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Categorical Formulation of Finite-dimensional C*-algebras

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Abstract

We develop the concept of an *involution monoid*, and use it to show that finite-dimensional C*-algebras are the same as special unitary †-Frobenius monoids in the category of finite-dimensional complex Hilbert spaces. This gives a new, geometrical definition of finite-dimensional C*-algebras, contrasting with the conventional algebraic one.

Keywords: C*-algebras, Frobenius algebras

1 Introduction

The main purpose of this paper is to describe how †-Frobenius monoids are the correct tool for describing finite-dimensional C*-algebras in a categorical way. Since †-Frobenius monoids have entirely geometrical axioms, this gives a new way to look at these traditionally algebraic objects.

This difference in perspective can be thought of as moving from an 'internal' to an 'external' viewpoint. Traditionally, we formulate a C*-algebra as the set of elements of a vector space, along with extra structure that tells you how to multiply elements, find a unit element, apply an involution and take norms. This is an 'internal' view, since we are dealing directly with the elements of the set. The 'external' alternative is to 'zoom out' in perspective: we can no longer discern the individual elements of the C*-algebra, but we can see more clearly how it relates to

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other vector spaces, and these relationships give an alternative way to completely define the C*-algebra.

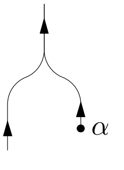
Of course, this metaphor is made completely precise by category theory, and the passage between these two viewpoints is very familiar. Category-theoretically, our main result can be stated as follows: in theorem 3.26 we show that finite-dimensional C*-algebras are the same as special unitary †-Frobenius monoids in **Hilb**, the category of finite-dimensional complex Hilbert spaces.

It is expected that these results will have interesting ramifications for information theory. C*-algebraic techniques are often useful for exploring the properties of quantum information; a good example of this is the CBH theorem [3]. The description of C*-algebras given in this paper gives a new, more abstract way to approach this subject.

These results also suggest a new, more abstract route into investigations of physical applications of C*-algebras. The most immediate application, which we do not discuss in this paper, is to the study of unitary topological quantum field theories. We also note that the special unitary Frobenius monoids that we are concerned with in this paper have already been shown to give rise to conformal field theories [9]; the results of this paper then suggest that such theories should be thought of as generalised C*-algebras.

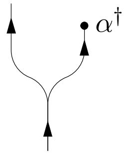
1.1 Why †-Frobenius monoids?

An insight into why †-Frobenius monoids are the correct structures to choose is contained in the following observation, due to Coecke, Pavlovic and the author [5]. Let (V, m, u) be an associative, unital algebra on a complex vector space V, with multiplication map $m : V \otimes V \longrightarrow V$ and unit map $u : \mathbb{C} \longrightarrow V$. We can map any element $\alpha \in V$ into the algebra of operators on V by constructing its right action, a linear map $R_\alpha := m \circ (\mathrm{id}_A \otimes \alpha) : V \longrightarrow V$. We draw this right action in the following way:

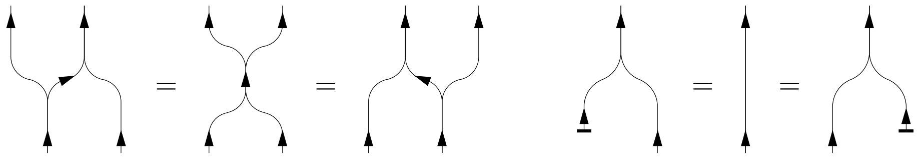


The diagram is read from bottom to top. This is a direct representation of our definition of R_α: vertical lines represent the vector space V, the dot represents preparation of the state α, and the merging of the two lines represents the multiplication operation $m : V \otimes V \longrightarrow V$. If V is in fact a Hilbert space we can then construct the adjoint map ${R_\alpha}^\dagger : V \longrightarrow V$. Will this adjoint also be the right action of some element of V?

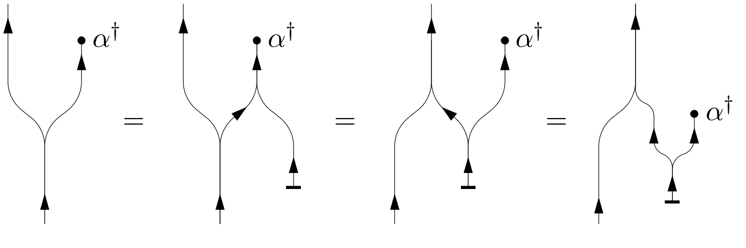
In the case that (V, m, u) is in fact a †-*Frobenius monoid*, the answer is yes. We draw the adjoint ${R_\alpha}^\dagger$ by flipping the diagram on a horizontal axis, but keeping the arrows pointing in their original direction:



The splitting of the line into two represents the adjoint to the multiplication, and the dot represents the linear map $\alpha^\dagger : V \longrightarrow \mathbb{C}$. The multiplication and unit morphisms of the †-Frobenius monoid, along with their adjoints, must obey the following equations (see definition 3.3):

