A TRACE PRESERVING OPERATOR AND APPLICATIONS

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ABSTRACT. We construct a trace preserving operator which improves the integrability of functions in Sobolev classes refining the ones available in literature. As applications, we prove a $C^{1,\alpha}$ partial regularity result for local minimizers of quasiconvex integral functionals satisfying non standard (p, q) growth conditions in the borderline case $p = n - 1$ and $q = n$, and a global integrability result for weak solutions to a nonlinear elliptic system.
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1. INTRODUCTION AND STATEMENTS

The aim of this paper is to construct a linear operator from $W^{1,p}$ into $W^{1,p}$ which preserves the boundary values and improves the integrability of the functions and of their distributional gradients. A suitable regularity assumption on the boundary allows us to obtain an improved integrability result that, in certain aspects, refines those available in literature (see in particular [23]). Our construction is similar to an approach used by Gagliardo and we obtain the integral bounds expoliting an idea used by Hedberg for proving boundedness of Riesz potentials. It relies on a convolution kernel operator that leads us to extend a function defined on $\mathbb{R}^{n-1}$, considered as the boundary hyperplane of the n-dimensional upper half space $\mathbb{R}^n_+$, to the whole of $\mathbb{R}^n_+$. Thus

$$\mathbb{R}^n_+ = \{(x, y) : x \in \mathbb{R}^{n-1}, y > 0\}, \quad \text{and} \quad \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\} = \{(x, 0) : x \in \mathbb{R}^{n-1}\}.$$

In addition we denote by $\mathcal{B}^{n-1}$ the unit ball of $\mathbb{R}^{n-1}$,

$$\mathcal{B}^{n-1} = \{x \in \mathbb{R}^{n-1} : |x| \le 1\}$$

and consider a convolution kernel $\mathcal{K} \in C^\infty_c(\mathcal{B}^{n-1})$ such that

$$\int_{\mathbb{R}^{n-1}} \mathcal{K} d\mathcal{H}^{n-1} = 1.$$

Put, for $x \in \mathbb{R}^{n-1}$ and $y > 0$,

$$\mathcal{K}_y(x) = y^{1-n}\mathcal{K}\Big(\frac{x}{y}\Big)$$

and define the operator

$$\mathcal{E}^+ : \ f(x) \mapsto F(x, y) = (\mathcal{K}_y * f)(x), \tag{1.1}$$

where $*$ denotes convolution of functions on $\mathbb{R}^{n-1}$ (defined component-wise when f is vector-valued). Let us recall the definition of the homogeneous Sobolev space $\mathcal{W}^{1,p} = \mathcal{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ for $1 \le p < \infty$ as the space of locally integrable maps $v : \mathbb{R}^n_+ \to \mathbb{R}^N$ such that $Dv \in L^p(\mathbb{R}^n_+, \mathbb{R}^{N\times n})$. Our main result is the following:

Proposition 1.1. *The linear extension operator $\mathcal{E}^+$ defined by* (1.1) *maps $W^{1,p}(\mathbb{R}^{n-1}, \mathbb{R}^N)$ continuously into $\mathcal{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$, $p \ge 1$. Moreover we have for $p \in (1, \infty)$:*

i) *If $f \in W^{1,p}(\mathbb{R}^{n-1}, \mathbb{R}^N) \cap L^r(\mathbb{R}^{n-1}, \mathbb{R}^N)$ and $q = p + \frac{r}{n-1+r}$, then there exists a positive constant c depending only on n, N, r, p such that*

$$\|F\|^q_{L^{q^\star}(\mathbb{R}^n_+, \mathbb{R}^N)} + \|DF\|^q_{L^q(\mathbb{R}^n_+, \mathbb{R}^{N\times n})} \le c\|Df\|^p_{L^p(\mathbb{R}^{n-1}, \mathbb{R}^{N\times(n-1)})} \|f\|^{\frac{r}{n-1+r}}_{L^r(\mathbb{R}^{n-1}, \mathbb{R}^N)}. \tag{1.2}$$

ii) *If $f \in W^{1,p}(\mathbb{R}^{n-1},\mathbb{R}^N)\cap BMO(\mathbb{R}^{n-1},\mathbb{R}^N)$ and $q=p+1$, then there exists a positive constant c depending only on n, N, p such that*

$$\|F\|^q_{L^{q^\star}(\mathbb{R}^n_+,\mathbb{R}^N)}+\|DF\|^q_{L^q(\mathbb{R}^n_+,\mathbb{R}^{N\times n})}\le c\|Df\|^p_{L^p(\mathbb{R}^{n-1},\mathbb{R}^{N\times(n-1)})}\|f\|_{BMO(\mathbb{R}^{n-1},\mathbb{R}^N)}\,. \tag{1.3}$$

iii) *If $f \in W^{1,p}(\mathbb{R}^{n-1},\mathbb{R}^N)\cap C^{0,\alpha}(\mathbb{R}^{n-1};\mathbb{R}^N)$ and $q=p+\frac{1}{1-\alpha}$ then there exists a positive constant c depending only on n, N, α, p such that*

$$\|F\|^q_{L^{q^\star}(\mathbb{R}^n_+,\mathbb{R}^N)}+\|DF\|^q_{L^q(\mathbb{R}^n_+,\mathbb{R}^{N\times n})}\le c\|Df\|^p_{L^p(\mathbb{R}^{n-1},\mathbb{R}^{N\times(n-1)})}\|f\|^{\frac{1}{1-\alpha}}_{C^{0,\alpha}(\mathbb{R}^{n-1},\mathbb{R}^N)}\,, \tag{1.4}$$

where as usual $q^\star=\frac{nq}{n-q}$ denotes the Sobolev conjugate exponent of q if $q<n$, or any finite exponent if $q\ge n$.

We transfer this construction to functions defined on spheres in $\mathbb{R}^n$, as usual by the use of the stereographic projections. This will allow us to connect two $W^{1,p}$ functions in an annulus with a $W^{1,q}$ function, for some $q\ge p$. It is well known that, when dealing with integral functionals with (p,q)-growth conditions, one of the main difficulties is to construct suitable test functions and this is the motivation for our result.
Here, as an application, we will establish the $C^{1,\alpha}$ partial regularity of the local minimizers of quasiconvex integrals of the form

$$\mathcal{I}(v,O)=\int_O G(Dv(x))\,\mathrm{d}x,$$

where O is an open subset of some fixed bounded and open subset Ω of $\mathbb{R}^n$, $n>2$, and $v\colon\Omega\to\mathbb{R}^N$, $N\ge 2$. We consider integrands $G\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ satisfying for some positive constants L, $\nu>0$ the following hypotheses:

$$G\colon \mathbb{R}^{N\times n}\to\mathbb{R} \text{ is } C^2 \tag{H1}$$

$$0\le G(\xi)\le L(1+|\xi|^n) \tag{H2}$$

$$\int_\Omega G(\xi+D\varphi(x))\,\mathrm{d}x\ge\int_\Omega\Big(G(\xi)+\nu(1+|D\varphi(x)|^2)^{\frac{n-3}{2}}|D\varphi(x)|^2\Big)\,\mathrm{d}x \tag{H3}$$

for every $\xi\in\mathbb{R}^{N\times n}$ and for all $\varphi\in W^{1,n-1}_0(\Omega,\mathbb{R}^N)$. An example of a functional that is not convex and satisfies the above assumptions in dimensions $n=N>2$ is given by

$$\int_O(1+|Du|^2)^{\frac{n-1}{2}}+|\mathrm{Adj}Du|^{\frac{n}{n-1}}.$$

The quasiconvexity condition (H3) is a uniform strict form of the $W^{1,n-1}$-quasiconvexity condition considered in [3]. See also [32].
The notion of quasiconvexity, originally introduced in [42] as a condition, that under suitable growth conditions, is equivalent to sequential weak lower semicontinuity, has nowadays become a common condition in the vectorial calculus of variations.
The study of regularity of minimizers of quasiconvex integrals started with the celebrated paper by Evans [20] and has subsequently been studied by many authors (see for instance [28, 1, 11, 7, 12, 13, 30, 17] for results under standard growth condition). We recall that for vector valued minimizers the regularity results are only partial, meaning that they are obtained outside a negligible relatively closed subset of Ω. This state of affairs is even unavoidable for the case of regular variational problems (i.e. strongly convex integrands) by well-known examples, see for instance [15, 43, 48, 41].

The assumptions (H1)– (H3) clearly entail a (p,q) growth condition with $p=n-1$ and $q=n$. The study of the regularity of minimizers of such functionals started with the celebrated papers by Marcellini (see in particular [36, 38]) and has since attracted much attention. From very early on it has been clear that no regularity can be expected if the coercivity and growth exponents, denoted p and q, respectively, are too far apart (see [27, 37]). On the other hand, many regularity results are available for convex integrands if the ratio q/p is bounded above by a suitable constant depending on the dimension n, and converging to 1 when n tends to infinity (incl.[2, 4, 5, 6, 8, 16, 18, 19, 21, 34, 39, 40, 44]).

The first papers regarding the non convex (p,q) growth case are due to Fusco and Hutchinson ([24, 25]). More precisely, they considered polyconvex integral functionals satisfying a (p,q) growth condition of the type $n-1<p<q=n$. We emphasize that their techniques can not be used to treat the general quasiconvex case.

More recently, the general quasiconvex case has been addressed in a series of papers by Schmidt ([45, 46, 47]) under a (p,q) growth assumption of the type $p<q<p+\min\{\frac{1}{n},\frac{p}{2n}\}$. Here, as an application of the trace preserving operator, we give an extension of one of the results by Schmidt ([45]) to the case $p=n-1$, $q=n$ and, in this way, we cover what appears to be a borderline case. At least this is so for the corresponding situation in the Lebesgue-Serring-Marcelini set-up, where the functional is defined by relaxation from smooth maps (see the example of Malý [35]).

