and write $p - \sigma_{s \in S}^x X_s$ for the subspace of $p - \prod_{s \in S} X_s$ such that points of $p - \sigma_{s \in S}^x X_s$ have all but finitely many co-ordinates equal to x . We call $p - \sigma_{s \in S}^x X_s$ the p -sigma product of the X_s 's. For any z in a p -sigma product, $p - \sigma_{s \in S}^x X_s$ say, we write $I(z)$ for the set $\{s \in S: z(s) \neq x(s)\}$. One way of motivating study of p -sigma products is to recall that Borges [3] has shown that a p -sigma product of stratifiable spaces is stratifiable when $p = \mathbb{P}(S)$ (so $p - \sigma_{s \in S}^x X_s$ is a σ -subspace of a box product).

In order to determine when a p -sigma product is monotonically normal, we make the following definitions. Fix p , a free filter on a set S . The filter p is said to be κ -linear, for some cardinal κ , if there is a subcollection \mathcal{C} of p which has order type κ with respect to reverse inclusion, and $\bigcap \mathcal{C} = \emptyset$. The filter p , respectively is linear if it is κ -linear for some κ . Further, p is tangle free provided there is a function $G: S \rightarrow p$ such that $t \in G(s)$ implies $s \notin G(t)$, for all $s, t \in S$. More generally, a pair (p, q) of free filters, on sets S and T , respectively, are said to be pairwise tangle free whenever there are functions $G: S \rightarrow q$ and $H: T \rightarrow p$ so that $t \in G(s) \Leftrightarrow s \notin H(t)$, for all $s \in S$ and $t \in T$. That linearity and tangle freeness of filters capture the behaviour of interest to us is made clear by the next three results.

Lemma 5. Let $(X_s)_{s \in S}$ be a family of nontrivial spaces, let x be in $\prod_{s \in S} X_s$, and let p be a free filter on S . Then $X(p)$ embeds as a closed subspace in $p - \sigma_{s \in S}^x X_s$.

Proof. For each $s \in S$, set $x_s^1 = x(s)$, and pick a distinct $x_s^2 \in X_s$. Define $\phi: X(p) \rightarrow (p - \sigma_{s \in S}^x X_s)$ by

$$\phi(s)(t) = \begin{cases} x(t), & \text{if } s = p, \\ x_s^2, & \text{if } s = t, \\ x_s^1, & \text{if } s \neq p, s \neq t. \end{cases}$$

It is not difficult to check that ϕ is an embedding, and that the image of ϕ is closed in $p - \sigma_{s \in S}^x X_s$. \square

In order to state the next result in full generality, we recall that a space is κ -metrizable if it has a compatible uniformity which has a base of size no more than κ linearly ordered by set inclusion. Each κ -metrizable space is κ -stratifiable, and a space is \aleph_0 -metrizable if and only if it is metrizable.

Theorem 6. Let p be a free filter on S , let $(X_s)_{s \in S}$ be a family of nontrivial spaces, and let x be a point of $\prod_{s \in S} X_s$.

Then,

- (1) p is κ -linear if and only if $X(p)$ is κ -stratifiable;
- (2) if $p - \sigma_{s \in S}^x X_s$ is κ -stratifiable, then $X(p)$ is κ -stratifiable; and
- (3) if all the X_s 's are κ -metrizable, and p is κ -linear, then $p - \sigma_{s \in S}^x X_s$ is κ -stratifiable.

Proof. Let us start with the first claim. So let $\mathcal{C} = \{C_\alpha: \alpha < \kappa\}$ be a subset of p witnessing κ -linearity of p .

Define, for closed H in $X(p)$, and $\alpha < \kappa$, $G(\alpha, H) = H \cup C_\alpha$ if $p \in H$ and $G(\alpha, H) = H$ if $p \notin H$. It is straightforward to check that

$$\begin{aligned} G(\alpha, H) \text{ is closed and open, } G(\alpha, H) &\subseteq G(\alpha, K), \quad \text{if } H \subseteq K, \\ G(\beta, H) &\subseteq G(\alpha, H), \quad \text{if } \alpha \leq \beta, \text{ and } \bigcap_{\alpha < \kappa} G(\alpha, H) = H. \end{aligned}$$

Thus $X(p)$ is κ -stratifiable. For the converse, suppose $X(p)$ is κ -stratifiable, with κ -stratification operator $G(\cdot, \cdot)$. Set $C_\alpha = G(\alpha, \{p\}) \cap S$, and $\mathcal{C} = \{C_\alpha: \alpha < \kappa\}$. Then \mathcal{C} witnesses κ -linearity of p .

