Analytical and Topological Aspects of Signatures

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THESIS

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Abstract

In both physical and social sciences, we usually use controlled differential equation to model various continuous evolving system; describing how a response y relates to another process x called control. For regular controls x, the unique existence of the response y is guaranteed while it would never be the case for non-smooth controls via the classical approach. Besides, uniform closeness of controls may not imply closeness of their corresponding responses. Theory of rough paths provides a solution to both concerns. Since the creation of rough path theory, it enjoys a fruitful development and finds wide applications in stochastic analysis. In particular, rough path theory provides an effective method to study irregularity of curves and its geometric consequences in relation to integration of differential forms. In the chapter 2, we demonstrate the power of rough path theory in classical complex analysis by showing the rough path nature of the boundaries of a class of Hölder's domains; as an immediate application, we extend the classical Gauss-Green's theorem.

Until recently, there has been only limited research on applications of theory of rough paths to high dimensional geometry. It is clear to us that many geometric objects, in some senses appearing as solids, are actually comprised of filaments. In the chapter 3, two basic results in the theory of rough paths which will motivate later development of my thesis has been included. In the chapters 4 and 5, we identify a sensible way to do geometric calculus via those filaments (more precisely, space-filling rough paths) in dimension 3.

In a recent joint work of Hambly and Lyons, they have shown that every rectifiable path can be completely characterized, up to tree-like deformation, by an algebraic object called the signature, tensor of all iterated integrals, of the path. It is clear that all tree-like deformation of the path would not change its topological features. For instance, the number of times a planar loop of finite length winds around a point (not lying on the path) is unaltered if one deforms the path in tree-like ways. Therefore, it should be plausible to extract this topological information out from the signature of the loop since the signature is a complete algebraic invariant. In the chapter 6, we express the winding number of a nice loop (respectively linking number of a pair of nice loops) as a linear functional of the signature of the loop (respectively signatures of the pair of loops).

Contents

Chapter 1

Introduction

Let V and W be two Banach spaces. In modeling continuous evolving systems, we usually encounter differential equations in the form:

$$\begin{aligned} dy_t &= \sum_{i=1}^{n} f_i(y_t)\, dx_t^i \\ y_0 &= \xi_0, \end{aligned} \tag{1.1}$$

where $f_i(\cdot)$ are smooth enough vector fields, $y_\cdot \in C([0,1], W)$ is the dependent process (called the response) under the influence of another continuous process $x_\cdot \in C([0,1], V)$ (called the control). For bounded variation controls $x_\cdot$, the unique existence of solutions is obvious. However, this is not the case for non-smooth yet continuous controls $x_\cdot$; indeed, we even have a problem to interpret the "differentials dx_t^i" properly.

1.1 Non-smooth controls

Suppose $(W, < \cdot, \cdot >)$ is now a Hilbert space with induced norm $||\cdot||$. With regard to non-smooth controls, the first successful attempt to establish both the existence and uniqueness of (1.1) was done by K. Ito [1944], [1946]. Indeed, Ito is the first who formulated an integral calculus against Brownian motions: given a filtered probability space, $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P}\right)$, an $n-$dimensional $\mathcal{F}_t$ adapted Brownian motion $\mathbb{B}_\cdot$ and an $\mathcal{F}_t-$adapted process $\alpha_\cdot : [0,1] \to L(\mathbb{R}^n, W)$ (where $L(\mathbb{R}^n, W)$ is the space of bounded W-valued linear maps on $\mathbb{R}^n$) such that

$$\mathbb{P}\left(\int_0^1 ||\alpha_s||^2 \, ds < \infty\right) = 1,$$

Ito constructed a stochastic integral of $\alpha.$ against $\mathbb{B}.$ as the limit in probability of Riemann sums over dyadic partitions

$$\int_s^t \alpha_u \left(d\mathbb{B}_u\right) = prob - \lim_{n\to\infty} \sum_{s^n_\cdot \in D^n \cap [s,t]} \alpha_{s_k^n} \left(\mathbb{B}_{s_{k+1}^n} - \mathbb{B}_{s_k^n}\right),$$

where we denote $s_k^n = \frac{k}{2^n}$ and $D^n = \{s_k^n\}_{k=0}^{2^n}$.

Using Picard's iteration argument and a localization principle in probability, one can deduce the almost sure existence and uniqueness of solutions up to explosion of equation (1.1) for locally Lipschitz vector fields when the control is Brownian motion. Extension, by using a similar approach, to semi-martingales had been found by a number of scholars including H. Kunita and S. Watanabe [1967] and P. A. Meyer [1976].

1.2 Discontinuity of response

Even though for regular controls $x.$, the unique existence of solutions of (1.1) is ensured, uniform closeness of controls may not imply that of their responses. An example borrowed from Lyons and Qian [2002] is to consider a differential equation

$$\begin{aligned} dy_t &= y_t dx_t^1 + dx_t^2 \\ y_t &= \xi_0. \end{aligned}$$

Its solution can be expressed as a series of iterated integrals:

$$\begin{aligned} y_t &= \xi_0 + \int_{0<t_1<t} dx_{t_1}^2 - \int_{0<t_1<t_2<t} dx_{t_1}^1 dx_{t_2}^2 \\ &\quad + \sum_{k=2}^{\infty} (-1)^k \int_{0<t_1<\cdots<t_k<s<t} dx_{t_1}^1 \ldots dx_{t_k}^1 dx_s^2. \end{aligned}$$

Consider a sequence of controls $x(n)_t \triangleq \left(\frac{1}{n}\cos n^2 t, \frac{1}{n}\sin n^2 t\right)$. It is clear that $x(n)_\cdot$ converges uniformly to 0 on compacta; however, by simple calculation

$$\begin{aligned}\int_{0<t_1<t} dx(n)^2_{t_1} &= \frac{1}{n}\sin n^2 t \to 0 \\ \int_{0<t_1<t_2<t} dx^1_{t_1} dx^2_{t_2} &= \frac{t}{2} + \frac{1}{4n^2}\sin 2n^2 t - \frac{1}{n^2}\sin n^2 t \to \frac{t}{2} \\ \int_{0<t_1<\cdots<t_k<s<t} dx^1_{t_1}\dots dx^1_{t_k} dx^2_s &= \frac{1}{k!n^{k-1}}\int_0^t \left(\cos n^2 s - 1\right)^k \cos n^2 s ds,\end{aligned}$$

while the last term has modulus bounded by $\left(\frac{2}{n}\right)^{k-1}\frac{1}{k!}$, therefore

$$\begin{aligned}\left|\sum_{k=2}^{\infty}(-1)^k\int_{0<t_1<\cdots<t_k<s<t} dx^1_{t_1}\dots dx^1_{t_k}dx^2_s\right| &\le 2t\sum_{k=2}^{\infty}\left(\frac{2}{n}\right)^{k-1}\frac{1}{k!} \\ &= t\left(\frac{\exp\left(\frac{2}{n}\right)-1-\frac{2}{n}}{\frac{2}{n}}\right)\to 0\end{aligned}$$

Hence, the responses $y(n)_t$, respectively corresponding to the control $x(n)_t$, converge to $\xi_0 - \frac{t}{2}$ but not ξ_0! Meanwhile, we also notice the trouble is essentially coming from the iterated integral

$$\int_{0<t_1<t_2<t} dx^1_{t_1} dx^2_{t_2}.$$

1.3 Alternative interpretation of (1.1) and signatures

Both issues described in Sections 1.1 and 1.2 of the controlled differential equations can be resolved by introducing a metric finer than uniform one on an extension of the space of continuous paths $C([0,1], V)$. Before this, it is better to view our controlled differential equations from a different, but a natural, perspective.

Suppose $V = \mathbb{R}^n$ and all vector fields $\{f_i\}_{i=1}^n$ are linear. The solution of equation (1.1) can be expressed as

$$y_t = \xi_0 + \sum_{k=1}^{\infty}\sum_{i_j=1,\dots,n} f_{i_k}\left(\dots f_{i_2}\left(f_{i_1}(\xi_0)\right)\right)\int_{0<t_1<\cdots<t_{k-1}<t} dx^{i_1}_{t_1}dx^{i_2}_{t_2}\dots dx^{i_k}_{t_k}.$$

Define $V^{\otimes 0} = \mathbb{R}$ and $V^{\otimes 1} = V$. Let $T(V)$ be the tensor algebra $\oplus_{k=0}^{\infty} V^{\otimes k}$. For $k \in \mathbb{N}$, define $\pi_k : T(V) \to V^{\otimes k}$ to be the projection operator onto $V^{\otimes k}$ and linear maps $F^k(\cdot) : V^{\otimes k} \to L(W, W)$ such that

$$\begin{aligned} F^0(\cdot) &: 1 \in \mathbb{R} \mapsto \iota : W \to W, \\ F^k(\cdot) &: e^{i_1} \otimes \cdots \otimes e^{i_k} \mapsto (f_{i_k} \circ \cdots \circ f_{i_2} \circ f_{i_1})(\cdot), \ k = 1, 2, \ldots, \end{aligned}$$

where $\{e^i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. For simplicity, for any $X \in T(V)$, we denote $\pi_k(X)$ by $X^{(k)}$. For each $k \in \mathbb{N}$, define two linear maps from $T(V)$ to $L(W, W)$,

$$\begin{aligned} \Pi_k(*)(\cdot) &\triangleq F^k(*) \circ \pi_k(\cdot) \\ \Pi(*)(\cdot) &\triangleq \sum_{k=0}^{\infty} \Pi_k(*)(\cdot), \end{aligned}$$

also define a linear map $F^{r_1} \otimes \cdots \otimes F^{r_k}(\cdot)$ from $V^{\otimes r_1} \otimes \cdots \otimes V^{\otimes r_k}$ to $L\left(W, W^{\otimes k}\right)$ such that

$$F^{r_1} \otimes \cdots \otimes F^{r_k}(\cdot)(w_1 \otimes \cdots \otimes w_k) = F^{r_1}(\cdot)(w_1) \otimes \cdots \otimes F^{r_k}(\cdot)(w_k),$$

for any $w_i \in V^{\otimes r_i}$ for $i = 1, \ldots, k$.

Definition 1.1 *The signature of a bounded variation path $x_{\cdot}$ over $[s, t]$, denoted by $S(x)_{s,t}$, is the tensor of iterated integrals over $[s, t]$, i.e.*

$$S(x)_{s,t} \triangleq 1 + \int_{s<t_1<t} dx_{t_1} + \cdots + \int_{s<t_1<\cdots<t_k<t} dx_{t_1} \otimes \cdots \otimes dx_{t_k} + \ldots. \quad (1.2)$$

It should be remarked that, for any path $x_{\cdot}$, $S(x)_{\cdot}$ is continuous in two variables s, t and satisfies an algebraic identity called Chen's identity (Chen [1958]):

$$S(x)_{s,t} = S(x)_{s,u} \otimes S(x)_{u,t}, \quad (1.3)$$

for any $s < u < t$.

Now, the solution of (1.1) can be written neatly as

$$y_t = \Pi(\xi_0)\left(S(x)_{0,t}\right),$$

which is a convergent series since the factorial decay of moduli of the tensors of iterated integrals compensates the exponential growth of the norms of the maps F^k. So in the case of linear vector fields, the solution of (1.1) can be regarded as a linear map of the signature of the control $x_{\cdot}$.

In general, the signature of the response $y_{\cdot}$ is also a linear map of the control $x_{\cdot}$; indeed, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
&\int_{0<t_1<\cdots<t_k<t} dy_{t_1} \otimes \cdots \otimes dy_{t_k} \\
= &\int_{0<t_1<\cdots<t_k<t} \Pi(\xi_0)\left(dS(x)_{0,t_1}\right) \otimes \cdots \otimes \Pi(\xi_0)\left(dS(x)_{0,t_k}\right) \\
= &\sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} \int_{0<t_1<\cdots<t_k<t} \Pi_{r_1}(\xi_0)\left(dS(x)_{0,t_1}\right) \otimes \cdots \otimes \Pi_{r_k}(\xi_0)\left(dS(x)_{0,t_k}\right) \\
= &\sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} \int_{0<t_1<\cdots<t_k<t} F^{r_1}(\xi_0)\left(dS(x)_{0,t_1}^{(r_1)}\right) \otimes \cdots \otimes F^{r_k}(\xi_0)\left(dS(x)_{0,t_k}^{(r_k)}\right) \\
= &\sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} F^{r_1} \otimes \cdots \otimes F^{r_k}(\xi_0)\left(\int_{0<t_1<\cdots<t_k<t} dS(x)_{0,t_1}^{(r_1)} \otimes \cdots \otimes dS(x)_{0,t_k}^{(r_k)}\right) \\
= &\sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} F^{r_1} \otimes \cdots \otimes F^{r_k}(\xi_0)\left(\sum_{\sigma \in OS(r_1,\ldots,r_k)} \pi^{\sigma}\left(S(x)_{0,t}\right)\right),
\end{aligned}
$$

where $OS(r_1, \ldots, r_k)$ is all the permutations σ of $u_1 < u_2 < \cdots < u_{r_1+\cdots+r_k}$ such that

$$
\begin{aligned}
u_{\sigma(1)} &< \cdots < u_{\sigma(r_1)} \\
u_{\sigma(r_1+1)} &< \cdots < u_{\sigma(r_1+r_2)} \\
&\vdots \\
u_{\sigma(r_1+\cdots+r_{k-1}+1)} &< \cdots < u_{\sigma(r_1+\cdots+r_k)}
\end{aligned}
$$

and

$$
u_{\sigma(r_1)} < u_{\sigma(r_1+r_2)} < \cdots < u_{\sigma(r_1+\cdots+r_k)}.
$$

While π^{σ} is a linear map from $T(V)$ to $V^{\otimes r_1} \otimes \cdots \otimes V^{\otimes r_k}$ such that

$$
\pi^{\sigma}(X) = \sum_{i_j} X^{(r_1+\cdots+r_k), i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(r_1+\cdots+r_k)}} e^{i_1} \otimes \cdots \otimes e^{i_{r_1+\cdots+r_k}}.
$$

We now interpret the controlled differential equation (1.1) as a nonlinear map sending the signature of the control $x_{\cdot}$ to the signature of the response

$y.$ augmented with the control $x.$:

$$\begin{aligned} S(x)_{0,t} &\in T(V) \longmapsto S(x+y)_{0,t} \in T(V \oplus W) \\ dy_t &= \sum_{i=1}^{n} f_i(y_t)\, dx_t^i \\ y_0 &= \xi_0. \end{aligned}$$

1.4 p-variation topology and Universal limit theorem

Notice that a signature is a formal object in the tensor algebra which is not stable with respect to limit taking. Fortunately, a sufficient number of low order iterated integrals, together with Chen's identity, can determine the whole profile of the signature.

For $m \in \mathbb{N} \cup \{\infty\}$, let $T^{(m)}(V)$ be the quotient algebra of $T(V)$ by the ideal $\oplus_{k=m+1}^{\infty} V^{\otimes k}$ and $g^{(m)}(V)$ the Lie subalgebra of $T^{(m)}(V)$ generated by V, i.e.

$$g^{(m)}(V) = \oplus_{k=1}^{m} V_k$$

where

$$\begin{aligned} V_1 &= V \\ V_{k+1} &= [V, V_k]. \end{aligned}$$

where for $A, B \subset T(V)$, $[A,B] \triangleq \Big\{[a,b] \triangleq a \otimes b - b \otimes a : a \in A, b \in B\Big\}$. Also let $G^{(m)}(V) = \exp\left(g^{(m)}(V)\right)$ and so, for any regular path $x.$, $S(x)_{.} \in G^{(\infty)}(V)$. Let $\delta_\lambda : T(V) \to T(V)$ be a dilation operator such that for any $X \in T(V)$,

$$\delta_\lambda(X) = X^{(0)} + \lambda X^{(1)} + \cdots + \lambda^m X^{(m)} + \ldots.$$

Definition 1.2 *Let $m \in \mathbb{N}$. A symmetric homogeneous norm $||\cdot||_m$ on $G^{(m)}(V)$ is a norm such that for any $g, h \in G^{(m)}(V)$,*

1. $||g||_m = 0$ *if and only if* $g = 1$,
2. $||\delta_\lambda(g)||_m = |\lambda| \cdot ||g||_m$ *for any* $\lambda \in \mathbb{R}$,

3. $||g \otimes h||_m \leq ||g||_m + ||h||_m$,

4. $||g^{-1}||_m = ||g||_m$.

For example, for any $p \geq 1$,

$$||g||_{m,p} \triangleq \max_{i=1,\dots,m} \left(\left(\frac{i}{p} \right)! \left| g^i \right| \right)^{\frac{1}{i}}$$

is a symmetric homogeneous norm on $G^{(m)}(V)$. Note that if V is finite dimensional, all homogeneous norm are equivalent, see Goodman [1977]. Now, $d_{m,p}(X, Y) = ||X^{-1} \otimes Y||_{m,p}$ defines a metric on $G^{(m)}(V)$.

Denote by $C_0\left([0,1], G^{(m)}(V)\right)$ the set of all continuous $G^{(m)}(V)-$ valued paths $Z_\cdot : [0,1] \to G^{(m)}(V)$ such that $Z_0 = 1$. Define $Z_{s,t} = Z_s^{-1} \otimes Z_t$, then

$$Z_{s,u} = Z_{s,t} \otimes Z_{u,t}.$$

For any $X_\cdot, Y_\cdot \in C_0\left([0,1], G^{(m)}(V)\right)$, define

$$d_{m,p-var}(X_\cdot, Y_\cdot)_{s,t} \triangleq \sup_{\mathcal{D} \subset [s,t]} \left(\sum_{t_i} d_{m,p}\left(X_{t_i,t_{i+1}}, Y_{t_i,t_{i+1}}\right)^p \right)^{\frac{1}{p}}$$

where the supremum is taken over all partition $\mathcal{D}$ of $[s,t]$. Denote $d_{m,p-var}(X_\cdot, 1)_{[s,t]}$ by $||X_\cdot||_{m,p-var,[s,t]}$. Note that $d^p_{m,p-var}(\cdot,\cdot)_{s,t}$ is a control (see the Definition 1.6 and also see Chapter 3 for a proof). Now, $d_{m,p-var,[0,1]}$ defines a metric on $C_0\left([0,1], G^{(m)}(V)\right)$ called the $p-$ variation metric.

Proposition 1.3 *(Extension theorem) Fix three real numbers $p \geq 1$, $m \geq \lfloor p \rfloor$ and*

$$\beta \geq p^2 \left(1 + 2^{\frac{\lfloor p \rfloor + 1}{p}} \left(\zeta\left(\frac{\lfloor p \rfloor + 1}{p} \right) - 1 \right) \right).$$

Suppose $X_\cdot \in C_0\left([0,1], G^{(m)}(V)\right)$ with $||X_\cdot||_{m,p-var,[0,1]} < \infty$, define

$$\omega(s,t) \triangleq \left(\beta \, ||X_\cdot||_{m,p-var,[s,t]} \right)^p.$$

Then there is a unique $\widetilde{X}_\cdot : [0,1] \to G^{(\infty)}(V)$ such that

1. $\pi_{G^{(m)}(V)}\left(\widetilde{X}_\cdot\right) = X_\cdot$.

2. For any $n \geq m$,

$$\left\| \pi_{G^{(n)}(V)} \left(\widetilde{X}_{\cdot} \right) \right\|_{n,p-var,[s,t]} \leq \omega(s,t)^{\frac{1}{p}}.$$

Proof. As an immediate consequence of the definition, the result is a reformulation of the extension theorem in Lyons [1998]. ∎

Let $G\Omega_p(V) \subset C_0\left([0,1], G^{(\lfloor p \rfloor)}(V)\right)$ be the closure of all $\pi_{G^{(\lfloor p \rfloor)}(V)}\left(S(\gamma)_{0,\cdot}\right)$, for some $V-$ valued smooth path $\gamma_{\cdot}$, under the $p-$ variation metric $d_{\lfloor p \rfloor, p-var}(\cdot,\cdot)_{[0,1]}$. Elements in $G\Omega_p(V)$ are called $p-$geometric rough paths over V.

Theorem 1.4 *(Universal limit theorem) Suppose all vector fields f_i are smooth. Consider a map governed by the controlled differential equation*

$$\begin{aligned} S(x)_{0,\cdot} &\in G\Omega_p(V) \longmapsto S(x+y)_{0,\cdot} \in G\Omega_p(V \oplus W) \\ dy_t &= \sum_{i=1}^{n} f_i(y_t)\, dx_t^i \\ y_0 &= \xi_0, \end{aligned}$$

with finite $p-$variation control $x_{\cdot}$. Then the map is continuous with respect to the $p-$variation metrics $d_{\lfloor p \rfloor, p-var,[0,1]}$ on $G^{(\lfloor p \rfloor)}(V)$ and $G^{(\lfloor p \rfloor)}(V \oplus W)$ respectively.

Proof. See Lyons [1998] for details. ∎

As a consequence, we can be continuously extend the notion of solution of controlled differential equation driven by all $p-$geometric rough paths in $G\Omega_p(V)$.

1.5 Main results

1.5.1 Rough path nature of boundaries of Hölder domains

Before our work [24], most of the continuous paths (see [23] and the references therein) with infinite variation which were shown that can be lifted as geometric rough paths (should possess self-similar property (at least in statistical sense). Actually, self-similar structure of a path is sufficient to provide easily computed controls (see Definition 3.4) for the approximants of the path and the path itself.

Definition 1.5 *A simply connected planar domain $D \subset \mathbb{C}$ is called an $\alpha-$Hőlder domain if D is the conformal image of a unique univalent analytic function $\phi : \mathbb{D} \to \mathbb{C}$ with*

$$\phi\left(\mathbb{S}^1\right) = \partial D,$$

such that $\phi : \mathbb{S}^1 \to \mathbb{C}$ is also $\alpha-$Hölder continuous.

It is clear that no boundary of any $\alpha-$Hölder domain is self-similar. Chapter 2 is a detailed introduction to our work [24]; indeed, we shall demonstrate that pure analytic property of the boundaries of certain $\alpha-$Hölder domains is enough to establish the rough path nature of the boundaries. In the chapter, we prove that the boundaries of $\alpha(> \frac{1}{3})-$Hölder domain are $\left(\frac{1}{\alpha} + \varepsilon\right)-$ geometric rough paths for any $\varepsilon > 0$. We establish a construction of canonical area processes for the boundaries. Under the restriction that $\alpha > \frac{1}{3}$, it happens with a sophisticated application of isoperimetric inequality and standard arguments in harmonic analysis, including the Hardy-Littlewood lemma, that a canonical area process exists. As an immediate application, we extend the classical Gauss-Green's formula to this class of fractal planar domains

1.5.2 Two results in rough path theory

Definition 1.6 *A non-negative continuous function ω on Δ_T is called a control if*

1. *ω is superadditive:*

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u)$$

   for $(s,t), (t,u) \in \Delta_T$,

2. *$\omega(t,t) = 0$ for all $t \in [0,T]$.*

Definition 1.7 *Let $X.$ be a $p-$rough path. Define, for any $(s,t) \in \Delta_T$,*

$$\omega_X(s,t) \triangleq \sup_{1\leq i\leq [p]} \sup_{D\subset[s,t]} \sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}}, \tag{1.4}$$

where the supremum runs over all finite partitions D of the interval $[s,t]$. We call $\omega_X(s,t)$ the $p-$variation of $X.$ over $[s,t]$.

In Chapter 3, we collect two results (and their proofs) which are fundamental in nature in the theory of rough paths. The Extension theorem in the theory of rough paths (Proposition 1.3 in Chapter 1 or its equivalence, Theorem 2.1.1 in Lyons [1998]) states that any $p-$rough path $X_\cdot : \Delta_T \to T^{(\lfloor p \rfloor)}(V)$ can be naturally enhanced as a multiplicative functional of finite $p-$variation on $T^{(N)}(V)$ for any $N > \lfloor p \rfloor$; in other words, all high order tensors of "iterated integrals" of a rough path are completely determined by its tensors of "iterated integrals" of order less than p. In order to prove the Extension theorem, one has to show that a particular sequence of functionals $\{X(n)_\cdot : \Delta_T \to T^{(N)}(V)\}_{n \in \mathbb{N}}$ is Cauchy by applying the maximal inequality (see Lyons [1998]) and the fact that there is a control for the rough path $X_\cdot$. Unfortunately, the assertion that there is a natural control for any continuous multiplicative functional with finite $p-$variation is only a mathematics folklore without a proper proof; even there is none in [23] as claimed. In Section 3.1, I shall provide an elementary proof of the claim which is quoted in Theorem 3.6.

In the theory of rough paths, it admits that the signature (recall Definition 1.1) of a geometric rough path completely characterizes the path itself in the context of control theory. Along the line of thought, if a space-filling curve γ which can be shown to be a geometric rough path and fills up a geometric object $\mathcal{M}$, it is reasonable to expect that we can extract the analytical properties of $\mathcal{M}$ by decoding the information contained in the signature of γ; this is precisely our concern in Chapter 5. In regard to extracting information embedded in the signature of a path, it is natural to achieve our goal by integrating, with aids of different one-forms, against the signature (which is now considered as a path itself). Therefore, it is essential to ask if the signature of a geometric rough path, or in general any functional of signature, can still be treated as a geometric rough path. In Section 3.2, we shall prove Proposition 3.34 which states that any functional of the log-signature of a $p-$geometric rough path can be canonically enhanced as a $p-$geometric rough path.

1.5.3 Space-filling rough paths

In the theory of rough paths, it admits that the signature (Definition 1.1) of a geometric rough path completely characterizes the path itself in the sense of controlling any arbitrary controlled differential equation. Along the same line of thought, it is interesting to ask if one can find quantities, which are analogous to the signature of a path, that can characterize a high dimensional geometric object $\mathcal{M}$ in the sense of integrating differential forms on $\mathcal{M}$. It is expected that the established theory of rough paths may help us to resolve our curiosity; indeed, one plausible approach is to first use path(s) to represent a high dimensional geometric object $\mathcal{M}$, and then regard a differential form ω as an one-form $\widetilde{\omega}$ over tensors so that integrating the differential form ω on $\mathcal{M}$ is the same as integrating the one-form $\widetilde{\omega}$ against the path(s). More precisely, given a nice high dimensional geometric object $\mathcal{M}$, can one find a space-filling curve γ for $\mathcal{M}$ which can "naturally" be enhanced as a geometric rough path? If the answer were positive, could we also find a way so that integrating a differential form ω on $\mathcal{M}$ is equivalent to integrating an one-form $\widetilde{\omega}$ against the enhanced path of γ? In Chapter 4, we shall show that a class of space-filling curves are actually geometric rough paths; while in Chapter 5, we shall discuss the issue about integrating a differential form on a geometric object as contracting a one-form against its space-filling rough paths.

Since many nice geometric objects can be arbitrarily closely approximated by a finite number of hypercubes, it is more tractable to commence our program of research by first looking for a class of space-filling curves for hypercubes that can be lifted as geometric rough paths. In particular, we make a conjecture that for each $d \in \mathbb{N}$, the space-filling curve $F^{[d]}$ (Definition 4.11) for the $d-$dimensional unit hypercube can be naturally (but not in a unique way) enhanced as a $p\,(> d)-$geometric rough path. We shall prove our claim when $d = 3$ in this chapter; unfortunately, it is still open for $d > 4$ because iterated integrals of order not less than 4 would need to be investigated which seems to be far from trivial.

1.5.4 Integral of a 3-form α as an integral of a spinor $(\widetilde{\alpha}, -\widetilde{\alpha})$

Once we establish the fact (Theorem 4.82) that there is a class $\mathcal{C}$ of space-filling rough paths with their $\mathbb{R}^3-$projections being all the same and filling up a three-dimensional unit cube; similar results can be extended to those $3-$dimensional geometric object (nice chainlet) $\mathcal{N}$ which can be well-approximated by cubes. In this respect, one may expect that analytical properties of $\mathcal{N}$ can be extracted by decoding the information contained in the signatures of those space-filling rough paths in the corresponding class $\mathcal{C}$

for $\mathcal{N}$. In Chapter 5, we shall show how we can answer our concern, first for cubes and then for some nice chainlets (see Harrison [1998]), in the dimension 3; indeed, a special pair of space-filling rough paths in $\mathcal{C}$ for $\mathcal{N}$ will do the job. Moreover, we shall identify any differential form ω as an one-form $\widetilde{\omega}$ (see Lemma 5.4) over tensors so that integrating the differential form ω on $\mathcal{N}$ is equivalent to (see (5.43)) integrating the one-form (a spinor) $(\widetilde{\omega}, -\widetilde{\omega})$ against the pair in $\mathcal{C}$ with respect to a properly chosen integrator (see Section 5.2).

1.5.5 Winding and linking numbers as a functional of signatures

Recently, Hambly and Lyons (2006) proved a significant result that all continuous paths of finite variation can be characterized, up to tree-like deformation, by their respective signatures. As a remark, it should be mentioned that Chen also acquired similar result if the underlying paths are piecewise-smooth [Chen (1958)]. It should be obvious that any tree-like deformation of a path would not alter its topological features. For instance, for almost every point z in the plane, the number of times a finite variation planar loop $\gamma : I \to \mathbb{R}^2$ winds about z remains unchanged if one deforms the path γ in tree-like ways without crossing the point z. In Chapter 6, based on our recent study [Lyons and Yam (2007)], we successfully extract low dimensional topological information, e.g. winding and linking numbers, from the signatures of the loops under consideration. Furthermore, under mild regularity conditions on the loops, we have also shown that the winding number $\eta(\gamma_\cdot, z)$ about a point $z \in \mathbb{C}$ (respectively the linking numbers) of an arbitrary loop γ (respectively a pair of loops γ_1 and γ_2) can be expressed as a linear functional of the signature $S(\gamma_\cdot)_{0,1}$ of γ (respectively the signatures $S(\gamma_1)_{0,1}$ and $S(\gamma_2)_{0,1}$ of both loops).

Chapter 2

Boundaries of Hölder domains as rough paths

Before our work [24], most of the continuous paths (see [23] and the references therein) with infinite variation which were shown that can be lifted as geometric rough paths (should possess self-similar property (at least in statistical sense). Actually, self-similar structure of a path is sufficient to provide easily computed controls (see Definition 3.4) for the approximants of the path and the path itself. It is clear that no boundary of any $\alpha-$Hölder domain is self-similar. This chapter is a detailed introduction to our work [24]; indeed, we shall demonstrate that pure analytic property of the boundaries of certain $\alpha-$Hölder domains is enough to establish the rough path nature of the boundaries. In particular, we are aim to establish the fact that, if $\alpha > \frac{1}{3}$ and $\epsilon > 0$, the boundary $\gamma.$ of a planar $\alpha-$Hölder domain can be canonically enhanced as a $\frac{1}{\alpha} + \epsilon$ geometric rough path. The key step to prove our claim is Lemma 2.30 which aids us to settle the problem in Section 2.4 about the convergence of geometric Levy processes (see Definition 2.20) of approximants of $\gamma.$ to that of $\gamma.$. In order to estimate the sizes of various regions arisen in Lemma 2.30, we applied the Pohl-Banchoff inequality (Proposition 2.13) and consequences of Hardy-Littlewood lemma (Lemma 2.3) as listed in Section 2.1. Finally, as an immediate application, we extend the classical Green-Gauss' formula to the mentioned class of fractal planar domains.

In this chapter, we use C and M to denote constants

2.1 Preliminary definitions and results

In this section, we formulate the notion of planar Hölder domains and introduce preliminary results from complex analysis which relates the smoothness of the boundary of a Hölder domain to an analytic condition in the interior of the domain.

Definition 2.1 (Pommerenke [1992]) *Let $\alpha \in (0,1]$ and A be a connected subset in $\mathbb{C}$. A function $\phi : A \to \mathbb{C}$ is said to be $\alpha-$Hölder continuous if there is $M \in \mathbb{R}^+$ such that $\forall z_1, z_2 \in A$,*

$$|\phi(z_1) - \phi(z_2)| \leq M |z_1 - z_2|^{\alpha}. \tag{2.1}$$

Suppose $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ with $0 \leq \theta_1, \theta_2 < 2\pi$. To say that ϕ is $\alpha-$Hölder continuous over the unit circle $\mathbb{S}^1$ is the same as saying that ϕ satisfies the condition:

$$|\phi(z_1) - \phi(z_2)| \leq M' |\arg(z_2 - z_1)|^{\alpha} \leq M' |\theta_1 - \theta_2|^{\alpha} \tag{2.2}$$

for some $M' > 0$. Denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$.

Definition 2.2 (Pommerenke [1992]) *A simply connected planar domain $D \subset \mathbb{C}$ is called an $\alpha-$Hölder domain if D is the conformal image of a unique univalent analytic function $\phi : \mathbb{D} \to \mathbb{C}$ with*

$$\phi\left(\mathbb{S}^1\right) = \partial D$$

such that $\phi : \mathbb{S}^1 \to \mathbb{C}$ is also $\alpha-$Hölder continuous.

Next, we introduce a lemma, first obtained by Hardy and Littlewood [1932], which characterizes the rate of growth of the derivative of ϕ when z approaches to $\mathbb{S}^1$. For an account of this result and its further applications, see Duren [1970] and Pommerenke [1992]. We here include its proof since it will motivate further results in this chapter.

Lemma 2.3 (Hardy-Littlewood) *Let $\alpha \in (0,1)$. Suppose ϕ is a univalent analytic function in the interior of the unit disc $\mathbb{D}$. Then $\phi : \mathbb{D} \to \mathbb{C}$ is continuous and $\phi : \mathbb{S}^1 \to \mathbb{C}$ is also $\alpha-$Hölder continuous if, and only if, there is $C > 0$ such that for any $z \in \mathbb{C}$,*

$$|\phi'(z)| \leq \frac{C}{(1 - |z|)^{1-\alpha}}. \tag{2.3}$$

Proof. Suppose that ϕ is continuous in $\mathbb{D}$ and $\phi(e^{i\theta})$ is α–Hölder continuous in θ. According to the Cauchy formula, $\forall z \in \mathbb{D}^o$ and $n \in \mathbb{N}$,

$$\begin{aligned}\phi^{(n)}(z) &= \frac{n!}{2\pi i}\oint_{\mathbb{S}^1}\frac{\phi(\zeta)d\zeta}{(\zeta - z)^{n+1}}\\ &= \frac{n!}{2\pi}\int_0^{2\pi}\frac{\phi(e^{it})\,e^{it}dt}{(e^{it}-z)^{n+1}}.\end{aligned}$$

Expressing $z = re^{i\theta}$ for some $r < 1$ and $-\pi < \theta \leq \pi$, and taking into account that for any $n \geq 1$,

$$\oint_{\mathbb{S}^1}\frac{d\zeta}{(\zeta - z)^{n+1}} = \int_0^{2\pi}\frac{e^{it}dt}{(e^{it}-z)^{n+1}} = 0,$$

we then have

$$\begin{aligned}\phi^{(n)}(z) &= \frac{n!}{2\pi}\int_0^{2\pi}\frac{(\phi(e^{it}) - \phi(e^{i\theta}))\,e^{it}dt}{(e^{it}-re^{i\theta})^{n+1}}\\ &= \frac{n!}{2\pi}\int_{-\theta}^{2\pi-\theta}\frac{(\phi(e^{i(t+\theta)}) - \phi(e^{i\theta}))}{(e^{it}-r)^{n+1}}\,e^{i(t+\theta)}e^{-(n+1)i\theta}dt\\ |\phi^{(n)}(z)| &\leq \frac{n!}{2\pi}\int_{-\theta}^{2\pi-\theta}\frac{|\phi(e^{i(t+\theta)}) - \phi(e^{i\theta})|}{|e^{it}-r|^{n+1}}dt.\end{aligned}$$

Note that ϕ is periodic,

$$\begin{aligned}&\int_{-\theta}^{2\pi-\theta}\frac{|\phi(e^{i(t+\theta)}) - \phi(e^{i\theta})|}{|e^{it}-r|^{n+1}}dt\\ =& \int_{-\theta}^{\pi}\frac{|\phi(e^{i(t+\theta)}) - \phi(e^{i\theta})|}{|e^{it}-r|^{n+1}}dt + \int_{\pi}^{2\pi-\theta}\frac{|\phi(e^{i(t+\theta-2\pi)}) - \phi(e^{i\theta})|}{|e^{i(t-2\pi)}-r|^{n+1}}dt\\ =& \int_{-\theta}^{\pi}\frac{|\phi(e^{i(t+\theta)}) - \phi(e^{i\theta})|}{|e^{it}-r|^{n+1}}dt + \int_{-\pi}^{-\theta}\frac{|\phi(e^{i(t+\theta)}) - \phi(e^{i\theta})|}{|e^{it}-r|^{n+1}}dt\\ =& \int_{-\pi}^{\pi}\frac{|\phi(e^{i(t+\theta)}) - \phi(e^{i\theta})|}{((1-r)^2+4r\sin^2\frac{t}{2})^{\frac{n+1}{2}}}dt.\end{aligned}$$

Using the Hölder condition of ϕ and the fact that

$$\forall t,\ |t| \leq \pi \Rightarrow \sin^2\frac{t}{2} \geq (\frac{t}{\pi})^2,$$

we then obtain:

$$\begin{aligned}
\left|\phi^{(n)}(z)\right| &\leq \frac{n!}{2\pi}\int_{-\pi}^{\pi}\frac{M'\cdot|t|^\alpha}{((1-r)^2+\frac{4r}{\pi^2}t^2)^{\frac{n+1}{2}}}dt \\
&= \left(\frac{n!}{2\pi(1-r)^{n-\alpha}}\right)\int_{-\frac{\pi}{1-r}}^{\frac{\pi}{1-r}}\frac{|u|^\alpha}{(1+\frac{4r}{\pi^2}u^2)^{\frac{n+1}{2}}}du \\
\left|\phi^{(n)}(z)\right| &\leq \left(\frac{n!}{2\pi(1-r)^{n-\alpha}}\right)\int_{-\infty}^{\infty}\frac{|u|^\alpha}{(1+\frac{4r}{\pi^2}u^2)^{\frac{n+1}{2}}}du, \qquad (2.4)
\end{aligned}$$

where the last integral is clearly finite when $\alpha < 1$; and hence

$$\left|\phi^{(n)}(z)\right| = O\left(\frac{1}{(1-|z|)^{n-\alpha}}\right).$$

For instance, we also have

$$\phi'(z) \leq \frac{C}{(1-|z|)^{1-\alpha}}.$$

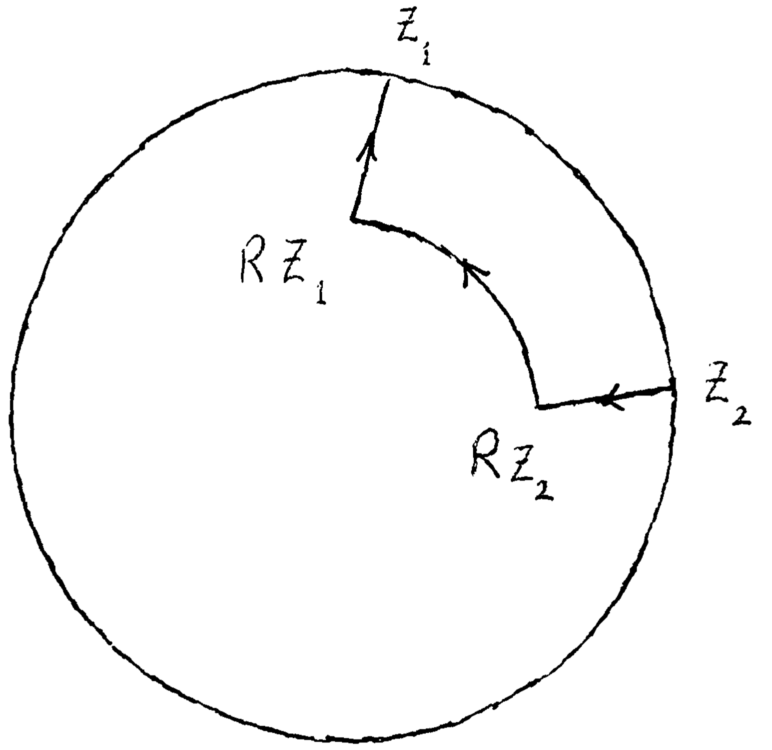
Conversely, suppose $\phi'(z) = O(\frac{1}{(1-|z|)^{1-\alpha}})$, then for each $\theta \in [0, 2\pi)$,

$$\left|\int_0^{1-}\phi'(re^{i\theta})dr\right| \leq \int_0^{1-}|\phi'(re^{i\theta})|dr \leq \int_0^{1-}\frac{C}{(1-r)^{1-\alpha}}dr < \infty,$$

and hence

$$\phi(0) + \lim_{r\uparrow 1}\int_0^r \phi'(\rho e^{i\theta})d\rho \text{ exists.}$$

So it is possible to extend our domain of definition of ϕ to the whole unit disc $\mathbb{D}$. For any $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$ and $R \leq 1$, consider the path γ starting from z_2 running radially inward to Rz_2, then moving along the circular arc with radius R to Rz_1, and finally reaching radially outward to z_1.



The modulus of the difference

$$\begin{aligned}
|\phi(z_1) - \phi(z_2)| &= \left|\int_\gamma \phi'(\zeta)\, d\zeta\right| \\
&\le \int_R^{1-} \left|\phi'(re^{i\theta_2})\right| dr + \int_{\theta_1}^{\theta_2} \left|\phi'(Re^{i\theta})\right| d\theta + \int_R^{1-} \left|\phi'(re^{i\theta_1})\right| dr \\
&\le \frac{2C}{\alpha}(1-R)^\alpha + \frac{C}{(1-R)^{1-\alpha}} |\theta_2 - \theta_1| .
\end{aligned}$$

1. Suppose $|\theta_2 - \theta_1| < 1$. Since R is arbitrary, we can choose $R = 1 - |\theta_2 - \theta_1|$ and consequently, we have

$$|\phi(z_1) - \phi(z_2)| \le C\left(1 + \frac{2}{\alpha}\right) |\theta_2 - \theta_1|^\alpha .$$

2. On the other hand, if $|\theta_2 - \theta_1| \ge 1$, choosing R to be arbitrarily close to 0, then

$$\begin{aligned}
|\phi(z_1) - \phi(z_2)| &\le \frac{2C}{\alpha} |\theta_2 - \theta_1|^\alpha + C|\theta_2 - \theta_1| \\
&\le C\left(\min(|\theta_2 - \theta_1|, 2\pi)^{1-\alpha} + \frac{2}{\alpha}\right) |\theta_2 - \theta_1|^\alpha .
\end{aligned}$$

Therefore, as a whole, we have

$$|\phi(z_1) - \phi(z_2)| \le C\left(\max\left(1, \min(|\theta_2 - \theta_1|, 2\pi)^{1-\alpha}\right) + \frac{2}{\alpha}\right) |\theta_2 - \theta_1|^\alpha ,$$

or simply

$$|\phi(z_1) - \phi(z_2)| \le C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right) |\theta_2 - \theta_1|^\alpha , \tag{2.5}$$

i.e. ϕ is $\alpha-$Hölder continuous on $\mathbb{S}^1$. Applying the fact that the solution to any well-posed Dirichlet problem is unique, we have

$$\phi(z) = \int_0^{2\pi} P(e^{i\theta}, z)\phi(e^{i\theta})d\theta,$$

where $P(\zeta, z) = \frac{1}{2\pi}\frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$ is the Poisson kernel; that is to say ϕ is a Poisson integral over its boundary values, and hence ϕ is continuous over the whole unit disc $\mathbb{D}$. ■

For every $r \leq 1$, define

$$\phi_r(z) \triangleq \phi(rz)$$

for all $z \in \mathbb{D}$. We next establish an upper bound of the length of the line segment joining the end-points of the image under $\phi_r(\cdot)$ of an arc of the unit circle.

Lemma 2.4 *Suppose that $\phi : \mathbb{S}^1 \to \mathbb{C}$ is $\alpha-$Hőlder continuous. For every $0 \leq r \leq 1$, $\phi_r(e^{i\theta})$ is also $\alpha-$Hőlder continuous in θ; in particular, we have*

$$\left|\phi_r\left(e^{i\theta_1}\right) - \phi_r\left(e^{i\theta_2}\right)\right| \leq Cr\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)|\theta_2 - \theta_1|^{\alpha}. \tag{2.6}$$

Proof. According to the *Hardy-Littlewood* lemma, for every $z \in \mathbb{D}$,

$$|\phi'(z)| \leq \frac{C}{(1-|z|)^{1-\alpha}},$$

for some $C > 0$. Differentiating ϕ_r with respect to z, we get $\phi_r{}'(z) = r\phi'(rz)$, and hence

$$|\phi_r'(z)| \leq \frac{Cr}{(1-r|z|)^{1-\alpha}} \leq \frac{Cr}{(1-|z|)^{1-\alpha}}. \tag{2.7}$$

Applying the converse part of the *Hardy-Littlewood* lemma 2.3 and its lines of proof, we deduce our desired result. ■

As a consequence, we also obtain an upper bound of the length of the line segment joining the end-points of the image under $\phi(\cdot)$ of an arc of any inner concentric circle.

Corollary 2.5 *Let $0 \leq r \leq 1$. For any $z_1, z_2 \in \mathbb{C}$ with $|z_1| = |z_2| = r$, we have*

$$|\phi(z_1) - \phi(z_2)| \leq \frac{C\pi^{\alpha}r^{1-\alpha}}{2^{\alpha}}\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)|z_2 - z_1|^{\alpha}. \tag{2.8}$$

Proof. It is clear that one can express $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$ with $|\theta_1 - \theta_2| \leq \pi$; using the inequality (2.6), we conclude our claim. ■

In addition, we also have an upper bound of the length of the line segment joining the end-points of the image under $\phi(\cdot)$ of a radial segment of the unit circle.

Lemma 2.6 *Let $0 \le \rho \le 1$. The increment of ϕ along any radial direction θ is bounded above as:*

$$\left|\phi\left(e^{i\theta}\right) - \phi\left(\rho e^{i\theta}\right)\right| \le \frac{C}{\alpha}(1-\rho)^{\alpha} \tag{2.9}$$

Proof.

$$\begin{aligned}\left|\phi\left(e^{i\theta}\right) - \phi\left(\rho e^{i\theta}\right)\right| &= \lim_{q\uparrow 1}\left|\int_{\rho}^{q}\phi'(re^{i\theta})d\left(re^{i\theta}\right)\right| \\ &\le \int_{\rho}^{1-}\frac{C}{(1-r)^{1-\alpha}}dr \\ &= \frac{C}{\alpha}(1-\rho)^{\alpha}.\end{aligned}$$

■

2.2 Isoperimetric inequality

Consider a metric space (X, d) and once again we denote the unit interval $[0,1]$ by I. In this section, we introduce a version of the isoperimetric inequality which will help proving further results in the rest of the chapter; for example, the inequality can assist us to estimate the sizes of two regions arisen in Lemma 2.30. We first recall the topological notion of winding number of a planar loop.

Definition 2.7 *A loop $\gamma_\cdot : I \to X$ is a continuous path in X such that $\gamma(0) = \gamma(1)$.*

Suppose $X = \mathbb{C}$. Given a loop $\gamma_\cdot : I \to \mathbb{C}$ in $\mathbb{C}$ and a point $z \in \mathbb{C}/\gamma(I)$, we set

$$g_z^{\gamma}(s) = \frac{\gamma_s - z}{||\gamma_s - z||}.$$

Then $g_z^{\gamma} : I \to \mathbb{S}^1$ is a loop in the unit circle $\mathbb{S}^1$. Let $p : \mathbb{R} \to \mathbb{S}^1$ with $p(\cdot) = e^{2\pi\cdot}$, the standard covering map of $\mathbb{S}^1$, one can show that there is a lifting $\tilde{g}_z^{\gamma} : I \to \mathbb{R}$ of g_z^{γ} from $\mathbb{S}^1$ to $\mathbb{R}$ such that

$$p \circ \tilde{g}_z^{\gamma} = g_z^{\gamma}.$$

Lemma 2.8 *For any loop* $\gamma_{\cdot} : I \to \mathbb{C}$ *and* $z \in \mathbb{C}/\gamma(I)$*, the difference* $\tilde{g}_z^{\gamma}(1) - \tilde{g}_z^{\gamma}(0)$ *is an integer.*

Proof. One can consult the book by Munkres [1999] for details. ■

Definition 2.9 *The winding number of* $\gamma_{\cdot} : I \to \mathbb{C}$ *with respect to* $z \notin \gamma(I)$*, denoted by* $\eta(\gamma_{\cdot}, z)$*, is defined to be the difference* $\tilde{g}_z^{\gamma}(1) - \tilde{g}_z^{\gamma}(0)$.

Proposition 2.10 *The winding number* $\eta(\gamma_{\cdot}, z)$ *is well-defined, i.e. its value is independent of the choice of the lifting of* g_z^{γ}*; in particular, if* $\tilde{g}_z^{\gamma}$ *is one of such liftings of* g_z^{γ}*, any other lifting of* g_{γ} *has the form* $\tilde{g}_z^{\gamma}(\cdot) + m$*, for some* $m \in \mathbb{Z}$*.*

Proof. *Again, see Munkres* [1999] *for details.* ■

It should be noted that the notion of winding number is well-defined even for loops of infinite variation. On the other hand, it is a fact that for any rectifiable loop $\gamma_{\cdot} : I \to \mathbb{C}$, $\gamma_{\cdot}^c$ is a countable union of connected open components; since $\eta(\gamma_{\cdot}, \cdot)$ is constant on each of these components, $\eta(\gamma_{\cdot}, \cdot)$ is therefore a measurable function. We next introduce the topological invariant nature of winding number of a loop $\gamma_{\cdot}$ around a point z with respect to any continuous deformation of $\gamma_{\cdot}$ without crossing z.

Definition 2.11 *Consider two loops* $\gamma_{\cdot}$ *and* $\gamma'_{\cdot}$ *in* X*. Suppose that there is a continuous map* $\Gamma : I \times I \to X$ *such that*

1. $\gamma_{\cdot} \equiv \Gamma(\cdot, 0)$ *and* $\gamma'_{\cdot} \equiv \Gamma(\cdot, 1)$.
2. *For any* $t \in I$, $\Gamma(0, t) = \Gamma(1, t)$.

Then we say that $\gamma_{\cdot}$ *is* $\Gamma-$ *freely homotopic to* $\gamma'_{\cdot}$.

Lemma 2.12 *Consider two loops* $\gamma_{\cdot}$ *and* $\gamma'_{\cdot}$ *in* $\mathbb{C}/\{z\}$*. If, for a continuous map* $\Gamma : I \times I \to \mathbb{C}$, $\gamma_{\cdot}$ *is* $\Gamma-$*freely homotopic to* $\gamma'_{\cdot}$ *such that* $\Gamma(I \times I) \subset \mathbb{C}/\{z\}$*, then* $\eta(\gamma_{\cdot}, z) = \eta(\gamma'_{\cdot}, z)$.

Proof. Define $G_z : I \times I \to \mathbb{S}^1$ by

$$G_z(s,t) = \frac{\Gamma(s,t) - z}{||\Gamma(s,t) - z||}$$

for $(s,t) \in I \times I$. Let $\tilde{G}_z$ be a lifting of G_z to $\mathbb{R}$. Then $\tilde{G}_z(1,t) - \tilde{G}_z(0,t)$ is an integer for every t. Since $\tilde{G}_z(1,\cdot) - \tilde{G}_z(0,\cdot)$ is continuous, hence the image of $\tilde{G}_z$ is connected because I is, and therefore it is a constant. ∎

Next, we will introduce a generalized version of the isoperimetric inequality - the so-called Pohl-Banchoff inequality, first proven by Pohl and Banchoff [1971/1972], and then Vogt [1981] provided a simpler proof for the planar case. In addition, we point out the relationships, which was first discovered by Rado [1936], between the Levy area for a rectifiable loop $\gamma.$ and the integral of winding numbers of $\gamma.$ over all points in $\mathbb{C}$. We here include an even simpler proof than that by Vogt [1981].

Proposition 2.13 *Denote $\lambda(\cdot)$ to be the two dimensional Lebesgue measure over $\mathbb{C}$. Let $\gamma. : I \longrightarrow \mathbb{C}$ be a rectifiable loop of length l. Then, one has*

1. *the Pohl-Banchoff inequality*

$$4\pi \int\int_{\mathbb{C}} \eta^2(\gamma.,\zeta)\,\lambda(d\underline{x}) \leq l^2, \tag{2.10}$$

where the equality holds if and only if $\gamma. = z_0 + R\,e^{2n\pi i\cdot}$ for some $n \in \mathbb{Z}$, $z_0 \in \mathbb{C}$ and $R > 0$.

2.

$$\int\int_{\mathbb{C}} \eta(\gamma.,z)\,\lambda(d\underline{x}) = \frac{1}{2}\left(\int_\gamma x_s dy_s - y_s dx_s\right). \tag{2.11}$$

To establish the claim, we first prove the result for simple polygonal loop and then for any polygonal loop. The most general result can be obtained by interpolating loop by a sequence of polygonal loops and by an application of Fatou's lemma.

Lemma 2.14 *Both the results* (2.10) *and* (2.11) *hold for any simple polygonal loop $\gamma.$.*

Proof. We prove the claim by induction on number of vertices ν of $\gamma.$. For $\nu = 3$, the polygonal loop $\gamma.$ is a triangle. Furthermore, if $\gamma.$ is non-degenerate, $\gamma.$ is clearly a Jordan loop and so $\eta(\gamma,\cdot)$ takes value 1 in the interior and 0 in the exterior. Applying the classical isoperimetric inequality (in do Carmo [1976]), both (2.10) and (2.11) follow.

Suppose that both (2.10) and (2.11) hold for all simple polygonal loops with $\nu \leq n$. Consider a polygonal loop $\gamma_. : I \to \mathbb{C}$ with $n+1$ vertices at times $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$. Again, by using the classical isoperimetric inequality (in do Carmo [1976]), result (2.10) follows immediately. We now add a chord $\gamma(t_0) \to \gamma(t_2)$, then γ is composed of two loops

$$\begin{aligned}
\gamma_1 &: \gamma(t_0) \to \gamma(t_1) \to \gamma(t_2) \to \gamma(t_0), \\
\gamma_2 &: \gamma(t_0) \to \gamma(t_2) \to \cdots \to \gamma(t_n) \to \gamma(t_{n+1}) = \gamma(t_0).
\end{aligned}$$

It is clear that $\eta(\gamma, \cdot) = \eta(\gamma_1, \cdot) + \eta(\gamma_2, \cdot)$ and γ_1, γ_2 are both polygonal loops with respective number of vertices $\leq n$. Using the induction hypothesis, we therefore have

$$\begin{aligned}
\int\int_{\mathbb{C}} \eta(\gamma_., z)\, \lambda(dA) &= \int\int_{\mathbb{C}} \eta(\gamma_1, z)\, \lambda(dA) + \int\int_{\mathbb{C}} \eta(\gamma_2, z)\, \lambda(dA) \\
&= \frac{1}{2}\left(\int_{\gamma_1} x_s dy_s - y_s dx_s\right) + \frac{1}{2}\left(\int_{\gamma_2} x_s dy_s - y_s dx_s\right) \\
&= \frac{1}{2}\left(\int_{\gamma} x_s dy_s - y_s dx_s\right).
\end{aligned}$$

■

Lemma 2.15 *Both the results* (2.10) *and* (2.11) *also hold for any polygonal loop* $\gamma_.$.

Proof. We again prove the claim by induction on number of vertices ν of $\gamma_.$. Suppose that the result is true for $v \leq n$ and $\gamma : I \to \mathbb{C}$ is a non-simple loop with $n+1$ vertices. Firstly, there must be two times $s, s' \in I$ such that

$$t_{j_1} \leq s < t_{j_1+1} < \cdots < t_{j_2} \leq s' < t_{j_2+1}$$

for some $j_1 < j_2$ and

$$\gamma(s) = \gamma(s').$$

$\gamma_.$ can now be decomposed into two polygonal loops

$$\begin{aligned}
\gamma_1 &: \gamma(t_0) \to \cdots \to \gamma(t_{j_1}) \to \gamma(s) \\
&= \gamma(s') \to \gamma(t_{j_2+1}) \to \cdots \to \gamma(t_{n+1}) = \gamma(t_0), \\
\gamma_2 &: \gamma(s) \to \gamma(t_{j_1+1}) \to \cdots \to \gamma(t_{j_2}) \to \gamma(s') = \gamma(s).
\end{aligned}$$

1. If there is exactly one t_j in between s and s', i.e. $t_{j_1+1} = t_{j_2}$. Both $\gamma([t_{j_1}, t_{j_1+1}])$ and $\gamma([t_{j_1+1}, t_{j_1+2}])$ are straight lines, and we have either

$$\gamma([t_{j_1}, t_{j_1+1}]) \subseteq \gamma([t_{j_1+1}, t_{j_1+2}]) \text{ and } \gamma(t_{j_1}) = \gamma(s) = \gamma(s')$$

or

$$\gamma([t_{j_1}, t_{j_1+1}]) \supseteq \gamma([t_{j_1+1}, t_{j_1+2}]) \text{ and } \gamma(s) = \gamma(s') = \gamma(t_{j_1+2}).$$

In the former case, γ_1 is a loop of only n vertices while γ_2 is just a straight line, hence $\eta(\gamma, \cdot) = \eta(\gamma_1', \cdot)$ almost everywhere in $\mathbb{C}$. Now,

$$4\pi \int\int_{\mathbb{C}} \eta^2(\gamma., \zeta)\, \lambda(dA) = 4\pi \int\int_{\mathbb{C}} \eta^2(\gamma_1, \zeta)\, \lambda(dA) \leq l(\gamma_1)^2 \leq l(\gamma)^2$$

and

$$\begin{aligned}
\int\int_{\mathbb{C}} \eta(\gamma., z)\, \lambda(dA) &= \int\int_{\mathbb{C}} \eta(\gamma_1, z)\, \lambda(dA) + \int\int_{\mathbb{C}} \eta(\gamma_2, z)\, \lambda(dA) \\
&= \frac{1}{2}\left(\int_{\gamma_1} x_s dy_s - y_s dx_s\right) + 0 \\
&= \frac{1}{2}\left(\int_{\gamma_1} x_s dy_s - y_s dx_s\right) + \frac{1}{2}\left(\int_{\gamma_2} x_s dy_s - y_s dx_s\right) \\
&= \frac{1}{2}\left(\int_{\gamma} x_s dy_s - y_s dx_s\right).
\end{aligned}$$

The second case can be dealt with similarly.

2. If there is more than one but $\leq n-1$ t_j in between s and s', both γ_1 and γ_2 have their number of vertices $\leq n$. Again $\eta(\gamma, \cdot) = \eta(\gamma_1, \cdot) + \eta(\gamma_2, \cdot)$, according to the induction hypothesis, we also have

$$\begin{aligned}
\|\eta(\gamma, \cdot)\|_2 &\triangleq \left(\int\int_{\mathbb{C}} \eta^2(\gamma., \zeta)\, \lambda(dA)\right)^{\frac{1}{2}} \\
&\leq \|\eta(\gamma_1, \cdot)\|_2 + \|\eta(\gamma_2, \cdot)\|_2 \\
&\leq \frac{l(\gamma_1)}{\sqrt{4\pi}} + \frac{l(\gamma_2)}{\sqrt{4\pi}} = \frac{l(\gamma)}{\sqrt{4\pi}}
\end{aligned}$$

and

$$\begin{aligned}\int\int_{\mathbb{C}} \eta(\gamma_{\cdot}, z)\lambda(dA) &= \int\int_{\mathbb{C}} \eta(\gamma_1, z)\lambda(dA) + \int\int_{\mathbb{C}} \eta(\gamma_2, z)\lambda(dA) \\ &= \frac{1}{2}\left(\int_{\gamma_1} x_s dy_s - y_s dx_s\right) + \frac{1}{2}\left(\int_{\gamma_2} x_s dy_s - y_s dx_s\right) \\ &= \frac{1}{2}\left(\int_{\gamma} x_s dy_s - y_s dx_s\right).\end{aligned}$$

3. If there are exactly n t_j in between s and s', the same line of proof as in the case (1) can be adopted, with the roles of γ_1 and γ_2 interchanged, to conclude our induction step.

■

Consider a family of partitions

$$\mathcal{D}^{(m)} \triangleq \left\{0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = 1\right\}$$

of I for $m \in \mathbb{Z}^+$ such that (1) $\mathcal{D}^{(m)} \subset \mathcal{D}^{(m+1)}$ and (2) their mesh sizes tend to zero. For $m \in \mathbb{Z}^+$ and any rectifiable loop γ, let $\gamma_{\cdot}^{(m)}$ be a polygonal loop with vertices $\{\gamma(t)\}_{t\in\mathcal{D}^{(m)}}$. It is clear that because $\gamma_{\cdot}$ is continuous with compact image, the sequence $\left\{\gamma_{\cdot}^{(m)}\right\}$ converges uniformly to $\gamma_{\cdot}$.

Lemma 2.16 *Let $\Omega \triangleq \mathbb{C}/\gamma(I) \cup_{m=1}^{\infty} \gamma^{(m)}(I)$. For any $z \in \Omega$,*

$$\eta(\gamma, z) = \lim_{m\to\infty} \eta\left(\gamma^{(m)}, z\right).$$

Proof. For any $z_0 \in \Omega$ with $d(z_0, \gamma(I)) > \epsilon > 0$, all but except finitely many $\gamma^{(m)}(I)$ lie inside the $\epsilon/2-$ neighborhood of $\gamma(I)$ which excludes z_0. For all large enough m, $\gamma_{\cdot}^{(m)}$ is freely homotopic to $\gamma_{\cdot}$, by Lemma 2.12, we therefore have

$$\eta\left(\gamma^{(m)}, z_0\right) = \eta(\gamma, z_0).$$

Since z_0 is arbitrary, we conclude that for all $z \in \Omega$,

$$\eta(\gamma, z) = \lim_{m\to\infty} \eta\left(\gamma^{(m)}, z\right).$$

■

Lemma 2.17 *$\eta(\gamma,\cdot)$ is a Lebesgue measurable function.*

Proof. All $\gamma(I)$ and $\gamma^{(m)}(I)$ are rectifiable, so

$$\lambda(\gamma(I)) = \lambda\left(\gamma^{(m)}(I)\right) = \lambda\left(\gamma(I)\cup_{m=1}^{\infty}\gamma^{(m)}(I)\right) = 0,$$

and hence $\lambda(\Omega^c) = 0$. As a consequence, being an almost everywhere pointwise limit of $\eta\left(\gamma^{(m)},\cdot\right)$, $\eta(\gamma,\cdot)$ is a measurable function. ■

Proof of Proposition 2.13. Using Lemma 2.15, for each m

$$4\pi\int\int_{\mathbb{C}}\eta^2\left(\gamma^{(m)},\zeta\right)\lambda(dA) \le l\left(\gamma^{(m)}\right)^2.$$

Applying Fatou's lemma, we immediately have

$$\begin{aligned}4\pi\int\int_{\mathbb{C}}\eta^2(\gamma,\zeta)\lambda(dA) &= 4\pi\int\int_{\mathbb{C}}\liminf_{m\to\infty}\eta^2\left(\gamma^{(m)},\zeta\right)\lambda(dA)\\ &\le 4\pi\liminf_{m\to\infty}\int\int_{\mathbb{C}}\eta^2\left(\gamma^{(m)},\zeta\right)\lambda(dA)\\ &\le \liminf_{m\to\infty} l\left(\gamma^{(m)}\right)^2 = l(\gamma)^2.\end{aligned}$$

Since all γ and $\gamma^{(m)}$ lie in a compact set, therefore we can apply L^2- convergence theorem to conclude that

$$\begin{aligned}\int\int_{\mathbb{C}}\eta(\gamma.,z)\lambda(dA) &= \lim_{m\to\infty}\int\int_{\mathbb{C}}\eta\left(\gamma^{(m)},z\right)\lambda(dA)\\ &= \frac{1}{2}\left(\lim_{m\to\infty}\int_{\gamma^{(m)}}x_s dy_s - y_s dx_s\right)\\ &= \frac{1}{2}\left(\int_{\gamma}x_s dy_s - y_s dx_s\right),\end{aligned}$$

where the last equality follows from the standard definition of Riemann's integrals. ■

Let $\frac{1}{3} < \alpha \le 1$ and for any $T > 0$

$$\Delta_T \triangleq \{(\theta,\varphi) : 0 \le \theta \le \varphi \le T\}.$$

In the remaining sections, for any $\varepsilon > 0$, we aim to show that $\phi_r\left(e^{i\cdot}\right)$ and its associated Levy area process converge in $\frac{1}{\alpha}+\varepsilon-$ variation topology to $\phi\left(e^{i\cdot}\right)$ with its geometric Levy area process (see Definition 2.20) as $r \to 1$ over $\mathbb{Q}$. For the case $\alpha = 1$, the claim is trivial; from now on, we assume $\frac{1}{3} < \alpha < 1$.

2.3 Convergence for the increment processes

In this section, we aim to establish the convergence of $\phi_r\left(e^{i\cdot}\right)$ to $\phi\left(e^{i\cdot}\right)$ in $p\left(>\frac{1}{\alpha}\right)$ – variation topology at level one.

Lemma 2.18 *For any $0 \leq \rho \leq 1$, $\varepsilon > 0$ and $(\theta, \varphi) \in \Delta_{2\pi}$, we have*

$$\begin{aligned} &\left|\phi_\rho\left(e^{i\theta}\right) - \phi_\rho\left(e^{i\varphi}\right) - \left(\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right)\right|^{\frac{1}{\alpha}+\epsilon} \\ \leq\; & 2^{\left(\frac{1}{\alpha}-1\right)+\epsilon} \frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon}\left(1+\rho^{\frac{1}{\alpha}}\right)\left((2\pi)^{1-\alpha}+\frac{2}{\alpha}\right)^{\frac{1}{\alpha}} \cdot (1-\rho)^{\alpha\epsilon} \cdot |\varphi - \theta| . \end{aligned}$$

Proof. For any $\beta, \delta > 0$ and $\eta = \beta + \delta$, the modulus of the difference

$$\begin{aligned} &\left|\phi_\rho\left(e^{i\theta}\right) - \phi_\rho\left(e^{i\varphi}\right) - \left(\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right)\right|^{\eta} \\ \leq\; & 2^{\beta+\delta-2}\left(\left|\phi\left(\rho e^{i\theta}\right) - \phi\left(e^{i\theta}\right)\right|^{\beta} + \left|\phi\left(\rho e^{i\varphi}\right) - \phi\left(e^{i\varphi}\right)\right|^{\beta}\right) \\ & \cdot\left(\left|\phi\left(\rho e^{i\theta}\right) - \phi\left(\rho e^{i\varphi}\right)\right|^{\delta} + \left|\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right|^{\delta}\right). \end{aligned}$$

Using inequalities (2.6) and (2.9), we obtain

$$\begin{aligned} &\left|\phi_\rho\left(e^{i\theta}\right) - \phi_\rho\left(e^{i\varphi}\right) - \left(\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right)\right|^{\eta} \\ \leq\; & 2^{\eta-1}\frac{C^\eta}{\alpha^\beta}\left(1+\rho^\delta\right)\left((2\pi)^{1-\alpha}+\frac{2}{\alpha}\right)^{\delta} \cdot (1-\rho)^{\alpha\beta} \cdot |\theta-\varphi|^{\alpha\delta} . \end{aligned}$$

Our desired inequality follows if we choose $\beta = \epsilon$ and $\delta = \frac{1}{\alpha}$. ■

Corollary 2.19 *For any partition $\mathcal{D} = \{0 = \theta_0 < \cdots < \theta_n = 2\pi\}$ of $[0, 2\pi]$, the partial sum*

$$\begin{aligned} &\sum_{i=0}^{n-1}\left|\phi_\rho\left(e^{i\theta_i}\right) - \phi_\rho\left(e^{i\theta_{i+1}}\right) - \left(\phi\left(e^{i\theta_i}\right) - \phi\left(e^{i\theta_{i+1}}\right)\right)\right|^{\frac{1}{\alpha}+\epsilon} \\ \leq\; & 2^{\frac{1}{\alpha}+\epsilon}\pi\frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon}\left(1+\rho^{\frac{1}{\alpha}}\right)\left((2\pi)^{1-\alpha}+\frac{2}{\alpha}\right)^{\frac{1}{\alpha}}(1-\rho)^{\alpha\epsilon} . \end{aligned}$$

Proof. The result is an immediate consequence of Lemma 2.18:

$$\begin{aligned}
&\sum_{i=0}^{n-1} \left|\phi_\rho\left(e^{i\theta_i}\right) - \phi_\rho\left(e^{i\theta_{i+1}}\right) - \left(\phi\left(e^{i\theta_i}\right) - \phi\left(e^{i\theta_{i+1}}\right)\right)\right|^{\frac{1}{\alpha}+\epsilon} \\
\leq\ & 2^{\left(\frac{1}{\alpha}-1\right)+\epsilon} \frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon} \left(1+\rho^{\frac{1}{\alpha}}\right) \left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)^{\frac{1}{\alpha}} (1-\rho)^{\alpha\epsilon} \cdot \sum_{i=0}^{n-1} |\theta_{i+1} - \theta_i| \\
=\ & 2^{\frac{1}{\alpha}+\epsilon} \pi \frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon} \left(1+\rho^{\frac{1}{\alpha}}\right) \left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)^{\frac{1}{\alpha}} (1-\rho)^{\alpha\epsilon}.
\end{aligned}$$

■

Therefore,

$$\begin{aligned}
&\sup_{\mathcal{D}} \sum_{i=0}^{n-1} \left|\phi_\rho\left(e^{i\theta_i}\right) - \phi_\rho\left(e^{i\theta_{i+1}}\right) - \left(\phi\left(e^{i\theta_i}\right) - \phi\left(e^{i\theta_{i+1}}\right)\right)\right|^{\frac{1}{\alpha}+\epsilon} \\
\leq\ & 2^{\frac{1}{\alpha}+\epsilon} \pi \frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon} \left(1+\rho^{\frac{1}{\alpha}}\right) \left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)^{\frac{1}{\alpha}} (1-\rho)^{\alpha\epsilon},
\end{aligned}$$

which converges to zero as ρ tends to 1.

