



Second weak order explicit stabilized methods for stiff stochastic differential equations

by

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February 20, 2012

Abstract

We introduce a new family of explicit integrators for stiff Itô stochastic differential equations (SDEs) of weak order two. These numerical methods belong to the class of one-step stabilized methods with extended stability domains and do not suffer from stepsize reduction that standard explicit methods face. The family is based on the classical stabilized methods of order two for deterministic problems and its construction relies on the strategy of modified equations recently introduced for SDEs. The convergence and the stability properties of the methods are analyzed. Numerical experiments, including applications to nonlinear SDEs and parabolic stochastic partial differential equations (SPDEs), are presented and confirm the theoretical results.

Keywords: Stiff SDEs, explicit stochastic methods, stabilized methods, Runge-Kutta Chebyshev, S-ROCK, modified equations.

AMS subject classification (2010): 65C30, 60H35, 65L20

1 Introduction

Multiscale differential equations arise in the modelling of many important problems in science and engineering. For the numerical simulation of such problems the use of explicit methods is often expensive because of the time step reduction due to stability issues. The problems under consideration (called stiff) are mean-square stable problems [16] with multiple scales for which classical explicit methods face a severe step size restriction [14, 19]. Such problems are usually solved numerically by (semi)-implicit methods, since classical explicit methods, for example the well-known Euler-Maruyama method, face severe time step reduction. This comes at the cost of solving linear or nonlinear algebraic systems at each step, which can be expensive for large systems and difficult to implement for complex problems. Recently, a new class of explicit stabilized methods called S-ROCK¹ has been introduced for stiff problems [2, 3, 5]. On one hand, these methods (fully explicit) are as easy to implement as the Euler-Maruyama method. On the other hand, their extended mean-square stability domains (for

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¹The acronym ROCK comes from the orthogonal Runge-Kutta Chebyshev introduced in [6, 1] and based on the coupling of stabilization stages and a finishing procedure to obtain the required accuracy. Analogously the S-ROCK are based on stabilization stages and a finishing procedure for the simulation of the appropriate noise.

suitable test problems) make them much more efficient than classical explicit methods for stiff problems.

Up to now, with the exception of [12], only weak first order stabilized methods have been proposed for stiff stochastic problems. In [12] an attempt to generalize the S-ROCK methods to second weak order has been proposed. However, this generalization involves the solution of a large number of order conditions and the resulting methods appear to have less favorable stability properties than the methods proposed in [2, 3, 5]. Departing from the traditional methodology which consists in comparing Taylor series of exact and numerical solutions [25, 35, 22, 21, 27], using the theory of rooted trees [10, 31], or using ideas based on extrapolation [36, 20], we use here the methodology of modified equations recently introduced for SDEs [4] to derive two new families of weak second order explicit stabilized methods, what we call S-ROCK2A and S-ROCK2B. This allows to by-pass the order conditions theory [31] to construct weak second order methods. The idea of modified equations is also important for deriving methods with certain additional properties as for example stabilization as proposed here, or conservation of first integrals as derived in [4]. Indeed, for a given SDE integrator (the basic method), the methodology of modified equations allows to construct a more accurate integrator based on the basic method. If this basic method has some additional properties (e.g., as listed above), then the more accurate integrator can take advantage of them. We close this introduction by mentioning that in the same spirit as our S-ROCK2 methods, a partitioned method based on RKC was recently proposed in [41] with extended stability domains in the imaginary axis for the time-integration of deterministic diffusion-advection-reaction problems.

This paper is organized as follows. In Section 2, we present all the material needed for introducing the concept of stabilized methods for stiff SDEs. In Section 3, we present our new weak second order explicit stabilized integrators and discuss their mean-square stability properties, while in Section 4 we present various numerical experiments in multiple dimensions, both for linear and non-linear stiff SDEs.

2 Stabilized stochastic methods

In this section we recap all the material needed for introducing the concept of stabilised methods for stiff SDEs. We start by briefly introducing the concepts of weak convergence and mean-square stability for stochastic integrators in Sections 2.1, 2.2 respectively. We then recall in Section 2.3 the construction of explicit stabilized methods for ordinary differential equations (ODEs). Finally, in Section 2.4 we briefly discuss the construction of first order weak S-ROCK methods.

2.1 Weak stochastic integrators

Given a probability space $(\Omega, \mathcal{F}, P)$, with a filtration $\mathcal{F}_{t \geq 0}$ fulfilling the usual conditions, we consider the Itô stochastic system of differential equations

$$dX = f(X)dt + g(X)dW(t), \quad X(0) = X_0, \quad (1)$$

where $X(t)$ is a random variable with values in $\mathbb{R}^d$, $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the drift term, $g : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ is the diffusion term with $d \times m$ matrix values, and $W(t) = (W_{[1]}(t), \dots, W_{[m]}(t))^T$ is a standard m -dimensional Wiener process. The drift and diffusion functions are assumed smooth enough, Lipschitz continuous and to satisfy a growth bound in order to ensure a

unique (mean-square bounded) solution of (1) [7, 19]. For the numerical approximation of (1) we consider the discrete map

$$X_{n+1} = \Psi(f, g, X_n, h, \xi_n), \quad (2)$$

where $\Psi(f, g, \cdot, h, \xi_n) : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $X_n \in \mathbb{R}^d$ for $n \geq 0$, h denotes the timestep size, and ξ_n denotes a random vector. We recall two concepts of accuracy and stability for the numerical integration of SDEs. A numerical approximation (2), starting from the exact initial condition X_0 of (1) is said to have weak order r if there exists a constant C such that

$$|\mathbb{E}(\phi(X_N)) - \mathbb{E}(\phi(X(t_N)))| \leq Ch^r, \quad (3)$$

and to have strong order r if there exists a constant C such that

$$\mathbb{E}|X_N - X(t_N)| \leq Ch^r. \quad (4)$$

Both (3) and (4) must hold for any $t_N = Nh \in [0, \tau]$ for a fixed $\tau > 0$, for all h sufficiently small. In addition (3) must hold for all functions $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \in C_P^{2(r+1)}(\mathbb{R}^d, \mathbb{R})$, with a constant C independent of h . Here and in what follows, $C_P^\ell(\mathbb{R}^d, \mathbb{R})$ denotes the space of ℓ times continuously differentiable functions $\mathbb{R}^d \rightarrow \mathbb{R}$ with all partial derivatives with polynomial growth.

Remark 2.1. A well-known theorem of Milstein [26] allows to infer the weak order from the error after one step. Assuming that f, g are Lipschitz continuous and satisfy $f, g \in C_P^{2(r+1)}(\mathbb{R}^d, \mathbb{R}^d)$, that the moments of the exact solution of the SDE (1) exist and are bounded (up to a sufficiently high order), and that $\phi \in C_P^{2(r+1)}(\mathbb{R}^d, \mathbb{R})$, then, the local error bound

$$|\mathbb{E}(\phi(X_1)) - \mathbb{E}(\phi(X(t_1)))| \leq Ch^{r+1} \quad (5)$$

for all initial values $X(0) = X_0$ and for all h sufficiently small implies the global error bound (3). Here the constant C is again independent of h . For the strong convergence we have the following result [24]: if

$$\mathbb{E}|X_1 - X(t_1)| \leq Ch^{r+1/2}, \quad \text{and} \quad |\mathbb{E}(X_1) - \mathbb{E}(X(t_1))| \leq Ch^{r+1}, \quad (6)$$

then the global error bound (4) holds.

The simplest method to approximate solutions to (1) is the so-called Euler-Maruyama method

$$X_{n+1} = X_n + hf(X_n) + g(X_n)\Delta W_n, \quad (7)$$

where ΔW_n is an m dimensional vector, whose components $\Delta W_{n,[j]} = W_{[j]}(t_{n+1}) - W_{[j]}(t_n)$ are independent Wiener increments. This method has strong order 1/2 and weak order 1 [23].

2.2 Mean-square stable stiff integrators

In practice it is not only the order of convergence that will guarantee an efficient approximation of an SDE, but also the long-time behavior of the solution. Stability properties of the exact solution and the numerical method are important to understand this behavior. Widely

used characterizations of stability for SDEs are mean-square stability and asymptotic stability (in the large) [7, 15]. The former measures the stability of moments, the latter measures the overall behavior of sample paths. In this paper, we focus on mean-square stability (for linear autonomous system of SDEs it implies asymptotic stability in the large [7, 15]).

The steady solution $X \equiv 0$ of the SDE (1) with $f(t, 0) = g(t, 0) = 0$ is said to be mean-square stable if there exists $\delta > 0$ such that

$$\lim_{t \rightarrow \infty} \mathbb{E}|X(t)|^2 = 0, \text{ for all } |X_0| < \delta. \quad (8)$$

Scalar linear SDEs. To gain insight on the stability behavior of a numerical method, we consider a class of linear scalar test problems widely used in the literature [33, 16, 11, 37, 8]

$$dX = \lambda X dt + \mu X dW(t), \quad X(0) = 1, \quad (9)$$

in dimensions $d = m = 1$, with fixed complex scalar parameters λ, μ . The solution of (9), given by $X(t) = \exp((\lambda + \frac{1}{2}\mu^2)t + \mu W(t))$, is found to be mean-square stable if and only if

$$\lim_{t \rightarrow \infty} \mathbb{E}(|X(t)|^2) = 0 \iff (\lambda, \mu) \in \mathcal{S}_{\text{SDE}} := \{(\lambda, \mu) \in \mathbb{C}^2 ; \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0\}. \quad (10)$$

We will call the domain $\mathcal{S}_{\text{SDE}}$ the stability domain of the test equation (9). If we apply a numerical method (2) to (9), square the result and take the expectation, we obtain after one step a relation of the form

$$\mathbb{E}(|X_{n+1}|^2) = R(p, q)\mathbb{E}(|X_n|^2), \quad (11)$$

where $p = h\lambda, q = \sqrt{h}\mu$, and $R(p, q)$ is called the stability function of the method. We say that a numerical method is mean-square stable for the test problem (9) if and only if

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n|^2) = 0 \iff (h\lambda, \sqrt{h}\mu) \in \mathcal{S}_{\text{num}} := \{(p, q) \in \mathbb{C}^2 ; R(p, q) < 1\}. \quad (12)$$

In order to be able to visualize the stability region, we now restrict ourself in our numerical studies to the case $\lambda, \mu \in \mathbb{R}$. For example, for the Euler-Maruyama method (7) we have

$$R(p, q) = |1 + p|^2 + q^2, \quad \mathcal{S}_{\text{EM}} := \{(p, q) \in \mathbb{C}^2 ; |1 + p|^2 + q^2 < 1\}, \quad (13)$$

with a stability domain (for $\lambda, \mu \in \mathbb{R}$) being a disk of radius 1 centred at $(p, q) = (-1, 0)$. It is possible to construct implicit methods such that $\mathcal{S}_{\text{SDE}} \subseteq \mathcal{S}_{\text{num}}$ [16], i.e., to construct implicit numerical methods that are mean-square stable whenever the exact solution is mean-square stable. This is impossible for explicit methods as their stability domains are finite. To characterize the stability domains of explicit methods we thus introduce the following “portion of the true stability domain”

$$\mathcal{S}_{\text{SDE}, \ell} = \{(p, q) \in [-\ell, 0) \times \mathbb{R} ; |q| < \sqrt{-2p}\}, \quad (14)$$

and define for a given method

$$L = \sup\{\ell > 0 ; \mathcal{S}_{\text{SDE}, \ell} \subset \mathcal{S}_{\text{num}}\}. \quad (15)$$

For the Euler-Maruyama method we have $L_{\text{EM}} \simeq 1/4$ [5]. The S-ROCK methods introduced in [2, 3] and studied in [5] for Itô-SDEs are explicit methods with much larger values of L . This family of methods (briefly described in Section 2.4) are constructed with various number

of stages s . The value of L can then be shown to increase quadratically with s , i.e., $L \simeq Cs^2$, while the work (measured in term of function evaluations) only increases linearly with s .

