

On lower bounds for the matching number of subcubic graphs

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Abstract

We give a complete description of the set of triples (α, β, γ) of real numbers with the following property. There exists a constant K such that $\alpha n_3 + \beta n_2 + \gamma n_1 - K$ is a lower bound for the matching number $\nu(G)$ of every connected subcubic graph G , where n_i denotes the number of vertices of degree i for each i .

Keywords: matching, subcubic graph, polyhedron

1 Introduction

A graph is said to be *subcubic* if its maximum degree is at most three. In this paper we consider lower bounds for the maximum size $\nu(G)$ of a matching in subcubic graphs G .

Various lower bounds on $\nu(G)$ for subcubic graphs G appear in the literature. For example, the following theorem is due to Biedl, Demaine, Duncan, Fleischer and Kobourov [1]. Here n_i denotes the number of vertices of degree i in G , and ℓ_2 denotes the number of end-blocks in the block-cutvertex tree of G .

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Theorem 1. *Let G be a connected graph with n vertices.*

1. *If G is cubic then $\nu(G) \geq 4(n-1)/9$.*
2. *If G is subcubic then $\nu(G) \geq n_3/2 + n_2/3 + n_1/2 - \ell_2/3$, and $\nu(G) \geq (n-1)/3$.*

They also asked whether $\nu(G) \geq (3n + n_2)/9$ for every subcubic graph. It will turn out below that this is not the case.

Generalisations of [1] to regular graphs of higher degree were given by Henning and Yeo in [5] (see also O and West [7]). Lower bounds in terms of other parameters of G have been given, for example, in [7] and [4].

Our aim in this paper is to give a *complete* description of the set L of 3-tuples of real coefficients (α, β, γ) for which there exists a constant K such that $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - K$ for every connected subcubic graph G . (Note that this is equivalent to saying $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - Kc(G)$ for every subcubic graph G , where $c(G)$ denotes the number of components of G .) Our work here is similar in spirit to a result of Chvátal and McDiarmid [2], who addressed a similar question for cover numbers of hypergraphs in terms of their number of vertices and number of edges. We will find, as in [2], that L is a convex set, but in contrast to [2] where the number of extreme points is infinite, in our case L is a certain 3-dimensional polyhedron with a relatively simple description.

We define the polyhedron $P \subset \mathbb{R}^3$ to be the intersection of the six half-spaces

$$\begin{aligned} x_3 &\leq 4/9, \\ x_2 &\leq 1/2, \\ x_3 + x_1 &\leq 2/3, \\ x_3 + 3x_2/2 &\leq 1, \\ x_3 + x_2 + x_1 &\leq 1, \\ x_3 + x_2/6 &\leq 1/2. \end{aligned}$$

We let P_+ be the intersection of P with the nonnegative orthant $[0, \infty)^3$ in \mathbb{R}^3 . It is easily seen that P is unbounded. However, it follows from the first three inequalities above that P_+ is a bounded subset of the nonnegative orthant.

The main aim of this paper is to prove the following theorem.

Theorem 2. $P = L$.

We will prove that $P \subseteq L$ in Section 2, and $L \subseteq P$ in Section 4.

Our proof that $P \subseteq L$ will need the fact that five specific points belong to L . This is a consequence of the following stronger result, which we prove in Section 3.

Theorem 3. *Let G be a subcubic graph with $c = c(G)$ components. Then*

$$\nu(G) \geq n_2/2 + n_1/2 - c/2, \quad (1)$$

$$\nu(G) \geq n_2/3 + 2n_1/3 - c, \quad (2)$$

$$\nu(G) \geq n_3/4 + n_2/2 + n_1/4 - c/2, \quad (3)$$

$$\nu(G) \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - c/8, \quad (4)$$

$$\nu(G) \geq 4n_3/9 + n_2/3 + 2n_1/9 - c/9. \quad (5)$$

All five of these bounds are sharp: (4) is attained by the triangle, (1) and (3) by any odd cycle, and (1), (2) and (5) by the claw $K_{1,3}$. Furthermore, for a subcubic graph G , each of the bounds is sharp for G if and only if it is sharp for every component of G . We will give further connected, sharp examples for (1), (2), (3), (5) in Section 4. The proof of Theorem 3 is given in Section 3, where we will also note the following corollary concerning the constant K from the definition of L .

Corollary 4. *Let (α, β, γ) be an element of P .*

1. *If $\alpha \geq 0$ then $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1$ for every connected subcubic graph G .*
2. *If $\alpha < 0$ then $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - (2|\alpha| + 1)$ for every connected subcubic graph G .*

Note in particular that if G is a connected subcubic graph then $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1$ for every $(\alpha, \beta, \gamma) \in P_+$. Note also that if we consider $G = K_{1,3}$ and $(\alpha, \beta, \gamma) = (-\lambda, 0, \lambda + 2/3)$ (which is in P for all $\lambda \geq 0$), then the first bound in Lemma 4 is sharp for $\lambda = 0$, and the second is sharp for all $\lambda > 0$.

In the other direction, the fact that $L \subseteq P$ is a consequence of the following result, which we will prove in Section 4.