2.4 Convergence for the Levy area processes

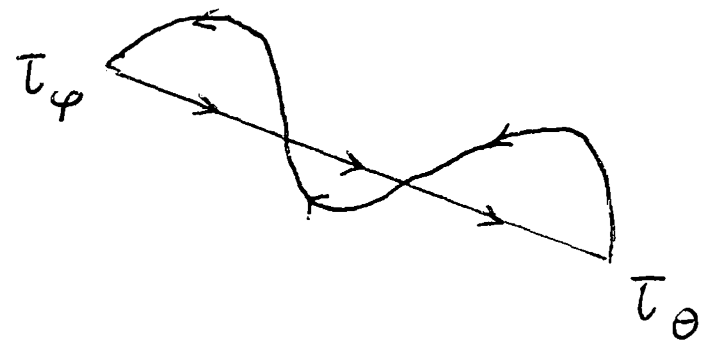
In this section, we aim to establish the convergence of $\phi_r\left(e^{i\cdot}\right)$ to $\phi\left(e^{i\cdot}\right)$ in $p\left(> \frac{1}{\alpha}\right)-$ variation topology at level two. That is to say, we want to show that the sequence of Levy area processes of $\phi_r\left(e^{i\cdot}\right)$ converges as $r \to 1$ over $\mathbb{Q}$ in $\frac{1}{\alpha} + \varepsilon-$ variation topology to the geometric Levy area process (see Definition 2.20) of $\phi\left(e^{i\cdot}\right)$.

2.4.1 Preliminaries

For simplicity of notation, for $0 \leq r \leq 1$, we denote $\phi_r\left(e^{i\cdot}\right)$ and $\phi\left(e^{i\cdot}\right)$ by $\gamma_\cdot^r$ and $\gamma_\cdot$ respectively; in particular, $\gamma_\cdot^1 \equiv \gamma_\cdot$.

Given a loop $\tau_\cdot : I \to \mathbb{C}$ and $(\theta, \varphi) \in \Delta_{2\pi}$, we denote the directed arc of $\tau_\cdot$ from θ to φ by $\tau_{\theta,\varphi}$ and the directed loop starting from the point τ_θ to τ_φ along the arc $\tau_{\theta,\varphi}$ and concatenating with the directed chord from τ_φ and τ_θ

by $\underleftarrow{\tau_{\theta,\varphi}}$.



Definition 2.20 *Given a loop $\tau : I \to \mathbb{C}$ such that $\lambda(\tau(I)) = 0$. For any $(\theta, \varphi) \in \Delta_{2\pi}$, we call the signed area, i.e. the signed two dimensional Lebesgue measure, of the region bounded by the directed loop $\underleftarrow{\tau_{\theta,\varphi}}$ (being positive when the loop is oriented anticlockwise), the geometric Levy area of τ. over $[\theta, \varphi]$.*

Lemma 2.21 *The loop $\gamma(I)$ has Lebesgue measure zero.*

Proof. In their paper, Jones and Markarov [1995] proved that the Hausdorff dimension (or the Minkowski dimension) of the boundary of the image D of the unit disc $\mathbb{D}$ under a univalent function ϕ, which is also α-Hölder continuous on $\mathbb{S}^1$, does not exceed $2 - C\alpha$, for a universal constant C. For each $\epsilon > 0$, let the $2 - C\alpha + \varepsilon$ - Hausdorff measure $H^{2-C\alpha+\epsilon}(\gamma(I))$ of $\gamma(I)$ be M_ϵ. Then there are

1. a decreasing sequence $\{\delta_n\}_{n\in\mathbb{Z}^+}$ with $\delta_n \to 0$,
2. for each $n \in \mathbb{Z}^+$, a family of open-balls $\{U_{i,n}\}_{i\in\mathbb{Z}^+}$ covering $\gamma(I)$

such that $diam(U_{\cdot,n}) \le \delta_n$ and $\sum_{i=1}^{\infty} diam(U_{i,n})^{2-C\alpha+\epsilon} \le M_\epsilon + \epsilon$. By definition, we have

$$\begin{aligned}
\lambda(\gamma(I)) &\le \sum_{i=1}^{\infty} \lambda(U_{i,n}) \le \frac{\pi}{4} \sum_{i=1}^{\infty} diam(U_{i,n})^2 \\
&\le \frac{\pi}{4} \delta_n^{C\alpha-\epsilon} \sum_{i=1}^{\infty} diam(U_{i,n})^{2-C\alpha+\epsilon} \\
&\le \frac{\pi}{4} (M_\epsilon + \epsilon) \delta_n^{C\alpha-\epsilon}.
\end{aligned}$$

Choosing $\epsilon < C\alpha$ and passing n to ∞, one deduces that $\lambda(\gamma(I)) = 0$. ∎

Corollary 2.22 *For any $0 \leq r \leq 1$ and $(\theta, \varphi) \in \Delta_{2\pi}$, define*

$$\Omega_{\theta,\varphi} \triangleq \mathbb{C} / \left(\cup_{r \in \mathbb{Q} \cap [0,1]} \underleftarrow{\gamma^r_{\theta,\varphi}} \right).$$

Then the Lebesgue measure $\lambda \left(\Omega^c_{\theta,\varphi} \right) = 0$.

Proof. For $0 \leq r < 1$, the loop $\gamma^r_\cdot$ is rectifiable and therefore its Lebesgue measure is zero. Using Lemma 2.21, we conclude our result. ■

Lemma 2.23 *For any $(\theta, \varphi) \in \Delta_{2\pi}$ and $z \in \Omega_{\theta,\varphi}$, we have*

$$\eta \left(\underleftarrow{\gamma_{\theta,\varphi}}, z \right) = \lim_{r \in \mathbb{Q},\, r \to 1} \eta \left(\underleftarrow{\gamma^r_{\theta,\varphi}}, z \right). \tag{2.12}$$

Proof. Suppose the distance $d \left(z, \underleftarrow{\gamma_{\theta,\varphi}} \right) > \epsilon > 0$. Since ϕ is continuous in $\mathbb{D}$ and $\gamma_{\theta,\varphi}$ is compact, $\left\{ \underleftarrow{\gamma^r_{\theta,\varphi}} \right\}$ uniformly converges to $\underleftarrow{\gamma_{\theta,\varphi}}$; therefore for all rational r sufficiently close to 1, we have $\underleftarrow{\gamma^r_{\theta,\varphi}}$ lying inside the $\epsilon/2-$neighborhood of $\underleftarrow{\gamma_{\theta,\varphi}}$ which excludes z. Hence, for all large enough r, $\underleftarrow{\gamma^r_{\theta,\varphi}}$ is freely homotopic to $\underleftarrow{\gamma_{\theta,\varphi}}$; using Lemma 2.12, we have $\eta \left(\underleftarrow{\gamma^r_{\theta,\varphi}}, z \right) = \eta \left(\underleftarrow{\gamma_{\theta,\varphi}}, z \right)$. ■

Corollary 2.24 *For every $(\theta, \varphi) \in \Delta_{2\pi}$, $\eta \left(\underleftarrow{\gamma_{\theta,\varphi}}, \cdot \right)$ is a Lebesgue measurable function on $\mathbb{C}$.*

Proof. $\eta \left(\underleftarrow{\gamma_{\theta,\varphi}}, \cdot \right)$ is an almost everywhere pointwise limit of $\eta \left(\underleftarrow{\gamma^r_{\theta,\varphi}}, \cdot \right)$ as $r \to 1$ over $\mathbb{Q}$. ■

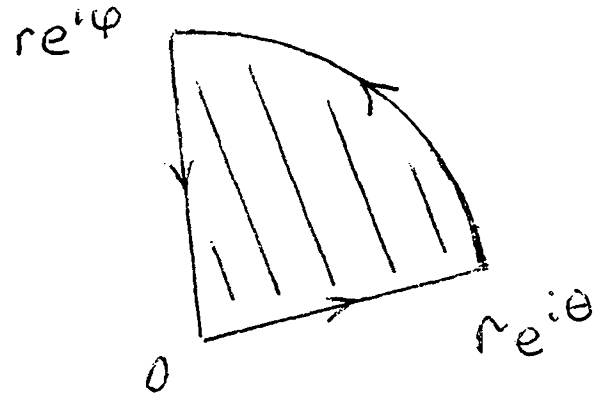
2.4.2 Pointwise convergence of $A^r_{\theta,\varphi}$

In this subsection, we establish the sequence of Levy area processes of $\phi_r \left(e^{i\cdot} \right)$ converges as $r \to 1$ over $\mathbb{Q}$ in pointwise sense to the geometric Levy area process of $\phi \left(e^{i\cdot} \right)$. We first introduce a few notations for various regions arisen in Lemma 2.30.

Definition 2.25 *Let $0 \le r \le R \le 1$ and $(\theta, \varphi) \in \Delta_{2\pi}$, we denote $S^r_{\theta,\varphi}$ to be the positive sector*

$$\left\{ z \in \mathbb{C} : z = \rho e^{ix},\ \text{where}\ 0 \le \rho \le r\ \text{and}\ \theta \le x \le \varphi \right\}$$

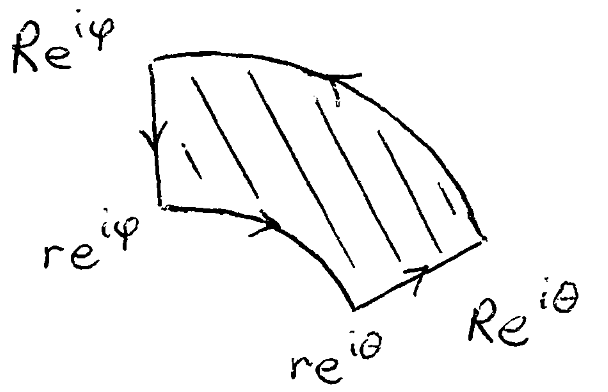
with its boundary having anticlockwise orientation.



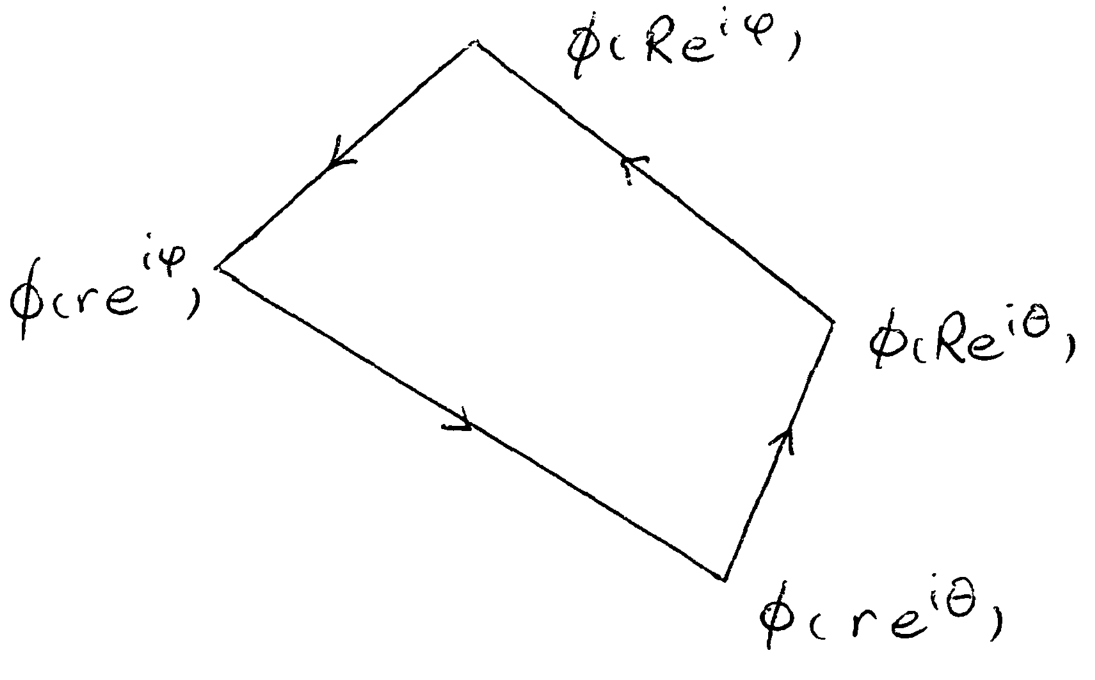
Definition 2.26 *Define*

$$D^{r,R}_{\theta,\varphi} \triangleq S^R_{\theta,\varphi} / S^r_{\theta,\varphi}$$

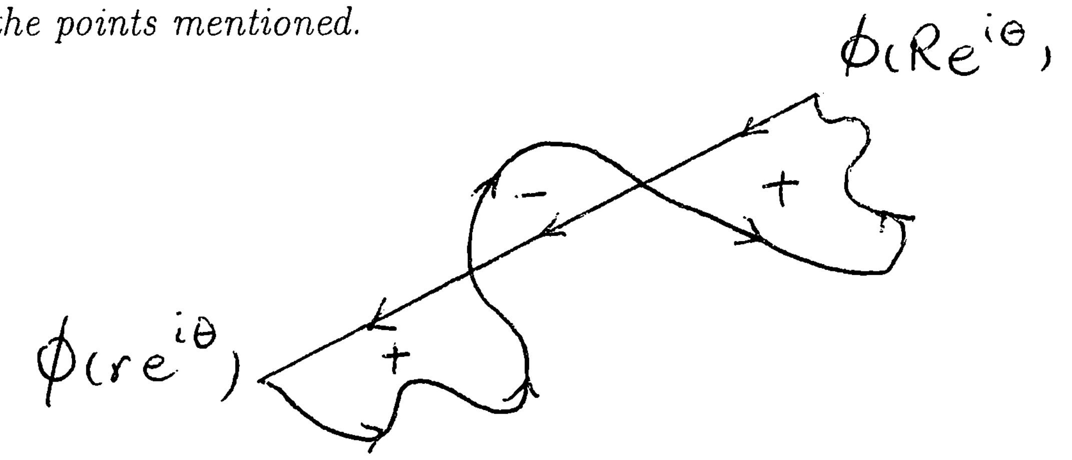
which has a sign induced by those of $S^r_{\theta,\varphi}$ and $S^R_{\theta,\varphi}$.



Definition 2.27 *Let $0 \le r \le R \le 1$ and $(\theta, \varphi) \in \Delta_{2\pi}$, define $Q^{r,R}_{\theta,\varphi}$ to be the signed quadrilateral with vertices $\phi\left(re^{i\theta}\right)$, $\phi\left(Re^{i\theta}\right)$, $\phi\left(Re^{i\varphi}\right)$ and $\phi\left(re^{i\varphi}\right)$ such that its orientation is taken in the same order.*



Definition 2.28 *Let $0 \leq r \leq R \leq 1$ and $(\theta, \varphi) \in \Delta_{2\pi}$, define $W_{\theta}^{r,R}$ to be the signed region bounded by the directed loop starting from $\phi\left(re^{i\theta}\right)$ running outward to $\phi\left(Re^{i\theta}\right)$ along the arc $\phi\left(\cdot e^{i\theta}\right)$ and concatenating with the directed chord from $\phi\left(Re^{i\theta}\right)$ to $\phi\left(re^{i\theta}\right)$ such that the orientation is taken in the same order of the points mentioned.*



For any signed region $A \subset \mathbb{C}$, we use ∂A to denote the oriented boundary of A.

Remark 2.29 *Consider an arbitrary simply connected signed region $A \subset \mathbb{D}$ with oriented Jordan boundary ∂A. The image $\phi(A)$ has the boundary*

$$\partial\phi(A) = \phi(\partial A)$$

which is oriented in a way which is also consistent with ∂A.

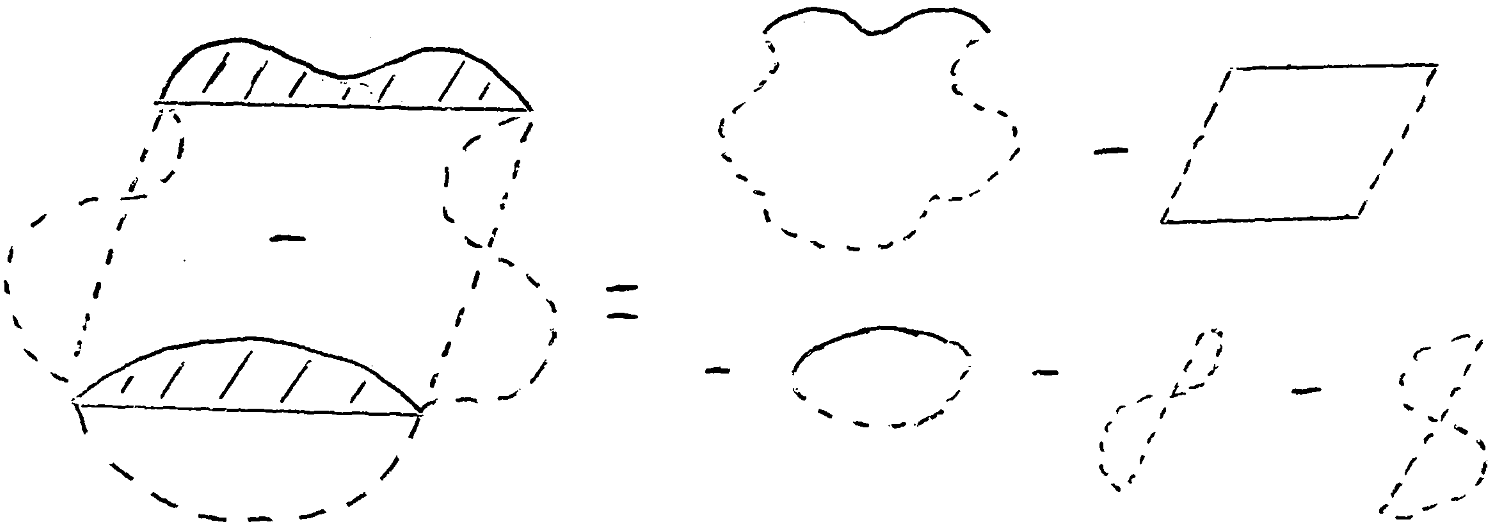
For each $r \in [0,1)$, define the Levy area of γ^r over $[\theta, \varphi]$ as

$$A_{\theta,\varphi}^{r} \triangleq \frac{1}{2}\int_{\theta<u_1<u_2<\varphi} d\left(\gamma^r\right)_{u_1}^{1} d\left(\gamma^r\right)_{u_2}^{2} - d\left(\gamma^r\right)_{u_1}^{2} d\left(\gamma^r\right)_{u_2}^{1}.$$

We now investigate the difference of Levy areas $A_{\theta,\varphi}^{r_2} - A_{\theta,\varphi}^{r_1}$ for any $r_1, r_2 \in \mathbb{Q} \cap [0,1]$ with $r_1 \leq r_2$; namely, we want to express the difference in terms of the areas of the regions listed in Definitions 2.25 to 2.28.

Lemma 2.30 *Let $0 < \rho < r_1$. We have a decomposition of the difference of Levy areas $A_{\theta,\varphi}^{r_2} - A_{\theta,\varphi}^{r_1}$ as:*

$$\begin{aligned} 2\left(A_{\theta,\varphi}^{r_2} - A_{\theta,\varphi}^{r_1}\right) &= \left(\int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_2}\right)} - \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right)} - \int_{\partial Q_{\theta,\varphi}^{r_1,r_2}}\right)\left(x_s dy_s - y_s dx_s\right) \\ &\quad - \left(\int_{\partial W_{\theta}^{r_1,r_2}} + \int_{\partial W_{\varphi}^{r_1,r_2}}\right)\left(x_s dy_s - y_s dx_s\right). \end{aligned}$$



Proof. In accordance with the definitions of the regions involved, the identity is clear by studying the corresponding geometric configuration. ■

We now estimate the sizes of various regions arisen in Lemma 2.30 by applying the Pohl-Banchoff inequality (Proposition 2.13) and consequences of Hardy-Littlewood lemma (Lemma 2.3) as listed in Section 2.1.

Lemma 2.31 *Let* $0 \leq r_1 \leq r_2 \leq 1$ *and* $(\theta, \varphi) \in \Delta_{2\pi}$,

$$\left| \int_{\partial Q_{\theta,\varphi}^{r_1,r_2}} x_s dy_s - y_s dx_s \right| \leq \frac{1}{2} \frac{C^2}{\alpha} \left((2\pi)^{1-\alpha} + \frac{2}{\alpha} \right) (r_1 + r_2)(r_2 - r_1)^{\alpha} |\varphi - \theta|^{\alpha}.$$

Proof. Decompose the region $Q_{\theta,\varphi}^{r_1,r_2}$ into two oriented triangles $(T_1)_{\theta,\varphi}^{r_1,r_2}$ and $(T_2)_{\theta,\varphi}^{r_1,r_2}$ with respective vertices

$$\left\{ \phi\left(r_1 e^{i\theta}\right), \phi\left(r_1 e^{i\varphi}\right), \phi\left(r_2 e^{i\varphi}\right) \right\}$$

and

$$\left\{ \phi\left(r_1 e^{i\theta}\right), \phi\left(r_2 e^{i\theta}\right), \phi\left(r_2 e^{i\varphi}\right) \right\}.$$

Using (2.8) and (2.9) to bound the sides of triangles, we have

$$\begin{aligned} & \left| \int_{\partial Q_{\theta,\varphi}^{r_1,r_2}} x_s dy_s - y_s dx_s \right| \\ \leq & \left| \int_{\partial (T_1)_{\theta,\varphi}^{r_1,r_2}} x_s dy_s - y_s dx_s \right| + \left| \int_{\partial (T_1)_{\theta,\varphi}^{r_1,r_2}} x_s dy_s - y_s dx_s \right| \\ \leq & \frac{1}{2} \frac{C^2}{\alpha} \left((2\pi)^{1-\alpha} + \frac{2}{\alpha} \right) (r_1 + r_2)(r_2 - r_1)^{\alpha} |\varphi - \theta|^{\alpha}. \end{aligned}$$

■

Lemma 2.32 *Let* $0 \leq r_1 \leq r_2 \leq 1$ *and* $\theta \in [0, 2\pi]$,

$$\left| \int_{\partial W_\theta^{r_1, r_2}} x_s dy_s - y_s dx_s \right| \leq \frac{1}{\sqrt{\pi}} \frac{C}{\alpha} \sqrt{\lambda(\phi(\mathbb{D}))} (r_2 - r_1)^\alpha .$$

Proof. Note that, using (2.7), the length

$$\begin{aligned} l\left(\phi\left(\cdot e^{i\theta}\right)\big|_{[r_1, r_2]}\right) &= \int_{\frac{r_1}{r_2}}^1 \left| d\phi_{r_2}\left(\rho e^{i\theta}\right) \right| \\ &\leq \int_{\frac{r_1}{r_2}}^1 \frac{C r_2}{(1-\rho)^{1-\alpha}} d\rho \\ &\leq \frac{C}{\alpha} (r_2 - r_1)^\alpha . \end{aligned}$$

Using the Pohl-Banchoff inequality in Proposition 2.13, the fact that the image $\phi(\mathbb{D})$ is compact and Lemma 2.6, we have

$$\begin{aligned} \left| \int_{\partial W_\theta^{r_1, r_2}} x_s dy_s - y_s dx_s \right| &\leq \int \int_{\mathbb{C}} \left| \eta\left(\partial W_\theta^{r_1, r_2}, z\right) \right| \lambda(dA) \\ &\leq \left\| \eta\left(\partial W_\theta^{r_1, r_2}, \cdot\right) \right\|_2 \cdot \sqrt{\lambda(\phi(\mathbb{D}))} \\ &\leq \frac{1}{\sqrt{4\pi}} l\left(\partial W_\theta^{r_1, r_2}\right) \cdot \sqrt{\lambda(\phi(\mathbb{D}))} \\ &\leq \frac{2}{\sqrt{4\pi}} \frac{C}{\alpha} (r_2 - r_1)^\alpha \cdot \sqrt{\lambda(\phi(\mathbb{D}))}. \end{aligned}$$

■

Corollary 2.33 *Let* $0 \leq r_1 \leq r_2 \leq 1$ *and* $(\theta, \varphi) \in \Delta_{2\pi}$,

$$\begin{aligned} & 2\left| A_{\theta,\varphi}^{r_2} - A_{\theta,\varphi}^{r_1} \right| \\ \leq & \left| \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho, r_2}\right)} - \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho, r_1}\right)} (x_s dy_s - y_s dx_s) \right| \\ & + \left(\frac{C^2}{\alpha} \left(2\pi + \frac{2}{\alpha} (2\pi)^\alpha \right) + \frac{2}{\sqrt{\pi}} \frac{C}{\alpha} \sqrt{\lambda(\phi(\mathbb{D}))} \right) (r_2 - r_1)^\alpha . \end{aligned}$$

Proof. Combining the inequalities in Lemmas 2.30 and 2.31 with the identity in Lemma 2.29, we then obtain our claim. ■

Lemma 2.34 *Let* $0 \leq \rho \leq 1$ *and* $(\theta, \varphi) \in \Delta_{2\pi}$*,*

$$\int\int_{\mathbb{C}} \eta\left(\partial\phi\left(D^{\rho,1}_{\theta,\varphi}\right), z\right) \lambda(dA) = \lim_{r\in\mathbb{Q},\, r\to 1} \int\int_{\mathbb{C}} \eta\left(\partial\phi\left(D^{\rho,r}_{\theta,\varphi}\right), z\right) \lambda(dA).$$

Proof. Note that ϕ is conformal over the interior of $\mathbb{D}$, i.e. $\forall z \in \mathbb{D}^o$,

$$\phi'(z) \neq 0.$$

Applying the mean value theorem to the smooth function $|\phi|^2$, we see that ϕ is injective over $\mathbb{D}^o$. As a consequence, for any $0 \leq r < R \leq 1$, $\phi\left(D^{r,R}_{\theta,\varphi}\right)$ is a Jordan domain. Hence, we have

$$\eta\left(\partial\phi\left(D^{r,R}_{\theta,\varphi}\right), z\right) = \begin{cases} 1, & \text{if } z \in \phi\left(D^{r,R}_{\theta,\varphi}\right)^o, \\ 0, & \text{if } z \in \phi\left(D^{r,R}_{\theta,\varphi}\right)^c, \end{cases}$$

and so

$$\left|\eta\left(\partial\phi\left(D^{r,R}_{\theta,\varphi}\right), \cdot\right)\right| \leq 1.$$

Moreover, the image $\phi(\mathbb{D})$ is compact, and therefore

$$\int\int_{\mathbb{C}} \eta\left(\partial\phi\left(D^{r,1}_{\theta,\varphi}\right), z\right) \lambda(dA)$$

is well-defined and uniformly bounded for all r. Because $\partial\phi\left(D^{\rho,r}_{\theta,\varphi}\right)$ uniformly converges to $\partial\phi\left(D^{\rho,1}_{\theta,\varphi}\right)$, using similar arguments as in Lemma 2.23, we have, $\forall z \in \mathbb{C}$,

$$\eta\left(\partial\phi\left(D^{\rho,1}_{\theta,\varphi}\right), z\right) = \lim_{r\in\mathbb{Q},\, r\to 1} \eta\left(\partial\phi\left(D^{\rho,r}_{\theta,\varphi}\right), z\right).$$

Now our claim is an immediate consequence of the classical bounded convergence theorem. ■

We are now ready to establish the pointwise convergence of Levy area processes.

Proposition 2.35 *Let* $(\theta, \varphi) \in \Delta_{2\pi}$ *and* $r \in \mathbb{Q} \cap [0,1]$*,* $A^r_{\theta,\varphi}$ *converges to a limit as* r *tends to* 1.

Proof. For any $\varepsilon > 0$, there is $\delta > 0$, such that whenever $1-\delta < r_1 < r_2 \le 1$, we have

$$\left|\int\int_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right), z\right)\lambda(dA) - \int\int_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,r_2}\right), z\right)\lambda(dA)\right| < \varepsilon.$$

Using Proposition 2.13, we obtain

$$\begin{aligned} & \left|\int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_2}\right)} - \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right)} (x_s dy_s - y_s dx_s)\right| \\ = & \left|\int\int_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,r_2}\right), z\right) - \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right), z\right)\lambda(dA)\right| \\ < & \ \varepsilon. \end{aligned}$$

In accordance with Corollary 2.32, we deduce our claim. ■

Definition 2.36 *For any $(\theta,\varphi) \in \Delta_{2\pi}$, define*

$$A_{\theta,\varphi} \triangleq \lim_{r\in\mathbb{Q},\ r\to 1} A_{\theta,\varphi}^{r}.$$

Proposition 2.37 *$A_{\theta,\varphi}$ equals to the geometric Levy area of the directed loop $\underleftarrow{\gamma_{\theta,\varphi}}$.*

Proof. In accordance with Lemma 2.21, the geometric Levy area $GA_{\theta,\varphi}$ of the directed loop $\underleftarrow{\gamma_{\theta,\varphi}}$ is well-defined in accordance with Definition 2.20. Applying Lemma 2.30, for any $(\rho, r) \in \Delta_1$,

$$\begin{aligned} 2\left(GA_{\theta,\varphi} - A_{\theta,\varphi}^{r}\right) &= \left(\int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,1}\right)} - \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r}\right)} - \int_{\partial Q_{\theta,\varphi}^{r,1}}\right)(x_s dy_s - y_s dx_s) \\ &\quad - \left(\int_{\partial W_{\theta}^{r,1}} + \int_{\partial W_{\varphi}^{r,1}}\right)(x_s dy_s - y_s dx_s). \end{aligned}$$

Passing r to 1 and using Lemmas 2.31 and 2.32 and Proposition 2.35, we obtain our claim. ■

2.4.3 Equicontinuity of $A^r_{\theta,\varphi}$

We are going to show that the difference $\left|A^r_{\theta_1,\varphi} - A^r_{\theta_2,\varphi}\right|$ is uniformly small for all $r \in \mathbb{Q}$ when θ_1 and θ_2 are close enough. Proof for the general case that $A^r_{\cdot}$ is equicontinuous is similar.

Lemma 2.38 *Let $(\theta_1, \theta_2), (\theta_2, \varphi) \in \Delta_{2\pi}$, denote $x^r_{\cdot} = Re\left(\phi\left(re^{i\cdot}\right)\right)$ and $y^r_{\cdot} = Im\left(\phi\left(re^{i\cdot}\right)\right)$.*

$$\begin{aligned} A^r_{\theta_1,\varphi} - A^r_{\theta_2,\varphi} &= A^r_{\theta_1,\theta_2} \qquad (2.13) \\ &+\frac{1}{2}\left(\left(x^r_{\theta_2} - x^r_{\theta_1}\right)\left(y^r_{\varphi} - y^r_{\theta_1}\right) - \left(x^r_{\varphi} - x^r_{\theta_2}\right)\left(y^r_{\theta_2} - y^r_{\theta_1}\right)\right). \end{aligned}$$

Proof. It is a direct consequence of Chen's identity (1.3). ■

We now estimate the moduli of two terms on the right hand side of (2.13).

Lemma 2.39 *Let $(\theta_1, \theta_2), (\theta_2, \varphi) \in \Delta_{2\pi}$,*

$$\begin{aligned} &\left|\left(x^r_{\theta_2} - x^r_{\theta_1}\right)\left(y^r_{\varphi} - y^r_{\theta_1}\right) - \left(x^r_{\varphi} - x^r_{\theta_2}\right)\left(y^r_{\theta_2} - y^r_{\theta_1}\right)\right| \\ \leq\ &\left(C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\right)^2 (2\pi)^{\alpha} \left|\theta_2 - \theta_1\right|^{\alpha}. \end{aligned}$$

Proof. Using the inequality (2.6),

$$\begin{aligned} &\left|\left(x^r_{\theta_2} - x^r_{\theta_1}\right)\left(y^r_{\varphi} - y^r_{\theta_1}\right) - \left(x^r_{\varphi} - x^r_{\theta_2}\right)\left(y^r_{\theta_2} - y^r_{\theta_1}\right)\right| \\ \leq\ &\left|\phi\left(re^{i\varphi}\right) - \phi\left(re^{i\theta_2}\right)\right| \left|\phi\left(re^{i\theta_2}\right) - \phi\left(re^{i\theta_1}\right)\right| \\ \leq\ &\left(C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\right)^2 (2\pi)^{\alpha} \left|\theta_2 - \theta_1\right|^{\alpha}. \end{aligned}$$

■

Lemma 2.40 *Let $(\theta_1, \theta_2) \in \Delta_{2\pi}$ and $\rho = 1 - \frac{\theta_2 - \theta_1}{2\pi}$. $\phi\left(D^{\rho r, r}_{\theta_1,\theta_2}\right)$ is contained in the disc centered at $\phi\left(re^{i\theta_1}\right)$ with radius $C\left((2\pi)^{1-\alpha} + \frac{3}{\alpha}\right)\left|\theta_2 - \theta_1\right|^{\alpha}$.*

Proof. For any $\rho < 1$, $z = |z|\, e^{it} \in D^{\rho r, r}_{\theta_1,\theta_2}$, using (2.6), (2.7) and (2.9), the difference

$$\begin{aligned} \left|\phi(z) - \phi\left(re^{i\theta_1}\right)\right| &\leq \left|\phi\left(|z|\, e^{it}\right) - \phi\left(re^{it}\right)\right| + \left|\phi\left(re^{it}\right) - \phi\left(re^{i\theta_1}\right)\right| \\ &\leq \frac{C}{\alpha} r (1-\rho)^{\alpha} + Cr\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\left|\theta_2 - \theta_1\right|^{\alpha}. \end{aligned}$$

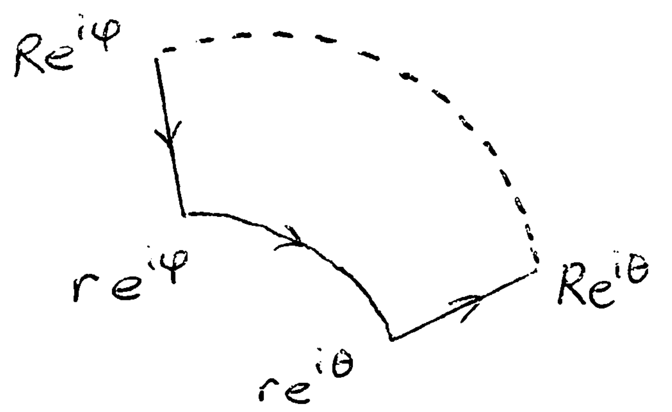
Choosing $\rho = 1 - \frac{\theta_2-\theta_1}{2\pi}$, we obtain

$$\left|\phi(z) - \phi\left(re^{i\theta_1}\right)\right| \leq C\left((2\pi)^{1-\alpha} + \frac{3}{\alpha}\right)|\theta_2-\theta_1|^\alpha.$$

■

Definition 2.41 *Let $(r,R) \in \Delta_1$ and $(\theta,\varphi) \in \Delta_{2\pi}$. Define $\Gamma_{\theta,\varphi}^{r,R}$ to be the path consisting with three curves concatenating in order:*

1. *A directed line segment running from $Re^{i\varphi}$ radially inward to $re^{i\varphi}$,*
2. *A directed circular arc with radius r from $re^{i\varphi}$ to $re^{i\theta}$,*
3. *A directed line segment running from $re^{i\theta}$ radially outward to $Re^{i\theta}$.*



Lemma 2.42 *Let $(\theta_1,\theta_2) \in \Delta_{2\pi}$, $\rho = 1 - \frac{\theta_2-\theta_1}{2\pi}$ and $r \in I$. the length of $\phi\left(\Gamma_{\theta_1,\theta_2}^{\rho r,r}\right)$*

$$l\left(\phi\left(\Gamma_{\theta_1,\theta_2}^{\rho r,r}\right)\right) \leq \left(\frac{2C}{\alpha} + C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\right)|\theta_2-\theta_1|^\alpha.$$

Proof. Using the inequalities (2.6) and (2.9),

$$\begin{aligned} l\left(\phi\left(\Gamma_{\theta_1,\theta_2}^{\rho r,r}\right)\right) &\leq 2\frac{C}{\alpha}(r-\rho r)^\alpha + C\rho r\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)|\theta_2-\theta_1|^\alpha \\ &\leq \left(\frac{2C}{\alpha} + C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\right)|\theta_2-\theta_1|^\alpha. \end{aligned}$$

■

Denote the chord joining $\phi\left(re^{i\theta_2}\right)$ and $\phi\left(re^{i\theta_1}\right)$ by $\overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}$.

Corollary 2.43 *Let* $(\theta_1, \theta_2) \in \Delta_{2\pi}$ *and* $r \in I$.

$$\left|A^r_{\theta_1,\theta_2}\right| \leq C^2 \left((2\pi)^{1-\alpha} + \frac{3}{\alpha}\right)^2 \left(\pi (2\pi)^\alpha + \pi^{-1}\right) |\theta_2 - \theta_1|^{2\alpha}. \tag{2.14}$$

Proof. Because of the conformal nature of ϕ, $\phi\left(D^{\rho r,r}_{\theta_1,\theta_2}\right)$ is a Jordan domain. Using the Pohl-Banchoff inequality in Proposition 2.13, we deduce that

$$\begin{aligned}
\left|A^r_{\theta_1,\theta_2}\right| &= \left|\left(\int_{\partial\phi\left(D^{\rho r,r}_{\theta_1,\theta_2}\right)} - \int_{\phi\left(\Gamma^{\rho r,r}_{\theta_1,\theta_2}\right)\cup\overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}}\right)(x_s dy_s - y_s dx_s)\right| \\
&\leq \int\int_{\mathbb{C}} \left|\eta\left(\partial\phi\left(D^{\rho r,r}_{\theta_1,\theta_2}\right), z\right)\right| \lambda(dA) \\
&\quad + \int\int_{\mathbb{C}} \left(\eta\left(\phi\left(\Gamma^{\rho r,r}_{\theta_1,\theta_2}\right) \cup \overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}, z\right)\right)^2 \lambda(dA) \\
&\leq \int\int_{\phi\left(D^{\rho r,r}_{\theta_1,\theta_2}\right)} 1\cdot\lambda(dA) + \frac{1}{4\pi} l\left(\phi\left(\Gamma^{\rho r,r}_{\theta_1,\theta_2}\right) \cup \overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}\right)^2 \\
&\leq C^2 \left((2\pi)^{1-\alpha} + \frac{3}{\alpha}\right)^2 \left(\pi (2\pi)^\alpha + \pi^{-1}\right) |\theta_2 - \theta_1|^{2\alpha},
\end{aligned}$$

where the last inequality follows by applying Lemmas 2.40 and 2.42. ■

Therefore, we can now obtain an upper bound of the modulus of the difference on the left hand side of (2.13).

Corollary 2.44 *Let* $(\theta_1, \theta_2), (\theta_2, \varphi) \in \Delta_{2\pi}$,

$$\left|A^r_{\theta_1,\varphi} - A^r_{\theta_2,\varphi}\right| \leq K |\theta_2 - \theta_1|^\alpha \tag{2.15}$$

where

$$K = \left(C^2 \left((2\pi)^{1-\alpha} + \frac{3}{\alpha}\right)^2 \left(\pi (2\pi)^\alpha + \pi^{-1}\right) + \frac{1}{2}\left(C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\right)^2\right)(2\pi)^\alpha.$$

Proof. By substituting the inequalities in both Lemma 2.39 and Corollary 2.43 into Equation (2.13), we conclude our claim. ■

Remark 2.45 *Note that the inequalities* (2.14) *and* (2.15) *hold uniformly for all* $r \in \mathbb{Q}$. *Therefore, by passing to the pointwise limit, we also have*

$$\left|A_{\theta_1,\varphi} - A_{\theta_2,\varphi}\right| \leq K |\theta_2 - \theta_1|^\alpha \tag{2.16}$$

and

$$\left|A_{\theta_1,\theta_2}\right| \leq C^2\left(\left(2\pi\right)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi\left(2\pi\right)^{\alpha}+\pi^{-1}\right)\left|\theta_2-\theta_1\right|^{2\alpha}. \qquad (2.17)$$

As a consequence, we can now obtain the $\frac{1}{2\alpha}+\varepsilon$–variations of the Levy area processes of $\gamma_\cdot^r$ and $\gamma_\cdot$.

Corollary 2.46 *Consider a partition* $\mathcal{D}=\{0=\theta_0\leq\cdots\leq\theta_n=2\pi\}$ *of* $[0,2\pi]$. *Let* $\varepsilon\geq 0$, $r\in I$. *Then we have*

$$\begin{aligned}\sum_{i=1}^{n}\left|A^r_{\theta_i,\theta_{i+1}}\right|^{\frac{1}{2\alpha}+\varepsilon} &\leq 2\pi\left(C^2\left(\left(2\pi\right)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi\left(2\pi\right)^{\alpha}+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}+\varepsilon}\\ &\quad\cdot\max\left|\theta_{i+1}-\theta_i\right|^{2\alpha\varepsilon}. \qquad (2.18)\end{aligned}$$

Proof. Using (2.14), we have

$$\begin{aligned}\sum_{i=1}^{n}\left|A^r_{\theta_i,\theta_{i+1}}\right|^{\frac{1}{2\alpha}+\varepsilon} &\leq \left(C^2\left(\left(2\pi\right)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi\left(2\pi\right)^{\alpha}+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}+\varepsilon}\\ &\quad\cdot\sum_{i=1}^{n}\left|\theta_{i+1}-\theta_i\right|^{1+2\alpha\varepsilon}\\ &\leq 2\pi\left(C^2\left(\left(2\pi\right)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi\left(2\pi\right)^{\alpha}+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}+\varepsilon}\\ &\quad\cdot\max\left|\theta_{i+1}-\theta_i\right|^{2\alpha\varepsilon}.\end{aligned}$$

■

Similarly, using (2.17), we also have

$$\begin{aligned}\sum_{i=1}^{n}\left|A_{\theta_i,\theta_{i+1}}\right|^{\frac{1}{2\alpha}+\varepsilon} &\leq 2\pi\left(C^2\left(\left(2\pi\right)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi\left(2\pi\right)^{\alpha}+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}+\varepsilon}\\ &\quad\cdot\max\left|\theta_{i+1}-\theta_i\right|^{2\alpha\varepsilon}. \qquad (2.19)\end{aligned}$$

By applying Arzela-Ascoli lemma, we can now conclude the uniform convergence, up to a subsequence, of Levy area processes of $\phi_r\left(e^{i\cdot}\right)$ to the geometric Levy area process of $\phi\left(e^{i\cdot}\right)$.

Theorem 2.47 *There is a subsequence* $\{A_{\cdot}^{r_k}\}_{k=1}^{\infty}$ *converges uniformly to* $\{A_{\cdot}\}$ *such that*

$$
\begin{aligned}
&\sup_{\mathcal{D}} \sum_{i=1}^{n} \left| A_{\theta_i,\theta_{i+1}}^{r_k} - A_{\theta_i,\theta_{i+1}} \right|^{\frac{\frac{1}{\alpha}+\varepsilon}{2}} \\
\leq\ & 4\pi \left(C^2 \left((2\pi)^{1-\alpha} + \frac{3}{\alpha} \right)^2 \left(\pi (2\pi)^{\alpha} + \pi^{-1} \right) \right)^{\frac{1}{2\alpha}} \sup_{(\theta,\varphi)\in\Delta_{2\pi}} \left| A_{\theta,\varphi}^{r_k} - A_{\theta,\varphi} \right|^{\frac{\varepsilon}{2}}
\end{aligned}
$$

In particular, the sum converges to 0 *as* $k \to \infty$.

Proof. According to the Arzela-Ascoli lemma, there is a subsequence $\{r_k \in \mathbb{Q}\}_{k=1}^{\infty}$ such that $\{A_{\cdot}^{r_k}\}_{k=1}^{\infty}$ converges uniformly to $\{A_{\cdot}\}$. The sum

$$
\begin{aligned}
&\sum_{i=1}^{n} \left| A_{\theta_i,\theta_{i+1}}^{r_k} - A_{\theta_i,\theta_{i+1}} \right|^{\frac{\frac{1}{\alpha}+\varepsilon}{2}} \\
\leq\ & \max \left| A_{\theta_i,\theta_{i+1}}^{r_k} - A_{\theta_i,\theta_{i+1}} \right|^{\frac{\varepsilon}{2}} \left(\sum_{i=1}^{n} \left| A_{\theta_i,\theta_{i+1}}^{r_k} \right|^{\frac{1}{2\alpha}} + \sum_{i=1}^{n} \left| A_{\theta_i,\theta_{i+1}} \right|^{\frac{1}{2\alpha}} \right) \\
\leq\ & 4\pi \left(C^2 \left((2\pi)^{1-\alpha} + \frac{3}{\alpha} \right)^2 \left(\pi (2\pi)^{\alpha} + \pi^{-1} \right) \right)^{\frac{1}{2\alpha}} \max \left| A_{\theta_i,\theta_{i+1}}^{r_k} - A_{\theta_i,\theta_{i+1}} \right|^{\frac{\varepsilon}{2}},
\end{aligned}
$$

where the last inequality is followed by the application of (2.18) and (2.19). ■

2.5 Main theorems

Let $\frac{1}{3} < \alpha \leq 1$. Consider an $\alpha-$Hölder planar domain D with an associated conformal map ϕ over $\mathbb{D}$. Again, we denote $\phi_r(e^{i\cdot})$ and $\phi(e^{i\cdot})$ by $\gamma_{\cdot}^{r} = (x_{\cdot}^{r}, y_{\cdot}^{r})$ and $\gamma_{\cdot} = (x_{\cdot}, y_{\cdot})$ respectively. In this section, we show that the boundary of D can be canonically enhanced as a $p\left(> \frac{1}{\alpha}\right)-$ geometric rough path. In addition, we also extend the classical Green-Gauss' formula to D.

Theorem 2.48 *Let* $p > \frac{1}{\alpha}$. *The boundary* $\gamma_{\cdot}$ *of* D *together with its geometric Levy area* $A_{\cdot}$ *form a* $p-$*geometric rough path. In particular, there is a sequence of rationals* $\{r_k\}_{k\in\mathbb{N}}$ *such that truncated signatures of* $\gamma_{\cdot}^{r_k}$,

$$
S\left(\gamma_{\cdot}^{r_k}\right)_{s,t}^{(2)} = \left(1, S\left(\gamma_{\cdot}^{r_k}\right)_{s,t}^{1}, S\left(\gamma_{\cdot}^{r_k}\right)_{s,t}^{2}\right),
$$

converge in p–variation topology to a tensor

$$X(\gamma)^{(2)}_{s,t} = \left(1, X(\gamma)^1_{s,t}, X(\gamma)^2_{s,t}\right)$$

in $T(\mathbb{R}^2)$ *where*

$$\begin{aligned} X(\gamma)^1_{s,t} &= \gamma_t - \gamma_s \\ X(\gamma)^2_{s,t} &= \frac{1}{2}\begin{pmatrix} (x_t - x_s)^2 & (x_t - x_s)(y_t - y_s) \\ (x_t - x_s)(y_t - y_s) & (y_t - y_s)^2 \end{pmatrix} \\ &\quad + \frac{1}{2}\begin{pmatrix} 0 & A_{s,t} \\ -A_{s,t} & 0 \end{pmatrix} \end{aligned}$$

for any $(s,t) \in \Delta_{2\pi}$.

Proof. In accordance with the notion of geometric rough paths as defined in Section 1.4, the claim is a consequence of Corollary 2.19 and Theorem 2.47. ■

According to the extension theorem, Proposition 1.3, we can uniquely extend our tensor $X(\gamma)^{(2)}_{\cdot}$ to $X(\gamma)_{\cdot}$ in $G^{(\infty)}(\mathbb{R}^2)$ so that the Chen identity is still satisfied.

Definition 2.49 *Let* $(\theta, \varphi) \in \Delta_{2\pi}$, *we call* $X(\gamma)_{\theta,\varphi}$ *to be the signature of the boundary curve* γ. *over* $[\theta, \varphi]$.

Theorem 2.50 (Generalised Gauss-Green) *Consider an one-form* $\omega = \omega_1 dx_1 + \omega_2 dx_2$ *such that for* $i = 1, 2$,

1. *If* $\frac{1}{2} < \alpha \leq 1$, $\omega_i \in C^1_b(\mathbb{C})$.
2. *If* $\frac{1}{3} < \alpha \leq \frac{1}{2}$, $\omega_i \in C^1_b(\mathbb{C})$ *and all its first order partial derivatives are* β*–Hölder continuous with* $\beta > \frac{1}{\alpha} - 2$.

We have the generalized Gauss-Green's formula

$$\pi_1\left(\int \omega(X(\gamma))\, dX(\gamma)\right) = \int\int_D d\omega. \tag{2.20}$$

Remark 2.51 *The integral $\int \omega(X(\gamma))\,dX(\gamma)$ is in the sense of rough path integral while the double integral is in the usual Lebesgue sense. Note that value*

$$\pi_1\left(\int \omega(X(\gamma))\,dX(\gamma)\right)$$

is independent of the choices of both the parametrization of γ and the roughness p.

Proof of the theorem 2.50. Fixed a $p > 1/\alpha$, according to Theorem 2.47, there is a sequence of rationals $\{r_k\}_{k\in\mathbb{N}}$ such that truncated signatures of $\gamma_\cdot^{r_k}$,

$$S\left(\gamma_\cdot^{r_k}\right)_{s,t}^{(2)} = \left(1, S\left(\gamma_\cdot^{r_k}\right)_{s,t}^{1}, S\left(\gamma_\cdot^{r_k}\right)_{s,t}^{2}\right),$$

converge in $p-$ variation topology to the tensor

$$X(\gamma)_{s,t}^{(2)} = \left(1, X(\gamma)_{s,t}^{1}, X(\gamma)_{s,t}^{2}\right).$$

Applying the universal limit theorem - Theorem 1.4, we have

$$\int \omega(X(\gamma))\,dX(\gamma) = \lim_{k\to\infty}\int \omega\left(S\left(\gamma^{r_k}\right)\right)dS\left(\gamma^{r_k}\right).$$

Using the fact that π_1 is continuous, we obtain

$$\begin{aligned}
\pi_1\left(\int \omega(X(\gamma))\,dX(\gamma)\right) &= \lim_{k\to\infty}\pi_1\left(\int \omega\left(X\left(\gamma^{r_k}\right)\right)dX\left(\gamma^{r_k}\right)\right)\\
&= \lim_{k\to\infty}\int_0^{2\pi}\omega\left(\gamma_u^{r_k}\right)\left(d\gamma_u^{r_k}\right)\\
&= \lim_{k\to\infty}\oint_{\gamma_\cdot^{r_k}}\omega.
\end{aligned}$$

On the other hand, the classical Green's formula for domains with C^1- boundaries implies

$$\oint_{\gamma_\cdot^{r_k}}\omega = \int\int_{D^{r_k}} d\omega \tag{2.21}$$

where D^{r_k} is the Jordan domain bounded by $\gamma_\cdot^{r_k}$. Since the Hausdorff dimension of D is strictly less than 2 and $\gamma_\cdot^{r_k}$ uniformly converges to $\gamma_\cdot$ from inside,

$$\int\int_D d\omega = \lim_{k\to\infty}\int\int_{D^{r_k}} d\omega,$$

and therefore the claim follows. ■

Chapter 3

Two results in the theory of rough paths

In this chapter, we collect two results (and their proofs) which are fundamental in nature in the theory of rough paths. The Extension theorem in the theory of rough paths (Proposition 1.3 in Chapter 1 or its equivalence, Theorem 2.1.1 in Lyons [1998]) states that any $p-$rough path $X_{\cdot}:\Delta_T\to T^{(\lfloor p\rfloor)}(V)$ can be naturally enhanced as a multiplicative functional of finite $p-$variation on $T^{(N)}(V)$ for any $N>\lfloor p\rfloor$; in other words, all high order tensors of "iterated integrals" of a rough path are completely determined by its tensors of "iterated integrals" of order less than p. In order to prove the Extension theorem, one has to show that a particular sequence of functionals $\left\{X(n)_{\cdot}:\Delta_T\to T^{(N)}(V)\right\}_{n\in\mathbb{N}}$ is Cauchy by applying the maximal inequality (see Lyons [1998]) and the fact that there is a control (Definition 3.4 below) for the rough path $X_{\cdot}$. Unfortunately, the assertion that there is a natural control for any continuous multiplicative functional with finite $p-$variation is only a mathematics folklore without a proper proof; even there is none in [23] as claimed. In Section 3.1, I shall provide an elementary proof of the claim which is quoted in Theorem 3.6. The key idea of the proof is to establish Corollaries 3.16 and 3.17, i.e. the uniform continuity of our proposed control ω_X in (3.4) with respect to either the first or second argument of ω_X.

In the theory of rough paths, it admits that the signature (recall Definition 1.1) of a geometric rough path completely characterizes the path itself in the context of control theory. Along the line of thought, if a space-filling curve γ which can be shown to be a geometric rough path and fills up a geometric object $\mathcal{M}$, it is reasonable to expect that we can extract the analytical properties of $\mathcal{M}$ by decoding the information contained in the signature of γ; this is precisely our concern in Chapter 5. In regard to extracting information embedded in the signature of a path, it is natural to achieve our goal by integrating, with aids of different one-forms, against the signature (which is now considered as a path itself). Therefore, it is essential to ask if the signature of a geometric rough path, or in general any functional of signature, can still be treated as a geometric rough path. In Section 3.2, we shall prove Proposition 3.34 which states that any functional of the log-signature of a $p-$geometric rough path can be canonically enhanced as a $p-$geometric rough path.

3.1 Existence of controls for rough paths

The purpose of this section is to establish Theorem 3.6, in particular, the joint continuity of the $p-$variation ω_X in (3.4) of a $p-$rough path $X_\cdot$. Denote $\Delta_T \triangleq \{(s,t) : 0 \leq s \leq t \leq T\}$.

3.1.1 Preliminary definitions

In this subsection, we shall recall the notion of rough paths and control functions. We first give the definition of multiplicative functionals of finite $p-$variation.

Definition 3.1 *A map $X_\cdot : \Delta_T \to T^{(N)}(V)$ is said to be a multiplicative functional of degree $N \in \mathbb{Z}^+$ if*

1. *For each $(s,t) \in \Delta_T$,*
$$X_{s,t} = (1, X^1_{s,t}, \ldots, X^N_{s,t}), \tag{3.1}$$
where $X^k_{s,t} \in V^{\otimes k}$ for $k = 1, \ldots, N$.

2. *$X_\cdot$ is continuous on Δ_T.*

3. *$X_\cdot$ satisfies Chen's identity:*
$$X_{s,t} \otimes X_{t,u} = X_{s,u}, \tag{3.2}$$
for $(s,t), (t,u), (s,u) \in \Delta_T$.

Definition 3.2 *A multiplicative functional* $X_{\cdot} = (1, X_{\cdot}^1, \ldots, X_{\cdot}^N)$ *on* $T^{(N)}(V)$, *is said to be of finite* $p-$*variation if*

$$\sup_{D[0,T]} \sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} < +\infty, \quad \forall i = 1, \ldots, N. \tag{3.3}$$

where the supremum runs over all finite partitions $D[0,T] = \{0 = t_0 \leq \cdots \leq t_r = T\}$ *of* $[0,T]$.

Now, a rough path is just a special type of multiplicative functionals, namely:

Definition 3.3 *A* $p-$*rough path is a multiplicative functional of degree* $\lfloor p \rfloor$ *with finite* $p-$*variation.*

Next, we define the notion of control function which is a superadditive continuous function that vanishes along the main diagonal.

Definition 3.4 *A non-negative continuous function* $\omega : \Delta_T \to \mathbb{R}$ *is called a control if*

1. ω *is superadditive:*

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u),$$

for $(s,t), (t,u) \in \Delta_T$.

2. $\omega(t,t) = 0$ *for all* $t \in [0,T]$.

3.1.2 Joint continuity of the $p-$ variation of a $p-$ rough path

The purpose of this subsection is to prove the following Theorem 3.6 which is essential in the theory of rough paths as we pointed out in the introduction of this chapter (for further details, one can consult the work [22] and [23]). It should be remarked that a proper proof of the statement is not yet available in the literature before I drafted this chapter. We begin this subsection by first defining a suitable control for a rough path.

Definition 3.5 *Suppose $X_\cdot$ is a $p-$rough path, we define for any $(s,t) \in \Delta_T$,*

$$\omega_X(s,t) \triangleq \sup_{1\leq i\leq [p]} \sup_{D[s,t]} \sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}}, \tag{3.4}$$

where the supremum runs over all finite partitions of the interval $[s,t]$. We call $\omega_X(s,t)$ the $p-$variation of $X_\cdot$ over $[s,t]$.

Now, we have the statement of our main result in this section, namely: the joint continuity of the $p-$variation ω_X in (3.4) of a $p-$rough path $X_\cdot$.

Theorem 3.6 *The $p-$variation $\omega_X(\cdot,\cdot)$ of a $p-$rough path $X_\cdot$ is a control.*

Note that $X_\cdot$ is uniformly continuous: for any $\varepsilon > 0$, there is $\delta_\varepsilon > 0$, such that for all $i = 1,\ldots,\lfloor p\rfloor$ and any $u < v$ with $|u-v| < \delta_\varepsilon$,

$$\left|X^i_{u,v}\right| < \varepsilon.$$

In order to prove Theorem 3.6, we have to first establish a number of lemmas. The key lemmas are Corollaries 3.16 and 3.17, i.e. the uniform continuity of the $p-$variation ω_X in (3.4) with respect to both the first and second arguments of ω_X respectively.

Lemma 3.7 *Let $\alpha > 1$. For any $a, b \in \mathbb{R}^+$ with $b \leq 1$, we have*

$$0 < (a+b)^\alpha - a^\alpha < \max\left\{2\alpha a^{\alpha-1}, 2^\alpha\right\}\cdot b.$$

Proof. Note that for $x \in [0,1]$,

$$(1+x)^\alpha \leq 1 + 2\alpha x.$$

1. For $a > b$,

$$(a+b)^\alpha - a^\alpha = a^\alpha\left\{\left(1+\frac{b}{a}\right)^\alpha - 1\right\} \leq a^\alpha\left(2\alpha\frac{b}{a}\right) = 2\alpha a^{\alpha-1}b.$$

2. For $a \leq b \leq 1$,

$$(a+b)^\alpha - a^\alpha \leq 2^\alpha b^\alpha \leq 2^\alpha b.$$

■

We next show that in the limit the $(p/i)^{th}$ −power of the modulus of the tensor $X^i_{\cdot}$ is additive.

Lemma 3.8 *For each* $i = 1, \ldots, \lfloor p \rfloor$,

$$\lim_{\varepsilon \to 0} \sup_{(u,t),(t,v) \in \Delta_T,\ |t-v|<\delta_\varepsilon} \left| \left|X^i_{u,v}\right|^{\frac{p}{i}} - \left|X^i_{u,t}\right|^{\frac{p}{i}} - \left|X^i_{t,v}\right|^{\frac{p}{i}} \right| = 0.$$

Proof. Firstly, using Chen's identity (3.2), we have for each $i = 1, \ldots, \lfloor p \rfloor$,

$$X^i_{u,v} = X^i_{u,t} + \sum_{k=1}^{i-1} X^{i-k}_{u,t} \otimes X^k_{t,v} + X^i_{t,v}.$$

Therefore,

$$\begin{aligned} & \left| \left|X^i_{u,v}\right|^{\frac{p}{i}} - \left|X^i_{u,t}\right|^{\frac{p}{i}} - \left|X^i_{t,v}\right|^{\frac{p}{i}} \right| \\ = & \left| \left| X^i_{u,t} + \sum_{k=1}^{i-1} X^{i-k}_{u,t} \otimes X^k_{t,v} + X^i_{t,v} \right|^{\frac{p}{i}} - \left|X^i_{u,t}\right|^{\frac{p}{i}} - \left|X^i_{t,v}\right|^{\frac{p}{i}} \right| \\ \leq & \left| \left| X^i_{u,t} + \sum_{k=1}^{i-1} X^{i-k}_{u,t} \otimes X^k_{t,v} + X^i_{t,v} \right|^{\frac{p}{i}} - \left|X^i_{u,t}\right|^{\frac{p}{i}} \right| + \left|X^i_{t,v}\right|^{\frac{p}{i}}. \end{aligned}$$

On the other hand, we have for any $(t,v) \in \Delta_T$ such that $|t-v| < \delta_\varepsilon$,

$$\begin{aligned} \left| \left| X^i_{u,t} + \sum_{k=1}^{i-1} X^{i-k}_{u,t} \otimes X^k_{t,v} + X^i_{t,v} \right| - \left|X^i_{u,t}\right| \right| & \leq \left|X^i_{t,v}\right| + \lfloor p \rfloor \varepsilon \left(\omega_X(0,T) \vee 1\right) \\ & \leq K\varepsilon, \end{aligned}$$

where $K = (1 + \lfloor p \rfloor (\omega_X(0,T) \vee 1))$. Hence, using Lemma 3.7, for any $(t,v) \in \Delta_T$ with $|t-v| < \delta_\varepsilon$,

$$\left| \left| X^i_{u,t} + \sum_{k=1}^{i-1} X^{i-k}_{u,t} \otimes X^k_{t,v} + X^i_{t,v} \right|^{\frac{p}{i}} - \left|X^i_{u,t}\right|^{\frac{p}{i}} \right| \leq \max\left\{2p\left(\omega_X(0,T) \vee 1\right), 2^p\right\} K\varepsilon$$

and hence, we deduce that

$$\begin{aligned} &\sup_{(u,t),(t,v)\in\Delta_T,\ |t-v|<\delta_\varepsilon} \left| \left|X^i_{u,v}\right|^{\frac{p}{i}} - \left|X^i_{u,t}\right|^{\frac{p}{i}} - \left|X^i_{t,v}\right|^{\frac{p}{i}} \right| \\ \leq\ & \max\left\{2p\cdot\omega_X\left(0,T\right),2^p\right\}K\varepsilon + \varepsilon^{\frac{p}{i}}. \end{aligned}$$

■

We now give a symbol for the measure for deviation from additivity of the i^{th} component of $p-$variation ω_X at the order triple (s,t,v).

Definition 3.9 *Let $(s,t),(t,v)\in\Delta_T$, define*

$$D^i_{s,t,v} \triangleq \sup_{D[s,v]}\left(\sum_l \left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}}\right) - \sup_{D[s,t]}\left(\sum_l \left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}}\right) - \sup_{D[t,v]}\left(\sum_l \left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}}\right).$$

Next, we show that the measure for deviation from additivity $D^i_{s,t,v}$ uniformly approaches to zero as v approaches to t from above; in other words, in the limit, the i^{th} component of $p-$variation ω_X is uniformly additive.

Corollary 3.10 *For each $i\in\{1,2,\ldots,\lfloor p\rfloor\}$,*

$$\lim_{v\downarrow t}\sup_{(s,t)\in\Delta_T} D^i_{s,t,v} = 0.$$

Proof. Fixed $\varepsilon>0$ and $0<\Delta t<\delta_\varepsilon$. For any partition

$$\mathcal{D}\triangleq\{s=t_0,t_1,\ldots,t_n=t+\Delta t\}$$

of $[s,t+\Delta t]$, where $t_j\leq t<t_{j+1}$, we have

$$\sum_l \left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}} = \sum_{l=1}^{j-1}\left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}} + \left|X^i_{t_j,t_{j+1}}\right|^{\frac{p}{i}} + \sum_{l=j+2}^{n}\left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}}.$$

Therefore, we obtain

$$\begin{aligned} &\left| \sum_{t_l\in\mathcal{D}\cap[s,t+\Delta t]}\left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}} - \sum_{t_l\in\mathcal{D}\cap[s,t]}\left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}} - \sum_{t_l\in\mathcal{D}\cap[t,t+\Delta t]}\left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}} \right| \\ =\ & \left| \left|X^i_{t_j,t_{j+1}}\right|^{\frac{p}{i}} - \left|X^i_{t_j,t}\right|^{\frac{p}{i}} - \left|X^i_{t,t_{j+1}}\right|^{\frac{p}{i}} \right|. \end{aligned}$$

Hence, by Definition 3.9, we have for any $v \in [t, t + \Delta t]$,

$$\begin{aligned} 0 &\leq \sup_{(s,t)\in\Delta_T} D^i_{s,t,v} \\ &\leq \sup_{(u,t),(t,v)\in\Delta_T,\ |t-v|<\delta_\varepsilon} \left| \left|X^i_{u,v}\right|^{\frac{p}{i}} - \left|X^i_{u,t}\right|^{\frac{p}{i}} - \left|X^i_{t,v}\right|^{\frac{p}{i}} \right|. \end{aligned}$$

Using Lemma 3.8, the result follows. ∎

Now, we have continuity of the $p-$variation ω_X with respect to the second argument.