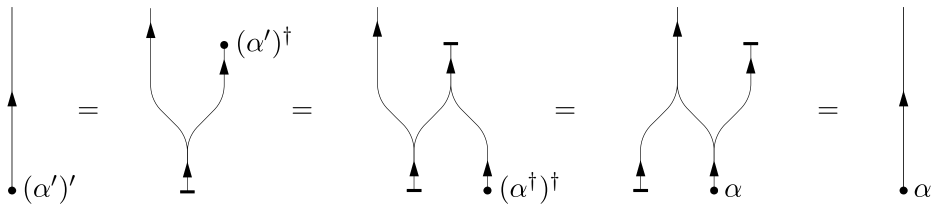


On the left are the Frobenius equations, and on the right are the unit equations. The short horizontal bar in the unit equations represents the unit for the monoid, and the straight vertical line represents the identity homomorphism on the monoid. In fact, we also have two extra equations, since we can take the adjoint of the unit equations. We can use a unit equation and a Frobenius equation to redraw the graphical representation of ${R_\alpha}^\dagger$ in the following way:



We therefore see that the adjoint of R_α is indeed a right-action of some element: ${R_\alpha}^\dagger = R_{\alpha'}$, for $\alpha' = (\mathrm{id}_A \otimes \alpha^\dagger) \circ m^\dagger \circ u$.

To better understand this transformation $\alpha \longmapsto \alpha'$ we apply it twice to evaluate $(\alpha')'$, using the Frobenius and unit equations and the fact that the †-functor is an involution:



We see that $(\alpha')' = \alpha$, and so the operation $\alpha \longmapsto \alpha'$ is an involution. Since taking the adjoint $R_\alpha \longmapsto R_\alpha^\dagger$ is also clearly an involution, the mapping of elements of the monoid into the ring of operators on V is therefore *involution-preserving*, as it maps one involution into another. We shall see that the mapping is injective and preserves the multiplication and unit of (V, m, u), so in fact we have a fully-fledged involution-preserving monoid embedding as described by lemmas 3.19 and 3.20.

This observation is one reason why †-Frobenius monoids are such powerful tools. In fact, given that the algebra of operators on V is a C*-algebra with $*$-involution given by operator adjoint, and since any involution-closed subalgebra of a C*-algebra is also a C*-algebra, we have already shown that every †-Frobenius monoid in **Hilb** can be given a C*-algebra norm.

2 Structures in †-categories

2.1 The †-functor

Of all the categorical structures that we will make use of, the most fundamental is the †-*functor*. It is an axiomatisation of the operation of taking the adjoint of a linear map between two Hilbert spaces, and since knowing the adjoints of all maps $\mathbb{C} \longrightarrow H$ is equivalent to knowing the inner product on H, it also serves as an axiomatisation of the inner product.

Definition 2.1 A †-*functor* on a category $\mathbf{C}$ is a contravariant endofunctor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, which is the identity on objects and which satisfies $\dagger \circ \dagger = \mathrm{id}_{\mathbf{C}}$.

Definition 2.2 A †-*category* is a category equipped with a particular choice of †-functor.

We denote the action of a †-functor on a morphism $f : A \longrightarrow B$ as $f^\dagger : B \longrightarrow A$, and by convention we refer to the morphism $f^\dagger$ as the *adjoint* of f. We can now make the following straightforward definitions:

Definition 2.3 In a †-category, a morphism $f : A \longrightarrow B$ is an *isometry* if $f^\dagger \circ f = \mathrm{id}_A$; in other words, if $f^\dagger$ is a retraction of f.

Definition 2.4 In a †-category, a morphism $f : A \longrightarrow B$ is *unitary* if $f^\dagger \circ f = \mathrm{id}_A$ and $f \circ f^\dagger = \mathrm{id}_B$; in other words, if f is an isomorphism and $f^{-1} = f^\dagger$.

2.2 Monoidal †-categories with duals

We now investigate appropriate compatibility conditions in the case that our monoidal category has both duals and a †-functor.

Definition 2.5 A *monoidal †-category* is a monoidal category equipped with a †-functor, such that the associativity and unit natural isomorphisms are unitary. If the monoidal category is equipped with braiding natural isomorphisms, then these must also be unitary.

A good reference for the essentials of monoidal category theory is [7].

In a monoidal †-category we can give abstract definitions of some important terminology normally associated with Hilbert spaces.

Definition 2.6 In a monoidal category, the *scalars* are the monoid $\mathrm{Hom}(I, I)$. In a monoidal †-category, the scalars form a monoid with involution.

Definition 2.7 In a monoidal †-category, a *state* of an object A is a morphism $\phi : I \longrightarrow A$.

Definition 2.8 In a monoidal †-category, the *squared norm* of a state $\phi : I \longrightarrow A$ is the scalar $\phi^\dagger \circ \phi : I \longrightarrow I$.

If our †-category also has a zero object, we note that it is quite possible for the squared norm of a non-zero state to be zero. For this reason, as it stands, definition 2.8 seems a poor abstraction of the notion of the squared norm on a vector space. In [11] we describe a way to overcome this problem, but it will not affect us here.

