Essays in Macroeconomic Modelling with Frictions and Rigidities

LUK, Sheung Kan

Lady Margaret Hall, University of Oxford

Thesis submitted for the degree of Doctor of Philosophy in Economics

Trinity Term 2013

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Abstract

This thesis presents three dynamic stochastic general equilibrium models to answer three macroeconomic questions. In each model, I impose one or more frictions or constraints and analyse how these frictions affect macroeconomic dynamics.

Chapter 2 studies the coordination of fiscal and monetary policies under optimal commitment and discretion policies under a New Keynesian framework. The chapter shows that when there is indexation in price setting which depends on the lagged output gap as in Steinsson (2003), under the optimal commitment policy, both fiscal and monetary policies have active roles in inflation stabilisation, even although debt follows a unit-root process. Under the optimal discretion policy, both fiscal and monetary policies have active roles in inflation stabilisation to drive debt back to the pre-shock level, consistent with Leith and Wren-Lewis (2008). Extending the model to include capital accumulation does not alter these results.

Chapter 3 presents a microfounded two-country model of global imbalances and debt deleveraging. During global imbalances a sustained rise in saving in one country can lead to a worldwide fall in the interest rates and an accumulation of debt in the other country. When an ensuing deleveraging shock occurs as a result of the global financial crisis, the interest rates are forced further down. I show that in the presence of a liquidity trap the deleveraging country may face a combination of a large fall in output, deflation and real exchange rate appreciation, as a result of debt deflation.

Chapter 4 adds a highly-leveraged financial sector to the Ramsey model and shows that this augments the macroeconomic effects of aggregate productivity shocks. My model is built on the financial-accelerator approach of Bernanke, Gertler and Gilchrist (BGG), in which leveraged goods-producers borrow from a competitive financial sector. In this chapter, by contrast, financial institutions are leveraged and subject to idiosyncratic productivity shocks. They obtain funds by paying an interest rate above the risk free rate, and this risk premium is anti-cyclical, and so amplifies the shocks. My parameterisation, based on US data, is one in which the leverage of the financial sector is two and a half times that of the goods-producers in the BGG model. This causes a much more significant augmentation of aggregate productivity shocks than that found in the BGG model.

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Chapter 1

Introduction

In this thesis I construct three dynamic stochastic general equilibrium models to answer three distinct macroeconomic questions.

Dynamic stochastic general equilibrium (henceforth DSGE) models have become the norm in macroeconomic modelling. Since Kydland and Prescott (1982) pioneered the study of real business cycle theory with a DSGE model, a large quantity of DSGE models have been constructed by researchers which also yielded substantial insights in macroeconomic dynamics.

Macroeconomic modelling using DSGE models has several advantages. First, DSGE models are usually based on optimising behaviours among the economic agents. Having clearly specified microfoundations, DSGE models are more immune to the Lucas critique (Lucas, 1976) than *ad hoc* or non-structural models in policy analysis. Second, the general equilibrium implies that demand and supply in all markets must clear in equilibrium and that Walras' law must hold. The general equilibrium nature of the model means that the budget constraints have to hold and the dynamic nature of the model means that these hold over time and thus that the model is stock-flow consistent. Third, the general equilibrium framework allows one to identify and separate the transmission channels and compare the rela-

tive importance of different effects in a coherent way. Finally, recent studies such as Christiano, Eichenbaum and Evans (2005) and Smets and Wouters (2007) show that estimated medium-scale DSGE models augmented with real and nominal rigidities have a potential to explain observed data satisfactorily and to predict out-of-sample as well as Bayesian vector-autoregressive (VAR) models.

In the models constructed in each of the chapters of the thesis, I impose one or more frictions or constraints in the system, and analyse how these frictions affect macroeconomic dynamics. In each of the models I make a new contribution compared with what is already established in the literature. In the following I make clear what these contributions are.

1.1 Optimal Monetary and Fiscal Policy Co-ordination

In chapter 2, I consider a model with nominal rigidities in the price setting behaviour of the firms. Nominal rigidities are a common assumption in New Keynesian models. Nominal rigidities have the effects of distorting the allocation of resources in the monopolistically competitive production sector and of creating a gap between actual output and the natural output, distorting the intratemporal tradeoff between working and consumption in the household sector relative to a world in which nominal rigidities are not present. Monetary and fiscal policies may have a role to minimise the welfare loss caused by the distortion. Benigno and Woodford (2004) and Leith and Wren-Lewis (2008) use the linear-quadratic approach to derive the optimal fiscal and monetary policies under commitment and discretion in a New Keynesian model without capital accumulation and inflation persistence. In particular Leith and Wren-Lewis (2008) show that in order to stabilise inflation fiscal policy plays no active role under commitment, but adjusts actively under discretion. Cheng (2009) finds that this conclusion is robust to the introduction of capital accumulation.

Stehn and Vines (2007) extends Leith and Wren-Lewis (2008) by adding inflation persistence with output indexation and show that in this case government spending does play an active role in stabilisation under commitment.

Chapter 2 contributes to this literature by studying the optimal monetary and fiscal policy coordination for a general kind of New Keynesian model which allows for both backward price-indexation and capital accumulation. In this chapter, optimal policies under both commitment and discretion are derived and the intuitions are explained in detail. This chapter yields two major results. First, under commitment, the core findings are that fiscal policy does not contribute to the stabilisation of the economy. Both backward-lookingness and indexation of past output in price setting are necessary to make fiscal policy active in inflation stabilisation. Adding capital does not alter this result in meaningful sense. Under discretion, fiscal and monetary policies are both active, as long as the accumulation of debt is endogenous. Hence the findings in the existing literature remain valid in my model. Second, when the steady-state debt-to-output ratio is high, it may be optimal to cut the interest rates for one period in response to a positive cost-push shock under discretion, violating the Taylor principle, and to raise the interest rates to fight inflation only after the first period. This finding is consistent with Stehn and Vines (2007).

1.2 Global Imbalances and the Global Financial Crisis

Chapter 3 presents a two-country DSGE model to study global imbalances and the global financial crisis. I model global imbalances as a persistent rise in patience in one of the countries. To model the global financial crisis, two constraints are imposed in the model. The first refers to a deleveraging constraint imposed on the country which has overspent, *i.e.* has run a current account deficit. The second

constraint is the liquidity trap, which prevents the (net) nominal interest rate from falling below zero. The liquidity trap makes this model different from standard open-macro models such as Clarida, Gali and Gertler (2001) in which a standard Taylor rule (without a zero bound constraint), together with the IS curve and the Phillips curve, ensures stability.

The contribution of this chapter is as follows. First, this is the first formal model that studies the connections of global imbalances and the global financial crisis. The model shows that a rise in patience in one country leads to a depreciation of the real exchange rate and a current account surplus in that country. Moreover, the interest rates in both countries fall. On top of this, when the overspent economy is hit by a persistent deleveraging shock, the interest rates fall further. The reduction in the interest rates supports Bernanke's (2005) 'global saving glut' hypothesis. Second, I show that when the deleveraging in the overspent country is so sharp that its interest rate falls below the zero bound, there will be a substantial fall in output and deflation. By comparing this outcome with that in a system without a zero bound, I show that the zero bound in the interest rate makes the system highly non-linear. The simulation further shows that, with such a model specification, in the beginning of the deleveraging phase, the real exchange rate of the deleveraging country appreciates, making the recession even worse. This is in contrast with views that the unwinding of global imbalances would imply a steady decline of the US real exchange rate.

1.3 Financial Frictions in Financial Institutions

In chapter 4, I impose financial frictions in the banking sector in a real business cycle model. The financial friction can be thought of, in reduced-form, as a wedge between the deposit rate depositors receive and the return to capital of financial institutions.

This wedge is a positive function of the leverage ratio of financial institutions, which is anti-cyclical in nature. I derive this wedge based on the financial-accelerator approach due to Bernanke, Gertler and Gilchrist (1999) and Hirakata, Sudo and Ueda (2009). The financial sector is put into a general equilibrium framework, augmented with Tobin's q theory of investment.

This chapter has the following contributions. The key innovation is that the financial friction is present between depositors and financial institutions, in contrast to Bernanke, Gerlter and Gilchrist (1999), and there is no information asymmetry between entrepreneurs and financial institutions who purchase the entrepreneurs' shares, in contrast to Hirakata, Sudo and Ueda (2009). The simulation results show three things. First, my model can produce larger amplification in response to aggregate productivity shocks than in Bernanke, Gertler and Gilchrist (1999), because the leverage of the financial sector is larger than that in the entrepreneurial sector. Second, given the same leverage, the amplification in my model is bigger than that in Hirakata, Sudo and Ueda (2009). This is because their assumption that financial institutions lend money to entrepreneurs, rather than buying their shares, dampens the amplification in their system in a way which does not happen in our model. Third, I show that the extent of amplification depends on the extent to which goods-producers are externally financed through the issue of shares, rather than self-financed.

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Chapter 2

Optimal Monetary and Fiscal Policy in an Economy with Capital Accumulation and Inflation Persistence

2.1 Introduction

This chapter studies the coordination of fiscal and monetary policy under optimal commitment and discretion policies under the New Keynesian framework. New Keynesian models have become an important tool for macroeconomic analysis. Developments in techniques within the New Keynesian framework, such as the linear-quadratic approach (Woodford 2003a), have made possible analytical understanding of the nature of optimal policies. Recent events have shown some policymakers want to use active monetary and fiscal policy to minimise business cycle fluctuations. In light of these developments, it is crucial to understand both analytically and intuitively the optimal policy in a realistic model.

This chapter intends to provide a unified setup which encompasses the existing models of fiscal and monetary policy coordination in the New Keynesian framework. In this chapter, I construct four models and consider the optimal commitment and discretion policies of these models. I begin by setting out a baseline model. The model has monopolistically competitive firms which adjust prices subject to nominal rigidities a la Calvo (1983). Consumers consume and supply labour, which is the only factor input for production. The policymaker has access to a lump sum tax and balances its budget every period. Analytical results show that under both commitment and discretion, the optimal fiscal policy should not supplement the monetary policy. The result relies on two arguments. First, consumption is more efficient than government spending to control inflation. This is because a fall in consumption not only reduces output, but it also affects consumers' marginal rate of substitution between consumption and labour, lowering the real wage, whereas a fall in government spending can only reduce aggregate demand. Second, the tradeoffs between consumption and inflation in both of the policy rules have identical slopes so that once the optimal consumption is chosen, the effects of consumption and inflation on the choice of the optimal fiscal spending cancel each other out. This exact cancelation means that government spending need not move. These arguments are presented in Stehn and Vines (2009) and will be used throughout the chapter to study the more general models.

I generalise the model in three steps, adding one feature at one time. First, I replace the lump-sum tax with distortionary income tax and a law of motion for the government debt. Second, I modify the price setting assumption from Calvo (1983) to one in which a fraction of price setter uses a backward-looking indexation rule. The indexation rule follows Steinsson (2003) and depends on lagged inflation and lagged output gap. Finally, I endogenise capital accumulation in the model. Capital accumulation is an important component in a Ramsey model. In the Ramsey model,

the marginal product of capital has to equal to the real interest rate minus depreciation. In the New Keynesian literature, there are models which study monetary policy in an environment without capital. Examples include Gali and Monacelli (2005), Benigno and Woodford (2004), to name a few. These models thus ignore the interest rate effect to investment. By including capital accumulation, I intend to capture this effect. In each of these models, I study the optimal momentary and fiscal policy under both commitment and discretion, and in each case, I consider economies with low debt and high debt.

The main results are summarised as follows. In the model with distortionary tax and no inflation persistence, under the optimal commitment policy, monetary policy is still the instrument of choice to control business cycle fluctuations. Government debt follows a random walk, consistent with Benigno and Woodford (2004) and Schmitt-Grohe and Uribe (2004a), and government spending falls on impact but reaches the new steady state after the first period, which ensures debt does not explode. I show analytically that the lack of response of government spending in the model is related to the 'exact cancelation' argument discussed above. The permanent deviation of government spending from the first-best is due to smoothing intertemporally over the social loss function. In the high-debt economy, we see a smaller increase in the interest rate and a larger fiscal response in the first period, both of which help to contain the rise in debt. With the same model, under optimal discretion policy, I show that debt returns to the pre-shock steady state. (Otherwise, the policymaker will have a constant temptation to unexpectedly inflate away the debt.) As a result, the optimal response is one in which both fiscal and monetary policies have active roles in stabilisation. The same is shown in Leith and Wren-Lewis (2008) and Stehn and Vines (2007). In the economy with high debt, the interest rate falls initially in response to a positive mark-up shock, violating the Taylor principle. This is because the interest rate has become more effective in

controlling debt, and the cost of reducing debt is the lowest in the first period due to the stabilisation bias. The impulse response shows oscillations in all variables. These results are known – Benigno and Woodford (2004) and Leith and Wren-Lewis (2008) have studied the optimal commitment and discretion policies respectively.

In the model in which inflation persistence and output indexation of the backward-looking price-setters are added, the fiscal policy has an active role to play in the optimal commitment policy. While monetary policy is still more efficient than fiscal policy in controlling inflation, the dependence on the lagged output gap in price setting changes the Phillips curve and the social loss function in a way that overturns the exact cancelation argument. Under the optimal discretion policy, the disinflation process is more persistent than in the model without inflation persistence for obvious reasons. The persistence removes the oscillation in the high-debt economy, but the Taylor principle is still violated. This model was first analysed in Stehn and Vines (2007). The authors, however, did not discuss their results in relation to the other models. In particular, they did not discuss the importance of indexation in driving their results. I provide a detailed analysis of the role of indexation in overturning the exact cancelation result.

In the last model, I show that including capital accumulation in the model does not modify the main results significantly. When there is inflation persistence with lagged-output indexation, fiscal policy again helps to stabilise the real business cycle. But when I turn off inflation persistence or lagged output indexation in the Phillips curve, the optimal government spending under commitment policy provides little additional macroeconomic stabilisation. Under the optimal discretion policy, it remains true that both monetary policy and fiscal policy are active, and that the interest rate falls in response to a positive mark-up shock in the high-debt economy.

2.1.1 Related Literature

A number of recent papers have sought to consider optimal monetary and fiscal policy, in the context of models with sticky prices.

Schmitt-Grohe and Uribe (2007) consider a medium-scale model with capital accumulation and a number of real rigidities. They use a numerical second-order perturbation technique developed in Schmitt-Grohe and Uribe (2004b) to find the Ramsey-optimal policy. Benigno and Woodford (2004) develop a linear-quadratic approach to study the problem of policy coordination. These papers find that under the optimal commitment policy, government debt follows a unit root process. However, the questions that these papers study differ from ours. They treat government spending as an exogenous disturbance and use distortionary income taxation and monetary policy as policy instruments when lump-sum subsidies to firms are not available. Instead, I treat the tax rate as exogenous and focus on the adjustments in fiscal spending, the reason being that fiscal spending is an instrument that influences demand for output directly. In addition, Schmitt-Grohe and Uribe (2007) do not derive the optimal policy analytically and hence the intuitions of their results are not very clear.

Stehn and Vines (2007, 2009), Eser, Leith and Wren-Lewis (2009) and Leith and Wren-Lewis (2008) study the coordination of government spending and monetary policy. Eser, Leith and Wren-Lewis (2009) consider a model of a small open economy with sticky prices and wages. They study optimal policies using monetary policy and government spending as policy instruments. In their benchmark model, in which taxation is lump-sum and government spending is fully financed by tax, optimal government spending should play no role in the stabilisation of shocks under commitment. This is true even when allowing for inflation inertia in price and wage-setting, as long as there is no discrepancy between the private and social eval-

uation of the marginal rate of substitution between consumption and leisure. They do not consider price indexation, however. In the presence of debt, Eser, Leith and Wren-Lewis (2009) show that government spending will deviate from the optimal steady state but only to the extent required to cover the deficit, not to provide any additional macroeconomic stabilisation. My finding confirms this. Leith and Wren-Lewis (2008) consider the nature of the time-inconsistency involved and its implication for discretionary policymaking. Using a New-Keynesian model with distortionary taxes and government debt, they show that under discretionary policy debt will return to the initial steady state, overturning the random walk result in the commitment policy. Stehn and Vines (2007) consider a model similar to this chapter, with backward-looking price-setters. They derive optimal monetary and fiscal policies under both commitment and discretion. However, these models do not consider the role of capital and investment in the economy. My model is general enough to nest the models in these papers.

Cheng (2009) analyses optimal monetary and fiscal policy under both commitment and discretion in a model with endogenous capital accumulation. This chapter improves on Cheng (2009) on three aspects. First, my model allows for backward-lookingness in firms' price setting and indexation of lagged output as a Steinsson (2003), nesting Cheng (2009). This turns out to make optimal fiscal policy under commitment active. Second, my model uses the more conventional capital adjustment costs due to Hayashi (1982), and I simplify the derivation of the second-order approximation of the social welfare function. Third, I provide both analytical and intuitive explanations of the optimal monetary and fiscal policies.

2.1.2 Plan of this Chapter

The chapter proceeds as follows. In Section 2.2, to set the stage, I outline a simple model as in Stehn and Vines (2009) and discuss why government spending plays

no role in the optimal policy. Section 2.3 formally sets up a New Keynesian model with distortionary tax and inflation persistence. Section 2.4 considers a special case in which there is no persistence in price setting. Both optimal commitment and discretion policies are analysed. I show analytically why fiscal policy does not play an active role in stabilisation in the optimal commitment policy. Section 2.5 studies a more general case with persistence in price setting and output indexation and explains why output indexation overturns the previous result that government spending is not active under commitment. In Section 2.6, I expand the model to allow for capital accumulation. I simulate the model under optimal commitment and discretion policies, and show that the main conclusions in the previous sections remain unchanged. Section 2.7 concludes.

2.2 A Simple Model

In this section, I use the Stein and Vines (2009) model to reproduce the argument that fiscal policy plays no role in stabalisation under discretion. I also show that the same argument can be applied to a policymaker under commitment. Although the results in this section are not new, the purpose of this section is to lay down the argument clearly with this simple model. In subsequent sections, this argument will be used repeatedly to evaluate against models with more complex features.

The system in Stehn and Vines (2009) is entirely forward-looking. The system is

$$\sigma c_t = \sigma E_t c_{t+1} - (r_t - E_t \pi_{t+1}), \tag{2.1}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\sigma c_t + \varphi y_t) + \kappa \hat{\mu}_t, \tag{2.2}$$

$$y_t = \theta c_t + (1-\theta) g_t. \tag{2.3}$$

where, c_t denotes consumption, g_t denotes government spending, y_t denotes output, r_t is the nominal interest rate and π_t is inflation. All these variables are log-deviations from the flex-price economy. $\hat{\mu}_t$ is a cost-push shock. These equations are the consumption Euler equation, the Phillips curve and the national income identity. These equations are standard and the derivations from the microfoundations will be deferred to the next section when I introduce a more general model.[1]

The microfounded social loss function is

$$L_t = \frac{1}{2} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{\epsilon}{\kappa} \pi_s^2 + \varphi y_s^2 + \sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 \right). \tag{2.4}$$

The consumption Euler equation is not binding because the monetary policy can be used freely to control consumption. The Lagrangian for the policymaker is:

$$\begin{aligned} \mathscr{L}_t &= E_t \sum_{s=t}^{\infty} \beta^{s-t} [\frac{1}{2} \left(\frac{\epsilon}{\kappa} \pi_s^2 + \varphi y_s^2 + \sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 \right) \\ &\quad + \lambda_{s+1}^{\pi} (\pi_s - \beta\pi_{s+1} - \kappa((\sigma + \varphi\theta) c_s + \varphi(1-\theta) g_s) - \kappa\hat{\mu}_s) \\ &\quad + \lambda_s^y (\theta c_s + (1-\theta) g_s - y_s)]. \end{aligned} \tag{2.5}$$

Under commitment, the first order conditions are

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t} = \frac{\epsilon}{\kappa} \pi_t + \lambda_{t+1}^{\pi} - \lambda_t^{\pi} = 0, \tag{2.6}$$

$$\frac{\partial \mathscr{L}_t}{\partial y_t} = \varphi y_t - \lambda_t^y = 0, \tag{2.7}$$

$$\frac{\partial \mathscr{L}_t}{\partial c_t} = \sigma\theta c_t - \kappa(\sigma + \varphi\theta) \lambda_{t+1}^{\pi} + \theta \lambda_t^y = 0, \tag{2.8}$$

$$\frac{\partial \mathscr{L}_t}{\partial g_t} = \sigma(1-\theta) g_t - \kappa\varphi(1-\theta) \lambda_{t+1}^{\pi} + (1-\theta) \lambda_t^y = 0. \tag{2.9}$$

[1] Specifically, Equation (2.1) is identical to Equation (2.39), which is the log-linearised version of the consumption Euler equation (2.18). Equation (2.2) is a special case of the Phillips curve Equation (2.30), in which the indexation parameter ω is set to zero. Equation (2.3) is identical to Equation (2.41), which is the log-linearised version of the aggregate demand Equation (2.33).

In order to derive the result that the government spending does not change under the optimal policy, one only needs to use the last three equations. The optimal fiscal and monetary policies are

$$c_t = \frac{\kappa}{\theta}\lambda^{\pi}_{t+1} - \frac{\varphi(1-\theta)}{(\sigma+\varphi\theta)}g_t, \tag{2.10}$$

$$g_t = \frac{\kappa\varphi}{\sigma+\varphi(1-\theta)}\lambda^{\pi}_{t+1} - \frac{\varphi\theta}{\sigma+\varphi(1-\theta)}c_t. \tag{2.11}$$

where λ^{π}_{t+1} is the Lagrange multiplier for the Phillips curve which measures the marginal social loss of raising inflation by one unit. Combining these two policy rules, one gets

$$c_t = \frac{\kappa}{\theta}\lambda^{\pi}_{t+1}, \tag{2.12}$$

$$g_t = 0. \tag{2.13}$$

As is discussed in Stehn and Vines (2009), the absence of fiscal stabilisation in the optimal policy relies on two arguments. The first is a comparative advantage argument. Fiscal policy is inefficient compared with monetary policy in influencing inflation. Monetary policy can induce a fall in consumption, which has two effects to inflation. The fall in consumption reduces demand. *Ceteris paribus*, it also increases labour supply, reducing the real wage. This latter effect is not present with fiscal policy.

The second argument is the exact cancelation argument. In the following I consider the marginal social benefits and losses from the monetary and fiscal policies respectively. A fall in consumption has the benefit of reducing inflation in two ways (the marginal benefit of reducing inflation is summarised by the Lagrange multiplier λ^{π}_{t+1}.). Observing the Phillips curve Equation (2.2), the direct effect of a fall in consumption to inflation (a fall in consumption reduces the real wage required to

supply the same amount of labour) is $\kappa\theta$ and the marginal benefit of the fall in inflation to the loss is $\kappa\theta\lambda^{\pi}_{t+1}$. There is an indirect effect: a fall in consumption reduces output by θ (which reduces demand for labour and the real wage). For a given level of government spending, this reduces inflation by $\kappa\theta\varphi$ and the marginal benefit in welfare terms is $\kappa\theta\varphi\lambda^{\pi}_{t+1}$. The total effect is $\kappa(\sigma+\varphi\theta)\lambda^{\pi}_{t+1}$.

On the other hand, a fall in consumption increases the loss function in two ways. The direct effect of a marginal fall in consumption is $\sigma\theta c_t$. Again, there is an indirect effect to the loss through output. A marginal fall in consumption reduces output by θ, which increases the loss by $\theta\varphi y_t$. For a given level of government spending, this effect is $\theta^2\varphi c_t$. The total effect of a marginal fall in consumption to the loss is $(\sigma+\varphi\theta)\theta c_t$.

The optimal monetary policy rule Equation (2.10) requires that the marginal benefits and losses of a fall in consumption are equal. In this rule, for a given government spending, the partial derivative $\partial c_t/\partial\lambda^{\pi}_{t+1}$ is a simple expression κ/θ because the direct and indirect effects exactly cancel each other out.

Next, consider government spending. Unlike consumption, a fall in government spending affects inflation only indirectly through output. From the Phillips curve, a unit fall in g_t reduces inflation by $\kappa\varphi(1-\theta)$ and improves welfare by $\kappa\varphi(1-\theta)\lambda^{\pi}_{t+1}$. On the other hand, a fall in government spending increases the loss function in two ways. The direct effect of a marginal fall in government spending is $\sigma(1-\theta)g_t$. There is also an indirect effect through output. A marginal fall in government spending reduces output by $(1-\theta)$, which increases the loss by $(1-\theta)\varphi y_t$. Suppose c_t falls by one unit, then for a given level of government spending, the marginal loss of a fall in government spending is $\theta(1-\theta)\varphi c_t$. Then according to the optimal fiscal rule, if it is the case that upon a fall in consumption, the optimal government spending does not change, it must be such that, at this level of government spending, the fall in consumption increases the marginal loss by as much as the increase in marginal

benefit of reducing inflation, which means that $\theta(1-\theta)\varphi c_t = \kappa\varphi(1-\theta)\lambda^{\pi}_{t+1}$. In other words, the partial derivative $\partial c_t/\partial\lambda^{\pi}_{t+1}$ in the fiscal rule is also κ/θ.

Therefore, if consumption is cut by precisely the amount that it would be cut if there were no change in government spending, *i.e.* $c_t = \frac{\kappa}{\theta}\lambda^{\pi}_{t+1}$, then the effect of c_t on the optimal choice of g_t will exactly cancel the effect of λ^{π}_{t+1} on the optimal choice of g_t. As a result, there will be no change in g_t.

To complete the description of the optimal commitment policy, the evolution of the state variable, λ^{π}_t, is given by Equation (2.6). The initial condition is $\lambda^{\pi}_0 = 0$ and it is to ensure the optimality of the adjustment path.

Under discretion, the model has no persistence and $E_t(\pi_{t+1}) = 0$. The optimal discretion policies is described by Equations (2.7), (2.8), (2.9) and

$$\lambda^{\pi}_{t+1} = -\frac{\epsilon}{\kappa}\pi_t. \tag{2.14}$$

This means that

$$c_t = -\frac{\epsilon}{\theta}\pi_t, \qquad g_t = 0. \tag{2.15}$$

Hence, for this simple model, it is true that under both discretion and commitment government spending does not have a role in inflation stabilisation. This is true because the optimal policies under the two environments differ only to the extent that the marginal social cost of inflation, λ^{π}_{t+1}, are different. Under both cases, Equations (2.7), (2.8) and (2.9) are identical.

Figure 2.1 shows the simulation of a 1% positive mark-up shock. The parameter values are reported in Table 2.1. In this model, under commitment, the policymaker promises a low inflation in future, which helps to reduce current inflation. Promising low inflation in future is helpful because the loss function is convex and there is discounting of future losses. The policy is time-inconsistent because after the initial period, there is no more shock. The policymaker can do better by reneg-

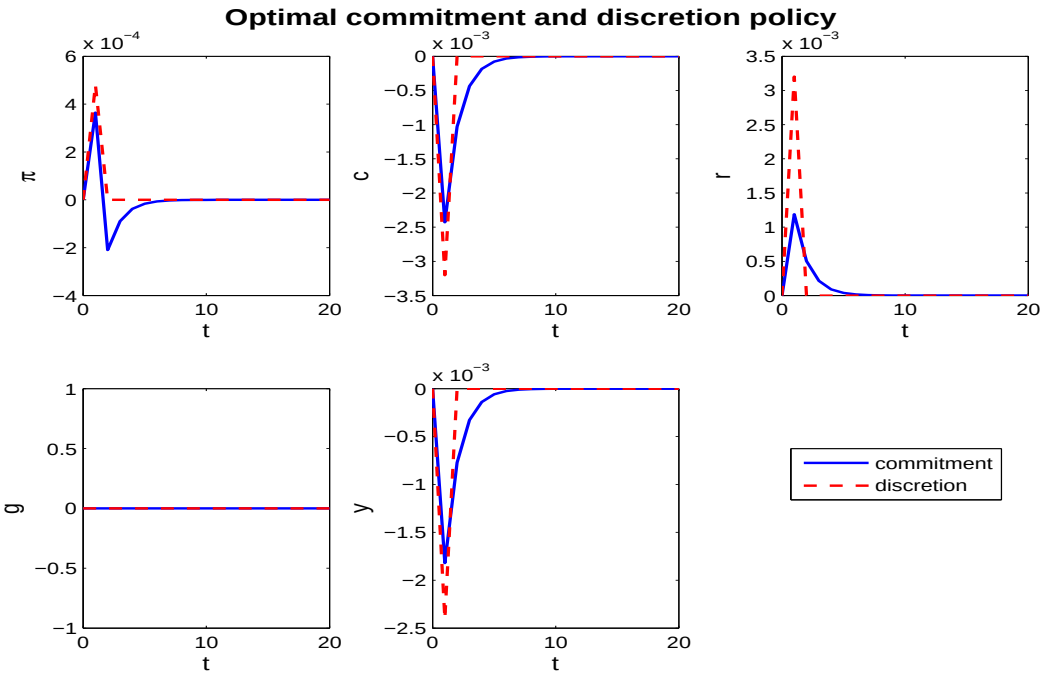


Figure 2.1: A 1% positive mark-up shock in the Stehn and Vines (2009) model.

ing, choosing no inflation. Under discretion, the policymaker is unable to promise a time-inconsistent inflation path in future. In the current environment, the policymaker has to raise the interest rates more drastically on impact. This results in the classic inflation stabilisation bias of Currie and Levine (1987, 1993). Inflation jumps up for one period and output down, and in the next period, both inflation and output return to the equilibrium as there is no persistence in the system.

2.3 A Model with Debt

In this section, I add government debt and inflation persistence to the model. The aim is to study under what conditions, and why, government spending may have an effect in the optimal monetary and fiscal policy.

2.3.1 The Model

The model is a standard New Keynesian model similar to that in Woodford (2003a). It differs from Woodford (2003a) in two ways. First, this setup includes fiscal policy with government spending and dynamic debt accumulation. The government can use lump-sum tax only to cover the production subsidy of the firms, which in turn is introduced to ensure the steady state is efficient. Second, there are backward-looking price-setters who use a price-indexation rule to set price so that the Phillips curve has a lagged inflation component. Below I derive the model making clear the microfoundations.

Consumers

Consumers are homogeneous. Assume that consumers have the following additively separable utility which is increasing in consumption and government spending and decreasing in labour:

$$U_t = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{C_s^{1-\sigma}}{1-\sigma} + \chi \frac{G_s^{1-\sigma}}{1-\sigma} - \frac{N_s^{1+\varphi}}{1+\varphi} \right), \tag{2.16}$$

where C_t denotes consumption in period t, G_t denotes government expenditure and N_t the labour supply. E_t is the expectation operator conditional on the information at time t.

Consumers supply labour, consume and save. They face the following budget constraint:

$$P_t C_t + E_t(Q_{t,t+1} D_{t+1}) = D_t + (1-\tau)(W_t N_t + \Omega_t) - T_t, \tag{2.17}$$

where P_t and W_t denote the price level and the nominal wage respectively. I assume τ is an exogenous labour income tax rate. Ω_t is firm profit and T_t is a lump-sum

tax to finance a production subsidy which offsets distortions caused by monopolistic competition among firms. This is a common short-cut to ensure that the steady state of the model is efficient. $Q_{t,t+1}$ is the stochastic discount factor which determines the price in period t to the individual of being able to carry a state-contingent amount D_{t+1} of wealth into period $t+1$. Its relation with the risk free interest rate is $E_t(Q_{t,t+1}) = R_t^{-1}$. The usual transversality condition applies.

The first order conditions for the consumers' utility optimisation problem are

$$R_t^{-1} = \beta E_t \left(\frac{C_t^{\sigma} P_t}{C_{t+1}^{\sigma} P_{t+1}} \right), \tag{2.18}$$

$$C_t^{\sigma} N_t^{\varphi} = (1-\tau)\frac{W_t}{P_t}. \tag{2.19}$$

The first equation is the consumption Euler equation which equates the marginal rate of substitution of consumption between period t and $t+1$ with the risk free interest rate. The second equation is the intratemporal optimal condition which equates the marginal rate of substitution between consumption and leisure on the left hand side of Equation (2.19) and the after-tax real wage on the right hand side.

Firms

On the production side, I assume intermediate goods firms produce only with labour. The production function for firm i is

$$Y_t(i) = A_t N_t(i). \tag{2.20}$$

A_t is the aggregate productivity shock with mean one in the steady state, and

$$\hat{A}_{t+1} = \rho_A \hat{A}_t + \xi_{A,t+1}, \tag{2.21}$$

where 'hat' denotes deviation from the steady state and $\xi_{A,t+1}$ is a mean zero, $i.i.d$ process. Aggregate labour is defined as $N_t \equiv \int_0^1 N_t(i)di$.

There is a final goods firm which mixes intermediate goods using a Dixit-Stiglitz (1977) aggregator:

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\epsilon_t - 1}{\epsilon_t}} di \right)^{\frac{\epsilon_t}{\epsilon_t - 1}}, \tag{2.22}$$

and the demand for each type of good is

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon_t} Y_t, \tag{2.23}$$

where $P_t \equiv \left[\int_0^1 P_t(i)^{1-\epsilon_t} di \right]^{\frac{1}{1-\epsilon_t}}$.

Price Setting

I model price setting as a mix of Calvo (1983) contracting and rule-of-thumb behaviour. Agents recalculate their prices with fixed probability $(1-\gamma)$. If prices are recalculated then a proportion ω of the price resetting agents use a rule of thumb to set their price and proportion $(1-\omega)$ calculate the optimum price. With probability γ prices are not recalculated and are assumed to rise at the average rate of inflation.

Allowing for indexation in the price setting is a simple and appealing way to take care of both theoretical consistency and data concerns. In the microfounded New Keynesian Phillips curve, there is no indexation, which means that inflation is entirely forward-looking. However, empirically the New Keynesian Phillips curve does not do a good job to match the observed data with high persistence (Mankiw (2001), Roberts (2001)). Fuhrer and Moore (1995) derive a hybrid Phillips curve with both backward-looking and forward-looking inflation terms similar to ours. They estimate the Phillips curve and conclude that it fits US data better than a purely forward-looking Phillips curve.

Specifically, I assume backward-looking agents set their prices P_t^{BL} using the rule of thumb:

$$P_t^{BL} = P_{t-1}^* \Pi_{t-1} \left(\frac{Y_{t-1}}{Y_{t-1}^n} \right)^{\eta}, \tag{2.24}$$

where $\Pi_t = P_t/P_{t-1}$ and Y_t^n is the flexible-price equilibrium of output. I use superscript * to denote the average price reset by the firms. The average price is a weighted average between forward-looking (P_t^{FL}) and backward-looking prices (P_t^{BL}):

$$P_t^* = (P_t^{FL})^{1-\omega} (P_t^{BL})^{\omega}. \tag{2.25}$$

This rule of thumb is identical to the formulation in Steinsson (2003), and when η is set to 0 the rule reduces to Gali and Gertler (1999).[2] The coefficient η defines the relative weight of output considerations in the rule of thumb. By including the rule (2.24), I have made two changes relative to Leith and Wren-Lewis (2008) who use the New-Keynesian Phillips curve a la Calvo (1983). First, I include backward-looking inflation. Second, I include lagged output indexation in the rule. I will show below that the latter makes government spending active in inflation stabilisation under the optimal commitment policy.

The forward-looking price-setters solve the first order conditions for profit maximisation and obtain the optimal solution. The first order condition is:

$$0 = E_t \sum_{s=t}^{\infty} Q_{t,s} \gamma^{s-t} \left(\frac{P_t^{FL}}{P_s} \right)^{-\epsilon_s} Y_s \left[\frac{P_t^{FL}}{P_S} - \frac{\mu_s}{\mu_w} \frac{C_s^{\sigma} N_s^{\varphi}}{(1-\tau) A_s} P_s \right], \tag{2.26}$$

where the stochastic mark-up of firms $\mu_s = \epsilon_s/(\epsilon_s - 1)$. The log-linearised condition is

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1 - \gamma\beta)(\sigma c_t + \varphi y_t + \hat{\mu}_t), \tag{2.27}$$

[2] As Steinsson (2003) shows, when price-setters are purely backward-looking, *i.e.* $\omega = 1$, the Gali and Gertler (1999) Phillips curve would be vertical rather than the accelerationist Phillips curve, and the optimal monetary policy would imply complete inflation stabilisation and explosive output response, which is undesirable. Therefore, I follow Steinsson (2003).

where $\hat{p}_t^{FL}$ denotes the log-deviation of the relative price set by forward-looking price-setters, that is $\hat{p}_t^{FL} = \hat{P}_t^{FL} - \hat{P}_t$, and $\pi_t \equiv \log \Pi_t$.

I assume the cost-push shock is an AR(1) process:

$$\hat{\mu}_{t+1} = \rho_\mu \hat{\mu}_t + \xi_{\mu,t+1}, \tag{2.28}$$

and $\xi_{\mu,t+1}$ is a mean zero, *i.i.d* process.

In the flexible price world, the real marginal cost is unity. This implies that a government subsidy of $\mu_w = \mu/(1-\tau)$ is necessary to ensure that the steady state is efficient.

The rest of the prices will rise at the steady-state rate of inflation $\Pi = 1$ with probability γ. I can write the price equation for the economy as a whole as:

$$P_t = \left(\gamma(\Pi P_{t-1})^{1-\epsilon_t} + (1-\gamma)(1-\omega)(P_t^{FL})^{1-\epsilon_t} + (1-\gamma)\omega(P_t^{BL})^{1-\epsilon_t}\right)^{\frac{1}{1-\epsilon_t}}. \tag{2.29}$$

Combining the evolution of the aggregate price level, Equation (2.29), and the price setting behaviour of the price-setters, Equations (2.27) and (2.24), I obtain a hybrid Phillips curve[3]

$$\pi_t = \chi^{FL}\beta E_t \pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c c_t + \kappa_{y0} y_t + \kappa_{y1} y_{t-1} + \kappa_\mu \hat{\mu}_t, \tag{2.30}$$

where the coefficients are defined as:

$$\begin{aligned}
\chi^{FL} &= \frac{\gamma}{\gamma + \omega(1-\gamma+\gamma\beta)}, \qquad \chi^{BL} = \frac{\omega}{\gamma + \omega(1-\gamma+\gamma\beta)}, \\
\kappa_c &= \frac{\sigma(1-\gamma)(1-\omega)(1-\gamma\beta)}{\gamma + \omega(1-\gamma+\gamma\beta)}, \qquad \kappa_{y0} = \frac{\varphi}{\sigma}\kappa_c - \frac{(1-\gamma)\gamma\beta\omega\eta}{\gamma + \omega(1-\gamma+\gamma\beta)}, \\
\kappa_{y1} &= \frac{(1-\gamma)\omega\eta}{\gamma + \omega(1-\gamma+\gamma\beta)}, \qquad \kappa_\mu = \frac{1}{\sigma}\kappa_c.
\end{aligned}$$

[3] A detailed derivation is provided in Appendix 2.8.2.

Notice the backward-looking inflation term in Equation (2.30) as a result of the backward-looking price-setters.

Government

The government buys goods, taxes income with a constant income tax rate τ and issues nominal debt $\bar{B}$. The lump sum tax pays for the production subsidy. The evolution of nominal debt is given by:

$$\frac{\bar{B}_{t+1}}{P_t} = R_t \left(\frac{\bar{B}_t}{P_{t-1}} \frac{P_{t-1}}{P_t} + G_t - \tau Y_t \right). \tag{2.31}$$

The debt evolution equation is log-linearised as follow:

$$\hat{B}_{t+1} = \hat{R}_t + \frac{1}{\beta} \left(\hat{B}_t - \pi_t + \frac{(1-\theta)}{B} \hat{G}_t - \frac{\tau}{B} \hat{Y}_t \right). \tag{2.32}$$

where the real debt stock is defined as $B_t \equiv \bar{B}_t / P_{t-1}$ and B is the steady-state debt to output ratio.

Aggregate Demand

Goods are demanded by consumers and the government. The national income identity is

$$Y_t = C_t + G_t. \tag{2.33}$$

In the steady state $G = (1-\theta)Y$, where θ is the share of private consumption in output. This completes the description of the model.

2.3.2 The Case of Flexible Price

It is clear that the source of friction considered in this model is the nominal rigidities. The rigidities lead to price distortions and welfare losses. Monetary and fiscal policies

are employed to minimise the distortions caused by the nominal rigidities. In the following, I compare this world with the first-best world in which there is no nominal rigidities, that is the flexible price world. To do this, I first solve the model in the flexible price world. One can imagine a social planner maximises consumers' welfare subject to the budget constraint. The maximisation problem is

$$\max_{C_t, G_t} \frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma}}{1-\sigma} - \frac{1}{1+\varphi} \left(\frac{Y_t}{A_t} \right)^{1+\varphi}, \tag{2.34}$$

and it is subject to the market clearing condition. The first order conditions are

$$(C_t^n)^{-\sigma} = \chi \, (G_t^n)^{-\sigma}, \tag{2.35}$$

$$(C_t^n)^{\sigma} \, (Y_t^n)^{\varphi} = A_t^{1+\varphi}. \tag{2.36}$$

The superscript 'n' denotes the efficient flexible-price level. Together with the market clearing condition, one can solve for C_t^n, G_t^n and Y_t^n in terms of the productivity shock. The log-linearised solutions are:

$$\hat{C}_t^n = \hat{G}_t^n = \hat{Y}_t^n = \frac{1+\varphi}{\sigma+\varphi} \hat{A}_t. \tag{2.37}$$

Finally, the natural interest rate is pinned down by the ratio of the marginal utilities of consumption:

$$\hat{R}_t^n = E_t C_{t+1}^n - C_t^n = E_t \hat{A}_{t+1} - \hat{A}_t = -\sigma(1-\rho_A) \frac{1+\varphi}{\sigma+\varphi} \hat{A}_t. \tag{2.38}$$

I am interested in to what extent nominal rigidities make the macroeconomic variables deviate from the flexible price level. Following Benigno and Woodford (2004), I write the system in terms of 'gap variables', where the gap is the difference

between actual and natural levels (that is $x_t \equiv \hat{X}_t - \hat{X}_t^n$).

$$\begin{aligned}
\sigma c_t &= \sigma E_t c_{t+1} - (r_t - E_t \pi_{t+1}), && (2.39)\\
\pi_t &= \chi^{FL} \beta E_t \pi_{t+1} + \chi^{BL} \pi_{t-1} + \kappa_c c_t + \kappa_{y0} y_t + \kappa_{y1} y_{t-1} + \kappa_\mu \hat{\mu}_t, && (2.40)\\
y_t &= \theta c_t + (1-\theta) g_t, && (2.41)\\
\hat{B}_{t+1} &= r_t + \frac{1}{\beta}\left(\hat{B}_t - \pi_t + \frac{(1-\theta)}{B} g_t - \frac{\tau}{B} y_t\right) + \hat{\nu}_t, && (2.42)
\end{aligned}$$

where $\hat{\nu}_t = \left(\frac{(1-\theta)-\tau}{\beta B} - (1-\rho_A)\right) \frac{1+\varphi}{\sigma+\varphi} \hat{A}_t$.

2.3.3 The Social Welfare Function

I now turn to the loss function for the policymaker. Following Woodford (2003a) and Steinsson (2003) the policymaker is concerned with maximising a second-order approximation of the expected utility of the households around the steady state. The loss function is[4]

$$L_t = -\frac{C^{1-\sigma}}{2\theta} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 + \epsilon Var_i(\hat{P}_s(i)) \right) + t.i.p. \quad (2.43)$$

where the price dispersion $Var_i(\hat{P}_t(i))$ is

$$Var_i(\hat{P}_t(i)) = \frac{1}{\kappa}\pi_t^2 + 2\frac{1}{\kappa}\phi_1 \Delta\pi_t y_{t-1} + \frac{1}{\kappa}\phi_2 \Delta\pi_t^2 + \frac{1}{\kappa}\phi_3 y_{t-1}^2 \quad (2.44)$$

and

$$\kappa = \frac{(1-\gamma\beta)(1-\gamma)}{\gamma}, \qquad \phi_1 = -\frac{\omega\eta(1-\gamma)}{\gamma(1-\omega)}, \qquad \phi_2 = \frac{\omega}{\gamma(1-\omega)}, \qquad \phi_3 = \frac{(1-\gamma)^2\omega\eta^2}{\gamma(1-\omega)},$$

[4]Detailed derivation of the loss function is presented in Appendix 2.8.3.

and $t.i.p$ stands for terms that are independent of policy and hence irrelevant for ranking alternative policies. The social loss function contains quadratic terms associated with lagged inflation and lagged output gap. This is because rule-of-thumb price-setters adjust their prices according to lagged inflation and lagged output gap, which affects the price dispersion in the economy. If the rule of thumb does not depend on lagged output, that is if $\eta = 0$, then the terms associated with the lagged output gap will disappear.

2.3.4 Calibration

Each period in the model is a quarter. The impatience parameter, β, is set to be 0.99 so that the steady state rate of interest is approximately 4.2%. Following Kirsanova and Wren-Lewis (2011), the ratio of consumption to output, θ, is set to 0.75, which also determines the relative preference for government spending in utility. The coefficient of relative risk aversion σ is 1, which implies logarithmic utility. The inverse of Frisch labour supply elasticity, φ, is set to be 0.5. The steady state elasticity of substitution across varieties of consumption goods, ϵ, is set to be 5. The probability of resetting price, γ is calibrated to be 0.75, which implies that prices are on average set once a year. These parameters are calibrated following the recent literature (*e.g.* Rotemberg and Woodford 1997). On the government side, I consider two cases regarding the steady state debt to output ratio. In the 'low debt' case, the debt to output ratio is 0.1. In the 'high debt' case, the steady state debt to output ratio is 2.4, that is 60% debt to output ratio. This number follows Benigno and Woodford (2004) and Kirsanova and Wren-Lewis (2011) and is close to the level of debt in a number of European economies.[5] The tax rate τ is pinned

[5]For instance, using the data from Reinhart and Rogoff (2011), the average of total gross general domestic government debt to GDP ratio from 2001-2010 in France, Germany, United Kingdom and United States are 66.7%, 66.7%, 48% and 66.6% respectively. After the global finance crisis in 2008, the government debt to GDP ratios in these countries are substantially higher than the 10-year average. For instance, in 2010, the number for these countries are 84.2%, 78.8%, 76.7% and 92.7%

Table 2.1: Parameter values.

Parameter	β	σ	φ	ϵ	γ	θ	B	ρ_μ
Value	0.99	1	0.5	5	0.75	0.75	0.1	0

down by the steady state debt to output ratio, *i.e.* $\tau = B(1-\beta) + (1-\theta)$. As for the shock process, for simplicity, I study one-off shocks to the mark-up and set ρ_μ to zero. The parameter values are summarised in Table 2.1.

2.4 Optimal Policies in the Model without Inflation Persistence

In this section, I consider a special case in which there is no backward-looking price-setter. I do this by setting ω to zero. This case has important insights about monetary and fiscal policy coordination under the optimal commitment and discretion policies. In particular, I show analytically that government spending will jump to its long run level after the first period under the optimal commitment policy. These results have already been shown in Eser, Leith and Wren-Lewis (2009) and Leith and Wren-Lewis (2008). Here I focus on understanding these results in relation to the exact cancelation argument.

With no backward-looking price-setters, the Phillips curve collapses to the Calvo (1983) Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\sigma c_t + \varphi y_t) + \kappa \hat{\mu}_t. \tag{2.45}$$

respectively.

The loss function does not contain any lagged terms:

$$L_t = -\frac{C^{1-\sigma}}{2\theta} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 + \frac{\epsilon}{\kappa} \pi_s^2 \right) + t.i.p. \tag{2.46}$$

In the following, I study the optimal commitment policy first, followed by the optimal discretion policy.

2.4.1 Optimal Policy under Commitment

The policymaker choose r_t and g_t subject to the above constraints. The Lagrangian for the policymaker is:[6]

$$\begin{aligned} \mathscr{L}_t \;=\; & E_t \sum_{t=s}^{\infty} \beta^{s-t} [\frac{1}{2} \left(\sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 + \frac{\epsilon}{\kappa} \pi_s^2 \right) \\ & + \lambda_{s+1}^{\pi} \left(\pi_s - \beta \pi_{s+1} - \kappa \left((\sigma + \varphi\theta) c_s + (1-\theta)\varphi g_s \right) - \kappa \hat{\mu}_s \right) \\ & + \lambda_s^{y} \left(\theta c_s + (1-\theta) g_s - y_s \right) \\ & + \lambda_{s+1}^{c} \left(\sigma c_s - \sigma c_{s+1} + r_s - \pi_{s+1} \right) \\ & + \lambda_{s+1}^{b} \left(r_s + \frac{1}{\beta} \left(\hat{B}_s - \pi_s + \frac{(1-\theta)}{B} g_s - \frac{\tau}{B} y_s \right) + \hat{\nu}_s - \hat{B}_{s+1} \right)]. \end{aligned} \tag{2.47}$$

[6] Unlike the simple model in Section 2.2, the consumption Euler equation is binding and has to be included in the Lagrangian.

The first order conditions are:

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t} = \frac{\epsilon}{\kappa}\pi_t + \lambda^{\pi}_{t+1} - \lambda^{\pi}_t - \beta^{-1}\lambda^{c}_t - \beta^{-1}\lambda^{b}_{t+1} = 0, \tag{2.48}$$

$$\frac{\partial \mathscr{L}_t}{\partial y_t} = \varphi y_t - \lambda^{y}_t - \frac{1}{\beta}\frac{\tau}{B}\lambda^{b}_{t+1} = 0, \tag{2.49}$$

$$\frac{\partial \mathscr{L}_t}{\partial c_t} = \sigma\theta c_t - \kappa(\sigma + \varphi\theta)\lambda^{\pi}_{t+1} + \theta\lambda^{y}_t + \sigma\lambda^{c}_{t+1} - \beta^{-1}\sigma\lambda^{c}_t = 0, \tag{2.50}$$

$$\frac{\partial \mathscr{L}_t}{\partial \hat{B}_t} = -\beta^{-1}\lambda^{b}_{t+1} + \beta^{-1}\lambda^{b}_t = 0, \tag{2.51}$$

$$\frac{\partial \mathscr{L}_t}{\partial r_t} = \lambda^{c}_{t+1} + \lambda^{b}_{t+1} = 0, \tag{2.52}$$

$$\frac{\partial \mathscr{L}_t}{\partial g_t} = \sigma(1-\theta)g_t - \kappa\varphi(1-\theta)\lambda^{\pi}_{t+1} + (1-\theta)\lambda^{y}_t + \frac{1}{\beta}\frac{(1-\theta)}{B}\lambda^{b}_{t+1} = 0. \tag{2.53}$$

The optimal commitment policy is fully described by Equations (2.39), (2.41), (2.42), (2.45), (2.48) – (2.53), the shock process and the initial values for the state variables. The state variables of this system include $\hat{B}_t$ and the Lagrange multipliers for the jump variables $\lambda^{\pi}_t, \lambda^{c}_t$. I assume that the economy is in the interior steady state before any shock occurs, coinciding the flexible price equilibrium. Since the equations describing the private sector of the economy are not binding initially, the Lagrange multipliers λ^{π}_0 and λ^{c}_0 are set to zero to make sure that the initial policy is optimal.

Two things can be seen directly from these conditions. First, a rise in the interest rate has the benefit of reducing current consumption, but has the cost of raising debt. The first order condition for the interest rate states that these marginal social benefits and losses must be identical in the optimal policy. Second, the Lagrange multiplier for debt, λ^{b}_t, follows a random walk, consistent with Benigno and Woodford (2004) and Schmitt-Grohe and Uribe (2004a). Taken together, these mean that $\lambda^{c}_t = \lambda^{c}_{t+1} = -\lambda^{b}_{t+1}$ for $t > 1$.

Solving the system, the optimal consumption and government spending are:

$$c_t = \frac{\kappa}{\theta}\lambda^{\pi}_{t+1} - \frac{\varphi(1-\theta)}{\sigma+\varphi\theta}g_t - \frac{\sigma}{\theta(\sigma+\varphi\theta)}\lambda^{c}_{t+1} + \frac{\sigma}{\beta\theta(\sigma+\varphi\theta)}\lambda^{c}_{t} + \frac{\tau}{\beta B}\frac{1}{(\sigma+\varphi\theta)}\lambda^{b}_{t+1} \quad (2.54)$$

$$g_t = \frac{\kappa\varphi}{\sigma+\varphi(1-\theta)}\lambda^{\pi}_{t+1} - \frac{\varphi\theta}{\sigma+\varphi(1-\theta)}c_t - \frac{1}{\sigma+\varphi(1-\theta)} \times \frac{(1-\tau)}{\beta B}\lambda^{b}_{t+1}. \quad (2.55)$$

Equations (2.54) and (2.55) equate the marginal cost and marginal benefit of a reduction in consumption and government spending respectively. The first two terms in the optimal consumption and government spending are identical to the optimal policy in the previous model in which government spending is financed by lump sum tax. The additional terms are associated with the Lagrange multipliers of the two additional constraints, the consumption Euler equation and the accumulation of debt. Solving (2.54) and (2.55), and using the fact that $\lambda^{c}_{t} = \lambda^{c}_{t+1} = -\lambda^{b}_{t+1}$ for $t \geq 1$, one can write

$$\begin{aligned} g_t &= \frac{1}{\sigma+\varphi}\left(-\frac{\varphi}{\beta} + \varphi + \frac{\sigma(1-\tau)+\varphi\theta}{\sigma\beta B}\right)\lambda^{c}_{t}, \qquad \text{for } \ t \geq 1, \\ g_t &= \frac{1}{\sigma+\varphi}\left(\varphi + \frac{\sigma(1-\tau)+\varphi\theta}{\sigma\beta B}\right)\lambda^{c}_{t}, \qquad \text{for } \ t = 0. \end{aligned} \quad (2.56)$$

Since λ^{c}_{t+1} follows a random walk, government spending for $t \geq 1$ also follows a random walk, playing no role in stabilising inflation. In the first period, government spending falls more than the long run level because $\lambda^{c}_{0} = 0$ initially, different from its new steady state. This in turn is because the shock is unexpected and the private sector sets $E_{-1}\pi_0 = E_{-1}c_0 = 0$. Leith and Wren-Lewis (2008) also find this result.

