

Quantifying the hydrodynamic limit of Vlasov-type equations with alignment and nonlocal forces

José A. Carrillo*

*Department of Mathematics,
Mathematical Institute,
University of Oxford, Oxford OX2 6GG, UK
carrillo@maths.ox.ac.uk*

Young-Pil Choi

*Department of Mathematics, Yonsei University,
50 Yonsei-Ro, Seodaemun-Gu,
Seoul 03722, Republic of Korea
ypchoi@yonsei.ac.kr*

Jinwook Jung

*Research Institute of Basic Sciences,
Seoul National University,
Seoul 08826, Republic of Korea
warp100@snu.ac.kr*

Received 16 September 2020

Revised 21 November 2020

Accepted 24 November 2020

Published 24 February 2021

Communicated by N. Bellomo

In this paper, we quantify the asymptotic limit of collective behavior kinetic equations arising in mathematical biology modeled by Vlasov-type equations with nonlocal interaction forces and alignment. More precisely, we investigate the hydrodynamic limit of a kinetic Cucker–Smale flocking model with confinement, nonlocal interaction, and local alignment forces, linear damping and diffusion in velocity. We first discuss the hydrodynamic limit of our main equation under strong local alignment and diffusion regime, and we rigorously derive the isothermal Euler equations with nonlocal forces. We also analyze the hydrodynamic limit corresponding to strong local alignment without diffusion. In this case, the limiting system is pressureless Euler-type equations. Our analysis includes the Coulomb interaction potential for both cases and explicit estimates on the distance towards the limiting hydrodynamic equations. The relative entropy method is

*Corresponding author.

This is an Open Access article published by World Scientific Publishing Company. It is distributed under the terms of the Creative Commons Attribution 4.0 (CC BY) License which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

the crucial technology in our main results, however, for the case without diffusion, we combine a modulated macroscopic kinetic energy with the bounded Lipschitz distance to deal with the nonlocality in the interaction forces. The existence of weak and strong solutions to the kinetic and fluid equations is also obtained. We emphasize that the existence of global weak solution with the needed free energy dissipation for the kinetic model is established.

Keywords: Hydrodynamic limit; Euler equations; Vlasov equation; relative entropy; bounded Lipschitz distance.

AMS Subject Classification 2020: 35B40, 82C40

1. Introduction

Collective self-organized motions of autonomous individuals, such as flocks of birds, crowd dynamics, and aggregation of bacteria, etc, appear in many applications in the field of engineering, biology, and sociology,^{2–4,25,36,39,40,42,45} we refer to Refs. 10 and 19 and references therein for recent surveys. Mathematical modeling of such behaviors is based on Individual-Based Models (IBMs) which are microscopic descriptions, and it includes three basic effects, a short-range repulsion, a long-range attraction, and an alignment in certain spatial regions. These IBMs lead to continuum description by means of mean-field limit,^{5,7,8,22,31,32} and in particular a second-order N -particle system converges toward a kinetic equation as the number of particles N goes to infinity. In this paper, we study a class of such kinetic-type models which are typically Vlasov-type equations with nonlocal forces. More precisely, let $f = f(x, v, t)$ be the one-particle distribution function at $(x, v) \in \Omega \times \mathbb{R}^d$ and at time $t > 0$, where Ω is either \mathbb{T}^d or \mathbb{R}^d with $d \geq 1$, then our main equation is given by

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot ((\gamma v + \lambda(\nabla_x V + \nabla_x W \star \rho))f) + \alpha \nabla_v \cdot (F[f]f) = \mathcal{N}_{\text{FP}}[f], \quad (1.1)$$

with $(x, v, t) \in \Omega \times \mathbb{R}^d \times \mathbb{R}_+$ subject to the initial data

$$f(x, v, t)|_{t=0} =: f_0(x, v), \quad (x, v) \in \Omega \times \mathbb{R}^d,$$

where $\rho = \rho(x, t)$ and $u = u(x, t)$ are the local particle density and velocity given by

$$\rho = \int_{\mathbb{R}^d} f \, dv \quad \text{and} \quad u = \frac{\int_{\mathbb{R}^d} v f \, dv}{\int_{\mathbb{R}^d} f \, dv},$$

respectively, $V : \Omega \rightarrow \mathbb{R}$ and $W : \Omega \rightarrow \mathbb{R}$ are the confinement and the interaction potentials with a positive coefficient λ , respectively. Here \mathcal{N}_{FP} denotes the nonlinear Fokker–Planck operator⁴⁶ given by

$$\mathcal{N}_{\text{FP}}[f](x, v) := \nabla_v \cdot (\beta(v - u)f + \sigma \nabla_v f) = \sigma \nabla_v \cdot \left(f \nabla_v \log \frac{f}{M_u} \right)$$

with the local Maxwellian

$$M_u := \frac{\beta^{d/2}}{(2\pi\sigma)^{d/2}} \exp\left(-\frac{\beta|u - v|^2}{2\sigma}\right),$$

and positive constants β and σ . F represents the velocity alignment force fields, where the local average of relative velocities weighted by the function ϕ , given by

$$F[f](x, v) := \int_{\Omega \times \mathbb{R}^d} \phi(x - y)(w - v)f(y, w) dy dw,$$

where $\phi : \Omega \rightarrow \mathbb{R}_+$ is called a communication weight. The confinement and interaction potentials are assumed to be symmetric in the sense $V(x) = V(-x)$ and $W(x) = W(-x)$ on Ω due to the action–reaction principle by Newton’s third law. The weight function ϕ is usually assumed to be radially symmetric, i.e. $\phi(x) = \hat{\phi}(|x|)$ for some $\hat{\phi} : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+$, and $\hat{\phi}$ is decreasing such that the closer particles have more stronger influence than the further ones. The right-hand side of (1.1) consists of the local alignment forces and the diffusion term in velocity. Throughout this paper, we also assume that f is a probability density, i.e. $\|f(\cdot, \cdot, t)\|_{L^1} = 1$ for $t \geq 0$, since the total mass is preserved in time.

In this work, we are interested in the asymptotic analysis of (1.1) by considering singular parameters. More specifically, we deal with hydrodynamic limits to isothermal/pressureless Euler equations with nonlocal forces.

1.1. Formal derivation from kinetic to isothermal/pressureless Euler equations

Taking into account the moments on the kinetic equation (1.1), we find that the local density ρ and local velocity u satisfy

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \cdot \left(\int_{\mathbb{R}^d} (v - u) \otimes (v - u) f(x, v, t) dv \right) \\ &= -\gamma \rho u - \lambda \rho (\nabla_x V + \nabla_x W \star \rho) - \alpha \rho \int_{\Omega} \phi(x - y)(u(x) - u(y)) \rho(y) dy. \end{aligned}$$

We note that the above system is not closed in the sense that it cannot be written only in terms of ρ and u . On the other hand, if we consider the singular parameters $\beta = \sigma = 1/\varepsilon$ in (1.1), i.e. the local alignment and diffusive forces are very strong and consider the limit $\varepsilon \rightarrow 0$, then at the formal level, we expect that $\mathcal{N}_{\text{FP}} \simeq 0$, and this leads that the particle density behaves like

$$f^\varepsilon(x, v, t) \simeq \frac{\rho(x, t)}{(2\pi)^{d/2}} \exp\left(-\frac{|v - u(x, t)|^2}{2}\right) \quad \text{for } \varepsilon \ll 1, \quad (1.2)$$

where f^ε denotes the corresponding solution of (1.1) with $\beta = \sigma = 1/\varepsilon$. This formal procedure gives the isothermal Euler equations with interaction forces

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \quad (x, t) \in \Omega \times \mathbb{R}_+, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \rho &= -\gamma \rho u - \lambda \rho (\nabla_x V + \nabla_x W \star \rho) \\ &\quad - \alpha \rho \int_{\Omega} \phi(x - y)(u(x) - u(y)) \rho(y) dy. \end{aligned} \quad (1.3)$$

Let us now take into account the hydrodynamic limit without diffusion, i.e. Eq. (1.1) with $\beta = 1/\varepsilon$ and $\sigma = 0$. Then, for the similar reason, we find that

$$f^\varepsilon(x, v, t) \simeq \rho(x, t) \otimes \delta_{u(x, t)}(v) \quad \text{for } \varepsilon \ll 1, \quad (1.4)$$

and this induces the following pressureless Euler equations with interaction forces:

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \quad (x, t) \in \Omega \times \mathbb{R}_+, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) &= -\gamma \rho u - \lambda \rho (\nabla_x V + \nabla_x W \star \rho) \\ &\quad - \alpha \rho \int_{\Omega} \phi(x - y)(u(x) - u(y)) \rho(y) dy. \end{aligned} \quad (1.5)$$

Some previous works closely related to the above asymptotic analysis can be summarized as follows. The asymptotic analysis for the kinetic Cucker–Smale model with a strong local alignment force and a strong diffusion, i.e. (1.1) with $\gamma = 0$, $\lambda, \alpha > 0$, $V, W \equiv 0$, $\sigma = \beta = 1/\varepsilon$, is investigated in Ref. 35. In this regime, the isothermal Euler system with the nonlocal velocity alignment forces, (1.3) with $\gamma = \lambda = 0$ and $\alpha > 0$ is rigorously derived, see also Ref. 18 for the global regularity of classical solutions of that system. In this work, the relative entropy method is employed, and the presence of the pressure term in the limiting system plays an important role in their strategy: it gives the convexity of the entropy with respect to the density ρ ; see Sec. 3 for details. For the diffusionless case, in Ref. 28, the velocity alignment term $F[f]$ is taken into account in the hydrodynamic limit, i.e. Eq. (1.1) with $V, W \equiv 0$, $\gamma = \sigma = 0$, $\alpha > 0$, and $\beta = 1/\varepsilon$ in the periodic spatial domain, and the pressureless Euler equations with the velocity alignment forces, (1.5) with $\lambda = \gamma = 0$ and $\alpha > 0$, which is also referred to *Euler alignment system* in Ref. 11, are rigorously derived. In that work, the modulated macroscopic energy combined with the second-order Wasserstein distance is used. This strategy is improved in a recent work⁶ where the whole space case is considered, see also Ref. 17 for the relation between modulated macroscopic kinetic energy and the p th-order Wasserstein distance. It is worth noting that the interaction potential W is not taken into account in Refs. 28 and 35, and it is not clear that the strategies used in that work can be applied to the case with the interaction potential W when W has a rather weak regularity, see Ref. 6 for the case with regular interaction potentials W . On the other hand, for the Coulomb interactions W , i.e. $-\Delta_x W \star \rho = \rho$, the hydrodynamic limit of Vlasov–Poisson equation with strong local alignment forces, which corresponds to (1.1) with $\gamma = \alpha = \sigma = 0$, $V \equiv 0$, $\beta = 1/\varepsilon$, is discussed in Ref. 33.

The main purpose of this work is to consider the most general form of kinetic swarming models (1.1) and identify regimes where the Euler-type equations (1.3) or (1.5) are well approximated by the kinetic equation (1.1) in a quantifiable way. We first deal with Eq. (1.1) with strong local alignment and diffusive forces, that is, we consider a singular parameter in the nonlinear Fokker–Planck operator \mathcal{N}_{FP} . In this case, as mentioned above, the limiting system is expected to be the isothermal

Euler-type system (1.3). We estimate the relative entropy functional together with the free energy to have the quantitative error estimate between solutions f^ε of (1.1) and (ρ, u) of (1.3). In particular, we make the formal observation (1.2) completely rigorous with a quantitative bound in terms of ε , see Corollary 2.2. Due to the presence of pressure, L^∞ bound assumptions for both the interaction potential W and the communication weight function ϕ are sufficient to have that estimate of hydrodynamic limit. We are also able to deal with the Coulomb potential for W .

In the case without diffusion, the limiting system is a pressureless Euler system (1.5), thus the corresponding macroscopic kinetic energy is not strictly convex with respect to ρ . In this respect, it is not clear to have the quantitative bound error estimate between solutions by means of the estimate of modulated kinetic energy only. For that reason, we combine the modulated kinetic energy estimate and the bounded Lipschitz distance between local particle density ρ^ε and the fluid density ρ . Note that the bounded Lipschitz distance and the first-order Wasserstein distance are equivalent in the set of probability measure with a bounded first moment. Thus, our result improves the previous works,^{6,28} where the second-order Wasserstein distance is used as mentioned above. We show that the bounded Lipschitz distance between densities can be bounded from the above by the modulated macroscopic kinetic energy, see Lemma 4.1. Compared to the case with pressure, we need rather stronger assumptions for W and ϕ , bounded and Lipschitz continuity. Combining these observations, we close the modulated kinetic energy estimates and obtain the quantitative error estimates between solutions f^ε of (1.1) with $\beta = 1/\varepsilon$, $\sigma = 0$ and (ρ, u) of (1.5). As we expected from the formal derivation (1.4), the particle distribution function f^ε converges to the monokinetic ansatz in the sense of distributions also quantified in terms of the bounded Lipschitz distance, see Corollary 2.3 and the proofs in Sec. 2.3. Even in the pressureless case, we are also able to take into account the Coulomb interaction potential W and establish the same convergence estimates with the regular interaction potential case.

Our main mathematical tool is based on the weak-strong uniqueness principle,²⁶ and thus for the rigorous asymptotic analysis mentioned above, the existence of weak solutions of the kinetic equation (1.1) and strong solutions to the limiting systems (1.3) and (1.5) should be obtained at least locally in time. We emphasize that we have also constructed global-in-time weak solutions of (1.1) for any dimension and general interaction potentials satisfying the free energy estimate. Note that existing results^{28,33,35} did not include the nonlinear interaction potential W and the confining potential V terms.

Here we introduce several notations used throughout this work. For functions, $f(x, v)$ and $g(x)$, $\|f\|_{L^p}$ and $\|g\|_{L^p}$ represent the usual $L^p(\Omega \times \mathbb{R}^d)$ - and $L^p(\Omega)$ -norms, respectively. We denote by C a generic positive constant which may differ from line to line. For simplicity, we often drop x -dependence of differential operators, that is, $\nabla f := \nabla_x f$ and $\Delta f := \Delta_x f$. For any nonnegative integer k and $p \in [1, \infty]$, $\mathcal{W}^{k,p} = \mathcal{W}^{k,p}(\Omega)$ stands for the k th order L^p Sobolev space. In particular, if $p = 2$, we denote by $H^k = H^k(\Omega) = \mathcal{W}^{k,2}(\Omega)$. $C^k([0, T]; E)$ is the set of k -times

continuously differentiable functions from an interval $[0, T] \subset \mathbb{R}$ into a Banach space E , and $L^p(0, T; E)$ is the set of measurable functions from an interval $(0, T)$ to a Banach space E , whose p th power of the E -norm is Lebesgue measurable. For $\Pi \subseteq \mathbb{R}^n$, $\mathcal{C}_c^\infty(\Pi)$ denotes the set of infinitely differentiable functions whose support is compact and contained in Π . ∇^k denotes any partial derivative ∂^α with multi-index α , $|\alpha| = k$.

The rest of this paper is organized as follows. In the next section, we provide several *a priori* estimates of free energy inequalities. We also give precise statements of our main results on the asymptotic analysis of (1.1). In Sec. 3, we consider our main equation (1.1) in the regime of strong local alignment and diffusion, i.e. $\beta = \sigma = 1/\varepsilon$. We show that the weak solution to the kinetic equation (1.1) strongly converges to the strong solution to the isothermal Euler equations with nonlocal interaction forces (1.3). Section 4 is devoted to the asymptotic analysis for the diffusionless case, i.e. (1.1) with $\sigma = 0$. In this case, we consider the strong local alignment regime, $\beta = 1/\varepsilon$ and provide the rigorous convergence estimates of solutions f^ε to the pressureless Euler system with nonlocal interactions forces (1.5). Finally, in Secs. 5 and 6, we provide the details on the global-in-time existence of weak solutions for the kinetic equation (1.1) satisfying the free energy estimate and the local-in-time existence and uniqueness of classical solutions to the isothermal/pressureless Euler equations (1.3) and (1.5).

2. Preliminaries and Main Results

2.1. Free energy estimates

In this part, we provide free energy estimates. For this, we introduce the free energy \mathcal{F} and the associated dissipations \mathcal{D}_1 , \mathcal{D}_2 , and \mathcal{D}_3 as follows:

$$\begin{aligned}\mathcal{F}(f) &:= \int_{\Omega \times \mathbb{R}^d} \frac{\sigma}{\beta} f \log f \, dx dv + \frac{1}{2} \int_{\Omega \times \mathbb{R}^d} |v|^2 f \, dx dv \\ &\quad + \frac{\lambda}{2} \int_{\Omega \times \Omega} W(x-y) \rho(x) \rho(y) \, dx dy + \lambda \int_{\Omega} V \rho \, dx, \\ \mathcal{D}_1(f) &:= \int_{\Omega \times \mathbb{R}^d} \frac{1}{f} \left| \frac{\sigma}{\beta} \nabla_v f - f(u-v) \right|^2 \, dx dv, \\ \mathcal{D}_2(f) &:= \frac{1}{2} \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |v-w|^2 f(x, v) f(y, w) \, dx dy dv dw,\end{aligned}$$

and

$$\mathcal{D}_3(f) := \int_{\Omega \times \mathbb{R}^d} |v|^2 f \, dx dv,$$

respectively.

Then, we have the following free energy estimate.

Lemma 2.1. Suppose that f is a solution of (1.1) with sufficient integrability. Then we have

$$\frac{d}{dt}\mathcal{F}(f) + \beta\mathcal{D}_1(f) + \alpha\mathcal{D}_2(f) + \gamma\mathcal{D}_3(f) = \frac{\sigma\gamma d}{\beta} + \frac{\sigma\alpha d}{\beta} \int_{\Omega \times \Omega} \phi(x-y)\rho(x)\rho(y) dx dy.$$

In particular, we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega \times \mathbb{R}^d} |v|^2 f dx dv + \frac{\lambda}{2} \int_{\Omega \times \Omega} W(x-y)\rho(x)\rho(y) dx dy + \lambda \int_{\Omega} V\rho dx \right) \\ &= -\beta \int_{\Omega \times \mathbb{R}^d} f|u-v|^2 dx dv - \alpha\mathcal{D}_2(f) - \gamma\mathcal{D}_3(f), \end{aligned} \quad (2.1)$$

when $\sigma = 0$.

Proof. A straightforward computation gives

$$\begin{aligned} \frac{d}{dt} \int_{\Omega \times \mathbb{R}^d} \frac{\sigma}{\beta} f \log f dx dv &= \frac{\sigma}{\beta} \int_{\Omega \times \mathbb{R}^d} \nabla_v \cdot ((\gamma v + \lambda(\nabla V + \nabla W \star \rho))f) \log f dx dv \\ &\quad - \frac{\sigma\alpha}{\beta} \int_{\Omega \times \mathbb{R}^d} \nabla_v \cdot (F[f]f) \log f dx dv \\ &\quad + \frac{\sigma}{\beta} \int_{\Omega \times \mathbb{R}^d} \nabla_v \cdot (\beta(v-u)f + \sigma \nabla_v f) \log f dx dv \\ &=: \sum_{i=1}^3 I_i, \end{aligned}$$

where $I_i, i = 1, 2, 3$, can be estimated as follows:

$$\begin{aligned} I_1 &= \frac{\sigma}{\beta} \int_{\Omega \times \mathbb{R}^d} \nabla_v \cdot (\gamma v + \lambda(\nabla V + \nabla W \star \rho))f dx dv = \frac{\sigma\gamma d}{\beta}, \\ I_2 &= -\frac{\sigma\alpha}{\beta} \int_{\Omega \times \mathbb{R}^d} \nabla_v \cdot (F[f]f) dx dv = \frac{\sigma\alpha d}{\beta} \int_{\Omega \times \Omega} \phi(x-y)\rho(x)\rho(y) dx dy, \\ I_3 &= -\frac{\sigma}{\beta} \int_{\Omega \times \mathbb{R}^d} (\beta(v-u)f + \sigma \nabla_v f) \cdot \frac{\nabla_v f}{f} dx dv. \end{aligned}$$

We also estimate the kinetic energy as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega \times \mathbb{R}^d} |v|^2 f dx dv \\ &= -\gamma \int_{\Omega \times \mathbb{R}^d} |v|^2 f dx dv - \frac{\lambda}{2} \frac{d}{dt} \int_{\Omega \times \Omega} W(x-y)\rho(x)\rho(y) dx dy - \lambda \frac{d}{dt} \int_{\Omega} V\rho dx \\ &\quad - \frac{\alpha}{2} \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y)|v-w|^2 f(x,v)f(y,w) dx dy dv dw \\ &\quad - \beta \int_{\Omega \times \mathbb{R}^d} |v-u|^2 f dx dv - \sigma \int_{\Omega \times \mathbb{R}^d} v \cdot \nabla_v f dx dv. \end{aligned}$$

Combining the above estimates yields

$$\frac{d}{dt}\mathcal{F}(f) + \beta\mathcal{D}_1(f) + \alpha\mathcal{D}_2(f) + \gamma\mathcal{D}_3(f) = \frac{\sigma\lambda d}{\beta} + \frac{\sigma\alpha d}{\beta} \int_{\Omega \times \Omega} \phi(x-y)\rho(x)\rho(y) dx dy.$$

□

Lemma 2.1 shows that the linear damping in velocity and nonlocal velocity generate the free energy increase. In the proposition below, we show that they are controlled by the dissipations and the free energy.