Multidimensional linear SDEs. The extension of the previous stability study for multidimensional linear systems

$$dX = AXdt + \sum_{r=1}^m B_r X dW_{[r]}(t), \quad X(0) = X_0, \quad (16)$$

where A is a $d \times d$ matrix, B_i are $d \times d$ matrices and $dW_{[r]}(t)$ are independent one-dimensional Wiener processes, is difficult in general. Indeed, while for $B \equiv 0$ (deterministic case) transformation into canonical form (Jordan or diagonal) justifies the use of the scalar linear test equation for studying the linear stability of numerical integrators (at least in the asymptotic regime), for SDEs no such transformation is possible in general as one cannot simultaneously put into diagonal form the matrices A and $B_1, \dots, B_r$ if these matrices do not commute. This well-known issue has triggered extensive work by many authors. A first generalization has been considered in [34] to study the mean-square stability of the Euler-Maruyama method applied to linear systems of the form (16) in dimensions $d = 2, m = 1$,

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \quad (17)$$

with $\lambda_1, \lambda_2, \sigma_{ij} \in \mathbb{R}$, $i, j = 1, 2$ and a single Wiener process. The approach in [34] is based on the study of the deterministic matrix differential system $Y'(t) = \Omega Y(t)$, where $Y(t)$ is obtained from a suitable transformation of the matrix $\mathbb{E}(X(t)X(t)^T)$ and the mean-square stability of the system (16) is guaranteed if a suitable matrix logarithmic norm of Ω is negative. This approach has been generalized in [30] for higher dimensional system, but does not allow an easy characterization of a stability criterion (see e.g., [30, Equ. (37)]). A second attempt to generalize the linear test equation was presented recently in [9]. Inspired by the theory of stochastic stabilization and destabilization, it is proposed to consider the following two sets of linear test equations of the type (16), in dimensions $d = m = 2$ and $d = m = 3$, respectively, with matrices given by

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}, \quad (18)$$

and

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \quad (19)$$

where $\lambda, \sigma, \varepsilon \in \mathbb{R}$. We notice that for both systems (18)-(19) the matrix A commutes with any of the matrices B_i . In fact the matrices in (18) can even be put simultaneously into diagonal form, and thus this system is a special case of the scalar linear test equation

$$dX = \lambda dX + \mu_1 X dW_{[1]}(t) + \mu_2 X dW_{[2]}(t).$$

The findings in [9] for the θ -Maruyama method show that the mean-square stability results for the scalar test equation (9) give the essential behavior (up to the noise intensity) for the method applied to (16) with (18) or (19). For the new S-ROCK methods developed in this paper, the test equation corresponding to (17) seems too complicated to give a guidance on the construction of stabilized methods. We will however see that the study of (9), (18) and (19) is not sufficient to give a guidance on the behavior of the S-ROCK methods for multi-dimensional linear systems. Our strategy is as follows: we will develop two-classes of weak second order stabilized methods, called S-ROCK2A and S-ROCK2B with optimized mean-square stability properties with respect to the test equation (9). Both families of methods will share *comparable* values of L , the length measure (15) of the stability domain. We will then show by considering a linear system of the type (16), that there is a fundamental issue with the family of methods S-ROCK2A for general systems of linear equations, when the matrices A and B_r do not commute. This issue is removed with the family of methods S-ROCK2B and various numerical tests with multi-dimensional systems of SDEs indicate that these methods are efficient also for multi-dimensional systems of stiff SDEs.

2.3 Deterministic stabilized methods

We start by briefly reviewing the use of stabilized methods (also called Chebyshev methods) in the case of ODEs. Such methods are explicit Runge-Kutta (RK) methods with extended stability domains along the negative real axis. These methods are suitable for large systems of stiff dissipative ODEs, as they do not need the solution of large linear or nonlinear systems at each step size (as e.g., implicit methods). At the same time, due to their extended stability domains along the negative real axis, stabilized methods have less severe step size restriction than classical explicit methods when solving stiff (dissipative) problems and permit the use of large timesteps. The earliest development goes back to the sixties and various types of stabilized methods have been proposed since then (see e.g., [14]). Useful for our new S-ROCK methods are the following types of stabilized methods introduced in [39].

First order Chebyshev methods. Consider a system of ordinary differential equations

$$\frac{dX}{dt} = f(X), \quad X(0) = X_0. \quad (20)$$

Given an integer $s \geq 1$ (called the number of stages) and a real parameter $\eta \geq 0$ (called the damping parameter), we define the following Runge-Kutta method (first order Chebyshev method) with step size h

$$X_{n+1} = \varphi_{h,s}(X_n), \quad (21)$$

by the following explicit recursion

$$\begin{aligned} K_0 &= X_n, \\ K_1 &= X_n + h \frac{\omega_1}{\omega_0} f(K_0), \\ K_j &= 2h \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} f(K_{j-1}) + 2\omega_0 \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} K_{j-1} - \frac{T_{j-2}(\omega_0)}{T_j(\omega_0)} K_{j-2}, \quad j = 2, \dots, s, \\ X_{n+1} &= K_s, \end{aligned}$$

with

$$\omega_0 = 1 + \frac{\eta}{s^2}, \quad \omega_1 = \frac{T_s(\omega_0)}{T'_s(\omega_0)}.$$

We note here that η is usually fixed (and small) for deterministic problems, but it shall depend on s for S-ROCK methods.

Applied to the linear test problem $dX(t)/dt = \lambda X(t)$ the method (21) gives $X_{n+1} = R_s(z)X_n$, where $z = \lambda h$ and where $R_s(z)$, called the stability function (polynomial) of the method, is given by

$$R_s(z) = \frac{T_s(\omega_0 + \omega_1 z)}{T_s(\omega_0)}. \quad (22)$$

We emphasize that (21) denotes in fact a family of methods indexed by the stage number s . A crucial property of the methods (21) is that

$$|R_s(z)| \leq 1 \quad \text{for all } z \in [-l_s, 0], \quad (23)$$

with $l_s \simeq C \cdot s^2$, for s large enough, where C depends on the damping parameter η (for $\eta = 0$, $C = 2$). Thus the stability domain

$$\mathcal{S} := \{z \in \mathbb{C}; |R(z)| \leq 1\} \quad (24)$$

of the methods increase quadratically with s on the negative real axis.

For a stiff problem (20) whose Jacobian has eigenvalues close to the real negative axis with a spectral radius given by Λ , the stability condition on the stepsize h of a classical explicit method is $h \leq \hat{C}/\Lambda$. Assuming that the requirement on the accuracy dictates a step size Δt , the aforementioned stability condition leads to $\Delta t/h = \Delta t\Lambda/\hat{C}$ function evaluations per stepsize Δt . For a Chebyshev method with a stability interval along the negative real axis of length $l_s = C \cdot s^2$ we can choose $\Delta t\Lambda = C \cdot s^2$ which gives $s = \sqrt{\Delta t\Lambda/C}$ function evaluations, the square root of the cost of the classical explicit method (the constants $C, \hat{C}$ are both of moderate size and comparable, e.g., for the first order Chebyshev method without damping and for the Euler explicit method we have $C = \hat{C} = 2$).

Second order Chebyshev method. The above Chebyshev method can be extended to a second order method [39]. The method denoted

$$X_{n+1} = \tilde{\varphi}_{h,s}(X_n), \quad (25)$$

is defined by the recursion

$$\begin{aligned} \tilde{K}_0 &= X_n, \\ \tilde{K}_1 &= X_n + \kappa_1 h f(\tilde{K}_0), \\ \tilde{K}_j &= X_n + \mu_j (\tilde{K}_{j-1} - X_n) + \nu_j (\tilde{K}_{j-2} - X_n) + \kappa_j h f(\tilde{K}_{j-1}) - a_{j-1} \kappa_j h f(X_n), \\ X_{n+1} &= \tilde{K}_s, \end{aligned}$$

where $j = 2, \dots, s$ and

$$\begin{aligned} \omega_0 &= 1 + \frac{\eta}{s^2}, \quad \omega_1 = \frac{T'_s(\omega_0)}{T''_s(\omega_0)}, \quad \kappa_1 = \omega_1 \frac{T''_2(\omega_0)}{(T'_2(\omega_0))^2} = \frac{\omega_1}{4\omega_0^2}, \\ b_j &= \frac{T''_j(w_0)}{(T'_j(w_0))^2}, \quad \mu_j = \frac{2b_j w_0}{b_{j-1}}, \quad \nu_j = \frac{-b_j}{b_{j-2}}, \quad \kappa_j = \frac{2b_j w_1}{b_{j-1}}, \quad j = 2, \dots, s \\ b_0 &= b_1, \quad b_1 = b_2, \quad a_j = 1 - b_j T_j(w_0), \quad j = 1, \dots, s. \end{aligned}$$

The stability function of the method (25) is given by

$$\tilde{R}_s(z) = a_s + b_s T_s(\omega_0 + \omega_1 z). \quad (26)$$

As for the family of methods (21), the stability domains of (25) increase quadratically along the negative real axis with the stage number s and we have again $l_s \simeq C \cdot s^2$. The constant C has a smaller value for second order methods [39], and for $\eta = 0$ its value is $C \simeq 2/3$.

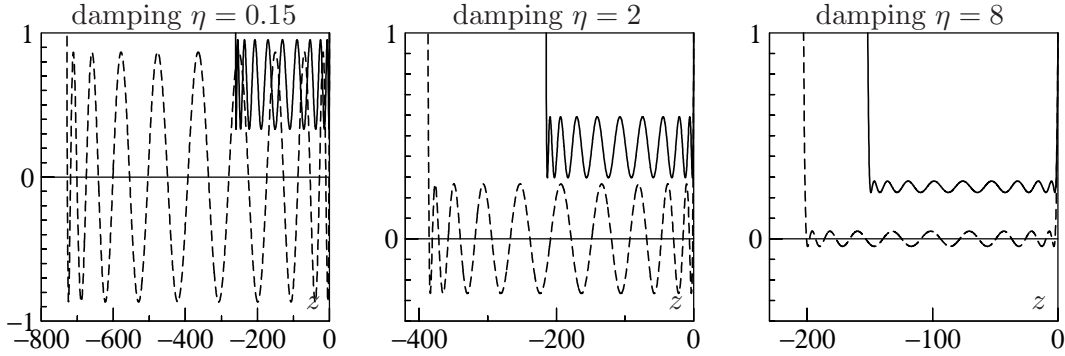


Figure 1: Comparison of the stability functions $R_s(z)$ (dashed lines) and $\tilde{R}_s(z)$ (solid lines) with $s = 20$ stages for various values of the damping parameter η .

The idea for the construction of S-ROCK methods is to consider large values of the damping parameter η , instead of small values traditionally considered for deterministic integrators. In Figure 1, we plot the stability functions $R_s, \tilde{R}_s$ given in (22),(26) as a function of $z \in \mathbb{R}^-$ with $s = 20$ stages for various values of the damping parameter, while in Figure 2 we plot the stability domains (24) in the complex plane for $s = 20$ and different values of the damping parameter. It can be observed in Figure 1 that the amplitude of the oscillations decreases in both cases as the damping parameter increases. However, we notice that R_s oscillates around zero (see dashed lines), while $\tilde{R}_s$ oscillates around the positive constant a_s (see solid lines). We will later see that this behaviour of $\tilde{R}_s$ has serious implications on the construction of higher order methods suitable for the integration of stochastic stiff problems.

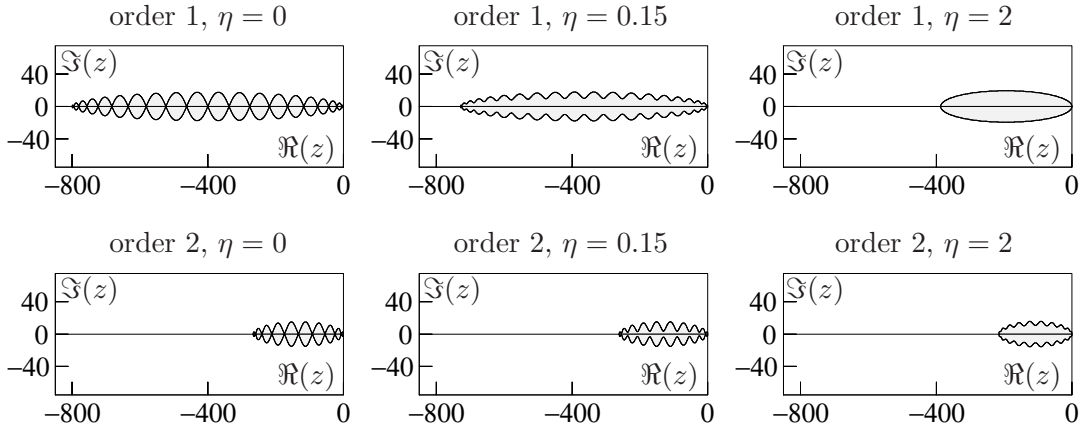


Figure 2: Comparison of the stability domains $\mathcal{S}$ for the deterministic Chebyshev methods of order 1 (top pictures) and order 2 (bottom pictures), defined in (21),(25), for various values of the damping parameter η with $s = 20$.

2.4 Stochastic stabilized S-ROCK methods of weak order one

We recall here the weak order one S-ROCK methods introduced in [5], and based on the first order Chebychev method (21). The main idea is to consider large values of the damping parameter η . The simplest method considered in [5] of strong order 1/2 and weak order 1, denoted here S-ROCK(1/2,1) is given by ²

$$X_{n+1} = \varphi_{h,s}(X_n) + g(\varphi_{h,s})\Delta W_n, \quad (27)$$

Another method considered in [5] of strong order 1 and weak order 1, denoted here S-ROCK(1,1), is given by

$$X_{n+1} = \varphi_{h,s}(X_n) + g(\varphi_{h,s})\Delta W_n + M(\varphi_{h,s}). \quad (28)$$

The Milstein term $M(x)$ (a vector of dimension d) is defined as

$$M_{[i]}(x) = \sum_{j,k=1}^m \sum_{l=1}^d \frac{\partial g_{[i,j]}}{\partial x_l}(x) g_{[l,k]}(x) I_{(j,k)}, \quad \text{for all } i = 1, \dots, d. \quad (29)$$

where $I_{(j,k)}$ is the multiple stochastic integral

$$I_{(j,k)} = \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^s dW_k(t) \right) dW_j(s), \quad (30)$$

and $g_{[i,j]}$ is the (i, j) -th entry in the $d \times m$ matrix g .