It is possible to understand the lack of response of government spending in this model using the arguments in Section 2.2. First, the comparative advantage argument remains valid. Since the Phillips curve is unchanged, it remains true that monetary policy is more efficient than fiscal. Second, the exact cancelation argument also holds in this model. The optimal consumption and government spending rules

in this model Equations (2.54) and (2.55) differ from the ones in the previous model (Equations (2.10) and (2.11)) only by the terms associated with the Lagrange multipliers associated with debt. This means that the optimal rules still have the same partial derivative $\partial c_t/\partial \lambda^{\pi}_{t+1}$. This means that if consumption is cut by precisely the amount that it would be cut if there were no change in government spending, the effect of c_t on the optimal choice of g_t will exactly cancel the effect of λ^{π}_{t+1} on the optimal choice of g_t Therefore, government spending does not respond to a change in the Lagrange multiplier for inflation. However, government does respond to changes in the Lagrange multiplier associated with debt.

After a positive inflation shock, the interest rate rises to control inflation, but the increase in the interest rate raises the government debt. It is not optimal to implement a temporary fiscal austerity (except in the first period) to drive debt back to the initial equilibrium. This is because debt does not enter the loss function directly and, since the loss is quadratic in government spending, the cost of temporarily reducing government spending to reduce debt is larger than the discounted cost of letting government spending gap stay below zero infinitely. Thus, government debt (as well as the Lagrange multiplier λ^{b}_{t+1}) follows a unit root process. The policymaker thus chooses and commits to an interest rate path that balances the need to cut inflation and the cost of raising debt by too much.

In the long run, government debt is higher than the pre-shock level and government spending lower. As consumers hold more government debt, they consume more in the new steady state and supply less labour, and output is lower.

To obtain numerical results, I use the generalised Schur decomposition to solve the model.[7] See Klein (2000) and Appendix 2.8.1. I simulate a one-off 1% positive mark-up shock in the economy. The optimal commitment policy in the low-debt economy is shown in Figure 2.2. The optimal commitment policy in the high-debt

[7]One has to count the unit eigenvalue as a stable one, due to the random-walk property of λ^{c}_{t}.

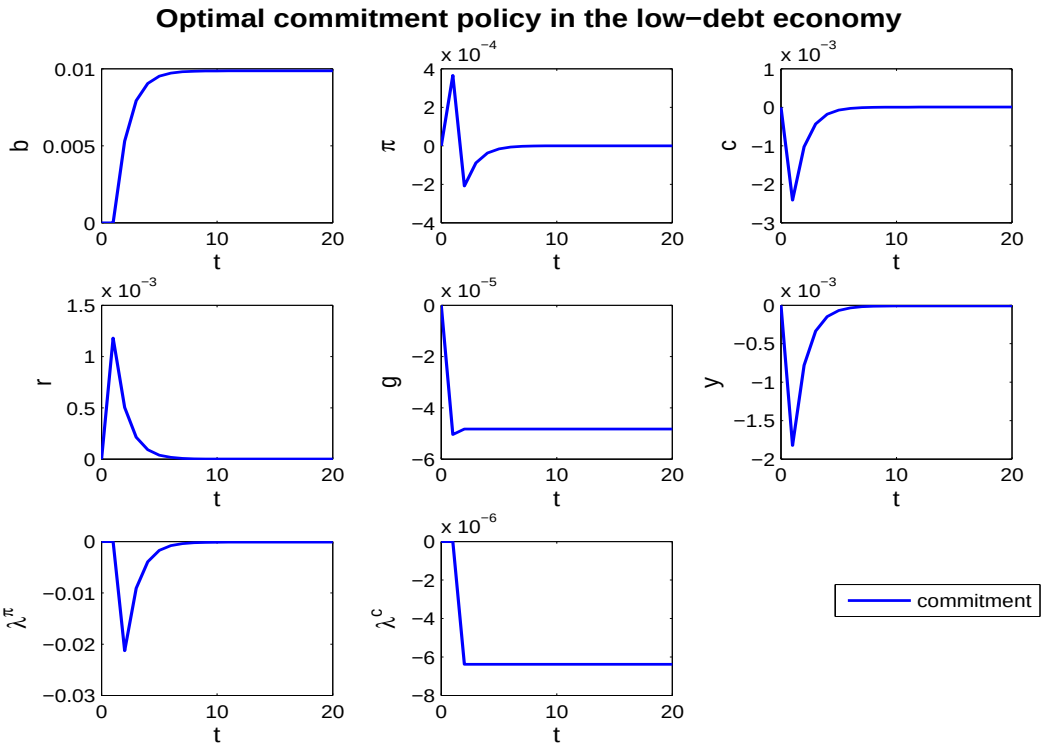


Figure 2.2: Response of a 1% positive mark-up shock in the low-debt economy in a model without inflation persistence under the optimal commitment policy.

economy is shown in Figure 2.3.

The optimal commitment policy in this model is time-inconsistent for two reasons. The first source of time-inconsistency is the standard inflation stabilisation bias of Currie and Levine (1987,1993) and Woodford (2003b), which in this context means that by promising a low inflation in future, the private sector forms low inflation expectation which helps reduce inflation in the current period, but in future periods, after inflation is stabilised, the policymaker has an incentive to renege by not creating the promised recession to bring about a fall in inflation. The second source of time-inconsistency is related to debt and is discussed in Leith and Wren-Lewis (2008). In the new equilibrium in this model under the optimal commitment policy, government spending is too low and consumption too high relative to the original equilibrium before any shock. Hence, the policymaker has an incentive to

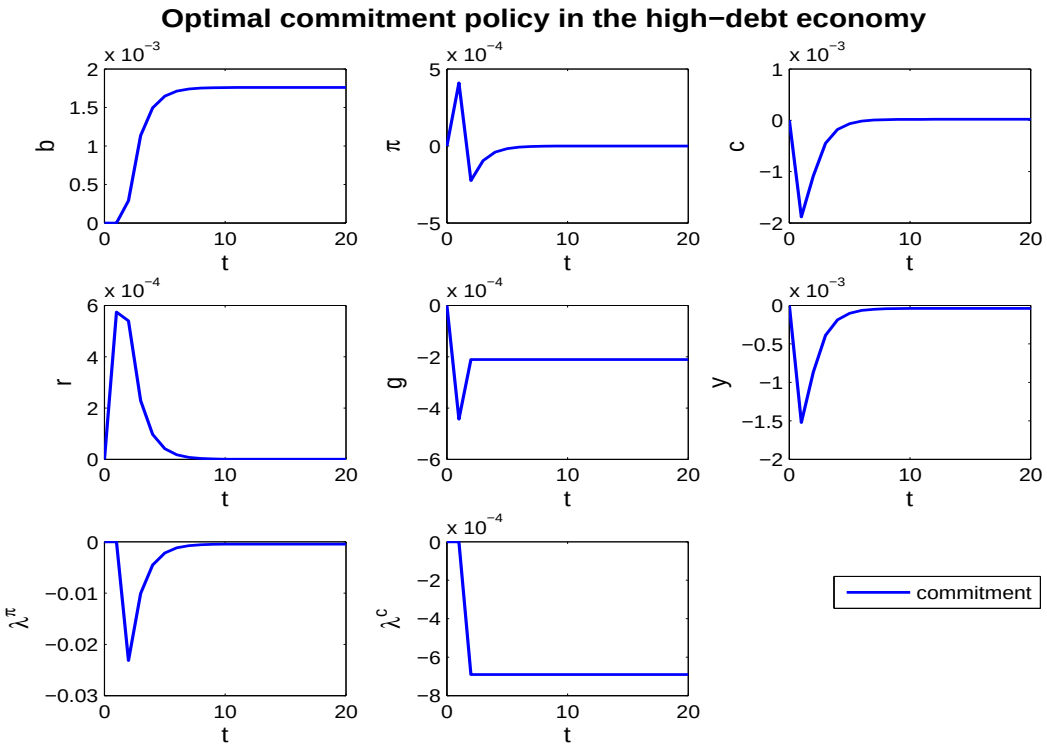


Figure 2.3: Response of a 1% positive mark-up shock in the high-debt economy in a model without inflation persistence under the optimal commitment policy.

inflate away the debt (by cutting the interest rate) unexpectedly in every period in the future. For the policymaker inflating unexpectedly is beneficial: since the private sector would not have expected a high inflation before the implementation of such policy, inflation would not have risen before the renegation.

The temptation to renege can be quantified by the Lagrange multipliers. Once optimal path for current and future monetary and fiscal policies has been found at time $t = 0$, and the Lagrange multipliers to the jump variables are set to zero, *i.e.* $\lambda_0^\pi = \lambda_0^c = 0$, the optimal policy defines a path for these Lagrange multipliers that are not necessarily zero for $t > 1$. This means that, given a chance to re-optimise at time t, the policymaker will renege and choose a path such that $\lambda_t^\pi = \lambda_t^c = 0$. Thus, the magnitude of these Lagrange multipliers captures the extent of the time-inconsistency problem.

From Figure 2.2, the Lagrange multiplier for inflation is the largest in the first period and decays back to a value close to zero over time. The incentive to renege from the optimal path of inflation is the largest in the first period because immediately after the cost-push shock, the policymaker has to promise a low inflation in order to disinflate, but after the shock has dissipated there is a temptation to not implement the low inflation. Subsequently, as inflation moves gradually back to zero, the incentive to renege falls over time, but the Lagrange multiplier for inflation, λ_t^π, does not fall back to exact zero in the new steady state because the policymaker has a temptation to inflate away the debt.

The Lagrange multiplier for consumption, λ_t^c, and the Lagrange multiplier for debt, λ_t^b, follow random walks and stay permanently away from zero. Again, this is because the social benefit of cutting debt, leading to higher permanent government spending and lower consumption, is constant over time. Since the Lagrange multiplier for debt follows a random walk, government debt must be permanently different from the initial level.

The impulse response of the high-debt economy is reported in Figure 2.3. The responses of the high debt economy are similar to the low debt economy. As in the low-debt economy, the policymaker commits to a low inflation in future, which is implemented by creating a downturn. The only important difference is that less debt is accumulated in the high-debt economy. There are two reasons. First, the interest rate rises by less; second, government spending falls by more in the initial period. Despite these policies, the long-run government spending in the high-debt economy is lower, compared with the low-debt economy.

2.4.2 Optimal Policy under Discretion

This subsection studies the optimal discretion policy – something that has been done before by Leith and Wren-Lewis (2008). Under discretion, the policymaker can no longer make time-inconsistent promises about the future. However, the policymaker understands that his decision today affects the state of the economy (the predetermined variables) tomorrow and hence the optimal policy tomorrow. Following Soderlind (2003), I adopt a Bellman equation approach to solve for the optimal discretion policy. The solution method is presented in Appendix 2.8.1.

Figure 2.4 shows the simulations to a 1% positive mark-up shock in the low-debt economy. Discretion policy has to be time-consistent. This has two implications to the model. First, the policymaker cannot influence inflation expectations of the private sector by making time-inconsistent promises, so he has to raise the interest rates strongly in the first period, compared with the optimal policy under commitment. Second, the policymaker cannot promise not to inflate away the debt in future. The only time-consistent solution is one in which the policymaker has no incentive to reduce debt unexpectedly. Hence, in this model debt will not follow a unit root process under discretion, but will return to its pre-shock level.[8] The

[8]This result is first found in Leith and Wren-Lewis (2008).

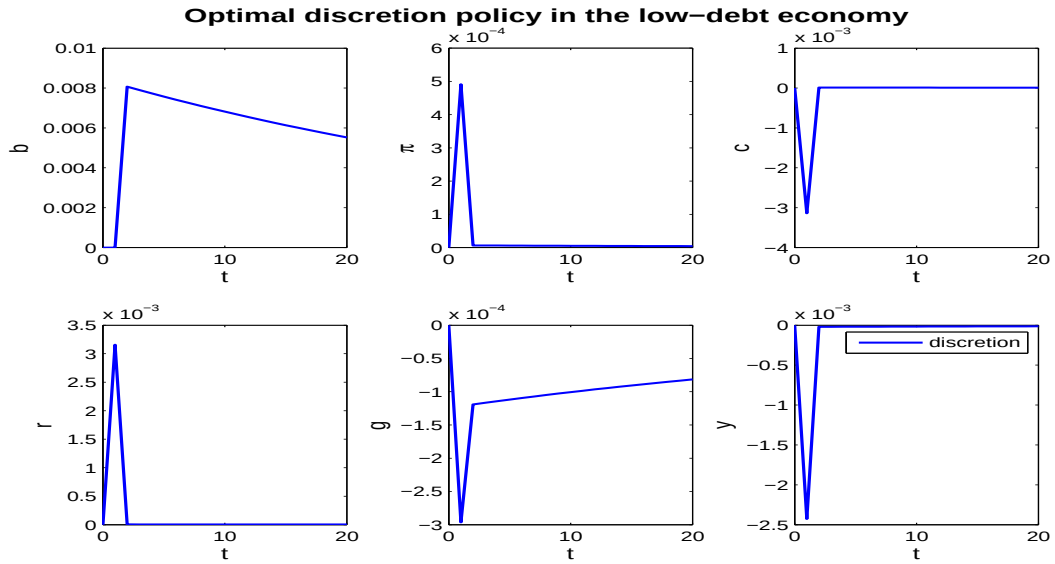


Figure 2.4: Response of a 1% positive mark-up shock in the low-debt economy in a model without inflation persistence under the optimal discretion policy.

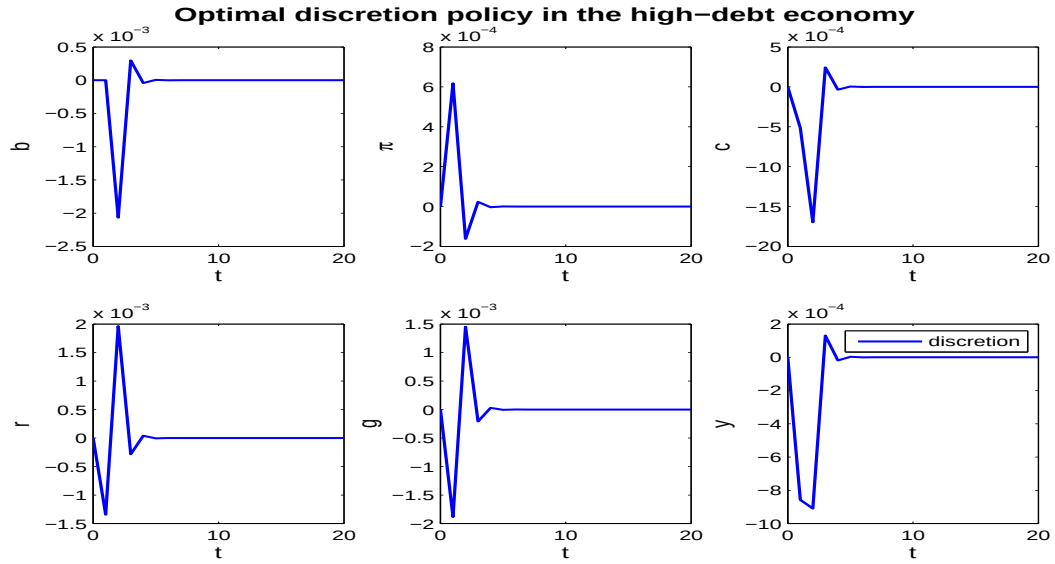


Figure 2.5: Response of a 1% positive mark-up shock in the high-debt economy in a model without inflation persistence under the optimal discretion policy.

reason is the following: if debt is different from its pre-shock level in equilibrium, the welfare loss is larger than the loss in the equilibrium in which debt is at its pre-shock level. Therefore, the policymaker will reduce debt unexpectedly back to the pre-shock level, but this means that debt cannot be different from its pre-shock level in an equilibrium.

As a consequence, in my model the optimal policy under discretion is one in which both monetary policy and fiscal policy will have active roles in stabilising inflation and in steering debt back to its pre-shock level. Otherwise, suppose fiscal policy is inactive. Given the fact that debt has to return to the pre-shock level, fiscal policy must be unchanged after the cost-push shock. In my calibration for the 'low debt' economy for instance, the interest rates will rise initially to control inflation, but this will raise the interest payment for debt and cause debt to rise. In order to reduce debt, the interest rates will have to fall in subsequent periods which create inflation. This policy has large welfare costs because future inflationary expectations make controlling inflation today harder. The discussion above implies that the policymaker in this model will choose to do a significant part of debt adjustment in the first period when the inflationary costs of doing this is the smallest. Therefore, government spending falls substantially in the first period.

In subsequent periods, government spending will move gradually back to zero. The argument is the following. As debt will fall back to the pre-shock level, government spending will eventually return to zero. As argued above, monetary policy alone cannot drive debt back to the pre-shock level because this will be inflationary. Hence, government spending will be low during the adjustment. Due to the convex nature of the loss function, the optimal adjustment path will involve smoothing of government spending across periods.

The optimal discretion policy for the high-debt economy is reported in Figure 2.5. The impulse responses show a zigzag path for all macroeconomic variables. In

response to a positive mark-up shock, government spending falls in period 1. The interest rate in period 1 also *falls*, violating the Taylor principle, as found in Leith and Wren-Lewis (2008) and Stehn and Vines (2007).[9] This is because, as explained, the policymaker needs to bring the deviation of debt back to zero in the long-run and the inflationary cost of reducing debt is the smallest in the first period. In the high-debt economy, fiscal policy is less effective than monetary policy to control debt (From the law of motion for debt, Equation (2.42), a fall in government spending by one unit reduces debt by $(1-\theta)/\beta B$ unit, which is decreasing in the steady state level of debt B.) Hence, the policymaker is forced to cut rate to reduce debt in period 1.

The monetary and fiscal policies together make debt undershoots in period 2. This is to allow for a future rise in debt when the policymaker deals with inflation. Inflation in period 1 does not explode for two reasons. First, government spending, as well as consumption, is low in this period. Second, consumers expect the policymaker in period 2 to raise the interest rate to curb inflation. This expectation is time-consistent.

In period 2, the interest rate rises to control inflation as expected. Curiously, government spending overshoots and debt overshoots. This is to say the dynamics of this system is such that $\hat{B}_{t+1} = const \times \hat{B}_t$ where $const < 0$. In order to understand this counter-intuitive result, compare this optimal path to a path in which debt returns to zero in period 3. This path involves lower government spending in period 2. Along that path, the economy will return to the steady state from period 3 onwards (since debt is the only state variable and the shock is one-shot.). But this path is not optimal because the output in period 2 is too low that the loss

[9] In the simulations of Stehn and Vines (2007), the optimal discretion policy in the 'high-debt' economy violates the Taylor principle, but does not follow a zigzag path. This is because they use a steady-state debt to output ratio of 0.4, which translates to a debt to annual GDP ratio of 10%, which is low.

associated with the low output along this path exceeds the discounted future losses in all subsequent periods along the optimal path.

Admittedly, the optimal discretion policy in this model gives rise to an unrealistic path with oscillations. In the next section, I will show that when inflation persistence is allowed for, this feature will disappear.

2.5 Optimal Policy in the Model with Debt and Backward-looking Price-setters

In this section, I study the model with backward-looking price-setters, *i.e.* $\omega > 0$. Stehn and Vines (2007) analysed this model, but they did not discuss the role of indexation in driving their results. Equipped with the insights from the previous section, I show analytically that when the backward-looking price-setters use a rule which depends on the lagged output gap (*i.e.* $\eta > 0$), the exact cancelation result is overturned and the optimal commitment policy will be one in which both monetary and fiscal policies have active roles in inflation stabilisation.

2.5.1 Calibration of the Model with Inflation Persistence

The parameter values are kept the same as those in Table 2.1. There are two additional variables, ω, the fraction of rule-of-thumb price-setters, and η, the indexation of the rule-of-thumb price-setters on the lagged output gap. I use $\omega = 0.75$ following Stehn and Vines (2007), which follows the empirical results in Mehra (2004) and Rudebusch (2002). I follow Steinsson (2003) to set η so that it is equal to the coefficient on y_t in the Phillips curve when $\omega = 0$. This means that $\eta = \kappa$.

2.5.2 Optimal Policy under Commitment

The policymaker chooses r_t and g_t subject to the constraints Equations (2.39)–(2.42). The loss function is derived in Equation (2.43) following Woodford (2003a) and Steinsson (2003), assuming a labour subsidy makes the steady state efficient. The Lagrangian for the policymaker is:

$$\begin{aligned}
\mathscr{L}_t &= E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[\frac{1}{2} \begin{pmatrix} \sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 \\ + \frac{\epsilon}{\kappa}\pi_s^2 + 2\frac{\epsilon}{\kappa}\phi_1 \Delta\pi_s y_{s-1} + \frac{\epsilon}{\kappa}\phi_2 \Delta\pi_s^2 + \frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2 \end{pmatrix} \right. \\
&\quad + \lambda_{s+1}^{\pi} \begin{pmatrix} \pi_s - \chi^{FL}\beta\pi_{s+1} - \chi^{BL}\pi_{s-1} - (\kappa_c + \theta\kappa_{y0}) c_s \\ -(1-\theta)\kappa_{y0} g_s - \theta\kappa_{y1} c_{s-1} - (1-\theta)\kappa_{y1} g_{s-1} - \kappa_\mu \hat{\mu}_s \end{pmatrix} \\
&\quad + \lambda_s^{y} \left(\sigma\theta c_s + \sigma(1-\theta) g_s - y_s \right) \\
&\quad + \lambda_{s+1}^{c} \left(c_s - c_{s+1} + r_s - \pi_{s+1} \right) \\
&\quad \left. + \lambda_{s+1}^{b} \left(r_s + \frac{1}{\beta} \left(\hat{B}_s - \pi_s + \frac{(1-\theta)}{B} g_s - \frac{\tau}{B} y_s \right) + \hat{\nu}_s - \hat{B}_{s+1} \right) \right]. \quad (2.57)
\end{aligned}$$

The first order conditions are

$$\begin{aligned}
\frac{\partial \mathscr{L}_t}{\partial \pi_t} &= \frac{\epsilon}{\kappa}\pi_t + \frac{\epsilon}{\kappa}\phi_1 y_{t-1} - \beta\frac{\epsilon}{\kappa}\phi_1 y_t + \frac{\epsilon}{\kappa}\phi_2(\pi_t - \pi_{t-1}) \\
&\quad - \frac{\epsilon}{\kappa}\phi_2\beta(\pi_{t+1} - \pi_t) + \lambda_{t+1}^{\pi} - \chi^{FL}\lambda_t^{\pi} - \chi^{BL}\beta\lambda_{t+2}^{\pi} - \frac{1}{\beta}\lambda_t^{c} - \frac{1}{\beta}\lambda_{t+1}^{b} = 0, \quad (2.58) \\
\frac{\partial \mathscr{L}_t}{\partial y_t} &= \varphi y_t + \beta\frac{\epsilon}{\kappa}\phi_1(\pi_{t+1} - \pi_t) + \beta\frac{\epsilon}{\kappa}\phi_3 y_t - \lambda_t^{y} - \frac{\tau}{\beta B}\lambda_{t+1}^{b} = 0, \quad (2.59) \\
\frac{\partial \mathscr{L}_t}{\partial c_t} &= \sigma\theta c_t - (\kappa_c + \theta\kappa_{y0})\lambda_{t+1}^{\pi} - \beta\theta\kappa_{y1}\lambda_{t+2}^{\pi} + \theta\lambda_t^{y} + \sigma\lambda_{t+1}^{c} - \sigma\frac{1}{\beta}\lambda_t^{c} = 0, \quad (2.60) \\
\frac{\partial \mathscr{L}_t}{\partial \hat{B}_t} &= -\beta^{-1}\lambda_{t+1}^{b} + \beta^{-1}\lambda_t^{b} = 0, \quad (2.61) \\
\frac{\partial \mathscr{L}_t}{\partial g_t} &= \sigma(1-\theta) g_t - (1-\theta)\kappa_{y0}\lambda_{t+1}^{\pi} - \beta(1-\theta)\kappa_{y1}\lambda_{t+2}^{\pi} \\
&\quad + (1-\theta)\lambda_t^{y} + \frac{(1-\theta)}{\beta B}\lambda_{t+1}^{b} = 0, \quad (2.62) \\
\frac{\partial \mathscr{L}_t}{\partial r_t} &= \lambda_{t+1}^{c} + \lambda_{t+1}^{b} = 0. \quad (2.63)
\end{aligned}$$

The optimal commitment policy is described by Equations (2.39)– (2.42), (2.58) – (2.63), the shock process and the initial values for the state variables $\hat{B}_0, \pi_{-1}, c_{-1}, \lambda_0^\pi, \lambda_0^c$. The initial values λ_0^π and λ_0^c are set to 0 to ensure optimality.

The first order conditions for government debt and for the interest rates are identical to the ones in the model without inflation persistence. The Lagrange multiplier for government debt, λ_t^b, continues to follow a random walk. Hence, debt will continue to follow a unit root process.

After substituting out the indirect effects of output, the optimal consumption and government spending are

$$c_t = \frac{1}{\theta\left(\sigma+\theta\varphi+\beta\theta\frac{\epsilon}{\kappa}\phi_3\right)}\begin{bmatrix}\left(\kappa_c(1+\frac{\theta\varphi}{\sigma})-\frac{(1-\gamma)\gamma\theta\beta\omega\eta}{\gamma+\omega(1-\gamma+\gamma\beta)}\right)\lambda_{t+1}^\pi+\beta\theta\kappa_{y1}\lambda_{t+2}^\pi\\ -\theta(1-\theta)\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)g_t-\beta\theta\frac{\epsilon}{\kappa}\phi_1(\pi_{t+1}-\pi_t)\\ -\sigma\lambda_{t+1}^c+\sigma\frac{1}{\beta}\lambda_t^c+\frac{\theta\tau}{\beta B}\lambda_{t+1}^b\end{bmatrix} \quad (2.64)$$

$$g_t = \frac{1}{\sigma+(1-\theta)\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)}\begin{bmatrix}\kappa_{y0}\lambda_{t+1}^\pi+\beta\kappa_{y1}\lambda_{t+2}^\pi-\beta\frac{\epsilon}{\kappa}\phi_1(\pi_{t+1}-\pi_t)\\ -\theta\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)c_t-\frac{(1-\tau)}{\beta B}\lambda_{t+1}^b\end{bmatrix}. \quad (2.65)$$

These two equations can be combined to solve for g_t:

$$g_t = \frac{1}{\sigma+\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)}\begin{bmatrix}\left[\kappa_{y0}-\frac{1}{\sigma}\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)\kappa_c\right]\lambda_{t+1}^\pi+\beta\kappa_{y1}\lambda_{t+2}^\pi\\ -\beta\frac{\epsilon}{\kappa}\phi_1(\pi_{t+1}-\pi_t)+\frac{\left[\sigma(1-\tau)+\theta\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)\right]}{\sigma\beta B}\lambda_{t+1}^c\\ +\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)\lambda_{t+1}^c-\frac{1}{\beta}\left(\varphi+\beta\frac{\epsilon}{\kappa}\phi_3\right)\lambda_t^c\end{bmatrix} \quad (2.66)$$

This key equation shows analytically that government spending in this model does have an active role in inflation stabilisation. (Stehn and Vines (2007) do not show this analytically.) As I will explain below, the role of government spending comes from the assumption of the lagged output gap in the rule of thumb of the backward-looking price-setters, Equation (2.24).

The simulation results of the optimal commitment policy in the low-debt econ-

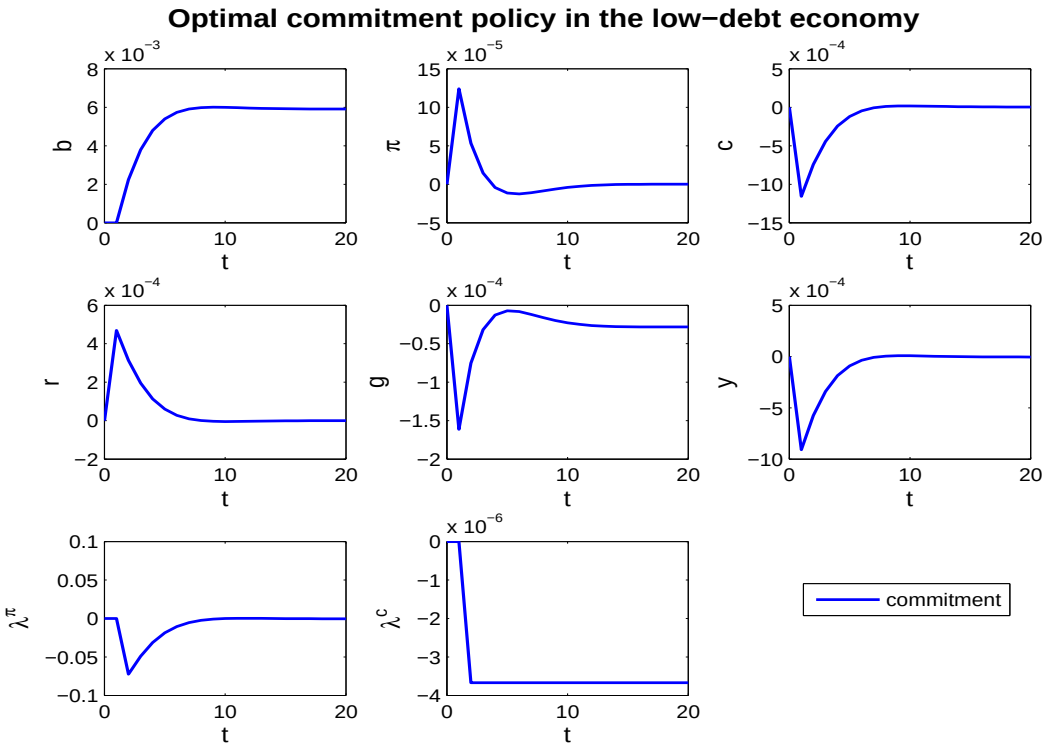


Figure 2.6: Response of a 1% positive mark-up shock in the low-debt economy in a model with inflation persistence under the optimal commitment policy.

omy are reported in Figure 2.6.

The intuitions from the previous models can be applied to understand why government spending has an active role in inflation stabilisation. It remains true that monetary policy is more efficient than government spending in controlling inflation. This can be seen from the first order condition of the forward-looking price-setters, Equation (2.27). A fall in consumption induced by a rise in the interest rates affects inflation through three channels. First, the marginal utility of consumption rises and the real wage falls, *ceteris paribus*. Second, output falls, so labour demand is reduced and real wage is driven down. Third, there is an additional indirect channel through which a fall in output (induced by monetary or fiscal policy) affects the relative forward-looking price, $\hat{p}_t^{FL}$, which is the ratio between the forward-looking prices, P_t^{FL}, and the aggregate price level, P_t. This effect comes from the expected

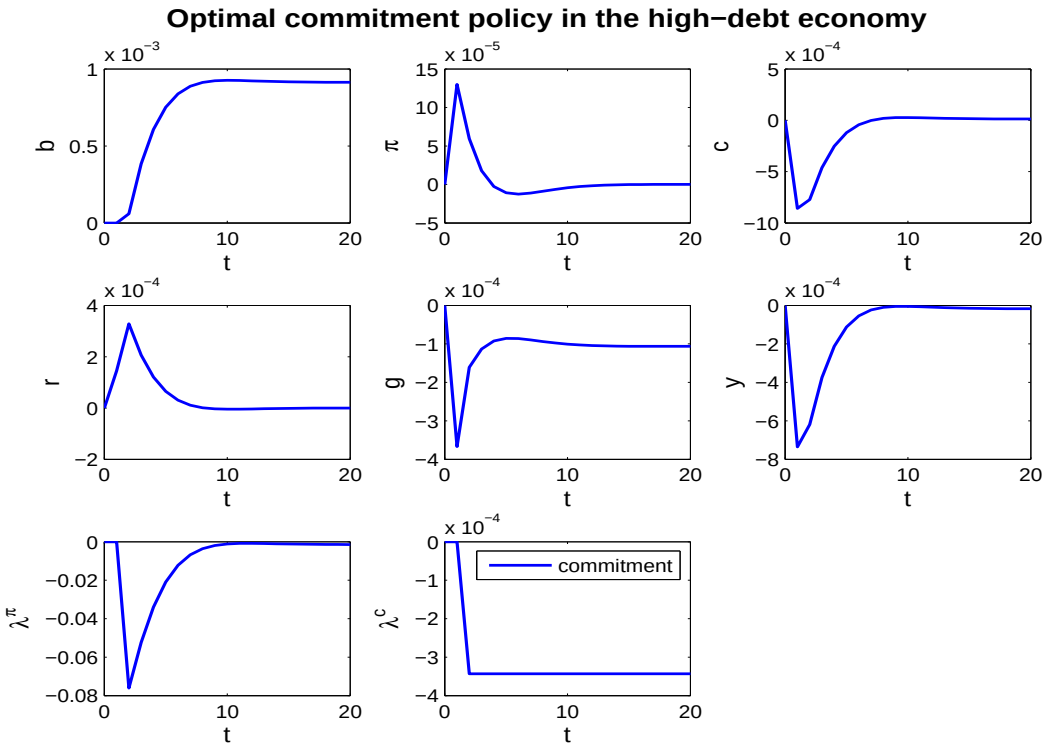


Figure 2.7: Response of a 1% positive mark-up shock in the high-debt economy in a model with inflation persistence under the optimal commitment policy.

relative forward-looking price, $E_t\hat{p}^{FL}_{t+1}$. This is because a fall in output gap at time t reduces the expected backward-looking price in period $t+1$, according to the rule of thumb. This in turn reduces the expected aggregate price level in period $t+1$, and therefore increases the expected relative forward-looking price. This effect, which has not been pointed out elsewhere in the literature, partly offsets the other two effects.

A fall in government spending affects inflation only through the second and third channels – there is no direct effect through labour supply. This means that the effect working through labour supply becomes relatively more important, making monetary policy even more powerful than fiscal policy relative to the previous models.

The fact that government spending under optimal commitment policy is inactive in previous models hinges on the argument that, in the optimal consumption rule, the marginal benefits of reducing inflation exactly cancels the direct marginal costs of a reduction in consumption and the indirect marginal costs of a fall in output caused by the reduction in consumption. This exact cancelation argument is no longer true for this model with inflation persistence for two reasons. First, the marginal benefits of reducing inflation in period t caused by a reduction in consumption in the same period does not exactly cancel with the marginal costs in period t of a reduction in consumption. This is because the relative forward-looking price channel, as explained, reduces the marginal benefit of reducing inflation.

This argument can be understood with reference to the optimal consumption rule, Equation (2.64). The direct marginal costs in period t of a reduction in consumption $\sigma\theta c_t$ and the indirect marginal costs of a fall in output caused by the reduction in consumption $\varphi\theta^2 c_t$. These are the first two terms in the denominator in front of the square brackets. The marginal benefits of reducing inflation in period t caused by a reduction of consumption in the same period is $\left(\kappa_c(1+\frac{\theta\varphi}{\sigma}) - \frac{(1-\gamma)\gamma\theta\beta\omega\eta}{\gamma+\omega(1-\gamma+\gamma\beta)}\right)\lambda^{\pi}_{t+1}$, where $\kappa_c\lambda^{\pi}_{t+1}$ is the labour supply effect which means that a fall in consumption in-

creases the marginal utility of consumption and reduce the real wage, $\kappa_c\theta\varphi\lambda^{\pi}_{t+1}$ is the demand effect which means that a reduction in output leads to lower labour demand and lower real wage, and $-\frac{(1-\gamma)\gamma\theta\beta\omega\eta}{\gamma+\omega(1-\gamma+\gamma\beta)}\lambda^{\pi}_{t+1}$ is the effect working through the relative forward-looking price. The last effect prevents the exact cancelation of marginal benefits and marginal costs.

Second, there are terms associated with the lagged output gap in both the Phillips curve and the social loss function. This is because the price indexation schedule, Equation (2.24), depends on lagged output gap, y_{t-1}. A rise in the lagged output gap in period $t-1$ increases the backward-looking price in period t and hence both inflation and price dispersion. The intertemporal effects – the discounted marginal benefit of a fall in inflation in period $t+1$ caused by a reduction in consumption in period t and the discounted marginal cost of a negative output gap in period $t+1$ caused by a reduction in consumption in period t – do not cancel in the optimal commitment policy. In the optimal consumption rule, Equation (2.64), these terms refer to terms associated with coefficients κ_{y1}, ϕ_1 and ϕ_3. Hence, fiscal policy becomes helpful to inflation stabilisation.

From these arguments, the indexation of the lagged output gap is essential to give fiscal policy a role in inflation stailibsation. Having backward-looking price-setters *per se* does not necessarily give rise to active fiscal policy. To see this, suppose the backward-looking price-setters do not index on past output, that is, $\eta = 0$. Then in the price setting rule for the forward-looking price-setters, Equation (2.27), the relative forward-looking price no longer depends on the lagged output, which means that a fall in consumption only affects inflation through the labour supply effect and the demand effect. Moreover, the price dispersion only depends on present and past inflation, so the loss function in each period does not depend on lagged output. (To see this, substitute $\eta = 0$ into the expression for price dispersion (2.44).) In this case, the exact cancellation argument applies and there is no active role in fiscal

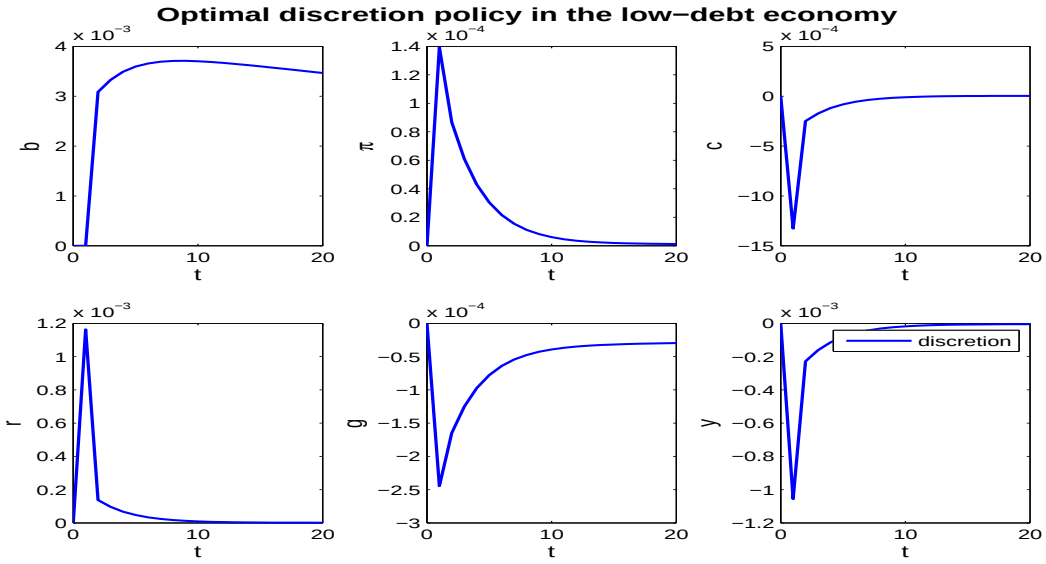


Figure 2.8: Response of a 1% positive mark-up shock in the low-debt economy in a model with inflation persistence under the optimal discretion policy.

stabilisation. This point is confirmed analytically: when $\eta = 0$, $\kappa_{y1} = \phi_1 = \phi_3 = 0$ and the optimal rules reduce to (2.54) and (2.55).

Figure 2.7 reports the simulation results of the high-debt economy. The intuitions are similar to the low-debt economy. As before, in the high-debt economy, government spending falls by more initially and the interest rate rises by less so that less debt (in terms of the deviation from the steady state level of debt) is accumulated.

2.5.3 Optimal Policy under Discretion

The numerical solution for the optimal policy under discretion is again solved using the Bellman equation approach. Figure 2.8 shows the simulation results of the model with inflation persistence under the optimal discretion policy in the low-debt economy. Figure 2.9 shows the simulation results in the high-debt economy.

Compared with the optimal policy in a model without inflation persistence (Figure 2.4), the evolution of the interest rates, government spending, inflation and other

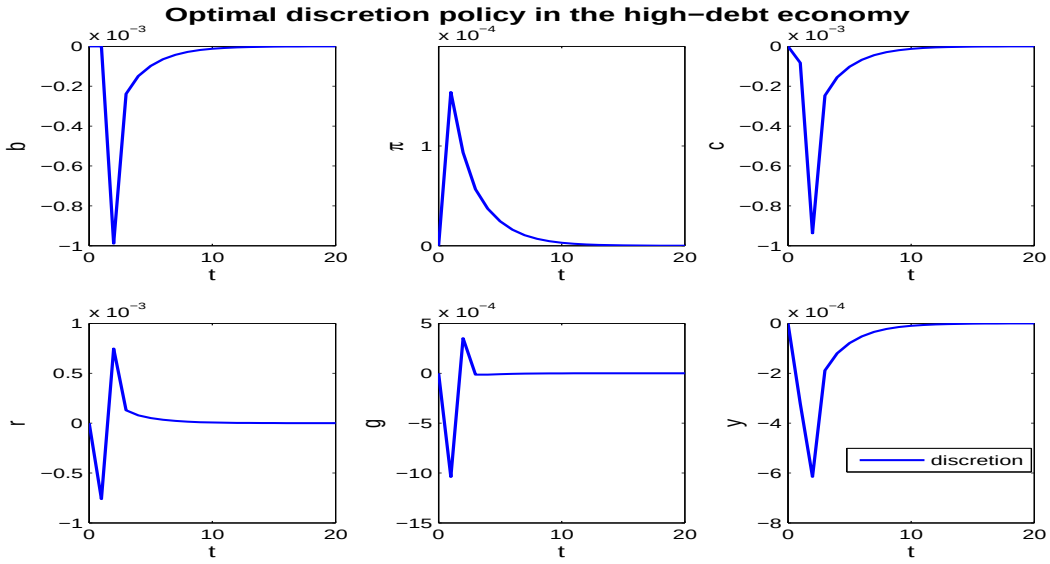


Figure 2.9: Response of a 1% positive mark-up shock in the high-debt economy in a model with inflation persistence under the optimal discretion policy.

macroeconomic variables are more gradual in the model with inflation persistence. Backward-looking price-setters in the model makes inflation harder to control. As the interest rate remains high after the shock and returns to the equilibrium only gradually, debt will continue to accumulate and peak at about 10 periods after the shock before it returns very gradually to its pre-shock level. This is in contrast to the model without inflation persistence in which debt rises as the shock hits and falls back gradually afterwards.

In the high-debt economy (Figure 2.9), the interest rate and government spending fall in the first period. This feature is also present in the model with no inflation persistence. As explained, the violation of the Taylor principle is crucial to bring debt down in the first period so that debt can rise back to the steady state when the interest rate rises subsequently to control inflation. With inflation persistence, the fall in inflation is gradual, and therefore, the interest rate remains contractionary after the first period. The convergence path is smooth after the first period, contrary to the oscillating path in the model without inflation persistence.

2.6 A Model with Capital Accumulation

In this section, I expand the model to allow for endogenous capital accumulation. The other features of the model are unchanged. The purpose is to show that the conclusions in the previous sections still hold true in this more general model.

2.6.1 The Model

Following the previous section, the utility of the homogeneous consumers is given by Equation (2.16). Consumers supply labour, consume, save by buying government bonds and invest to accumulate capital. The budget constraint of the representative consumer is modified to allow for capital accumulation:

$$P_t I_t + P_t C_t + E_t(Q_{t,t+1} D_{t+1}) = D_t + (1-\tau)(W_t N_t + R_t^K K_t + \Omega_t) - T_t, \quad (2.67)$$

where K_t denotes the aggregate capital stock and I_t denotes investment. R_t^K denotes the return on capital through renting the capital to the firms. The other variables are defined as in the model in the previous section.

Capital investment is now subject to a capital adjustment cost. The formulation of the adjustment cost is due to Hayashi (1982).[10] The capital accumulation equation is:

$$K_{t+1} = (1-\delta)K_t + J\left(\frac{I_t}{K_t}\right)K_t, \quad (2.68)$$

where δ is the capital depreciation rate. The function J is a quadratic adjustment cost function which satisfies $J(\delta) = \delta, J'(\delta) = 1$ and J'' is a negative constant. It turns out that the capital adjustment cost affects the optimal policy in an important

[10] I choose the Hayashi (1982) capital adjustment cost instead of following Cheng (2009) for two reasons. First, the Hayashi (1982) formulation is common in the macro-DSGE literature. Second, Wang and Wen (2010) show that the adjustment cost can be microfounded by *ex ante* homogeneous firms facing idiosyncratic investment efficiency shocks and subject to Kiyotaki and Moore (1997)-type collateral constraints and irreversible investment. They also show that the microfounded adjustment cost matches both firm-level and aggregate data well.

way. See below for the details.

Consumers maximise the utility subject to the budget constraint and the capital accumulation equation. The first order conditions for consumers' utility maximisation problem are the usual consumption Euler equation and labour supply curve (Equations (2.18) and (2.19), plus the following first order condition for the investment decision:

$$E_t\left(R_t\frac{P_t}{P_{t+1}}\right) = J'\left(\frac{I_t}{K_t}\right)E_t\left[\frac{(1-\tau)R^K_{t+1}}{P_{t+1}} + \frac{1-\delta+J\left(\frac{I_{t+1}}{K_{t+1}}\right)-\frac{I_{t+1}}{K_{t+1}}J'\left(\frac{I_{t+1}}{K_{t+1}}\right)}{J'\left(\frac{I_{t+1}}{K_{t+1}}\right)}\right] \tag{2.69}$$

The optimal decision for investment equates the marginal costs and benefits of acquiring an additional unit of investment. The marginal cost of investment is the forgone consumption in the current period. The marginal benefits are the value of the after-tax rental income in the next period, plus the value of next period's capital which accounts for depreciation and the change in capital adjustment costs.

On the production side, I assume the intermediate goods firms are monopolistic competitive. Each firm i produces with capital and labour. The production function for firm i is

$$Y_t(i) = A_t N_t(i)^{1-\alpha} K_t(i)^{\alpha}. \tag{2.70}$$

The productivity shock process is given by Equation (2.21).

Aggregate labour is defined as $N_t \equiv \int_0^1 N_t(i)di$. There is a final goods firm which mixes intermediate goods using a Dixit-Stiglitz (1977) aggregator identical to Equation (2.22). The demand for each type of good is given by Equation (2.23).

In each period, each intermediate goods firm chooses labour and capital input to maximise profit. The profit of firm i in period t is

$$\Omega_t(i) = P_t(i)Y_t(i) - \frac{1}{\mu_w}(W_t N_t(i) + R^K_t K_t(i)). \tag{2.71}$$

Firms receive a steady-state subsidy μ_w from the government which ensures that the steady state is efficient. The subsidy is paid for by a lump sum tax, T_t. I denote the Lagrange multiplier to the production function as $P_t MC_t(i)$, where $MC_t(i)$ also has the interpretation as the real marginal cost of firm i. The labour and capital demands are

$$N_t(i) = (1-\alpha)\mu_w MC_t(i)\frac{Y_t(i)}{W_t/P_t}, \tag{2.72}$$

$$K_t(i) = \alpha\mu_w MC_t(i)\frac{Y_t(i)}{R_t^K/P_t}. \tag{2.73}$$

Since the factor prices are equal across firms, the capital-to-labour ratio is identical across firms because

$$\frac{N_t(i)}{K_t(i)} = \frac{(1-\alpha)R_t^K/P_t}{\alpha W_t/P_t} = \frac{N_t}{K_t}, \tag{2.74}$$

where $K_t \equiv \int_0^1 K_t(i)di$.

The marginal cost is also identical across firms, such that

$$MC_t(i) = \frac{\left(\frac{W_t}{P_t}\right)^{1-\alpha}\left(\frac{R_t^K}{P_t}\right)^{\alpha}}{A_t\mu_w(1-\alpha)^{1-\alpha}\alpha^{\alpha}} \equiv MC_t. \tag{2.75}$$

The aggregate production function is

$$Y_t = \frac{A_t K_t^{\alpha} N_t^{1-\alpha}}{\Delta_t}, \tag{2.76}$$

where $\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon_t} di$ is the price dispersion.[11] Hence, when prices are more dispersed, that is when $Var_i(P_t(i))$ rises, the price dispersion Δ_t also increases. Conditional on the aggregate capital and labour, the aggregate output decreases. Price distortion causes some firms to produce more than the others and hence reduces

[11] To derive the aggregate production function, integrate the demand for goods, Equation (2.23), and the production function for each variety of goods across all varieties, noting that the capital-to-labour ratio is constant across firms.

welfare relative to the flexible price equilibrium.

Price-setting is modelled as a mix of Calvo (1983) contracting and rule-of-thumb behaviour as in Section 2.3.1. The average price set by the firms that can reset their price P_t^* is given by Equation (2.25). The backward-looking agents use Equation (2.24) as their rule of thumb, following Steinsson (2003). The inclusion of backward-looking behaviour and indexation behaviour makes my model more general than Cheng (2009), and the indexation turns out to have important effects to the optimal fiscal policy under commitment. The forward-looking price-setters solve the first order conditions for profit maximisation and obtain the optimal solution. The Lagrangian for the price setting problem is:

$$\begin{aligned} Lagr_t &= E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} \left(\frac{P_t(i)}{P_{t+s}} \left(\frac{P_t(i)}{P_{t+s}} \right)^{-\epsilon_{t+s}} Y_{t+s} - \frac{1}{\mu_w} \left(\frac{W_{t+s}}{P_{t+s}} N_{t+s}(i) + \frac{R^K_{t+s}}{P_{t+s}} K_{t+s}(i) \right) \right. \\ &\quad \left. + MC_{t+s} \left(A_{t+s} N_{t+s}(i)^{1-\alpha} K_{t+s}(i)^{\alpha} - \left(\frac{P_t(i)}{P_{t+s}} \right)^{-\epsilon_{t+s}} Y_{t+s} \right) \right). \end{aligned} \tag{2.77}$$

Since there is no state variable in the price-setting problem, all forward-looking price-setters choose the same price P_t^{FL} to satisfy the following first order condition:

$$E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} \left(\frac{P_t^{FL}}{P_{t+s}} \right)^{-\epsilon_{t+s}} Y_{t+s} \left(\frac{P_t^{FL}}{P_{t+s}} - \mu_{t+s} MC_{t+s} \right) = 0, \tag{2.78}$$

where $\mu_s = \epsilon_s / (\epsilon_s - 1)$.

In Appendix 2.8.2, I show that the log-linearised first order condition is

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta) \left(\sigma c_t + \frac{\alpha+\varphi}{1-\alpha} y_t - \frac{\alpha(1+\varphi)}{1-\alpha} k_t + \hat{\mu}_t \right), \tag{2.79}$$

where $\hat{p}_t^{FL} = \hat{P}_t^{FL} - \hat{P}_t$ is the forward-looking price relative to the aggregate price level, $\pi_t = \hat{P}_t - \hat{P}_{t-1}$ is inflation. This equation is analogous to Equation (2.27) in the model without capital. It shows that monetary policy is more effective than

fiscal policy in stabilising inflation.

The Phillips curve can be written as

$$\pi_t = \chi^{FL}\beta E_t\pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c c_t + \kappa_{y0}y_t + \kappa_{y1}y_{t-1} - \kappa_k k_t + \kappa_\mu \hat{\mu}_t, \tag{2.80}$$

where

$$\begin{aligned}
\chi^{FL} &= \frac{\gamma}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \chi^{BL} = \frac{\omega}{\omega(1-\gamma+\gamma\beta)+\gamma}, \\
\kappa_c &= \frac{\sigma(1-\gamma)(1-\omega)(1-\gamma\beta)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \kappa_{y0} = \frac{1}{\sigma}\frac{\alpha+\varphi}{1-\alpha}\kappa_c - \frac{\gamma\beta\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \\
\kappa_{y1} &= \frac{\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \kappa_k = \frac{\alpha(1+\varphi)}{\sigma(1-\alpha)}\kappa_c, \qquad \kappa_\mu = \frac{1}{\sigma}\kappa_c.
\end{aligned}$$

The cost-push shock process is given by Equation (2.28).