Proposition 2.1. *Suppose that f is a solution of (1.1) with sufficient integrability. Then we have*

$$\begin{aligned} \mathcal{F}(f) + \int_0^t \left(\frac{\beta}{2} \mathcal{D}_1(f) + \gamma \int_{\Omega} \rho |u|^2 dx \right) ds \\ + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \phi(x-y) |u(x) - u(y)|^2 \rho(x) \rho(y) dx dy ds \\ \leq \mathcal{F}(f_0) \exp \left(\frac{C}{\beta} (1 + \gamma^2) T \right). \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} \mathcal{F}(f) + \int_0^t \left(\frac{\beta}{2} \mathcal{D}_1(f) + \gamma \int_{\Omega} \rho |u|^2 dx \right) ds \\ + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \phi(x-y) |u(x) - u(y)|^2 \rho(x) \rho(y) dx dy ds \\ \leq \mathcal{F}(f_0) + \left(\frac{C}{\beta} (1 + \gamma^2) \right), \end{aligned} \quad (2.2)$$

where $C > 0$ depends only T , f_0 and $\|\phi\|_{L^\infty}$.

Proof. It follows from Proposition 2.1 in Ref. 35 or Lemma 7.3 in Ref. 34 that

$$\begin{aligned} \frac{\alpha}{2} \int_{\Omega \times \Omega} \phi(x-y) |u(x) - u(y)|^2 \rho(x) \rho(y) dx dy - \frac{\beta}{4} \mathcal{D}_1(f) \\ \leq \frac{C}{\beta} \mathcal{F}(f) + \alpha \mathcal{D}_2(f) - \frac{\sigma\alpha d}{\beta} \int_{\Omega \times \Omega} \phi(x-y) \rho(x) \rho(y) dx dy, \end{aligned}$$

where C depends only on T , $\|\phi\|_{L^\infty}$. On the other hand, a straightforward computation gives

$$\begin{aligned} \frac{\gamma\sigma d}{\beta} &= \gamma \int_{\Omega \times \mathbb{R}^d} v \cdot \left(f(u-v) - \frac{\sigma}{\beta} \nabla_v f \right) dx dv - \gamma \int_{\Omega \times \mathbb{R}^d} v \cdot f(u-v) dx dv \\ &=: J_1 + J_2, \end{aligned}$$

where J_2 can be estimated as

$$J_2 = \gamma \int_{\Omega \times \mathbb{R}^d} f |v|^2 dx dv - \gamma \int_{\Omega} \rho |u|^2 dx.$$

For the estimate of J_1 , we use Hölder inequality to get

$$\begin{aligned} J_1 &= \gamma \int_{\Omega \times \mathbb{R}^d} \sqrt{f} v \cdot \frac{1}{\sqrt{f}} \left(f(u-v) - \frac{\sigma}{\beta} \nabla_v f \right) dx dv \\ &\leq \gamma \left(\int_{\Omega \times \mathbb{R}^d} |v|^2 f dx dv \right)^{1/2} \mathcal{D}_1(f)^{1/2} \\ &\leq \frac{\gamma^2}{\beta} \int_{\Omega \times \mathbb{R}^d} |v|^2 f dx dv + \frac{\beta}{4} \mathcal{D}_1(f), \end{aligned}$$

i.e.

$$J_1 \leq \frac{C\gamma^2}{\beta} \mathcal{F}(f) + \frac{\beta}{4} \mathcal{D}_1(f).$$

Thus, we have

$$\frac{\gamma \sigma d}{\beta} \leq \frac{C\gamma^2}{\beta} \mathcal{F}(f) + \frac{\beta}{4} \mathcal{D}_1(f) + \gamma \mathcal{D}_3(f) - \gamma \int_{\Omega} \rho |u|^2 dx.$$

Now we combine the above estimates together with Lemma 2.1 to obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(f) + \frac{\beta}{2} \mathcal{D}_1(f) + \gamma \int_{\Omega} \rho |u|^2 dx + \frac{\alpha}{2} \int_{\Omega \times \Omega} \phi(x-y) |u(x) - u(y)|^2 \rho(x) \rho(y) dx dy \\ \leq \frac{C}{\beta} (1 + \gamma^2) \mathcal{F}(f). \end{aligned}$$

Applying Gronwall's inequality to the above concludes the desired first result. The second inequality just follows from the first result and the above inequality. \square

2.2. Main results

For the hydrodynamic limit to isothermal/pressureless Euler system with nonlocal forces, we use the relative entropy argument. For this, we need to establish the existence of weak solutions to Eq. (1.1) and the existence of the unique strong solution to the system (1.3) and (1.5) at least locally in time. Thus, we first present a notion of weak solutions of Eq. (1.1).

Definition 2.1. For a given $T \in (0, \infty)$, we say that f is a weak solution to Eq. (1.1) if the following conditions are satisfied:

- (i) $f \in L^\infty(0, T; (L^1_+ \cap L^\infty)(\Omega \times \mathbb{R}^d))$,
- (ii) for any $\varphi \in \mathcal{C}_c^\infty(\Omega \times \mathbb{R}^d \times [0, T])$,

$$\begin{aligned} &\int_0^t \int_{\Omega \times \mathbb{R}^d} f (\partial_s \varphi + v \cdot \nabla \varphi - (\gamma v + \lambda(\nabla V + \nabla W \star \rho)) \cdot \nabla_v \varphi) dx dv ds \\ &\quad + \int_0^t \int_{\Omega \times \mathbb{R}^d} f ((\alpha F[f] + \beta(u-v)) \cdot \nabla_v \varphi + \sigma \Delta_v \varphi) dx dv ds \\ &\quad + \int_{\Omega \times \mathbb{R}^d} f_0 \varphi(x, v, 0) dx dv = 0. \end{aligned}$$

We next state definitions of strong solutions to the systems (1.3) and (1.5) below.

Definition 2.2. For given $T \in (0, \infty)$, the pair (ρ, u) is a strong solution of (1.3) on the time interval $[0, T]$ if and only if the following conditions are satisfied:

- (i) $(\rho, u) \in \mathcal{C}([0, T]; L_+^1(\Omega)) \times \mathcal{C}([0, T]; \mathcal{W}^{1,\infty}(\Omega))$,
- (ii) (ρ, u) satisfies the following free energy estimate in the sense of distributions:

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} \rho |u|^2 dx + \int_{\Omega} \rho \log \rho dx + \lambda \int_{\Omega} \rho V dx + \frac{\lambda}{2} \int_{\Omega} (W \star \rho) \rho dx \right) \\ = -\gamma \int_{\Omega} \rho |u|^2 dx - \frac{\alpha}{2} \int_{\Omega \times \Omega} \phi(x-y) |u(x) - u(y)|^2 \rho(x) \rho(y) dx dy, \end{aligned}$$

- (iii) (ρ, u) satisfies the system (1.3) in the sense of distributions.

Definition 2.3. For given $T \in (0, \infty)$, the pair (ρ, u) is a strong solution of (1.5) on the time interval $[0, T]$ if and only if the following conditions are satisfied:

- (i) $(\rho, u) \in \mathcal{C}([0, T]; L_+^1(\Omega)) \times \mathcal{C}([0, T]; \mathcal{W}^{1,\infty}(\Omega))$,
- (ii) (ρ, u) satisfies the following free energy estimate in the sense of distributions:

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} \rho |u|^2 dx + \lambda \int_{\Omega} \rho V dx + \frac{\lambda}{2} \int_{\Omega} (W \star \rho) \rho dx \right) \\ = -\gamma \int_{\Omega} \rho |u|^2 dx - \frac{\alpha}{2} \int_{\Omega \times \Omega} \phi(x-y) |u(x) - u(y)|^2 \rho(x) \rho(y) dx dy, \end{aligned}$$

- (iii) (ρ, u) satisfies the system (1.5) in the sense of distributions.

Before providing our results on the hydrodynamic limits, we list our main assumptions on the initial data below.

(H1) The initial data related to the entropy are well-prepared

$$\begin{aligned} \int_{\Omega} (\rho_0^\varepsilon (\log \rho_0^\varepsilon - \log \rho_0) + (\rho_0 - \rho_0^\varepsilon)) dx = \mathcal{O}(\sqrt{\varepsilon}) \quad \text{and} \\ \int_{\Omega} \left(\int_{\mathbb{R}^d} f_0^\varepsilon \log f_0^\varepsilon dv - \rho_0 \log \rho_0 \right) dx = \mathcal{O}(\sqrt{\varepsilon}). \end{aligned}$$

(H2) The initial data related to the kinetic energy part in the entropy are well-prepared

$$\begin{aligned} \int_{\Omega} \rho_0^\varepsilon |u_0 - u_0^\varepsilon|^2 dx = \mathcal{O}(\sqrt{\varepsilon}) \quad \text{and} \\ \int_{\Omega} \left(\int_{\mathbb{R}^d} f_0^\varepsilon |v|^2 dv - \rho_0 |u_0|^2 \right) dx = \mathcal{O}(\sqrt{\varepsilon}). \end{aligned}$$

(H3) The bounded Lipschitz distance between initial local densities satisfies

$$d_{\text{BL}}^2(\rho_0^\varepsilon, \rho_0) = \mathcal{O}(\sqrt{\varepsilon}),$$

where the bounded Lipschitz distance d_{BL} for probability measures is defined by

$$d_{\text{BL}}(\mu, \nu) := \sup \left\{ \left| \int_{\mathbb{R}^d} \phi d\mu - \int_{\mathbb{R}^d} \phi d\nu \right| : \begin{aligned} &\|\phi\|_{L^\infty(\mathbb{R}^d)} \leq 1, \\ &\|\phi\|_{\text{Lip}} := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|} \leq 1. \end{aligned} \right\}$$

Remark 2.1. If we choose the initial data f_0^ε as

$$f_0^\varepsilon(x, v) = \frac{\rho_0(x)}{(2\pi)^{d/2}} \exp\left(-\frac{|u_0(x) - v|^2}{2}\right) \quad \text{for all } \varepsilon > 0,$$

then we obtain

$$\rho_0^\varepsilon = \int_{\mathbb{R}^d} f_0^\varepsilon dv = \rho_0 \quad \text{and} \quad \rho_0^\varepsilon u_0^\varepsilon = \int_{\mathbb{R}^d} v f_0^\varepsilon dv = \int_{\mathbb{R}^d} u_0 f_0^\varepsilon dv = \rho_0 u_0.$$

Let us define the classical relative entropy between two probability densities ρ_1 and ρ_2 as

$$\mathcal{H}(\rho_1|\rho_2) = \int_{\rho_2}^{\rho_1} \frac{\rho_1 - z}{z} dz = \rho_1 \log \rho_1 - \rho_2 \log \rho_2 - (1 + \log \rho_2)(\rho_1 - \rho_2), \quad (2.3)$$

and analogously for two densities f_1 and f_2 in phase space as

$$\mathcal{H}(f_1|f_2) = \int_{f_2}^{f_1} \frac{f_1 - z}{z} dz = f_1 \log f_1 - f_2 \log f_2 - (1 + \log f_2)(f_1 - f_2).$$

Remark 2.2. The first assumptions in **(H1)** and **(H2)** imply that

$$\int_{\Omega} \frac{\rho_0^\varepsilon}{2} |u_0^\varepsilon - u_0|^2 dx + \int_{\Omega} \mathcal{H}(\rho_0^\varepsilon|\rho_0) dx = \mathcal{O}(\sqrt{\varepsilon}).$$

Theorem 2.1. *Let f^ε be a weak solution to Eq. (1.1) with $\beta = \sigma = 1/\varepsilon$ in the sense of Definition 2.1 and (ρ, u) be a strong solution to the system (1.3) in the sense of Definition 2.2 up to the time $T^* > 0$. Suppose that the assumptions **(H1)**–**(H2)** hold. Then we have the following inequalities for $0 < \varepsilon \leq 1$ and $t \leq T^*$:*

(i) *Coulomb case $\Delta W = -\delta_0$:*

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + \int_{\Omega} \mathcal{H}(\rho^\varepsilon|\rho) dx + \frac{\lambda}{2} \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx \\ &\quad + \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2 dx ds \\ &\quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) \end{aligned}$$

$$\begin{aligned} & \times |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ & \leq C\sqrt{\varepsilon} + C \int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx. \end{aligned}$$

(ii) *Weakly regular case* $\nabla W \in L^\infty(\Omega)$:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + \int_{\Omega} \mathcal{H}(\rho^\varepsilon | \rho) dx + \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2 dx ds \\ & + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) \\ & - u(y))|^2 dx dy ds \\ & \leq C\sqrt{\varepsilon}. \end{aligned}$$

Here $C > 0$ is a positive constant independent of ε .

Remark 2.3. Coulomb interaction potential on \mathbb{R}^d is explicitly given by

$$W(x) = \begin{cases} -\frac{|x|}{2} & \text{for } d = 1, \\ -\frac{1}{2\pi} \log |x| & \text{for } d = 2, \\ \frac{1}{(d-2)|B(0,1)|} \frac{1}{|x|^{d-2}} & \text{for } d \geq 3. \end{cases}$$

Here $|B(0,1)|$ denotes the volume of unit ball $B(0,1)$ in \mathbb{R}^d , i.e. $|B(0,1)| = \pi^{d/2}/\Gamma(d/2+1)$, where $\Gamma = \Gamma(\cdot)$ denotes the gamma function.

Corollary 2.1. *Suppose that all the assumptions in Theorem 2.1 hold. Then, we have the following convergences hold for the weakly regular case (ii):*

$$\begin{aligned} & \rho^\varepsilon \rightarrow \rho \quad \text{a.e. and in } L^\infty(0, T^*; L^1(\Omega)), \\ & \rho^\varepsilon u^\varepsilon \rightarrow \rho u \quad \text{a.e. and in } L^\infty(0, T^*; L^1(\Omega)), \\ & \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon \rightarrow \rho u \otimes u \quad \text{a.e. and in } L^\infty(0, T^*; L^1(\Omega)), \quad \text{and} \\ & \int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv \rightarrow \rho u \otimes u + \rho \mathbb{I}_{d \times d} \quad \text{a.e. and in } L^p(0, T^*; L^1(\Omega)) \quad \text{for } 1 \leq p \leq 2 \end{aligned} \quad (2.4)$$

as $\varepsilon \rightarrow 0$. The same convergences for the Coulomb case (i) can be obtained if

$$\int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

In the corollary below, under suitable assumptions we provide the convergence of f^ε towards the local Maxwellian $M_{\rho,u}$ given by

$$M_{\rho,u} := \frac{\rho}{(2\pi)^{d/2}} e^{-\frac{|u-v|^2}{2}},$$

where (ρ, u) is the strong solution to the system (1.3).

Corollary 2.2. *Suppose that all the assumptions in Theorem 2.1 hold. Moreover, we assume that the confinement potential V satisfies $|\nabla V(x)|^2 \leq C|V(x)|$ for some $C > 0$. Then for $t \leq T^*$, we have*

$$\|f^\varepsilon - M_{\rho,u}\|_{L^1} \leq C \left(\int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f_0^\varepsilon | M_{\rho_0,u_0}) dx dv \right)^{1/2} + C\varepsilon^{1/8},$$

for the weakly regular potential case (ii), and

$$\begin{aligned} \|f^\varepsilon - M_{\rho,u}\|_{L^1} &\leq C \left(\int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f_0^\varepsilon | M_{\rho_0,u_0}) dx dv \right)^{1/2} + C\varepsilon^{1/8} \\ &\quad + C \left(\min \left\{ 1, \int_{\Omega} |\nabla W \star (\rho_0^\varepsilon - \rho_0)|^2 dx \right\} \right)^{1/4}, \end{aligned}$$

for the Coulomb potential case (i), where $C > 0$ is independent of $\varepsilon > 0$. In particular, if the right-hand side of the above inequality converges to zero, then we have

$$f^\varepsilon \rightarrow M_{\rho,u} := \frac{\rho}{(2\pi)^{d/2}} e^{-\frac{|u-v|^2}{2}} \quad \text{in } L^\infty(0, T^*; L^1(\Omega))$$

as $\varepsilon \rightarrow 0$.

Proof. Since this proof is lengthy and technical, we postpone it to Appendix A. \square

Remark 2.4. Note that the assumption on V in Corollary 2.2 holds for the quadratic confinement potential $V(x) = |x|^2/2$.

Theorem 2.2. *Let f^ε be a weak solution to Eq. (1.1) with $\beta = 1/\varepsilon$ and $\sigma = 0$ in the sense of Definition 2.1 and (ρ, u) be a strong solution to the system (1.5) in the sense of Definition 2.3 up to the time $T^* > 0$. Suppose that the assumptions (H2)–(H3) hold. Then, we have the following inequalities for $0 < \varepsilon \leq 1$ and $t \leq T^*$:*

(i) *Coulomb case $\Delta W = -\delta_0$:*

$$\begin{aligned} &\int_{\Omega} \frac{\rho^\varepsilon}{2} |u^\varepsilon - u|^2 dx + \frac{\lambda}{2} \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx + d_{\text{BL}}^2(\rho^\varepsilon, \rho) \\ &\quad + \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2 dx ds \\ &\quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) \\ &\quad \times |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ &\leq C\sqrt{\varepsilon} + C \int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx, \end{aligned}$$

(ii) *Strongly regular case* $\nabla W \in \mathcal{W}^{1,\infty}(\Omega)$:

$$\begin{aligned} & \int_{\Omega} \frac{\rho^\varepsilon}{2} |u^\varepsilon - u|^2 dx + d_{\text{BL}}^2(\rho^\varepsilon, \rho) + \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2 dx ds \\ & \quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) \\ & \quad \times |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ & \leq C \sqrt{\varepsilon}. \end{aligned}$$

Here $C > 0$ is a positive constant independent of ε .

Remark 2.5. Compared to Theorem 2.1(ii), the pressureless case requires higher regularity for W , like $\nabla W \in \mathcal{W}^{1,\infty}(\Omega)$ due to the lack of convexity of the entropy with respect to ρ .

Corollary 2.3. *Suppose that all the assumptions in Theorem 2.2 hold. If*

$$\int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

for Coulomb interaction case, then the following convergences hold:

$$\begin{aligned} \rho^\varepsilon u^\varepsilon &\rightharpoonup \rho u \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}), \\ \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon &\rightharpoonup \rho u \otimes u \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}), \\ \int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv &\rightharpoonup \rho u \otimes u \quad \text{weakly in } L^1(0, T^*; \mathcal{M}), \quad \text{and} \\ f^\varepsilon &\rightharpoonup \rho \otimes \delta_u \quad \text{weakly in } L^p(0, T^*; \mathcal{M}) \end{aligned}$$

as $\varepsilon \rightarrow 0$, for $1 \leq p \leq 2$. Here \mathcal{M} is the space of signed Radon measures.

Remark 2.6. The convergence of $d_{\text{BL}}(\rho^\varepsilon, \rho)$ directly gives

$$\rho^\varepsilon \rightharpoonup \rho \quad \text{weakly in } L^\infty(0, T^*; \mathcal{M}).$$

Remark 2.7. Our results on the hydrodynamic limit also hold in a bounded domain with the specular reflection boundary condition. In this case, the limiting system has a kinematic boundary condition. For the hydrodynamic limit estimate, we refer to Ref. 21 where the hydrodynamic limit of nonlinear Vlasov–Fokker–Planck/Navier–Stokes equations in a bounded domain is discussed.

2.3. Proofs of Corollaries 2.1 and 2.3

Before proceeding, for the readers' convenience, we provide the details of proofs of convergences in Corollaries 2.1 and 2.3. In fact, we provide quantitative bounds of convergences.