Theorem 5. *If $(\alpha, \beta, \gamma) \notin P$ then for every constant K there exists a connected subcubic graph G such that $\nu(G) < \alpha n_3 + \beta n_2 + \gamma n_1 - K$.*

Our results generalize previous work. For example, the first bound in Theorem 1 is a special case of (5); the bound $\nu \geq (n-1)/3$ follows from a convex combination of (2) and (5). On the other hand, the answer to the question of Biedl, Demaine, Duncan, Fleischer and Kobourov [1] as to whether $\nu(G) \geq (3n+n_2)/9$ for every subcubic graph is negative by Theorem 2: the vector $(1/3, 4/9, 1/3)$ is not in P as it violates the inequality $x_1 + x_2 + x_3 \leq 1$, and Example 3 in Section 4 is a counterexample.

2 $P \subseteq L$

In this section we prove one direction of Theorem 2, namely that $P \subseteq L$ (leaving aside the proof of Theorem 3, which we defer to the next section). We will prove that $P \subseteq L$ in two steps. We first show that it is enough to consider just P_+ , and then prove that $P_+ \subseteq L$.

We begin with the following simple but useful observation.

Lemma 6. *In any connected subcubic graph G we have $n_3 \geq n_1 - 2$.*

Proof. Let T be a spanning tree of G , and let t_i denote the number of vertices of degree i in T . Then $t_1 \geq n_1$, $t_3 \leq n_3$, and $t_1 = t_3 + 2$. Thus $n_3 \geq n_1 - 2$. \square

Next we note some closure properties of L .

Lemma 7. *1. L is convex.*

2. L is downward closed: if $(a_3, a_2, a_1) \in L$ and $b_i \leq a_i$ for all i then $(b_3, b_2, b_1) \in L$.

3. If $(x_3, x_2, x_1) \in L$ then $(x_3 - \lambda, x_2, x_1 + \lambda) \in L$ for all $\lambda \geq 0$.

Proof. Suppose that $\mathbf{a} = (a_3, a_2, a_1)$, $\mathbf{b} = (b_3, b_2, b_1)$ lie in L , with associated constants K_a, K_b . Thus for every subcubic graph G , say with parameters $\mathbf{n} = (n_3, n_2, n_1)$ and matching number ν , we have $\mathbf{a} \cdot \mathbf{n} \leq \nu + K_a$ and $\mathbf{b} \cdot \mathbf{n} \leq \nu + K_b$. Suppose that $\lambda \in [0, 1]$ and $\mathbf{c} = \lambda \mathbf{a} + (1 - \lambda) \mathbf{b}$. Then

$$\begin{aligned} \mathbf{c} \cdot \mathbf{n} &= \lambda \mathbf{a} \cdot \mathbf{n} + (1 - \lambda) \mathbf{b} \cdot \mathbf{n} \\ &\leq \lambda(\nu + K_a) + (1 - \lambda)(\nu + K_b) \\ &= \nu + \lambda K_a + (1 - \lambda) K_b. \end{aligned}$$

It follows that $\mathbf{c} \in L$, with associated constant $\lambda K_a + (1 - \lambda)K_b$. Thus L is convex.

For the second claim, simply note that if $\mathbf{a} \in P$ with associated constant K , then for every subcubic graph G , say with parameters $\mathbf{n} = (n_3, n_2, n_1)$ and matching number ν , we have $\mathbf{b} \cdot \mathbf{n} \leq \mathbf{a} \cdot \mathbf{n} \leq \nu + K$, so $\mathbf{b} \in L$ with associated constant K .

Now for the final part. Let K be such that $\nu(G) \geq x_3 n_3 + x_2 n_2 + x_1 n_1 - K$ for every connected subcubic graph G . By Lemma 6 we have $n_3 \geq n_1 - 2$, and so $(x_3 - \lambda)n_3 + x_2 n_2 + (x_1 + \lambda)n_1 - (K + 2\lambda) \leq x_3 n_3 + x_2 n_2 + x_1 n_1 - K \leq \nu(G)$, which shows that $(x_3 - \lambda, x_2, x_1 + \lambda) \in L$. \square

The next lemma will allow us to restrict our attention to P_+ .

Lemma 8. *If $P_+ \subseteq L$ then $P \subseteq L$.*

Proof. Consider $x = (x_3, x_2, x_1) \in P \setminus L$. Our aim is to find a point in $P_+ \setminus L$. If each x_i is non-negative then x is such a point, so we assume the contrary.

First suppose $x_2 < 0$. We claim that $x' = (x_3, 0, x_1) \in P$. Since $x \in P$, the first and third inequalities defining P are immediate for x' , and the second is trivial. The fourth and sixth inequalities follow from the first, and the fifth follows from the third. Therefore $x' \in P$. Now if $x' \in L$ then $x \in L$ because L is downward closed, contradicting our choice of x . Thus $x' \in P \setminus L$.

Therefore we may assume that $x_2 \geq 0$. Next we consider the case in which $x_3 < 0$. Set $\lambda = -x_3$ and let $x' = (x_3 + \lambda, x_2, x_1 - \lambda) = (0, x_2, x_1 + x_3)$. We claim that $x' \in P$. The first inequality for P is trivial, and the second, third and fifth are true because $x \in P$. The fourth and sixth inequalities are implied by the second. Thus $x' \in P$. If $x' \in L$ then by Lemma 7 the point $(x_3 + \lambda - \lambda, x_2, x_1 - \lambda + \lambda) = x \in L$, contradicting our choice of x . Therefore $x' \in P \setminus L$ and we may assume $x_3 \geq 0$.