Goods are demanded for consumption, government spending and investment in this model. This means that

$$Y_t = C_t + I_t + G_t. \tag{2.81}$$

In the steady state assume $C = \theta_c Y$, $G = \theta_g Y$ and $I = \theta_i Y$ where $\theta_c, \theta_g, \theta_i$ are the shares of private consumption, government spending and investment in output.

The government buys goods, taxes income with a constant income tax rate τ and issues nominal debt $\bar{B}$. In each period, the production subsidy is fully funded by the lump-sum tax. The evolution of nominal debt is identical to Equation (2.31) and the log-linearised version is

$$\hat{B}_{t+1} = \hat{R}_t + \frac{1}{\beta}\left(\hat{B}_t - \pi_t + \frac{\theta_g}{B}\hat{G}_t - \frac{\tau}{B}\hat{Y}_t\right). \tag{2.82}$$

This completes the description of the model with capital accumulation.

2.6.2 The Case of Flexible Price and the System in 'Gap' Form

In order to study the welfare loss caused by the distortions associated with nominal rigidities, we write the system with nominal rigidities in the form of deviation from the efficient flexible price level. To do this, we first need to solve for the system with flexible prices.

In the case of flexible price, the social planner maximises utility, Equation (2.16), subject to the goods market clearing condition, the evolution of capital and the production function. The first order conditions are

$$(C_t^n)^{-\sigma} = \chi (G_t^n)^{-\sigma}, \tag{2.83}$$

$$(C_t^n)^{\sigma} = (1-\alpha)\frac{Y_t^n}{(N_t^n)^{1+\varphi}}, \tag{2.84}$$

$$E_t \frac{\left(C_{t+1}^n\right)^{\sigma}}{\beta (C_t^n)^{\sigma}} = J'\left(\frac{I_t^n}{K_t^n}\right) E_t \left[\alpha \frac{Y_{t+1}^n}{K_{t+1}^n} + \frac{1-\delta+J\left(\frac{I_{t+1}^n}{K_{t+1}^n}\right) - \frac{I_{t+1}^n}{K_{t+1}^n} J'\left(\frac{I_{t+1}^n}{K_{t+1}^n}\right)}{J'\left(\frac{I_{t+1}^n}{K_{t+1}^n}\right)}\right] \tag{2.85}$$

where the superscript 'n' denotes the efficient flexible price level. I follow Neiss and Nelson (2003) to define the natural level of output such that the natural level of output in a given period depends on the capital stock consistent with the flexible-price model (rather than the capital stock that actually exists in the economy in that period, as proposed by Woodford (2003a)). Edge (2003) and Cheng (2009) show that with this assumption, the second-order approximated utility-based welfare function can be completely characterised in terms of the 'gap' variables.

Next, we log-linearise the full flexible price system above. The log-linearised

system is

$$\hat{C}_t^n = \hat{G}_t^n, \tag{2.86}$$

$$\hat{Y}_t^n = \sigma\hat{C}_t^n + (1+\varphi)\hat{N}_t^n, \tag{2.87}$$

$$\sigma\hat{C}_t^n = E_t\left[\sigma\hat{C}_{t+1}^n - J''\Delta\hat{K}_{t+1}^n + \beta J''\Delta\hat{K}_{t+2}^n - (1-(1-\delta)\beta)\left(\hat{Y}_{t+1}^n - \hat{K}_{t+1}^n\right)\right], \tag{2.88}$$

$$\hat{Y}_t^n = \hat{A}_t + \alpha\hat{K}_t^n + (1-\alpha)\hat{N}_t^n, \tag{2.89}$$

$$\hat{Y}_t^n = \theta_c\hat{C}_t^n + \theta_i\hat{I}_t^n + \theta_g\hat{G}_t^n, \tag{2.90}$$

$$\hat{K}_{t+1}^n = (1-\delta)\hat{K}_t^n + \delta\hat{I}_t^n. \tag{2.91}$$

This set of equations can be solved using standard methods (such as the generalised-Schur decomposition) to yield the transition paths of the macroeconomic variables in the following form:

$$\begin{aligned}\hat{K}_{t+1}^n &= \psi_K\hat{K}_t^n + \zeta_K\hat{A}_t,\\ \hat{X}_t^n &= \psi_X\hat{K}_t^n + \zeta_X\hat{A}_t,\end{aligned}$$

where $X_t = \{Y_t,\ C_t,\ I_t,\ N_t,\ G_t\}$, and $\hat{A}_t$ follows an exogenous shock process. Note that the capital stock and the productivity are the only predetermined variables in the system. Finally, the natural rate of interest $\hat{R}_t^n$ has to equal to the ratio of marginal utilities of consumption:

$$\hat{R}_t^n = \sigma\hat{C}_t^n - \sigma E_t\hat{C}_{t+1}^n = \sigma\psi_C(1-\psi_K)\hat{K}_t^n + \sigma[\zeta_C(1-\rho_A) - \psi_C\zeta_K]\hat{A}_t. \tag{2.92}$$

After some manipulations, the model with nominal rigidities can be written in

terms of the deviations from the natural levels as follows (Define $x_t \equiv \hat{X}_t - \hat{X}_t^n$):

$$
\begin{aligned}
0 &= \sigma c_t - \sigma E_t c_{t+1} + (r_t - E_t \pi_{t+1}), && (2.93)\\
0 &= (1-(1-\delta)\beta) E_t \left(\frac{1+\varphi}{1-\alpha} y_{t+1} + \sigma c_{t+1} - \frac{1+\varphi\alpha}{1-\alpha} k_{t+1} \right) \\
&\quad + J'' E_t(\Delta k_{t+1}) - \beta J'' E_t(\Delta k_{t+2}) - (r_t - E_t \pi_{t+1}), && (2.94)\\
\pi_t &= \chi^{FL} \beta E_t \pi_{t+1} + \chi^{BL} \pi_{t-1} + \kappa_c c_t + \kappa_{y0} y_t + \kappa_{y1} y_{t-1} - \kappa_k k_t + \kappa_c \hat{\mu}_t, && (2.95)\\
y_t &= \theta_c c_t + \theta_i \left(\frac{1}{\delta} k_{t+1} - \frac{(1-\delta)}{\delta} k_t \right) + \theta_g g_t, && (2.96)\\
\hat{B}_{t+1} &= r_t + \frac{1}{\beta} \left(\hat{B}_t - \pi_t + \frac{\theta_g}{B} g_t - \frac{\tau}{B} y_t \right) + \hat{\nu}_t. && (2.97)
\end{aligned}
$$

where $\hat{\nu}_t = \left(\sigma \psi_C (1-\psi_K) + \frac{\theta_g}{\beta B} \psi_C - \frac{\tau}{\beta B} \psi_Y \right) \hat{K}_t^n + \left(\sigma \left(\zeta_C (1-\rho_A) - \psi_C \zeta_K \right) + \frac{\theta_g}{\beta B} \zeta_C - \frac{\tau}{\beta B} \zeta_Y \right) \hat{A}_t$.

2.6.3 Social Welfare Function

I derive a welfare-based loss function for the policymaker using a second-order approximation of the utility function around the efficient steady state. The derivation, given in Appendix 2.8.3, simplifies substantially Cheng's (2009) procedure in approximating the capital adjustment cost to the second order. The loss function is

$$
\begin{aligned}
L_t &= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + \frac{\alpha+\varphi}{(1-\alpha)} y_s^2 \\ -2\frac{\alpha(1+\varphi)}{(1-\alpha)} y_s k_s + \frac{\alpha(1+\alpha\varphi)}{(1-\alpha)} k_s^2 - \frac{\theta_i}{\delta} J'' (\Delta k_{s+1})^2 \\ + \frac{\epsilon}{\kappa} \pi_s^2 + 2\frac{\epsilon}{\kappa} \phi_1 (\Delta \pi_s) y_{s-1} + \frac{\epsilon}{\kappa} \phi_2 (\Delta \pi_s)^2 + \frac{\epsilon}{\kappa} \phi_3 y_{s-1}^2 \end{pmatrix} \\
&\quad + t.i.p + O(3), && (2.98)
\end{aligned}
$$

where

$$
\kappa = \frac{(1-\gamma\beta)(1-\gamma)}{\gamma}, \qquad \phi_1 = -\frac{\omega\eta(1-\gamma)}{\gamma(1-\omega)}, \qquad \phi_2 = \frac{\omega}{\gamma(1-\omega)}, \qquad \phi_3 = \frac{(1-\gamma)^2 \omega \eta^2}{\gamma(1-\omega)}.
$$

Table 2.2: Parameter values in the model with capital.

Parameter	β	σ	φ	ϵ	γ	θ_c	θ_g	α	ω	δ	J''	B	ρ_μ
Value	0.99	1	0.5	5	0.75	0.56	0.2	0.34	0.75	0.025	-10	0.1	0

This social welfare function is general enough to encompass a model without capital. By setting $\alpha = 0$ so that $n_t = y_t$, the social welfare function reduces to Equation (2.43). It also encompasses a model without backward-looking price-setters, when $\omega = 0$.

2.6.4 Calibration

The parameter values are given in Table 2.2. I follow the parameter values in the model without capital whenever possible. With capital and investment, the share of consumption and government spending to output are set to be 0.56 and 0.2 respectively. These numbers are the same as Cheng (2009). The share of capital in the production function is set to 0.34. Capital depreciates at a rate of $\delta = 2.5\%$ per quarter and the curvature of the capital adjustment cost, J'' is set to -10, which is the same as Bernanke, Gertler and Gilchrist (1999).

2.6.5 Optimal Policy under Commitment

The policymaker choose r_t and g_t subject to the above constraints. The Lagrangian for the policymaker is:

$$\begin{aligned}
\mathscr{L}_t \;=\; & E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[\frac{1}{2} \begin{pmatrix} \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + \frac{\alpha+\varphi}{(1-\alpha)} y_s^2 \\ -2\frac{\alpha(1+\varphi)}{(1-\alpha)} y_s k_s + \frac{\alpha(1+\alpha\varphi)}{(1-\alpha)} k_s^2 - \frac{\theta_i}{\delta} J'' \left(\Delta k_{s+1}\right)^2 \\ +\frac{\epsilon}{\kappa}\pi_s^2 + 2\frac{\epsilon}{\kappa}\phi_1 \left(\Delta\pi_s\right) y_{s-1} + \frac{\epsilon}{\kappa}\phi_2 \left(\Delta\pi_s\right)^2 + \frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2 \end{pmatrix} \right. \\
& +\lambda_{s+1}^{\pi} \left(\pi_s - \chi^{FL}\beta\pi_{s+1} - \chi^{BL}\pi_{s-1} - \kappa_c c_s - \kappa_{y0} y_s - \kappa_{y1} y_{s-1} + \kappa_k k_s - \kappa_\mu \hat{\mu}_s \right) \\
& +\lambda_s^{y} \left(\theta_c c_s + \theta_g g_s - \frac{\theta_i(1-\delta)}{\delta} k_s + \frac{\theta_i}{\delta} k_{s+1} - y_s \right) \\
& +\lambda_{s+1}^{c} \left(\sigma c_s - \sigma c_{s+1} + r_s - \pi_{s+1} \right) \\
& +\lambda_{s+1}^{b} \left(r_s + \frac{1}{\beta} \left(\hat{B}_s - \pi_s + \frac{\theta_g}{B} g_s - \frac{\tau}{B} y_s \right) + \hat{\nu}_s - \hat{B}_{s+1} \right) \\
& \left. +\lambda_{s+2}^{k} \begin{pmatrix} (1-(1-\delta)\beta) \left(\frac{1+\varphi}{1-\alpha} y_{s+1} + \sigma c_{s+1} - \frac{1-\alpha\varphi}{1-\alpha} k_{s+1} \right) \\ +(1+\beta) J'' k_{s+1} - J'' k_s - r_s + \pi_{s+1} - \beta J'' k_{s+2} \end{pmatrix} \right].
\end{aligned} \tag{2.99}$$

The first order conditions are:

$$\begin{aligned}
\frac{\partial \mathscr{L}_t}{\partial \pi_t} &= \frac{\epsilon}{\kappa}\pi_t + \frac{\epsilon}{\kappa}\phi_1 y_{t-1} - \beta\frac{\epsilon}{\kappa}\phi_1 y_t + \frac{\epsilon}{\kappa}\phi_2\Delta\pi_t - \beta\frac{\epsilon}{\kappa}\phi_2\Delta\pi_{t+1} + \lambda^{\pi}_{t+1} \\
&\quad -\chi^{FL}\lambda^{\pi}_t - \beta\chi^{BL}\lambda^{\pi}_{t+2} - \beta^{-1}\lambda^{c}_t - \beta^{-1}\lambda^{b}_{t+1} + \beta^{-1}\lambda^{k}_{t+1} = 0, \qquad (2.100) \\
\frac{\partial \mathscr{L}_t}{\partial y_t} &= \frac{\alpha+\varphi}{1-\alpha}y_t - \frac{\alpha(1+\varphi)}{1-\alpha}k_t + \beta\frac{\epsilon}{\kappa}\phi_1\Delta\pi_{t+1} + \beta\frac{\epsilon}{\kappa}\phi_3 y_t - \kappa_{y0}\lambda^{\pi}_{t+1} \\
&\quad -\beta\kappa_{y1}\lambda^{\pi}_{t+2} - \lambda^{y}_t - \frac{\tau}{\beta B}\lambda^{b}_{t+1} + \frac{(1-(1-\delta)\beta)}{\beta}\left(\frac{1+\varphi}{1-\alpha}\right)\lambda^{k}_{t+1} = 0, \qquad (2.101) \\
\frac{\partial \mathscr{L}_t}{\partial c_t} &= \sigma\theta_c c_t - \kappa_c\lambda^{\pi}_{t+1} + \sigma\theta_c\lambda^{y}_t + \sigma\lambda^{c}_{t+1} - \frac{\sigma}{\beta}\lambda^{c}_t + \frac{(1-(1-\delta)\beta)}{\beta}\lambda^{k}_{t+1} = 0, \qquad (2.102) \\
\frac{\partial \mathscr{L}_t}{\partial k_t} &= -\frac{\alpha(1+\varphi)}{1-\alpha}y_t + \frac{\alpha(1+\alpha\varphi)}{1-\alpha}k_t + \kappa_k\lambda^{\pi}_{t+1} - \frac{\theta_i(1-\delta)}{\delta}\lambda^{y}_t + \beta^{-1}\frac{\theta_i}{\delta}\lambda^{y}_{t-1} \\
&\quad -J''\lambda^{k}_{t+2} + \beta^{-1}(1+\beta)J''\lambda^{k}_{t+1} - \beta^{-1}J''\lambda^{k}_t = 0, \qquad (2.103) \\
\frac{\partial \mathscr{L}_t}{\partial \hat{B}_t} &= \beta^{-1}\lambda^{b}_{t+1} - \beta^{-1}\lambda^{b}_t = 0, \qquad (2.104) \\
\frac{\partial \mathscr{L}_t}{\partial r_t} &= \lambda^{c}_{t+1} + \lambda^{b}_{t+1} - \lambda^{k}_{t+2} = 0, \qquad (2.105) \\
\frac{\partial \mathscr{L}_t}{\partial g_t} &= \sigma\theta_g g_t + \theta_g\lambda^{y}_t + \frac{\theta_g}{\beta B}\lambda^{b}_{t+1} = 0. \qquad (2.106)
\end{aligned}$$

The solution for the optimal commitment policy is fully described by Equations (2.93) – (2.97), (2.100) – (2.106), the shock process and the initial condition of the state variables. The state variables of the system include $\pi_{t-1}, y_{t-1}, \hat{B}_t, k_t$ and the Lagrange multipliers for π_t, y_t, c_t, k_{t+1}.[12] To ensure optimality, the initial conditions for the Lagrange multipliers for the jump variables are set so that $\lambda^{\pi}_0 = \lambda^{y}_0 = \lambda^{c}_0 = \lambda^{k}_{-1} = 0$.

An important feature of this model is that the interest rate has three functions. A rise in the interest rate discourages consumption, raises the debt interest payments

[12]Note that in the way I write down the system (Equations (2.93) – (2.97)), π_t, k_t, y_t have both backward-looking and forward-looking components. Backward- and forward-looking price-setters lead to the backward- and forward-looking components in inflation. Capital has a jump component because the demand for capital depends on $E_t k_{t+2}$ due to the capital adjustment cost. Output has a jump component because demand for capital depends on the expected future marginal product of capital which is a function of output. Output also has a predetermined component due to output indexation of backward-looking price-setters.

and hence raises the required return on capital. This reduces the demand for capital by the firms. The first order condition for the interest rate, Equation (2.105), states that with the optimal commitment policy, the marginal social benefit of raising the interest rate to control inflation has to compensate for the marginal social costs of an increase in interest payment on debt and of a fall in capital stock.

The first order condition for government debt is identical to the one in the model without capital. The Lagrange multiplier for government debt follows a random walk. This means that government debt continues to follow a unit root process.

After substituting out the indirect effects of output, the optimal consumption and government spending are:

$$c_t = \frac{1}{\theta_c\left(\sigma+\theta_c\left(\frac{\alpha+\varphi}{1-\alpha}+\beta\frac{\epsilon}{\kappa}\phi_3\right)\right)}\begin{bmatrix}\left(\left(1+\frac{\theta_c}{\sigma}\frac{\alpha+\varphi}{1-\alpha}\right)\kappa_c-\frac{\theta_c\gamma\beta\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}\right)\lambda^{\pi}_{t+1}\\ -\theta_c\theta_g\left(\frac{\alpha+\varphi}{1-\alpha}+\beta\frac{\epsilon}{\kappa}\phi_3\right)g_t-\frac{\theta_c\theta_i}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha}+\frac{\beta\epsilon}{\kappa}\phi_3\right)k_{t+1}\\ +\theta_c\left(\frac{\theta_i(1-\delta)}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha}+\beta\frac{\epsilon}{\kappa}\phi_3\right)+\frac{\alpha(1+\varphi)}{1-\alpha}\right)k_t\\ -\beta\theta_c\frac{\epsilon}{\kappa}\phi_1\Delta\pi_{t+1}+\beta\theta_c\kappa_{y1}\lambda^{\pi}_{t+2}+\frac{\tau\theta_c}{\beta B}\lambda^{b}_{t+1}\\ -\frac{(1-(1-\delta)\beta)}{\beta}\left(\sigma+\theta_c\frac{1+\varphi}{1-\alpha}\right)\lambda^{k}_{t+1}-\sigma\lambda^{c}_{t+1}+\frac{\sigma}{\beta}\lambda^{c}_{t}\end{bmatrix} \tag{2.107}$$

$$g_t = \frac{1}{\sigma+\theta_g\left(\frac{\alpha+\varphi}{1-\alpha}+\beta\frac{\epsilon}{\kappa}\phi_3\right)}\begin{bmatrix}-\theta_c\left(\frac{\alpha+\varphi}{1-\alpha}+\beta\frac{\epsilon}{\kappa}\phi_3\right)c_t-\frac{\theta_i}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha}+\beta\frac{\epsilon}{\kappa}\phi_3\right)k_{t+1}\\ +\left(\frac{\theta_i(1-\delta)}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha}+\beta\frac{\epsilon}{\kappa}\phi_3\right)+\frac{\alpha(1+\varphi)}{1-\alpha}\right)k_t\\ -\beta\frac{\epsilon}{\kappa}\phi_1\Delta\pi_{t+1}+\kappa_{y0}\lambda^{\pi}_{t+1}+\beta\kappa_{y1}\lambda^{\pi}_{t+2}-\frac{(1-\tau)}{\beta B}\lambda^{b}_{t+1}\\ -\beta^{-1}(1-(1-\delta)\beta)\frac{1+\varphi}{1-\alpha}\lambda^{k}_{t+1}\end{bmatrix} \tag{2.108}$$

These two equations are combined to solve for g_t:

$$g_t = \frac{1}{\sigma + (\theta_c + \theta_g)\left(\frac{\alpha+\varphi}{1-\alpha} + \frac{\beta\epsilon}{\kappa}\phi_3\right)} \left[\begin{array}{c} \left(\kappa_{y0} - \frac{\kappa_c}{\sigma}\left(\frac{\alpha+\varphi}{1-\alpha} + \frac{\beta\epsilon}{\kappa}\phi_3\right)\right)\lambda^{\pi}_{t+1} - \frac{\theta_i}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha} + \frac{\beta\epsilon}{\kappa}\phi_3\right)k_{t+1} \\ + \left(\frac{\theta_i(1-\delta)}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha} + \frac{\beta\epsilon}{\kappa}\phi_3\right) + \frac{\alpha(1+\varphi)}{1-\alpha}\right)k_t - \frac{\beta\epsilon}{\kappa}\phi_1\Delta\pi_{t+1} \\ +\beta\kappa_{y1}\lambda^{\pi}_{t+2} + \frac{1}{\sigma\beta B}\left(\sigma(1-\tau) + \theta_c\left(\frac{\alpha+\varphi}{1-\alpha} + \frac{\beta\epsilon}{\kappa}\phi_3\right)\right)\lambda^{c}_{t+1} \\ + \left(\frac{\alpha+\varphi}{1-\alpha} + \frac{\beta\epsilon}{\kappa}\phi_3\right)\lambda^{c}_{t+1} - \beta^{-1}\left(\frac{\alpha+\varphi}{1-\alpha} + \frac{\beta\epsilon}{\kappa}\phi_3\right)\lambda^{c}_{t} \\ -\frac{(1-(1-\delta)\beta)}{\beta}\left(1 - \frac{\beta\epsilon}{\kappa}\phi_3\right)\lambda^{k}_{t+1} \end{array}\right] \quad (2.109)$$

This fiscal policy rule (not derived in Cheng (2009)) shows analytically what makes government spending active. Again, I make use of the intuitions explained earlier in this model to understand this. First, the comparative advantage argument holds: monetary policy is more efficient than government spending in controlling inflation for the same reasons. A fall in consumption induced by a rise in the interest rates affects inflation through three channels, namely through labour supply, demand and the relative forward-looking price. Both the labour supply and demand channels are disinflationary whereas the relative forward-looking price channel partially offsets these forces. On the other hand, a fall in government spending affects inflation only through the labour demand channel and the relative forward-looking price channel.

Second, the marginal benefits of reducing inflation do not exactly cancel with the marginal costs associated with the fall in consumption. There are two reasons, both of which are related to the fact that the backward-looking price-setters index their prices to lagged output gap. First, from the first order condition of consumption, Equation (2.107), the marginal benefits of reducing inflation caused by a fall in consumption within the same period is $\left(\left(1 + \frac{\theta_c}{\sigma}\frac{\alpha+\varphi}{1-\alpha}\right)\kappa_c - \frac{\theta_c\gamma\beta\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}\right)\lambda^{\pi}_{t+1}$, where the last term is the relative forward-looking price effect. This effect prevents the total marginal benefits to cancel with the total marginal cost (within the same period), $\theta_c\left(\sigma + \theta_c\frac{\alpha+\varphi}{1-\alpha}\right)c_t$. Second, a fall in consumption today, through lagged output, re-

duces inflation and increases price dispersion tomorrow. With these intertemporal effects in the optimal consumption rule together with the relative forward-looking price channel, even when consumption follows the optimal rule, government spending still depends on the Lagrange multiplier for inflation, meaning that government spending plays an active role in stabilisation.

In addition, in this model with endogenous capital accumulation, the government spending rule Equation (2.109) depends on the stock of capital and the Lagrange multiplier for capital. Hence, even without the backward-looking price-setters or the indexation of the lagged output gap, in which case there is exact cancelation and the optimal government spending does not depend on the Lagrange multiplier for inflation, government spending will not jump immediately to the long-run level after the first period. For instance, when the share of backward-looking price-setters, ω, is zero, the optimal rule for government spending is

$$g_t = \frac{1}{\sigma + (\theta_c + \theta_g)\frac{\alpha+\varphi}{1-\alpha}} \begin{bmatrix} -\frac{\theta_i}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha}\right) k_{t+1} + \left(\frac{\theta_i(1-\delta)}{\delta}\left(\frac{\alpha+\varphi}{1-\alpha}\right) + \frac{\alpha(1+\varphi)}{1-\alpha}\right) k_t \\ -\frac{1}{\sigma\beta B}\left(\sigma(1-\tau) + \theta_c\frac{\alpha+\varphi}{1-\alpha}\right)\lambda^b_{t+1} + \left(\frac{\alpha+\varphi}{1-\alpha}\right)\lambda^c_{t+1} \\ -\beta^{-1}\left(\frac{\alpha+\varphi}{1-\alpha}\right)\lambda^c_t - \beta^{-1}\left(1-(1-\delta)\beta\right)\lambda^k_{t+1} \end{bmatrix}. \quad (2.110)$$

This rule is independent from λ^{π}_{t+1} as discussed. Using Equation (2.105), I write λ^c_t in terms of λ^b_t and λ^k_{t+1}, and λ^b_t follows a random walk. Hence, with endogenous capital accumulation, government spending will adjust according to the evolution of capital. However, in practice the capital is a predetermined variable and its accumulation is gradual. As will be confirmed in the simulation, this effect coming from capital is negligible.

In the long run, debt will settle at a higher level, and government spending will have to be permanently lower. As consumers hold more debt, they will consume more and supply less labour. Hence, output and capital are lower in the new equilibrium. The Lagrange multiplier for consumption and inflation will not return to zero in

the long run, even after inflation returns to zero. The policymaker has a permanent temptation to inflate away the government debt.

I simulate this model with a 1% cost-push shock. The simulations results of the optimal commitment policy in the low-debt economy under various price setting assumptions are reported in Figure 2.10. Figure 2.11 simulates the optimal commitment policy in the high-debt economy. Notice that when there is no backward-looking price-setters or no output indexation, the government spending is nearly flat, in both the low-debt and high-debt economy, confirming the claim that the effect of capital is small.

In the high-debt economy (Figure 2.11), the government spending is cut more aggressively to ensure that government debt does not rise by too much in the new equilibrium. The responses of the other variables are quantitatively similar to those in the case with low debt.

Changing the Capital Adjustment Cost

Why is the capital adjustment cost important? To answer this question, I also simulate the model without adjustment cost (*i.e.* $J'' = 0$). The impulse responses of a cost-push shock with the optimal commitment policy are reported in Figure 2.12.

Figure 2.12 shows that when the capital adjustment cost is absent, the fall in capital stock in response to a cost-push shock is large. The supply side effect of the fall in capital stock alone is more than enough to disinflate. In the optimal commitment policy, the nominal interest rate falls. This not only limits the fall in capital, but also encourages consumption. This is beneficial as the loss function contains quadratic terms in consumption and also capital, so the policymaker wants to minimise the volatility of these variables. In addition, a fall in the interest rate means that government debt does not rise initially, and the initial fall in government

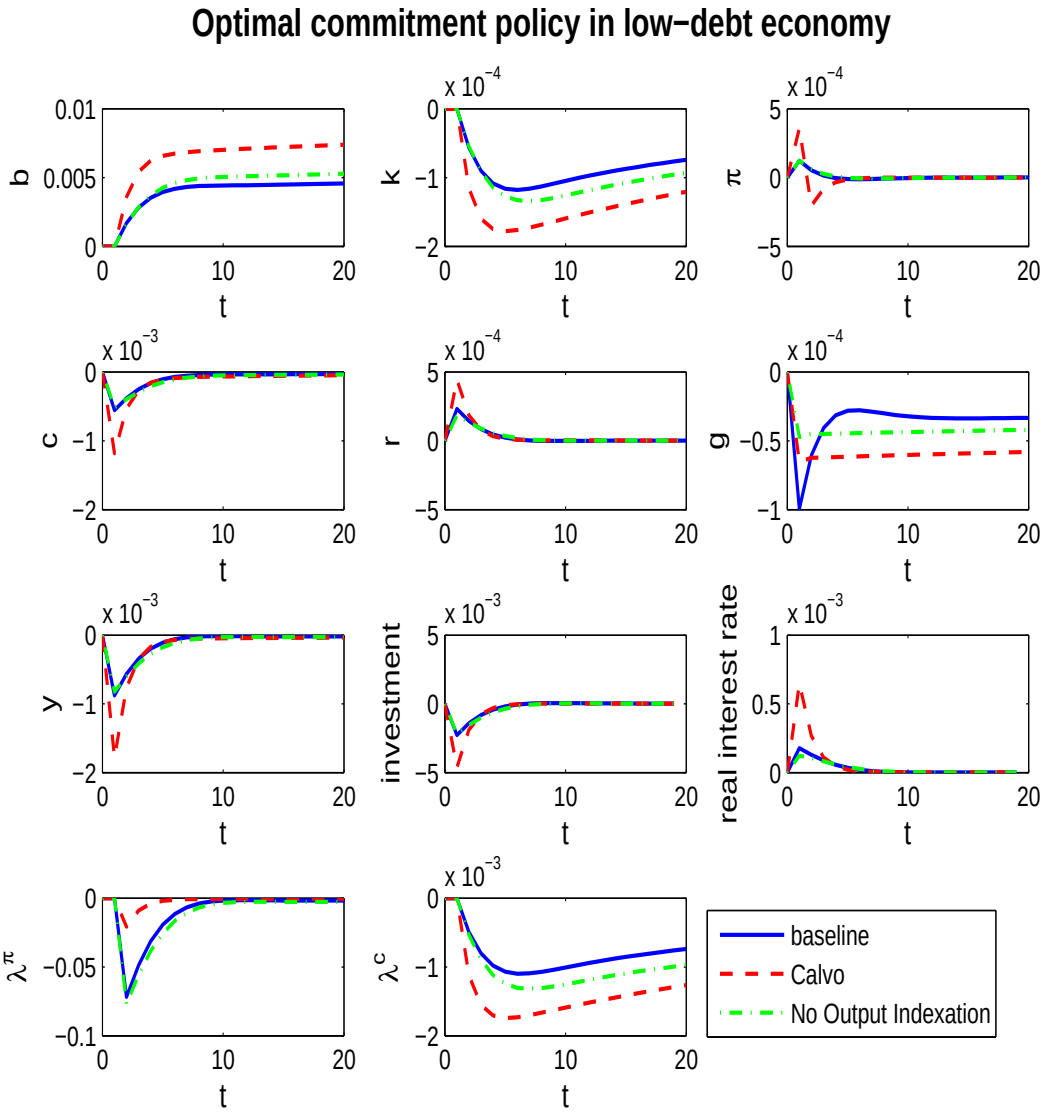


Figure 2.10: Response of a 1% positive mark-up shock in the low-debt economy in the model with capital under the optimal commitment policy.

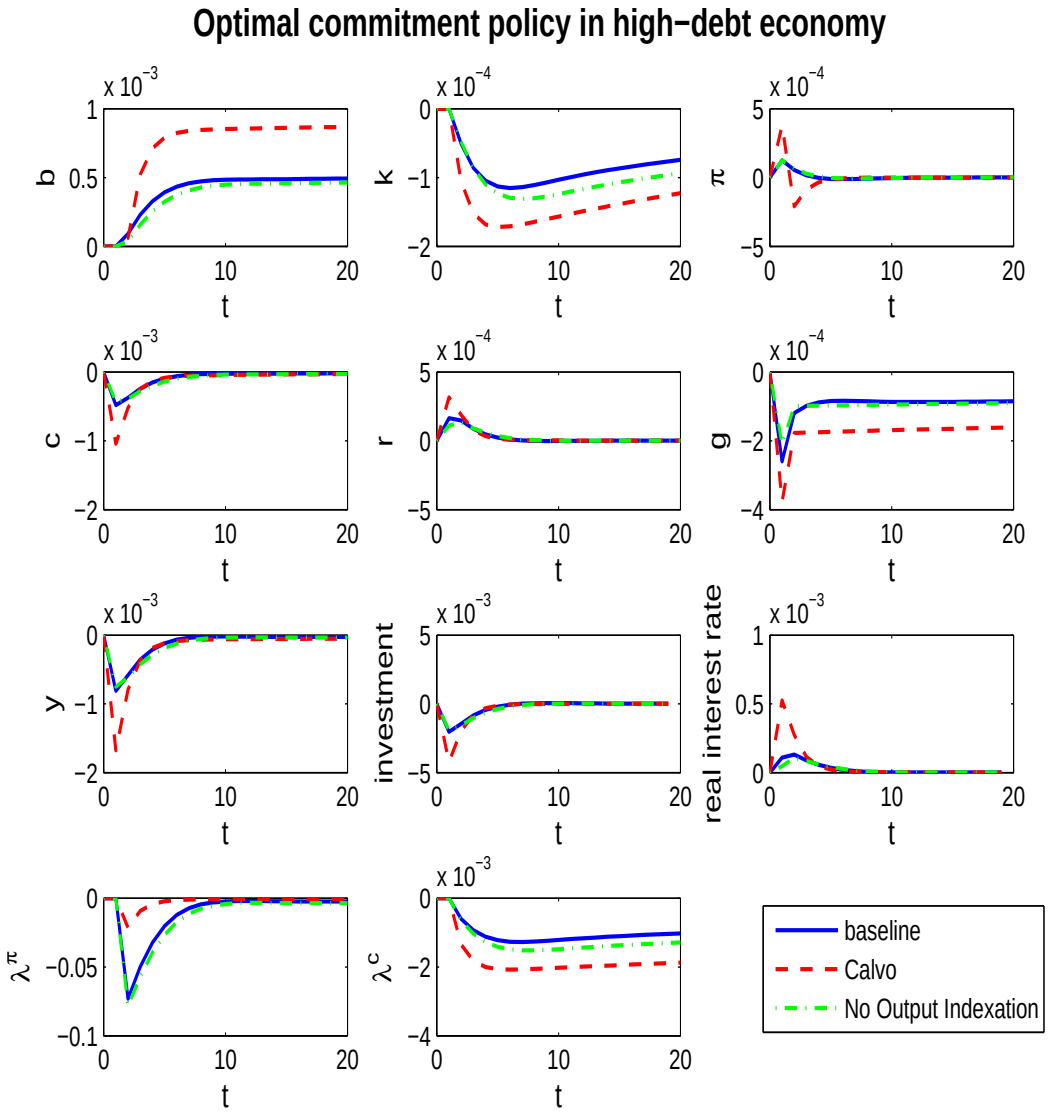


Figure 2.11: Response of a 1% positive mark-up shock in the high-debt economy in the model with capital under the optimal commitment policy.

spending is small compared with the model in which the capital adjustment cost is present.

In contrast, in the setup in which the capital adjustment cost is present, the fall in capital is more gradual. A rise in the nominal interest rate is necessary to reduce consumption and investment in order to drive inflation down. The rise in the interest rate leads to an increase in debt, and, in response, government spending falls more initially.

2.6.6 Optimal Policy under Discretion

Figure 2.13 shows the optimal policy under discretion for different assumptions on sticky prices. Compared to the simulations under discretion of the models with no capital (Figure 2.4 and 2.8), one can see that adding capital does not change the main features of the policies.

2.7 Conclusion

In this chapter, I studied the optimal monetary and fiscal policy co-ordination in an extended class of New Keynesian models which allow for backward-looking price-setters, lagged-output indexation and capital accumulation. I examined to what extent the results in Eser, Leith and Wren-Lewis (2009) under commitment and Leith and Wren-Lewis (2008) under discretion remained true.

Under commitment, I derived analytically the optimal interest rate and government expenditure. Our result confirmed that government debt followed a unit-root process in all the models considered. As for the behaviour of the government expenditure, I replicated the Leith and Wren-Lewis (2008) result that government expenditure jumped to the new steady state after the first period in a system with the New Keynesian Phillips curve and no capital accumulation. Furthermore, I

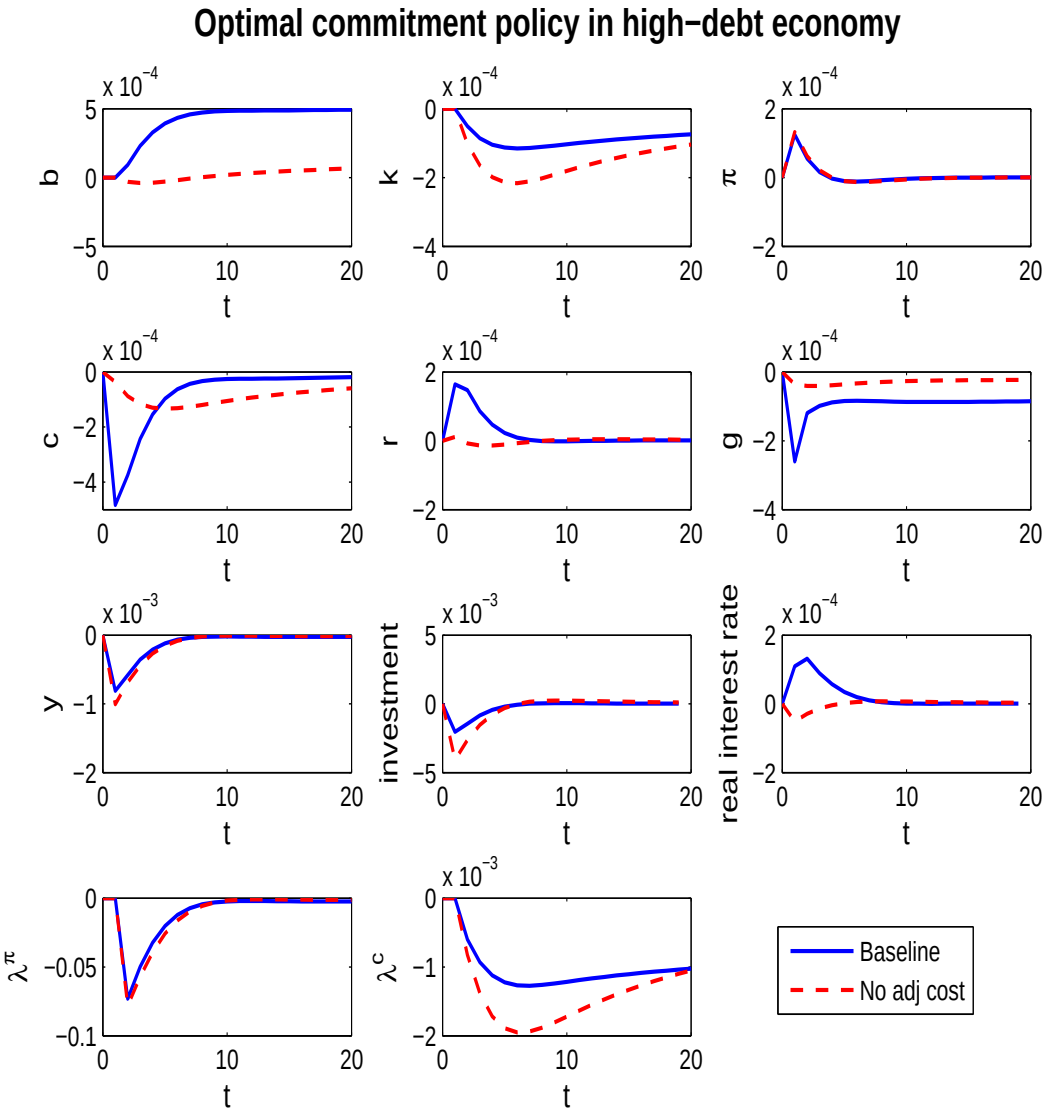


Figure 2.12: Response of a 1% positive mark-up shock in the high-debt economy in the model with capital under the optimal commitment policy.

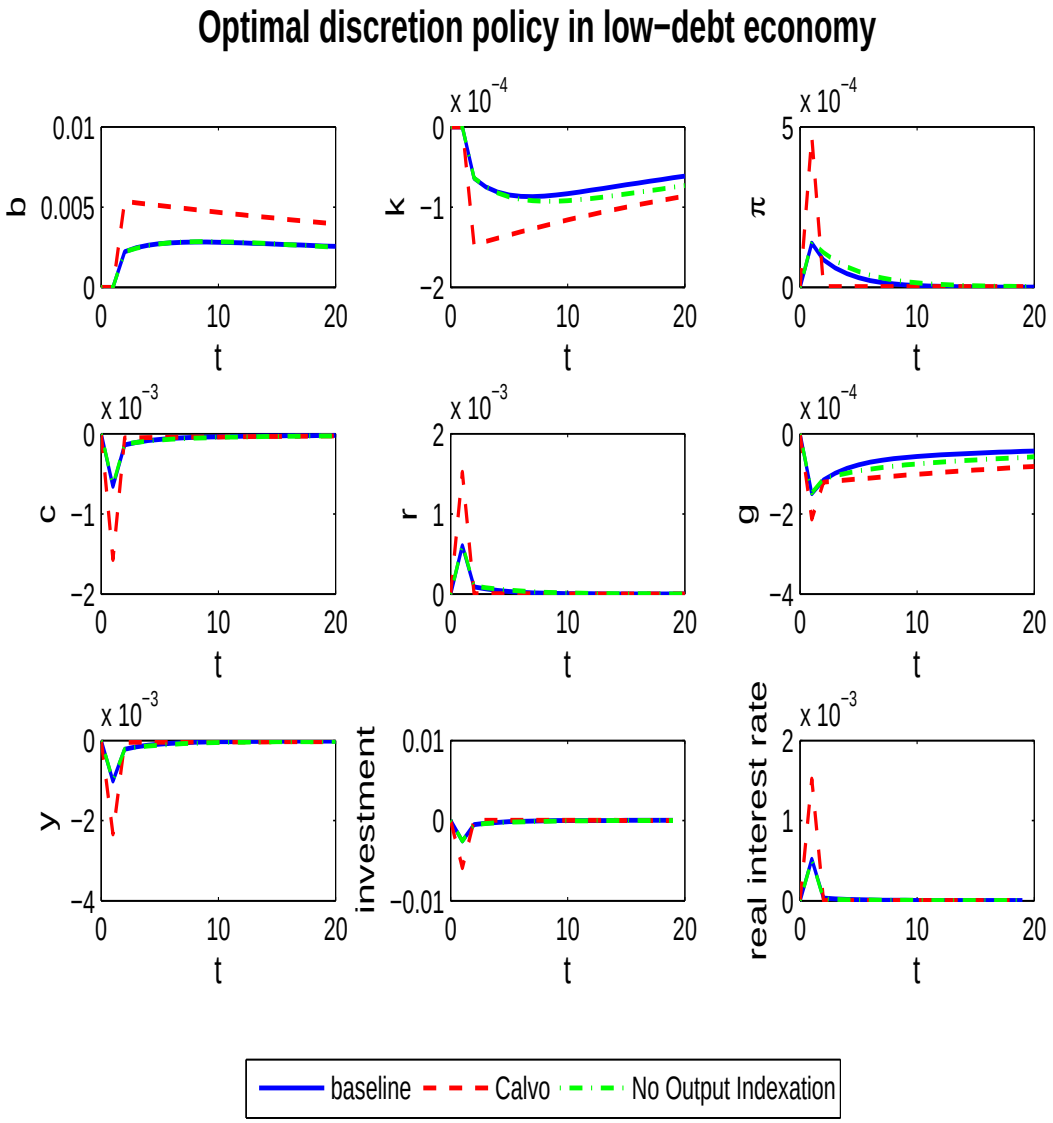


Figure 2.13: Response of a 1% positive mark-up shock in the low-debt economy in the model with capital under the optimal discretion policy.

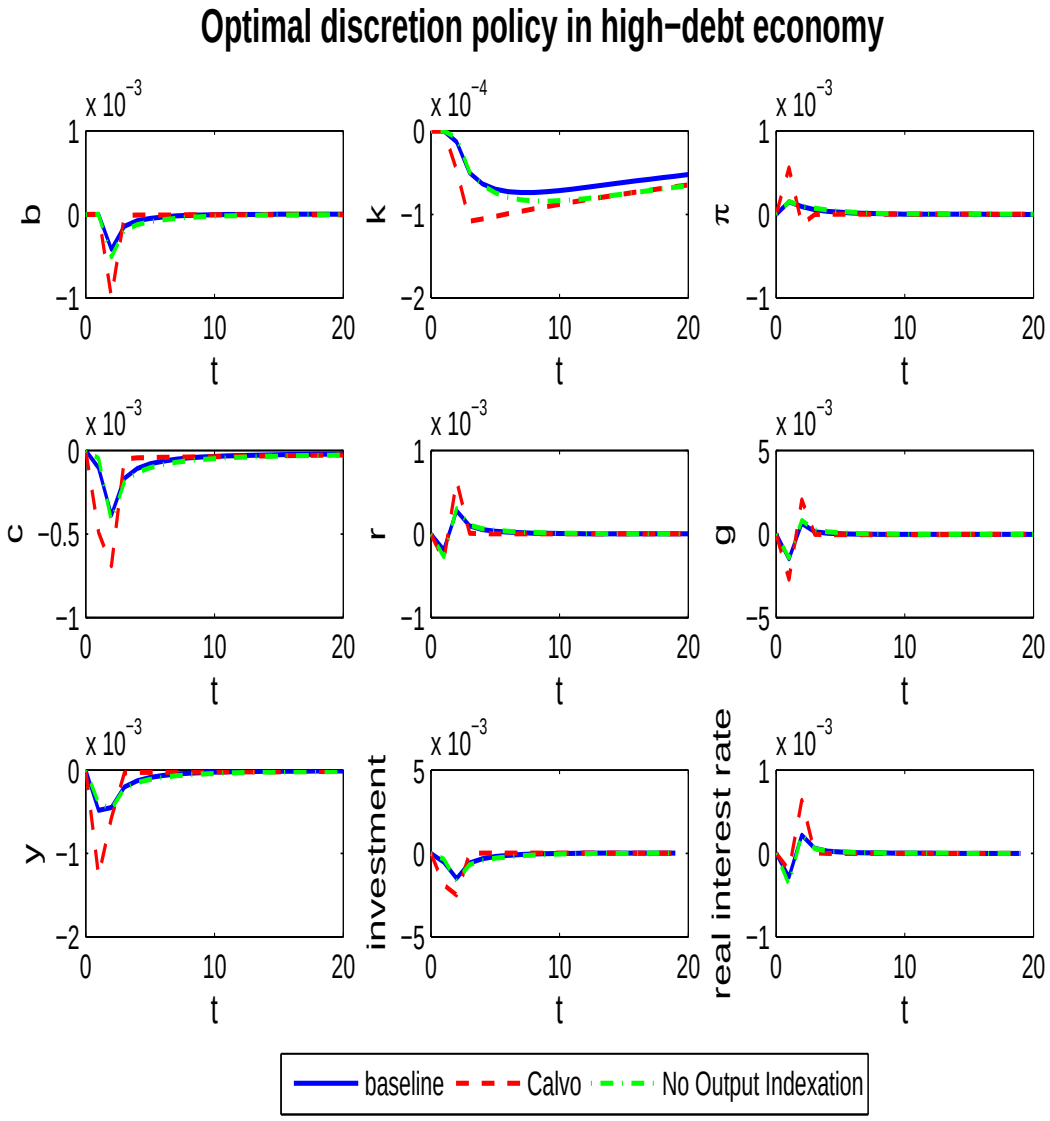


Figure 2.14: Response of a 1% positive mark-up shock in the high-debt economy in the model with capital under the optimal discretion policy.

showed that adding capital accumulation did not alter the result in any meaningful way. However, when I replaced the New Keynesian Phillips curve with the Steinsson (2003) Phillips curve with backward-looking price-setters and when I allowed for the indexation of the lagged output gap, government expenditure played an active role in stabilisation. Moreover, with a higher steady-state level of debt, the initial increase in the rate of interest would be smaller because monetary policy becomes more efficient to control debt.

Under discretion, the policy has to be time-consistent. As a result, the responses in my models under the optimal discretion policy is different from the commitment outcomes in two ways. First, since the policymaker could no longer promise a low inflation (in a time-inconsistent way) to disinflate, the initial response of the interest rate becomes more aggressive. Second, the policymaker could not promise not to inflate debt in future. The optimal policy under discretion in my models would require that debt returns to the pre-shock steady state. The latter implies that government expenditure plays an active role in inflation stabilisation. I showed that these intuitions developed in a simple New-Keynesian model by Leith and Wren-Lewis (2008) remained true in my models with inflation persistence and endogenous capital accumulation.

2.8 Appendix

2.8.1 Solution Methods

The Problem

The policymaker under both commitment and discretion chooses the policy to minimise the quadratic loss function subject to the linear constraints of the economy. The loss function is

$$V_t = E_t \sum_{s=0}^{\infty} \beta^s \begin{bmatrix} x'_{t+s} & u'_{t+s} \end{bmatrix} \begin{bmatrix} Q & U \\ U' & R \end{bmatrix} \begin{bmatrix} x_{t+s} \\ u_{t+s} \end{bmatrix},$$

where $x_t = [x'_{1t}\ x'_{2t}]'$. x_{1t} is a $n_1 \times 1$ vector of predetermined variables and x_{2t} an $n_2 \times 1$ vector of forward-looking variables. Q and R are symmetric matrices.

The constraints are[13]

$$\begin{bmatrix} I & \mathbf{0}_{n_1 \times n_2} \\ \mathbf{0}_{n_2 \times n_1} & H \end{bmatrix} \begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_t + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0}_{n_2 \times 1} \end{bmatrix},$$

where u_t is $n_u \times 1$, and where x_{10} is given.

It is helpful to distinguish between the coefficients in the loss function that are associated with the predetermined and forward-looking variables. I partition the matrices Q and U such that

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q'_{12} & Q_{22} \end{bmatrix}, \qquad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

where Q_{11} is $n_1 \times n_1$, Q_{12} is $n_1 \times n_2$, Q_{22} is $n_2 \times n_2$, U_1 is $n_1 \times n_u$ and U_2 is $n_2 \times n_u$.

[13] This formulation is more general than Soderlind (2003). In Soderlind, H is restricted to be $I_{n_2 \times n_2}$.

Solution under Commitment Policy

Under commitment, the policymaker chooses an optimal path from time t onwards and the public believes that the policymaker will stick to this path. Since the policymaker can affect the public's expectation of future variables, the optimal commitment policy is the solution of the standard Lagrangian problem.

For this problem, I find it convenient to write the constraints in the following form:

$$\bar{H}\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \\ E_t u_{t+1} \end{bmatrix} = \bar{A}\begin{bmatrix} x_{1t} \\ x_{2t} \\ u_t \end{bmatrix} + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0}_{n_2\times 1} \\ \mathbf{0}_{n_u\times 1} \end{bmatrix}, \tag{2.111}$$

where I define

$$\bar{H} \equiv \begin{bmatrix} I & \mathbf{0}_{n_1\times n_2} & \mathbf{0}_{n_1\times n_u} \\ \mathbf{0}_{n_2\times n_1} & H & \mathbf{0}_{n_2\times n_u} \end{bmatrix}, \qquad \bar{A} \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix}, \qquad W \equiv \begin{bmatrix} Q & U \\ U' & R \end{bmatrix}.$$

The Lagrangian is

$$\max_{[x_{1s}',x_{2s}',u_s']'} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{bmatrix} x_{1s} \\ x_{2s} \\ u_s \end{bmatrix}' W \begin{bmatrix} x_{1s} \\ x_{2s} \\ u_s \end{bmatrix} + \begin{bmatrix} \lambda_{s+1}^{x_1} \\ \lambda_{s+1}^{x_2} \end{bmatrix}' \left(\bar{H}\begin{bmatrix} x_{1s+1} \\ x_{2s+1} \\ u_{s+1} \end{bmatrix} - \bar{A}\begin{bmatrix} x_{1s} \\ x_{2s} \\ u_s \end{bmatrix} \right) \right),$$

where $\lambda_{s+1}^{x_1}$ and $\lambda_{s+1}^{x_2}$ are the Lagrange multipliers for the predetermined variables and the forward-looking variables respectively.

The first order conditions are simply[14]

$$2W \begin{bmatrix} x_{1t} \\ x_{2t} \\ u_t \end{bmatrix} + \frac{1}{\beta}\bar{H}' \begin{bmatrix} \lambda_t^{x_1} \\ \lambda_t^{x_2} \end{bmatrix} - \bar{A}' \begin{bmatrix} \lambda_{t+1}^{x_1} \\ \lambda_{t+1}^{x_2} \end{bmatrix} = \mathbf{0}_{(n_1+n_2+n_u)\times 1}. \tag{2.112}$$

The Lagrange multipliers for the predetermined variables are forward-looking variables and the Lagrange multipliers for the forward-looking variables are predetermined. I set the Lagrange multipliers for the forward-looking variables in initial period t to be zero ($\lambda_0^{x_2} = 0$). This ensures that the initial policy is optimal and that the policymaker has no incentive to renege from this policy at time 0.

Then, the constraints (2.111), the first order conditions (2.112), together with the initial conditions for the predetermined variables fully describe the optimal commitment solution. The system can be written succinctly as follows:

$$\begin{bmatrix} \bar{H} & \mathbf{0} \\ \mathbf{0} & \bar{A}' \end{bmatrix} \begin{bmatrix} x_{1t+1} \\ x_{2t+1} \\ u_{t+1} \\ \lambda_{t+1}^{x_1} \\ \lambda_{t+1}^{x_2} \end{bmatrix} = \begin{bmatrix} \bar{A} & \mathbf{0} \\ 2W & \frac{1}{\beta}\bar{H}' \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ u_t \\ \lambda_t^{x_1} \\ \lambda_t^{x_2} \end{bmatrix} + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where the predetermined variables are $\{x_{1t},\ \lambda_t^{x_2}\}$ and the forward-looking variables are $\{x_{2t},\ u_t,\ \lambda_t^{x_1}\}$. The solution can be found by generalised Schur decomposition. See Sims (2003) for example.