Definition 1.1. *A map $u\in W^{1,n-1}_{\rm loc}(\Omega,\mathbb{R}^N)$ is a local G-minimizer if*

$$\int_O G(Du)\,\mathrm{d}x\le\int_O G(Dv)\,\mathrm{d}x$$

for any $O\Subset\Omega$ and any $v\in W^{1,n-1}_u(O,\mathbb{R}^N)$.

Recall that quasiconvexity in connection with the growth condition (H2) implies

$$|DG(\xi)|\le c(1+|\xi|^{n-1}). \tag{H4}$$

(for the proof see [36], Step 2, p. 6).
Moreover, the strict uniform quasiconvexity condition, stated in (H3), implies the strong Legendre–Hadamard condition (see [42]), that is:

$$D^2G(A)(b\otimes a,b\otimes a)\ge\gamma|b|^2|a|^2 \tag{1.5}$$

for all $b\in\mathbb{R}^N$, $a\in\mathbb{R}^n$, where as usual $b\otimes a$ is the $N\times n$ matrix with entry b^ra^c in row r, column c.

Remark 1.1. *[The Euler-Lagrange system for $q\le p+1$.] If u is a local minimizer of the functional $\mathcal{I}$ and $\phi\in C^1_c(\Omega,\mathbb{R}^N)$ we get by use of (H4) and the minimality condition that for any $\varepsilon>0$:*

$$0\le\int_\Omega[G(Du+\varepsilon D\phi)-G(Du)]\,\mathrm{d}x=\varepsilon\int_\Omega\int_0^1\frac{\partial G}{\partial\xi^i_\alpha}(Du+\varepsilon tD\phi)D_\alpha\phi^i\,\mathrm{d}t\,\mathrm{d}x\,,$$

where the usual summation convention is in force. Dividing this inequality by ε, and letting $\varepsilon\searrow 0$, we infer from (H4) since $q\le p+1$, that

$$\int_\Omega\frac{\partial G}{\partial\xi^i_\alpha}(Du)D_\alpha\phi^i\,\mathrm{d}x\ge 0.$$

Consequently, u is a weak solution to the Euler-Lagrange system for $\mathcal{I}$:

$$\int_\Omega\frac{\partial G}{\partial\xi^i_\alpha}(Du)D_\alpha\phi^i\,\mathrm{d}x=0\qquad\forall\phi\in C^1_c(\Omega,\mathbb{R}^N).$$

We recall that the connection between G-extremality and G-minimality has been considered in [9], [10] for convex integrands, under more general assumptions on the growth conditions of G. Actually in these papers it has been proven that minimizers satisfy the Euler-Lagrange system assuming that the convex integrand is only superlinear at ∞ without any growth assumption from above.
Notice that we do not assume any growth condition on the second derivatives of G and that they do not follow from our hypotheses (H1)–(H3). Instead we observe that since G is C^2 we have for each $M>0$ the bound

$$\sup\Big\{|D^2G(A)|:\,|A|\le M\Big\}=:K_M<\infty. \tag{1.6}$$

We will prove the following $C^{1,\alpha}$- partial regularity result for local G-minimizers.

Theorem 1.1. *Let G satisfy the assumptions (H1), (H2), (H3). If $u\in W^{1,n-1}_{\rm loc}(\Omega,\mathbb{R}^N)$ is a local G-minimizer, then there exists an open subset Ω_0 of Ω such that*

$$\mathrm{meas}(\Omega\setminus\Omega_0)=0$$

and

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N) \qquad \text{for all} \quad \alpha < 1\,.$$

The second application of the trace preserving operator concerns integral bounds for weak solutions to a nonlinear elliptic system and is presented in Section 4.

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2. An extension operator

The aim of this section is to give the proof of our main result from which we will deduce that two $W^{1,p}$ functions can be connected in an annulus with a $W^{1,q}$ function, for some $q > p$ (see Lemmas 2.1 and 2.2 below).

Proof of Proposition 1.1. Because $F(x,y) \to 0$ uniformly in $x \in \mathbb{R}^{n-1}$ as $y \to \infty$, it follows from the Poincaré-Sobolev inequality (when $q < n$) that

$$\|F\|^q_{L^{q^\star}(\mathbb{R}^n_+,\mathbb{R}^N)} \leq c(n,N,q)\|DF\|^q_{L^q(\mathbb{R}^n_+,\mathbb{R}^{N\times n})}.$$

Obviously, when $q = n$ we will have a BMO bound and when $q > n$ a $(1-\frac{n}{q})$-Hölder bound. Hence we can confine ourselves to estimating only the derivatives of F. Carrying out the differentiation gives

$$\begin{aligned} D_{x_i}F(x,y) &= (\mathcal{K}_y * D_{x_i}f)(x) \\ &= \frac{1}{y}\Big((D_{x_i}\mathcal{K})_y * f\Big)(x), \end{aligned} \tag{2.1}$$

where clearly $D_{x_i}\mathcal{K} \in C^\infty_c(\mathcal{B}^{n-1})$ and $\int_{\mathbb{R}^{n-1}} D_{x_i}\mathcal{K} d\mathcal{H}^{n-1} = 0$. Likewise we may write

$$D_yF(x,y) = \frac{1}{y}(\tilde{\mathcal{K}}_y * f)(x) \ \text{ with } \tilde{\mathcal{K}} = -(n-1)K(x) - DK(x).x. \tag{2.2}$$

Note that $\tilde{\mathcal{K}} \in C^\infty_c(\mathcal{B}^{n-1})$ and $\int_{\mathbb{R}^{n-1}} \tilde{\mathcal{K}} d\mathcal{H}^{n-1} = 0$. Moreover, the definition of F and a simple change of variables yield

$$D_yF(x,y) = -\int_{\mathbb{R}^{n-1}} \mathcal{K}(z)\langle Df(x-yz), z\rangle \, d\mathcal{H}^{n-1}(z). \tag{2.3}$$

First we observe that one easily gets that the operator $\mathcal{E}^+$ maps $W^{1,p}(\mathbb{R}^{n-1},\mathbb{R}^N)$ continuously into $\mathcal{W}^{1,p}(\mathbb{R}^n_+,\mathbb{R}^N)$. Namely, for each fixed $x \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned} \int_0^\infty |D_yF(x,y)|^p \, \mathrm{d}y &= \int_0^1 \left| \int_{\mathbb{R}^{n-1}} \mathcal{K}(z)\langle Df(x-yz), z\rangle d\mathcal{H}^{n-1}(z) \right|^p \mathrm{d}y \\ &\quad + \int_1^\infty \left| \int_{\mathbb{R}^{n-1}} \frac{1}{y}(\tilde{\mathcal{K}}_y(z) f(x-z) d\mathcal{H}^{n-1}(z) \right|^p \mathrm{d}y \\ &\leq c(n)\|\mathcal{K}\|^p_{L^\infty} M(|Df|)^p(x) + c(n)\|\tilde{\mathcal{K}}\|^p_{L^\infty} M(|f|)^p(x). \end{aligned} \tag{2.4}$$

Here and in the following we denote by $M(g)$ the Hardy-Littlewood maximal function of g. Integrating (2.4) over $x \in \mathbb{R}^{n-1}$ and using the maximal theorem we obtain

$$\int_{\mathbb{R}^n_+} |D_yF(x,y)|^p \, \mathrm{d}x \, \mathrm{d}y \leq c(n,N,p) \int_{\mathbb{R}^{n-1}} (|Df(x)|^p + |f(x)|^p) \, \mathrm{d}x. \tag{2.5}$$

Similar bounds hold for the other partial derivatives. Next step is to prove inequalities (1.2)–(1.4). To this aim we prove a corresponding inequality for each fixed $x \in \mathbb{R}^{n-1}$ and then we integrate the result with respect to x.

We will examine only the y-derivative of F, since the computations for the other partial derivatives are analogous. First write

$$\begin{aligned}\int_0^\infty |D_y F(x,y)|^q \,\mathrm{d}y &= \int_0^\delta \left| \int_{\mathbb{R}^{n-1}} \mathcal{K}(z)\langle Df(x-yz), z\rangle d\mathcal{H}^{n-1}(z)\right|^q \mathrm{d}y \\ &\quad + \int_\delta^\infty \left| \int_{\mathbb{R}^{n-1}} \frac{1}{y}\tilde{\mathcal{K}}_y(z) f(x-z)\, d\mathcal{H}^{n-1}(z)\right|^q \mathrm{d}y \\ &=: I + II. \end{aligned} \tag{2.6}$$

In order to estimate the term I it suffices to observe that

$$\begin{aligned} I &\leq \int_0^\delta \left(\int_{\mathcal{B}^{n-1}} |\mathcal{K}(z)||Df(x-yz)| d\mathcal{H}^{n-1}(z)\right)^q \mathrm{d}y \\ &\leq c(n)\|\mathcal{K}\|_{L^\infty}^q M(|Df|)^q(x)\delta, \end{aligned} \tag{2.7}$$

where $M(Df)$ is the (Hardy-Littlewood) maximal function of Df. We shall estimate the term II in three separate cases.