Lemma 2.2. *There exists a positive constant C depending only on $\|u\|_{\mathcal{W}^{1,\infty}}$ such that the following inequalities hold:*

(i) *Error estimate between moments:*

$$\|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^1} \leq \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} + \|u\|_{L^\infty} \|\rho^\varepsilon - \rho\|_{L^1}$$

and

$$d_{\text{BL}}(\rho^\varepsilon u^\varepsilon, \rho u) \leq \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} + C d_{\text{BL}}(\rho^\varepsilon, \rho).$$

(ii) *Error estimate between convections:*

$$\begin{aligned} & \|\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u\|_{L^1} \\ & \leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + 2\|u\|_{L^\infty} \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} \\ & \quad + 3\|u\|_{L^\infty}^2 \|\rho^\varepsilon - \rho\|_{L^1} \end{aligned}$$

and

$$\begin{aligned} & d_{\text{BL}}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon, \rho u \otimes u) \\ & \leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + C \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} + C d_{\text{BL}}(\rho^\varepsilon, \rho). \end{aligned}$$

(iii) *Error estimate between particle distribution and monokinetic ansatz:*

$$\begin{aligned} d_{\text{BL}}(f^\varepsilon, \rho \otimes \delta_u) & \leq \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega \times \mathbb{R}^d} |v - u^\varepsilon|^2 f^\varepsilon dx dv \right)^{1/2} \\ & \quad + \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} + C d_{\text{BL}}(\rho^\varepsilon, \rho). \end{aligned}$$

Proof. For any $\psi \in (L^\infty \cap \text{Lip})(\Omega)$, where $\text{Lip}(\Omega)$ stands for the space of Lipschitz continuous functions, we get

$$\begin{aligned} & \left| \int_{\Omega} \psi(x) ((\rho^\varepsilon u^\varepsilon)(x) - (\rho u)(x)) dx \right| \\ & = \left| \int_{\Omega} \psi(x) (\rho^\varepsilon(x)(u^\varepsilon - u)(x) - (\rho^\varepsilon - \rho)(x)u(x)) dx \right| \\ & \leq \|\psi\|_{L^\infty} \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u| dx + \left| \int_{\Omega} (\rho^\varepsilon - \rho)(x)(\psi u)(x) dx \right| \\ & \leq \|\psi\|_{L^\infty} \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} + \|\psi u\|_{L^\infty \cap \text{Lip}} d_{\text{BL}}(\rho^\varepsilon, \rho). \end{aligned}$$

This asserts the inequality (i). For the estimate of (ii), we note that

$$\begin{aligned}\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u &= \rho^\varepsilon (u^\varepsilon - u) \otimes (u^\varepsilon - u) + u \otimes (\rho^\varepsilon u^\varepsilon - \rho u) \\ &\quad + (\rho^\varepsilon u^\varepsilon - \rho u) \otimes u - (\rho^\varepsilon - \rho) u \otimes u.\end{aligned}$$

Using this identity, we obtain

$$\begin{aligned}& \left| \int_{\Omega} \psi(x) ((\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon)(x) - (\rho u \otimes u)(x)) dx \right| \\ & \leq \left| \int_{\Omega} \psi(x) \rho^\varepsilon(x) (u^\varepsilon - u)(x) \otimes (u^\varepsilon - u)(x) dx \right| \\ & \quad + \left| \int_{\Omega} \psi(x) u(x) \otimes (\rho^\varepsilon u^\varepsilon - \rho u)(x) dx \right| \\ & \quad + \left| \int_{\Omega} \psi(x) (\rho^\varepsilon u^\varepsilon - \rho u)(x) \otimes u(x) dx \right| + \left| \int_{\Omega} \psi(x) (\rho^\varepsilon - \rho)(x) u(x) \otimes u(x) dx \right| \\ & \leq \|\psi\|_{L^\infty} \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + 2\|\psi u\|_{L^\infty \cap Lip} d_{BL}(\rho^\varepsilon u^\varepsilon, \rho u) \\ & \quad + \|\psi u \otimes u\|_{L^\infty \cap Lip} d_{BL}(\rho^\varepsilon, \rho).\end{aligned}$$

This yields

$$\begin{aligned}d_{BL}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon, \rho u \otimes u) &\leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + C \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} \\ &\quad + C d_{BL}(\rho^\varepsilon, \rho),\end{aligned}$$

where $C > 0$ depends only on $\|u\|_{W^{1,\infty}}$. For (iii), we find for any $\varphi \in (L^\infty \cap Lip)(\Omega \times \mathbb{R}^d)$ that

$$\begin{aligned}& \left| \int_{\Omega \times \mathbb{R}^d} \varphi(x, v) (f^\varepsilon(x, v) - \rho(x) \otimes \delta_{u(x)}(v)) dx dv \right| \\ & = \left| \int_{\Omega \times \mathbb{R}^d} \varphi(x, v) f^\varepsilon(x, v) dx dv - \int_{\Omega} \varphi(x, u(x)) \rho(x) dx \right| \\ & \leq \int_{\Omega \times \mathbb{R}^d} |\varphi(x, v) - \varphi(x, u^\varepsilon(x))| f^\varepsilon dx dv + \int_{\Omega} |\varphi(x, u^\varepsilon) - \varphi(x, u)| \rho^\varepsilon dx \\ & \quad + \left| \int_{\Omega} \varphi(x, u(x)) (\rho^\varepsilon - \rho) dx \right| \\ & \leq \|\varphi\|_{Lip} \int_{\Omega \times \mathbb{R}^d} |v - u^\varepsilon| f^\varepsilon dx dv + \|\varphi\|_{Lip} \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u| dx \\ & \quad + \|\varphi\|_{Lip} \|u\|_{Lip} d_{BL}(\rho^\varepsilon, \rho)\end{aligned}$$

$$\begin{aligned} &\leq \|\varphi\|_{\text{Lip}} \|\rho^\varepsilon\|_{L^1}^{1/2} \left(\left(\int_{\Omega \times \mathbb{R}^d} |v - u^\varepsilon|^2 f^\varepsilon dx dv \right)^{1/2} + \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} \right) \\ &\quad + \|\varphi\|_{\text{Lip}} \|u\|_{\text{Lip}} d_{\text{BL}}(\rho^\varepsilon, \rho). \end{aligned}$$

This concludes the inequality (iii). \square

Proof of Corollary 2.1. We first obtain

$$\|\rho^\varepsilon - \rho\|_{L^1} \leq C \left(\int_{\Omega} \mathcal{H} \rho^\varepsilon |\rho| dx \right)^{1/2},$$

where $C > 0$ depends only on $\|\rho^\varepsilon\|_{L^1}$ and $\|\rho\|_{L^1}$, see (3.4) for details. This together with Lemma 2.2 yields

$$\begin{aligned} &\|\rho^\varepsilon - \rho\|_{L^1} + \|\rho^\varepsilon u^\varepsilon - \rho u\|_{L^1} + \|\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u\|_{L^1} \\ &\leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + C \left(\int_{\Omega} \mathcal{H}(\rho^\varepsilon |\rho|) dx \right)^{1/2} \\ &\leq C\varepsilon^{1/4} + C \left(\int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \right)^{1/2} \rightarrow 0 \end{aligned} \quad (2.5)$$

as $\varepsilon \rightarrow 0$, where $C > 0$ is independent of ε . Note that

$$\begin{aligned} &\int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - (\rho u \otimes u + \rho \mathbb{I}_{d \times d}) \\ &= \int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + \rho^\varepsilon \mathbb{I}_{d \times d}) + \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u + (\rho^\varepsilon - \rho) \mathbb{I}_{d \times d}. \end{aligned}$$

On the other hand, we find from Lemma 4.8 in Ref. 35 or Sec. 3 in Ref. 21 that

$$\begin{aligned} &\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v + \mathbb{I}_{d \times d}) f^\varepsilon dv \\ &= \int_{\mathbb{R}^d} u^\varepsilon \sqrt{f^\varepsilon} \otimes \left((u^\varepsilon - v) \sqrt{f^\varepsilon} - 2 \nabla_v \sqrt{f^\varepsilon} \right) dv \\ &\quad + \int_{\mathbb{R}^d} \left((u^\varepsilon - v) \sqrt{f^\varepsilon} - 2 \nabla_v \sqrt{f^\varepsilon} \right) \otimes v \sqrt{f^\varepsilon} dv. \end{aligned} \quad (2.6)$$

This yields

$$\begin{aligned} &\left\| \int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + \rho^\varepsilon \mathbb{I}_{d \times d}) \right\|_{L^1} \\ &\leq \left(\int_{\Omega \times \mathbb{R}^d} f^\varepsilon |u^\varepsilon|^2 + f^\varepsilon |v|^2 dx dv \right)^{1/2} \\ &\quad \times \left(\int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon|^2 dx dv \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq C\sqrt{\varepsilon} \sup_{0 \leq t \leq T} \left(\int_{\Omega \times \mathbb{R}^d} f^\varepsilon |v|^2 dx dv \right)^{1/2} \\
&\quad \times \left(\frac{1}{2\varepsilon} \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v)f^\varepsilon|^2 dx dv \right)^{1/2} \\
&\leq C\sqrt{\varepsilon} \left(\frac{1}{2\varepsilon} \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v)f^\varepsilon|^2 dx dv \right)^{1/2}.
\end{aligned}$$

Combining this, (2.5), and Proposition 2.1 with $\beta = \sigma = 1/\varepsilon$, we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - (\rho u \otimes u + \rho \mathbb{I}_{d \times d}) \right\|_{L^2(0, T^*; L^1(\Omega))} \\
&\leq \left\| \int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + \rho^\varepsilon \mathbb{I}_{d \times d}) \right\|_{L^2(0, T; L^1(\Omega))} \\
&\quad + \|\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u\|_{L^2(0, T; L^1(\Omega))} + \|\rho^\varepsilon - \rho\|_{L^2(0, T; L^1(\Omega))} \\
&\leq C\varepsilon^{1/4} + C \left(\int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \right)^{1/2} \rightarrow 0.
\end{aligned}$$

This completes the proof. \square

Proof of Corollary 2.3. A simple combination of inequalities in Lemma 2.2 together with Theorem 2.2 gives

$$\begin{aligned}
&d_{\text{BL}}(\rho^\varepsilon u^\varepsilon, \rho u) + d_{\text{BL}}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon, \rho u \otimes u) \\
&\leq C \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx + C \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} + C d_{\text{BL}}(\rho^\varepsilon, \rho) \\
&\leq C\varepsilon^{1/4} + C \left(\int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \right)^{1/2}.
\end{aligned}$$

This asserts the first two convergences. Note that

$$\int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon (u^\varepsilon - v) \otimes (u^\varepsilon - v) dv,$$

thus we get

$$\int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - \rho u \otimes u = \int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv - \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u.$$

This yields

$$\begin{aligned}
d_{\text{BL}} \left(\int_{\mathbb{R}^d} f^\varepsilon v \otimes v dv, \rho u \otimes u \right) &\leq \int_{\Omega \times \mathbb{R}^d} f^\varepsilon |u^\varepsilon - v|^2 dx dv \\
&\quad + d_{\text{BL}}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon, \rho u \otimes u).
\end{aligned} \tag{2.7}$$

On the other hand, it follows from (2.1) with $\beta = 1/\varepsilon$ that

$$\int_0^t \int_{\Omega \times \mathbb{R}^d} f^\varepsilon |u^\varepsilon - v|^2 dx dv ds \leq C\varepsilon.$$

This together with (2.7) implies the third assertion. We also use Lemma 2.2 and (2.1) with $\beta = 1/\varepsilon$ to conclude that for $1 \leq p \leq 2$

$$\begin{aligned} & \int_0^t d_{\text{BL}}^p(f^\varepsilon(\cdot, \cdot, s), \rho(\cdot, s) \otimes \delta_{u(\cdot, s)}) ds \\ & \leq C\varepsilon^{1/4} + C \left(\int_0^t \int_{\Omega \times \mathbb{R}^d} |v - u^\varepsilon|^2 f^\varepsilon dx dv ds \right)^{1/2} \\ & \quad + C \left(\int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \right)^{1/2} \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$. □

3. Hydrodynamic Limit from Kinetic to Isothermal Euler Equations

In this section, we study the rigorous derivation of the isothermal Euler equations (1.3) from the kinetic equation (1.1) with $\beta = \sigma = 1/\varepsilon$ as $\varepsilon \rightarrow 0$. As mentioned before, we use the relative entropy argument based on the weak-strong uniqueness principle to have the quantitative error estimates between the kinetic equation and the limiting system.

3.1. Relative entropy inequality

We rewrite Eq. (1.3) as a conservative form

$$\partial_t U + \nabla \cdot A(U) = F(U), \quad (3.1)$$

where

$$U := \begin{pmatrix} \rho \\ m \end{pmatrix} \quad \text{with } m = \rho u, \quad A(U) := \begin{pmatrix} m & 0 \\ \frac{m \otimes m}{\rho} & \rho \mathbb{I}_{d \times d} \end{pmatrix},$$

and

$$F(U) := \begin{pmatrix} 0 \\ \alpha \rho \int_{\Omega} \phi(x - y)(u(y) - u(x)) \rho(y) dy - \gamma \rho u - \lambda \rho (\nabla V + \nabla W \star \rho) \end{pmatrix}.$$

Here $\mathbb{I}_{d \times d}$ denotes the $d \times d$ identity matrix. The free energy of the above system is given by

$$E(U) := \frac{|m|^2}{2\rho} + \rho \log \rho. \quad (3.2)$$

We now define the relative entropy functional \mathcal{E} between two states of the system U and \bar{U} as follows:

$$\mathcal{E}(\bar{U} | U) := E(\bar{U}) - E(U) - DE(U)(\bar{U} - U) \quad \text{with } \bar{U} := \begin{pmatrix} \bar{\rho} \\ \bar{m} \end{pmatrix}, \quad \bar{m} = \bar{\rho}\bar{u}, \quad (3.3)$$

where $DE(U)$ denotes the derivation of E with respect to ρ, m , i.e.

$$\begin{aligned} -DE(U)(\bar{U} - U) &= - \begin{pmatrix} -\frac{|m|^2}{2\rho^2} + \log \rho + 1 \\ \frac{m}{\rho} \end{pmatrix} \begin{pmatrix} \bar{\rho} - \rho \\ \bar{m} - m \end{pmatrix} \\ &= \frac{\bar{\rho}|u|^2}{2} - \frac{\rho|u|^2}{2} + (\rho - \bar{\rho})(\log \rho + 1) + \rho|u|^2 - \bar{\rho}u \cdot \bar{u}. \end{aligned}$$

This yields

$$\begin{aligned} \mathcal{E}(\bar{U} | U) &= \frac{\bar{\rho}|\bar{u}|^2}{2} - \frac{\rho|u|^2}{2} + \bar{\rho} \log \bar{\rho} - \rho \log \rho + \frac{\bar{\rho}|u|^2}{2} - \frac{\rho|u|^2}{2} \\ &\quad + (\rho - \bar{\rho})(\log \rho + 1) + \rho|u|^2 - \bar{\rho}u \cdot \bar{u} \\ &= \frac{\bar{\rho}}{2} |\bar{u} - u|^2 + \mathcal{H}(\bar{\rho} | \rho), \end{aligned}$$

where $\mathcal{H}(\bar{\rho} | \rho)$ is the relative entropy between densities given by (2.3). By Taylor's theorem, we readily see

$$\mathcal{H}(\bar{\rho} | \rho) \geq \frac{1}{2} \min \left\{ \frac{1}{\bar{\rho}}, \frac{1}{\rho} \right\} (\rho - \bar{\rho})^2,$$

and moreover, we get

$$\begin{aligned} \|\bar{\rho} - \rho\|_{L^1} &= \int_{\Omega} \min \{ (\sqrt{\bar{\rho}})^{-1}, (\sqrt{\rho})^{-1} \} \max \{ \sqrt{\bar{\rho}}, \sqrt{\rho} \} |\bar{\rho} - \rho| dx \\ &\leq \left(\frac{1}{2} \int_{\Omega} \min \{ (\bar{\rho})^{-1}, \rho^{-1} \} (\rho - \bar{\rho})^2 dx \right)^{1/2} \left(2 \int_{\Omega} \max \{ \bar{\rho}, \rho \} dx \right)^{1/2} \\ &\leq \left(\int_{\Omega} \mathcal{H}(\bar{\rho} | \rho) dx \right)^{1/2} (2(\|\bar{\rho}\|_{L^1} + \|\rho\|_{L^1}))^{1/2}. \end{aligned}$$

Thus, we obtain

$$\|\bar{\rho} - \rho\|_{L^1}^2 \leq C \int_{\Omega} \mathcal{H}(\bar{\rho} | \rho) dx \leq C \int_{\Omega} \mathcal{E}(\bar{U} | U) dx, \quad (3.4)$$

where $C > 0$ only depends on $\|\bar{\rho}\|_{L^1}$ and $\|\rho\|_{L^1}$.

Remark 3.1. The free energy of the system (3.1) is given by

$$\tilde{E}(U) = \frac{|m|^2}{2\rho} + \rho \log \rho + \lambda \rho V + \frac{\lambda}{2} \rho W \star \rho,$$

and we can also define its modulated energy, also often called the relative entropy, as

$$\tilde{\mathcal{E}}(\bar{U}|U) := \tilde{E}(\bar{U}) - \tilde{E}(U) - D\tilde{E}(U)(\bar{U} - U).$$

A straightforward computation shows

$$\tilde{\mathcal{E}}(\bar{U}|U) = \frac{\bar{\rho}}{2} |\bar{u} - u|^2 + \mathcal{H}(\bar{\rho}|\rho) + \frac{\lambda}{2} (\rho - \bar{\rho}) W \star \rho + \frac{\lambda}{2} \bar{\rho} W \star (\bar{\rho} - \rho),$$

and by symmetry of W , we obtain

$$\int_{\Omega} \tilde{\mathcal{E}}(\bar{U}|U) dx = \int_{\Omega} \frac{\bar{\rho}}{2} |\bar{u} - u|^2 dx + \int_{\Omega} \mathcal{H}(\bar{\rho}|\rho) dx + \frac{\lambda}{2} \int_{\Omega} (\rho - \bar{\rho}) W \star (\rho - \bar{\rho}) dx.$$

This functional $\tilde{\mathcal{E}}$ is used in the study of large friction limit of Euler equations with nonlocal forces,^{14,37,38} see also Ref. 17 for the pressureless case. However, we employ the form (3.2) to use the estimates in Ref. 35 providing the relation between $E(U)$ and the flux $A(U)$, see the estimate of I_3 in the proof of Lemma 3.1 below.

Lemma 3.1. *The relative entropy \mathcal{E} defined in (3.3) satisfies the following equality:*

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \mathcal{E}(\bar{U}|U) dx + \frac{\alpha}{2} \int_{\Omega \times \Omega} \bar{\rho}(x) \bar{\rho}(y) \phi(x-y) \\ & \quad \times |(\bar{u}(x) - u(x)) - (\bar{u}(y) - u(y))|^2 dx dy \\ &= \int_{\Omega} \partial_t E(\bar{U}) dx - \int_{\Omega} \nabla(DE(U)) : A(\bar{U}|U) dx \\ & \quad - \int_{\Omega} DE(U) [\partial_t \bar{U} + \nabla \cdot A(\bar{U}) - F(\bar{U})] dx \\ & \quad + \frac{\alpha}{2} \int_{\Omega \times \Omega} \bar{\rho}(x) \bar{\rho}(y) \phi(x-y) |\bar{u}(x) - \bar{u}(y)|^2 dx dy \\ & \quad - \alpha \int_{\Omega \times \Omega} \bar{\rho}(x) (\rho(y) - \bar{\rho}(y)) \phi(x-y) (\bar{u}(x) - u(x)) \cdot (u(y) - u(x)) dx dy \\ & \quad - \gamma \int_{\Omega} \bar{\rho} |\bar{u} - u|^2 - \bar{\rho} |\bar{u}|^2 dx + \lambda \int_{\Omega} \nabla V \cdot \bar{\rho} \bar{u} dx \\ & \quad + \lambda \int_{\Omega} \bar{\rho} (\bar{u} - u) \cdot \nabla W \star (\rho - \bar{\rho}) + \bar{\rho} \bar{u} \cdot \nabla W \star \bar{\rho} dx, \end{aligned}$$

where $A : B := \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij}$ for $A, B \in \mathbb{R}^{mn}$ and $A(\bar{U}|U)$ is the relative flux functional given by

$$A(\bar{U}|U) := A(\bar{U}) - A(U) - DA(U)(\bar{U} - U).$$

Proof. It follows from (3.3) that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \mathcal{E}(\bar{U}|U) dx &= \int_{\Omega} \partial_t E(\bar{U}) dx - \int_{\Omega} DE(U)(\partial_t \bar{U} + \nabla \cdot A(\bar{U}) - F(\bar{U})) dx \\ &\quad + \int_{\Omega} D^2 E(U) \nabla \cdot A(U)(\bar{U} - U) + DE(U) \nabla \cdot A(\bar{U}) dx \\ &\quad - \int_{\Omega} D^2 E(U) F(U)(\bar{U} - U) + DE(U) F(\bar{U}) dx \\ &=: \sum_{i=1}^4 I_i. \end{aligned}$$

We first use the integration by parts [35, Lemma 4.1] to get

$$\int_{\Omega} D^2 E(U) \nabla \cdot A(U)(\bar{U} - U) dx = \int_{\Omega} \nabla DE(U) : DA(U)(\bar{U} - U) dx.$$

Furthermore, we use the following identity [35, Proof of Proposition 4.2]

$$\int_{\Omega} \nabla DE(U) : A(U) dx = 0$$

to yield

$$\begin{aligned} I_3 &= \int_{\Omega} (\nabla DE(U)) : (DA(U)(\bar{U} - U) - A(\bar{U})) dx \\ &= - \int_{\Omega} (\nabla DE(U)) : (A(\bar{U}|U) + A(U)) dx \\ &= - \int_{\Omega} (\nabla DE(U)) : A(\bar{U}|U) dx. \end{aligned}$$

For the estimate I_4 , we note that

$$DE(U) = \begin{pmatrix} -\frac{|m|^2}{2\rho^2} + \log \rho + 1 \\ \frac{m}{\rho} \end{pmatrix} \quad \text{and} \quad D^2 E(U) = \begin{pmatrix} * & -\frac{m}{\rho^2} \\ * & \frac{1}{\rho} \end{pmatrix}.$$

Then, by direct calculation, we find

$$\begin{aligned} D^2 E(U) F(U)(\bar{U} - U) &= \bar{\rho}(x)(\bar{u}(x) - u(x)) \cdot \left[\alpha \int_{\Omega} \phi(x-y)(u(y) - u(x)) \rho(y) dy \right. \\ &\quad \left. - \lambda(u + \nabla V + \nabla W \star \rho) \right] \end{aligned}$$

and

$$DE(U) F(\bar{U}) = \bar{\rho} u \cdot \left[\alpha \int_{\Omega} \phi(x-y)(\bar{u}(y) - \bar{u}(x)) \bar{\rho}(y) dy - \lambda(\bar{u} + \nabla V + \nabla W \star \bar{\rho}) \right].$$