Finally suppose $x_1 < 0$. Then we claim $x' = (x_3, x_2, 0) \in P \setminus L$. To check $x' \in P$ observe that the first, second, fourth and sixth inequalities are true because $x \in P$. The third follows from the first and the fifth follows from the first and second. Again we may conclude $x' \notin L$ because L is downward closed. Hence $x' \in P \setminus L$ as required, completing the proof that $P_+ \subseteq L$ implies $P \subseteq L$. \square

It is therefore enough to prove that $P_+ \subseteq L$. Since L is a convex set, it is enough to show that the extreme points of P_+ all belong to L . The extreme

points of P_+ (written as (x_3, x_2, x_1)) are

$$\begin{aligned} & \{(0, 1/2, 1/2), (0, 1/3, 2/3), (1/4, 1/2, 1/4), (7/16, 3/8, 3/16), \\ & (4/9, 1/3, 2/9), (1/4, 1/2, 0), (7/16, 3/8, 0), (0, 1/2, 0), (4/9, 0, 0), \\ & (0, 0, 0), (4/9, 1/3, 0), (0, 0, 2/3), (4/9, 0, 2/9)\}. \end{aligned}$$

This can be verified by hand, or (as we did) by using a computational package such as *polymake* [3].

Our aim is then to show that all thirteen extreme points of P_+ belong to L . Since L is downward closed, it is enough to consider the points that do not lie below any others: for instance, $(7/16, 3/8, 0)$ lies below $(7/16, 3/8, 3/16)$, so $(7/16, 3/8, 3/16) \in L$ implies that $(7/16, 3/8, 0) \in L$. This leaves us with the following five points:

$$\{(0, 1/2, 1/2), (0, 1/3, 2/3), (1/4, 1/2, 1/4), (7/16, 3/8, 3/16), (4/9, 1/3, 2/9)\}.$$

The fact that these points all belong to L follows from Theorem 3, which we prove in the next section. We conclude that $P \subseteq L$.

3 Proofs of Theorem 3 and Corollary 4

First we show how Corollary 4 follows from Theorem 3.

Proof. Let G be a connected subcubic graph. Observe that by Theorem 3 and monotonicity, we have $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1$ for each extreme point (α, β, γ) of P_+ . By convexity, the same inequality holds for every point $(\alpha, \beta, \gamma) \in P_+$.

Now suppose $(\alpha, \beta, \gamma) \in P$ and $\alpha \geq 0$. Then (arguing as in the proof of Lemma 8) we know that $(\alpha, \beta', \gamma') \in P_+$ where $\beta' = \max\{\beta, 0\}$ and $\gamma' = \max\{\gamma, 0\}$. Hence

$$\nu(G) \geq \alpha n_3 + \beta' n_2 + \gamma' n_1 - 1 \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1.$$

If $\alpha < 0$ then set $\lambda = |\alpha|$. Then as in the proof of Lemma 8 we find that $(\alpha + \lambda, \beta, \gamma - \lambda) = (0, \beta, \gamma - \lambda) \in P$. Hence by the previous paragraph $\nu(G) \geq \beta n_2 + (\gamma - \lambda) n_1 - 1$. By Lemma 6 we have $2\lambda \geq \lambda n_1 - \lambda n_3$. Summing these two inequalities and rearranging gives $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - (2\lambda + 1)$ as required. \square

The remainder of this section is devoted to the proof of Theorem 3.

Lemma 9. *Let G be a connected subcubic graph with n vertices. Suppose $\nu(G) \geq (n-1)/2$. Then G satisfies Theorem 3.*

Proof. Bounds (1) and (3) are immediate. Bound (4) holds unless $7n/16 - 1/8 > n/2 - 1/2$, which implies $n \leq 5$. If (5) fails to hold then $4n/9 - 1/9 > n/2 - 1/2$, which means $n \leq 6$. These cases are easily checked. For (2), using Lemma 6 we find $n_1 \leq n_3 + 2 \leq n - n_1 + 2$, and hence $n_1 \leq 1 + n/2$. Thus $n_2/3 + 2n_1/3 - 1 \leq n/3 + n_1/3 - 1 \leq n/2 + 1/3 - 1$. \square

In particular, if G has a perfect matching or if G is hypomatchable (meaning $G - v$ has a perfect matching for every $v \in V(G)$) then Theorem 3 holds.

In our proof we will make use of the Gallai-Edmonds structure theorem (see, for instance, [6]). In the statement below, the sets A , B and C are defined as follows (here $\Gamma(A)$ denotes the neighbourhood of A).

- $A = \{v \in V(G) : \nu(G - v) = \nu(G)\},$
- $B = \Gamma(A) \setminus A,$
- $C = V(G) \setminus (A \cup B).$

Theorem 10. (*Gallai-Edmonds*) *Let G be a graph. Then*

1. *every component of $G[A]$ is hypomatchable,*
2. *every component of $G[C]$ has a perfect matching,*
3. *every $X \subseteq B$ has neighbours in at least $|X| + 1$ components of $G[A]$.*

One consequence of Theorem 10 is that we may assume $B \neq \emptyset$, otherwise each component of G has a perfect matching or is hypomatchable, in which case we are done by Lemma 9. Note also that Part (3) implies that each vertex of B has degree at least two.