Remark 2.2. *In the case $n = 1$, we have $M(x) = \frac{1}{2}g'(x)g(x)(\Delta W^2 - h)$, since then the calculation of (30) becomes trivial. In higher dimensions, however, the multiple integral matrix (30) is difficult to evaluate numerically in general and needs to be approximated. The choice behind this approximation relies on the type of convergence that one is interested in, see [19]. Notice that one can also replace the Gaussian variables $\Delta W_{n,[j]} \sim \mathcal{N}(0, h)$ by appropriate discrete random variables and still retain the weak second order. Furthermore, it is possible to obtain derivative free versions of the S-ROCK methods (28), (27) by approximating the derivatives by appropriate finite differences as in [32].*

Applied to the test problem (9) the stability functions in (11) of the methods (27) and (28) are respectively given by

$$R_{s,(1/2,1)}(p, q) = R_s^2(p)(1 + q^2), \quad (31)$$

$$R_{s,(1,1)}(p, q) = R_s^2(p) \left(1 + q^2 + \frac{q^4}{2} \right). \quad (32)$$

It should be noted that the methods (27) and (28) depend on the value of the damping parameter η . Consider the stability domain $\mathcal{S}_{num}$ in (12). Then by increasing the value of η one can increase the width of the stability region of the methods (27) and (28) in the q direction [5]. The task is then to optimize the value of η for each stage number s to obtain the largest portion (defined by the parameter L in (15)) of the true stability domain $\mathcal{S}_{SDE,\ell}$

²We notice that the methods considered in [5] had a slightly different form with $g(K_{s-1})$ instead of $g(\varphi_{h,s})$, where K_{s-1} is the internal stage of index $s - 1$ defined in (22). This modification does not affect the order of convergence and the mean-square stability property of the methods.

as defined in (14) included in the stability domain of the considered numerical method (see also the discussion in Section 3.3.1). The values of L (depending on s) for the methods (27) or (28) are given by

$$L_s \simeq 0.33 \cdot s^2, \quad L_s \simeq 0.19 \cdot s^2,$$

for all s large enough. One thus sees that the portion of the true stability domain included in the stability domains of the first order S-ROCK methods increases quadratically (along the p direction) with s , and thus these methods are much more efficient than classical explicit one-step methods for stiff stochastic SDEs (we refer to [5] for numerical tests and comparisons).

3 S-ROCK methods of weak order two

To construct higher weak order S-ROCK methods we need to address two issues

- satisfy the order conditions for weak second order methods;
- construct a method with good mean-square stability properties.

On the one hand, starting from a general Runge-Kutta methods, numerous order conditions have to be fulfilled to achieve weak second order (see [32, Thm. 5.1]). On the other hand, to obtain good mean-square stability properties, one needs to integrate suitable damped Chebyshev polynomials in the drift and diffusion terms as shown in [2, 5] for the first order methods.

3.1 Preliminaries

To address both aforementioned issues, we will use our recently introduced framework on modified equations [4].

Integrators based on modified equations. We recall the basic idea for the Milstein method and for a one-dimensional SDE with scalar noise. Consider the Milstein method

$$X_{n+1} = X_n + hf(X_n) + g(X_n)\Delta W_n + \frac{1}{2}g'(X_n)g(X_n)((\Delta W_n)^2 - h), \quad (33)$$

where ΔW_n are independent $\mathcal{N}(0, h)$ distributed random variables. The Milstein method has weak order one. To construct a second order method with this basic method as a building block, the idea is as follows. We consider the modified equation

$$dX = [f(X) + hf_1(X)] dt + [g(X) + hg_1(X)] dW(t), \quad X(0) = X_0. \quad (34)$$

According to [4, Thm. 2.3] if we choose

$$f_1(x) = \frac{1}{2}f'(x)f(x) + \frac{1}{4}f''(x)g^2(x), \quad (35a)$$

$$g_1(x) = \frac{1}{2}f'(x)g(x) + \frac{1}{2}g'(x)f(x) + \frac{1}{4}g^2(x)g''(x), \quad (35b)$$

then the integrator (33) applied to (34), i.e.,

$$X_{n+1} = X_n + hf_{h,1}(X_n) + g_{h,1}(X_n)\Delta W_n + \frac{1}{2}g'(X_n)g(X_n)((\Delta W_n)^2 - h), \quad (36)$$

where

$$f_{h,1}(x) = f(x) + hf_1(x), \quad g_{h,1}(x) = g(x) + hg_1(x),$$

yields a weak second order method for the original equation (1). Notice that for obtaining the weak order two, it is not required to perturb the diffusion function g in the Milstein term $\frac{1}{2}g'(X_n)g(X_n)$. The advantage of this approach of modified equations for constructing weak high order integrators, is that it permits to fulfil automatically the weak order conditions.

Weak second order S-ROCK methods based on modified equations. The weak second order (explicit) method (36), known as the classical Milstein-Talay method, has only a small stability domain (12). As mentioned earlier, to increase its mean-square stability domain, we have to stabilize the method, and we introduce Chebyshev polynomials with damping. The first idea is simply to replace (33) by the first weak order S-ROCK method (28), which yields the basic integrator

$$X_{n+1} = \varphi_{h,s}(X_n) + g(\varphi_{h,s}(X_n))\Delta W_n + \frac{1}{2}g'(\varphi_{h,s}(X_n))g(\varphi_{h,s}(X_n))((\Delta W_n)^2 - h), \quad (37)$$

where we recall that $\varphi_{h,s}(X_n) = X_n + hf(X_n) + \mathcal{O}(h^2)$. But applying this method to the corresponding modified problem (34) to achieve weak order two yields stability polynomials in the recursion (21) that have poor stability properties along the negative real axis due to the term $f'f$ in (35a). This term, that encodes the second order for deterministic problems has to be built in the basic method in order to control the deterministic stability behavior. The next idea is to consider

$$X_{n+1} = \tilde{\varphi}_{h,s}(X_n) + g(\tilde{\varphi}_{h,s}(X_n))\Delta W_n + \frac{1}{2}g'(\tilde{\varphi}_{h,s}(X_n))g(\tilde{\varphi}_{h,s}(X_n))((\Delta W_n)^2 - h), \quad (38)$$

where $\tilde{\varphi}_{h,s}$ is the second order Chebyshev method (25). Again, by applying this method to the corresponding modified problem (34), weak order two can be obtained. But since $\tilde{R}_s$ oscillates around the positive constant a_s (see Figure 1) this method does not have enough damping to include large portions (14) of the true stability domain (10) in the stability domain of the stochastic method. It has however large stability domains along the p axis (growing quadratically with the degree s) and the method (38) could still be useful for problems with small noise.

3.2 The S-ROCK2A methods

We are now ready to construct our new weak second order S-ROCK methods. We consider here multi-dimensional SDEs (1), and replace the Milstein term $\frac{1}{2}g'(x)g(x)((\Delta W_n)^2 - h)$ by its multidimensional expression (29). Here ΔW_n denotes an m -dimensional vector with components being independent $\mathcal{N}(0, h)$ distributed random variables. We start with the S-ROCK2A method. In view of the discussion in Section 3.1 we consider the following basic integrator

$$X_{n+1} = \tilde{\varphi}_{h,s_2}(X_n) + g(\varphi_{\theta h,s_1}(X_n))\Delta W_n + M(\varphi_{\theta h,s_1}(X_n)), \quad (39)$$

where we take different Chebyshev methods for the drift and diffusion terms. We also allow for different stage numbers s_1, s_2 and stepsizes in the Chebyshev methods and we consider for this purpose a fixed parameter $0 \leq \theta \leq 1$. It is easy to verify that the method (39) has weak and strong order 1.

Lemma 3.1. *For a fixed θ , the numerical integrator (39) applied to the modified SDE*

$$dX = (f(X) + hf_1(X))dt + (g(X) + hg_1(X))dW(t), \quad (40)$$

yields a numerical integrator of weak order two for (1), where f_1, g_1 are given (component-wise) by

$$f_{1,[i]} = \frac{1}{4}gg^T : f''_{[i]}, \quad g_{1,[i,j]} = \frac{1}{2}(f'g)_{[i,j]} + \left(\frac{1}{2} - \theta\right)g'_{[i,j]}f + \frac{1}{4}gg^T : g''_{[i,j]}. \quad (41)$$

for all $i = 1, \dots, d$ and $j = 1, \dots, m$.

Here f, f_1 are column vectors of size d (with i th component denoted by $\cdot_{[i]}$) and g, g_1 matrices of size $d \times m$ (with entries denoted by $\cdot_{[i,j]}$), and we consider in (41) the usual scalar product on square matrices $A : B = \text{Trace}(A^T B)$. Also, $f''_{[i]}$ and $g''_{[i,j]}$ (sizes $d \times d$) denote usual Hessian matrices with respect to x .

Proof. Observing that

$$\tilde{\varphi}_{h,s_2}(x) = x + hf(x) + \frac{h^2}{2}f'(x)f(x) + \mathcal{O}(h^3), \quad \varphi_{\theta h,s_1}(x) = x + \theta hf(x) + \mathcal{O}(h^2),$$

and following the lines of the derivation of a weak second order method based on modified equation in [4, Sect. 3.1.2] gives the proof. $\square$

An immediate consequence of the above lemma is the following result.

Theorem 3.2. *For a fixed parameter θ , the scheme*

$$X_{n+1} = \tilde{\varphi}_{h,s_2}(X_n) + g(\varphi_{\theta h,s_1}(X_n))\Delta W_n + M(\varphi_{\theta h,s_1}(X_n)) + U(\varphi_{\theta h,s_1}(X_n)), \quad (42)$$

where $U(x) = h^2 f_1(x) + hg_1(x)\Delta W_n$, and f_1, g_1 defined in (41) has weak order two for (1).

Proof. A straightforward calculation shows that the scheme of Theorem 3.2 has the same local error $r + 1 = 3$ in (5) as the scheme of Lemma 3.1. The proof then follows from Remark 2.1. $\square$

Remark 3.3. *Observe that both in Lemmas 3.1 and Theorem 3.2 we also obtain a second order method if we replace $M(\varphi_{\theta h,s_1}(X_n))$ with $M(\psi_h(X_n))$ and $U(\varphi_{\theta h,s_1}(X_n))$ with $U(\psi_h(X_n))$, for any smooth function $\psi_h(x)$ such that $\psi_h(x) = x + \mathcal{O}(h)$ (in particular $\psi_h(x) = x$). The motivation to evaluate these terms at $\varphi_{\theta h,s_1}(X_n)$ comes from stability issues that are discussed below.*

Mean-square stability (scalar linear test problem) The method (42) applied to the scalar linear test problem (9) yields after one step

$$X_{n+1} = \left(\tilde{R}_{s_2}(p) + qR_{s_1}(\theta p)V_n + \frac{1}{2}q^2R_{s_1}(\theta p)(V_n^2 - 1) + pq(1 - \theta)R_{s_1}(\theta p)V_n \right) X_n, \quad (43)$$

where for a one-dimensional Wiener increment ΔW_n we write $\Delta W_n = \sqrt{h}V_n$ and V_n is a $\mathcal{N}(0,1)$ random variable. Here $R_{s_2}, \tilde{R}_{s_1}$ are defined in (22), (26), respectively and $p = \lambda h, q = \mu\sqrt{h}$. Squaring and taking the expectation gives

$$\begin{aligned}\mathbb{E}(|X_{n+1}|^2) &= \mathbb{E}\left(|\tilde{R}_{s_2}(p)X_n|^2\right) \\ &+ \mathbb{E}\left(\left|q + \frac{1}{2}(pq - \theta pq)R_{s_1}(p)X_n\right|^2\right) \\ &+ \frac{1}{2}\mathbb{E}(|q^2 R_{s_1}(\theta p)X_n|^2),\end{aligned}\tag{44}$$

where we have used $\mathbb{E}(V_n^2) = 1, \mathbb{E}(V_n^4) = 3$ and $\mathbb{E}(V_n^j) = 0$ for j odd. We thus obtain for the stability function

$$R_{\theta A}(p, q) = \tilde{R}_{s_2}(p)^2 + q^2 \left(\frac{1}{2}q^2 + (1 + (1 - \theta)p)^2 \right) R_{s_1}(\theta p)^2,\tag{45}$$

We observe that if we want $\mathcal{S}_{SDE, \ell}$ to be included in the stability domain of the numerical method (42) we need $R_{s_1}(\theta p)^2 r(p, q) \leq 1$ for all $(p, q) \in \mathcal{S}_{SDE, \ell}$, where $r(p, q) = \frac{1}{2}q^4 + q^2(1 + (1 - \theta)p)^2$. As the border of the true stability domain (10) scales as $|p| = \frac{1}{2}q^2$ we see that the dominant term in the polynomial $r(p, q)$ for $|p|$ large would be $(1 - \theta)q^2 p^2$ if $\theta \neq 1$. We therefore choose $\theta = 1$ (that eliminates the aforementioned term).