Solution under Discretion Policy

This subsection is largely adapted from Soderlind (2003). Under discretion, the policymaker cannot make time-inconsistent promises. Moreover, the objective function

[14] I have used $\frac{d}{dx}Mx = M'$ and $\frac{d}{dx}x'Ax = 2Ax$ where A is a symmetric matrix.

depends on the forward-looking variables, x_{2t}, which are determined endogenously – and depends on expected future values of x_{1t} and x_{2t}. When the policymaker decides today's policy, although he cannot manipulate the expectations of the private sector, he has to consider that the policy will affect the predetermined variables tomorrow, and hence tomorrow's optimal policy as well as forward-looking variables. To capture this, I use a Bellman equation approach together with the method of undetermined coefficient.

I first write the objective function as a Bellman equation

$$V_t(x_{1t}) = \min_{u_t}\{q_t + \beta E_t V_{t+1}(x_{1t+1})\},$$

where

$$q_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}' \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + 2 \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}' \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} u_t + u_t' R u_t.$$

Given the linear-quadratic nature of the problem it is reasonable to conjecture the forward-looking variables are linear combinations of the predetermined variables:

$$x_{2t} = C x_{1t}, \tag{2.113}$$

where C is an unknown $n_2 \times n_1$ matrix. I will verify that this conjecture is indeed correct.

The relation (2.113) can be taken one period forward and substituted with the

constraints of the economy as follows:

$$\begin{aligned} E_t x_{2t+1} &= CE_t x_{1t+1}, \\ HE_t x_{2t+1} &= HCE_t x_{1t+1}, \\ A_{21}x_{1t} + A_{22}x_{2t} + B_2 u_t &= HC\left(A_{11}x_{1t} + A_{12}x_{2t} + B_1 u_t\right), \\ \left(A_{22} - HCA_{12}\right)x_{2t} &= \left(HCA_{11} - A_{21}\right)x_{1t} + \left(HCB_1 - B_2\right)u_t. \end{aligned}$$

Assuming $(A_{22} - HCA_{12})$ is invertible, then

$$x_{2t} = Dx_{1t} + Gu_t, \tag{2.114}$$

where

$$D \equiv \left(A_{22} - HCA_{12}\right)^{-1}\left(HCA_{11} - A_{21}\right), \tag{2.115}$$

$$G \equiv \left(A_{22} - HCA_{12}\right)^{-1}\left(HCB_1 - B_2\right), \tag{2.116}$$

where D is $n_2 \times n_1$ and G is $n_2 \times n_u$. Equation (2.114) can be used to eliminate the forward-looking variables in the constraints as well as the objective function.

First, rewrite the evolution of the predetermined variables, using the constraints of the economy, substituting out the forward-looking variables:

$$\begin{aligned} x_{1t+1} &= A_{11}x_{1t} + A_{12}x_{2t} + B_1 u_t + \xi_{t+1}, \\ &= A^* x_{1t} + B^* u_t + \xi_{t+1}, \end{aligned}$$

where

$$A^* \equiv A_{11} + A_{12}D, \tag{2.117}$$

$$B^* \equiv B_1 + A_{12}G. \tag{2.118}$$

A^* is $n_1 \times n_1$ and B^* is $n_1 \times n_u$.

Second, the period t loss function can be rewritten as

$$\begin{aligned} q_t &= \begin{bmatrix} x_{1t} \\ Dx_{1t} + Gu_t \end{bmatrix}' \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ Dx_{1t} + Gu_t \end{bmatrix} + 2 \begin{bmatrix} x_{1t} \\ Dx_{1t} + Gu_t \end{bmatrix}' \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} u_t \\ &\quad + u_t' R u_t, \\ &= x_{1t}' Q^* x_{1t} + 2 x_{1t}' U^* u_t + u_t' R^* u_t, \end{aligned}$$

where

$$Q^* \equiv Q_{11} + Q_{12}D + D'Q_{21} + D'Q_{22}D, \tag{2.119}$$

$$R^* \equiv R + G'Q_{22}G + G'U_2 + U_2'G, \tag{2.120}$$

$$U^* \equiv Q_{12}G + D'Q_{22}G + U_1 + D'U_2, \tag{2.121}$$

where Q^* is $n_1 \times n_1$, U^* is $n_1 \times n_u$, and R^* is $n_u \times n_u$. R^* is symmetric.

I have now reformulated the optimisation problem in terms of the predetermined variables and the control variables only. The problem is

$$\begin{aligned} V(x_1) &= \min_u \{x_1' Q^* x_1 + 2x_1' U^* u + u' R^* u + \beta E(\tilde{V})\}, \\ s.t \qquad \tilde{x}_1 &= A^* x_1 + B^* u + \tilde{\xi}, \end{aligned}$$

where I use a 'tilde' to denote the variables in the next period. (I drop the time subscripts since the problem has a recursive structure.)

This problem is now a standard linear-quadratic problem. From the standard literature, *e.g.* Ljungqvist and Sargent (2003), the policy function is linear and the value function is quadratic. I conjecture that

$$V = x_1' P x_1 + \nu, \tag{2.122}$$

where P is an unknown $n_1 \times n_1$ matrix and ν is an unknown scalar. This means that

$$V = \min_u \{x_1' Q^* x_1 + 2x_1' U^* u + u' R^* u + \beta E(\tilde{x}_1' P \tilde{x}_1 + \nu)\}.$$

I now show that the solution indeed has the above form. Using the transition law to eliminate next period's state, the Bellman equation becomes

$$V = \min_u \left\{ \begin{array}{c} x_1' Q^* x_1 + 2x_1' U^* u + u' R^* u \\ +\beta E\left(\left(A^* x_1 + B^* u + \tilde{\xi}\right)' P \left(A^* x_1 + B^* u + \tilde{\xi}\right) + \nu\right) \end{array} \right\}.$$

The first order conditions are

$$\begin{aligned} 0 &= 2U^{*'} x_1 + 2R^{*'} u + E\left(2\beta B^{*'} P \left(A^* x_1 + B^* u + \tilde{\epsilon}\right)\right), \\ &= U^{*'} x_1 + R^* u + \beta B^{*'} P \left(A^* x_1 + B^* u\right), \\ &= \left(U^{*'} + \beta B^{*'} P A^*\right) x_1 + \left(R^* + \beta B^{*'} P B^*\right) u. \end{aligned}$$

This means that

$$u = -F x_1, \tag{2.123}$$

where

$$F \equiv \left(R^* + \beta B^{*'} P B^*\right)^{-1} \left(U^{*'} + \beta B^{*'} P A^*\right). \tag{2.124}$$

The policy rule does not depend on future shocks – the certainty equivalence result. Also the policy rule is a linear combination of the predetermined variables. Thus, substituting the policy rule into Equation (2.114), I obtain

$$x_2 = D x_1 + G u = D x_1 - G F x_1 = (D - GF) x_1 \equiv C x_1. \tag{2.125}$$

This confirms the conjecture regarding the forwarding looking variables in Equation (2.113).

Next, to verify the value function has the quadratic form as in Equation (2.122), I substitute the policy rule back into the value function:

$$\begin{aligned} V &= x_1' Q^* x_1 + 2x_1' U^* (-Fx_1) + (-Fx_1)' R^* (-Fx_1) \\ &\quad + \beta E \left(\left(A^* x_1 + B^* (-Fx_1) + \tilde{\xi} \right)' P \left(A^* x_1 + B^* (-Fx_1) + \tilde{\xi} \right) + \nu \right), \\ &= x_1' \left(Q^* - U^* F - F' U^{*'} + F' R^* F + \beta \left(A^* - B^* F \right)' P \left(A^* - B^* F \right) \right) x_1 \\ &\quad + \beta E_t \left[\left(\tilde{\xi}' P \tilde{\xi} \right) + \nu \right]. \end{aligned}$$

This equation is of the same form as the value function in Equation (2.122), where

$$P = Q^* - U^* F - F' U^{*'} + F' R^* F + \beta \left(A^* - B^* F \right)' P \left(A^* - B^* F \right), \quad (2.126)$$

and

$$\begin{aligned} \nu &= \beta E_t \left[\left(\tilde{\xi}' P \tilde{\xi} \right) + \nu \right], \\ \therefore \nu &= \frac{\beta}{1-\beta} E_t \left(\tilde{\xi}' P \tilde{\xi} \right) = \frac{\beta}{1-\beta} \mathrm{tr} \left(E_t \left(\tilde{\xi}' P \tilde{\xi} \right) \right) = \frac{\beta}{1-\beta} \mathrm{tr} \left(P \Sigma \right). \end{aligned}$$

where Σ is the covariance matrix of the shocks. I have used the fact that $E_t \left(\tilde{\xi}' P \tilde{\xi} \right)$ is a scalar and that $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$ where A, B, C are conformable matrices as needed.

Hence, the time invariant solution is

$$\begin{aligned} u_t &= -Fx_{1t}, \\ x_{2t} &= Cx_{1t}, \\ x_{1t+1} &= A_{11} x_{1t} + A_{12} x_{2t} + B_1 u_t + \xi_{t+1}, \end{aligned}$$

and the value function is

$$V_t = x_{1t}' P x_{1t} + \frac{\beta}{1-\beta} \text{tr}\left(P\Sigma\right).$$

The matrices F, C and P can be computed by a recursive algorithm as follows:

1. Start with some guesses of C and a symmetric and positive definite P.
2. Find $D, G, A^*, B^*, Q^*, U^*, R^*, F$ and new C and P.
3. Iterate until F, C and P converge. The equations used in the iterations are Equations (2.115), (2.116), (2.117), (2.118), (2.119), (2.120), (2.121), (2.124), (2.125) and (2.126).

My models

I write the models in this chapter in the form above. In the model without capital, the vectors of variables $\{x_{1t}, x_{2t}, u_t, \xi_t\}$, the matrices

$\{A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, Q_{11}, Q_{12}, Q_{22}, U_1, U_2, R\}$ and sizes $\{n_1, n_2, n, k\}$ are:

$$
\begin{aligned}
x_{1t} &= \left[\hat{\mu}_t\ \pi_{t-1}\ y_{t-1}\ \hat{B}_t\right]', \quad x_{2t} = [\pi_t\ c_t]', \quad u_t = [r_t\ g_t]', \quad \xi_t = [\epsilon_{\mu,t}\ 0\ 0\ 0]', \quad H = I_{2\times 2}, \\
A_{11} &= \begin{bmatrix} \rho_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \theta \\ -\frac{1}{\beta} & -\frac{\theta\tau}{\beta B} \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -\frac{\kappa_\mu}{\chi^{FL}\beta} & -\frac{\chi^{BL}}{\chi^{FL}\beta} & -\frac{\kappa_{y1}}{\chi^{FL}\beta} & 0 \\ \frac{\kappa_\mu}{\sigma\chi^{FL}\beta} & \frac{\chi^{BL}}{\sigma\chi^{FL}\beta} & \frac{\kappa_{y1}}{\sigma\chi^{FL}\beta} & 0 \end{bmatrix}, \\
A_{22} &= \begin{bmatrix} \frac{1}{\chi^{FL}\beta} & -\frac{\kappa_c+\theta\kappa_{y0}}{\chi^{FL}\beta} \\ -\frac{1}{\sigma\chi^{FL}\beta} & \left(1+\frac{\kappa_c+\theta\kappa_{y0}}{\sigma\chi^{FL}\beta}\right) \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1-\theta \\ 1 & \frac{(1-\theta)(1-\tau)}{\beta B} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & -\frac{(1-\theta)\kappa_{y0}}{\chi^{FL}\beta} \\ 1 & \frac{(1-\theta)\kappa_{y0}}{\chi^{FL}\beta} \end{bmatrix}, \\
Q_{11} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{\kappa}\phi_2 & -\frac{\epsilon}{\kappa}\phi_1 & 0 \\ 0 & -\frac{\epsilon}{\kappa}\phi_1 & \frac{\epsilon}{\kappa}\phi_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 0 & 0 \\ -\frac{\epsilon}{\kappa}\phi_2 & 0 \\ \frac{\epsilon}{\kappa}\phi_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} \frac{\epsilon}{\kappa}(1+\phi_2) & 0 \\ 0 & \theta(\sigma+\varphi\theta) \end{bmatrix}, \\
U_1 &= \mathbf{0}_{4\times 2}, \quad U_2 = \begin{bmatrix} 0 & 0 \\ 0 & \varphi\theta(1-\theta) \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & (1-\theta)(\sigma+\varphi(1-\theta)) \end{bmatrix}, \\
n_1 &= 4, \quad n_2 = 2, \quad n_u = 2.
\end{aligned}
$$

For the model with capital, the vectors of variables $\{x_{1t}, x_{2t}, u_t, \xi_t\}$, the matrices

$\{A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, Q_{11}, Q_{12}, Q_{22}, U_1, U_2, R\}$ and sizes $\{n_1, n_2, n, k\}$ are:

$$x_{1t} = \left[\hat{\mu}_t\ \pi_{t-1}\ y_{t-1}\ \hat{B}_t\ k_t\right]', \qquad x_{2t} = [\pi_t\ y_t\ c_t\ k_{t+1}]',\ \ u_t = [r_t\ g_t]',\ \ \xi_t = [\epsilon_{\mu,t}\ 0\ 0\ 0]',$$

$$H = \begin{bmatrix} -\chi^{FL}\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -\sigma & 0 \\ -1 & -(1-\beta(1-\delta))\frac{1+\varphi}{1-\alpha} & -\sigma(1-\beta(1-\delta)) & \beta J'' \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} \rho_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\beta} & -\frac{\tau}{\beta B} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} \kappa_\mu & \chi^{BL} & \kappa_{y1} & 0 & -\kappa_k \\ 0 & 0 & 0 & 0 & -\frac{\theta_i(1-\delta)}{\delta} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -J'' \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -1 & \kappa_{y0} & \kappa_c & 0 \\ 0 & -1 & \theta_c & \frac{\theta_i}{\delta} \\ 0 & 0 & -\sigma & 0 \\ 0 & 0 & 0 & -\left((1-\beta(1-\delta))\left(\frac{1+\alpha\varphi}{1-\alpha}\right) - (1+\beta)J''\right) \end{bmatrix},$$

$$
\begin{aligned}
B_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & \frac{\theta_g}{\beta B} \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & \theta_g \\ -1 & 0 \\ -1 & 0 \end{bmatrix}, \\
Q_{11} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{\kappa}\phi_2 & -\frac{\epsilon}{\kappa}\phi_1 & 0 & 0 \\ 0 & -\frac{\epsilon}{\kappa}\phi_1 & \frac{\epsilon}{\kappa}\phi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha(1+\alpha\varphi)}{1-\alpha} - J''\frac{\theta_i}{\delta} \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{\epsilon}{\kappa}\phi_2 & 0 & 0 & 0 \\ \frac{\epsilon}{\kappa}\phi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1+\varphi)}{1-\alpha} & 0 & J''\frac{\theta_i}{\delta} \end{bmatrix}, \\
Q_{22} &= \begin{bmatrix} \frac{\epsilon}{\kappa}(1+\phi_2) & 0 & 0 & 0 \\ 0 & \frac{\alpha+\varphi}{1-\alpha} & 0 & 0 \\ 0 & 0 & \sigma\theta_c & 0 \\ 0 & 0 & 0 & -J''\frac{\theta_i}{\delta} \end{bmatrix}, \quad U_1 = \mathbf{0}_{5\times 2}, \quad U_2 = \mathbf{0}_{4\times 2}, \\
R &= \begin{bmatrix} 0 & 0 \\ 0 & \sigma\theta_g \end{bmatrix}, \quad n_1 = 5, \quad n_2 = 4, \quad n_u = 2.
\end{aligned}
$$

2.8.2 Derivation of the Hybrid Phillips Curve in the Model with Capital

Define

$$
p_t^{FL} = \frac{P_t^{FL}}{P_t}, \qquad p_t^{BL} = \frac{P_t^{BL}}{P_t}, \qquad p_t^* = \frac{P_t^*}{P_t}.
$$

Then (2.25) can be written as

$$
\begin{aligned}
p_t^* &= (p_t^{FL})^{1-\omega}(P_t^{BL})^{\omega}, \\
\hat{p}_t^* &= (1-\omega)\hat{p}_t^{FL} + \omega\hat{p}_t^{BL}. \qquad (2.127)
\end{aligned}
$$

Next, (2.24) can be written as

$$\begin{aligned}\frac{P_t^{BL}}{P_t} &= \frac{P_{t-1}^*}{P_{t-1}}\frac{P_{t-1}}{P_t}\Pi_{t-1}\left(\frac{Y_{t-1}}{Y_{t-1}^n}\right)^{\eta}, \\ \hat{p}_t^{BL} &= \hat{p}_{t-1}^* - \pi_t + \pi_{t-1} + \eta y_{t-1}, \end{aligned} \tag{2.128}$$

where y_t is the deviation from the natural level of output.

The evolution of price is written as

$$\hat{p}_t^{BL} = \frac{\gamma}{(1-\gamma)\omega}\pi_t - \frac{(1-\omega)}{\omega}\hat{p}_t^{FL}. \tag{2.129}$$

Next, I eliminate $\hat{p}_t^{BL}$ and $\hat{p}_t^*$.

$$\begin{aligned}\hat{p}_t^{BL} &= \left[(1-\omega)\hat{p}_{t-1}^{FL} + \omega\hat{p}_{t-1}^{BL}\right] - \pi_t + \pi_{t-1} + \eta y_{t-1}, \\ \frac{\gamma}{(1-\gamma)\omega}\pi_t &= \frac{(1-\omega)}{\omega}\hat{p}_t^{FL} + (1-\omega)\hat{p}_{t-1}^{FL} + \omega\left(\frac{\gamma}{(1-\gamma)\omega}\pi_{t-1} - \frac{(1-\omega)}{\omega}\hat{p}_{t-1}^{FL}\right) \\ &\quad -\pi_t + \pi_{t-1} + \eta y_{t-1}, \\ \left(1 + \frac{\gamma}{(1-\gamma)\omega}\right)\pi_t &= \frac{(1-\omega)}{\omega}\hat{p}_t^{FL} + \left(\frac{\gamma}{1-\gamma} - 1\right)\pi_{t-1} + \eta y_{t-1}, \\ \hat{p}_t^{FL} &= \frac{\omega(1-\gamma)+\gamma}{(1-\gamma)(1-\omega)}\pi_t - \frac{\omega}{(1-\gamma)(1-\omega)}\pi_{t-1} - \frac{\omega\eta}{1-\omega}y_{t-1}. \end{aligned} \tag{2.130}$$

The forward-looking price-setters will set their prices to maximise their expected discounted future profits. The first order condition is

$$E_t\sum_{s=0}^{\infty}\gamma^s Q_{t,t+s}\left(\frac{P_t^{FL}}{P_{t+s}}\right)^{-\epsilon_{t+s}} Y_{t+s}\left(\frac{P_t^{FL}}{P_{t+s}} - \mu_{t+s}MC_{t+s}\right) = 0, \tag{2.131}$$

where $\mu_s = \epsilon_s/(\epsilon_{s-1})$. When prices are fully flexible and there is no fluctuation in monopolistic power, the above first order condition becomes

$$\mu MC_t = 1. \tag{2.132}$$

Using the labour demand equation, Equation (2.72), and the labour supply equation, Equation (2.19), I can write the real marginal cost in the steady state as

$$\frac{1}{\mu} = MC_t = \frac{N_t \times \frac{W_t}{P_t}}{(1-\alpha)\mu_w Y_t} = \frac{N_t^{1+\varphi} C_t}{(1-\tau)(1-\alpha)\mu_w Y_t}. \tag{2.133}$$

Compare this equation with the labour supply equation in the efficient flexible-price steady state, Equation (2.84). These equations show that if the production subsidy is $\mu_w = \mu/(1-\tau)$, the sticky-price economy will have the same steady state as the efficient flexible-price steady state.

The terms in the brackets of the first order condition, Equation (2.131), is log-linearised as follows:

$$\frac{P_t^{FL}}{P_{t+s}} - \mu_{t+s} MC_{t+s} = \hat{P}_t^{FL} - \hat{P}_{t+s} - \hat{MC}_{t+s} - \hat{\mu}_{t+s} = \hat{p}_t^{FL} - \sum_{k=1}^{s} \pi_{t+k} - \hat{MC}_{t+s} - \hat{\mu}_{t+s}.$$

Hence, the first order condition, Equation (2.131), is log-linearised as follows:

$$\begin{aligned}
0 &= E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \left(\hat{p}_t^{FL} - \sum_{k=1}^{s-t} \pi_{t+k} - \hat{MC}_s - \hat{\mu}_s \right), \\
\hat{p}_t^{FL} &= (1-\gamma\beta) E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \left(\sum_{k=1}^{s-t} \pi_{t+k} + \hat{MC}_s + \hat{\mu}_s \right), \\
\hat{p}_t^{FL} &= (1-\gamma\beta) E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \sum_{k=1}^{s-t} \pi_{t+k} + (1-\gamma\beta) E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \left(\hat{MC}_s + \hat{\mu}_s \right), \\
\hat{p}_t^{FL} &= E_t \sum_{s=1}^{\infty} (\gamma\beta)^{s} \pi_{t+s} + (1-\gamma\beta) E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \left(\sigma \hat{C}_s + (1+\varphi)\hat{N}_s - \hat{Y}_s + \hat{\mu}_s \right)
\end{aligned} \tag{2.134}$$

The first order condition in quasi-difference form can be written as follows:

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta) \left(\sigma \hat{C}_t + (1+\varphi)\hat{N}_t - \hat{Y}_t + \hat{\mu}_t \right). \tag{2.135}$$

To continue to derive the Phillips curve, first write the above equation in the

form of deviations from the efficient steady state as follows:

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta)\left(\sigma c_t + (1+\varphi) n_t - y_t + \hat{\mu}_t\right). \tag{2.136}$$

I also eliminate labour using the production function:

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta)\left(\sigma c_t + \frac{\alpha+\varphi}{1-\alpha} y_t - \frac{\alpha(1+\varphi)}{1-\alpha} k_t + \hat{\mu}_t\right). \tag{2.137}$$

I then use Equation (2.130) to substitute the relative forward-looking price. Hence,

$$\pi_t = \chi^{FL}\beta E_t \pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c c_t + \kappa_{y0} y_t + \kappa_{y1} y_{t-1} - \kappa_k k_t + \kappa_\mu \hat{\mu}_t, \tag{2.138}$$

where

$$\begin{aligned}
\chi^{FL} &= \frac{\gamma}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \chi^{BL} = \frac{\omega}{\omega(1-\gamma+\gamma\beta)+\gamma}, \\
\kappa_c &= \frac{\sigma(1-\gamma)(1-\omega)(1-\gamma\beta)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \kappa_{y0} = \frac{1}{\sigma}\frac{\alpha+\varphi}{1-\alpha}\kappa_c - \frac{\gamma\beta\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \\
\kappa_{y1} &= \frac{\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \kappa_k = \frac{\alpha(1+\varphi)}{\sigma(1-\alpha)}\kappa_c, \qquad \kappa_\mu = \frac{1}{\sigma}\kappa_c.
\end{aligned}$$

This is Equation (2.80) in the main text.

In the model without capital, set $\alpha = 0$ to get

$$\pi_t = \chi^{FL}\beta E_t \pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c c_t + \kappa_{y0} y_t + \kappa_{y1} y_{t-1} + \kappa_\mu \hat{\mu}_t. \tag{2.139}$$

This is Equation (2.30) in the main text.

2.8.3 Second-order Approximation of the Social Welfare Function

The utility function is

$$U_t = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{C_s^{1-\sigma}}{1-\sigma} + \chi \frac{G_s^{1-\sigma}}{1-\sigma} - \frac{N_s^{1+\varphi}}{1+\varphi} \right). \tag{2.140}$$

The second-order approximation of $\ln C_s$ is

$$\frac{C_s^{1-\sigma}}{1-\sigma} = \frac{C^{1-\sigma}}{1-\sigma} + C^{1-\sigma} \left(\hat{C}_s + \frac{1}{2}(1-\sigma)\hat{C}_s^2 \right) + O(3). \tag{2.141}$$

A similar expression can be derived for the second-order approximation of the utility from government spending.

The disutility of labour can be approximated as follows:

$$\begin{aligned} \frac{N_t^{1+\varphi}}{1+\varphi} &= \frac{N^{1+\varphi}}{1+\varphi} + N^{\varphi} \left(\frac{N_t - N}{N} \right) N + \frac{1}{2} N^{\varphi-1} \left(\frac{N_t - N}{N} \right)^2 N^2 + O(3), \\ &= \frac{N^{1+\varphi}}{1+\varphi} + N^{1+\varphi} \left(\hat{N}_t + \frac{1}{2}\hat{N}_t^2 + \frac{1}{2}\varphi \hat{N}_t^2 \right) + O(3), \\ &= \frac{N^{1+\varphi}}{1+\varphi} + N^{1+\varphi} \left(\hat{N}_t + \frac{1}{2}(1+\varphi)\hat{N}_t^2 \right). \end{aligned} \tag{2.142}$$

Therefore,

$$\begin{aligned} U_t = E_t \sum_{s=t}^{\infty} \beta^{s-t} \Bigg(& C^{1-\sigma} \left(\hat{C}_s + \frac{1}{2}(1-\sigma)\hat{C}_s^2 \right) + \chi G^{1-\sigma} \left(\hat{G}_s + \frac{1}{2}(1-\sigma)\hat{G}_s^2 \right) \\ & - N^{1+\varphi} \left(\hat{N}_s + \frac{1}{2}(1+\varphi)\hat{N}_s^2 \right) \Bigg) + t.i.p + O(3), \end{aligned} \tag{2.143}$$

where *t.i.p* stands for terms that are independent of policy and hence irrelevant for ranking alternative policies. In this model, I have assumed that a subsidy makes the

steady state efficient. Using the steady-state versions of Equations (2.83) to (2.84),

$$\frac{1}{C^{\sigma}} = \frac{\chi}{G^{\sigma}} = \frac{N^{1+\varphi}}{(1-\alpha)Y}. \tag{2.144}$$

This means that

$$\chi = \frac{G^{\sigma}}{C^{\sigma}}, \qquad N^{1+\varphi} = \frac{(1-\alpha)}{\theta_c} C^{1-\sigma}.$$

The social welfare function can be re-written as

$$\begin{aligned} U_t &= \frac{C^{1-\sigma}}{\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\theta_c \hat{C}_s + \theta_g \hat{G}_s + \frac{1}{2}(1-\sigma)\left(\theta_c \hat{C}_s^2 + \theta_g \hat{G}_s^2\right) \right. \\ &\quad \left. -(1-\alpha)\left(\hat{N}_s + \frac{1}{2}(1+\varphi)\hat{N}_s^2\right)\right) + t.i.p + O(3). \end{aligned} \tag{2.145}$$

Next, I remove the first order terms from the social welfare function. I use the second-order approximated goods market clearing condition and the production function to eliminate these terms. First the aggregate production functions can be approximated up to the second order as follows:

$$\begin{aligned} \Delta_t Y_t &= A_t K_t^{\alpha} N_t^{1-\alpha}, \\ Y_t - Y &= -Y(\Delta_t - \Delta) + \alpha \frac{Y}{K}(K_t - K) + (1-\alpha)\frac{Y}{N}(N_t - N) \\ &\quad + \frac{\alpha(\alpha-1)}{2}\frac{Y}{K^2}(K_t - K)^2 - \frac{\alpha(1-\alpha)}{2}\frac{Y}{N^2}(N_t - N)^2 + \alpha\frac{Y}{K}(K_t - K)(A_t - A) \\ &\quad + \alpha\frac{Y}{K}(K_t - K)(A_t - A) + \alpha(1-\alpha)\frac{Y}{KN}(K_t - K)(N_t - N) + t.i.p + O(3), \\ \hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 &= -\hat{\Delta}_t + \alpha\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) + (1-\alpha)\left(\hat{N}_t + \frac{1}{2}\hat{N}_t^2\right) - \frac{\alpha(1-\alpha)}{2}\hat{K}_t^2 \\ &\quad - \frac{\alpha(1-\alpha)}{2}\hat{N}_t^2 + \hat{A}_t\left(\alpha\hat{K}_t + (1-\alpha)\hat{N}_t\right) + \alpha(1-\alpha)\hat{K}_t\hat{N}_t \\ &\quad + t.i.p + O(3). \end{aligned} \tag{2.146}$$

It can be shown (see Woodford 2003a, chapter 6) that

$$\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon_t} di = 1 + \frac{\epsilon}{2} Var_i(\hat{P}_t(i)) + O(3). \tag{2.147}$$

The goods market clearing condition can be approximated to the second order as follows:

$$\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 = \theta_c \left(\hat{C}_t + \frac{1}{2}\hat{C}_t^2\right) + \theta_g \left(\hat{G}_t + \frac{1}{2}\hat{G}_t^2\right) + \theta_i \left(\hat{I}_t + \frac{1}{2}\hat{I}_t^2\right) + O(3), \tag{2.148}$$

where the investment terms can be written in terms of capital, using the second-order approximation of the capital accumulation equation:

$$\begin{aligned} \hat{K}_{t+1} + \frac{1}{2}\hat{K}_{t+1}^2 &= (1-\delta)\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) + \delta\left(\hat{I}_t + \frac{1}{2}\hat{I}_t^2\right) + \frac{1}{2}J''\delta^2\left(\hat{I}_t - \hat{K}_t\right)^2 + O(3), \\ &= (1-\delta)\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) + \delta\left(\hat{I}_t + \frac{1}{2}\hat{I}_t^2\right) + \frac{1}{2}J''\left(\Delta\hat{K}_{t+1}\right)^2 + O(3) \end{aligned} \tag{2.149}$$

Equations (2.146), (2.148) and (2.149) are combined to obtain

$$\begin{aligned} & \theta_c\hat{C}_t + \theta_g\hat{G}_t - (1-\alpha)\hat{N}_t \\ = \; & \alpha\hat{K}_t + \frac{1}{2}\alpha^2\hat{K}_t^2 + \frac{(1-\alpha)^2}{2}\hat{N}_t^2 + \hat{A}_t\left(\alpha\hat{K}_t + (1-\alpha)\hat{N}_t\right) + \alpha(1-\alpha)\hat{K}_t\hat{N}_t \\ & -\frac{\epsilon}{2}Var_i(\hat{P}_t(i)) - \frac{\theta_c}{2}\hat{C}_t^2 - \frac{\theta_g}{2}\hat{G}_t^2 - \frac{\theta_i}{\delta}\left(\hat{K}_{t+1} + \frac{1}{2}\hat{K}_{t+1}^2\right) + \frac{\theta_i(1-\delta)}{\delta}\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) \\ & + \frac{\theta_i}{2\delta}J''\left(\Delta\hat{K}_{t+1}\right)^2 + t.i.p + O(3). \end{aligned} \tag{2.150}$$

This expression is substituted into (2.145) as follows:

$$U_t = \frac{C^{1-\sigma}}{\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \alpha\hat{K}_s + \frac{1}{2}\alpha^2\hat{K}_s^2 + \frac{(1-\alpha)^2}{2}\hat{N}_s^2 + \hat{A}_s\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s\right) \\ +\alpha(1-\alpha)\hat{K}_s\hat{N}_s - \frac{\theta_c}{2}\hat{C}_s^2 - \frac{\theta_g}{2}\hat{G}_s^2 - \frac{\theta_i}{\delta}\left(\hat{K}_{s+1} + \frac{1}{2}\hat{K}_{s+1}^2\right) \\ +\frac{\theta_i(1-\delta)}{\delta}\left(\hat{K}_s + \frac{1}{2}\hat{K}_s^2\right) - \frac{1}{2}(1-\alpha)(1+\varphi)\hat{N}_s^2 - \frac{\epsilon}{2}Var_i(\hat{P}_s(i)) \\ +\frac{\theta_i}{2\delta}J''\left(\Delta\hat{K}_{s+1}\right)^2 + \frac{1}{2}(1-\sigma)\left(\theta_c\hat{C}_s^2 + \theta_g\hat{G}_s^2\right) \end{pmatrix} + t.i.p + O(3), \tag{2.151}$$

$$= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} -2\left[\left(\alpha + \frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s - \frac{\theta_i}{\delta}\hat{K}_{s+1}\right] - \left(\alpha^2 + \frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s^2 \\ -2\hat{A}_s\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s\right) - 2\alpha(1-\alpha)\hat{K}_s\hat{N}_s \\ +\sigma\theta_c\hat{C}_s^2 + \sigma\theta_g\hat{G}_s^2 + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 + (1-\alpha)(\alpha+\varphi)\hat{N}_s^2 \\ +\epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i}{\delta}J''\left(\Delta\hat{K}_{s+1}\right)^2 \end{pmatrix} + t.i.p + O(3). \tag{2.152}$$

Notice that

$$\left(\alpha + \frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s - \frac{\theta_i}{\delta}\hat{K}_{s+1} = \frac{\theta_i}{\delta}\left[\left(\frac{\alpha\delta}{\theta_i} + 1 - \delta\right)\hat{K}_s - \hat{K}_{s+1}\right] = \frac{\theta_i}{\delta}\left[\frac{1}{\beta}\hat{K}_s - \hat{K}_{s+1}\right],$$

where the last equality follows from the steady-state version of Equation (2.85). Hence,

$$E_t\sum_{s=t}^{\infty}\beta^{s-t}\left[\left(\alpha + \frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s - \frac{\theta_i}{\delta}\hat{K}_{s+1}\right] = E_t\sum_{s=t}^{\infty}\beta^{s-t}\left[\frac{1}{\beta}\hat{K}_s - \hat{K}_{s+1}\right] = \frac{1}{\beta}\hat{K}_t, \tag{2.153}$$

where the initial stock of capital K_t is independent of the policy.

Moreover, I can eliminate $\hat{A}_t$ using the production function and the intratemporal

tradeoff between consumption and labour supply as follows:

$$\begin{aligned}\hat{A}_t &= \hat{Y}_t^n - \alpha\hat{K}_t^n - (1-\alpha)\hat{N}_t^n, \\ &= \left(\sigma\hat{C}_t^n + (1+\varphi)\hat{N}_t^n\right) - \alpha\hat{K}_t^n - (1-\alpha)\hat{N}_t^n, \\ &= (\alpha+\varphi)\hat{N}_t^n + \sigma\hat{C}_t^n - \alpha\hat{K}_t^n. \end{aligned} \tag{2.154}$$

Hence,

$$\begin{aligned} & -2\hat{A}_s\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s\right) \\ = \; & -2\left[(\alpha+\varphi)\hat{N}_s^n + \sigma\hat{C}_s^n - \alpha\hat{K}_s^n\right]\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s\right), \\ = \; & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s\right)\sigma\hat{C}_s^n, \\ = \; & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\left(\hat{Y}_s - \hat{A}_s\right)\sigma\hat{C}_s^n, \\ = \; & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\left(\theta_c\hat{C}_s + \frac{\theta_i}{\delta}(\hat{K}_{s+1} - (1-\delta)\hat{K}_s) + \theta_g\hat{G}_s\right)\sigma\hat{C}_s^n + t.i.p, \\ = \; & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\sigma\theta_c\hat{C}_s\hat{C}_s^n - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1} - (1-\delta)\hat{K}_s)\hat{C}_s^n - 2\sigma\theta_g\hat{G}_s\hat{G}_s^n + t.i.p. \end{aligned} \tag{2.155}$$

Equation (2.155) is substituted into the social welfare function (2.152) as follows:

$$U_t = -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\begin{pmatrix} -\left(\alpha^2+\frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s^2 - 2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n \\ -2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ -2\sigma\theta_c\hat{C}_s\hat{C}_s^n - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1}-(1-\delta)\hat{K}_s)\hat{C}_s^n - 2\sigma\theta_g\hat{G}_s\hat{G}_s^n \\ -2\alpha(1-\alpha)\hat{K}_s\hat{N}_s + \sigma\theta_c\hat{C}_s^2 + \sigma\theta_g\hat{G}_s^2 + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 \\ +(1-\alpha)(\alpha+\varphi)\hat{N}_s^2 + \epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i}{\delta}J''\left(\Delta\hat{K}_{t+1}\right)^2 \end{pmatrix}$$
$$+t.i.p + O(3), \tag{2.156}$$

$$= -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\begin{pmatrix} -\frac{\theta_i(1-\delta)}{\delta}\hat{K}_s^2 - \alpha^2\left(\hat{K}_s^2 - 2\hat{K}_s\hat{K}_s^2\right) - 2\alpha(1+\varphi)\hat{K}_s\hat{N}_s^n \\ +(1-\alpha)(\alpha+\varphi)\left(\hat{N}_s^2 - 2\hat{N}_s\hat{N}_s^n\right) - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1}-(1-\delta)\hat{K}_s)\hat{C}_s^n \\ -2\alpha(1-\alpha)\left(\hat{K}_s\hat{N}_s - \hat{K}_s\hat{N}_s^n - \hat{N}_s\hat{K}_s^n\right) + \sigma\theta_c\left(\hat{C}_s^2 - 2\hat{C}_s\hat{C}_s^n\right) \\ +\sigma\theta_g\left(\hat{G}_s^2 - 2\hat{G}_s\hat{G}_s^2\right) + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 + \epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i}{\delta}J''\left(\Delta\hat{K}_{t+1}\right)^2 \end{pmatrix}$$
$$+t.i.p + O(3), \tag{2.157}$$

$$= -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\begin{pmatrix} \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + (1-\alpha)(\alpha+\varphi)n_s^2 - \alpha^2 k_s^2 - 2\alpha(1-\alpha)k_s n_s \\ +\epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i(1-\delta)}{\delta}\hat{K}_s^2 - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1}-(1-\delta)\hat{K}_s)\hat{C}_s^n \\ -2\alpha(1+\varphi)\hat{K}_s\hat{N}_s^n + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 - \frac{\theta_i}{\delta}J''\left(\Delta\hat{K}_{t+1}\right)^2 \end{pmatrix}$$
$$+t.i.p + O(3). \tag{2.158}$$

Finally, using Equation (2.87), we can write

$$(1+\varphi)\hat{K}_s\hat{N}_s^n = \hat{K}_s\hat{Y}_s^n - \sigma\hat{K}_s\hat{C}_s^n. \tag{2.159}$$

This means that

$$
\begin{aligned}
& E_t \sum_{s=t}^{\infty} \beta^{s-t} 2\alpha(1+\varphi)\hat{K}_s \hat{N}_s^n \\
= \;& E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha \hat{K}_s \hat{Y}_s^n - 2\sigma\alpha \hat{K}_s \hat{C}_s^n \right), \\
= \;& E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha\beta \hat{K}_{s+1} \hat{Y}_{s+1}^n - 2\sigma\alpha \hat{K}_s \hat{C}_s^n \right) + 2\alpha \hat{K}_t \hat{Y}_t^n, \\
= \;& E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha\beta \hat{K}_{s+1} \hat{Y}_{s+1}^n - 2\alpha\beta \hat{K}_{s+1} \hat{K}_{s+1}^n + 2\alpha\beta \hat{K}_{s+1} \hat{K}_{s+1}^n - 2\sigma\alpha \hat{K}_s \hat{C}_s^n \right) + t.i.p, \\
= \;& E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha\beta \hat{K}_{s+1} \left(\hat{Y}_{s+1}^n - \hat{K}_{s+1}^n \right) + 2\alpha \hat{K}_s \hat{K}_s^n - 2\sigma\alpha \hat{K}_s \hat{C}_s^n \right) - 2\alpha \hat{K}_t \hat{K}_t^n + t.i.p, \\
= \;& E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{array}{c} \frac{2\alpha\beta}{1-(1-\delta)\beta} \hat{K}_{s+1} \left(\sigma \hat{C}_{s+1}^n - \sigma \hat{C}_s^n - J'' \Delta \hat{K}_{t+1}^n + \beta J'' \Delta \hat{K}_{t+2}^n \right) \\ +2\alpha \hat{K}_s \hat{K}_s^n - 2\sigma\alpha \hat{K}_s \hat{C}_s^n \end{array} \right) \\
& + t.i.p,
\end{aligned} \tag{2.160}
$$

where the final equality follows from Equation (2.88). Moreover, using that fact that $\theta_i = \frac{\delta\alpha\beta}{1-(1-\delta)\beta}$,

$$
\frac{2\alpha\beta}{1-(1-\delta)\beta} = \frac{2\theta_i}{\delta}.
$$

Equation (2.160) is substituted into the social welfare function (2.158) as follows:

$$
\begin{aligned}
U_t \;=\; & -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{array}{c} \theta_c c_s^2 + \theta_g g_s^2 + (1-\alpha)(\alpha+\varphi) n_s^2 - \alpha^2 k_s^2 - 2\alpha(1-\alpha) k_s n_s \\ +\epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i(1-\delta)}{\delta} \hat{K}_s^2 - 2\sigma \frac{\theta_i}{\delta} (\hat{K}_{s+1} - (1-\delta)\hat{K}_s) \hat{C}_s^n \\ -\frac{2\theta_i}{\delta} \hat{K}_{s+1} \left(\sigma \hat{C}_{s+1}^n - \sigma \hat{C}_s^n - J'' \Delta \hat{K}_{s+1}^n + \beta J'' \Delta \hat{K}_{s+2}^n \right) \\ -2\alpha \hat{K}_s \hat{K}_s^n + 2\sigma\alpha \hat{K}_s \hat{C}_s^n + \frac{\theta_i}{\delta} \hat{K}_{s+1}^2 - \frac{\theta_i}{\delta} J'' \left(\Delta \hat{K}_{s+1} \right)^2 \end{array} \right) \\
& + t.i.p + O(3),
\end{aligned} \tag{2.161}
$$

$$
\begin{aligned}
&= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \theta_c c_s^2 + \theta_g g_s^2 + (1-\alpha)(\alpha+\varphi) n_s^2 - \alpha^2 k_s^2 - 2\alpha(1-\alpha) k_s n_s \\ +\epsilon Var_i(\hat{P}_s(i)) + 2\sigma \frac{\theta_i}{\delta} \left[\left((1-\delta) + \frac{\alpha\delta}{\theta_i} \right) \hat{K}_s \hat{C}_s^n - \hat{K}_{s+1} \hat{C}_{s+1}^n \right] \\ -\frac{\theta_i}{\delta} \left[\left((1-\delta) + \frac{\alpha\delta}{\theta_i} \right) \hat{K}_s^2 - \hat{K}_{s+1}^2 \right] + \alpha \hat{K}_s^2 - 2\alpha \hat{K}_s \hat{K}_s^n \\ -\frac{\theta_i}{\delta} J'' \left(2\hat{K}_{s+1} \left(\beta \Delta \hat{K}_{s+2}^n - \Delta \hat{K}_{s+1}^n \right) + \left(\Delta \hat{K}_{s+1} \right)^2 \right) \end{pmatrix} \\
&\quad + t.i.p + O(3). \qquad (2.162)
\end{aligned}
$$

And notice that

$$
\begin{aligned}
& E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\hat{K}_{s+1} \left(\beta \Delta \hat{K}_{s+2}^n - \Delta \hat{K}_{s+1}^n \right) + \left(\Delta \hat{K}_{s+1} \right)^2 \right), \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\hat{K}_s \left(\Delta \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \left(\Delta \hat{K}_{s+1}^n \right) + \left(\Delta \hat{K}_{s+1} \right)^2 \right) - 2\hat{K}_t \left(\Delta \hat{K}_{t+1}^n \right), \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\hat{K}_s \left(\Delta \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \left(\Delta \hat{K}_{s+1}^n \right) + \hat{K}_{s+1}^2 - 2\hat{K}_s \hat{K}_{s+1} + \hat{K}_s^2 \right) + t.i.p, \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} 2\hat{K}_s \left(\Delta \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \left(\Delta \hat{K}_{s+1}^n \right) \\ + \left(\hat{K}_{s+1} - \hat{K}_{s+1}^n \right)^2 + 2\hat{K}_{s+1} \hat{K}_{s+1}^n \\ -2 \left(\hat{K}_s - \hat{K}_s^n \right) \left(\hat{K}_{s+1} - \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \hat{K}_s^n - 2\hat{K}_s \hat{K}_{s+1}^n \\ + \left(\hat{K}_s - \hat{K}_s^n \right)^2 + 2\hat{K}_s \hat{K}_s^n \end{pmatrix} + t.i.p, \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\Delta k_{s+1} \right)^2 + t.i.p. \qquad (2.163)
\end{aligned}
$$

The utility-based welfare function is just

$$
\begin{aligned}
&= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + (1-\alpha)(\alpha+\varphi) n_s^2 - 2\alpha(1-\alpha) k_s n_s \\ +\alpha(1-\alpha) k_s^2 - \frac{\theta_i}{\delta} J'' \left(\Delta k_{s+1} \right)^2 + \epsilon Var_i(\hat{P}_s(i)) \end{pmatrix} \\
&\quad + t.i.p + O(3). \qquad (2.164)
\end{aligned}
$$

The derivation of Equation (2.164) a simplification of Cheng (2009).

Thus the second-order approximated utility-based welfare function can be writ-

ten completely in gap form. The welfare function can be represented in terms of output. The labour term is eliminated using the production function:

$$\begin{aligned} & (1-\alpha)(\alpha+\varphi)n_s^2 + \alpha(1-\alpha)k_s^2 - 2\alpha(1-\alpha)k_s n_s \\ = \; & \frac{\alpha+\varphi}{(1-\alpha)}y_s^2 - 2\frac{\alpha(1+\varphi)}{(1-\alpha)}y_s k_s + \frac{\alpha(1+\alpha\varphi)}{(1-\alpha)}k_s^2. \end{aligned} \tag{2.165}$$

Hence the utility-based welfare function can be written as:

$$\begin{aligned} U_t \;=\; & -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\left(\begin{array}{c}\sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + \frac{\alpha+\varphi}{(1-\alpha)}y_s^2 - 2\frac{\alpha(1+\varphi)}{(1-\alpha)}y_s k_s \\ +\frac{\alpha(1+\alpha\varphi)}{(1-\alpha)}k_s^2 - \frac{\theta_i}{\delta}J''\left(\Delta k_{s+1}\right)^2 + \epsilon Var_i(\hat{P}_s(i))\end{array}\right) \\ & +t.i.p + O(3). \end{aligned} \tag{2.166}$$

Price dispersion

In the hybrid Phillips curve, the assumption is that a fraction $(1-\gamma)$ of the households in the economy are able to change their prices in each period. Consequently, the distribution of prices, $\{P_t(i)\}$, at time t consists of γ times the distribution of prices at time $t-1$, plus two atoms of size $(1-\gamma)(1-\omega)$ and $(1-\gamma)\omega$ at the two new prices, P_t^{FL} and P_t^{BL}, respectively. Define

$$\bar{P}_t \;\equiv\; E_i \ln P_t(i).$$

Using the recursive characterization of the distribution of prices at date t that

$$\begin{aligned}
\bar{P}_t - \bar{P}_{t-1} &= E_i(\ln P_t(i) - \bar{P}_{t-1}), \\
&= \gamma E_i(\ln P_{t-1}(i) - \bar{P}_{t-1}) + (1-\gamma)(1-\omega)(\ln P_t^{FL} - \bar{P}_{t-1}) \\
&\quad +(1-\gamma)\omega(\ln P_t^{BL} - \bar{P}_{t-1}), \\
&= \gamma(\bar{P}_{t-1} - \bar{P}_{t-1}) + (1-\gamma)(1-\omega)(\ln P_t^{FL} - \bar{P}_{t-1}) \\
&\quad +(1-\gamma)\omega(\ln P_t^{BL} - \bar{P}_{t-1}), \\
&= (1-\gamma)(1-\omega)(\ln P_t^{FL} - \bar{P}_{t-1}) + (1-\gamma)\omega(\ln P_t^{BL} - \bar{P}_{t-1}), \\
&= (1-\gamma)(\ln P_t^{*} - \bar{P}_{t-1}), \\
\ln P_t^{*} &= \bar{P}_{t-1} + \frac{1}{(1-\gamma)}\pi_t, \qquad (2.167)
\end{aligned}$$

where $\bar{P}_t - \bar{P}_{t-1} = \pi_t$ up to a first order approximation. The definition of $\ln P_t^*$ is

$$\ln P_t^{*} \equiv (1-\omega)\ln P_t^{FL} + \omega \ln P_t^{BL}. \qquad (2.168)$$

Next, notice that

$$Var_i(\hat{P}_t(i)) = Var_i(\ln P_t(i) - \ln P) = Var_i(\ln P_t(i)). \qquad (2.169)$$

Hence, we may derive an expression for $Var_i(\hat{P}_t(i))$:

$$\begin{aligned}
Var_i(\hat{P}_t(i)) &= Var_i[\ln P_t(i) - \bar{P}_{t-1}], \\
&= E_i\left([\ln P_t(i) - \bar{P}_{t-1}]^2\right) - [E_i \ln P_t(i) - \bar{P}_{t-1}]^2. \qquad (2.170)
\end{aligned}$$

Again, using the recursive characterization of the distribution of prices we can see

that the first term on the right-hand side can be rewritten as

$$\begin{aligned} E_i\left([\ln P_t(i) - \bar{P}_{t-1}]^2\right) &= \gamma E_i\left([\ln P_{t-1}(i) - \bar{P}_{t-1}]^2\right) + (1-\gamma)(1-\omega)[\ln P_t^{FL} - \bar{P}_{t-1}]^2 \\ &\quad +(1-\gamma)\omega[\ln P_t^{BL} - \bar{P}_{t-1}]^2, \\ &= \gamma Var_i(\hat{P}_{t-1}(i)) + (1-\gamma)(1-\omega)[\ln P_t^{FL} - \bar{P}_{t-1}]^2 \\ &\quad +(1-\gamma)\omega[\ln P_t^{BL} - \bar{P}_{t-1}]^2. \end{aligned} \tag{2.171}$$

Using the log-linear backward-looking pricing rule

$$\ln P_t^{BL} = \ln P_{t-1}^* + \pi_{t-1} + \eta y_{t-1}, \tag{2.172}$$

and recall

$$\ln P_t = E_i \ln P_t(i) + O(2) = \bar{P}_t + O(2),$$

I further develop the last two terms on the right-hand side of equation (2.171):

$$\begin{aligned} \ln P_t^{BL} - \bar{P}_{t-1} &= \ln P_{t-1}^* + \pi_{t-1} + \eta y_{t-1} - \bar{P}_{t-1}, \\ &= \ln P_{t-1}^* - \bar{P}_{t-2} - (\bar{P}_{t-1} - \bar{P}_{t-2}) + \pi_{t-1} + \eta y_{t-1}, \\ &= \ln P_{t-1}^* - \bar{P}_{t-2} + \eta y_{t-1} + O(2), \\ &= \frac{1}{1-\gamma}\pi_{t-1} + \eta y_{t-1} + O(2), \end{aligned} \tag{2.173}$$

where the final equation uses Equation (2.167). Using this,

$$\begin{aligned}
\ln P_t^{FL} - \bar{P}_{t-1} &= \frac{1}{1-\omega}\ln P_t^* - \frac{\omega}{1-\omega}\ln p_t^{BL} - \bar{P}_{t-1}, \\
&= \frac{1}{1-\omega}\ln P_t^* - \frac{\omega}{1-\omega}(\ln P_{t-1}^* + \pi_{t-1} + \eta y_{t-1}) - \bar{P}_{t-1}, \\
&= \frac{1}{1-\omega}(\ln P_t^* - \bar{P}_{t-1}) - \frac{\omega}{1-\omega}(\ln P_{t-1}^* + \pi_{t-1} + \eta y_{t-1} - \bar{P}_{t-1}), \\
&= \frac{1}{1-\omega}(\ln P_t^* - \bar{P}_{t-1}) - \frac{\omega}{1-\omega}(\ln P_{t-1}^* - \bar{P}_{t-2} + \eta y_{t-1}) + O(2), \\
&= \frac{1}{(1-\gamma)(1-\omega)}\pi_t - \frac{\omega}{1-\omega}(\frac{1}{1-\gamma}\pi_{t-1} + \eta y_{t-1}) + O(2). \quad (2.174)
\end{aligned}$$

After that, I substitute the above expressions into equation (2.170):

$$\begin{aligned}
Var_i(\hat{P}_t(i)) &= \gamma Var_i(\hat{P}_{t-1}(i)) + (1-\gamma)(1-\omega)[\ln P_t^{FL} - \bar{P}_{t-1}]^2 \\
&\quad +(1-\gamma)\omega[\ln P_t^{BL} - \bar{P}_{t-1}]^2 - [\bar{P}_t - \bar{P}_{t-1}]^2, \\
&= \gamma Var_i(\hat{P}_{t-1}(i)) \\
&\quad +(1-\gamma)(1-\omega)\left(\frac{1}{(1-\gamma)(1-\omega)}\pi_t - \frac{\omega}{1-\omega}\left(\frac{1}{1-\gamma}\pi_{t-1} + \eta y_{t-1}\right)\right)^2 \\
&\quad +(1-\gamma)\omega\left(\frac{1}{1-\gamma}\pi_{t-1} + \eta y_{t-1}\right)^2 - \pi_t^2 + O(3), \\
&= \gamma Var_i(\hat{P}_{t-1}(i)) + \frac{\gamma}{1-\gamma}\pi_t^2 - \frac{2\omega\eta}{1-\omega}\Delta\pi_t y_{t-1} \\
&\quad +\frac{\omega}{(1-\gamma)(1-\omega)}\Delta\pi_t^2 + \frac{(1-\gamma)\omega\eta^2}{1-\omega}y_{t-1}^2 + O(3), \quad (2.175)
\end{aligned}$$

where $\Delta\pi_t = \pi_t - \pi_{t-1}$, *i.e.* $\Delta\pi_t$ is the acceleration of the price level at time t. Setting

$$\Omega_t \equiv \frac{\gamma}{1-\gamma}\pi_t^2 - \frac{2\omega\eta}{1-\omega}\Delta\pi_t y_{t-1} + \frac{\omega}{(1-\gamma)(1-\omega)}\Delta\pi_t^2 + \frac{(1-\gamma)\omega\eta^2}{1-\omega}y_{t-1}^2, \quad (2.176)$$

then, by iterating backwards, I get

$$Var_i(\hat{P}_t(i)) \;=\; \gamma^{t+1} Var_i(\hat{P}_{-1}(i)) + \sum_{s=0}^{t} \gamma^{t-s}\Omega_s + O(3). \tag{2.177}$$

I can now take the discounted present value of these terms for all periods $t \geq 0$:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t Var_i(\hat{P}_{t-1}(i)) &= \sum_{t=0}^{\infty} \beta^t \left(\gamma^{t+1} Var_i(\hat{P}_{-1}(i)) + \sum_{s=0}^{t} \gamma^{t-s}\Omega_s \right) + O(3), \\
&= \sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{t} \gamma^{t-s}\Omega_s + t.i.p + O(3), \\
&= \sum_{t=0}^{\infty} \beta^t (\Omega_0 \gamma^t + \Omega_1 \gamma^{t-1} + ... + \Omega_{t-1}\gamma + \Omega_t) + t.i.p + O(3), \\
&= \sum_{t=0}^{\infty} (\Omega_0 (\gamma\beta)^t + \Omega_1 \beta (\gamma\beta)^{t-1} + ... + \Omega_{t-1}\beta^{t-1}(\gamma\beta) + \Omega_t \beta^t) \\
&\quad + t.i.p + O(3), \\
&= \frac{1}{1-\gamma\beta} \sum_{t=0}^{\infty} \beta^t \Omega_t + t.i.p + O(3), \\
&= \frac{1}{1-\gamma\beta} \sum_{t=0}^{\infty} \beta^t \begin{pmatrix} \frac{\gamma}{1-\gamma}\pi_t^2 - \frac{2\omega\eta}{1-\omega}\Delta\pi_t y_{t-1} \\ + \frac{\omega}{(1-\gamma)(1-\omega)}\Delta\pi_t^2 + \frac{(1-\gamma)\omega\eta^2}{1-\omega} y_{t-1}^2 \end{pmatrix} \\
&\quad + t.i.p + O(3).
\end{aligned} \tag{2.178}$$

Hence the social loss function L is

$$\begin{aligned}
L_t \;=\; & -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + \frac{\alpha+\varphi}{(1-\alpha)} y_s^2 \\ -2\frac{\alpha(1+\varphi)}{(1-\alpha)} y_s k_s + \frac{\alpha(1+\alpha\varphi)}{(1-\alpha)} k_s^2 - \frac{\theta_i}{\delta} J'' (\Delta k_{s+1})^2 \\ + \frac{\epsilon}{\kappa}\pi_s^2 + 2\frac{\epsilon}{\kappa}\phi_1 (\Delta\pi_s) y_{s-1} + \frac{\epsilon}{\kappa}\phi_2 (\Delta\pi_s)^2 + \frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2 \end{pmatrix} \\
& + t.i.p + O(3),
\end{aligned} \tag{2.179}$$

where

$$\kappa = \frac{(1-\gamma\beta)(1-\gamma)}{\gamma}, \qquad \phi_1 = -\frac{\omega\eta(1-\gamma)}{\gamma(1-\omega)}, \qquad \phi_2 = \frac{\omega}{\gamma(1-\omega)}, \qquad \phi_3 = \frac{(1-\gamma)^2\omega\eta^2}{\gamma(1-\omega)}.$$

To derive the social welfare function in the model without capital, substitute $\alpha = 0,\ \theta_i = 0,\ \theta_c = \theta$ and $\theta_g = (1-\theta)$ to obtain:

$$\begin{aligned} L_t &= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{array}{c} \sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 + \frac{\epsilon}{\kappa}\pi_s^2 \\ +2\frac{\epsilon}{\kappa}\phi_1 \left(\Delta\pi_s\right) y_{s-1} + \frac{\epsilon}{\kappa}\phi_2 \left(\Delta\pi_s\right)^2 + \frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2 \end{array} \right) \\ &\quad + t.i.p + O(3). \end{aligned} \tag{2.180}$$

This is Equation (2.43) in the main text.