The second claim follows immediately from Lemma 5, and the fact that κ -stratifiability is hereditary.

So it remains to establish the third claim. Suppose, then, that all the X_s 's are κ -metrizable. The proofs are given separately for the cases when $\kappa = \omega$ or $\kappa > \omega$.

Proof for $\kappa = \omega$. Let d_s be a metric for X_s ($s \in S$), and let $\{C_n: n < \omega\} \subseteq p$ be such that $C_{n+1} \subseteq C_n$, and $\bigcap_{n \in \omega} C_n = \emptyset$. Define, for each $z \in p - \sigma_{s \in S}^x X_s$, $i(z) = \min\{n \in \omega: I(z) \cap C_n = \emptyset\}$. Observe that a point z in $p - \sigma_{s \in S}^x X_s$ has basic neighborhoods of the form

$$B(z, F, (\varepsilon_s)_{s \notin F}) = \{w \in p - \sigma_{s \in S}^x X_s: d_s(z(s), w(s)) < \varepsilon_s, \forall s \notin F\}.$$

where $F \in p$ and $\varepsilon_s > 0$ ($s \notin F$).

Define

$$g_r(z) = B\left(z, C(r, z), (\varepsilon(r, z)_s)_{s \notin C(r, z)}\right),$$

where

$$C(r, z) = C_{\max(i(z), r)} \quad \text{and} \quad \varepsilon(r, z)_s = 2^{-r} \min(1, d_s(1, d_s(z(s), x(s)))).$$

Set $G(r, H) = \bigcup_{z \in H} g_r(z)$. Clearly, $G(r, H)$ is open, $H \subseteq G(r, H)$, $G(r, H) \subseteq G(r, K)$ if $H \subseteq K$, and $G(r, H) \subseteq G(s, H)$ if $r \geq s$. To establish stratifiability it is sufficient to show that $\bigcap_{n \in \omega} \overline{G(n, H)} = H$. This follows from Claim 1.

Claim 1. *If $w \in (p - \sigma_{s \in S}^x X_s) \setminus H$, then there is an open V containing w , and an $r \in \omega$, such that $V \cap g_r(z) = \emptyset$, for all $z \in H$.*

Pick a basic $B(w, F, (\varepsilon_s)_{s \notin F})$ contained in $p - \sigma_{s \in S}^x X_s \setminus H$. Now we define $V = B(w, F, ((1/2)\delta_s)_{s \notin F})$ where $\delta_s = 2^{-m}$ if $s \notin I(w)$, $\delta_s = \varepsilon_s$ if $s \in I(w)$, and $m \in \omega$ is chosen so that $2^{-m} < \varepsilon_s$ for all $s \in I(w)$. Set $r = \max(m, i(w)) + 1$.

Fix $z \in H$ and $v \in g_r(z)$. Since $z \notin B(w, F, (\varepsilon_s)_{s \notin F})$, for some $s \in S$, $d_s(w(s), z(s)) \geq \varepsilon_s$. Two cases now arise.

Case 1. $s \notin C_{i(w)}$.

Since $s \notin C_{i(w)}$ and $C_{i(w)} \supseteq C_{\max(i(w), r)}$,

$$d_s(z(s), v(s)) < 2^{-r} \min(1, d_s(z(s), x(s))) \leq (1/2)\varepsilon_s \quad (\text{because } r > m).$$

So we must have $d_s(v(s), w(s)) \geq (1/2)\varepsilon_s \geq (1/2)\delta_s$ (because $s \notin C_{i(w)}$). Hence $v \notin V$.

Case 2. $s \in C_{i(w)}$.

Since $d_s(w(s), x(s)) = 0$, $s \notin C_{\max(i(z), r)}$, and hence

$$d_s(z(s), v(s)) < 2^{-r} \min(1, d_s(z(s), x(s))) \leq (1/2)\varepsilon_s.$$

So we must have $d_s(v(s), x(s)) \geq (1/2)\varepsilon_s = (1/2)\delta_s$. Hence $v \notin V$.