Thus, we obtain

$$\begin{aligned}
 -I_4 &= \alpha \int_{\Omega \times \Omega} \bar{\rho}(x) \phi(x-y) (\bar{u}(x) - u(x)) \cdot (u(y) - u(x)) \rho(y) dy dx \\
 &\quad + \alpha \int_{\Omega \times \Omega} \bar{\rho}(x) \phi(x-y) u(x) \cdot (\bar{u}(y) - \bar{u}(x)) \bar{\rho}(y) dy dx \\
 &\quad - \int_{\Omega} \bar{\rho}(x) (\bar{u}(x) - u(x)) \cdot (\gamma u(x) + \lambda (\nabla V(x) + (\nabla W \star \rho)(x))) dx \\
 &\quad - \lambda \int_{\Omega} \bar{\rho}(x) u(x) \cdot (\bar{u}(x) + \nabla V(x) + (\nabla W \star \bar{\rho})(x)) dx \\
 &=: \sum_{i=1}^4 I_4^i.
 \end{aligned}$$

Here we follow the same argument as in Ref. 35 to get

$$\begin{aligned}
 I_4^1 + I_4^2 &= \frac{\alpha}{2} \int_{\Omega \times \Omega} \bar{\rho}(x) \bar{\rho}(y) \phi(x-y) |(\bar{u}(x) - u(x)) - (\bar{u}(y) - u(y))|^2 dx dy \\
 &\quad - \frac{\alpha}{2} \int_{\Omega \times \Omega} \bar{\rho}(x) \bar{\rho}(y) \phi(x-y) |\bar{u}(x) - \bar{u}(y)|^2 dx dy \\
 &\quad + \alpha \int_{\Omega \times \Omega} \bar{\rho}(x) (\rho(y) - \bar{\rho}(y)) \phi(x-y) (\bar{u}(x) - u(x)) \cdot (u(y) - u(x)) dx dy.
 \end{aligned}$$

We next estimate $I_4^3 + I_4^4$ as

$$\begin{aligned}
 I_4^3 + I_4^4 &= \gamma \int_{\Omega} \bar{\rho} |\bar{u} - u|^2 - \bar{\rho} |\bar{u}|^2 dx - \lambda \int_{\Omega} \nabla V \cdot \bar{\rho} \bar{u} dx \\
 &\quad - \lambda \int_{\Omega} \bar{\rho} (\bar{u} - u) \cdot \nabla W \star (\rho - \bar{\rho}) + \bar{\rho} \bar{u} \cdot \nabla W \star \bar{\rho} dx.
 \end{aligned}$$

Combining the above estimates yields

$$\begin{aligned}
 I_4 &= -\frac{\alpha}{2} \int_{\Omega \times \Omega} \bar{\rho}(x) \bar{\rho}(y) \phi(x-y) |(\bar{u}(x) - u(x)) - (\bar{u}(y) - u(y))|^2 dx dy \\
 &\quad + \frac{\alpha}{2} \int_{\Omega \times \Omega} \bar{\rho}(x) \bar{\rho}(y) \phi(x-y) |\bar{u}(x) - \bar{u}(y)|^2 dx dy \\
 &\quad - \alpha \int_{\Omega \times \Omega} \bar{\rho}(x) (\rho(y) - \bar{\rho}(y)) \phi(x-y) (\bar{u}(x) - u(x)) \cdot (u(y) - u(x)) dx dy \\
 &\quad - \gamma \int_{\Omega} \bar{\rho} |\bar{u} - u|^2 - \bar{\rho} |\bar{u}|^2 dx + \lambda \int_{\Omega} \nabla V \cdot \bar{\rho} \bar{u} dx \\
 &\quad + \lambda \int_{\Omega} \bar{\rho} (\bar{u} - u) \cdot \nabla W \star (\rho - \bar{\rho}) + \bar{\rho} \bar{u} \cdot \nabla W \star \bar{\rho} dx.
 \end{aligned}$$

This completes the proof. \square

We now set

$$m^\varepsilon = \rho^\varepsilon u^\varepsilon \quad \text{and} \quad U^\varepsilon = \begin{pmatrix} \rho^\varepsilon \\ m^\varepsilon \end{pmatrix} \quad \text{with} \quad \rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon dv, \quad m^\varepsilon = \int_{\mathbb{R}^d} v f^\varepsilon dv,$$

where f^ε is a weak solution to Eq. (1.1).

Proposition 3.1. *Let f^ε be a global weak solution to Eq. (1.1) and (ρ, u) be a strong solution to the system (1.3) on the time interval $[0, T]$. Then we have*

$$\begin{aligned} & \int_{\Omega} \mathcal{E}(U^\varepsilon | U) dx + \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2 dx ds \\ & \quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x-y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ & \leq C\sqrt{\varepsilon} + C(1+\alpha) \int_0^t \int_{\Omega} \mathcal{E}(U^\varepsilon | U) dx ds \\ & \quad + \lambda \int_0^t \int_{\Omega} \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) dx ds \end{aligned}$$

for $0 < \varepsilon \leq 1$, where $C > 0$ is independent of ε .

Proof. It follows from Lemma 3.1 that

$$\begin{aligned} & \int_{\Omega} \mathcal{E}(U^\varepsilon | U) dx + \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2 dx ds \\ & \quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x-y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ & = \int_{\Omega} \mathcal{E}(U_0^\varepsilon | U_0) dx + \int_{\Omega} E(U^\varepsilon) - E(U_0^\varepsilon) dx \\ & \quad - \int_0^t \int_{\Omega} \nabla(DE(U)) : A(U^\varepsilon | U) dx ds \\ & \quad - \int_0^t \int_{\Omega} DE(U) [\partial_s U^\varepsilon + \nabla \cdot A(U^\varepsilon) - F(U^\varepsilon)] dx ds \\ & \quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x-y) |u^\varepsilon(x) - u^\varepsilon(y)|^2 dx dy ds \\ & \quad - \alpha \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) (\rho(y) - \rho^\varepsilon(y)) \phi(x-y) \\ & \quad \times (u^\varepsilon(x) - u(x)) \cdot (u(y) - u(x)) dx dy ds \\ & \quad + \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x)|^2 dx ds + \lambda \int_0^t \int_{\Omega} \nabla V(x) \cdot \rho^\varepsilon(x) u^\varepsilon(x) dx ds \end{aligned}$$

$$\begin{aligned}
 & + \lambda \int_0^t \int_{\Omega} \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) \\
 & + \rho^\varepsilon(x) u^\varepsilon(x) \cdot (\nabla W \star \rho^\varepsilon)(x) dx ds \\
 & =: \sum_{i=1}^8 J_i^\varepsilon.
 \end{aligned}$$

Here $J_i^\varepsilon, i = 1, \dots, 8$ can be estimated as follows.

Estimate of J_1^ε : By the assumptions **(H1)** and **(H2)** on the initial data, we get

$$J_1^\varepsilon = \int_{\Omega} \mathcal{E}(U_0^\varepsilon | U_0) dx = \frac{1}{2} \int_{\Omega} \rho_0^\varepsilon |u_0^\varepsilon - u_0|^2 dx + \int_{\Omega} \mathcal{H}(\rho_0^\varepsilon | \rho_0) dx = \mathcal{O}(\sqrt{\varepsilon}).$$

Estimate of J_2^ε : Note that

$$\begin{aligned}
 E(U^\varepsilon) &= \frac{1}{2} \int_{\Omega} \rho^\varepsilon |u^\varepsilon|^2 dx + \int_{\Omega} \rho \log \rho dx \\
 &\leq \frac{1}{2} \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\varepsilon dx dv + \int_{\Omega \times \mathbb{R}^d} f^\varepsilon \log f^\varepsilon dx dv =: K(f^\varepsilon).
 \end{aligned}$$

Thus, by adding and subtracting the functional $K(f^\varepsilon)$, we find

$$\begin{aligned}
 J_2^\varepsilon &= \int_{\Omega} E(U^\varepsilon) dx - K(f^\varepsilon) + K(f^\varepsilon) - K(f_0^\varepsilon) + K(f_0^\varepsilon) - \int_{\Omega} E(U_0^\varepsilon) dx \\
 &\leq 0 + K(f^\varepsilon) - K(f_0^\varepsilon) + \mathcal{O}(\sqrt{\varepsilon}).
 \end{aligned}$$

Estimate of J_3^ε : It follows from Lemma 4.3 in Ref. 35 that

$$A(U^\varepsilon | U) = \begin{pmatrix} 0 & 0 \\ \rho^\varepsilon(u^\varepsilon - u) \otimes (u^\varepsilon - u) & 0 \end{pmatrix}.$$

This yields

$$\begin{aligned}
 |J_3^\varepsilon| &= \left| \int_0^t \int_{\Omega} \nabla u : \rho^\varepsilon(u^\varepsilon - u) \otimes (u^\varepsilon - u) dx ds \right| \\
 &\leq \|\nabla u\|_{L^\infty} \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \\
 &\leq \|\nabla u\|_{L^\infty} \int_0^t \int_{\Omega} \mathcal{E}(U^\varepsilon | U) dx ds.
 \end{aligned}$$

Estimate of J_4^ε : Note that U^ε satisfies

$$\partial_t U^\varepsilon + \nabla \cdot A(U^\varepsilon) - F(U^\varepsilon) = \begin{pmatrix} 0 \\ \nabla \cdot \left(\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v + \mathbb{I}_{d \times d}) f^\varepsilon dv \right) \end{pmatrix}.$$

This implies

$$\begin{aligned} J_4^\varepsilon &= - \int_0^t \int_\Omega u \cdot \left(\nabla \cdot \left(\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v + \mathbb{I}_{d \times d}) f^\varepsilon dv \right) \right) dx ds \\ &= \int_0^t \int_\Omega \nabla u : \left(\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v + \mathbb{I}_{d \times d}) f^\varepsilon dv \right) dx ds \\ &\leq \|\nabla u\|_{L^\infty} \int_0^t \int_\Omega \left| \int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v + \mathbb{I}_{d \times d}) f^\varepsilon dv \right| dx ds. \end{aligned}$$

On the other hand, we recall (2.6) that

$$\begin{aligned} &\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v + \mathbb{I}_{d \times d}) f^\varepsilon dv \\ &= \int_{\mathbb{R}^d} u^\varepsilon \sqrt{f^\varepsilon} \otimes \left((u^\varepsilon - v) \sqrt{f^\varepsilon} - 2 \nabla_v \sqrt{f^\varepsilon} \right) dv \\ &\quad + \int_{\mathbb{R}^d} \left((u^\varepsilon - v) \sqrt{f^\varepsilon} - 2 \nabla_v \sqrt{f^\varepsilon} \right) \otimes v \sqrt{f^\varepsilon} dv. \end{aligned}$$

By this and Proposition 2.1, we obtain

$$\begin{aligned} J_4^\varepsilon &\leq \|\nabla u\|_{L^\infty} \int_0^t \left(\int_{\Omega \times \mathbb{R}^d} f^\varepsilon |u^\varepsilon|^2 + f^\varepsilon |v|^2 dx dv \right)^{1/2} (\mathcal{D}_1(f^\varepsilon)(s))^{1/2} ds \\ &\leq 2 \|\nabla u\|_{L^\infty} \sqrt{\varepsilon} \int_0^t \left(\int_{\Omega \times \mathbb{R}^d} f^\varepsilon |v|^2 dx dv \right)^{1/2} \left(\frac{1}{2\varepsilon} \mathcal{D}_1(f^\varepsilon)(s) \right)^{1/2} ds \\ &\leq C \sqrt{\varepsilon} \sup_{0 \leq t \leq T} \left(\int_{\Omega \times \mathbb{R}^d} f^\varepsilon |v|^2 dx dv \right)^{1/2} \left(\int_0^t \frac{1}{2\varepsilon} \mathcal{D}_1(f^\varepsilon)(s) ds \right)^{1/2} \\ &\leq C \sqrt{\varepsilon}, \end{aligned}$$

where we used Hölder inequality to find

$$|u^\varepsilon|^2 = \left| \frac{\int_{\mathbb{R}^d} v f^\varepsilon dv}{\int_{\mathbb{R}^d} f^\varepsilon dv} \right|^2 \leq \frac{\int_{\mathbb{R}^d} |v|^2 f^\varepsilon dv}{\rho^\varepsilon}, \quad \text{i.e. } \rho^\varepsilon |u^\varepsilon|^2 \leq \int_{\mathbb{R}^d} |v|^2 f^\varepsilon dv. \quad (3.5)$$

Here $C > 0$ depends on T , $\|\nabla u\|_{L^\infty}$. It is worth emphasizing that our estimate gives that the constant C depends on $\|\nabla u\|_{L^\infty}$, while Lemma 4.8 in Ref. 35 provides that it also depends on $\|\nabla \log \rho\|_{L^\infty}$.

Estimate of J_5^ε : We again use (3.5) to get

$$\begin{aligned} &\int_{\Omega \times \Omega} \phi(x-y) |u^\varepsilon(x) - u^\varepsilon(y)|^2 \rho^\varepsilon(x) \rho^\varepsilon(y) dx dy \\ &= \int_{\Omega \times \Omega} \phi(x-y) (|u^\varepsilon(x)|^2 - 2u^\varepsilon(x) \cdot u^\varepsilon(y) + |u^\varepsilon(y)|^2) \rho^\varepsilon(x) \rho^\varepsilon(y) dx dy \\ &\leq \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\varepsilon(x, v) f^\varepsilon(y, w) dx dv dy dw. \end{aligned}$$

Thus, we have

$$J_5^\varepsilon \leq \frac{\alpha}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\varepsilon(x, v) f^\varepsilon(y, w) dx dv dy dw ds.$$

Estimate of J_6^ε : A straightforward computation gives

$$\begin{aligned} J_6^\varepsilon &\leq 2\alpha \|u\|_{L^\infty} \|\phi\|_{L^\infty} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) |\rho(y) - \rho^\varepsilon(y)| |u^\varepsilon(x) - u(x)| dx dy ds \\ &= 2\alpha \|u\|_{L^\infty} \|\phi\|_{L^\infty} \int_0^t \|\rho - \rho^\varepsilon\|_{L^1} \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)| dx ds \\ &\leq 2\alpha \|u\|_{L^\infty} \|\phi\|_{L^\infty} \left(\int_0^t \|(\rho - \rho^\varepsilon)(\cdot, s)\|_{L^1}^2 ds \right)^{1/2} \\ &\quad \times \left(\int_0^t \int_{\Omega} \rho^\varepsilon(x, s) |u^\varepsilon(x, s) - u(x, s)|^2 dx ds \right)^{1/2}. \end{aligned}$$

We then use (3.4) to have

$$J_6^\varepsilon \leq C\alpha \int_0^t \int_{\Omega} \mathcal{E}(U^\varepsilon | U) dx ds,$$

where $C > 0$ depends on $\|u\|_{L^\infty}$, $\|\rho^\varepsilon\|_{L^1}$, and $\|\rho\|_{L^1}$.

Estimate of J_7^ε : Integrating by parts gives

$$\begin{aligned} \lambda \int_0^t \int_{\Omega} \nabla V(x) \cdot \rho^\varepsilon(x) u^\varepsilon(x) dx ds &= -\lambda \int_0^t \int_{\Omega} V(x) \nabla \cdot (\rho^\varepsilon(x, s) u^\varepsilon(x, s)) dx ds \\ &= \lambda \int_0^t \int_{\Omega} V(x) \partial_s \rho^\varepsilon(x, s) dx ds \\ &= \lambda \int_{\Omega} V(x) \rho^\varepsilon(x, t) dx - \lambda \int_{\Omega} V(x) \rho_0^\varepsilon(x) dx. \end{aligned}$$

Thus, we get

$$J_7^\varepsilon = \gamma \int_0^t \int_{\Omega} \rho^\varepsilon(x) |u^\varepsilon(x)|^2 dx ds + \lambda \int_{\Omega} V(x) \rho^\varepsilon(x, t) dx - \lambda \int_{\Omega} V(x) \rho_0^\varepsilon(x) dx.$$

Estimate of J_8^ε : Note that

$$\begin{aligned} \lambda \int_0^t \int_{\Omega} \rho^\varepsilon(x, s) u^\varepsilon(x, s) \cdot (\nabla W \star \rho^\varepsilon)(x, s) dx ds \\ &= \lambda \int_0^t \int_{\Omega} \partial_s (\rho^\varepsilon(x, s)) (W \star \rho^\varepsilon)(x, s) dx ds \\ &= \frac{\lambda}{2} \int_0^t \frac{\partial}{\partial s} \left(\int_{\Omega \times \Omega} W(x-y) \rho^\varepsilon(x, s) \rho^\varepsilon(y, s) dx dy \right) ds \\ &= \frac{\lambda}{2} \left(\int_{\Omega \times \Omega} W(x-y) \rho^\varepsilon(x, t) \rho^\varepsilon(y, t) dx dy - \int_{\Omega \times \Omega} W(x-y) \rho_0^\varepsilon(x) \rho_0^\varepsilon(y) dx dy \right). \end{aligned}$$

This yields

$$\begin{aligned} J_8^\varepsilon &= \lambda \int_0^t \int_\Omega \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) \, dx ds \\ &\quad + \frac{\lambda}{2} \left(\int_{\Omega \times \Omega} W(x-y) \rho^\varepsilon(x, t) \rho^\varepsilon(y, t) \, dx dy \right. \\ &\quad \left. - \int_{\Omega \times \Omega} W(x-y) \rho_0^\varepsilon(x) \rho_0^\varepsilon(y) \, dx dy \right). \end{aligned}$$

We now combine the estimates $J_i^\varepsilon, i = 2, 5, 7, 8$ to get

$$\begin{aligned} \sum_{i \in \{2, 5, 7, 8\}} J_i^\varepsilon &= \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{F}(f) - \mathcal{F}(f_0) \\ &\quad + \frac{\alpha}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\varepsilon(x, v) f^\varepsilon(y, w) \, dx dv dy dw ds \\ &\quad + \lambda \int_0^t \int_\Omega \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) \, dx ds \\ &\quad + \gamma \int_0^t \int_\Omega \rho^\varepsilon(x) |u^\varepsilon(x)|^2 \, dx ds. \end{aligned}$$

We then use Proposition 2.1 to find

$$\begin{aligned} \sum_{i \in \{2, 5, 7, 8\}} J_i^\varepsilon &\leq \mathcal{O}(\sqrt{\varepsilon}) + C(1 + \gamma^2)\varepsilon \\ &\quad + \lambda \int_0^t \int_\Omega \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) \, dx ds. \end{aligned}$$

We finally combine all the above estimates to conclude the proof. \square

3.2. Singular/weakly regular interactions: $\Delta W = -\delta_0$ & $\phi \in L^\infty(\Omega)$

In this part, we consider the Coulomb interactions W satisfying $\Delta W = -\delta_0$. Motivated from Refs. 13, 37 and 38, we use a particular structure of the Poisson equation.

Lemma 3.2. *Suppose that the interaction potential W satisfies $\Delta W = -\delta_0$. Then we have*

$$\frac{\lambda}{2} \frac{d}{dt} \int_\Omega |\nabla W \star (\rho - \rho^\varepsilon)|^2 \, dx = \lambda \int_\Omega \nabla W \star (\rho - \rho^\varepsilon) \cdot ((\rho u) - (\rho^\varepsilon u^\varepsilon)) \, dx$$

for $t \in [0, T]$.

Proof. Using the continuity equations of ρ and ρ^ε , we find

$$\begin{aligned} \frac{\lambda}{2} \frac{d}{dt} \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx &= \lambda \int_{\Omega} (\nabla W \star (\rho - \rho^\varepsilon)) \cdot (\nabla W \star (\partial_t \rho - \partial_t \rho^\varepsilon)) dx \\ &= -\lambda \int_{\Omega} (\Delta W \star (\rho - \rho^\varepsilon)) (W \star (\partial_t \rho - \partial_t \rho^\varepsilon)) dx \\ &= \lambda \int_{\Omega} (\rho - \rho^\varepsilon) (W \star (\partial_t \rho - \partial_t \rho^\varepsilon)) dx. \end{aligned} \quad (3.6)$$

We then use the symmetry of W to get

$$\begin{aligned} \int_{\Omega} (\rho - \rho^\varepsilon) (W \star (\partial_t \rho - \partial_t \rho^\varepsilon)) dx &= - \int_{\Omega \times \Omega} (\rho - \rho^\varepsilon)(x) W(x-y) (\nabla_y \cdot (\rho u)(y) - \nabla_y \cdot (\rho^\varepsilon u^\varepsilon)(y)) dx dy \\ &= \int_{\Omega \times \Omega} (\rho - \rho^\varepsilon)(x) \nabla_y (W(x-y)) \cdot ((\rho u)(y) - (\rho^\varepsilon u^\varepsilon)(y)) dx dy \\ &= - \int_{\Omega \times \Omega} (\rho - \rho^\varepsilon)(x) \nabla_x W(x-y) \cdot ((\rho u)(y) - (\rho^\varepsilon u^\varepsilon)(y)) dx dy \\ &= \int_{\Omega \times \Omega} (\rho - \rho^\varepsilon)(y) \nabla_x W(x-y) \cdot ((\rho u)(x) - (\rho^\varepsilon u^\varepsilon)(x)) dx dy \\ &= \int_{\Omega} \nabla W \star (\rho - \rho^\varepsilon) \cdot ((\rho u) - (\rho^\varepsilon u^\varepsilon)) dx. \end{aligned} \quad (3.7)$$

We finally combine (3.6) and (3.7) to conclude the proof. \square

Then, we are now ready to provide the details of the proof of Theorem 2.1 for the singular interactions case. Since the strong convergences (2.4) can be obtained from the inequalities in Theorem 2.1 and it is also already discussed in Ref. 35, we skip the details of that proof.