It is easy to check that all the bounds in Theorem 3 hold for graphs with at most three vertices, so we assume G has $n \geq 4$ vertices and that the theorem is true for graphs with fewer than n vertices. Since we may consider each component separately, we may assume G is connected. Choose a vertex $v \in B$, and consider $G - v$. Since $v \notin A$ we know $\nu(G - v) = \nu(G) - 1$. Let t_i denote the number of neighbours of v of degree i for $i = 1, 2, 3$. Let U denote

the set of neighbours of v of degree 1, so $|U| = t_1$. Then $G' = G - v - U$ satisfies $\nu(G') = \nu(G) - 1$.

Let n'_i denote the number of vertices of degree i in G' . Since each degree-3 neighbour of v becomes a degree-2 vertex, the number of degree-3 vertices drops by t_3 , plus one more if v itself has degree 3. Thus $n'_3 = n_3 - t_3 - (d(v) - 2) = n_3 - t_3 - (t_1 + t_2 + t_3 - 2) = n_3 - 2t_3 - t_2 - t_1 + 2$. Each degree-2 neighbour of v becomes a degree-1 vertex, and if v has degree 2 then the number of degree-2 vertices drops by one more. Hence $n'_2 = n_2 + t_3 - t_2 - (3 - d(v)) = n_2 + t_3 - t_2 - (3 - t_1 - t_2 - t_3) = n_2 + 2t_3 + t_1 - 3$. Finally $n'_1 = n_1 - t_1 + t_2$, and $c' \leq t_3 + t_2$. Then by the induction hypothesis,

1. $\nu(G') \geq n'_2/2 + n'_1/2 - c'/2$
 $\geq n_2/2 + (2t_3 + t_1 - 3)/2 + n_1/2 + (t_2 - t_1)/2 - (t_3 + t_2)/2$
 $= n_2/2 + n_1/2 - 1/2 + (t_3 - 2)/2,$
2. $\nu(G') \geq n'_2/3 + 2n'_1/3 - c'$
 $\geq n_2/3 + (2t_3 + t_1 - 3)/3 + 2n_1/3 + 2(t_2 - t_1)/3 - (t_3 + t_2)$
 $= n_2/3 + 2n_1/3 - 1 - (t_3 + t_2 + t_1)/3,$
3. $\nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - c'/2$
 $\geq n_3/4 + (2 - 2t_3 - t_2 - t_1)/4 + n_2/2 + (2t_3 + t_1 - 3)/2 + n_1/4$
 $\quad + (t_2 - t_1)/4 - (t_3 + t_2)/2$
 $= n_3/4 + n_2/2 + n_1/4 - 1/2 - (t_2 + 1)/2,$
4. $\nu(G') \geq 7n'_3/16 + 3n'_2/8 + 3n'_1/16 - c'/8$
 $\geq 7n_3/16 + 7(2 - 2t_3 - t_2 - t_1)/16 + 3n_2/8 + 3(2t_3 + t_1 - 3)/8$
 $\quad + 3n_1/16 + 3(t_2 - t_1)/16 - (t_3 + t_2)/8$
 $= 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/4 - t_3/4 - 3t_2/8 - t_1/4$
 $= [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - (4t_3 + 6t_2 + 4t_1 + 2)/16,$
5. $\nu(G') \geq 4n'_3/9 + n'_2/3 + 2n'_1/9 - c'/9$
 $\geq 4n_3/9 + 4(2 - 2t_3 - t_2 - t_1)/9 + n_2/3 + (2t_3 + t_1 - 3)/3$
 $\quad + 2n_1/9 + 2(t_2 - t_1)/9 - (t_3 + t_2)/9$
 $= 4n_3/9 + n_2/3 + 2n_1/9 - 1/9 - (t_3 + t_2 + t_1)/3.$

Since $\nu(G) = \nu(G') + 1$ and $t_3 + t_2 + t_1 \leq 3$ it follows from the calculations above that bounds (1), (2) and (5) hold for G . (In fact (2) alternatively follows from (5) together with Lemma 7(3)).

We now focus on bounds (3) and (4). Note that in these cases, our inductive statement gives

$$\nu(G') \geq n_3/4 + n_2/2 + n_1/4 - 1/2 - (t_2 + 1)/2,$$

and

$$\nu(G') \geq [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - (4t_3 + 6t_2 + 4t_1 + 2)/16.$$

First we note some consequences of Theorem 10 and the above calculations.

- Lemma 11.**
1. *Every $v \in B$ has at least two neighbours in A .*
 2. *If $x \in A$ has exactly two neighbours u and w , and if $u \in B$, then $w \in B$ as well.*
 3. *If (4) fails for G then every $v \in B$ has degree 3.*
 4. *If one of (3) and (4) fails for G then every $v \in B$ has at least two degree-2 neighbours.*

Proof. We have already noted that the first statement is immediate from Theorem 10(3). To verify the second claim, observe that if $w \in A$ then u and w are both in a component H of $G[A]$, which is hypomatchable by Theorem 10. But x has degree 1 in H , which is not possible in a hypomatchable component. Thus $w \in B$.