Proof for $\kappa > \omega$. Let $\{C_\alpha: \alpha < \kappa\} \subseteq p$ be such that $C_\alpha \subseteq C_\beta$ if $\alpha \geq \beta$, and $\bigcap_{\alpha < \kappa} C_\alpha = \emptyset$. For z in $p - \sigma_{s \in S}^x X_s$ set $i(z) = \min\{\alpha \in \kappa: I(z) \cap C_\alpha = \emptyset\}$. Nyikos [13] has shown that if X is κ -metrizable, and $\kappa > \omega$, then X has a symmetric function $d: X^2 \rightarrow \kappa + 1$ such that $d(x, x) = \kappa$, $d(x, z) \geq \min(d(x, y), d(y, z))$, and for each $x \in X$ and $\alpha < \kappa$, $B(x, \alpha) = \{x' \in X: d(x, x') \geq \alpha\}$ is basic neighbourhood of x .

Pick such a function d_s for each X_s ($s \in S$). Observe that for each z in $p - \sigma_{s \in S}^x X_s$, a basic neighbourhood is of the form $B(z, F, (\alpha_s)_{s \notin F})$ which equals $\{w: d_s(z(s), w(s)) \geq \alpha_s, \forall s \notin F\}$.

Define

$$g_\alpha(z) = B\left(z, C(\alpha, z), (\varepsilon(\alpha, z)_s)_{s \notin C(\alpha, z)}\right),$$

where

$$C(\alpha, z) = C_{\max(\alpha, i(z))} \quad \text{and} \quad \varepsilon(\alpha, z)_s = d_s(z(s), x(s)).$$

Set $G(\alpha, H) = \bigcup_{z \in H} g_\alpha(z)$. Then, as in the case when $\kappa = \omega$, it is sufficient to establish the following Claim 2.

Claim 2. *If $w \in (p - \sigma_{s \in S}^x X_s \setminus H)$, then there is an open V containing w , and an $\alpha < \kappa$, such that $V \cap g_\alpha(z) = \emptyset$, for all $z \in H$.*

Pick a basic $B(w, F, (\beta_s)_{s \notin F})$ contained in $p - \sigma_{s \in S}^x X_s \setminus H$. Define V to be $B(w, F, (\gamma_s + 1)_{s \notin F})$ where $\gamma_s = (\max_{s \in I(w)} \beta_s) + 1$, and if $s \notin I(w)$, $\gamma_s = \beta_s$ if $s \in I(w)$. Set $\alpha = \max(\{\beta_s : s \in I(w)\} \cup \{i(w)\}) + 1$.

Now the proof follows the same course (with some simplifications) as in the case when $\kappa = \omega$. Consequently it is omitted. \square

Theorem 7. *Let p be a free filter on S , let $(X_s)_{s \in S}$ be a family of nontrivial spaces, and let x be a point of $\prod_{s \in S} X_s$.*

Then,

(1) *the following are equivalent*

- (i) *p is tangle free,*
- (ii) *$X(p)^2 \setminus \Delta$ is normal, and*
- (iii) *$X(p)^n$ is monotonically normal for all $n \in \mathbb{N}$;*

(2) *if $p - \sigma_{s \in S}^x X_s$ is monotonically normal, then p is tangle free; and*

(3) *if all the X_s 's are discrete, and p is tangle free, then all finite powers of $p - \sigma_{s \in S}^x X_s$ are monotonically normal.*

Proof. We start with part (1). Let us suppose $G : S \rightarrow p$ witnesses tangle freeness of p . Define $V(\cdot, \cdot)$ on points and basic neighborhoods of $X(p)^2$ by

$$V(\langle s_1, s_2 \rangle, U_1 \times U_2) = \begin{cases} \{\langle s_1, s_2 \rangle\}, & \text{if } s_1, s_2 \in S, \\ \{s_1\} \times ((G(s_1) \cup \{p\}) \cap U_2), & \text{if } s_1 \in S, s_2 = p, \\ ((G(s_2) \cup \{p\}) \cap U_1) \times \{s_2\}, & \text{if } s_1 = p, s_2 \in S, \\ U_1 \times U_2, & \text{if } s_1 = p = s_2. \end{cases}$$

Clearly $V(\langle s_1, s_2 \rangle, U_1 \times U_2)$ is a basic open neighbourhood of $\langle s_1, s_2 \rangle$ contained in $U_1 \times U_2$.