Lemma 3.11 *Let* $(s,t) \in \Delta_T$. *Then we have*

$$\lim_{\Delta t \to 0^+} \omega_X(s, t + \Delta t) = \omega_X(s,t).$$

In particular,

$$\lim_{\Delta s \to 0^+} \omega_X(s, s + \Delta s) = 0.$$

Proof. Note that for every s, $\omega_X(s,\cdot)$ is non-decreasing on $[s,T]$. We shall prove our claim by contradiction. Suppose that it is not the case, there would be a pair of $(s_0,t_0) \in \Delta_T$ such that $\lim_{\Delta t\to 0^+} \omega_X(s_0, t_0 + \Delta t) - \omega_X(s_0,t_0) > 0$; this is only possible if there is $i_0 \in \{1, 2, \ldots, \lfloor p \rfloor\}$ such that

$$\eta \triangleq \lim_{\Delta t \to 0^+} \sup_{D[s_0,t_0+\Delta t]} \left(\sum_l \left| X^{i_0}_{t_{l-1},t_l} \right|^{\frac{p}{i_0}} \right) - \sup_{D[s_0,t_0]} \left(\sum_l \left| X^{i_0}_{t_{l-1},t_l} \right|^{\frac{p}{i_0}} \right) > 0.$$

As a consequence of Corollary 3.10, for any $\varepsilon > 0$, there is a $\Delta t > 0$ such that for any $v \in (t_0, t_0 + \Delta t)$,

$$\left| \sup_{D[t_0,v]} \left(\sum_l \left| X^{i_0}_{t_{l-1},t_l} \right|^{\frac{p}{i_0}} \right) - \eta \right| \leq \varepsilon. \tag{3.5}$$

On the other hand, for any $u \in [t_0, v]$, by definition,

$$\sup_{D[t_0,u]} \left(\sum_l \left| X^{i_0}_{t_{l-1},t_l} \right|^{\frac{p}{i_0}} \right) + \sup_{D[u,v]} \left(\sum_l \left| X^{i_0}_{t_{l-1},t_l} \right|^{\frac{p}{i_0}} \right) \leq \sup_{D[t_0,v]} \left(\sum_l \left| X^{i_0}_{t_{l-1},t_l} \right|^{\frac{p}{i_0}} \right),$$

and consequently, using (3.5), for all $t_0 < u \leq v < t_0 + \Delta t$,

$$0 \leq \sup_{D[u,v]} \left(\sum_l \left| X^{i_0}_{t_{l-1},t_l} \right|^{\frac{p}{i_0}} \right) \leq 2\varepsilon. \tag{3.6}$$

For any partition $\mathcal{D} \triangleq \{t_0 = v_0 < v_1 < \cdots < v_m = v\}$ of $[t_0, v]$,

$$\begin{aligned}\sum_{v_l \in \mathcal{D}} \left|X^{i_0}_{v_{l-1},v_l}\right|^{\frac{p}{i_0}} &\le \left|X^{i_0}_{t_0,v_1}\right|^{\frac{p}{i_0}} + \sum_{l=2}^{m} \left|X^{i_0}_{v_{l-1},v_l}\right|^{\frac{p}{i_0}} \\ &\le \sup_{u_1 \in (t_0,v)} \left|X^{i_0}_{t_0,u_1}\right|^{\frac{p}{i_0}} + \sup_{D[v_1,v]} \left(\sum_l \left|X^{i_0}_{t_{l-1},t_l}\right|^{\frac{p}{i_0}}\right),\end{aligned}$$

therefore, using (3.6), we deduce that for any $v \in (t_0, t_0 + \Delta t)$,

$$\sup_{D[t_0,v]} \left(\sum_l \left|X^{i_0}_{t_{l-1},t_l}\right|^{\frac{p}{i_0}}\right) \le \sup_{u_1 \in (t_0,v)} \left|X^{i_0}_{t_0,u_1}\right|^{\frac{p}{i_0}} + 2\varepsilon.$$

Using (3.5), we also have

$$\lim_{v \downarrow t_0} \sup_{u_1 \in (t_0,v)} \left|X^{i_0}_{t_0,u_1}\right|^{\frac{p}{i_0}} \ge \eta - 3\varepsilon.$$

Since ε is arbitrary, we conclude that

$$\lim_{v \downarrow t_0} \sup_{u_1 \in (t_0,v)} \left|X^{i_0}_{t_0,u_1}\right|^{\frac{p}{i_0}} \ge \eta > 0,$$

which contradicts the continuity of X.. ■

As a corollary, we obtain a partial result on the uniform continuity of the $p-$variation ω_X with respect to the second argument.

Corollary 3.12 *For each $t \in [0, T]$,*

$$\lim_{\delta \downarrow 0} \sup_{(s,t) \in \Delta_T,\, v \in [t,t+\delta]} \left|\omega_X(s,v) - \omega_X(s,t)\right| = 0.$$

Proof. Using Lemma 3.11, we have, for every $\varepsilon > 0$, there is $\delta > 0$, such that for any $v \in [t, t+\delta]$,

$$0 \le \omega_X(t,v) \le \varepsilon,$$

and

$$\max_i \sup_{(s,t) \in \Delta_T} D^i_{s,t,v} \le \varepsilon.$$

For all $s \leq t$ and $i = 1, 2, \ldots, \lfloor p \rfloor$,

$$\begin{aligned} & \left| \sup_{D[s,v]} \left(\sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \right) - \sup_{D[s,t]} \left(\sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \right) \right| \\ \leq \; & \sup_{D[t,v]} \left(\sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \right) + \varepsilon \leq 2\varepsilon. \end{aligned}$$

■

Definition 3.13 *For any* $(s,t) \in \Delta_T$, *define*

$$\widehat{X}_{s,t} \triangleq \widehat{X}_{T-t,T-s}.$$

Using Chen's identity (3.2), for any $(s,t), (t,u) \in \Delta_T$,

$$\widehat{X}_{s,u} = \widehat{X}_{t,u} \otimes \widehat{X}_{s,t}.$$

Since all the derivations in the previous lemmas do not rely on the non-commutative nature of $\otimes$, without further effort, we can immediately deduce the continuity of the $p-$variation ω_X with respect to the first argument.

Lemma 3.14 *For any* $(s,t) \in \Delta_T$,

$$\lim_{\Delta s \to 0^+} \omega_X(s - \Delta s, t) = \omega_X(s,t).$$

In particular,

$$\lim_{\Delta s \to 0^+} \omega_X(s - \Delta s, s) = 0.$$

Proof. Note that the arguments in the proofs of Lemmas 3.8, 3.10 and 3.11 do not depend on the noncommutativity of $\otimes$. Following the same lines of proof by replacing $X_\cdot$ by $\widehat{X}_\cdot$, we attain our result. ■

Similarly, we also obtain a partial result on the uniform continuity of the $p-$variation ω_X with respect to the first argument.

Lemma 3.15 *For each* $t \in [0,T]$,

$$\lim_{\delta \downarrow 0} \sup_{(s,u) \in \Delta_T,\, u \in [t-\delta, t]} |\omega_X(s,t) - \omega_X(s,u)| = 0.$$

Proof. For all $i = 1, 2, \ldots, \lfloor p \rfloor$, using the fact that $X.$ is continuous and Lemma 3.14, for any $\varepsilon > 0$, there is a $\delta > 0$, such that for $u \in [t - \delta, t]$,

$$\left| X^i_{u,t} \right| < \varepsilon$$

and

$$\sup_{D[u,t]} \left(\sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \right) < \varepsilon.$$

Consider a partition $\mathcal{D} \triangleq \{s = t_0, t_1, \ldots, t_n = t\}$ of $[s,t]$ such that $t_j < u < t_{j+1}$. Then we have

$$\begin{aligned}
& \sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \\
\leq\ & \sum_{l=1}^{j} \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} + \left| X^i_{t_j,u} + \sum_{k=1}^{i} X^{i-k}_{t_j,u} \otimes X^k_{u,t_{j+1}} \right|^{\frac{p}{i}} + \sum_{l=j+2}^{n} \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \\
\leq\ & \sum_{l=1}^{j} \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} + \left\{ \left| X^i_{t_j,u} \right| + \sum_{k=1}^{i} \varepsilon \left| X^{i-k}_{t_j,u} \right| \right\}^{\frac{p}{i}} + \sum_{l=j+2}^{n} \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \\
\leq\ & \sup_{D[s,u]} \left(\sum_l \left| X^i_{u_{l-1},u_l} \right|^{\frac{p}{i}} \right) - \left| X^i_{t_j,u} \right|^{\frac{p}{i}} \\
& + \left\{ \left| X^i_{t_j,u} \right| + \varepsilon \lfloor p \rfloor \left(\omega(0,T) \vee 1 \right) \right\}^{\frac{p}{i}} + \sup_{D[t_{j+2},t]} \left(\sum_l \left| X^i_{u_{l-1},u_l} \right|^{\frac{p}{i}} \right),
\end{aligned}$$

and so

$$\begin{aligned}
& \sup_{D[s,t]} \left(\sum_l \left| X^i_{t_{l-1},t_l} \right|^{\frac{p}{i}} \right) \\
\leq\ & \sup_{D[s,u]} \left(\sum_l \left| X^i_{u_{l-1},u_l} \right|^{\frac{p}{i}} \right) + \sup_{D[u,t]} \left(\sum_l \left| X^i_{u_{l-1},u_l} \right|^{\frac{p}{i}} \right) \\
& + \sup_{s \leq v \leq u} \left\{ \left\{ \left| X^i_{v,u} \right| + \varepsilon \lfloor p \rfloor \left(\omega_X(0,T) \vee 1 \right) \right\}^{\frac{p}{i}} - \left| X^i_{v,u} \right|^{\frac{p}{i}} \right\} \\
\leq\ & \sup_{D[s,u]} \left(\sum_l \left| X^i_{u_{l-1},u_l} \right|^{\frac{p}{i}} \right) + \sup_{D[u,t]} \left(\sum_l \left| X^i_{u_{l-1},u_l} \right|^{\frac{p}{i}} \right) \\
& + K^i_{p,\omega_X} \cdot \varepsilon
\end{aligned}$$

where

$$K^i_{p,\omega_X} \triangleq \lfloor p \rfloor (\omega_X (0,T) \vee 1) \cdot \max \left\{ \frac{2p}{i} \omega_X (0,T)^{1-\frac{i}{p}}, 2^{\frac{p}{i}} \right\}.$$

The last inequality holds because of Lemma 3.7 and the fact that for all $(s,t) \in \Delta_T$,

$$\left| X^i_{s,t} \right| \leq \omega (0,T)^{\frac{i}{p}} < \infty.$$

Hence

$$0 \leq \omega_X (s,t) - \omega_X (s,u) \leq \omega_X (u,t) + \max_{i=1,\ldots,\lfloor p \rfloor} K^i_{p,\omega_X} \cdot \varepsilon.$$

■

By combining our previous claims, namely Corollaries 3.12 and 3.15, we can now deduce the uniform continuity of the $p-$variation ω_X with respect to the second and first arguments respectively.

Corollary 3.16 *For each* $t \in [0,T]$,

$$\lim_{\delta \downarrow 0} \sup_{(s,t),(s,v) \in \Delta_T,\ v \in [t-\delta, t+\delta]} |\omega_X (s,t) - \omega_X (s,v)| = 0.$$

Proof. Combining the results in Lemmas 3.12 and 3.15, we obtain our desired result. ■

Corollary 3.17 *For each* $s \in [0,T]$,

$$\lim_{\delta \downarrow 0} \sup_{(s,t),(u,t) \in \Delta_T, u \in [s-\delta, s+\delta]} |\omega_X (s,t) - \omega_X (u,t)| = 0.$$

Proof. Again note that arguments leading to Corollary 3.16 do not depend on the noncommutativity of $\otimes$. By replacing all $X.$ by $\overleftarrow{X}.$, we conclude the result. ■

We are now ready to settle our Theorem 3.6 as follow.

Proof of Theorem 3.6. It is clear that ω_X is both superadditve and vanishing on the diagonal. What remains is to establish the joint continuity of ω_X. Let $(s,t) \in \Delta_T$ with $s < t$. Using Corollaries 3.16 and 3.17, for $\varepsilon > 0$, there is a positive $\delta < \frac{t-s}{2}$ such that for $s' \in (s-\delta, s+\delta)$ and $t' \in (t-\delta, t+\delta)$,

$$|\omega_X (s,t) - \omega_X (s',t)| < \varepsilon$$

and

$$\left|\omega_X\left(s',t\right)-\omega_X\left(s',t'\right)\right|<\varepsilon.$$

Therefore

$$\left|\omega_X\left(s,t\right)-\omega_X\left(s',t'\right)\right|<2\varepsilon.$$

On the other hand, we consider the case (s,s) lying on the diagonal. Note that $\omega_X(s,s)=0$. According to Lemmas 3.11 and 3.14 and the fact that $X.$ is continuous, for any $\varepsilon>0$, there is $\delta'>0$, such that for $u\in[s-\delta',s]$, $v\in[s,s+\delta']$ and $i=1,\ldots,\lfloor p\rfloor$,

$$\omega_X\left(u,s\right),\omega_X\left(s,v\right),\left|X_{u,v}^i\right|^{\frac{p}{i}}\leq\varepsilon$$

therefore,

$$\begin{aligned}\omega_X\left(u,v\right) &\leq \omega_X\left(u,s\right)+\sup_{i=1,\ldots,\lfloor p\rfloor}\sup_{s-\delta'\leq u\leq s\leq v\leq s+\delta'}\left|X_{u,v}^i\right|^{\frac{p}{i}}+\omega_X\left(s,v\right)\\ &\leq 3\varepsilon.\end{aligned}$$

Finally, in general, for any $(u,v)\in\Delta_T$, $s-\delta'<u<v<s+\delta'$, we have $\omega_X(u,v)\leq\omega_X(s\wedge u,s\vee v)\leq 3\varepsilon$. ■

3.2 Functions of signatures as geometric rough paths

In this section, we first recall some elementary notion in the theory of rough paths. We then establish Proposition 3.31 which states that a special family of smooth vector fields induces a sequence of recursively defined continuous paths such that each of the paths can naturally be enhanced as a geometric rough path. As an immediate consequence, we conclude our claim that any nice functionals of the log-signature of a $p-$geometric rough path are also $p-$geometric rough paths.

3.2.1 Preliminary definitions and results

This subsection is partially equivalent to Section 1.4 but from a different perspective which can ease the development leading to our desired results in Chapter 5. We shall first recall the notion of signatures of geometric rough paths, and then introduce an alternative yet equivalent definition of

$p-$variation topology and finally we state another formulation of Universal Limit Theorem in the theory of rough paths.

Consider a separable Banach space $(V, |\cdot|_V)$ and its family of algebraic tensor products $V^{\otimes_a k} \equiv V \otimes_a \cdots \otimes_a V$ (total of k copies) with tensor norms $|\cdot|_k$ such that:

1. $|\cdot|_1 = |\cdot|_V$.
2. $|v \otimes w|_{k,l} \leq |v|_k \cdot |w|_l$, where $v \in V^{\otimes_a k}$ and $w \in V^{\otimes_a l}$.

The completion of the algebraic tensor product $V^{\otimes_a k}$ under the norm $|\cdot|_k$ is denoted by $(V^{\otimes k}, |\cdot|_k)$ or $V^{\otimes k}$ for short if the tensor norms are clear.

Definition 3.18 *For each $N \in \mathbb{N}$, the (truncated) tensor algebra on V, $(T^{(N)}(V), \otimes)$, is the direct sum of all tensor products up to order N:*

$$T^{(N)}(V) \equiv \oplus_{k=0}^{N} V^{\otimes k},$$

where $V^{\otimes 0} \equiv \mathbb{R}$ and $V^{\otimes 1} \equiv V$.

Definition 3.19 *The multiplication in $T^{(N)}(V)$ is defined as: $\forall \xi = (\xi^0, \xi^1, \ldots, \xi^N)$, $\eta = (\eta^0, \eta^1, \ldots, \eta^N) \in T^{(N)}(V)$, their product $\zeta = (\zeta^0, \zeta^1, \ldots, \zeta^N)$ is such that for $k = 1, 2, \ldots, N$,*

$$\zeta^k = \sum_{i=0}^{k} \xi^i \otimes \eta^{k-i}.$$

Lemma 3.20 *Define a norm $|\cdot|_{(N)}$ on $T^{(N)}(V)$ such that $\forall \xi \in T^{(N)}(V)$,*

$$|\xi|_{(N)} = \sum_{k=0}^{N} |\xi^k|_k.$$

Then, $\left(T^{(N)}(V), |\cdot|_{(N)}\right)$ is a Banach algebra.

Define a partial order in Δ_T so that $(s,t) \leq (u,v)$ if both $s \leq u$ and $t \leq v$. Consider a finite variation continuous path $\gamma : [0,1] \to V$. Let $1 \leq k \in \mathbb{Z}^+$, the tensor of the k^{th}-order iterated integral, as a function from Δ_1 to $V^{\otimes k}$, of γ:

$$S(\gamma)_{s,t}^k \triangleq \int \cdots \int_{s<u_1<\cdots<u_k<t} d\gamma_{u_1} \otimes \cdots \otimes d\gamma_{u_k}$$

is well-defined and continuous on Δ_1.

Definition 3.21 *For any $(s,t) \in \Delta_T$, we define*

$$S(\gamma)_{s,t} \triangleq \left(1, S(\gamma)^1_{s,t}, \ldots, S(\gamma)^k_{s,t}, \ldots\right)$$

to be the signature of γ over $[s,t]$.

According to the work of Lyons and Hambly [2006], the signature $S(\gamma)_{s,t}$ characterizes the finite variation path γ over $[s,t]$ up to tree-like pieces. We now prepare to extend the notion of signatures to geometric rough paths.

Fix $N \in \mathbb{N}$. We use $C\left(\Delta_1, T^{(N)}(V)\right)$ to denote the space of all continuous functions $X_\cdot : \Delta_1 \to \left(T^{(N)}(V), |\cdot|_{(N)}\right)$.

Definition 3.22 *Define $\Omega^N(V)$ to be the subset of $C\left(\Delta_1, T^{(N)}(V)\right)$ containing all elements $X_\cdot$ having properties:*

1. $X^0 \equiv 1$.
2. $X_\cdot$ satisfies Chen's identity (3.2):

$$X_{s,t} = X_{s,u} \otimes X_{u,t}$$

for any $(s,u) \leq (u,t) \in \Delta_1$.

Definition 3.23 *Define $G\Omega^N(V)$ to be the subset of $\Omega^N(V)$ so that $\forall Y_\cdot : \Delta_1 \to T^{(N)}(V) \in G\Omega^N(V)$, there is a finite variation continuous path $\gamma : [0,1] \to V$ such that for any $k = 0, \ldots, N$,*

$$Y^k_\cdot = S(\gamma)^k_\cdot.$$

Definition 3.24 *Define the $p-$variation metric $d^N_p(\cdot,\cdot)$ on $\Omega^N(V)$ so that $\forall X_\cdot, Y_\cdot \in \Omega^N(V)$,*

$$d^N_p(X_\cdot, Y_\cdot) \triangleq \max_{1\leq k\leq N} \sup_{\mathcal{D}} \left(\sum_i \left|X^k_{t_{i-1},t_i} - Y^k_{t_{i-1},t_i}\right|^{p/k}\right)^{k/p},$$

where the supremum runs through all partitions $\mathcal{D}$ of $[0,1]$. The corresponding topology induced is called the $p-$variation topology.

Remark 3.25 *According to the work of Friz and Victoir* [2006], *the induced* $p-$ *variation topology in Definition* 3.24 *is equivalent to the one mentioned in Section* 1.4.

Fix a real number $p \geq 1$.

Definition 3.26 *The set of* $p-$*geometric rough paths* $G\Omega_p(V)$ *is the completion of* $G\Omega^{\lfloor p \rfloor}(V)$ *in* $\Omega^{\lfloor p \rfloor}(V)$ *under the* $p-$*variation metric* $d_p^{\lfloor p \rfloor}$.

Definition 3.27 *Let* $X_\cdot \in G\Omega_p(V)$ *and* $0 \leq t \leq 1$. *Define* $X_{0,t}$ *as the signature and* $log X_{0,t}$ *as the log-signature of the path* $X^1_{0,\cdot} : [0,1] \to V$ *over* $[0,t]$.

Denote the space of all finite variation $V-$valued paths $X_\cdot : [0,1] \to V$ by $G\Omega^1(V)$. Next, we state an alternative formulation of Universal Limit Theorem in the theory of rough paths.

Proposition 3.28 *Let* V *and* W *be two Banach spaces and* $f : V \to Lip[\alpha, W, W]$ *be a linear map from* V *to* $\alpha-$*Lipschitz vector fields (see Stein* [1970] *for a definition) on* W. *Define the Itô map* $\mathcal{I}_f : G\Omega^1(V) \to G\Omega^1(V \oplus W)$ *so that* $\forall X_\cdot \in G\Omega^1(V)$,

$$Z_\cdot \triangleq \mathcal{I}_f(X_\cdot) = X_\cdot + Y_\cdot,$$

where, for some fixed $a \in W$,

$$\begin{aligned} dY_t &= f(Y_t)\, dX_t, \\ Y_0 &= a. \end{aligned}$$

Also define a linear map $h : V \oplus W \to Lip[1, V \oplus W, V \oplus W]$ *such that* $\forall x, v \in V$ *and* $y, w \in W$,

$$h(x+y)(v+w) = h(y)(v+w) = v + f(y)\, v.$$

Suppose $1 \leq p < \alpha$. *Then for every geometric rough path* $\mathbf{X}_\cdot \in G\Omega_p(V)$ *such that*

$$\pi_V(\mathbf{X}_\cdot) = X_\cdot,$$

there is exactly one geometric rough path $\mathbf{Z}_\cdot \in G\Omega_p(V \oplus W)$ *such that*

1. *We can express*

$$\pi_{V \oplus W}(\mathbf{Z}_{0,\cdot}) = X_\cdot + Y_\cdot.$$

2. $\mathbf{Z}_{\cdot}$ *satisfies the rough differential equation*

$$\delta \mathbf{Z}_{\cdot} = h\left(Y_t + a\right)\left(\delta \mathbf{Z}_{\cdot}\right).$$

3. *The map* $\mathcal{I}_h : \mathbf{X}_{\cdot} \longmapsto \mathbf{Z}_{\cdot}$ *is the unique continuous extension of the Itô map* $\mathcal{I}_f : G\Omega^1\left(V\right) \rightarrow G\Omega^1\left(V \oplus W\right)$ *with respect to* $p-$*variation topology in such a way that for every finite variation path* $X_{\cdot}$ *with signature* $\mathbf{X}_{\cdot}$,

$$\pi_W\left(\mathcal{I}_h\left(\mathbf{X}\right)_{s,t}\right) = \int_s^t f\left(Y_u\right) dX_u.$$

Proof. For the detail of a proof, one can consult the paper by Lyons [1998] on page 298. ■

3.2.2 Main results

For $m \in \mathbb{N}$, consider a family of linear maps f_m from $\left(\bigoplus_{k=1}^{m} V^{\otimes k}, \left|\cdot\right|_{(m)}\right)$ to the smooth vector fields on $\left(\bigoplus_{k=1}^{m+1} V^{\otimes k}, \left|\cdot\right|_{(m+1)}\right)$.

Definition 3.29 *Let* $\mathbf{X} = X^0 + X^1 + \cdots \in T\left(V\right)$ *with* $X^m \in V^{\otimes m}$ *for* $m = 0, 1, \ldots$. *For every* $k \in \mathbb{N}$, *we define*

$$X^{\{k\}} \triangleq X^1 + \cdots + X^k$$

Given a finite variation path $\gamma : [0,1] \rightarrow V$ with signature $X\left(\gamma\right)_{s,t}$ over $[s,t]$ for all $(s,t) \in \Delta_1$, we construct a sequence of paths recursively.

Definition 3.30 *Define* $\gamma_{\cdot}^{f_1}$ *such that*

$$\gamma_{\cdot}^{f_1} \triangleq \int_0^{\cdot} f_1\left(\gamma_u^{f_1}\right)\left(dX_{0,u}^{\{1\}}\left(\gamma\right)\right) \in \bigoplus_{k=1}^{2} V^{\otimes k}.$$

For every $m \geq 1$, *we also define* $\gamma_{\cdot}^{f_m}$ *to be the solution to the integral equation:*

$$\gamma_{\cdot}^{f_m} = \int_0^{\cdot} f_m\left(\gamma_u^{f_m}\right)\left(d\gamma_u^{f_{m-1}}\right) \in \bigoplus_{k=1}^{m+1} V^{\otimes k}.$$

It is clear that each $\gamma_{\cdot}^{f_m}$ is both continuous and of finite variation; we now attempt to extend this result in the rough path setting. Fix $p \geq 1$.

Proposition 3.31 *For every $m \in \mathbb{Z}^+$, there is a unique map $\mathcal{J}^{f_m}$ from $G\Omega_p(V)$ to $G\Omega_p\left(\bigoplus_{k=1}^{m+1} V^{\otimes k}\right)$ such that $\mathcal{J}^{f_m}$ is continuous with respect to $p-$variation topology and for every finite variation path $\gamma_\cdot$,*

$$\mathcal{J}^{f_m}\left(X(\gamma)_\cdot\right)_\cdot = S\left(\gamma_\cdot^{f_m}\right)_\cdot .$$

Proof. Uniqueness of the map is clear as a continuous extension of a definite map. Let $\gamma : [0,1] \to V$ be a finite variation continuous path. For $m = 1$, by definition, $X_{0,u}^{\{m\}}(\gamma) = X_{0,u}^{1}(\gamma) \in V^{\otimes 1}$. Define a one-form h_1 such that $\forall v^{\{1\}} \in V^{\otimes 1}$ and $w^{\{1\}} + w^{\{2\}} \in V^{\otimes 1} \bigoplus \left(\bigoplus_{k=1}^{2} V^{\otimes k}\right)$,

$$h_1\left(w^{\{1\}} + w^{\{2\}}\right)\left(v^{\{1\}}\right) = v^{\{1\}} + f_1\left(w^{\{2\}}\right)\left(v^{\{1\}}\right).$$

Consider the rough differential equation

$$\delta Z_\cdot^{\{2\}} = h_1\left(Z_\cdot^{\{2\}} + 0\right)\left(\delta Z_\cdot^{\{2\}}\right).$$

According to Proposition 3.28, the Ito map $\mathcal{I}_{h_1}$ is a continuous extension with respect to $p-$ variation topology of $\mathcal{I}_{f_1}$ such that

$$\pi_{G\Omega_p\left(\bigoplus_{k=1}^{2} V^{\otimes k}\right)}\left(\mathcal{I}_{h_1}\left(X(\gamma)_\cdot\right)\right)_\cdot = S\left(\gamma_\cdot^{f_1}\right)_\cdot .$$

So we take

$$\mathcal{J}^{f_1} \triangleq \pi_{G\Omega_p\left(\bigoplus_{k=1}^{2} V^{\otimes k}\right)} \circ \mathcal{I}_{h_1}.$$

For each $m \in \mathbb{Z}^+$, consider a finite variation path $\phi^m : [0,1] \to \bigoplus_{k=1}^{m} V^{\otimes k}$. Define an one-form h_m such that $\forall v^{\{m\}} \in \bigoplus_{k=1}^{m} V^{\otimes k}$, $w^{\{m\}} + w^{\{m+1\}} \in \left(\bigoplus_{k=1}^{m} V^{\otimes k}\right) \oplus \left(\bigoplus_{k=1}^{m+1} V^{\otimes k}\right)$

$$h_m\left(w^{\{m\}} + w^{\{m+1\}}\right)\left(v^{\{m\}}\right) = v^{\{m\}} + f_m\left(w^{\{m+1\}}\right)\left(v^{\{m\}}\right).$$

As above, using Proposition 3.28 again, the Ito map $\mathcal{I}_{h_m}$ is a continuous extension with respect to the $p-$ variation topology of $\mathcal{I}_{f_m}$ such that

$$\pi_{G\Omega_p\left(\bigoplus_{k=1}^{m+1} V^{\otimes k}\right)}\left(\mathcal{I}_{h_m}\left(X(\phi^m)_\cdot\right)\right)_\cdot = S\left(\mathcal{I}_{f_m}(\phi^m)_{0,\cdot}\right)_\cdot ,$$

where $\mathcal{I}_{f_m}(\phi^m)_{0,\cdot}$ is the solution to the integral equation $y_\cdot = \int_0^\cdot f_m(y_u)(d\phi_u^m) \in \bigoplus_{k=1}^{m+1} V^{\otimes k}$. We now take

$$\mathcal{J}^{f_m} \triangleq \pi_{G\Omega_p\left(\bigoplus_{k=1}^{m+1} V^{\otimes k}\right)} \circ \mathcal{I}_{h_m} \circ \cdots \circ \pi_{G\Omega_p\left(\bigoplus_{k=1}^{2} V^{\otimes k}\right)} \circ \mathcal{I}_{h_1}.$$

■

Suppose V is finite dimensional. Fix an integer $N \geq 1$. As a corollary, we deduce that the log-signature of a $p-$geometric rough path can naturally be enhanced as a $p-$geometric rough path.

Corollary 3.32 *Given $X_\cdot \in G\Omega_p(V)$, the truncated log-signature $(logX)_{0,\cdot}^{\{N\}} \in \bigoplus_{k=1}^N V^{\otimes k}$ can naturally be enhanced as a $p-$geometric rough path. In particular, if $\{\mathbf{X}(m)_\cdot\}_{m=1}^\infty$ is a sequence of smooth rough paths converging to $\mathbf{X}_\cdot$ in $G\Omega_p(V)$, then $(\log \mathbf{X}(m))_{0,\cdot}^{\{N\}}$ also converges to $(\log \mathbf{X})_{0,\cdot}^{\{N\}}$ in $G\Omega_p\left(\bigoplus_{k=1}^N V^{\otimes k}\right)$.*

Proof. By suitably choosing the sequence of vector fields $\{f_m\}$ in Proposition 3.31, we obtain our result. For example, suppose V has a basis $\{e_i\}_{i=1}^{\dim V}$, choose f_1 such that for any $v^{\{1\}} \in V^{\otimes 1}$, $w^{\{1\}}+w^{\{2\}} \in V^{\otimes 1} \bigoplus \left(\bigoplus_{k=1}^2 V^{\otimes k}\right)$

$$\begin{aligned} f_1\left(w^{\{1\}}+w^{\{2\}}\right)\left(v^{\{1\}}\right) &= \frac{1}{2}\sum_{i<j\leq \dim(V)} \pi_{<e_i>}\left(w^{\{2\}}\right)\pi_{<e_j>}\left(v^{\{1\}}\right)e_i\otimes e_j \\ &\quad -\pi_{<e_j>}\left(w^{\{2\}}\right)\pi_{<e_i>}\left(v^{\{1\}}\right)e_j\otimes e_i \end{aligned}$$

then for any Lipschitz path $\gamma:[0,1]\to V$, $(\log \mathbf{X}(\gamma))_{0,\cdot}^{\{2\}} = X(\gamma)_{0,\cdot}^1 + A(\gamma)_{0,\cdot}$,

$$\mathcal{J}^{f_1}\left(X(\gamma)_\cdot\right)_\cdot = S\left(X(\gamma)_{0,\cdot}^1 + A(\gamma)_{0,\cdot}\right)_\cdot$$

where $A(\gamma)_{0,\cdot}$ is the Levy area process of $\gamma_\cdot$. The continuity result also holds because of Propositions 3.28 and 3.31. ■

Lemma 3.33 *Let V and W be two Banach spaces and $F: V\to W$ be an $\alpha\,(>p+1)-$Lipschitz function. Suppose $X_\cdot \in G\Omega_p(V)$, then $F\left(X_{0,\cdot}^1\right)$ can naturally be lifted to a $p-$geometric rough path in $G\Omega_p(W)$.*

Proof. For any Lipschitz path $\gamma:[0,1]\to V$, by regarding $F\left(X(\gamma)_{0,\cdot}^1\right)$ as a solution of the differential equation

$$\begin{cases} dy_\cdot = DF\left(X_{0,\cdot}^1\right)dX_{0,\cdot}^1 \\ y_0 = F(0) \end{cases}$$

and using Proposition 3.28, we obtain our claim. ■

In general, we can now obtain our main theorem that any functional of the log-signature of a geometric rough path can naturally be enhanced as a geometric rough path.

Proposition 3.34 *Fix an integer $N \geq 1$. Suppose that V is a finite dimensional and W is a separable Banach space. For any smooth function $F : V \to W$, $X_\cdot \in G\Omega_p(V)$,*

$$F\left((logX)_{0,\cdot}^{\{N\}}\right)$$

can naturally be enhanced as a $p-$geometric rough path in $G\Omega_p(W)$. In particular, the truncated signature

$$X_{0,\cdot}^{\{N\}} = \exp (logX)_{0,\cdot}^{\{N\}}$$

is also a $p-$geometric rough path in $G\Omega_p\left(\bigoplus_{k=1}^{N} V^{\otimes k}\right)$.

Proof. As an immediate consequence of Proposition 3.31 and Lemma 3.33. ■

Chapter 4

Space-filling rough paths

In the theory of rough paths, it admits that the signature (Definition 1.1) of a geometric rough path completely characterizes the path itself in the sense of controlling any arbitrary controlled differential equation. Along the same line of thought, it is interesting to ask if one can find quantities, which are analogous to the signature of a path, that can characterize a high dimensional geometric object $\mathcal{M}$ in the sense of integrating differential forms on $\mathcal{M}$. It is expected that the established theory of rough paths may help us to resolve our curiosity; indeed, one plausible approach is to first use path(s) to represent a high dimensional geometric object $\mathcal{M}$, and then regard a differential form ω as an one-form $\widetilde{\omega}$ over tensors so that integrating the differential form ω on $\mathcal{M}$ is the same as integrating the one-form $\widetilde{\omega}$ against the path(s). More precisely, given a nice high dimensional geometric object $\mathcal{M}$, can one find a space-filling curve γ for $\mathcal{M}$ which can "naturally" be enhanced as a geometric rough path? If the answer were positive, could we find a way so that integrating a differential form ω on $\mathcal{M}$ is equivalent to integrating an one-form $\widetilde{\omega}$ against the enhanced path of γ? In Chapter 4, we shall show that a class of space-filling curves are actually geometric rough paths; while in Chapter 5, we shall discuss the issue about integrating a differential form on a geometric object as contracting a one-form against its space-filling rough paths.

Since many nice geometric objects can be arbitrarily closely approximated by a finite number of hypercubes, it is more tractable to commence our program of research by first looking for a class of space-filling curves for hypercubes that can be lifted as geometric rough paths. In particular, we make a conjecture that for each $d \in \mathbb{N}$, the space-filling curve $F^{[d]}$ (Definition

4.11) for the $d-$dimensional unit hypercube can be naturally (but not in a unique way) enhanced as a $p\,(>d)-$geometric rough path. We shall prove our claim when $d=3$ in this chapter; unfortunately, it is still open for $d>4$ because iterated integrals of order not less than 4 would need to be investigated which seems to be far from trivial.

In Section 4.1, we shall construct the family of space-filling curves $F^{[d]}_{\cdot}$ for hypercubes and show that $F^{[d]}_{\cdot}$ is $1/d-$Hölder continuous. It should be noted that the first such construction for $d=2$ had been done by Buckley [1996]. In Section 4.2 and 4.3, we shall establish respectively the self-similar and reversible nature (Definition 4.30) of $F^{[d]}_{\cdot}$. The key idea to settle our conjecture for $d=3$ is to first develop recursive relations between tensors of iterated integrals of successive approximants as in Subsection 4.4.2. Secondly, in Subsection 4.4.3, we use the reversibility of $F^{[d]}_{\cdot}$ to simplify the tensor of third order iterated integral of each approximant as an integral of the Levy area of the approximant against its increment. Finally, in Subsection 4.4.4 and 4.4.5, we establish conditions that suffice for an application of Ascoli-Azela lemma to conclude our Theorem 4.82 in Subsection 4.4.6.

Let $T>0$. In the following, we again use I to denote $[0,1]$ and Δ_T to denote $\{(s,t):0\leq s\leq t\leq T\}$. Also use C and M to denote constants. We begin with some elementary notion.

Definition 4.1 *Let V be a vector space. Define recursively, for every $n\in\mathbb{N}$,*

$$\begin{aligned}\mathcal{L}^1(V) &= V,\\ \mathcal{L}^{n+1}(V) &= [V,\mathcal{L}^n(V)]\end{aligned}$$

and

$$\mathcal{L}(V)=\oplus_{k=1}^{\infty}\mathcal{L}^k(V).$$

Definition 4.2 *Let $d\in\mathbb{Z}^+$ and $\gamma_\cdot:I\to\mathbb{R}^d$ be a continuous path of finite variation in $\mathbb{R}^d$. For every $(s,t)\in\Delta_1$, define*

$$L^1(\gamma_\cdot)_{s,t} \triangleq \gamma_t-\gamma_s=\sum_{1\leq i\leq d}\left(\gamma_t^i-\gamma_s^i\right)e^i, \tag{4.1}$$

$$L^2(\gamma_\cdot)_{s,t} \triangleq A(\gamma_\cdot)_{s,t}=\sum_{1\leq i<j\leq d}A(\gamma_\cdot)_{s,t}^{i,j}\left[e^i,e^j\right], \tag{4.2}$$

where $\{e^k\}_{k=1}^d$ *is the standard basis of* $\mathbb{R}^d$ *and*

$$A(\gamma.)_{s,t}^{i,j} = \frac{1}{2}\int_{s<u_1<u_2<t} d\gamma_{u_1}^i d\gamma_{u_2}^j - d\gamma_{u_1}^j d\gamma_{u_2}^i, \tag{4.3}$$

is called the Levy area of $\gamma.$ *over* $[s,t]$.

Note that $A(\gamma.)$ can be shown to be independent of the choice of the basis of V.

Theorem 4.3 (Chen) *Consider a finite variation continuous path* $\gamma. : [0,1] \to V$. *Denote its signature over* $[s,t]$ *by* $S(\gamma.)_{s,t}$. *Let* $(s,t) \in \Delta_1$ *and* $n \in \mathbb{N}$. *Then the truncated log-signature*

$$\log\left(S(\gamma)_{s,t}\right)^{(n)} \in \mathcal{L}^{(n)}(V).$$

Furthermore,

$$\log\left(S(\gamma)_{s,t}\right)^{(1)} = L(\gamma.)_{s,t}, \tag{4.4}$$

$$\log\left(S(\gamma)_{s,t}\right)^{(2)} = A(\gamma.)_{s,t}. \tag{4.5}$$

Proof. One can consult the work done by Chen [1957], [1958] for details. ■

4.1 A class of space-filling curves

Let $d \in \mathbb{Z}^+$. In this section, we introduce a $\frac{1}{d}$−Hölder continuous space-filling curve for the unit hypercube I^d. A construction for the two dimensional case can be found in Buckley [1996]. Our construction given here is the extension of the work done by Buckley, namely based on digit construction. We first introduce some basic notion and a few elementary results.

Definition 4.4 *Define* Ω *to be the set of all* $\mathbb{Z}_3$−*valued sequences,*

$$\Omega \triangleq \{(\omega_i)_{i=1}^{\infty} : \omega_i \in \{0,1,2\},\ i = 1,2,\dots\}.$$

We now define a map that contracts any infinite sequence from Ω as a real number in I.

Definition 4.5 *Define a map* $\theta : \Omega \to I$ *such that* $\forall (\omega_i)_{i=1}^{\infty} \in \Omega$,

$$\theta\left((\omega_i)_{i=1}^{\infty}\right) \triangleq (\omega_1\omega_2\dots)_3 = \sum_{i=1}^{\infty} \frac{\omega_i}{3^i}.$$

It is clear that θ is surjective but not injective; for instance, both $(1, 2, 2, \dots)$ and $(2, 0, 0, \dots)$ are mapped to $\frac{2}{3}$ by θ. We next give a notation for the preimage under θ^{-1} of a real number from I.

Definition 4.6 *Define* $\Gamma \subset I$ *to be the set of all numbers having a base-3 expansion* (ω_i) *such that* $\omega_i = 0$ *for all but finitely many* i. *For each* $x \in \Gamma$, *we denote*

$$V_x \triangleq \theta^{-1}(x).$$

We then have an elementary result about the cardinality of a set V_x.

Lemma 4.7 *Every element in* Γ *has exactly two base-3 expansions, i.e.* $\forall x \in \Gamma$,

$$card(V_x) = 2.$$

Proof. Let $x \in \Gamma$, by definition, there is $n_x \in \mathbb{N}$ such that x can be expressed as

$$x = \sum_{i=1}^{\infty} \frac{\omega_i}{3^i},$$

where $\omega_j = 0$ for all $j > n_x$ with $\omega_{n_x} \neq 0$. Hence, we have

$$V_x = \{(\omega_i), (\omega_1, \dots, \omega_{n_x} - 1, 2, 2, \dots)\}.$$

■

Lemma 4.8 *Every element in* I/Γ *has exactly one base-3 expansion, i.e.* $\forall x \in I/\Gamma$,

$$card(V_x) = 1.$$

Proof. Note that for every $x = (\nu_i) \in I/\Gamma$, there should be infinitely many $\nu_i = 1$. Let $(\omega_i) \in \Omega$ so that there are infinitely many $j_k \in \mathbb{Z}^+$ such that $\omega_{j_k} = 1$. For every $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{i=m}^{\infty} \frac{\omega_i}{3^i} &\leq \sum_{i=m}^{\infty} \frac{2}{3^i} - \sum_{j_k \geq m} \frac{1}{3^{j_k}} \\ &< \frac{1}{3^{m-1}}.\end{aligned}$$

Therefore, every $x \in I/\Gamma$ can only have exactly one base-3 expansion. ■

We now introduce a notion for the family of maps $\phi : I \longrightarrow \Omega$ such that $\theta \circ \phi$ is the identity map on I.

Definition 4.9 *Let Φ be the family of all mappings $\phi : I \longrightarrow \Omega$ such that $\forall x \in I$,*

$$\phi(x) \in V_x.$$

We next introduce a way to construct a d−dimensional point in Ω^d out from a point in Ω. The digit swapping mechanism (4.8) ensures $F^{[d]}$ constructed below is a continuous path.

Definition 4.10 *Let $d,\ i \in \mathbb{N}$. For each $r \in \{1, \ldots, d\}$, let*

$$\Delta(d)_{i,r} = \{k \in \mathbb{N} : 1 \leq k \leq i \cdot d + r\} / (d\mathbb{Z} + r). \tag{4.6}$$

Define a map $G^{[d]} : \Omega \longrightarrow \Omega^d$ such that $\forall (\omega_i) \in \Omega$

$$G^{[d]}((\omega_i)) = \left(\left(u_i^{(r)}\right)_{i=1}^{\infty}\right)_{r=1}^{d}, \tag{4.7}$$

where for each $i \in \mathbb{N}$,

$$u_i^{(r)} = \begin{cases} \omega_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} \omega_k \equiv 0 \pmod 2, \\ 2 - \omega_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} \omega_k \equiv 1 \pmod 2. \end{cases} \tag{4.8}$$

Definition 4.11 *For each $\phi \in \Phi$, we define a map $F_\phi^{[d]} : I \longrightarrow I^d$ such that $\forall x \in I$,*

$$F_\phi^{[d]}(x) = \left(\left(\theta \circ \pi_r \circ G^{[d]} \circ \phi\right)(x)\right)_{r=1}^{d},$$

where $\pi_r : \Omega^d \longrightarrow \Omega$ is the projection operator such that

$$\pi_r\left(\left(\left(\omega_i^{(k)}\right)_{i=1}^{\infty}\right)_{k=1}^{d}\right) = \left(\omega_i^{(r)}\right)_{i=1}^{\infty},$$

for every $r \in \{1, \ldots, d\}$.

We shall now show that the definition of $F_\phi^{[d]}$ is independent of the choice of ϕ.

Lemma 4.12 *For each $d \in \mathbb{N}$, the definition of $F_\phi^{[d]}$ is independent of the choice of $\phi \in \Phi$. That is to say, $\forall \phi_1, \phi_2 \in \Phi$ and $x \in I$,*

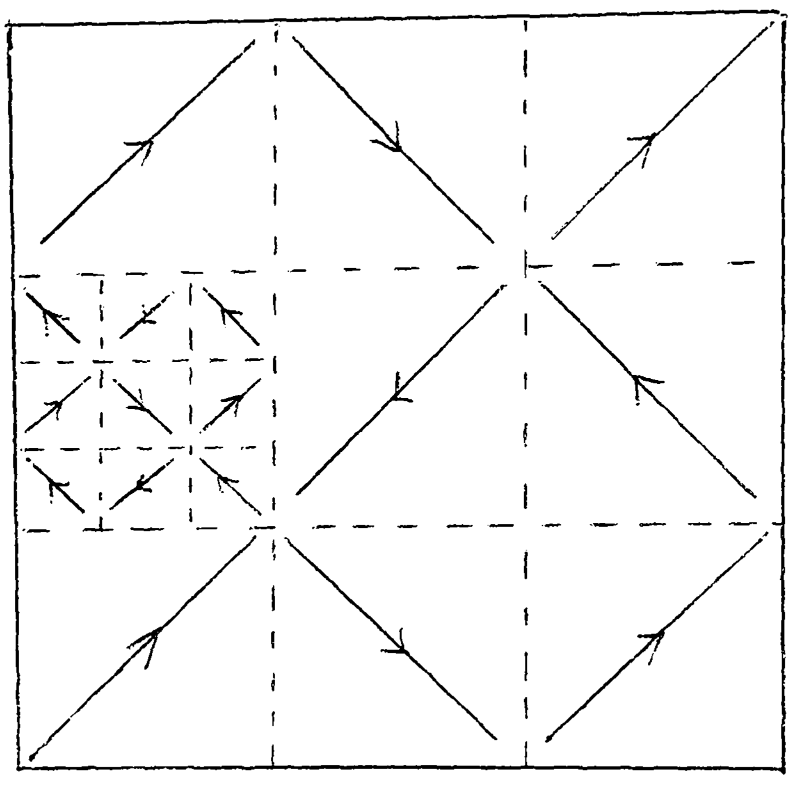
$$F_{\phi_1}^{[d]}(x) = F_{\phi_2}^{[d]}(x).$$

Definition 4.13 *Define*

$$F^{[d]} \triangleq F_\phi^{[d]} \tag{4.9}$$

for some $\phi \in \Phi$.

For a graphical illustration of $F^{[d]}$ in the case of $d = 2$, it is depicted as below:



Proof of lemma 4.12. For each $x \in I/\Gamma$, $card(V_x) = 1$ and it is clear that $F_\phi^{[d]}(x)$ is the same for all choices of ϕ. Without loss of generality, we now consider $x \in \Gamma$ such that its base-3 expansion is

$$x = (\omega_1\omega_2 \ldots \omega_{jd}100\ldots)_3 = (\omega_1\omega_2 \ldots \omega_{jd}022\ldots)_3,$$

for some $j \in \mathbb{N}$. Observe that

$$G^{[d]}((\omega_1\omega_2 \ldots \omega_{jd}100\ldots)_3) = \left(\left(u_i^{(r)}\right)_{i=1}^{\infty}\right)_{r=1}^{d},$$

where

$$\left(u_i^{(1)}\right) = \left(\omega_1, \omega_{d+1}, \ldots, \omega_{(j-1)d+1}, \begin{cases} 1,0,0,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,1}} \omega_k \equiv 0 \ (mod\ 2), \\ 1,2,2,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,1}} \omega_k \equiv 1 \ (mod\ 2). \end{cases}\right.$$

$$\left(u_i^{(r)}\right) = \left(\omega_r, \omega_{d+r}, \ldots, \omega_{(j-1)d+r}, \begin{cases} 2,2,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,r}} \omega_k \equiv 0 \ (mod\ 2), \\ 0,0,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,r}} \omega_k \equiv 1 \ (mod\ 2). \end{cases}\right.$$

for every $r \in \{2, \ldots, d\}$. On the other hand,

$$G^{[d]}\left((\omega_1\omega_2\ldots\omega_{jd}022\ldots)_3\right) = \left(\left(v_i^{(r)}\right)_{i=1}^{\infty}\right)_{r=1}^{d},$$

where

$$\left(v_i^{(1)}\right) = \left(\omega_1, \omega_d, \ldots, \omega_{(j-1)d+1}, \begin{cases} 0,2,2,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,1}} \omega_k \equiv 0 \ (mod\ 2), \\ 2,0,0,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,1}} \omega_k \equiv 1 \ (mod\ 2). \end{cases}\right.$$

$$\left(v_i^{(r)}\right) = \left(\omega_r, \omega_{d+r}, \ldots, \omega_{(j-1)d+r}, \begin{cases} 2,2,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,r}} \omega_k \equiv 0 \ (mod\ 2), \\ 0,0,\ldots), & \text{if } \sum_{k\in\Delta(d)_{j-1,r}} \omega_k \equiv 1 \ (mod\ 2). \end{cases}\right.$$

for every $r \in \{2, \ldots, d\}$. Therefore, we have

$$\theta \circ \pi_r \circ G^{[d]}\left((\omega_1, \ldots, \omega_{jd}, 1, 0, \ldots)_3\right) = \theta \circ \pi_r \circ G^{[d]}\left((\omega_1, \ldots, \omega_{jd}, 0, 2, \ldots)_3\right),$$

and hence, we conclude that all $F_\phi^{[d]}(\cdot)$ agree at x. Other cases can similarly be dealt with. ■

Corollary 4.14 $F^{[d]} : I \to I^d$ *is surjective.*

Proof. For each $k \in \mathbb{N}$ and $r \in \{1, \ldots, d\}$,

$$\sum_{i=1}^{k} u_i^{(r)} = \sum_{i=1}^{k} \omega_{(i-1)d+r} \ (mod\ 2).$$

Together with the fact that elements in $\left\{u_i^{(r)}\right\}_{i=1}^{\infty}$ are defined recursively, the claim that $F^{[d]}$ is surjective is immediate. ■

Definition 4.15 *For each $d \in \mathbb{N}$, define a map $\rho : \Omega^d \to \Omega$ such that $\forall \left(\left(u_i^{(r)}\right)_{i=1}^{\infty}\right)_{r=1}^{d} \in \Omega^d$,*

$$\rho\left(\left(\left(u_i^{(r)}\right)_{i=1}^{\infty}\right)_{r=1}^{d}\right) \triangleq \left(u_1^{(1)}, u_1^{(2)}, \ldots, u_1^{(d)}, u_2^{(1)}, u_2^{(2)}, \ldots\right).$$

We next prove our main result in this section, namely: the Hölder continuity of $F^{[d]}$. The idea of the proof is that our Definition 4.10 of $G^{[d]}$ ensures every point $t \in \left[\frac{k}{3^d}, \frac{k+1}{3^d}\right]$, for $k = 0, \ldots, 3^d - 1$, with its image $F^{[d]}(t)$ to lie in the sub-hypercube with its main diagonal having endpoints $F^{[d]}\left(\frac{k}{3^d}\right)$ and $F^{[d]}\left(\frac{k+1}{3^d}\right)$.

Proposition 4.16 *For each $d \in \mathbb{N}$, $F^{[d]} : I \to I^d$ is $\frac{1}{d}$–Hölder continuous.*

Proof. Without loss of generality, consider $x = (x_i)$, $y = (y_i) \in \Omega$ such that

$$\theta x < \theta y,$$

and $\rho \circ G^{[d]}(x)$, $\rho \circ G^{[d]}(y)$ are different from the $(dm)^{th}$ term on. According to the definition of ρ and Lemma 4.12, we have

$$\begin{aligned} & \left\|F^{[d]}(\theta x) - F^{[d]}(\theta y)\right\|^2 \\ \leq\ & (d-1)\left(\frac{2}{3^{m+1}} + \frac{2}{3^{m+2}} + \ldots\right)^2 + \left(\frac{2}{3^m} + \frac{2}{3^{m+1}} + \ldots\right)^2 \\ =\ & (d-1)\left(\frac{1}{3^m}\right)^2 + \left(\frac{1}{3^{m-1}}\right)^2, \end{aligned}$$

and therefore,

$$\left\|F^{[d]}(\theta x) - F^{[d]}(\theta y)\right\| \leq \frac{1}{3^m}\sqrt{d+8}.$$

In the case that

$$|\theta x - \theta y| \geq 3^{-dm},$$

it is clear that

$$\left\|F^{[d]}(\theta x) - F^{[d]}(\theta y)\right\| \leq \sqrt{d+8}\ |\theta x - \theta y|^{\frac{1}{d}}.$$

Suppose that

$$|\theta x - \theta y| < 3^{-dm}.$$

For if there is a $j \in \mathbb{N}$ with $0 < j < dm$ so that $\forall i < j$,

$$x_i = y_i \text{ but } x_j \neq y_j.$$

Under our assumption that $\theta x < \theta y$, it is either $(1) : y_j = 2$ and $x_j = 0$ or $(2) : y_j = x_j + 1$. In the former case,

$$\begin{aligned} |\theta x - \theta y| &\geq \frac{2}{3^j} - \left(\frac{2}{3^{j+1}} + \frac{2}{3^{j+2}} + \dots\right) \\ &= 3^{-j} \\ &\geq 3^{-dm}, \end{aligned}$$

which contradicts to the assumption that $|\theta x - \theta y| < 3^{-dm}$. On the other hand, in the second case that

$$y_j = x_j + 1,$$

we should have $\forall i \in \mathbb{N}$ with $j < i \leq dm$,

$$x_i = 2, \ y_i = 0;$$

otherwise, one can find a $j' \in \mathbb{N}$ with $j < j' \leq dm$ such that

$$y_{j'} - x_{j'} \geq -1,$$

and

$$\begin{aligned} \theta y - \theta x &\geq \frac{1}{3^j} - \left(\frac{2}{3^{j+1}} + \dots + \frac{2}{3^{j'-1}} + \frac{1}{3^{j'}} + \frac{2}{3^{j'+1}} + \dots\right) \\ &= 3^{-j'} \\ &\geq 3^{-dm}, \end{aligned}$$

which again contradicts our assumption that $|\theta x - \theta y| < 3^{-dm}$. However, even in the present case, the condition still leads to a contradiction to the assumption that $\rho \circ G^{[d]}(x)$, $\rho \circ G^{[d]}(y)$ differ from the $(dm)^{th}$ term on. Therefore, we can only have, $\forall i < dm$,

$$x_i = y_i.$$

Since $\theta x < \theta y$ and so $y_{dm} > x_{dm}$, it is either $(1) : y_{dm} = 2$ and $x_{dm} = 0$ or $(2) : y_{dm} = x_{dm} + 1$; in the former case, we again have

$$|\theta x - \theta y| \geq 3^{-dm}.$$

In conclusion, we have $x_i = y_i$ for all $i < dm$ and $y_{dm} = x_{dm} + 1$. Consider $z = (z_i), \widetilde{z} = (\widetilde{z}_i) \in \Omega$ such that

$$\begin{cases} z_i = y_i, & \text{for } i \leq dm, \\ z_i = 0, & \text{for } i > dm, \end{cases}$$

and

$$\begin{cases} \widetilde{z}_i = x_i, & \text{for } i \leq nm, \\ \widetilde{z}_i = 2, & \text{for } i > nm. \end{cases}$$

It is clear that because $y_{dm} = x_{dm} + 1$, we have

$$\theta x \leq \theta \widetilde{z} = \theta z \leq \theta y.$$

If y and z are different from the k^{th} $(> dm)$ term on, then

$$\begin{aligned} \theta y - \theta x &\geq \theta y - \theta z \\ &\geq 3^{-k+1}, \end{aligned}$$

and

$$\left\| F^{[d]}(\theta z) - F^{[d]}(\theta y) \right\| \leq 3\sqrt{d}\left(3^{-\lceil \frac{k}{d} \rceil}\right).$$

Similarly, if x and $\widetilde{z}$ are different from the l^{th} $(> dm)$ term on, then

$$\begin{aligned} \theta y - \theta x &\geq \theta \widetilde{z} - \theta x \\ &\geq 3^{-l+1}, \end{aligned}$$

and

$$\left\| F^{[d]}(\theta \widetilde{z}) - F^{[d]}(\theta x) \right\| \leq 3\sqrt{d}\left(3^{-\lceil \frac{l}{d} \rceil}\right).$$

Using Lemma 4.12 again, we can find ϕ and $\widetilde{\phi} \in \Phi$ such that

$$\begin{aligned} (\phi \circ \theta)(z) &= z, \\ \left(\widetilde{\phi} \circ \theta\right)(\widetilde{z}) &= \widetilde{z}. \end{aligned}$$

hence we get

$$\begin{aligned} F^{[d]}(\theta z) &= F^{[d]}_{\phi}(\theta z) = F^{[d]}_{\phi}(\theta \widetilde{z}) = F^{[d]}_{\widetilde{\phi}}(\theta \widetilde{z}) \\ &= F^{[d]}(\theta \widetilde{z}). \end{aligned}$$

Finally, we have

$$
\begin{aligned}
& \left\| F^{[d]}(\theta y) - F^{[d]}(\theta x) \right\| \\
\leq\ & \left\| F^{[d]}(\theta y) - F^{[d]}(\theta z) \right\| + \left\| F^{[d]}(\theta z) - F^{[d]}(\theta \widetilde{z}) \right\| + \left\| F^{[d]}(\theta \widetilde{z}) - F^{[d]}(\theta x) \right\| \\
\leq\ & 3\sqrt{d}\left(3^{-\lceil \frac{k}{d} \rceil}\right) + 0 + 3\sqrt{d}\left(3^{-\lceil \frac{l}{d} \rceil}\right) \\
\leq\ & 6\sqrt{d}\left(3^{-\min\left(\frac{k}{d}, \frac{l}{d}\right)}\right).
\end{aligned}
$$

Note that $\theta y - \theta x \geq 3^{-\min(k-1, l-1)}$, and hence,

$$
\left\| F^{[d]}(\theta y) - F^{[d]}(\theta x) \right\| \leq 6\left(3^{-\frac{1}{d}}\sqrt{d}\right)|\theta y - \theta x|^{\frac{1}{d}}.
$$

■

We now conclude our main claim:

Corollary 4.17 *$F^{[d]} : I \longrightarrow I^d$ is a space-filling curve for I^d.*

Proof. By combining the results in Corollary 4.14 and Proposition 4.16, we conclude our claim. ■

4.2 Self-similarity of $F_{\cdot}^{[d]}$

In this section, we aim to establish the self-similarity of the space-filling curve $F_{\cdot}^{[d]}$ for the unit hypercube I^d. We first study properties and the cardinality of the set of all orthogonal transformations induced by increments of $F_{\cdot}^{[d]}$ (see Corollary 4.22).

Definition 4.18 *For every $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, define a linear transformation $Q_v : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $\forall w \in \mathbb{R}^d$,*

$$
Q_v(w) = \sum_{i=1}^{d} (v_i \cdot w_i)\, e^i,
$$

where $\{e^i\}_{i=1}^d$ is the standard basis for $\mathbb{R}^d$.

Lemma 4.19 *Let $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ with $|v_i| = 1$ for all $i = 1, \ldots, d$. Then, $Q_v : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is an orthogonal transformation.*

Definition 4.20 *For every $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, 3^{d \cdot k}\}$, define*

$$t_j^k(d) \triangleq \frac{j}{3^{d \cdot k}} \tag{4.10}$$

and a partition of I,

$$\mathcal{D}^k(d) \triangleq \left\{t_j^k(d) : 0 \leq j \leq 3^{d \cdot k}\right\}.$$

We first show that $F^{[d]}\left(t_j^1(d)\right)$ and $F^{[d]}\left(t_{j+1}^1(d)\right)$ are endpoints of the main diagonal of a sub-hypercube.

Lemma 4.21 *For every $d \in \mathbb{N}$ and $j \in \{0, \ldots, 3^d - 1\}$, we express*

$$\begin{aligned} & F^{[d]}\left(t_{j+1}^1(d)\right) - F^{[d]}\left(t_j^1(d)\right) \\ \triangleq & \sum_{i=1}^{d} \left(F^{[d]}\left(t_{j+1}^1(d)\right) - F^{[d]}\left(t_j^1(d)\right)\right)_i e^i. \end{aligned} \tag{4.11}$$

Suppose that

$$t_j^1(d) = \sum_{i=1}^{d} \frac{\omega_i}{3^i}.$$

Then, for every $l \in \{1, \ldots, d\}$,

$$\begin{aligned} & \left(F^{[d]}\left(t_{j+1}^1(d)\right) - F^{[d]}\left(t_j^1(d)\right)\right)_l \\ = & \begin{cases} \frac{1}{3}, & \text{if } \sum_{k \in \Delta(d)_{1,l}} \omega_k \equiv 0 \pmod 2, \\ -\frac{1}{3}, & \text{if } \sum_{k \in \Delta(d)_{1,l}} \omega_k \equiv 1 \pmod 2. \end{cases} \end{aligned} \tag{4.12}$$

Proof. Without loss of generality, we consider the case that for some $m < d$,

$$t_j^1(d) = \sum_{i=1}^{d} \frac{\omega_i}{3^i}$$

with $\omega_i = 0$ for all $i = m+1, \ldots, d$. Note that

$$t_{j+1}^1(d) = \sum_{i=1}^{d-1} \frac{\omega_i}{3^i} + \frac{1}{3^d}.$$

According to Definition 4.10, for every $l \in \{1, \ldots, d\}$,

$$F^{[d]}\left(t_j^1(d)\right)_l = (u_l, \begin{cases} 0,0,\ldots)_3, & \text{if } \sum_{k\in\Delta(d)_{1,l}} \omega_k \equiv 0 \ (mod\ 2), \\ 2,2,\ldots)_3, & \text{if } \sum_{k\in\Delta(d)_{1,l}} \omega_k \equiv 1 \ (mod\ 2), \end{cases}$$

where

$$u_l = \begin{cases} \omega_l, & \text{if } \sum_{k\le l-1} \omega_k \equiv 0 \ (mod\ 2), \\ 2-\omega_l, & \text{if } \sum_{k\le l-1} \omega_k \equiv 1 \ (mod\ 2). \end{cases}$$

On the other hand, for each $m \in \{1, \ldots, d-1\}$,

$$F^{[d]}\left(t_{j+1}^1(d)\right)_m = (u_m, \begin{cases} 2,2,\ldots)_3, & \text{if } \sum_{k\in\Delta(d)_{1,m}} \omega_k \equiv 0 \ (mod\ 2), \\ 0,0,\ldots)_3, & \text{if } \sum_{k\in\Delta(d)_{1,m}} \omega_k \equiv 1 \ (mod\ 2), \end{cases}$$

$$F^{[d]}\left(t_{j+1}^1(d)\right)_d = (1, \begin{cases} 0,0,\ldots)_3, & \text{if } \sum_{k\le d-1} \omega_k \equiv 0 \ (mod\ 2), \\ 2,2,\ldots)_3, & \text{if } \sum_{k\le d-1} \omega_k \equiv 1 \ (mod\ 2). \end{cases}$$

Therefore, for every $l \in \{1, \ldots, d\}$,

$$\begin{aligned} &\left(F^{[d]}\left(t_{j+1}^1(d)\right) - F^{[d]}\left(t_j^1(d)\right)\right)_l \\ &= \begin{cases} \sum_{k=2}^{\infty} \frac{2}{3^k}, & \text{if } \sum_{k\in\Delta(d)_{1,l}} \omega_k \equiv 0 \ (mod\ 2), \\ -\sum_{k=2}^{\infty} \frac{2}{3^k}, & \text{if } \sum_{k\in\Delta(d)_{1,l}} \omega_k \equiv 1 \ (mod\ 2). \end{cases} \end{aligned}$$

Other cases can similarly be dealt with. ∎

We now deduce our claim that each increment $F^{[d]}\left(t_{j+1}^1(d)\right) - F^{[d]}\left(t_j^1(d)\right)$ induces an orthogonal transformation.

Corollary 4.22 *For every $d \in \mathbb{N}$ and $j \in \left\{0, \ldots, 3^d - 1\right\}$, define*

$$\delta_j^{[d]} \triangleq F^{[d]}\left(t_{j+1}^1(d)\right) - F^{[d]}\left(t_j^1(d)\right). \tag{4.13}$$

Then

$$Q_{3\delta_j^{[d]}} : \mathbb{R}^d \to \mathbb{R}^d$$

is an orthogonal transformation on $\mathbb{R}^d$.

Proof. The claim is an immediate consequence of Lemmas 4.19 and 4.21. ∎

For every $j \in \{1, \ldots, 3^{d-1}\}$, if we express $t_j^1(d)$ as:

$$t_j^1(d) = \sum_{k=1}^{d} \frac{\omega\left(t_j^1(d)\right)_k}{3^k},$$

it is clear that for each $i \in \{1, \ldots, d\}$,

$$\sum_{k \in \Delta_{0,i}} \omega\left(t_j^1(d)\right)_k = \sum_{k=1}^{d} \omega\left(t_j^1(d)\right)_k - \omega\left(t_j^1(d)\right)_i.$$

We now count the total number of different $Q_{3\delta_j^{[d]}}$.

Corollary 4.23 *For every $d \in \mathbb{N}$, define*

$$Q(d) \triangleq \left\{Q_{3\delta_j^{[d]}} : j = 0, \ldots, 3^d - 1\right\}. \tag{4.14}$$

Then

$$card\left(Q(d)\right) = \begin{cases} 2^d, & if\ d \equiv 0 \pmod 2, \\ 2^{d-1}, & if\ d \equiv 1 \pmod 2. \end{cases} \tag{4.15}$$

Proof. Let

$$s \triangleq (s_1, \ldots, s_{d-1}) \in \{-1, 1\}^{d-1}.$$

Choose $\omega_k \in \{0, 1, 2\}$ for $k \in \{1, \ldots, d\}$ such that

$$\sum_{k=1}^{d} \omega_k = 1 \pmod 2,$$

and for every $k \in \{1, \ldots, d-1\}$,

$$\omega_k = 1 - \frac{1 + s_k}{2}.$$

Using Lemma 4.21, there is j_1 with

$$t_{j_1}^1(d) = \sum_{k=1}^{d} \frac{\omega_k}{3^k}.$$

and

$$\delta_{j_1}^{[d]} = \frac{1}{3}\left(s_1, \ldots, s_{d-1}, s_d\right),$$

where

$$s_d = 2\left\{(d-1) - \sum_{k=1}^{d-1} \frac{1+s_k}{2} \ (mod\ 2)\right\} - 1.$$

Also choose $\nu_k \in \{0,1,2\}$, $k = 1,\ldots,d$ such that

$$\sum_{k=1}^{d} \nu_k = 0 \ (mod\ 2),$$

and for every $k \in \{1,\ldots,d-1\}$,

$$\nu_k = -\frac{1+s_k}{2}.$$

Using Lemma 4.21 again, there is j_2 with

$$t_{j_2}^1(d) = \sum_{k=1}^{d} \frac{\nu_k}{3^k}.$$

and

$$\delta_{j_1}^{[d]} = \frac{1}{3}(s_1,\ldots,s_{d-1},s_d'),$$

where

$$s_d' = 2\left\{-\sum_{k=1}^{d-1} \frac{1+s_k}{2} \ (mod\ 2)\right\} - 1.$$

It is clear that $s_d = s_d'$ if and only if d is odd; hence the result follows. ∎

Finally, we show that each $Q_{3\delta_j^{[d]}}$ is positively oriented when d is an odd number. We first have a lemma:

Lemma 4.24 *Let d be an odd number. For every $j \in \{0,\ldots,3^d-1\}$,*

$$\prod_{l=1}^{d} sgn\left(\delta_j^{[d]}\right)_l = 1.$$

Proof. According to Lemma 4.21,

$$sgn\left(\delta_j^{[d]}\right)_l = \begin{cases} 1, & \text{if } \sum_{k\in\Delta(d)_{1,l}} \omega_k \equiv 0 \ (mod\ 2), \\ -1, & \text{if } \sum_{k\in\Delta(d)_{1,l}} \omega_k \equiv 1 \ (mod\ 2), \end{cases}$$

where

$$t_j^1(d) = \sum_{k=1}^{d} \frac{\omega_k}{3^k}.$$

Let m and n be the number of "0" and "2" and the number of "1" respectively and so $d = m + n$. We have two cases:

1. For even m and odd n, there are an even number of $sgn\left(\delta_j^{[d]}\right)_l = -1$ and an odd number of $sgn\left(\delta_j^{[d]}\right)_l = 1$.

2. For odd m and even n, we again have an even number of $sgn\left(\delta_j^{[d]}\right)_l = -1$ and an odd number of $sgn\left(\delta_j^{[d]}\right)_l = 1$.

■

Corollary 4.25 *Let d be an odd number. For every $j \in \{0, \ldots, 3^d - 1\}$, $Q_{3\delta_j^{[d]}} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ has a unit determinant, i.e.*

$$\det\left(Q_{3\delta_j^{[d]}}\right) = 1. \tag{4.16}$$

Finally, we conclude this section with our main result that $F^{[d]}$ is self-similar which eases our computation of controls for the approximants of $F^{[d]}$ in the course of establishing the geometric rough path nature of $F^{[d]}$.