If (3) fails then $t_2 \geq 2$; if (4) fails then $4t_3 + 6t_2 + 4t_1 \geq 15$ and so (as $d(v) \leq 3$) we have $t_2 \geq 2$ and $t_1 + t_2 + t_3 = 3$. The last two assertions follow immediately, as the same calculation holds for any vertex of B . \square

Next we derive some elementary facts about the neighbours of degree-2 vertices.

Lemma 12. *Suppose G fails to satisfy one of (3) and (4). Then no two degree-2 vertices of G are adjacent. Furthermore every vertex of B has degree 3.*

Proof. Recall our assumption that G has at least four vertices. If G is a 4-cycle then (3) and (4) are satisfied (by Lemma 9), so let us assume otherwise. Suppose u and w are adjacent degree-2 vertices.

If u and w are not in a triangle or 4-cycle then suppressing u and w (i.e. if u' and v' are the other neighbours of u, v then we replace the path $u'uvv'$ by the edge $u'v'$) gives a connected graph G' with $\nu(G') = \nu(G) - 1$, $n'_3 = n_3$, $n'_2 = n_2 - 2$, and $n'_1 = n_1$. Then by the induction hypothesis for (3), $\nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - 1/2 = n_3/4 + n_2/2 + n_1/4 - 1/2 - 1$, showing G satisfies (3). For (4) we have by induction $\nu(G') \geq 7n'_3/16 + 3n'_2/8 + 3n'_1/16 - 1/8 = 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 6/8$, which also suffices.

If uwv is a triangle then form G' by removing u and w . Then $\nu(G') = \nu(G) - 1$, $n'_3 = n_3 - 1$, $n'_2 = n_2 - 2$, $n'_1 = n_1 + 1$, and $c' = 1$. For (3) we get $\nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - 1/2 = n_3/4 - 1/4 + n_2/2 - 1 + n_1/4 + 1/4 - 1/2 = [n_3/4 + n_2/2 + n_1/4 - 1/2] - 1$, showing G satisfies (3). For (4) we have by induction $\nu(G') \geq 7n_3/16 - 7/16 + 3n_2/8 - 6/8 + 3n_1/16 + 3/16 - 1/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - 1$, as needed.

If u and w are in a 4-cycle $uwvz$ then by assumption (say) x has degree 3. Form G' by removing u and w , so that $\nu(G') = \nu(G) - 1$. If $d(z) = 3$ then G' has $n'_3 = n_3 - 2$, $n'_2 = n_2$, $n'_1 = n_1$, and $c' = 1$. Then using induction for (3) we find $\nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - 1/2 = (n_3 - 2)/4 + n_2/2 + n_1/4 - 1/2 = n_3/4 + n_2/2 + n_1/4 - 1/2 - 1/2$, which suffices. For (4) we get $\nu(G') \geq 7n_3/16 - 14/16 + 3n_2/8 + 3n_1/16 - 1/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - 14/16$ as required.

If $d(z) = 2$ the parameters become $n'_3 = n_3 - 1$, $n'_2 = n_2 - 2$, and $n'_1 = n_1 + 1$, giving for (3) $\nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - 1/2 = (n_3 - 1)/4 + (n_2 - 2)/2 + n_1/4 = 1/4 - 1/2 + n_3/4 + n_2/2 + n_1/4 - 1/2 - 1$ as needed. For (4) we get $\nu(G') \geq 7n_3/16 - 7/16 + 3n_2/8 - 6/8 + 3n_1/16 + 3/16 - 1/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - 1$. This completes the proof of the first statement. The second statement now follows using Lemma 11(3),(4). \square

Lemma 13. *Suppose G fails to satisfy one of (3) and (4). Then each degree-2 vertex w has two degree-3 neighbours.*

Proof. Lemma 12 tells us that w has no degree-2 neighbours. Suppose for a contradiction that w has a degree-1 neighbour x . Then (recalling G has at least four vertices) $G' = G - \{w, x\}$ has $\nu(G') = \nu(G) - 1$, $n'_3 = n_3 - 1$, $n'_2 = n_2$, $n'_1 = n_1 - 1$, and $c' = 1$. Then using induction for (3) gives $\nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - c'/2 \geq n_3/4 - 1/4 + n_2/2 + n_1/4 - 1/4 - 1/2 = [n_3/4 + n_2/2 + n_1/4 - 1/2] - 1/2$, which suffices. For (4) we get $\nu(G') \geq$

$$7n'_3/16 + 3n'_2/8 + 3n'_1/16 - c'/8 \geq 7n_3/16 - 7/16 + 3n_2/8 + 3n_1/16 - 3/16 - 1/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - 10/16. \quad \square$$

Call a degree-3 vertex $v \in G$ *good* if it has two degree-2 neighbours that do not have a common neighbour different from v . Observe that if v has three degree-2 neighbours then either v is good, or $G = K_{2,3}$, in which case (3) and (4) hold.