So suppose $V(\langle s_1, s_2 \rangle, U_1 \times U_2) \cap V(\langle \hat{s}_1, \hat{s}_2 \rangle, \hat{U}_1 \times \hat{U}_2) \neq \emptyset$. Then the only case where the above condition does not automatically imply either $\langle s_1, s_2 \rangle \in \hat{U}_1 \times \hat{U}_2$ or $\langle \hat{s}_1, \hat{s}_2 \rangle \in U_1 \times U_2$, occurs when precisely one of s_1, s_2 is p and precisely one of \hat{s}_1, \hat{s}_2 is p . However, the definition of tangle freeness specifically excludes this possibility. Therefore, $V(\cdot, \cdot)$ is a monotone normality operator for $X(p)^2$. From Theorem 1, we deduce that all finite powers of $X(p)$ are also monotonically normal.

Evidently, if $X(p)^2$ is monotonically normal, then condition (ii) holds. So suppose $X(p)^2 \setminus \Delta$ is normal. Then the two disjoint closed sets $S \times \{p\}$ and $\{p\} \times S$ can be separated by disjoint open sets U_1 and U_2 .

As U_1, U_2 are open, for each s in S , we may pick F_s^1 and F_s^2 in p so that $(F_s^1 \cup \{p\}) \times \{s\} \subseteq U_1$ and $\{s\} \times (F_s^2 \cup \{p\}) \subseteq U_2$. Define $G : S \rightarrow p$ by $G(s) = F_s^1 \cap F_s^2$. Suppose,

if possible, that $t \in G(s)$ and $s \in G(t)$ for some $s, t \in S$. Then $(s, t) \in U_1 \cap U_2$, contradicting U_1 and U_2 disjoint. Therefore, p is tangle free.

Now for part (2). Since each X_s is nontrivial, and since monotone normality is hereditary, we may suppose that each X_s contains precisely two elements, x_s^1 and x_s^2 , and that $x = (x_s^1)_{s \in S}$. Let ϕ be the natural embedding of $X(p)$ into $p - \sigma_{s \in S}^x X_s$, as in Lemma 5. Let us observe that a basic neighbourhood of z in $p - \sigma_{s \in S}^x X_s$ is of the form

$$B(z, F) = \{w \in p - \sigma_{s \in S}^x X_s : w(s) = z(s), \forall s \notin F\}, \quad \text{for } F \in p.$$

Since S is a discrete subspace of $X(p)$, by monotone normality of $p - \sigma_{s \in S}^x X_s$, for each s in S , we may select $F_s \in p$ so that $\{B(\phi(s), F_s)\}_{s \in S}$ is a collection of pairwise disjoint open subsets of $p - \sigma_{s \in S}^x X_s$. Suppose s and s' were distinct elements of S such that $s \in F_{s'}$ and $s' \in F_s$. Then $B(\phi(s), F_s) \cap B(\phi(s'), F_{s'}) \neq \emptyset$. This contradiction implies that the map $s \mapsto F_s$ witnesses that p is tangle free.

It remains to show that if all the X_s 's are discrete, and p is tangle free, then all finite powers $p - \sigma_{s \in S}^x X_s$ are monotonically normal. In fact we may avoid the task of showing that the square of $p - \sigma_{s \in S}^x X_s$ is monotonically normal (which is sufficient by Theorem 1). For, suppose that, under our hypotheses, we can at least show that $p - \sigma_{s \in S}^x X_s$ is monotonically normal. Then $(p - \sigma_{s \in S}^x X_s)^2$ is homeomorphic to $(p \oplus p) - \sigma_{s \in S \oplus S}^x X_s$, where $S \oplus S$ is the disjoint union of two copies of S and $p \oplus p$ is the natural filter induced on $S \oplus S$. In the paragraph preceeding Proposition 13 in the next section, a formal definition of these objects is given, and from Proposition 13 it is clear that, since p is tangle free, so to is $p \oplus p$. Thus, $(p \oplus p) - \sigma_{s \in S \oplus S}^x X_s$, and the square of $p - \sigma_{s \in S}^x X_s$, are monotonically normal.

Let $G: S \rightarrow p$ witness tangle freeness of p . For each $z \in p - \sigma_{s \in S}^x X_s$ set

$$F(z) = \bigcap_{s \in I(z)} G(s).$$

The following fact is used repeatedly. By tangle freeness of p , s is not in $G(s)$ so, $I(z)$ and $F(z)$ are disjoint. Define an operator $V_1(\cdot, \cdot)$ on pairs of points and basic neighborhoods by $V_1(z, B(z, F)) = B(z, F \cap F(z))$. We show that V_1 is a monotone normality operator for $p - \sigma_{s \in S}^x X_s$.