Proof of Theorem 2.1(i). Using the integration by parts, we estimate

$$\begin{aligned} \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot (\nabla W \star (\rho - \rho^\varepsilon)) dx &+ \int_{\Omega} \nabla W \star (\rho - \rho^\varepsilon) \cdot ((\rho u) - (\rho^\varepsilon u^\varepsilon)) dx \\ &= \int_{\Omega} \nabla W \star (\rho - \rho^\varepsilon) \cdot u (\rho - \rho^\varepsilon) dx \\ &= - \int_{\Omega} \nabla W \star (\rho - \rho^\varepsilon) \cdot u (\Delta W \star (\rho - \rho^\varepsilon)) dx \\ &= -\frac{\lambda}{2} \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 \nabla \cdot u dx \\ &\quad + \lambda \int_{\Omega} \nabla W \star (\rho - \rho^\varepsilon) \otimes \nabla W \star (\rho - \rho^\varepsilon) : \nabla u dx, \end{aligned}$$

i.e.

$$\left| \int_{\Omega} \rho^{\varepsilon} (u^{\varepsilon} - u) \cdot (\nabla W \star (\rho - \rho^{\varepsilon})) dx + \int_{\Omega} \nabla W \star (\rho - \rho^{\varepsilon}) \cdot ((\rho u) - (\rho^{\varepsilon} u^{\varepsilon})) dx \right| \\ \leq \frac{3\lambda}{2} \|\nabla u\|_{L^{\infty}} \int_{\Omega} |\nabla W \star (\rho - \rho^{\varepsilon})|^2 dx.$$

This together with Lemma 3.2 and Proposition 3.1 yields

$$\int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx + \frac{\lambda}{2} \int_{\Omega} |\nabla W \star (\rho - \rho^{\varepsilon})|^2 dx + \gamma \int_0^t \int_{\Omega} \rho^{\varepsilon}(x) |u^{\varepsilon}(x) - u(x)|^2 dx ds \\ + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^{\varepsilon}(x) \rho^{\varepsilon}(y) \phi(x - y) |(u^{\varepsilon}(x) - u(x)) - (u^{\varepsilon}(y) - u(y))|^2 dx dy ds \\ \leq C\sqrt{\varepsilon} + \frac{\lambda}{2} \int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^{\varepsilon})|^2 dx \\ + C(1 + \alpha) \int_0^t \int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx ds + C\lambda \int_0^t \int_{\Omega} |\nabla W \star (\rho - \rho^{\varepsilon})|^2 dx ds.$$

We finally apply Grönwall's lemma to the above to conclude the desired result. This completes the proof. \square

Remark 3.2. The convergence

$$\int_{\Omega} |\nabla W \star (\rho - \rho^{\varepsilon})|^2 dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

implies

$$\rho^{\varepsilon} \rightarrow \rho \quad \text{in } L^{\infty}(0, T; H^{-1}(\Omega)).$$

Indeed, we can easily find

$$\|\rho^{\varepsilon} - \rho\|_{H^{-1}} \leq \|\nabla W \star (\rho - \rho^{\varepsilon})\|_{L^2}.$$

3.3. Weakly regular interactions: $\nabla W \in L^{\infty}(\Omega)$ & $\phi \in L^{\infty}(\Omega)$

In this part, we deal with the weakly singular interactions case.

Lemma 3.3. Suppose that the interaction potential W satisfies $\nabla W \in L^{\infty}(\Omega)$. Then, we have

$$\left| \int_{\Omega} \rho^{\varepsilon}(x) (u^{\varepsilon}(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^{\varepsilon}))(x) dx \right| \leq 2\|\nabla W\|_{L^{\infty}} \int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx.$$

Proof. We use Hölder inequality to get

$$\begin{aligned} & \int_{\Omega} \rho^{\varepsilon}(x)(u^{\varepsilon}(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^{\varepsilon}))(x) dx \\ & \leq \|\nabla W \star (\rho - \rho^{\varepsilon})\|_{L^{\infty}} \left(\int_{\Omega} \rho^{\varepsilon} |u^{\varepsilon} - u|^2 dx \right)^{1/2} \\ & \leq \|\nabla W\|_{L^{\infty}} \|\rho - \rho^{\varepsilon}\|_{L^1} \left(\int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx \right)^{1/2}. \end{aligned}$$

On the other hand, L^1 -norm of $\rho - \rho^{\varepsilon}$ can be estimated as

$$\begin{aligned} \int_{\Omega} |\rho - \rho^{\varepsilon}| dx &= \int_{\Omega} \min \left\{ \frac{1}{\rho}, \frac{1}{\rho^{\varepsilon}} \right\}^{1/2} \max\{\rho, \rho^{\varepsilon}\}^{1/2} |\rho - \rho^{\varepsilon}| dx \\ &\leq \left(\int_{\Omega} \min \left\{ \frac{1}{\rho}, \frac{1}{\rho^{\varepsilon}} \right\} (\rho - \rho^{\varepsilon})^2 dx \right)^{1/2} \left(\int_{\Omega} \max\{\rho, \rho^{\varepsilon}\} dx \right)^{1/2} \\ &\leq 2 \left(\int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx \right)^{1/2}, \end{aligned}$$

due to (3.4). Thus we have

$$\left| \int_{\Omega} \rho^{\varepsilon}(x)(u^{\varepsilon}(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^{\varepsilon}))(x) dx \right| \leq 2 \|\nabla W\|_{L^{\infty}} \int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx. \quad \square$$

Proof of Theorem 2.1(ii). By combining Lemma 3.3 and Proposition 3.1, we find

$$\begin{aligned} & \int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx + \gamma \int_0^t \int_{\Omega} \rho^{\varepsilon}(x) |u^{\varepsilon}(x) - u(x)|^2 dx ds \\ & \quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^{\varepsilon}(x) \rho^{\varepsilon}(y) \phi(x - y) |(u^{\varepsilon}(x) - u(x)) - (u^{\varepsilon}(y) - u(y))|^2 dx dy ds \\ & \leq C \sqrt{\varepsilon} + C(1 + \gamma + \alpha) \int_0^t \int_{\Omega} \mathcal{E}(U^{\varepsilon} | U) dx ds. \end{aligned}$$

We complete the proof by using the Gronwall's inequality to the above. \square

4. Hydrodynamic Limit from Kinetic to Pressureless Euler Equations

In this section, we consider the hydrodynamic limit from (1.1) with $\sigma = 0$ to the pressureless Euler equations with nonlocal interaction forces (1.5). Similarly as before, we rewrite the limiting system (1.5) as the following conservative form:

$$\partial_t U + \nabla \cdot \hat{A}(U) = F(U),$$

where

$$m = \rho u, \quad U := \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad \hat{A}(U) := \begin{pmatrix} m \\ \frac{m \otimes m}{\rho} \end{pmatrix},$$

and

$$F(U) := \begin{pmatrix} 0 \\ \alpha \rho \int_{\Omega} \phi(x-y)(u(y) - u(x))\rho(y) dy - \gamma \rho u - \lambda \rho (\nabla V + \nabla W \star \rho) \end{pmatrix}.$$

We then consider the kinetic energy of the above system

$$\hat{E}(U) := \frac{|m|^2}{2\rho}.$$

Note that the entropy defined above is not strictly convex with respect to ρ . We also define the modulated kinetic energy as

$$\begin{aligned} \hat{\mathcal{E}}(\bar{U}|U) &:= \hat{E}(\bar{U}) - \hat{E}(U) - D\hat{E}(U)(\bar{U} - U) \\ &= \frac{\bar{\rho}|\bar{u}|^2}{2} - \frac{\rho|u|^2}{2} - \frac{|u|^2}{2}(\rho - \bar{\rho}) - u \cdot (\bar{\rho}\bar{u} - \rho u) \\ &= \frac{\bar{\rho}}{2}|\bar{u} - u|^2 \quad \text{with } \bar{U} := \begin{pmatrix} \bar{\rho} \\ \bar{m} \end{pmatrix}. \end{aligned}$$

Compared to the previous diffusive case, our functional $\hat{\mathcal{E}}$ does not include the relative pressure, and as a consequence we cannot deal with the L^1 -norm of the $\bar{\rho} - \rho$. Thus, we need to estimate the nonlocal interaction forces in a different way. For this, we will use a bounded Lipschitz distance for local densities, and this requires a higher regularity for the communication weight ϕ .

4.1. Modulated kinetic energy inequality

In the proposition below, we provide the modulated kinetic energy estimate.

Proposition 4.1. *Let $T > 0$, f^ε be a global weak solution to Eq. (1.1) with $\sigma = 0$, and let (ρ, u) be a strong solution to the system (1.3) on the time interval $[0, T]$. Then, we have*

$$\begin{aligned} &\int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx + \gamma \int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\ &\quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x-y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ &\leq \int_{\Omega} \hat{\mathcal{E}}(U_0^\varepsilon | U_0) dx + \hat{K}(f_0^\varepsilon) - \int_{\Omega} \hat{E}(U_0^\varepsilon) dx \end{aligned}$$

$$\begin{aligned}
 & + (\|\nabla u\|_{L^\infty} + C\alpha^2) \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\
 & + \|\nabla u\|_{L^\infty} \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds + C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds \\
 & + \lambda \int_0^t \int_\Omega \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) dx ds
 \end{aligned} \tag{4.1}$$

for $t \in [0, T]$, where $\hat{K}(f)$ denotes the kinetic energy for the kinetic equation, i.e.

$$\hat{K}(f) := \frac{1}{2} \int_{\Omega \times \mathbb{R}^d} |v|^2 f dx dv.$$

Proof. Employing almost the same arguments as in Lemma 3.1 and Proposition 3.1, we find

$$\begin{aligned}
 & \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx + \gamma \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\
 & + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\
 & = \int_\Omega \hat{\mathcal{E}}(U_0^\varepsilon | U_0) dx + \int_\Omega \hat{E}(U^\varepsilon) - \hat{E}(U_0) dx \\
 & - \int_0^t \int_\Omega \nabla(D\hat{E}(U)) : A(U^\varepsilon | U) dx ds \\
 & - \int_0^t \int_\Omega D\hat{E}(U) [\partial_s U^\varepsilon + \nabla \cdot \hat{A}(U^\varepsilon) - F(U^\varepsilon)] dx ds \\
 & + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) |u^\varepsilon(x) - u^\varepsilon(y)|^2 dx dy ds \\
 & - \alpha \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) (\rho(y) - \rho^\varepsilon(y)) \phi(x - y) \\
 & \times (u^\varepsilon(x) - u(x)) \cdot (u(y) - u(x)) dx dy ds \\
 & + \gamma \int_0^t \int_\Omega \rho^\varepsilon(x) |u^\varepsilon(x)|^2 dx ds + \lambda \int_0^t \int_\Omega \nabla V(x) \cdot \rho^\varepsilon(x) u^\varepsilon(x) dx ds \\
 & + \lambda \int_0^t \int_\Omega \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) dx ds \\
 & + \lambda \int_0^t \int_\Omega \rho^\varepsilon(x) u^\varepsilon(x) \cdot (\nabla W \star \rho^\varepsilon)(x) dx ds \\
 & \leq \int_\Omega \hat{\mathcal{E}}(U_0^\varepsilon | U_0) dx + \hat{K}(f_0^\varepsilon) - \int_\Omega \hat{E}(U_0^\varepsilon) dx + \|\nabla u\|_{L^\infty} \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds
 \end{aligned}$$

$$\begin{aligned}
& - \int_0^t \int_{\Omega} D\hat{E}(U)[\partial_s U^\varepsilon + \nabla \cdot \hat{A}(U^\varepsilon) - F(U^\varepsilon)] dx ds \\
& - \alpha \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x)(\rho(y) - \rho^\varepsilon(y))\phi(x-y) \\
& \times (u^\varepsilon(x) - u(x)) \cdot (u(y) - u(x)) dx dy ds \\
& + \lambda \int_0^t \int_{\Omega} \rho^\varepsilon(x)(u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) dx ds.
\end{aligned}$$

We note that

$$\begin{aligned}
\partial_t U^\varepsilon + \nabla \cdot \hat{A}(U^\varepsilon) - F(U^\varepsilon) &= \begin{pmatrix} 0 \\ \nabla \cdot \left(\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v) f^\varepsilon dv \right) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \nabla \cdot \left(\int_{\mathbb{R}^d} (u^\varepsilon - v) \otimes (v - u^\varepsilon) f^\varepsilon dv \right) \end{pmatrix},
\end{aligned}$$

and this yields

$$\begin{aligned}
& \left| \int_0^t \int_{\Omega} D\hat{E}(U)[\partial_s U^\varepsilon + \nabla \cdot \hat{A}(U^\varepsilon) - F(U^\varepsilon)] dx ds \right| \\
& \leq \|\nabla u\|_{L^\infty} \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds.
\end{aligned}$$

For the term with the communication weight function ϕ , we denoted it by I^ε and split into two terms:

$$\begin{aligned}
I^\varepsilon &= -\alpha \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x)(\rho(y) - \rho^\varepsilon(y))\phi(x-y)(u^\varepsilon(x) - u(x)) \cdot u(y) dx dy ds \\
& \quad + \alpha \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x)(\rho(y) - \rho^\varepsilon(y))\phi(x-y)(u^\varepsilon(x) - u(x)) \cdot u(x) dx dy ds \\
& =: I_1^\varepsilon + I_2^\varepsilon,
\end{aligned}$$

where I_1^ε can be estimated as

$$\begin{aligned}
|I_1^\varepsilon| &= \alpha \left| \int_0^t \int_{\Omega} \left(\int_{\Omega} (\rho(y) - \rho^\varepsilon(y))\phi(x-y)u(y) dy \right) \cdot \rho^\varepsilon(x)(u^\varepsilon(x) - u(x)) dx ds \right| \\
&\leq C\alpha \int_0^t d_{\text{BL}}(\rho^\varepsilon, \rho) \int_{\Omega} \rho^\varepsilon(x)|u^\varepsilon(x) - u(x)| dx dt \\
&\leq C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds + C\alpha^2 \int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds.
\end{aligned}$$

Here we used the fact that $y \mapsto \phi(\cdot, y)u(y)$ is bounded and Lipschitz continuous. Similarly, we can also show that

$$|I_2^\varepsilon| \leq C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds + C\alpha^2 \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds,$$

and this yields

$$|I^\varepsilon| \leq C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds + C\alpha^2 \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds,$$

where $C > 0$ is independent of $\varepsilon > 0$. This completes the proof. \square

4.2. Singular/strongly regular interactions: $\Delta W = -\delta_0$ \mathcal{E} $\phi \in \mathcal{W}^{1,\infty}(\Omega)$

In this section, we consider the Coulomb interaction potential W , i.e. W satisfies $\Delta W = -\delta_0$.

We first note from Lemma 3.2, see also proof of Theorem 2.1(i), that the last term on the right-hand side of (4.1) can be bounded from above by

$$\begin{aligned} & -\frac{\lambda}{2} \int_\Omega |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx + \frac{\lambda}{2} \int_\Omega |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \\ & + \frac{3\lambda}{2} \|\nabla u\|_{L^\infty} \int_0^t \int_\Omega |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx ds. \end{aligned}$$

We next use the free energy estimate (2.1) to show

$$\int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds \leq C\varepsilon,$$

where $C > 0$ is independent of ε . Combining those observations with Proposition 4.1 yields the following proposition.

Proposition 4.2. *Let $T > 0$, f^ε be a global weak solution to Eq. (1.1) with $\sigma = 0$, and let (ρ, u) be a strong solution to the system (1.3) on the time interval $[0, T]$. Then we have*

$$\begin{aligned} & \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx + \frac{\lambda}{2} \int_\Omega |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx + \gamma \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\ & + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) \\ & \quad \times |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ & \leq \int_\Omega \hat{\mathcal{E}}(U_0^\varepsilon | U_0) dx + \hat{K}(f_0^\varepsilon) - \int_\Omega \hat{E}(U_0^\varepsilon) dx \\ & \quad + \frac{\lambda}{2} \int_\Omega |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx + C\varepsilon \end{aligned}$$

$$\begin{aligned}
& + C \int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds + C \int_0^t \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx ds \\
& + C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds
\end{aligned}$$

for $t \in [0, T]$, where $C > 0$ is independent of $\varepsilon > 0$.

In order to close the modulated kinetic energy inequality, we show that the bounded and Lipschitz distance d_{BL} between local densities can be bounded from above by the modulated kinetic energy, which directly gives the quantitative error estimate between ρ and ρ^ε .

Lemma 4.1. *Let f^ε be a global weak solution to Eq. (1.1) with $\sigma = 0$ and (ρ, u) be a strong solution to the system (1.5) on the time interval $[0, T]$. Then we have*

$$d_{\text{BL}}(\rho(t), \rho^\varepsilon(t)) \leq C d_{\text{BL}}(\rho_0, \rho_0^\varepsilon) + C \left(\int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \right)^{1/2}$$

for $0 \leq t \leq T$, where $C > 0$ is independent of $\varepsilon > 0$.

Although it has been already studied in Ref. 17, see also Refs. 6, 28, we give the details of proof for the completeness of our work. Let us define forward characteristics $X(t) := X(t; 0, x)$ which solves

$$\partial_t X(t) = u(X(t), t) \quad \text{with } X(0) = x \in \Omega. \quad (4.2)$$

Then $X(t)$ uniquely exists on the time interval $[0, T]$ since u is bounded and Lipschitz continuous. Moreover, the solution ρ can be determined as the push-forward of its initial densities through the flow maps X , i.e. $\rho(t) = X(t; 0, \cdot) \# \rho_0$. On the other hand, we cannot consider the characteristic for the continuity equation of ρ^ε due to the lack of regularity of u^ε . Regarding this problem, we recall the following proposition from Theorem 8.2.1 in Ref. 1, see also Proposition 3.3 in Ref. 28. Before we proceed, we denote $\mathcal{P}_p(\Omega)$ by the space of Borel probability measures with finite p th moments.

Proposition 4.3. *Let $T > 0$ and $\rho : [0, T] \rightarrow \mathcal{P}_p(\Omega)$ be a narrowly continuous solution of (4.2), that is, ρ is continuous in the duality with continuous bounded functions, for a Borel vector field u satisfying*

$$\int_0^T \int_{\Omega} |u(x, t)|^p \rho(x, t) dx dt < \infty \quad (4.3)$$

for some $p > 1$. Let $\Gamma_T : [0, T] \rightarrow \Omega$ denote the space of continuous curves. Then there exists a probability measure η on $\Gamma_T \times \Omega$ satisfying the following properties:

- (i) η is concentrated on the set of pairs (γ, x) such that γ is an absolutely continuous curve satisfying

$$\dot{\gamma}(t) = u(\gamma(t), t)$$

for almost everywhere $t \in (0, T)$ with $\gamma(0) = x \in \Omega$.

(ii) ρ satisfies

$$\int \varphi(x) \rho \, dx = \int_{\Gamma_T \times \Omega} \varphi(\gamma(t)) \, d\eta(\gamma, x)$$

for all $\varphi \in \mathcal{C}_b(\Omega)$, $t \in [0, T]$.

Proof of Lemma 4.1. It follows from Lemma 2.1 that

$$\begin{aligned} \int_{\Omega} |u^\varepsilon|^2 \rho^\varepsilon \, dx &\leq \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\varepsilon \, dx dv \\ &\leq \int_{\Omega \times \mathbb{R}^d} |v|^2 f_0^\varepsilon \, dx dv + \lambda \int_{\Omega \times \Omega} W(x-y) \rho_0^\varepsilon(x) \rho^\varepsilon(y) \, dx dy \\ &\quad + 2\lambda \int_{\Omega} V \rho_0^\varepsilon(x) \, dx \\ &< \infty, \end{aligned}$$

thus, the integrability condition (4.3) holds for $p = 2$, and thus by Proposition 4.3, we obtain a probability measure η^ε in $\Gamma_T \times \mathbb{R}$ concentrated on the set of pairs (γ, x) such that γ is a solution of

$$\dot{\gamma}(t) = u^\varepsilon(\gamma(t), t) \quad (4.4)$$

with $\gamma(0) = x \in \Omega$. Moreover, ρ^ε satisfies Proposition 4.3(ii), i.e.

$$\int_{\Omega} \varphi(x) \rho^\varepsilon(x, t) \, dx = \int_{\Gamma_T \times \Omega} \varphi(\gamma(t)) \, d\eta^\varepsilon(\gamma, x) \quad (4.5)$$

for all $\varphi \in \mathcal{C}_b(\Omega)$, $t \in [0, T]$. We now consider the push-forward of ρ_0^ε through the flow map X and denote it by $\bar{\rho}^\varepsilon$, i.e. $\bar{\rho}^\varepsilon = X \# \rho_0^\varepsilon$.