Definition of the S-ROCK2A methods Choosing $\theta = 1$ in (43), we define the family of S-ROCK2A methods by

$$X_{n+1} = \tilde{\varphi}_{h, s_2}(X_n) + g(\varphi_{h, s_1}(X_n))\Delta W_n + M(\varphi_{h, s_1}(X_n)) + U(\varphi_{h, s_1}(X_n)),\tag{46}$$

where $U(x) = h^2 f_1(x) + hg_1(x)\Delta W_n$, with f_1, g_1 defined in (41) (with $\theta = 1$) with the corresponding stability function given by

$$R_A(p, q) = \tilde{R}_{s_2}(p)^2 + \left(q^2 + \frac{1}{2}q^4 \right) R_{s_1}(p)^2.\tag{47}$$

To emphasize that the stability function (47) depends on s_1, s_2, η_1, η_2 we will sometimes use the notation $R_{A(s_1, s_2, \eta_1, \eta_2)}(p, q)$. The damping properties of the polynomial $R_{s_1}(p)^2$ will allow to control the growth of the polynomial $r(p, q) = q^2 + \frac{1}{2}q^4$ and ensure good mean-square stability properties of the S-ROCK2A method for scalar problems as shown below. The mean-square stability domains of the S-ROCK2A methods are given by

$$\mathcal{S}_{A(s_1, s_2, \eta_1, \eta_2)} = \{(p, q) \in \mathbb{C}^2 ; R_{A(s_1, s_2, \eta_1, \eta_2)}(p, q) < 1\}.\tag{48}$$

The task is now to optimize numerically the parameters s_1, s_2, η_1, η_2 such that $\mathcal{S}_{SDE, \ell} \subset \mathcal{S}_{A(s_1, s_2, \eta_1, \eta_2)}$ with ℓ as large as possible.

Remark 3.4. Notice that $R_A(p, q)$ in (47) is an increasing function of $q \geq 0$. This implies that when searching for the damping parameters s_1, s_2, η_1, η_2 that maximize L defined in (15), we can replace $q = \sqrt{-2p}$ in (47) and use the simpler formula

$$L = \inf\{\ell > 0; R_A(-\ell, \sqrt{2\ell}) > 1\}.$$

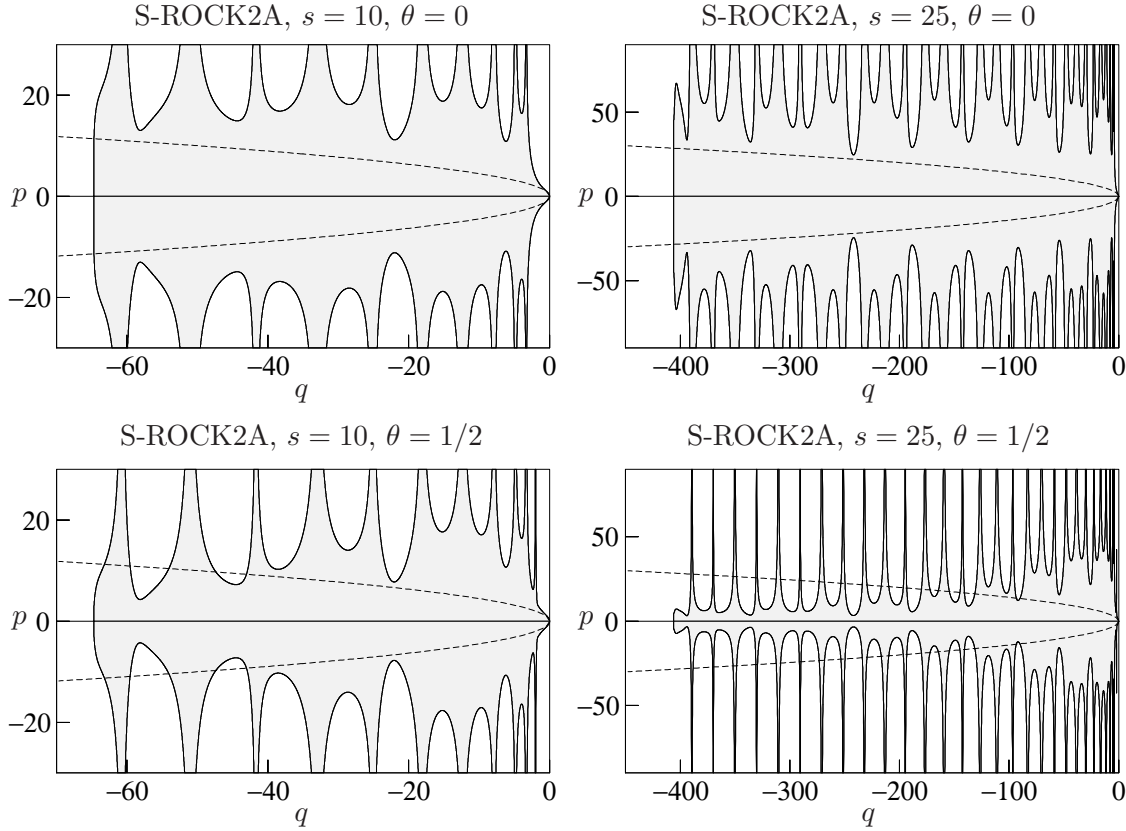


Figure 3: Stability domains (10) associated to (43) for the S-ROCK2A method for $\theta = 1/2$ and $\theta = 1$ with number of stages $s_1 = s$, $s_2 = 2s$, with $s = 10, 25$, respectively. Damping parameters are $\eta_1 = 2/13$ (all cases) and $\eta_2 = 20.0, 31.0$ respectively.

Examples of stability domains for the integrator (42) with $\theta = 1$ are shown in Figure 3. We also illustrate for the same values of s_1, s_2, η_1, η_2 as in (48) the stability domains of the integrator (42) with $\theta \neq 1$. As we can see, the portion of the stability domain covered for $\theta \neq 1$, is significantly smaller than in the case $\theta = 1$ as the number of stages increase. Furthermore, it can be observed that only when $\theta = 1$ would L defined in (15) have the right asymptotic growth as a function of a number of stages s_1, s_2 (see also the discussion in Section 3.3.1). Thus from now on we only focus on the S-ROCK2A integrator (46), which is the integrator (42) when $\theta = 1$.

Multi-dimensional linear systems We notice that for multi-dimensional linear problems of the form (16), if the matrix A commutes with all matrices $B_1, \dots, B_r$, then the choice $\theta = 1$ implies that the function f_1 and g_1 in (41) vanish. In this situation (and for the particular choice of $\theta = 1$) the method (39) is in fact already of weak second order for the linear test equation (9). This is however not true for general multi-dimensional linear problems or more generally for nonlinear problems. Indeed, consider the following linear system of SDEs

$$dX = AXdt + BXdW(t), \quad (49)$$

where A, B are general $d \times d$ matrices and $dW(t)$ is a standard one-dimensional Wiener process. Then applying the method (42) to (49) gives

$$\begin{aligned} X_{n+1} = & \left[\tilde{R}_{s_2}(hA) + \left(\sqrt{h} V_n B \right. \right. \\ & \left. \left. + \frac{1}{2} \left(hA\sqrt{h}B - \sqrt{h}BhA \right) V_n + \frac{1}{2} (V_n^2 - 1) hB^2 \right) R_{s_1}(hA) \right] X_n, \end{aligned} \quad (50)$$

and

$$\begin{aligned} \mathbb{E}(|X_{n+1}|^2) = & \mathbb{E} \left(|\tilde{R}_{s_2}(P)X_n|^2 \right) \\ & + \mathbb{E} \left(\left| Q + \frac{1}{2}(PQ - QP) R_{s_1}(P)X_n \right|^2 \right) \\ & + \frac{1}{2} \mathbb{E} (|Q^2 R_{s_1}(P)X_n|^2), \end{aligned} \quad (51)$$

with matrices $P = hA, Q = \sqrt{h}B$. If P and Q can be simultaneously put into diagonal form (in particular if A, B are symmetric and commute), then the scalar theory with the stability function (47) can be applied: the S-ROCK2A method will be stable for (49) (provided adequate stage number). For the test problems introduced in [9] (i.e., the SDE (16) with matrices A and B_j given by (18) or (19)) the S-ROCK2A will still exhibit similar mean-square stability behavior as for the scalar linear test equation. Indeed in (18) or (19) the matrix A commutes with each of the B_j matrices and terms of the form $PQ - QP$ vanish in (51). This is however not the case when the matrices P and Q do not commute, and our numerical findings (see the numerical examples in Section 4.1) show that the S-ROCK2A methods do no longer, in general, inherit the favorable mean-square stability properties constructed for the linear scalar test equation. We notice that this issue with the terms of the form $PQ - QP$ entering the stability function is only seen for second order methods. A numerical illustration (see Sect. 4.1) shows that while S-ROCK2A blows up for a stiff linear system (49) the first order S-ROCK methods remains stable even though both methods have similar mean-square stability domains for the linear test problem (9). As mentioned in Section 2.2, for problems (49) a stability analysis of the S-ROCK2A method is difficult. To illustrate the difficulties arising with the problem (49) with non-commutative matrices A and B , we consider the method (43) with $\theta \neq 1$. The stability function (47) then contains a factor of the type $q^2 p^2$ which mimics terms of the form $|\frac{1}{2}(PQ - QP) \dots|^2$ in (51). As already seen in Figure 3, this term destroys the good mean-square stability properties of the S-ROCK2A method.

We summarize our findings:

- For the S-ROCK2A method, neither the scalar test problems (9) nor the multi-dimensional linear test problems in [9] give the qualitative behavior of the mean-square stability of the method when applied to general linear systems;
- the expression $AB - BA$ (or its scaled version $PQ - QP$) in the stability function (for noncommutative matrices A, B) arising from the terms $f'g$ or $g'f$ that need to be evaluated for weak second order methods destroy in general the extended mean-square stability domains of the S-ROCK2A methods.

These findings motivate the introduction of another S-ROCK2 method, one which besides showing similar behavior to the S-ROCK2A on the scalar test problem, also works for classes of multi-dimensional mean-square problems.

3.3 The S-ROCK2B methods

We are now in position to define the main new family of weak second order stabilized integrators introduced in this paper, denoted S-ROCK2B. Let us again consider the integrator (39). We have seen in Section 3.2 that the terms $\frac{1}{2}(f'g)_{[i,j]} + (\frac{1}{2} - \theta) g'_{[i,j]}f$ in (41) needed for the modified integrators (42) are problematic for the mean-square stability of the methods. We now choose $\theta = 1/2$ to kill the second term and thus consider

$$X_{n+1} = \tilde{\varphi}_{h,s_2}(X_n) + g(\varphi_{h/2,s_1}(X_n))\Delta W_n + M(\psi_h(X_n)), \quad (52)$$

where we have replaced the term $M(\varphi_{\theta h,s_1}(X_n))$ with $M(\psi_h(X_n))$, where ψ_h is an arbitrary smooth function satisfying $\psi_h(x) = x + \mathcal{O}(h)$. The precise form of ψ_h will be discussed later. We know that this integrator applied to the modified problem (40) yields a weak second order method (recall also from Remark 3.3 that the choice of ψ does not affect the weak second order of accuracy). We still need to take care of the $\frac{1}{2}(f'g)_{[i,j]}$ term. For that, we modify the first term in (52) and consider $\tilde{\varphi}_{h,s_2}(X_n + \frac{1}{2}G(X_n))$, where $G(x) = g(\varphi_{s_1,h/2}(x))\Delta W_n$. Recalling that $\tilde{\varphi}_{h,s_2}(x) = x + hf(x) + \frac{h^2}{2}f'(x)f(x) + \mathcal{O}(h^3)$ we obtain

$$\begin{aligned} \tilde{\varphi}_{h,s_2}\left(x + \frac{1}{2}G(x)\right) &= \tilde{\varphi}_{h,s_2}(x) + \frac{1}{2}G(x) + \frac{h}{2}f'(x)G(x) \\ &\quad + \frac{h}{8}f''(x)(G(x), G(x)) + R_{\tilde{\varphi}}(x, W_n), \end{aligned} \quad (53)$$

where the rest $R_{\tilde{\varphi}}(x, W_n)$ has mean and variance of size $\mathcal{O}(h^3)$. Here $f''(x)(\cdot, \cdot)$ denotes the bilinear form associated to the second derivative of f . Since $\varphi_{h/2,s_1}(x) = x + \frac{1}{2}hf(x) + \mathcal{O}(h^2)$ we next observe that

$$G(x) = g(x)\Delta W_n + \frac{h}{2}g'(x)f(x)\Delta W_n + R_G(x, W_n), \quad (54)$$

where $R_G(x, W_n)$ has mean and variance of size $\mathcal{O}(h^3)$. Summarizing our derivation we obtain the following theorem.

Theorem 3.5. *The scheme*

$$X_{n+1} = \tilde{\varphi}_{h,s_2}\left(X_n + \frac{1}{2}G(X_n)\right) + \frac{1}{2}G(X_n) + M(\psi_h(X_n)) + U(\varphi_{h/2,s_1}(X_n)), \quad (55)$$

where $M(x)$ is defined in (29),

$$G(x) = g(\varphi_{s_1,h/2}(x))\Delta W_n, \quad U(x) = h^2f_1(x) + hg_1(x)\Delta W_n - \frac{h}{8}f''(x)(G(x), G(x)),$$

and

$$f_{1,[i]} = \frac{1}{4}gg^T : f''_{[i]}, \quad g_{1,[i,j]} = \frac{1}{4}gg^T : g''_{[i,j]}, \quad (56)$$

for all $i = 1, \dots, d$ and $j = 1, \dots, m$, has weak order two for (1).