Definition 2.9 A *monoidal †-category with duals* is a monoidal †-category such that each object A has an assigned left- and right-dual object A^*, with this assignment satisfying $(A^*)^* = A$, and assigned left and right duality morphisms for each object, such that these assignments are compatible with the †-functor in the following way:

$$\epsilon_A^{\mathrm{L}} = \eta_A^{\mathrm{R}\dagger} = \eta_{A^*}^{\mathrm{L}}{}^\dagger = \epsilon_{A^*}^{\mathrm{R}} \qquad \eta_A^{\mathrm{L}} = \epsilon_A^{\mathrm{R}\dagger} = \epsilon_{A^*}^{\mathrm{L}}{}^\dagger = \eta_{A^*}^{\mathrm{R}} \qquad (-)^{*\mathrm{L}} = (-)^{*\mathrm{R}} \tag{1}$$

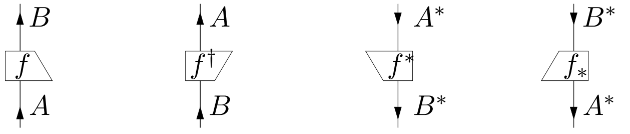
Since the left and right duality morphisms can be obtained from each other using the †-functor, from now on we will only refer directly to the left-duality morphisms, defining $\epsilon_A := \epsilon_A^{\mathrm{L}}$ and $\eta_A := \eta_A^{\mathrm{L}}$. Also, since the duality functors $(-)^{*\mathrm{L}}$ and $(-)^{*\mathrm{R}}$ are the same, we use the simpler notation $(-)^*$.

In fact, this duality functor is an involution, and commutes with the †-functor. The composite of the duality and †-functors will therefore also be an involution.

Definition 2.10 In a monoidal †-category with duals, the *conjugation functor* $(-)_*$ is defined on all morphisms f by $f_* = (f^*)^\dagger = (f^\dagger)^*$.

Since the †-functor is the identity on objects, we have $A_* = A^*$ for all objects A. To make this equality clear we will write A^* exclusively, and the A_* form will not be used.

For any morphism $f : A \longrightarrow B$ we can use these functors to construct $f_* : A^* \longrightarrow B^*$, $f^* : B^* \longrightarrow A^*$ and $f^\dagger : B \longrightarrow A$, and it will be important to be able to easily distinguish between these graphically. We will use an approach originally due to Selinger [10], in the form adopted by Coecke and Pavlovic [4]:

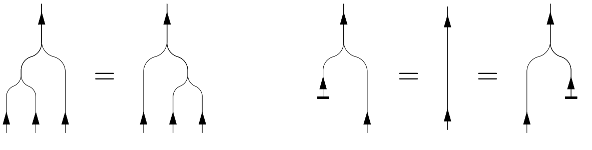


In other work, an important notion is that of a *strongly compact-closed category* [1,2]. Using the definitions given here, this is equivalent to a symmetric monoidal †-category with duals.

2.3 Involution monoids

An important tool in functional analysis is the $*$-*algebra*: a complex, associative, unital algebra equipped with an antilinear involutive homomorphism from the algebra to itself which reverses the order of multiplication. Category-theoretically, such a homomorphism is not very convenient to work with, since morphisms in a category of vector spaces are usually chosen to be the *linear* maps. However, if the vector space has an inner product, this induces a canonical antilinear isomorphism from the vector space to its dual. Composing this with the antilinear self-involution, we obtain a *linear* isomorphism from the vector space to its dual. This style of isomorphism is much more useful from a categorical perspective, and we use it to define the concept of an *involution monoid*. We will demonstrate that this is equivalent to a conventional $*$-algebra when applied in a category of complex Hilbert spaces. The natural setting for the study of these categorical objects is a category with a conjugation functor, as defined above.

Definition 2.11 In a monoidal category, a *monoid* is an ordered triple (A, m, u) consisting of an object A, a *multiplication* morphism $m : A \otimes A \longrightarrow A$ and a *unit* morphism $u : I \longrightarrow A$, which satisfy *associativity* and *unit* equations:



(2)

Definition 2.12 In a monoidal †-category with duals, an *involution monoid* $(A, m, u; s)$ is an internal monoid (A, m, u) equipped with a morphism $s : A \longrightarrow A^*$ called the *linear involution*, which is a morphism of monoids with respect to the monoid structure (A^*, m_*, u_*) on A^*, and which satisfies the *involution condition*

$$s_* \circ s = \mathrm{id}_A. \tag{3}$$

It follows from this definition that s and s_* are mutually inverse morphisms, since applying the conjugation functor to the involution condition gives $s \circ s_* = \mathrm{id}_{A^*}$. We also note that for any such involution monoid $s : A \longrightarrow A^*$ and $s^* : A \longrightarrow A^*$ are parallel morphisms, but they are not necessarily the same.

Definition 2.13 In a monoidal †-category with duals, given involution monoids $(A, m, u; s_A)$ and $(B, n, v; s_B)$, a morphism $f : A \longrightarrow B$ is a *homomorphism of involution monoids* if it is a morphism of monoids, and if it satisfies the *involution-preservation condition*

$$s_B \circ f = f_* \circ s_A. \tag{4}$$

If an object B is self-dual, it is possible for the involution $s_B : B \longrightarrow B$ to be the identity. Let $(B, n, v; \mathrm{id}_B)$ be such an involution monoid. In this case, it is sometimes possible to find an embedding $f : (A, m, u; s_A) \hookrightarrow (B, n, v; \mathrm{id}_B)$ of involution monoids even when the linear involution s_A is *not* trivial! We will see an example of this in the next section.