We first estimate the error between $\bar{\rho}^\varepsilon$ and ρ^ε in bounded Lipschitz distance. By the disintegration theorem of measures, see Ref. 1, we can write

$$d\eta^\varepsilon(\gamma, x) = \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x) \, dx,$$

where $\{\eta_x^\varepsilon\}_{x \in \Omega}$ is a family of probability measures on Γ_T concentrated on solutions of (4.4). By using this newly introduced measure η^ε , we define a measure ν^ε on $\Gamma_T \times \Gamma_T \times \Omega$ by

$$d\nu^\varepsilon(\gamma, \sigma, x) := \eta_x^\varepsilon(d\gamma) \otimes \delta_{X(\cdot; 0, x)}(d\sigma) \otimes \rho_0^\varepsilon(x) \, dx.$$

We further take into account an evaluation map $E_t : \Gamma_T \times \Gamma_T \times \Omega \rightarrow \Omega \times \Omega$ defined as $E_t(\gamma, \sigma, x) = (\gamma(t), \sigma(t))$. Then, we find that measure $\pi_t^\varepsilon := (E_t) \# \nu^\varepsilon$ on $\Omega \times \Omega$ has marginals $\rho^\varepsilon(x, t) \, dx$ and $\bar{\rho}^\varepsilon(y, t) \, dy$ for $t \in [0, T]$, see (4.5). This implies

$$\begin{aligned} d_{\text{BL}}(\rho^\varepsilon(t), \bar{\rho}^\varepsilon(t)) &\leq \int_{\Omega \times \Omega} |x - y| \, d\pi_t^\varepsilon(x, y) \\ &= \int_{\Gamma_T \times \Gamma_T \times \Omega} |\sigma(t) - \gamma(t)| \, d\nu^\varepsilon(\gamma, \sigma, x) \\ &= \int_{\Gamma_T \times \Omega} |X(t; 0, x) - \gamma(t)| \, d\eta^\varepsilon(\gamma, x). \end{aligned} \quad (4.6)$$

We note from (4.2) and (4.4) that

$$\begin{aligned}
 & |X(t; 0, x) - \gamma(t)| \\
 &= \left| \int_0^t u(X(s; 0, x)) - u^\varepsilon(\gamma(s), s) ds \right| \\
 &\leq \int_0^t |u(X(s; 0, x)) - u(\gamma(s), s)| ds + \int_0^t |u(\gamma(s), s) - u^\varepsilon(\gamma(s), s)| ds \\
 &\leq \|\nabla_x u\|_{L^\infty} \int_0^t |X(s; 0, x) - \gamma(s)| ds + \int_0^t |u(\gamma(s), s) - u^\varepsilon(\gamma(s), s)| ds.
 \end{aligned}$$

We then apply Grönwall's lemma to the above to yield

$$|X(t; 0, x) - \gamma(t)| \leq C \int_0^t |u(\gamma(s), s) - u^\varepsilon(\gamma(s), s)| ds,$$

where $C > 0$ is independent of $\varepsilon > 0$. Putting this into (4.6) entails

$$\begin{aligned}
 d_{\text{BL}}(\rho^\varepsilon(t), \bar{\rho}^\varepsilon(t)) &\leq C \int_0^t \int_{\Gamma_T \times \Omega} |u(\gamma(s), s) - u^\varepsilon(\gamma(s), s)| d\eta^\varepsilon(\gamma, x) ds \\
 &\leq C \int_0^t \int_\Omega |u(x, s) - u^\varepsilon(x, s)| \rho^\varepsilon(x, s) dx ds \\
 &\leq C\sqrt{T} \left(\int_0^t \int_\Omega |u^\varepsilon(x, s) - u(x, s)|^2 \rho^\varepsilon(x, s) dx ds \right)^{1/2} \\
 &= C \left(\int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \right)^{1/2}, \tag{4.7}
 \end{aligned}$$

where $C > 0$ is independent of $\varepsilon > 0$, and we used (4.5).

We next estimate the bounded Lipschitz distance between $\bar{\rho}^\varepsilon$ and ρ . For bounded Lipschitz function ϕ , we find

$$\begin{aligned}
 \left| \int_\Omega \phi(x)(\rho(x) - \bar{\rho}^\varepsilon(x)) dx \right| &= \left| \int_\Omega \phi(X(t))(\rho_0(x) - \rho_0^\varepsilon(x)) dx \right| \\
 &\leq C d_{\text{BL}}(\rho_0, \rho_0^\varepsilon), \tag{4.8}
 \end{aligned}$$

where $C > 0$ is independent of ε , and we used the bounded Lipschitz continuity of $\phi(X(t; 0, \cdot))$. More precisely, we have

$$\begin{aligned}
 |X(t; 0, x) - X(t; 0, y)| &\leq |x - y| + \int_0^t |u(X(s; 0, x)) - u(X(s; 0, y))| ds \\
 &\leq |x - y| + \|\nabla_x u\|_{L^\infty} \int_0^t |X(s; 0, x) - X(s; 0, y)| ds,
 \end{aligned}$$

and applying Grönwall's lemma to the above yields the Lipschitz continuity of the characteristic flow $X(t; 0, x)$ in x . Furthermore, we have

$$\begin{aligned} |\phi(X(t; 0, x)) - \phi(X(t; 0, y))| &\leq \|\phi\|_{\text{Lip}} |X(t; 0, x) - X(t; 0, y)| \\ &\leq \|\phi\|_{\text{Lip}} \|X\|_{\text{Lip}} |x - y|, \end{aligned}$$

where $\|\cdot\|_{\text{Lip}}$ denotes the Lipschitz constant given by

$$\|\phi\|_{\text{Lip}} := \sup_{x \neq y \in \mathbb{R}^d} \frac{|\phi(x) - \phi(y)|}{|x - y|}.$$

This together with (4.8) implies

$$d_{\text{BL}}(\rho(t), \bar{\rho}^\varepsilon(t)) \leq C d_{\text{BL}}(\rho_0, \rho_0^\varepsilon)$$

for $t \in [0, T]$, where $C > 0$ is independent of $\varepsilon > 0$. Finally, we combine this with (4.7) to conclude

$$\begin{aligned} d_{\text{BL}}(\rho(t), \rho^\varepsilon(t)) &\leq d_{\text{BL}}(\rho(t), \bar{\rho}^\varepsilon(t)) + d_{\text{BL}}(\rho^\varepsilon(t), \bar{\rho}^\varepsilon(t)) \\ &\leq C d_{\text{BL}}(\rho_0, \rho_0^\varepsilon) + C \left(\int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \right)^{1/2}, \end{aligned}$$

where $C > 0$ is independent of $\varepsilon > 0$. □

Proof of Theorem 2.2(i). Applying Lemma 4.1 to Proposition 4.2 yields

$$\begin{aligned} &\int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx + \frac{\lambda}{2} \int_\Omega |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx + d_{\text{BL}}^2(\rho^\varepsilon, \rho) + \gamma \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\ &\quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x - y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ &\leq \int_\Omega \hat{\mathcal{E}}(U_0^\varepsilon | U_0) dx + \hat{K}(f_0^\varepsilon) - \int_\Omega \hat{E}(U_0^\varepsilon) dx + \frac{\lambda}{2} \int_\Omega |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \\ &\quad + C d_{\text{BL}}^2(\rho_0^\varepsilon, \rho_0) + C \sqrt{\varepsilon} + C \int_0^t \int_\Omega \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\ &\quad + C \int_0^t \int_\Omega |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx ds + C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds. \end{aligned}$$

We then use Grönwall's lemma to the above and the assumptions **(H2)**–**(H3)** to conclude the desired result. □

4.3. Strongly regular interactions: $\nabla W \in \mathcal{W}^{1,\infty}(\Omega)$ & $\phi \in \mathcal{W}^{1,\infty}(\Omega)$

As a direct consequence of Proposition 4.1 together with the assumptions **(H2)**–**(H3)**, we have the following proposition.

Proposition 4.4. *Let f^ε be a global weak solution to Eq. (1.1) with $\sigma = 0$ and (ρ, u) be a strong solution to the system (1.5) on the time interval $[0, T]$. Then we have*

$$\begin{aligned} & \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx + \gamma \int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\ & \quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x-y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ & \leq C\sqrt{\varepsilon} + C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds + C \int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\ & \quad + \lambda \int_0^t \int_{\Omega} \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) dx ds, \end{aligned}$$

where $C > 0$ is independent of ε .

Proof of Theorem 2.2(ii). We first claim that

$$\begin{aligned} & \lambda \left| \int_{\Omega} \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) dx \right| \\ & \leq C d_{\text{BL}}^2(\rho^\varepsilon, \rho) + C\lambda^2 \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx. \end{aligned}$$

Indeed, since $\nabla W \in \mathcal{W}^{1,\infty}(\Omega)$, we get

$$\left| \int_{\Omega} \nabla W(x-y) (\rho(y) - \rho^\varepsilon(y)) dy \right| \leq C d_{\text{BL}}(\rho^\varepsilon, \rho).$$

This yields

$$\begin{aligned} & \lambda \left| \int_{\Omega} \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \cdot (\nabla W \star (\rho - \rho^\varepsilon))(x) dx \right| \\ & \leq C\lambda d_{\text{BL}}(\rho^\varepsilon, \rho) \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} \leq C d_{\text{BL}}^2(\rho^\varepsilon, \rho) + C\lambda^2 \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx. \end{aligned}$$

This together with Proposition 4.4 provides

$$\begin{aligned} & \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx + d_{\text{BL}}^2(\rho^\varepsilon, \rho) + \gamma \int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds \\ & \quad + \frac{\alpha}{2} \int_0^t \int_{\Omega \times \Omega} \rho^\varepsilon(x) \rho^\varepsilon(y) \phi(x-y) |(u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y))|^2 dx dy ds \\ & \leq C\sqrt{\varepsilon} + C \int_0^t d_{\text{BL}}^2(\rho^\varepsilon, \rho) ds + C \int_0^t \int_{\Omega} \hat{\mathcal{E}}(U^\varepsilon | U) dx ds. \end{aligned}$$

Hence, by applying Gronwall's lemma to the above, we complete the proof. \square

5. Global Existence of Weak Solutions to the Kinetic Equation (1.1)

In this section, we provide the global-in-time existence of weak solutions to the system (1.1). For notational simplicity, we set $\gamma = \lambda = \alpha = \beta = 1$. We also only consider the Coulomb case since the weakly singular case $\nabla_x W \in L^\infty(\Omega)$ can be easily obtained similarly as in Ref. 34. Here, the domain of our interest is $\Omega = \mathbb{T}^d$ or \mathbb{R}^d with $d \geq 3$. Since the analysis on \mathbb{T}^d is almost similar to the \mathbb{R}^d case, we mostly consider the case $\Omega = \mathbb{R}^d$.

Theorem 5.1. *Let $T > 0$. Suppose that f_0 satisfies*

$$f_0 \in (L^1_+ \cap L^\infty)(\Omega \times \mathbb{R}^d) \quad \text{and} \quad (|v|^2 + V + W \star \rho_0)f_0 \in L^1(\Omega \times \mathbb{R}^d).$$

Then there exists a weak solution of Eq. (1.1) in the sense of Definition 2.1 satisfying

$$f \in \mathcal{C}([0, T]; L^1(\Omega \times \mathbb{R}^d)) \cap L^\infty(\Omega \times \mathbb{R}^d \times (0, T)) \quad \text{and} \\ (|v|^2 + V + W \star \rho)f \in L^\infty(0, T; L^1(\Omega \times \mathbb{R}^d)).$$

Furthermore, f satisfies the entropy inequality (2.2).

5.1. Regularized kinetic equation

For the existence of weak solutions to (1.1), we first regularize the system with respect to regularization parameters $\eta := (R, \zeta, \varepsilon)$ as follows:

$$\begin{aligned} \partial_t f^\eta + v \cdot \nabla f^\eta - \nabla_v \cdot ((v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta) f^\eta) + \nabla_v \cdot (F[f^\eta] f^\eta) \\ = \nabla_v \cdot ((v - \chi_\zeta(u^\eta_\varepsilon)) f^\eta + \sigma \nabla_v f^\eta), \end{aligned} \quad (5.1)$$

subject to initial data

$$f_0^\eta = f^\eta(0, x, v) := f_0(x, v) \mathbb{1}_{\{|v| \leq \zeta\}},$$

where

$$\rho^\eta := \int_{\mathbb{R}^d} f^\eta dv, \quad \rho^\eta u^\eta := \int_{\mathbb{R}^d} v f^\eta dv, \quad u^\eta_\varepsilon := \frac{\rho^\eta u^\eta}{\rho^\eta + \varepsilon},$$

and $W^\varepsilon = W^\varepsilon(x)$ is given as

$$W^\varepsilon(x) := c_d(\varepsilon + |x|^2)^{-(d-2)/2}$$

with $d \geq 3$, where c_d is a normalization constant. Moreover, χ_ζ is given as

$$\chi_\zeta(v) = v \mathbb{1}_{\{|v| \leq \zeta\}},$$

and V^R is given as

$$V^R(x) := V(x) M\left(\frac{x}{R}\right).$$

Here $M(x) \in C_c^\infty(\mathbb{R}^d)$ is a smooth function given by

$$M(x) = \begin{cases} 1 & |x| \leq 1, \\ 0 < M(x) < 1 & 1 < |x| < 2, \\ 0 & |x| \geq 2. \end{cases}$$

Now, we partially linearize (5.1) as follows:

$$\begin{aligned} \partial_t f^\eta + v \cdot \nabla f^\eta - \nabla_v \cdot ((v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta) f^\eta) + \nabla_v \cdot (F[f^\eta] f^\eta) \\ = \nabla_v \cdot ((v - \chi_\zeta(\tilde{u})) f^\eta + \sigma \nabla_v f^\eta), \end{aligned} \quad (5.2)$$

where \tilde{u} is in $\mathcal{S} := L^2(\Omega \times (0, T))$. Once we note that ∇W^ε is bounded and Lipschitz continuous, the existence of weak solutions to (5.2) comes from almost the same argument in Theorem 6.3 in Ref. 34. Moreover, we estimate

$$\begin{aligned} \frac{d}{dt} \int_{\Omega \times \mathbb{R}^d} (f^\eta)^p dx dv \\ = (p-1) \int_{\Omega \times \mathbb{R}^d} (f^\eta)^p \nabla_v \cdot (2v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta - \chi_\zeta(\tilde{u}) - F[f^\eta]) dx dv \\ - \sigma p(p-1) \int_{\Omega \times \mathbb{R}^d} (f^\eta)^{p-2} |\nabla_v f^\eta|^2 dx dv \\ = (p-1) \int_{\Omega \times \mathbb{R}^d} (f^\eta)^p (2d + d\phi \star \rho^\eta) dx dv \\ - \frac{4\sigma(p-1)}{p} \int_{\Omega \times \mathbb{R}^d} |\nabla_v (f^\eta)^{p/2}|^2 dx dv \end{aligned}$$

for $p \in [1, \infty)$. This together with Grönwall's lemma gives

$$\begin{aligned} \|f^\eta(\cdot, \cdot, t)\|_{L^p}^p + \frac{4\sigma(p-1)}{p} \int_0^t e^{d(p-1)(2+\|\phi\|_{L^\infty}\|f_0^\eta\|_{L^1})(t-s)} \|\nabla_v (f^\eta)^{p/2}(\cdot, \cdot, s)\|_{L^2}^2 ds \\ \leq \|f_0^\eta\|_{L^p}^p e^{d(p-1)(2+\|\phi\|_{L^\infty}\|f_0^\eta\|_{L^1})t}. \end{aligned}$$

In particular, we have

$$\|f^\eta(\cdot, \cdot, t)\|_{L^1} \leq \|f_0^\eta\|_{L^1} \leq \|f_0\|_{L^1} = 1, \quad \|f^\eta(\cdot, \cdot, t)\|_{L^\infty} \leq \|f_0^\eta\|_{L^\infty} e^{d(2+\|\phi\|_{L^\infty})t}$$

for $t \in [0, T]$.

We next estimate higher-order velocity moments and entropy inequality of solutions to (5.2).

Lemma 5.1. *For a weak solution f^η to (5.2), its velocity moments satisfy the following boundedness condition:*

$$\sup_{t \in (0, T)} \int_{\Omega \times \mathbb{R}^d} |v|^N f dx dv \leq C(d, \eta, N, T), \quad \forall N \geq 0.$$

Proof. For a weak solution f^η to (5.2) and $N \geq 2$, we let

$$m_N(f) := \int_{\Omega \times \mathbb{R}^d} |v|^N f \, dx dv.$$

Then, we estimate

$$\begin{aligned} & \frac{d}{dt} m_N(f^\eta) \\ &= -N \int_{\Omega \times \mathbb{R}^d} (v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta) \cdot v f^\eta |v|^{N-2} \, dx dv \\ & \quad + N \int_{\Omega \times \mathbb{R}^d} F[f^\eta] f^\eta \cdot v |v|^{N-2} \, dx dv - N \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot v f^\eta |v|^{N-2} \, dx dv \\ & \quad - \sigma N \int_{\Omega \times \mathbb{R}^d} \nabla_v f^\eta \cdot v |v|^{N-2} \, dx dv \\ &= -2N m_N(f^\eta) - N \int_{\Omega \times \mathbb{R}^d} (\nabla V^R + \nabla W^\varepsilon \star \rho^\eta) \cdot v f^\eta |v|^{N-2} \, dx dv \\ & \quad + N \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y)(w-v) \cdot v f^\eta(y, w) f^\eta(x, v) |v|^{N-2} \, dx dy dv dw \\ & \quad + N \int_{\Omega \times \mathbb{R}^d} \chi_\zeta(\tilde{u}) \cdot v f^\eta |v|^{N-2} \, dx dv + \sigma N(N-2+d) m_{N-2}(f^\eta) \\ &\leq -N m_N(f^\eta) + \frac{N}{2} \int_{\Omega \times \mathbb{R}^d} (\nabla V^R + \nabla W^\varepsilon \star \rho^\eta)^2 f^\eta |v|^{N-2} \, dx dv \\ & \quad + N \phi_M \int_{\Omega^2 \times \mathbb{R}^{2d}} |w| |v| f^\eta(y, w) f^\eta(x, v) |v|^{N-2} \, dx dy dv dw \\ & \quad + \frac{N}{2} \int_{\Omega \times \mathbb{R}^d} (\chi_\zeta(\tilde{u}))^2 f^\eta |v|^{N-2} \, dx dv + \sigma N(N-2+d) m_{N-2}(f^\eta) \\ &\leq \frac{N}{2} \int_{\Omega \times \mathbb{R}^d} (\nabla V^R + \nabla W^\varepsilon \star \rho^\eta)^2 f^\eta |v|^{N-2} \, dx dv + \frac{N(\phi_M)^2}{2} m_2(f^\eta) m_{N-2}(f^\eta) \\ & \quad + \frac{N}{2} \int_{\Omega \times \mathbb{R}^d} (\chi_\zeta(\tilde{u}))^2 f^\eta |v|^{N-2} \, dx dv + \sigma N(N-2+d) m_{N-2}(f^\eta) \\ &\leq C(m_N(f^\eta) + m_{N-2}(f^\eta)), \end{aligned}$$

where $C = C(d, \eta, N, T)$ is a positive constant and we used Young's inequality. Since $m_0(f^\eta)$ is just $\|f^\eta\|_{L^1} = \|f_0^\eta\|_{L^1}$, one uses Grönwall's lemma and induction argument to conclude that

$$\sup_{t \in (0, T)} \int_{\Omega \times \mathbb{R}^d} |v|^N f \, dx dv \leq C(d, \eta, N, T), \quad \forall N = 0, 2, 4, \dots$$

Moreover, for $N \in \mathbb{R}_+ \setminus \{0, 2, 4, \dots\}$, we can find $l \in \mathbb{N} \cup \{0\}$ that satisfies $0 < N - 2l < 2$, and this gives

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^d} |v|^{N-2l} f \, dx dv &\leq \left(\int_{\Omega \times \mathbb{R}^d} |v|^2 f \, dx dv \right)^{\frac{N-2l}{2}} \left(\int_{\Omega \times \mathbb{R}^d} f \, dx dv \right)^{\frac{2+2l-N}{2}} \\ &\leq C(d, \eta, N, T). \end{aligned}$$

This asserts our desired result. \square

Proposition 5.1. *For a weak solution f^η to (5.2), it satisfies the following relation:*

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V^R + \sigma \log f^\eta \right) f^\eta \, dx dv + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) \, dx dy \right) \\ &\quad + \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(\tilde{u})) f^\eta|^2 \, dx dv + \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\eta \, dx dv \\ &\quad + \frac{1}{2} \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) \, dx dy dv dw \\ &= \int_{\Omega \times \mathbb{R}^d} (\chi_\zeta(\tilde{u}) - v) \cdot \chi_\zeta(\tilde{u}) f^\eta \, dx dv \\ &\quad + \sigma d \|f_0^\eta\|_{L^1} + \sigma d \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) f^\eta(y, w) f^\eta(x, v) \, dx dy dv dw. \end{aligned}$$

Proof. First, it directly follows from Lemma 5.1 that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega \times \mathbb{R}^d} |v|^2 f^\eta \, dx dv \right) \\ &= - \int_{\Omega \times \mathbb{R}^d} (v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta) \cdot v f^\eta \, dx dv \\ &\quad + \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) (w-v) \cdot v f^\eta(y, w) f^\eta(x, v) \, dx dy dv dw \\ &\quad - \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot v f^\eta \, dx dv - \sigma \int_{\Omega \times \mathbb{R}^d} v \cdot \nabla_v f^\eta \, dx dv \\ &= - \int_{\Omega \times \mathbb{R}^d} (v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta) \cdot v f^\eta \, dx dv \\ &\quad - \frac{1}{2} \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) \, dx dy dv dw \\ &\quad - \int_{\Omega \times \mathbb{R}^d} |v - \chi_\zeta(\tilde{u})|^2 f^\eta \, dx dv - \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot \chi_\zeta(\tilde{u}) f^\eta \, dx dv \\ &\quad - \sigma \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot \nabla_v f^\eta \, dx dv. \end{aligned}$$