Case 1: $f \in W^{1,p}(\mathbb{R}^{n-1}, \mathbb{R}^N) \cap L^r(\mathbb{R}^{n-1}, \mathbb{R}^N)$.
Thanks to Hölder's inequality and the assumptions on $\tilde{\mathcal{K}}$ and f, we have that

$$\begin{aligned} II &\leq \|f\|_{L^r}^q \int_\delta^\infty y^{-nq} \left(\int_{\mathbb{R}^{n-1}} \tilde{\mathcal{K}}^{\frac{r}{r-1}}\left(\frac{z}{y}\right) d\mathcal{H}^{n-1}(z)\right)^{q\frac{r-1}{r}} \mathrm{d}y \\ &\leq c(n)\|\tilde{\mathcal{K}}\|_{L^\infty}^q \|f\|_{L^r}^q \int_\delta^\infty y^{-nq} y^{(n-1)q\frac{r-1}{r}}\,\mathrm{d}y \\ &\leq c(n,q,r)\|\tilde{\mathcal{K}}\|_{L^\infty}^q \|f\|_{L^r}^q \delta^{-nq+1+(n-1)q\frac{r-1}{r}}, \end{aligned} \tag{2.8}$$

where we used that, since the support of $\tilde{\mathcal{K}}$ is contained in the unit ball of $\mathbb{R}^{n-1}$, then

$$\int_{\mathbb{R}^{n-1}} \tilde{\mathcal{K}}^{\frac{r}{r-1}}\left(\frac{z}{y}\right) d\mathcal{H}^{n-1}(z) \leq c(n)\|\tilde{\mathcal{K}}\|_{L^\infty} y^{n-1},$$

and also that, by virtue of the assumption on the exponent q, we have that $-nq+1+(n-1)q\frac{r-1}{r} < 0$. Now, if we choose

$$\delta = \left(\frac{\|f\|_{L^r}}{M(|Df|)(x)} \right)^{\frac{r}{n-1+r}}$$

in (2.7) and in (2.8), we obtain that

$$I + II = \int_0^\infty |D_y F(x,y)|^q \,\mathrm{d}y \leq c(n,q,r,\|\tilde{\mathcal{K}}\|_{L^\infty})\|f\|_{L^r}^{\frac{r}{n-1+r}} M(|Df|)^{q-\frac{r}{n-1+r}}(x) \tag{2.9}$$

Integrating (2.9) with respect to $x \in \mathbb{R}^{n-1}$, we get

$$\int_{\mathbb{R}^n_+} |D_y F(x,y)|^q \,\mathrm{d}x\,\mathrm{d}y \leq c(n,N,q,r,\|\tilde{\mathcal{K}}\|_{L^\infty})\|f\|_{L^r}^{\frac{r}{n-1+r}} \int_{\mathbb{R}^{n-1}} |Df|^p(x)\,\mathrm{d}x \tag{2.10}$$

provided $q = p + \frac{r}{n-1+r}$, by virtue of the maximal theorem.

Case 2: $f \in W^{1,p}(\mathbb{R}^{n-1}, \mathbb{R}^N) \cap BMO(\mathbb{R}^{n-1}, \mathbb{R}^N)$.
Denoting by f_B the mean value of f on the ball $B(0,|y|)$, we have that

$$\begin{aligned} II &= \int_\delta^\infty \left| \int_{\mathbb{R}^{n-1}} \frac{1}{y}(\tilde{\mathcal{K}}_y(z)(f(x-z) - f_B) d\mathcal{H}^{n-1}(z)\right|^q \mathrm{d}y \\ &\leq c(n)\|\tilde{\mathcal{K}}\|_{L^\infty}^q \int_\delta^\infty \frac{1}{y^q}\left(\fint_{B(0,|y|)} |f(x-z) - f_B| d\mathcal{H}^{n-1}(z)\right)^q \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&\leq c(n)\|\tilde{\mathcal{K}}\|^q_{L^\infty}\|f\|^q_{BMO}\int_\delta^\infty y^{-q}\,\mathrm{d}y \\
&\leq c(n,q)\|\tilde{\mathcal{K}}\|^q_{L^\infty}\|f\|^q_{BMO}\delta^{1-q}. \qquad (2.11)
\end{aligned}
$$

Then, inserting (2.7) and (2.11) in (2.6), we get

$$
\begin{aligned}
\int_0^\infty |D_yF(x,y)|^q\,\mathrm{d}y \leq\ & c(n)\|\mathcal{K}\|^q_{L^\infty}M(|Df|)^q(x)\delta \\
& +c(n,q)\|\tilde{\mathcal{K}}\|^q_{L^\infty}\|f\|^q_{BMO}\delta^{1-q}\,. \qquad (2.12)
\end{aligned}
$$

It is easy to verify that for

$$\delta\ =\ \|f\|_{BMO}M(|Df|)^{-1}(x)$$

the following equality

$$M(|Df|)^q(x)\delta\ =\ \|f\|^q_{BMO}\delta^{1-q}$$

is satisfied. Then inserting such δ in (2.12) we obtain

$$\int_0^\infty |D_yF(x,y)|^q\,\mathrm{d}y \leq c(n,q,\|\tilde{\mathcal{K}}\|_{L^\infty})\|f\|_{BMO}M(|Df|)^{q-1}(x). \qquad (2.13)$$

Now, integrating both sides of (2.13) with respect to $x\in\mathbb{R}^{n-1}$ and using the maximal theorem, we get

$$
\begin{aligned}
\int_{\mathbb{R}^n_+}|D_yF(x,y)|^q\,\mathrm{d}x\,\mathrm{d}y \leq\ & c(n,q)\|\mathcal{K}\|^q_{L^\infty}\|f\|_{BMO}\int_{\mathbb{R}^{n-1}}M(|Df|)^{q-1}(x)\,\mathrm{d}x \\
\leq\ & c(n,N,q)\|\mathcal{K}\|^q_{L^\infty}\|f\|_{BMO}\int_{\mathbb{R}^{n-1}}|Df|^p(x)\,\mathrm{d}x, \qquad (2.14)
\end{aligned}
$$

provided $q=p+1$.

Case 3: $f\in W^{1,p}(\mathbb{R}^{n-1},\mathbb{R}^N)\cap C^{0,\alpha}(\mathbb{R}^{n-1},\mathbb{R}^N)$.
Thanks to our assumptions on $\mathcal{K}$, we have that

$$
\begin{aligned}
II\ =\ & \int_\delta^\infty\left|\int_{\mathbb{R}^{n-1}}\frac{1}{y}(\tilde{\mathcal{K}}_y(z)(f(x-z)-f(x))d\mathcal{H}^{n-1}(z)\right|^q\mathrm{d}y \\
\leq\ & \|f\|^q_{C^{0,\alpha}}\int_\delta^\infty\left(\int_{\mathbb{R}^{n-1}}|\tilde{\mathcal{K}}_y(z)|\,d\mathcal{H}^{n-1}(z)\right)^q y^{(\alpha-1)q}\,\mathrm{d}y \\
\leq\ & c(\alpha,n,q)\|\tilde{\mathcal{K}}\|^q_{L^\infty}\|f\|^q_{C^{0,\alpha}}\delta^{1+(\alpha-1)q}. \qquad (2.15)
\end{aligned}
$$

Then, inserting (2.7) and (2.15) in (2.6), we get

$$
\begin{aligned}
\int_0^\infty |D_yF(x,y)|^q\,\mathrm{d}y \leq\ & c(n)\|\mathcal{K}\|^q_{L^\infty}M(|Df|)^q(x)\delta \\
& +c(\alpha,n,q)\|\tilde{\mathcal{K}}\|^q_{L^\infty}\|f\|^q_{C^{0,\alpha}}\delta^{1+(\alpha-1)q}. \qquad (2.16)
\end{aligned}
$$

If we choose

$$\delta\ =\ M(|Df|)^{\frac{1}{\alpha-1}}(x)\|f\|^{\frac{1}{1-\alpha}}_{C^{0,\alpha}}$$

then

$$M(|Df|)^q(x)\delta\ =\ \|f\|^q_{C^{0,\alpha}}\delta^{1+(\alpha-1)q}.$$

Thus inserting such δ in (2.16) we obtain

$$\int_0^\infty |D_yF(x,y)|^q\,\mathrm{d}y \leq c(\alpha,n,q,\|\tilde{\mathcal{K}}\|_{L^\infty})\|f\|^{\frac{1}{1-\alpha}}_{C^{0,\alpha}}M(|Df|)^{q+\frac{1}{\alpha-1}}(x) \qquad (2.17)$$

Now, integrating both sides of (2.17) with respect to $x\in\mathbb{R}^{n-1}$ and using the maximal theorem, we get

$$\int_{\mathbb{R}^n_+}|D_yF(x,y)|^q\,\mathrm{d}x\,\mathrm{d}y \leq c(\alpha,n,q,\|\tilde{\mathcal{K}}\|_{L^\infty})\|f\|^{\frac{1}{1-\alpha}}_{C^{0,\alpha}}\int_{\mathbb{R}^{n-1}}M(|Df|)^{q+\frac{1}{\alpha-1}}(x)\,\mathrm{d}x$$

$$\leq \quad c(\alpha, n, N, q, \|\tilde{\mathcal{K}}\|_{L^\infty})\|f\|_{C^{0,\alpha}}^{\frac{1}{1-\alpha}} \int_{\mathbb{R}^{n-1}} |Df|^p(x)\,\mathrm{d}x, \tag{2.18}$$

provided $q = p + \frac{1}{1-\alpha}$.
In order to conclude the proof it is enough to observe that estimates (2.10), (2.14), and (2.18) hold true for all the derivatives of $F(x, y)$, then summing up we conclude with (1.2), (1.3), and (1.4), respectively. □