The following lemma establishes that the traditional concept of $*$-algebra and the categorical concept of an involution monoid are the same, in an appropriate context. We demonstrate the equivalence for finite-dimensional algebras, since the category of finite-dimensional complex vector spaces forms a category with duals.

Lemma 2.14 *For a unital, associative algebra on a finite-dimensional complex Hilbert space V, there is a correspondence between the following structures:*

(i) *antilinear maps $t : V \longrightarrow V$ which are involutions, and which are order-reversing algebra homomorphisms;*
(ii) *linear maps $s : V \longrightarrow V^*$ where V^* is the dual space of V, satisfying $s_* \circ s = \mathrm{id}_V$, and which are algebra homomorphisms to the conjugate algebra on V^*.*

Furthermore, the natural notions of homomorphism for these structures are also equivalent.

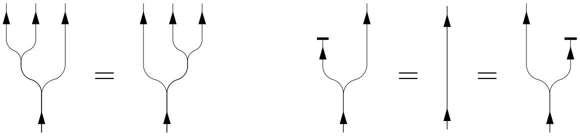
For reasons of space, we omit the proof.

3 Results on †-Frobenius monoids

3.1 Introducing †-Frobenius monoids

We begin with definitions of the important concepts.

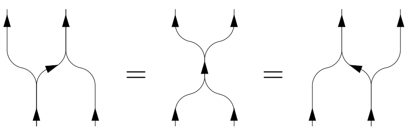
Definition 3.1 In a monoidal category, a *comonoid* is the dual concept to a monoid; that is, it is an ordered triple $(A, n, v)_\times$ consisting of an object A, a *comultiplication* $n : A \longrightarrow A \otimes A$ and a *counit* $v : A \longrightarrow I$, which satisfy *coassociativity* and *counit* equations:



(5)

If an object has both a chosen monoid structure and a chosen comonoid structure, then there is an important way in which these might be compatible with each other.

Definition 3.2 In a monoidal category, a *Frobenius structure* is a choice of monoid (A, m, u) and comonoid $(A, n, v)_\times$ for some object A, such that the multiplication m and the comultiplication n satisfy the following equations:



(6)

Reading these diagrams from bottom to top, the splitting of a line represents the comultiplication n, and merging of two lines represents the multiplication m. This geometrical definition of a Frobenius structure, although well-known, is quite different from the 'classical' definition in terms of an exact pairing. A good discussion of the different possible definitions is given in the book by Kock [6]. An important property of a Frobenius structure is that it can be used to demonstrate that the underlying object is self-dual.

If we are working in a †-category, from any monoid (A, m, u) we can canonically obtain an 'adjoint' comonoid $(A, m^\dagger, u^\dagger)_\times$, and it is then natural to make the following definition.

Definition 3.3 In a monoidal †-category, a monoid (A, m, u) is a *†-Frobenius monoid* if it forms a Frobenius structure with its adjoint $(A, m^\dagger, u^\dagger)_\times$.

Given a †-Frobenius monoid (A, m, u), we refer to $m^\dagger$ as its comultiplication and to $u^\dagger$ as its counit.

3.2 Involutions on †-Frobenius monoids

We now look at the relationship between †-Frobenius monoids and the involution monoids of section 2. We will see that a †-Frobenius monoid can be given the structure of an involution monoid in two canonical ways, which in general will be different.

Definition 3.4 In a monoidal †-category with duals, a †-Frobenius monoid (A, m, u) has a *left involution* $s_{\mathrm{L}} : A \longrightarrow A^*$ and *right involution* $s_{\mathrm{R}} : A \longrightarrow A^*$

defined as follows:

(7)

$$s_{\mathrm{L}} := \big((u^\dagger \circ m) \otimes \mathrm{id}_{A^*}\big) \circ \big(\mathrm{id}_A \otimes \epsilon_{A^*}\big) \quad s_{\mathrm{R}} := \big(\mathrm{id}_{A^*} \otimes (u^\dagger \circ m)\big) \circ \big(\epsilon_A \otimes \mathrm{id}_A\big)$$

In each case the second picture is just a convenient shorthand, which should literally be interpreted as the first picture. These involutions interact with the conjugation and transposition functors in interesting ways, as we explore in the next lemma.

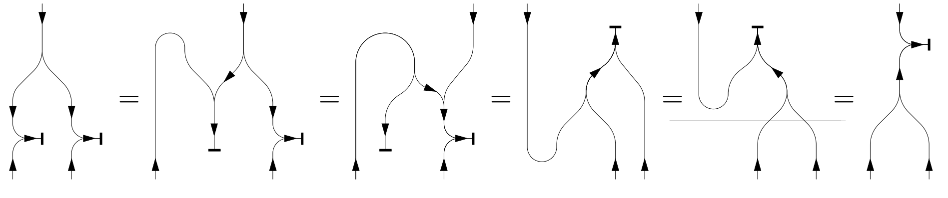
Lemma 3.5 *In a monoidal †-category with duals, the left and right involutions of a †-Frobenius monoid satisfy the following equations:*

$$s_{\mathrm{L}}{}^* = s_{\mathrm{R}}, \quad s_{\mathrm{R}}{}^* = s_{\mathrm{L}} \tag{8}$$
$$s_{\mathrm{L}*} = s_{\mathrm{L}}^{-1}, \quad s_{\mathrm{R}*} = s_{\mathrm{R}}^{-1} \tag{9}$$
$$s_{\mathrm{L}}^{-1} = s_{\mathrm{R}}{}^\dagger, \quad s_{\mathrm{R}}^{-1} = s_{\mathrm{L}}{}^\dagger \tag{10}$$

We now combine these results on involutions of †-Frobenius monoids with the concept of an involution monoid from section 2.