On the other hand, we get

$$\frac{d}{dt} \left(\int_{\Omega \times \mathbb{R}^d} V^R f^\eta dx dv \right) = \int_{\Omega \times \mathbb{R}^d} v \cdot \nabla V^R f^\eta dx dv$$

and

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \right) \\ &= \int_{\Omega \times \Omega} W^\varepsilon(x-y) \partial_t \rho^\eta(x) \rho^\eta(y) dx dy \\ &= - \int_{\Omega \times \Omega} W^\varepsilon(x-y) \nabla \cdot (\rho^\eta u^\eta)(x) \rho^\eta(y) dx dy \\ &= \int_{\Omega \times \Omega} \nabla W^\varepsilon(x-y) \rho^\eta(y) \cdot (\rho^\eta u^\eta)(x) dx dy \\ &= \int_{\Omega} (\nabla W^\varepsilon \star \rho^\eta) \cdot (\rho^\eta u^\eta) dx \\ &= \int_{\Omega \times \mathbb{R}^d} (\nabla W^\varepsilon \star \rho^\eta) \cdot v f^\eta dx dv. \end{aligned}$$

This yields

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\eta dx dv + \int_{\Omega \times \mathbb{R}^d} V^R f^\eta dx dv \right. \\ & \quad \left. + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \right) \\ &= - \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\eta dx dv \\ & \quad - \frac{1}{2} \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) dx dy dv dw \\ & \quad - \int_{\Omega \times \mathbb{R}^d} |v - \chi_\zeta(\tilde{u})|^2 f^\eta dx dv - \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot \chi_\zeta(\tilde{u}) f^\eta dx dv \\ & \quad - \sigma \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot \nabla_v f^\eta dx dv. \end{aligned}$$

We then combine the previous estimates with the following entropy estimate

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega \times \mathbb{R}^d} \sigma f^\eta \log f^\eta dx dv \right) \\ &= \int_{\Omega \times \mathbb{R}^d} \sigma (\partial_t f^\eta) \log f^\eta dx dv \\ &= -\sigma \int_{\Omega \times \mathbb{R}^d} (v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta) \cdot \nabla_v f^\eta dx dv \end{aligned}$$

$$\begin{aligned}
& + \sigma \int_{\Omega \times \mathbb{R}^d} F[f^\eta] \cdot \nabla_v f^\eta \, dx dv \\
& - \sigma \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot \nabla_v f^\eta - \sigma^2 \int_{\Omega \times \mathbb{R}^d} \frac{|\nabla_v f^\eta|^2}{f^\eta} \, dx dv \\
& = \sigma d \|f_0^\eta\|_{L^1} + \sigma d \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) f^\eta(y, w) f^\eta(x, v) \, dx dy dv dw \\
& - \sigma \int_{\Omega \times \mathbb{R}^d} (v - \chi_\zeta(\tilde{u})) \cdot \nabla_v f^\eta \, dx dv - \sigma^2 \int_{\Omega \times \mathbb{R}^d} \frac{|\nabla_v f^\eta|^2}{f^\eta} \, dx dv
\end{aligned}$$

to conclude the desired result. \square

5.2. Existence of the regularized kinetic equation

Now, we provide the existence of weak solutions to (5.1) and their energy estimates. Similarly as in Refs. 9, 41, we define the mapping $\mathcal{T} : \mathcal{S} \rightarrow \mathcal{S}$, where $\mathcal{S} = L^2(\Omega \times (0, T))$ by

$$\tilde{u} \mapsto \mathcal{T}(\tilde{u}) := u_\varepsilon^\eta = \frac{\rho^\eta u^\eta}{\rho^\eta + \varepsilon}.$$

First, we prove that the operator \mathcal{T} is well-defined.

Lemma 5.2. *For a weak solution f^η to (5.2), the averaged quantities $(\rho^\eta, \rho^\eta u^\eta)$ satisfy*

$$\rho^\eta \in L^p(\Omega), \quad \rho^\eta u^\eta \in L^p(\Omega), \quad \forall p \in [1, \infty),$$

and as a consequence, \mathcal{T} is well-defined.

Proof. Since the proof for the first assertion can be found in Ref. 34, we omit its proof. Since ρ^η is bounded, it suffices to show the boundedness of u_ε^η . Obviously,

$$|u_\varepsilon^\eta| = \left| \frac{\rho^\eta u^\eta}{\rho^\eta + \varepsilon} \right| \leq \frac{1}{\varepsilon} |\rho^\eta u^\eta|,$$

and since $\rho^\eta u^\eta$ is bounded, \mathcal{T} is well-defined. \square

Next, we discuss the compactness of \mathcal{T} . Here, we state the velocity averaging lemma from Ref. 43.

Lemma 5.3. *Let $\{f^m\}$ be bounded in $L_{\text{loc}}^p(\mathbb{R}^{2d+1})$ with $1 < p < \infty$ and $\{G^m\}$ be bounded in $L_{\text{loc}}^p(\mathbb{R}^{2d+1})$. If f^m and G^m satisfy*

$$f_t^m + v \cdot \nabla f^m = \nabla_v^\alpha G^m, \quad f^m|_{t=0} = f_0 \in L^p(\mathbb{R}^{2d})$$

for some multi-index α and $\varphi \in \mathcal{C}_c^{|\alpha|}(\mathbb{R}^{2d})$, then

$$\left\{ \int_{\mathbb{R}^d} f^m \varphi \, dv \right\}$$

is relatively compact in $L^p_{\text{loc}}(\mathbb{R}^{d+1})$.

We then use the previous lemma to obtain the following result, which is very similar to Lemma 2.7 in 34.

Lemma 5.4. *Let $\{f^m\}$ and $\{G^m\}$ be in Lemma 5.3 and assume that for $r \geq 2$,*

$$\sup_{m \in \mathbb{N}} \|f^m\|_{L^\infty(\mathbb{R}^{2d+1})} + \sup_{m \in \mathbb{N}} \|(|v|^r + |x|^2)f^m\|_{L^\infty(0,T;L^1(\mathbb{R}^{2d}))} < \infty. \quad (5.3)$$

Then, for any $\varphi(v)$ satisfying $|\varphi(v)| \leq c|v|$, the sequence

$$\left\{ \int_{\mathbb{R}^d} f^m \varphi \, dv \right\}$$

is relatively compact in $L^q(\mathbb{R}^{d+1})$ for any $q \in (1, \frac{d+r}{d+1})$.

Thanks to Lemma 5.4, we can prove the compactness of \mathcal{T} .

Corollary 5.1. *For a uniformly bounded sequence \tilde{u}^m in \mathcal{S} , the sequence $\mathcal{T}(\tilde{u}^m) = (u_\varepsilon^\eta)^m$ converges strongly in \mathcal{S} , up to a subsequence.*

Proof. For the convergence of $\{(u_\varepsilon^\eta)^m\}$, we set

$$f^m := (f^\eta)^m,$$

$$G^m := (\sigma \nabla_v (f^\eta)^m + (2v + \nabla V^R + \nabla W^\varepsilon \star (\rho^\eta)^m - F[(f^\eta)^m] - \chi_\zeta(\tilde{u}))(f^\eta)^m),$$

then it is easy to see $G^m \in L^p_{\text{loc}}(\mathbb{R}^{2d+1})$. Let us choose r appeared in (5.3) sufficiently large. Then, we set $\varphi(v) = 1$ and $\varphi(v) = v$ in Lemma 5.4, respectively, and obtain the following strong convergence up to a subsequence:

$$(\rho^\eta)^m \rightarrow \rho^\eta \quad \text{in } L^2(\Omega \times (0, T)) \quad \text{and a.e.,}$$

$$(\rho^\eta)^m (u^\eta)^m \rightarrow \rho^\eta u^\eta \quad \text{in } L^2(\Omega \times (0, T)),$$

and consequently, it gives the convergence of $\{(u_\varepsilon^\eta)^m\}$ up to a subsequence. \square

Remark 5.1. Lemma 5.2 and Corollary 5.1 imply that the operator \mathcal{T} is well-defined, continuous and compact. Thus, we can use the Schauder's fixed point theorem to obtain a weak solution f^η to (5.1).

From the previous fixed point argument, the following energy inequality associated with (5.1) is obvious.

Corollary 5.2. *Let f^η be a weak solution to (5.1). Then, it satisfies the following energy inequality:*

$$\frac{d}{dt} \left(\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V^R + \sigma \log f^\eta \right) f^\eta \, dx dv + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) \, dx dy \right)$$

$$\begin{aligned}
& + \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(u_\varepsilon^\eta)) f^\eta|^2 dx dv + \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\eta dx dv \\
& + \frac{1}{2} \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) dx dy dv dw \\
& \leq \sigma d \|f_0^\eta\|_{L^1} + \sigma d \int_{\Omega} (\phi \star \rho^\eta) \rho^\eta dx.
\end{aligned}$$

Proof. From the existence of the fixed point of \mathcal{T} and Proposition 5.1, it is obvious that

$$\begin{aligned}
& \frac{d}{dt} \left(\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V^R + \sigma \log f^\eta \right) f^\eta dx dv + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \right) \\
& + \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(u_\varepsilon^\eta)) f^\eta|^2 dx dv + \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\eta dx dv \\
& + \frac{1}{2} \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) dx dy dv dw \\
& = \int_{\Omega \times \mathbb{R}^d} (\chi_\zeta(u_\varepsilon^\eta) - v) \cdot \chi_\zeta(u_\varepsilon^\eta) f^\eta dx dv \\
& + \sigma d \|f_0^\eta\|_{L^1} + \sigma d \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) f^\eta(y, w) f^\eta(x, v) dx dy dv dw.
\end{aligned}$$

Here, we note that

$$\begin{aligned}
\int_{\mathbb{R}^d} (\chi_\zeta(u_\varepsilon^\eta) - v) \cdot \chi_\zeta(u_\varepsilon^\eta) f^\eta dv & = \rho^\eta (\chi_\zeta(u_\varepsilon^\eta) - u^\eta) \cdot \chi_\zeta(u_\varepsilon^\eta) \\
& = \rho^\eta \left(\frac{\rho^\eta u^\eta}{\rho^\eta + \varepsilon} - u^\eta \right) \cdot \frac{\rho^\eta u^\eta}{\rho^\eta + \varepsilon} \mathbf{1}_{\{|u_\varepsilon^\eta| \leq \zeta\}} \\
& = -\varepsilon \left| \frac{\rho^\eta u^\eta}{\rho^\eta + \varepsilon} \right| \mathbf{1}_{\{|u_\varepsilon^\eta| \leq \zeta\}} \leq 0,
\end{aligned}$$

which implies our desired estimate. \square

5.3. Proof of Theorem 5.1

Now, we provide the existence of a weak solution to (1.1) based on the energy inequality, compactness argument and velocity averaging lemma.

5.3.1. Convergences $R \rightarrow \infty$ and $\zeta \rightarrow \infty$

First, we set $R = \zeta$, and we will tend R to infinity.

• (Step A: Uniform bound estimates) As an initial step, we derive an upper-bound estimate which is uniform in R and ζ from Corollary 5.2. For technical reason, we also estimate

$$\frac{d}{dt} \left(\int_{\Omega \times \mathbb{R}^d} \frac{|x|^2}{2} f^\eta dx dv \right) = \int_{\Omega \times \mathbb{R}^d} x \cdot v f^\eta dx dv \leq \int_{\Omega \times \mathbb{R}^d} \left(\frac{|x|^2}{2} + \frac{|v|^2}{2} \right) f^\eta dx dv,$$

and combine this with Corollary 5.2 to get

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + V^R + \sigma \log f^\eta \right) f^\eta dx dv \\ & + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \\ & + \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(u_\varepsilon^\eta)) f^\eta|^2 dx dv ds \\ & + \frac{1}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) dx dy dv dw ds \\ & \leq \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + V^R + \sigma \log f_0^\eta \right) f_0^\eta dx dv \\ & + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) dx dy + \sigma dt \|f_0^\eta\|_{L^1} \\ & + \sigma d \int_0^t \int_{\Omega} (\phi \star \rho^\eta) \rho^\eta dx ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{|x|^2}{2} f^\eta dx dv ds. \end{aligned} \quad (5.4)$$

Here, we recall the following inequality from the classical result¹⁵:

$$\begin{aligned} 2 \int_{\Omega \times \mathbb{R}^d} f^\varepsilon \log_- f^\varepsilon dx dv & \leq \int_{\Omega \times \mathbb{R}^d} f^\varepsilon \left(\frac{|x|^2}{2} + \frac{|v|^2}{2} \right) dx dv \\ & + \frac{1}{e} \int_{\Omega \times \mathbb{R}^d} e^{-\frac{|v|^2}{4} - \frac{|x|^2}{4}} dx dv, \end{aligned}$$

where $\log_- g(x) := \max\{0, -\log g(x)\}$. We apply the aforementioned inequality to (5.4) and obtain

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + V^R + \sigma |\log f^\eta| \right) f^\eta dx dv \\ & + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \\ & + \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(u_\varepsilon^\eta)) f^\eta|^2 dx dv ds \\ & + \frac{1}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) dx dy dv dw ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + V^R + \sigma |\log f_0^\eta| \right) f_0^\eta dx dv \\
&\quad + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) dx dy + \sigma dt \|f_0^\eta\|_{L^1} \\
&\quad + \sigma d \|\phi\|_{L^\infty} t \|f_0^\eta\|_{L^1}^2 + \int_0^t \int_{\Omega \times \mathbb{R}^d} (|v|^2 + |x|^2) f^\eta dx dv ds + C,
\end{aligned}$$

where $C = C(T)$ is a positive constant independent of η . Thus, we use $\|f_0^\eta\|_{L^1} \leq \|f_0\|_{L^1}$ and $W^\varepsilon(x) \leq W(x)$, and apply Grönwall's lemma to yield, for $t \in [0, T]$,

$$\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} \right) f^\eta dx dv \leq C,$$

where $C = C(T)$ is a positive constant independent of η . Since f^η satisfies

$$\begin{aligned}
&\|f^\eta(\cdot, \cdot, t)\|_{L^p}^p + \frac{4\sigma(p-1)}{p} \int_0^t e^{d(p-1)(2+\|\phi\|_{L^\infty}\|f_0^\eta\|_{L^1})(t-s)} \|\nabla_v (f^\eta)^{p/2}(\cdot, \cdot, s)\|_{L^2}^2 ds \\
&\leq \|f_0^\eta\|_{L^p}^p e^{d(p-1)(2+\|\phi\|_{L^\infty}\|f_0^\eta\|_{L^1})t} \\
&\leq \|f_0\|_{L^p}^p e^{d(p-1)(2+\|\phi\|_{L^\infty}\|f_0^\eta\|_{L^1})t}
\end{aligned}$$

for every $p \in [1, \infty]$, these uniform bounds yield the following estimates:

$$\|f^\eta\|_{L^\infty(0,T;L^p(\Omega \times \mathbb{R}^d))} + \|\rho^\eta\|_{L^\infty(0,T;L^{q_1}(\Omega))} + \|\rho^\eta u^\eta\|_{L^\infty(0,T;L^{q_2}(\Omega))} \leq C(T),$$

where $p \in [1, \infty]$, $q_1 \in [1, (d+2)/d)$, $q_2 \in [1, (d+2)/(d+1))$ and $C = C(T)$ is a positive constant independent of η . This uniform estimate implies the following weak convergence as $R \rightarrow \infty$ up to a subsequence:

$$\begin{aligned}
f^\eta &\rightharpoonup f^\varepsilon && \text{in } L^\infty(0, T; L^p(\Omega \times \mathbb{R}^d)), \quad p \in [1, \infty], \\
\rho^\eta &\rightharpoonup \rho^\varepsilon && \text{in } L^\infty(0, T; L^p(\Omega)), \quad p \in [1, (d+2)/d), \\
\rho^\eta u^\eta &\rightharpoonup \rho^\varepsilon u^\varepsilon && \text{in } L^\infty(0, T; L^p(\Omega)), \quad p \in [1, (d+2)/(d+1)).
\end{aligned}$$

Moreover, once we choose $p \in (1, (d+2)/(d+1))$ and write G^η as

$$G^\eta := \sigma \nabla_v f^\eta + (2v + \nabla V^R + \nabla W^\varepsilon \star \rho^\eta - F[f^\eta] - \chi_\zeta(u_\varepsilon^\eta)) f^\eta,$$

then we can see that $G^\eta \in L_{\text{loc}}^p(\Omega \times \mathbb{R}^d \times (0, T))$. Indeed, if we consider a bounded region $\mathcal{D} \subset \Omega \times \mathbb{R}^d$, the boundedness of $\nabla V^R f^\eta$ in $L^p(\mathcal{D})$ follows since

$$\begin{aligned}
|\nabla V^R f^\eta| &= \left| (\nabla V)(x) M\left(\frac{x}{R}\right) f^\eta + \frac{1}{R} V(x) (\nabla M)\left(\frac{x}{R}\right) f^\eta \right| \\
&\leq |\nabla V(x)| f^\eta + \frac{\|\nabla M\|_{L^\infty}}{R} V(x) f^\eta,
\end{aligned}$$

and we will consider the regime $R \rightarrow \infty$. The boundedness of the others naturally follows. Thus, we let G^η as above, set $r = 2$ and apply them to Lemma 5.4 to

obtain, for $p \in (1, (d+2)/(d+1))$,

$$\begin{aligned}\rho^\eta &\rightarrow \rho^\varepsilon && \text{in } L^p(\Omega \times (0, T)) \text{ and a.e.,} \\ \rho^\eta u^\eta &\rightarrow \rho^\varepsilon u^\varepsilon && \text{in } L^p(\Omega \times (0, T))\end{aligned}$$

as $R \rightarrow \infty$, up to a subsequence.

• (Step B: Existence of weak solutions and entropy inequality) Now, it remains to show that the limit f^ε satisfies the following equation in distributional sense:

$$\begin{aligned}\partial_t f^\varepsilon + v \cdot \nabla f^\varepsilon - \nabla_v \cdot ((v + (\nabla V + \nabla W^\varepsilon \star \rho^\varepsilon))f^\varepsilon) + \nabla_v \cdot (F[f^\varepsilon]f^\varepsilon) \\ = \nabla_v \cdot ((v - u_\varepsilon^\varepsilon)f^\varepsilon + \sigma \nabla_v f^\varepsilon),\end{aligned}\quad (5.5)$$

and also the entropy inequality:

$$\begin{aligned}&\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f^\varepsilon \right) f^\varepsilon dx dv + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\varepsilon(x) \rho^\varepsilon(y) dx dy \\ &+ \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\sigma \nabla_v f^\varepsilon - (v - u_\varepsilon^\varepsilon)f^\varepsilon|^2 dx dv ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\varepsilon dx dv ds \\ &+ \frac{1}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\varepsilon(y, w) f^\varepsilon(x, v) dx dy dv dw ds \\ &\leq \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f_0^\varepsilon \right) f_0^\varepsilon dx dv \\ &+ \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho_0^\varepsilon(x) \rho_0^\varepsilon(y) dx dy \\ &+ \sigma dt \|f_0\|_{L^1} + \sigma d \int_0^t \int_{\Omega} (\phi \star \rho^\varepsilon) \rho^\varepsilon dx ds.\end{aligned}\quad (5.6)$$

For f^ε to be a weak solution to (5.5), it suffices to show the following convergence in distribution sense since the others are obvious:

$$\begin{cases} \text{(i)} & \nabla W^\varepsilon \star \rho^\eta \rightarrow \nabla W^\varepsilon \star \rho^\varepsilon, \\ \text{(ii)} & F[f^\eta]f^\eta \rightarrow F[f^\varepsilon]f^\varepsilon, \\ \text{(iii)} & \chi_R(u_\varepsilon^\eta)f^\eta \rightarrow u_\varepsilon^\varepsilon f^\varepsilon.\end{cases}\quad (5.7)$$

◇ (Step B-1: Convergence of (5.7)(i)) We choose $\Psi \in \mathcal{C}_c^\infty(\Omega \times \mathbb{R}^d \times [0, T])$ and write

$$\rho_\Psi^\eta := \int_{\mathbb{R}^d} f^\eta(v) \Psi(v) dv.$$

Then, we have

$$\begin{aligned} & \int_0^t \int_{\Omega \times \mathbb{R}^d} [(\nabla W^\varepsilon \star \rho^\eta) f^\eta - (\nabla W^\varepsilon \star \rho^\varepsilon) f^\varepsilon] \Psi \, dx dv ds \\ &= \int_0^t \int_{\Omega} \nabla W^\varepsilon \star (\rho^\eta - \rho^\varepsilon) \rho_\Psi^\eta \, dx ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} (\nabla W^\varepsilon \star \rho^\varepsilon) (f^\eta - f^\varepsilon) \Psi \, dx dv ds \\ &=: K_1^1 + K_1^2. \end{aligned}$$

For K_1^1 , since f^η is uniformly bounded in $L^\infty(\Omega \times \mathbb{R}^d \times (0, T))$ and Ψ is compactly supported, it is obvious that

$$\rho_\Psi^\eta \in L^p(0, T; L^q(\Omega)) \quad \text{for any } p, q \in [1, \infty], \quad \text{uniformly in } \eta.$$

Note that $\rho_{|\Psi|}^\eta := \int_{\mathbb{R}^d} f^\eta(v) |\Psi|(v) \, dv$ also satisfies the above estimate. Thus, we have

$$\begin{aligned} K_1^1 &= \int_0^t \int_{\Omega} (\nabla W^\varepsilon \mathbb{1}_{\{|\cdot| \leq 1\}}) \star (\rho^\eta - \rho^\varepsilon) \rho_\Psi^\eta \, dx ds \\ &\quad + \int_0^t \int_{\Omega} (\nabla W^\varepsilon \mathbb{1}_{\{|\cdot| > 1\}}) \star (\rho^\eta - \rho^\varepsilon) \rho_\Psi^\eta \, dx ds \\ &\leq \| |\nabla W(\cdot)| \mathbb{1}_{\{|\cdot| \leq 1\}} \|_{L^1(\Omega)} \|\rho^\eta - \rho^\varepsilon\|_{L^p(\Omega \times (0, T))} \|\rho_\Psi^\eta\|_{L^{p'}(\Omega \times (0, T))} \\ &\quad + \| |\nabla W(\cdot)| \mathbb{1}_{\{|\cdot| > 1\}} \cdot \mathbb{1}_{\text{supp}_x \Psi} \|_{L^{p'}(\Omega)} \|\rho^\eta - \rho^\varepsilon\|_{L^p(\Omega \times (0, T))} \|\rho_\Psi^\eta\|_{L^{p'}(0, T; L^1(\Omega))}, \end{aligned}$$

where $p \in (1, (d+2)/(d+1))$, p' is the Hölder conjugate of p (thus $p' > d+2$), $\text{supp}_x \Psi$ denotes the support of Ψ in x which is compact

$$\text{supp}_x \Psi := \overline{\{x \in \Omega \mid \Psi(x, v) \neq 0 \text{ for some } v \in \mathbb{R}^d\}},$$

and we used Young's convolution inequality

$$\int_{\Omega} (f \star g) h \, dx \leq \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

Thus, thanks to the strong convergence of ρ^η , we have $K_1^1 \rightarrow 0$ as $R \rightarrow \infty$.