Suppose $w \in V_1(z, B(z, F)) \cap V_1(z', B(z', F'))$, so that

$$\begin{aligned} w(s) &= z(s) & \forall s \notin F \cap F(z), \\ w(s) &= z'(s) & \forall s \notin F' \cap F(z'). \end{aligned}$$

Assume $z \notin B(z', F')$, so there is an $s_0 \notin F'$ with $z(s_0) \neq z'(s_0)$. Hence $w(s_0) = z'(s_0)$, $s_0 \in F \cap F(z)$, and thus $z(s_0) = x(s_0)$.

It remains to verify that $z' \in B(z, F)$, in other words, if $t \notin F$, then $z'(t) = z(t)$. To this end, take any $t \in S \setminus F$.

Case 1. $t \notin I(z)$.

As $t \notin I(z)$, $z(t) = x(t)$. And from $t \notin F$, we see that $z(t) = w(t)$. But $z'(t) = w(t)$ when $t \in I(z')$, and $z'(t) = x(t)$ if $t \notin I(z')$. In either case, $z'(t) = z(t)$.

Case 2. $t \in I(z)$.

Since $w(s_0) = z'(s_0)$ and $z(s_0) \neq z'(s_0)$, $z(s_0) \neq w(s_0)$. Hence $s_0 \in F \cap F(z) \subseteq G(t)$, and $t \notin G(s_0)$. As $z'(s_0) \neq x(s_0)$, $s_0 \in I(z')$, thus $t \in (S \setminus G(s_0))$, and $w(t) = z'(t)$. Also, $t \in I(z)$, so $w(t) = z(t)$. Therefore, $z'(t) = w(t) = z(t)$. \square

Theorem 7 has an application to more general spaces.

Corollary 8. *If a space X has monotonically normal square then every neighborhood filter of a point x in X , considered as a free filter on $X \setminus \{x\}$, is tangle free.*

Proof. Let X be a space whose square is monotonically normal, x a point of X , $S = X \setminus \{x\}$, and let p be the trace of the neighbourhood filter of x on S .

Then $X(p)$ can be identified, as a set, with X , and as a topological space has a finer topology than X .

Since X^2 is monotonically normal, $X^2 \setminus \Delta$ is normal, so the sets $S \times \{p\}$ and $\{p\} \times S$, which are closed in this space, can be separated by disjoint open sets U_1 and U_2 . Then U_1 and U_2 are disjoint open sets in $X(p)^2 \setminus \Delta$ separating $S \times \{p\}$ from $\{p\} \times S$. Therefore, by part (1) of Theorem 7, p is tangle free. \square

As an aside, we note that tangle free filters have been considered in other contexts. Indeed, in light of Theorem 7, Dowker's Problem [5] can be rephrased in purely topological terms as follows.

Dowker's Problem. Does there exist a free filter p on a set S such that

- (1) $X(p)^2$ is not monotonically normal, but
- (2) all subspaces of the form $(A \cup \{p\}) \times ((S \cup \{p\}) \setminus A)$ of $X(p)^2$ are monotonically normal?

Recently Balogh and Gruenhage [1] have shown that there is a model of ZFC containing such a filter.

4. A tangle free but nonlinear filter

Comparing the first statements of each of Theorems 6 and 7, we see that our search for a space all of whose finite powers are monotonically normal, but which is not linearly stratifiable, is at an end once we can construct a filter which is tangle free but not linear. Such a construction is presented in this section. The following, and final section, demonstrate how to create a topological group with the same properties. First a constraint on such a filter.

Proposition 9. *If p is a tangle free filter on a set S of cardinality $\leq \omega_1$, then p is linear.*

Proof. Let p be a tangle free filter on S , where $|S| \leq \omega_1$. Let $G: S \rightarrow p$ witness tangle freeness. Two cases arise.

Case 1. For every countable $C \subseteq S$, $S \setminus C \in p$.

Well order $S = \{s_\alpha: \alpha < \kappa\}$, for some cardinal $\kappa \leq \omega_1$. Set $C_\alpha = S \setminus \{s_\beta: \beta < \alpha\}$, and $C = \{C_\alpha: \alpha < \kappa\}$. Then each $C_\alpha \in p$, $C_\beta \subseteq C_\alpha$ if $\alpha \leq \beta$, and $\bigcap_{\alpha < \kappa} C_\alpha = \emptyset$. Thus p is linear.