Lemma 3.6 *In a monoidal †-category with duals, given a †-Frobenius monoid (A, m, u) we can canonically obtain two involution monoids $(A, m, u; s_{\mathrm{L}})$ and $(A, m, u; s_{\mathrm{R}})$, where s_{L} and s_{R} are respectively the left and right involutions associated to the monoid.*

Proof. We deal with the right-involution case; the left-involution case is analogous. We must show that $s_R : A \longrightarrow A_*$ is a morphism of monoids, and that it satisfies the involution condition. We first show that it preserves multiplication, employing the Frobenius, unit and associativity laws:



We omit the proof that s_R preserves the unit, as it is straightforward. The involution condition $s_{R*} \circ s_R = \mathrm{id}_A$ follows from one of the equations (9) in lemma 3.5. □

This leads us to the following definition.

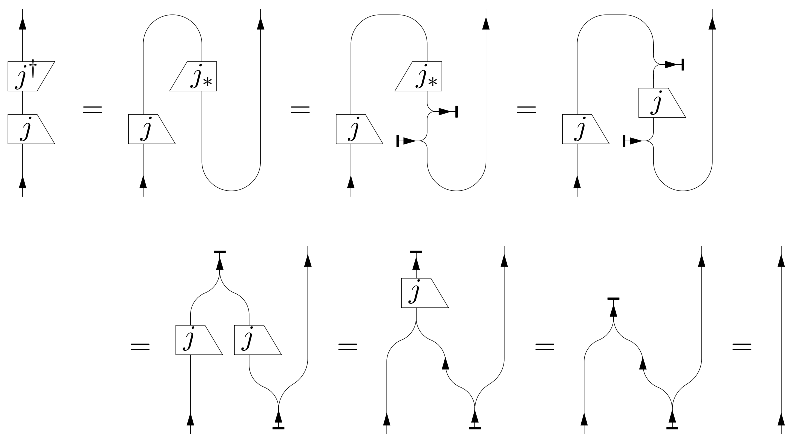
Definition 3.7 In a monoidal †-category with duals, a †-*Frobenius left-* (or *right-*) *involution monoid* is an involution monoid $(A, m, u; s)$ such that the monoid

(A, m, u) is †-Frobenius, and such that the involution s is the left (or right) involution of the †-Frobenius monoid in the manner described by definition 3.4.

A useful property of †-Frobenius right-involution monoids is described by the following lemma, which gives a necessary and sufficient algebraic condition for a monoid homomorphism to be an isometry.

Lemma 3.8 *In a monoidal †-category with duals, a homomorphism of †-Frobenius right-involution monoids is an isometry if and only if it preserves the counit.*

Proof. Let $j : (A, m, u) \longrightarrow (B, n, v)$ be a homomorphism between †-Frobenius right-involution monoids. Assuming that j preserves the counit, we show that it is an isometry by the following graphical argument. The third step uses the fact that j preserves the involution, the fifth that it is a homomorphism of monoids, and the sixth that it preserves the counit.



Now instead assume that j is an isometry. It is a homomorphism, so we have the unit-preservation equation $j \circ u = v$, and therefore $j^\dagger \circ j \circ u = u = j^\dagger \circ v$. Applying the †-functor to this we obtain $u^\dagger = v^\dagger \circ j$, which is the counit preservation condition.□

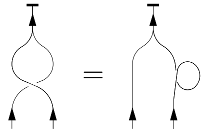
3.3 Special unitary †-Frobenius monoids

We will mostly be interested in the case when the two involutions are the same, and we now explore under what conditions this holds.

Definition 3.9 In a monoidal †-category with duals, a †-Frobenius monoid is *unitary* if the left involution, or equivalently the right involution, is unitary.

That these are equivalent follows from lemma 3.5.

Definition 3.10 In a braided monoidal †-category with duals, a †-Frobenius monoid is *balanced-symmetric* if the following equation is satisfied:



(11)

The term *symmetric* is standard (for example, see [6, section 2.2.9]), and describes a similar property that lacks the 'balancing loop' on one of the legs of the right-hand side of the equation. In **Hilb** this loop is the identity and so the concepts are the same, but this may not be the case in other categories of interest.

Lemma 3.11 *In a monoidal †-category with duals, the following properties of a †-Frobenius monoid are equivalent:*

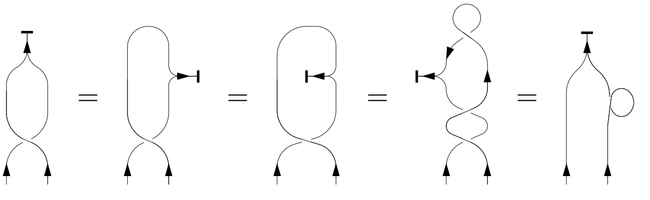
(i) *it is unitary;*

(ii) *it is balanced-symmetric;*

(iii) *the left and right involutions are the same;*

where property 2 only applies if the monoidal structure has a braiding.