When there is no persistence in inflation, further substitute $\omega = \phi_1 = \phi_2 = \phi_3 = 0$ to obtain:

$$\begin{aligned} L_t &= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 + \frac{\epsilon}{\kappa}\pi_s^2 \right) \\ &\quad + t.i.p + O(3). \end{aligned} \tag{2.181}$$

This is Equation (2.4) in the main text.

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Chapter 3

Responses to Global Imbalances: Debt Deleveraging, Zero Bound and Exchange Rate Behaviour

3.1 Introduction

In this chapter, I build a simple, tractable model to study the causes of global imbalances in the last fifteen years. It is apparent that these imbalances were interconnected with what happened during the great moderation, and with what has happened since the global financial crisis of 2008. The model in this chapter attempts to link these events in a straightforward manner.

The initial component of this story is that of the 'saving glut' made famous by Bernanke (2005). He argued that this was partly caused by a rise in excess saving of China and other emerging-market economies following export-led growth in the aftermath of the Asian financial crisis in 1997-98. This rapid rise in saving contributed to a fall in world interest rates which fuelled the asset market bubbles in the developed economies (Obstfeld and Rogoff 2005).

The second component of the story is that of deleveraging in the developed economies. The asset market bubbles and property-backed leveraged developed in the great moderation went into reverse as soon as house prices started to decline, which they did, in the US, in 2006. The subsequent collapse of an over-leveraged financial system and cut-backs on construction and household spending in affected economies generated a deep recession. That collapse then led to an even bigger *ex ante* saving glut and so to a further decline in the real interest rate on safe assets to the depressed levels we see today. See, for example, Wolf (2012). I seek to understand the connections between these events in a simple formal model.

Blanchard and Milesi-Ferretti (2011) study these events with a static, non-microfounded model. Their model has the following three equations:

$$\begin{aligned} \text{Home output} \quad &: \quad \bar{Y} = Y(S, R), \;\; Y_S < 0, \; Y_R < 0. \\ \text{Foreign output} \quad &: \quad \bar{Y}^* = Y^*(S, R^*), \;\; Y_S > 0, \; Y_R < 0. \\ \text{Perfect capital mobility} \quad &: \quad R = R^*. \end{aligned}$$

The first equation is the home market clearing condition. Demand for home goods falls when the real exchange rate S appreciates (where S is the price of home goods in terms of foreign goods) and when the real interest rate rises. $\bar{Y}$ is an exogenous endowment of home goods. An analogous condition holds in the foreign country, where a 'star' denotes a foreign variable. Finally, there is perfect capital mobility, so home and foreign interest rates are equal.

Figure 3.1 shows the iso-output loci and the global equilibrium of this model. Suppose there is a rise in desired saving in the foreign country. This means that the iso-output locus of the foreign country shifts to the right. Also suppose that the foreign country ensures full employment by depreciation. In the world in which there is no zero bound, the foreign exchange rate depreciates to attract demand

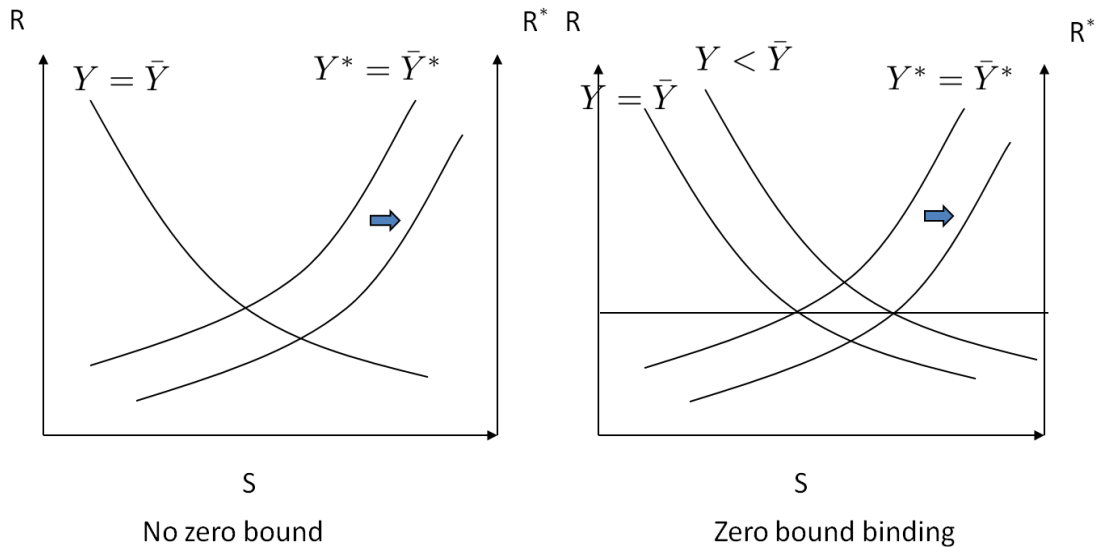


Figure 3.1: Blanchard and Milesi-Ferretti (2011): A rise in preference to save in the foreign country. Left panel: no zero bound. Right panel: zero bound binds.

from the rest of the world, and world interest rates fall, as in the left panel. Both countries are in full employment. However, in the world in which the zero bound binds, as in the right panel, a rise in desired saving in the foreign country causes the foreign country to depreciate its exchange rate, 'stealing' demand from the home country. The home country is unable to reduce interest rates any further and suffers from a rise in unemployment. This argument suggests that the saving glut becomes a problem if the real interest rates are not free to move.

How much of these results survive in a microfounded, intertemporal world? To answer this question, I build a formal two-country model. In the model, homogeneous consumers in each country supply labour and consume both kinds of goods and save. There is home bias in consumption. Demand for goods comes only from consumers. The only way in which a country can save for the future is by running a current account surplus. In equilibrium, the interest rates and real exchange rates must move in such a way that the goods markets and debt markets must clear. I

will call the two countries the US and China.[1]

I consider three events in the model, corresponding to the global saving glut, the global financial crisis up to the present, and a time in the future when external debts will be repaid. Specifically, I first represent the global saving glut as a persistent rise in the subjective discount factor in China. Next, I study the global financial crisis. The crisis is a 'deleveraging shock' in the US on top of the saving shock in China. The shock imposes a binding debt limit to the US so that consumption by the US consumers are constrained. In addition, I will study a situation in which I impose a 'zero-bound' constraint such that the net nominal interest rate in each country cannot fall below zero.

My main results are the following. First, in response to the Chinese increasing their saving, the real interest rates fall in both China and the US. Overall, saving rises in China and falls in the US, which gives rise to global imbalances, and the Chinese terms of trade worsen initially – *i.e.* the Chinese real exchange rate depreciates. In my model, the terms of trade in China worsen because US consumers who prefer US goods consume more in this period. However, in the new equilibrium, the Chinese real exchange rate appreciates vis-a-vis the US dollar. This is because in the long run US consumers will have accumulated positive external debt and pay interest to Chinese consumers.

Second, in response to the deleveraging shock, US consumers, who are impatient compared with those in China, become leverage-constrained and are forced to cut back their consumption. Clearly the less is the US able to borrow, the lower are the interest rates in the world. This is important because the interest rates are

[1]The model applies to other countries whose saving increased substantially during the great moderation. These countries include the Middle East oil exporters and the developing Asian countries. According to IMF World Economic Outlook (WEO) data, the gross national savings as a percentage of national GDP in the Middle East rose from about 25% in 2000 to over 40% in 2007. During the same period, the gross national savings in developing Asia rose from 30% to about 45%.

already reduced before the crisis as a result of the previous 'saving glut' shock. When the nominal interest rate in the US hits the zero lower bound, the US will have a substantial underemployment of resources. The reason is that when the interest rate can no longer fall to clear the goods market, firms adjust their prices downwards. The deflation makes the deleveraging constraint more binding, leading to lower consumption and more deflation. With my specification of the preferences, output also falls significantly, causing an exchange rate *appreciation*, consistent with what was observed in the US immediately after the Lehman crisis in 2008. The real exchange rate appreciation in my model is caused by a rise in the real interest rate, which is an alternative explanation to Blanchard and Milesi Ferretti's (2011) 'demand stealing' channel and the popular 'flight-to-safety' idea.

3.1.1 Related Literature

There are plenty of descriptive accounts of the connection between global imbalances and the global financial crisis. See, for instance, Obstfeld and Rogoff (2009), Bean (2009), Krugman (2009) and Truman (2009). However, to my knowledge there is no discussion on this connection which contains a formal model. I identify below two strands of theoretical literature, the first on global imbalance, and the second on deleveraging and the zero bound. The aim of this chapter is to bring these two strands of the modelling literature together.

This chapter is related to a number of other models studying global imbalances. Caballero, Farhi and Gourinchas (2008) develop a two-country model to match the low real interest rate, a current account deficit in the US and a rise in the fraction of US assets in global portfolios. They explain global imbalances by the inability of the financial systems of the emerging economies to supply assets to use all the savings available. In their model, global imbalances are caused by a collapse in the asset markets in the emerging economies in the Asian financial crisis and the fast growth

of China. There are two main differences between Caballero et al. (2008) and the present study. First, I abstract from the details of the asset market and model the aftermath of the Asian financial crisis as an exogenous rise in Chinese consumers' preference to save. This is a common approach, shared by Artige and Cavenaile (2011) and Choi, Mark and Sul (2008). Second, I use the model to demonstrate the connections between global imbalances and the global financial crisis, giving a formal representation to the interconnections discussed in Obstfeld and Rogoff (2009).

This chapter is also related to the literature that studies deleveraging and the zero interest rate lower bound. Early models such as Eggertsson and Woodford (2003) and Christiano, Eichenbaum and Rebelo (2010) study an exogenous rise in discount factor which pushes the interest rate to the zero bound. They assume in each period there is a constant probability that the discount factor falls back to normal and stays there forever. This allow them to evaluate the forward-looking expectations and solve their models at the zero bound. Fernandez-Villaverde et al. (2012) model the zero lower bound as an occasionally binding constraint and study conventional AR(1) shocks. These models do not consider deleveraging. Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2011) and Philippon and Midrigan (2011) model deleveraging and consider the zero lower bound in closed economy. In particular, Eggertsson and Krugman (2012) study a model with patient and impatient consumers. Impatient consumers borrow from patient consumers up to an exogenous debt limit. Eggertsson and Krugman analyse a deleveraging shock in which this exogenous debt limit for impatient consumers falls. To restore full employment, the interest rate must fall to induce the patient consumption smoothers to spend. If the fall in the debt limit is so severe that the zero lower bound for the interest rate binds, some goods are not demanded. My model can be thought of as an international version of Eggertsson and Krumgan (2012), in which the developed world is analogous to impatient consumers and the saving-glut countries

to the patient. Benigno and Romei (2012) build a two-country model of Eggertsson and Krugman (2012) similar to ours, but they focus on the implications to welfare of fixed exchange rate regimes and the monetary unions. Fornaro (2012) study a heterogeneous multi-country monetary union model with aggregate uncertainty, analysing the way in which deleveraging in a liquidity trap gives rise to a union-wide recession. My work differs from Benigno and Romei (2012) and Fornaro (2012) in considering adjustments in a flexible exchange rate setting whereas they study adjustment without exchange rate flexibility.

3.1.2 The Plan of this Chapter

The rest of the chapter is structured as follows. Section 3.2 sets out the two-country model to study what happens when the world is hit by a 'saving glut' shock in China, followed by a deleveraging shock in the US. I calibrate and simulate the model and show that these two shocks make the interest rates in both of the two countries fall. This means that deleveraging may cause the interest rates to fall to the zero bound. In section 3.3 I formally study deleveraging at the zero bound by extending the model to include nominal rigidities. I compare the response of the deleveraging shock with the model without zero bound and show that a zero bound makes the fall in output more severe. Section 3.4 concludes.

3.2 The Model

The model follows closely those of open-economy macroeconomics literature such as Obstfeld and Rogoff (2005) and Benigno (2009). Homogenous consumers in each of the two countries supply labour, consume and save. Consumers exhibit consumption home bias, so a change in relative wealth of consumers in the two countries affects the relative demand for goods and the real exchange rate. Goods are produced

in each of the two countries by perfectly competitive firms which turn labour into goods. For simplicity I assume there is no investment by firms in any capital stock. There is an international debt market. In this section, I build the benchmark model without nominal rigidities.

3.2.1 The Allocation of Consumption between Home and Foreign Goods

Home and foreign consumers consume composite goods C_t and C_t^* respectively using a CES aggregator with home bias.

$$C_t = \left(\alpha^{\frac{1}{\eta}} C_{Ht}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} C_{Ft}^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}, \tag{3.1}$$

$$C_t^* = \left((1-\alpha)^{\frac{1}{\eta}} \left(C_{Ht}^*\right)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} \left(C_{Ft}^*\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}, \tag{3.2}$$

where C_H and C_F are the home consumption of home and foreign goods, and C_H^* and C_F^* are the foreign counterparts. I denote P_{Ht} and P_{Ft} the price of home and foreign goods in the home country; and similarly, I denote P_{Ht}^* and P_{Ft}^* the price of home and foreign goods in the foreign country. The home bias assumption implies that $\alpha > 1/2$. Optimisation between domestically produced and imported goods implies

$$C_{Ht} = \alpha \left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t, \qquad C_{Ft} = (1-\alpha)\left(\frac{P_{Ft}}{P_t}\right)^{-\eta} C_t,$$

$$C_{Ht}^* = (1-\alpha)\left(\frac{P_{Ht}^*}{P_t^*}\right)^{-\eta} C_t^*, \qquad C_{Ft}^* = \alpha\left(\frac{P_{Ft}^*}{P_t^*}\right) C_t^*,$$

where the home and foreign aggregate price levels are

$$P_t \equiv \left(\alpha P_{Ht}^{1-\eta} + (1-\alpha) P_{Ft}^{1-\eta}\right)^{\frac{1}{1-\eta}}, \tag{3.3}$$

$$P_t^* \equiv \left((1-\alpha)\left(P_{Ht}^*\right)^{1-\eta} + \alpha\left(P_{Ft}^*\right)^{1-\eta}\right)^{\frac{1}{1-\eta}}. \tag{3.4}$$

I assume that the law of one price holds, *i.e.* after conversion at the ruling nominal exchange rate, e_t, each good sells at the same price in each country:

$$e_t = \frac{P_{Ht}^*}{P_{Ht}} = \frac{P_{Ft}^*}{P_{Ft}}. \tag{3.5}$$

A rise in e_t is a depreciation of the foreign currency.

I let S_t denote the terms of trade (from the foreign country's perspective),

$$S_t \equiv \frac{P_{Ht}^*}{P_{Ft}^*} = \frac{P_{Ht}}{P_{Ft}}. \tag{3.6}$$

A rise in S is a strengthening of the terms of trade in the home country.

3.2.2 Intertemporal Choice

Consumers in each of the two countries are homogeneous. I assume home consumers have the following utility:

$$U_{Ht} = E_t \sum_{s=0}^{\infty} \left(\Pi_{v=1}^{s} \beta_{Ht+v}\right) u(C_{t+s}, L_{t+s}), \tag{3.7}$$

where the period utility depends on consumption C and labour L. I allow the discount factor $\beta_{H,t}$ to be variable across time.

Consumers are assumed to have Greenwood-Hercowitz-Huffman (1988) (henceforth GHH) preferences. It is well-known that the utility can generate a labour supply schedule that only depends on the real wage. Moreover, Correia, Neves and

Rebelo (1995) show that GHH preferences are better suited to match the second moments of open economies. Raffo (2006) also shows that GHH preferences can improve the empirical performance of two-country models by generating sufficient volatility in consumption. I will explain in detail the choice of the preference and its implications in the discussion below. Specifically, the preferences used are the following:

$$u(C_t, L_t) = \ln\left(C_t - \chi\frac{L_t^{1+\varphi}}{1+\varphi}\right).$$

Consumers maximise their utility subject to the following budget constraint:

$$P_t C_t = W_t L_t + \frac{D_{t+1}}{1+i_t} - D_t. \tag{3.8}$$

In each period, home consumers earn their wage income, repay their (one-period) nominal debt D_t from the last period, obtain new borrowing amount to $D_{t+1}/(1+i_t)$ and purchase consumption goods. I assume that nominal debt is the only financial asset in the system for the tractability of the model.[2] A little manipulation of the budget constraint yields

$$C_t = \frac{W_t}{P_t}L_t + \frac{D_{t+1}}{P_{t+1}}\frac{1}{(1+r_t)} - \frac{D_t}{P_t}, \tag{3.9}$$

where I define $(1+r_t) \equiv \frac{P_t(1+i_t)}{P_{t+1}}$ as the real interest rate.

Consumers maximise their utility subject to the budget constraint and the standard transversality condition associated with debt. This yields the following first

[2]My analysis of global imbalances and the global financial crisis abstracts from an elaborate financial market with multiple financial assets earning different returns. Despite the popularity of the topic, few models consider this. Exceptions include Blanchard, Giavazzi and Sa (2005) and Gourinchas, Rey and Govillot (2010), where the former model relies on imperfect substitutability across assets and the latter studies endogenous portfolio choices under disaster shocks.

order conditions:

$$\frac{1}{\left(C_t - \chi\frac{L_t^{1+\varphi}}{1+\varphi}\right)} = E_t\left(\beta_{Ht+1}(1+i_t)\frac{P_t}{P_{t+1}}\frac{1}{\left(C_{t+1} - \chi\frac{L_{t+1}^{1+\varphi}}{1+\varphi}\right)}\right), \tag{3.10}$$

$$\chi L_t^{\varphi} = \frac{W_t}{P_t}. \tag{3.11}$$

The first equation is the consumption Euler equation. The second equation is the intratemporal tradeoff between consumption and leisure. Notice that this labour supply curve has no wealth effect, which is a feature of the GHH preferences.

Foreign consumers have a symmetric structure to home consumers. Their utility is

$$U_{Ft} = E_t \sum_{s=0}^{\infty} \left(\Pi_{v=1}^{s}\beta_{Ft+v}\right) \ln\left(C_{t+s}^* - \chi\frac{\left(L_{t+s}^*\right)^{1+\varphi}}{1+\varphi}\right). \tag{3.12}$$

The budget constraint for foreign consumers is

$$P_t^* C_t^* = W_t^* L_t^* + \frac{e_t D_{t+1}^*}{1+i_t} - e_t D_t^*. \tag{3.13}$$

where D_t^* is the nominal debt acquired by foreign consumers. e_t is the nominal exchange rate, defined in Equation (3.5). For the world as a whole, debt has a zero net supply, which means that

$$D_t + D_t^* = 0. \tag{3.14}$$

I substitute the debt market clearing condition to the foreign budget constraint to obtain:

$$C_t^* = \frac{W_t^*}{P_t^*}L_t^* - \frac{Q_t}{1+r_t}\frac{D_{t+1}}{P_{t+1}} + Q_t\frac{D_t}{P_t}, \tag{3.15}$$

where Q_t is the real exchange rate, defined as $Q_t \equiv \frac{e_t P_t}{P_t^*}$.

The first order conditions for foreign consumers' utility maximisation problem with respect to consumption C_t^* and labour L_t^* are again the consumption Euler

equation and the labour supply curve.

$$\frac{1}{\left(C_t^* - \chi\frac{(L_t^*)^{1+\varphi}}{1+\varphi}\right)} = E_t\left(\beta_{Ft+1}(1+r_t^*)\frac{1}{\left(C_{t+1}^* - \chi\frac{\left(L_{t+1}^*\right)^{1+\varphi}}{1+\varphi}\right)}\right), \quad (3.16)$$

$$\chi\left(L_t^*\right)^{\varphi} = \frac{W_t^*}{P_t^*}. \quad (3.17)$$

The uncovered interest parity holds so that

$$(1+r_t^*) = E_t\left[(1+r_t)\frac{Q_{t+1}}{Q_t}\right]. \quad (3.18)$$

3.2.3 Production

Home goods are produced in home firms. In this benchmark model, I assume that firms are perfectly competitive and there is no nominal rigidities. Firms produce with a simple linear technology:

$$Y_t = L_t. \quad (3.19)$$

Profit maximisation means that price equals the marginal cost ($W_t = P_{Ht}$) and the profit of the industry is zero. Production in the foreign economy is assumed to have an analogous structure.

3.2.4 Goods Market Clearing

Consumption of home goods by home and foreign consumers should add up to the home output:

$$Y_t = \alpha\left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t + (1-\alpha)\left(\frac{P_{Ht}^*}{P_t^*}\right)^{-\eta} C_t^*. \quad (3.20)$$

An analogous goods market clearing condition holds for the foreign economy:

$$Y_t^* = (1-\alpha)\left(\frac{P_{Ft}}{P_t}\right)^{-\eta} C_t + \alpha\left(\frac{P_{Ft}^*}{P_t^*}\right)^{-\eta} C_t^*. \tag{3.21}$$

This completes the description of the model.

3.2.5 Model Summary

I summarise the model as follows:

The home and foreign budget constraints are

$$C_t = \Theta_t^{-\frac{1}{1-\eta}} Y_t + \frac{D_{t+1}}{P_{t+1}}\frac{1}{(1+r_t)} - \frac{D_t}{P_t}, \tag{3.22}$$

$$C_t^* = \Xi_t^{-\frac{1}{1-\eta}} Y_t^* - \frac{Q_t}{1+r_t}\frac{D_{t+1}}{P_{t+1}} + Q_t\frac{D_t}{P_t}. \tag{3.23}$$

The home and foreign consumption Euler equations are

$$\frac{1}{\left(C_t - \chi\frac{Y_t^{1+\varphi}}{1+\varphi}\right)} = \beta_{Ht+1}(1+r_t)\frac{1}{\left(C_{t+1} - \chi\frac{Y_{t+1}^{1+\varphi}}{1+\varphi}\right)}, \tag{3.24}$$

$$\frac{1}{\left(C_t^* - \chi\frac{(Y_t^*)^{1+\varphi}}{1+\varphi}\right)} = \beta_{Ft+1}(1+r_t^*)\frac{1}{\left(C_{t+1}^* - \chi\frac{\left(Y_{t+1}^*\right)^{1+\varphi}}{1+\varphi}\right)}. \tag{3.25}$$

The uncovered interest parity is

$$(1+r_t^*) = (1+r_t)\frac{Q_{t+1}}{Q_t}. \tag{3.26}$$

The home and foreign intratemporal tradeoffs are

$$\Theta_t^{-\frac{1}{1-\eta}} = \chi Y_t^{\varphi}, \tag{3.27}$$

$$\Xi_t^{-\frac{1}{1-\eta}} = \chi (Y_t^*)^{\varphi}. \tag{3.28}$$

The home and foreign goods market clearing conditions are

$$Y_t = \alpha\Theta_t^{\frac{\eta}{1-\eta}} C_t + (1-\alpha) S_t^{-\eta} \Xi_t^{\frac{\eta}{1-\eta}} C_t^*, \tag{3.29}$$

$$Y_t^* = (1-\alpha) S_t^{\eta} \Theta_t^{\frac{\eta}{1-\eta}} C_t + \alpha \Xi_t^{\frac{\eta}{1-\eta}} C_t^*, \tag{3.30}$$

where Θ_t, Ξ_t, Q_t are functions of the terms of trade S_t:

$$\Theta_t \equiv \alpha + (1-\alpha) S_t^{-(1-\eta)}, \tag{3.31}$$

$$\Xi_t \equiv \alpha + (1-\alpha) S_t^{1-\eta}, \tag{3.32}$$

$$Q_t \equiv Q_t \equiv \frac{e_t P_t}{P_t^*} = S_t \left(\frac{\Theta_t}{\Xi_t} \right)^{\frac{1}{1-\eta}}. \tag{3.33}$$

The system contains 9 equations (Equations (3.22) - (3.30)) and solves for 8 variables

$$\{C_t,\ C_t^*,\ Y_t,\ Y_t^*,\ S_t,\ r_t,\ r_t^*,\ D_t/P_t\}.$$

given the initial level of real debt, D_0/P_0 and the shock process which will be described below. By Walras' law, one of the equations is redundant. Real debt is the only state variable in the system.

3.2.6 The Steady State

Suppose before hitting by any shocks, the model is initially at the steady state. In the following, I characterise the steady state of the model. Throughout this chapter, I assume that the home economy has a constant discount factor ($\beta_{Ht} = \beta_H$ for all t). In the steady state, consumers in both countries choose consumption according to their consumption Euler equations. This means that

$$\beta_H = \frac{1}{(1+\bar{r})} = \frac{1}{(1+\bar{r}^*)} = \bar{\beta}_F, \tag{3.34}$$

where $\bar{\beta}_F$ denotes the steady state for the foreign discount factor and $\bar{r}$ and $\bar{r}^*$ denote the steady state for home and foreign real interest rates. This equation implies that the discount factor in the foreign economy cannot deviate from the discount factor in the home country permanently. Otherwise, suppose foreign consumers are more patient than home consumers permanently. According to their consumption Euler equation, foreign consumers will be delaying their consumption in the steady state, which is a contradiction. This property for open-economy models is well-known in the literature.[3]

There is a related problem concerning debt. As the steady-state version of both the home and foreign consumption Euler equations solve for the steady-state real interest rate, there is an additional degree of freedom for the other variables in the system. Hence, this model belongs to a class of open economy models that features a steady state that depends on initial conditions and equilibrium dynamics that possess a random walk component (Schmitt-Grohe and Uribe, 2003). After a temporary shock hits the system, the system may not return to the pre-shock steady state. Instead, the new steady state is endogenously determined, jointly with the transition dynamics.

3.2.7 Calibration and Simulation

I study a temporary rise in patience in the foreign country. In other words, I assume that China has a saving glut shock. Specifically, I assume that the shock lasts for 60 periods – or 15 years, and after that the discount factor falls back to the equilibrium level. I assume that the rise in patience is initially unanticipated, but after the first period, the length of the shock is known to the consumers. There are no stochastic shocks in the system. Consumers are assumed to have perfect foresight after the initial shock. In other words, the expectation operators in the model can be omitted.

[3]See, for instance, Mendoza (1991), Mendoza and Uribe (2000) and Neumeyer and Perri (2001).

The calibration of the parameters are presented as follows. I assume that the world is initially symmetric so that both home and foreign economies inherit zero real debt in the beginning of the world. I use $\alpha = 0.7$ for the home bias. The elasticity of substitution between home and foreign goods, η, is set to 2. These parameters are the same as Obstfeld and Rogoff (2005). Each period in my model is a quarter. I use the home discount factor $\beta_H = 0.99$, which implies that the annualised steady-state interest rate to be approximately 4.1%, a commonly used value in the macroeconomic literature. The utility weight of disutility of labour, χ is set to 1. This is to keep the output in the initial steady state at unity for tractability. The inverse of Frisch elasticity of labour supply, φ, is set to 1, in line with evidence by Kimball and Shapiro (2008). The values used are common in the literature. The calibrations are summarised in Table 3.1 in Appendix 3.5.1.

It is easy then to verify that in the initial steady state, before any shock occurs, both outputs and consumptions are unity $Y_{init} = Y^*_{init} = C_{init} = C^*_{init} = 1$. The symmetry of the model means that the real exchange rate S_{init} is also unity initially.

The saving glut shock is calibrated as follows:

$$\beta_{Ft} = \begin{cases} \bar{\beta}_F \times (1 + \epsilon_{\beta t}), & t = 1, 2, ...60. \\ \bar{\beta}_F, & thereafter. \end{cases} \tag{3.35}$$

The magnitude of the shock is calibrated so that in 30 periods, (that is, as will be discussed in the next subsection, when the deleveraging shock sets in), the home external debt to GDP ratio is 20%, following Fornaro (2012). This requires a shock to the foreign discount factor, $\epsilon_{\beta t}$, of 0.37% above the steady-state level, at a quarterly frequency. This means that the value of the foreign discount factor when the 'saving glut' shock is in place is 0.9937.[4]

[4]The choice is consistent with calibrations in the zero lower bound literature. Fernandez-Villaverde et al. (2012) and Bernigno and Romei (2012), among others, calibrate their models so that the steady-state annual interest rate is 2.5%, which implies $\beta = 0.9938$. Those models,

I simulate the model using the reverse shooting method. Given the random-walk property of debt discussed above, it is inappropriate to solve the log-linearised version of the model because we do not know the new steady state (and as my simulation will show the new steady state is far from the old one). For this reason, any local approximation methods such as higher-order perturbation methods are not suitable.[5] This means that we have to use non-linear solution methods. Since the external debt position is the only state variable in the system and there exists a steady state for any level of debt, the system reaches the steady state once the shock has dissipated. Under these circumstances, the reverse shooting method is simple to implement under perfect foresight. Besides, the method can be implemented in levels, preserving the non-linearities of the system. Detailed description of the solution algorithm is presented in Appendix 3.5.3.

The simulation results of the saving glut shock are displayed in the dashed line in Figure 3.2. When foreign consumers become more patient, they demand fewer goods initially. In response, the interest rates fall to shift demand back to the present. Since home consumers are more willing to spend and their spending is biased towards home goods, the home terms of trade has to strengthen in order to clear the home and foreign goods markets. As a result the home country accumulates external debt for the periods which foreign consumers are more patient than home. The US real exchange rate in the new steady state is depreciated compared with the initial steady state, due to the accumulation of debt.

One feature of the impulse responses shown in Figure 3.2 is that consumption and output move in the same direction, as required by the data. Backus and Kehoe (1992), for instance, find that consumption is uniformly pro-cyclical using a dataset that covers ten advanced economy (including the US) for at least a century.

however, take the low world interest rates during the saving glut periods as exogenous.

[5]Due to this problem, we cannot use the perturbation method introduced by Holden and Paetz (2012) to analyse the model in the next section with a zero bound.

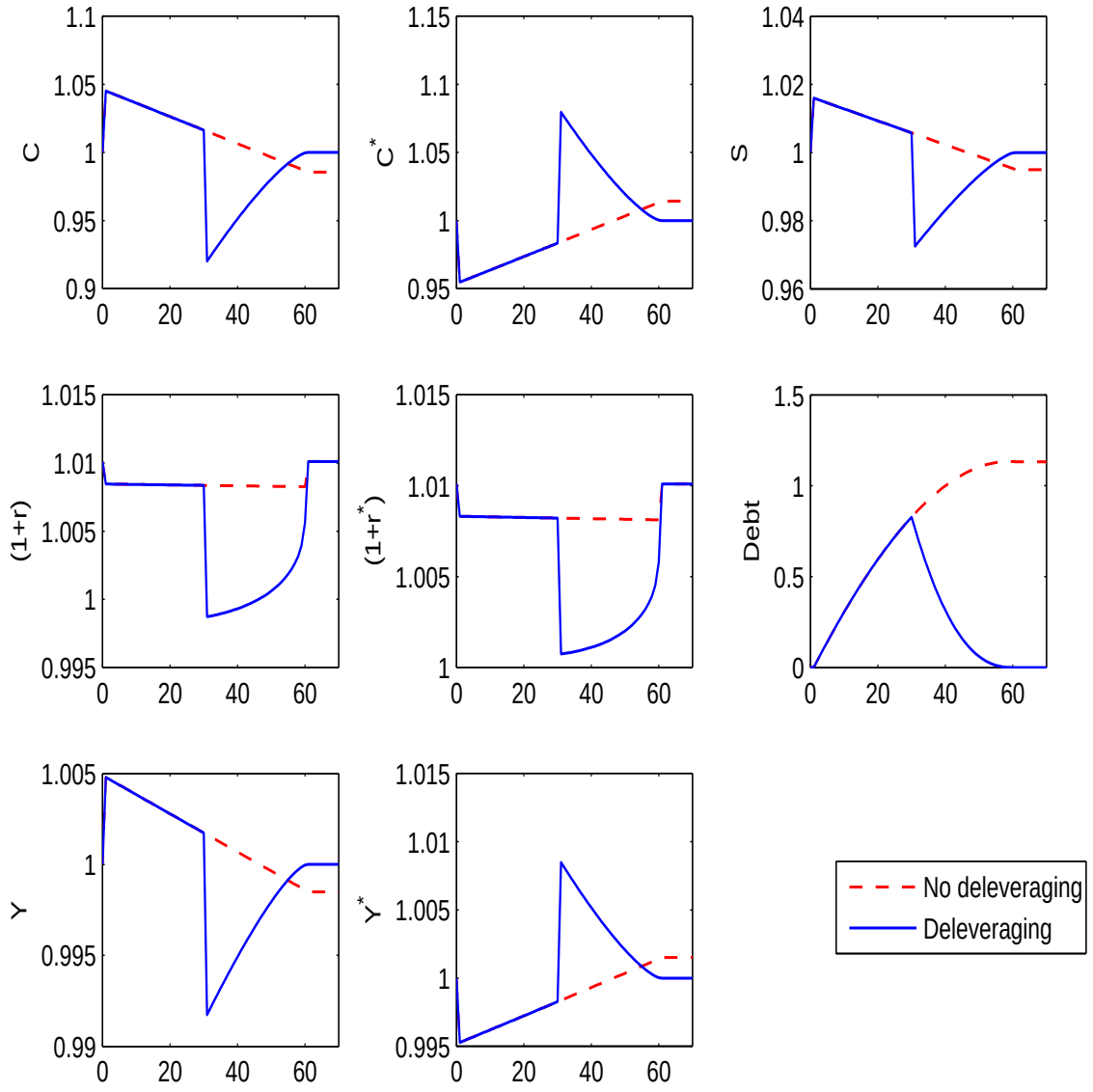


Figure 3.2: Impulse response of a saving glut shock and deleveraging in the benchmark model.

Aguiar and Gopinath (2007) also find co-movement between these two variables in 13 emerging economies, with an average correlation of 0.72. To generate such co-movements, I use GHH preferences in the model. This is because the additively separable preferences cannot generate positive co-movements between output and consumption in my open economy model, even although it can generate positive co-movements in a closed economy.[6] In a closed economy, after a saving shock, consumption falls and so does output. Output must move in the same direction as consumption because the real wage falls by more than consumption so that labour supply falls according to the labour supply condition. However, in my open economy model, with additively separable preferences and reasonable calibrations, the fall in the real wage is lessened because it induces a real exchange rate depreciation which increases foreign demand for domestic goods by so much that output and consumption move in opposite directions. With GHH preferences, the labour supply does not depend on the marginal utility of consumption, so when a saving shock occurs both consumption and output fall (as noted in Fernandez-Villaverde et al., 2011). Moreover, output falls by less than consumption due to a depreciation of the real exchange rate. It is therefore still the case that the 'saving glut' country exhibits a rise in export surplus.

The simulation results are broadly consistent with what we observed between the US and China during the great moderation period. There was a fall in world interest rates, low saving rates in the advanced economies and sustained current account deficits in the US.[7][8] My finding is in line with Bernanke's (2005) 'global

[6] An example of the additively separable preferences is $u(C_t, L_t) = \ln C_t - \chi L_t^{1+\varphi}/(1+\varphi)$.

[7] The US 10-year TIPS rate and the UK 10-year inflation-indexed government bond yields, common market-based proxies of the long-term real interest rate, declined steadily from around 3.5% in 1997 to around 2% in 2003 and stayed flat until 2008. After the crisis the interest rates declined further to below zero. (Source: Global Financial Data)

[8] The US current account deficit in 1998 was about 200 billion USD, or some 3% of US GDP. The deficit quadrupled to 800 billions USD, or 6% of USD GDP in 2006. During the same period, the current account surplus in developing East Asia (which includes China) went up from 2% of GDP in 2001 to 10% in 2007, according to IMF WDI data.

saving glut' hypothesis. However, my simulation results suggest that the saving shock alone does not generate large movements in the real exchange rate of the magnitude observed in reality. Neither does it generate a rise in output in the foreign country. One explanation is that developing countries such as China also experienced sustained rise in productivity during the same period.[9] Including non-traded goods, as stressed by Obstfeld and Rogoff (2005), can as well improve the impulse response of inflation.

Furthermore, Figure 3.2 shows that the level of net external debt in the new steady state is about 28% of annual output, a large number relative to US net foreign debt-to-GDP ratio in 2007 of 17%[10]. Such a large stock of debt could provide a motivation for deleveraging which I will discuss in the next section.

3.2.8 Deleveraging

I study deleveraging in the following way. Suppose as a result of the accumulation of debt there is an unanticipated deleveraging shock in the home country. Suppose this occurs in period 30, the middle of the saving glut period. Also suppose the deleveraging phase lasts for 30 periods and at the end of this phase, the debt in the home country goes back to zero. This assumption is somewhat arbitrary but convenient because at the end of this phase, the impatience shock in the foreign country also ends which means that the system will return to its pre-shock symmetric steady state with no debt.

Specifically, I impose an exogenous debt limit $\overline{\left(\frac{D_t}{P_t}\right)}^{lim}$ to the overspent home

[9]Zhu (2012) finds that the TFP growth in China in the post-1978 period is around 3.2% a year. The US TFP growth is estimated to be 2.3% from 1995-2006. The narrowing of the productivity gap may account for the exchange rate depreciation in China.

[10]This number comes from the updated and extended version of dataset constructed by Lane and Milesi-Ferretti (2007).

country as follows:

$$\frac{D_t}{P_t} \leq \overline{\left(\frac{D_t}{P_t}\right)}^{lim}. \tag{3.36}$$

The debt limit can be thought of as a collateral constraint in reduced form. Collateral constraints are first analysed in DSGE models by Kiyotaki and Moore (1997). In their model, the borrower in a loan contract can only pledge a fraction κ of his future income and this will be the maximum amount the lender is willing to lend. Similar constraints have been used in open-economy models such as Aoki et al. (2010) and Mendoza (2010). However, the fraction of pledgable income κ in these models is not well microfounded and usually treated as an exogenous shock in the financial-frictions literature. The imposition of the debt limit in Equation (3.36) can be thought of as an unexpected fall in κ in a financial crisis.

For simplicity, I consider a case in which the debt limit evolves following an exogenous path similar to Eggertsson and Krugman (2012) and Benigno and Romei (2012):

$$\overline{\left(\frac{D_t}{P_t}\right)}^{lim} = \left(\frac{60-t}{30}\right)^{\varsigma} \frac{D_{30}}{P_{30}}, \qquad \text{for } t = 30, 31...60, \tag{3.37}$$

where ς determines the speed of deleveraging. I assume the deleveraging shock is initially anticipated, but after the initial period, the subsequent path of debt evolution is known by the consumers.

Equation (3.37) implies that debt is decaying throughout the deleveraging periods. During these periods foreign consumers are more patient than home consumers, which implies that the debt limit Equation (3.36) is always binding. This means that consumers in the home country can no longer smooth their consumption according to their consumption Euler equation (3.10). Instead, consumption will be governed by the budget constraint so that when consumers are required to deleverage more quickly, they are forced to cut their consumption by more.

I assume debt decays more than linearly (*i.e.* $\varsigma > 1$), contrary to Benigno

and Romei (2012) and Fornaro (2012), for two reasons. First, it is reasonable that creditors require debt to be deleveraged more quickly when the stock of debt is high. Second, a linear decay of debt as in Bernigno and Romei (2012) and Fornaro (2012) means that when the system exits the deleveraging phase, consumption of the deleveraging country is no longer constrained, which means that the interest rate has to jump up sharply for one period, as required by the consumption Euler equation. I calibrate the speed of deleveraging parameter, ς, to be 2.38. (I leave the discussion of the calibration of this parameter to the next section.)

The solid line in Figure 3.2 shows the adjustment of home and foreign consumption, output and the real interest rates together with the real exchange rates and the net foreign debt after imposing the deleveraging constraint. In response to the deleveraging shock, home consumption is forced to drop immediately. The interest rates decrease to stimulate demand by foreign consumers even although they are relatively more patient. Because of consumption home bias, the relative price of home goods falls, *i.e.* there is an immediate and large depreciation of the home real exchange rate. In the long run, external debt position returns to zero according to the exogenous deleveraging path, Equation (3.37). The system returns to the initial symmetric steady state.

The home interest rate falls significantly, consistent with empirical data. The reason is as follows. First, the foreign interest rate falls to clear the goods markets. Second, the home and foreign interest rates are linked by the uncovered interest parity, Equation (3.18). Along the adjustment path, debt is falling more gradually, which means that home consumption is rising, increasing the demand for home goods. As a result, the US real exchange rate is appreciating along the adjustment path. This means that the home interest rate has to fall initially by more than the foreign interest rate to allow for the exchange rate appreciation. Hence, real exchange rate movements in this open economy model lead to a larger reduction in

the home interest rate in response to the deleveraging shock, compared with closed economy models such as Eggertsson and Krugman (2011).

Output co-moves with consumption in response to the deleveraging shock because consumers have GHH preferences. The reason why additively separable preferences cannot produce the co-movements as required by the data is explained in the previous section: when the deleveraging shock hits the home economy, consumption falls. But with reasonable parameterisation, the depreciation in the real exchange rate ensures that foreign consumers buy more home goods so that the real wage in the home country falls by less than consumption and home output rises. To produce the required co-movement between output and consumption with additively separable preferences, adjustments in the real exchange rate has to be limited. Benigno and Romei (2012) and Fornaro (2012) study the adjustments in fixed exchange rate regimes and monetary unions with nominal rigidities. I choose GHH preferences not only because it is simple, but also because it can produce co-movements between consumption and output for economies with flexible exchange rates in a way that is consistent with observed data.

The fall in the home interest rate during the deleveraging periods is on top of the reduction in the interest rate caused by the saving shock in the foreign economy. One can imagine that if these shocks are large enough, it is possible that the nominal interest rate is pushed to zero and cannot fall further. In the next section, I will extend this model to include nominal rigidity to formally analyse what happens when there is a zero bound in the home economy.

3.3 Model with Nominal Rigidities and the Zero Bound

The previous section shows that when the world is hit by a saving shock and a deleveraging shock, the real interest rates fall to clear the goods markets. This gives rise to the possibility that the nominal interest rate may fall to zero.[11] In this section, I study this situation explicitly by adding nominal rigidities to the model. I show that when the high-debt economy is hit by a large deleveraging shock, the nominal interest rate may hit the zero lower bound. And, when this happens, the fall in output will be severe compared with a world without the zero bound.

In order to study nominal rigidities, I now assume that there are differentiated goods in each country and monopolistically competitive firms set prices in a staggered manner. Specifically, I assume on the supply side there is an infinite number of intermediate goods firms, indexed by $i \in (0,1)$, in each country. Home firms produce with a linear production function:

$$Y_t(i) = L_t(i). \tag{3.38}$$

Aggregate labour is defined as $L_t \equiv \int_0^1 L_t(i)di$. The firms are monopolistically competitive. There is a final goods firm in each country which combines the varieties of goods into a final output using a Dixit-Stiglitz aggregator:

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}. \tag{3.39}$$

[11] And this indeed happened after Lehman's collapse. The Federal funds rate indeed has stayed at 0% to 0.25% per annum since 16 December 2008. See also Figure 3.4 below.

The demand for each variety of goods is

$$Y_t(i) = \left(\frac{P_{Ht}(i)}{P_{Ht}}\right)^{-\theta} Y_t, \tag{3.40}$$

and the home production price aggregate is defined as $P_{Ht} \equiv \left(\int_0^1 P_t(i)^{1-\theta} di\right)^{\frac{1}{1-\theta}}$. Hence, the aggregate production function is

$$Y_t = L_t \times \left[\int_0^1 \left(\frac{P_{Ht}(i)}{P_{Ht}}\right)^{-\theta} di\right]^{-1}. \tag{3.41}$$

I model price setting following Calvo (1983) contracting. Specifically, in each period, there is a probability $(1-\gamma)$ that an intermediate goods firm can re-optimise the price of its goods. This probability is independent across firms and time. With probability γ the price is not re-optimised and assumed to rise at steady-state rate of inflation $\bar{\Pi}_H = 1$, where $\Pi_{Ht} \equiv P_{Ht}/P_{Ht-1}$.

The profit maximisation problem for an intermediate goods firm i which is able to reset price in period t is:

$$Profit(i)_t = \max_{P_{Ht}(i)} E_t \sum_{s=0}^{\infty} \Lambda_{Ht,t+s} \gamma^s \left(\frac{P_{Ht}(i)}{P_{Ht+s}} Y_{t+s}(i) - \frac{1}{\mu_w} \frac{W_{t+s}}{P_{Ht+s}} L_{t+s}(i)\right), \tag{3.42}$$

where $\Lambda_{Ht,t+s} = (\Pi_{v=1}^{s} \beta_{Ht+v})\, u_c(C_{t+s}, L_{t+s})/u_c(C_t, L_t)$ is the stochastic discount factor of the home economy. Firms maximise their profits subject to the production function and the demand (3.40). The term $\mu_w = \theta/(\theta-1)$ is a subsidy to ensure that the steady state is efficient and identical to the benchmark model in the last section so that these models are comparable. The subsidy is paid for by consumers with a lump-sum tax. Since the optimisation problem is entirely forward-looking, every optimising firm chooses the same price, denoted by $\tilde{P}_{Ht}$. The first order condition

is given by

$$\frac{\tilde{P}_{Ht}}{P_{Ht}} = \frac{K_t}{F_t}, \tag{3.43}$$

where

$$K_t = \frac{\theta}{\theta - 1}\frac{1}{\mu_w} E_t \sum_{s=0}^{\infty} \Lambda_{Ht,t+s}\gamma^s \frac{W_{t+s}}{P_{Ht+s}} \left(\frac{P_{Ht+s}}{P_{Ht}}\right)^{\theta} Y_{t+s}, \tag{3.44}$$

$$F_t = E_t \sum_{s=0}^{\infty} \Lambda_{Ht,t+s}\gamma^s \left(\frac{P_{Ht+s}}{P_{Ht}}\right)^{\theta-1} Y_{t+s}. \tag{3.45}$$

Since only a fraction $(1-\gamma)$ can re-optimise their prices, the aggregate price level can be written as follows

$$P_{Ht}^{1-\theta} = \gamma P_{Ht-1}^{1-\theta} + (1-\gamma)\tilde{P}_{Ht}^{1-\theta}. \tag{3.46}$$

Foreign firms are also subject to nominal rigidities in price setting. They have an analogous structure to the home firms.

In addition, I need to specify the monetary policy by the central banks. I suppose the two countries can use monetary policy to replicate the flexible price economy outcome as long as the nominal interest rates are not constrained by the zero lower bound. However, when the nominal interest rate implied by this rule is below zero, then there is a zero lower bound, and the nominal interest rate is set to zero. Specifically, the monetary policy rules are

$$(1+i_t) = \max\left(1 + r_{flex,t}, 1\right), \tag{3.47}$$

$$(1+i_t^*) = \max\left(1 + r_{flex,t}^*, 1\right). \tag{3.48}$$

where $r_{flex,t}$ and $r_{flex,t}^*$ are the interest rates that prevail in the flexible price economy. This is an extreme assumption to make this model comparable to the model without price rigidity.

Finally, the path of the terms of trade adjusts to ensure that the uncovered interest parity holds so that

$$(1 + i_t^*) = (1 + i_t)\frac{\Pi_{Ft+1}^*}{\Pi_{Ht+1}}\frac{S_{t+1}}{S_t}. \tag{3.49}$$

The rest of the model comprises the home and foreign budget constraints, the first order conditions for home and foreign utility maximisation with respect to consumption and labour supply and the goods market clearing conditions. They are mostly identical to the benchmark model in the previous section. Appendix 3.5.2 reports the full system with nominal rigidities.

The parameters are set to be the same as in the previous section. Two additional parameters related to nominal rigidities have to be calibrated in this model. The price stickiness, γ, is set to 0.75, which means that on average prices last for one year. I calibrate the elasticity of substitution between goods varieties, θ, to 6, implying a steady-state mark-up of 20%. These are within the range of standard values in the literature. Given the assumptions on monetary policy and the calibration of the parameters, the only differences between this system and the system in the previous section are the nominal rigidities and the imposition of the zero lower bound in the nominal interest rates.

Figure 3.3 shows the simulation of the model with nominal rigidities. As before, I assume initially that the saving glut shock occurs in the foreign country according to Equation (3.35). The shock reduces the interest rates in both the home and foreign economies. However, with my calibration the shock is not large enough to push the nominal rate up against the zero bound. Given my assumption of the monetary policies, the nominal interest rates are set so that the real interest rates coincide with the natural interest rates. There is no change in prices and inflation. The behaviour of the system is therefore identical to the system in the previous section.