Proposition 4.26 (Self-similarity of $F^{[d]}$) *Let $j \in \{0, \ldots, 3^d - 1\}$. For any $t \in \left[t_j^1(d), t_{j+1}^1(d)\right]$, we have*

$$F^{[d]}(t) = F^{[d]}\left(t_j^1(d)\right) + Q_{3\delta_j^{[d]}} \circ \delta_{\frac{1}{3}}\left(F^{[d]}\left(3^d\left(t - t_j^1(d)\right)\right)\right). \tag{4.17}$$

Proof. Let

$$t_j^1(d) = \sum_{i=1}^{d} \frac{\omega_i}{3^i}$$

and

$$t = \sum_{i=1}^{d} \frac{\omega_i}{3^i} + \sum_{i=1}^{\infty} \frac{v_i}{3^{d+i}}.$$

Then

$$3^d\left(t - t_j^1(d)\right) = \sum_{i=1}^{\infty} \frac{v_i}{3^i},$$

by Definition 4.11 and Lemma 4.12, we have

$$F^{[d]}\left(3^d\left(t - t_j^1(d)\right)\right) = \left(\left(u_j^{(m)}\right)_3\right)_{m=1}^d,$$

where

$$u_j^{(m)} = \begin{cases} v_{(j-1)d+m}, & \text{if } \sum_{k\in\Delta(d)_{j-1,m}} v_k \equiv 0 \ (mod\ 2), \\ 2 - v_{(j-1)d+m}, & \text{if } \sum_{k\in\Delta(d)_{j-1,m}} v_k \equiv 1 \ (mod\ 2). \end{cases}$$

On the other hand, we also have

$$F^{[d]}(t) = \left(\left(a_j^{(m)}\right)_3\right)_{m=1}^d,$$

where

$$a_1^{(m)} = \begin{cases} \omega_m, & \text{if } \sum_{k\leq m-1} \omega_k \equiv 0 \ (mod\ 2), \\ 2 - \omega_m, & \text{if } \sum_{k\leq m-1} \omega_k \equiv 1 \ (mod\ 2). \end{cases}$$

and for each $j \in \mathbb{N}$ with $j \geq 2$,

$$\begin{aligned} a_j^{(m)} &= \begin{cases} v_{(j-2)d+m}, & \text{if } \sum_{k\neq m} \omega_k + \sum_{k\in\Delta(d)_{j-2,m}} v_k \equiv 0 \ (mod\ 2), \\ 2 - v_{(j-2)d+m}, & \text{if } \sum_{k\neq m} \omega_k + \sum_{k\in\Delta(d)_{j-2,m}} v_k \equiv 1 \ (mod\ 2). \end{cases} \\ &= \begin{cases} u_{j-1}^{(m)}, & \text{if } \sum_{k\neq m} \omega_k \equiv 0 \ (mod\ 2), \\ 2 - u_{j-1}^{(m)}, & \text{if } \sum_{k\neq m} \omega_k \equiv 1 \ (mod\ 2). \end{cases} \end{aligned}$$

Therefore, for every $m \in \{1, \ldots, d\}$,

$$F^{[d]}(t)_m = F^{[d]}\left(t_j^1(d)\right)_m + \frac{(-1)^{\sum_{k\neq m}\omega_k}}{3} \cdot F^{[d]}\left(3^d\left(t - t_j^1(d)\right)\right)_m.$$

Using Lemma 4.21, we conclude our claim. ■

Definition 4.27 *For every $d \in \mathbb{Z}^+$ and $j \in \{0, \ldots 3^d - 1\}$, define an affine map $P_j^{[d]} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\forall v \in \mathbb{R}^d$,*

$$P_j^{[d]}(v) \triangleq Q_{3\delta_j^{[d]}} \circ \delta_{\frac{1}{3}}(v), \tag{4.18}$$

where, for any $\lambda > 0$, $\delta_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ is the $\lambda-$dilation map so that $\forall v \in \mathbb{R}^d$,

$$\delta_\lambda(v) \triangleq \lambda \cdot v.$$

Definition 4.28 *Let $r, k \in \mathbb{Z}^{+}$. For every $j \in \{0, \ldots, 3^{dk}-1\}$, we ex all $t_j^k(d) \in [0,1]$ as:*

$$t_j^k(d) \triangleq \sum_{i=1}^{kd} \frac{\omega_i\left(t_j^k(d)\right)}{3^i}.$$

Define a map $n_r : \mathcal{D}^r(d) \rightarrow \mathbb{N}$ such that

$$n_r\left(t_j^k(d)\right) = 3^d \left(\sum_{i=1}^{d} \frac{\omega_{(r-1)d+i}\left(t_j^k(d)\right)}{3^i} \right). \qquad ($$

Corollary 4.29 *Let $k \in \mathbb{Z}^{+}$. For any $t \in \left[t_j^k(d), t_{j+1}^k(d)\right]$, we can exp*

$$\begin{aligned}
& F^{[d]}(t) \\
= \; & F^{[d]}\left(\frac{n_1\left(t_j^k(d)\right)}{3^d}\right) \\
& + \sum_{r=1}^{k-1} P^{[d]}_{n_1\left(t_j^k(d)\right)} \circ \cdots \circ P^{[d]}_{n_r\left(t_j^k(d)\right)} \left(F^{[d]}\left(\frac{n_{r+1}\left(t_j^k(d)\right)}{3^d}\right)\right) \\
& + P^{[d]}_{n_1\left(t_j^k(d)\right)} \circ \cdots \circ P^{[d]}_{n_k\left(t_j^k(d)\right)} \left(F^{[d]}\left(3^{kd}\left(t - t_j^k(d)\right)\right)\right). \qquad (
\end{aligned}$$

Proof. We shall prove the result by induction. In accordance with P: sition 4.26, the result is true for the case $k = 1$. Assume it is true f $k \leq n$. Note that for every $k \geq 2$ and $r \geq 1$,

$$n_{r+1}\left(t_j^k(d)\right) = n_r\left(3^d\left(t_j^k(d) - \frac{n_1\left(t_j^k(d)\right)}{3^d}\right)\right),$$

and consequently, using the induction hypothesis, we have

$$\begin{aligned}
& F^{[d]}\left(3^d\left(t - \frac{n_1\left(t_j^{n+1}(d)\right)}{3^d}\right)\right) \\
= \; & F^{[d]}\left(\frac{n_2\left(t_j^{n+1}(d)\right)}{3^d}\right) \\
& + \sum_{r=1}^{n-1} P^{[d]}_{n_2\left(t_j^{n+1}(d)\right)} \circ \cdots \circ P^{[d]}_{n_{r+1}\left(t_j^{n+1}(d)\right)} \left(F^{[d]}\left(\frac{n_{r+2}\left(t_j^{n+1}(d)\right)}{3^d}\right)\right), \\
& + P^{[d]}_{n_2\left(t_j^{n+1}(d)\right)} \circ \cdots \circ P^{[d]}_{n_k\left(t_j^{n+1}(d)\right)} \left(F^{[d]}\left(3^{nd} \cdot 3^d\left(t - t_j^{n+1}(d)\right)\right)\right).
\end{aligned}$$

Using Proposition 4.26 again, we have

$$F^{[d]}(t) = F^{[d]}\left(\frac{n_1\left(t_j^{n+1}(d)\right)}{3^d}\right) + P^{[d]}_{n_1\left(t_j^{n+1}(d)\right)}\left(F^{[d]}\left(3^d\left(t - \frac{n_1\left(t_j^{n+1}(d)\right)}{3^d}\right)\right)\right).$$

After substitution, we deduce our claim. ■

4.3 Reversible paths

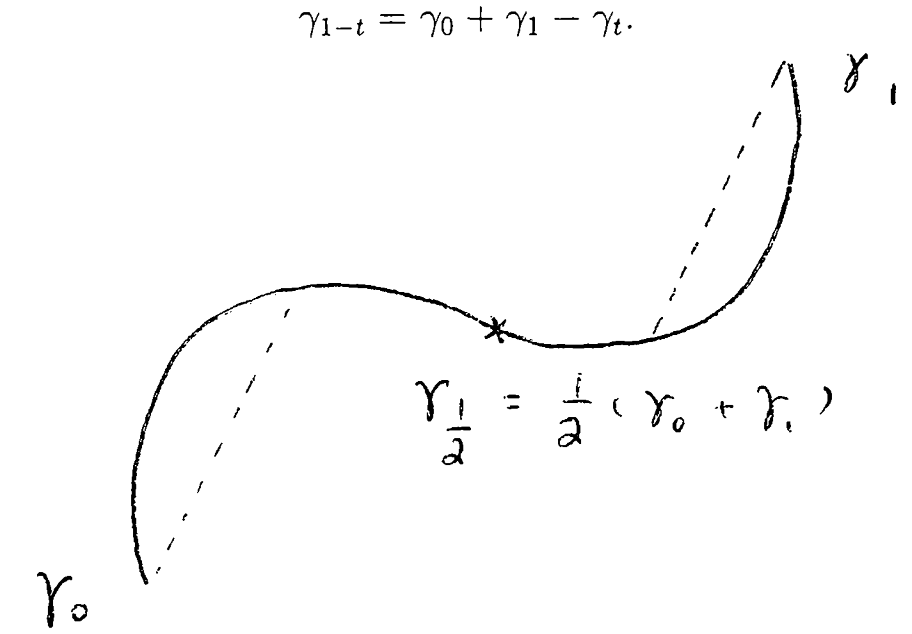
In this section, we introduce the notion of reversible paths and then show that both $F^{[d]}_{\cdot}$ and the path defined in Definition 4.34 are reversible paths. In Section 4.4, we shall use the reversibility of $F^{[d]}_{\cdot}$ to simplify the level three component of the signature of each approximant of $F^{[d]}_{\cdot}$ as an integral of the Levy area of the approximant against its increment.

Definition 4.30 *Any continuous path $\gamma_{\cdot} : I \longrightarrow \mathbb{R}^d$ is said to be reversible if $\forall t \in I$,*

$$\gamma_t - \gamma_0 = -\left(\gamma_{1-t} - \gamma_1\right), \tag{4.21}$$

or

$$\gamma_{1-t} = \gamma_0 + \gamma_1 - \gamma_t. \tag{4.22}$$



Notice that reversibility of a path is preserved under any affine transformation.

Lemma 4.31 *Let $A : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an affine map and $\gamma_{\cdot} : I \longrightarrow \mathbb{R}^d$ be a reversible path. Then, $A \circ \gamma_{\cdot} : I \longrightarrow \mathbb{R}^d$ is also reversible.*

Proof. The linear relation in the definition of reversibility of a path is preserved under any affine transformation. ■

Before establishing a common property of reversible paths, we first establish the claim that $F^{[d]}$ is a reversible path. Once again, Definition 4.10 of $G^{[d]}$ plays a key role in our argument leading to the claim.

Lemma 4.32 *$F^{[d]} : I \to I^d$ is a reversible path.*

Proof. Let $t \in I$ be such that

$$t = \sum_{k=1}^{\infty} \frac{\omega_k}{3^k}.$$

By construction, if $F^{[d]}(t) = \left(F^{[d]}(t)_1, \ldots, F^{[d]}(t)_d\right)$, then for every $r \in \{1, \ldots, d\}$,

$$F^{[d]}(t)_r = \sum_{k=1}^{\infty} \frac{u_k^{(r)}}{3^k},$$

such that for all $i \in \mathbb{Z}^+$,

$$u_i^{(r)} = \begin{cases} \omega_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} \omega_k \equiv 0 \ (mod\ 2), \\ 2 - \omega_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} \omega_k \equiv 1 \ (mod\ 2). \end{cases}$$

Note that $F^{[d]}(0) = (0, \ldots, 0)$ and $F^{[d]}(1) = (1, \ldots, 1)$. Now,

$$\begin{aligned} F^{[d]}(0)_r + F^{[d]}(1)_r - F^{[d]}(t)_r &= \sum_{i=1}^{\infty} \frac{2 - u_i^{(r)}}{3^i} \\ &= \sum_{i=1}^{\infty} \frac{v_i^{(r)}}{3^i}, \end{aligned}$$

where for every $i \in \mathbb{Z}^+$,

$$\begin{aligned} v_i^{(r)} &= \begin{cases} 2 - \omega_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} \omega_k \equiv 0 \ (mod\ 2), \\ \omega_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} \omega_k \equiv 1 \ (mod\ 2). \end{cases} \\ &= \begin{cases} w_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} w_k \equiv 0 \ (mod\ 2), \\ 2 - w_{(i-1)d+r}, & \text{if } \sum_{k \in \Delta(d)_{i-1,r}} w_k \equiv 1 \ (mod\ 2), \end{cases} \end{aligned}$$

such that $\forall n \in \mathbb{N}$, $w_n = 2 - \omega_n$. It is clear that

$$\sum_{k=1}^{\infty} \frac{w_k}{3^k} = 1 - \sum_{k=1}^{\infty} \frac{\omega_k}{3^k} = 1 - t,$$

hence for every $r \in \{1, \dots, d\}$,

$$F^{[d]}(0)_r + F^{[d]}(1)_r - F^{[d]}(t)_r = F^{[d]}(1-t)_r .$$

■

Consider a reversible path $\gamma_\cdot : I \longrightarrow \mathbb{R}^d$. According to Chen's theorem, Theorem 4.3, for every $(s,t) \in \Delta_1$, we can express the signature

$$S(\gamma_\cdot)_{s,t} = \exp\left(\sum_{i=1}^{\infty} l(\gamma_\cdot)_{s,t}^i\right),$$

for some $l(\gamma_\cdot)_{s,t}^i \in \mathcal{L}^i(\mathbb{R}^d)$, $i \in \mathbb{N}$. We then have the result that every even order component of the signature over $[0,1]$ is vanished.

Lemma 4.33 *Let $\gamma_\cdot : I \longrightarrow \mathbb{R}^d$ be a reversible path. Then for every $n \in \mathbb{N}$, we have*

$$l(\gamma_\cdot)_{0,1}^{2n} = 0.$$

Proof. Since $\gamma_{1-\cdot}$ is the path running backwards, so

$$S(\gamma_{1-\cdot})_{0,1} = S(\gamma_\cdot)_{0,1}^{-1} = \exp\left(-\sum_{i=1}^{\infty} l(\gamma_\cdot)_{0,1}^i\right). \tag{4.23}$$

Since $\gamma_\cdot : I \longrightarrow \mathbb{R}^d$ is reversible, we therefore have

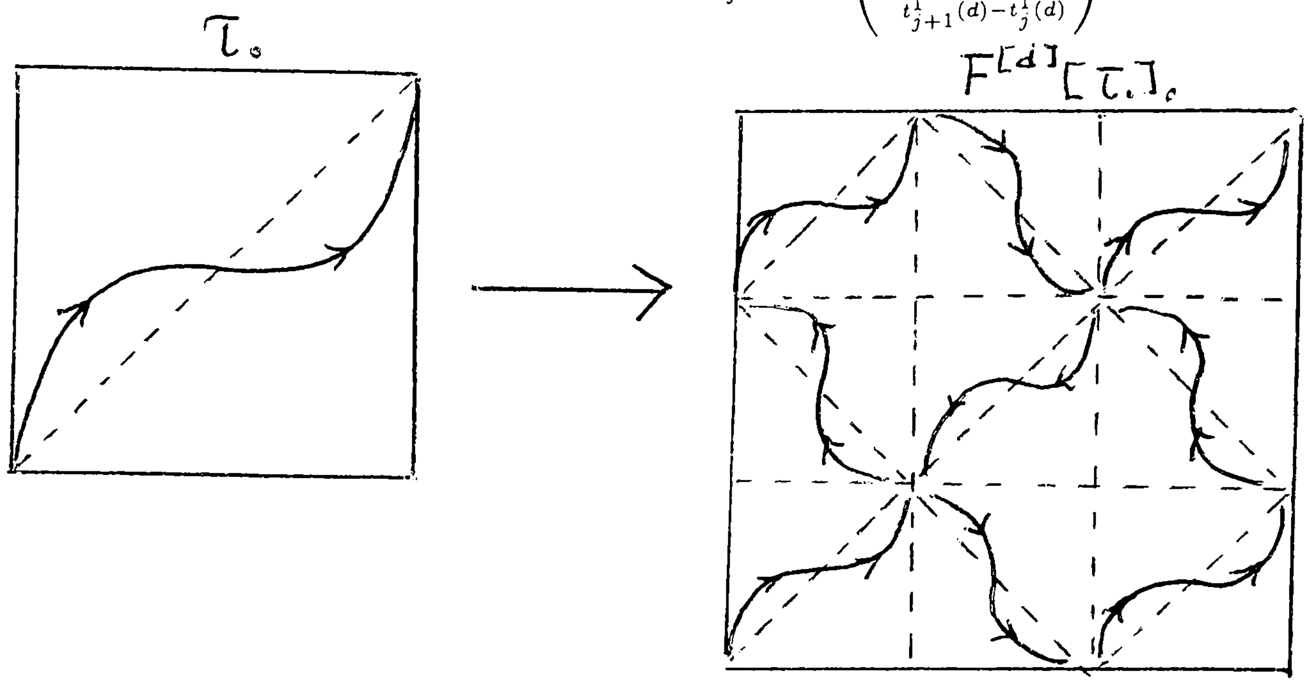
$$S(\gamma_{1-\cdot})_{0,1} = \exp\left(\sum_{i=1}^{\infty} (-1)^i \cdot l(\gamma_\cdot)_{0,1}^i\right). \tag{4.24}$$

By equating the two expressions (4.23) and (4.24), we deduce our claim. ■

It would be interesting to ask if the converse of Lemma 4.33 still holds; unfortunately, before the submission of this thesis, the problem still remains unsolved. We next generalize our claim in Lemma 4.32 to more general class of paths which are basically induced by $F_\cdot^{[d]}$. We shall later show in our main theorem, Theorem 4.82, that each of various $F^{[d]}[\tau_\cdot]_\cdot$ would induce a sequence of paths that will converge to different geometric rough path yet having a common increment process $F_\cdot^{[d]}$.

Definition 4.34 *Let $\tau_\cdot : I \longrightarrow I^d$ be a continuous path such that $\tau_0 = (0,\ldots,0)$ and $\tau_1 = (1,\ldots,1)$. Define a continuous path $F^{[d]}[\tau_\cdot]_\cdot : I \longrightarrow I^d$ so that for every $j \in \{0,\ldots,3^d\}$ and $t \in \left[t_j^1(d), t_{j+1}^1(d)\right]$,*

$$F^{[d]}[\tau_\cdot]_t \triangleq F^{[d]}\left(t_j^1(d)\right) + Q_{3\delta_j^{[d]}} \circ \delta_{\frac{1}{3}}\left(\tau_{\frac{t-t_j^1(d)}{t_{j+1}^1(d)-t_j^1(d)}}\right). \tag{4.25}$$



Lemma 4.35 *Suppose $\tau_\cdot : I \longrightarrow I^d$ is a reversible path such that $\tau_0 = (0,\ldots,0)$ and $\tau_1 = (1,\ldots,1)$. Then $F^{[d]}[\tau_\cdot]_\cdot : I \longrightarrow I^d$ is also reversible.*

Proof. Denote $(1,\ldots,1)$ by $\mathbf{1}$. For every $j \in \{0,\ldots,3^d\}$ and $t \in \left[t_j^1(d), t_{j+1}^1(d)\right]$, let

$$\begin{aligned} u_j(t) &\triangleq \frac{t - t_j^1(d)}{t_{j+1}^1(d) - t_j^1(d)}, \\ v_j(t) &\triangleq \frac{1 - t - \left(1 - t_{j+1}^1(d)\right)}{\left(1 - t_j^1(d)\right) - \left(1 - t_{j+1}^1(d)\right)}. \end{aligned}$$

It is clear that

$$1 - u_j(t) = v_j(t).$$

Since $\tau_\cdot$ is reversible, hence

$$\begin{aligned} -\tau_{u_j(t)} &= \tau_{1-u_j(t)} - \mathbf{1} \\ &= \tau_{v_j(t)} - \mathbf{1}. \end{aligned}$$

Note that, using the fact that $F_\cdot^{[d]}$ is reversible, we have

$$\begin{aligned}\delta_j^{[d]} &= F^{[d]}\left(t_{j+1}^1(d)\right) - F^{[d]}\left(t_j^1(d)\right)\\ &= \left(\mathbf{1} - F^{[d]}\left(1 - t_{j+1}^1(d)\right)\right) - \left(\mathbf{1} - F^{[d]}\left(1 - t_j^1(d)\right)\right)\\ &= F^{[d]}\left(1 - t_j^1(d)\right) - F^{[d]}\left(1 - t_{j+1}^1(d)\right).\end{aligned}$$

Using Proposition 4.26, we also have

$$\begin{aligned}& F^{[d]}\left(1 - t_{j+1}^1(d)\right)\\ = {}& F^{[d]}\left(1 - t_j^1(d)\right) - Q_{3\left(F^{[d]}\left(1-t_j^1(d)\right)-F^{[d]}\left(1-t_{j+1}^1(d)\right)\right)} \circ \delta_{\frac{1}{3}}(\mathbf{1}).\end{aligned}$$

Now, using the fact that $F_\cdot^{[d]}$ is reversible again, we have

$$\begin{aligned}& F^{[d]}[\tau_\cdot]_0 + F^{[d]}[\tau_\cdot]_1 - F^{[d]}[\tau_\cdot]_t\\ = {}& \mathbf{1} - F^{[d]}[\tau_\cdot]_t\\ = {}& \mathbf{1} - F^{[d]}\left(t_j^1(d)\right) + Q_{3\delta_j^{[d]}} \circ \delta_{\frac{1}{3}}\left(-\tau_{u_j(t)}\right)\\ = {}& F^{[d]}\left(1 - t_j^1(d)\right) + Q_{3\left(F^{[d]}\left(1-t_j^1(d)\right)-F^{[d]}\left(1-t_{j+1}^1(d)\right)\right)} \circ \delta_{\frac{1}{3}}\left(\tau_{v_j(t)} - \mathbf{1}\right)\\ = {}& F^{[d]}\left(1 - t_{j+1}^1(d)\right) + Q_{3\left(F^{[d]}\left(1-t_j^1(d)\right)-F^{[d]}\left(1-t_{j+1}^1(d)\right)\right)} \circ \delta_{\frac{1}{3}}\left(\tau_{v_j(t)}\right)\\ = {}& F^{[d]}[\tau_\cdot]_{1-t}.\end{aligned}$$

■

4.4 Enhancing $F_\cdot^{[3]}$ as a geometric rough path

In this section, we establish our main theorem, Theorem 4.82, that each of various sequences of $\mathbb{R}^3$−valued paths as defined in Definition 4.46 converges to a $p\,(>3)$ −geometric rough path so that, in any case, its increment process is still equal to $F_\cdot^{[3]}$. The key idea to settle our claim is to first develop recursive relations between tensors of iterated integrals of successive paths as defined in Definition 4.46 by using the self-similarity of $F_\cdot^{[3]}$. Secondly, in Subsection 4.4.3, we shall apply the reversibility of $F_\cdot^{[d]}$ and the recursive relations developed in Subsection 4.4.2 to simplify the third level component of the signature over I of each member of a sequence of paths as defined in Definition 4.4.6 as an integral of the Levy area of the member against its increment. Finally, from Subsection 4.4.4 to 4.4.6, we shall establish the uniform boundedness and equicontinuity of the signatures (as functions on Δ_T) of paths as defined in Definition 4.4.6 and these conditions will suffice for an application of Ascoli-Azela lemma to conclude our Theorem 4.82 in Subsection 4.4.6. We first enclose some preliminary definitions and results that will motive our desired results in later sections.

4.4.1 Preliminary definitions and results

In this subsection, we show that the $p-$variation of a group-valued path can be controlled by the suitably chosen partial sums. The result is a generalization of Theorem 4.1.1 on page 62 in the book by Lyons and Qian [2002]. The idea of the proof is the same as that leading to the well-known Kolmogorov's continuity theorem in the context of probability theory.

Let $e \leq d \in \mathbb{N}$. For every $k \in \mathbb{N}$ and $j \in \{0, \ldots, 3^d\}$, we again define

$$t_j^k(d) \triangleq \frac{j}{3^{d \cdot k}}$$

and

$$\mathcal{D}^k(d) \triangleq \left\{t_j^k(d) : 0 \leq j \leq 3^{d \cdot k}\right\}$$

as a partition of I. We first recall the notion of symmetric homogeneous norms on $G^{(e)}(\mathbb{R}^d)$ as stated in Section 1.4.

Definition 4.36 *Define a symmetric homogeneous norm* $||\cdot||_e$ *on* $G^{(e)}(\mathbb{R}^d)$ *so that* $\forall g \triangleq 1 + g^1 + \cdots + g^e \in G^{(e)}(\mathbb{R}^d)$,

$$||g||_e \triangleq \max_{i=1,\ldots,e} \left(i! \left|g^i\right|_i\right)^{\frac{1}{i}}. \tag{4.26}$$

Also define a metric $d_e(\cdot,\cdot)$ *on* $G^{(e)}(\mathbb{R}^d)$ *so that* $\forall g, h \in G^{(e)}(\mathbb{R}^d)$,

$$d_e(g,h) \triangleq \left|\left|g^{-1} \otimes h\right|\right|_e. \tag{4.27}$$

Next we make a remark on the invariant nature of the symmetric homogeneous norm $||\cdot||_e$.

Lemma 4.37 *Let* V *be a Hilbert space with tensor norms* $|\cdot|_i$ *on* $V^{\otimes i}$. *Suppose all* $|\cdot|_i$ *are invariant with respect to an orthogonal transformation* $Q : V \to V$. *Then the symmetric homogeneous norm* $||\cdot||_e$ *is also invariant with respect to* Q.

In the rest of this chapter, we assume that every symmetric homogeneous norm $\|\cdot\|_e$ under consideration is invariant with respect to all orthogonal transformations on $\mathbb{R}^d$. We next recall the notion of $p-$variation.

Definition 4.38 *Denote the set of all $\left(G^{(e)}\left(\mathbb{R}^d\right), d_e\left(\cdot,\cdot\right)\right)$ − valued continuous paths by Λ^d.*

Definition 4.39 *For every $X_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right) \in \Lambda^d$, define*

$$X_{s,t} \triangleq X_s^{-1} \otimes X_t. \tag{4.28}$$

Definition 4.40 *Define a $p-$variation metric $d_e\left(\cdot,\cdot\right)$ on Λ^d so that $\forall X_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right), Y_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right) \in \Lambda^d$,*

$$d_{e,p}\left(X_\cdot, Y_\cdot\right) \triangleq \sup_{\mathcal{D}}\left(\sum_i d_e\left(X_{t_i,t_{i+1}}, Y_{t_i,t_{i+1}}\right)^p\right)^{\frac{1}{p}}, \tag{4.29}$$

where the supremum runs through all possible partitions $\mathcal{D}$ of I.

We next establish a way on how to effectively partition any interval (s,t) by points $t_j^k\left(d\right)$.

Lemma 4.41 *Let $0 \leq s < t \leq 1$ so that there is an $m \in \mathbb{N}$ with*

$$3^{-d(m+1)} \leq t - s < 3^{-dm}.$$

Then there are $k_{m+1}, r_{m+1} \in \mathbb{N}$ with $0 \leq r_{m+1} < 3^d$ and four sequences of natural numbers $\{\alpha_r\}_{r=m+1}^{\infty}, \{\beta_r\}_{r=m+1}^{\infty}, \{\gamma_r\}_{r=m+1}^{\infty}$ and $\{\delta_r\}_{r=m+1}^{\infty}$ such that:

1. *$\alpha_{m+1} = \gamma_{m+1} = k_{m+1}$, $\beta_{m+1} = \delta_{m+1} = r_{m+1}$ and both k_{m+1} and r_{m+1} are chosen in such a way that r_{m+1} is the largest number so that*

$$s \leq t_{k_{m+1}}^{m+1}\left(d\right) < t_{k_{m+1}+1}^{m+1}\left(d\right) < \cdots < t_{k_{m+1}+r_{m+1}}^{m+1}\left(d\right) \leq t.$$

2. *$\forall r = m+1, m+2, \ldots,$*

$$\begin{aligned} 0 &\leq \beta_r < 3^d, \\ 0 &\leq \delta_r < 3^d. \end{aligned}$$

3. $\forall r = m+2, m+3, \ldots$, *both* α_r *and* β_r *are chosen in such a way that* β_r *is the largest number so that*

$$s \leq t^{r+1}_{\alpha_{r+1}+\beta_{r+1}}(d) = t^r_{\alpha_r}(d) < t^r_{\alpha_r+1}(d) < \cdots < t^r_{\alpha_r+\beta_r}(d) = t^{r-1}_{\alpha_{r-1}}(d).$$

4. $\forall r = m+2, m+3, \ldots$, *both* γ_r *and* δ_r *are chosen in such a way that* δ_r *is the largest number such that*

$$t^{r-1}_{\gamma_{r-1}+\delta_{r-1}}(d) = t^r_{\gamma_r}(d) < t^r_{\gamma_r+1}(d) < \cdots < t^r_{\gamma_r+\delta_r}(d) = t^{r+1}_{\gamma_{r+1}}(d) \leq t.$$

Proof. Similar arguments to the one used in proving Theorem 4.1.1 on page 62 in Lyons and Qian [2002] can be adopted. ■

We next provide an effective bound for the norm of the signature a path over (s,t) by the norms of the signatures of the path over $\left(t^k_j(d), t^k_{j+1}(d)\right)$.

Corollary 4.42 *Let* $X_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right) \in \Lambda^d$. *For every* $(s,t) \in \Delta_1$ *so that for some* $m \in \mathbb{N}$,

$$3^{-d(m+1)} \leq t - s < 3^{-dm},$$

we have

$$\begin{aligned} \|X_{s,t}\|_e \leq & \sum_{l=0}^{r_1-1} \left\| X_{t^{m+1}_{k_{m+1}+l}(d), t^{m+1}_{k_{m+1}+l+1}(d)} \right\|_e + \sum_{r=m+2}^{\infty} \sum_{k=0}^{\beta_r-1} \left\| X_{t^r_{\alpha_r+k}(d), t^r_{\alpha_r+k+1}(d)} \right\|_e \\ & + \sum_{r=m+2}^{\infty} \sum_{k=0}^{\delta_r-1} \left\| X_{t^r_{\gamma_r+k}(d), t^r_{\gamma_r+k+1}(d)} \right\|_e . \end{aligned} \quad (4.30)$$

Proof. By combining Lemma 4.41 and the triangle inequality satisfied by the homogeneous norm $\|\cdot\|_e$, we deduce our claim. ■

Lemma 4.43 *For any sequence of* m *non-negative numbers* $a_1, \ldots, a_m$, *we have*

$$\left(\sum_{i=1}^m a_i\right)^p \leq m^{p-1} \sum a_i^p.$$

Proof. This is a special case of Hölder's inequality. ■

Corollary 4.44 *Let* $X_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right) \in \Lambda^d$. *For every* $p \geq d$ *and* $\lambda > p-1$,

$$\begin{aligned} \|X_{s,t}\|_e^p \leq\ & c(p,d,m,\lambda) \left\{ \sum_{l=0}^{r_1-1} \left\| X_{t^{m+1}_{k_{m+1}}(d), t^{m+1}_{k_{m+1}+l+1}(d)} \right\|_e^p \right. \\ & + \sum_{r=m+2}^{\infty} r^{\lambda} \sum_{k=0}^{\beta_r-1} \left\| X_{t^r_{\alpha_r+k}(d), t^r_{\alpha_r+k+1}(d)} \right\|_e^p \\ & \left. + \sum_{r=m+2}^{\infty} r^{\lambda} \sum_{k=0}^{\delta_r-1} \left\| X_{t^r_{\gamma_r+k}(d), t^r_{\gamma_r+k+1}(d)} \right\|_e^p \right\}, \end{aligned} \tag{4.31}$$

where

$$c(p,d,m,\lambda) = 3^{p-1}\left(3^d-1\right)^{p-1}\left(1 \vee \left(\sum_{r=m+2}^{\infty} r^{-\frac{\lambda}{p-1}}\right)\right) \tag{4.32}$$

Proof. As a consequence of Corollary 4.42 and Lemma 4.43, using Hölder's inequality,

$$\begin{aligned} \|X_{s,t}\|_e^p \leq\ & 3^{p-1} \left\{ \left(\sum_{l=0}^{r_1-1} \left\| X_{t^{m+1}_{k_{m+1}}(d), t^{m+1}_{k_{m+1}+l+1}(d)} \right\|_e \right)^p \right. \\ & + \left(\sum_{r=m+2}^{\infty} r^{-\frac{\lambda}{p}} \cdot r^{\frac{\lambda}{p}} \sum_{k=0}^{\beta_r-1} \left\| X_{t^r_{\alpha_r+k}(d), t^r_{\alpha_r+k+1}(d)} \right\|_e \right)^p \\ & \left. + \left(\sum_{r=m+2}^{\infty} r^{-\frac{\lambda}{p}} \cdot r^{\frac{\lambda}{p}} \sum_{k=0}^{\delta_r-1} \left\| X_{t^r_{\gamma_r+k}(d), t^r_{\gamma_r+k+1}(d)} \right\|_e \right)^p \right\} \end{aligned}$$

$$\begin{aligned} \leq\ & 3^{p-1} \left\{ \left(3^d-1\right)^{p-1} \sum_{l=0}^{r_1-1} \left\| X_{t^{m+1}_{k_{m+1}}(d), t^{m+1}_{k_{m+1}+l+1}(d)} \right\|_e^p \right. \\ & + \left(\sum_{r=m+2}^{\infty} r^{-\frac{\lambda}{p-1}} \right)^{p-1} \sum_{r=m+2}^{\infty} r^{\lambda} \left(\sum_{k=0}^{\beta_r-1} \left\| X_{t^r_{\alpha_r+k}(d), t^r_{\alpha_r+k+1}(d)} \right\|_e \right)^p \\ & \left. + \left(\sum_{r=m+2}^{\infty} r^{-\frac{\lambda}{p-1}} \right)^{p-1} \sum_{r=m+2}^{\infty} r^{\lambda} \left(\sum_{k=0}^{\delta_r-1} \left\| X_{t^r_{\gamma_r+k}(d), t^r_{\gamma_r+k+1}(d)} \right\|_e \right)^p \right\} \end{aligned}$$

$$\leq 3^{p-1}\left\{\left(3^d-1\right)^{p-1}\sum_{l=0}^{r_1-1}\left|\left|X_{t^{m+1}_{k_{m+1}}(d),t^{m+1}_{k_{m+1}+l+1}(d)}\right|\right|_e^p\right.$$
$$+\left(\sum_{r=m+2}^{\infty}r^{-\frac{\lambda}{p-1}}\right)^{p-1}\sum_{r=m+2}^{\infty}r^{\lambda}\left(3^d-1\right)^{p-1}\sum_{k=0}^{\beta_r-1}\left|\left|X_{t^r_{\alpha_r+k}(d),t^r_{\alpha_r+k+1}(d)}\right|\right|_e^p$$
$$\left.+\left(\sum_{r=m+2}^{\infty}r^{-\frac{\lambda}{p-1}}\right)^{p-1}\sum_{r=m+2}^{\infty}r^{\lambda}\left(3^d-1\right)^{p-1}\sum_{k=0}^{\delta_r-1}\left|\left|X_{t^r_{\gamma_r+k}(d),t^r_{\gamma_r+k+1}(d)}\right|\right|_e^p\right\}.$$

■

Finally, we deduce our claim that the $p-$variation of the signature of a path over I is bounded above by a sum involving the norms of the signatures of the path over $\left(t_j^k(d), t_{j+1}^k(d)\right)$.

Proposition 4.45 *Let $X_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right) \in \Lambda^d$ and $\lambda > p-1$. Then we have,*

$$d_{e,p}(X_\cdot,1) \leq c(p,d,1,\lambda)\sum_{r=0}^{\infty}(1+r)^{\lambda}\sum_{t_l\in\mathcal{D}^r(d)}\left|\left|X_{t_l,t_{l+1}}\right|\right|_e^p. \tag{4.33}$$

Proof. For every partition $\mathcal{D} \triangleq \{0=t_0<t_1<\cdots<t_n=1\}$ of I, since for any $i \neq j$, $(t_i,t_{i+1})\cap(t_j,t_{j+1})=\emptyset$, using Corollary 4.44, we have

$$\begin{aligned}d_{e,p}(X_\cdot,1) &\triangleq \sup_{\mathcal{D}}\sum\left|\left|X_{t_i,t_{i+1}}\right|\right|_e^p\\ &\leq c(p,d,1,\lambda)\sum_{r=0}^{\infty}(1+r)^{\lambda}\sum_{t_l\in\mathcal{D}^r(d)}\left|\left|X_{t_l,t_{l+1}}\right|\right|_e^p.\end{aligned}$$

■

4.4.2 Linear recursive relations between $\left(logX_\tau(n)_{0,1}\right)^3$

In this subsection, all the results hold for any integral value of d. We shall establish recursive relations between tensors of iterated integrals of successive paths as defined in Definition 4.46 by using the self-similarity of $F^{[3]}$. We next use the self-similar maps $P_j^{[d]}$ as defined in Definition 4.27 to construct a sequence of reversible paths:

Definition 4.46 *Let $\tau_{\cdot} : I \to I^d$ be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. In accordance with Lemma 4.35, define a sequence of reversible paths $\{\tau(n)_{\cdot} : I \to I^d\}$ so that $\forall n \in \mathbb{N}$,*

$$\begin{aligned} \tau(0)_{\cdot} &= \tau_{\cdot}, \\ \tau(n+1)_{\cdot} &= F^{[d]}[\tau(n)_{\cdot}]_{\cdot}. \end{aligned} \tag{4.34}$$

For simplicity, we also denote, $\forall t \in I$,

$$X_\tau(n)_t \triangleq S(\tau(n))_{0,t}. \tag{4.35}$$

Definition 4.47 *Let V be a vector space. For every $k \in \mathbb{N}$, define*

$$\mathcal{I}_k \triangleq \oplus_{i=k}^{\infty} V^{\otimes i}.$$

In accordance with Chow's theorem, for every path $\tau_{\cdot} : I \to V$, define $L^k(\tau_{\cdot}) \in \mathcal{L}^k(V)$ so that

$$\log S(\tau_{\cdot})_{0,1} = \sum_{k=1}^{\infty} L^k(\tau_{\cdot}). \tag{4.36}$$

In the rest of this subsection, we shall establish linear recursive relations for low order Lie elements $L^k(\tau(n)_{\cdot})$. We first have a result that Lie elements $k^n \in \mathcal{L}^n(V)$ are in the centre of $\mathcal{I}_n$.

Lemma 4.48 *Let V be a vector space. For every $n \in \mathbb{Z}^+$ and $i \in \{1, \ldots, n\}$, let $l^i, k^i \in \mathcal{L}^i(V)$. Then, we have*

$$\begin{aligned} \exp\left(l^1 + \cdots + l^n + k^n\right) &\equiv \exp\left(l^1 + \cdots + l^n\right)\exp\left(k^n\right) \ (mod\ \mathcal{I}_{n+1}) \\ &\equiv \exp\left(k^n\right)\exp\left(l^1 + \cdots + l^n\right) \ (mod\ \mathcal{I}_{n+1}). \end{aligned}$$

Definition 4.49 *Define $\Upsilon : I \to I^d$ to be a path such that $\forall t \in I$,*

$$\Upsilon_t = t\left(\sum_{k=1}^{d} e_k\right), \tag{4.37}$$

where $\{e_k\}_{k=1}^d$ is the standard basis of $\mathbb{R}^d$.

Lemma 4.50 *Let $\gamma_\cdot : I \to I^d$ be a reversible path with $\gamma_0 = O$ and $\gamma_1 = (1, \ldots, 1)$. For every $n \in \mathbb{N}$, we have*

$$\begin{aligned} & S\left(\gamma(n+1)_\cdot\right)_{0,1} \\ \equiv\ & S\left(F^{[d]}(\Upsilon_\cdot)_\cdot\right)_{0,1} \otimes \exp\left(\sum_{j=0}^{3^d-1}\left(P_j^{[d]}\right)^{\otimes 3}\left(L^3\left(\gamma(n)_\cdot\right)\right)\right) \ (mod\, \mathcal{I}_4). \end{aligned}$$

Proof. Using Lemma 4.35 and by induction, for every $n \in \mathbb{N}$, $\gamma(n)$ is reversible. According to Lemma 4.33, we have

$$L^2\left(\gamma(n)_\cdot\right) = A\left(\gamma(n)_\cdot\right)_{0,1} = 0,$$

and hence, using Lemma 4.48,

$$\begin{aligned} S\left(\gamma(n)_\cdot\right)_{0,1} &\equiv \exp\left(L^1\left(\gamma(n)_\cdot\right)_{0,1} + L^3\left(\gamma(n)_\cdot\right)\right) \ (mod\, \mathcal{I}_4) \\ &\equiv \exp\left(L^1\left(\gamma(n)_\cdot\right)_{0,1}\right)\exp\left(L^3\left(\gamma(n)_\cdot\right)\right) \ (mod\, \mathcal{I}_4). \end{aligned}$$

Note that for every $n \in \mathbb{N}$,

$$L^1\left(\gamma(n)_\cdot\right)_{0,1} = L^1\left(\Upsilon_\cdot\right)_{0,1},$$

and moreover, for every $j \in \{0, \ldots, 3^d - 1\}$,

$$S\left(P_j^{[d]}(\Upsilon_\cdot)\right)_{0,1} = \exp\left(P_j^{[d]}\left(L^1\left(\Upsilon_\cdot\right)_{0,1}\right)\right).$$

Using Lemma 4.48 again, we also have

$$\begin{aligned} & S\left(\gamma(n+1)_\cdot\right)_{0,1} \\ \equiv\ & \exp\left(P_0^{[d]}\left(L^1\left(\gamma(n)_\cdot\right)_{0,1}\right)\right) \otimes \cdots \otimes \exp\left(P_{3^d-1}^{[d]}\left(L^1\left(\gamma(n)_\cdot\right)_{0,1}\right)\right) \\ & \otimes \exp\left(\sum_{j=0}^{3^d-1}\left(P_j^{[d]}\right)^{\otimes 3}\left(L^3\left(\gamma(n)_\cdot\right)\right)\right) \ (mod\, \mathcal{I}_4). \end{aligned} \tag{4.38}$$

Note that by definition and Chen's identity (1.3),

$$\begin{aligned} & S\left(F^{[d]}(\Upsilon_\cdot)_\cdot\right)_{0,1} \\ =\ & S\left(P_0^{[d]}(\Upsilon_\cdot)\right)_{0,1} \otimes \cdots \otimes S\left(P_{3^d-1}^{[d]}(\Upsilon_\cdot)\right)_{0,1} \\ =\ & \exp\left(P_0^{[d]}\left(L^1\left(\Upsilon_\cdot\right)_{0,1}\right)\right) \otimes \cdots \otimes \exp\left(P_{3^d-1}^{[d]}\left(L^1\left(\Upsilon_\cdot\right)_{0,1}\right)\right) \\ =\ & \exp\left(P_0^{[d]}\left(L^1\left(\gamma(n)_\cdot\right)_{0,1}\right)\right) \otimes \cdots \otimes \exp\left(P_{3^d-1}^{[d]}\left(L^1\left(\gamma(n)_\cdot\right)_{0,1}\right)\right), \end{aligned}$$

therefore, after substitution into (4.38), the claim follows. ∎

We finally attain the recursive relations between tensors of iterated integrals of successive members as defined in Definition 4.46.

Proposition 4.51 *Let $\gamma_\cdot : I \to I^d$ be a reversible path with $\gamma_0 = O$ and $\gamma_1 = (1, \ldots, 1)$. For every $n \in \mathbb{N}$,*

$$L^1(\gamma(n+1)_\cdot) = L^1\left(F^{[d]}(\Upsilon_\cdot)_\cdot\right), \tag{4.39}$$

$$L^2(\gamma(n+1)_\cdot) = L^2\left(F^{[d]}(\Upsilon_\cdot)_\cdot\right) = 0, \tag{4.40}$$

$$L^3(\gamma(n+1)_\cdot) = L^3\left(F^{[d]}(\Upsilon_\cdot)_\cdot\right) + \sum_{j=0}^{3^d-1}\left(P_j^{[d]}\right)^{\otimes 3}\left(L^3(\gamma(n)_\cdot)\right). \tag{4.41}$$

Proof. In accordance with the fact that every $\gamma(n)$ is reversible, we again have, for $\forall n \in \mathbb{N}$,

$$L^1(\gamma(n)_\cdot)_{0,1} = L^1(\Upsilon_\cdot)_{0,1},$$

$$A(\gamma(n)_\cdot)_{0,1} = 0.$$

The claim is now an immediate consequence of Definition 4.47 and Lemmas 4.48 and 4.50. ∎

4.4.3 Expressing coefficients of $L^3(\tau_\cdot)$ in terms of increment and Levy area

In this subsection, we assume that all paths under consideration are of finite variation. Let $d \in \mathbb{N}$ and $\tau : I \to I^d$ be a reversible path. We shall apply the reversibility of $\tau_\cdot$ to simplify the expression of the third order Lie element $L^3(\tau_\cdot)$ as an integral of the Levy area of $\tau_\cdot$ against the increment of $\tau_\cdot$.

Lemma 4.52 *Let $\tau_\cdot : I \to I^d$ be a reversible path. Then we have*

$$L^3(\tau_\cdot) = S(\tau_\cdot)_{0,1}^3 - \frac{1}{3!}L^1(\tau_\cdot)^{\otimes 3}. \tag{4.42}$$

Proof. According to Lemma 4.33,

$$S(\tau_\cdot)_{0,1} \equiv \exp\left(L^1(\tau_\cdot) + L^3(\tau_\cdot)\right) \ (mod\, \mathcal{I}_4),$$

therefore

$$S(\tau_\cdot)^3_{0,1} = \frac{1}{3!}L^1(\tau_\cdot)^{\otimes 3} + L^3(\tau_\cdot).$$

■

Taking $\{e^i\}_{i=1}^d$ to be the standard basis of $\mathbb{R}^d$. Let $\gamma_\cdot : I \to \mathbb{R}^d$ be a continuous path, we express

$$L^3(\gamma_\cdot) \triangleq \sum_{1\le i,j,k\le d} L^3(\gamma_\cdot)^{i,j,k}\left(e^i \otimes e^j \otimes e^k\right). \tag{4.43}$$

Corollary 4.53 *Let $\tau_\cdot : I \to I^d$ be a reversible path. For any $i,j,k \in \{1,\ldots,d\}$,*

$$\begin{aligned} L^3(\tau_\cdot)^{i,j,k} &= \int_{0<u_1<u_2<u_3<1} d\tau^i_{u_1} d\tau^j_{u_2} d\tau^k_{u_3} \\ &\quad -\frac{1}{3!}\left(\int_0^1 d\tau^i_{u_1}\right)\left(\int_0^1 d\tau^j_{u_2}\right)\left(\int_0^1 d\tau^k_{u_3}\right). \end{aligned} \tag{4.44}$$

Lemma 4.54 *Let $\tau_\cdot : I \to I^d$ be a reversible path. For any $i,j,k \in \{1,\ldots,d\}$,*

$$\int_{0<u_1<u_2<u_3<1} d\tau^i_{u_1} d\tau^j_{u_2} d\tau^k_{u_3} = \int_{0<u_1<u_2<u_3<1} d\tau^k_{u_1} d\tau^j_{u_2} d\tau^i_{u_3}. \tag{4.45}$$

Proof. Since $\tau_\cdot$ is reversible, for any $u \in I$,

$$d\tau_u = d(-\tau_{1-u}).$$

Hence, we have

$$\begin{aligned} &\int_{0<u_1<u_2<u_3<1} d\tau^i_{u_1} d\tau^j_{u_2} d\tau^k_{u_3} \\ =& \int_{0<1-u_3<1-u_2<1-u_1<1} d\left(-\tau^i_{1-u_1}\right) d\left(-\tau^j_{1-u_2}\right) d\left(-\tau^k_{1-u_3}\right) \\ =& \int_{0<v_1<v_2<v_3<1} d\tau^i_{v_3} d\tau^j_{v_2} d\tau^k_{v_1}. \end{aligned}$$

■

Corollary 4.55 *Let $\tau_\cdot : I \to I^d$ be a reversible path. For any $i,j,k \in \{1,\ldots,d\}$,*

$$\begin{aligned}
&\left(\int_0^1 d\tau_{u_1}^i\right)\left(\int_0^1 d\tau_{u_2}^j\right)\left(\int_0^1 d\tau_{u_3}^k\right) \\
&= 2\left(\int_{0<u_1<u_2<u_3<1} d\tau_{u_1}^i d\tau_{u_2}^j d\tau_{u_3}^k + \int_{0<u_1<u_2<u_3<1} d\tau_{u_1}^j d\tau_{u_2}^k d\tau_{u_3}^i \right. \\
&\quad \left. + \int_{0<u_1<u_2<u_3<1} d\tau_{u_1}^k d\tau_{u_2}^i d\tau_{u_3}^j \right). \qquad (4.46)
\end{aligned}$$

Proof. After expanding the product of integrals as a shuffle product, the result is an immediate consequence of Lemma 4.54. ∎

Lemma 4.56 *Let $\tau_\cdot : I \to I^d$ be a reversible path. For any $i,j,k \in \{1,\ldots,d\}$,*

$$\int_{0<u<1} A(\tau_\cdot)_{u,1}^{i,j}\, d\tau_u^k = -\int_{0<u<1} A(\tau_\cdot)_{0,u}^{i,j}\, d\tau_u^k.$$

Proof. Note that for any reversible path $\tau_\cdot$ and $u \in I$, as an immediate consequence of the definition of Levy area $A_\cdot$ in (4.2),

$$A(\tau_\cdot)_{u,1} = -A(\tau_\cdot)_{0,1-u}.$$

Therefore, we deduce that

$$\begin{aligned}
\int_{0<u<1} A(\tau_\cdot)_{u,1}^{i,j}\, d\tau_u^k &= \int_{0<1-u<1} \left(-A(\tau_\cdot)_{0,1-u}^{i,j}\right) d\left(-\tau_{1-u}^k\right) \\
&= -\int_{0<u<1} A(\tau_\cdot)_{0,u}^{i,j}\, d\tau_u^k.
\end{aligned}$$

∎

Finally, we have our desired expression:

Proposition 4.57 *Let $\tau_\cdot : I \to I^d$ be a reversible path. For any $i,j,k \in \{1,\ldots,d\}$,*

$$L^3(\tau_\cdot)^{i,j,k} = \frac{2}{3}\left(\int_{0<u<1} A(\tau_\cdot)_{0,u}^{i,j}\, d\tau_u^k - \int_{0<u<1} A(\tau_\cdot)_{0,u}^{j,k}\, d\tau_u^i\right). \qquad (4.47)$$

Proof. Applying Corollaries 4.53 and 4.55 and Lemma 4.54, we have

$$\begin{aligned} L^3(\tau.)^{i,j,k} &= \frac{2}{3}\int_{0<u_1<u_2<u_3<1} d\tau^i_{u_1} d\tau^j_{u_2} d\tau^k_{u_3} - \frac{1}{3}\left(\int_{0<u_1<u_2<u_3<1} d\tau^j_{u_1} d\tau^k_{u_2} d\tau^i_{u_3} \right. \\ &\left. + \int_{0<u_1<u_2<u_3<1} d\tau^k_{u_1} d\tau^i_{u_2} d\tau^j_{u_3}\right) \\ &= \frac{2}{3}\int_{0<u_1<u_2<u_3<1} d\tau^i_{u_1} d\tau^j_{u_2} d\tau^k_{u_3} - \frac{1}{3}\left(\int_{0<u_1<u_2<u_3<1} d\tau^i_{u_1} d\tau^k_{u_2} d\tau^j_{u_3} \right. \\ &\left. + \int_{0<u_1<u_2<u_3<1} d\tau^j_{u_1} d\tau^i_{u_2} d\tau^k_{u_3}\right) \\ &= \frac{1}{3}\int_{0<u_1<u_2<u_3<1} \left(d\tau^i_{u_1} d\tau^j_{u_2} - d\tau^j_{u_1} d\tau^i_{u_2}\right) d\tau^k_{u_3} \\ &+ \frac{1}{3}\int_{0<u_1<u_2<u_3<1} d\tau^i_{u_1}\left(d\tau^j_{u_2} d\tau^k_{u_3} - d\tau^i_{u_2} d\tau^k_{u_3}\right) \\ &= \frac{2}{3}\int_{0<u<1} A(\tau.)^{i,j}_{0,u}\, d\tau^k_u + \frac{2}{3}\int_{0<u<1} A(\tau.)^{j,k}_{u,1}\, d\tau^i_u. \end{aligned}$$

Applying Lemma 4.56, we conclude our claim. ■

Lemma 4.58 *Let $\tau. : I \to I^d$ be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. For every $i, j \in \{1, \ldots, d\}$ and $t \in \left[0, \frac{1}{2}\right]$, we have*

$$A(\tau.)^{i,j}_{0,t} = A(\tau.)^{i,j}_{0,\frac{1}{2}} - A(\tau.)^{i,j}_{t,\frac{1}{2}} + \frac{1}{4}\left(\tau^j_t - \tau^i_t\right). \tag{4.48}$$

For any $t \in \left[\frac{1}{2}, 1\right]$, we have

$$A(\tau.)^{i,j}_{0,t} = A(\tau.)^{i,j}_{0,\frac{1}{2}} + A(\tau.)^{i,j}_{\frac{1}{2},t} + \frac{1}{4}\left(\tau^j_t - \tau^i_t\right). \tag{4.49}$$

Proof. For any $t \in \left[0, \frac{1}{2}\right]$, using Chen's identity (1.3), we have

$$A(\tau.)_{0,\frac{1}{2}} = A(\tau.)_{0,t} + A(\tau.)_{t,\frac{1}{2}} + \frac{1}{2}\left[L^1(\tau.)_{0,t}, L^1(\tau.)_{t,\frac{1}{2}}\right].$$

Note that since $\tau.$ is reversible,

$$\begin{aligned} \tau_{\frac{1}{2}} &= \frac{1}{2}\tau_1, \\ \tau_{\frac{1}{2}-t} - \tau_{\frac{1}{2}} &= -\left(\tau_{\frac{1}{2}+t} - \tau_{\frac{1}{2}}\right). \end{aligned}$$

Hence, we have

$$A(\tau.)^{i,j}_{0,\frac{1}{2}} = A(\tau.)^{i,j}_{0,t} + A(\tau.)^{i,j}_{t,\frac{1}{2}} + \frac{1}{2}\left(\tau^i_t\left(\frac{1}{2} - \tau^j_t\right) - \tau^j_t\left(\frac{1}{2} - \tau^i_t\right)\right).$$

After rearranging the terms we obtain our first identity (4.48); similar calculations deduce the second identity (4.49). ∎

Furthermore, we can have an alternative expression of the integrals on the right hand side of formula (4.47); we first need a lemma:

Lemma 4.59 *Let $\tau. : I \longrightarrow I^d$ be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. For any $t \in \left[0, \frac{1}{2}\right]$,*

$$A(\tau.)_{\frac{1}{2},\frac{1}{2}+t} = -A(\tau.)_{\frac{1}{2}-t,\frac{1}{2}}. \tag{4.50}$$

Proof. Again using the fact that if $\tau.$ is reversible,

$$\begin{aligned} \tau_{\frac{1}{2}} &= \frac{1}{2}\tau_1, \\ \tau_{\frac{1}{2}-t} - \tau_{\frac{1}{2}} &= -\left(\tau_{\frac{1}{2}+t} - \tau_{\frac{1}{2}}\right), \end{aligned}$$

the result is immediate in accordance with the definition of Levy area $A(\tau.)$ in (4.2). ∎

Corollary 4.60 *Let $\tau. : I \longrightarrow I^d$ be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. For any $i, j \in \{1, \ldots, d\}$ and $t \in \left[0, \frac{1}{2}\right]$,*

$$A(\tau.)^{i,j}_{0,\frac{1}{2}-t} + A(\tau.)^{i,j}_{0,\frac{1}{2}+t} = 2\left(A(\tau.)^{i,j}_{0,\frac{1}{2}} + A(\tau.)^{i,j}_{\frac{1}{2},\frac{1}{2}+t}\right). \tag{4.51}$$

Proof. Note that if $\tau.$ is reversible and $t \in \left[0, \frac{1}{2}\right]$,

$$\tau_{\frac{1}{2}-t} + \tau_{\frac{1}{2}+t} = 2\tau_{\frac{1}{2}} = 1.$$

Using this fact and Lemma 4.59, summing the two identities in Lemma 4.58, we deduce our identity (4.51). ∎

Proposition 4.61 *Let $\tau. : I \longrightarrow I^d$ be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. For any $i, j, k \in \{1, \ldots, d\}$,*

$$\begin{aligned} \int_{0<u<1} A(\tau.)^{i,j}_{0,u}\, d\tau^k_u &= A(\tau.)^{i,j}_{0,\frac{1}{2}} + 2\int_{0<u<\frac{1}{2}} A(\tau.)^{i,j}_{\frac{1}{2},\frac{1}{2}+u}\, d\tau^k_{\frac{1}{2}+u} && (4.52) \\ &= -2\int_{0<u<\frac{1}{2}} \left(\tau^k_{\frac{1}{2}+u} - \tau^k_{\frac{1}{2}}\right) dA(\tau.)^{i,j}_{\frac{1}{2},\frac{1}{2}+u}. && (4.53) \end{aligned}$$

Proof. Firstly, we have

$$\begin{aligned}\int_{0<u<1} A(\tau.)_{0,u}^{i,j}\, d\tau_u^k &= \int_{0<u<\frac{1}{2}} A(\tau.)_{0,u}^{i,j}\, d\tau_u^k + \int_{0<u<\frac{1}{2}} A(\tau.)_{0,\frac{1}{2}+u}^{i,j}\, d\tau_{\frac{1}{2}+u}^k \\ &= -\int_{0<u<\frac{1}{2}} A(\tau.)_{0,\frac{1}{2}-u}^{i,j}\, d\tau_{\frac{1}{2}-u}^k + \int_{0<u<\frac{1}{2}} A(\tau.)_{0,\frac{1}{2}+u}^{i,j}\, d\tau_{\frac{1}{2}+u}^k \\ &= \int_{0<u<\frac{1}{2}} \left(A(\tau.)_{0,\frac{1}{2}-u}^{i,j} + A(\tau.)_{0,\frac{1}{2}+u}^{i,j} \right) d\tau_{\frac{1}{2}+u}^k,\end{aligned}$$

after substituting the expression (4.51) from Corollary 4.60, we obtain our first identity. Furthermore, applying integration by parts to the first identity (4.52), we also obtain our second formula (4.53). ■

4.4.4 Uniform boundedness of $\left\{ \left\| X_\tau(n)_{0,1}^e \right\|_e \right\}_{n\in\mathbb{N}}$ when $d = e = 3$

In this subsection, we shall establish the uniform boundedness of the symmetric homogeneous norms (4.26) of the truncated signatures of members of a sequence as defined in Definition 4.46. More precisely, we aim to show that

$$\sup \left\| X_\tau(n)_{0,1}^3 \right\|_3 < \infty.$$

In addition, we also find an explicit expression for $L^3(\tau(n).)$.

Let $d = 3$. Recall that we define $\Upsilon : I \to I^3$ to be the path so that $\forall t \in I$,

$$\Upsilon_t = t\left(\sum_{k=1}^{3} e_k\right),$$

where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$. We first introduce an algebraic lemma which provides us an effective basis for $\mathcal{L}^3(\mathbb{R}^3)$.

Lemma 4.62 *The set $\mathcal{B}$ of the following 8 elements constitutes a basis for $\mathcal{L}^3(\mathbb{R}^3)$:*

$$\begin{aligned}&[e^1,[e^1,e^2]],\ [e^1,[e^1,e^3]],\ [e^2,[e^2,e^3]],\\ &[e^2,[e^2,e^1]],\ [e^3,[e^3,e^1]],\ [e^3,[e^3,e^2]],\\ &[e^1,[e^2,e^3]],\ [e^2,[e^3,e^1]].\end{aligned}$$

Proof. A direct calculation implies our claim. ■

Define $\mathcal{A} \triangleq \{[e^1,[e^2,e^3]], [e^2,[e^3,e^1]]\}$. Let us now find the coefficients of those dominant terms in the representation of $L^3(\tau.)$ with respect to the basis $\mathcal{B}$.

Lemma 4.63 *Let $\tau. : I \to I^3$ be a reversible path. If we express*

$$L^3(\tau.) \triangleq a(\tau.)_{123}\left[e^1,\left[e^2,e^3\right]\right] + a(\tau.)_{231}\left[e^2,\left[e^3,e^1\right]\right] + \ldots, \tag{4.54}$$

then we have

$$a(\tau.)_{123} = L^3(\tau.)^{1,2,3}, \tag{4.55}$$

$$a(\tau.)_{231} = -L^3(\tau.)^{3,1,2}. \tag{4.56}$$

Proof. Note that

$$\begin{aligned}\left[e^1,\left[e^2,e^3\right]\right] &= e^1\otimes e^2\otimes e^3 - e^1\otimes e^3\otimes e^2 - e^2\otimes e^3\otimes e^1 + e^3\otimes e^2\otimes e^1,\\ \left[e^2,\left[e^3,e^1\right]\right] &= e^2\otimes e^3\otimes e^1 - e^2\otimes e^1\otimes e^3 - e^3\otimes e^1\otimes e^2 + e^1\otimes e^3\otimes e^2.\end{aligned}$$

It is clear that

$$span\mathcal{A}\cap span(\mathcal{B}/\mathcal{A}) = \{0\},$$

hence the result follows. ■

Actually, both $a(\tau.)_{123}$ and $a(\tau.)_{231}$ are vanished; we first need a critical lemma:

Lemma 4.64

$$\begin{aligned}\int_0^1 A\left(F^{[3]}(\Upsilon.)\right)_{0,u}^{1,2} dF^{[3]}(\Upsilon.)_u^3 &= \int_0^1 A\left(F^{[3]}(\Upsilon.)\right)_{0,u}^{2,3} dF^{[3]}(\Upsilon.)_u^1\\ &= \int_0^1 A\left(F^{[3]}(\Upsilon.)\right)_{0,u}^{3,1} dF^{[3]}(\Upsilon.)_u^2\\ &= 0.\end{aligned}$$

Proof. Define a map $J:\mathbb{R}^2\to\mathbb{R}$ such that $\forall\alpha,\beta\in\mathbb{R}$,

$$J(\alpha,\beta) = \int_0^1\left(\alpha+\beta t-\frac{1}{2}\right)dt.$$

Using the formula (4.53) in Proposition 4.61, we can reduce the evaluation of

$$\int_0^1 A\left(F^{[3]}\left(\Upsilon_\cdot\right)\right)_{0,u}^{1,2} dF^{[3]}\left(\Upsilon_\cdot\right)_u^3$$

to calculating a sum of 9 values of $J(\alpha,\beta)$ at specific values of α and β; indeed, for some non-zero constant c,

$$\begin{aligned}
& c\int_0^1 A\left(F^{[3]}\left(\Upsilon_\cdot\right)\right)_{0,u}^{1,2} dF^{[3]}\left(\Upsilon_\cdot\right)_u^3 \\
= & \frac{1}{2}J\left(1,-\frac{1}{3}\right)-\frac{1}{2}J\left(\frac{2}{3},-\frac{1}{3}\right)+\frac{1}{2}J\left(\frac{1}{3},-\frac{1}{3}\right) \\
& +J\left(0,\frac{1}{3}\right)-J\left(\frac{1}{3},\frac{1}{3}\right)+J\left(\frac{2}{3},\frac{1}{3}\right) \\
& +\frac{1}{2}J\left(1,-\frac{1}{3}\right)+\frac{1}{2}J\left(\frac{2}{3},-\frac{1}{3}\right)+\frac{1}{2}J\left(\frac{1}{3},-\frac{1}{3}\right) \\
= & \left\{\frac{1}{6}+0+\left(-\frac{1}{6}\right)\right\}+\left\{\left(-\frac{1}{3}\right)+0+\frac{1}{3}\right\} \\
& +\left\{\frac{1}{6}+0+\left(-\frac{1}{6}\right)\right\} \\
= & 0.
\end{aligned}$$

Similar calculations lead to the conclusion that the other two integrals also vanish. ■

Corollary 4.65

$$a\left(F^{[3]}\left(\Upsilon_\cdot\right)_\cdot\right)_{123}=a\left(F^{[3]}\left(\Upsilon_\cdot\right)_\cdot\right)_{231}=0.$$

Proof. This is an immediate consequence of Proposition 4.57 and Lemmas 4.63 and 4.64. ■

For every $i\in\{1,2,3\}$, define an orthogonal transformation $Q_i:\mathbb{R}^3\to\mathbb{R}^3$ so that $\forall j\in\{1,2,3\}$,

$$Q_i\left(e^j\right)=(-1)^{1-\delta_i(j)}\cdot e^j,$$

where $\delta_i(\cdot):\mathbb{N}\to\mathbb{N}$ is the Kronecker delta function. In accordance with Corollary 4.23,

$$card\left(Q\left(3\right)\right)=2^{3-1}=4;$$

indeed, we have

1. 9 of $Q_{3\delta_j^{[3]}}$ are the identity map $I : \mathbb{R}^3 \to \mathbb{R}^3$.

2. 6 of $Q_{3\delta_j^{[3]}}$ are Q_1.

3. 6 of $Q_{3\delta_j^{[3]}}$ are Q_2.

4. 6 of $Q_{3\delta_j^{[3]}}$ are Q_3.

We finally have the formula for the third level Lie element of $\tau(n)_{\cdot}$ by applying the recursive relations obtained in Subsection 4.4.2.

Proposition 4.66 *Let $\tau_{\cdot} : I \longrightarrow I^3$ be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. Then we have*

$$\begin{aligned}
& L^3(\tau(n)_{\cdot}) \\
= & \left(1 + \frac{1}{3^2} + \cdots + \left(\frac{1}{3^2}\right)^{n-1}\right) L^3\left(F^{[3]}(\Upsilon_{\cdot})_{\cdot}\right) + \left(\frac{1}{3^2}\right)^n L^3(\tau(0)_{\cdot}) \\
& + \left(1 - \left(\frac{1}{3^2}\right)^n\right) \left\{a(\tau(0)_{\cdot})_{123}\left[e^1, \left[e^2, e^3\right]\right]\right. \\
& \left. + a(\tau(0)_{\cdot})_{231}\left[e^2, \left[e^3, e^1\right]\right]\right\}, \qquad (4.57)
\end{aligned}$$

and therefore $\{L^3(\tau(n)_{\cdot})\}_{n\in\mathbb{N}}$ is a convergent sequence.

Proof. Note that, by direct calculation, for every $v \in \mathcal{B}/\mathcal{A}$,

$$\sum_{j=0}^{3^3-1}\left(P_j^{[3]}\right)^{\otimes 3}(v) = \frac{1}{3^2}v;$$

however, for every $w \in \mathcal{A}$,

$$\sum_{j=0}^{3^3-1}\left(P_j^{[3]}\right)^{\otimes 3}(w) = w.$$

Nevertheless, in accordance with Proposition 4.51 and Corollary 4.65,

$$\begin{aligned}
a(\tau(1)_{\cdot})_{123} &= a\left(F^{[3]}(\Upsilon_{\cdot})_{\cdot}\right)_{123} + a(\tau(0)_{\cdot})_{123} \\
&= a(\tau(0)_{\cdot})_{123}\,.
\end{aligned}$$

Similarly, we also have

$$a(\tau(1)_\cdot)_{231} = a(\tau(0)_\cdot)_{231}.$$

By induction, we have, for every $n \in \mathbb{N}$,

$$\begin{aligned} a(\tau(n)_\cdot)_{123} &= a(\tau(0)_\cdot)_{123}, \\ a(\tau(n)_\cdot)_{231} &= a(\tau(0)_\cdot)_{231}. \end{aligned}$$

Using (4.41) in Proposition 4.51 and by induction, we can deduce our claim. ∎

Let $m \leq d \in \mathbb{N}$. For any $g = \exp(l^1 + \cdots + l^m)$ with $l^i \in \mathcal{L}^i(\mathbb{R}^d)$, define

$$\|g\|_{\mathcal{L}^m(\mathbb{R}^d)} \triangleq \max_{i=1,\dots,m} \left|l^i\right|_i^{\frac{1}{i}}.$$

Note that $\|\cdot\|_{\mathcal{L}^m(\mathbb{R}^d)}$ is a symmetric homogeneous norm on $G^{(m)}(\mathbb{R}^d)$; according to the work of Goodman [1977], $\|\cdot\|_{\mathcal{L}^m(V)}$ is equivalent to $\|\cdot\|_m$. As a consequence of Proposition 4.66, we now have the uniform boundedness of the moduli of the truncated signatures $X_\tau(n)_{0,1}^3$.