Lemma 14. *Suppose G fails to satisfy one of (3) and (4). Then every good vertex v of G has three degree-2 neighbours, all of which are in different components of $G - v$.*

Proof. Let w and x be degree-2 neighbours that are not adjacent and have no common neighbour other than v . As before, we write t_i for the number of degree i neighbours of v , and U for the set of degree 1 neighbours of v . Let G' be the graph obtained by removing $\{v\} \cup U$ and identifying w and x into a new vertex of degree 2. Then $\nu(G') = \nu(G) - 1$, $n'_3 = n_3 - t_3 - 1$, $n'_2 = n_2 - t_2 + t_3 + 1$, $n'_1 = n_1 - t_1 + t_2 - 2$, and $c' \leq 2 - t_1$.

The computation for (3) becomes $\nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - c'/2 \geq n_3/4 - t_3/4 - 1/4 + n_2/2 + (t_3 + 1 - t_2)/2 + n_1/4 + (t_2 - t_1 - 2)/4 - (2 - t_1)/2 = [n_3/4 + n_2/2 + n_1/4 - 1/2] + t_3/4 - t_2/4 + t_1/4 - 3/4$. Then (3) holds unless $t_2 = 3$ and $c' = 2$.

For (4) we get $\nu(G') \geq 7n'_3/16 + 3n'_2/8 + 3n'_1/16 - c'/8 \geq 7n_3/16 - 7t_3/16 - 7/16 + 3n_2/8 + 3(t_3 + 1 - t_2)/8 + 3n_1/16 + 3(t_2 - t_1 - 2)/16 - (2 - t_1)/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - (t_3 + 3t_2 + t_1 + 9)/16$, so (4) holds unless $t_2 = 3$ and $c' = 2$.

Hence in both cases we may assume that $t_2 = 3$ and so $c' = 2$. Let y be the third neighbour of v . Since $c' = 2$ we know that y is in a different component of G' (and hence of $G - v$) to w and x . In particular, y is not adjacent to w or x and does not share a second common neighbour with either of them. Thus we could apply the above argument with w and y and find that x is in a different component of $G - v$ from both w and y . This completes the proof. \square

We may now complete the proof for (3).

Lemma 15. *G satisfies (3).*

Proof. Suppose the contrary. If any degree-3 vertex has another degree-3 vertex in its neighbourhood, then we may verify (3) by considering the

graph G' obtained by deleting an edge joining two degree-3 vertices. In this case $n'_3 = n_3 - 2$, $n'_2 = n_2 + 2$, $n'_1 = n_1$ and $c' \leq 2$. Hence using induction we get $\nu(G) \geq \nu(G') \geq n'_3/4 + n'_2/2 + n'_1/4 - 2/2 = n_3/4 + n_2/2 + n_1/4 - 1/2$, proving (3) as required.

Thus we may assume no two degree-3 vertices are adjacent. Next we check that no degree-3 vertex has two degree-1 neighbours. If on the contrary x has degree-1 neighbours v and w , and a third neighbour z (which necessarily has degree 2, or else G is $K_{1,3}$ and satisfies (3)), form G' by removing v , w , and x . Then $n'_3 = n_3 - 1$, $n'_2 = n_2 - 1$, $n'_1 = n_1 - 1$, $c' = 1$ and $\nu(G) = \nu(G') + 1$. Therefore by induction $\nu(G) \geq n'_3/4 + n'_2/2 + n'_1/4 - c'/2 + 1 \geq [n_3/4 + n_2/2 + n_1/4 - 1/2] - 1/4 - 1/2 - 1/4 + 1$, showing (3) holds. Thus every degree-3 vertex has at least two degree-2 neighbours.

Suppose a degree-2 vertex w has neighbours v and z (which both have degree 3 by Lemma 13). If v is good then z is also good, since otherwise every other degree-2 neighbour of z (at least one of which exists) is also a degree-2 neighbour of v , and would therefore be in the same component of $G - v$ as w , contradicting Lemma 14. Therefore there are no good vertices at all, since otherwise (since G has at least one degree-3 vertex, in B) by Lemma 14 we would find that G is a subdivision of a connected 3-regular graph, but removing any degree-3 vertex results in 3 components. This is not possible since, in particular, every connected graph has a vertex whose removal leaves a connected graph.

Since G has no good vertices, in particular no degree-3 vertex can have three degree-2 neighbours. So every degree 3 vertex has exactly two degree 2 neighbours. It follows that G is a cycle (of even length) with a pendant edge attached to every second vertex (these are the graphs $G_3(t)$ in Example 3 in the next section). But (3) holds for this graph, completing the proof. \square

We are left to verify (4). We need one more technical lemma.

Lemma 16. *No vertex in B is good.*

Proof. Suppose on the contrary that B contains good vertices. Let $v \in B$ be a good vertex. Let W be the union of the vertex sets of all paths of the form $vw_1w_2 \dots w_r$ where $r \geq 1$, each w_i with i odd is a degree-2 vertex in A , and each w_i with i even is in B . Let H be the subgraph of G induced by W . Then H is connected.