Remark 2.1. *Observe that letting $r \nearrow \infty$ in* (1.2) *we obtain an estimate analogous to* (1.3)*, where $\|f\|_{BMO}$ is replaced by $\|f\|_{L^\infty}$. Moreover, if the function $f \in W^{1,p}(\mathbb{R}^{n-1}, \mathbb{R}^N)$ for a $p < n-1$, but does not satisfy any other regularity assumption, then by the Sobolev embedding theorem we know that $f \in L^r(\mathbb{R}^{n-1}, \mathbb{R}^N)$, with $r = \frac{(n-1)p}{n-1-p}$. In this case the exponent of improved integrability is given by $q = \frac{np}{n-1}$, which is exactly the borderline case determined by Lemma 2.4 in* [22]*. However, note that our result holds true also for the borderline exponent $q = \frac{np}{n-1}$, whereas for the result in* [22] *the exponent q has to be strictly less than $\frac{np}{n-1}$. The borderline case was also discussed in* [31].
We also note that if $f \in W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^N)$, then Sobolev's embedding theorem implies that $f \in BMO(\mathbb{R}^{n-1}, \mathbb{R}^N)$. Hence any map $f \in W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^N)$ can be extended to a map F belonging to $\mathcal{W}^{1,n}(\mathbb{R}^n_+, \mathbb{R}^N)$ and

$$\|DF\|^n_{L^n(\mathbb{R}^n_+, \mathbb{R}^{N\times n})} \leq C(n, N)\|Df\|^n_{L^{n-1}(\mathbb{R}^{n-1}, \mathbb{R}^{N\times(n-1)})}.$$

For further needs we now prove the following

Proposition 2.1. *The linear extension operator $\mathcal{E}^+$ defined by* (1.1) *maps $W^{1,2}(\mathbb{R}^{n-1}, \mathbb{R}^N)$ continuously into $\mathcal{W}^{1,2}(\mathbb{R}^n_+, \mathbb{R}^N)$. Moreover we have that there exists a positive constant c depending on n, N such that*

$$\|DF\|^2_{L^2(\mathbb{R}^n_+, \mathbb{R}^{N\times n})} \leq c\|Df\|^2_{L^2(\mathbb{R}^{n-1}, \mathbb{R}^{N\times(n-1)})}. \tag{2.19}$$

Proof. By the definition of $F(x, y)$ and by virtue of the equality (2.2) we have that

$$\begin{aligned}
\int_{\mathbb{R}^n_+} |D_y F|^2\,\mathrm{d}x\mathrm{d}y &= \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} |\tilde{\mathcal{K}}_y \star f|^2 \frac{\mathrm{d}y}{y^2}\mathrm{d}x \\
&= \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} |\widehat{\tilde{\mathcal{K}}_y}|^2 |\widehat{f}|^2 \frac{\mathrm{d}y}{y^2}\mathrm{d}x \\
&= \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} |\widehat{\tilde{\mathcal{K}}}(y\xi)|^2 |\widehat{f}(\xi)|^2 \frac{\mathrm{d}y}{y^2} d\xi \\
&\leq c\int_0^{\frac{1}{|\xi|}}\int_{\mathbb{R}^{n-1}} |y\xi|^2 |\widehat{f}(\xi)|^2 \frac{\mathrm{d}y}{y^2} d\xi + c\int_{\frac{1}{|\xi|}}^{+\infty}\int_{\mathbb{R}^{n-1}} |y\xi|^{-2} |\widehat{f}(\xi)|^2 \frac{\mathrm{d}y}{y^2} d\xi \\
&\leq c\int_{\mathbb{R}^{n-1}} |\xi||\widehat{f}(\xi)|^2\,d\xi \leq c\|f\|_{W^{1/2,2}(\mathbb{R}^{n-1}, \mathbb{R}^N)} \leq c\|Df\|_{L^2(\mathbb{R}^{n-1}, \mathbb{R}^{N\times(n-1)})},
\end{aligned}$$

where the symbol $\widehat{h}$ denotes the Fourier transform of h and we used that

$$|\widehat{\tilde{\mathcal{K}}}(\eta)| \leq c\min\left\{|\eta|, \frac{1}{|\eta|}\right\}.$$

□

It is now routine to construct linear extension operators from smooth hypersurfaces in $\mathbb{R}^n$ with similar mapping properties. Here we shall confine ourselves to spheres where the previous construction can be transferred by use of two stereographic projections and a standard covering argument. Our results can be translated into the following

Corollary 2.1. *Let $v \in W^{1,p}(B_R, \mathbb{R}^N)$, with $p > 1$. Then there exists a function $\tilde{v} \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ such that*

$$\tilde{v} = v \qquad \text{on} \quad \partial B_R$$

and

j) *if $v \in L^r(\partial B_R, \mathbb{R}^N)$ with $\frac{(n-1)p}{n-1-p} \leq r < +\infty$, then*

$$\|\tilde{v}\|^q_{L^{q^\star}(\mathbb{R}^n\setminus B_R, \mathbb{R}^N)} + \|D\tilde{v}\|^q_{L^q(\mathbb{R}^n\setminus B_R, \mathbb{R}^{N\times n})} \leq c(n, N, p, r)\|Dv\|^p_{L^p(\partial B_R, \mathbb{R}^{N\times(n-1)})}\|v\|^{q-p}_{L^r(\partial B_R, \mathbb{R}^N)} \tag{2.20}$$

where $q = p + \frac{r}{n-1+r}$.

jj) *If* $v \in BMO(\partial B_R, \mathbb{R}^N)$, *then*

$$\|\tilde v\|^q_{L^{q^\star}(\mathbb{R}^n\setminus B_R,\mathbb{R}^N)} + \|D\tilde v\|^q_{L^q(\mathbb{R}^n\setminus B_R,\mathbb{R}^{N\times n})} \le c(n,N,p)\|Dv\|^p_{L^p(\partial B_R,\mathbb{R}^{N\times(n-1)})}\|v\|_{BMO(\partial B_R,\mathbb{R}^N)}, \tag{2.21}$$

where $q = p + 1$.

jjj) *If* $v \in C^{0,\alpha}(\partial B_R, \mathbb{R}^N)$, *then*

$$\|\tilde v\|^q_{L^{q^\star}(\mathbb{R}^n\setminus B_R,\mathbb{R}^N)} + \|D\tilde v\|^q_{L^q(\mathbb{R}^n\setminus B_R,\mathbb{R}^{N\times n})} \le c(n,N,\alpha,p)\|Dv\|^p_{L^p(\partial B_R,\mathbb{R}^{N\times(n-1)})}\|v\|^{\frac{1}{1-\alpha}}_{C^{0,\alpha}(\partial B_R,\mathbb{R}^N)}, \tag{2.22}$$

where $q = p + \frac{1}{1-\alpha}$.

Via the inversion in the sphere we get also

Corollary 2.2. *Let* $w \in W^{1,p}(B_R, \mathbb{R}^N)$ *with* $p > 1$. *Then there exists a function* $\tilde w \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ *such that*

$$\tilde w = w \qquad \text{on} \quad \partial B_R$$

and

h) *if* $w \in L^r(\partial B_R, \mathbb{R}^N)$ *with* $\frac{(n-1)p}{n-1-p} \le r < +\infty$ *then, for every* $\gamma \in (0,1)$

$$\|\tilde w\|^q_{L^{q^\star}(B_R\setminus B_{\gamma R},\mathbb{R}^N)} + \|D\tilde w\|^q_{L^q(B_R\setminus B_{\gamma R},\mathbb{R}^{N\times n})} \le \frac{1}{\gamma^2}c(n,N,p,r)\|Dw\|^p_{L^p(\partial B_R,\mathbb{R}^{N\times(n-1)})}\|w\|^{q-p}_{L^r(\partial B_R,\mathbb{R}^N)}, \tag{2.23}$$

where $q = p + \frac{r}{n-1+r}$.

hh) *If* $w \in BMO(\partial B_R, \mathbb{R}^N)$ *then, for every* $\gamma \in (0,1)$

$$\|\tilde w\|^q_{L^{q^\star}(B_R\setminus B_{\gamma R},\mathbb{R}^N)} + \|D\tilde w\|^q_{L^q(B_R\setminus B_{\gamma R},\mathbb{R}^{N\times n})} \le \frac{1}{\gamma^2}c(n,N,p)\|Dw\|^p_{L^p(\partial B_R,\mathbb{R}^{N\times(n-1)})}\|w\|_{BMO(\partial B_R,\mathbb{R}^N)}, \tag{2.24}$$

where $q = p + 1$.

hhh) *If* $v \in C^{0,\alpha}(\partial B_R, \mathbb{R}^N)$ *then, for every* $\gamma \in (0,1)$

$$\|\tilde w\|^q_{L^{q^\star}(B_R\setminus B_{\gamma R},\mathbb{R}^N)} + \|D\tilde w\|^q_{L^q(B_R\setminus B_{\gamma R},\mathbb{R}^{N\times n})} \le \frac{1}{\gamma^2}c(n,N,\alpha,p)\|Dw\|^p_{L^p(\partial B_R,\mathbb{R}^{N\times(n-1)})}\|w\|^{\frac{1}{1-\alpha}}_{C^{0,\alpha}(\partial B_R,\mathbb{R}^N)}, \tag{2.25}$$

where $q = p + \frac{1}{1-\alpha}$.