Proof. We first give a graphical proof that $3 \Rightarrow 2$, using property 3 to transform the second expression into the third:



A similar argument shows that $2 \Rightarrow 3$. From equations (10) of lemma 3.5 it follows that $1 \Leftrightarrow 3$, and so all three properties are equivalent. □

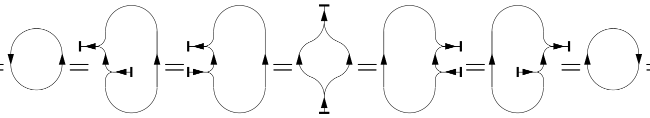
We will mostly use the term 'unitary' to refer to these equivalent properties, since it is more obviously in keeping with the general philosophy of †-categories, that all structural isomorphisms should be unitary. We also note that if a †-Frobenius left- or right-involution monoid is unitary then we can simply refer to it as a '†-Frobenius involution monoid', as the left and right involutions coincide in that case.

One particularly nice feature of unitary †-Frobenius monoids is that we can canonically obtain an abstract 'dimension' of their underlying space from the multiplication, unit, comultiplication and counit, as the following lemma shows. They also force this dimension to be well-behaved. In a category of vector spaces and linear maps, this dimension will correspond to the dimension of the vector space.

Definition 3.12 In a monoidal †-category, the *dimension* of an object A is given by the scalar ${\epsilon_A}^\dagger \circ \epsilon_A : I \longrightarrow I$, and is denoted $\dim(A)$.

Lemma 3.13 *In a monoidal †-category with duals, given a unitary †-Frobenius monoid (A, m, u), $\dim(A) = u^\dagger \circ m \circ m^\dagger \circ u$; that is, the dimension of A is equal to the squared norm of $m^\dagger \circ u$. Also, $\dim(A) = \dim(A)^*$.*

Proof. We demonstrate this with the following series of pictures:

$\dim(A) =$  $= \dim(A)^*$

The central diagram is $u^\dagger \circ m \circ m^\dagger \circ u$, so this proves the lemma. □

We now introduce one final property of a †-Frobenius monoid.

Definition 3.14 In a monoidal †-category, a †-Frobenius monoid (A, m, u) is *special* if $m \circ m^\dagger = \mathrm{id}_A$; that is, if the comultiplication is an isometry.

This property simplifies the expression for the dimension of the underlying space.

Lemma 3.15 *In a monoidal †-category with duals, a special unitary †-Frobenius monoid (A, m, u) has $\dim(A) = u^\dagger \circ u$; that is, the dimension of A is equal to the squared norm of u.*

Proof. Straightforward from lemma 3.13. □

3.4 Endomorphism monoids

Given a Hilbert space H, it is often useful to consider the algebra of bounded linear operators on H. These give the prototypical examples of C*-algebras, with the $*$-involution given by taking the operator adjoint. In a monoidal category with duals we can construct *endomorphism monoids*, which are categorical analogues of these algebras of bounded linear operators. We will see that they form an important class of †-Frobenius monoids, and that they have particularly nice properties.

Definition 3.16 In a monoidal category, for an object A with a left dual $A^{*\mathrm{L}}$, the *endomorphism monoid* $\mathrm{End}(A)$ is defined by

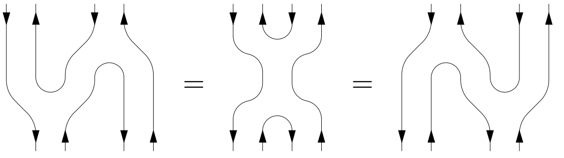
$$\mathrm{End}(A) := \big(A^{*\mathrm{L}} \otimes A, \mathrm{id}_{A^{*\mathrm{L}}} \otimes \eta_A^{\mathrm{L}} \otimes \mathrm{id}_A, \epsilon_A^{\mathrm{L}}\big). \qquad (12)$$

The following lemma describes a well-known connection between categorical duality and Frobenius structures.

Lemma 3.17 *In a monoidal †-category with duals, an endomorphism monoid is a †-Frobenius monoid.*

Proof. That the †-Frobenius property holds for an endomorphism monoid $\mathrm{End}(A)$

is clear from its graphical representation, which we give here:

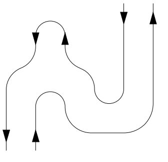


□

They are examples of the unitary monoids discussed in the previous section.

Lemma 3.18 *In a monoidal †-category with duals, endomorphism monoids are unitary.*

Proof. Following equation (7) for the left involution associated to a †-Frobenius monoid, we obtain the following:

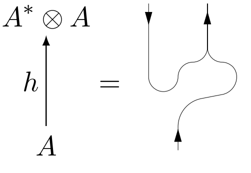


This is clearly the identity on $A^* \otimes A$. The right involution is also the identity, by the conjugate of this picture. By lemma 3.11 the †-Frobenius monoid must therefore unitary. □

The following lemma is a formal description of the intuitive notion that an algebra should have a homomorphism into the algebra of operators on the underlying space, given by taking the right action of each element.