Proof. Let us denote by X_A and X_B the schemes (46) and (52), respectively, after one step, starting from x . In view of (53) and (54) we see that the difference $X_A - X_B = R(x, W_n)$, where $R(x, W_n)$ has mean and variance of size $\mathcal{O}(h^3)$. In particular, Theorem 3.2 implies that X_A has local error $r + 1 = 3$ in (5) and thus X_B has also local error $r + 1 = 3$. The proof then follows from Remark 2.1. $\square$

We still need to choose the function ψ_h . The choice $\psi_h(x) = \varphi_{h,s_1}(x)$ as for the S-ROCK2A method could be possible. But as $\varphi_{h/2,s_1}(x)$ is already needed in the definition of the method (55) we rather choose

$$\psi_h(x) = \varphi_{h/2,s_1} \circ \varphi_{h/2,s_1}(x).$$

Notice that the other choice $\psi_h(x) = \varphi_{h/2,2s_1}(x)$, with an equivalent computational complexity as above by using $2s_1$ stages, would be sufficient to damp the q^4 term in the stability function for large values of q , but it would yield a gap in the stability domain near the origin.

Definition of the S-ROCK2B methods The family of S-ROCK2B methods of weak order two is defined by

$$X_{n+1} = \tilde{\varphi}_{h,s_2}\left(X_n + \frac{1}{2}G(X_n)\right) + \frac{1}{2}G(X_n) + M(\varphi_{h/2,s_1} \circ \varphi_{h/2,s_1}(X_n)) + U(\varphi_{h/2,s_1}(X_n)) \quad (57)$$

where $M(x)$ is defined in (29),

$$G(x) = g(\varphi_{s_1,h/2}(x))\Delta W_n, \quad U(x) = h^2 f_1(x) + hg_1(x)\Delta W_n - \frac{h}{8}f''(x)(G(x), G(x)),$$

and f_1, g_1 are defined in (56). The values of the number of stages s_1, s_2 and damping parameters η_1, η_2 for the Chebyshev methods $\varphi_{h,s_1}, \tilde{\varphi}_{h,s_2}$ in (57) shall be discussed in the next Section 3.3.1.

Remark 3.6. *Similar arguments as given in Theorem 3.5 show that the scheme*

$$X_{n+1} = \varphi_{h/2}\left(\varphi_{h/2}(X_n) + g(\varphi_{h/2}(X_n))\Delta W_n + M(\varphi_{h/2}(X_n)) + U(\varphi_{h/2}(X_n))\right), \quad (58)$$

where $U(x) = h^2 f_1(x) + hg_1(x)\Delta W_n - h/2 f''(x)(g(x)\Delta W_n, g(x)\Delta W_n)$, has weak order two of accuracy, provided that φ_h is a scheme of order two for ODEs (20). To build stabilized methods, φ_h should in addition have extended stability around the negative real axis and good damping properties. For example, the ROCK2 method [6] has both of the aforementioned properties (recall that as its stability function oscillates around zero, it has much better damping properties than the RKC method (25)). Thus, it should be possible to construct S-ROCK2 methods using the ROCK2 methods as a building block, instead of having to use two different stability functions as in the S-ROCK2B method (57), where we use the second order RKC method $\tilde{\varphi}_{h,s_2}$ to have order two for the deterministic terms and the first order Chebyshev method (with polynomials oscillating around zero) to have sufficient damping for the diffusion terms. The only (technical) issue with ROCK2 is that varying the damping is more involved than with the RKC method. For each value of the damping an iterative procedure to obtain the appropriate polynomials has to be used. However, as this has only to be done once to optimize the parameters (stage number and damping) of the methods, it does not constitute a fundamental issue.

3.3.1 Mean-square stability and optimal parameters

The method (57) applied to the linear test problem yields (9)

$$X_{n+1} = \left(\tilde{R}_{s_2}(p) + \frac{1}{2}R_{s_1}(p/2)q(\tilde{R}_{s_2}(p) + 1)V_n + \frac{1}{2}(V_n^2 - 1)q^2 R_{s_1}^2(p/2) \right) X_n,$$

where V_n is a $\mathcal{N}(0, 1)$ random variable and $R_{s_1}, \tilde{R}_{s_2}$ are defined in (22), (26), respectively. Squaring the above expression and taking the expectation gives

$$\begin{aligned}\mathbb{E}(|X_{n+1}|^2) &= \mathbb{E}\left(|\tilde{R}_{s_2}(p)X_n|^2\right) \\ &+ \frac{1}{4}\mathbb{E}\left(\left|R_{s_1}(p/2)q(\tilde{R}_{s_2}(p) + 1)X_n\right|^2\right) \\ &+ \frac{1}{2}\mathbb{E}\left(|q^2R_{s_1}^2(p)X_n|^2\right).\end{aligned}\tag{59}$$

We deduce that the corresponding mean-square stability function is given by

$$R_B(p, q) = \tilde{R}_{s_2}^2(p) + \left(q^2 \frac{(1 + \tilde{R}_{s_2}(p))^2}{4} R_{s_1}^2(p/2) + \frac{1}{2} q^4 R_{s_1}^4(p/2)\right).\tag{60}$$

We optimize the parameters s_1, s_2, η_1, η_2 in the SROCK method (57) in order to obtain the most efficient stiff integrators. We emphasize that the S-ROCK2B methods (similarly to the S-ROCK2A methods) need the same number of random number generation and diffusion function evaluations (per time-step) than a standard second order explicit method. The mean-square stability domains of the S-ROCK2B method are given by

$$\mathcal{S}_{B(s_1, s_2, \eta_1, \eta_2)} = \{(p, q) \in \mathbb{C}^2 ; R_{B(s_1, s_2, \eta_1, \eta_2)}(p, q) < 1\},\tag{61}$$

where the notation $R_{B(s_1, s_2, \eta_1, \eta_2)}(p, q)$ emphasizes the dependence of the stability function $R_B(p, q)$ on the parameters s_1, s_2, η_1, η_2 (recall that s_1 and s_2 represent the stage numbers of $\tilde{R}_{s_2}(p)$ and $R_{s_1}(p)$, respectively, and η_1, η_2 the values of their damping). As for the S-ROCK2A methods, we optimize the parameters s_1, s_2, η_1, η_2 such that $\mathcal{S}_{SDE, \ell} \subset \mathcal{S}_{B(s_1, s_2, \eta_1, \eta_2)}$ with ℓ as large as possible. To search in this large parameter space, we first fix $\eta_2 = 2/13$, which is the usual damping parameter considered in the deterministic case for RKC. Numerical experiments indicate that for a fixed η_2 and a fixed computational budget, the choice $s_1 = s_2 = s$ gives optimal values of L . We next fix $cost = s_1 + 2s_2$ which is the total number of drift function evaluations and with the choice $s_1 = s_2$ we have $cost = 3s$. We notice that with a parallel implementation (on two processors) the evaluation of $\tilde{\varphi}_{h, s_2}$ and the second step of the composition $\varphi_{h/2, s_1} \circ \varphi_{h/2, s_1}$ can be done in parallel. In this case, and with the choice $s_1 = s_2$ we have $cost = 2s$. Let us define

$$L(s, \eta_1(s)) := \sup\{\ell > 0 ; \mathcal{S}_{SDE, \ell} \subset \mathcal{S}_{B(s, \eta_1(s))}\}.\tag{62}$$

In view of Remark 3.4 it is enough to optimize

$$L(s, \eta_1(s)) = \inf\{\ell > 0 ; R_{B(s, \eta_1(s))}(-\ell, \sqrt{2\ell}) > 1\}.$$

We next compute numerically for $\eta_2 = 2/13$ the optimal parameter $\eta_1(s)$ that maximizes $L(s, \eta_1(s))$ with the choice $s_1 = s_2 = s$ and for all values $s = 2, 3, 4, \dots, 300$. For each s we thus obtain a couple $L(s), \eta_1(s)$, where $L(s) \simeq L(s, \eta_1(s))$. Using a continuous piecewise polynomial interpolations we then find that

$$\begin{aligned}s(L) &= \begin{cases} \alpha_1 L + \alpha_2 \sqrt{L} + \alpha_3, & \text{if } L \leq L_0, \\ \alpha_4 \sqrt{L} + \alpha_5, & \text{if } L > L_0, \end{cases} \\ \eta_1(s(L)) &= \alpha_6 s(L)^2 + \alpha_7 s(L) + \alpha_8,\end{aligned}\tag{63}$$

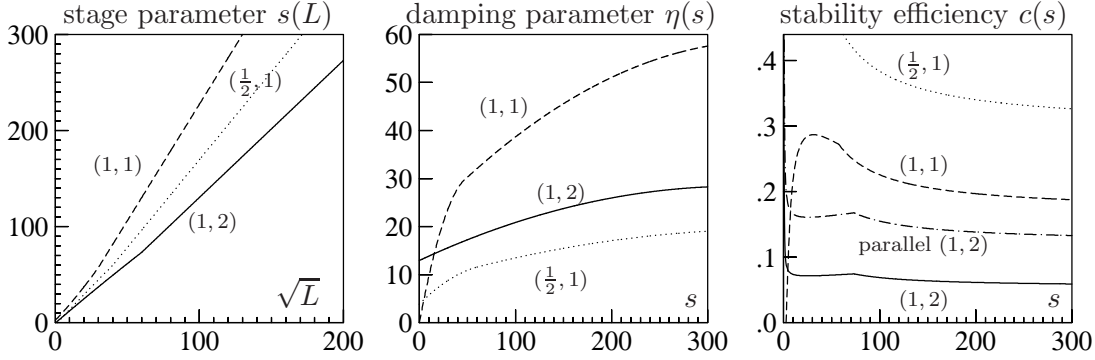


Figure 4: Comparison of S-ROCK2B (solid lines) and the S-ROCK methods of orders (1, 1) (dashed lines), $(\frac{1}{2}, 1)$ (dotted lines): optimal stage parameter s , optimal damping parameter $\eta(s)$ and stability efficiency $c(s) = L/\text{cost}^2$, where L is the length of the stochastic stability domain and cost is the number of function evaluations.

where

$$\begin{aligned} \alpha_1 &= -0.000979873, & \alpha_2 &= 1.286345, & \alpha_3 &= -0.413239, & \alpha_4 &= 1.427199, \\ \alpha_5 &= -12.39198, & \alpha_6 &= -0.000139944, & \alpha_7 &= 0.0930229, & \alpha_8 &= 12.95990, & \sqrt{L_0} &= 60. \end{aligned}$$

In Figure 4, we plot the optimal number of stages $s_1 = s_2 = s(L)$ (left picture), the optimal damping parameter $\eta_2 = \eta(s)$ given in (63) (middle picture). It can be seen that we obtain a relation of the form $L = c(s) \cdot \text{cost}^2$ with $c(s)$ that we call the efficiency factor, approaching a limit value $c(\infty)$ as s increases. For comparison, we also include the results for the S-ROCK methods of orders (1, 1) and $(1/2, 1)$. We also plot the corresponding stability domains for the S-ROCK2B method when $s_1 = s_2 = 10, 25, 50, 100$ in Figure 5. In the right picture of Figure 4 we observe that the stability efficiencies $c(s)$ of all considered methods are asymptotically decreasing to positive constants reported in Table 1. Thus, for all methods the lengths of the

Table 1: Stability efficiency $c(\infty)$ of the considered S-ROCK methods.

method	$c(\infty)$
S-ROCK2B	≈ 0.059
S-ROCK2B (parallel)	≈ 0.13
S-ROCK(1, 1)	≈ 0.19
S-ROCK(1/2, 1)	≈ 0.33

stability domain satisfies asymptotically the following quadratic growth

$$L = c(s)\text{cost}^2 \geq c(\infty)\text{cost}^2.$$

Thus for getting a given portion L of the true stability domain $\mathcal{S}_{SDE}$ the cost is

$$\text{cost} \leq \sqrt{\frac{L}{c(\infty)}}.$$

This is sharp contrast with the cost for standard explicit methods such as the Euler-Maruyama or the Milstein methods, for which we would get (by applying repeatedly the method with

maximum step size dictated by its stability region)

$$cost \leq \frac{L}{c},$$

with c a small constant (for example, for the Euler-Maruyama method, we have $c \approx 1/4$ [5]).