Proof. Let us consider the inversion in the sphere $S^{n-1}_{x_0,R} = \partial B_{x_0,R}$, i.e., the map defined as

$$\Pi(x) = x_0 + R^2\frac{x-x_0}{|x-x_0|^2} \qquad x \in \mathbb{R}^n \setminus \{x_0\}.$$

Recall that Π is a conformal mapping of the Riemann sphere $\bar{\mathbb{R}}^n$ onto itself. Furthermore

$$|\Pi(x) - x_0| = \frac{R^2}{|x-x_0|} \tag{2.26}$$

and

$$|\Pi(x) - \Pi(y)| = \frac{R^2|x-y|}{|x-x_0||y-x_0|}. \tag{2.27}$$

Thanks to (2.27), Π is a bilipschitzian mapping of any ring domain $B_R \setminus B_{\gamma R}$ $(0 < \gamma < 1)$ onto itself and the Lipschitz constant is $\frac{1}{\gamma^2}$. At this point it suffices to set

$$\tilde w(y) = \bar w(\Pi(y)),$$

where $\bar{w}$ is the function determined by Corollary 2.1 corresponding to w and observe that

$$|D\tilde{w}| \leq \frac{C}{\gamma^2}|D\bar{w}|.$$

□

Now, if we combine estimates (2.20) and (2.23) in Corollary 2.1 and 2.2 respectively, we can prove the following

Lemma 2.1. *Let $v, w \in W^{1,p}(B_1(0), \mathbb{R}^N) \cap L^r(B_1(0), \mathbb{R}^N)$, $p > 1$ and $0 < s < t < 1$. If $p < q \leq p+1$, there exist a function $z \in W^{1,p}(B_1(0), \mathbb{R}^N)$ and numbers $0 < s < s' < t' < t < 1$, depending on v, w such that*

$$\frac{t-s}{8} \leq t' - s' \leq t - s$$

and

$$z = \begin{cases} v & \text{on } B_{s'} \\ w & \text{on } B_1 \setminus B_{t'} \end{cases}$$

Moreover

$$\int_{B_{t'} \setminus B_{s'}} |Dz|^q \, \mathrm{d}x \quad \leq \quad C\frac{t^2 \|v\|^{q-p}_{L^r(\partial B_{s'}, \mathbb{R}^N)}}{s^2(t-s)^{1+\frac{q(n-1)}{n}}} \left(\int_{B_t \setminus B_s} |Dv|^p \, \mathrm{d}x \right) + C\frac{t^2 \|w\|^{q-p}_{L^r(\partial B_{s'}, \mathbb{R}^N)}}{s^2(t-s)^{1+\frac{q(n-1)}{n}}} \left(\int_{B_t \setminus B_s} |Dw|^p \, \mathrm{d}x \right)$$

and

$$\left(\int_{B_{t'} \setminus B_{s'}} |z|^{q^*} \, \mathrm{d}x \right)^{\frac{q}{q^*}} \quad \leq \quad C\frac{t^2 \|v\|^{q-p}_{L^r(\partial B_{s'})}}{s^2(t-s)} \left(\int_{B_t \setminus B_s} |Dv|^p \, \mathrm{d}x \right) + C\frac{t^2 \|w\|^{q-p}_{L^r(\partial B_{s'})}}{s^2(t-s)} \left(\int_{B_t \setminus B_s} |Dw|^p \, \mathrm{d}x \right),$$

where $C = C(n, N, q, p)$ and $q = p + \frac{r}{n-1+r}$.

Proof. Fix $0 < s < t < 1$ and define the set

$$E_{s,t} = \left\{ \rho \in (s,t) : \int_{\partial B_\rho} |Dv(x)|^p \, d\mathcal{H}^{n-1}(x) \quad \leq \quad \frac{4}{t-s} \int_{B_t \setminus B_s} |Dv(x)|^p \, \mathrm{d}x \right.$$
$$\left. \text{and} \int_{\partial B_\rho} |Dw(x)|^p \, d\mathcal{H}^{n-1}(x) \quad \leq \quad \frac{4}{t-s} \int_{B_t \setminus B_s} |Dw(x)|^p \, \mathrm{d}x \right\}.$$

One can easily check that $|E_{s,t}| \geq \frac{t-s}{2}$ and therefore we can fix radii $s', t' \in E_{s,t}$ such that $s' < t'$ and $t' - s' > \frac{t-s}{8}$. Using Corollary 2.1 there exists a function $\tilde{v} \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ such that

$$\tilde{v} = v \qquad \text{on} \quad \partial B_{s'}$$

and

$$\|\tilde{v}\|^q_{L^{q^\star}(\mathbb{R}^n \setminus B_{s'}, \mathbb{R}^N)} + \|D\tilde{v}\|^q_{L^q(\mathbb{R}^n \setminus B_{s'}, \mathbb{R}^{N\times n})} \leq c(n,p,r)\|Dv\|^p_{L^p(\partial B_{s'}, \mathbb{R}^{N\times n})}\|v\|^{q-p}_{L^r(\partial B_{s'}, \mathbb{R}^N)}.$$

Therefore, by the definition of $E_{s,t}$, the previous estimate in particular gives

$$\left(\int_{B_{t'} \setminus B_{s'}} |\tilde{v}|^{q^\star} \, dx \right)^{\frac{q}{q^\star}} \quad + \quad \int_{B_{t'} \setminus B_{s'}} |D\tilde{v}|^q \, \mathrm{d}x$$
$$\leq \quad \|v\|^{q-p}_{L^r(\partial B_{s'}, \mathbb{R}^N)} \left(\frac{C}{t-s} \int_{B_t \setminus B_s} |Dv|^p \, \mathrm{d}x \right). \tag{2.28}$$

On the other hand by Corollary 2.2 we also have that there exists a function $\tilde{w} \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ such that

$$\tilde{w} = w \qquad \text{on} \quad \partial B_{t'}$$

and

$$\|\tilde{w}\|^q_{L^{q^\star}(B_{t'} \setminus B_{\gamma t'}, \mathbb{R}^N)} + \|D\tilde{w}\|^q_{L^q(B_{t'} \setminus B_{\gamma t'}, \mathbb{R}^{N\times n})} \leq \frac{1}{\gamma^2} c(n,N,r,p)\|Dw\|^p_{L^p(\partial B_{t'}, \mathbb{R}^{N\times(n-1)})}\|w\|^{q-p}_{L^r(\partial B_{t'}, \mathbb{R}^N)},$$

for every $\gamma \in (0,1)$. Hence, again by the definition of $E_{s,t}$, the previous estimate in particular yields

$$\begin{aligned} \left(\int_{B_{t'}\setminus B_{s'}} |\tilde{w}|^{q^\star}\,\mathrm{d}x\right)^{\frac{q}{q^\star}} &+ \int_{B_{t'}\setminus B_{s'}} |D\tilde{w}|^q\,\mathrm{d}x \\ &\le \|w\|^{q-p}_{L^r(\partial B_{t'},\mathbb{R}^N)} \left(\frac{C}{t-s}\int_{B_t\setminus B_s} |Dw|^p\,\mathrm{d}x\right). \end{aligned} \tag{2.29}$$

Define the map

$$z = \begin{cases} v & \text{on } B_{s'} \\ \frac{(t'-|x|)\tilde{v}+(|x|-s')\tilde{w}}{t'-s'} & \text{on } B_{t'}\setminus B_{s'} \\ w & \text{on } B_1\setminus B_{t'}. \end{cases}$$

A direct computation shows that in $B_{t'}\setminus B_{s'}$

$$|Dz|^q \le C\Big(|D\tilde{v}|^q + |D\tilde{w}|^q + \frac{|\tilde{v}-\tilde{w}|^q}{(t'-s')^q}\Big). \tag{2.30}$$

Then combining inequality (2.30) with estimates (2.28), (2.29), using Hölder's inequality and the fact that $t'-s' > \frac{t-s}{8}$, we easily get

$$\begin{aligned} \int_{B_{t'}\setminus B_{s'}} |Dz|^q\,\mathrm{d}x &\le C\frac{t^2}{s^2(t-s)^{1+\frac{q(n-1)}{n}}}\|v\|^{q-p}_{L^r(\partial B_{s'},\mathbb{R}^N)}\int_{B_t\setminus B_s}|Dv|^p\,\mathrm{d}x \\ &+ C\frac{t^2}{s^2(t-s)^{1+\frac{q(n-1)}{n}}}\|w\|^{q-p}_{L^r(\partial B_{t'},\mathbb{R}^N)}\int_{B_t\setminus B_s}|Dw|^p\,\mathrm{d}x. \end{aligned} \tag{2.31}$$

The other estimate follows similarly. □

Remark 2.2. *Notice that arguing in the same way we can connect two functions in* $W^{1,p}(B_1(0),\mathbb{R}^N)\cap L^\infty(B_1(0),\mathbb{R}^N)$ *or* $W^{1,p}(B_1(0),\mathbb{R}^N)\cap BMO(B_1(0),\mathbb{R}^N)$ *or in* $W^{1,p}(B_1(0),\mathbb{R}^N)\cap C^{0,\alpha}(B_1(0),\mathbb{R}^N)$ *with a function* $W^{1,q}$ *in a ring domain of* $B_1(0)$*, where q is determined by Theorem 1.1.*

In particular, when the two maps belong to $W^{1,n-1}(B_1(0),\mathbb{R}^N)$, the connection between them is a $W^{1,n}$ map in a ring domain of $B_1(0)$. In this case the regularity assumption on the traces is guaranteed by the borderline Sobolev embedding theorem. Namely, we have

Lemma 2.2. *Let* $v, w \in W^{1,n-1}(B_1(0),\mathbb{R}^N)$ *and* $0<s<t<1$. *There exist a function* $z\in W^{1,n}(B_1(0),\mathbb{R}^N)$ *and* $0<s<s'<t'<r<1$*, depending on* v,w *such that*

$$\frac{t-s}{8} \le t'-s' \le t-s$$

and

$$z = \begin{cases} v & \text{on } B_{s'} \\ w & \text{on } B_1\setminus B_{t'} \end{cases}$$

Moreover

$$\begin{aligned} \int_{B_{t'}\setminus B_{s'}} |Dz|^n\,\mathrm{d}x &\le C\frac{t^2}{s^2(t-s)^{n+\frac{1}{n-1}}}\Big(\int_{B_t\setminus B_s}|Dv|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}} \\ &\quad +C\frac{t^2}{s^2(t-s)^{n+\frac{1}{n-1}}}\Big(\int_{B_t\setminus B_s}|Dw|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}} \end{aligned}$$

and

$$\begin{aligned} \int_{B_{t'}\setminus B_{s'}} |z|^n\,\mathrm{d}x &\le C\frac{t^2}{s^2(t-s)^{\frac{n}{n-1}}}\Big(\int_{B_t\setminus B_s}|Dv|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}} \\ &\quad +C\frac{t^2}{s^2(t-s)^{\frac{n}{n-1}}}\Big(\int_{B_t\setminus B_s}|Dw|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}}, \end{aligned}$$

where $C=C(n,N)$.