Lemma 3.19 *Let (A, m, u) be a monoid in a monoidal category in which the object A has a left dual. Then (A, m, u) has a monic homomorphism into the endomorphism monoid of A.*

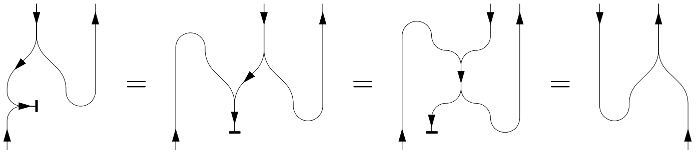
The embedding morphism has the following graphical representation:



As we saw in the introduction, for the case of †-Frobenius monoids this embedding has a special property: it preserves an involution. We establish this formally in the following lemma.

Lemma 3.20 *Let $(A, m, u; s_R)$ be a †-Frobenius right-involution monoid. Then the canonical embedding of $(A, m, u; s_R)$ into the †-Frobenius involution monoid* $\mathrm{End}(A)$ *is a morphism of involution monoids.*

Proof. By lemma 3.19 the embedding must be a morphism of monoids. Note that we do not need to specify whether we are using the left or right involution of $\mathrm{End}(A)$, since by lemma 3.18 they are both the identity. We must show that this embedding morphism $k : A \hookrightarrow A^* \otimes A$ satisfies the involution condition $k = k_* \circ s_R$ given in definition 2.13. The proof uses the Frobenius law and the unit law.



□

3.5 Special unitary †-Frobenius monoids in **Hilb**

From now on we will mainly work in **Hilb**, the category of finite-dimensional complex Hilbert spaces and continuous linear maps, which is a symmetric monoidal †-category with duals. Special unitary †-Frobenius monoids have particularly good properties in this setting.

The following lemma contains the important insight due to Coecke, Pavlovic and the author, as described in the introduction and in [5].

Lemma 3.21 *In* **Hilb***, a †-Frobenius right-involution monoid admits a norm making it into a C*-algebra.*

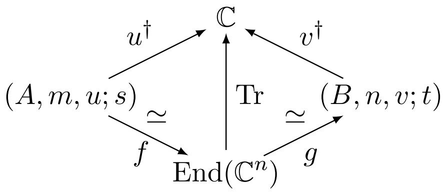
Proof. By lemma 3.20 a †-Frobenius right-involution monoid (A, m, u) has an involution-preserving embedding into $\mathrm{End}(A)$, which is a C*-algebra when equipped with the operator norm. The involution monoid (A, m, u) therefore admits a C*-algebra norm, taken from the norm on $\mathrm{End}(A)$ under the embedding. □

We will also require the following important result, which is demonstrates a crucial abstract property of the category **Hilb**.

Lemma 3.22 *In* **Hilb***, isomorphisms of special unitary †-Frobenius involution monoids preserve the counit.*

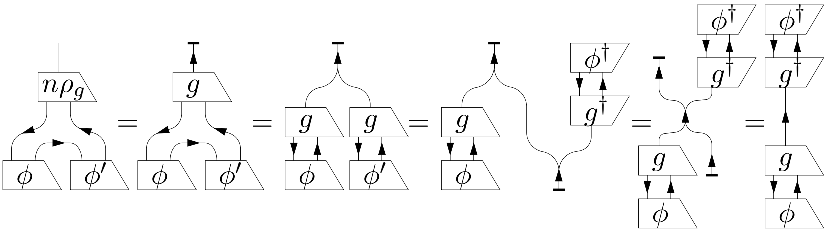
Proof. Any special unitary †-Frobenius involution monoid is in particular a †-Frobenius right-involution monoid, and so admits a norm with which it becomes a C*-algebra by lemma 3.21. Finite-dimensional C*-algebras are semisimple, and are therefore isomorphic to finite direct sums of matrix algebras in a canonical way; an isomorphism between two finite-dimensional C*-algebras is then given by a direct sum of pairwise isomorphisms of matrix algebras. We therefore need only show that the lemma is true for special unitary †-Frobenius involution monoids which are matrix algebras, with involution given by matrix adjoint.

Let $(A, m, u; s)$ and $(B, n, v; t)$ be special unitary †-Frobenius involution monoids which are both isomorphic to some matrix algebra $\mathrm{End}(\mathbb{C}^n)$. Any isomorphism between them must have some decomposition into isomorphisms $f : (A, m, u; s) \longrightarrow \mathrm{End}(\mathbb{C}^n)$ and $g : \mathrm{End}(\mathbb{C}^n) \longrightarrow (B, n, v; t)$. The statement that $g \circ f$ preserves the counit is equivalent to the statement that the outside diamond of the following diagram commutes:



(13)

We will show that each triangle separately commutes, and therefore that the entire diagram commutes. We focus on the triangle involving the isomorphism g; the treatment of the other triangle is analogous. Our strategy is to show that $\rho_g := \frac{1}{n} \cdot v^\dagger \circ g$ is a tracial state of $\mathrm{End}(\mathbb{C}^n)$. It takes the unit to 1, since $\frac{1}{n} \cdot v^\dagger \circ g \circ \epsilon^{\mathrm{L}}_B = \frac{1}{n} \cdot v^\dagger \circ v = \frac{1}{n} \cdot \dim(B) = \frac{1}{n} \cdot n = 1$, where we used the fact that g is a homomorphism and lemma 3.15; this is the reason that we require the †-Frobenius monoid to be special. We can simplify the action of ρ_g on positive elements in the following way, where $\phi : I \longrightarrow \mathbb{C}^{n*} \otimes \mathbb{C}^n$ is an arbitrary nonzero state of $\mathrm{End}(\mathbb{C}^n)$, and ϕ' is the result of applying the involution to this state:



The expression on the right-hand side is the squared norm of $g \circ \phi$, which is positive because the inner product in **Hilb** is nondegenerate and ϕ is nonzero; this shows that ρ_g takes positive elements to nonnegative real numbers, and so is a state of $\mathrm{End}(\mathbb{C}^n)$. By lemma 3.11 the involution monoid $\mathrm{End}(A)$ is balanced-symmetric, and since we are in **Hilb**, the balancing loop can be neglected; this means that $\rho_g \circ (a \otimes b) = \rho_g \circ (b \otimes a)$ for all $a, b \in \mathrm{End}(A)$, and so ρ_g is tracial. Altogether ρ_g is a tracial state of a matrix algebra. However, it is a standard result that the matrix algebra on a complex n-dimensional vector space has a unique tracial state given by $\frac{1}{n}\mathrm{Tr}$ (for example, see [8, Example 6.2.1]). The triangle therefore commutes as required. □

We can combine this with an earlier lemma to obtain a very useful result.

Lemma 3.23 *In* **Hilb***, isomorphisms of special unitary †-Frobenius involution monoids are unitary.*

Proof. Straightforward from lemmas 3.8 and 3.22. □

Given a †-Frobenius monoid in **Hilb**, scaling the inner product on the underlying complex vector space produces a family of new †-Frobenius monoids. We first note the following relationship between scaling inner products and adjoints to linear maps.

Lemma 3.24 *Let V be a complex vector space with inner product $(-,-)_V$ and let $f : V^{\otimes n} \longrightarrow V^{\otimes m}$ a linear map, with the adjoint $f^\dagger$ under this inner product. If the inner product is scaled to $\alpha \cdot (-,-)$ for α a positive real number, the adjoint to f becomes $\alpha^{m-n} f^\dagger$.*

Lemma 3.25 *For a †-Frobenius monoid (A, m, u), scaling the inner product on A by any positive real number gives rise to a new †-Frobenius monoid. Moreover, this scaling preserves unitarity.*

We are now ready to prove our main correspondence theorem between finite-dimensional C*-algebras and symmetric unitary †-Frobenius monoids.

Theorem 3.26 *In* **Hilb**, *the following properties of an involution monoid are equivalent:*

(i) *it admits a norm making it a C*-algebra;*

(ii) *it admits an inner product making it a special unitary †-Frobenius involution monoid;*

(iii) *it admits an inner product making it a †-Frobenius right-involution monoid.*

Furthermore, if these properties hold, then the structures in 1 and 2 are admitted uniquely.

Proof. First, we point out that the norm of property 1 is *not* directly related to the inner products of properties 2 or 3, in the usual way by which a norm can be obtained from an inner product, and sometimes vice-versa. In fact, the norm of a C*-algebra will usually not satisfy the parallelogram identity, and so cannot arise directly from any inner product.

We begin by showing 1 ⇒ 2. We first decompose our finite-dimensional C*-algebra into a finite direct sum of matrix algebras. For any such matrix algebra, an inner product is given by $(a, b) := \mathrm{Tr}(a^\dagger b)$. This gives an endomorphism monoid $\mathrm{End}(\mathbb{C}^n)$ in **Hilb** for some n, which is a unitary †-Frobenius monoid as described by lemmas 3.17 and 3.18. Such a monoid is not special unless it is one-dimensional; we have $m \circ m^\dagger = n \cdot \mathrm{id}_{A^* \otimes A}$, where m is the multiplication for the endomorphism monoid. We rescale the inner-product, replacing it with $((a, b)) := n\,\mathrm{Tr}(a^\dagger b)$. As described by lemma 3.24, writing the adjoint of m under this new inner product as $m^\ddagger$, we will have $m^\ddagger = \frac{1}{n} m^\dagger$, and $m \circ m^\ddagger = \mathrm{id}_{A^* \otimes A}$. By lemma 3.25 this preserves the involution and the unitarity of the monoid, and so we obtain a special unitary †-Frobenius monoid with the same underlying algebra and involution as the original matrix algebra. Taking the direct sum of these for each matrix algebra in the decomposition gives a special unitary †-Frobenius involution monoid, with the same underlying algebra and involution as the original C*-algebra.

The implication 2 ⇒ 3 is trivial, and the implication 3 ⇒ 1 is contained in lemma 3.21, so the three properties are therefore equivalent.

We now show that, if these properties hold, the norm and inner product in properties 1 and 2 are admitted uniquely. It is well-known that a C*-algebra admits a unique norm. Now assume that a finite-dimensional complex ∗-algebra has two distinct inner products, which give rise to two special unitary †-Frobenius involution monoids. Since these monoids have the same underlying set of elements and the same involution, there is an obvious involution-preserving isomorphism between them given by the identity on this set. But by lemma 3.23 any isomorphism of special unitary †-Frobenius involution monoids in **Hilb** is necessarily an isometry, and therefore unitary, and so the inner products on the two monoids are in fact the same. □

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