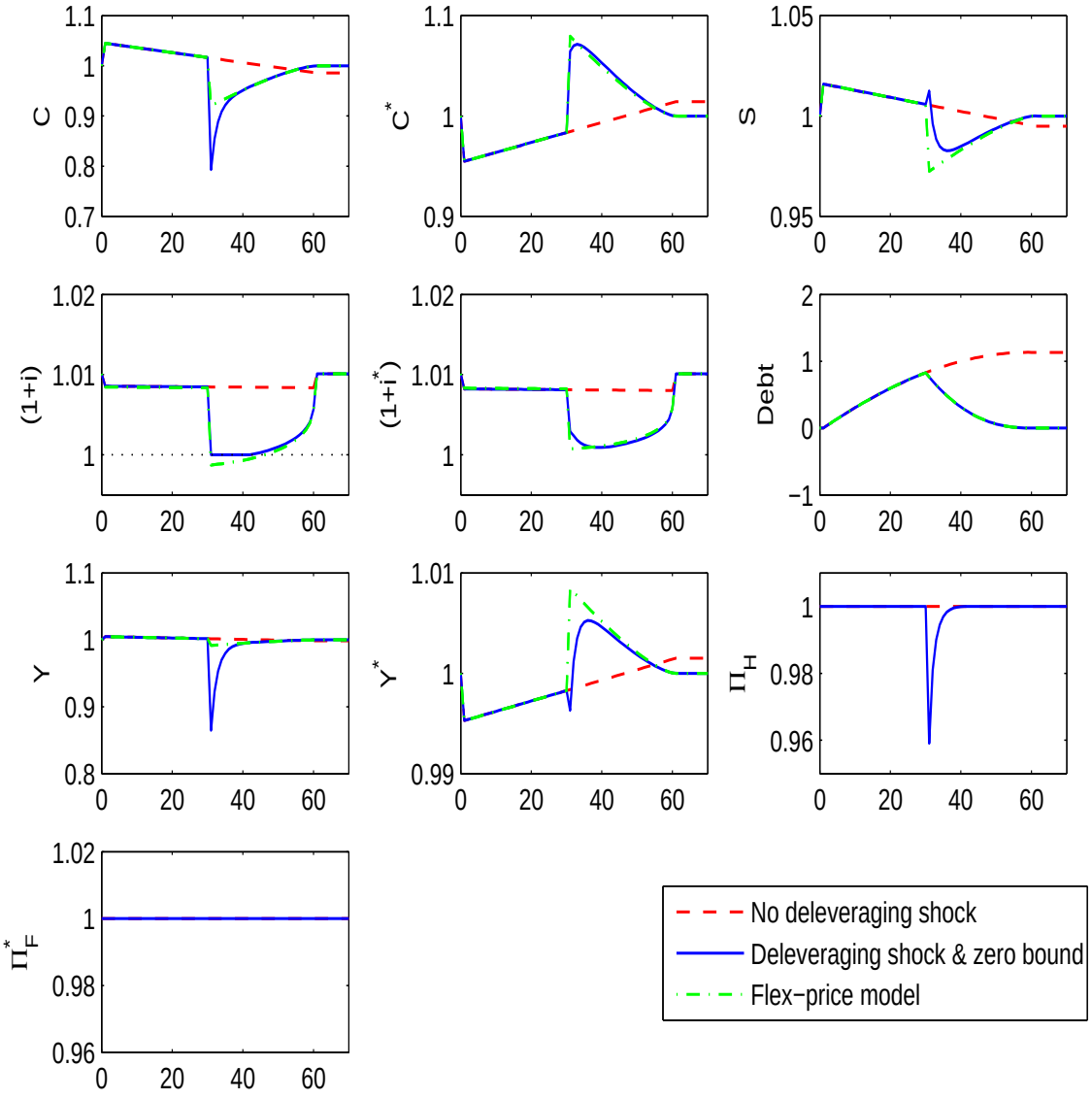


Figure 3.3: Impulse response of a saving glut shock and deleveraging in a model with price stickiness. (In the flex-price model I show the real interest rates.)

The dashed line in Figure 3.3 shows the simulation with the saving glut shock. The impulse responses before deleveraging occurs are identical to the dashed-dotted line, which shows the impulse response in the flexible price model. (Note that in the flex-price model I plot the real interest rates.)

What is more interesting is when I impose the second shock – the deleveraging shock – to this system. As in the previous section, the deleveraging shock in the US follows a process described by Equation (3.37). In particular, I choose the speed of deleveraging parameter, ς, so that the deleveraging country stays at the zero bound for a reasonable amount of time. There is, however, little consensus as to how long on average a liquidity trap lasts. These numbers range from 4 quarters in Fornaro (2012) and 5 quarters on average in Christiano et al. (2010) to 12 quarters in Benigno and Romei (2012). Eggertsson and Woodford (2003) and Eggertsson and Krugman (2010) assume 10 quarters. Besides, Fernandez-Villaverde (2012) find that a zero bound on average lasts for 2.06 quarters with a standard deviation of 1.82 quarters. However, the variance increases monotonically conditional on the number of quarters already at the zero bound. In reality, since the outbreak of the global financial crisis in 2008, the US interest rates have stayed at a level close to zero for more than 3 years. Therefore, I choose the speed of deleveraging parameter so that the zero bound lasts for 12 quarters in my simulation. I use $\varsigma = 2.38$ – the same value as in the flexible price model.

The solid line in Figure 3.3 shows the impulse responses when the deleveraging shock is imposed in period 30.[12] In response to the deleveraging shock, consumption in the home country falls. The central bank lowers the nominal interest rate according to the Taylor rule, in an attempt to bring inflation back to target. Given the size of the deleveraging shock, the central bank's attempt is not successful because it would require a negative nominal interest rate to hit the inflation target. As a

[12]Details of the solution algorithm are presented in Appendix 3.5.3.

result, the central bank brings the nominal interest rate down to zero, and the zero interest rate bound binds. Since the interest rate does not fall enough to equilibrate the supply and demand in the goods markets, firms who can reset prices cut their prices, creating a deflation in the home country.

The deflation triggers Fisher's debt deflation spiral: a combination of the zero bound in the nominal interest rates and a fall in home price level leads to a rise in the real interest rate. This means that the debt interest repayment for home consumers rises, which further depresses home consumption demand and output and causes more deflation. The zero bound in the nominal interest rate also explains the US real exchange rate appreciation immediately after the deleveraging shock in Figure 3.3. In such circumstance, one would normally expect the real exchange rate to depreciate to encourage demand for domestic goods. This happens eventually once the zero bound disappears. But immediately after the shock, deflation in the home country means that real interest rates are higher in the home country than abroad. This effect is large enough to cause the real exchange rate to appreciate initially, further strengthening the deflationary pressure. The overall effect on home output and inflation is substantial. In my simulation, the debt deflation spiral causes around 4% fall in inflation in the home country on impact. Home output falls by more than 10%, which also causes foreign output to fall.

This initial appreciation of the real exchange rate again depends on the GHH preferences. The reason is the following. As discussed previously, the GHH preferences generate positive co-movement between output and consumption in my model, while the additively separable preferences generate negative co-movement driven by the labour supply effect. This means that, in the absence of the zero bound, deleveraging with normal preferences leads to larger real exchange rate depreciation along the adjustment path, compared with the case with GHH preferences. In the presence of the zero bound, deleveraging causes the zero bound to bind in the home economy

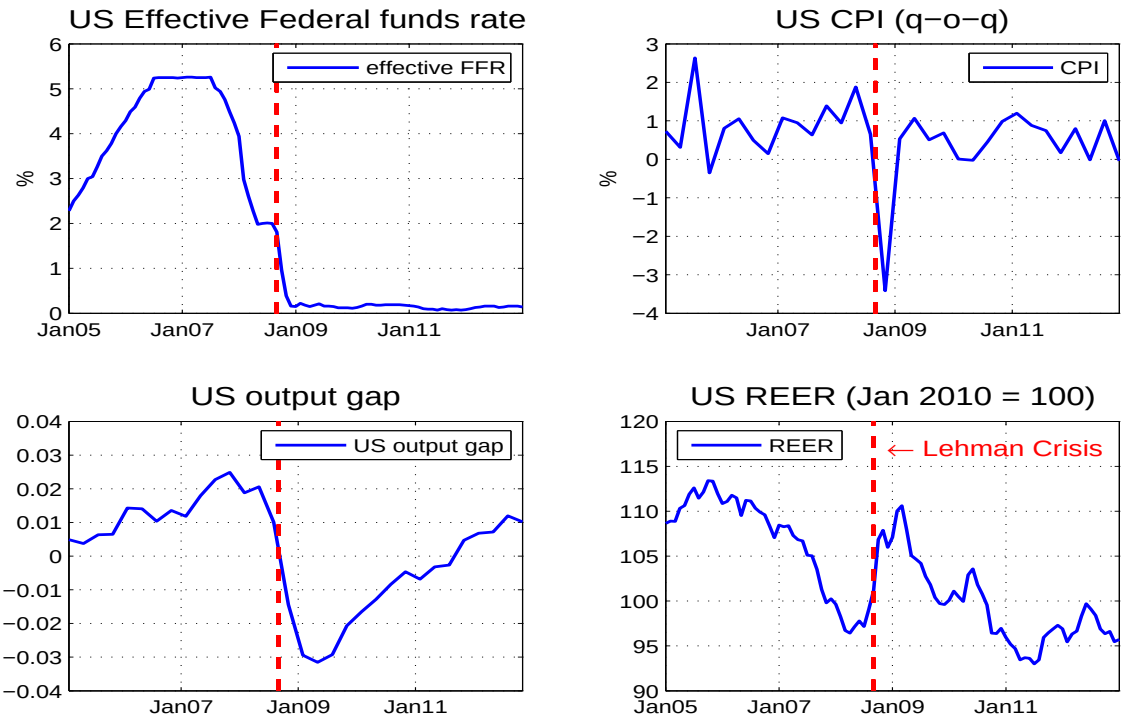


Figure 3.4: US data.

initially, which leads to deflation and a rise in the home real interest rate for both types of preferences. Therefore, according to the uncovered interest parity, the real exchange rate is depreciating when the home real interest rate is higher than the foreign, which roughly corresponds to the periods in which the zero bound binds; and is appreciating after the home economy leaves the zero bound. But with additively separable preferences the real exchange rate has to be more depreciated after the home interest rate leaves the zero bound than with GHH preferences, as a result of the negative co-movement between consumption and output. This prevents the real exchange rate from appreciating initially with additively separable preferences.

I compare the observed data with the simulation results. Figure 3.4 shows the observed US data.[13] I use the collapse of Lehman Brothers in September 2008 as

[13]Data source: Bank of International Settlements, Bureau of Economic Analysis (US), Board of Governors of the Federal Reserve System and Federal Reserve Bank of Cleveland. Output gap shows the difference between the quarterly real GDP series and the hp-filtered series with a

a proxy for the beginning of the deleveraging phase, indicated by the red dotted line. After the collapse, the nominal interest rate immediately dropped to zero, a clear indication of the binding zero bound. The sharp deflation was also immediate, with a quarter-to-quarter drop of more than 3%, which indicated a rise in the real interest rate. The deflation was short-lived, compared with my simulation. One possible reason is that commodity prices remained high after the global financial crisis. Quantitative easing might also have an effect on inflation (and the real exchange rate) which is not captured in this simple model. The output gap in the actual data also fell sharply, with a peak-to-trough drop of about 5%, but this was smaller than in my simulation. Given the simplicity of my setup, I can only explain part of what happened during that period.

Finally, there was an immediate exchange rate appreciation in the actual data. Such behaviour of the exchange rate was in contrast to many observers who had predicted a steady fall in the dollar as a likely macroeconomic outcome in the process of the unwinding of global imbalances. See, for instance, Blanchard, Giavazzi and Sa (2005), Kuralbayeva and Vines (2009) and Krugman (2007). Various explanations have been put forward to explain this phenomenon. First, Blanchard and Milesi-Ferretti (2011) and Adam, Subacchi and Vines (2012) argue that some countries in the rest of the world may follow a 'beggar-thy-neighbour' policy to keep a depreciated exchange rate in order to achieve full employment in their own countries. Second, the sharp increase in demand for US government bonds – the global safe asset – as a result of the 'flight to safety' in a time of heightened volatility is an important aspect emphasised by the macro-finance literature (for instance Caballero and Krishnamurthy, 2008, Gourinchas, Rey and Govillot, 2010, and Maggiori, 2013). I provide an alternative explanation to the exchange rate appreciation, based on the zero bound and debt deflation. The combined effect of these factors led

smoothing parameter λ=1600.

to an exchange rate appreciation of about 10% in the actual data before a subsequent depreciation.

3.4 Conclusion

I have set out a two-country model for understanding the relationship between global imbalances and the global financial crisis. The model suggests that during the great moderation, a rise in saving in China led to a fall in the interest rates in both the US and China, together with a rise in current account deficits in the US. These features are consistent with the description by Bernanke (2005) of the global savings glut. I then connect this analysis with the global financial crisis. I assume that as a consequence of debt accumulation, US consumers face a deleveraging shock. The shock leads to forced saving in the US and to a further drop in the world interest rate. In the presence of nominal rigidities in goods prices, a large deleveraging shock can reduce the nominal interest rate to zero. Firms begin to set lower prices, leading to a deflation, which raises the real interest rate and the real value of debt, further tightening the debt constraint. Moreover, with the GHH preferences, our simulation shows that the resulting rise in the real interest rate causes an initial appreciation of the dollar. I have shown that deleveraging and a zero bound can together have a major impact on output in the US, due to non-linearity in responses in these circumstances.

My model addresses the issue raised by Blanchard and Milesi-Ferretti (2011). It suggests that that even if policymakers can adjust the interest rates and the real exchange rates so as to maintain full employment, it would be a policy mistake to ignore the imbalances. This is because large countries which have high indebtedness are susceptible to deleveraging shocks. If interest rates are already low, then the consequence can be a liquidity trap, debt deflation and macroeconomic instability.

One major limitation of the analysis is its reliance on an exogenous debt deleveraging constraint, a limitation shared with other recent works in the field. In reality, the debt limit is likely to depend in part on economic fundamentals in ways that are not fully understood. Reinhart and Rogoff (2010) provide empirical evidence suggesting that a public debt to GDP ratio of 90% is a threshold above which median growth is one percent lower than otherwise.[14] But the transmission mechanisms are not clear. More work has to be done to understand the determinants of the debt limit.

Another direction for future research is related to the role of fiscal expenditure in countering the recession when monetary policy is stuck at the zero bound. The present analysis assumes no government, but could be extended to include one. Christiano, Eichenbaum and Rebelo (2010) and Eggertsson and Krugman (2012) have shown in the closed economy that the fiscal multiplier is significantly above unity when the interest rate is at the zero bound, which suggests that this might be important. See also Nakata (2012) who studies optimal fiscal and monetary policy with an occasionally binding zero-bound constraint for a closed economy; this work needs to be extended to an open economy.

[14] A recent critique by Herndon, Ash and Pollin (2014) discovered coding errors and selective exclusion of available data in Reinhart and Rogoff (2010), casting doubts to the original claim about the nonlinear relationship between public debt and growth.

3.5 Appendix

3.5.1 Parameter Values

The following table shows the parameter values I used to calibrate the model:

Table 3.1: Parameter values.

Definition	Parameter	Target/ Source
Consumption home bias	$\alpha = 0.7$	Obstfeld and Rogoff (2005)
Elasticity of substitution between home and foreign goods	$\eta = 2$	Obstfeld and Rogoff (2005)
Steady-state discount factor	$\beta_H = 0.99$	Interest rate $r = 4.1\%$
Utility weight of labour disutility	$\chi = 1$	Output in initial steady state = 1
Inverse of Frisch elasticity	$\varphi = 1$	Kimball and Shapiro (2008)
Magnitude of saving glut shock	$\epsilon_t = 0.0037$	Deleveraging shock begins in 7.5 years, at which home debt-to-GDP ratio = 20% (Fornaro, 2012)
Speed of deleveraging	$\varsigma = 2.38$	Zero bound at home binds for 3 years
Probability firm reoptimises prices	$\gamma = 0.75$	Prices fixed for 4 quarters on average
Elasticity of substitution between good varieties	$\theta = 6$	Steady-state mark-up = 20%

3.5.2 Full System with Nominal Rigidities

In this Appendix, I present the full system with nominal rigidities.

The home and foreign budget constraints are

$$C_t = \Theta_t^{-\frac{1}{1-\eta}} \Delta_t Y_t + \frac{1}{1+i_t} \frac{D_{t+1}}{P_{t+1}} \Pi_{Ht+1} \left(\frac{\Theta_{t+1}}{\Theta_t} \right)^{\frac{1}{1-\eta}} - \frac{D_t}{P_t}, \tag{3.50}$$

$$C_t^* = \Xi_t^{-\frac{1}{1-\eta}} \Delta_t^* Y_t^* - Q_t \frac{1}{1+i_t} \frac{D_{t+1}}{P_{t+1}} \Pi_{Ht+1} \left(\frac{\Theta_{t+1}}{\Theta_t} \right)^{\frac{1}{1-\eta}} + Q_t \frac{D_t}{P_t}. \tag{3.51}$$

The utility maximisation conditions for the home and foreign consumers are

$$
\begin{aligned}
1 &= \Lambda_{Ht,t+1}(1+i_t)\frac{1}{\Pi_{Ht+1}}\left(\frac{\Theta_t}{\Theta_{t+1}}\right)^{\frac{1}{1-\eta}}, && (3.52)\\
\frac{W_t}{P_t} &= \chi Y_t^{\varphi}\Delta_t^{\varphi}, && (3.53)\\
1 &= \Lambda^*_{Ft,t+1}(1+i_t)\frac{1}{\Pi_{Ht+1}}\left(\frac{\Theta_t}{\Theta_{t+1}}\right)^{\frac{1}{1-\eta}}\frac{Q_{t+1}}{Q_t}, && (3.54)\\
\frac{W^*_t}{P^*_t} &= \chi\left(Y^*_t\right)^{\varphi}\left(\Delta^*_t\right)^{\varphi}, && (3.55)
\end{aligned}
$$

where the stochastic discount factors are

$$
\begin{aligned}
\Lambda_{Ht,t+1} &= \beta_{Ht+1}\left(C_t-\chi\frac{(\Delta_t Y_t)^{1+\varphi}}{1+\varphi}\right)\left(C_{t+1}-\chi\frac{(\Delta_{t+1}Y_{t+1})^{1+\varphi}}{1+\varphi}\right)^{-1},\\
\Lambda^*_{Ft,t+1} &= \beta_{Ft+1}\left(C^*_t-\chi\frac{(\Delta^*_t Y^*_t)^{1+\varphi}}{1+\varphi}\right)\left(C^*_{t+1}-\chi\frac{(\Delta^*_{t+1}Y^*_{t+1})^{1+\varphi}}{1+\varphi}\right)^{-1}.
\end{aligned}
$$

The price setting behaviour of home firms are given by

$$
\begin{aligned}
\frac{\tilde{P}_{Ht}}{P_{Ht}}F_t &= K_t, && (3.56)\\
F_t &= Y_t+\Lambda_{Ht,t+1}\gamma\Pi^{\theta-1}_{Ht+1}F_{t+1}, && (3.57)\\
K_t &= \frac{W_t}{P_t}\Theta_t^{\frac{1}{1-\eta}}Y_t+\Lambda_{Ht,t+1}\gamma\Pi^{\theta}_{Ht+1}K_{t+1}, && (3.58)\\
1 &= \gamma\Pi_{Ht}^{-(1-\theta)}+(1-\gamma)\left(\frac{\tilde{P}_{Ht}}{P_{Ht}}\right)^{1-\theta}. && (3.59)
\end{aligned}
$$

The price setting behaviour of foreign firms are given by

$$\frac{\tilde{P}^*_{Ft}}{P^*_{Ft}}F^*_t = K^*_t, \tag{3.60}$$

$$F^*_t = Y^*_t + \Lambda^*_{Ft,t+1}\gamma\left(\Pi^*_{Ft+1}\right)^{\theta-1}F^*_{t+1}, \tag{3.61}$$

$$K^*_t = \frac{W^*_t}{P^*_t}\Xi_t^{\frac{1}{1-\eta}}Y^*_t + \Lambda^*_{Ft,t+1}\gamma\left(\Pi^*_{Ft+1}\right)^{\theta}K^*_{t+1}, \tag{3.62}$$

$$1 = \gamma\left(\Pi^*_{Ft}\right)^{-(1-\theta)} + (1-\gamma)\left(\frac{\tilde{P}^*_{Ft}}{P^*_{Ft}}\right)^{1-\theta}, \tag{3.63}$$

and the evolution of the price dispersions Δ_t and Δ^*_t are

$$\Delta_t = \gamma\Delta_{t-1}\Pi^{\theta}_{Ht} + (1-\gamma)\left(\frac{\tilde{P}_{Ht}}{P_{Ht}}\right)^{-\theta}, \tag{3.64}$$

$$\Delta^*_t = \gamma\Delta^*_{t-1}\left(\Pi^*_{Ft}\right)^{\theta} + (1-\gamma)\left(\frac{\tilde{P}^*_{Ft}}{P^*_{Ft}}\right)^{-\theta}, \tag{3.65}$$

where the price dispersions are defined as

$$\Delta_t \equiv \int_0^1\left(\frac{P_{Ht}(i)}{P_{Ht}}\right)^{-\theta}di,$$

$$\Delta^*_t \equiv \int_0^1\left(\frac{P^*_{Ft}(i)}{P^*_{Ft}}\right)^{-\theta}di.$$

The home and foreign goods market clearing conditions are

$$\Delta_t Y_t = \alpha\Theta_t^{\frac{\eta}{1-\eta}}C_t + (1-\alpha)S_t^{-\eta}\Xi_t^{\frac{\eta}{1-\eta}}C^*_t, \tag{3.66}$$

$$\Delta^*_t Y^*_t = (1-\alpha)S_t^{\eta}\Theta_t^{\frac{\eta}{1-\eta}}C_t + \alpha\Xi_t^{\frac{\eta}{1-\eta}}C^*_t, \tag{3.67}$$

The uncovered interest parity is written as

$$(1+i^*_t) = (1+i_t)\frac{\Pi^*_{Ft+1}}{\Pi_{Ht+1}}\frac{S_{t+1}}{S_t}. \tag{3.68}$$

Lastly, the monetary policy rules for the home and foreign economies are described in Equation (3.47) and (3.48). In practice, I assume the monetary policies are Taylor rules as follows:

$$(1+i_t) = \max\left(\frac{1}{\beta_H}\left(\frac{\Pi_{Ht}}{\Pi_H}\right)^{\phi_H}, 1\right), \tag{3.69}$$

$$(1+i_t^*) = \max\left(\frac{1}{\beta_F}\left(\frac{\Pi_{Ft}^*}{\Pi_F^*}\right)^{\phi_F}, 1\right). \tag{3.70}$$

where I set the inflation elasticities to be high ($\phi_H = \phi_F = 10^8$). This ensures that in normal times inflation is close to zero and the nominal interest rates are set to replicate the flexible price economy.

Moreover, Θ_t, Ξ_t, Q_t are defined in Equations (3.31), (3.32) and (3.33). The process for β_{Ft} follows Equation (3.35).

The system contains 21 equations (Equations (3.50) - (3.70)) and solves for 20 variables

$$C_t,\ C_t^*,\ Y_t,\ Y_t^*,\ S_t,\ i_t,\ i_t^*,\ D_t/P_t,\ \Pi_{Ht},\ \Pi_{Ft}^*,\ \frac{W_t}{P_t},\ \frac{W_t^*}{P_t^*},$$
$$K_t,\ K_t^*,\ F_t,\ F_t^*,\ \frac{\tilde{P}_{Ht}}{P_{Ht}},\ \frac{\tilde{P}_{Ft}^*}{P_{Ft}^*},\ \Delta_t,\ \Delta_t^*.$$

given the initial level of debt D_0/P_0 and the price dispersions Δ_t, Δ_t^* and the exogenous shock process for β_{Ft}, which follows Equation (3.35). By Walras' law, one of the equations is redundant. After deleveraging, the home consumption Euler equation is dropped. Debt follows the exogenous debt deleveraging process (3.37).

After the deleveraging shock hits the home economy in period 30, home consumers are forced to deleverage. The consumption Euler equation (3.52) no longer applies, and is replaced with the exogenous debt deleveraging process (3.37).

3.5.3 Solution Algorithm

This appendix discusses the reverse shooting algorithm employed in this chapter to solve the model. As consumers have perfect foresight, the shooting algorithm is a convenient technique to solve the model. In the following, I first describe the solution method in the flexible price model, and then the model with nominal rigidities. For each model, I first describe the simulation to the saving shock, and then the simulation to the deleveraging shock.

In the flexible price model, the only state variable is the real debt D/P. In addition, it is known that debt follows a unit-root process in this model. Without any disturbances, for any given value of debt, there is an equilibrium associated with this level of debt. Steady-state values of other variables are found by solving the steady-state version of the system (Equation (3.22) – (3.30)). I assume that the savings shock only lasts for 60 periods, and after that there will be no further disturbances. The above discussion implies that at the end of period 60, debt reaches a new steady state and stays there forever. The reverse shooting algorithm involves guessing the level of debt in the new steady state and updating the guess until convergence as follows (See Judd (1998) for further details):

1. Select an upper bound for the guess of the level of debt in the final steady state. This upper bound has to be sufficiently large. Call this upper bound $\left(\frac{D}{P}\right)^{high}$. Also set a lower bound at $\left(\frac{D}{P}\right)^{low} = 0$. The true level of debt in the final steady state has to be bounded by the two.

2. Guess the level of debt in the final steady state. Call this guess $\left(\frac{D}{P}\right)_{60}^{(j)}$ where the subscript means period 60 and the superscript denotes the j^{th} guess.

3. Compute the steady-state values of $C_{60}^{(j)}$, $(C_{60}^{*})^{(j)}$, $Y_{60}^{(j)}$, $(Y_{60}^{*})^{(j)}$, $S_{60}^{(j)}$, $r_{60}^{(j)}$, $(r_{60}^{*})^{(j)}$ implied by $\left(\frac{D}{P}\right)_{60}^{(j)}$, using the steady state version of Equation (3.22) – (3.30).

4. Compute the values of $C_{59}^{(j)},\ (C_{59}^{*})^{(j)},\ Y_{59}^{(j)},\ (Y_{59}^{*})^{(j)},\ S_{59}^{(j)},\ r_{59}^{(j)},\ (r_{59}^{*})^{(j)},\ \left(\frac{D}{P}\right)_{59}^{(j)}$, using Equation (3.22) – (3.30), where $C_{60}^{(j)},\ (C_{60}^{*})^{(j)},\ Y_{60}^{(j)},\ (Y_{60}^{*})^{(j)},\ S_{60}^{(j)},\ \left(\frac{D}{P}\right)_{60}^{(j)}$ are known and $\beta_{F,59}$ is given by Equation (3.35).

5. Repeat the last step for $t = 58, 57...$ until we obtain $C_{0}^{(j)},\ (C_{0}^{*})^{(j)},\ Y_{0}^{(j)},\ (Y_{0}^{*})^{(j)},\ S_{0}^{(j)},\ r_{0}^{(j)},\ (r_{0}^{*})^{(j)},\ \left(\frac{D}{P}\right)_{0}^{(j)}$.

6. If $|\left(\frac{D}{P}\right)_{0}^{(j)}| < threshold$, we have found the transition path and stop. Otherwise, update the guess for the level of debt in the final steady state using method of bisection. Specifically, if $\left(\frac{D}{P}\right)_{0}^{(j)} > 0$, then $\left(\frac{D}{P}\right)_{60}^{(j+1)} = (\left(\frac{D}{P}\right)_{60}^{(j)} + \left(\frac{D}{P}\right)^{low})/2$, and update the lower bound to $\left(\frac{D}{P}\right)^{low} = \left(\frac{D}{P}\right)_{60}^{(j)}$. Otherwise, if $\left(\frac{D}{P}\right)_{0}^{(j)} < 0$, then $\left(\frac{D}{P}\right)_{60}^{(j+1)} = (\left(\frac{D}{P}\right)_{60}^{(j)} + \left(\frac{D}{P}\right)^{high})/2$, and update the upper bound to $\left(\frac{D}{P}\right)^{high} = \left(\frac{D}{P}\right)_{60}^{(j)}$.

7. Repeat from step 3 and iterate until convergence.

In the simulations I choose the upper bound $\left(\frac{D}{P}\right)^{high}$ to be 8, a debt-to-GDP ratio of 200%. The initial guess of the final steady state is $(D/P)_{60}^{(1)} = 0.1$. The stopping threshold is chosen to be 10^{-6}. The algorithm takes 21 iterations to converge.

The algorithm for the simulation of the deleveraging shock in the flexible price model is described as follows:

1. The dynamics of debt is given by Equation (3.36).

2. The system reaches the steady state at the end of period 60 with $\left(\frac{D}{P}\right)_{60} = 0$. The steady-state values of other variables are obtained by solving the steady state version of Equation (3.22) – (3.30).

3. Use Equation (3.22), (3.23), (3.25) – (3.30) to solve for $C_{59},\ C_{59}^{*},\ Y_{59},\ Y_{59}^{*},\ S_{59},\ r_{59},\ r_{59}^{*}$. Note that the dynamics of debt is known and $\beta_{F,59}$ is given by Equation (3.35).

4. Repeat the previous step for $t = 58, 57...$ until we reach period 30.

The system with nominal rigidities is more complicated because there are three state variables, namely $\Delta_t, \Delta_t^*, D_t/P_t$. With multiple state variables, this system does not reach the steady state immediately after period 60. I approximate this system by fixing the price dispersions, Δ_t and Δ_t^*, at unity and eliminate Equations (3.64) and (3.65). The benefit of the approximation is that this eliminates two predetermined variables so that external debt D/P is the only state variable remaining in the system. This makes it possible to reverse-shoot from period 60. And, according to Woodford (2003), the price dispersions are second-order terms, so the accuracy cost of this approximation is low. The alternative, multi-dimensional shooting method developed by Atolia and Buffie (2009) is computationally intensive and time-consuming. I do not pursue this approach in this chapter.

I apply the same approach as in the flexible price model to solve for the dynamics for the saving shock in China, assuming the the zero bound constraint does not bind in home and foreign country. As before, I choose $\left(\frac{D}{P}\right)^{high} = 8$, $(D/P)_{60}^{(1)} = 0.1$, $threshold = 10^{-6}$. The algorithm converges at $j = 23$ iterations. The simulation result confirms that the zero bound does not bind with the saving shock alone.

In the simulation for the deleveraging shock, I have to allow for the zero bound. The algorithm is the same as the one used in the flexible price model, except for step 3, which is changed as follows:

3. Assume the zero bound does not bind in both the home and foreign countries. Use Equations (3.50), (3.51), (3.53) – (3.63), (3.66) – (3.70) to solve for $C_t,\ C_t^*,\ Y_t,\ Y_t^*,\ S_t,\ i_t,\ i_t^*,\ \Pi_{Ht},\ \Pi_{Ft}^*,\ \frac{W_t}{P_t},\ \frac{W_t^*}{P_t^*}, K_t,\ K_t^*,\ F_t,\ F_t^*,\ \frac{\tilde{P}_{Ht}}{P_{Ht}},\ \frac{\tilde{P}_{Ft}^*}{P_{Ft}^*}$. If the solution is such that $i_t > 0$, $i_t^* > 0$ then go to the next step. If the solution is such that $i_t < 0$ then the zero bound constraint is binding, then set $i_t = 0$ and use the system equations to solve for the other variables. If the solution is such that $i_t^* > 0$, go to the next step. If this results in $i_t^* < 0$, impose the zero bound for the foreign country as well and use the system equations to solve

for the other variables.

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Chapter 4

Financial-Friction Macroeconomics with Highly Leveraged Financial Institutions[1]

4.1 Introduction

4.1.1 Financial-Friction Macroeconomics

We still do not understand enough about the macroeconomic amplification of the global financial crisis. This chapter adds to attempts to understand this. The crisis began in the sub-prime mortgage market in the US, which is rather small, valued at the time at around $1.2 trillion, or 12% of the US residential mortgage market (Lockhart 2008, Weaver 2008). Sub-prime-related losses are estimated to have been the rather small sum of about $300-400 billion (Blundell-Wignall, 2008; Roubini, 2008). By contrast, according to the IMF's Global Financial Stability Report, the

[1]This chapter is being done jointly with Professor David Vines. Professor Vines has certified that I have played a major part of bring this chapter to completion. If necessary, Professor Vines can be asked to confirm this.

banking system write-downs totalled about $2.3 trillion world-wide (IMF, 2010). This suggests an amplification of the shock within the financial system of about 6-8 times. This chapter shows how leverage in the financial system can help to explain such magnification.

Our analysis builds on the analysis of Bernanke Gertler and Gilchrist (1999), henceforth BGG. In that paper final-goods producers are leveraged, and subject to idiosyncratic productivity shocks, so that loans to them are subject to a risk premium. Amplification happens because this risk premium is anti-cyclical, giving rise to a financial accelerator. But in BGG the financial sector obtains its financing from depositors without financial frictions. This assumption makes that model inadequate as a way of studying the recent financial crisis.

By contrast, in the present chapter final-goods producers raise finance from financial institutions by issuing shares. As a result, financial institutions can diversify away the risk coming from idiosyncratic shocks affecting the final-goods producers, and so these do not cause any risk premium in the model. By contrast financial institutions are leveraged, and – because they are also subject to idiosyncratic productivity shocks – risky. Consumers are unable to diversify away this risk. Financial institutions can therefore only obtain funds by paying an interest rate to consumers which is above the risk free rate; we show that this risk premium is anti-cyclical, and so augments the effects of productivity shocks. The financial-accelerator effect in our model operates between depositors and financial institutions, rather than, as in BGG, between financial institutions and the final-goods producers. By assumption consumers are unable to invest directly in final-goods producers because it is assumed that they lack the ability to diversify away the risk attached to each final-goods producer.

Our model of a leveraged financial system is based on the analysis of Hirakata, Sudo, and Ueda (2009), henceforth HSU. That paper supposes that financial in-

stitutions are leveraged but also that entrepreneurs are leveraged so that financial institutions lend money to entrepreneurs, rather than assuming like us that financial institutions buy shares issued by entrepreneurs. In effect the HSU setup contains two BGG-like contracts, joined together, giving rise to what they call chained credit contracts. Our analysis is simpler than theirs. In the run-up to the crisis a large proportion of the financial system consisted of investment banks, which provided credit to final-goods producers by purchasing asset-backed securities, rather than by providing loans.

Since the 2008 global financial crisis, many people have tried to embed a financial system into a macroeconomic model. We will review alternative approaches in modelling the financial sector, and argue that our way has some advantages than other modelling approaches.

Our setup provides two insights into what caused the amplification which happened during the financial crisis. First, our setup produces a stronger amplification of shocks than BGG. This is because of the way the model is calibrated, but that choice of calibration is deliberate and reflects reality. In BGG the firms in the leveraged sector of the model, the final-goods producers, have a leverage of 2, a number which the authors based on US data. By contrast in our model, the firms in the leveraged sector of the model, the financial sector, have a leverage of 5, a number which we take from HSU who obtained it from US data.[2] Second, we show that

[2] Adrian and Shin (2008) examine the balance sheets of what were the five largest US investment banks with the data from the Securities and Exchange Commission. The leverage (defined by total assets divided by book equity) between 1992Q1 and 2008Q1 was about 30. Kalemli-Ozcan et al. (2011) use bank data from BANKSCOPE between 2000-2009 and obtained a similar number for aggregate leverage for US investment banks. In fact, leverage rose to 40 in 2007 during the crisis and fell down to 20 in 2009. The effective leverage before the crisis may have been even higher than this figure suggests, since the balance sheets do not include off-balance sheet items such as asset-backed commercial paper (ABCP) conduits and structured investment vehicles (SIVs), the sizes of which are harder to estimate. (See Acharya and Schnabl, 2009.) Nevertheless we take the much lower number of 5 to avoid exaggerating our results. This number was obtained by HSU as the leverage for the whole of the financial system, much of which is much less highly leveraged than the institutions just described.

our setup leads to a greater amplification of shocks than HSU. That is because their assumption that financial institutions lend money to entrepreneurs, rather than buying their shares, dampens the amplification in their system in a way which does not happen in our model.[3]

4.1.2 The Financial Accelerator, Financial Leverage and Macroeconomic Outcomes in this Model

In the BGG model, consumers do not directly invest in the goods-producing firms, but instead deposit their money with 'investors'. These investors allocate their money across financial institutions, and offer consumers interest on their deposits. Financial institutions make loans to goods-producing firms, who are leveraged and so need to borrow to finance part of their capital investment. Aggregate productivity is risky, a lower level of aggregate productivity will reduce the net worth of final-goods producers and increase their leverage, meaning that a larger proportion of them are likely to go bankrupt; financial institutions are assumed not to be able to hedge such risks. They therefore charge a risk premium on the funds which they lend to goods-producing firms. But the financial system is competitive and consumers are able to diversify away the effects of risks specific to particular financial institutions. As a result, in the BGG model, consumers lend to financial institutions at a risk free rate. But the cost of capital to firms is higher than the risk free rate because of the risk premium charged by financial institutions.

Any particular fall in aggregate productivity will increase the leverage of final-goods producing firms and mean that a larger proportion of them are likely to go bankrupt. Financial institutions therefore charge a higher risk premium on the funds which they lend to to goods-producing firms after such a shock. As a result, this

[3]We will explain below why this is the case.

risk premium behaves in an anti-cyclical manner in response to productivity shocks. This gives rise to a financial accelerator.

In the model presented below, consumers deposit their money with 'investors' who allocate their money across financial institutions, and offer consumers interest on their deposits, as in the BGG model. Financial institutions purchase shares in goods-producing firms, and can diversify away their risks by holding a range of shares, and so do not charge a risk premium.[4] But financial institutions are leveraged and are themselves risky – they suffer from idiosyncratic shocks and also from the risk attached to the aggregate value of shares. As a result investors charge a risk premium to these financial institutions. In this model as in the BGG model, the cost of capital to firms is higher than the risk free rate, but now because of the risk premium charged by investors on their loans to financial institutions.

We will see that any particular fall in aggregate productivity will reduce the net worth financial institutions by a larger proportion than the fall in their net worth and so increase their leverage; meaning that a larger proportion of them are likely to go bankrupt. Investors therefore charge a higher risk premium on the funds which they lend to financial institutions after such a shock. As a result, this risk premium behaves in an anti-cyclical manner in response to aggregate productivity shocks. This again gives rise to a financial accelerator.

The full macroeconomic model which we present is built, like the BGG model, on a Ramsey model with adjustment costs in capital formation. In the latter kind of model, the financial sector costlessly allocates the saving of consumers for use as investment by final-goods producers, and there is only one interest rate - the

[4]In our model, consumers are assumed not to directly purchase the shares of goods-producing firms. In reality this may be because many individual consumers lack sufficient knowledge of goods-producing firms, or because they are too small to diversify risk individually, and so buy funds managed by financial institutions. For whatever reason a significant proportion of financial intermediation has been achieved by means of consumers depositing money in financial institutions which purchase shares on their behalf.

interest rate received by consumers is equal to the cost of capital to investors. Investment is driven by Tobin's Q. When Q is greater than unity, this leads to an increase in investment (and vice versa), by an amount which depends on the size of adjustment costs. For obvious reasons, we call this the Q-Ramsey model. When, in this Q-Ramsey model, there is a sustained negative shock to the aggregate level of productivity, aggregate supply falls. This has implications for aggregate demand and the interest rate. The level of the capital stock will fall, leading to an immediate reduction in investment, by an amount depending on the size of adjustment costs. Moreover, the sustainable level of current consumption will fall, and, in the interests of consumption smoothing, consumers will cut their level of consumption immediately. If the adjustment costs of capital investment are sufficiently low, the fall in demand will be larger than the fall in supply. This means that the rate of interest will need to fall to ensure that aggregate demand remains equal to the lower level of aggregate supply. We assume a sufficiently low level of adjustment costs in what follows.

In the BGG model, if there is a sustained negative shock to the aggregate level of productivity, investment falls and consumption falls. As in Q-Ramsey model, the interest rate received by consumers - the risk free rate in the BGG model - must fall to stimulate consumption. But, as explained above, the risk premium rises, because of the higher interest rate charged by financial institutions to goods-producing firms. This will further depress investment. As a result, the risk free rate must fall by further to stimulate consumption and investment. Overall the fall in investment will be larger than in the Q-Ramsey model - that is the BGG financial accelerator at work.

The same is true in our model. Now the risk premium rises because of the higher interest rate charged by investors to financial institutions. As in the BGG model this will magnify the fall in investment. This is a financial-accelerator effect at work

in our model.

4.1.3 Plan of this Chapter

The next section reviews recent developments in modelling financial market frictions.

Section 4.3 sets out our model of the financial system. We outline the nature of the credit contracts in the economy. We set out the nature of the loan contract between investors and financial institutions, and the nature of the share contract between financial institutions and final-goods producers . These analyses are *micro*economic analyses, in that they take the price of shares, and the risk free rate of interest, as given.

Section 4.4 presents the *macro*economic model in which the financial system is embedded. It is a basic Ramsey model with adjustment costs attached to capital investment, in which there are financial frictions between investors and financial institutions, and in which financial institutions purchase shares of goods-producing firms rather than lending to them. These features give rise to financial-accelerator properties in the full macro system.

Section 4.5 describes the calibration of the model and then discusses simulation results. We simulate the effects of a temporary negative productivity shock in the goods-producing sector, and compare the results provided by our model with those provided by the BGG and HSU models. This shock causes investment to fall and the risk free interest rate to fall in all models. However, in our model, the risk premium also rises a great deal. This means that investment falls by more in our model and that there will be a larger fall in the risk free interest rate, and so a smaller fall of consumption in order that resources remain fully utilised. At the end of this section we also show that the larger the proportion of investment in goods-producing firms that is financed externally, the larger the amplification of the shock.

Section 4.6 presents brief conclusions.

4.2 Literature Review: Modelling Banking Frictions in a DSGE Framework

The Modigliani and Miller (1958) theorem states that the financial structure of economic agents does not matter in a world with complete market. This implies that the source of funding of the financial institutions is not important. After the financial crisis in 2007-08, the macroeconomic profession as well as policymakers come to realise that financial markets are far from complete. Financial frictions may amplify real shocks, increasing the volatility of the real variables in the business cycle. In addition, shocks originated in the financial sector may have real consequences. It is therefore important to develop macroeconomic models with financial frictions to understand how the dynamics of the models will be affected by these frictions. In this section, we review the literature of DSGE models with financial features. In particular, we focus on the various approaches to introducing financial frictions in the models.

A common feature in many of models with financial frictions is that the cost of funds for the lenders, R, is lower than the expected return on capital for the borrowers, $E(R^k)$. Moreover, the difference between the two rates, the credit spread, depends on the balance sheet of the borrowers in the following way:

$$E(R^k)/R = f(\text{leverage of the borrower}), \qquad f' > 0. \tag{4.1}$$

Leverage is defined as total assets divided by total liabilities. Since this equation relates the credit spread to the leverage of the borrower, we sometimes refer to this relation as the leverage supply function.

Compared with a frictionless economy in which the return on capital for the borrowers equals to the creditors' cost of funds, the credit spread $E(R^k)/R$ can be

seen as a wedge in the financial market. The presence of the wedge means that the borrowers of the funds demand less capital relative to what they demand in the efficient economy. In addition, the credit spread is endogenous in the models with financial frictions.

When a negative shock hits the economy, the realised return on capital falls. The financial position of the borrowers worsens, and the leverage rises. According to the leverage supply function, the credit spread widens, which means that the shadow price of capital has to fall further initially relative to the frictionless model. As a result, a model with financial friction usually produces larger macroeconomic volatility than a model without friction.

Since financial transactions are mostly intertemporal, financial contracting is more susceptible to agency problems than the transaction of goods. For instance, the borrower may renege on his promises. Also, if the return of the loan is state-contingent, then there may be information asymmetry so that the lender may not observe the true return. If the return of the loan depends on the effort of the borrower, a moral hazard problem exists because the borrower may not exert the efficient level of effort.

In what follows, we review four strategies in the existing literature to derive the leverage supply function. we will discuss four modelling approaches, which include costly financial intermediation, limited enforcement, hidden action and costly state verification.[5] Apart from costly financial intermediation, all other modelling approaches involve endogenous agency problems in which the set of feasible and incentive-compatible financial contracts is smaller than what would have been in a perfect financial market.[6]

[5]Other than these approaches, Stiglitz and Weiss (1981), Christiano and Ikeda (2011), Boissay (2011) and Boissay, Collard and Smets (2012) model adverse selection where banks have asymmetric information about the counterparty's type.

[6]In addition to imperfect financial markets, in a model with financial frictions, either there are more than one kind of agents, or in the case that there is only one kind of agents, they have to be

4.2.1 Costly Financial Intermediation

The first approach which we call the costly financial intermediation approach introduces an exogenous cost to the banking industry. The representative bank chooses the amount of deposits and lending to maximise its profits subject to the cost. For instance, Curdia and Woodford (2009) assume that there is an intermediation cost which is an increasing and convex function of the loans intermediated. Profit maximization of the bank gives rise to a positive interest rate premium on loans. However, the interest rate premium derived in this way is pro-cyclical: a rise in output increases the demand for loans, raises the bank intermediation cost and widens the spread, which dampens the rise in output. The pro-cyclicity of the interest rate spread and a lack of amplification of shocks is clearly at odds with empirical findings. Gerali et al. (2010) and Benes et al. (2009) overcome this difficulty by assuming that a bank incurs a quadratic cost when its leverage ratio, defined as the lending divided by the bank's net worth, deviates from an exogenous target level. The target level can be viewed as a penalty charged by the regulatory authority. When the leverage ratio is high, the interest rate spread in equilibrium must be high so that the marginal benefit of lending compensates for the marginal cost of having leverage different from its target level. This means that the interest rate spread is an increasing function of bank leverage. However, the estimated model in Gerali et al. (2009) suggests that the quadratic penalty is quantitatively unimportant.

This modelling approach is relatively popular in central bank models.[7] One advantage is that the balance sheet of the banks are transparent, which clarifies the transmission effects of interbank borrowing and the related risks. Gerali et al. (2009), for instance, introduce monopolistically competitive banks and sticky

heterogeneous. Otherwise, there is no scope for borrowing and lending. Quadrini (2011) elaborates on these two points.

[7]See Roger and Vlcek (2012) for a survey of the models of major central banks in the developed countries.

interest rate movements to allow for interest rate pass-through and maturity mismatch between bank assets and liabilities, which add realism to the model. The approach also makes the analysis of macro-prudential policies (such as reserve requirements, capital adequacy ratios and leverage restrictions) possible. Angelini, Neri and Panetta (2011) and Roger and Vlcek (2011) study these policies. But the disadvantage is that this approach lacks rigorous microfoundation of the bank intermediation costs.

4.2.2 Limited Enforcement

Limited enforcement is developed in Kiyotaki and Moore (1997) and is based on the idea that lenders cannot force borrowers to produce once the borrower reneges on the debt, *i.e.* at least a part of the borrowers' income is not pledgable. To prevent the borrowers from reneging, the borrowers face a binding collateral constraint, which keep their credit under the desired and optimal level.[8] The amplification mechanism is as follows: a negative shock lowers the value of the collateral, which makes the collateral constraint of the borrowers more binding. Hence, the asset demand of these constrained borrowers falls which further lowers the asset price. There is an additional, intertemporal effect: as demand falls, the net worth of the borrowers in future periods falls, which makes the collateral constraints in future periods more binding, reducing the asset price and investment today even further. A partial list of works in this literature includes Kocherlakota (2000), Cooley, Marimon and Quadrini (2004), Cordoba and Ripoll (2004), Lorenzoni and Walentin (2007), Iacoviello and Neri (2010), Liu, Wang and Zha (2011) and Jermann and Quadrini (2011). While this framework is successful in analysing financial frictions in the housing market (Iacoviello 2005), the fact that the leverage of the borrower is an exogenous constant

[8]Note that the constraint in this approach is on the quantity of loans instead of the price of loans, so the leverage supply function does not apply.

given by the binding collateral constraint is not consistent with the evolution of bank leverage over the business cycle in reality.

Gertler and Karadi (2011), Gertler and Kiyotaki (2011) and Gertler, Kiyotaki and Queralto (2011) adopt the limited commitment idea and apply it to the banking sector. A bank takes deposits and lends to firms. The bank manager has an option to divert a fraction of available funds. If he does, the bank is bankrupt and depositors suffer a loss. The bank manager chooses to cheat when the quantity of divertible funds is larger than the continuing value of the bank, which is the expected discounted profit of the bank. To prevent the bank manager from cheating, depositors limit their deposits. This setup means that, for any given bank net worth, a higher lending rate relative to the cost of funds increases the banks continuing value, and so means that depositors are willing to lend more, leading to an upward-sloping leverage supply function. See also Christiano and Zha (2010) and Gertler and Kiyotaki (2012) for related models.

Miao and Wang (2012) study another limited commitment problem in the banking sector. They assume that a bank manager may take the deposits and renege. If he does so, the depositors are left with a constant fraction of bank net worth, reorganise the bank and keep it running. The bank manager then renegotiates with the depositors to keep the bank. Suppose the depositors have no bargaining power, the bank manager only pays the continuing value of the bank with the fraction of capital that the depositors are left with. Hence, the bank manager chooses to default when the deposits are bigger than the continuing value of what is left. To prevent the bank manager from defaulting, depositors limit their deposits in a way similar to Gertler and Karadi (2009). When the credit spread widens, for any given bank net worth, the continuing value rises, the incentive constraint relaxes and deposits rise. Provided that there is no bubble in the economy, the leverage supply function must be upward-sloping.

4.2.3 Hidden Action

Meh and Moran (2010) and Christensen, Meh and Moran (2011) develop a general equilibrium model using a 'double moral hazard' approach due to Holstrom and Tirole (1997) and Chen (2001). In their setup, a bank takes deposits and lends to entrepreneurs who produce. However, the production outcome depends on the entrepreneurs' effort which is not observed by the banks and depositors. Entrepreneurs can choose high, middle or low effort, where effort is privately costly for them. The return of the project is non-decreasing in effort. Banks own a costly and imperfect monitoring technology which can detect between middle and low effort. As monitoring is costly for the banks, depositors limit their deposits so that banks put down enough of their own capital in funding entrepreneurial projects. In this way, banks have a private incentive to monitor *every* entrepreneur in equilibrium. To ensure entrepreneurs exert high effort, the optimal contract has to give entrepreneurs at least a certain share of the project return, and this limits the size of the project. What is important for the bank is that when the spread between the deposit rate and the lending rate widens, the bank has more private incentive to monitor the entrepreneurs, and the depositors are willing to increase their deposits, raising the bank leverage.

4.2.4 Costly State Verification

The costly state verification approach, which we follow, assumes that the return of the debt is subject to an idiosyncratic risk. The financial contract is not perfect due to information asymmetry. The borrower observes the realisation of the return, whereas the lender needs to pay a monitoring cost to observe it. As it is too costly to monitor every contract, in the optimal contract, if the realisation of shock is above a cutoff threshold, the borrower pays a constant interest rate and keeps the remainder;

if the realisation of shock is below the cutoff threshold, the borrower is bankrupt. In the latter case, the lender pays the monitoring cost and takes the borrower's entire wealth. BGG (1998) use this approach. The borrower chooses the level of loans to maximise its profits and a contractual interest rate that satisfies the participation constraint of the lender. The solution implies an upward-sloping leverage supply function. The lender demands a higher return than the cost of fund in order to compensate for the monitoring cost. When the borrower's leverage is higher, he is more likely to default, which means that the lender will need to monitor more often. As a result, the lender requires a high risk premium.

This 'financial accelerator' framework is commonly used to model the financial friction in firms. A partial list of models using this approach includes Carlstrom and Fuerst (1997), Christiano, Motto and Rostango (2010), Cespedes, Chang and Velasco (2004), Gertler, Gilchrist and Natalucci (2007), Curdia (2008), Christensen and Dib (2008) and Gilchrist and Zakrajsek (2012).

There are three recent innovation related to modelling banking fractions under this framework. First, Hirakata, Sudo and Ueda (2011) and Ueda (2012) modify the BGG model to introduce risk in the financial sector with a so-called 'chained credit contract'. In this contract, depositors lend to financial institutions in a loan contract and financial institutions lend to entrepreneurs in a loan contract. Financial institutions maximise their profits subject to the participation constraints of depositors and entrepreneurs. The resulting financial contract implies that the risk premium is a function of the capital-to-net-worth ratio of both financial institutions and entrepreneurs.

Second, Benes and Kumhof (2011), following Calstrom, Fuerst and Paustian (2011) and Calstrom et al. (2012), assumed that in the debt contract between banks and entrepreneurs, the contractual rate is fixed *ex ante*, which means that after an aggregate shock, the participation constraint of the banks may be violated and in

this case banks will make a loss. This provides some theoretical underpinnings to study bank capital adequacy requirements.

Third, Nowobilski (2012) study another modification of the BGG model to introduce banks and the interbank market. In his model, some depositors observe an imperfect signal of the idiosyncratic shock to the bank. After that they have an option to withdraw their deposits before the shock realises. Early redemption intensifies contractions in credit, further amplifying the volatility in the economy.