3. A PARTIAL REGULARITY RESULT

In this section we shall prove the partial regularity result of Theorem 1.1. As usual, we shall use a linearization argument aimed to establish a decay estimate for the so called excess functional, that measures how the gradient of a local minimizer is far from being constant in a ball.

3.1. **Caccioppoli type estimates.**

The aim of this section is to prove a Caccioppoli type inequality for minimizers of the rescaled functionals defined in Lemma 3.1 below. The inequality can be viewed as a perturbation of the classical Caccioppoli inequality. For our purposes the important point is that the additional terms do not affect the blow-up procedure.

We start by observing that the weak bound on the second derivative at (1.6), via the following result from [1], it suffices to control the growth of certain rescaled functionals that appear in our proof.

Lemma 3.1. *Let $n \geq 2$ and let $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be an integrand of class C^2 satisfying*

$$|G(\xi)| \leq L(1+|\xi|^n) \qquad \text{and} \qquad |DG(\xi)| \leq c(1+|\xi|^{n-1})$$

Then for every $M > 0$ there exists a constant $\beta = \beta(M)$ such that, if we define for $\lambda > 0$ and $A \in \mathbb{R}^{N\times n}$ with $|A| \leq M$ the shifted and rescaled integrand

$$G_{A,\lambda} = \frac{1}{\lambda^2}[G(A+\lambda\xi) - G(A) - \lambda DG(A)\xi] \tag{3.1}$$

then

$$\begin{aligned} |G_{A,\lambda}(\xi)| &\leq \beta(|\xi|^2 + \lambda^{n-2}|\xi|^n) \\ |DG_{A,\lambda}(\xi)| &\leq \beta(|\xi| + \lambda^{n-2}|\xi|^{n-1})\,. \end{aligned} \tag{3.2}$$

Now, we are ready to prove the following

Theorem 3.1. *Let $G_{A,\lambda}$ be a the function defined in* (3.1) *and let $v \in W^{1,n-1}_{\rm loc}(B_1(0),\mathbb{R}^N)$ be a local $G_{A,\lambda}$–minimizer. Then there exists a positive constant $c = c(n,N,\beta,\nu)$ such that*

$$\fint_{B_\tau}\Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big)\,\mathrm{d}x \;\leq\; c\fint_{B_{2\tau}}\frac{|v|^2}{\tau^2}\,\mathrm{d}x + c(\tau)\lambda^{\frac{2}{n-1}}\Big(\lambda^{n-3}\int_{B_{2\tau}}|Dv|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}},$$

for every $0 < \tau < \frac{1}{2}$.

Proof. Fix $0 < \frac{\tau}{2} < s < t < \tau < \frac{1}{2}$. By Lemma 2.2 there exist a function $z \in W^{1,n}(B_1(0),\mathbb{R}^N)$ and $s < s' < t' < t$, depending on v such that

$$\frac{t-s}{8} \leq t' - s' \leq t - s$$

and

$$z = \begin{cases} v & \text{on } B_{s'} \\ v & \text{on } B_1 \setminus B_{t'} \end{cases}$$

Moreover

$$\int_{B_{t'}\setminus B_{s'}}|Dz|^n\,\mathrm{d}x \;\leq\; c\frac{1}{(t-s)^{n+\frac{1}{n-1}}}\Big(\int_{B_t\setminus B_s}|Dv|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}}, \tag{3.3}$$

and

$$\int_{B_{t'}\setminus B_{s'}}|z|^n\,\mathrm{d}x \;\leq\; c\frac{1}{(t-s)^{\frac{n}{n-1}}}\Big(\int_{B_t\setminus B_s}|Dv|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}}, \tag{3.4}$$

where $c = c(n,N)$. Fix now a cut-off function $\eta \in C^\infty_0(B_{t'})$, such that $\eta \equiv 1$ in $B_{s'}$ and $|\nabla\eta| \leq \frac{c}{t'-s'}$. Since $\eta z \in W^{1,n-1}_0(B_{t'})$, the strict uniform quasiconvexity of G stated in (H3), implies

$$\nu\int_{B_{s'}}\Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big)\,\mathrm{d}x \leq \int_{B_{t'}} G_{A,\lambda}(D(\eta z))\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{B_{s'}} G_{A,\lambda}(Dv)\,\mathrm{d}x + \int_{B_{t'}\setminus B_{s'}} G_{A,\lambda}(D(\eta z))\,\mathrm{d}x \\
&\leq \int_{B_{t'}} G_{A,\lambda}(Dv)\,\mathrm{d}x - \int_{B_t'\setminus B_{s'}} G_{A,\lambda}(Dv) + \beta \int_{B_{t'}\setminus B_{s'}} \Big(|D(\eta z)|^2 + \lambda^{n-2}|D(\eta z)|^n\Big)\,\mathrm{d}x \\
&\leq \int_{B_{t'}} G_{A,\lambda}(D((1-\eta)z))\,\mathrm{d}x - \int_{B_{t'}\setminus B_{s'}} G_{A,\lambda}(Dv)\,\mathrm{d}x \\
&\quad +\beta \int_{B_{t'}\setminus B_{s'}} \Big(|D(\eta z)|^2 + \lambda^{n-2}|D(\eta z)|^n\Big)\,\mathrm{d}x \\
&\leq \beta \int_{B_{t'}\setminus B_{s'}} \Big(|D((1-\eta)z)|^2 + \lambda^{n-2}|D((1-\eta)z)|^n\Big)\,\mathrm{d}x \\
&\quad +\beta \int_{B_{t'}\setminus B_{s'}} \Big(|D(\eta z)|^2 + \lambda^{n-2}|D(\eta z)|^n\Big)\,\mathrm{d}x \\
&\quad - \int_{B_{t'}\setminus B_{s'}} G_{A,\lambda}(Dv)\,\mathrm{d}x \\
&=: I + II + III. \qquad (3.5)
\end{aligned}
$$

In previous estimate we used that $\eta z \equiv v$ on the ball $B_{s'}$, the minimality of v and the growth condition (3.2). Using the properties of η, we get

$$
I + II \leq \beta \int_{B_{t'}\setminus B_{s'}} \Big(|Dz|^2 + \lambda^{n-2}|Dz|^n\Big)\mathrm{d}x + c\int_{B_{t'}\setminus B_{s'}} \Big(\frac{|z|^2}{(t'-s')^2} + \lambda^{n-2}\frac{|z|^n}{(t'-s')^n}\Big)\,\mathrm{d}x
$$

Reasoning as in Corollary 2.2, thanks to Proposition 2.1, we evaluate the L^2–norm of Dz, as follows

$$
\int_{B_{t'}\setminus B_{s'}} |Dz|^2 \leq c\int_{B_t\setminus B_s} |Dv|^2 \qquad (3.6)
$$

and

$$
\int_{B_{t'}\setminus B_{s'}} |z|^2 \leq c\int_{B_t\setminus B_s} |v|^2. \qquad (3.7)
$$

Using (3.3), (3.4), (3.6) and (3.7), we get

$$
\begin{aligned}
I + II &\leq c\int_{B_t\setminus B_s} |Dv|^2\,\mathrm{d}x + c\int_{B_\tau} \frac{|v|^2}{(t-s)^2}\,\mathrm{d}x \\
&\quad + \frac{c}{(t-s)^{\frac{n^2}{n-1}}}\lambda^{n-2}\Big(\int_{B_\tau} |Dv|^{n-1}\,\mathrm{d}x\Big)^{\frac{n}{n-1}}. \qquad (3.8)
\end{aligned}
$$

In order to estimate III, we recall that, by virtue of definition of G at (3.1), we have

$$
G_{A,\lambda}(\xi) = \int_0^1\int_0^1 \langle D^2G(A+st\lambda\xi)s\xi,\xi\rangle dsdt
$$

Therefore, we have

$$
\begin{aligned}
III &= -\int_{(B_{t'}\setminus B_{s'})\cap\{|\lambda Dv|\leq 1\}} G_{A,\lambda}(Dv) - \int_{(B_{t'}\setminus B_{s'})\cap\{|\lambda Dv|>1\}} G_{A,\lambda}(Dv) \\
&= -\int_{(B_{t'}\setminus B_{s'})\cap\{|\lambda Dv|\leq 1\}} \int_0^1\int_0^1 \langle D^2G(A+st\lambda Dv)sDv, Dv\rangle dsdt\mathrm{d}x \\
&\quad -\frac{1}{\lambda^2}\int_{(B_{t'}\setminus B_{s'})\cap\{|\lambda Dv|>1\}} G(A+\lambda Dv) - G(A) - DG(A)\lambda Dv \\
&\leq k_M \int_{(B_t\setminus B_s)\cap\{|\lambda Dv|\leq 1\}} |Dv|^2 \\
&\quad +\frac{1}{\lambda^2}\int_{(B_t\setminus B_s)\cap\{|\lambda Dv|>1\}} |G(A)| + |DG(A)||\lambda Dv|
\end{aligned}
$$

$$\leq \; k_M \int_{(B_t\setminus B_s)} |Dv|^2 \,\mathrm{d}x + \frac{1}{\lambda^2} \int_{(B_t\setminus B_s)\cap\{|\lambda Dv|>1\}} (|G(A)| + |DG(A)|)\lambda^2 |Dv|^2 \,dx$$

$$\leq \; k_M \int_{B_t\setminus B_s} |Dv|^2. \tag{3.9}$$

Inserting (3.8) and (3.9) in (3.5), we get

$$\nu \int_{B_s} \Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big) \,\mathrm{d}x \leq \nu \int_{B_{s'}} \Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big) \,\mathrm{d}x$$

$$\leq \; c(n,M) \int_{B_t\setminus B_s} |Dv|^2 \,\mathrm{d}x + c \int_{B_\tau} \frac{|v|^2}{(t-s)^2} \,\mathrm{d}x$$

$$+ \frac{c}{(t-s)^{\frac{n^2}{n-1}}} \lambda^{n-2} \Big(\int_{B_\tau} |Dv|^{n-1} \,\mathrm{d}x \Big)^{\frac{n}{n-1}} \tag{3.10}$$