Corollary 4.67 *Let $\tau_\cdot : I \to I^3$ be a finite variation reversible path with $\tau_0 = O$ and $\tau_1 = (1, \dots, 1)$. Then we have*

$$\sup_{n\in\mathbb{N}} \left\|X_\tau(n)_{0,1}^3\right\|_3 < \infty. \tag{4.58}$$

Proof. In accordance with Proposition 4.51, for every $n \in \mathbb{N}$,

$$\begin{aligned} L^1(\tau(n)_\cdot) &= L^1\left(F^{[d]}(\Upsilon_\cdot)_\cdot\right) = 1, \\ L^2(\tau(n)_\cdot) &= L^2\left(F^{[d]}(\Upsilon_\cdot)_\cdot\right) = 0. \end{aligned}$$

As a result of Proposition 4.66, we therefore have, for some $C > 0$,

$$\begin{aligned} \sup_{n\in\mathbb{N}} \left\|X_\tau(n)_{0,1}^3\right\|_3 &\leq C \sup_{n\in\mathbb{N}} \left\|X_\tau(n)_{0,1}^3\right\|_{\mathcal{L}^3(\mathbb{R}^3)} \\ &\triangleq C \sup_{n\in\mathbb{N}} \max_{i=1,2,3} \left|L^i(\tau(n)_\cdot)\right|^{\frac{1}{i}} \\ &< \infty. \end{aligned}$$

∎

4.4.5 On uniform boundedness of $\{d_e(X_\tau(n)_\cdot^e, 1)\}_{n\in\mathbb{N}}$

Let $e \le d \in \mathbb{N}$. In this subsection, we assume $\tau : I \to I^d$ to be a finite variation reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. We shall establish a condition (4.65) under which we can obtain uniform boundedness of $p(>3)$ −variations of the group-valued paths $X_\tau(n)_\cdot^e$ for all $n \in \mathbb{N}$. As an application, we shall prove that

$$\sup_{n\in\mathbb{N}} d_{3,p}\left(X_\tau(n)_\cdot^3, 1\right) < \infty.$$

Definition 4.68 *For every $k \in \mathbb{N}$ and $p \ge d$, define*

$$var_k^p(X_\tau(n)_\cdot^e) \triangleq \sum_{t_i^k\in\mathcal{D}^k(d)} \left\|X_\tau(n)_{t_i^k,t_{i+1}^k}^e\right\|_e^p. \tag{4.59}$$

Lemma 4.69 *For every $k \in \mathbb{Z}^+$ and $p \ge d$,*

$$var_k^p(X_\tau(n+1)_\cdot^e) = \frac{1}{3^{p-d}} \cdot var_{k-1}^p(X_\tau(n)_\cdot^e). \tag{4.60}$$

Proof. Using the invariant nature of $\|\cdot\|_e$ with respect to all orthogonal transformations, we have

$$\begin{aligned}
& var_k^p(X_\tau(n+1)_\cdot^e) \\
= & \sum_{t_i^k\in\mathcal{D}^k(d)} \left\|X_\tau(n+1)_{t_i^k,t_{i+1}^k}^e\right\|_e^p \\
= & \sum_{t_j^1\in\mathcal{D}^1(d)} \sum_{t_i^k\in\mathcal{D}^k(d)\cap[t_j^1,t_{j+1}^1]} \left\|X_\tau(n+1)_{t_i^k,t_{i+1}^k}^e\right\|_e^p \\
= & \sum_{t_j^1\in\mathcal{D}^1(d)} \sum_{t_i^k\in\mathcal{D}^k(d)\cap[t_j^1,t_{j+1}^1]} \left\|\delta_{\frac{1}{3}}\left(X_\tau(n)_{3^d(t_i^k-t_j^1),3^d(t_{i+1}^k-t_j^1)}^e\right)\right\|_e^p \\
= & \sum_{t_j^1\in\mathcal{D}^1(d)} \sum_{t_i^{k-1}\in\mathcal{D}^{k-1}(d)} \left(\frac{1}{3}\right)^p \left\|X_\tau(n)_{t_i^{k-1},t_{i+1}^{k-1}}^e\right\|_e^p \\
= & \left(\frac{1}{3}\right)^{p-d} \sum_{t_i^{k-1}\in\mathcal{D}^{k-1}(d)} \left\|X_\tau(n)_{t_i^{k-1},t_{i+1}^{k-1}}^e\right\|_e^p.
\end{aligned}$$

■

Corollary 4.70 *Let $n \in \mathbb{Z}^+$. For any $1 \leq k \leq n$ and $p \geq d$,*

$$var_k^p\left(X_\tau(n)_\cdot^e\right) = \left(\frac{1}{3^{p-d}}\right)^k var_0^p\left(X_\tau(n-k)_\cdot^e\right). \tag{4.61}$$

For any $k > n$,

$$var_k^p\left(X_\tau(n)_\cdot^e\right) = \left(\frac{1}{3^{p-d}}\right)^n var_{k-n}^p\left(X_\tau(0)_\cdot^e\right). \tag{4.62}$$

Proof. Applying Lemma 4.69 and by an induction argument, we conclude our claim. ■

Lemma 4.71 *Suppose that $\tau' : I \to \mathbb{R}^d$ is piecewise continuous. For any $k > n$, we have*

$$var_k^p\left(X_\tau(n)_\cdot^e\right) \leq \sup_{t\in I}|\tau_t'|^p \left(\frac{1}{3^{p-d}}\right)^n \left(\frac{1}{3^{pd-d}}\right)^{k-n}. \tag{4.63}$$

Proof. Let $(s,t) \in \Delta_1$. Note that the length $l(\tau_\cdot)_{s,t}$ of $\tau_\cdot$ over (s,t) is bounded above as:

$$l(\tau_\cdot)_{s,t} \leq \sup_{t\in I}|\tau_t'| \cdot |s-t|.$$

Also

$$\left|\left|X_\tau(0)_{s,t}^e\right|\right|_e \leq l(\tau_\cdot)_{s,t}.$$

Now, for any $p > d \geq 1$,

$$\begin{aligned} var_{k-n}^p\left(X_\tau(0)_\cdot^e\right) &= \sum_{t_i^{k-n}\in\mathcal{D}^{k-n}(d)} \left|\left|X_\tau(0)_{t_i^{k-n},t_{i+1}^{k-n}}^e\right|\right|_e^p \\ &\leq \sum_{t_i^{k-n}\in\mathcal{D}^{k-n}(d)} \left(l(\tau_\cdot)_{t_i^{k-n},t_{i+1}^{k-n}}\right)^p \\ &\leq \max_{t_i^{k-n}\in\mathcal{D}^{k-n}(d)} \left(l(\tau_\cdot)_{t_i^{k-n},t_{i+1}^{k-n}}\right)^{p-1} l(\tau_\cdot)_{0,1} \\ &\leq \left(\sup_{t\in I}|\tau_t'| \cdot \left|\frac{1}{3^{d(k-n)}}\right|\right)^{p-1} \cdot \sup_{t\in I}|\tau_t'| \end{aligned}$$

Using (4.62) in Corollary 4.70, we obtain our inequality. ■

Proposition 4.72 *Let $\tau' : I \to \mathbb{R}^d$ be piecewise continuous. For any $p > d$ and $\lambda > p-1$, we have*

$$\begin{aligned} d_{e,p}\left(X_\tau(n)^e_\cdot, 1\right) &\leq c(p,d,1,\lambda)\left\{\sum_{r=0}^{n}(1+r)^\lambda\left(\frac{1}{3^{p-d}}\right)^r \left\|X_\tau(n-r)^e_{0,1}\right\|_e^p \right. \\ &\quad \left. + \sup_{t\in I}|\tau_t'|^p \cdot \sum_{r=0}^{\infty}(1+r)^\lambda\left(\frac{1}{3^{p-d}}\right)^r\right\}. \end{aligned} \tag{4.64}$$

Furthermore, if

$$N_e(\tau) \triangleq \sup_{k\in\mathbb{N}}\left\|X_\tau(k)^e_{0,1}\right\|_e < \infty, \tag{4.65}$$

then we also have

$$\begin{aligned} M_e(\tau) &\triangleq \sup_{n\in\mathbb{N}} d_{e,p}\left(X_\tau(n)^e_\cdot, 1\right) \\ &\leq c(p,d,1,\lambda)\left(N_e(\tau) + \sup_{t\in I}|\tau_t'|^p\right)\sum_{r=0}^{\infty}(1+r)^\lambda\left(\frac{1}{3^{p-d}}\right)^r. \end{aligned} \tag{4.66}$$

Proof. Using Proposition 4.45, Corollary 4.70 and Lemma 4.71, we have

$$\begin{aligned} d_{e,p}\left(X_\tau(n)^e_\cdot, 1\right) &\triangleq \sup_{\mathcal{D}}\sum\left\|X_\tau(n)^e_{t_i,t_{i+1}}\right\|_e^p \\ &\leq c(p,d,1,\lambda)\left\{\sum_{r=0}^{n}(1+r)^\lambda\left(\frac{1}{3^{p-d}}\right)^r var_0^p\left(X_\tau(n-r)^e_\cdot\right)\right. \\ &\quad \left. + \sum_{r=0}^{\infty}(1+r)^\lambda \sup_{t\in I}|\tau_t'|^p\left(\frac{1}{3^{p-d}}\right)^n\left(\frac{1}{3^{pd-d}}\right)^{r-n}\right\}. \end{aligned}$$

Note that by definition,

$$var_0^p\left(X_\tau(n-r)^e_\cdot\right) = \left\|X_\tau(n-r)^e_{0,1}\right\|_e^p,$$

and for any $d \geq 1$,

$$\frac{1}{3^{pd-d}} \leq \frac{1}{3^{p-d}}.$$

After simplification and substitution, our claims are immediate. ■

Corollary 4.73 *Let $\tau : I \to I^3$ be a finite variation reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$ such that $\tau' : I \to \mathbb{R}^3$ is piecewise continuous. For any $p > 3$,*

$$\sup_{n\in\mathbb{N}} d_{3,p}\left(X_\tau(n)^3_\cdot, 1\right) < \infty.$$

Proof. This is an immediate consequence of Corollary 4.67 and Proposition 4.72. ∎

4.4.6 On equicontinuity of $X_\tau(n)^e_\cdot$

Let $e \leq d \in \mathbb{N}$. In this subsection, we again assume $\tau : I \to I^d$ to be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$. We shall establish a condition as stated in Proposition 4.76 under which we can obtain equicontinuity of the group-valued paths $X_\tau(n)^e_\cdot$ for all $n \in \mathbb{N}$. As an application, using the idea of proof leading the well-known Ascoli-Azela lemma, we shall prove that $\left\{X_\tau(n)^3_\cdot\right\}_{n\in\mathbb{N}}$ is Cauchy.

Lemma 4.74 *Let $k, n \in \mathbb{N}$ with $k \leq n$. For any $i \in \left\{0, \ldots, 3^{kd} - 1\right\}$, we have*

$$\sup_{t_j\in\mathcal{D}\cap\left[t_i^k, t_{i+1}^k\right]} \sum \left|\left|X_\tau(n)^e_{t_j,t_{j+1}}\right|\right|_e^p = \frac{1}{3^p} \sup_{t_j\in\mathcal{D}\cap\left[t_0^{k-1}, t_1^{k-1}\right]} \sum \left|\left|X_\tau(n-1)^e_{t_j,t_{j+1}}\right|\right|_e^p.$$

By induction, we also have

$$\sup_{t_j\in\mathcal{D}\cap\left[t_i^k, t_{i+1}^k\right]} \sum \left|\left|X_\tau(n)^e_{t_j,t_{j+1}}\right|\right|_e^p = \frac{1}{3^{pk}} \sup_{t_j\in\mathcal{D}\cap[0,1]} \sum \left|\left|X_\tau(n-k)^e_{t_j,t_{j+1}}\right|\right|_e^p.$$

Proof. This is an immediate consequence of the self-similar construction of $\{\tau(n)_\cdot\}_{n\in\mathbb{N}}$ as defined in Definition 4.46. ∎

Corollary 4.75 *Let $k, n \in \mathbb{N}$ with $k \leq n$. For any $i \in \left\{0, \ldots, 3^{kd} - 1\right\}$, we have*

$$\sup_{t_j\in\mathcal{D}\cap\left[t_i^k, t_{i+1}^k\right]} \sum \left|\left|X_\tau(n)^e_{t_j,t_{j+1}}\right|\right|_e^p \leq \frac{1}{3^{pk}} \cdot M_e(\tau),$$

where

$$M_e(\tau) \triangleq \sup_{n\in\mathbb{N}} d_{e,p}\left(X_\tau(n)^e_\cdot, 1\right).$$

Proposition 4.76 *Suppose that*

$$M_e(\tau) \triangleq \sup_{n\in\mathbb{N}} d_{e,p}\left(X_\tau(n)^e_\cdot, 1\right) < \infty.$$

The family of $G^{(e)}\left(\mathbb{R}^d\right)$ *– valued continuous paths*

$$\left\{X_\tau(n)^e_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right)\right\}_{n\in\mathbb{N}}$$

is equicontinuous.

Proof. Given $\varepsilon > 0$, we choose $k_0 \in \mathbb{N}$ such that

$$\left(\frac{1}{3}\right)^{pk_0} M_e(\tau) \leq \left(\frac{\varepsilon}{2}\right)^p.$$

Using Corollary 4.75, $\forall m \geq k_0,\ s,t \in \left[t^k_i, t^k_{i+1}\right]$ and $i \in \left\{0, \ldots, 3^{kd} - 1\right\}$,

$$\left|\left|X_\tau(m)^e_{s,t}\right|\right|^p_e \leq \frac{1}{3^{pk_0}} \cdot M_e(\tau).$$

In general, for any $(s,t) \in \Delta_1$ with $|s-t| \leq \frac{1}{3^{dk_0}}$, there is $t^{k_0}_i \in [0,1]$ such that

$$s \leq t^{k_0}_i \leq t,$$

and so

$$\begin{aligned} \left|\left|X_\tau(m)^e_{s,t}\right|\right|_e &\leq \left|\left|X_\tau(m)^e_{s,t^{k_0}_i}\right|\right|_e + \left|\left|X_\tau(m)^e_{t^{k_0}_i,t}\right|\right|_e \\ &\leq 2\left\{\left(\frac{1}{3}\right)^{pk_0} M_e(\tau)\right\}^{\frac{1}{p}} \leq \varepsilon. \end{aligned}$$

On the other hand, since all paths $X_\tau(n)^e_\cdot : I \to G^{(e)}\left(\mathbb{R}^d\right)$ are continuous, one can choose $\delta > 0$ such that for all $0 \leq n < k_0$, $\forall (s,t) \in \Delta_1$ with $|s-t| \leq \delta$,

$$\left|\left|X_\tau(n)^e_{s,t}\right|\right|_e \leq \varepsilon.$$

Combining the results, we conclude that $\forall n \in \mathbb{N}$, $(s,t) \in \Delta_1$ with $|s-t| \leq \min\left(\delta, \frac{1}{3^{dk_0}}\right)$, we have

$$\left|\left|X_\tau(n)^e_{s,t}\right|\right|_e \leq \varepsilon.$$

■

Lemma 4.77 *Let $\tau : I \to I^3$ be a finite variation reversible path with $\tau_0 = O$ and $\tau_1 = (1, \dots, 1)$. For any $k \in \mathbb{N}$ and $t_j^k(3) \in \mathcal{D}^k(3)$, the limit*

$$\lim_{n\to\infty} X_\tau(n)^3_{t_j^k(3)} \text{ exists.}$$

Proof. For any $(s,t) \in \Delta_1$, we define

$$X_\tau(n)_{s,t} \equiv \exp\left(L(n)^1_{s,t} + L(n)^2_{s,t} + L(n)^3_{s,t}\right) \ (mod\ I_4).$$

Fix a $k \in \mathbb{N}$. For every $d \in \mathbb{N}$ and $j \in \{0, \dots, 3^{dk} - 1\}$, define

$$\delta(k)^{[d]}_j \triangleq L(k)^1_{t_j^k(d), t_{j+1}^k(d)}.$$

Also define an affine map $P(k)^{[d]}_j : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\forall v \in \mathbb{R}^3$,

$$P(k)^{[d]}_j(v) \triangleq Q_{3^k \delta(k)^{[d]}_j} \circ \delta_{\frac{1}{3^k}}(v).$$

Note that because of the self-similar construction and the reversibility of $\tau(n)_\cdot$, using Lemma 4.48 and Proposition 4.51, for any $n \geq k$, $j \in \{0, \dots, 3^{dk}\}$,

$$\begin{aligned}
& X_\tau(n)_{t_j^k(d)} \\
\equiv\ & \exp\left(P(k)^{[d]}_0\left(L(0)^1_{0,1}\right)\right) \otimes \cdots \otimes \exp\left(P(k)^{[d]}_{j-1}\left(L(0)^1_{0,1}\right)\right) \\
& \otimes \exp\left(\sum_{i=0}^{j-1}\left(P(k)^{[d]}_i\right)^{\otimes 3}\left(L(n-k)^3_{0,1}\right)\right) \ (mod\ \mathcal{I}_4).
\end{aligned}$$

Therefore, for any $m, n \geq k$, we have

$$\begin{aligned}
& \left(X_\tau(m)_{t_j^k(d)}\right)^{-1} \otimes X_\tau(n)_{t_j^k(d)} \\
\equiv\ & \exp\left(\sum_{i=0}^{j-1}\left(P(k)^{[d]}_i\right)^{\otimes 3}\left(L(n-k)^3_{0,1} - L(m-k)^3_{0,1}\right)\right) \ (mod\ \mathcal{I}_4).
\end{aligned}$$

Now,

$$\begin{aligned}
&\left\|\left(X_\tau(m)^3_{0,t^k_j(d)}\right)^{-1}\otimes X_\tau(n)^3_{0,t^k_j(d)}\right\|^3_{\mathcal{L}^3(\mathbb{R}^3)} \\
\leq\; & \sum_{i=0}^{j-1}\left|\left(P(k)^{[d]}_i\right)^{\otimes 3}\left(L(m-k)^3_{0,1}-L(n-k)^3_{0,1}\right)\right|_3 \\
=\; & \frac{j}{3^{dk}}\left|L(m-k)^3_{0,1}-L(n-k)^3_{0,1}\right|_3 \\
\leq\; & \left|L(m-k)^3_{0,1}-L(n-k)^3_{0,1}\right|_3
\end{aligned}$$

For $d=3$, in accordance with Proposition 4.66, we deduce that

$$\left\{X_\tau(n)^3_{t^k_j(3)}\right\}_{n\in\mathbb{N}}$$

is Cauchy with respect to $d_3(\cdot,\cdot)$. ■

Corollary 4.78 *Let $\tau : I \longrightarrow I^3$ be a finite variation reversible path with $\tau_0 = O$ and $\tau_1 = (1,\ldots,1)$ such that $\tau' : I \longrightarrow \mathbb{R}^3$ is piecewise continuous. $\left\{X_\tau(n)^3_{0,\cdot}\right\}_{n\in\mathbb{N}}$ converges uniformly on I, i.e. $\forall\varepsilon>0$, there is $k\in\mathbb{N}$ such that $\forall m,n\geq k$,*

$$d_\infty\left(X_\tau(m)^3_\cdot, X_\tau(n)^3_\cdot\right) \triangleq \sup_{t\in I} d_3\left(X_\tau(m)^3_{0,t}, X_\tau(n)^3_{0,t}\right)\leq\varepsilon.$$

Proof. Under the given condition, the result in Corollary 4.73 holds and so $\left\{X_\tau(n)^3_{0,\cdot}\right\}_{n\in\mathbb{N}}$ is equicontinuous in accordance with the Proposition 4.76. Combining with Lemma 4.77, we conclude our claim. ■

Definition 4.79 *Let $\tau : I \longrightarrow I^3$ be a finite variation reversible path with $\tau_0 = O$ and $\tau_1 = (1,\ldots,1)$ such that $\tau' : I \longrightarrow \mathbb{R}^3$ is piecewise continuous. In accordance with Corollary 4.78, we define the unique continuous path $X_\tau(\infty)^3_\cdot : I \longrightarrow G^{(3)}(\mathbb{R}^3)$ such that*

$$X_\tau(\infty)^3_t \triangleq \lim_{n\to\infty} X_\tau(n)^3_t.$$

As a consequence,

$$\lim_{n\to\infty} d_\infty\left(X_\tau(n)^3_\cdot, X_\tau(\infty)^3_\cdot\right)=0.$$

4.4.7 Main theorem

Finally, we prove our main theorem that for any smooth enough reversible path $\tau : I \to I^3$ with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$, $X_\tau(\infty)^3_\cdot$ is a $p\,(> 3)-$ geometric rough path. That is to say the space-filling curve $F^{[3]}_\cdot$ can be enhanced as a geometric rough path in many different 'natural' ways. We first recall two useful results in Friz and Victoir [2006].

Proposition 4.80 *Given a Banach space V. Let $X_\cdot : I \to G^{(e)}(\mathbb{R}^d)$ and $X(n)_\cdot : I \to G^{(e)}(\mathbb{R}^d)$ be group-valued paths. Then we have*

$$\lim_{n\to\infty} d_\infty (X(n)_\cdot, X_\cdot) = \lim_{n\to\infty} \sup_{t\in I} d_e (X(n)_t, X_t) = 0$$

if and only if

$$\lim_{n\to\infty} \sup_{(s,t)\in\Delta_1} d_e \left(X(n)_{s,t}, X_{s,t} \right) = 0.$$

Proposition 4.81 *Let $1 \le p \le p' < \infty$. Then for all $G^{(e)}(\mathbb{R}^d)-$valued paths Y, Z,*

$$d_{e,p'}(Y, Z) \le \sup_{(s,t)\in\Delta_1} d_e (Y, Z)^{1-\frac{p}{p'}} \cdot d_{e,p'}(Y, Z)^{\frac{p}{p'}}.$$

In particular, if $\{Y(n)\}_{n\in\mathbb{N}}$ converges uniformly to Y and

$$\sup_{n\in\mathbb{N}} d_{e,p}(Y(n)_\cdot, 1) < \infty,$$

then we have

$$\lim_{n\to\infty} d_{e,p'}(Y(n), Z) = 0.$$

Theorem 4.82 *Let $\tau : I \to I^3$ be a finite variation reversible path with $\tau_0 = O$ and $\tau_1 = (1, \ldots, 1)$ such that $\tau' : I \to \mathbb{R}^3$ is piecewise continuous. For any $\varepsilon > 0$, $X_\tau(\infty)^3_\cdot : I \to G^{(3)}(\mathbb{R}^3)$ is a $(3+\varepsilon)-$ geometric rough path such that $\forall t \in I$,*

$$\pi_1 \left(X_\tau (\infty)^3_t \right) = F^{[3]}_t. \tag{4.67}$$

Proof. Under the given condition, using Corollaries 4.78 and Propositions 4.80 and 4.81, we deduce that $\left\{ X_\tau(n)^3_t \right\}_{n\in\mathbb{N}}$ converges to $X_\tau(\infty)^3_t$ in $3+\varepsilon-$variation. Furthermore, using Proposition 4.26, we also deduce the equality of (4.67). ■

Before writing up the thesis, the author can only prove our claim for the case $d = 3$; it is still open for $d > 4$ because iterated integrals of order not less than 4 would need to be investigated which seems to be far from trivial.

Chapter 5

Integral of a 3-form as integral of a spinor

In the theory of rough paths, it admits that the signature (Definition 1.1) of a geometric rough path completely characterizes the path itself in the sense of controlling an arbitrary controlled differential equation. Along the same line of thought, it is interesting to ask whether one can find quantities, which are analogous to the signature of a path, that can characterize a high dimensional geometric object $\mathcal{M}$ in the sense of integrating differential forms on $\mathcal{M}$. Recall that, in Chapter 4, we have already established the fact (Theorem 4.82) that there is a class $\mathcal{C}$ of space-filling rough paths with their $\mathbb{R}^3-$projections being all the same and filling up a three-dimensional unit cube; similar results can be extended to those $3-$dimensional geometric object (nice chainlet) $\mathcal{N}$ which can be well-approximated by cubes. In this respect, one may expect that analytical properties of $\mathcal{N}$ can be extracted by decoding the information contained in the signatures of those space-filling rough paths in the corresponding class $\mathcal{C}$ for $\mathcal{N}$. In this chapter, we shall show how we can answer our concern, first for cubes and then for some nice chainlets (see Harrison [1998]), in the dimension 3; indeed, a special pair of space-filling rough paths in $\mathcal{C}$ for $\mathcal{N}$ will do the job. Moreover, we shall identify any differential form ω as an one-form $\widetilde{\omega}$ (Lemma 5.4) over tensors so that integrating the differential form ω on $\mathcal{N}$ is equivalent to (see (5.43)) integrating the one-form (a spinor) $(\widetilde{\omega}, -\widetilde{\omega})$ against the pair in $\mathcal{C}$ with respect to a properly chosen integrator (see Section 5.2).

Let V be a Banach space. In the following, we shall adopt all the notation in Chapters 3 and 4. In Section 5.1, we first introduce some algebraic results (e.g. Proposition 5.8) which suggest any $n-$form $\widetilde{\omega}$ would annihilate all Lie elements of order greater than 2; in other words, if we want to integrate $\widetilde{\omega}$ against a polynomial of iterated integrals of a path $\gamma_{\cdot}$, only increment and Levy area processes of $\gamma_{\cdot}$ would come to play. Besides, we shall also point out that why the signature of a path fails to be an integrator.

Section 5.2 is rather technical; in this section, we shall suggest a feasible integrator, namely:

$$A\left(\gamma_{\cdot}\right)_{-1,t}\otimes dL^{1}\left(\gamma_{\cdot}\right)_{-1,t},$$

where $A\left(\gamma_{\cdot}\right)_{-1,t}$ and $L^{1}\left(\gamma_{\cdot}\right)_{-1,t}$ are respectively the Levy area and increment processes of a path $\gamma_{\cdot}$ over $[-1,t]$. In accordance with an immediate consequence (Lemma 5.23) of Chen's identity, we can split the integrator into three items as, for any $s\in[0,t]$:

$$\begin{aligned}
&A\left(\gamma_{\cdot}\right)_{-1,t}\otimes dL^{1}\left(\gamma_{\cdot}\right)_{-1,t}\\
=\;&A\left(\gamma_{\cdot}\right)_{-1,s}\otimes dL^{1}\left(\gamma_{\cdot}\right)_{-1,t}+\frac{1}{2}\left[L^{1}\left(\gamma_{\cdot}\right)_{-1,s},L^{1}\left(\gamma_{\cdot}\right)_{s,t}\right]\otimes dL^{1}\left(\gamma_{\cdot}\right)_{-1,t}\\
&+A\left(\gamma_{\cdot}\right)_{s,t}\otimes dL^{1}\left(\gamma_{\cdot}\right)_{0,t}.
\end{aligned}$$

Suppose that $\widetilde{\alpha}$ is a smooth $3-$form (see Definition 5.17 for a notion of smoothness). In the first part of Section 5.2, we shall find an expression of the difference between two integrals $\widetilde{\alpha}$ over a subinterval $\left[t_{j}^{k},t_{j+1}^{k}\right]\subset[0,1]$ against each item in the similar splitting of $A\left(\gamma^{\tau}\left(n\right)_{\cdot}\right)_{-1,t}\otimes dL^{1}\left(\gamma^{\tau}\left(n\right)_{\cdot}\right)_{-1,t}$ with respect to two reversible paths τ_{1} and τ_{2}. In the second half of this section, we shall look for a simplification of the formula in Lemma 5.22 which leads to a subtle implication (see Corollary 5.27) that the difference of integrals:

$$\begin{aligned}
&\int_{t_{j}^{k}}^{t_{j+1}^{k}}\widetilde{\alpha}\left(\gamma^{\tau_{1}}\left(n\right)_{t}\right)\left(A\left(\gamma^{\tau_{1}}\left(n\right)\right)_{t_{j}^{k},t}\otimes dL^{1}\left(\gamma^{\tau_{1}}\left(n\right)\right)_{-1,t}\right)\\
&\quad-\int_{t_{j}^{k}}^{t_{j+1}^{k}}\widetilde{\alpha}\left(\gamma^{\tau_{2}}\left(n\right)_{t}\right)\left(A\left(\gamma^{\tau_{2}}\left(n\right)\right)_{t_{j}^{k},t}\otimes dL^{1}\left(\gamma^{\tau_{2}}\left(n\right)\right)_{-1,t}\right),
\end{aligned}$$

can describe the difference between the local actions of $\widetilde{\alpha}$ on

$$\int_{0}^{1}A\left(\tau_{i}\right)_{0,t}\otimes dL^{1}\left(\tau_{i}\right)_{0,t},$$

for $i = 1, 2$.

In Section 5.3, we shall find, for any $k \in \mathbb{N}$ and $j = 0, \ldots, 3^{3k}$, the asymptotic value (see Corollary 5.37) of

$$3^{3k} \cdot J\left(\gamma^{\tau}\left(k+r\right)\right)_{t_j^k, t_{j+1}^k}$$

in (5.29) as r goes to infinity. Note that the quantity $3^{3k} \cdot J\left(\gamma^{\tau}\left(k+r\right)\right)_{t_j^k, t_{j+1}^k}$ plays a key role in the derivation of our main result Theorem 5.45 in Section 5.4. The key idea of seeking for the asymptotic value is to first identify the expression $3^{3k} \cdot J\left(\gamma^{\tau}\left(k+r\right)\right)_{t_j^k, t_{j+1}^k}$, by using self-similar nature of the definition of γ^{τ}, as an expected value of some functional of a irreducible aperiodic Markov Chain as defined in Definition 5.28; and then an easy application of a celebrated result (Theorem 5.30) in ergodic theory leads to the expression of the asymptotic value.

In Section 5.4, we first use the asymptotic value of $3^{3k} \cdot J\left(\gamma^{\tau}\left(k+r\right)\right)_{t_j^k, t_{j+1}^k}$ obtained in Section 5.3 to further simplify each of the differences of integrals $\widetilde{\alpha}$ obtained in Section 5.2. Finally, we shall establish our main result Theorem 5.45 which relates a difference (5.40) of two rough path integrals of two space-filling rough paths from C for a unit cube to a few Lebesgue integrals of some functions induced by $\widetilde{\alpha}$ over the unit cube.

In Section 5.5, we shall identity some choices of pairs of space-filling rough paths for a unit cube so that those integrals against $\widetilde{\alpha}^{(1)}$ and $\widetilde{\alpha}^{(2)}$ in (5.40) would vanish while the remaining difference of two Lebesgue integrals would become the ordinary integral of 3$-$form α. Together with the theory of chainlets by Harrison (1998), we extend our result to nice chainlets at the end of the section and conclude that any 3$-$form α on a nice chainlet $\mathcal{N}$ can be expressed as a limit of a sequence of integrals against a spinor $(\widetilde{\alpha}, -\widetilde{\alpha})$.

5.1 Algebraic background

In this section, we shall provide a few algebraic results (e.g. Proposition 5.8) which suggest that any $n-$form $\widetilde{\omega}$ annihilates all Lie elements of order > 2; that is to say, whenever we contract $\widetilde{\omega}$ against a polynomial of iterated integrals of a path $\gamma_{\cdot}$, only increment and Levy area processes of $\gamma_{\cdot}$ are needed. At the end of this section, we shall make a remark that the signature of a path fails to be a sensible integrator. We first introduce the notion about how to canonically identify a multi-linear map as a linear map on tensor algebra. Recall that the tensor algebra of V is the enveloping algebra of V.

Definition 5.1 *Impose the natural Lie bracket $[\cdot,\cdot]$ on $T(V)$ such that, $\forall u, v \in T(V)$,*

$$[u, v] = u \otimes v - v \otimes u. \tag{5.1}$$

For each $k \in \mathbb{N}$, we denote by B_k the ideal generated by elements of the form

$$[\ldots [v_1, v_2], \ldots, v_m]$$

for $m \geq k$, $v_1, \ldots, v_m \in V$.

Definition 5.2 *For $n \in \mathbb{N}$, define*

$$\begin{aligned} \mathcal{L}^1(V) &= V \\ \mathcal{L}^{n+1}(V) &= [V, \mathcal{L}^n(V)] \end{aligned}$$

and

$$\mathcal{L}(V) = \oplus_{k=1}^{\infty} \mathcal{L}^k(V).$$

Proposition 5.3 *$\mathcal{L}(V)$ is a free Lie algebra while $T(V)$ is its enveloping algebra.*

Proof. One can find the details of a proof in the book by Reutenauer [1993]. ∎

Let $n \in \mathbb{N}$ and consider a multilinear map $\omega : V^n \to \mathbb{R}$.

Lemma 5.4 *There is a unique linear map $\widetilde{\omega} : V^{\otimes n} \to \mathbb{R}$ such that for any $v_1, \ldots, v_n \in V$,*

$$\widetilde{\omega}(v_1 \otimes \cdots \otimes v_n) = \omega(v_1, \ldots, v_n). \tag{5.2}$$

Proof. It is an immediate consequence of the universal property of the tensor products of V. ∎

We can now tell how to canonically treat any $n-$form on V as an one-form over $T(V)$.

Definition 5.5 *A multilinear map $\omega : V^n \longrightarrow \mathbb{R}$ is said to be $n-$alternating if for any permutation σ of $\{1, 2, \ldots, n\}$, $v_1, \ldots, v_n \in V$,*

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(n)}\right) = (-1)^{sgn\sigma} \cdot \omega\left(v_1, \ldots, v_n\right). \tag{5.3}$$

The corresponding linear map $\widetilde{\omega} : V^{\otimes n} \longrightarrow \mathbb{R}$ is also said to be $n-$alternating. Define $Alt^n(V)$ to be the set of all $n-$alternating multilinear maps $\omega : V^n \longrightarrow \mathbb{R}$. Also define $\widetilde{Alt^n}(V)$ to be the set of all $n-$alternating linear maps $\widetilde{\omega} : V^{\otimes n} \longrightarrow \mathbb{R}$.

Note that any $n-$alternating ω is also alternating with respect to any of the $m\,(\leq n)$ components.

Definition 5.6 *For any open set $U \subset V$, we call a continuous map $\omega(\cdot) : U \longrightarrow Alt^n(V)$ an $n-$form over U.*

Without loss of ambiguity, we may identify $\omega(\cdot) : U \longrightarrow Alt^n(V)$ with the induced map $\widetilde{\omega}(\cdot) : U \longrightarrow \widetilde{Alt^n}(V)$ in accordance with Definition 5.5; and so we would also call $\widetilde{\omega}(\cdot)$ an $n-$form over U. We are ready to introduce our main result in this section that $\widetilde{\omega}$ annihilates all Lie elements of order greater than 2.

Lemma 5.7 *Let $m \leq n \in \mathbb{N}$, $v^{n-m} \in V^{\otimes(n-m)}$. Define a linear map $\widetilde{\omega}^m : V^m \longrightarrow \mathbb{R}$ such that $\forall v_1, \ldots, v_m \in V$*

$$\widetilde{\omega}^m\left(v_1, \ldots, v_m\right) = \widetilde{\omega}\left(v_1 \otimes \cdots \otimes v_m \otimes v^{n-m}\right).$$

Then $\widetilde{\omega}^m$ is also $m-$alternating.

Proof. For any $1 \leq i < j \leq m$,

$$\widetilde{\omega}\left(\cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v^{n-m}\right) = -\widetilde{\omega}\left(\cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v^{n-m}\right).$$

■

Proposition 5.8 *For $3 \leq n \in \mathbb{N}$, if $\omega : V^n \longrightarrow \mathbb{R}$ is $n-$alternating, then $\widetilde{\omega}$ annihilates all elements in $B_3 \cap V^{\otimes n}$.*

Proof. For $m \geq 3$, consider an element of the form $[\ldots[v_1, v_2], \ldots, v_m] \otimes v^{n-m}$ where $v^{n-m} \in V^{\otimes(n-m)}$. By definition,

$$\begin{aligned}
[[v_1, v_2], v_3] &= (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_3 - v_3 \otimes (v_1 \otimes v_2 - v_2 \otimes v_1) \\
&= v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_1 \otimes v_2 + v_3 \otimes v_2 \otimes v_1.
\end{aligned}$$

Using Lemma 5.7 and the fact that $\widetilde{\omega}(\cdot)$ is linear, we have

$$\begin{aligned}
&\widetilde{\omega}\left([\ldots[v_1, v_2], \ldots, v_m] \otimes v^{n-m}\right) \\
=\ &\widetilde{\omega}\left([\ldots[v_1 \otimes v_2 \otimes v_3, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
&-\widetilde{\omega}\left([\ldots[v_2 \otimes v_1 \otimes v_3, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
&-\widetilde{\omega}\left([\ldots[v_3 \otimes v_1 \otimes v_2, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
&+\widetilde{\omega}\left([\ldots[v_3 \otimes v_2 \otimes v_1, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
=\ &\widetilde{\omega}\left([\ldots[v_1 \otimes v_2 \otimes v_3, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
&+\widetilde{\omega}\left([\ldots[v_1 \otimes v_2 \otimes v_3, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
&-\widetilde{\omega}\left([\ldots[v_1 \otimes v_2 \otimes v_3, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
&-\widetilde{\omega}\left([\ldots[v_1 \otimes v_2 \otimes v_3, v_4], \ldots, v_m] \otimes v^{n-m}\right) \\
=\ &0.
\end{aligned}$$

■

Definition 5.9 *Define D to be the ideal generated by double commutators of the form*

$$[[u_1, u_2], [v_1, v_2]]$$

for some $u_j, v_j \in V$, $j = 1, 2$.

Corollary 5.10 *For $4 \leq n \in \mathbb{N}$, given an $n-$alternating multilinear map $\omega : V^n \longrightarrow \mathbb{R}$, $\widetilde{\omega}$ annihilates all elements in $D \cap V^{\otimes n}$.*

Proof. Note that for $u_j, v_j \in V$, $j = 1, 2$,

$$[[u_1, u_2], [v_1, v_2]] = [[[v_1, v_2], u_2], u_1] - [[[v_1, v_2], u_1], u_2].$$

The result is now an immediate consequence of Proposition 5.8. ■

Unfortunately, the converse of Proposition 5.8 may not be valid. To see this, we first recall a fundamental result in the theory of free Lie algebras:

Proposition 5.11 (Poincare-Birkhoff-Witt) *There is an ordered basis*

$$\mathcal{B} \triangleq (v_i)_{i\in\mathcal{I}}$$

for $\mathcal{L}(V)$ as an $\mathbb{R}$−module such that

1. *For each $n\in\mathbb{N}$, $\mathcal{B}\cap\mathcal{L}^n(V)\neq\emptyset$.*
2. *$\mathcal{C} \triangleq \{v_{i_1}\otimes\cdots\otimes v_{i_k} : k\in\mathbb{N},\ v_{i_1},\ldots,v_{i_k}\in\mathcal{B}\}$ constitutes a basis for $T(V)$ as an $\mathbb{R}$−module.*

Proof. One can find the details of a proof in the book by Reutenauer [1993]. ■

Proposition 5.12 *The converse of Proposition* 5.8 *is not true.*

Proof. We construct a counter-example. For each $n\in\mathbb{N}$, let

$$\mathcal{C}^n \triangleq \mathcal{C}\cap V^{\otimes n}.$$

Define a map $\widetilde{\omega}:\mathcal{C}^n\longrightarrow\mathbb{R}$ such that $n=i_1+\cdots+i_k$,

$$\widetilde{\omega}\left(v_{i_1}\otimes\cdots\otimes v_{i_k}\right)=\begin{cases}1, & \forall v_{i_j}\in\mathcal{L}^n(V),\ n=1,2\\ 0, & \exists v_{i_j}\in\mathcal{L}^n(V),\ n\geq 3\end{cases}.$$

Now, we extend $\widetilde{\omega}$ linearly to $V^{\otimes n}$. Using the universal property of $T(V)$, we conclude that the extended map induces a multilinear map $\omega: V^n\longrightarrow\mathbb{R}$ which is not alternating but $\widetilde{\omega}$ annihilates $B_3\cap V^{\otimes n}$. ■

Finally, we make a remark that the signature of a path cannot serve as a candidate of plausible integrators as mentioned in Introduction of this chapter, especially when the underlying space is of odd dimension; indeed, any constant form annihilates the signature of an arbitrary path when $\dim(V)$ is odd (see Corollary 5.15). Nevertheless, in Section 5.2, we shall suggest a sensible integrator for our later work.

Lemma 5.13 *For $m\in\mathbb{N}$, $j=1,2$, $k=1,\ldots,m$, let $v_k^j\in\mathcal{L}^j(V)$. Then*

$$\begin{aligned}&\exp\left(v_1^1+v_1^2\right)\otimes\cdots\otimes\exp\left(v_m^1+v_m^2\right)\\ =\ &\exp\left(\sum_{k=1}^m v_k^1+\sum_{k=1}^m v_k^2+\frac{1}{2}\sum_{1\leq i<j\leq m}\left[v_i^1,v_j^1\right]\right)\ (mod\ B_3).\end{aligned}$$

Proof. For $m = 2$, using Campbell-Baker-Hausdorff's formula (see Reutenauer [1993]), we have

$$\begin{aligned} & \exp\left(v_1^1 + v_1^2\right) \otimes \exp\left(v_2^1 + v_2^2\right) \\ = & \exp\left(v_1^1 + v_2^1 + v_1^2 + v_2^2 + \frac{1}{2}\left[v_1^1 + v_1^2, v_2^1 + v_2^2\right]\right) \ (mod\ B_3) \\ = & \exp\left(v_1^1 + v_2^1 + v_1^2 + v_2^2 + \frac{1}{2}\left[v_1^1, v_2^1\right]\right) \ (mod\ B_3), \end{aligned}$$

where the last equality holds because for any $x \in \mathcal{L}^i(V)$ and $y \in \mathcal{L}^j(V)$ with $i + j \geq 3$,

$$[x, y] \in B_3.$$

The general result follows by induction. ■

Corollary 5.14 *Let $m \in \mathbb{N}$, consider m finite variation continuous paths $\gamma_k : I \to V$, $k = 1, \ldots, m$. Let $\omega : V^n \to \mathbb{R}$ be $n-$alternating. In accordance with Definition 5.5, for any $(s,t) \in \Delta_1$,*

$$\begin{aligned} & \widetilde{\omega}\left(\left(S(\gamma_1)_{s,t} \otimes \cdots \otimes S(\gamma_m)_{s,t}\right)^{(n)}\right) \\ = & \widetilde{\omega}\left(\exp\left(\sum_{k=1}^m L(\gamma_k)_{s,t} + \sum_{k=1}^m A(\gamma_k)_{s,t} + \frac{1}{2}\sum_{1\leq k<l\leq m}\left[L(\gamma_k)_{s,t}, L(\gamma_l)_{s,t}\right]\right)^{(n)}\right). \end{aligned}$$

Proof. Using Lemma 5.13, we have

$$\begin{aligned} & S(\gamma_1)_{s,t} \otimes \cdots \otimes S(\gamma_m)_{s,t} \\ = & \exp\left(\sum_{k=1}^m L(\gamma_k)_{s,t} + \sum_{k=1}^m A(\gamma_k)_{s,t} + \frac{1}{2}\sum_{1\leq k<l\leq m}\left[L(\gamma_k)_{s,t}, L(\gamma_l)_{s,t}\right]\right) \ (mod\ B_3 \end{aligned}$$

According to Proposition 5.8, we deduce our result. ■

Corollary 5.15 *Let n be an odd integer. Consider an $n-$alternating multilinear map $\omega : V^n \to \mathbb{R}$ and a Lipschitz loop $\gamma_. : I \to V$, i.e. $\gamma_0 = \gamma_1$,*

$$\widetilde{\omega}\left(S(\gamma_.)_{0,1}^{(n)}\right) = 0.$$

Proof. Since $L(\gamma.)_{0,1} = 0$, we have

$$\begin{aligned} \widetilde{\omega}\left(S(\gamma.)_{0,1}^{(n)}\right) &= \widetilde{\omega}\left(\exp\left(A(\gamma.)_{0,1}\right)^{(n)}\right) \\ &= 0, \end{aligned}$$

because $\exp\left(A(\gamma.)_{0,1}\right)$ can only have non-zero even order tensors. ∎

5.2 An integrator $A(\gamma.)_{-1,t} \otimes dL^1(\gamma.)_{-1,t}$

In this section, we suggest a plausible integrator, namely:

$$A(\gamma.)_{-1,t} \otimes dL^1(\gamma.)_{-1,t},$$

where $A(\gamma.)_{-1,t}$ and $L^1(\gamma.)_{-1,t}$ are respectively the Levy area and increment processes of a path $\gamma.$ over $[-1,t]$. Suppose that $\widetilde{\alpha}$ is a smooth 3−form (see Definition 5.17 for a notion of smoothness). In the first part of this section, we shall find an expression of the difference between two integrals $\widetilde{\alpha}$ over a subinterval $\left[t_j^k, t_{j+1}^k\right] \subset [0,1]$ against each item in the following splitting of $A(\gamma^\tau(n).)_{-1,t} \otimes dL^1(\gamma^\tau(n).)_{-1,t}$ with respect to two reversible paths τ_1 and τ_2:

$$\begin{aligned} & A(\gamma^\tau(n).)_{-1,t} \otimes dL^1(\gamma^\tau(n).)_{-1,t} \\ = \; & A(\gamma^\tau(n).)_{-1,s} \otimes dL^1(\gamma^\tau(n).)_{-1,t} \\ & + \frac{1}{2}\left[L^1(\gamma^\tau(n).)_{-1,s}, L^1(\gamma^\tau(n).)_{s,t}\right] \otimes dL^1(\gamma^\tau(n).)_{-1,t} \\ & + A(\gamma^\tau(n).)_{s,t} \otimes dL^1(\gamma^\tau(n).)_{0,t}. \end{aligned}$$

In the second half of this section, we shall establish a simplification of Lemma 5.22 which leads to a subtle implication that the difference of integrals:

$$\begin{aligned} & \int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}(\gamma^{\tau_1}(n)_t)\left(A(\gamma^{\tau_1}(n))_{t_j^k,t} \otimes dL^1(\gamma^{\tau_1}(n))_{-1,t}\right) \\ & - \int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}(\gamma^{\tau_2}(n)_t)\left(A(\gamma^{\tau_2}(n))_{t_j^k,t} \otimes dL^1(\gamma^{\tau_2}(n))_{-1,t}\right), \end{aligned}$$

can describe the difference between the local actions of $\widetilde{\alpha}$ on

$$\int_0^1 A(\tau_i)_{0,t} \otimes dL^1(\tau_i)_{0,t},$$

for $i = 1, 2..$ Let $\tau_{\cdot} : I \to I^3$ be a reversible path with $\tau_0 = O$ and $\tau_1 = (1, 1, 1)$. Fixed two Lie elements $l^m \in \mathcal{L}^m(\mathbb{R}^3)$ for $m = 1, 2$. First of all, for the sake of convenience, we shall introduce some useful notation:

Definition 5.16 *For each $n \in \mathbb{N}$, define a continuous path of finite variation $\gamma^\tau(n) : [-1, 1] \to \mathbb{R}^d$ such that*

$$
\begin{aligned}
L^1(\gamma^\tau(n))_{-1,0} &= l^1, && (5.4)\\
A(\gamma^\tau(n))_{-1,0} &= l^2. && (5.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma^\tau(0)_{\cdot} &= \tau_{\cdot},\\
\gamma^\tau(n+1)_{\cdot} &= F^{[3]}(\gamma^\tau(n)). && (5.6)
\end{aligned}
$$

We again denote $\Upsilon : I \to I^d$ to be the path such that for $t \in I$,

$$\Upsilon_t = t\left(\sum_{k=1}^{d} e_k\right). \tag{5.7}$$

We next assume that all one-form $\widetilde{\alpha}(\cdot)$ under consideration will be smooth in the following sense:

Definition 5.17 (Stein [1970]) *Let $d \in \mathbb{Z}^+$, $k \in \mathbb{N}$ and F be a closed subset of $\mathbb{R}^d$. Equip $\widetilde{Alt^3}(\mathbb{R}^d)$ with operator norm. A continuous 3-form $\widetilde{\alpha}(\cdot) : F \to \widetilde{Alt^3}(\mathbb{R}^d)$ is said to be Lip(λ, F), $k < \lambda \leq k+1$ if there are symmetric multilinear functions $\widetilde{\alpha}^{(j)} : (\mathbb{R}^d)^{\otimes j} \to \widetilde{Alt^3}(\mathbb{R}^d)$, $j = 0, \ldots, k$, such that for any continuous path $x : I \to \mathbb{R}^3$ with $x(I) \subset F$,*

$$
\begin{aligned}
\widetilde{\alpha}(x_t) &= \sum_{j=0}^{k} \widetilde{\alpha}^{(j)}(x_s)\left(\int_{s<u_1<\cdots<u_j<t} dx_{u_1} \otimes \cdots \otimes dx_{u_j}\right) + R_k(x_t, x_s),\\
\widetilde{\alpha}^{(0)}(\cdot) &= \widetilde{\alpha}(\cdot), && (5.8)
\end{aligned}
$$

for any $(s,t) \in \Delta_1$, and

$$|||\widetilde{\alpha}||| \triangleq \max_{j=0,\ldots,k} \sup_{x \in F} \left|\left|\widetilde{\alpha}^{(j)}(x)\right|\right| \vee \sup_{x,y \in F} \frac{||R_k(x,y)||}{|x-y|^\lambda} < \infty.$$

Let $k \in \mathbb{N}$. For simplicity, we denote for $j = 0, \ldots, 3^{3k}$,

$$t_j^k = \frac{j}{3^{3k}}; \tag{5.9}$$

for any reversible path $\tau : I \longrightarrow I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$, we denote, for $u \in I$,

$$\theta(\gamma^\tau(n))_u^{t_i^k} = L^1(\gamma^\tau(n))_{t_i^k, t_i^k + (t_{i+1}^k - t_i^k)u}. \tag{5.10}$$

We first have an expression of the difference between two integrals $\widetilde{\alpha}$ over $[t_j^k, t_{j+1}^k]$ against

$$A(\gamma^\tau(n))_{-1,t_j^k} \otimes dL^1(\gamma^\tau(n))_{-1,t}$$

with respect to two reversible paths τ_1 and τ_2:

Lemma 5.18 *Let $\varepsilon > 0$. Suppose that a 3−form $\widetilde{\alpha}(\cdot) : I^3 \longrightarrow \widetilde{Alt^3}(\mathbb{R}^d)$ is $Lip(2+\varepsilon, I^3)$. For $m = 1, 2$, consider two finite variation reversible paths $\tau_m : I \longrightarrow I^3$ with $(\tau_m)_0 = O$ and $(\tau_m)_1 = (1,1,1)$. For $k, r \in \mathbb{N}$ with $n = k+r$, the difference*

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}(\gamma^{\tau_1}(n)_t)\left(A(\gamma^{\tau_1}(n))_{-1,t_j^k} \otimes dL^1(\gamma^{\tau_1}(n))_{-1,t}\right) \\
&- \int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}(\gamma^{\tau_2}(n)_t)\left(A(\gamma^{\tau_2}(n))_{-1,t_j^k} \otimes dL^1(\gamma^{\tau_2}(n))_{-1,t}\right) \\
= & \sum_{t_j^k \le t_i^{k+r} < t_{j+1}^k} \int_0^{\frac{1}{2}} \widetilde{\alpha}^{(2)}\left(F^{[3]}(t_j^k)\right)\left(\left(\theta(\gamma^{\tau_1}(n))_u^{t_i^{k+r}} - \theta(\gamma^{\tau_1}(n))_{\frac{1}{2}}^{t_i^{k+r}}\right)^{\otimes 2}\right) \\
&\left(A(\gamma^{\tau_1}(n))_{-1,t_j^k} \otimes d\theta(\gamma^{\tau_1}(n))_u^{t_i^{k+r}}\right) \\
&- \sum_{t_j^k \le t_i^{k+r} < t_{j+1}^k} \int_0^{\frac{1}{2}} \widetilde{\alpha}^{(2)}\left(F^{[3]}(t_j^k)\right)\left(\left(\theta(\gamma^{\tau_2}(n))_u^{t_i^{k+r}} - \theta(\gamma^{\tau_2}(n))_{\frac{1}{2}}^{t_i^{k+r}}\right)^{\otimes 2}\right) \\
&\left(A(\gamma^{\tau_2}(n))_{-1,t_j^k} \otimes d\theta(\gamma^{\tau_2}(n))_u^{t_i^{k+r}}\right) \\
&+ O\left(\frac{1}{3^{3k}} \cdot 3^{2r-\varepsilon k} \cdot \sup_{t\in[0,1]} \left|A(\gamma^{\tau_1}(n))_{-1,t}\right| \cdot l(\tau_1)\right) \\
&+ O\left(\frac{1}{3^{3k}} \cdot 3^{2r-\varepsilon k} \cdot \sup_{t\in[0,1]} \left|A(\gamma^{\tau_2}(n))_{-1,t}\right| \cdot l(\tau_2)\right),
\end{aligned} \tag{5.11}$$

where $l(\tau)$ denotes the length of τ.

Proof. For $k, r, n \in \mathbb{N}$ with $n = k + r$,

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(A\left(\gamma^\tau(n)\right)_{-1,t_j^k} \otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right) \\
= &\sum_{t_j^k \le t_i^{k+r} < t_{j+1}^k} \int_0^1 \widetilde{\alpha}\left(\gamma^\tau(n)_{\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}} + \left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\right) \\
&\left(A\left(\gamma^\tau(n)\right)_{-1,t_j^k} \otimes d\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}\right).
\end{aligned}$$

Since $\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}$ is reversible for each $n \in \mathbb{N}$, for $u \in \left[0, \frac{1}{2}\right]$,

$$\left(\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}+u}^{t_i^{k+r}} - \theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right) = -\left(\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}-u}^{t_i^{k+r}} - \theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right),$$

consequently, we have

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(A\left(\gamma^\tau(n)\right)_{-1,t_j^k} \otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right) \\
= &\sum_{t_j^k \le t_i^{k+r} < t_{j+1}^k} \int_0^{\frac{1}{2}} \left\{\widetilde{\alpha}\left(\gamma^\tau(n)_{\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}} + \left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\right)\right. \\
&\left. + \widetilde{\alpha}\left(\gamma^\tau(n)_{\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}} - \left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\right)\right\} \\
&\left(A\left(\gamma^\tau(n)\right)_{-1,t_j^k} \otimes d\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}\right).
\end{aligned}$$

By using (5.8) in Definition 5.17, we can express

$$\begin{aligned}
&\widetilde{\alpha}\left(\gamma^\tau(n)_{\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}} + \left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\right) \\
&+\widetilde{\alpha}\left(\gamma^\tau(n)_{\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}} - \left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\right) \\
= \;&2\widetilde{\alpha}\left(F^{[3]}\left(t_j^k\right)\right) \\
&+2\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(F^{[3]}\left(\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}\right) - F^{[3]}\left(t_j^k\right)\right)
\end{aligned}$$

$$+\widetilde{\alpha}^{(2)}\left(F^{[3]}\left(t_j^k\right)\right)\left(\left(F^{[3]}\left(\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}\right)-F^{[3]}\left(t_j^k\right)\right)^{\otimes 2}\right)$$
$$+\widetilde{\alpha}^{(2)}\left(F^{[3]}\left(t_j^k\right)\right)\left(\left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)^{\otimes 2}\right)$$
$$+O\left(\left|\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right|^{2+\varepsilon}\right)$$
$$+O\left(\left|F^{[3]}\left(\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}\right)-F^{[3]}\left(t_j^k\right)\right|^{2+\varepsilon}\right).$$

Note that because of the self-similar construction in Definition 4.46,

$$A\left(\gamma^{\tau_1}(n)\right)_{-1,t_j^k}=A\left(\gamma^{\tau_2}(n)\right)_{-1,t_j^k},$$

and

$$\sum_{t_j^k\le t_i^{k+r}<t_{j+1}^k}\int_0^{\frac{1}{2}} O\left(\left|\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right|^{2+\varepsilon}\right)$$
$$\left(A\left(\gamma^\tau(n)\right)_{-1,t_j^k}\otimes d\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}\right)$$
$$= O\left(\frac{1}{3^{3k}}\cdot 3^{2r-\varepsilon k}\cdot\sup_{t\in[0,1]}\left|A\left(\gamma^\tau(n)\right)_{-1,t}\right|\cdot l(\tau)\right).$$

Hence, we obtain our identity after substitution. ∎

For if $\widetilde{\alpha}$ were constant, we even have a simple expression of the integrals $\widetilde{\alpha}$ over $\left[t_j^k,t_{j+1}^k\right]$ against

$$A\left(\gamma^\tau(n)\right)_{-1,t_j^k}\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}$$

with respect to a paths τ:

Corollary 5.19 *Let $\widetilde{\alpha}(\cdot): I^3\longrightarrow\widetilde{Alt^3}\left(\mathbb{R}^d\right)$ be a constant 3−form, then for any finite variation reversible paths $\tau: I\longrightarrow I^3$ with $\tau_0=O$ and $\tau_1=(1,1,1)$, then $\forall k,r\in\mathbb{N}$ with $n=k+r$,*

$$\widetilde{\alpha}(O)\left(\int_{t_j^k}^{t_{j+1}^k}A\left(\gamma^\tau(n)\right)_{-1,t_j^k}\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right)$$
$$=\widetilde{\alpha}(O)\left(\int_{t_j^k}^{t_{j+1}^k}A\left(\gamma^\Upsilon(n)\right)_{-1,t_j^k}\otimes dL^1\left(\gamma^\Upsilon(n)\right)_{-1,t}\right).\qquad(5.12)$$

Proof. The result is a special case of Lemma 5.18. ■

We next derive an expression of the difference between two integrals $\widetilde{\alpha}$ over $\left[t_j^k, t_{j+1}^k\right]$ against

$$\left[L^1\left(\gamma^{\tau_1}(n)\right)_{-1,t_j^k}, L^1\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\right] \otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}$$

with respect to two reversible paths τ_1 and τ_2:

Lemma 5.20 *Let $\varepsilon > 0$. Suppose that a 3–form $\widetilde{\alpha}(\cdot) : I^3 \longrightarrow \widetilde{Alt^3}\left(\mathbb{R}^d\right)$ is $Lip\left(1+\varepsilon, I^3\right)$. For $m = 1, 2$, consider two finite variation reversible paths $\tau_m : I \longrightarrow I^3$ with $(\tau_m)_0 = O$ and $(\tau_m)_1 = (1,1,1)$. For $k, r \in \mathbb{N}$ with $n = k+r$, the difference*

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_1}(n)_t\right)\left(\left[L^1\left(\gamma^{\tau_1}(n)\right)_{-1,t_j^k}, L^1\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\right] \otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}\right) \\
&-\int_{t_j^k}^{t_{j+i}^k} \widetilde{\alpha}\left(\gamma^{\tau_2}(n)_t\right)\left(\left[L^1\left(\gamma^{\tau_2}(n)\right)_{-1,t_j^k}, L^1\left(\gamma^{\tau_2}(n)\right)_{t_j^k,t}\right] \otimes dL^1\left(\gamma^{\tau_2}(n)\right)_{-1,t}\right) \\
= {} & 4 \sum_{t_j^k \le t_i^{k+r} < t_{j+1}^k} \int_0^{\frac{1}{2}} \widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(\theta\left(\gamma^{\tau_1}(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^{\tau_1}(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right) \\
&\left(L^1\left(\gamma^{\tau_1}(n)\right)_{-1,t_j^k} \otimes \left(\theta\left(\gamma^{\tau_1}(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^{\tau_1}(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right) \otimes d\theta\left(\gamma^{\tau_1}(n)\right)_u^{t_i^{k+r}}\right) \\
&-4 \sum_{t_j^k \le t_i^{k+r} < t_{j+1}^k} \int_0^{\frac{1}{2}} \widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(\theta\left(\gamma^{\tau_2}(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^{\tau_2}(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right) \\
&\left(L^1\left(\gamma^{\tau_2}(n)\right)_{-1,t_j^k} \otimes \left(\theta\left(\gamma^{\tau_2}(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^{\tau_2}(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right) \otimes d\theta\left(\gamma^{\tau_2}(n)\right)_u^{t_i^{k+r}}\right) \\
&+O\left(\frac{1}{3^{3k}} \cdot 3^{2r-\varepsilon k} \cdot \sup_{t\in[0,1]} \left|L^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}\right| \cdot l\left(\tau_1\right)\right) \\
&+O\left(\frac{1}{3^{3k}} \cdot 3^{2r-\varepsilon k} \cdot \sup_{t\in[0,1]} \left|L^1\left(\gamma^{\tau_2}(n)\right)_{-1,t}\right| \cdot l\left(\tau_2\right)\right), \qquad (5.13)
\end{aligned}$$

where $l(\tau)$ denotes the length of a path τ.

Proof. For $v \in \mathbb{R}^3$, define

$$\begin{aligned}
&\mu\left(t_j^k, t_i^{k+r}, v\right) \\
\triangleq {} & 2\int_0^{\frac{1}{2}} \widetilde{\alpha}\left(\gamma^{\tau}(n)_{t_j^k} + v\right)\left(L^1\left(\gamma^{\tau}(n)\right)_{-1,t_j^k} \otimes v \otimes d\theta\left(\gamma^{\tau}(n)\right)_u^{t_i^{k+r}}\right).
\end{aligned}$$

For $k,r,n\in\mathbb{N}$ with $n=k+r$, by noting that, for $u\in\left[0,\frac{1}{2}\right]$,

$$\left(\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}+u}^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)=-\left(\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}-u}^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right),$$

we also have,

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(\left[L^1\left(\gamma^\tau(n)\right)_{-1,t_j^k},L^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\right]\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right)\\
=\;&2\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(L^1\left(\gamma^\tau(n)\right)_{-1,t_j^k}\otimes L^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right)\\
=\;&\sum_{t_j^k\le t_i^{k+r}<t_{j+1}^k}\mu\left(t_j^k,t_i^{k+r},L^1\left(\gamma^\tau(n)\right)_{t_j^k,\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}}+\left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\right)\\
&+\mu\left(t_j^k,t_i^{k+r},L^1\left(\gamma^\tau(n)\right)_{t_j^k,\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}}-\left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\right).
\end{aligned}$$

Using the Taylor's expansion for $\widetilde{\alpha}$ as in Lemma 5.18,

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(\left[L^1\left(\gamma^\tau(n)\right)_{-1,t_j^k},L^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\right]\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right)\\
=\;&4\sum_{t_j^k\le t_i^{k+r}<t_{j+1}^k}\int_0^{\frac{1}{2}}\widetilde{\alpha}\left(F^{[3]}\left(t_j^k\right)\right)\left(L^1\left(\gamma^\tau(n)\right)_{-1,t_j^k}\right.\\
&\left.\otimes\left(F^{[3]}\left(\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}\right)-F^{[3]}\left(t_j^k\right)\right)\otimes d\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}\right)\\
&+4\sum_{t_j^k\le t_i^{k+r}<t_{j+1}^k}\int_0^{\frac{1}{2}}\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(F^{[3]}\left(\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}\right)-F^{[3]}\left(t_j^k\right)\right)\\
&\left(L^1\left(\gamma^\tau(n)\right)_{-1,t_j^k}\otimes\left(F^{[3]}\left(\frac{t_i^{k+r}+t_{i+1}^{k+r}}{2}\right)-F^{[3]}\left(t_j^k\right)\right)\otimes d\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}\right)\\
&+4\sum_{t_j^k\le t_i^{k+r}<t_{j+1}^k}\int_0^{\frac{1}{2}}\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\\
&\left(L^1\left(\gamma^\tau(n)\right)_{-1,t_j^k}\otimes\left(\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}-\theta\left(\gamma^\tau(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)\otimes d\theta\left(\gamma^\tau(n)\right)_u^{t_i^{k+r}}\right)
\end{aligned}$$

$$+O\left(\frac{1}{3^{3k}}\cdot 3^{2r-\varepsilon k}\cdot \sup_{t\in[0,1]}\left|L^1\left(\gamma^{\tau}(n)\right)_{-1,t}\right|\cdot l(\tau)\right);$$

together with the fact that

$$L^1\left(\gamma^{\tau_1}(n)\right)_{-1,t_j^k}=L^1\left(\gamma^{\tau_2}(n)\right)_{-1,t_j^k},$$

we conclude our identity (5.13). ■

For if $\widetilde{\alpha}$ were constant, we even have a simple expression of the integrals $\widetilde{\alpha}$ over $\left[t_j^k,t_{j+1}^k\right]$ against

$$\left[L^1\left(\gamma^{\tau_1}(n)\right)_{-1,t_j^k},L^1\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\right]\otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}$$

with respect to a reversible paths τ:

Corollary 5.21 *Let $\widetilde{\alpha}(\cdot):I^3\longrightarrow\widetilde{Alt^3}\left(\mathbb{R}^d\right)$ be a constant 3−form, then for any finite variation reversible paths $\tau:I\longrightarrow I^3$ with $\tau_0=O$ and $\tau_1=(1,1,1)$, and $\forall k,r\in\mathbb{N}$ with $n=k+r$,*

$$\begin{aligned}&\widetilde{\alpha}(O)\left(\int_{t_j^k}^{t_{j+1}^k}\left[L^1\left(\gamma^{\tau}(n)\right)_{-1,t_j^k},L^1\left(\gamma^{\tau}(n)\right)_{t_j^k,t}\right]\otimes dL^1\left(\gamma^{\tau}(n)\right)_{-1,t}\right)\\=\;&\widetilde{\alpha}(O)\left(\int_{t_j^k}^{t_{j+1}^k}\left[L^1\left(\gamma^{\Upsilon}(n)\right)_{-1,t_j^k},L^1\left(\gamma^{\Upsilon}(n)\right)_{t_j^k,t}\right]\otimes dL^1\left(\gamma^{\Upsilon}(n)\right)_{-1,t}\right).\end{aligned}\tag{5.14}$$

Proof. The result is a special case of Lemma 5.20. ■

We next derive an expression of the difference between two integrals $\widetilde{\alpha}$ over $\left[t_j^k,t_{j+1}^k\right]$ against

$$A\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}$$

with respect to two reversible paths τ_1 and τ_2:

Lemma 5.22 *Let $\varepsilon>0$. Suppose that a 3−form $\widetilde{\alpha}(\cdot):I^3\longrightarrow\widetilde{Alt^3}\left(\mathbb{R}^d\right)$ is $Lip\left(\varepsilon,I^3\right)$. For $m=1,2$, consider two finite variation reversible paths*

$\tau_m : I \longrightarrow I^3$ with $(\tau_m)_0 = O$ and $(\tau_m)_1 = (1,1,1)$. For $k, r \in \mathbb{N}$ with $n = k + r$, the difference

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_1}(n)_t\right)\left(A\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}\right) \\
&- \int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_2}(n)_t\right)\left(A\left(\gamma^{\tau_2}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau_2}(n)\right)_{-1,t}\right) \\
= \; & \widetilde{\alpha}\left(\gamma^{\tau_1}(n)_{t_j^k}\right)\left(\int_{t_j^k}^{t_{j+1}^k} A\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\right) \\
&- \widetilde{\alpha}\left(\gamma^{\tau_2}(n)_{t_j^k}\right)\left(\int_{t_j^k}^{t_{j+1}^k} A\left(\gamma^{\tau_2}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau_2}(n)\right)_{t_j^k,t}\right) \\
&+ O\left(\frac{1}{3^{3k}} \cdot 3^{2r-\varepsilon k} \cdot \sup_{t\in[0,1]}\left|A\left(\gamma^{\tau_1}(r)\right)_{0,t}\right| \cdot l(\tau_1)\right) \\
&+ O\left(\frac{1}{3^{3k}} \cdot 3^{2r-\varepsilon k} \cdot \sup_{t\in[0,1]}\left|A\left(\gamma^{\tau_2}(r)\right)_{0,t}\right| \cdot l(\tau_2)\right), \qquad (5.15)
\end{aligned}$$

where $l(\tau)$ denotes the length of a path τ.

Proof. For any reversible path $\tau : I \to I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$.

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau}(n)_t\right)\left(A\left(\gamma^{\tau}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau}(n)\right)_{-1,t}\right) \\
= &\int_{t_j^k}^{t_{j+1}^k} \left(\widetilde{\alpha}\left(\gamma^{\tau}(n)_{t_j^k}\right) + R_0\left(L^1\left(\gamma^{\tau}(n)\right)_{t_j^k,t}\right)\right)\left(A\left(\gamma^{\tau}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau}(n)\right)_{t_j^k,t}\right)
\end{aligned}$$

According to the self similar construction in Definition 4.46, we note that for any $t \in \left[t_j^k, t_{j+1}^k\right]$,

$$\begin{aligned}
R_0\left(L^1\left(\gamma^{\tau}(n)\right)_{t_j^k,t}\right) &= O\left(\frac{1}{3^{\varepsilon k}}\right) \\
A\left(\gamma^{\tau}(n)\right)_{t_j^k,t} &= O\left(\frac{1}{3^{2k}} \cdot \sup_{t\in[0,1]}\left|A\left(\gamma^{\tau}(r)\right)_{0,t}\right|\right)
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} R_0\left(L^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\right)\left(A\left(\gamma^\tau(n)\right)_{t_j^k,t}\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right)\\
=\; & O\left(\frac{1}{3^{\varepsilon k}}\cdot\frac{1}{3^{2k}}\cdot\sup_{t\in[0,1]}\left|A\left(\gamma^\tau(r)\right)_{0,t}\right|\sum_{t_j^k\le t_i^{k+r}<t_{j+1}^k}\int_{t_i^{k+r}}^{t_{i+1}^{k+r}}\left|dL^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\right|\right),
\end{aligned}$$

where

$$\begin{aligned}
\sum_{t_j^k\le t_i^{k+r}<t_{j+1}^k}\int_{t_i^{k+r}}^{t_{i+1}^{k+r}}\left|dL^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\right| &\le 3^{3r}\frac{1}{3^n}l(\tau)\\
&= \frac{3^{2r}}{3^k}l(\tau),
\end{aligned}$$

and hence the result follows. ■

Recall a result which is a direct consequence of Chen's identity (1.3).