Case 2. There is a countable $C \subseteq S$ so that $C \cap F \neq \emptyset$ for all $F \in p$.

Enumerate $C = \{s_n\}_{n \in \omega}$. Let $C_n = \bigcap_{i=0}^n G(s_i)$. Then each $C_n \in p$, and $C_{n+1} \subseteq C_n$. Further, if $s \in \bigcap_{n \in \omega} C_n = \bigcap_{n \in \omega} G(s_n)$, then $s_n \notin G(s)$, for all $n \in \omega$. Thus $C \cap G(s) = \emptyset$, contradicting $G(s) \in p$. Therefore, $\bigcap_{n \in \omega} C_n = \emptyset$, and p is again linear. \square

The construction of our example uses the idea of pairwise tangle freeness introduced in the preceding section. Every filter p has a *dual* filter, denoted p^* , such that p and p^* are pairwise tangle free. For, give a filter p on a set S , define the filter p^* on the set p to be the filter with base $B(F) = \{F' \in p: F' \subseteq F\}$, where F is in p . Note that $B(F_0) \cap B(F_1) = B(F_0 \cap F_1)$, so the $B(F)$'s do form a filter base.

Lemma 10. *Suppose p is a free filter on a set S , and $p \neq \mathbb{P}(S)$. Then p^* is free, and (p, p^*) are pairwise tangle free.*

Proof. To see that p^* is free, fix F in p , and pick \hat{F} a proper subset of F , then $F \notin B(\hat{F})$. For pairwise tangle freeness, define $G: S \rightarrow p^*$ and $H: p \rightarrow p$ by, $G(s) = B(S \setminus \{s\})$ and $H(F) = F$. An easy check shows that $F \in G(s) \Leftrightarrow s \notin F = H(F)$. \square

Mimicking the proof of part (1) of Theorem 7, one can easily prove the following, which gives us another interesting example concerning monotone normality in products.

Proposition 11. *Let p and q be free filters. Then the following are equivalent:*

- (1) (p, q) are pairwise tangle free,
- (2) $X(p) \times X(q)$ is monotonically normal.

Example 12. There is a compact nonmetrizable space K and a nondiscrete stratifiable space X , such that $K \times X$ is monotonically normal.

Proof. Let $S = \omega_1$, and let p be the co-finite filter on S . Set $K = X(p)$ and $X = X(p^*)$. Then K is compact but not metrizable, and (p, p^*) are pairwise tangle free, thus $K \times X$ is monotonically normal by Proposition 11. Moreover, K contains a convergent sequence, hence by Lemma 3, X is stratifiable. \square

Given filters, p and q on sets S and T , respectively, we define two new filters. Without loss of generality, assume that S and T are disjoint. On $S \cup T$ set $p \oplus q = \{F \cup F': F \in p \text{ and } F' \in q\}$. Clearly, $p \oplus q$ is a filter, and is free provided p and q are free. Let $p \otimes q$ be the filter on $S \times T$ with base consisting of all $F \times F'$, for $F \in p$ and $F' \in q$. Again, it is easy to see that $p \otimes q$ is free whenever both p and q are free.

Proposition 13. *If the free filters p , q , are tangle free and pairwise tangle free, then $p \oplus q$ is tangle free.*

Proof. Let $G: S \rightarrow p$, $H: T \rightarrow q$ witness tangle freeness of p and q . Further, let $J: S \rightarrow q$, $K: T \rightarrow p$ witness pairwise tangle freeness of p and q .

Define $L: S \cup T \rightarrow p \oplus q$ by

$$L(s) = G(s) \cup J(s), \quad \text{if } s \in S,$$

$$L(t) = H(t) \cup K(t), \quad \text{if } t \in T.$$

A straightforward, case by case analysis, will show that if $a \in L(b)$, then $b \notin L(a)$. Thus L shows that $p \oplus q$ is tangle free. \square

Proposition 14. *If p and q are free filters, with q tangle free, then $p \otimes q$ is tangle free.*

Proof. Suppose p is a free filter on S , q is a free filter on T , and $G: T \rightarrow q$ witnesses tangle freeness of q . Define $H: S \times T \rightarrow p \otimes q$ by $H(s, t) = S \times G(t)$. If $(s', t') \in H(s, t)$, then $t' \in G(t)$, hence $t \notin G(t')$, and $(s, t) \notin S \times G(t') = H(s', t')$. Thus we see that $p \otimes q$ is tangle free. \square

So to the counter-example.