We fill the hole by adding to both sides of (3.10) the term

$$c(n,M) \int_{B_s} \Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big) \,\mathrm{d}x$$

thus obtaining

$$\int_{B_s} \Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big) \,\mathrm{d}x$$

$$\leq \; \theta \int_{B_t} \Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big) \,\mathrm{d}x + c \int_{B_\tau} \frac{|v|^2}{(t-s)^2} \,\mathrm{d}x$$

$$+ \frac{c}{(t-s)^{\frac{n^2}{n-1}}} \lambda^{n-2} \Big(\int_{B_\tau} |Dv|^{n-1} \,\mathrm{d}x \Big)^{\frac{n}{n-1}} \tag{3.11}$$

with $\theta < 1$. A standard iteration Lemma (see [29]) yields

$$\int_{B_{\frac{\tau}{2}}} \Big(|Dv|^2 + \lambda^{n-3}|Dv|^{n-1}\Big) \,\mathrm{d}x \quad \leq \quad c \int_{B_\tau} \frac{|v|^2}{\tau^2} \,\mathrm{d}x + c(\tau) \lambda^{\frac{2}{n-1}} \Big(\lambda^{n-3} \int_{B_\tau} |Dv|^{n-1} \,\mathrm{d}x \Big)^{\frac{n}{n-1}}.$$

We conclude the proof, dividing previous inequality by $|B_\tau|$. □

3.2. **The decay estimate.** As usual, to get the partial regularity result stated in Theorems 1.1, it suffices to have a decay estimate for the excess function $U(x_0, r)$ defined as

$$U(x_0, r) = \fint_{B_{r(x_0)}} \Big(|Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^{n-1} \Big) \,\mathrm{d}x.$$

The desired decay estimate is established in the next Proposition.

Proposition 3.1. *Assume that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local minimizer for an integrand $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying hypotheses (H1), (H2), (H3).*
Fix $M > 0$. There exists a constant $C_M > 0$ such that for every $0 < \tau < \frac{1}{4}$ there exists $\varepsilon = \varepsilon(\tau, M)$ with the following property. If

$$|(Du)_{x_0,r}| \leq M \qquad \text{and} \qquad U(x_0, r) \leq \varepsilon,$$

then

$$U(x_0, \tau r) \leq C_M \tau^2 U(x_0, r) \quad . \tag{3.12}$$

Proof. Fix M and τ. We shall determine C_M later. We argue by contradiction assuming that there exists a sequence of balls $B_{r_h}(x_h)$ satisfying

$$B_{r_h}(x_h) \subset \Omega, \qquad |(Du)_{x_h,r_h}| \leq M, \qquad \lim_h U(x_h, r_h) = 0,$$

but

$$U(x_h, \tau r_h) > C^*_M \tau^2 U(x_h, r_h) \quad . \tag{3.13}$$

Set

$$A_h = (Du)_{x_h,r_h} \qquad a_h = (u)_{x_h,r_h} \qquad \lambda_h^2 = U(x_h, r_h)$$

Step 1 (Blow up): We rescale the function u in each $B_{r_h}(x_h)$ to obtain a sequence of functions on $B_1 = B_1(0)$. Set

$$v_h(y) = \frac{1}{\lambda_h r_h}[u(x_h + r_h y) - a_h - r_h A_h y], \quad y \in B_1.$$

Then $v_h \in W^{1,p}(B_1)$ and

$$Dv_h(y) = \frac{1}{\lambda_h}[Du(x_h + r_h y) - A_h] \quad .$$

Clearly we have

$$(Dv_h)_{0,1} = 0, \quad |A_h| \le M,$$

and

$$\frac{U(x_h, r_h)}{\lambda_h^2} = \fint_{B_1} \Big(|Dv_h|^2 + \lambda_h^{n-3}|Dv_h|^{n-1}\Big) \mathrm{d}y = 1. \tag{3.14}$$

Then, passing possibly to a subsequence, we may suppose that

$$v_h \rightharpoonup v \qquad \text{weakly} \quad \text{in} \quad W^{1,2}(B_1) \tag{3.15}$$

and

$$A_h \to A. \tag{3.16}$$

Step 2 (v solves a linear system): Now we show that

$$\int_{B_1(0)} \frac{\partial^2 G}{\partial \xi_\alpha^i \partial \xi_\beta^j}(A) D_\beta v^j D_\alpha \phi^i dy = 0 \qquad \forall \phi \in C_0^1(B_1, \mathbb{R}^N). \tag{3.17}$$

The minimizer u is a weak solution to the Euler-Lagrange system (see Remark 1.1). Then, rescaling in each $B_{r_h}(x_h)$, we get for any $\phi \in C_0^1(B_1, \mathbb{R}^N)$ and each $1 \le i \le N$

$$\int_{B_1} \frac{\partial G}{\partial \xi_\alpha^i}(A_h + \lambda_h Dv_h) D_\alpha \phi^i \, dy = 0,$$

and therefore

$$\frac{1}{\lambda_h} \int_{B_1(0)} \Big[\frac{\partial G}{\partial \xi_\alpha^i}(A_h + \lambda_h Dv_h) - \frac{\partial G}{\partial \xi_\alpha^i}(A_h)\Big] D_\alpha \phi^i \, \mathrm{d}y = 0. \tag{3.18}$$

Let us write

$$B_1 = E_h^+ \cup E_h^- = \{y \in B_1 : \lambda_h |Dv_h(y)| > 1\} \cup \{y \in B_1 : \lambda_h |Dv_h(y)| \le 1\} \quad ,$$

then by (3.14) we get

$$|E_h^+| \le \int_{E_h^+} \lambda_h^2 |Dv_h|^2 dy \le \lambda_h^2 \int_{B_1(0)} |Dv_h|^2 dy \le c\lambda_h^2. \tag{3.19}$$

Now, by (H4), we have that

$$\begin{aligned} & \frac{1}{\lambda_h} \left| \int_{E_h^+} \Big[DG(A_h + \lambda_h Dv_h) - DG(A_h)\Big] D\phi dy \right| \\ \le & \left(\frac{c}{\lambda_h} |E_h^+| + c\lambda_h^{n-2} \int_{E_h^+} |Dv_h|^{n-1} \, \mathrm{d}y \right) \|D\phi\|_{L^\infty} \\ \le & \quad c\lambda_h \|D\phi\|_{L^\infty} \,, \end{aligned} \tag{3.20}$$

where we used (3.14) again. From this it follows that

$$\lim_h \frac{1}{\lambda_h} \int_{E_h^+} \Big[DG(A_h + \lambda_h Dv_h) - DG(A_h)\Big] D\phi \, \mathrm{d}y = 0. \tag{3.21}$$

On E_h^- we have

$$
\begin{aligned}
& \frac{1}{\lambda_h}\int_{E_h^-}\Big[DG(A_h+\lambda_h Dv_h)-DG(A_h)\Big]D\phi\,\mathrm{d}y \\
= & \int_{E_h^-}\int_0^1 D^2G(A_h+s\lambda_h Dv_h)Dv_h D\phi \mathrm{d}s\mathrm{d}y \\
= & \int_{E_h^-}\int_0^1 \Big[D^2G(A_h+s\lambda_h Dv_h)-D^2G(A_h)\Big]Dv_h D\phi\,\mathrm{d}s\,\mathrm{d}y \\
& +\int_{E_h^-} D^2G(A_h)Dv_h D\phi\,\mathrm{d}y.
\end{aligned}
$$

Note that (3.19) ensures that $\chi_{E_h^-}\to\chi_{B_1}$ in $L^r(B_1)$ for all $r<\infty$ and by (3.14) we have,

$$\lambda_h Dv_h(y)\to 0 \qquad \text{in measure on} \quad B_1 \quad .$$

Then, by (3.15), (3.16) and the uniform continuity of D^2F on bounded sets, we get

$$
\begin{aligned}
\lim_h \frac{1}{\lambda_h}\int_{E_h^-}&[DG(A_h+\lambda_h Dv_h)-DG(A_h)]D\phi\,\mathrm{d}y \\
&=\int_{B_1} D^2G(A)DvD\phi\,\mathrm{d}y \quad .
\end{aligned}
$$

By (3.18), (3.21) and the above equality, we infer that v solves system (3.17). Since $D^2F(A)$ satisfies the Legendre-Hadamard condition by (H3), we can apply the classical regularity results for the solution v (see for example [29]). Consequently, for any $0<\tau<1/2$, we have

$$\fint_{B_\tau}|Dv-(Dv)_\tau|^2\mathrm{d}y\le c\tau^2\fint_{B_1}|Dv-(Dv)_1|^2\mathrm{d}y\le c\tau^2.$$

and

$$|(Dv)_{2\tau}-(Dv)_\tau|^2\le c\tau^2, \tag{3.22}$$

where $c=c(M)$. Moreover we have

$$v\in C^\infty(B_1,\mathbb{R}^N).$$

Step 3 (Conclusion): We set

$$G_{A_h,\lambda_h}(\xi)=\frac{1}{\lambda_h^2}[G(A_h+\lambda_h\xi)-G(A_h)-\lambda_h DG(A_h)\xi]$$

and for every $r<1$

$$I_{h,r}(w)=\int_{B_r}G_{A_h,\lambda_h}(Dw)dy\ .$$

One can easily check that v_h minimizes the functional $I_{h,r}$ for every h.
Fix $\tau\in(0,\frac{1}{4})$, set $b_h=(v_h)_{B_{2\tau}}$, $B_h=(Dv_h)_{B_\tau}$ and define

$$w_h(y)=v_h(y)-b_h-B_h y.$$

Note that w_h minimizes

$$J_{h,r}(w)=\int_{B_r}\tilde{G}_h(Dw)\,\mathrm{d}y,$$

where

$$\tilde{G}_h(\xi)=\frac{1}{\lambda_h^2}[G(A_h+\lambda_h(\xi+B_h))-G(A_h+\lambda_h B_h)-\lambda_h G(A_h+\lambda_h B_h)\xi]$$