Lemma 5.23 *For any* $(s,t),(t,u)\in[-1,1]$,

$$A_{s,u}=A_{s,t}+\frac{1}{2}\left[L_{s,t}^1,L_{t,u}^1\right]+A_{t,u}. \tag{5.16}$$

Corollary 5.24 *Let* $\widetilde{\alpha}(\cdot):I^3\longrightarrow\widetilde{Alt^3}\left(\mathbb{R}^d\right)$ *be a* $3-$*form. For any finite variation reversible path* $\tau:I\longrightarrow I^3$ *with* $\tau_0=O$ *and* $\tau_1=(1,1,1)$,

$$\begin{aligned}
&\int_0^1\widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(A\left(\gamma^\tau(n)\right)_{-1,t}\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right)\\
=\;&\frac{1}{2}\sum_{t_j^k}\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(\left[L^1\left(\gamma^{\tau_1}(n)\right)_{-1,t_j^k},L^1\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\right]\otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}\right)\\
&+\sum_{t_j^k}\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(A\left(\gamma^\tau(n)\right)_{-1,t_j^k}\otimes dL^1\left(\gamma^\tau(n)\right)_{-1,t}\right)\\
&+\sum_{t_j^k}\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^\tau(n)_t\right)\left(A\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}\right).
\end{aligned}\tag{5.17}$$

In the remaining part of this section, we shall look for a simplification of Lemma 5.22 which leads to a subtle implication that the difference of integrals:

$$\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_1}(n)_t\right)\left(A\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}\right)$$
$$-\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_2}(n)_t\right)\left(A\left(\gamma^{\tau_2}(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^{\tau_2}(n)\right)_{-1,t}\right),$$

can describe the difference between the local actions of $\widetilde{\alpha}$ on

$$\int_0^1 A(\tau_i)_{0,t} \otimes dL^1(\tau_i)_{0,t},$$

for $i = 1, 2$.

Lemma 5.25 *Let $\widetilde{\alpha}(\cdot) : I^3 \to \widetilde{Alt^3}\left(\mathbb{R}^d\right)$ be a constant 3–form. For any $n \in \mathbb{Z}^+$,*

1.

$$\widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^\Upsilon(n)\right)_{0,t} \otimes dL^1\left(\gamma^\Upsilon(n)\right)_{0,t}\right) = 0.$$

2.

$$\begin{aligned}
&\widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^\Upsilon(n)\right)_{-1,t} \otimes dL^1\left(\gamma^\Upsilon(n)\right)_{-1,t}\right) \\
=\ &\widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^\Upsilon(1)\right)_{-1,t} \otimes dL^1\left(\gamma^\Upsilon(1)\right)_{-1,t}\right). \qquad (5.18)
\end{aligned}$$

Proof. Note that for any $t \in \left[t_j^k, t_{j+1}^k\right]$,

$$A\left(\gamma^\Upsilon(n)\right)_{t_j^k,t} = 0.$$

1) In accordance with Lemma 4.64, for any constant 3–form,

$$\widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^\Upsilon(1)\right)_{0,t} \otimes dL^1\left(\gamma^\Upsilon(1)\right)_{0,t}\right) = 0.$$

In general, for any $n \geq 2$,

$$\begin{aligned}
& \widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^{\Upsilon}(n)\right)_{0,t} \otimes dL^1\left(\gamma^{\Upsilon}(n)\right)_{0,t}\right) \\
= & \widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^{\Upsilon}(1)\right)_{0,t} \otimes dL^1\left(\gamma^{\Upsilon}(1)\right)_{0,t}\right) \\
& +\widetilde{\alpha}(O)\left(\sum_{0\leq t_j^i\leq 1}\int_{t_j^1}^{t_{j+1}^1} A\left(\gamma^{\Upsilon}(n-1)\right)_{t_j^1,t} \otimes dL^1\left(\gamma^{\Upsilon}(n-1)\right)_{t_j^1,t}\right) \\
= & \sum_{0\leq t_j^i\leq 1}\widetilde{\alpha}(O)\left(\left(P_j^{[3]}\right)^{\otimes 3}\left(\int_0^1 A\left(\gamma^{\Upsilon}(n-1)\right)_{0,t} \otimes dL^1\left(\gamma^{\Upsilon}(n-1)\right)_{0,t}\right)\right) \\
= & 0,
\end{aligned}$$

hence the result follows by induction.

2) Applying the claim in part (1), we have,

$$\begin{aligned}
& \widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^{\Upsilon}(n)\right)_{-1,t} \otimes dL^1\left(\gamma^{\Upsilon}(n)\right)_{-1,t}\right) \\
= & \sum_{0\leq t_j^1\leq 1}\widetilde{\alpha}(O)\left(\int_{t_j^1}^{t_{j+1}^1} A\left(\gamma^{\Upsilon}(n)\right)_{t_j^1,t} \otimes dL^1\left(\gamma^{\Upsilon}(n)\right)_{t_j^1,t}\right) \\
& +\widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^{\Upsilon}(1)\right)_{-1,t} \otimes dL^1\left(\gamma^{\Upsilon}(1)\right)_{-1,t}\right) \\
= & \widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^{\Upsilon}(1)\right)_{-1,t} \otimes dL^1\left(\gamma^{\Upsilon}(1)\right)_{-1,t}\right).
\end{aligned}$$

■

Corollary 5.26 *Let $\widetilde{\alpha}(\cdot) : I^3 \to \widetilde{Alt^3}\left(\mathbb{R}^d\right)$ be a constant 3−form. For any finite variation reversible paths $\tau : I \to I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$, $\forall n \in \mathbb{Z}^+$,*

1.

$$\begin{aligned}
& \widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^{\tau}(n)\right)_{-1,t} \otimes dL^1\left(\gamma^{\tau}(n)\right)_{-1,t}\right) \\
= & \widetilde{\alpha}(O)\left(\int_0^1 A(\tau)_{0,t} \otimes dL^1(\tau)_{0,t}\right).
\end{aligned}$$

2.

$$
\begin{aligned}
&\widetilde{\alpha}(O)\left(\int_0^1 A(\gamma^\tau(n))_{-1,t} \otimes dL^1(\gamma^\tau(n))_{-1,t}\right) \\
=\ &\widetilde{\alpha}(O)\left(\int_0^1 A(\tau)_{0,t} \otimes dL^1(\tau)_{0,t}\right) \\
&+\widetilde{\alpha}(O)\left(\int_0^1 A(\gamma^\tau(1))_{-1,t} \otimes dL^1(\gamma^\tau(1))_{-1,t}\right). \qquad (5.19)
\end{aligned}
$$

Proof. In accordance with Corollaries 5.19, 5.21 and 5.24, we have

$$
\begin{aligned}
&\widetilde{\alpha}(O)\left(\int_0^1 A(\gamma^\tau(n))_{-1,t} \otimes dL^1(\gamma^\tau(n))_{-1,t}\right) \\
&-\widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^\Upsilon(n)\right)_{-1,t} \otimes dL^1\left(\gamma^\Upsilon(n)\right)_{-1,t}\right) \\
=\ &\sum_{0\le t_j^1\le 1} \widetilde{\alpha}(O)\left(\int_{t_j^1}^{t_{j+1}^1} A(\gamma^\tau(n))_{t_j^1,t} \otimes dL^1(\gamma^\tau(n))_{t_j^1,t}\right) \\
&-\sum_{0\le t_j^1\le 1} \widetilde{\alpha}(O)\left(\int_{t_j^1}^{t_{j+1}^1} A\left(\gamma^\Upsilon(n)\right)_{t_j^1,t} \otimes dL^1\left(\gamma^\Upsilon(n)\right)_{t_j^1,t}\right).
\end{aligned}
$$

Recall the notion of $P_j^{[d]}$ in Definition 4.27 for $d = 3$. In accordance with the self-similar construction in Definition 4.46 and Lemma 5.25, we have

$$
\begin{aligned}
&\widetilde{\alpha}(O)\left(\int_0^1 A(\gamma^\tau(n))_{-1,t} \otimes dL^1(\gamma^\tau(n))_{-1,t}\right) \\
&-\widetilde{\alpha}(O)\left(\int_0^1 A\left(\gamma^\Upsilon(1)\right)_{-1,t} \otimes dL^1\left(\gamma^\Upsilon(1)\right)_{-1,t}\right) \\
=\ &\sum_{0\le t_j^1\le 1} \widetilde{\alpha}(O)\left(\left(P_j^{[3]}\right)^{\otimes 3}\left(\int_0^1 A(\gamma^\tau(n-1))_{0,t} \otimes dL^1(\gamma^\tau(n-1))_{0,t}\right)\right) \\
&-\sum_{0\le t_j^1\le 1} \widetilde{\alpha}(O)\left(\left(P_j^{[3]}\right)^{\otimes 3}\left(\int_0^1 A\left(\gamma^\Upsilon(n-1)\right)_{0,t} \otimes dL^1\left(\gamma^\Upsilon(n-1)\right)_{0,t}\right)\right) \\
=\ &\sum_{0\le j\le 3^3-1} \widetilde{\alpha}(O)\left(\left(P_j^{[3]}\right)^{\otimes 3}\left(\int_0^1 A(\gamma^\tau(n-1))_{0,t} \otimes dL^1(\gamma^\tau(n-1))_{0,t}\right)\right)
\end{aligned}
$$

$$= \sum_{0\le j\le 3^3-1} \frac{1}{3^3}\widetilde{\alpha}(O)\left(\int_0^1 A(\gamma^\tau(n-1))_{0,t}\otimes dL^1(\gamma^\tau(n-1))_{0,t}\right)$$
$$= \widetilde{\alpha}(O)\left(\int_0^1 A(\gamma^\tau(n-1))_{0,t}\otimes dL^1(\gamma^\tau(n-1))_{0,t}\right)$$
$$= \widetilde{\alpha}(O)\left(\int_0^1 A(\tau)_{0,t}\otimes dL^1(\tau)_{0,t}\right).$$

■

Corollary 5.27 *Let $\varepsilon > 0$. Suppose that a 3−form $\widetilde{\alpha}(\cdot) : I^3 \longrightarrow \widetilde{Alt^3}(\mathbb{R}^d)$ is $Lip(\varepsilon, I^3)$. For $m = 1, 2$, consider two finite variation reversible paths $\tau_m : I \longrightarrow I^3$ with $(\tau_m)_0 = O$ and $(\tau_m)_1 = (1,1,1)$. For $k, r \in \mathbb{N}$ with $n = k + r$, the difference*

$$\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}(\gamma^{\tau_1}(n)_t)\left(A(\gamma^{\tau_1}(n))_{t_j^k,t}\otimes dL^1(\gamma^{\tau_1}(n))_{-1,t}\right)$$
$$-\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}(\gamma^{\tau_2}(n)_t)\left(A(\gamma^{\tau_2}(n))_{t_j^k,t}\otimes dL^1(\gamma^{\tau_2}(n))_{-1,t}\right)$$
$$= \frac{1}{3^{3k}}\widetilde{\alpha}\left(\gamma^{\tau_1}(n)_{t_j^k}\right)\left(\int_0^1 A(\tau_1)_{0,t}\otimes dL^1(\tau_1)_{0,t}\right)$$
$$-\frac{1}{3^{3k}}\widetilde{\alpha}\left(\gamma^{\tau_2}(n)_{t_j^k}\right)\left(\int_0^1 A(\tau_2)_{0,t}\otimes dL^1(\tau_2)_{0,t}\right)$$
$$+O\left(\frac{1}{3^{3k}}\cdot 3^{2r-\varepsilon k}\cdot \sup_{t\in[0,1]}\left|A(\gamma^{\tau_1}(r))_{0,t}\right|\cdot l(\tau_1)\right)$$
$$+O\left(\frac{1}{3^{3k}}\cdot 3^{2r-\varepsilon k}\cdot \sup_{t\in[0,1]}\left|A(\gamma^{\tau_2}(r))_{0,t}\right|\cdot l(\tau_2)\right),$$

where $l(\tau)$ denotes the length of a path τ.

Proof. According to Corollary 4.29, we have

$$\int_{t_j^k}^{t_{j+1}^k} A(\gamma^\tau(n))_{t_j^k,t}\otimes dL^1(\gamma^\tau(n))_{t_j^k,t}$$
$$= \left(P^{[3]}_{n_1(t_j^k)}\right)^{\otimes 3}\circ\cdots\circ\left(P^{[3]}_{n_k(t_j^k)}\right)^{\otimes 3}\left(\int_0^1 A(\gamma^\tau(r))_{0,t}\otimes dL^1(\gamma^\tau(r))_{0,t}\right).$$

For any constant 3−form $\widetilde{\omega}$ and a finite variation reversible path $\tau : I \to I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$,

$$\begin{aligned}
&\widetilde{\omega}\left(\int_{t_j^k}^{t_{j+1}^k} A\left(\gamma^\tau(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\right) \\
= \;& \widetilde{\omega}\left(\left(P^{[3]}_{n_1(t_j^k)}\right)^{\otimes 3} \circ \cdots \circ \left(P^{[3]}_{n_k(t_j^k)}\right)^{\otimes 3}\left(\int_0^1 A\left(\gamma^\tau(r)\right)_{0,t} \otimes dL^1\left(\gamma^\tau(r)\right)_{0,t}\right)\right) \\
= \;& \frac{1}{3^{3k}}\widetilde{\omega}\left(\int_0^1 A\left(\gamma^\tau(r)\right)_{0,t} \otimes dL^1\left(\gamma^\tau(r)\right)_{0,t}\right).
\end{aligned}$$

Using Corollary 5.26, we deduce that

$$\begin{aligned}
&\widetilde{\omega}\left(\int_{t_j^k}^{t_{j+1}^k} A\left(\gamma^\tau(n)\right)_{t_j^k,t} \otimes dL^1\left(\gamma^\tau(n)\right)_{t_j^k,t}\right) \\
= \;& \frac{1}{3^{3k}}\widetilde{\omega}\left(\int_0^1 A(\tau)_{0,t} \otimes dL^1(\tau)_{0,t}\right). \qquad (5.20)
\end{aligned}$$

By substituting (5.20) in Lemma 5.22, we conclude our result. ■

5.3 An ergodic result

In this section, we shall find, for any $k \in \mathbb{N}$ and $j = 0, \ldots, 3^{3k}$, the asymptotic value (see Corollary 5.37) of

$$3^{3k} \cdot J\left(\gamma^\tau(k+r)\right)_{t_j^k, t_{j+1}^k}$$

in (5.29) as r goes to infinity. Note that the quantity $3^{3k} \cdot J\left(\gamma^\tau(k+r)\right)_{t_j^k, t_{j+1}^k}$ plays a key role in the derivation of our main result Theorem 5.45 in Section 5.4. The key idea of finding out the asymptotic value is to first identify the expression $3^{3k} \cdot J\left(\gamma^\tau(k+r)\right)_{t_j^k, t_{j+1}^k}$, by using self-similar nature of the definition of γ^τ, as an expected value of some functional of a irreducible aperiodic Markov Chain as defined in Definition 5.28; and then an easy application of a celebrated result (Theorem 5.30) in ergodic theory leads to the expression of the asymptotic value.

We first introduce the mentioned Markov chain. For $i = 1, 2, 3$, define $Q_i : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Q_i\left(e^j\right) = (-1)^{1-\delta_i(j)} \cdot e^j$ where $\delta_i(\cdot) : \mathbb{N} \to \mathbb{N}$ is the

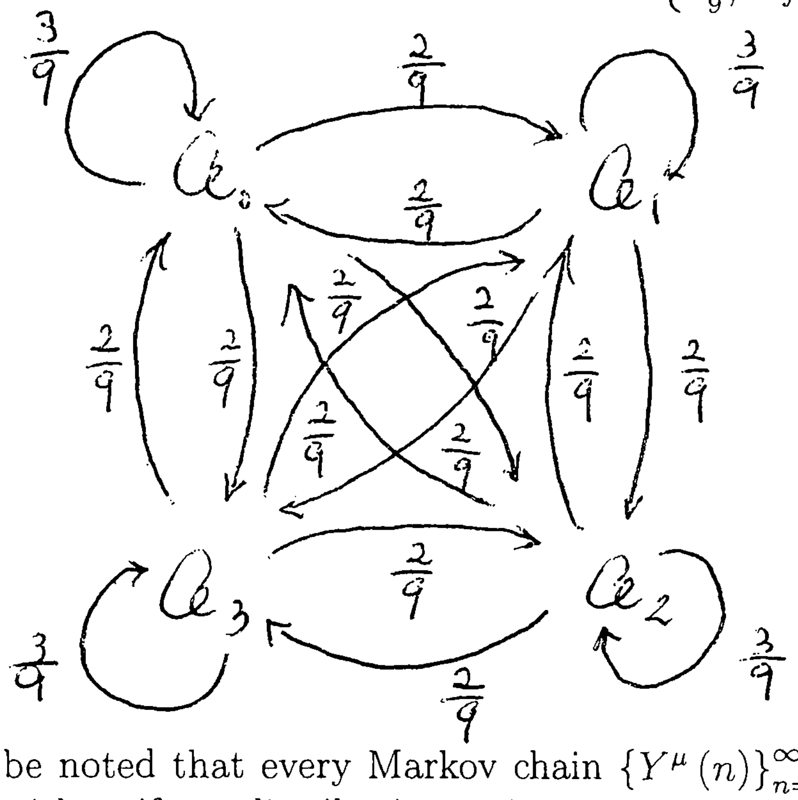
Kronecker delta function. In accordance with Corollary 4.23,

$$card\,(Q\,(3)) = 2^{3-1} = 4;$$

indeed, 9 of $Q_{3\delta_j^{[3]}}$ are the identity map $Q_0 \triangleq I : \mathbb{R}^3 \to \mathbb{R}^3$, 6 of $Q_{3\delta_j^{[3]}}$ are Q_1, another 6 of $Q_{3\delta_j^{[3]}}$ are Q_2 and the remaining 6 of $Q_{3\delta_j^{[3]}}$ are Q_3.

Definition 5.28 *Define $\{Y^{\mu}(n)\}_{n=0}^{\infty}$ to be a discrete time-homogeneous Markov chain with states space $G \triangleq \{Q_0, Q_1, Q_2, Q_3\}$, an initial distribution μ and transition probability such that for any $i, j = 0, \ldots, 3$, $n \in \mathbb{N}$,*

$$\mathbb{P}\,(Y\,(n+1) = Q_j | Y\,(n) = Q_i) = \begin{cases} \frac{3}{9}, & \text{if } j = i, \\ \frac{2}{9}, & \text{if } j \neq i. \end{cases} \tag{5.21}$$



It should be noted that every Markov chain $\{Y^{\mu}(n)\}_{n=0}^{\infty}$ is irreducible and aperiodic with uniform distribution as its invariant measure.

Lemma 5.29 *$G \triangleq \{Q_0, Q_1, Q_2, Q_3\}$ is an abelian group under matrix multiplication such that*

$$Q_i^2 = Q_0 = I \tag{5.22}$$

for all $i \in \{0, 1, 2, 3\}$ and

$$Q_i Q_j = Q_k \tag{5.23}$$

for distinct $i, j, k \in \{0, 1, 2, 3\}$.

We next recall a celebrated result in ergodic theory:

Theorem 5.30 (Ergodic behavior of Markov chains) *Let S be a Borel state space. For any irreducible, aperiodic Markov Chain $\{X(n)\}_{n=1}^{\infty}$ in S, exactly one of these cases holds:*

1. *There exists a unique invariant measure ν, the latter satisfies $\nu(i) > 0$ for all $i \in S$ and for any initial distribution μ,*

$$\lim_{n\to\infty} \left|\left|\mathbb{P}^{\mu} \circ X(n)^{-1} - \nu\right|\right| = 0, \tag{5.24}$$

where for any signed measure λ on S,

$$||\lambda|| \triangleq \sup_{A\in\sigma(A)} ||\lambda(A)||. \tag{5.25}$$

2. *No invariant measure exists and we also have*

$$\lim_{n\to\infty} p_{ij}^{n} = 0, \tag{5.26}$$

where p_{ij}^{n} is the n^{th}-step transition probability from state i to j, for any $i, j \in S$.

Proof. One can refer to Theorem 8.18 in Kallenberg [2002] for details. ∎

Corollary 5.31 *Let V be a Banach space. Suppose that the states space S is finite. Then for any $f : S \to V$,*

$$\lim_{n\to\infty} \left(\sup_{\mu} \left|\mathbb{E}^{\mu}(f(X(n))) - \nu(f)\right|\right) = 0. \tag{5.27}$$

Proof. Let $x \in S$ and δ_x is the Dirac measure with unit mass at x. As a consequence of Theorem 5.30, we deduce that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}^{\delta_x}(f(X(n))) &= \lim_{n\to\infty} \sum_{y\in S} f(y)\,\mathbb{P}^{x}(X(n) = y) \\
&= \sum_{y\in S} f(y) \left(\lim_{n\to\infty} \mathbb{P}^{x}(X(n) = y)\right) \\
&= \sum_{y\in S} f(y) \cdot \nu(y) \\
&= \nu(f).
\end{aligned}$$

Since S is finite, every measure μ on S can be expressed as

$$\mu(\cdot) = \sum w_y \cdot \delta_y(\cdot),$$

where $0 \leq w_y \leq 1$. Now, we have

$$\left|\mathbb{E}^{\mu}(f(X(n))) - \nu(f)\right| \leq \sum_{x\in S} w_x \left|\mathbb{E}^{\delta_x}(f(X(n))) - \nu(f)\right|,$$

so

$$\sup_{\mu} \left|\mathbb{E}^{\mu}(f(X(n))) - \nu(f)\right| \leq \sum_{x\in S} \left|\mathbb{E}^{\delta_x}(f(X(n))) - \nu(f)\right|,$$

where the last sum tends to zero as n approaches to infinity since S is finite. ■

Recall the notations in Definitions 4.27 and 4.28.

Lemma 5.32 *Let $m \in \mathbb{N}$ and $i = 0, \ldots, 3^{3m} - 1$. Consider a reversible path $\tau : I \longrightarrow I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$. For any $u \in I$,*

$$\begin{aligned} &\int_0^{\frac{1}{2}} \left(\theta(\gamma^{\tau}(m))_u^{t_i^m} - \theta(\gamma^{\tau}(m))_{\frac{1}{2}}^{t_i^m}\right)^{\otimes 2} \otimes d\theta(\gamma^{\tau}(m))_u^{t_i^m} \\ = &\left(P^{[3]}_{n_1(t_i^m)}\right)^{\otimes 3} \circ \cdots \circ \left(P^{[3]}_{n_m(t_i^m)}\right)^{\otimes 3} \left(\int_0^{\frac{1}{2}} \left(\tau_u - \tau_{\frac{1}{2}}\right)^{\otimes 2} \otimes d\tau_u\right). \end{aligned} \tag{5.28}$$

Proof. As a consequence of Corollary 4.29, we have

$$\begin{aligned} &\theta(\gamma^{\tau}(m))_u^{t_i^m} - \theta(\gamma^{\tau}(m))_{\frac{1}{2}}^{t_i^m} \\ = &\gamma^{\tau}(m)_{t_i^m + u\left(t_{i+1}^m - t_i^m\right)} - \gamma^{\tau}(m)_{\frac{t_i^m + t_{i+1}^m}{2}} \\ = &P^{[3]}_{n_1(t_i^m)} \circ \cdots \circ P^{[3]}_{n_m(t_i^m)} \left(\tau_u - \tau_{\frac{1}{2}}\right), \end{aligned}$$

therefore the result follows. ■

Corollary 5.33 *Let $k, r \in \mathbb{N}$, $n = k + r$ and $j = 0, \ldots, 3^{3k}$, consider a* *reversible path $\tau : I \to I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$.*

$$\begin{aligned}
& J\left(\gamma^{\tau}(n)\right)_{t_j^k, t_{j+1}^k} \\
\triangleq & \sum_{t_j^k \le t_i^{k+r} < t_{j+1}^k} \int_0^{\frac{1}{2}} \left(\theta\left(\gamma^{\tau}(n)\right)_u^{t_i^{k+r}} - \theta\left(\gamma^{\tau}(n)\right)_{\frac{1}{2}}^{t_i^{k+r}}\right)^{\otimes 2} \otimes d\theta\left(\gamma^{\tau}(n)\right)_u^{t_i^{k+r}} \\
= & \left(P_{n_1(t_j^k)}^{[3]}\right)^{\otimes 3} \circ \cdots \circ \left(P_{n_k(t_j^k)}^{[3]}\right)^{\otimes 3} \\
& \left(\sum_{0 \le t_i^r < 1} \left(P_{n_1(t_i^r)}^{[3]}\right)^{\otimes 3} \circ \cdots \circ \left(P_{n_r(t_i^r)}^{[3]}\right)^{\otimes 3} \left(\int_0^{\frac{1}{2}} \left(\tau_u - \tau_{\frac{1}{2}}\right)^{\otimes 2} \otimes d\tau_u\right)\right).
\end{aligned} \tag{5.29}$$

Proof. In accordance with Definition 4.28, for $l = 1, \ldots, k$, $m = 1, \ldots, r$, $\beta = 0, \ldots, 3^{3r} - 1$,

$$\begin{aligned}
n_l\left(t_i^{k+r}\right) &= n_l\left(t_j^k\right), \\
n_{k+m}\left(t_{3^{3(k+r)} \cdot t_j^k + \beta}^{k+r}\right) &= n_m\left(t_\beta^r\right).
\end{aligned}$$

Applying Lemma 5.32, we conclude our result. ■

Lemma 5.34 *Let $r \in \mathbb{N}$,*

$$\sum_{0 \le t_i^r < 1} \left(P_{n_1(t_i^r)}^{[3]}\right)^{\otimes 3} \circ \cdots \circ \left(P_{n_r(t_i^r)}^{[3]}\right)^{\otimes 3} = \mathbb{E}\left(Y(r)^{\otimes 3} \mid Y(0) = Q_0\right). \tag{5.30}$$

Proof. By Definition 4.27, we have

$$\begin{aligned}
& \sum_{0 \le t_i^r < 1} \left(P_{n_1(t_i^r)}^{[3]}\right)^{\otimes 3} \circ \cdots \circ \left(P_{n_r(t_i^r)}^{[3]}\right)^{\otimes 3} \\
= & \frac{1}{3^{3r}} \sum_{i_1, \ldots, i_r = 0}^{3^3 - 1} \left(Q_{3\delta_{i_1}^{[3]}}^{[3]}\right)^{\otimes 3} \circ \cdots \circ \left(Q_{3\delta_{i_r}^{[3]}}^{[3]}\right)^{\otimes 3} \\
= & \left(\frac{3}{9} Q_0^{\otimes 3} + \frac{2}{9} Q_1^{\otimes 3} + \frac{2}{9} Q_2^{\otimes 3} + \frac{2}{9} Q_3^{\otimes 3}\right)^{\circ r} \\
= & \mathbb{E}\left(Y(r)^{\otimes 3} \mid Y(0) = Q_0\right)
\end{aligned}$$

where the last equality holds because of Definition 5.28 and Lemma 5.29. ∎

We are now ready to find out the asymptotic value of $3^{3k} \cdot J\left(\gamma^{\tau}\left(k+r\right)\right)_{t_j^k, t_{j+1}^k}$ as r goes to infinity.

Proposition 5.35 *Consider a reversible path $\tau : I \longrightarrow I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$.*

$$\begin{aligned} &\lim_{r\to\infty} \sup_{k,j} \left| 3^{3k} \cdot J\left(\gamma^{\tau}\left(k+r\right)\right)_{t_j^k, t_{j+1}^k} - \frac{1}{4} \sum_{i=0}^{3} Q_i^{\otimes 3} \left(\int_0^{\frac{1}{2}} \left(\tau_u - \tau_{\frac{1}{2}} \right)^{\otimes 2} \otimes d\tau_u \right) \right| \\ = \ & 0. \end{aligned} \tag{5.31}$$

Proof. Note that for any $k \in \mathbb{N}$, using Lemma 5.29,

$$\begin{aligned} & 3^{3k} \cdot \left(P^{[3]}_{n_1(t_i^k)} \right)^{\otimes 3} \circ \cdots \circ \left(P^{[3]}_{n_k(t_i^k)} \right)^{\otimes 3} \\ = & \left(Q^{[3]}_{3\delta^{[3]}_{n_1(t_i^k)}} \right)^{\otimes 3} \cdots \left(Q^{[3]}_{3\delta^{[3]}_{n_k(t_i^k)}} \right)^{\otimes 3} \in G^{\otimes 3} \end{aligned}$$

According to Corollary 5.31, we have

$$\lim_{r\to\infty} \sup_{p\in\{0,\ldots,3\}} \left| \mathbb{E}\left(Y\left(r\right)^{\otimes 3} \middle| Y\left(0\right) = Q_p \right) - \frac{1}{4} \sum_{i=0}^{3} Q_i^{\otimes 3} \right| = 0, \tag{5.32}$$

since the invariant measure of $\{Y(n)\}_{n\in\mathbb{N}}$ is the uniform one. The result follows by substituting expression (5.30) into Corollary 5.33 and combining with (5.32). ∎

Lemma 5.36 *Let $\{e^i\}_{i=1}^3$ be the standard basis of $\mathbb{R}^3$, then*

$$\begin{aligned} & \frac{1}{4} \sum_{i=0}^{3} Q_i^{\otimes 3} \left(e^a \otimes e^b \otimes e^c \right) \\ = & \begin{cases} 0, & \text{if any two of } a,b,c \text{ are the same,} \\ e^a \otimes e^b \otimes e^c, & \text{if all } a,b,c \text{ are distinct.} \end{cases} \end{aligned} \tag{5.33}$$

Proof. By permutation symmetry of G with respect to $\{e^i\}_{i=1}^3$, we only have to check the validity of our claim for the following six cases

$$e^1 \otimes e^1 \otimes e^1,\ e^1 \otimes e^2 \otimes e^1,\ e^1 \otimes e^3 \otimes e^1,$$
$$e^2 \otimes e^2 \otimes e^1,\ e^2 \otimes e^3 \otimes e^1,\ e^3 \otimes e^3 \otimes e^1.$$

■

For any $a, b, c \in \{1, 2, 3\}$, we define

$$I^\tau(a,b,c) \triangleq \int_0^{\frac{1}{2}} \left(\tau_u^a - \tau_{\frac{1}{2}}^a\right)\left(\tau_u^b - \tau_{\frac{1}{2}}^b\right) d\tau_u^c. \tag{5.34}$$

Corollary 5.37 *Consider a reversible path $\tau : I \longrightarrow I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$. We have*

$$\begin{aligned}&\limsup_{r\to\infty}\sup_{k,j}\left|3^{3k}\cdot J\left(\gamma^\tau(n)\right)_{t_j^k,t_{j+1}^k} - \sum_{\text{all distinct } a,b,c} I^\tau(a,b,c)\cdot e^a\otimes e^b\otimes e^c\right| \\ =\ & 0.\end{aligned}$$

Proof. By taking into account Lemma 5.36, the result follows as an immediate corollary of Proposition 5.35. ■

5.4 Integrating 1-form against space-filling rough paths

In this section, we shall first use the asymptotic value of $3^{3k}\cdot J\left(\gamma^\tau(k+r)\right)_{t_j^k,t_{j+1}^k}$ obtained in Section 5.3 to further simplify each of the differences of integrals $\widetilde{\alpha}$ obtained in Section 5.2. Finally, we shall establish our main result Theorem 5.45 which relates a difference (5.40) of two rough path integrals of two space-filling rough paths for a unit cube in accordance with Theorem 4.82 to a few Lebesgue integrals of some functions induced by $\widetilde{\alpha}$ over the unit cube. In the following, we fix an arbitrary $\varepsilon > 0$ and consider two finite variation reversible paths $\tau_m : I \longrightarrow I^3$ with $(\tau_m)_0 = O$ and $(\tau_m)_1 = (1,1,1)$ for $m = 1, 2$. In spite of Corollary 5.37, there is $r_\varepsilon \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $j = 0, \ldots, 3^{3k} - 1$,

$$\left|J\left(\gamma^{\tau_m}(n)\right)_{t_j^k,t_{j+1}^k} - \frac{1}{3^{3k}}\sum_{\text{all distinct } a,b,c} I^{\tau_m}(a,b,c)\cdot e^a\otimes e^b\otimes e^c\right| \le \frac{\varepsilon}{3^{3k}}. \tag{5.35}$$

For $k \in \mathbb{N}$, we denote $n_\varepsilon(k) = k + r_\varepsilon$. We first have a simplification of the expression (5.11) in Lemma 5.18:

Lemma 5.38 *Let $\varepsilon' > 0$. Let $\widetilde{\alpha}(\cdot) : I^3 \to \widetilde{Alt^3}(\mathbb{R}^d)$ be a $Lip(2+\varepsilon', I^3)$ 3−form. The difference*

$$\begin{aligned}
&\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_1}(n_\varepsilon(k))_t\right)\left(A\left(\gamma^{\tau_1}(n_\varepsilon(k))\right)_{-1,t_j^k} \otimes dL^1\left(\gamma^{\tau_1}(n_\varepsilon(k))\right)_{-1,t}\right) \\
&- \int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_2}(n_\varepsilon(k))_t\right)\left(A\left(\gamma^{\tau_2}(n_\varepsilon(k))\right)_{-1,t_j^k} \otimes dL^1\left(\gamma^{\tau_2}(n_\varepsilon(k))\right)_{-1,t}\right) \\
= &\sum_{\text{all distinct } a,b,c} \left(I^{\tau_1}(a,b,c) - I^{\tau_2}(a,b,c)\right) \\
&\cdot \widetilde{\alpha}^{(2)}\left(F^{[3]}\left(t_j^k\right)\right)\left(e^a \otimes e^b\right)\left(A\left(\gamma^{\Upsilon}(n_\varepsilon(k))\right)_{-1,t_j^k} \otimes e^c\right) \cdot \frac{1}{3^{3k}} \\
&+ O\left(\frac{1}{3^{3k}} \cdot 3^{2r_\varepsilon - \varepsilon k} \cdot \sup_{t\in[0,1]} \left|A\left(\gamma^{\tau_1}(n_\varepsilon(k))\right)_{-1,t}\right| \cdot l(\tau_1)\right) \\
&+ O\left(\frac{1}{3^{3k}} \cdot 3^{2r_\varepsilon - \varepsilon k} \cdot \sup_{t\in[0,1]} \left|A\left(\gamma^{\tau_2}(n_\varepsilon(k))\right)_{-1,t}\right| \cdot l(\tau_2)\right) \\
&+ O\left(\frac{\varepsilon}{3^{3k}} |||\widetilde{\alpha}||| \cdot \sup_{t\in[0,1]} \left|A\left(\gamma^{\Upsilon}(n_\varepsilon(k))\right)_{-1,t}\right|\right),
\end{aligned} \tag{5.36}$$

where $l(\tau)$ denotes the length of the path τ.

Proof. Note that using the reversibility of $\gamma^{\tau_m}(h)$ on $\left[t_j^k, t_{j+1}^k\right]$ for all $k \le h \in \mathbb{N}$, we have

$$A\left(\gamma^{\tau_m}(h)\right)_{-1,t_j^k} = A\left(\gamma^{\Upsilon}(h)\right)_{-1,t_j^k}.$$

By substituting our estimate (5.35) of $J\left(\gamma^{\tau_m}(n)\right)_{t_j^k, t_{j+1}^k}$ into the Lemma 5.18, we obtain our claim. ■

We next have a simplification of (5.13) in Lemma 5.20:

Lemma 5.39 *Let $\varepsilon' > 0$ and $\widetilde{\alpha}(\cdot) : I^3 \to \widetilde{Alt^3}(\mathbb{R}^d)$ be a $Lip(1+\varepsilon', I^3)$*

3−form. The difference

$$\begin{aligned}
& \int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_1}\left(n_\varepsilon(k)\right)_t\right)\left(\left[L^1\left(\gamma^{\tau_1}\left(n_\varepsilon(k)\right)\right)_{-1,t_j^k}, L^1\left(\gamma^{\tau_1}\left(n_\varepsilon(k)\right)\right)_{t_j^k,t}\right]\right. \\
& \left.\otimes dL^1\left(\gamma^{\tau_1}\left(n_\varepsilon(k)\right)\right)_{-1,t}\right) \\
& -\int_{t_j^k}^{t_{j+1}^k} \widetilde{\alpha}\left(\gamma^{\tau_2}\left(n_\varepsilon(k)\right)_t\right)\left(\left[L^1\left(\gamma^{\tau_2}\left(n_\varepsilon(k)\right)\right)_{-1,t_j^k}, L^1\left(\gamma^{\tau_2}\left(n_\varepsilon(k)\right)\right)_{t_j^k,t}\right]\right. \\
& \left.\otimes dL^1\left(\gamma^{\tau_2}\left(n_\varepsilon(k)\right)\right)_{-1,t}\right) \\
= \; & 4 \sum_{\text{all distinct } a,b,c} \left(I^{\tau_1}(a,b,c) - I^{\tau_2}(a,b,c)\right) \\
& \widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(e^a\right)\left(\left(F^{[3]}\left(t_j^k\right)-\gamma_{-1}\right)\otimes e^b \otimes e^c\right)\cdot \frac{1}{3^{3k}} \\
& +O\left(\frac{1}{3^{3k}}\cdot 3^{2r_\varepsilon-\varepsilon k}\cdot \sup_{t\in[0,1]}\left|L^1\left(\gamma^{\Upsilon}\left(n_\varepsilon(k)\right)\right)_{-1,t}\right|\cdot\left(l(\tau_1)+l(\tau_2)\right)\right) \\
& +O\left(\frac{\varepsilon}{3^{3k}}\,|||\widetilde{\alpha}|||\sup_{t\in[0,1]}\left|L^1\left(\gamma^{\Upsilon}\left(n_\varepsilon(k)\right)\right)_{-1,t}\right|\right), \qquad (5.37)
\end{aligned}$$

where $l(\tau)$ denotes the length of the path τ.

Proof. Note that using the self-similar construction in Definition 4.46 of $\gamma^{\tau_m}(h)$ on $\left[t_j^k, t_{j+1}^k\right]$ for all $k \le h \in \mathbb{N}$, we have

$$\begin{aligned}
L^1\left(\gamma^{\tau_m}(h)\right)_{-1,t_j^k} &= L^1\left(\gamma^{\Upsilon}(h)\right)_{-1,t_j^k} \\
&= F^{[3]}\left(t_j^k\right)-\gamma_{-1}.
\end{aligned}$$

By substituting our estimate (5.35) of $J\left(\gamma^{\tau_m}(n)\right)_{t_j^k,t_{j+1}^k}$ into Lemma 5.20, we obtain our claim. ∎

We are now ready to establish our main result Theorem 5.45 which relates a difference (5.40) of two rough path integrals against the integrator $A(\gamma_\cdot)_{-1,t}\otimes dL^1(\gamma_\cdot)_{-1,t}$, more precisely, against $\widehat{\mathbf{X}}_\cdot$ (see (5.39)) of two space-filling rough paths $\mathbf{X}_\cdot$ for a unit cube in accordance with Theorem 4.82 to a few Lebesgue integrals of some functions induced by $\widetilde{\alpha}$ over the unit cube. We first introduce a notion of the first hitting time of the space-filling curve $F^{[3]}$ at a point x.

Definition 5.40 *Let $k \in \mathbb{N}$ and $j = 0, \ldots 3^{3k}$. Define a map $\Phi_j^k : I^3 \to I^3$ such that*

$$\Phi_j^k(\cdot) \triangleq F^{[3]}\left(t_j^k\right) + Q_{3^k\left(F^{[3]}\left(t_{j+1}^k\right) - F^{[3]}\left(t_j^k\right)\right)} \circ \delta_{\frac{1}{3^k}}(\cdot)$$

For each $x \in I^3$, we define

$$\nu_k(x) \triangleq \min\left\{t_j^k : x \in \Phi_j^k\left(I^3\right)\right\}.$$

Lemma 5.41 *For each $x \in I^3$, $\nu_k(x)$ is increasing in $k \in \mathbb{N}$. Hence the limit*

$$\lim_{k\to\infty} \nu_k(x)$$

exists; as a consequence,

$$\nu(\cdot) \triangleq \lim_{k\to\infty} \nu_k(\cdot)$$

is measurable on I^3.

Proof. According to Corollary 4.29, we note that for any $r \in \mathbb{N}$, $t_j^k \leq t_i^{k+r} < t_{j+1}^k$,

$$\Phi_i^{k+r}\left(I^3\right) \subset \Phi_j^k\left(I^3\right).$$

Therefore, for any $x \in \Phi_i^{k+r}\left(I^3\right)$,

$$\nu_k(x) \leq \nu_{k+r}(x).$$

The measurability of $\lim_{k\to\infty} \nu_k(\cdot)$ is a consequence of the fact that every $\nu_k(\cdot)$, as a piecewise constant map, is measurable. ■

Lemma 5.42 (The first hitting time of $F^{[3]}$ at x) *For any $x \in I^3$,*

$$\nu(x) = \inf\left\{t : x = F^{[3]}(t)\right\}.$$

Proof. Again, in accordance with Corollary 4.29, we note that

$$F^{[d]}\left(\left[t_j^k, t_{j+1}^k\right]\right)^\circ \subseteq \Phi_j^k\left(I^3\right).$$

Together with Theorem 4.82 for the case $\tau_\cdot = \Upsilon_\cdot$, we conclude our result. ■

Definition 5.43 *Let $p > 3$. Suppose γ is a $p-$geometric rough path on $[-1, 0]$. In accordance with Theorem 4.82, for any finite variation reversible paths $\tau : I \to I^3$ with $\tau_0 = O$ and $\tau_1 = (1,1,1)$, we define the $p-$geometric rough path*

$$S\left(\gamma^{\tau_1}(\infty)\right)_{-1,\cdot} = \lim_{n\to\infty} S\left(\gamma^{\tau_1}(n)\right)_{-1,\cdot}. \tag{5.38}$$

We next treat a path $\gamma_\cdot$ and the corresponding integrator $A(\gamma_\cdot)_{-1,t} \otimes dL^1(\gamma_\cdot)_{-1,t}$ as a unity:

Lemma 5.44 *Let V be a Banach space. Given $3 \leq p < 4$, for every $p-$geometric rough path $\mathbf{X}_\cdot \in G\Omega_p(V)$, consider a path*

$$\widehat{\mathbf{X}}_t \triangleq \left(X_{0,t}, A_{0,t}, \int_0^t A_{0,s} \otimes dX_{0,s}\right) \tag{5.39}$$

in $V \oplus V^{\otimes 2} \oplus V^{\otimes 3}$. Then $\widehat{\mathbf{X}}_\cdot$ can be lifted as a $p-$geometric rough path in $G\Omega_p(V \oplus V^{\otimes 2} \oplus V^{\otimes 3})$, in such a way that if $\mathbf{X}(m)_{s,t}$'s are smooth rough paths converging to $\mathbf{X}_{s,t}$ in $G\Omega_p(V)$ with $p-$variation topology, then $\left\{\widehat{\mathbf{X}}(m)_\cdot\right\}$ also converges to $\widehat{\mathbf{X}}_\cdot$ in $G\Omega_p(V \oplus V^{\otimes 2} \oplus V^{\otimes 3})$ with corresponding $p-$variation topology.

Proof. This is a special case of Proposition 3.31. ■

Theorem 5.45 *Let $3 \leq p < 4$. Assume that $\gamma_\cdot$ can be enhanced as a $p-$geometric rough path on $[-1,0]$. Consider a $Lip(q, I^3)$ $3-$form $\widetilde{\alpha}(\cdot): I^3 \to \widetilde{Alt^3}(\mathbb{R}^d)$, where $q > p$. Let $(0,0,\widetilde{\alpha}(\cdot))$ be the one-form on $T^{(3)}(\mathbb{R}^3)$ such that $\forall v_i \in (\mathbb{R}^3)^{\otimes i}$ for $i = 1,2,3$,*

$$(0,0,\widetilde{\alpha}(\cdot))(v_1 + v_2 + v_3) = \widetilde{\alpha}(\cdot)(v_3)$$

Then we have

$$\begin{aligned}
&\pi_1\left(\int_{S(\widehat{\gamma^{\tau_1}(\infty)})_{-1,\cdot}} (0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3})\right) - \pi_1\left(\int_{S(\widehat{\gamma^{\tau_2}(\infty)})_{-1,\cdot}} (0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3})\right)\\
&= \int_{I^3} \widetilde{\alpha}(x)\left(\int_0^1 A(\tau_1)_{0,t} \otimes dL^1(\tau_1)_{0,t}\right) d\lambda \\
&\quad - \int_{I^3} \widetilde{\alpha}(x)\left(\int_0^1 A(\tau_2)_{0,t} \otimes dL^1(\tau_2)_{0,t}\right) d\lambda \\
&\quad + \sum_{\text{all distinct } a,b,c} \left(I^{\tau_1}(a,b,c) - I^{\tau_2}(a,b,c)\right) \qquad (5.40)\\
&\quad \cdot\left(2\int_{I^3} \widetilde{\alpha}^{(1)}\left(F^{[3]}(\nu(x))\right)(e^a)\left(\left(F^{[3]}(\nu(x)) - \gamma_{-1}\right)\otimes e^b \otimes e^c\right) d\lambda\right.\\
&\quad \left. + \int_{I^3} \widetilde{\alpha}^{(2)}\left(F^{[3]}(\nu(x))\right)\left(e^a \otimes e^b\right)\left(A\left(\gamma^{\Upsilon}(\infty)\right)_{-1,\nu(x)} \otimes e^c\right) d\lambda\right),
\end{aligned}$$

where $\lambda(\cdot)$ *is the usual three-dimensional Lebesgue measure.*

Before proving Theorem 5.45, we shall first find out the expressions of limits of several Riemann sums.

Lemma 5.46 *Using the notation in Lemma* 5.38, *we have*

$$\lim_{k\to\infty}\sum_{t_j^k}\frac{1}{3^{3k}}\widetilde{\alpha}^{(2)}\left(F^{[3]}\left(t_j^k\right)\right)\left(e^a\otimes e^b\right)\left(A\left(\gamma^{\Upsilon}\left(n_\varepsilon(k)\right)\right)_{-1,t_j^k}\otimes e^c\right)$$
$$=\int_{I^3}\widetilde{\alpha}^{(2)}\left(F^{[3]}\left(\nu(x)\right)\right)\left(e^a\otimes e^b\right)\left(A\left(\gamma^{\Upsilon}(\infty)\right)_{-1,\nu(x)}\otimes e^c\right)d\lambda.$$

Proof. Note that by definition of $\nu_k(x)$,

$$\sum_{t_j^k}\frac{1}{3^{3k}}\widetilde{\alpha}^{(2)}\left(F^{[3]}\left(t_j^k\right)\right)\left(e^a\otimes e^b\right)\left(A\left(\gamma^{\Upsilon}\left(n_\varepsilon(k)\right)\right)_{-1,t_j^k}\otimes e^c\right)$$
$$=\int_{I^3}\widetilde{\alpha}^{(2)}\left(F^{[3]}\left(\nu_k(x)\right)\right)\left(e^a\otimes e^b\right)$$
$$\left(A\left(\gamma^{\Upsilon}\left(n_\varepsilon(k)\right)\right)_{-1,\nu_k(x)}\otimes e^c\right)d\lambda$$

According to Theorem 4.82 and Lemma 5.41, we see that

$$\begin{aligned}\lim_{k\to\infty}F^{[3]}\left(\nu_k(x)\right) &= F^{[3]}\left(\nu(x)\right),\\ \lim_{k\to\infty}A\left(\gamma^{\Upsilon}\left(n_\varepsilon(k)\right)\right)_{-1,\nu_k(x)} &= A\left(\gamma^{\Upsilon}(\infty)\right)_{-1,\nu(x)}.\end{aligned}$$

Using Corollary 4.73, we also have

$$\sup_{k,x}\left|F^{[3]}\left(\nu_k(x)\right)\right|+\sup_{k,x}\left|A\left(\gamma^{\Upsilon}\left(n_\varepsilon(k)\right)\right)_{-1,\nu_k(x)}\right|<\infty.$$

Therefore, by applying the usual dominated convergence theorem, we conclude our result. ■

Corollary 5.47

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{t_j^k}\left(\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^{\tau_1}\left(n_\varepsilon\left(k\right)\right)_t\right)\left(A\left(\gamma^{\tau_1}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t_j^k}\otimes dL^1\left(\gamma^{\tau_1}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t}\right)\right.\\
&\left.-\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^{\tau_2}\left(n_\varepsilon\left(k\right)\right)_t\right)\left(A\left(\gamma^{\tau_2}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t_j^k}\otimes dL^1\left(\gamma^{\tau_2}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t}\right)\right)\\
=&\sum_{all\ distinct\ a,b,c}\left(I^{\tau_1}\left(a,b,c\right)-I^{\tau_2}\left(a,b,c\right)\right)\\
&\cdot\int_{I^3}\widetilde{\alpha}^{(2)}\left(F^{[3]}\left(\nu\left(x\right)\right)\right)\left(e^a\otimes e^b\right)\left(A\left(\gamma^{\Upsilon}\left(\infty\right)\right)_{-1,\nu(x)}\otimes e^c\right)d\lambda\\
&+O\left(\varepsilon\right). \qquad (5.41)
\end{aligned}$$

Proof. In spite of Corollary 4.73, we have

$$\sup_n\sup_{t\in[0,1]}\left|A\left(\gamma^{\Upsilon}\left(n\right)\right)_{-1,t}\right|<\infty.$$

Combining the limit result in Lemma 5.46 with Lemma 5.38, we conclude our claim. ■

Lemma 5.48 *Using the notation in Lemma* 5.39, *we have*

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{t_j^k}\frac{1}{3^{3k}}\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(e^a\right)\left(\left(F^{[3]}\left(t_j^k\right)-\gamma_{-1}\right)\otimes e^b\otimes e^c\right)\\
=&\int_{I^3}\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(\nu\left(x\right)\right)\right)\left(e^a\right)\left(\left(F^{[3]}\left(\nu\left(x\right)\right)-\gamma_{-1}\right)\otimes e^b\otimes e^c\right)d\lambda.
\end{aligned}$$

Proof. Again by the definition of $\nu_k\left(x\right)$,

$$\begin{aligned}
&\sum_{t_j^k}\frac{1}{3^{3k}}\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(t_j^k\right)\right)\left(e^a\right)\left(\left(F^{[3]}\left(t_j^k\right)-\gamma_{-1}\right)\otimes e^b\otimes e^c\right)\\
=&\int_{I^3}\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(\nu_k\left(x\right)\right)\right)\left(e^a\right)\left(\left(F^{[3]}\left(\nu_k\left(x\right)\right)-\gamma_{-1}\right)\otimes e^b\otimes e^c\right)d\lambda
\end{aligned}$$

According to Theorem 4.82 and Lemma 5.41, we see that

$$\lim_{k\to\infty}F^{[3]}\left(\nu_k\left(x\right)\right)=F^{[3]}\left(\nu\left(x\right)\right),$$

Using Corollary 4.73, we also have

$$\sup_{k,x}\left|F^{[3]}\left(\nu_k\left(x\right)\right)\right|<\infty.$$

Therefore, applying the usual dominated convergence theorem, we conclude our claim. ■

Corollary 5.49 *Using the notation in Corollary* 5.27,

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{t_j^k}\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^{\tau_1}\left(n_\varepsilon\left(k\right)\right)_t\right)\left(\left[L^1\left(\gamma^{\tau_1}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t_j^k},L^1\left(\gamma^{\tau_1}\left(n_\varepsilon\left(k\right)\right)\right)_{t_j^k,t}\right]\right.\\
&\left.\otimes dL^1\left(\gamma^{\tau_1}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t}\right)\\
&-\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^{\tau_2}\left(n_\varepsilon\left(k\right)\right)_t\right)\left(\left[L^1\left(\gamma^{\tau_2}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t_j^k},L^1\left(\gamma^{\tau_2}\left(n_\varepsilon\left(k\right)\right)\right)_{t_j^k,t}\right]\right.\\
&\left.\otimes dL^1\left(\gamma^{\tau_2}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t}\right)\\
=\;&4\sum_{all\ distinct\ a,b,c}\left(I^{\tau_1}\left(a,b,c\right)-I^{\tau_2}\left(a,b,c\right)\right)\\
&\int_{I^3}\widetilde{\alpha}^{(1)}\left(F^{[3]}\left(\nu\left(x\right)\right)\right)\left(e^a\right)\left(\left(F^{[3]}\left(\nu\left(x\right)\right)-\gamma_{-1}\right)\otimes e^b\otimes e^c\right)d\lambda\\
&+O\left(\varepsilon\right).
\end{aligned}\tag{5.42}$$

Proof. In spite of Corollary 4.73, we have

$$\sup_n\sup_{t\in[0,1]}\left|L^1\left(\gamma^{\Upsilon}\left(n_\varepsilon\left(k\right)\right)\right)_{-1,t}\right|<\infty.$$

Combining the limit result in Lemma 5.48 with Lemma 5.39, we conclude our result. ■

Lemma 5.50

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{t_j^k}\left(\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^{\tau_1}(n)_t\right)\left(A\left(\gamma^{\tau_1}(n)\right)_{t_j^k,t}\otimes dL^1\left(\gamma^{\tau_1}(n)\right)_{-1,t}\right)\right.\\
&\left.-\int_{t_j^k}^{t_{j+1}^k}\widetilde{\alpha}\left(\gamma^{\tau_2}(n)_t\right)\left(A\left(\gamma^{\tau_2}(n)\right)_{t_j^k,t}\otimes dL^1\left(\gamma^{\tau_2}(n)\right)_{-1,t}\right)\right)\\
=&\int_{I^3}\widetilde{\alpha}(x)\left(\int_0^1 A(\tau_1)_{0,t}\otimes dL^1(\tau_1)_{0,t}\right)d\lambda\\
&-\int_{I^3}\widetilde{\alpha}(x)\left(\int_0^1 A(\tau_2)_{0,t}\otimes dL^1(\tau_2)_{0,t}\right)d\lambda. \qquad (5.43)
\end{aligned}$$

Proof. The result is an immediate consequence of the definition of Riemann's integration together with the application of Corollary 5.27. ■

Proof of Theorem 5.45. According to Definition 5.43 and the universal limit theorem in Section 1.4, we have for $i = 1, 2$,

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^1\widetilde{\alpha}\left(\gamma^{\tau_i}(n)_t\right)\left(A\left(\gamma^{\tau_i}(n)\right)_{-1,t}\otimes dL^1\left(\gamma^{\tau_i}(n)\right)_{-1,t}\right)\\
=&\ \pi_1\left(\int_{S(\widehat{\gamma^{\tau_1}(\infty)})_{-1,\cdot}}(0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3})\right).
\end{aligned}$$

Recall Corollary 5.24, by passing limit on both sides of (5.17) and using the expressions in Corollaries 5.47 and 5.49 and Lemma 5.50, we deduce that

$$\begin{aligned}
&\pi_1\left(\int_{S(\widehat{\gamma^{\tau_1}(\infty)})_{-1,\cdot}}(0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3})\right)-\pi_1\left(\int_{S(\widehat{\gamma^{\tau_2}(\infty)})_{-1,\cdot}}(0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3})\right)\\
=&\int_{I^3}\widetilde{\alpha}(x)\left(\int_0^1 A(\tau_1)_{0,t}\otimes dL^1(\tau_1)_{0,t}\right)d\lambda\\
&-\int_{I^3}\widetilde{\alpha}(x)\left(\int_0^1 A(\tau_2)_{0,t}\otimes dL^1(\tau_2)_{0,t}\right)d\lambda\\
&+\sum_{\text{all distinct } a,b,c}\left(I^{\tau_1}(a,b,c)-I^{\tau_2}(a,b,c)\right)\\
&\cdot\left(2\int_{I^3}\widetilde{\alpha}^{(1)}\left(F^{[3]}(\nu(x))\right)(e^a)\left(\left(F^{[3]}(\nu(x))-\gamma_{-1}\right)\otimes e^b\otimes e^c\right)d\lambda\right.
\end{aligned}$$

$$+\int_{I^3} \widetilde{\alpha}^{(2)} \left(F^{[3]}\left(\nu\left(x\right)\right)\right)\left(e^a \otimes e^b\right)\left(A\left(\gamma^{\Upsilon}\left(\infty\right)\right)_{-1,\nu(x)} \otimes e^c\right) d\lambda\Bigg)$$
$$+O\left(\varepsilon\right).$$

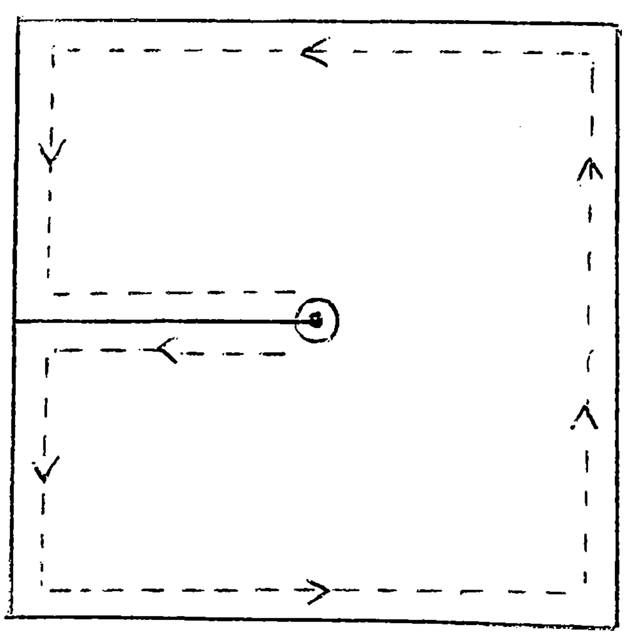
Note that ε is arbitrary, therefore the result is proved by passing ε to zero. ■

5.5 Integral of a 3-form as integral of a spinor

In this section, we shall identity some choices of pairs of space-filling rough paths for a unit cube so that those integrals against $\widetilde{\alpha}^{(1)}$ and $\widetilde{\alpha}^{(2)}$ in (5.40) would vanish while the remaining difference of two Lebesgue integrals would become the ordinary integral of 3−form α. Together with the theory of chainlets by Harrison (1998), we extend our result to nice chainlets at the end of this section and conclude that any 3−form α on a nice chainlet $\mathcal{N}$ can be expressed as a limit of a sequence of integrals against a spinor $(\widetilde{\alpha}, -\widetilde{\alpha})$. We first find a possible pair as shown below:

Definition 5.51 *For $\delta \in \mathbb{R}^+$, define a path $\Theta(\delta)_\cdot : I \longrightarrow \mathbb{R}^3$ such that*

$$\Theta\left(\delta\right)_t = \begin{cases} -\frac{10\delta}{2} t e^1, & for\, t \in \left[0, \frac{1}{10}\right], \\ -\frac{\delta}{2}e^1 - \frac{10\delta}{2}\left(t-\frac{1}{10}\right)e^2, & for\, t \in \left[\frac{1}{10}, \frac{2}{10}\right], \\ \frac{10\delta}{2}\left(t-\frac{3}{10}\right)e^1 - \frac{\delta}{2}e^2, & for\, t \in \left[\frac{2}{10}, \frac{4}{10}\right], \\ \frac{\delta}{2}e^1 + \frac{10\delta}{2}\left(t-\frac{5}{10}\right)e^2, & for\, t \in \left[\frac{4}{10}, \frac{6}{10}\right], \\ -\frac{10\delta}{2}\left(t-\frac{7}{10}\right)e^1 + \frac{\delta}{2}e^2, & for\, t \in \left[\frac{6}{10}, \frac{8}{10}\right], \\ -\frac{\delta}{2}e^1 - \frac{10\delta}{2}\left(t-\frac{9}{10}\right)e^2, & for\, t \in \left[\frac{8}{10}, \frac{9}{10}\right], \\ \frac{10\delta}{2}\left(t-1\right)e^1, & for\, t \in \left[\frac{9}{10}, 1\right]. \end{cases} \tag{5.44}$$



Definition 5.52 *For $r \in \mathbb{R}^+$, define a path $\Gamma(r)_\cdot : I \to \mathbb{R}^3$ such that*

$$\Gamma(r)_t = rte^3, \text{ for } t \in [0,1].$$

We next recall the concept of concatenating paths:

Definition 5.53 (Concatenation of paths) *Let V be a normed space. Consider two finite variation paths $\gamma : I \to V$ and $\tau : I \to V$, we define the concatenation of γ with τ as the path $\gamma * \tau : I \to V$ such that*

$$\gamma * \tau_t = \begin{cases} \gamma(2t), & \text{for } t \in \left[0, \frac{1}{2}\right], \\ \gamma(1) + \tau(2t-1), & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{5.45}$$

Denote, for any $0 \leq \theta < 2\pi$, $R(\theta)$ to be the matrix representing a rotation of θ about the $z-$axis:

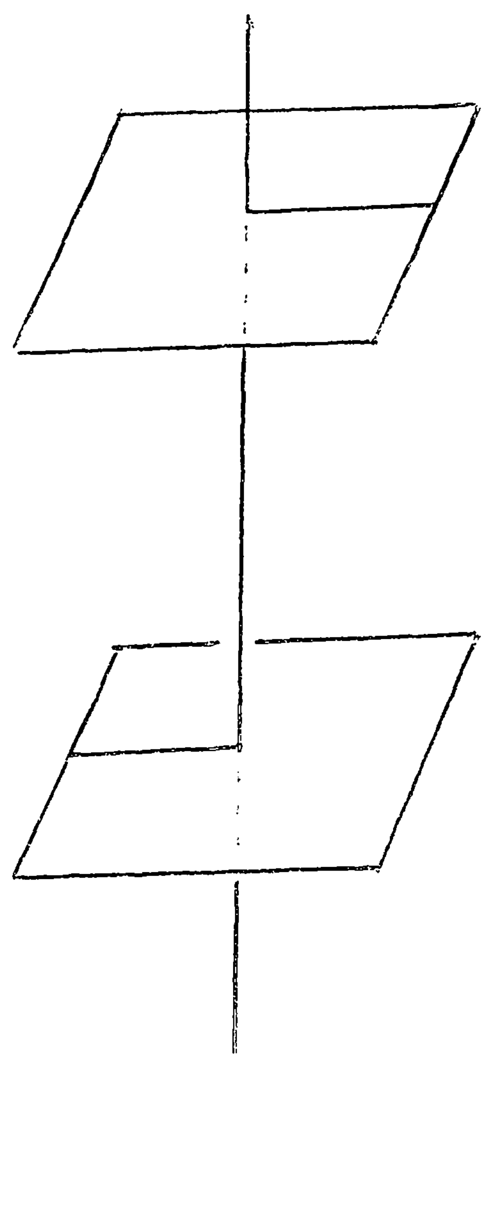
$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 5.54 *Let $0 < r < \sqrt{3}$ and $\delta_1, \ldots, \delta_n \in \mathbb{R}^+$, we define*

$$\Psi(r, (\delta_1, \theta_1), \ldots, (\delta_n, \theta_n)) : I \to \mathbb{R}^3$$

to be the reversible path such that for any $t \in [0, 1/2]$,

$$\begin{aligned} & \Psi(r, (\delta_1, \theta_1), \ldots, (\delta_n, \theta_n))_t \\ \triangleq\ & \Gamma(r) * (R(\theta_1) \circ \Theta(\delta_1)) * \cdots * (R(\theta_n) \circ \Theta(\delta_n)) * \Gamma\left(\sqrt{3} - r\right)_{2t-1}. \end{aligned} \tag{5.46}$$



Definition 5.55 *Let* $0 < r < \frac{\sqrt{3}}{2}$ *and* $\delta_1, \ldots, \delta_n \in \mathbb{R}^+$. *Denote*

$$A = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}.$$

Define a reversible path

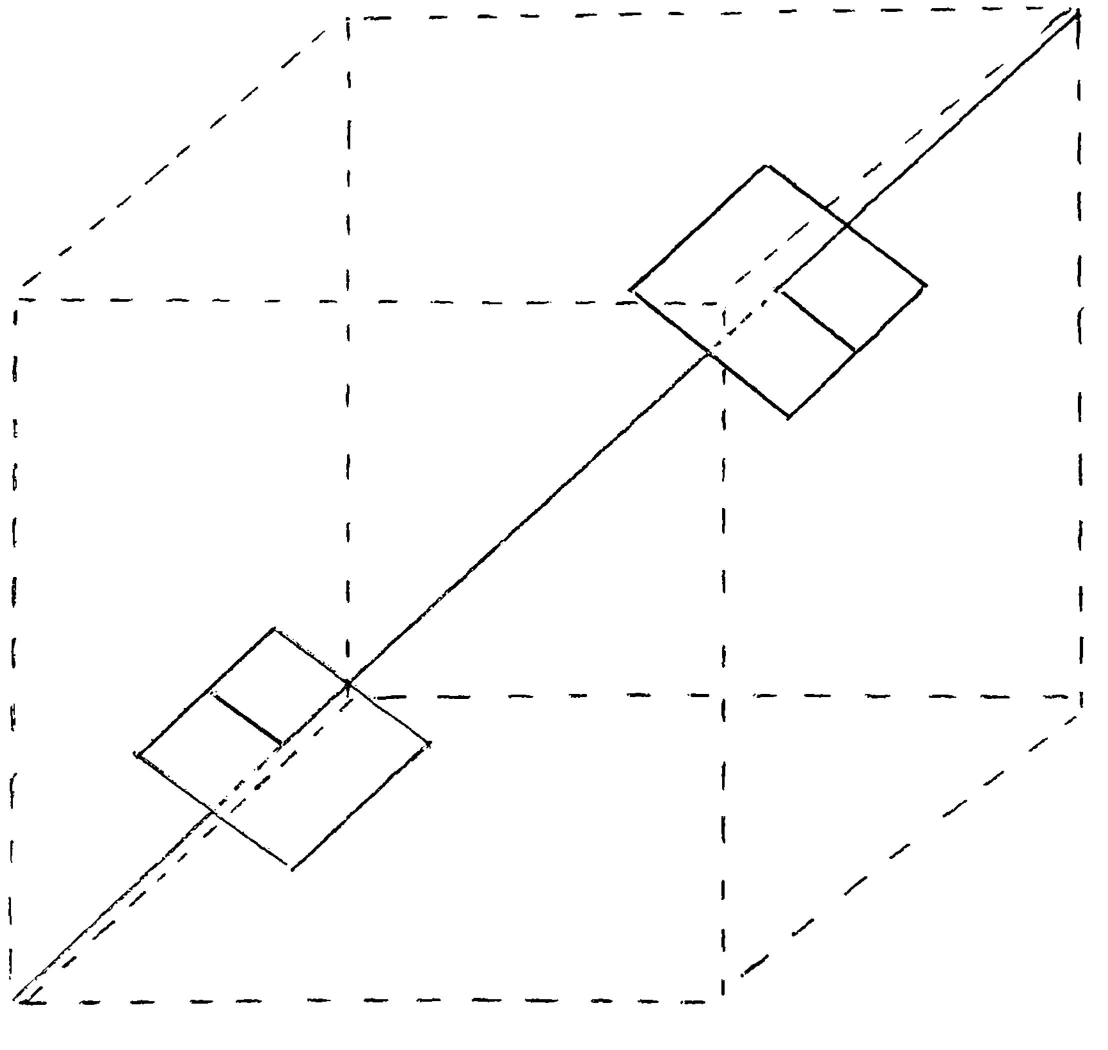
$$\tau\left(r, (\delta_1, \theta_1), \ldots, (\delta_n, \theta_n)\right) : I \to \mathbb{R}^3$$

such that

$$\begin{aligned} & \tau\left(r, (\delta_1, \theta_1), \ldots, (\delta_n, \theta_n)\right). \\ = \; & AB\Psi\left(r, (\delta_1, \theta_1), \ldots, (\delta_n, \theta_n)\right). \end{aligned} \tag{5.47}$$

and

$$\begin{aligned} \tau\left(r, (\delta_1, \theta_1), \ldots, (\delta_n, \theta_n)\right)_0 &= O, \\ \tau\left(r, (\delta_1, \theta_1), \ldots, (\delta_n, \theta_n)\right)_1 &= (1, 1, 1). \end{aligned}$$