4.2.5 Our Work

The financial friction considered in our model arises out of costly state verification in the contract linking consumers and financial institutions. We have chosen this framework for two reasons. First, bank default in this framework is endogenous, whereas in all of the other approaches, bankruptcy of banks is ruled out by the incentive constraints. Second, Christiano et al. (2010) show how a rise in volatility in the idiosyncratic shocks can, through the cutoffs, have first-order effects to macroeconomic variables. This corresponds to the observation during the 2007-08 financial crisis that increased uncertainty aggravates the information asymmetry between lenders and borrowers and, consequently, worsens credit conditions. Specifically, in the following sections, we simplify the HSU (2011) chained credit contract by assuming that there is no friction between financial institutions and entrepreneurs. We show that the credit contract collapses to one which is similar to the BGG contract, but with higher volatility.

4.3 Credit Contracts, and the Evolution of the Net Worth of Financial Institutions and Entrepreneurs

We present our model in this section and the next. This section discusses the credit contracts between investors, financial institutions and entrepreneurs in a partial equilibrium framework. The main purpose of the section is to discuss the effects of financial frictions in the financial intermediation process. We take as given the net worths, the price of shares, the return on capital and the risk free interest rate. Each firm will maximise profits by choosing its optimal stock of capital, but for the economy as a whole, these choices will need to be consistent with the stock of capital in existence. In the subsequent section we endogenise these variables within a full general equilibrium macroeconomic system.

Three types of agents, namely investors, financial institutions and entrepreneurs (i.e the producers of goods), participate in the financial market. Financial institutions and entrepreneurs earn positive profits, accumulating net worth. Denote $N_{i,t}^F$ the net worth of the financial institution indexed i, where $i \in \{1, 2, ..., \infty\}$. The aggregate net worth of financial institutions is denoted as

$$N_t^F \equiv \sum_i N_{i,t}^F.$$

Each financial institution i buys the shares of the entrepreneurs indexed $j_i \in \{1, 2, ..., \infty\}$. The net worth of the entrepreneur indexed j_i is $N_{j_i,t}^E$. Denote $N_{i,t}^E$ as the sum of the net worths of the entrepreneurs whose shares are bought by financial institution i.

$$N_{i,t}^E \equiv \sum_{j_i} N_{j_i,t}^E, \qquad \text{for each } i.$$

The aggregate net worth of entrepreneurs is denoted as

$$N_t^E \equiv \sum_i N_{i,t}^E.$$

The net worths of financial institutions alone are not enough to finance the entrepreneurs' purchase of capital, so financial institutions take loans from investors. Investors are risk neutral agents who collect the deposits from consumers, paying them the risk free rate R_{t+1}, and play no interesting part in the model.

In any period of time, entrepreneurs sell used capital and choose capital for use in the following period. This means that at time t, entrepreneur j_i purchases its desired quantity of used capital and combines with investment goods to produce new capital for use at $t+1$. The quantity of capital to be used by this entrepreneur for production in period $t+1$ is denoted $K_{j_i,t+1}$, with the subscript denoting the period in which the capital is actually used, and the subscript j_i denoting the entrepreneur. Denote $K_{i,t+1}$ the sum of capital purchased by entrepreneurs whose shares are bought by financial institution i. Then

$$K_{i,t+1} \equiv \sum_{j_i} K_{j_i,t+1}, \qquad \text{for each } i.$$

Denote the aggregate capital stock in period $t+1$ as K_{t+1}, then

$$K_{t+1} \equiv \sum_i K_{i,t+1}.$$

Both entrepreneurs and financial institutions face idiosyncratic shocks to their return to capital. In each period, each entrepreneur draws from a distribution of ω^E, where $\ln \omega^E$ follows a normal distribution with mean $-\frac{1}{2}(\sigma^E)^2$ and variance $(\sigma^E)^2$, so that $E(\omega^E) = 1$. The realised gross return on capital from entrepreneur i is $\omega_{j_i}^E R_{t+1}^E$, where $\omega_{j_i}^E$ is an idiosyncratic disturbance to entrepreneur j_i's return and R_{t+1}^E is

the realised aggregate return to capital of entrepreneurs. Similarly, in each period, each financial institution draws from a distribution of ω^F, where $\ln \omega^F$ follows a normal distribution with mean $-\frac{1}{2}(\sigma^F)^2$ and variance $(\sigma^F)^2$, so that $E(\omega^F) = 1$. The realised gross return on capital from financial institution i is $\omega_i^F R_{t+1}^F$, where ω_i^F is an idiosyncratic disturbance to financial institution i's return and R_{t+1}^F is the realised aggregate return to capital of financial institutions. We further assume that ω^E and ω^F are independent across time and across entrepreneurs and financial institutions. Lenders can diversify away the idiosyncratic risks of the borrowers by investing in a lot of borrowers.

4.3.1 The Loan Contract for Each Financial Institution

We first describe the contract between consumers and financial institutions, which is a loan contract.

We follow the financial frictions approach pioneered by BGG, which assumes costly state verification as a way of motivating financial frictions. The return of each financial institution is subject to an idiosyncratic shock ω^F of the kind described above. If investors want to observe the shock for a specific financial institution, they need to pay a monitoring cost, which is assumed to be a fixed fraction μ^F of the financial institution's total capital.

The solution to this costly state verification problem is the standard debt contract due to Townsend (1979). Here is why. Due to the monitoring cost, it would be too costly for investors to monitor every contract. Then there is a cutoff value of the shock $\bar{\omega}_i^F$ for each financial institution i above which the financial institution will make positive profits and below which it is bankrupt. Above the cutoff value, financial institution i gets a return sufficient to pay investors the contractual rate of interest Z_i^F and keep what is left for itself; investors have no incentive to monitor the financial institution's true return. Below the cutoff value, the financial institution is

bankrupt, and investors pay the monitoring cost and take the financial institution's entire wealth. The existence of such a cost is what drives the external finance premium on lending to financial institutions.

Since each financial institution receives a different realisation of the random shock every period, financial institutions will have different net worths. We now investigate the optimal contracting problem facing the financial institution indexed i with a net worth of $N^F_{i,t}$.

Financial institution i purchases shares from entrepreneurs. These entrepreneurs together wish to hold a capital stock of $K_{i,t+1}$, for reasons discussed in the macroeconomic part of the model. They finance $N^E_{i,t+1}$ of such holdings of capital from their own net worths. If Q_t is the price of shares in capital then such entrepreneurs will need to sell $(Q_tK_{i,t+1} - N^E_{i,t+1})$ shares. Financial institution i purchases these shares. We assume that financial intermediation is costless. Then this financial institution will need to borrow $(Q_tK_{i,t+1} - N^E_{i,t+1} - N^F_{i,t+1})$ from investors.

Now suppose that financial institution i is one whose idiosyncratic shock is exactly $\bar{\omega}^F_i$. The returns to this financial institution will be exactly equal to the contractual rate of interest $Z^F_{i,t+1}$, in the contract between the financial institution and its investors, times what has been borrowed from its investors $(Q_tK_{i,t+1} - N^E_{i,t+1} - N^F_{i,t+1})$. It is useful for expositional reasons to describe $(Q_tK_{i,t+1} - N^E_{i,t+1})$ as the expenditure on shares by financial institution i. Since the returns to this financial institution are exactly $\bar{\omega}^F_{i,t+1}R^F_{t+1}(Q_tK_{i,t+1} - N^E_{i,t+1})$, this means that

$$\bar{\omega}^F_{i,t+1}R^F_{t+1}(Q_tK_{i,t+1} - N^E_{i,t+1}) = Z^F_{i,t+1}(Q_tK_{i,t+1} - N^E_{i,t+1} - N^F_{i,t+1}). \tag{4.2}$$

It is clear from this equation that there is a one-to-one mapping between $Z^F_{i,t+1}$, and $\bar{\omega}^F_{i,t+1}$; a higher contractual rate of interest will cause more financial institutions to go bankrupt and so cause $\bar{\omega}^F_{i,t+1}$ to rise.

To be perfectly insured against the idiosyncratic shocks of financial institutions, investors sign contracts with an infinite number of financial institutions. The solution of the loan contract defines how the investment revenue is split between the lenders (investors) and the borrowers (financial institutions) under all possible realisations of future state-spaces. Suppose the idiosyncratic shock has a cumulative distribution function $F^F(\omega^F)$. Let $\Gamma^F(\bar{\omega}_i^F)$ be the share that goes to the lender, then Γ^F is given by

$$\Gamma^F(\bar{\omega}_i^F) \quad = \quad G^F(\bar{\omega}_i^F) + [1 - F^F(\bar{\omega}_i^F)]\bar{\omega}_i^F. \tag{4.3}$$

where $G^F(\bar{\omega}_i^F)$ is defined as[9]

$$G^F(\bar{\omega}_i^F) = \int_0^{\bar{\omega}_i^F} \omega^F dF^F(\omega^F).$$

The first term on the right hand side of Equation (4.3) accounts for what investors get in expectation (before monitoring) from financial institution i if it defaults, and the second term refers to the fixed interest payment investors get (expressed in the share of profits) from the financial institution if it does not default, with probability $1 - F^F(\bar{\omega}_i^F)$. One can easily show that $\Gamma_\omega^F(\bar{\omega}_i^F) = 1 - F^F(\bar{\omega}_i^F) > 0$, that is the higher the cutoff value, the larger is the share that goes to the lender. The borrower gets the share $(1 - \Gamma^F(\bar{\omega}_i^F))$. Figure 4.1 shows the Γ^F function diagrammatically.

After taking into account the monitoring cost, the net share that goes to the lender is

$$\Psi^F(\bar{\omega}_i^F) = \Gamma^F(\bar{\omega}_i^F) - \mu^F G^F(\bar{\omega}_i^F), \tag{4.4}$$

[9]Given the log-normal assumption, it can be shown that

$$F^F(\bar{\omega}^F) = \Phi\left(\frac{\ln \bar{\omega}^F + \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right), \text{ and} \qquad G^F(\bar{\omega}^F) = \Phi\left(\frac{\ln \bar{\omega}^F - \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. Using (4.3) and (4.4), Γ^F and Ψ^F can similarly be written in terms of Φ.

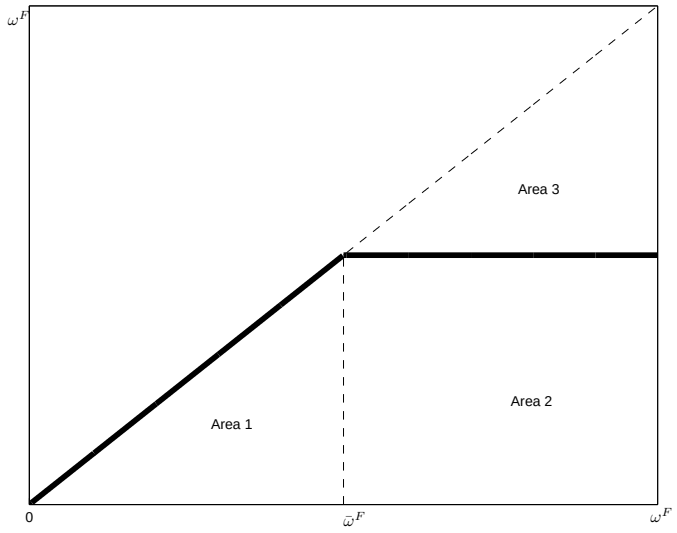


Figure 4.1: The Γ^F function. Area 1 = $G^F(\bar{\omega}^F) = \int_0^{\bar{\omega}^F} \omega^F dF^F$. Area 2 = $(1 - F^F(\bar{\omega}^F))\bar{\omega}^F$. Area 3 = 1- Area 1 - Area 2. The share of revenue retained by investors = $\Gamma^F(\bar{\omega}^F)$ = Area under solid line = Area 1 + Area 2. The share of revenue retained by financial institutions = Area 3. It is easy to see that $\Gamma^F(\bar{\omega}^F)$ is increasing in the cutoff $\bar{\omega}^F$.

where the last term on the right hand side accounts for the resources wasted in monitoring. The sign of Ψ^F_ω is ambiguous. However, in the optimal contract, it remains true that the optimal $\bar{\omega}^F_i$ will be such that $\Psi^F_\omega(\bar{\omega}^F_i) > 0$.[10]

We assume that investors are perfectly competitive, making zero profit. As the idiosyncratic risk is totally diversifiable for investors, the required return to investors is the risk free rate R_{t+1}. The participation constraint for these investors requires that these investors' expected earnings on their loans $E_t[[\Gamma^F(\bar{\omega}^F_{i,t+1}) - \mu^F G^F(\bar{\omega}^F_{i,t+1})]R^F_{t+1}](Q_t K_{i,t+1} - N^E_{i,t+1})$ are exactly equal to the risk free rate, R_{t+1}, (which these investors pay to consumers) times what they lend to this financial institution which is equal to $(Q_t K_{i,t+1} - N^E_{i,t+1} - N^F_{i,t+1})$. Therefore we can write the 'expected participation constraint' as

$$E_t[[\Gamma^F(\bar{\omega}^F_{i,t+1}) - \mu^F G^F(\bar{\omega}^F_{i,t+1})]R^F_{t+1}](Q_t K_{i,t+1} - N^E_{i,t+1}) = R_{t+1}(Q_t K_{i,t+1} - N^E_{i,t+1} - N^F_{i,t+1}), \tag{4.5}$$

The variables Q and R are determined macroeconomically and are given to each financial institution. The net worths of financial institutions and of entrepreneurs are predetermined, and the expected value of the return on share-holdings, R^F, is also given to individual financial institutions.[11] The demand for shares will be chosen by each financial institution in light of this.[12]

[10] The proof is the following: for the log-normal distribution, $\bar{\omega}^F h^F(\bar{\omega}^F)$ is increasing in $\bar{\omega}^F$, where $h^F(\bar{\omega}^F) = f^F(\bar{\omega}^F)/(1 - F^F(\bar{\omega}^F))$ is the hazard rate. Therefore,

$$\begin{aligned} G^F_\omega(\bar{\omega}^F) &= \bar{\omega}^F f^F(\bar{\omega}^F) = \bar{\omega}^F h^F(\bar{\omega}^F)(1 - F^F(\bar{\omega}^F)) > 0, \\ \Gamma^F_\omega(\bar{\omega}^F) &= G^F_\omega(\bar{\omega}^F) + (1 - F^F(\bar{\omega}^F)) - \bar{\omega}^F f^F(\bar{\omega}^F) = 1 - F^F(\bar{\omega}^F) > 0. \end{aligned}$$

There exists an $\bar{\omega}^{F*}$ such that $\Psi^F_\omega(\bar{\omega}^F) = \Gamma^F_\omega(\bar{\omega}^F) - \mu^F G^F_\omega(\bar{\omega}^F) = (1 - F^F(\bar{\omega}^F))(1 - \mu^F \bar{\omega}^F h^F(\bar{\omega}^F)) > 0$ for $\bar{\omega}^F < \bar{\omega}^{F*}$ and $\Psi^F_\omega(\bar{\omega}^F) < 0$ for $\bar{\omega}^F > \bar{\omega}^{F*}$. For any $\bar{\omega}^F_1$ such that $\bar{\omega}^F_1 > \bar{\omega}^{F*}$, there exist a $\bar{\omega}^F_2$ such that $\bar{\omega}^F_2 < \bar{\omega}^{F*} < \bar{\omega}^F_1$ and $\Psi^F(\bar{\omega}^F_2) = \Psi^F(\bar{\omega}^F_1)$. Since the smaller $\bar{\omega}^F_2$ implies a smaller bankruptcy rate for the borrower than $\bar{\omega}^F_1$ while keeping the lender's share of profit unchanged, the borrower will never choose any $\bar{\omega}^F_1 > \bar{\omega}^{F*}$. Hence, $\bar{\omega}^F$ has an interior solution and in the optimal contract $\Psi^F_\omega(\bar{\omega}^F) > 0$.

[11] We will discuss in detail how it is determined macroeconomically in the next section.

[12] As will be discussed in the next section, macroeconomically these demands must add up to the outstanding stock of shares issued by entrepreneurs.

Since financial institutions are risk-neutral, financial institution i will maximise its expected profit, $E_t[(1-\Gamma^F(\bar{\omega}^F_{i,t+1}))R^F_{t+1}(Q_tK_{i,t+1}-N^E_{i,t+1})]$, by choosing a desired expenditure on shares, $Q_tK_{i,t+1}-N^E_{i,t+1}$, in period t, and choosing the value of $\bar{\omega}^F_{t+1}$ in period $t+1$, given R^F_{t+1}, and subject to the participation constraint.

The Lagrangian for this problem is

$$\begin{aligned} L &= E_t\Big[(1-\Gamma^F(\bar{\omega}^F_{i,t+1}))R^F_{t+1}(Q_tK_{i,t+1}-N^E_{i,t+1}) \\ &\quad +\lambda_{i,t+1}\left(\begin{array}{c}(\Gamma^F(\bar{\omega}^F_{i,t+1})-\mu^F G^F(\bar{\omega}^F_{i,t+1}))R^F_{t+1}(Q_tK_{i,t+1}-N^E_{i,t+1}) \\ -R_{t+1}(Q_tK_{i,t+1}-N^E_{i,t+1}-N^F_{i,t+1})\end{array}\right)\Big], \end{aligned}$$

and the first order conditions are[13]

$$\lambda(\bar{\omega}^F_{i,t+1}) = \frac{\Gamma^F_\omega(\bar{\omega}^F_{i,t+1})}{\Gamma^F_\omega(\bar{\omega}^F_{i,t+1})-\mu^F G^F_\omega(\bar{\omega}^F_{i,t+1})}, \qquad \lambda_\omega(\bar{\omega}^F_{i,t+1})>0, \tag{4.6}$$

$$\frac{E_t(R^F_{t+1})}{R_{t+1}} = E_t\left(\frac{\lambda(\bar{\omega}^F_{i,t+1})}{\lambda(\bar{\omega}^F_{i,t+1})(\Gamma^F(\bar{\omega}^F_{i,t+1})-\mu^F G^F(\bar{\omega}^F_{i,t+1}))+(1-\Gamma^F(\bar{\omega}^F_{i,t+1}))}\right) \tag{4.7}$$

These first order conditions are similar to the contract in BGG and are easy to understand intuitively. The Lagrange multiplier to the participation constraint $\lambda_{i,t+1}$, derived in Equation (4.6), measures the shadow value of an additional unit of net worth (internal funds) to financial institution i. Equation (4.6) says that the optimal

[13] To derive the first derivative of λ with respect to $\bar{\omega}^F$, note that

$$\begin{aligned} G^F_{\omega\omega}(\bar{\omega}^F) &= \frac{d}{d\bar{\omega}^F}[\bar{\omega}^F h(\bar{\omega}^F)(1-F^F(\bar{\omega}^F))] = (1-F^F(\bar{\omega}^F))\frac{d(\bar{\omega}^F h^F(\bar{\omega}^F))}{d\bar{\omega}^F} - \bar{\omega}^F h^F(\bar{\omega}^F) f^F(\bar{\omega}^F), \\ \Gamma^F_{\omega\omega}(\bar{\omega}^F) &= -f^F(\bar{\omega}^F). \end{aligned}$$

Therefore,

$$\begin{aligned} &\Gamma^F_\omega(\bar{\omega}^F)G^F_{\omega\omega}(\bar{\omega}^F)-\Gamma^F_{\omega\omega}(\bar{\omega}^F)G^F_\omega(\bar{\omega}^F), \\ =\ &(1-F^F(\bar{\omega}^F))^2\frac{d(\bar{\omega}^F h^F(\bar{\omega}^F))}{d\bar{\omega}^F} - \bar{\omega}^F h^F(\bar{\omega}^F) f^F(\bar{\omega}^F)(1-F^F(\bar{\omega}^F)) - (-f^F(\bar{\omega}^F))\bar{\omega}^F h^F(\bar{\omega}^F)(1-F^F(\bar{\omega}^F)), \\ =\ &(1-F^F(\bar{\omega}^F))^2\frac{d(\bar{\omega}^F h^F(\bar{\omega}^F))}{d\bar{\omega}^F} > 0. \end{aligned}$$

And $\lambda_\omega(\bar{\omega}^F) = \frac{\mu^F}{(\Gamma^F_\omega(\bar{\omega}^F)-\mu^F G^F_\omega(\bar{\omega}^F))^2} \times [\Gamma^F_\omega(\bar{\omega}^F)G^F_{\omega\omega}(\bar{\omega}^F) - \Gamma^F_{\omega\omega}(\bar{\omega}^F)G^F_\omega(\bar{\omega}^F)] > 0.$

cutoff is chosen such that the marginal value to the borrower of reducing the cutoff is identical to the marginal cost of tightening the participation constraint. Equation (4.7) shows that the shadow value of internal funds, $\lambda_{i,t+1}$, will be larger than unity precisely because $E_t R^F_{t+1}$ is bigger than R_{t+1}, by an amount which depends on the size of the monitoring cost parameter μ^F. Funds obtained externally need to pay a higher return to cover the possibility that the lender will need to pay for the monitoring cost. We call $E_t(R^F_{t+1})/R_{t+1}$ the external finance premium.

The two first order conditions solve for an optimal expected cutoff value $E_t\bar{\omega}^F_{i,t+1}$. We use Equations (4.6) and (4.7) to eliminate the Lagrange multiplier $\lambda_{i,t+1}$ and get the 'combined FOC':

$$E_t\left(\frac{\frac{\Gamma^F_\omega(\bar{\omega}^F_{i,t+1})}{\Gamma^F_\omega(\bar{\omega}^F_{i,t+1})-\mu^F G^F_\omega(\bar{\omega}^F_{i,t+1})}}{\frac{\Gamma^F_\omega(\bar{\omega}^F_{i,t+1})}{\Gamma^F_\omega(\bar{\omega}^F_{i,t+1})-\mu^F G^F_\omega(\bar{\omega}^F_{i,t+1})}(\Gamma^F(\bar{\omega}^F_{i,t+1})-\mu^F G^F(\bar{\omega}^F_{i,t+1}))+(1-\Gamma^F(\bar{\omega}^F_{i,t+1}))}\right)=\frac{E_t(R^F_{t+1})}{R_{t+1}}. \tag{4.8}$$

This combined FOC equates the slope of the constant expected profits line and the slope of the participation constraint. This is shown diagrammatically in Figure 4.2 in $(Q_tK_{i,t+1}-N^E_{i,t+1}, E_t\bar{\omega}^F_{i,t+1})$ space.[14] Suppose financial institution i chooses a point on the left of the tangency point on the expected participation constraint. This point is not optimal because by demanding an additional unit of share purchase, and incurring a higher expected cutoff value, financial institution i can get higher expected profits. This is because, an additional unit of share purchase requires a rise in the expected cutoff by $\frac{1}{QK-N^E}\left(\frac{R/R^F-\Psi}{\Psi'}\right)$ unit; on the other hand, an additional unit of share purchase will increase the expected profits unless the expected cutoff value rises by $\frac{1}{QK-N^E}\left(\frac{1-\Gamma}{\Gamma'}\right)$ unit. Below the optimal point, the expected returns on shares $E_tR^F_{t+1}$ relative to the risk free rate R_{t+1} is more than enough to cover the

[14]The constant expected profits line is concave because a rise in the cutoff value increases the Γ function in a decreasing rate. Therefore, $(1-\Gamma)$ is increasing more than linearly, which means that the desired expenditure on shares has to rise in an increasing rate as the expected cutoff value rises linearly. The reasoning for concavity of the participation constraint is similar.

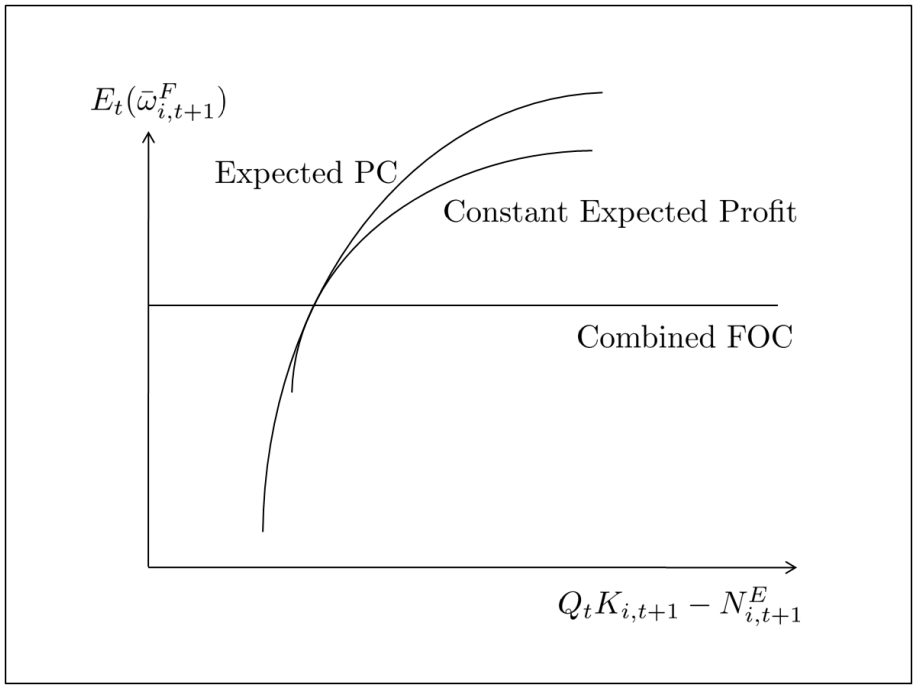


Figure 4.2: The constant expected profits line, the expected participation constraint and the combined first order condition of financial institution i's profit maximisation problem in the loan contract.

costs associated with the marginal increase in monitoring costs paid by investors, so it would be profitable for the financial institution to demand an additional unit of share purchase.

This combined FOC is scale independent, which comes from the fact that monitoring costs are a constant fraction μ^F of the financial institution's desired expenditure on shares. As a result, both the marginal rate of substitution in the constant expected profits line and the slope of the participation constraint are linear in the desired expenditure on shares. A rise in the desired expenditure on shares increases the slopes in an offsetting way, so the expected cutoff value is independent of the desired expenditure on shares.

Taken together, the participation constraint and the combined FOC fully specify the credit contract: they fully determine the purchase of shares by financial

institution i and the cutoff value.

4.3.2 Aggregation of the Loan Contract

We aggregate across financial institutions in this subsection. Consider again the combined first order condition Equation (4.8). The external finance premium on the right hand side is determined macroeconomically. This equation implies that every financial institution will choose the same expected cutoff value, $E_t(\bar{\omega}^F_{i,t+1})$, regardless of its net worth, so that $E_t(\bar{\omega}^F_{i,t+1}) = E_t(\bar{\omega}^F_{t+1})$ for all i. The combined FOC, Equation (4.8), can thus be replaced by the following equation for the economy as a whole as follows:

$$E_t\left(\frac{\frac{\Gamma^F_\omega(\bar{\omega}^F_{t+1})}{\Gamma^F_\omega(\bar{\omega}^F_{t+1})-\mu^F G^F_\omega(\bar{\omega}^F_{t+1})}}{\frac{\Gamma^F_\omega(\bar{\omega}^F_{t+1})}{\Gamma^F_\omega(\bar{\omega}^F_{t+1})-\mu^F G^F_\omega(\bar{\omega}^F_{t+1})}(\Gamma^F(\bar{\omega}^F_{t+1})-\mu^F G^F(\bar{\omega}^F_{t+1}))+(1-\Gamma^F(\bar{\omega}^F_{t+1}))}\right)=\frac{E_t(R^F_{t+1})}{R_{t+1}}. \tag{4.9}$$

Using the fact that financial institutions choose the same expected cutoff value, $E_t(\bar{\omega}^F_{t+1})$, the expected participation constraint, Equation (4.5), implies that financial institutions with different net worths will choose their desired expenditure on shares so that their leverages $(Q_tK_{i,t+1} - N^E_{i,t+1})/N^F_{i,t+1}$ are the same across all financial institutions. Hence the expected participation constraint can be aggregated to obtain

$$E_t[[\Gamma^F(\bar{\omega}^F_{t+1})-\mu^F G^F(\bar{\omega}^F_{t+1})]R^F_{t+1}](Q_tK_{t+1}-N^E_{t+1}) = R_{t+1}(Q_tK_{t+1}-N^E_{t+1}-N^F_{t+1}). \tag{4.10}$$

These two equations will hold at any point in time in which there had not been any unanticipated aggregate shocks.

However, after a shock, $Q_tK_{t+1} - N^E_{t+1}$ has already been fixed, consumers must continue to receive the risk free rate R_{t+1}. Taken together, these two things deter-

mines what the realised cutoff value $\bar{\omega}^F_{t+1}$ must be. It is this *realised* participation constraint,

$$[\Gamma^F(\bar{\omega}^F_t) - \mu^F G^F(\bar{\omega}^F_t)]R^F_t(Q_{t-1}K_t - N^E_t) = R_t(Q_{t-1}K_t - N^E_t - N^F_t), \qquad (4.11)$$

which is implemented in the model at any time t, along with the combined FOC, which contains expectations of what will happen in period $t+1$. See Equations (4.47) and (4.48).

These equations can be combined to obtain a 'leverage supply function', and we will use this to discuss our simulation results. The leverage supply function is:[15]

$$\frac{Q_t K_{t+1} - N^E_{t+1}}{N^F_{t+1}} = s\left(\frac{E_t R^F_{t+1}}{R_{t+1}}\right), \quad s' > 0. \qquad (4.12)$$

Note that the leverage is an increasing function of the external finance premium.

We use this leverage supply function to show the way in which $E_t R^F_{t+1}$ must be determined, macroeconomically. For the economy as a whole, the supply of capital is fixed at any point in time. Since the market for shares has to clear, this means that financial institutions need to be sufficiently leveraged to hold whatever capital is in existence. This means that, all terms on the left hand side of Equation (4.12) are exogenous to the financial system taken as a whole. As a result it must be the case that Equation (4.12) ends up determining $E_t R^F_{t+1}/R_{t+1}$. In other words, at any point in time the return on holding shares R^F_{t+1} must be sufficiently large to induce financial institutions to hold them. How this actually happens depends on the actual workings of the whole economy, and in particular on how Q is determined.

[15]To derive the leverage supply function, we first combine Equations (4.9) and (4.10) to write the leverage in terms of the expected cutoff value. Next, we differentiate the leverage with respect to the expected cutoff value and show that $d\left(\frac{Q_t K_{t+1} - N^E_{t+1}}{N^F_{t+1}}\right)/dE_t(\bar{\omega}^F_{t+1}) > 0$. Finally, we take the first derivative of Equation (4.9) with respect to the expected cutoff value to show that $dE_t(\bar{\omega}^F_{t+1})/d(\frac{E_t(R^F_{t+1})}{R_{t+1}}) > 0$. Hence, the leverage supply function, (4.12), has a positive slope.

4.3.3 The Share Contract

The contract between financial institutions and entrepreneurs is a share contract. As equity holders of entrepreneurs, financial institutions absorb the profits and losses of entrepreneurs. We assume there is no problem of asymmetric information and no monitoring takes place. Financial institutions and entrepreneurs split the profits according to the shares of their investments, regardless of the idiosyncratic shock to entrepreneurs ω^E. To diversify away from the entrepreneur-specific idiosyncratic risk, each financial institution will invest in an infinite number of entrepreneurs.

Consider financial institution i again. It finances a worth of $(Q_t K_{i,t+1} - N^E_{i,t+1})$ to entrepreneurs. Denote the aggregate return to capital as R^E_{t+1} and financial institution i's average share of profits from its share contracts (with an infinite number of entrepreneurs $j_i = \{1, 2, ..., \infty\}$) as $\Gamma^E_{i,t+1}$, then

$$\begin{aligned} \Gamma^E_{i,t+1} R^E_{t+1} Q_t K_{i,t+1} &= \frac{Q_t K_{i,t+1} - N^E_{i,t+1}}{Q_t K_{i,t+1}} \int_0^\infty \omega^E R^E_{t+1} Q_t K_{i,t+1} dF^E(\omega^E), \\ \therefore \qquad \Gamma^E_{i,t+1} &= \frac{Q_t K_{i,t+1} - N^E_{i,t+1}}{Q_t K_{i,t+1}}. \end{aligned} \tag{4.13}$$

The share Γ^E_i is independent of the idiosyncratic shock ω^E because there is no asymmetric information. Since entrepreneurs in the entire economy draw the idiosyncratic shocks from the same distribution, every financial institution should receive the same average share of profits. Denote Γ^E the aggregate share of profits in the share contracts that goes to financial institutions, then $\Gamma^E_{i,t} = \Gamma^E_t$. Hence,

$$\Gamma^E_t = \frac{Q_t K_{t+1} - N^E_{t+1}}{Q_t K_{t+1}}. \tag{4.14}$$

The split of the profits does not depend on the realisation of the idiosyncratic shock.

Finally, the realised return to financial institutions' investment equals the realised

return on entrepreneurs' investment:

$$R_t^E = R_t^F. \tag{4.15}$$

In Appendix 4.7.2, we describe the 'chained loan contract' in the Hirakata, Sudo and Ueda (2009) model. HSU assume the contract between financial institutions and entrepreneurs is also a loan contract. In the simulations we will compare our model with the HSU model.

4.3.4 Dynamic Behaviour of Net Worth

Next, we consider the evolution of the aggregate net worths of financial institutions and entrepreneurs. As discussed in the previous section, the nature of the loan contract is such that the leverage of financial institutions is independent of the net worth of the institutions N_t^F. In the share contract, the distribution of idiosyncratic shocks to entrepreneurs is symmetric and the Γ^E function depends only on aggregate variables. We only need to keep track of the aggregate levels of net worths in each period. Let V_t^E and V_t^F be the earnings from credit contracts of entrepreneurs and financial institutions, then

$$V_t^E = (1-\Gamma_t^E)R_t^E Q_{t-1}K_t = R_t^E N_t^E, \tag{4.16}$$

$$V_t^F = \Gamma_t^E(1-\Gamma^F(\bar{\omega}_t^F))R_t^E Q_{t-1}K_t = (1-\Gamma^F(\bar{\omega}_t^F))R_t^E(Q_{t-1}K_t - N_t^E). \tag{4.17}$$

There are two technical matters. First, it is necessary to start financial institutions and entrepreneurs off with some net worth in order to allow them to begin their operations. Following BGG and Calstrom and Fuerst (1996), we assume that in addition to operating firms, entrepreneurs and financial institutions supplement their income by working in the general labour market. Both inelastically supply one

unit of labour. The net worths come from earnings from the credit contracts and labour income. As in BGG, in the calibrations we keep the share of labour income going to financial institutions and entrepreneurs small, so that the modification has no significant direct effects on the results.

Second, in order to prevent financial institutions and entrepreneurs from growing out of the financial constraints, we assume that at the end of every period, constant fractions $(1-\gamma^E)$ and $(1-\gamma^F)$ of entrepreneurs and financial institutions respectively quit their businesses. When they quit, they consume their net worth C_t^E and C_t^F.

So, the laws of motion of net worths for entrepreneurs and financial institutions are

$$N_{t+1}^E = \gamma^E V_t^E + W_{Et}, \tag{4.18}$$

$$N_{t+1}^F = \gamma^F V_t^F + W_{Ft}, \tag{4.19}$$

where W_{Et} and W_{Ft} denote the labour incomes of entrepreneurs and financial institutions. Consumption of the quitting entrepreneurs and financial institutions are

$$C_t^E = (1-\gamma^E)V_t^E, \tag{4.20}$$

$$C_t^F = (1-\gamma^F)V_t^F. \tag{4.21}$$

4.4 The Full Macro Model

Our model adopts the same basic real business cycle framework with capital adjustment costs and adds in the credit contracts. By embedding the partial equilibrium in the general equilibrium framework, we endogenise the risk free rate R, gross return on capital R^E and the price of capital Q.

The economy consists of four types of agents, namely consumers, investors, fi-

nancial institutions and entrepreneurs. In the following we describe the aggregate behaviour for consumers and entrepreneurs.

4.4.1 Consumers

Our consumers are reasonably conventional. Homogeneous consumers consume, save and supply labour. They save by lending funds to investors. For simplicity, we assume log utility in consumption and separability between consumption utility and labour disutility. Consumers maximise the following utility

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \chi \frac{L_{Ct}^{1+\varphi}}{1+\varphi} \right), \tag{4.22}$$

subject to the budget constraint

$$C_t + B_{t+1} = W_t L_{Ct} + R_t B_t, \tag{4.23}$$

where C_t is consumption, L_{Ct} is labour supply by consumers, W_t is the real wage and B_{t+1} is the real amount of lending. R_t is the real return from lending from period $t-1$ to t. β is the discount factor, χ the utility weight on leisure and φ the inverse of Frisch elasticity of labour supply.

The first order conditions are standard:

$$1 = \beta E_t \left(\frac{C_t}{C_{t+1}} R_{t+1} \right), \tag{4.24}$$

$$W_t = \chi C_t L_{Ct}^{\varphi}. \tag{4.25}$$

The first equation is the Euler equation that governs intertemporal consumption decision, meaning that consumption is rising when the interest rate is high. The second equation is the condition of intratemporal substitution between consumption

and labour, meaning that, *ceteris paribus,* the labour supply rises when the real wage is high, and that the labour supply falls when consumption rises.

4.4.2 The Entrepreneurial Sector

Each entrepreneur produces goods using capital and labour using a Cobb-Douglas production function:

$$Y_{j_i,t} = A_t K_{j_i,t}^{\alpha} L_{j_i,t}^{1-\alpha}, \tag{4.26}$$

where A_t is the economy-wide level of technology of goods production. $Y_{j_i,t}$ is the output by entrepreneur j_i. $K_{j_i,t}$ is the capital used by the entrepreneur in goods production. $L_{j_i,t}$ is a composite of the labour supplied by consumers $L_{j_i,Ct}$, financial institutions $L_{j_i,Ft}$ and entrepreneurs $L_{j_i,Et}$ which has the following form:

$$L_{j_i,t} = L_{j_i,Ct}^{1-\Omega_E-\Omega_F} L_{j_i,Et}^{\Omega_E} L_{j_i,Ft}^{\Omega_F}. \tag{4.27}$$

We follow Lorenzoni and Walentin (2007) in the formulation of the investment adjustment cost. Every period after production takes place, capital depreciates and entrepreneurs has $K_{j_i,t}$ units of used capital. He can trade capital in the used capital market at price Q_t^o and choose to hold on to $K_{j_i,t}^o$ units of used capital.[16] Next, the entrepreneur chooses the capital stock for the next period, $K_{j_i,t+1}$, and pays $I_{j_i,t} = I(K_{j_i,t+1}, K_{j_i,t}^o)$ which includes both the cost of acquiring new capital and the installation (or uninstallation) cost. I is increasing and convex in $K_{j_i,t+1}$, decreasing in $K_{j_i,t}^o$, homogeneous of degree one and $\partial I(K_{j_i,t+1}, K_{j_i,t}^o)/\partial K_{j_i,t+1} = 1$ if $K_{j_i,t+1} = K_{j_i,t}^o$.

[16]The capital market in this setup only opens once in each period. An alternative setup is one in which there are goods producers and capital producers. In that setup, used capital are sold to capital producers who produce new capital within the same period and sell the new capital back to goods producers. The prices of capital in these two markets would in general be different and it is therefore incorrect to use a single Q_t to denote the prices in these two markets. Our setup does not suffer from this problem.

The maximisation problem for the entrepreneur is

$$\max E_0 \sum_{t=0}^{\infty} \frac{1}{\prod_{s=1}^{t} R_s^E} \begin{bmatrix} A_t K_{j_i,t}^{\alpha} L_{j_i,t}^{(1-\alpha)} + Q_t^o (K_{j_i,t} - K_{j_i,t}^o) - I(K_{j_i,t+1}, K_{j_i,t}^o) \\ -W_t L_{j_i,Ct} - W_{Et} L_{j_i,Et} - W_{Ft} L_{j_i,Ft} \end{bmatrix} \quad (4.28)$$

The entrepreneur chooses the quantity of old capital to buy, the new capital to produce and labour in the next period.

Given the prices $Q_t^o, W_t, W_{Et}, W_{Ft}$, we can define a new variable Q_t to satisfy the following conditions for any $K_{j_i,t}$ and $K_{j_i,t+1}$:[17]

$$Q_t K_{j_i,t+1} = \min_{K_{j_i,t}^o} [I(K_{j_i,t+1}, K_{j_i,t}^o) + Q_t^o K_{j_i,t}^o]. \quad (4.29)$$

Equation (4.29) minimises the cost of producing $K_{j_i,t+1}$ units of capital. Note that Q_t is the shadow price of new capital.

The entrepreneur's profit maximisation problem can be re-written as follows:

$$\max_{K_{j_i,t+1}, L_{j_i,Ct}, L_{j_i,Ft}, L_{j_i,Et}} E_0 \sum_{t=0}^{\infty} \frac{1}{\prod_{s=1}^{t} R_s^E} \begin{bmatrix} A_t K_{j_i,t}^{\alpha} L_{j_i,t}^{(1-\alpha)} + Q_t^o K_{j_i,t} - Q_t K_{j_i,t+1} \\ -W_t L_{j_i,Ct} - W_{Et} L_{j_i,Et} - W_{Ft} L_{j_i,Ft} \end{bmatrix} \quad (4.30)$$

[17] The left hand side of Equation (4.29) is linear in capital because $I(K_{j_i,t+1}, K_{j_i,t}^o)$ is homogeneous of degree 1. To see this, suppose $\tilde{K}_{j_i,t}^o$ minimises $I(1, K_{j_i,t}^o) + Q_t^o K_{j_i,t}^o$, then, using the fact that $I(K_{j_i,t+1}, K_{j_i,t}^o)$ is homogeneous of degree 1, we can write the following:

$$I(K_{j_i,t+1}, K_{j_i,t}^o) + Q_t^o K_{j_i,t}^o = I\left(1, \frac{K_{j_i,t}^o}{K_{j_i,t+1}}\right) K_{j_i,t+1} + Q_t^o \frac{K_{j_i,t}^o}{K_{j_i,t+1}} K_{j_i,t+1} = \left(I\left(1, \frac{K_{j_i,t}^o}{K_{j_i,t+1}}\right) + Q_t^o \frac{K_{j_i,t}^o}{K_{j_i,t+1}}\right) K_{j_i,t+1}.$$

Obviously, the above expression is minimised by $K_{j_i,t}^o = K_{j_i,t+1} \tilde{K}_{j_i,t}^o$. Therefore,

$$\min_{K_{j_i,t}^o} \{I(K_{j_i,t+1}, K_{j_i,t}^o) + Q_t^o K_{j_i,t}^o\} = \left(I\left(1, \tilde{K}_{j_i,t}^o\right) + Q_t^o \tilde{K}_{j_i,t}^o\right) K_{j_i,t+1} \equiv Q_t K_{j_i,t+1}.$$

The first order conditions for the labour demands are

$$W_t = (1-\alpha)(1-\Omega_E-\Omega_F)\frac{Y_{j_i,t}}{L_{j_i,Ct}} = (1-\alpha)(1-\Omega_E-\Omega_F)\frac{Y_t}{L_{Ct}}, \quad (4.31)$$

$$W_{Ft} = (1-\alpha)\Omega_F\frac{Y_{j_i,t}}{L_{j_i,Ft}} = (1-\alpha)\Omega_F Y_t, \quad (4.32)$$

$$W_{Et} = (1-\alpha)\Omega_E\frac{Y_{j_i,t}}{L_{j_i,Et}} = (1-\alpha)\Omega_E Y_t. \quad (4.33)$$

Entrepreneurs will choose the same output to labour ratio. Aggregate output is the sum of output of all entrepreneurs, that is $Y_t = \sum_i \sum_{j_i} Y_{j_i,t}$, and similarly for each type of labour. We have used the fact that both entrepreneurs and financial institutions supply one unit of labour inelastically.

The first order condition for $K_{j_i,t+1}$ is:

$$E_t R^E_{t+1} = E_t\left(\frac{\alpha\frac{Y_{t+1}}{K_{t+1}} + Q^o_{t+1}}{Q_t}\right). \quad (4.34)$$

This means that the expected return on capital in period t is the marginal product of capital in period $t+1$ plus the resale value of capital in period $t+1$ divided by the shadow price of capital in period t.

And the realised return on capital is

$$R^E_t = \frac{\alpha\frac{Y_t}{K_t} + Q^o_t}{Q_{t-1}}. \quad (4.35)$$

Next, we find the relation between the market price of capital Q^o_t and the shadow price of capital Q_t. The relation is given implicitly by Equation (4.29). The first order condition for $K_{j_i,t+1}$ is

$$Q^o_t = -\frac{\partial I(K_{j_i,t+1}, K^o_{j_i,t})}{\partial K^o_{j_i,t}} = -\frac{\partial I(1, K^o_{j_i,t}/K_{j_i,t+1})}{\partial\left(K^o_{j_i,t}/K_{j_i,t+1}\right)} = -\frac{\partial I(K_{t+1}, K_t)}{\partial K_t}. \quad (4.36)$$

The second last equality holds because the adjustment cost $I(.)$ is homogeneous of degree one. The last equality holds because every entrepreneur chooses the same $K^o_{j_i,t}/K_{j_i,t+1}$ ratio and because the used capital market clears so that $K_t = \sum_i \sum_{j_i} K^o_{j_i,t}$.

Q_t is defined by Equation (4.29), which means that

$$Q_t = \frac{I(K_{t+1}, K_t) - \frac{\partial I(K_{t+1},K_t)}{\partial K_t} K_t}{K_{t+1}}, \tag{4.37}$$

where the capital accumulation equation is just

$$I_t = I(K_{t+1}, K_t). \tag{4.38}$$

In the simulation we assume the following functional form for the adjustment costs: we now use the following function assumption for I:

$$I_t = I(K_{t+1}, K_t) = K_{t+1} - (1-\delta)K_t + \frac{\xi}{2}\frac{(K_{t+1} - K_t)^2}{K_t}. \tag{4.39}$$

It is easy to check that this function has all the properties described above.[18]

Using Equations (4.36), (4.37) and (4.39), we can solve for Q^o_t and Q_t analytically:

$$Q^o_t = (1-\delta) + \frac{\xi}{2}\left[\left(\frac{K_{t+1}}{K_t}\right)^2 - 1\right], \tag{4.40}$$

$$Q_t = 1 - \xi + \xi\frac{K_{t+1}}{K_t}. \tag{4.41}$$

The equations above allow us to express Q^o_t in terms of Q_t:

$$Q^o_t = (1-\delta) + (Q_t - 1) + \frac{1}{2\xi}(Q_t - 1)^2. \tag{4.42}$$

[18] We show that this adjustment cost is the same as the Hayashi (1982) adjustment cost $J(I/K)$ up to the second order when $\xi = -J''$ evaluated at the steady state.

Thus, we can write the realised return as

$$R_t^E = \frac{\alpha \frac{Y_t}{K_t} + (1-\delta) + (Q_t - 1) + \frac{1}{2\xi}(Q_t - 1)^2}{Q_{t-1}}. \tag{4.43}$$

Equations (4.26), (4.31), (4.32), (4.33), (4.39), (4.41) and (4.43) fully describe the aggregate behaviour of the entrepreneurial sector.

4.4.3 Aggregate Demand

To close the model, we need the goods market equilibrium:

$$Y_t = C_t + C_t^E + C_t^F + I_t + \mu^F G^F(\bar{\omega}_t^F) R_t^F (Q_{t-1} K_t - N_t^E). \tag{4.44}$$

The final term corresponds to the cost associated with the investors' monitoring.

4.4.4 The Full Macroeconomic Model

To summarise, the model equations are:

$$\begin{aligned}
\frac{1}{C_t} &= \beta E_t\left(\frac{1}{C_{t+1}}\right) R_{t+1}, && (4.45)\\
W_t &= \chi C_t L_{Ct}^{\varphi}, && (4.46)\\
1 &= \frac{[\Gamma^F(\bar{\omega}_t^F) - \mu^F G^F(\bar{\omega}_t^F)] R_t^F (Q_{t-1}K_t - N_t^E)}{R_t(Q_{t-1}K_t - N_t^E - N_t^F)}, && (4.47)\\
\frac{E_t(R_{t+1}^F)}{R_{t+1}} &= E_t\left(\frac{\lambda(\bar{\omega}_{t+1}^F)}{\lambda(\bar{\omega}_{t+1}^F)[\Gamma^F(\bar{\omega}_{t+1}^F) - \mu^F G^F(\bar{\omega}_{t+1}^F)] + (1-\Gamma^F(\bar{\omega}_{t+1}^F))}\right), && (4.48)\\
R_t^F &= R_t^E, && (4.49)\\
N_{t+1}^F &= \gamma^F(1-\Gamma^F(\bar{\omega}_t^F)) R_t^F (Q_{t-1}K_t - N_t^E) + (1-\alpha)\Omega_F Y_t, && (4.50)\\
N_{t+1}^E &= \gamma^E R_t^E N_t^E + (1-\alpha)\Omega_E Y_t, && (4.51)\\
C_t^F &= (1-\gamma^F)(1-\Gamma^F(\bar{\omega}_t^F)) R_t^F (Q_{t-1}K_t - N_t^E), && (4.52)\\
C_t^E &= (1-\gamma^E) R_t^E N_t^E, && (4.53)\\
Y_t &= A_t K_t^{\alpha} L_{Ct}^{(1-\alpha)(1-\Omega_E-\Omega_F)}, && (4.54)\\
W_t &= (1-\alpha)(1-\Omega_E-\Omega_F)\frac{Y_t}{L_{Ct}}, && (4.55)\\
R_t^E &= \frac{\alpha\frac{Y_t}{K_t} + (1-\delta) + (Q_t - 1) + \frac{1}{2\xi}(Q_t-1)^2}{Q_{t-1}}, && (4.56)\\
I_t &= K_{t+1} - (1-\delta)K_t + \frac{\xi}{2}\frac{(K_{t+1}-K_t)^2}{K_t}, && (4.57)\\
Q_t &= 1 - \xi + \xi\frac{K_{t+1}}{K_t}, && (4.58)\\
Y_t &= C_t + C_t^E + C_t^F + I_t + \mu^F G^F(\bar{\omega}_t^F) R_t^F (Q_{t-1}K_t - N_t^E), && (4.59)
\end{aligned}$$

where $\lambda(\bar{\omega}_{t+1}^F)$ is the shadow price of internal funds of financial institutions, defined in Equation (4.6).

The 15 equations above ensure that (i) consumers maximise utility given the relative prices; (ii) financial institutions and entrepreneurs maximise their respective profits given the relative prices; (iii) the resource constraints hold; (iv) and markets

Table 4.1: Parameter values.

Parameter	α	β	Ω_E	Ω_F	δ	χ	φ	ξ	A
Numerical Value	0.35	0.99	0.01	0.01	0.025	0.3	0.33	10	5.3958

clear.

Given the initial conditions for $\{K_0, N_0^E, N_0^F\}$ and realisation of productivity shocks $\{A_t\}_{t=0}^{\infty}$, the system solves

$$\{C_t, R_t, W_t, L_{Ct}, Y_t, K_{t+1}, N_{t+1}^F, N_{t+1}^E, C_t^F, C_t^E, R_t^E, Q_t, I_t, R_t^F, \bar{\omega}_t^F\}_{t=0}^{\infty}.$$

The log-linearised equations are given in Appendix 4.7.1.

4.5 Calibration and Simulation

4.5.1 Calibration

We discuss the calibration of our model here. We also discuss the calibrations for the models that we use to compare with our model.

The standard parameters of our model are set following HSU and BGG based on US data. We report these values in Table 4.1.[19]

In our model, the credit contract parameter values are set to satisfy the following steady-state conditions:

1. the steady-state external finance premium $R^E - R = 0.02/4$ (same as BGG and HSU);

[19] All parameters except the inverse of Frisch elasticity of labour supply are the same as BGG (1999). We use a different φ because we want to be consistent with the HSU model, in which the formulation of labour disutility is slightly different from BGG. BGG's period t utility of leisure is $\chi \ln(1 - L_{Ct})$. However, the parameter φ has no significant impact to the impulse responses.

Table 4.2: Credit contract parameters and steady-state values.

Parameter	μ^F	γ^E	γ^F	$\bar{\omega}^F$	σ^F	G^F	Ψ^F	Γ^F	λ
Value	0.0452	0.9837	0.9622	0.7974	0.1074	0.0153	0.7961	0.7968	1.0212

Table 4.3: Steady-state macroeconomic variables.

Parameter	Y	C	C^E	C^F	I	W	L_C	K
Value	100	67.2264	7.2401	3.3462	21.8199	26.8863	2.3692	872.796
Parameter	N^E	N^F	R	R^E	R^F	Z^F		
Value	436.398	87.2796	1.0101	1.0151	1.0151	1.0118		

2. the steady-state entrepreneurial leverage K/N^E is 2 times (same as BGG and HSU);

3. the steady-state financial institution capital-to-net-worth ratio K/N^F is 10 times (same as HSU);

4. financial institutions' failure rate is 2%, *i.e.* $F^F(\bar{\omega}^F) = 2\%$ (same as HSU).

We use the combined FOC, the participation constraint and the cumulative distribution function $F^F(.)$ in the steady state to solve for the parameters μ^F and σ^F and the steady-state cutoff value $\bar{\omega}^F$. The credit contract parameters and steady-state values are given in Table 4.2. The steady-state values of the macroeconomic variables are given in Table 4.3.