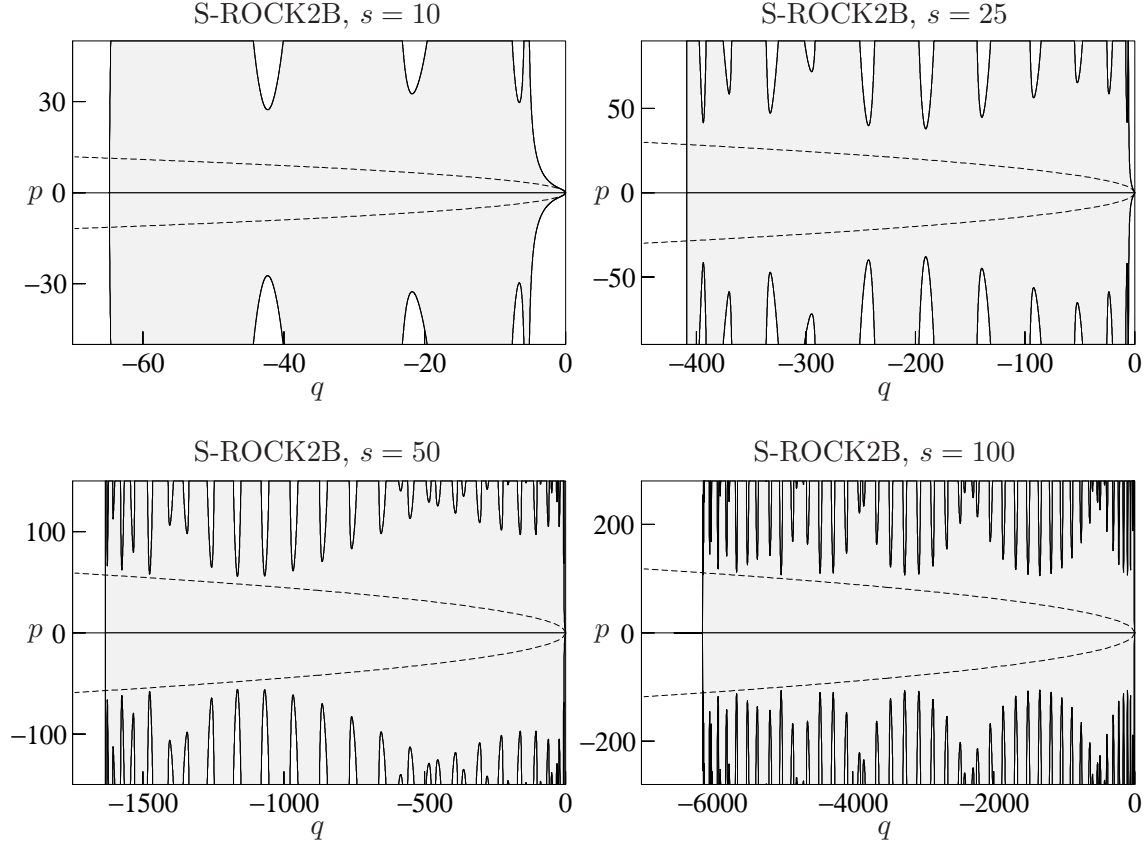


Figure 5: Stability domains of the S-ROCK2B method with stage numbers $s_1 = s_2 = s = 10, 25, 50, 100$, respectively. Chebyshev damping parameters are $\eta_1 = 2/13$ (in all cases) and $\eta_2 = 13.7, 15.1, 17.3, 21.3$, respectively.

Multi-dimensional linear systems We apply the S-ROCK2B method to the linear system of SDEs (49) and obtain

$$X_{n+1} = \left[F_1 + V_n F_2 + \frac{1}{2} (V_n^2 - 1) F_3 \right] X_n,$$

where V_n is a $\mathcal{N}(0, 1)$ Gaussian random variable, and

$$\begin{aligned} F_1 &= \tilde{R}_{s_2}(P), \\ F_2 &= \frac{1}{2} R_{s_1}(P/2) Q (\tilde{R}_{s_2}(P) + I), \\ F_3 &= Q^2 R_{s_1}^2(P/2), \end{aligned}$$

where we again denote the matrices $P = hA, Q = \sqrt{h}B$. We thus obtain

$$\begin{aligned}\mathbb{E}(|X_{n+1}|^2) &= \mathbb{E}\left(|\tilde{R}_{s_2}(P)X_n|^2\right) \\ &+ \frac{1}{4}\mathbb{E}\left(\left|R_{s_1}(P/2)Q(\tilde{R}_{s_2}(P) + I)X_n\right|^2\right) \\ &+ \frac{1}{2}\mathbb{E}\left(|Q^2R_{s_1}(P/2)^2X_n|^2\right).\end{aligned}\tag{64}$$

We see that while the first and third term in the right hand side of the equality in (64) have comparable growth to the first and third term in the right hand side of the equality (51) (S-ROCK2A methods), the second term of the right hand side in (51) or (64) are different. In contrast to (51) all the P matrices are now arguments of the polynomials $R_{s_1}, \tilde{R}_{s_2}$ and can therefore be damped by these polynomials. Also going from (64) to the one-dimensional case (59) does not allow for any cancellation. The same power of p, q remains in the scalar case (59). This is in sharp contrast with (51), where for the scalar case the term $pq - qp$ cancels in (47).

We close this section by considering a special class of multi-dimensional linear systems for which we can prove the mean-square stability of the S-ROCK2B method. Consider (49) with $d = 2, m = 1$ and

$$A = \begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix},\tag{65}$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2, \alpha$ are fixed real parameters.

Remark 3.7. *Following the methodology in [34] for studying (17), it can be checked that the solutions of (49) with matrices given by (65) are mean-square stable (10) for all initial condition $X(0) = X_0$ if and only if*

$$\lambda_j < -\frac{1}{2}|\mu_j|^2, \quad j = 1, 2.\tag{66}$$

Notice that this condition remains valid for $\lambda_1 = \lambda_2, \alpha \neq 0$ i.e. in the case of a non-normal drift matrix A .

Proposition 3.8. *Consider the S-ROCK2B method (42) applied to the problem (49) with matrices given by (65) with stepsize h satisfying $h\lambda < L$, where $\lambda = \max(|\lambda_1|, |\lambda_2|)$ and L is defined in (15). If (66) holds, then*

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n|^2) = 0.$$

Proof. Inspired by the standard approach for deterministic linear problems, we consider for $\varepsilon > 0$ the change of variable $Y_{[1]} = X_{[1]}, Y_{[2]} = \varepsilon^{-1}X_{[2]}$. We obtain that Y solves the SDE (49) with α replaced by $\varepsilon\alpha$. In view of (64) we have that the corresponding numerical solution Y_n satisfies

$$\mathbb{E}(|Y_{n+1}|^2) = \mathbb{E}(|F_1Y_n|^2 + \frac{1}{4}|F_2Y_n|^2 + \frac{1}{2}|F_3Y_n|^2)$$

where the triangular matrices F_1, F_2, F_3 are given by

$$\begin{aligned} F_1 &= \tilde{R}_{s_2}(hA) = \begin{pmatrix} \tilde{R}_{s_2}(p_1) & \varepsilon\alpha C_1 \\ 0 & \tilde{R}_{s_2}(p_2) \end{pmatrix}, \\ F_2 &= \sqrt{h}R_{s_1}(hA/2)B(\tilde{R}_{s_2}(hA) + I) \\ &= \begin{pmatrix} q_1R_{s_1}(p_1/2)(1 + \tilde{R}_{s_2}(p_1)) & \varepsilon\alpha C_2 \\ 0 & q_2R_{s_1}(p_2/2)(1 + \tilde{R}_{s_2}(p_2)) \end{pmatrix}, \\ F_3 &= hB^2R_{s_1}^2(hA/2) = \begin{pmatrix} q_1^2R_{s_1}^2(p_1/2) & \varepsilon\alpha C_3 \\ 0 & q_2^2R_{s_1}^2(p_2/2) \end{pmatrix}, \end{aligned}$$

where C_1, C_2, C_3 depend on $p_j = h\lambda_j, q_j = \sqrt{h}\mu_j, j = 1, 2$ but are independent of ε, α . Using the identity (60), a calculation then yields

$$\begin{aligned} \mathbb{E}(|Y_{n+1}|^2) &= \mathbb{E}\left(\sum_{j=1}^2 R_B(p_j, q_j)Y_{n,[j]}^2 + \sum_{k=1}^3 \beta_k(\varepsilon^2\alpha^2 C_k^2 Y_{n,[2]}^2 + 2\varepsilon\alpha C_k F_{k,[1,1]} Y_{n,[1]} Y_{n,[2]})\right) \\ &\leq \left(\max_{j=1,2} R_B(p_j, q_j) + C\varepsilon|\alpha|\right)\mathbb{E}(|Y_n|^2), \end{aligned}$$

for all ε small enough, where $F_{k,[1,1]}$ is the upper-left coefficient in matrix F_k , $\beta_1 = 1, \beta_2 = 1/4, \beta_3 = 1/2$, and the constant C is independent of ε, α (but depends on C_1, C_2, C_3). By assumption on $h, \lambda_1, \lambda_2, \mu_1, \mu_2$ we have $\max_{j=1,2} R_B(p_j, q_j) < 1$, thus there exists $\varepsilon > 0$ small enough such that $\mathbb{E}(|Y_{n+1}|^2) \leq \delta \mathbb{E}(|Y_n|^2)$ with $\delta < 1$. We deduce $\mathbb{E}(|Y_n|^2) \rightarrow 0$ for $n \rightarrow \infty$, which concludes the proof. $\square$

Convergence rates To illustrate numerically the convergence rates of the new integrator S-ROCK2B, we consider the linear test problem (9). We study the error at the final time $T = 1$ for various timesteps h . In Figure 6, we compare the weak and strong errors of S-ROCK2B (solid lines) which are nearly identical for $s = 10, 25, 50, 100$ stages, S-ROCK(1, 1) (dashed lines) for $s = 10, 100$ stages, and S-ROCK(1/2, 1) (dotted lines) for $s = 10, 100$ stages. We take $\lambda = 2, \mu = 0.1$ (top pictures) and $\lambda = 2, \mu = 0.2$ (bottom pictures). In order to make the Monte Carlo error negligible, the curves are the averages over 10^9 experiments. All the S-ROCK methods have been carefully implemented in FORTRAN, and for a fair comparison, we use for all methods the same set of random numbers, generated using the algorithm [28]. We observe in the left pictures of Figure 6 the expected curves of slope two for the weak error (first moment) of S-ROCK2B, and curves of slope one for S-ROCK(1/2, 1) and S-ROCK(1, 1). The weak accuracy of S-ROCK2B is the best among the considered methods for all timesteps, and it is nearly two magnitudes better for the smallest considered timestep $h = 1/128$. In the right pictures of Figure 6, we observe the expected lines of slope one for the strong error of S-ROCK2B, which again performs better than S-ROCK(1/2, 1) and S-ROCK(1, 1) with lines of slope 1/2 and one, especially for small timesteps and when the noise is small (compare the cases $\mu = 0.1$ and $\mu = 0.2$ in the right pictures of Figure 6).

4 Numerical experiments

In this section we present various different numerical experiments with our newly constructed methods. We start our investigations in Section 4.1 by testing the mean-square stability

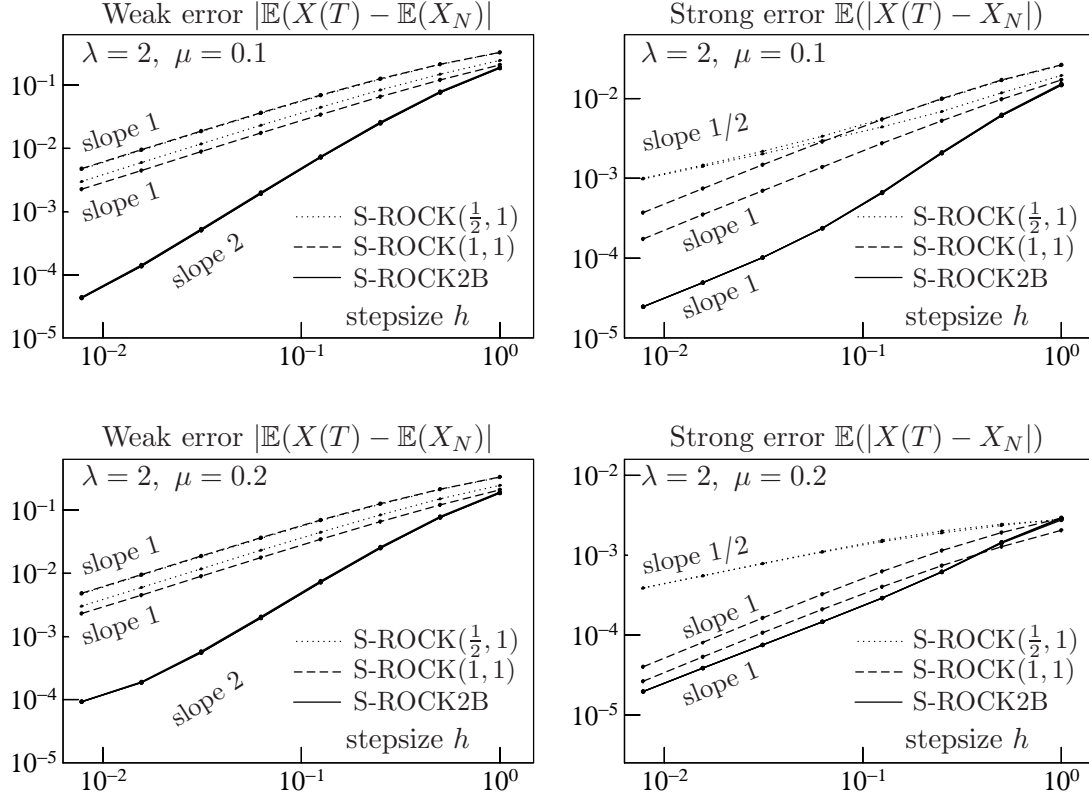


Figure 6: Weak and strong relative errors for the linear test problem (9). Comparison of S-ROCK2B with stage numbers $s = 10, 25, 50, 100$, S-ROCK(1, 1) with $s = 10, 100$, and S-ROCK(1/2, 1) with $s = 10, 100$.

properties of our methods for the three different linear test equations of the form (16) with (17)–(19), and we observe that it can be problem dependent. We then in Section 4.2 consider a two dimensional nonlinear stiff problem for which only the S-ROCK2B succeeds in capturing the correct limiting behaviour. Finally, in Section 4.3 we present numerical results for a stochastic PDE arising in neuroscience.