For the sake of convenience, in the following, we shall use abbreviations $\Psi(\cdot)_t$ in place of

$$\Psi\left(r,(\delta_1,\theta_1),\ldots,(\delta_n,\theta_n)\right)_t$$

while $\tau(\cdot)_t$ in place of

$$\tau\left(r,(\delta_1,\theta_1),\ldots,(\delta_n,\theta_n)\right)_t$$

without much loss of ambiguity. We next evaluate those Lebesgue integrals involving only $\widetilde{\alpha}$ in (5.40) when the pair of paths is $(\tau(\cdot)_., \Upsilon_.)$:

Lemma 5.56 *Let $0 < r < \frac{\sqrt{3}}{2}$, $\theta_1 = \cdots = \theta_n = 0$ and $\delta_1, \ldots, \delta_n \in \mathbb{R}^+$. Then we have,*

$$\begin{aligned}
\int_0^1 A\left(\Psi(\cdot)_.\right)_{0,t}^{3,1} d\Psi(\cdot)_t^2 &= -\sum_{i=1}^n\left(\frac{\sqrt{3}}{2}-r\right)\delta_i^2,\\
\int_0^1 A\left(\Psi(\cdot)_.\right)_{0,t}^{2,3} d\Psi(\cdot)_t^1 &= \sum_{i=1}^n\left(\frac{\sqrt{3}}{2}-r\right)\delta_i^2,\\
\int_0^1 A\left(\Psi(\cdot)_.\right)_{0,t}^{1,2} d\Psi(\cdot)_t^3 &= 2\sum_{i=1}^n\left(\frac{\sqrt{3}}{2}-r\right)\delta_i^2.
\end{aligned}$$

Proof. Let $d = \left(\sqrt{3}-2r\right)$ and $h_i = \frac{\delta_i}{2}$ for $i = 1, \ldots, n$. By direct calculation, we find that:

$$\begin{aligned}
&\int_0^1 A\left(\Psi(\cdot)_.\right)_{0,t}^{3,1} d\Psi(\cdot)_t^2\\
=\ &\sum_{i=1}^n\left\{\left(-\frac{1}{2}r\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right)+\left(\frac{1}{2}r\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right)+\left(\frac{1}{2}r\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right)\right.\\
&+\left(-\frac{1}{2}r\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right)+\left(\frac{1}{2}(r+d)\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right)\\
&+\left(-\frac{1}{2}(r+d)\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right)+\left(-\frac{1}{2}(r+d)\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right)\\
&\left.+\left(\frac{1}{2}(r+d)\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right)\right\}\\
=\ &-\sum_{i=1}^n 2d\left(\frac{\delta_i}{2}\right)^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_0^1 A\left(\Psi\left(\cdot\right)_{.}\right)_{0,t}^{2,3} d\Psi\left(\cdot\right)_t^1 \\
=\; &\sum_{i=1}^{n}\left\{\left(-\frac{1}{2}r\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right)+\left(-\frac{1}{2}r\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right)+\left(\frac{1}{2}r\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right)\right. \\
&+\left(\frac{1}{2}r\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right)+\left(-\frac{1}{2}\left(r+d\right)\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right) \\
&+\left(-\frac{1}{2}\left(r+d\right)\frac{\delta_i}{2}\right)\left(-\frac{\delta_i}{2}\right)+\left(\frac{1}{2}\left(r+d\right)\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right) \\
&\left.+\left(\frac{1}{2}\left(r+d\right)\frac{\delta_i}{2}\right)\left(\frac{\delta_i}{2}\right)\right\} \\
=\; &\sum_{i=1}^{n} 2d\left(\frac{\delta_i}{2}\right)^2.
\end{aligned}$$

Finally, we also have

$$\begin{aligned}
&\int_0^1 A\left(\Psi\left(\cdot\right)_{.}\right)_{0,t}^{1,2} d\Psi\left(\cdot\right)_t^3 \\
=\; &\sum_{i=1}^{n}\delta_i^2 d.
\end{aligned}$$

■

Corollary 5.57 *Let* $0<r<\frac{\sqrt{3}}{2}$, $0\le\theta_1,\ldots,\theta_n<2\pi$, $\delta_1,\ldots,\delta_n\in\mathbb{R}^+$ *and* α *be an alternating multilinear map,*

$$\begin{aligned}
&\widetilde{\alpha}\left(\int_0^1 A\left(\tau\left(\cdot\right)_{.}\right)_{0,t}\otimes dL^1\left(\tau\left(\cdot\right)_{.}\right)_{0,t}\right) \\
=\; &\widetilde{\alpha}\left(e^1\otimes e^2\otimes e^3\right)\cdot\sum_{i=1i}^{n}\delta_i^2\left(\sqrt{3}-2r\right).
\end{aligned} \tag{5.48}$$

Proof. Note that

$$det\left(AB\right)=1$$

Because α is alternating, we can express

$$\begin{aligned}
&\widetilde{\alpha}\left(\int_0^1 A\left(\tau\left(\cdot\right)_.\right)_{0,t}\otimes dL^1\left(\tau\left(\cdot\right)_.\right)_{0,t}\right)\\
=\ &\det\left(AB\right)\left(\widetilde{\alpha}\left(e^1\otimes e^2\otimes e^3\right)\int_0^1 A\left(\Psi\left(\cdot\right)_.\right)_{0,t}^{1,2} d\Psi\left(\cdot\right)_t^3\right.\\
&+\widetilde{\alpha}\left(e^2\otimes e^3\otimes e^1\right)\cdot\int_0^1 A\left(\Psi\left(\cdot\right)_.\right)_{0,t}^{2,3} d\Psi\left(\cdot\right)_t^1\\
&\left.+\ \widetilde{\alpha}\left(e^3\otimes e^1\otimes e^2\right)\cdot\int_0^1 A\left(\Psi\left(\cdot\right)_.\right)_{0,t}^{3,1} d\Psi\left(\cdot\right)_t^2\right)\\
=\ &\sum_{i=1}^n \widetilde{\alpha}\left(R\left(\theta_i\right)^{\otimes 3}\left(e^1\otimes e^2\otimes e^3\right)\right)\cdot\delta_i^2 d\\
&+\sum_{i=1}^n \widetilde{\alpha}\left(R\left(\theta_i\right)^{\otimes 3}\left(e^2\otimes e^3\otimes e^1\right)\right)\cdot\left(\frac{1}{2}\delta_i^2 d\right)\\
&+\sum_{i=1}^n \widetilde{\alpha}\left(R\left(\theta_i\right)^{\otimes 3}\left(e^3\otimes e^1\otimes e^2\right)\right)\cdot\left(-\frac{1}{2}\delta_i^2 d\right)\\
=\ &\widetilde{\alpha}\left(e^1\otimes e^2\otimes e^3\right)\cdot\sum_{i=1}^n \delta_i^2 d,
\end{aligned}$$

where the last equality holds by applying Lemma 5.56 and by noting that for any $0 \leq \theta < 2\pi$,

$$\det\left(R\left(\theta\right)\right) = 1.$$

■

We now find a pair of paths $\left(\tau\left(\cdot\right)_., \Upsilon_.\right)$ so that the coefficient $I^{\tau(\cdot)_.}\left(a,b,c\right) - I^{\Upsilon_.}\left(a,b,c\right)$ in (5.40) vanishes:

Lemma 5.58 *Let* $0 < r < \frac{\sqrt{3}}{2}$ *and* $\delta_1, \ldots, \delta_{3m} \in \mathbb{R}^+$. *For any* $a, b, c \in \{1,2,3\}$, *recall that, as in* (5.34), *we define*

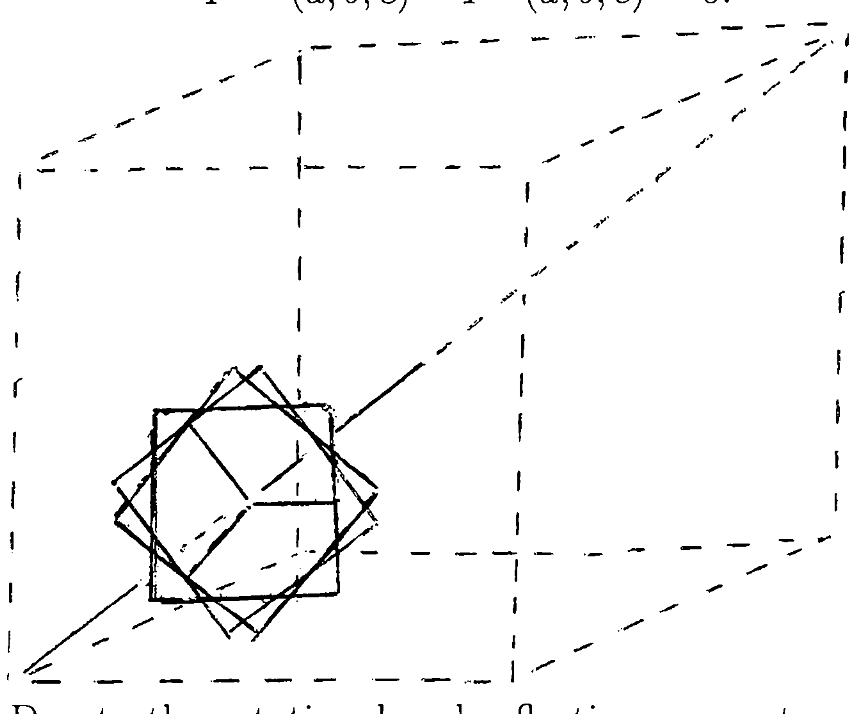
$$I^{\tau}\left(a,b,c\right) \triangleq \int_0^{\frac{1}{2}}\left(\tau_u^a - \tau_{\frac{1}{2}}^a\right)\left(\tau_u^b - \tau_{\frac{1}{2}}^b\right) d\tau_u^c.$$

Suppose that for $k = 1, \ldots, 3m$,

$$\theta_k = \begin{cases} 0, & k-1 \equiv 0 \ (mod\ 3), \\ \frac{2\pi}{3}, & k-1 \equiv 1 \ (mod\ 3), \\ \frac{4\pi}{3}, & k-1 \equiv 2 \ (mod\ 3). \end{cases}$$

Then we have $\forall a, b, c \in \{1, 2, 3\}$,

$$I^{\tau(\cdot)\cdot}(a, b, c) - I^{\Upsilon\cdot}(a, b, c) = 0. \tag{5.49}$$



Proof. Due to the rotational and reflection symmetry of $\tau(\cdot) : I \to I^3$ along its own axis, we only have to check the case for $a = 1$, $b = 2$ and $c = 3$,

$$\begin{aligned} & I^{\tau(\cdot)\cdot}(1,2,3) - I^{\Upsilon\cdot}(1,2,3) \\ = & \sum_{i=1}^{n} c_i \int_0^1 \left(\sqrt{\frac{2}{3}}r - \delta_n + \frac{c_n}{2}(1-2t)\right)\left(\sqrt{\frac{2}{3}}r + \delta_n + \frac{c_n}{2}(1-2t)\right) dt \\ & - c_i \int_0^1 \left(\sqrt{\frac{2}{3}}r + \delta_n + \frac{c_n}{2}(1-2t)\right)\left(\sqrt{\frac{2}{3}}r - \delta_n + \frac{c_n}{2}(1-2t)\right) dt \\ = & \ 0. \end{aligned}$$

for some positive constants c_i. ■

As a corollary, we can now express any 3−form α as a difference of 1−forms $\widetilde{\alpha}$ over a cube:

Theorem 5.59 *Let* $\tau(\cdot) : I \to \mathbb{R}^3$ *be such reversible path that satisfies the condition in Lemma* 5.58 *and has a range* $\tau(\cdot)(I) \subset I^3$ *with*

$$\sum_{i=1}^{n} \delta_i^2 \left(\sqrt{3} - 2r\right) = 1. \tag{5.50}$$

Then we have,

$$\int_{I^3} \alpha = \int_{\widehat{S(\gamma^{\tau(\cdot)}(\infty))}_{-1,\cdot}} (0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3}) - \int_{\widehat{S(\gamma^{\Upsilon}(\infty))}_{-1,\cdot}} (0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3}). \tag{5.51}$$

Proof. Note that $\forall t \in I$,

$$A(\Upsilon)_{0,t} = 0,$$

and hence

$$\int_0^1 A(\Upsilon)_{0,t} \otimes dL^1(\Upsilon)_{0,t} = 0.$$

By applying the expression in Corollary 5.57 and Lemma 5.58 to Theorem 5.45, we deduce that

$$\begin{aligned}
&\pi_1\left(\int_{\widehat{S(\gamma^{\tau(\cdot)}(\infty))}_{-1,\cdot}} (0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3})\right) - \pi_1\left(\int_{\widehat{S(\gamma^{\Upsilon}(\infty))}_{-1,\cdot}} (0,0,\widetilde{\alpha}\circ\pi_{\mathbb{R}^3})\right)\\
&= \int_{I^3} \widetilde{\alpha}(x)\left(e^1\otimes e^2\otimes e^3\right) d\lambda\\
&= \int_{I^3} \alpha(x)\left(e^1, e^2, e^3\right) d\lambda\\
&= \int_{I^3} \alpha.
\end{aligned}$$

■

Note that both the terms in (5.39) and the form $\widetilde{\alpha}$ are independent of the choice of the basis of $\mathbb{R}^3$; therefore, after rescaling, we can extend our claim to arbitrary unit cube in $\mathbb{R}^d$ for any $d \geq 3$: namely, for any unit cube $\mathbf{C} \subset \mathbb{R}^d$, there are two space-filling geometric rough paths τ_1 and τ_2 for $\mathbf{C}$ such that we can express any 3−form ω over the cube $\mathbf{C}$ so that

$$\int_{\mathbf{C}} \omega = \int_{\widehat{S(\tau_1)}_{0,\cdot}} (0,0,\widetilde{\omega}\circ\pi_{\mathbb{R}^d}) - \int_{\widehat{S(\tau_2)}_{0,\cdot}} (0,0,\widetilde{\omega}\circ\pi_{\mathbb{R}^d}). \tag{5.52}$$

Also notice that both $\gamma^{\tau(\cdot)}(\infty)$ and $\gamma^{\Upsilon}(\infty)$ in (5.51) have the same increment and Levy area over $[-1,1]$, i.e.

$$\begin{aligned}
L^1\left(\gamma^{\tau(\cdot)}(\infty)\right)_{-1,1} &= L^1\left(\gamma^{\Upsilon}(\infty)\right)_{-1,1},\\
A\left(\gamma^{\tau(\cdot)}(\infty)\right)_{-1,1} &= A\left(\gamma^{\Upsilon}(\infty)\right)_{-1,1};
\end{aligned}$$

therefore, by concatenating paths, the claim is still valid in general for any chain of finitely many cubes embedded in $\mathbb{R}^d$ for any $d \geq 3$.

A decade ago, Harrison (1998) first introduced the notion of flat norm on the space of simplicial complexes; the completed space is called the space of chainlets in which the classical Stokes' theorem remains valid.

Suppose that there is a sequence $\{\mathcal{N}_n\}_{n=1}^{\infty}$ of chains of finitely many cubes embedded in $\mathbb{R}^d$, for some $d \geq 3$, so that they are convergent to a chainlet $\mathcal{N}$ with respect to the flat norm. Also assume that for each $n \in \mathbb{Z}^+$, there is a pair of space-filling rough paths $\tau(n)_1$ and $\tau(n)_2$ such that

$$\int_C \omega = \pi_1 \left(\int_{\widehat{S(\tau(n)_1)}_{0,\cdot}} (0, 0, \widetilde{\omega} \circ \pi_{\mathbb{R}^d}) \right) - \pi_1 \left(\int_{\widehat{S(\tau(n)_2)}_{0,\cdot}} (0, 0, \widetilde{\omega} \circ \pi_{\mathbb{R}^d}) \right),$$

and both the sequences $\{\tau(n)_1\}_{n=1}^{\infty}$ and $\{\tau(n)_2\}_{n=1}^{\infty}$ are Cauchy in $p-$variation topology; hence there are two space-filling rough paths τ_1 and τ_2 for the chainlet $\mathcal{N}$ which are the limits of $\{\tau(n)_1\}_{n=1}^{\infty}$ and $\{\tau(n)_2\}_{n=1}^{\infty}$ respectively. In light of (5.52), we also have similar breakdown of any $3-$form ω on the chainlet $\mathcal{N}$, i.e.

$$\begin{aligned} \int_{\mathcal{N}} \omega &= \pi_1 \left(\int_{\widehat{S(\tau_1)}_{0,\cdot}} (0, 0, \widetilde{\omega} \circ \pi_{\mathbb{R}^d}) \right) - \pi_1 \left(\int_{\widehat{S(\tau_2)}_{0,\cdot}} (0, 0, \widetilde{\omega} \circ \pi_{\mathbb{R}^d}) \right). \\ &= \lim_{n \to \infty} \int_{\left(\widehat{S(\tau(n)_1)}_{0,\cdot}, \widehat{S(\tau(n)_2)}_{0,\cdot}\right)} (\widetilde{\omega}, -\widetilde{\omega}), \end{aligned} \tag{5.53}$$

where the last expression can be interpreted as a limit of a sequence of integrals of a spinor $(\widetilde{\omega}, -\widetilde{\omega})$ against a pair of paths $\tau(n)_1$ and $\tau(n)_2$.

Chapter 6

Some topological aspects of signatures

In the work of Hambly and Lyons [2006], they proved that finite variation paths can be characterized up to tree-like deformation by their respective signatures. It is clear that a tree-like deformation does not change the topological features of the path; for instance, for almost every point z in the plane, the number of times a finite variation planar path $\gamma_\cdot : I \longrightarrow \mathbb{R}^2$ winds about z remains unchanged if one deforms the path γ in tree-like ways without crossing the point z. In this chapter, we aim to extract low dimensional topological information, e.g. winding and linking numbers, from the signatures of paths under consideration.

In Section 6.1, we shall establish the preliminary result that the winding number $\eta(\gamma_\cdot, z)$ of a finite variation planar loop $\gamma_\cdot$ around a point z is a limit of a sequence of linear combinations of (but involving infinitely many) iterated integrals of $\gamma_\cdot$. We first recall an elementary result in complex analysis, namely: the winding number formula

$$\eta(\gamma_\cdot, z) = \frac{1}{2\pi i}\int_0^1 \frac{d\gamma_t}{\gamma_t - z}.$$

The key idea of the proof of the main result Theorem 6.35 in Section 6.1 is to first consider a family of random variable N_δ, where

$$N_\delta \triangleq \begin{cases} \eta(\gamma_\cdot, Z_\delta), & Z_\delta \in \mathbb{R}^2/\gamma([0,1]) \\ 0, & Z_\delta \in \gamma([0,1]) \end{cases}$$

and Z_δ is a two-dimensional Gaussian random variable on the plane with mean z and variance δ. On one hand, using Prohorov theorem, the expected values $\mathbb{E}(N_\delta)$ would converge to $\eta(\gamma_\cdot, z)$. On the other hand, using change of variables, rapidly decaying nature of any Gaussian kernel and Fubini's theorem, one can remove the singularity of the integrand in the winding number formula in the course of computation of each $\mathbb{E}(N_\delta)$; henceforth, we can express each $\mathbb{E}(N_\delta)$ as a linear combination of iterated integrals of $\gamma_\cdot$. In the appendix of Section 6.1, we shall take an excursion to study on how one can relate the signature of a logarithm of a planar path $\gamma_\cdot$ to the signature of $\gamma_\cdot$. Even though a logarithm of $\gamma_\cdot$ needs not to be unique, there is still exactly one signature (called logarithmic signature in Definition 6.51) that corresponds to any logarithm of $\gamma_\cdot$ (see Corollary 6.50 for details). More precisely, we shall establish the fact (Corollary 6.55) that the logarithmic signature will satisfies a tensor-valued differential equation driven by $\gamma_\cdot$. Moreover, we shall also express the logarithmic signature as a limit of a sequence of linear functions of the signature of $\gamma_\cdot$. The reason why the author interested in investigating in such relations is his belief that the relations can motivate the estimation of the distance between the path $\gamma_\cdot$ and a point $z \notin \gamma([0,1])$ in terms of the signature of $\gamma_\cdot$.

In Section 6.2, we shall show that, under a mild regularity condition on a planar loop $\gamma_\cdot$ (Condition 6.75), one can express the winding number of $\gamma_\cdot$ about a point z as a linear function which only involves finitely many of iterated integrals of $\gamma_\cdot$, in other words, the truncated signature of $\gamma_\cdot$; see Theorem 6.80 for details.

In Section 6.3, we shall also provide a numerical illustration for our Theorem 6.80; more specifically, we shall show that one may need iterated integrals of order not more than 40 to express the winding number of a unit circle around the origin. In terms of computation, it seems less complicated than those knot invariants first suggested by Kontsevich (one can consult the work by Bar-Natan (1995) for a brief summary); this is a reason why we guess that the signature of a loop may lead to a relatively simple and more computable expression for knot invariants, and we shall explore this topic in our future work.

Finally, in Section 6.4, we shall extend the method we used in Section 6.1 in order to give a sketch of arguments which suggests how one can express the linking number of a pair of 3−dimensional loops in terms of the joint iterated integrals of the pair of loops.

6.1 Winding numbers for loops of finite variation

In this section, we shall express the winding number of an arbitrary planar loop $\gamma_\cdot : I \longrightarrow \mathbb{R}^2$ as a functional of the signature of $\gamma_\cdot$. The key idea of the derivation of the expression is to first consider a family of random variable N_δ, where

$$N_\delta \triangleq \begin{cases} \eta(\gamma_\cdot, Z_\delta), & Z_\delta \in \mathbb{R}^2/\gamma([0,1]) \\ 0, & Z_\delta \in \gamma([0,1]) \end{cases}$$

and Z_δ is a two-dimensional Gaussian random variable on the plane with mean z and variance δ. Secondly, by applying Prohorov theorem, the expected values $\mathbb{E}(N_\delta)$ would converge to $\eta(\gamma_\cdot, z)$. Finally, using change of variables, rapidly decaying nature of any Gaussian kernel and Fubini's theorem, one can remove the singularity of the integrand in the winding number formula in the course of computation of each $\mathbb{E}(N_\delta)$; henceforth, we can express each $\mathbb{E}(N_\delta)$ as a linear combination of iterated integrals of $\gamma_\cdot$ while the coefficients are functions of γ_0 and z. In the rest of this chapter, we shall identify $\mathbb{C}$ with $\mathbb{R}^2$.

We first reduce our general problem to a simple one by shifting the path $\gamma_\cdot$:

Definition 6.1 *Let $\tau_\cdot : I \longrightarrow \mathbb{C}$ be a continuous path and $z \in \mathbb{C}$. Define $\tau_\cdot + z$ to be the path translated by z.*

Given an arbitrary loop $\gamma_\cdot : I \longrightarrow \mathbb{C}$ and a point $z \in \mathbb{C}$. We recall the notion of winding number of $\gamma_\cdot$ around z as defined in Section 2.2.

Lemma 6.2 *For every $z \in \mathbb{C}/\gamma(I)$, we have*

$$\eta(\gamma_\cdot, z) = \eta(\gamma_\cdot - z, O). \tag{6.1}$$

Without loss of generality, in the following, we only consider the winding number of $\gamma_\cdot$ around the origin $O \notin \gamma(I)$. Denote by $\lambda_2(\cdot)$ the usual Lebesgue measure on $\mathbb{R}^2$.

Lemma 6.3 *Let $1 \leq p < 2$ and $\gamma_{\cdot} : I \to \mathbb{C}$ be a loop of finite $p-$variation,*

$$\lambda_2 (\gamma (I)) = 0.$$

Consider a family of 2 dimensional Gaussian probability kernels $\{\phi_t\}_{t \in I/\{0\}}$ on $\mathbb{R}^2$ so that, $\forall z \in \mathbb{R}^2$,

$$\phi_t (z) \triangleq \frac{1}{2\pi t} \exp \left(-\frac{|z|^2}{2t} \right).$$

Definition 6.4 *For each $t \in I/\{0\}$, define Z_t to be the two-dimensional Gaussian random variable with probability density function ϕ_t.*

We now establish our first critical lemma:

Lemma 6.5 *Let $\gamma_{\cdot} : I \to \mathbb{C}$ be a loop of finite variation such that $O \notin \gamma (I)$. Then we have*

$$\eta (\gamma_{\cdot}, O) = \lim_{t \downarrow 0} \mathbb{E} (\eta (\gamma_{\cdot}, Z_t)). \tag{6.2}$$

Before showing our lemma, we first recall a basic result in measure theory:

Definition 6.6 *Let $(\Omega, \mathcal{B})$ be a measurable space. For any $F \in \mathcal{B}$, define $\partial F \triangleq \overline{F}/F^o$.*

Proposition 6.7 (Prohorov) *Let μ and $\{\mu_\alpha\}_{\alpha \in J}$ be probability measures on $(\Omega, \mathcal{B})$. Then we have the net $\{\mu_\alpha\}_{\alpha \in J}$ weakly converges to μ. over $(\Omega, \mathcal{B})$ if, and only if, $\forall F \in \mathcal{B}$ with $\mu (\partial F) = 0$,*

$$\lim_{\alpha} \mu_\alpha (F) = \mu (F). \tag{6.3}$$

Proof. See Kallenberg [2002] for details. ■

Let $z \in \mathbb{C}$ and $r > 0$. Denote the open ball with centre z and radius r by $B_z (r)$. Secondly, we need one more regularity lemma on ϕ_t in order to provide a proof for Lemma 6.5:

Lemma 6.8 *For any $\delta > 0$, we have*

$$\sup_{z \in \mathbb{C},\, t \in I/\{0\}} \left| \phi_t (z) \cdot 1_{B_O(\delta)^c} (z) \right| < \infty. \tag{6.4}$$

Proof of Lemma 6.5. It is clear that $\{\phi_t\}_{t\in I/\{0\}}$ weakly converges to the Dirac measure $\delta_O(\cdot)$. Note that the union of open components of $\gamma(I)^C$ is a Lebesgue measurable set since $\lambda_2(\gamma(I)) = 0$. Using Prohorov's Theorem, we conclude that for any $N \in \mathbb{Z}^+$,

$$(\eta(\gamma_\cdot, 0) \wedge N) \vee -N = \lim_{t\downarrow 0} \mathbb{E}\{(\eta(\gamma_\cdot, Z_t) \wedge N) \vee -N\}.$$

Let D be a bounded region containing the graph $\gamma(I)$. According to Proposition 2.13,

$$\eta(\gamma_\cdot, \cdot) \in \mathbb{L}^2(D, \lambda_2),$$

and so

$$\eta(\gamma_\cdot, \cdot) \in \mathbb{L}^1(D, \lambda_2).$$

Since $\forall z \in D^c$

$$\eta(\gamma_\cdot, z) = 0,$$

therefore we also have

$$\eta(\gamma_\cdot, \cdot) \in \mathbb{L}^1(\mathbb{R}^2, \lambda_2).$$

For every integer $N > |\eta(\gamma_\cdot, O)|$, using Lemma 6.8,

$$\begin{aligned}
& \sup_{t\in(0,1]} \left|\mathbb{E}_t\left((\eta(\gamma_\cdot, \cdot) - N)^+ - (-\eta(\gamma_\cdot, \cdot) - N)^+\right)\right| \\
\leq\ & \sup_{t\in(0,1]} \mathbb{E}_t\left(|\eta(\gamma_\cdot, \cdot)|\, 1_{\{|\eta(\gamma_\cdot,\cdot)|>N\}}\right) \\
=\ & \sup_{t\in(0,1]} \int\int_{\mathbb{C}} |\eta(\gamma_\cdot, z)|\, 1_{\{|\eta(\gamma_\cdot,\cdot)|>N\}}(z) \cdot 1_{B_O(\delta)^c}(z) \cdot \phi_t(z)\, \lambda_2(dz) \\
\leq\ & C \cdot \int\int_{\mathbb{C}} |\eta(\gamma_\cdot, z)|\, 1_{\{|\eta(\gamma_\cdot,\cdot)|>N\}}(z)\, \lambda_2(dz)
\end{aligned}$$

which converges to 0 as N tends to infinity since $\eta(\gamma_\cdot, \cdot) \in \mathbb{L}^1(\mathbb{R}^2, \lambda_2)$. By passing N to infinity, we deduce our desired claim. ■

Before we express each $\mathbb{E}(N_\delta)$ as a linear combination of iterated integrals of $\gamma_\cdot$, by applying the rapidly decaying nature of ϕ_t, we first establish the boundedness of the following integral

$$\oint_\gamma \widetilde{\phi}_t(w)\, |dw|$$

in (6.11) step-by-step:

Lemma 6.9 *For every* $t \in I/\{0\}$*, the function*

$$\widetilde{\phi}_t(\cdot) \triangleq \int_{\mathbb{C}} \frac{\phi_t(z)}{|z - \cdot|} dxdy \tag{6.5}$$

is well-defined, i.e. it is pointwise bounded; indeed, it is also uniformly bounded.

Proof. For every $w \in \mathbb{C}$,

$$\begin{aligned}
\widetilde{\phi}_t(w) &= \int_{\mathbb{C}} \frac{\phi_t(z+w)}{|z|} dxdy \\
&= \int_{\mathbb{C}} \frac{1}{|z|} \frac{1}{2\pi t} \exp\left(-\frac{|z+w|^2}{2t}\right) dxdy \\
&\leq \frac{1}{2\pi t} \int_{B_O(1)} \frac{1}{|z|} dxdy + \int_{B_O(1)^c} \frac{1}{2\pi t} \exp\left(-\frac{|z+w|^2}{2t}\right) dxdy \\
&\leq \frac{1}{2\pi t} \int_0^{2\pi} \int_0^1 \frac{1}{r} r dr d\theta + 1 \\
&\leq \frac{1}{t} + 1.
\end{aligned}$$

■

Corollary 6.10 *For each* $t \in I/\{0\}$*, the Cauchy transform of* ϕ_t*:*

$$\widehat{\phi}_t(\cdot) = \int_{\mathbb{C}} \frac{\phi_t(z)}{z - \cdot} dxdy \tag{6.6}$$

exists and is uniformly bounded on $\mathbb{C}$*.*

Definition 6.11 *Define the class* $U \subset L^1_{loc}(\mathbb{R}^2, \lambda_2)$ *of all functions* u *in* $L^1_{loc}(\mathbb{R}^2, \lambda_2)$ *such that there is a rational function* $R_u(\cdot)$ *so that*

$$\lim_{|z|\to\infty} \sup \left| \frac{u(z)}{R_u(z)} \right| < \infty. \tag{6.7}$$

Definition 6.12 *A function* $\varphi \in C^\infty(\mathbb{R}^2)$ *is called rapidly decreasing on* $\mathbb{R}^2$ *if* $\forall n \in \mathbb{Z}^+$*,*

$$\lim_{|z|\to\infty} \sup |z^n \cdot \varphi(z)| < \infty. \tag{6.8}$$

Lemma 6.13 *Let $\varphi \in C^{\infty}(\mathbb{R}^2)$ be rapidly decreasing and $u \in U$. Then $u * \varphi$ is continuous.*

Proof. Let $R_u(\cdot)$ be a rational function such that (6.7) holds. Note that $R_u(\cdot)$ has, at the most, finitely many isolated singularities s_i. Fix a point $z \in \mathbb{C}$. For every $n \in \mathbb{Z}^+$, let $K_z(n)$ be a compact set such that $B_z(n) \cup (\cup_i B_{s_i}(1)) \subset K_z(n)$. For any $z' \in B_z(1)$,

$$\begin{aligned} |u * \varphi(z) - u * \varphi(z')| &= \left| \int_{\mathbb{C}} u(w) (\varphi(z-w) - \varphi(z'-w)) \, dudv \right| \\ &\leq \|u\|_{1,K_z(n)} \|\varphi(z-\cdot) - \varphi(z'-\cdot)\|_{\infty,K_z(n)} \\ &\quad + 2M \sup_{\zeta \in B_z(1)} \int_{K_z(n)^c} \frac{dudv}{|\zeta - w|^k}, \end{aligned}$$

for some $M > 0$ and an integer $k > 2$. Note that as $n \to \infty$,

$$\lim_{n \to \infty} \sup_{\zeta \in B_z(1)} \int_{K_z(n)^c} \frac{dudv}{|\zeta - w|^k} = 0.$$

For each n, we also have

$$\|u\|_{1,K_z(n)} < \infty,$$

and

$$\lim_{\delta \downarrow 0} \sup_{|\zeta - z| \leq \delta} \|\varphi(z-\cdot) - \varphi(\zeta-\cdot)\|_{\infty,K_z(n)} = 0.$$

Therefore, we conclude our claim. ∎

Lemma 6.14 *Let $\varphi \in C^{\infty}(\mathbb{R}^2)$ be rapidly decreasing and $u \in U$. Also let $\{K(n)\}_{n\in\mathbb{N}}$ be a sequence of compact sets such that $\operatorname{diam} K(n) \geq n$. For any $\alpha > 1$ and $h > 0$, we have*

$$\lim_{n \to \infty} \int_{K(\lfloor \alpha n \rfloor)^c} \sup_{w \in K(n)} \sup_{\zeta \in B_w(h)} |u(z) \varphi(\zeta - z)| \, dxdy = 0. \tag{6.9}$$

Proof. Note that for some $M > 0$ and an integer $k > 2$,

$$
\begin{aligned}
& \int_{K(\lfloor \alpha n \rfloor)^c} \sup_{w \in K(n)} \sup_{\zeta \in B_w(h)} |u(z)\varphi(\zeta - z)|\, dxdy \\
\leq\ & M \int_{K(\lfloor \alpha n \rfloor)^c} \sup_{w \in K(n)} \sup_{\zeta \in B_w(h)} \frac{dxdy}{|\zeta - z|^k} \\
\leq\ & M \int_{K(\lfloor \alpha n \rfloor)^c} \frac{dxdy}{(\alpha - 1)^k |z|^k} \\
=\ & O\left(\frac{1}{((\alpha - 1) n)^{k-2}}\right).
\end{aligned}
$$

■

Proposition 6.15 *Let $v \in \mathbb{C}$ and $u \in U$. Then we have*

$$
\nabla_v (\phi_t * u)(\cdot) = (\nabla_v \phi_t) * u(\cdot). \tag{6.10}
$$

Proof. For any $h \in \mathbb{R}$, $w \in \mathbb{C}$ and a compact subset K of $\mathbb{C}$,

$$
\begin{aligned}
& \frac{\phi_t * u(w + hv) - \phi_t * u(w)}{h} - \nabla_v \phi_t * u(w) \\
=\ & \int_{\mathbb{C}} u(z) \left(\frac{\phi_t(w - z + hv) - \phi_t(w - z)}{h} - \nabla_v \phi_t (w - z) \right) dxdy \\
=\ & \int_K u(z) \left(\nabla_v \phi_t (w - z + \xi_h (w - z)) - \nabla_v \phi_t (w - z) \right) dxdy \\
& + \int_{K^c} u(z) \left(\nabla_v \phi_t (w - z + \xi_h (w - z)) - \nabla_v \phi_t (w - z) \right) dxdy,
\end{aligned}
$$

where $\xi_h(w - z) \in B_O(|h|)$. Note that $\nabla_v \phi_t$ is a rapidly decreasing function. Using Lemma 6.14, for any $\varepsilon > 0$, there is a compact set K_ε such that

$$
\begin{aligned}
& \left| \int_{K_\varepsilon^c} u(z) \left(\nabla_v \phi_t (w - z + \xi_h (w - z)) - \nabla_v \phi_t (w - z) \right) dxdy \right| \\
\leq\ & 2 \int_{K_\varepsilon^c} \sup_{\zeta \in B_w(|h|)} |u(z) \nabla_v \phi_t (\zeta - z)| \\
<\ & \varepsilon.
\end{aligned}
$$

Now, $\nabla_v \phi_t$ is uniformly continuous on K_ε, there is a $\delta > 0$ such that for all $|h| < \delta$,

$$\begin{aligned} & \left| \int_{K_\varepsilon} u(z) \left(\nabla_v \phi_t (w - z + \xi_h (w - z)) - \nabla_v \phi_t (w - z) \right) dxdy \right| \\ \leq\ & \varepsilon \|u\|_{1,K_\varepsilon} \end{aligned}$$

hence we obtain our claim. ■

Corollary 6.16 *For each $t \in I/\{0\}$, both $\widetilde{\phi}_t(\cdot)$ and $\widehat{\phi}_t(\cdot)$ are continuously differentiable.*

Proof. One can substitute $u(\cdot)$ by

$$\frac{1}{|z - \cdot|}$$

and

$$\frac{1}{z - \cdot}$$

respectively in Proposition 6.15, the claim can then be proved. ■

Let $R \geq 1$. Consider a continuous loop $\gamma_\cdot : I \to \mathbb{C}$ of finite variation such that

$$\gamma(I) \in B_O(R).$$

Lemma 6.17 *For each $t \in I/\{0\}$,*

$$\oint_\gamma \widetilde{\phi}_t(w) |dw| < \infty. \tag{6.11}$$

Proof. This is a consequence of the continuity of $\widetilde{\phi}_t(\cdot)$. ■

Henceforth, we conclude with the boundedness of $\widetilde{\phi}_t(\cdot)$ on the compact $\gamma(I)$. We can now apply the fact that the integral (6.11) is bounded and Fubini's theorem to express each $\mathbb{E}(N_\delta)$ as a linear combination of iterated integrals of $\gamma_\cdot$:

Lemma 6.18 *Let $t \in I/\{0\}$ and $1 < m \in \mathbb{Z}^+$. Then we have*

$$\begin{aligned} \mathbb{E}(\eta(\gamma_\cdot, Z_t)) &= -\frac{1}{2\pi i} \oint_\gamma \left(\int\int_{B_O(mR)} \frac{\phi_t(z+w)}{z} dxdy \right) dw \\ &\quad -\frac{1}{2\pi i} \oint_\gamma \left(\int\int_{B_O(mR)^c} \frac{\phi_t(z+w)}{z} dxdy \right) dw. \end{aligned} \tag{6.12}$$

Proof. According to Lemma 6.17, using Fubini's theorem,

$$\begin{aligned}\mathbb{E}\left(\eta\left(\gamma_{\cdot}, Z_t\right)\right) &= \int\int_{\mathbb{C}} \left(\frac{1}{2\pi i}\oint_{\gamma} \frac{dw}{w-z}\right) \phi_t(z)\, dxdy \\ &= \frac{1}{2\pi i}\oint_{\gamma}\left(\int\int_{\mathbb{C}} \frac{\phi_t(z)}{w-z} dxdy\right) dw \\ &= -\frac{1}{2\pi i}\oint_{\gamma}\left(\int\int_{\mathbb{C}} \frac{\phi_t(z+w)}{z} dxdy\right) dw.\end{aligned}$$

After splitting the integration domain, we obtain our claim. ■

Notice that the second integral in (6.12) tends to zero as t goes to zero: Given a finite variation path $\tau_{\cdot} : I \longrightarrow \mathbb{C}$, we denote the length of $\tau_{\cdot}$,

$$\int_0^1 |d\tau_u|,$$

by $l(\tau)$.

Lemma 6.19 *For $r \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$,*

$$\int\int_{\{|z|>r\}} \phi_t(z)^k\, dxdy = \frac{1}{k \cdot (2\pi t)^{k-1}} \exp\left(-\frac{k}{2t} r^2\right). \tag{6.13}$$

Lemma 6.20 *Let $\gamma_{\cdot} : I \longrightarrow \mathbb{C}$ be a loop of length $l(\gamma_{\cdot})$ and $1 < m \in \mathbb{Z}^+$,*

$$\begin{aligned}&\sup_{t\in(0,1]}\left|\frac{1}{2\pi i}\oint_{\gamma}\left(\int\int_{B_O(mR)^c} \frac{\phi_t(z+w)}{z} dxdy\right) dw\right| \\ &\le \frac{l(\gamma_{\cdot})}{2\pi m R} \exp\left(-\frac{(m-1)^2 R^2}{2}\right).\end{aligned} \tag{6.14}$$

Proof. For $t \in I/\{0\}$ and $w \in \gamma(I)$,

$$\begin{aligned}\left|\int\int_{B_O(mR)^c} \frac{\phi_t(z+w)}{z} dxdy\right| &\le \frac{1}{mR}\int\int_{B_O(mR)^c} \phi_t(z+w)\, dxdy \\ &\le \frac{1}{mR}\int\int_{\{|z|>(m-1)R\}} \phi_t(z)\, dxdy \\ &\le \frac{1}{mR}\exp\left(-\frac{(m-1)^2}{2t} R^2\right).\end{aligned}$$

Therefore, we obtain:

$$\sup_{t\in(0,1]}\left|\int\int_{B_O(mR)^c}\frac{\phi_t(z+w)}{z}dxdy\right|\leq\frac{1}{mR}\exp\left(-\frac{(m-1)^2}{2}R^2\right).$$

Hence,

$$\sup_{t\in(0,1]}\left|\frac{1}{2\pi i}\oint_\gamma\left(\iint_{B(0,mR)^c}\frac{\phi_t(z+w)}{z}dxdy\right)dw\right|\leq\frac{l(\gamma.)}{2\pi mR}\exp\left(-\frac{(m-1)^2}{2}R^2\right).$$

■

Lemma 6.21 *Let $n\in\mathbb{Z}$, if n is expressed as*

$$n=x+r$$

with $|r|<\frac{1}{2}$, then n is equal to the nearest integer to x.

Definition 6.22 *For each $\rho\in\mathbb{R}^+$, define the smallest positive integer $m(\rho)$ such that*

$$\frac{l(\gamma.)}{2\pi m(\rho)\cdot\rho}\exp\left(-\frac{(m(\rho)-1)^2}{2}\rho^2\right)\leq\frac{1}{4}. \tag{6.15}$$

Corollary 6.23 *The winding number of γ about the origin O, $\eta(\gamma.,O)$ is equal to the nearest integer to*

$$\lim_{t\downarrow 0}Re\left(\frac{i}{2\pi}\oint_\gamma\left(\int\int_{B_O(m(R)\cdot R)}\frac{\phi_t(z+w)}{z}dxdy\right)dw\right). \tag{6.16}$$

Proof. As a consequence of Lemmas 6.5, 6.18, 6.20 and 6.21 together with Definition 6.22, by passing t to zero, we obtain our claim. ■

Lemma 6.24 *Let $t\in I/\{0\}$. Suppose $z=x+iy$, $w=u+iv$ and $\gamma_0=u_0+iv_0$,*

$$\begin{aligned}\phi_t(z+w) &= \frac{1}{2\pi t}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{j=0}^{2r}\sum_{l=0}^{2(k-r)}\frac{(-1)^k}{(2t)^k\cdot k!}\binom{k}{r}\binom{2r}{j}\binom{2(k-r)}{l}\\ &\quad\cdot(x+u_0)^{2r-j}(y+v_0)^{2(k-r)-l}(u-u_0)^j(v-v_0)^l.\end{aligned} \tag{6.17}$$

Proof. From a Taylor expansion, we obtain:

$$\begin{aligned}\phi_t(z+w) &= \frac{1}{2\pi t}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2t)^k\cdot k!}\left[(x+u_0+(u-u_0))^2+(y+v_0+(v-v_0))^2\right]^k \\ &= \frac{1}{2\pi t}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2t)^k\cdot k!}\sum_{r=0}^{k}\binom{k}{r}\left(\sum_{j=0}^{2r}\binom{2r}{j}(x+u_0)^{2r-j}(u-u_0)^j\right) \\ &\quad\cdot\left(\sum_{l=0}^{2(k-r)}\binom{2(k-r)}{l}(y+v_0)^{2(k-r)-l}(v-v_0)^l\right).\end{aligned}$$

After simplification, the result follows. ■

Definition 6.25 *For each $t\in I/\{0\}$, define a sequence of functions $\{\phi_t^n(z)\}_{n\in\mathbb{N}}$,*

$$\phi_t^n(z)\triangleq\frac{1}{2\pi t}\sum_{k=0}^{n}\frac{(-1)^k}{(2t)^k\cdot k!}|z|^{2k}.$$

Lemma 6.26 *For each $t\in I/\{0\}$, $\{\phi_t^n(z)\}_{n\in\mathbb{N}}$ converges to ϕ_t uniformly on compacta.*

Corollary 6.27 *The winding number $\eta(\gamma,O)$ is equal to the nearest integer to*

$$\lim_{t\downarrow 0}\lim_{n\to\infty}Re\left(\frac{i}{2\pi}\oint_\gamma\left(\int\int_{B_O(m(R)\cdot R)}\frac{\phi_t^n(z+w)}{z}dxdy\right)(du+idv)\right). \quad (6.18)$$

Proof. Note that both $B_O(m(R)\cdot R)$ and $\gamma(I)$ are compact subsets of $\mathbb{C}$. ■

Remark 6.28 *The last integral can be expanded as*

$$\begin{aligned}&Re\left(\frac{i}{2\pi}\oint_\gamma\left(\int\int_{B_O(m(R)\cdot R)}\frac{x-iy}{x^2+y^2}\phi_t^n(z+w)\,dxdy\right)(du+idv)\right)\\ =\ &\frac{1}{2\pi}\left(\oint_\gamma\int\int_{B_O(m(R)\cdot R)}\frac{y}{x^2+y^2}\phi_t^n(z+w)\,dxdydu\right.\\ &\left.-\oint_\gamma\int\int_{B_O(m(R)\cdot R)}\frac{x}{x^2+y^2}\phi_t^n(z+w)\,dxdydv\right).\end{aligned}$$

We are now ready to express our winding number of $\gamma_{\cdot}$ about the origin O as a linear functional of its signature $S(\gamma_{\cdot})_{0,1}$.

Definition 6.29 *Given a finite Banach space V with a base $\{e_i\}_{i=1}^{\infty}$. We define*

$$\pi^{i_1,i_2,\dots,i_k} : T(V) \to < e_{i_1} \otimes \cdots \otimes e_{i_k} > \subset V^{\otimes k}$$

to be the projection operator on the tensor algebra $T(V)$ such that for any $X \in T(V)$,

$$\pi^{i_1,i_2,\dots,i_k}(X) \triangleq X^{(k),i_1,i_2,\dots,i_k}.$$

We recall again the definition of the signature as an immediate reference.

Definition 6.30 *Let V be a finite dimensional Banach space. Given a rectifiable path $\tau_{\cdot} : I \to \mathbb{R}$, the signature of $\tau_{\cdot}$ over $[s,t]$, denoted by $X(\tau_{\cdot})_{s,t}$, is the unique tensor such that*

1. $\pi_{\mathbb{R}}\left(X(\tau_{\cdot})_{s,t}\right) = 1.$

2. *For $k \in \mathbb{Z}^+$,* $\pi_{V^{\otimes k}}\left(X(\tau_{\cdot})_{s,t}\right) = \int \cdots \int_{s<u_1<\cdots<u_k<t} d\tau_{u_1} \otimes \cdots \otimes d\tau_{u_k}.$

Before giving our main result Theorem 6.35 in this section, we first introduce two particular functions:

Definition 6.31 *Let $\rho \in \mathbb{R}^+$, $u, v \in \mathbb{R}$ and $\delta, \xi \in \mathbb{N}$. Define two functions:*

$$\alpha^{\delta,\xi}(\rho,u,v) \triangleq \int\int_{B_O(\rho)} \frac{x}{x^2+y^2}(x+u)^{\delta}(y+v)^{\xi}\,dxdy \qquad (6.19)$$

$$\beta^{\delta,\xi}(\rho,u,v) \triangleq \int\int_{B_O(\rho)} \frac{y}{x^2+y^2}(x+u)^{\delta}(y+v)^{\xi}\,dxdy. \qquad (6.20)$$

Both families of functions $\alpha^{\delta,\xi}$ and $\beta^{\delta,\xi}$ can be further simplified.

Lemma 6.32 *Let $m, n \in \mathbb{N}$,*

$$\begin{aligned} \mathcal{I}_{m,n} &\triangleq \int_0^{2\pi} \cos^m\theta \sin^n\theta d\theta \\ &= \begin{cases} 0 & \text{if } (m-1)(n-1) = 0 \pmod 2, \\ \frac{(m-1)!!(n-1)!!}{(m+n)!!}(2\pi) & \text{if } (m-1)(n-1) = 1 \pmod 2. \end{cases} \end{aligned} \qquad (6.21)$$

Lemma 6.33 *Let $\rho \in \mathbb{R}^+$, $u, v \in \mathbb{R}$ and $\delta, \xi \in \mathbb{N}$,*

$$\begin{aligned} \alpha^{\delta,\xi}(\rho,u,v) &= 2\pi \sum_{m=0}^{\delta}\sum_{n=0}^{\xi} \frac{m!!\,(n-1)!!\chi_{m(n-1)(mod\ 2)}(1)}{(m+n+1)\cdot(m+n+1)!!} \\ &\quad \cdot \binom{\delta}{m}\binom{\xi}{n}\rho^{m+n+1}u^{\delta-m}v^{\xi-n}, \end{aligned} \tag{6.22}$$

and

$$\begin{aligned} \beta^{\delta,\xi}(\rho,u,v) &= 2\pi \sum_{m=0}^{\delta}\sum_{n=0}^{\xi} \frac{(m-1)!!n!!\chi_{(m-1)n(mod\ 2)}(1)}{(m+n+1)\cdot(m+n+1)!!} \\ &\quad \cdot \binom{\delta}{m}\binom{\xi}{n}\rho^{m+n+1}u^{\delta-m}v^{\xi-n}, \end{aligned} \tag{6.23}$$

where χ is the indicator function and

$$\chi_{(m-1)n\ (mod2)}(1) = \begin{cases} 0 & if\ (m-1)\,n \equiv 0\ (mod\ 2), \\ 1 & if\ (m-1)\,n \equiv 1\ (mod\ 2). \end{cases} \tag{6.24}$$

Proof. Using the binomial theorem,

$$\begin{aligned} \alpha^{\delta,\xi}(\rho,u,v) &\triangleq \int\int_{B_O(\rho)} \frac{x}{x^2+y^2}(x+u)^{\delta}(y+v)^{\xi}\,dxdy \\ &= \sum_{m=0}^{\delta}\sum_{n=0}^{\xi}\binom{\delta}{m}\binom{\xi}{n}u^{\delta-m}v^{\xi-n}\int\int_{B_O(\rho)}\frac{x}{x^2+y^2}x^m y^n\,dxdy \\ &= \sum_{m=0}^{\delta}\sum_{n=0}^{\xi}\binom{\delta}{m}\binom{\xi}{n}u^{\delta-m}v^{\xi-n} \\ &\quad \cdot \int_0^{2\pi}\left(\int_0^{\rho} r^{m+n}dr\right)\cos^{m+1}\theta\sin^n\theta d\theta \\ &= \sum_{m=0}^{\delta}\sum_{n=0}^{\xi}\frac{1}{m+n+1}\binom{\delta}{m}\binom{\xi}{n}\rho^{m+n+1}u^{\delta-m}v^{\xi-n} \\ &\quad \cdot \int_0^{2\pi}\cos^{m+1}\theta\sin^n\theta d\theta. \end{aligned}$$

Similarly, we also have

$$\begin{aligned}\beta^{\delta,\xi}(\rho,u,v) &= \sum_{m=0}^{\delta}\sum_{n=0}^{\xi}\frac{1}{m+n+1}\binom{\delta}{m}\binom{\xi}{n}\rho^{m+n+1}u^{\delta-m}v^{\xi-n}\\ &\quad\cdot\int_0^{2\pi}\cos^m\theta\sin^{n+1}\theta d\theta.\end{aligned}$$

After applying Lemma 6.32, we obtain our claim. ■

Definition 6.34 *For $j,l\in\mathbb{N}$, we define $C(j,l)$ to be the set of all possible permutations of j indistinguishable "1"'s and l indistinguishable "2"'s.*

Finally, we obtain our main result:

Theorem 6.35 *Given a rectifiable loop $\gamma_\cdot : I \to \mathbb{C}$ with length $l(\gamma)$. The winding number $\eta(\gamma_\cdot, O)$ is equal to the nearest integer to*

$$\begin{aligned}&\lim_{t\downarrow 0}\frac{1}{(2\pi)^2 t}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{j=0}^{2r}\sum_{l=0}^{2(k-r)}\frac{(-1)^k j!l!}{(2t)^k k!}\binom{k}{r}\binom{2r}{j}\binom{2(k-r)}{l}\\ &\cdot\Big(\beta^{2r-j,2(k-r)-l}\left(m(\rho_\gamma(u_0,v_0))\cdot\rho_\gamma(u_0,v_0),u_0,v_0\right)\\ &\cdot\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),1}\left(S(\gamma_\cdot)_{0,1}\right)\\ &-\alpha^{2r-j,2(k-r)-l}\left(m(\rho_\gamma(u_0,v_0))\cdot\rho_\gamma(u_0,v_0),u_0,v_0\right)\\ &\cdot\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),2}\left(S(\gamma_\cdot)_{0,1}\right)\Big),\end{aligned}\tag{6.25}$$

where

$$\rho_\gamma(u,v)\triangleq\sqrt{u^2+v^2}+\frac{l(\gamma_\cdot)}{2}.$$

In general, for any $z\triangleq x+iy\in\mathbb{C}$, the winding number $\eta(\gamma_\cdot,z)$ is equal *to the nearest integer to*

$$\lim_{t\downarrow 0}\frac{1}{(2\pi)^2 t}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{j=0}^{2r}\sum_{l=0}^{2(k-r)}\frac{(-1)^k j!l!}{(2t)^k k!}\binom{k}{r}\binom{2r}{j}\binom{2(k-r)}{l}$$
$$\cdot\Big(\beta^{2r-j,2(k-r)-l}\left(m\left(\rho_\gamma\left(u_0-x,v_0-y\right)\right)\cdot\rho_\gamma\left(u_0-x,v_0-y\right),u_0-x,v_0-y\right)$$
$$\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),1}\left(S\left(\gamma_\cdot\right)_{0,1}\right)$$
$$-\alpha^{2r-j,2(k-r)-l}\left(m\left(\rho_\gamma\left(u_0-x,v_0-y\right)\right)\cdot\rho_\gamma\left(u_0-x,v_0-y\right),u_0-x,v_0-y\right)$$
$$\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),2}\left(S\left(\gamma_\cdot\right)_{0,1}\right)\Big). \tag{6.26}$$

Remark 6.36 *For each $t\in(0,1]$, the series is absolutely convergent since the series in Definition 6.25 is absolutely convergent.*

Proof of Theorem 6.35. For $t\in I$, we let $\gamma_t=(u_t,v_t)$. For $j,l\in\mathbb{N}$,

$$\int_0^1\left(u_t-u_0\right)^j\left(v_t-v_0\right)^l du_t$$
$$= j!l!\int_{0<s<1}\left(\int\cdots\int_{0<s_1<\cdots<s_j<s}\pi^{\overbrace{1,\ldots,1}^{j}}\left(d\gamma_{s_1}\otimes\cdots\otimes d\gamma_{s_j}\right)\right)$$
$$\cdot\left(\int\cdots\int_{0<t_1<\cdots<t_l<s}\pi^{\overbrace{2,\ldots,2}^{l}}\left(d\gamma_{t_1}\otimes\cdots\otimes d\gamma_{t_j}\right)\right)du_s$$
$$= j!l!\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),1}\left(\int\cdots\int_{0<t_1<\cdots<t_{j+l+1}<1}d\gamma_{t_1}\otimes\cdots\otimes d\gamma_{t_{j+l+1}}\right).$$

Similarly, we also have

$$\int_0^1\left(u_t-u_0\right)^j\left(v_t-v_0\right)^l dv_t$$
$$= j!l!\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),2}\left(\int\cdots\int_{0<t_1<\cdots<t_{j+l+1}<1}d\gamma_{t_1}\otimes\cdots\otimes d\gamma_{t_{j+l+1}}\right).$$

After substitutions into (6.18) in Corollary 6.27 and choosing

$$\rho_\gamma(u_0, v_0) = \frac{l(\gamma.)}{2} + \sqrt{u_0^2 + v_0^2},$$

we obtain our desired formula (6.25). In accordance with Lemma 6.2, we also obtain (6.26). ■

6.1.1 Appendix: Logarithm of planar paths

In this appendix, we shall take an excursion to study the relation between the signature of a planar path $\gamma.$ with the signature of a logarithm of $\gamma.$. Notice that logarithm of $\gamma.$ needs not be unique but there is exactly one signature (logarithmic signature) that corresponds to any logarithm of $\gamma.$ (see Corollary 6.50 for details). More precisely, we shall establish the fact (Corollary 6.55) that the logarithmic signature will satisfies a tensor-valued differential equation driven by $\gamma.$. Finally, we shall also express the logarithmic signature as a limit of a sequence of linear functions of the signature of $\gamma.$.

Let $V = \mathbb{R}^2$. We first define a permutation operator $\hat{}$ on the tensor algebra $T(V)$ in Definition 6.41.

Definition 6.37 *For each $k \in \mathbb{N}$, let $|\cdot|_k$ be a norm on $V^{\otimes k}$. The family $\{|\cdot|_k\}_{k=0}^\infty$ of norms are said to be consistent if*

1. *$|\cdot|_0$ is the usual absolute value on reals.*
2. *For $k, l \in \mathbb{N}$, $x_k \in V^{\otimes k}$ and $x_l \in V^{\otimes l}$*

$$|x_k \otimes x_l|_{k+l} = |x_k|_k \cdot |x_l|_l \,. \tag{6.27}$$

Definition 6.38 *Given a consistent family of norms $\{|\cdot|_k\}_{k=0}^\infty$ on $\{V^{\otimes k}\}_{k=0}^\infty$. The family of norms is said to be symmetric if for any $k, r \in \mathbb{Z}^+$ with $k \geq r$, $l_1, \ldots, l_r \in \mathbb{N}$ with $l_1 + \cdots + l_r = k$, any permutation σ of $\{1, 2, \ldots, r\}$, we have for any $v_i \in V^{\otimes l_i}$, $i = 1, \ldots, r$,*

$$|v_1 \otimes \cdots \otimes v_r|_k = \left|v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}\right|_k \,. \tag{6.28}$$

Definition 6.39 *Let $\{|\cdot|_k\}_{k=0}^{\infty}$ be a consistent family of norms, define a norm $||\cdot||$ on $T(V)$ such that for any $x = x_1 + x_2 + \cdots \in T(V)$ with $x_k \in V^{\otimes k}$ for $k = 1, 2, \ldots,$*

$$||x|| \triangleq \sum_{k=0}^{\infty} |x_k|_k .$$

Note that the tensor algebra $T(V)$ is a Banach space when it is equipped with the norm $||\cdot||$.

Lemma 6.40 *Suppose the family of norms $\{|\cdot|_k\}_{k=0}^{\infty}$ is consistent. $(T(V), ||\cdot||)$ is a Banach space.*

We now our permutation operator $\hat{}$ which is also a norm 1 operator on $T(V)$ indeed.

Definition 6.41 *Define a linear operator $\hat{} : T(V) \to T(V)$ such that $\forall k \in \mathbb{Z}^+$, $v_i \in V$, $i = 1, 2, \ldots, k$,*

$$(v_1 \otimes \cdots \otimes v_k)\hat{} \triangleq (-1)^k v_k \otimes \cdots \otimes v_1.$$

Lemma 6.42 *Suppose that the family of norms $\{|\cdot|_k\}_{k=0}^{\infty}$ is consistent and symmetric. $\hat{}$ is a bounded linear operator with a norm*

$$||\hat{}|| = 1. \tag{6.29}$$

Also note that signature of a planar path lies in the Banach space $(T(V), ||\cdot||)$:

Lemma 6.43 *Given a finite variation continuous path $\tau. : I \to V$. For any $(s,t) \in \Delta_1$,*

$$S(\tau.)_{s,t} \in (T(V), ||\cdot||). \tag{6.30}$$

Proof. For $(s,t) \in \Delta_1$,

$$\begin{aligned} \left|\left|S(\tau.)_{s,t}\right|\right| &\leq \sum_{k=0}^{\infty} \int_{s<u_1<\cdots<u_k<t} |d\tau_{u_1}| \ldots |d\tau_{u_k}| \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_s^t |d\tau_u|\right)^k \\ &= \exp\left(\int_s^t |d\tau_u|\right) < \infty. \end{aligned}$$

■

We next relate the image of the signature of a planar path $\gamma.$ under permutation operator $\hat{}$ to the signature of the path which runs $\gamma.$ backwards.

Definition 6.44 *Let $\tau_\cdot : I \to V$ be a continuous path. Define its path running backwards $\overleftarrow{\tau}_\cdot : I \to V$ to be a path so that $\forall t \in I$,*

$$\overleftarrow{\tau}_t \triangleq \tau_{T-t}.$$

Lemma 6.45 *Given a finite variation continuous path $\tau_\cdot : I \to V$, for any $(s,t) \in \Delta_1$,*

$$\left(S\left(\tau_\cdot\right)_{s,t}\right)^{\hat{}} = S\left(\overleftarrow{\tau}_\cdot\right)_{T-t,T-s}.$$

Definition 6.46 *Define a linear map $J : V \to \mathbb{C}$ such that for any $(u,v) \in V$,*

$$J\left((u,v)\right) = u + iv.$$

Note that J is a bijection and hence invertible.

Lemma 6.47 *There is a unique linear map $\widetilde{J} : T(V) \to \mathbb{C}$ such that for any $k \in \mathbb{Z}^+$, $(u_j, v_j) \in V$, $j = 1, \ldots, k$,*

$$\widetilde{J}\left((u_1,v_1) \otimes \cdots \otimes (u_k,v_k)\right) = \Pi_{j=1}^{k}\left(u_j + iv_j\right).$$

Proof. This is a consequence of the universal property of the tensor algebra of V. ∎

Lemma 6.48 *Suppose that the family of norms $\{|\cdot|_k\}_{k=0}^{\infty}$ is consistent. The norm of $\widetilde{J}$*

$$\left\|\widetilde{J}\right\| \leq 1.$$

Given a planar path $\gamma_\cdot : I \to \mathbb{C}$ of finite variation with $O \notin \gamma(I)$. It can be shown that there is (non-unique) a finite variation continuous path $\theta_\cdot : I \to \mathbb{C}$ such that for any $t \in I$,

$$\gamma_t = \exp\left(\theta_t\right).$$

Lemma 6.49 *Suppose $\theta'_\cdot : I \to \mathbb{C}$ is another finite variation continuous path such that for any $t \in I$,*

$$\gamma_t = \exp\left(\theta'_t\right).$$

Then, for any $t \in I$,

$$\theta'_t = \theta_t + 2\pi n i$$

for some $n \in \mathbb{Z}$.

Despite of infinitely many choices of logarithm of a planar path $\gamma_.$, there is exactly one signature which corresponds to every logarithm of $\gamma_.$:

Corollary 6.50 *The signatures of* $J^{-1}(\theta_.)$ *and* $J^{-1}(\theta'_.)$ *are equal, i.e. for all* $(s,t) \in \Delta_T$,

$$S\left(J^{-1}(\theta_.)\right)_{s,t} = S\left(J^{-1}(\theta'_.)\right)_{s,t} \in T(V). \tag{6.31}$$

Definition 6.51 *Let* $(s,t) \in \Delta_1$. *The signature* $S\left(J^{-1}(\theta_.)\right)_{s,t}$ *is called the signature of the logarithm of* $\gamma_.$ *over* $[s,t]$.

$S\left(J^{-1}(\theta_.)\right)_.$ as a solution to a differential equation driven by $\gamma_.$

We next use a special interaction of $\hat{}$ and $\widetilde{J}$ (see (6.33)) to derive the differential equation satisfied by a logarithmic signature.

Lemma 6.52 *For* $(s,t) \in \Delta_1$,

$$\widetilde{J}\left(S\left(J^{-1}(\theta_.)\right)_{s,t}\right) = \gamma_s^{-1} \cdot \gamma_t. \tag{6.32}$$

Proof. For any $(s,t) \in \Delta_1$,

$$\begin{aligned}
\widetilde{J}\left(S\left(J^{-1}(\theta_.)\right)_{s,t}\right) &= 1 + \int_s^t d\theta_u + \cdots + \int_{s<u_1<\cdots<u_k<t} d\theta_{u_1} \ldots d\theta_{u_k} + \ldots \\
&= \exp\left(\int_s^t d\theta_u\right) \\
&= \exp\left(\theta_t - \theta_s\right).
\end{aligned}$$

■

Corollary 6.53 *For* $(s,t) \in \Delta_1$,

$$\widetilde{J}\left(\left(S\left(J^{-1}(\theta_.)\right)_{s,t}\right)\hat{}\right) = \gamma_t^{-1} \cdot \gamma_s. \tag{6.33}$$

Proof.

$$\begin{aligned}
\widetilde{J}\left(\left(S\left(J^{-1}(\theta_.)\right)_{s,t}\right)\hat{}\right) &= \widetilde{J}\left(S\left(J^{-1}\left(\overleftarrow{\theta}_.\right)\right)_{T-t,T-s}\right) \\
&= \sum_{k=0}^{\infty} (-1)^k \int_{s<u_1<\cdots<u_k<t} d\theta_{u_1} \ldots d\theta_{u_k} \\
&= \exp\left(-\left(\theta_t - \theta_s\right)\right).
\end{aligned}$$

■

Corollary 6.54 *For $t \in I$,*

$$d\theta_t = \frac{1}{\gamma_0}\widetilde{J}\left(\left(S\left(J^{-1}\left(\theta_\cdot\right)\right)_{0,t}\right)^\frown\right) d\gamma_t. \tag{6.34}$$

Eventually, we conclude that the logarithmic signature of $\gamma_\cdot$ satisfies the following tensor-valued differential equation (6.35) driven by $\gamma_\cdot$.

Corollary 6.55 *The signature $S\left(J^{-1}\left(\theta_\cdot\right)\right)_{0,\cdot}$ of the logarithm of $\gamma_\cdot$ satisfies the differential equation*

$$\begin{aligned} dU_t &= U_t \otimes J^{-1}\left(\frac{1}{\gamma_0}\widetilde{J}\left(\left(U_t\right)^\frown\right) d\gamma_t\right) \\ U_0 &= 1 \end{aligned} \tag{6.35}$$

for $t \in I$.

Proof. Note that for $t \in I$, by definition,

$$dS\left(J^{-1}\left(\gamma_\cdot\right)\right)_{0,t} = S\left(J^{-1}\left(\gamma_\cdot\right)\right)_{0,t} \otimes J^{-1}\left(d\gamma_t\right).$$

Using Corollary 6.54, we attain our result. ∎

$S\left(J^{-1}\left(\theta_\cdot\right)\right)_\cdot$ as a functional of $S\left(J^{-1}\left(\gamma_\cdot\right)\right)_\cdot$

In the remaining of this subsection, we shall express the logarithmic signature as a limit of a sequence of linear functions of the signature of $\gamma_\cdot$. In particular, we shall attempt to express the signature of $J^{-1}\left(\theta_\cdot\right)$ in terms of that of $J^{-1}\left(\gamma_\cdot\right)$. The idea of derivation is the same as the argument used to establish Theorem 6.35. Again let $\{\phi_t\}_{t\in I/\{0\}}$ be a family of two-dimensional Gaussian probability kernels on $\mathbb{C}$ such that $\forall x \in \mathbb{C}$,

$$\phi_t\left(x\right) \triangleq \frac{1}{2\pi t}\exp\left(-\frac{|x|^2}{2t}\right).$$

For each $t \in I/\{0\}$, define Z_t to be the two-dimensional Gaussian random variable with probability density function ϕ_t.

Lemma 6.56 *Let $w \in \mathbb{C}/\{0\}$. For $t \in I/\{0\}$,*

$$\mathbb{E}\left(\frac{1}{|w+Z_t|}\right) \leq \sqrt{\frac{2\pi}{t}}. \tag{6.36}$$

Proof. For each $t \in I/\{0\}$,

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{|w+Z_t|}\right) &= \int\int_{\mathbb{R}^2} \frac{1}{2\pi t\,|z|} \exp\left(-\frac{|z-w|^2}{2t}\right) dxdy \\
&\leq \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi t \cdot r} \exp\left(-\frac{(r-|w|)^2}{2t}\right) r dr d\theta \\
&= \frac{2\pi}{\sqrt{2\pi t}} \int_0^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r-|w|)^2}{2t}\right) dr \\
&\leq \sqrt{\frac{2\pi}{t}}.
\end{aligned}$$

■

Corollary 6.57 *For each $\delta \in I/\{0\}$,*

$$\mathbb{E}\left(\int_s^t \frac{d\gamma_u}{|\gamma_u + Z_\delta|}\right) \tag{6.37}$$

is well-defined.