We have that $\tilde{G}_h$ satisfies the inequalities in (3.2) of Lemma 3.1, for some constant that could depend on τ through $|\lambda_h B_h|$. But, given $\tau \in (0, \frac{1}{4})$, we may always choose h large enough to have $|\lambda_h B_h| < \frac{\lambda_h}{\tau^{\frac{n}{2}}} < 1$. Then we can apply to w_h the estimate of Theorem 3.1, thus obtaining

$$
\begin{aligned}
\frac{U(x_h, \tau r_h)}{\lambda_h^2} &= \fint_{B_\tau} \Big(|Dw_h|^2 + \lambda_h^{n-3} |Dw_h|^{n-1} \Big) \, \mathrm{d}y \\
&\le c \fint_{B_{2\tau}} \frac{|w_h|^2}{\tau^2} \, \mathrm{d}y + c(\tau) \lambda_h^{\frac{2}{n-1}} \Big(\lambda_h^{n-3} \int_{B_{2\tau}} |Dw_h|^{n-1} \, \mathrm{d}y \Big)^{\frac{n}{n-1}}. \qquad (3.23)
\end{aligned}
$$

Now, passing to the limit as $h \to \infty$ in (3.23), using Poincaré's inequality and estimate (3.22) we get

$$
\begin{aligned}
\lim_h \frac{U(x_h, \tau r_h)}{\lambda_h^2} &\le \frac{c}{\tau^2} \fint_{B_{2\tau}} (|v - (v)_{2\tau} - (Dv)_{2\tau} y|^2 \, \mathrm{d}y + \frac{c}{\tau^2} \fint_{B_{2\tau}} |(Dv)_{2\tau} - (Dv)_\tau|^2 y^2 \, \mathrm{d}y \\
&\le c \fint_{B_{2\tau}} |(Dv)_{2\tau} - (Dv)_\tau|^2) \, \mathrm{d}y \\
&\le C_M \tau^2, \qquad (3.24)
\end{aligned}
$$

which contradicts (3.13) if we choose C_M^* larger than C_M. □

It follows from our proof that the singular set can be explicitly described as follows:

$$
\Omega \setminus \Omega_0 \subseteq \Big\{ x \in \Omega \colon \ \liminf_{r \to 0} \fint_{B(x,r)} |V(Du) - V((Du)_{x,r})|^2 > 0 \ \text{ or } \ \limsup_{r \to 0} |V((Du)_{x,r})| = \infty \Big\}.
$$

4. A NONLINEAR $\mathcal{A}$-HARMONIC EXTENSION

Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain. Another application of our main result is an $\mathcal{A}$-harmonic extension of Sobolev functions defined on $\partial\Omega$.

More precisely, let us consider $\mathcal{A} \colon \Omega \times \mathbb{R}^{N \times n} \mapsto \mathbb{R}^{N \times n}$ a continuous mapping satisfying, for m, ℓ, L positive constants, a parameter $\mu \ge 0$ and an exponent $p \ge 2$, the following conditions

$$|\mathcal{A}(x,z) - \mathcal{A}(y,z)| \le m(\mu^2 + |z|^2)^{\frac{p-1}{2}} |x - y| \qquad \text{(C1)}$$

$$z \mapsto \mathcal{A}(x,z) \text{ is } C^1 \text{ and } (x,z) \mapsto D_z\mathcal{A}(x,z) \text{ is jointly continuous} \qquad \text{(C2)}$$

$$|D_z\mathcal{A}(x,z)| \le L(\mu^2 + |z|^2)^{\frac{p-2}{2}} \qquad \text{(C3)}$$

$$\langle D_z\mathcal{A}(x,z)\lambda, \lambda \rangle \ge \ell(\mu^2 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2, \qquad \text{(C4)}$$

for every $z, \lambda \in M$ and a.e. $x \in \Omega$.
Let us consider the boundary value problem

$$
\text{(P)} \qquad \begin{cases} -\mathrm{div}\mathcal{A}(x, Du) = 0 & \text{in } \Omega \\ \qquad\qquad u = f & \text{on } \partial\Omega \end{cases}
$$

where $f \in W^{1,p}(\partial\Omega; \mathbb{R}^N)$. It is well known by the Browder-Minty theory of monotone operators that problem (P) admits a unique solution $u \in W^{1,p}(\Omega; \mathbb{R}^N)$. Let

$$\mathcal{S}_{\mathcal{A}} \colon W^{1,p}(\partial\Omega; \mathbb{R}^N) \mapsto W^{1,p}(\Omega; \mathbb{R}^N)$$

denote the solution operator. The aim of this section is to prove the following

Theorem 4.1. *Let $\mathcal{A}$ satisfy the assumption (C1)–(C4). Then we have the following three statements*

k) *If $f \in W^{1,p}(\partial\Omega;\mathbb{R}^N) \cap L^r(\partial\Omega,\mathbb{R}^N)$ with $\frac{(n-1)p}{n-1-p} \leq r < +\infty$ then*

$$\|D\mathcal{S}_\mathcal{A}(f)\|^q_{L^q(\Omega,\mathbb{R}^{N\times n})} \leq c(\Omega,n,N,p,r,m,\ell,L)\|Df\|^p_{L^p(\partial\Omega,\mathbb{R}^{N\times(n-1)})}\|f\|^{q-p}_{L^r(\partial\Omega,\mathbb{R}^N)}, \tag{4.1}$$

where $q = \min\left\{p + \frac{r}{n-1+r}, \frac{pn}{n-2}\right\}$.

kk) *If $f \in W^{1,p}(\partial\Omega;\mathbb{R}^N) \cap \in BMO(\partial\Omega,\mathbb{R}^N)$ then*

$$\|D\mathcal{S}_\mathcal{A}(f)\|^q_{L^q(\Omega,\mathbb{R}^{N\times n})} \leq c(\Omega,n,N,p,m,\ell,L)\|Df\|^p_{L^p(\partial\Omega,\mathbb{R}^{N\times(n-1)})}\|f\|_{BMO(\partial\Omega,\mathbb{R}^N)}, \tag{4.2}$$

where $q = \min\left\{p+1, \frac{pn}{n-2}\right\}$.

kkk) *If $f \in W^{1,p}(\partial\Omega;\mathbb{R}^N) \cap C^{0,\alpha}(\partial\Omega,\mathbb{R}^N)$ then,*

$$\|D\mathcal{S}_\mathcal{A}(f)\|^q_{L^q(\Omega,\mathbb{R}^{N\times n})} \leq c(\Omega,n,N,p,\alpha,m,\ell,L)\|Df\|^p_{L^p(\partial\Omega,\mathbb{R}^{N\times(n-1)})}\|f\|^{\frac{1}{1-\alpha}}_{C^{0,\alpha}(\partial\Omega,\mathbb{R}^N)}, \tag{4.3}$$

where $q = \min\left\{p + \frac{1}{1-\alpha}, \frac{pn}{n-2}\right\}$.

Proof. Suppose that $\Omega = B_R$. We will only give details for the statement k), the others can be proven analogously. Since $f \in W^{1,p}(\partial\Omega;\mathbb{R}^N) \cap L^r(\partial\Omega,\mathbb{R}^N)$, by using Corollary 2.2, we get a function $\tilde f \in W^{1,q}(B_R \setminus B_{\frac{R}{4}})$, with $q = p + \frac{r}{n-1+r}$, such that

$$\|D\tilde f\|^q_{L^q(B_R\setminus B_{\frac{R}{4}},\mathbb{R}^{N\times n})} \leq c(n,N,p,r)\|Df\|^p_{L^p(\partial B_R,\mathbb{R}^{N\times n})}\|f\|^{q-p}_{L^r(\partial B_R,\mathbb{R}^N)}. \tag{4.4}$$

Now let us define

$$F = \begin{cases} 0 & \text{on } B_{\frac{R}{4}} \\ \frac{|x|-R/4}{R/4}\tilde f & \text{on } B_{\frac{R}{2}} \setminus B_{\frac{R}{4}} \\ \tilde f & \text{on } B_R \setminus B_{\frac{R}{2}}. \end{cases}$$

By (4.4), one can easily check that $F \in W^{1,q}(B_R;\mathbb{R}^N)$ and that

$$\|DF\|^q_{L^q(B_R,\mathbb{R}^{N\times n})} \leq c(n,N,p,r,R)\|Df\|^p_{L^p(\partial B_R,\mathbb{R}^{N\times n})}\|f\|^{q-p}_{L^r(\partial B_R,\mathbb{R}^N)}. \tag{4.5}$$

Since $\mathcal{S}_\mathcal{A}(f) = \mathcal{S}_\mathcal{A}(F)$, by applying Theorem 7.8 in [33] we have that $\mathcal{S}_\mathcal{A}(f) \in W^{1,q}(B_R)$, for every $q \leq \min\left\{p + \frac{r}{n-1+r}, \frac{pn}{n-2}\right\}$ and the following estimate

$$\|D\mathcal{S}_\mathcal{A}(f)\|_{L^q(B_R,\mathbb{R}^{N\times n})} \leq C(R,n,N,q,m,\ell,L)\|DF\|_{L^q(B_R,\mathbb{R}^{N\times n})}$$

holds. Combining previous estimate with (4.5) gives the conclusion on balls. The result for general C^2 domains, follows through a standard covering argument. □

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