We claim that each vertex of $W \cap B$ is good. To verify this, consider a good vertex $w \in W \cap B$ (for example $w = v$). By Lemma 11(1) we know w

has at least two neighbours u and x in A , and $d(u) = d(x) = 2$ by Lemma 14. Also, Lemma 11(2) implies that the other neighbour z of u is in B and hence is in $W \cap B$. Thus $d(z) = 3$ by Lemma 11(3). If z were not good then every degree-2 neighbour of z different from u (at least one of which exists, by Lemma 11(4)) would be a degree-2 neighbour of w , and would hence be in the same component of $G - w$ as u , contradicting Lemma 14. Hence z is good. Applying this observation repeatedly (moving along the paths used to define H) we find that every vertex of $W \cap B$ is good.

By Lemma 11(2) we know that $A \cap W$ is independent, and each $x \in A \cap W$ has exactly two neighbours in $B \cap W$. Since each $w \in B \cap W$ is good, it has three degree-2 neighbours in G by Lemma 14, at least two of which are in A by Lemma 11(1). So by Lemma 11(3) we know $B \cap W$ is independent. Therefore H is the subdivision of a connected subcubic graph J with vertex set $B \cap W$ and minimum degree at least 2. (Note that J has no multiple edges by Lemma 14 and the fact that each $w \in B \cap W$ is good.)

Since each $w \in B \cap W$ is good, the graph J has the property that $J - y$ has $d(y) \geq 2$ components for every vertex y of J . Such a graph cannot exist, so the proof is complete. \square

We may therefore assume that no vertex in B has three degree-2 neighbours. Choose $v \in B$. By Lemma 12 we have $d(v) = 3$, and by Lemma 11(4) we know that v has at least two degree-2 neighbours, say w and x . By Lemma 11(1) at least one of them, say w , is in A . Since v is not good, the other neighbour z of v is not a degree-2 vertex, and w and x have another common neighbour y . By Lemma 11(2) we know y is in B . Then by Lemma 11(3) we have that y has another neighbour u , and $d(u) \neq 2$ since y is not good. Since (4) holds for K_4 with one edge deleted, we may assume $u \neq v$. If G consists of a 4-cycle plus two pendant edges attached to non-adjacent vertices then (4) holds, so we may assume without loss of generality that z has degree 3.

If $z = u$ remove v, w, x, y . Then $n'_3 = n_3 - 3$, $n'_2 = n_2 - 2$, $n'_1 = n_1 + 1$, $c' = 1$ and $\nu(G') = \nu(G) - 2$. Then by induction $\nu(G) \geq \nu(G') + 2 \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 30/16 + 2$, which implies our result.

If u has degree 1 we remove u, v, w, x, y . Then $n'_3 = n_3 - 3$, $n'_2 = n_2 - 1$, $n'_1 = n_1 - 1$, $c' = 1$ and $\nu(G') = \nu(G) - 2$. Then by induction $\nu(G) \geq \nu(G') + 2 \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 30/16 + 2$, as needed.

Otherwise $z \neq u$, and $d(z) = d(u) = 3$. In this case we remove v, w, x, y . Then $n'_3 = n_3 - 4$, $n'_2 = n_2$, $n'_1 = n_1$, $c' \leq 2$ and $\nu(G') = \nu(G) - 2$. Then by

induction $\nu(G) \geq \nu(G') + 2 \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 30/16 + 2$, which completes the proof of Theorem 3.

4 $L \subseteq P$

The fact that $L \subseteq P$ is an immediate consequence of Theorem 5, which we prove in this section.

Suppose that $(x_3, x_2, x_1) \in L$, so there is some real number K such that

$$\nu(G) \geq x_3 n_3(G) + x_2 n_2(G) + x_1 n_1(G) - K \quad (6)$$

for every connected subcubic graph G (where $n_i(G)$ denotes the number of vertices of G of degree i). We fix a choice of (x_3, x_2, x_1) and K for the rest of this section.

We will consider six special families of graphs: each family will show that (x_3, x_2, x_1) must satisfy one of the inequalities in the definition of P . An example from each family is shown in the figures.

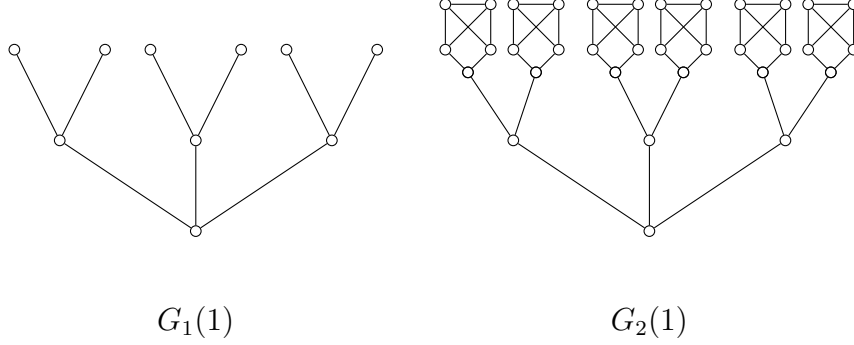
Example 1. Let t be an odd positive integer. The graph $G_1(t)$ is the tree with a root plus $t + 1$ levels, indexed by $i = 0, \dots, t$, in which level i contains $3 \cdot 2^i$ vertices, and all vertices except the leaves have degree 3. Thus $G_1(t)$ is (internally) a cubic tree and has depth $t + 1$. Then $n_1 = 3 \cdot 2^t$, $n_2 = 0$ and $n_3 = 1 + 3(2^t - 1) = 3 \cdot 2^t - 2$. Since $G_1(t)$ is bipartite with one partition class S formed by the vertices at levels $0, 2, \dots, t - 1$ we see $\nu(G_1(t)) \leq |S| = 3(4^{(t+1)/2} - 1)/3 = 2^{t+1} - 1$. By (6) we must have