We wish to compare our model with the HSU model. The HSU model is fully set out in Appendix 4.7.2. In the HSU model, not only the contract between investors and financial institutions is a loan contract, but the contract between financial institutions and entrepreneurs is also a loan contract. In this comparison, our objective is to compare the financial-accelerator effects which arise when financial institutions buy shares from entrepreneurs with what happens when financial institutions make

loans to entrepreneurs. A fair comparison between the two models is possible since our model is identical to the HSU model except for the way for the specification of credit contracts between financial institutions and goods-producing firms. In the calibration of the credit contract parameters in the HSU model, we require two additional parameters which come from the steady-state conditions in that model. These steady-state conditions are as follows: the failure rate of entrepreneurs is 2% per period; and the equilibrium spread between the contractual rate which financial institutions lend to entrepreneurs and the rate which they borrow from investors in the model is assumed to be 230 basis points annually. These additional steady-state conditions are necessary for the identification of the monitoring cost, the variance of the idiosyncratic shock and the steady-state cutoff value in the loan contract between financial intermediaries and entrepreneurs. Details are given in Appendix 4.7.2.

We will also compare our model with the BGG model. In that setup there is no friction between investors and financial institutions; the only financial friction comes from the loan contract between investors and entrepreneurs. To make a fair comparison between the models possible, we have constructed a version of the BGG model which keeps the macroeconomic setup of our model. This setup differs from the published BGG setup in four ways. First, we assume all goods prices are flexible. Second, we follow the formulation of labour disutility used in our model. Third, we set the macroeconomic parameter values equal to those in our model, presented in Table 4.1. Finally, we treat $\bar{\omega}^E$ as an endogenous variable, where $\bar{\omega}^E$ is the level of the idiosyncratic productivity shock ω^E below which entrepreneurs are bankrupt, as in Christiano et al. (2010).[20] We calibrate the credit contract parameters using

[20]In the published version of the BGG model, the authors treat the value of $\bar{\omega}^E$ as a constant, equal to its steady-state value. But in the BGG model, $\bar{\omega}^E$ should be endogenous, in exactly the way that it is in our model, for reasons which we have identified in the body of this chapter. In fact, BGG introduce a rather clever adjustment to the budget constraint of entrepreneurs which almost exactly offsets the effects of their erroneous simplification. But, in order to assist with our

the first two steady-state relations in our baseline calibration, that is (i) a steady-state external finance premium of 2% per annum; (ii) a steady-state entrepreneurial leverage of 2; and together with (iii) an entrepreneurial failure rate of 2%. By doing so, we make the slightest change to the macroeconomic environment (in particular, the steady-state values are about the same as the values in our baseline model) so that all the dissimilarities in the simulation results come from the nature of the credit contracts in the two setups. This version of the BGG model, and the calibration details, are reported in Appendix 4.7.3.

We will also compare our model with the Q-Ramsey model. We construct the Q-Ramsey model by removing the financial imperfection in our model. This is achieved by (1) fixing the external finance premium; and (2) fixing the cutoff value of the idiosyncratic shock at their steady-state levels instead of allowing them to respond to changes in the leverage supply function, Equation (4.12). This comparison is fair because the Q-Ramsey model has the same steady state as our model, but the financial accelerator has been turned off.

4.5.2 Simulated Outcomes

In this section, we simulate the impulse response of credit spreads and other macroeconomic variables to a 1% temporary negative shock to productivity. We assume aggregate productivity follows an AR(1) process:

$$A_t = \rho_A A_{t-1} + \epsilon_{At} \tag{4.60}$$

and set $\rho_A = 0.85$, following HSU (2009). The results are presented in Figure 4.3 and 4.4.

In response to a negative productivity shock, the demand for capital falls. The

comparison between different approaches to financial frictions, it is appropriate for us to use a version of the BGG model in which $\bar{\omega}^E$ is properly endogenous.

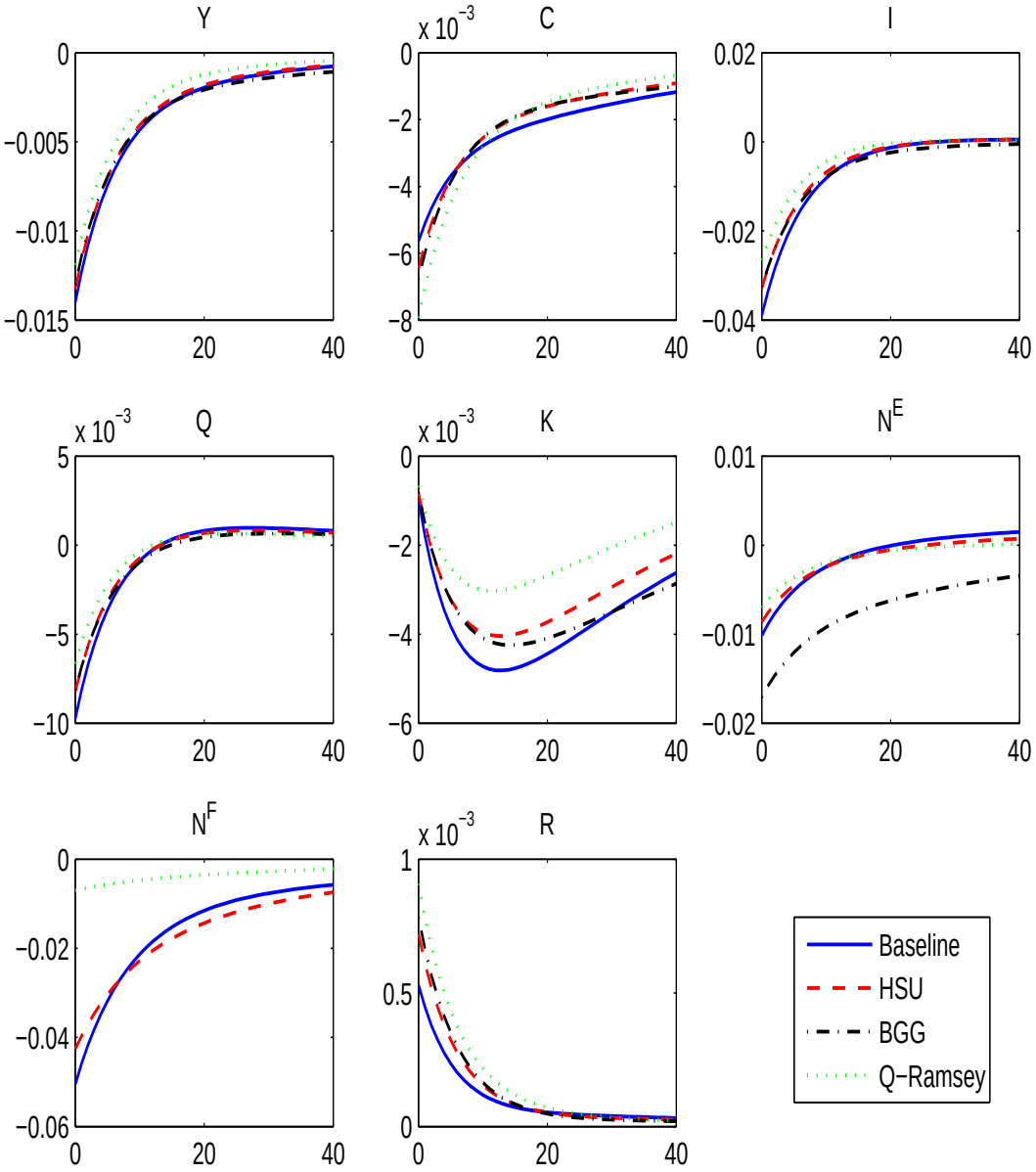


Figure 4.3: Impulse responses of our model, HSU and BGG models: temporary productivity shock 1.

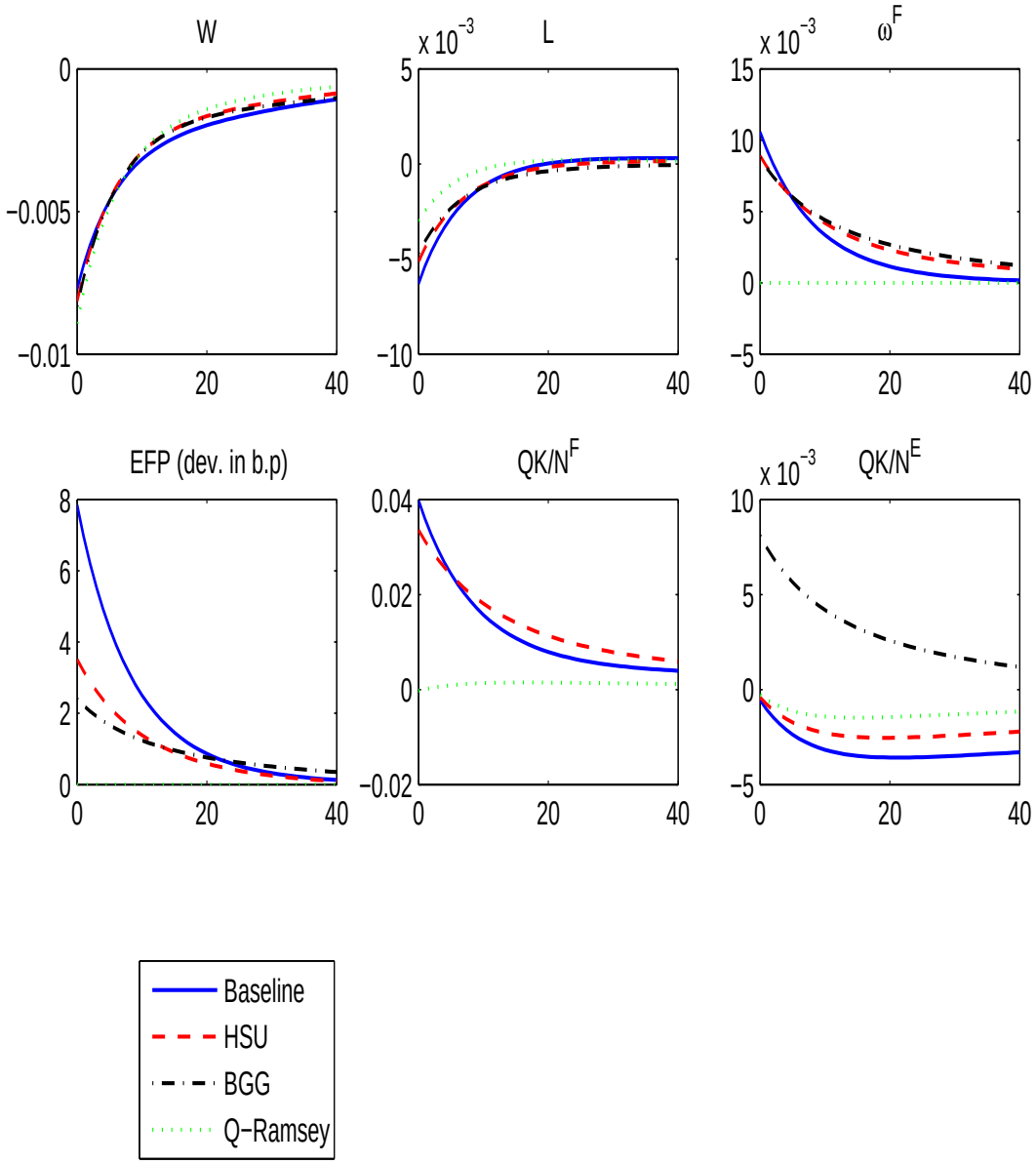


Figure 4.4: Impulse responses of our model, HSU and BGG models: temporary productivity shock 2.

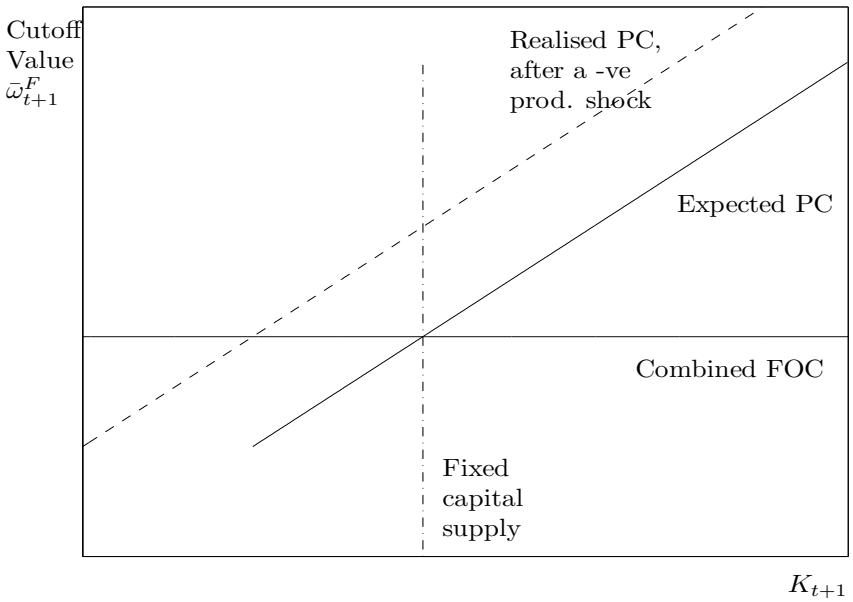


Figure 4.5: Partial equilibrium of our model.

share price Q jumps down so that along the adjustment path Q is rising, leading to capital gains and a higher expected return to entrepreneurs' capital $E(R^E_{t+1})$. (The extent to which the expected return to capital has to rise and the share price to fall depends on the cost of funds of entrepreneurs, as will be discussed.) This unanticipated fall in asset prices reduces the realised return on capital to entrepreneurs R^E_t in the period in which the unanticipated shock occurs. Since financial institutions hold the shares of entrepreneurs, their return R^F_t falls by the same amount. This shifts the realised participation constraint to the left in $(K_{t+1}, \bar{\omega}^F_{t+1})$ space relative to the expected participation constraint, raising the cutoff value for financial institutions relative to the expected cutoff value, as shown in Figure 4.5.

The aggregate net worth of financial institutions falls not only because the realised return is low relative to the expected return evaluated one period before, but also because a higher cutoff value increases bankruptcy and reduces the share of profit that goes to financial institutions in the loan contract, $(1 - \Gamma^F(\bar{\omega}^F_t))$. With

these two effects, the fall in net worth of financial institutions is proportionally bigger than the fall in share price, so their leverage increases. Investors are now exposed to bigger risks of default. To compensate for the rise in the monitoring costs, investors require a higher return from financial institutions, forcing up the external finance premium R^F/R. The transmission mechanism discussed above comes from the loan contract and is the same as BGG.

In the share contract between financial institutions and entrepreneurs, as the cost of funds of financial institutions rises, they require a higher return from entrepreneurs. Owing to the counter-cyclical external finance premium, the shadow price of capital Q has to fall further initially relative to a model without the financial accelerator to allow for bigger capital gains along the adjustment path. This leads to a larger fall in the demand for investment relative to a model without the financial accelerator. Q here has an interpretation of share price because the financial contract concerned is a share contract.

We compare the impulse responses of our model and the HSU model. The impacts on the aggregate variables are qualitatively similar, but our amplifications are quantitatively larger than HSU. In particular, the rise in external finance premium in our model is significantly larger than in the HSU model, leading to a larger financial-accelerator effect and a more pronounced fall in investment. To see why, we derive the following first order condition for the optimal contract for the financial institutions in the HSU model:[21]

$$\begin{aligned} E_t\frac{(R^E_{t+1})}{R_{t+1}} &= E_t\left(\frac{\lambda^E(\bar{\omega}^E_{t+1})}{(1-\Gamma(\bar{\omega}^E_{t+1}))+\lambda^E(\bar{\omega}^E_{t+1})(\Gamma(\bar{\omega}^E_{t+1})-\mu^E G(\bar{\omega}^E_{t+1}))}\right. \\ &\quad \left.\times\frac{\lambda^F(\bar{\omega}^F_{t+1})}{(1-\Gamma(\bar{\omega}^F_{t+1}))+\lambda^F(\bar{\omega}^F_{t+1})(\Gamma(\bar{\omega}^F_{t+1})-\mu^F G(\bar{\omega}^F_{t+1}))}\right), \quad (4.61) \end{aligned}$$

[21] See Appendix 4.7.2 for a more detailed analysis.

where $\lambda^F(\bar{\omega}^F_{t+1})$ and $\lambda^E(\bar{\omega}^E_{t+1})$ are given by

$$\begin{aligned}
\lambda^E(\bar{\omega}^E_{t+1}) &= \frac{\Gamma^E_\omega(\bar{\omega}^E_{t+1})}{\Gamma^E_\omega(\bar{\omega}^E_{t+1}) - \mu^E G^E_\omega(\bar{\omega}^E_{t+1})} > 1, \qquad \lambda^E_\omega(\bar{\omega}^E_{t+1}) > 0, \\
\lambda^F(\bar{\omega}^F_{t+1}) &= \frac{\Gamma^F_\omega(\bar{\omega}^F_{t+1})}{\Gamma^F_\omega(\bar{\omega}^F_{t+1}) - \mu^F G^F_\omega(\bar{\omega}^F_{t+1})} > 1, \qquad \lambda^F_\omega(\bar{\omega}^F_{t+1}) > 0.
\end{aligned}$$

Equation (4.61), which is not derived in the original HSU paper, shows that although financial institutions solve a profit maximisation problem subject to the joint constraints by both the entrepreneurs and investors, the external finance premium is influenced symmetrically by the cutoff values in the loan contract between financial institutions and investors and the cutoff values in the loan contract between financial institutions and investors. Furthermore, each of these two components has an identical structure with the external financial premium in the original BGG contract. It is therefore obvious that the external finance premium in the HSU contract is increasing in each of the two cutoff values, $\bar{\omega}^E$ and $\bar{\omega}^F$.

Subsequent to a negative shock hitting the economy, financial institutions in the HSU model are forced to set a lower cutoff value $\bar{\omega}^E$ in the loan contract between financial institutions and entrepreneurs in order not to violate the entrepreneurs' participation constraint, otherwise entrepreneurs will be better off using their own funds only and not participate in the credit contract. The fall in the cutoff value $\bar{\omega}^E$, *ceteris paribus*, dampens the amplification of the shock. In our setup, the contract between financial institutions and entrepreneurs is a share contract which does not involve a participation constraint for entrepreneurs, so the dampening effect described above is absent, leading to a larger amplification than the loans economy.

We compare the impulse responses of our model and the BGG model. From Figure 4.3 and 4.4, a negative productivity shock causes the entrepreneurial net worth in the BGG setup to jump down by more than in our model and HSU, leading to an immediate rise in leverage in the entrepreneurial sector. This is due to the fact

that the financial accelerator in BGG operates in the contract between investors and entrepreneurs, whereas in our model as well as HSU, the amplification comes from the financial friction between investors and financial institutions. The simulation results show that, the external finance premium in BGG is anti-cyclical, but its increase in response to a negative productivity shock is smaller than in our baseline setup and in HSU. This is simply a result of the calibration of the models which reflects the fact that, in the real world, the financial sector is much more highly leveraged than the entrepreneurial sector. Higher leverage makes the economy more volatile.

4.5.3 Varying the Size of Entrepreneurial Net Worth

We investigate how the size of the entrepreneurial sector affects the volatility of the economy. We proceed by reducing the steady-state entrepreneurial net worth N^E. Entrepreneurs will need less net worth of their own in order to get funding from the financial sector. We simulate the model with entrepreneurial sizes 50%, 30%, 10% and 0% of the capital stock, keeping the size of the financial sector constant at 10% of capital stock.

In the case in which entrepreneurial net worth falls to zero, entrepreneurs do not need any funds of their own in order to obtain funding from the banks. In other words financial institutions own the entire goods-producing sector. Therefore, the banks and the entrepreneurial sector can be thought of as one sector. In the full system, we collapse entrepreneurial net worth N^E and the entrepreneurial share of labour Ω_E to 0. It turns out that the system in this special case has an identical structure with the BGG model. The only distinction is in parameterisation. The steady-state leverage of the financial-entrepreneurial sector is 10 times in our setup, which is larger than 2 times in BGG's calibration.

We re-parameterise the steady state for each entrepreneurial size and simulate a

1% temporary negative productivity shock. The impulse responses of a permanent productivity shock are presented in Figure 4.6 and 4.7.

Compared with the other models, the model with entrepreneurial net worth equal to zero (*i.e.* the re-parameterised BGG model) is the most volatile system. As we increase the steady-state entrepreneurial net worth N^E and recalibrate the model, the volatility of the system falls. The reason is the following. When the size of entrepreneurial net worth is smaller in the steady state, a larger share of revenue goes to financial institutions. When the economy is hit by a negative productivity shock, financial institutions suffer a larger loss in revenue, and so, keeping the size of financial institutions fixed, a larger proportional fall in revenue, and so a larger proportional decrease in net worth. Therefore, the leverage of the financial sector $(QK - N^E)/N^F$ initially rises by more, which - for reasons analysed above - causes a bigger rise in the external finance premium in the contract between financial institutions and investors, and thus a bigger fall in investment. On the other hand, in the limit, if N^E approaches the value of the entire capital stock, the size of the financial sector shrinks to zero. The system then reduces to a simple Q-Ramsey economy in which there is no influence of financial frictions.

We have seen that a reduction in entrepreneurial net worth N^E relative to the stock of capital K reduces macroeconomic volatility. This effect could be counteracted by increasing the net worth of the financial sector. This is because a smaller N^E increases the leverage of the financial sector, $(QK - N^E)/N^F$, and a higher N^F reduces it. When a negative shock occurs, the larger self-financing of the financial sector acts as a buffer in the loan contract to the increase in volatility caused by the reduction in N^E. In particular, it can be shown that, if N^E falls, and at the same time N^F rises by an amount which keeps the steady-state leverage of the financial sector identical to that in the baseline model, then macroeconomic volatility will remain unchanged.

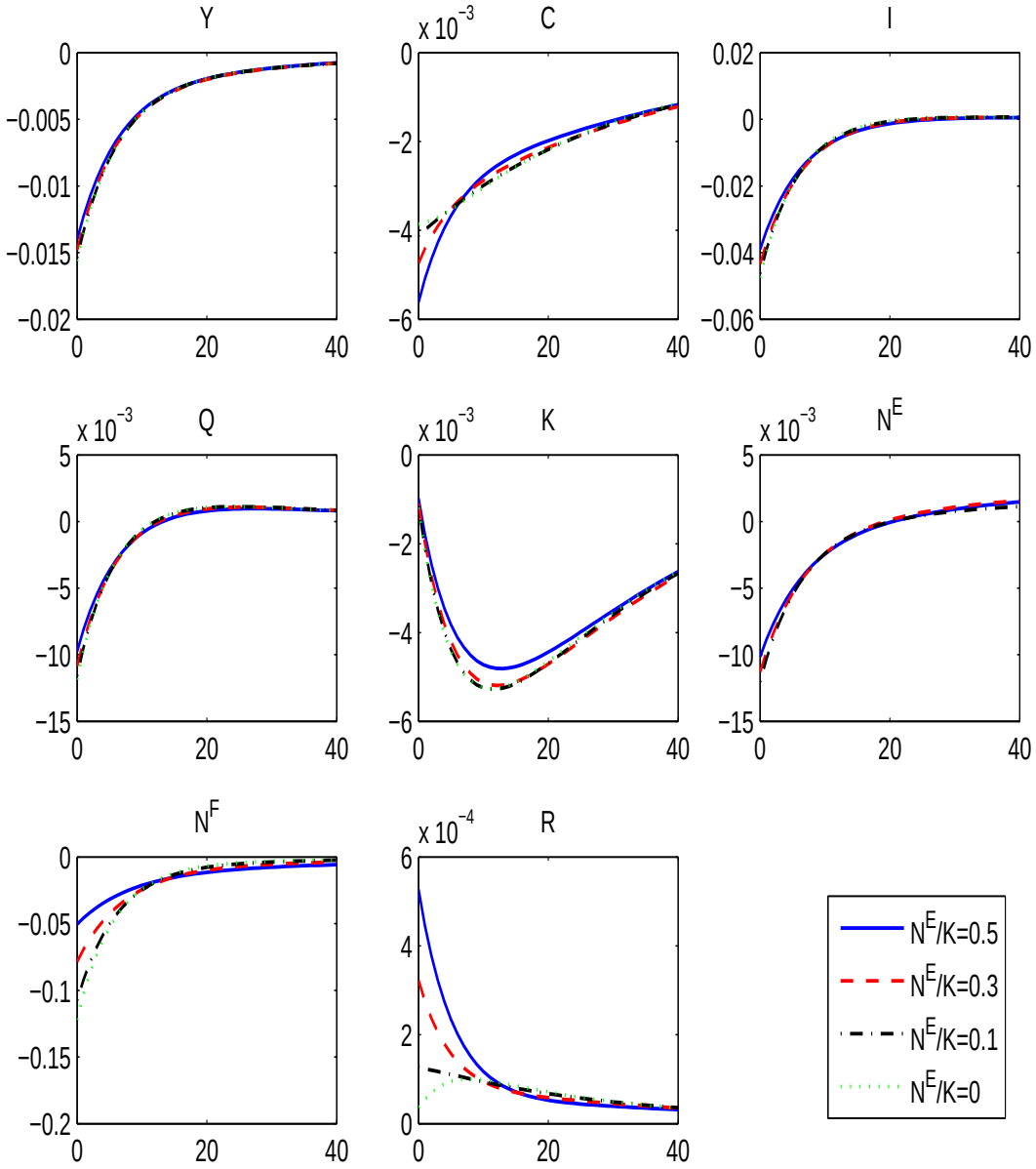


Figure 4.6: Impulse responses of our model with different entrepreneurial net worths N^E: temporary productivity shock 1.

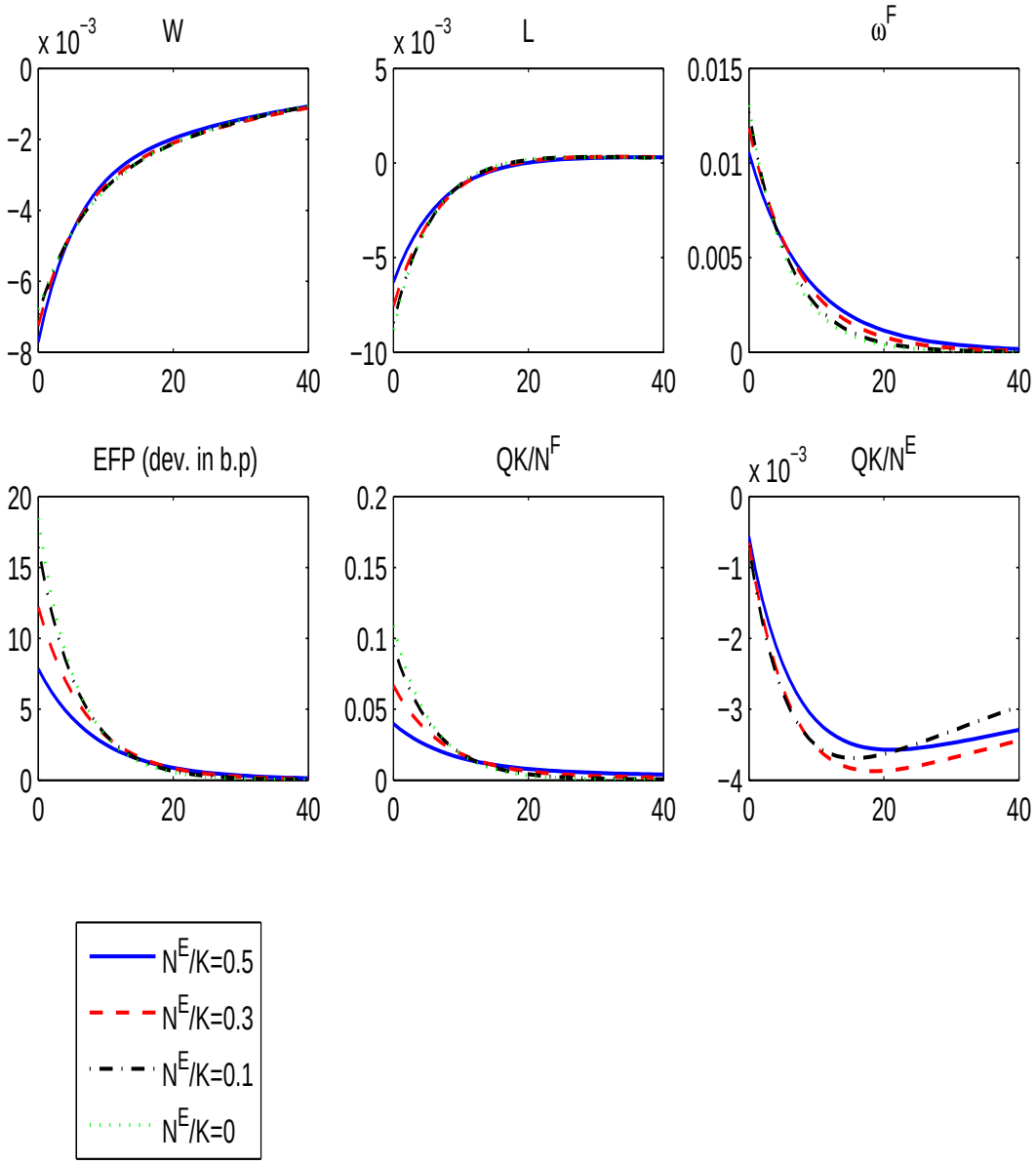


Figure 4.7: Impulse responses of our model with different entrepreneurial net worths N^E: temporary productivity shock 2.

4.6 Conclusion

This chapter has explored how leverage within the financial system can make the economy more volatile in response to shocks. In our model, each financial institution faces the possibility of bankruptcy. This means that the interest rate which consumers require from the financial institutions to which they lend is subject to a risk premium. This risk premium is anti-cyclical, and therefore disturbance-amplifying, to an extent which depends on the degree of leverage of the financial institutions. In a downturn, caused, say by a reduction in productivity, the share price of firms will fall, increasing the likelihood of bankruptcy of the financial institutions which hold their shares, and causing the risk premium to rise, which is disturbance-amplifying. We have compared our findings with those of both BGG and HSU, and have shown that our setup is more disturbance-amplifying than either of those other setups.

In conclusion it is worth mentioning another important approach to this problem, coming from a study of the way in which financial institutions are connected to each other. When the crisis hit, another thing that financial institutions did, when seeking to re-build their balance sheets, was to seek to rebuild their position by withdrawing loans to other financial institutions, rather than merely by attempting to sell the shares of the firms in which they had invested, as is assumed in this chapter. It has become apparent that 'network inter-connectedness' can mean that such a withdrawal of loans to other financial institutions can lead to a domino-effect operating within the financial system. This effect can operate as in the case of Northern Rock in which a withdrawal of funding led to the collapse of the institution. Or it can operate as in the case of Lehman Brothers, in which the collapse of one institution can lead to the collapse of other institutions. Recent studies (*e.g.* by Haldane, 2009), focus on the fact that financial systems which appear to be in a satisfactory equilibrium state can collapse after what seems to be a small shock, as

a result of such domino effects. Moore (2011) has produced similar results. Work by May et al. (2007) and Haldane and May (2010) has shown how epidemiological models may be used to study the spread of financial contagion in this way. Such work is also important in showing how a small shock can lead to a big crisis within the financial system.

Such features could be included in the model which we have presented in this chapter, but we have not yet done this.

4.7 Appendix

4.7.1 Log-linearised Model Equations

This appendix reports the log-linearised form of the equations of the model. The equations in levels are reported as Equations (4.45) to Equation (4.59) in the main text. We denote X the steady-state value of the variable X_t and $\hat{X}_t$ the percentage deviation of the variable from its steady state such that $\hat{X}_t = (X_t - X)/X$. The log-linearised equations are the following:

Consumption Euler equation:

$$\hat{C}_t - E_t\hat{C}_{t+1} + \hat{R}_{t+1} = 0. \tag{4.62}$$

Labour supply:

$$\hat{W}_t = \hat{C}_t + \varphi\hat{L}_{Ct}. \tag{4.63}$$

Realised participation constraint for investors:

$$\begin{aligned}\hat{R}^F_t - \hat{R}_t &= \left(\frac{1}{K - N^E - N^F} - \frac{1}{K - N^E}\right)[K(\hat{Q}_{t-1} + \hat{K}_t) - N^E\hat{N}^E_t] \\ &\quad -\frac{N^F}{K - N^E - N^F}\hat{N}^F_t - \frac{\Psi^F_\omega}{\Psi^F}\bar{\omega}^F\hat{\bar{\omega}}^F_t.\end{aligned} \tag{4.64}$$

First order condition of financial institutions' profit maximisation:

$$\begin{aligned}0 &= \frac{\Gamma^F_\omega}{\Gamma^F_\omega - \mu^F G^F_\omega}(E_t(\hat{R}^F_{t+1}) - \hat{R}_{t+1}) \\ &\quad -(1 - \Gamma^F)\frac{R^F}{R}\left(\frac{\Gamma^F_{\omega\omega}}{\Gamma^F_\omega} - \frac{\Gamma^F_{\omega\omega} - \mu^F G^F_{\omega\omega}}{\Gamma^F_\omega - \mu^F G^F_\omega}\right)\bar{\omega}^F E_t\hat{\bar{\omega}}^F_{t+1}.\end{aligned} \tag{4.65}$$

Return on holding shares:

$$\hat{R}^E_t = \hat{R}^F_t. \tag{4.66}$$

Law of motion of financial institutions' net worth:

$$\begin{aligned} N^F \hat{N}^F_{t+1} &= \gamma^F(1-\Gamma^F)R^F[(K-N^E)\hat{R}^F_t + K(\hat{Q}_{t-1}+\hat{K}_t) - N^E\hat{N}^E_t] \\ &\quad -\gamma^F R^F(K-N^E)\Gamma^F_\omega \bar{\omega}^F\hat{\bar{\omega}}^F_t + (1-\alpha)\Omega_F Y\hat{Y}_t. \end{aligned} \tag{4.67}$$

Law of motion of entrepreneurs' net worth:

$$\hat{N}^E_{t+1} = \gamma^E R^E(\hat{R}^E_t + \hat{N}^E_t) + (1-\gamma^E R^E)\hat{Y}_t. \tag{4.68}$$

Consumption by quitting financial institutions:

$$\hat{C}^F_t = \hat{R}^F_t + \frac{K}{K-N^E}(\hat{Q}_{t-1}+\hat{K}_t) - \frac{N^E}{K-N^E}\hat{N}^E_t - \frac{\Gamma^F_\omega}{1-\Gamma^F}\bar{\omega}^F\hat{\bar{\omega}}^F_t. \tag{4.69}$$

Consumption by quitting entrepreneurs:

$$\hat{C}^E_t = \hat{R}^E_t + \hat{N}^E_t. \tag{4.70}$$

Production function:

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_t + (1-\alpha)(1-\Omega_E-\Omega_F)\hat{L}_{Ct}. \tag{4.71}$$

Labour demand:

$$\hat{W}_t = \hat{Y}_t - \hat{L}_{Ct}. \tag{4.72}$$

Capital demand:

$$R^E(\hat{Q}_{t-1}+\hat{R}^E_t) = (R^E-(1-\delta))(\hat{Y}_t-\hat{K}_t) + \hat{Q}_t. \tag{4.73}$$

Capital accumulation:

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t. \tag{4.74}$$

Investment decision:

$$\hat{Q}_t = \xi\delta(\hat{I}_t - \hat{K}_t). \tag{4.75}$$

Aggregate demand:

$$\begin{aligned} Y\hat{Y}_t &= C\hat{C}_t + C^E\hat{C}_t^E + C^F\hat{C}_t^F + \delta K\hat{I}_t + \mu^F R^F(K-N^E)G_\omega^F\bar{\omega}^F\hat{\bar{\omega}}_t^F \\ &\quad +\mu^F G^F R^F[(K-N^E)\hat{R}_t^F + K(\hat{Q}_{t-1}+\hat{K}_t) - N^E\hat{N}_t^E]. \end{aligned} \tag{4.76}$$

The derivatives of the distribution functions are defined as follows:

$$F_\omega^F = \frac{1}{\sigma^F\bar{\omega}^F}\phi\left(\frac{\ln\bar{\omega}^F + \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right) > 0, \tag{4.77}$$

$$G_\omega^F = \frac{1}{\sigma^F\bar{\omega}^F}\phi\left(\frac{\ln\bar{\omega}^F - \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right) > 0, \tag{4.78}$$

$$\Gamma_\omega^F = 1 - F^F(\bar{\omega}^F) = 1 - \Phi\left(\frac{\ln\bar{\omega}^F + \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right) > 0, \tag{4.79}$$

$$\Gamma_{\omega\omega}^F = -F_\omega^F = -\frac{1}{\sigma^F\bar{\omega}^F}\phi\left(\frac{\ln\bar{\omega}^F + \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right) < 0, \tag{4.80}$$

where ϕ is the probability density function of the standard normal distribution. To derive $G_{\omega\omega}^F$, we make use of the fact that $\phi'(x) = -x\phi(x)$. Therefore,

$$\begin{aligned} G_{\omega\omega}^F &= -\frac{1}{\sigma^F(\bar{\omega}^F)^2}\phi\left(\frac{\ln\bar{\omega}^F - \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right) + \frac{1}{\sigma^F\bar{\omega}^F}\phi'\left(\frac{\ln\bar{\omega}^F - \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right)\frac{1}{\sigma^F\bar{\omega}^F}, \\ &= -\frac{1}{\sigma^F(\bar{\omega}^F)^2}\phi\left(\frac{\ln\bar{\omega}^F - \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right) - \frac{1}{\sigma^2(\bar{\omega}^F)^2}\phi\left(\frac{\ln\bar{\omega}^F - \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right)\left(\frac{\ln\bar{\omega}^F - \frac{1}{2}(\sigma^F)^2}{\sigma^F}\right). \end{aligned}$$

The numerical values of these steady-state parameters are reported in Table 4.4.

Table 4.4: Distributions regarding ω^F.

Parameter	G^F	Γ^F	Ψ^F	F^F_ω	G^F_ω	Γ^F_ω
Value	0.0153	0.7968	0.7961	0.5651	0.4506	0.98
Parameter	$\Gamma^F_{\omega\omega}$	$G^F_{\omega\omega}$	Ψ^F_ω	$\Psi^F_{\omega\omega}$		
Value	-0.5651	10.802	0.9596	-1.0538		

4.7.2 The HSU model

This appendix describes the credit contract and the general equilibrium of the HSU model which we use to compare with our baseline model. The HSU model features the 'chained financial contract'. It also involves investors, financial institutions and entrepreneurs. Financial institutions mediate funds between entrepreneurs and investors. However, in the HSU model, both contracts are loan contracts. Financial institutions need to pay a linear monitoring cost μ^E when an entrepreneur goes bankrupt. Financial institutions choose the contractual rates and loan amounts for both contracts subject to the participation constraints of investors and entrepreneurs.

The financial institutions' maximisation problem is the following (We drop the time subscripts in the following presentation for simplicity):

$$\max_{\bar{\omega}^F,\bar{\omega}^E,K}(1-\Gamma^F(\bar{\omega}^F))R^F(QK-N^E), \tag{4.81}$$

where $(1-\Gamma^F(\bar{\omega}^F))$ is the borrower's (expected) share of profit in the contract, and $R^F(QK-N^E)$ is the gross return of the contract, defined by

$$R^F(QK-N^E)=\Psi^E(\bar{\omega}^E)R^EQK. \tag{4.82}$$

On the right hand side, $\Psi^E(\bar{\omega}^E)$ is the lender's share of profit in the contract after

monitoring, and $R^E QK$ is the gross return of the entrepreneurs' production.

To ensure that investors participate, the zero profit condition for the perfectly competitive investor must be satisfied so that

$$\Psi^F(\bar{\omega}^F)R^F(QK - N^E) = R(QK - N^E - N^F), \tag{4.83}$$

where $\Psi^F(\bar{\omega}^F)$ is the lender's share of profit in the contract after monitoring, while the right hand side is the opportunity cost of the investors' lending. Investors sign contracts with a lot of banks, the idiosyncratic risks of the banks are diversifiable.

To ensure entrepreneurs participate, there is a participation constraint for them:

$$(1 - \Gamma^E(\bar{\omega}^E))R^E QK \geq R^E N^E. \tag{4.84}$$

The condition states that in the contract between financial institutions and entrepreneurs, the entrepreneurs' expected return is set to equal to the return from their alternative option – investing with just their own funds. This condition is always binding.

The banks' problem is therefore the maximisation of Equation (4.81) subject to Equation (4.82), (4.83) and (4.84). The problem can be simplified to

$$\max_{\bar{\omega}^F, \bar{\omega}^E, K} (1 - \Gamma^F(\bar{\omega}^F))\Psi^E(\bar{\omega}^E)R^E QK \tag{4.85}$$

such that

$$\begin{aligned} \Psi^F(\bar{\omega}^F)\Psi^E(\bar{\omega}^E)R^E QK &= R(QK - N^E - N^F), & (4.86) \\ (1 - \Gamma^E(\bar{\omega}^E))QK &= N^E. & (4.87) \end{aligned}$$

We write down the Lagrangian with λ_1 denoting the Lagrangian multiplier for the in-

vestors' zero profit constraint and λ_2 the Lagrangian multiplier for the entrepreneurs' participation constraint:

$$\begin{aligned} L &= (1-\Gamma^F(\bar{\omega}^F))\Psi^E(\bar{\omega}^E)R^EQK + \lambda_1[\Psi^F(\bar{\omega}^F)\Psi^E(\bar{\omega}^E)R^EQK - R(QK-N^E-N^F)] \\ &\quad +\lambda_2[(1-\Gamma^E(\bar{\omega}^E))QK - N^E]. \end{aligned} \tag{4.88}$$

The first order conditions are

$$\begin{aligned} K &: (1-\Gamma^F(\bar{\omega}^F))\Psi^E(\bar{\omega}^E)R^E + \lambda_1[R - \Psi^F(\bar{\omega}^F)\Psi^E(\bar{\omega}^E)R^E] = \lambda_2(1-\Gamma^E(\bar{\omega}^E)), \\ \bar{\omega}^F &: -\Gamma^F_\omega(\bar{\omega}^F) + \lambda_1\Psi^F_\omega(\bar{\omega}^F) = 0, \\ \bar{\omega}^E &: (1-\Gamma^F(\bar{\omega}^F))\Psi^E_\omega(\bar{\omega}^E)R^E + \lambda_1\Psi^F(\bar{\omega}^F)\Psi^E_\omega(\bar{\omega}^E)R^E - \lambda_2\Gamma^E_\omega(\bar{\omega}^E) = 0. \end{aligned}$$

Rearranging the last two equations to solve for the Lagrange multipliers,

$$\lambda_1 = \frac{\Gamma^F_\omega(\bar{\omega}^F)}{\Psi^F_\omega(\bar{\omega}^F)}, \tag{4.89}$$

$$\lambda_2 = \left(\frac{(1-\Gamma^F(\bar{\omega}^F))\Psi^E_\omega(\bar{\omega}^E)}{\Gamma^E_\omega(\bar{\omega}^E)} + \frac{\Gamma^F_\omega(\bar{\omega}^F)\Psi^F(\bar{\omega}^F)\Psi^E_\omega(\bar{\omega}^E)}{\Gamma^E_\omega(\bar{\omega}^E)\Psi^F_\omega(\bar{\omega}^F)}\right)R^E. \tag{4.90}$$

where λ_1 is the marginal value of the internal funds of financial institution, and λ_2 is the marginal increase in the profits of financial institutions per unit increase in entrepreneurial net worth. Substitute these back into the first order condition for K and rearrange, we obtain the following expression:

$$\begin{aligned} \frac{R^E}{R} &= \frac{\lambda^E(\bar{\omega}^E)}{(1-\Gamma(\bar{\omega}^E)) + \lambda^E(\bar{\omega}^E)(\Gamma(\bar{\omega}^E) - \mu^E G(\bar{\omega}^E))} \\ &\quad \times \frac{\lambda^F(\bar{\omega}^F)}{(1-\Gamma(\bar{\omega}^F)) + \lambda^F(\bar{\omega}^F)(\Gamma(\bar{\omega}^F) - \mu^F G(\bar{\omega}^F))}, \end{aligned} \tag{4.91}$$

where $\lambda^F(\bar{\omega}^F)$ and $\lambda^E(\bar{\omega}^E)$ are given by

$$\lambda^E(\bar{\omega}^E) = \frac{\Gamma^E_\omega(\bar{\omega}^E)}{\Gamma^E_\omega(\bar{\omega}^E) - \mu^E G^E_\omega(\bar{\omega}^E)} > 1, \qquad \lambda^E_\omega(\bar{\omega}^E) > 0, \tag{4.92}$$

$$\lambda^F(\bar{\omega}^F) = \frac{\Gamma^F_\omega(\bar{\omega}^F)}{\Gamma^F_\omega(\bar{\omega}^F) - \mu^F G^F_\omega(\bar{\omega}^F)} > 1, \qquad \lambda^F_\omega(\bar{\omega}^F) > 0. \tag{4.93}$$

The external finance premium can be written simply as the product of functions of $\bar{\omega}^E$ and $\bar{\omega}^F$. Notice that the structure of this first order condition is identical to that in the BGG contract except for the fact that Equation (4.91) contains one more shock that affects the external finance premium.[22] Hence, HSU refer to this contract as the 'chained financial contract'.

The loan contract between financial institutions and entrepreneurs is shock-dampening. To see this, refer to the simulation of a permanent negative shock to productivity in Figure 4.4. The shock reduces entrepreneurs' leverage QK/N^E in the HSU model along the adjustment path. From the participation constraint of entrepreneurs (4.84), a fall in entrepreneurs' leverage requires a rise in the share of profit that goes to entrepreneurs, so the cutoff value for entrepreneurs $\bar{\omega}^E$ has to fall. Since the $\bar{\omega}^E$ component on the right hand side of Equation (4.91) is increasing in $\bar{\omega}^E$, a fall in $\bar{\omega}^E$, *ceteris paribus*, reduces the external finance premium, which dampens a positive shock.

The net worth accumulation equations are

$$N^E_{t+1} = \gamma^E(1 - \Gamma^E(\bar{\omega}^E_t))R^E_t Q_{t-1}K_t + (1-\alpha)\Omega_E Y_t, \tag{4.94}$$

$$N^F_{t+1} = \gamma^F \Psi^E(\bar{\omega}^E_t)(1 - \Gamma^F(\bar{\omega}^F_t))R^E_t Q_{t-1}K_t + (1-\alpha)\Omega_F Y_t. \tag{4.95}$$

The aggregate demand has to include the resources used in the monitoring of

[22] HSU does not derive Equation (4.91) explicitly.

Table 4.5: Credit contract parameters and steady-State values in HSU.

Parameter	$\bar{\omega}^E$	$\bar{\omega}^F$	σ^E	σ^F	μ^E	μ^F	γ^E	γ^F
Value	0.5010	0.7974	0.3127	0.1074	0.0263	0.0241	0.9837	0.9629
Parameter	G^E	G^F	Ψ^E	Ψ^F	Γ^E	Γ^F	Ψ'^E	Ψ'^F
Value	0.009	0.0153	0.4998	0.7964	0.5	0.7968	0.9759	0.9692

Table 4.6: Steady-State variables in HSU.

Parameter	Y	C	C^E	C^F	I	W	L_C	K
Value	100	67.2264	7.2401	3.3407	21.8199	26.8863	2.3692	872.796
Parameter	N^E	N^F	R	R^E	R^F	Z^E	Z^F	
Value	436.398	87.2796	1.0101	1.0151	1.0146	1.0172	1.0113	

entrepreneurs:

$$\begin{aligned} Y_t &= C_t + C_t^E + C_t^F + I_t + \mu^E G^E(\bar{\omega}_t^E) R_t^E Q_{t-1} K_t \\ &\quad + \mu^F G^F(\bar{\omega}_t^F) R_t^F (Q_{t-1} K_t - N_t^E). \end{aligned} \tag{4.96}$$

The other equations in the general equilibrium are unchanged.

To make a fair comparison, the general equilibrium parameter values used in the HSU model are the same as ours and are reported in Table 4.1. We calibrate the credit contract parameter values to satisfy the same set of steady-state relations in our baseline model described in Section 4.5.1. Two more steady-state relations are required to calibrate the HSU model: (1) The annualised failure rate of entrepreneurs is 2%; and (2) the spread between the financial institutions' loan rate (Z^E) and the borrowing rate (Z^F) is 230 basis points annually. The credit contract parameters and steady-state values used in our calibration of the HSU model are reported in Table 4.5 and Table 4.6.

4.7.3 The BGG model

We describe the BGG model in this appendix. It is necessary to do this, even although this model is well known, because the BGG setup adopted here differs from the published BGG setup in four ways, all of which are necessary to make possible a fair comparison with our model. First, we assume all goods prices are flexible, since BGG's financial accelerator does not intrinsically depend on the existence of nominal rigidities and unlike BGG we are not interested in a monetary policy shock. Second, we follow the formulation of labour disutility used in our baseline model. Third, we set the macroeconomic parameter values equal to those in our baseline model. Finally, as explained, we change the way in which $\bar{\omega}^E$ is treated, where $\bar{\omega}^E$ is the level of the idiosyncratic productivity shock below which entrepreneurs are bankrupt. See footnote 20 in this chapter.

As explained in the main text, in the BGG model, there is no financial friction between investors and financial institutions. We may think of entrepreneurs as financed directly by loans from consumers. The external finance premium reflects only entrepreneurial riskiness, and the financial accelerator comes only from this effect.

In the BGG credit contract, entrepreneurs maximise their expected revenue as follows: (I drop the time subscripts to conserve notations.)

$$\max_{\bar{\omega}^E, K} (1 - \Gamma^E(\bar{\omega}^E)) R^E Q K \tag{4.97}$$

subject to the participation constraint of investors:

$$(\Gamma^E(\bar{\omega}^E) - \mu^E G(\bar{\omega}^E)) R^E Q K = R(QK - N^E). \tag{4.98}$$

The first order conditions are

$$\bar{\omega}^E \quad : \qquad \lambda(\bar{\omega}^E) = \frac{\Gamma^E_\omega(\bar{\omega}^E)}{\Gamma^E_\omega(\bar{\omega}^E) - \mu^E G^E_\omega(\bar{\omega}^E)}, \qquad \lambda_\omega(\bar{\omega}^E) > 0, \tag{4.99}$$

$$K \quad : \qquad \frac{R^E}{R} = \frac{\lambda(\bar{\omega}^E)}{\lambda(\bar{\omega}^E)(\Gamma^E(\bar{\omega}^E) - \mu^E G^E(\bar{\omega}^E)) + (1 - \Gamma^E(\bar{\omega}^E))}. \tag{4.100}$$

The nature of the BGG contract is identical to the loan contract presented in the main text, and the intuitions are the same. The only difference is that in our baseline model the loan contract occurs between investors and financial institutions whereas in the BGG model, the loan contract occurs between investors and entrepreneurs.

The net worth accumulation of entrepreneurs is

$$N^E_{t+1} = \gamma^E(1 - \Gamma^E(\bar{\omega}^E_{t+1}))R^E_t Q_{t-1} K_t + (1 - \alpha)\Omega_E Y_t, \tag{4.101}$$

where the second term is the 'entrepreneurial wage'. The consumption by quitting entrepreneurs is given by:

$$C^E_t = (1 - \gamma^E)(1 - \Gamma^E(\bar{\omega}^E_{t+1}))R^E_t Q_{t-1} K_t. \tag{4.102}$$

The aggregate demand equation now has to account for the consumption by quitting entrepreneurs and the resource wasted by monitoring in the BGG loan contract:

$$Y_t = C_t + C^E_t + I_t + \mu^E G^E(\bar{\omega}^E_t)R^E_t Q_{t-1} K_t. \tag{4.103}$$

Since there are no separate financial institutions, we no longer need any supply of labour by financial institutions. We set the parameter Ω_F to be zero. Therefore,

Table 4.7: Credit contract parameters and steady-state values in BGG.

Parameter	$\bar{\omega}^E$	σ^E	μ^E	γ^E	G^E	Γ^E	G^E_ω	Γ^E_ω
Value	0.4991	0.3143	0.0561	0.9798	0.0089	0.4980	0.1540	0.98

Table 4.8: Steady-state variables in BGG.

Parameter	Y	C	C^E	I	W
Value	100	68.7581	8.9774	21.82	27.4139
Parameter	L_C	K	N^E	R	R^E
Value	2.3473	872.80	436.40	1.0101	1.0151

the production function and the demand of labour are given by

$$Y_t = A_t K_t^{\alpha} L_{Ct}^{(1-\alpha)(1-\Omega_E)}, \tag{4.104}$$

$$W_t = (1-\alpha)(1-\Omega_E)\frac{Y_t}{L_{Ct}}. \tag{4.105}$$

The other equations in the general equilibrium are unchanged.

We calibrate the credit contract parameters using the first two steady-state relations in our baseline calibration, that is (i) a steady-state external finance premium of 2% per annum; (ii) a steady-state entrepreneurial leverage of 2; and together with (iii) an entrepreneurial failure rate of 2%. The credit contract parameters and steady-state values used in our calibration of the BGG model are reported in Table 4.7 and 4.8.

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