Proof. Using Lemma 6.56, since $\gamma_\cdot$ is of finite variation, we have

$$\int_s^t \mathbb{E}\left(\left|\frac{1}{\gamma_u + Z_\delta}\right|\right) |d\gamma_u| < \infty.$$

■

Lemma 6.58 *Let $r \in \mathbb{R}^+$.*

$$\mathbb{P}(|Z_1| > r) \leq \sqrt{\frac{2}{\pi}}\frac{1}{r} \exp\left(-\frac{r^2}{2}\right).$$

Proof. *The claim is immediate by simple calculation.* ■

Lemma 6.59 *Let $w \triangleq r_0 e^{i\theta_0} \in \mathbb{C}/\{0\}$ and $0 \in D \subset \mathbb{C}$ be an open set with compact closure $\overline{D}$. For each $\varepsilon\,(< r_0)$, we define*

$$F_\varepsilon(r_0, \theta_0) \triangleq \left\{re^{i\theta} : r \in (r_0 - \varepsilon, r_0 + \varepsilon), \theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)\right\}. \tag{6.38}$$

There is $\varepsilon_0 > 0$ such that

$$\lim_{t \downarrow 0} \sup_{\zeta \in F_{\varepsilon_0}(r_0,\theta_0)} \mathbb{E}\left(\frac{1}{|\zeta + Z_t|} \cdot 1_{\overline{D}^c}(Z_t)\right) = 0. \tag{6.39}$$

Proof. Let $\varepsilon > 0$ such that $0 \notin F_\varepsilon(r_0, \theta_0) \subset w + \overline{D}$.

$$
\begin{aligned}
& \sup_{\zeta \in F_{\frac{\varepsilon}{3}}(r_0,\theta_0)} \mathbb{E}\left(\frac{1}{|\zeta + Z_t|} \cdot 1_{\overline{D}^c}(Z_t)\right) \\
= & \sup_{\zeta \in F_{\frac{\varepsilon}{3}}(r_0,\theta_0)} \int\int_{(\zeta+\overline{D})^c} \frac{1}{2\pi t\,|z|} \exp\left(-\frac{|z-\zeta|^2}{2t}\right) dxdy \\
\leq & \int\int_{F_{\frac{\varepsilon}{3}}(r_0,\theta_0)^c} \frac{1}{2\pi t \cdot r} \exp\left(-\frac{\left|re^{i\theta} - r_0 e^{i\theta_0}\right|^2}{2\left(\frac{t}{2}\right)}\right) rdrd\theta \\
\leq & \; 2\pi \int_{\left[r_0 - \frac{\varepsilon}{3}, r_0 + \frac{\varepsilon}{3}\right]^c} \frac{1}{2\pi t} \exp\left(-\frac{(r-r_0)^2}{2\left(\frac{t}{2}\right)}\right) dr \\
& + \int_{[\theta_0 - \varepsilon, \theta_0 + \varepsilon]^c} \int_{r_0 - \frac{\varepsilon}{3}}^{r_0 + \frac{\varepsilon}{3}} \frac{1}{2\pi t} \exp\left(-\frac{(r_0-\varepsilon)^2 \varepsilon^2}{2\left(\frac{t}{2}\right)}\right) drd\theta \\
\leq & \sqrt{\frac{\pi}{t}} \mathbb{P}\left(|Z_1| > \frac{\sqrt{2}}{3}\frac{\varepsilon}{\sqrt{t}}\right) + \frac{\varepsilon(6\pi - 2\varepsilon)}{9\pi t} \exp\left(-\frac{(r_0-\varepsilon)^2\varepsilon^2}{t}\right) \qquad (6.40)
\end{aligned}
$$

Using Lemma 6.58, we deduce the convergence to zero. ■

Corollary 6.60 *Using the notation as in Lemma* 6.59, *there is* $\varepsilon_0 > 0$ *such that*

$$
\sup_{t \in (0,1]} \sup_{\zeta \in F_{\varepsilon_0}(r_0,\theta_0)} \mathbb{E}\left(\frac{1}{|\zeta + Z_t|} \cdot 1_{\overline{D}^c}(Z_t)\right) < \infty. \qquad (6.41)
$$

and hence

$$
\begin{aligned}
\sup_{t \in (0,1]} \sup_{u \in I} \mathbb{E}\left(\frac{1}{|\gamma_u + Z_t|} \cdot 1_{\overline{D}^c}(Z_t)\right) &< \infty, \qquad (6.42) \\
\limsup_{t \to 0} \sup_{u \in I} \mathbb{E}\left(\frac{1}{|\gamma_u + Z_t|} \cdot 1_{\overline{D}^c}(Z_t)\right) &= 0. \qquad (6.43)
\end{aligned}
$$

Proof. The result (6.41) is an immediate consequence of (6.40). Since $\gamma(I)$ is compact, there is a finite open cover $\{F_{\varepsilon_i}(r_i, \theta_i)\}_{i=1}^k$ for $\gamma(I)$. Now, we have

$$
\begin{aligned}
& \sup_{t \in I} \sup_{u \in I} \mathbb{E}\left(\frac{1}{|\gamma_u + Z_t|} \cdot 1_{\overline{D}^c}(Z_t)\right) \\
\leq & \max_{i=1,\ldots,k} \sup_{t \in I} \sup_{\zeta \in F_i(r_i,\theta_i)} \mathbb{E}\left(\frac{1}{|\zeta + Z_t|} \cdot 1_{\overline{D}^c}(Z_t)\right) \\
< & \; \infty,
\end{aligned}
$$

and

$$\begin{aligned}0 &\le \limsup_{t\to 0}\sup_{u\in I}\mathbb{E}\left(\frac{1}{|\gamma_u + Z_t|}\cdot 1_{\overline{D}^c}(Z_t)\right)\\ &\le \max_{i=1,\ldots,k}\lim_{t\to 0}\sup_{\zeta\in F_i(r_i,\theta_i)}\mathbb{E}\left(\frac{1}{|\zeta+Z_t|}\cdot 1_{\overline{D}^c}(Z_t)\right)\\ &= 0.\end{aligned}$$

■

For any $A\subset\mathbb{C}$, define

$$\gamma(I)+A\triangleq\{\gamma_t+z : t\in I, z\in A\}.$$

There is an open set $0\in G\subset\mathbb{C}$ with compact closure $\overline{G}$ such that $B_O\left(\frac{1}{2}\min_{t\in I}|\gamma_t|\right)\cap\left(\gamma(I)+\overline{G}\right)=\emptyset$ and ∂G has measure zero.

Corollary 6.61

$$\sup_{\delta\in(0,1]}\sup_{u\in I}\mathbb{E}\left(\left|\frac{1}{\gamma_u+Z_\delta}\right|\right)<\infty.$$

Proof. Note that for any $\delta\in(0,1]$,

$$\sup_{u\in I}\mathbb{E}\left(\frac{1_G}{\gamma_u+Z_\delta}\right)\le\frac{2}{\min_{t\in I}|\gamma_t|}.$$

$$\begin{aligned}&\sup_{u\in I}\mathbb{E}\left(\left|\frac{1}{\gamma_u+Z_\delta}\right|\right)\\ \le\ &\sup_{u\in I}\mathbb{E}\left(\frac{1_G}{\gamma_u+Z_\delta}\right)+\sup_{u\in I}\mathbb{E}\left(\frac{1_{G^c}}{\gamma_u+Z_\delta}\right)\\ \le\ &\frac{2}{\min_{t\in I}|\gamma_t|}+\sup_{\delta\in(0,1]}\sup_{u\in I}\mathbb{E}\left(\frac{1}{|\gamma_u+Z_\delta|}\cdot 1_{G^c}(Z_\delta)\right)\\ <\ &\infty.\end{aligned}$$

■

Lemma 6.62 *Let W be a Banach space and $\{f_\delta(\cdot):I\to Hom(\mathbb{C},W)\}_{\delta\in(0,1]}$ be a family of $Hom(\mathbb{C},W)-$ valued functions such that the limit function*

$$f(\cdot)\triangleq\lim_{\delta\to 0}f_\delta(\cdot)$$

exists and for each compact set K,

$$\sup_{\delta\in(0,1]}\sup_{z\in K}\|f_\delta(z)\|<\infty.$$

For $(s,t)\in\Delta_1$

$$\int_s^t f(u)(d\theta_u)=\lim_{\delta\downarrow 0}\int_s^t f_\delta(u)\left(\mathbb{E}\left(\frac{1}{\gamma_u+Z_\delta}\right)d\gamma_u\right). \tag{6.44}$$

Proof. For $\delta\in I/\{0\}$, in accordance with Corollary 6.57, using Fubini's theorem,

$$\begin{aligned}\mathbb{E}\left(\int_s^t\frac{f_\delta(u)\,d\gamma_u}{\gamma_u+Z_\delta}\right) &= \int_s^t f_\delta(u)\cdot\mathbb{E}\left(\frac{1_G}{\gamma_u+Z_\delta}\right)d\gamma_u\\ &\quad+\int_s^t f_\delta(u)\,\mathbb{E}\left(\frac{1_{G^c}}{\gamma_u+Z_\delta}\right)d\gamma_u\\ &= (I)+(II)\end{aligned}$$

Using Corollary 6.60 and the dominated convergence theorem, we deduce that the second integral (II) converges to zero as δ tends to zero. On the other hand,

$$\mathbb{E}\left(\frac{1_G}{\gamma_u+Z_\delta}\right)\le\frac{2}{\min_{t\in I}|\gamma_t|}.$$

The dominated convergence theorem ensures that

$$\begin{aligned}&\lim_{\delta\downarrow 0}\int_s^t f_\delta(u)\cdot\mathbb{E}\left(\frac{1_G}{\gamma_u+Z_\delta}\right)d\gamma_u\\ &=\int_s^t f_\delta(u)\cdot\lim_{\delta\downarrow 0}\mathbb{E}\left(\frac{1_G}{\gamma_u+Z_\delta}\right)d\gamma_u\\ &=\int_s^t f(u)\frac{d\gamma_u}{\gamma_u},\end{aligned}$$

while the last equality holds because of the weak convergence of $\{\phi_t\}_{t\in I}$ to δ_0. ■

Lemma 6.63 *Let* $w \triangleq u + iv, u_0 + iv_0 \in \mathbb{C}/\{0\}$,

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{w+Z_t}\right) \\
= &\lim_{\rho\to\infty}\frac{1}{2\pi t}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{j=0}^{2r}\sum_{l=0}^{2(k-r)}\frac{(-1)^{k+j+l}}{(2t)^k\cdot k!} \\
&\cdot\binom{k}{r}\binom{2r}{j}\binom{2(k-r)}{l} \\
&\cdot\Big(\alpha^{2r-j,2(k-r)-l}(\rho,-u_0,-v_0)\cdot(u-u_0)^j(v-v_0)^l \\
&-i\beta^{2r-j,2(k-r)-l}(\rho,-u_0,-v_0)\cdot(u-u_0)^j(v-v_0)^l\Big). \qquad (6.45)
\end{aligned}$$

Proof. Recall that for $t \in I/\{0\}$, $z \triangleq x + iy \in \mathbb{C}$,

$$\begin{aligned}
\phi_t(z+w) &= \frac{1}{2\pi t}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{j=0}^{2r}\sum_{l=0}^{2(k-r)}\frac{(-1)^k}{(2t)^k\cdot k!}\binom{k}{r}\binom{2r}{j}\binom{2(k-r)}{l} \\
&\cdot(x+u_0)^{2r-j}(y+v_0)^{2(k-r)-l}(u-u_0)^j(v-v_0)^l.
\end{aligned}$$

Substituting the expression into the following equation:

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{w+Z_t}\right) &= \int\int_{\mathbb{R}^2}\frac{1}{z}\phi_t(z-w)\,dxdy \\
&= \int\int_{\mathbb{R}^2}\frac{x-iy}{x^2+y^2}\phi_t(z-w)\,dxdy.
\end{aligned}$$

■

Definition 6.64 *For* $\delta,\rho \in \mathbb{R}^+$, $n \in \mathbb{N}$, $u,v \in \mathbb{R}$, *we define two linear functionals* $A_{u,v}^{\delta,\rho,n} : T(V) \to \mathbb{R}$ *and* $B_{u,v}^{\delta,\rho,n} : T(V) \to \mathbb{R}$ *such that*

$$\begin{aligned}
A_{u,v}^{\delta,\rho,n}(\cdot) &= \frac{1}{2\pi\delta}\sum_{k=0}^{n}\sum_{r=0}^{k}\sum_{j=0}^{2r}\sum_{l=0}^{2(k-r)}\frac{(-1)^{k+j+l}\cdot j!l!}{(2\delta)^k\cdot k!}\binom{k}{r}\binom{2r}{j}\binom{2(k-r)}{l} \\
&\Bigg(\alpha^{2r-j,2(k-r)-l}(\rho,u,v)\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),1}(\cdot) \\
&+\beta^{2r-j,2(k-r)-l}(\rho,u,v)\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),2}(\cdot)\Bigg), \qquad (6.46)
\end{aligned}$$

$$\begin{aligned} B_{u,v}^{\delta,\rho,n}(\cdot) &= \frac{1}{2\pi\delta}\sum_{k=0}^{n}\sum_{r=0}^{k}\sum_{j=0}^{2r}\sum_{l=0}^{2(k-r)}\frac{(-1)^{k+j+l}\cdot j!l!}{(2\delta)^k\cdot k!}\binom{k}{r}\binom{2r}{j}\binom{2(k-r)}{l} \\ &\quad\cdot\left(\alpha^{2r-j,2(k-r)-l}(\rho,u,v)\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),2}(\cdot)\right. \\ &\quad\left.-\beta^{2r-j,2(k-r)-l}(\rho,u,v)\sum_{\sigma\in C(j,l)}\pi^{\sigma(1),\ldots,\sigma(j+l),1}(\cdot)\right). \end{aligned} \tag{6.47}$$

Corollary 6.65 *For* $(s,t)\in\Delta_1$, $\gamma_0\triangleq u_0+iv_0\in\mathbb{C}$,

$$\int_s^t d\theta_u=\lim_{\delta\downarrow 0}\lim_{\rho\to\infty}\lim_{n\to\infty}\left(A_{-u_0,-v_0}^{\delta,\rho,n}\left(S\left(J^{-1}(\gamma_\cdot)\right)_{s,t}\right)+iB_{-u_0,-v_0}^{\delta,\rho,n}\left(S\left(J^{-1}(\gamma_\cdot)\right)_{s,t}\right)\right).$$

Proof. By substituting (6.45) into (6.44) in Lemma 6.62, we get our result. ■

Definition 6.66 *For* $\delta,\rho\in\mathbb{R}^+$, $n\in\mathbb{N}$, $u,v\in\mathbb{R}$, *we define a linear map* $\Pi_{u,v}^{\delta,\rho,n}:T(V)\to\mathbb{R}^2$ *such that*

$$\Pi_{u,v}^{\delta,\rho,n}(\cdot)=\left(A_{u,v}^{\delta,\rho,n}(\cdot),B_{u,v}^{\delta,\rho,n}(\cdot)\right).$$

For each $k\in\mathbb{N}$, *we also define the linear map* $\left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(k)}\triangleq\Pi_{u,v}^{\delta,\rho,n}\circ\pi_k$.

As an immediate consequence of Definition 6.64, we have

$$\Pi_{u,v}^{\delta,\rho,n}=\sum_{k=0}^{2n}\left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(k)}.$$

Corollary 6.67 *For* $(s,t)\in\Delta_1$, $\gamma_0\triangleq u_0+iv_0\in\mathbb{C}$,

$$\int_s^t dJ^{-1}(\theta_u)=\lim_{\delta\downarrow 0}\lim_{\rho\to\infty}\lim_{n\to\infty}\Pi_{-u_0,-v_0}^{\delta,\rho,n}\left(S\left(J^{-1}(\gamma_\cdot)\right)_{s,t}\right).$$

Definition 6.68 *Define a family of independent random elements*

$$\left\{\mathbf{Z}^j\triangleq\left\{Z_t^j\right\}_{t\in I}\right\}_{j\in\mathbb{N}}$$

such that each Z_t^j *is a Gaussian random variable with probability kernel* ϕ_t.

Lemma 6.69 *For $(s,t) \in \Delta_1$, $k \in \mathbb{N}$,*

$$\begin{aligned} & \int_{s<u_1<\cdots<u_k<t} d\theta_{u_1} \otimes \cdots \otimes d\theta_{u_k} \\ = & \lim_{\delta \downarrow 0} \mathbb{E}\left(\int_{s<u_1<\cdots<u_k<t} \left(\frac{d\gamma_{u_1}}{\gamma_{u_1} + Z_\delta^1} \right) \otimes \cdots \otimes \left(\frac{d\gamma_{u_k}}{\gamma_{u_k} + Z_\delta^k} \right) \right). \end{aligned}$$

Proof. We prove our lemma by induction. According to Lemma 6.62, it holds for the case $k=1$. Suppose the result is valid for $k=n$. Note that for each $i = 1, \ldots, n$,

$$\sup_{\delta \in (0,1]} \sup_{t \in I} \mathbb{E}\left(\left| \frac{1}{\gamma_t + Z_\delta^i} \right| \right) < \infty.$$

By definition,

$$\begin{aligned} & \sup_{\delta \in (0,1]} \sup_{(s,t) \in \Delta_1} \mathbb{E} \left| \int_{s<u_1<\cdots<u_n<t} \left(\frac{d\gamma_{u_1}}{\gamma_{u_1} + Z_\delta^1} \right) \otimes \cdots \otimes \left(\frac{d\gamma_{u_n}}{\gamma_{u_k} + Z_\delta^n} \right) \right| \\ \leq & \int_{0<u_1<\cdots<u_n<1} \left(\sup_{\delta \in (0,1]} \sup_{t \in I} \mathbb{E}\left(\left| \frac{1}{\gamma_t + Z_\delta} \right| \right) \right)^n |d\gamma_{u_1}| \ldots |d\gamma_{u_n}| \\ < & \infty. \end{aligned}$$

Applying the induction hypothesis and Lemma 6.62, we deduce our result. ■

Corollary 6.70 *For $(s,t) \in \Delta_1$, $k \in \mathbb{N}$,*

$$\begin{aligned} & \int_{s<u_1<\cdots<u_k<t} dJ^{-1}(\theta_{u_1}) \otimes \cdots \otimes dJ^{-1}(\theta_{u_k}) \\ = & \lim_{\delta \downarrow 0} \mathbb{E}\left(\int_{s<u_1<\cdots<u_k<t} J^{-1}\left(\frac{d\gamma_{u_1}}{\gamma_{u_1} + Z_\delta^1} \right) \otimes \cdots \otimes J^{-1}\left(\frac{d\gamma_{u_k}}{\gamma_{u_k} + Z_\delta^k} \right) \right). \end{aligned}$$

Recall the notation of tensors of linear maps and shuffle products in Section 1.3.

Proposition 6.71 *For $t \in I$, $k \in \mathbb{N}$,*

$$\int_{s<u_1<\cdots<u_k<t} dJ^{-1}(\theta_{u_1}) \otimes \cdots \otimes dJ^{-1}(\theta_{u_k})$$
$$= \lim_{\delta\downarrow 0} \lim_{\rho\to\infty} \lim_{n\to\infty} \sum_{r_1,\ldots r_k=0}^{2n} \left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(r_1)} \otimes$$
$$\cdots \otimes \left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(r_k)} \left(\sum_{\sigma\in OS(r_1,\ldots,r_k)} \pi^{\sigma}\left(S\left(J^{-1}(\gamma.)\right)_{s,t}\right)\right).$$

Proof. Applying the results in Section 1.3, we have

$$\int_{s<u_1<\cdots<u_k<t} dJ^{-1}(\theta_{u_1}) \otimes \cdots \otimes dJ^{-1}(\theta_{u_k})$$
$$= \lim_{\delta\downarrow 0} \lim_{\rho\to\infty} \lim_{n\to\infty} \int_{s<u_1<\cdots<u_k<t} \Pi_{-u_0,-v_0}^{\delta,\rho,n}\left(dS\left(J^{-1}(\gamma.)\right)_{s,u_1}\right) \otimes$$
$$\cdots \otimes \Pi_{-u_0,-v_0}^{\delta,\rho,n}\left(dS\left(J^{-1}(\gamma.)\right)_{s,u_k}\right)$$
$$= \lim_{\delta\downarrow 0} \lim_{\rho\to\infty} \lim_{n\to\infty} \sum_{r_1,\ldots r_k=0}^{2n} \left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(r_1)} \otimes$$
$$\cdots \otimes \left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(r_k)} \left(\int_{s<u_1<\cdots<u_k<t} dS\left(J^{-1}(\gamma.)\right)_{s,u_1}^{(r_1)} \otimes \cdots \otimes dS\left(J^{-1}(\gamma.)\right)_{s,u_k}^{(r_k)}\right)$$
$$= \lim_{\delta\downarrow 0} \lim_{\rho\to\infty} \lim_{n\to\infty} \sum_{r_1,\ldots r_k=0}^{2n} \left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(r_1)} \otimes$$
$$\cdots \otimes \left(\Pi_{u,v}^{\delta,\rho,n}\right)^{(r_k)} \left(\sum_{\sigma\in OS(r_1,\ldots,r_k)} \pi^{\sigma}\left(S\left(J^{-1}(\gamma.)\right)_{s,t}\right)\right).$$

■

6.2 Winding number in terms of truncated signature

In this section, we shall show our Theorem 6.80 that, under a mild regularity condition on a planar loop $\gamma.$ (Condition 6.75), one can express the winding number of $\gamma.$ about a point z as a linear function which involves only finitely many of iterated integrals of $\gamma.$, i.e. the truncated signature of $\gamma.$. We first recall a result in Hambly and Lyons [2006] which bounds the length of a finite variation path $\gamma.$ by an infinite sum involving the signature of $\gamma.$.

Definition 6.72 *Let V be a finite dimensional Banach space. Given a consistent family of norms $\{|\cdot|_k\}_{k=0}^{\infty}$ on $\{V^{\otimes k}\}_{k=0}^{\infty}$ and a continuous path $\tau. : I \to V$, for each $k \in \mathbb{N}$, we define*

$$b_k(\tau.) = k! \left| \int_{0<u_1<\cdots<u_k<1} d\tau_{u_1} \otimes \cdots \otimes d\tau_{u_k} \right|_k .$$

Proposition 6.73 *Consider a continuous path $\tau. : I \to V$ of finite length $l(\tau.)$ such that its derivative τ' being parameterized at unit speed is continuous. Then*

$$l(\tau.) = 1 + \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \left(e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} b_k(\tau.) \right).$$

Proof. For details, one can consult Theorem 5 in Hambly and Lyons [2006]. ∎

Next, we shall provide a sufficient condition under which one can only use finite number of iterated integrals of a path $\gamma.$ to estimate the length of $\gamma.$:

Lemma 6.74 *Given a continuous path $\tau. : I \to V$. Suppose that the derivative $\tau'_.$ when parameterized at unit speed is continuous with modulus of continuity*

$$\delta_\tau(\cdot) \triangleq \sup_{|s-t|<\cdot} |\tau'_s - \tau'_t|,$$

such that

$$\lim_{\varepsilon\to 0} \delta_\tau(\varepsilon) = 0.$$

Assume that for some $M > 0$, $\forall k \in \mathbb{N}$,

$$b_k(\tau.) \le M^k.$$

Then for $\varepsilon > 0$, there is $\alpha(\varepsilon)$ and $N(\varepsilon)$, such that

$$0 \le l(\tau.) - \frac{1}{\alpha(\varepsilon)} \log \left(\sum_{k=0}^{N(\varepsilon)} \frac{\alpha(\varepsilon)^k}{k!} b_k(\tau.) \right) \le \varepsilon.$$

Proof. In accordance with the work of Hambly and Lyons [2006], there are constants D_1, $D_2 > 0$, which are independent of the length $l(\tau.)$, such that $\forall \alpha \in \mathbb{R}^+$,

$$0 \leq l(\tau.) - \frac{1}{\alpha}\log\left(\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}b_k(\tau.)\right) \leq D_1\delta_\tau\left(\frac{D_2}{\alpha}\right)^2 l(\tau.).$$

Since $\lim_{\varepsilon\to 0}\delta_\tau(\varepsilon) = 0$, there is $\alpha' > 0$ such that $\inf_{\alpha\geq\alpha'}\left(1 - D_1\delta_\tau\left(\frac{D_2}{\alpha}\right)^2\right) > 0$, and hence

$$\begin{aligned} l(\tau.) &\leq \frac{1}{\left(1 - D_1\delta_\tau\left(\frac{D_2}{\alpha'}\right)^2\right)\alpha'}\log\left(\sum_{k=0}^{\infty}\frac{\alpha'^k}{k!}b_k(\tau.)\right) \\ &\leq \frac{M}{1 - D_1\delta_\tau\left(\frac{D_2}{\alpha'}\right)^2}. \end{aligned}$$

Therefore,

$$0 \leq l(\tau.) - \frac{1}{\alpha'}\log\left(\sum_{k=0}^{\infty}\frac{\alpha'^k}{k!}b_k(\tau.)\right) \leq \frac{MD_1\delta_\tau\left(\frac{D_2}{\alpha'}\right)^2}{1 - D_1\delta_\tau\left(\frac{D_2}{\alpha'}\right)^2}.$$

■

In the following, we consider a finite variation continuous planar loop $\gamma. : I \longrightarrow \mathbb{R}^2$ such that $O \notin \gamma(I)$. We can now state the mild condition under which one only needs a finite number of iterated integrals of a loop $\gamma.$ to express the winding number of $\gamma.$ about the origin.

Condition 6.75 *We impose three assumptions on a planar loop:*

1. *For some* r_O,
$$B(O, r_O) \subset \gamma(I)^c.$$

2. *When* $\gamma'_.$ *is parameterized at unit speed, it is continuous with modulus of continuity* $\delta_\gamma(\cdot)$ *such that*
$$\lim_{\varepsilon\to 0}\delta_\gamma(\varepsilon) = 0.$$

3. For some $\alpha(1) \in \mathbb{R}^+$, $N(1) \in \mathbb{Z}^+$,

$$l(\gamma.) \leq 1 + \frac{1}{\alpha(1)} \log \left(\sum_{k=0}^{N(1)} \frac{\alpha(1)^k}{k!} b_k(\gamma.) \right).$$

Lemma 6.76 *Suppose that* $\gamma.$ *satisfies Condition* 6.75. *Then we have*

$$|\eta(\gamma., O)| \leq \frac{1}{2\pi r_O} \left\{ 1 + \frac{1}{\alpha(1)} \log \left(\sum_{k=0}^{N(1)} \frac{\alpha(1)^k}{k!} b_k(\gamma.) \right) \right\}.$$

Proof. Note that for all $\zeta \in B(O, r_O)$, $\eta(\gamma., \zeta) \equiv \eta(\gamma., O)$. According to the Pohl-Banchoff inequality in Proposition 2.13,

$$\begin{aligned} 4\pi (\eta(\gamma., O))^2 (\pi r_O^2) &\leq 4\pi \iint_{\mathbb{C}} \eta^2(\gamma., \zeta) \lambda(dA) \leq l(\gamma.)^2 \\ |\eta(\gamma., O)| &\leq \frac{l(\gamma.)}{2\pi r_O}. \end{aligned}$$

Using Lemma 6.74 with $\varepsilon = 1$, the result follows. ■

Lemma 6.77 *Suppose that* $\gamma.$ *satisfies Condition* 6.75. *For each* $t \in I/\{0\}$, *we have*

$$\begin{aligned} & |\mathbb{E}(\eta(\gamma., Z_t)) - \eta(\gamma., O)| \\ \leq\ & \frac{1}{2\pi} \left\{ \frac{1}{2t^{\frac{1}{2}}} + \frac{1}{r_O} \right\} \exp\left(-\frac{r_O^2}{2t} \right) \left\{ 1 + \frac{1}{\alpha(1)} \log \left(\sum_{k=0}^{N(1)} \frac{\alpha(1)^k}{k!} b_k(\gamma.) \right) \right\}. \end{aligned}$$

Proof.

$$\begin{aligned}
& \left|\mathbb{E}\left(\eta\left(\gamma_{\cdot}, Z_t\right)\right)-\eta\left(\gamma_{\cdot}, O\right)\right| \\
\leq\ & \left|\mathbb{E}\left(\eta\left(\gamma_{\cdot}, Z_t\right) \cdot 1_{\left|Z_t\right|>r_O}\right)\right|+\left|\eta\left(\gamma_{\cdot}, O\right)\right| \cdot \operatorname{Pr}\left(\left|Z_t\right|>r_O\right) \\
\leq\ & \int \int_{\mathbb{C}} \eta\left(\gamma_{\cdot}, z\right) \cdot 1_{\left|Z_t\right|>r_O} \phi_t(z)\, d x d y+\left|\eta\left(\gamma_{\cdot}, O\right)\right| \cdot \operatorname{Pr}\left(\left|Z_t\right|>r_O\right) \\
\leq\ & \sqrt{\int \int_{\mathbb{C}} \eta\left(\gamma_{\cdot}, z\right)^2 d x d y} \cdot \sqrt{\int \int_{\{|z|>r_O\}} \phi_t(z)^2 d x d y} \\
& +\left|\eta\left(\gamma_{\cdot}, O\right)\right| \cdot \operatorname{Pr}\left(\left|Z_t\right|>r_O\right) \\
=\ & \sqrt{\frac{l\left(\gamma_{\cdot}\right)^2}{4 \pi}} \cdot \sqrt{\frac{1}{4 \pi t} \exp \left(-\frac{r_O^2}{t}\right)}+\left|\eta\left(\gamma_{\cdot}, O\right)\right| \cdot \exp \left(-\frac{r_O^2}{2 t}\right) \\
=\ & \left\{\frac{l\left(\gamma_{\cdot}\right)}{4 \pi t^{\frac{1}{2}}}+\left|\eta\left(\gamma_{\cdot}, O\right)\right|\right\} \exp \left(-\frac{r_O^2}{2 t}\right) .
\end{aligned}$$

Using Lemma 6.76, the result follows. ■

Definition 6.78 *Define a positive number $t(\gamma_{\cdot})$ such that*

$$\begin{aligned}
\tfrac{1}{9}\ & \geq \frac{1}{2 \pi}\left\{\frac{1}{2 t\left(\gamma_{\cdot}\right)^{\frac{1}{2}}}+\frac{1}{r_O}\right\} \exp \left(-\frac{r_O^2}{2 t\left(\gamma_{\cdot}\right)}\right) \\
& \cdot\left\{1+\frac{1}{\alpha(1)} \log \left(\sum_{k=0}^{N(1)} \frac{\alpha(1)^k}{k!} b_k\left(\gamma_{\cdot}\right)\right)\right\} .
\end{aligned} \tag{6.48}$$

For $\rho \in \mathbb{R}^{+}$, we also define $n(\rho)$ such that

$$\begin{aligned}
\tfrac{1}{9}\ & \geq \frac{1}{2 \pi n(\rho) \cdot \rho}\left\{1+\frac{1}{\alpha(1)} \log \left(\sum_{k=0}^{N(1)} \frac{\alpha(1)^k}{k!} b_k\left(\gamma_{\cdot}\right)\right)\right\} \\
& \cdot \exp \left(-\frac{(n(\rho)-1)^2}{2 t\left(\gamma_{\cdot}\right)} \rho^2\right) .
\end{aligned} \tag{6.49}$$

Definition 6.79 *For $\rho \in \mathbb{R}^{+}$, $\nu \in \mathbb{N}$, we define*

$$\begin{aligned}
\Phi_\gamma(\rho, \nu)\ & \triangleq\ \frac{((n(\rho)+1) \rho)^{2 \nu+1}}{2 \pi t\left(\gamma_{\cdot}\right) \cdot \nu!}\left\{1+\frac{1}{\alpha_1} \log \left(\sum_{k=0}^{N_1} \frac{\alpha_1^k}{k!} b_k\right)\right\} \\
& \cdot \exp \left(\frac{(n(\rho)+1)^2}{2 t\left(\gamma_{\cdot}\right)} \rho^2\right) .
\end{aligned} \tag{6.50}$$

Theorem 6.80 *Suppose* $\gamma_0 \triangleq u_0 + iv_0 \in \mathbb{C}$. *Under Condition* 6.75, *the winding number* $\eta(\gamma.,O)$ *is equal to the nearest integer to*

$$\frac{1}{(2\pi)^2 t(\gamma.)} \sum_{k=0}^{\nu} \sum_{r=0}^{k} \sum_{j=0}^{2r} \sum_{l=0}^{2(k-r)} \frac{(-1)^k j!l!}{(2t(\gamma.))^k k!} \binom{k}{r} \binom{2r}{j} \binom{2(k-r)}{l}$$
$$\cdot \left(\beta^{2r-j,2(k-r)-l}(n(\rho_0)\cdot\rho_0, u_0, v_0) \sum_{\sigma\in OS(j,l)} \pi^{\sigma(1),\ldots,\sigma(j+l),1}\left(S(\gamma.)_{0,1}\right) \right.$$
$$\left. -\alpha^{2r-j,2(k-r)-l}(n(\rho_0)\cdot\rho_0, u_0, v_0) \sum_{\sigma\in OS(j,l)} \pi^{\sigma(1),\ldots,\sigma(j+l),2}\left(S(\gamma.)_{0,1}\right) \right), \tag{6.51}$$

where

$$\rho_0 \triangleq \sqrt{u_0^2+v_0^2} + \frac{1}{2}\left\{ 1 + \frac{1}{\alpha(1)} \log\left(\sum_{k=0}^{N(1)} \frac{\alpha(1)^k}{k!} b_k(\gamma.) \right) \right\}, \tag{6.52}$$

and $\nu \in \mathbb{N}$ *such that*

$$\Phi_\gamma(\rho_0, \nu) < \frac{1}{9}. \tag{6.53}$$

Proof. Following the same lines of arguments starting from Lemma 6.20 leading to the claim in Theorem 6.35, using Stirling's formula, we deduce that

$$\begin{aligned}
&\left| \eta(\gamma.,O) - Re\left(\frac{i}{2\pi} \oint_\gamma \left(\int\int_{B_O(n(\rho_0)\cdot\rho_0)} \frac{\phi^\nu_{t(\gamma.)}(z+w)}{z} dxdy \right) dw \right) \right| \\
\leq\ & \left| \eta(\gamma.,O) - \mathbb{E}\left(\eta\left(\gamma., Z_{t(\gamma.)}\right)\right) \right| \\
&+ \left| \mathbb{E}\left(\eta\left(\gamma., Z_{t(\gamma.)}\right)\right) - \mathbb{E}\left(\eta\left(\gamma., Z_{t(\gamma.)}\right)\cdot 1_{B_O(n(\rho_0)\cdot\rho_0)}\right) \right| \\
&+ \left| \mathbb{E}\left(\eta\left(\gamma., Z_{t(\gamma.)}\right)\cdot 1_{B_O(n(\rho_0)\cdot\rho_0)}\right) \right. \\
&\left. - Re\left(\frac{i}{2\pi} \oint_\gamma \left(\int\int_{B_O(n(\rho_0)\cdot\rho_0)} \frac{\phi^\nu_{t(\gamma.)}(z+w)}{z} dxdy \right) dw \right) \right| \\
\leq\ & \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}.
\end{aligned}$$

Therefore the claim follows. ■

6.3 A numerical illustration: a circle

In this section, we shall provide a numerical illustration for our Theorem 6.80. In particular, we shall show that it is enough to have iterated integrals of order not more than 40 to express the winding number of a unit circle around the origin. In terms of computation, it seems less complicated than those knot invariants first suggested by Kontsevich (one can consult the work by Bar-Natan (1995) for a brief summary); this is a reason why we guess that the signature of a loop may lead to a relatively simple and more computable expression for knot invariants.

Consider the unit circle, $\gamma : [0,1] \to \mathbb{S}^1$, for $t \in I$,

$$\gamma_t \triangleq \gamma_t^1 + i\gamma_t^2 = \cos 2\pi t + i \sin 2\pi t.$$

Definition 6.81 *For $m, n \in \mathbb{N}$, we define*

$$\begin{aligned} J(m,n,1) &\triangleq \int_0^{2\pi} \left(\gamma_t^1 - 1\right)^m \left(\gamma_t^2\right)^n d\gamma_t^1 \\ &= \int_0^{2\pi} (\cos\theta - 1)^m \sin^n \theta d(\cos\theta), \end{aligned}$$

$$\begin{aligned} J(m,n,2) &\triangleq \int_0^{2\pi} \left(\gamma_t^1 - 1\right)^m \left(\gamma_t^2\right)^n d\gamma_t^2 \\ &= \int_0^{2\pi} (\cos\theta - 1)^m \sin^n \theta d(\sin\theta). \end{aligned}$$

Lemma 6.82 *For $m, n \in \mathbb{Z}^+$,*

$$\left\{ \begin{array}{c} J(m,n,1) = -\frac{n}{m+1} J(m+1, n-1, 2), \\ J(m,n-1,1) + J(m+1,n-1,1) + J(m,n,2) = 0. \end{array} \right. \quad \ldots (*)$$

Lemma 6.83 *For $m \in \mathbb{N}$,*

$$J(m,0,1) = 0.$$

Proof.

$$\begin{aligned} J(m,0,1) &= \int_0^{2\pi} (\cos\theta - 1)^m d(\cos\theta) = \left. \frac{(\cos\theta - 1)^{m+1}}{m+1} \right|_0^{2\pi} \\ &= 0, \end{aligned}$$

■

Lemma 6.84 *For* $m \in \mathbb{N}$,

$$\begin{aligned} K(m) &\triangleq \int_0^{\pi} \sin^m \phi d\phi \\ &= \begin{cases} 0 & \text{if } m = 1 \pmod 2 \\ \frac{(m-1)!!}{m!!}\pi & \text{if } m = 0 \pmod 2 \end{cases} \end{aligned}$$

Lemma 6.85 *For* $m \in \mathbb{N}$,

$$J(m,0,2) = (-2)^{m+1} \frac{(2m-1)!!}{(2m)!!} \left(\frac{m}{m+1}\right) \pi$$

Proof.

$$\begin{aligned} J(m,0,2) &= \int_0^{2\pi} (\cos\theta - 1)^{m+1} d\theta + \int_0^{2\pi} (\cos\theta - 1)^m d\theta \\ &= (-2)^{m+1} \int_0^{2\pi} \sin^{2(m+1)} \frac{\theta}{2} d\theta + (-2)^m \int_0^{2\pi} \sin^{2m} \frac{\theta}{2} d\theta \\ &= -(-2)^{m+2} K(2(m+1)) - (-2)^{m+1} K(2m). \end{aligned}$$

Applying Lemma 6.82, we obtain our formula. ■

Therefore, together with the recursive relations (*), we can determine $J(m,n,i)$ inductively for any $i = 1, 2$ and $m, n \in \mathbb{Z}$.

In our case, $l(\gamma.) = 2\pi$; we choose $\rho_0 = r_O = 1$. Choosing $t_0 = 0.3$, we have

$$\begin{aligned} & \frac{l(\gamma.)}{2\pi} \left(\frac{1}{2t_0^{\frac{1}{2}}} + \frac{1}{r_O} \right) \exp\left(-\frac{r_O^2}{2t_0}\right) l(\gamma.) \\ = & \left(\frac{1}{2t_0^{\frac{1}{2}}} + 1 \right) \exp\left(-\frac{1}{2t_0}\right) \doteq 0.361295. \end{aligned} \tag{6.54}$$

Choosing $n_0 = 2$, we also have

$$\begin{aligned} & \frac{l(\gamma.)}{2\pi n_0 \cdot \rho_0} \exp\left(-\frac{(n_0-1)^2}{2t_0}\rho_0^2\right) \\ = & \frac{1}{n_0} \exp\left(-\frac{(n_0-1)^2}{2t_0}\right) \doteq 0.094438. \end{aligned} \tag{6.55}$$

Finally, we choose $\nu_0 = 40$

$$\begin{aligned} & \frac{((n_0+1)\rho_0)^{2\nu+1}}{2\pi t_0 \cdot \nu_0!} l(\gamma.) \exp\left(\frac{(n_0+1)^2}{2t_0}\rho_0^2\right) \\ = & \frac{(n_0+1)^{2\nu+1}}{t_0 \cdot \nu_0!} \exp\left(\frac{(n_0+1)^2}{2t_0}\right) \doteq 0.005922. \end{aligned} \tag{6.56}$$

Then the sum of (6.54) + (6.55) + (6.56) $\doteq 0.461655 \leq \frac{1}{2}$. Hence, we get:

Proposition 6.86 *The winding number $\eta(\gamma., O) = 1$ is equal to the nearest integer to*

$$\begin{aligned} & \frac{1}{(2\pi)^2 t_0} \sum_{k=0}^{\nu_0} \sum_{r=0}^{k} \sum_{j=0}^{2r} \sum_{l=0}^{2(k-r)} \frac{(-1)^k j! l!}{(2t_0)^k k!} \binom{k}{r} \binom{2r}{j} \binom{2(k-r)}{l} \\ & \cdot \left(\beta^{2r-j, 2(k-r)-l}(n_0, 1, 0) J(j, l, 1) - \alpha^{2r-j, 2(k-r)-l}(n_0, 1, 0) J(j, l, 2)\right). \end{aligned}$$

Proof. By applying our Theorem 6.80 and recombining the shuffle products of signatures of $\gamma.$, we obtain our claim. ■

6.4 Linking numbers of a pair of loops

Finally, in this last section, we shall extend the method we used in Section 6.1 in order to give a sketch of arguments which suggests on how to express the linking number of a pair of 3−dimensional loops γ_1 and γ_2 in terms of the joint iterated integrals of the pair γ_1 and γ_2. We first recall a classical celebrated formula.

Theorem 6.87 (Gauss' formula) *Given two loops of finite variation γ_i, $i = 1, 2$ such that $\gamma_1(I) \cap \gamma_2(I) = \emptyset$. The linking number $N[\gamma_1, \gamma_2]$ between these loops is given by the formula:*

$$N[\gamma_1, \gamma_2] = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\langle d\gamma_1(s) \wedge d\gamma_2(t), \gamma_1(s) - \gamma_2(t) \rangle}{\|\gamma_1(s) - \gamma_2(t)\|^3}. \tag{6.57}$$

Proof. See Madsen and Tornehave [1997] for details. ■

Definition 6.88 *Define a family of* 3*−dimensional Gaussian kernels such that* $\forall\,(x_1,x_2,x_3)\in\mathbb{R}^3$,

$$\psi_t\,(x_1,x_2,x_3)=\frac{1}{(2\pi t)^{\frac{3}{2}}}\exp\left(-\frac{\sum_{i=1}^{3}x_i^2}{2t}\right), \tag{6.58}$$

where $t\in I/\{0\}$*; also for each* $t\in I/\{0\}$*, define* $Z_t\triangleq\left(Z_t^1,Z_t^2,Z_t^3\right)$ *to be the Gaussian random vector with the kernel* ψ_t.

Remark 6.89 $\{\psi_t\}_{t\in I/\{0\}}$ *is a net converging to the Dirac measure* $\delta_O(\cdot)$ *on* $\mathbb{R}^3$.

Denote the closed ball centered at p with radius r by $\mathcal{B}_O(r)$. We first establish a result which is analogous to Lemma 6.9.

Lemma 6.90

$$\sup_{(x^1,x^2,x^3)\in\mathbb{R}^3}\mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}\left(x^i+Z_t^i\right)^2}\right)\le\frac{\pi}{t}. \tag{6.59}$$

Proof. For any $(u^1,u^2,u^3)\in\mathbb{R}^3$, using the spherical polar-coordinates

$$\begin{aligned}u^1&=r\sin\varphi\cos\theta,\\u^2&=r\sin\varphi\sin\theta,\\u^3&=r\cos\varphi,\end{aligned}$$

where $0\le r$, $0\le\varphi\le\pi$ and $0\le\theta\le2\pi$, we can express

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}\left(x^i+Z_t^i\right)^2}\right)\\
=\;&\frac{1}{(2\pi t)^{\frac{3}{2}}}\int\!\!\int\!\!\int\frac{1}{\sum_{i=1}^{3}\left(u^i\right)^2}\exp\left(-\frac{\sum_{i=1}^{3}\left(u^i-x^i\right)^2}{2t}\right)du^1du^2du^3\\
\le\;&\frac{1}{(2\pi t)^{\frac{3}{2}}}\int\!\!\int\!\!\int\frac{1}{\sum_{i=1}^{3}\left(u^i\right)^2}\\
&\cdot\exp\left(-\frac{\left(\sqrt{\sum_{i=1}^{3}\left(-u^i\right)^2}-\sqrt{\sum_{i=1}^{3}\left(x^i\right)^2}\right)^2}{2t}\right)du^1du^2du^3\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_0^{2\pi} \int_0^{\pi} \int_{\mathbb{R}^+} \frac{1}{r^2} \exp\left(-\frac{\left(r - \|(x^1, x^2, x^3)\|\right)^2}{2t}\right) r^2 dr d\varphi d\theta \\ &= \frac{\pi}{t} \int_{\mathbb{R}^+} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(r - \|(x^1, x^2, x^3)\|\right)^2}{2t}\right) dr \\ &\leq \frac{\pi}{t}. \end{aligned} \tag{6.60}$$

■

We next point out a result which is analogous to Lemma 6.8.

Lemma 6.91 *For any $\delta > 0$, we have*

$$\sup_{p\in\mathbb{R}^3,\ t\in I/\{0\}} \left|\psi_t(p) \cdot 1_{\mathcal{B}_O(\delta)^c}(p)\right| < \infty. \tag{6.61}$$

Lemma 6.92 *For any $\delta > 0$,*

$$\sup_{t\in I/\{0\}} \sup_{p\triangleq(x^1,x^2,x^3)\in\mathbb{R}^3} \mathbb{E}\left(\frac{1}{\sum_{i=1}^3 \left(x^i + Z_t^i\right)^2} \cdot 1_{\mathcal{B}_O(\delta)^c}(Z_t)\right) < \infty, \tag{6.62}$$

$$\lim_{t\to 0} \sup_{p\triangleq(x^1,x^2,x^3)\in\mathbb{R}^3} \mathbb{E}\left(\frac{1}{\sum_{i=1}^3 \left(x^i + Z_t^i\right)^2} \cdot 1_{\mathcal{B}_O(\delta)^c}(Z_t)\right) = 0. \tag{6.63}$$

Proof. Using the substitution,

$$\begin{aligned} u^1 &= r\sin\varphi\cos\theta, \\ u^2 &= r\sin\varphi\sin\theta, \\ u^3 &= r\cos\varphi, \end{aligned}$$

we let

$$D_p(\delta) = \left\{(r,\varphi,\theta) : \left(u^1, u^2, u^3\right) \in \mathcal{B}_p(\delta)^c\right\}.$$

In the same way to derive (6.59), we also have, for $p = (x^1, x^2, x^3)$,

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}(x^i + Z_t^i)^2} \cdot 1_{\mathcal{B}_O(\delta)^c}(Z_t)\right) \\
\leq\ & \frac{1}{(2\pi t)^{\frac{3}{2}}} \int\int\int_{\mathcal{B}_p(\delta)^c} \frac{1}{\sum_{i=1}^{3}(u^i)^2} \exp\left(-\frac{\sum_{i=1}^{3}(u^i - x^i)^2}{2t}\right) du^1 du^2 du^3 \\
\leq\ & \frac{1}{(2\pi t)^{\frac{3}{2}}} \exp\left(-\frac{\delta^2}{4t}\right) \int\int\int_{\mathbb{R}^3} \frac{1}{\sum_{i=1}^{3}(u^i)^2} \\
&\cdot \exp\left(-\frac{\sum_{i=1}^{3}(u^i - x^i)^2}{4t}\right) du^1 du^2 du^3
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\sqrt{2}\pi}{t} \exp\left(-\frac{\delta^2}{4t}\right) \int_{\mathbb{R}^+} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(r - ||p||)^2}{4t}\right) dr \\
\leq\ & \frac{\sqrt{2}\pi}{t} \exp\left(-\frac{\delta^2}{4t}\right), \qquad (6.64)
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{p=(x^1,x^2,x^3)\in\mathbb{R}^3} \mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}(x^i + Z_t^i)^2} \cdot 1_{\mathcal{B}_O(\delta)^c}(Z_t)\right) \\
\leq\ & \frac{\sqrt{2}\pi}{t} \exp\left(-\frac{\delta^2}{4t}\right), \qquad (6.65)
\end{aligned}$$

which goes to zero as t tends to zero. ■

Corollary 6.93 *For any $\delta > 0$,*

$$\sup_{t\in I/\{0\}} \sup_{p\triangleq(x^1,x^2,x^3)\in\mathcal{B}_O(\delta)^c} \mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}(x^i + Z_t^i)^2}\right) < \infty. \qquad (6.66)$$

Proof. Note that for any $p = (x^1, x^2, x^3) \in \mathcal{B}_O(\delta)^c$,

$$\frac{1}{\sum_{i=1}^{3}(x^i + Z_t^i)^2} \cdot 1_{\mathcal{B}_O(\frac{\delta}{2})}(Z_t) \leq \frac{4}{\delta^2}.$$

And hence

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}\left(x^{i}+Z_{t}^{i}\right)^{2}}\right)\\
= &\mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}\left(x^{i}+Z_{t}^{i}\right)^{2}}\cdot 1_{\mathcal{B}_O\left(\frac{\delta}{2}\right)}\left(Z_{t}\right)\right)\\
&+\mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}\left(x^{i}+Z_{t}^{i}\right)^{2}}\cdot 1_{\mathcal{B}_O\left(\frac{\delta}{2}\right)^{c}}\left(Z_{t}\right)\right)\\
\leq &\frac{4}{\delta^{2}}+\sup_{t\in I/\{0\}}\sup_{p=\left(x^{1},x^{2},x^{3}\right)\in\mathbb{R}^{3}}\mathbb{E}\left(\frac{1}{\sum_{i=1}^{3}\left(x^{i}+Z_{t}^{i}\right)^{2}}\cdot 1_{\mathcal{B}_O\left(\frac{\delta}{2}\right)^{c}}\left(Z_{t}\right)\right).
\end{aligned}$$

■

We now have our main result which is analogous to Lemma 6.5.

Proposition 6.94 *Given two loops of finite variation γ_i, $i=1,2$ such that $\gamma_1(I)\cap\gamma_2(I)=\emptyset$.*

$$\begin{aligned}
&N\left[\gamma_{1},\gamma_{2}\right]\\
= &\lim_{\delta\to 0}\frac{1}{4\pi}\oint_{\gamma_{1}}\oint_{\gamma_{2}}\left\langle d\gamma_{1}(s)\wedge d\gamma_{2}(t),\mathbb{E}\left(\frac{\gamma_{1}(s)-\gamma_{2}(t)+Z_{\delta}}{\left\|\gamma_{1}(s)-\gamma_{2}(t)+Z_{\delta}\right\|^{3}}\right)\right\rangle.
\end{aligned}$$

Proof. By assumption and the compactness of $\gamma_1(I)$ and $\gamma_2(I)$, there is an $\varepsilon>0$ such that

$$\inf_{s,t\in I}\left|\gamma_{1}(s)-\gamma_{2}(t)\right|>\varepsilon.$$

According to Remark 6.89 and the uniform boundedness:

$$\left\|\frac{\gamma_{1}(s)-\gamma_{2}(t)+Z_{\delta}}{\left\|\gamma_{1}(s)-\gamma_{2}(t)+Z_{\delta}\right\|^{3}}\cdot 1_{\mathcal{B}_O\left(\frac{\varepsilon}{2}\right)}\left(Z_{\delta}\right)\right\|\leq\frac{4}{\varepsilon^{2}},$$

together with the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{\delta\to\infty}\mathbb{E}\left(\frac{\gamma_{1}(s)-\gamma_{2}(t)+Z_{\delta}}{\left\|\gamma_{1}(s)-\gamma_{2}(t)+Z_{\delta}\right\|^{3}}\cdot 1_{\mathcal{B}_O\left(\frac{\varepsilon}{2}\right)}\left(Z_{\delta}\right)\right)\\
= &\frac{\gamma_{1}(s)-\gamma_{2}(t)}{\left\|\gamma_{1}(s)-\gamma_{2}(t)\right\|^{3}}.
\end{aligned}$$

In accordance with Lemma 6.92,

$$\lim_{\delta\to 0}\sup_{s,t\in I}\left\|\mathbb{E}\left(\frac{\gamma_1(s)-\gamma_2(t)+Z_\delta}{\|\gamma_1(s)-\gamma_2(t)+Z_\delta\|^3}\cdot 1_{\mathcal{B}_O\left(\frac{\varepsilon}{2}\right)^c}(Z_\delta)\right)\right\|=0$$

Therefore,

$$\begin{aligned}&\lim_{\delta\to 0}\mathbb{E}\left(\frac{\gamma_1(s)-\gamma_2(t)+Z_\delta}{\|\gamma_1(s)-\gamma_2(t)+Z_\delta\|^3}\right)\\ =\ &\frac{\gamma_1(s)-\gamma_2(t)}{\|\gamma_1(s)-\gamma_2(t)\|^3}.\end{aligned}$$

Note that in accordance with Corollary 6.93, we have

$$\begin{aligned}&\sup_{\delta\in I/\{0\}}\sup_{s,t\in I}\left\|\mathbb{E}\left(\frac{\gamma_1(s)-\gamma_2(t)+Z_\delta}{\|\gamma_1(s)-\gamma_2(t)+Z_\delta\|^3}\right)\right\|\\ \leq\ &\sup_{\delta\in I/\{0\}}\sup_{s,t\in I}\mathbb{E}\left(\frac{1}{\|\gamma_1(s)-\gamma_2(t)+Z_\delta\|^2}\right)\\ <\ &\infty.\end{aligned}$$

Since both γ_1 and γ_2 are of finite variation, we deduce our result by applying the dominated convergence theorem again. ■

We next have a result which is analogous to Lemma 6.19.

Lemma 6.95 *Let $l(\tau)$ be the length of a continuous path of finite variation $\tau: I\to\mathbb{R}^3$. For any $r>2\sup_{s,t\in I}\|\gamma_1(s)-\gamma_2(t)\|$,*

$$\begin{aligned}&\sup_{\delta\in I/\{0\}}\left|\oint_{\gamma_1}\oint_{\gamma_2}\langle d\gamma_1(s)\wedge d\gamma_2(t),\right.\\ &\left.\mathbb{E}\left(\frac{\gamma_1(s)-\gamma_2(t)+Z_\delta}{\|\gamma_1(s)-\gamma_2(t)+Z_\delta\|^3}1_{\mathcal{B}_{-(\gamma_1(s)-\gamma_2(t))}(r)^c}(Z_\delta)\right)\rangle\right|\\ \leq\ &\sqrt{2}\pi(r-1)^{-2}\cdot l(\gamma_1)\cdot l(\gamma_2)\cdot\sup_{u\in[0,\infty)}u\exp\left(-\frac{u}{4}\right).\end{aligned}\tag{6.67}$$

Proof. Recall the inequality (6.64), for any $\delta\in I/\{0\}$,

$$\begin{aligned}&\sup_{p\triangleq(x^1,x^2,x^3)\in\mathbb{R}^3}\mathbb{E}\left(\frac{1}{\sum_{i=1}^3\left(x^i+Z_\delta^i\right)^2}\cdot 1_{\mathcal{B}_O(r)^c}(Z_\delta)\right)\\ \leq\ &\frac{\sqrt{2}\pi}{\delta}\exp\left(-\frac{r^2}{4\delta}\right)\\ \leq\ &\sqrt{2}\pi r^{-2}\cdot\sup_{u\in[0,\infty)}u\exp\left(-\frac{u}{4}\right).\end{aligned}\tag{6.68}$$

Therefore, for any $r > 2\sup_{s,t\in I} ||\gamma_1(s) - \gamma_2(t)||$,

$$\begin{aligned}
&\left|\left|\mathbb{E}\left(\frac{\gamma_1(s)-\gamma_2(t)+Z_\delta}{||\gamma_1(s)-\gamma_2(t)+Z_\delta||^3}1_{\mathcal{B}_{-(\gamma_1(s)-\gamma_2(t))}(r)^c}(Z_\delta)\right)\right|\right| \\
\leq\ & \mathbb{E}\left(\frac{1_{\mathcal{B}_{-(\gamma_1(s)-\gamma_2(t))}(r)^c}(Z_\delta)}{||\gamma_1(s)-\gamma_2(t)+Z_\delta||^2}\right) \\
\leq\ & \mathbb{E}\left(\frac{1_{\mathcal{B}_O(r-1)^c}(Z_\delta)}{||\gamma_1(s)-\gamma_2(t)+Z_\delta||^2}\right).
\end{aligned}$$

Also note that for any $a, b \in \mathbb{R}^3$,

$$||a\wedge b|| \leq ||a||\cdot||b||.$$

Hence, we have for any $\delta \in I/\{0\}$,

$$\begin{aligned}
&\left|\oint_{\gamma_1}\oint_{\gamma_2}\left\langle d\gamma_1(s)\wedge d\gamma_2(t),\right.\right. \\
&\left.\left.\mathbb{E}\left(\frac{\gamma_1(s)-\gamma_2(t)+Z_\delta}{||\gamma_1(s)-\gamma_2(t)+Z_\delta||^3}1_{\mathcal{B}_{-(\gamma_1(s)-\gamma_2(t))}(r)^c}(Z_\delta)\right)\right\rangle\right| \\
\leq\ & \oint_{\gamma_1}\oint_{\gamma_2}||d\gamma_1(s)||\cdot||d\gamma_2(t)||\cdot\mathbb{E}\left(\frac{1_{\mathcal{B}_O(r-1)^c}(Z_\delta)}{||\gamma_1(s)-\gamma_2(t)+Z_\delta||^2}\right) \\
\leq\ & \sqrt{2}\pi(r-1)^{-2}\cdot l(\gamma_1)\cdot l(\gamma_2)\cdot\sup_{u\in[0,\infty)} u\exp\left(-\frac{u}{4}\right),
\end{aligned}$$

which converges to zero as r tends to infinity. ■

Definition 6.96 *Define $\rho(\gamma_1,\gamma_2)$ to be the smallest positive number such that*

$$\left|\frac{\sqrt{2}l(\gamma_1)l(\gamma_2)}{4(\rho(\gamma_1,\gamma_2)-1)^2}\sup_{u\in[0,\infty)} u\exp\left(-\frac{u}{4}\right)\right| \leq \frac{1}{4}.$$

Corollary 6.97 *Given two loops of finite variation γ_i, $i = 1, 2$ such that $\gamma_1(I)\cap\gamma_2(I) = \emptyset$. The linking number $N[\gamma_1,\gamma_2]$ is equal to the nearest* *integer to*

$$\begin{aligned}
& \lim_{\delta\to\infty} \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \langle d\gamma_1(s) \wedge d\gamma_2(t), \\
& \mathbb{E}\left(\frac{\gamma_1(s) - \gamma_2(t) + Z_\delta}{\|\gamma_1(s) - \gamma_2(t) + Z_\delta\|^3} 1_{\mathcal{B}_{-(\gamma_1(s)-\gamma_2(t))}(\rho(\gamma_1,\gamma_2))}(Z_\delta) \right) \Big\rangle \\
= & \lim_{\delta\to\infty} \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \langle d\gamma_1(s) \wedge d\gamma_2(t), \\
& \int\int\int_{\mathcal{B}_O(\rho(\gamma_1,\gamma_2))} \frac{v}{\|v\|^3} \frac{1}{(2\pi\delta)^{\frac{3}{2}}} \exp\left(-\frac{\|v - (\gamma_1(s) - \gamma_2(t))\|^2}{2\delta} \right) \Big\rangle.
\end{aligned}$$

Proof. With the choice of $\rho(\gamma_1, \gamma_2)$ in Definition 6.96, we have

$$\begin{aligned}
& \sup_{\delta\in I/\{0\}} \Big| \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \langle d\gamma_1(s) \wedge d\gamma_2(t), \\
& \mathbb{E}\left(\frac{\gamma_1(s) - \gamma_2(t) + Z_\delta}{\|\gamma_1(s) - \gamma_2(t) + Z_\delta\|^3} 1_{\mathcal{B}_{-(\gamma_1(s)-\gamma_2(t))}(\rho(\gamma_1,\gamma_2))^c}(Z_\delta) \right) \Big\rangle \Big| \\
\leq & \frac{1}{4}.
\end{aligned}$$

Recall Lemma 6.21 and Proposition 6.94, we therefore obtain our claim. ■

Definition 6.98 *For $t \in I/\{0\}$, $n \in \mathbb{N}$ and $\forall x \triangleq (x_1, x_2, x_3) \in \mathbb{R}^3$, we define*

$$\psi_t^n(x_1, x_2, x_3) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \sum_{k=0}^{n} \frac{(-1)^k}{2^k t^k \cdot k!} \|x\|^{2k}.$$

Clearly, ψ_t^n converges uniformly to ψ_t on compacta.

Corollary 6.99 *Given two loops of finite variation γ_i, $i = 1, 2$ such that $\gamma_1(I) \cap \gamma_2(I) = \emptyset$.*

$$\begin{aligned}
& N[\gamma_1, \gamma_2] \\
= & \lim_{\delta\to\infty} \lim_{n\to\infty} \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \langle d\gamma_1(s) \wedge d\gamma_2(t), \\
& \int\int\int_{\mathcal{B}_O(\rho(\gamma_1,\gamma_2))} \frac{v}{(2\pi\delta)^{\frac{3}{2}} \|v\|^3} \sum_{k=0}^{n} \frac{(-1)^k}{2^k \delta^k \cdot k!} \|v - (\gamma_1(s) - \gamma_2(t))\|^{2k} \Big\rangle.
\end{aligned}$$

Proof. By substituting Definition 6.98 into Corollary 6.97, we immediately deduce our claim. ∎

Let $\gamma_\cdot : I \to (\mathbb{R}^3)^2$ be a path such that

$$\gamma_\cdot \triangleq (\gamma_{1\cdot}, \gamma_{2\cdot}).$$

By expansion, we can express the linking number as a sum in the form:

$$N[\gamma_1, \gamma_2] = \lim_{\delta\to\infty} \lim_{n\to\infty} \sum_{k=0}^{n} P_{k,\delta}(\rho(\gamma_1, \gamma_2), \gamma_1(0) - \gamma_2(0)) \Lambda_k(S(\gamma_\cdot)),$$

where $P_{k,\delta}(\rho(\gamma_1, \gamma_2), \gamma_1(0) - \gamma_2(0))$ is a polynomial in $\rho(\gamma_1, \gamma_2)$ and $\gamma_1(0) - \gamma_2(0)$ with each coefficient being a fraction of integrals of powers of trigonometric functions and powers of δ; $\Lambda_k(S(\gamma_\cdot))$ is a sum of projection operators from $T(V)$ to $< e^{i_1} \otimes \cdots \otimes e^{i_{n_k}} >$ for different $i_1, \ldots, i_{n_k}$ such that

$$\deg P_{k,\delta} + n_k = 2k + 3.$$

Furthermore, note that for any $z \in \mathbb{R}^3$ with

$$||z|| \le \frac{1}{2} \inf_{s,t\in I} ||\gamma_1(s) - \gamma_2(t)||,$$

we have

$$N[\gamma_1 + z, \gamma_2] = N[\gamma_1, \gamma_2].$$

Also note that

$$|N[\gamma_1, \gamma_2]| \le \frac{l(\gamma_1)\, l(\gamma_2)}{4\pi \cdot \inf_{s,t\in I} ||\gamma_1(s) - \gamma_2(t)||^2}. \tag{6.69}$$

Lemma 6.100 *For any* $r \in \mathbb{R}^+$,

$$\mathbb{P}(||Z_\delta|| > r) \le \left(\sqrt{\frac{\pi}{2\delta}}r + \frac{\sqrt{2\pi\delta}}{r}\right) \exp\left(-\frac{r^2}{2\delta}\right).$$

Proof.

$$\begin{aligned}
& \mathbb{P}(||Z_\delta|| > r) \\
\le & \int_0^\pi \int_0^{2\pi} \int_{\rho>r} \frac{1}{(2\pi\delta)^{\frac{3}{2}}} \exp\left(-\frac{\rho^2}{2\delta}\right) \rho^2 d\rho d\theta d\varphi \\
= & \frac{\pi}{\delta} \int_{\rho>r} \frac{1}{\sqrt{2\pi\delta}} \rho^2 \exp\left(-\frac{\rho^2}{2\delta}\right) d\rho \\
= & \frac{\pi}{\delta}\left(\frac{1}{\sqrt{2\pi\delta}} r\delta \exp\left(-\frac{r^2}{2\delta}\right) + \delta\mathbb{P}\left(|Z_1| > \frac{r}{\sqrt{\delta}}\right)\right),
\end{aligned}$$

combining with Lemma 6.58, we deduce our claim. ■

Corollary 6.101 *Given two loops of finite variation γ_i, $i = 1, 2$ such that $\gamma_1(I) \cap \gamma_2(I) = \emptyset$.*

$$\begin{aligned}
&\left| N[\gamma_1, \gamma_2] - \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \left\langle d\gamma_1(s) \wedge d\gamma_2(t), \mathbb{E}\left(\frac{\gamma_1(s) - \gamma_2(t) + Z_\delta}{\|\gamma_1(s) - \gamma_2(t) + Z_\delta\|^3} \right) \right\rangle \right| \\
\leq\ & \frac{l(\gamma_1)\, l(\gamma_2)}{4\pi \cdot \inf_{s,t\in I} \|\gamma_1(s) - \gamma_2(t)\|^2} \left(\sqrt{\frac{\pi}{2\delta}} r + \frac{\sqrt{2\pi\delta}}{r} \right) \exp\left(-\frac{r^2}{2\delta} \right) \\
&+ \frac{\sqrt{2}}{4\delta} l(\gamma_1) \cdot l(\gamma_2) \exp\left(-\frac{r^2}{4\delta} \right).
\end{aligned}$$

Proof. Let $r = \frac{1}{2} \inf_{s,t\in I} \|\gamma_1(s) - \gamma_2(t)\|$, we have

$$\begin{aligned}
&\left| N[\gamma_1, \gamma_2] - \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \left\langle d\gamma_1(s) \wedge d\gamma_2(t), \mathbb{E}\left(\frac{\gamma_1(s) - \gamma_2(t) + Z_\delta}{\|\gamma_1(s) - \gamma_2(t) + Z_\delta\|^3} \right) \right\rangle \right| \\
\leq\ & N[\gamma_1, \gamma_2]\, \mathbb{P}(\|Z_\delta\| > r) \\
&+ \frac{1}{4\pi} l(\gamma_1) \cdot l(\gamma_2) \sup_{p=(x^1,x^2,x^3)\in\mathbb{R}^3} \mathbb{E}\left(\frac{1}{\sum_{i=1}^3 (x^i + Z_\delta^i)^2} \cdot 1_{B_O(r)^c}(Z_\delta) \right) \\
\leq\ & N[\gamma_1, \gamma_2]\, \mathbb{P}(\|Z_\delta\| > r) + \frac{\sqrt{2}}{4\delta} l(\gamma_1) \cdot l(\gamma_2) \exp\left(-\frac{r^2}{4\delta} \right),
\end{aligned}$$

where we used (6.65) to deduce the last inequality. Using Lemma 6.100 and (6.69), we obtain our claim. ■

In accordance with Lemma 6.95, using the approach as in the proof of Theorem 6.80, we can also deduce that the linking number $N[\gamma_1, \gamma_2]$ can be expressed as a linear functional of the truncated signature of $S(\gamma_\cdot)_{0,1}$ (and hence computable though the complexity may not be trivial) given that both $\inf_{s,t\in I} \|\gamma_1(s) - \gamma_2(t)\|$ and $l(\gamma_i)$ for $i = 1, 2$ are known. The last condition can be replaced by the knowledge of the modulus of continuity of the derivative of γ_i when the later is parameterized at unit speed as Condition 6.75 we mentioned before.

In summary, this chapter is a preliminary study in how to extract topological information of a collection of loops from the joint signature of loops. Section 6.4 also suggested that some topological invariants of loops may be relatively easier to be computed in terms of their signatures. Both points of view motivate us to put a step forward to investigate the extension of the methods in Knot Theory in future work.

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