$$(3 \cdot 2^t - 2)x_3 + 3 \cdot 2^t x_1 - K \leq 2 \cdot 2^t - 1,$$

and so, dividing by $3 \cdot 2^t$ and letting $t \rightarrow \infty$, we see that

$$x_3 + x_1 \leq 2/3.$$

Example 2. Let J denote the graph obtained by subdividing one edge of K_4 , and let x denote the single vertex of degree 2 in J . We define the graph $G_2(t)$, again for odd t , by identifying each leaf in $G_1(t)$ with the vertex x in a copy of the graph J , such that all copies are disjoint from each other and the rest of the graph. Then for this graph $n_1 = n_2 = 0$, and $n_3 = 3 \cdot 2^t - 2 + 15 \cdot 2^t = 9 \cdot 2^{t+1} - 2$. The same set S as before now has the property that removing it leaves $1 + 3(2 + 2^3 + \dots + 2^t) = 1 + 6(4^{(t+1)/2} -$



$1)/3 = 2^{t+2} - 1$ odd components. Therefore any maximum matching in G must leave exposed at least $2^{t+2} - 1 - |S| = 2^{t+1}$ vertices. This tells us $\nu(G_2(t)) \leq (9 \cdot 2^{t+1} - 2 - 2^{t+1})/2 = (2^{t+4} - 2)/2 = 2^{t+3} - 1$. So (6) implies that

$$(9 \cdot 2^{t+1} - 2)x_3 - K \leq 2^{t+3} - 1.$$

Dividing by $9 \cdot 2^{t+1}$ and taking a limit gives

$$x_3 \leq 4/9.$$

Example 3. Let $t \geq 2$ be a positive integer. The graph $G_3(t)$ is obtained from the cycle with $2t$ vertices by attaching a pendant edge to every second vertex. Then $n_1 = n_2 = n_3 = t$. The graph is bipartite with one vertex class consisting of the vertices of degree 3, so $\nu(G_3(t)) \leq n_3 = t$. Thus

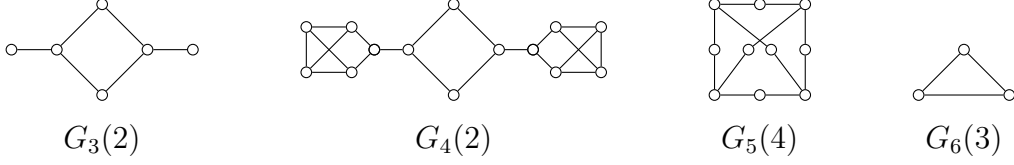
$$x_3 t + x_2 t + x_1 t - K \leq t.$$

Dividing by t and taking a limit gives

$$x_3 + x_2 + x_1 \leq 1.$$

Example 4. The graph $G_4(t)$ is obtained from $G_3(t)$ by adding t disjoint copies of J , identifying the vertex x in each copy with the leaf of a pendant edge. Then $n_1 = 0$, $n_2 = t$ and $n_3 = 6t$. The set of degree-3 vertices on the cycle has size t and leaves $2t$ odd components when deleted, showing $\nu(G_4(t)) \leq (7t - t)/2 = 3t$. Thus

$$6x_3 t + x_2 t - K \leq 3t.$$



Dividing by $6t$ and taking a limit gives

$$x_3 + x_2/6 \leq 1/2.$$

Example 5. For each even integer $t \geq 4$, let $G_5(t)$ be obtained from a cubic graph H on t vertices by subdividing every edge of H exactly once (for sake of definiteness, we may take H to be a cycle of length t with opposite vertices joined). Then $n_1 = 0$, $n_2 = e(H) = 3t/2$ and $n_3 = t$. Then $G_5(t)$ is bipartite with one vertex class $V(H)$ of size t , so $\nu(G) \leq t$. Thus

$$x_3t + 3x_2t/2 - K \leq t.$$

Dividing by t and taking a limit gives

$$x_3 + 3x_2/2 \leq 1.$$

Example 6. Finally, for odd integers $t \geq 3$, we let $G_6(t)$ be the odd cycle of length t . Then $n_1 = n_3 = 0$ and $n_2 = t$, while $\nu = (t-1)/2$. Thus

$$x_2t/2 - K \leq t/2 - 1/2.$$

Dividing by $t/2$ and taking a limit gives

$$x_2 \leq 1/2.$$

The proof of Theorem 5 is now immediate.

Proof of Theorem 5. If $(x_3, x_2, x_1) \notin P$ then it fails to satisfy one of the inequalities used to define P . Therefore, taking the example above that corresponds to this inequality (and noting that all the examples are connected) we see that by taking t large we can force K to be arbitrarily large. \square

In fact it is easy to see that equality holds in each expression bounding $\nu(G_i(t))$, but we do not need this fact. Finally, we note that Example 1 is sharp for (2) and (5); Example 2 is sharp for (5); and Example 6 is sharp for (1) and (3).

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