4.1 Multi-dimensional linear stiff problems

We start our investigation of the mean-square stability properties of our methods by considering the three different test problems discussed in Section 2.1. We recall here that all these problems are in the form of the general linear problem (16)

$$dX = AXdt + \sum_{r=1}^m B_r X dW_{[r]}(t), \quad X(0) = X_0,$$

where A, B_r are $d \times d$ matrices, and $W_{[r]}$ are one-dimensional Wiener processes. The set of parameters that we used can be found in Table 2, while the conditions under which solutions to (16) with respectively (17)–(19) are mean-square stable can be found in Table 3, together with references to the relevant literature. It can thus be easily checked that solutions corresponding to (17)–(19) with the parameters given in Table 2 are mean-square stable.

Table 2: Parameter values considered for equations (17)–(19)

Test equation	dimensions	parameter values		
equation (17)	$d = 2, m = 1$	$\lambda_1 = -1100$	$\lambda_2 = -4$	$\sigma_{11} = 46$
		$\sigma_{12} = 0.25$	$\sigma_{21} = 0.25$	$\sigma_{22} = 2.5$
equation (18)	$d = 2, m = 2$	$\lambda_1 = -1100$	$\sigma = 46.8$	$\varepsilon = 3$
equation (19)	$d = 3, m = 3$	$\lambda_1 = -1100$	$\varepsilon = 46.9$	

Table 3: Sufficient conditions on mean-square stability of problem (16)

Test equation	conditions	reference
equation (17)	$2\lambda_i + (\sigma_{i1} + \sigma_{i2})^2 < 0, i = 1, 2$	[34, Thm. 1]
equation (18)	$2\lambda_1 + (\sigma^2 + \varepsilon^2) < 0$	[9, Thm. 4.3]
equation (19)	$2\lambda_1 + \varepsilon^2 < 0$	[9, Thm. 4.4]
equation (65)	$2\lambda_i + \mu_i ^2, i = 1, 2$	Remark 3.7

For all our numerical investigation here we consider the method (46) with $s_1 = s = 10$, $s_2 = 2s$, $\theta = 1$ (see its stability domain in Figure 3, left picture) and the method (57) with $s_1 = s_2 = s = 10$ (see its stability domain in Figure 5, top-left picture). Furthermore, we consider the same stepsize $h = 0.05$, and the same set of random numbers for both methods. If we denote with μ_1 the largest eigenvalue of the diffusion matrix for all three equations (17)–(19), it can then be checked in Figures 3, 5 that the largest eigenvalue couple $(h\lambda_1, \sqrt{h\mu_1})$ belongs to the stability domains $\mathcal{S}_{SDE,L}$ of the two considered methods with stage parameter $s = 10$.

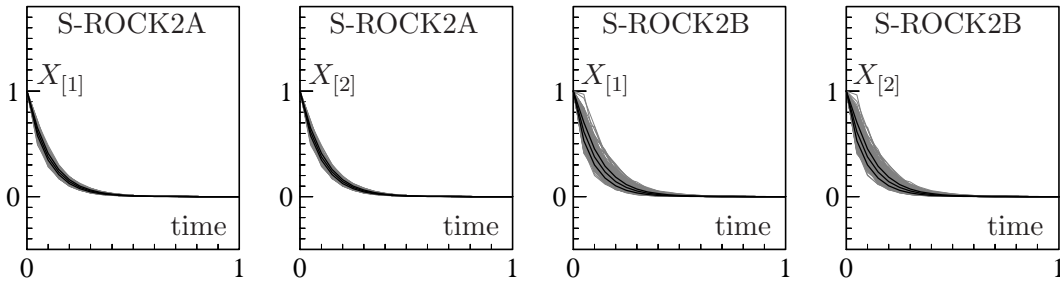


Figure 7: Comparison of S-ROCK2A and S-ROCK2B for the mean-square stable linear system (18) in dimensions $d = m = 2$ with parameters given in Table 2. Stage parameter $s = 10$ and stepsize $h = 0.05$. Gray curves: Components $X_{[1]}, X_{[2]}$ as a function of time $t_n = nh$ for 500 trajectories of the SDE. Black curves: associated mean and mean plus/minus the standard deviation (obtained as the average over 10^6 trajectories).

Notice that the problems corresponding to (18),(19) both involve the multiple integrals $I_{(1,2)}, I_{(2,1)}$ defined in (30). For (18), only the quantity $I_{(1,2)} + I_{(2,1)} = \Delta W_{n,[1]} \Delta W_{n,[2]}$ is needed, while for (19) we shall use the standard weak approximation $I_{(1,2)} \approx (\Delta W_{n,[1]} \Delta W_{n,[2]} + \xi_n)/2$, where $\mathbb{P}(\xi_n = \pm h) = 1/2$ (see Remark 2.2). The results of our numerical investigations

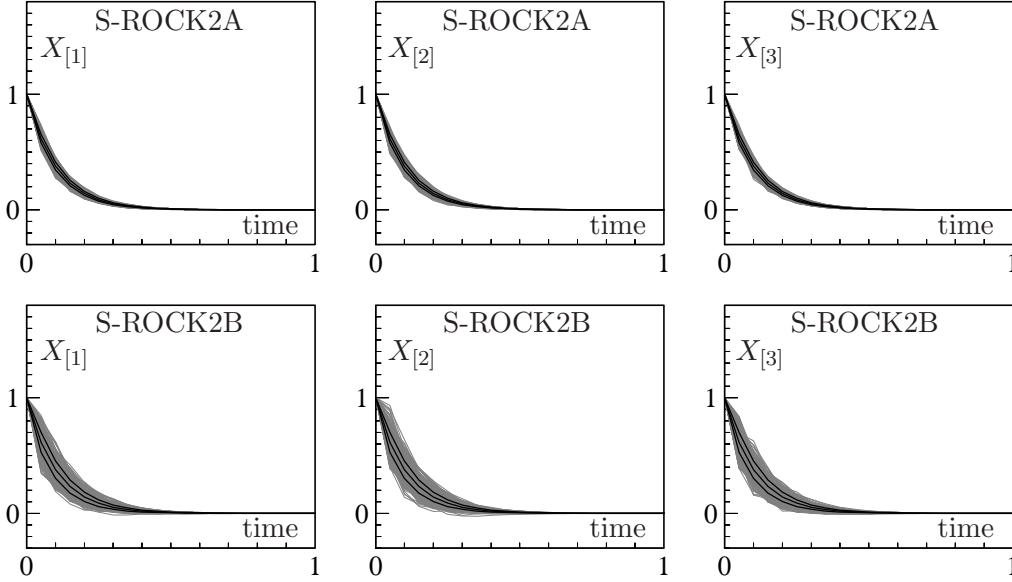


Figure 8: Comparison of S-ROCK2A and S-ROCK2B for the mean-square stable linear system (19) in dimensions $d = m = 3$ with parameters given in Table 2. Stage parameter $s = 10$ and stepsize $h = 0.05$. Gray curves: Components $X_{[1]}, X_{[2]}$ as a function of time $t_n = nh$ for 500 trajectories of the SDE. Black curves: associated mean and mean plus/minus the standard deviation (obtained as the average over 10^6 trajectories).

for equation (18) can be found in Figure 7, while the results for (19),(17) can be found in Figures 8, 9 respectively. In all of the figures we plot (gray curves) the components $X_{[i],n}$ as a function of time $t_n = nh$ for 500 trajectories of the SDE. We also include (black curves) the associated mean and mean plus/minus the standard deviation (obtained as the average over 10^6 trajectories)

$$\mathbb{E}(X_{[i],n}) \pm (\mathbb{E}(X_{[i],n}^2) - \mathbb{E}(X_{[i],n})^2)^{1/2}. \quad (67)$$

It can be observed that both S-ROCK2A and S-ROCK2B maintain mean-square stability for the test problems (18),(19). However, as we can see in Figure 9 this is not the case for the test problem (17). In particular, even though the analysis for the linear test problem (9) predicts that for the particular choice of parameters, both of the methods should be mean-square stable, S-ROCK2A fails to be so, since numerical solutions computed with it quickly blow up (Figures 9). Furthermore, both the weak first order methods S-ROCK(1/2, 1) and S-ROCK(1, 1) remain mean-square stable for the test problem (17). (We note here that for the S-ROCK(1, 1), we have used $s = 15$, since for $s = 10$ the method is not mean-square stable for $h = 0.05$). This illustrates the fact that the linear test problem (9) can inform about the mean-square stability properties of a weak first order method in multiple dimensions, but this does not have to be the case for weak second order methods.

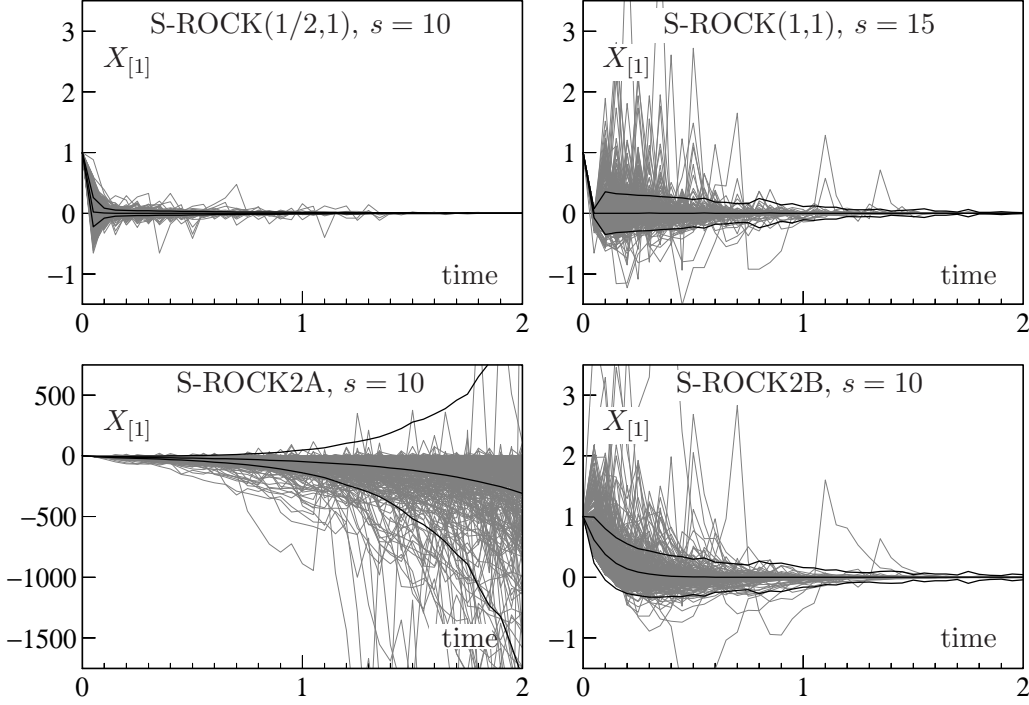


Figure 9: Comparison of S-ROCK methods for the mean-square stable linear system (17) in dimensions $d = 2$, $m = 1$, with parameters given in Table 2. Stepsize $h = 0.05$.

4.2 A multi-dimensional nonlinear stiff problem

We now consider the following nonlinear problem in dimensions $d = 2$, $m = 1$

$$dX = \begin{pmatrix} \alpha(X_{[2]} - 1) - \lambda_1 X_{[1]}(1 - X_{[1]}) \\ -\lambda_2 X_{[2]}(1 - X_{[2]}) \end{pmatrix} dt + \begin{pmatrix} -\mu_1 X_{[1]}(1 - X_{[1]}) \\ -\mu_2 X_{[2]}(1 - X_{[2]}) \end{pmatrix} dW(t), \quad X(0) = X_0, \quad (68)$$

which is inspired from a one-dimensional population dynamics model [5, Example 5.2]. Notice that if we linearise (68) around the stationary solution $X = (1, 1)^T$ we obtain the linear system (65). We take the initial condition $X(0) = (0.9, 0.9)^T$ and parameters

$$\lambda_1 = -1100, \quad \mu_1 = \sqrt{-2(\lambda_1 + 1)}, \quad \lambda_2 = -4, \quad \mu_2 = 2.5 \quad \alpha = 2. \quad (69)$$

We now solve (68) with S-ROCK2B with stage number $s_1 = s_2 = s = 10$ and $h = 0.05$ and plot 500 samples of realization of the SDE in Figure 10. As we can see, the numerical solution of (68) using S-ROCK2B converges quickly towards the asymptotic solution $X = (1, 1)^T$, in particular S-ROCK2B shows a correct good mean-square stability behavior. In contrast, numerical tests indicate that the method S-ROCK2A applied to the same problem (68)-(69) is severely unstable and the first component of the numerical solutions blows up rapidly after a few steps (it is thus not represented here). Notice that for $\alpha = 0$, S-ROCK2A (similarly to S-ROCK2B) would exhibit a good mean-square stability behavior, because in this case the stiff problem reduced to two decoupled one-dimensional problems.