Example 15. There is a free filter on a set of size ω_2 , which is tangle free but not linear.

Proof. Let S be ω_1 , and p the co-finite filter on S . Let $T = \omega_2$, and let q be the co- $< \omega_2$ filter on T . Set $r = p^* \oplus (p \otimes q)$. Thus r is a free filter on $p \cup (\omega_1 \times \omega_2)$, and this set is clearly of cardinality ω_2 .

Since q is ω_2 -linear, by Proposition 14, $p \otimes q$ is tangle free. Further, as was seen in Example 12, p^* is ω -linear. Hence by Proposition 13, r is tangle free.

That r is not linear follows from Claims 1 to 3 below.

Claim 1. *If $\mathcal{C} \subseteq p \otimes q$ and (\mathcal{C}, \supset) has order type ω , then $\bigcap \mathcal{C} \neq \emptyset$.*

This is immediate from the fact that neither p nor q is ω -linear.

Claim 2. *If $\mathcal{C} \subseteq p \otimes q$ and (\mathcal{C}, \supset) has order type ω_1 , then $\bigcap \mathcal{C} \neq \emptyset$.*

For each C in \mathcal{C} , pick $F_C \in p$, $F'_C \in q$, and $R_C \subseteq \omega_1 \times \omega_2$ such that $C = (F_C \times F'_C) \cup R_C$, and if $\alpha \notin F_C$, then $(\{\alpha\} \times \omega_2) \setminus R_C$ is co-final in $\{\alpha\} \times \omega_2$.

Observe that if $C, C' \in \mathcal{C}$ and $C \subseteq C'$, then $F_C \subseteq F_{C'}$. As (\mathcal{C}, \supset) has order type ω_1 , and infinite intersections of distinct elements of p are never in p , it follows that there is a $F_m \in p$ such that $F_m \subseteq F_C$ for all $C \in \mathcal{C}$. Pick $\alpha \in F_m$, and pick for each $C \in \mathcal{C}$ a β_C in F'_C . Set $\beta = \sup_{C \in \mathcal{C}} \beta_C$. Then $\langle \alpha, \beta \rangle \in \bigcap \mathcal{C}$.

Claim 3. *If $\mathcal{C} \subseteq \mathbb{P}(p \cup (\omega_1 \times \omega_2)) \setminus \{\emptyset\}$, (\mathcal{C}, \supset) has order type ω_2 , and $\bigcap \mathcal{C} = \emptyset$, then there is a $C \in \mathcal{C}$ such that $C \notin r$.*

For each $F \in p$ pick $C_F \in \mathcal{C}$ so that $F \not\subseteq C_F$. As $|p| = \omega_1$, but (\mathcal{C}, \supset) has order type ω_2 , there is a $C \in \mathcal{C}$ such that $C \cap p \subseteq (\bigcap_{F \in p} C_F) \cap p = \emptyset$. However, all elements of r meet p , so $C \in \mathcal{C}$ but $C \notin r$. \square

5. Topological groups and vector spaces

Let G be a topological group with identity element e , and let p be a free filter on a set S . Define $G(p)$ to be the topological space $p - \sigma_{s \in S}^x X_s$, where each X_s is a copy of G , and the s th co-ordinate of x is e . It is easy to check that $G(p)$ is a topological group when given the co-ordinatewise multiplication. In addition, if E is a locally convex topological vector space, then $E(p)$ is also a locally convex topological vector space.

This last observation gives us a way of constructing many stratifiable but nonmetrizable locally convex topological vector spaces. Yastchenko [17], who independently arrived at a similar construction (expressed in terms of ideals rather than filters) and proved part of Theorem 6 in this setting, has exploited this to construct a separable stratifiable locally convex topological vector space whose completion is not stratifiable.

However, for the purposes of the present paper, we are more interested in the case when G is discrete. For now we are able to give the promised Example 2.

Example 2*. Let G be any nontrivial discrete topological group. Let p be a tangle free filter which is not linear. Then $G(p)$ is a topological group all of whose finite powers are monotonically normal, but which is not linearly stratifiable.

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