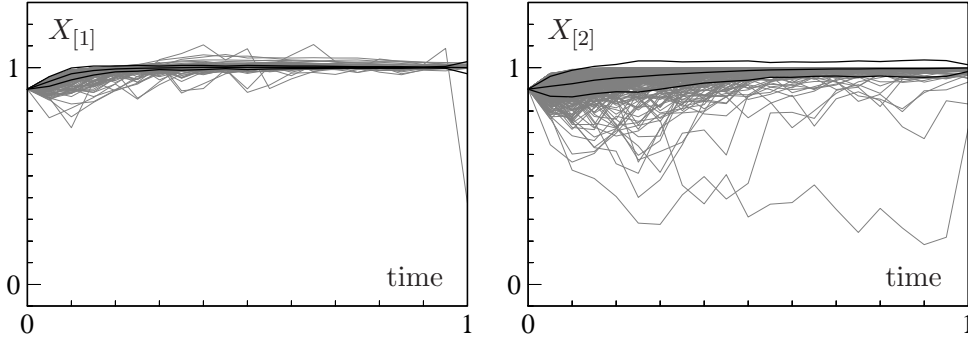


Figure 10: Numerical solution using S-ROCK2B of the nonlinear system (68) with parameters (69) and $\alpha = 2$. Stage parameter $s = 10$ and stepsize $h = 0.05$.

4.3 Electric potential in a neuron

We consider here the problem of the propagation of an electric potential $V(x, t)$ in a neuron [38]. This potential is governed by a system of non-linear PDEs called the Hodgkin-Huxley equations [17], but in certain ranges of values of V , this system of PDEs can be well approximated by the cable equation [18, 29]. In particular, if the neuron is subject to a uniform input current density over the dendrites and if certain geometric constraints are satisfied, then the electric potential satisfies the cable equation with uniform input current density.

$$\begin{aligned} \frac{\partial V}{\partial t}(t, x) &= \nu \frac{\partial^2 V}{\partial x^2}(t, x) - \beta V(t, x) + \sigma(V(t, x) + V_0) \dot{W}(t, x), \quad 0 \leq x \leq 1, \\ \frac{\partial V}{\partial t}(t, 0) &= \frac{\partial V}{\partial t}(t, 1) = 0, t > 0, \quad V(0, x) = V_0(x), \end{aligned} \quad (70)$$

where $\dot{W}(x, t) = \frac{\partial^2}{\partial x \partial t} w(x, t)$ is a space-time white noise meant in the Stratonovich sense. Here we have assumed that the distance between the origin (or soma) to the dendritic terminals is 1, and that the soma is located at $x = 0$. Furthermore, the white noise term is describing the effect of the arrival of random impulses and the multiplicative noise structure depicts the fact that the response of the neuron to a current impulse may depend on a local potential [40].

The quantity of interest is the threshold time

$$\tau = \inf\{t > 0; V(t, 0) > \lambda\}, \quad (71)$$

since when the potential at the soma (somatic depolarization) exceeds the threshold λ the neuron fires an action potential.

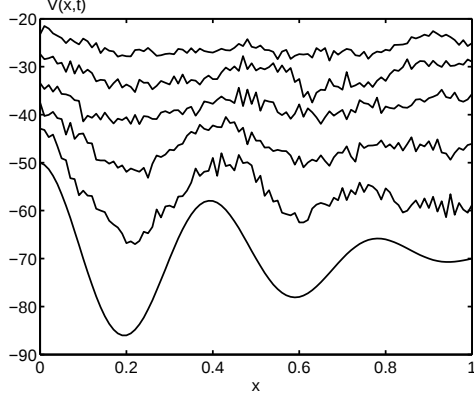
The SPDE (70) yields, after space discretization with finite differences [13] the following stiff system of SDE where $V(x_i, t) \approx u_i$, with $x_i = i\Delta x$, $\Delta x = 1/N$,

$$du_i = \nu \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} dt - \beta u_i dt + \sigma \frac{u_i + V_0}{\sqrt{\Delta x}} \circ dw_i, \quad i = 0, \dots, N \quad (72)$$

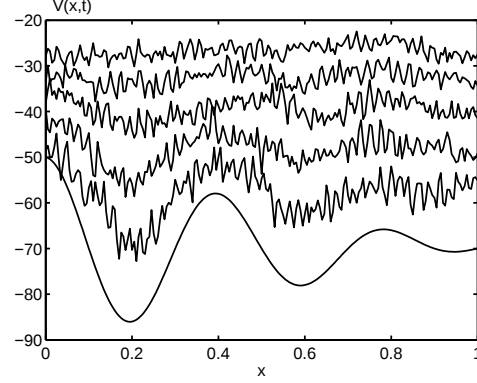
where the Neumann condition imposes $u_{-1} = u_0$ and $u_{N+1} = u_N$. Here $w_0, \dots, w_N$ are independent standard Wiener processes, and $\circ dw_i$ indicates Stratonovich noise. Converting this equation into an equivalent system of Itô SDEs, we obtain

$$du_i = \nu \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} dt + \left(\frac{\sigma^2}{\Delta x} (u_i + V_0) - \beta u_i \right) dt + \sigma \frac{u_i + V_0}{\sqrt{\Delta x}} dw_i, \quad i = 0, \dots, N \quad (73)$$

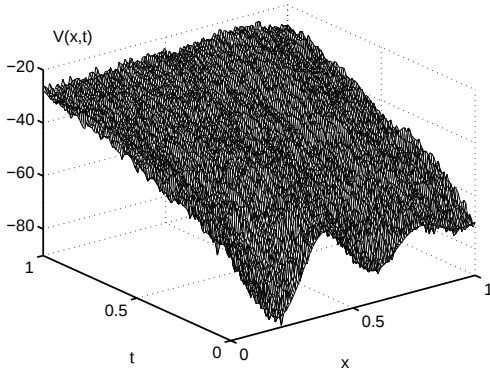
We consider the initial condition $V_0(x) = -70 + 20 \cos(5\pi x)(1-x)$ and the constants $\nu = 0.01$, $\sigma = 0.02$, $\beta = 1$, $V_0 = -10$, $\lambda = -40$. We consider the time interval $(0, T)$ with $T = 1$.



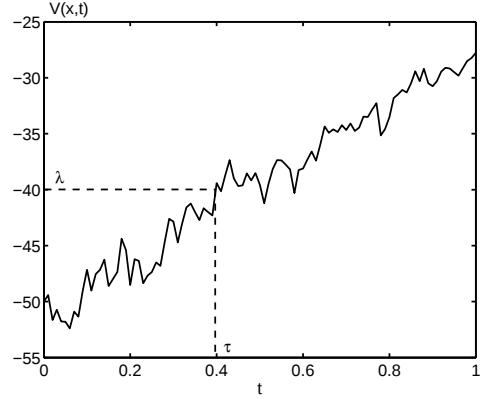
(a) $\Delta t = 1/100$, $\Delta x = 1/100$, fixed t .



(b) $\Delta t = 1/100$, $\Delta x = 1/200$, fixed t .



(c) $\Delta t = 1/100$, $\Delta x = 1/200$. Solution $V(x,t)$ as a function of x, t .



(d) $\Delta t = 1/100$, $\Delta x = 1/200$, fixed $x = 0$.

Figure 11: Samples of realisations of the discretized in space SPDE (70) using S-ROCK2B. Figures (a),(b): solutions as functions of x at fixed times $t = 0, 0.2, 0.4, \dots, 1.0$ (increasing with time, from bottom to top). Figure (d): solution as a function of t for $x = 0$.

Time slices of one realisation of the solution to (73) for different choices of the space-step Δx can be seen in Figures 11a and 11b. From the behaviour of the numerical solution for different number of discretisation points it is apparent that the effect of the noise is more significant in the case of higher space resolution. In Figure 11c we plot a complete realisation of $V(x,t)$, while in Figure 11d we plot $V(0,t)$ as a function of time and we see that for this particular realisation τ is about 0.4.

In Figure 12 we plot the empirical histograms for the threshold time τ calculated over 10^7 realisations of (73). Again we observe that the effect of the noise is stronger for higher space resolutions, since for the same value of time step Δt the empirical probability density function is wider around its mean for smaller Δx . It can also be observed that the variance is decreasing as Δt decreases. Furthermore, when comparing the histograms obtained with S-ROCK2B and the S-ROCK(1/2,1) methods, we observe that for the same time and space resolutions

the variance is larger with the S-ROCK2B method, while the empirical histogram obtained by the S-ROCK2B method is smoother than the one corresponding to S-ROCK(1/2, 1).

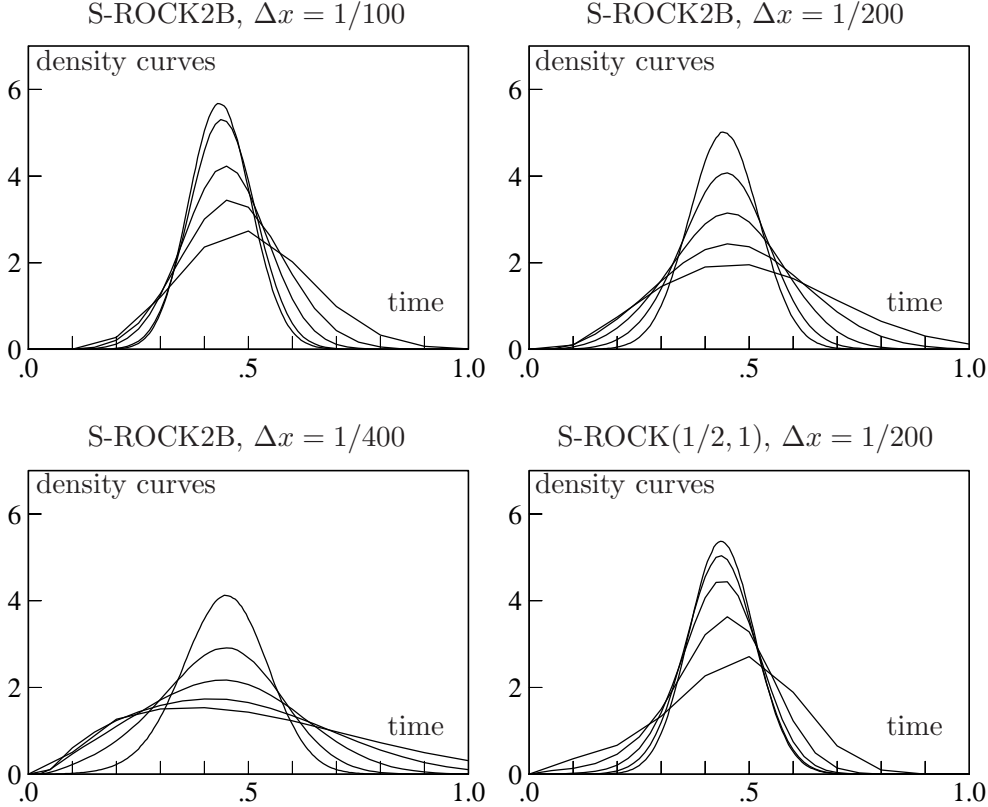


Figure 12: Density plots of the threshold time (71) in the SPDE neuron model (70) for various space mesh sizes $\Delta x = 1/100, 1/200, 1/400$. The five curves in each plot correspond respectively to $\Delta t = 1/10, 1/20, 1/40, 1/80, 1/200$ (the variances decrease when Δt decreases). Averages over 10 million samples.

5 Conclusion

In this paper, we introduced two new families of weak second order explicit stabilized methods, called S-ROCK2A and S-ROCK2B, well suited for the integration of stochastic stiff problems. These methods are shown to be much more efficient than standard explicit second order solvers for stiff (mean-square stable) problems. The framework based on modified equations, recently introduced in [4] to construct higher order weak methods proved to be useful to develop the methods proposed here. One of the benefits of this framework is that it simplifies the construction of higher order weak methods by avoiding to take care explicitly of the numerous order conditions for general weak second order schemes. Furthermore, this framework helps in retaining a particular qualitative behavior (here stability) of a basic integrator when extended to higher order as seen in Section 3.

One important characteristic of S-ROCK2B methods is that they seem to remain stable even in the case of multi-dimensional SDEs. This was verified by different numerical tests

both with linear and nonlinear examples (population model, stochastic PDE). Another important finding is that the linear test problems (18) and (19) proposed in [9] as a mean of generalizing the linear test problem (9) fail to discriminate between the S-ROCK2A and S-ROCK2B methods, despite the fact that these methods have different mean-square stability behavior for multi-dimensional problems. In contrast the test problem in (17) indicates different qualitative behavior of the S-ROCK2A and S-ROCK2B methods. The good stability property of the S-ROCK2B method and its efficiency in solving large stiff problems make the method also suitable for the numerical integration of SPDEs, as illustrated by the application and the numerical experiments for a stochastic model of neural response.

Acknowledgements

K.C.Z. was partially supported by Award No. KUK-C1-013-04 of the King Abdullah University of Science and Technology (KAUST).

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