For K_1^2 , it naturally follows from the compact support of Ψ that

$$(\nabla W^\varepsilon \star \rho^\varepsilon) \Psi \in L^1(0, T; L^p(\Omega)) \quad \text{for some } p \in (1, \infty).$$

Thus, due to the weak convergence of f^η , we get $K_1^2 \rightarrow 0$ as $R \rightarrow \infty$.

◇ (Step B-2: Convergence of (5.7) (ii)) We note that

$$\begin{aligned} F[f^\eta](x, v) &= \int_{\Omega \times \mathbb{R}^d} \phi(x - y) (w - v) f^\eta(y, w) \, dy dw \\ &= \int_{\Omega} \phi(x - y) ((\rho^\eta u^\eta)(y) - v \rho^\eta(y)) \, dy. \end{aligned}$$

Thus, for $\Psi \in \mathcal{C}_c^\infty([0, T] \times \Omega \times \mathbb{R}^d)$, we get

$$\begin{aligned} & \int_0^t \int_{\Omega \times \mathbb{R}^d} (F[f^\eta]f^\eta - F[f^\varepsilon]f^\varepsilon)\Psi \, dx dv ds \\ &= \int_0^t \int_{\Omega^2 \times \mathbb{R}^d} \phi(x-y)[((\rho^\eta u^\eta)(y) - v\rho^\eta(y)) - ((\rho^\varepsilon u^\varepsilon)(y) - v\rho^\varepsilon(y))] \\ & \quad \times (f^\eta \Psi)(x, v) \, dx dy dv ds \\ & \quad + \int_0^t \int_{\Omega^2 \times \mathbb{R}^d} \phi(x-y)((\rho^\varepsilon u^\varepsilon)(y) - v\rho^\varepsilon(y))(f^\eta - f^\varepsilon)(x, v)\Psi(x, v) \, dx dy dv \\ &=: K_2^1 + K_2^2. \end{aligned}$$

For K_2^1 , we use Young's convolution inequality, the uniform boundedness of ρ_Ψ^η and the compact support of Ψ to obtain

$$\begin{aligned} K_2^1 &= \int_0^t \int_{\Omega} \phi \star (\rho^\eta u^\eta - \rho^\varepsilon u^\varepsilon) \rho_\Psi^\eta \, dx ds \\ & \quad + \int_0^t \int_{\Omega \times \mathbb{R}^d} \phi \star (\rho^\varepsilon - \rho^\eta)(v f^\eta)(x, v) \cdot \Psi(x, v) \, dx dv ds \\ &\leq \|\phi\|_{L^\infty} \|\rho^\eta u^\eta - \rho^\varepsilon u^\varepsilon\|_{L^p(\Omega \times (0, T))} \|\rho_\Psi^\eta\|_{L^{p'}(\Omega \times (0, T))} \\ & \quad + \text{supp}_v \Psi \|\|\phi\|_{L^\infty} \|\rho^\eta - \rho^\varepsilon\|_{L^p(\Omega \times (0, T))} \|\rho_{|\Psi|}^\eta\|_{L^{p'}(\Omega \times (0, T))}, \end{aligned}$$

where $p \in (1, (d+2)/(d+1))$ and hence, $K_2^1 \rightarrow 0$ as $R \rightarrow \infty$.

For K_2^2 , we use the compact support of Ψ to get

$$(\phi \star (\rho^\varepsilon u^\varepsilon) - v\phi \star \rho^\varepsilon)\Psi \in L^1(0, T; L^p(\Omega \times \mathbb{R}^d))$$

for some $p \in (1, \infty)$, and we combine this with the weak convergence of f^η to yield $K_2^2 \rightarrow 0$ as $R \rightarrow \infty$.

◇ (Step B-3: Convergence of (5.7) (iii)) Again, for $\Psi \in \mathcal{C}_c^\infty(\Omega \times \mathbb{R}^d \times [0, T])$, we estimate

$$\begin{aligned} & \int_0^t \int_{\Omega \times \mathbb{R}^d} (\chi_R(u_\varepsilon^\eta) f^\eta - u_\varepsilon^\varepsilon f^\varepsilon) \Psi \, dx dv ds \\ &= \int_0^t \int_{\Omega \times \mathbb{R}^d} (u_\varepsilon^\eta f^\eta - u_\varepsilon^\varepsilon f^\varepsilon) \Psi \, dx dv ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} u_\varepsilon^\eta f^\eta \mathbf{1}_{\{|u_\varepsilon^\eta| > R\}} \Psi \, dx dv ds \\ &= \int_0^t \int_{\Omega} (u_\varepsilon^\eta - u_\varepsilon^\varepsilon) \rho_\Psi^\eta \, dx ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} u_\varepsilon^\varepsilon (f^\eta - f^\varepsilon) \Psi \, dx dv ds \\ & \quad + \int_0^t \int_{\Omega \times \mathbb{R}^d} u_\varepsilon^\eta f^\eta \mathbf{1}_{\{|u_\varepsilon^\eta| > R\}} \Psi \, dx dv ds \\ &=: K_3^1 + K_3^2 + K_3^3. \end{aligned}$$

For K_3^1 , we find

$$K_3^1 = \int_0^t \int_{\Omega} \left[\left(\frac{1}{\rho^\eta + \varepsilon} - \frac{1}{\rho^\varepsilon + \varepsilon} \right) (\rho^\varepsilon u^\varepsilon) + \frac{1}{\rho^\eta + \varepsilon} (\rho^\eta u^\eta - \rho^\varepsilon u^\varepsilon) \right] \rho_\Psi^\eta dx ds.$$

Here, since $\rho^\eta \rightarrow \rho^\varepsilon$ a.e. and $\rho_\Psi^\eta \in L^\infty((0, T) \times \Omega)$, we have

$$\left(\frac{1}{\rho^\eta + \varepsilon} - \frac{1}{\rho^\varepsilon + \varepsilon} \right) (\rho^\varepsilon u^\varepsilon) \rho_\Psi^\eta \rightarrow 0, \quad \text{a.e.}$$

as $R \rightarrow \infty$, and since

$$\left| \left(\frac{1}{\rho^\eta + \varepsilon} - \frac{1}{\rho^\varepsilon + \varepsilon} \right) (\rho^\varepsilon u^\varepsilon) \rho_\Psi^\eta \right| \leq \frac{2 \|\rho_\Psi^\eta\|_{L^\infty}}{\varepsilon} |\rho^\varepsilon u^\varepsilon| \leq C(\varepsilon) |\rho^\varepsilon u^\varepsilon|,$$

where $C = C(\varepsilon)$ is a positive constant independent of η , we use the dominated convergence theorem to get

$$\int_0^t \int_{\Omega} \left(\frac{1}{\rho^\eta + \varepsilon} - \frac{1}{\rho^\varepsilon + \varepsilon} \right) (\rho^\varepsilon u^\varepsilon) \rho_\Psi^\eta dx ds \rightarrow 0$$

as $R \rightarrow \infty$. Moreover, we estimate

$$\int_0^t \int_{\Omega} \frac{1}{\rho^\eta + \varepsilon} (\rho^\eta u^\eta - \rho^\varepsilon u^\varepsilon) \rho_\Psi^\eta dx ds \leq \frac{1}{\varepsilon} \|\rho^\eta u^\eta - \rho^\varepsilon u^\varepsilon\|_{L^p} \|\rho_\Psi^\eta\|_{L^{p'}},$$

where $p \in (1, (d+2)/(d+1))$, and hence we can get $K_3^1 \rightarrow 0$ as $R \rightarrow \infty$. For the estimate of K_3^2 , it is obvious that $u_\varepsilon^\varepsilon \Psi \in L^1(0, T; L^p(\Omega \times \mathbb{R}^d))$ for some $p \in (1, \infty)$. Thus, the weak convergence implies $K_3^2 \rightarrow 0$ as $R \rightarrow \infty$. Finally, we use

$$|u^\eta| \leq \left(\frac{\int_{\mathbb{R}^d} |v|^2 f^\eta dv}{\int_{\mathbb{R}^d} f^\eta dv} \right)^{1/2}$$

to get

$$K_3^3 \leq \frac{1}{R} \int_0^t \int_{\Omega \times \mathbb{R}^d} |u_\varepsilon^\eta|^2 f^\eta \Psi dx dv ds \leq \frac{\|\Psi\|_{L^\infty}}{R} \int_0^t \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\eta dx dv ds \rightarrow 0$$

as $R \rightarrow \infty$. Hence, we can find out that f^ε becomes a weak solution to (5.5).

◇ (Step B-4: Entropy inequality) For (5.6), we first take the liminf on the left-hand side of Corollary 5.2 and use convexity of the entropy and $\|f_0^\eta\|_{L^1}$ to get

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f^\varepsilon \right) f^\varepsilon dx dv + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho^\varepsilon(x) \rho^\varepsilon(y) dx dy \\ & + \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\sigma \nabla_v f^\varepsilon - (v - u_\varepsilon^\varepsilon) f^\varepsilon|^2 dx dv ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} |v|^2 f^\varepsilon dx dv ds \\ & + \frac{1}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\varepsilon(y, w) f^\varepsilon(x, v) dx dy dv dw ds \\ & \leq \liminf_{R \rightarrow 0} \left(\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f_0^\eta \right) f_0^\eta dx dv \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) \, dx dy \\
 & + \sigma dt \|f_0\|_{L^1} + \sigma d \liminf_{R \rightarrow 0} \int_0^t \int_{\Omega} (\phi \star \rho^\eta) \rho^\eta \, dx ds.
 \end{aligned}$$

Here, we use the reverse Fatou's lemma and the pointwise convergences $\rho_0^\eta \rightarrow \rho_0^\varepsilon = \rho_0$ and $f_0^\eta \rightarrow f_0^\varepsilon = f_0$ to get

$$\begin{aligned}
 & \liminf_{R \rightarrow 0} \left(\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f_0^\eta \right) f_0^\eta \, dx dv \right. \\
 & \quad \left. + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) \, dx dy \right) \\
 & \leq \limsup_{R \rightarrow 0} \left(\int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f_0^\eta \right) f_0^\eta \, dx dv \right. \\
 & \quad \left. + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) \, dx dy \right) \\
 & \leq \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f_0 \right) f_0 \, dx dv \\
 & \quad + \frac{1}{2} \int_{\Omega \times \Omega} W^\varepsilon(x-y) \rho_0(x) \rho_0(y) \, dx dy,
 \end{aligned}$$

and we claim that the following convergence holds:

$$\lim_{R \rightarrow 0} \int_0^t \int_{\Omega} (\phi \star \rho^\eta) \rho^\eta \, dx ds = \int_0^t \int_{\Omega} (\phi \star \rho^\varepsilon) \rho^\varepsilon \, dx ds.$$

For this, we present a theorem similar to Vitali convergence theorem whose proof is presented in Appendix B. Note that when $\Omega = \mathbb{T}^d$, the condition (ii) is unnecessary.

Theorem 5.2. *A sequence $\{h_n\}$ in $L^1(\Omega)$ converges to $h \in L^1(\Omega)$ in $L^1(\Omega)$ if the following three conditions hold:*

- (i) h_n converges to h almost everywhere.
- (ii) For every $\varepsilon > 0$, there exists $L > 0$ such that

$$\sup_{n \in \mathbb{N}} \int_{|x| > L} |h_n| \, dx < \varepsilon.$$

- (iii) For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $m(E) < \delta$,

$$\sup_{n \in \mathbb{N}} \int_E |h_n| \, dx < \varepsilon.$$

In our case, since $f^\eta \rightharpoonup f^\varepsilon$ in $L^1(\Omega \times \mathbb{R}^d \times (0, T))$ and $\phi \in L^\infty(\Omega)$, we obtain

$$\begin{aligned} \int_0^t \int_\Omega \phi(x-y) \rho^\eta(y, s) dy ds &= \int_0^t \int_{\Omega \times \mathbb{R}^d} \phi(x-y) f^\eta(y, w, s) dy dw ds \\ &\rightarrow \int_0^t \int_{\Omega \times \mathbb{R}^d} \phi(x-y) f^\varepsilon(y, w, s) dy dw ds \\ &= \int_0^t \int_\Omega \phi(x-y) \rho^\varepsilon(y, s) dy ds \end{aligned}$$

for each $x \in \Omega$ and $t \in (0, T)$. This implies the convergence of $\phi \star \rho^\eta$ to $\phi \star \rho^\varepsilon$ almost everywhere in $\Omega \times (0, t)$. On the other hand, we also know that $\rho^\eta \rightarrow \rho^\varepsilon$ almost everywhere, and thus we have the convergence of $(\phi \star \rho^\eta) \rho^\eta$ to $(\phi \star \rho^\varepsilon) \rho^\varepsilon$ almost everywhere in $\Omega \times (0, t)$.

For the condition (ii) in Theorem 5.2, we let $L > 0$ and

$$\begin{aligned} \int_{|x|>L} (\phi \star \rho^\eta) \rho^\eta dx &\leq \frac{1}{L} \int_{|x|>L} |x| (\phi \star \rho^\eta) \rho^\eta dx \\ &\leq \frac{\|\phi \star \rho^\eta\|_{L^\infty}}{L} \int_\Omega |x| \rho^\eta dx \leq \frac{\|\phi\|_{L^\infty}}{L} \left(\int_\Omega |x|^2 \rho^\eta dx \right)^{1/2}, \end{aligned}$$

and we can deduce the condition (ii) from the above. For the third condition (uniform integrability condition), we choose a measurable set $E \subset \Omega$ with $m(E) < \delta$. Then, we have

$$\int_E (\phi \star \rho^\eta) \rho^\eta dx \leq \|\phi \star \rho^\eta\|_{L^\infty} \|\rho^\eta\|_{L^p} m(E)^{1/p'} \leq C m(E)^{1/p'},$$

where $p \in (1, (d+2)/d)$ and C is a constant independent of η , and this implies the uniform integrability condition. This concludes our desired result.

5.3.2. Convergence $\varepsilon \rightarrow 0$

Finally, it remains to prove the convergence as $\varepsilon \rightarrow 0$. We note that the weak convergence of f^η to f^ε implies the following uniform upper bound estimate:

$$\begin{aligned} \|f^\varepsilon\|_{L^\infty(0,T;L^p(\Omega \times \mathbb{R}^d))}^p &+ \frac{4\sigma(p-1)}{p} \int_0^T \|\nabla_v (f^\varepsilon)^{p/2}(\cdot, \cdot, s)\|_{L^2}^2 ds \\ &\leq \|f_0\|_{L^p}^p e^{d(p-1)(2+\|\phi\|_{L^\infty})t} \end{aligned} \quad (5.8)$$

for $p \in [1, \infty]$. Thus, we combine (5.8) with the entropy inequality (5.6) to get the following weak convergence up to a subsequence as before:

$$\begin{aligned} f^\varepsilon &\rightharpoonup f && \text{in } L^\infty(0, T; L^p(\Omega \times \mathbb{R}^d)), \quad p \in [1, \infty], \\ \rho^\varepsilon &\rightharpoonup \rho && \text{in } L^\infty(0, T; L^p(\Omega)), \quad p \in [1, (d+2)/d], \\ \rho^\varepsilon u^\varepsilon &\rightharpoonup \rho u && \text{in } L^\infty(0, T; L^p(\Omega)), \quad p \in [1, (d+2)/(d+1)]. \end{aligned}$$

Moreover, applying the velocity averaging lemma, Lemma 5.4, asserts the strong convergence up to a subsequence for $p \in (1, (d+2)/(d+1))$:

$$\begin{aligned}\rho^\varepsilon &\rightarrow \rho && \text{in } L^p(\Omega \times (0, T)) \text{ and a.e.,} \\ \rho^\varepsilon u^\varepsilon &\rightarrow \rho u && \text{in } L^p(\Omega \times (0, T))\end{aligned}\quad (5.9)$$

as $\varepsilon \rightarrow 0$. Now, to show that f is a weak solution to (1.1), it suffices to show the following convergence in distribution sense, since others are obvious or can be obtained in the same way from the previous argument:

$$\begin{cases} \text{(i)} & (\nabla W^\varepsilon \star \rho^\varepsilon) f^\varepsilon \rightarrow (\nabla W \star \rho) f, \\ \text{(ii)} & u_\varepsilon^\varepsilon f^\varepsilon \rightarrow u f. \end{cases} \quad (5.10)$$

◇ (Convergence (5.10) (i)) We choose again $\Psi \in \mathcal{C}_c^\infty(\Omega \times \mathbb{R}^d \times [0, T])$ and get

$$\begin{aligned}& \int_0^t \int_{\Omega \times \mathbb{R}^d} [(\nabla W^\varepsilon \star \rho^\varepsilon) f^\varepsilon - (\nabla W \star \rho) f] \Psi \, dx dv ds \\&= \int_0^t \int_{\Omega} (\nabla(W^\varepsilon - W) \star \rho^\varepsilon) \rho_\Psi^\varepsilon \, dx ds \\&+ \int_0^t \int_{\Omega} \nabla W \star (\rho^\varepsilon - \rho) \rho_\Psi^\varepsilon \, dx ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} (\nabla W \star \rho) (f^\varepsilon - f) \Psi \, dx dv ds \\&=: K_4^1 + K_4^2 + K_4^3.\end{aligned}$$

Since the estimates for K_4^2 and K_4^3 are similar to those for K_1^1 and K_1^2 , respectively, we only need to show $K_4^1 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Still, thanks to the uniform-in- ε estimate for f^ε in $L^\infty(\Omega \times \mathbb{R}^d \times (0, T))$ and the compact support of Ψ , we find

$$\rho_\Psi^\varepsilon \in L^p(0, T; L^q(\Omega))$$

for any $p, q \in [1, \infty]$ uniformly in ε . This gives

$$\begin{aligned}K_4^1 &= \int_0^t \int_{\Omega} (\nabla(W^\varepsilon - W)(\cdot)) (\mathbf{1}_{\{|\cdot| \leq 1\}} + \mathbf{1}_{\{|\cdot| > 1\}}) \star \rho^\varepsilon \rho_\Psi^\varepsilon \, dx ds \\&\leq \|\nabla(W^\varepsilon - W)(\cdot) \mathbf{1}_{\{|\cdot| \leq 1\}}\|_{L^1(\Omega)} \|\rho^\varepsilon\|_{L^p(\Omega \times (0, T))} \|\rho_\Psi^\varepsilon\|_{L^{p'}(\Omega \times (0, T))} \\&+ \|\nabla(W^\varepsilon - W)(\cdot) \mathbf{1}_{\{|\cdot| > 1\}}\|_{L^{p'}(\Omega)} \|\rho^\varepsilon\|_{L^p(\Omega \times (0, T))} \|\rho_\Psi^\varepsilon\|_{L^{p'}(0, T; L^1(\Omega))},\end{aligned}$$

where $p \in (1, (d+2)/(d+1))$ and we used the uniform bound for ρ^ε , ρ_Ψ^ε and the dominated convergence theorem for the convergence of the interaction potential term.

◇ (Convergence of (5.10)(ii)) Although the proof is almost the same as that of Lemma 4.4 in Ref. 34, we present here for readers' convenience. First, consider a test function Ψ of the form $\Psi(x, v, t) := \psi(x, t) \varphi(v)$. Thus, $\varphi \in C_c^\infty(\mathbb{R}^d)$ and we similarly write $\rho_\varphi^\varepsilon := \int_{\mathbb{R}^d} f^\varepsilon \varphi(v) \, dv$. Then, we write

$$\int_0^t \int_{\Omega \times \mathbb{R}^d} f^\varepsilon u_\varepsilon^\varepsilon \Psi \, dx dv ds = \int_0^t \int_{\Omega} u_\varepsilon^\varepsilon \rho_\varphi^\varepsilon \psi \, dx ds.$$

Here, for $p \in (1, (d+2)/(d+1))$, we get

$$\begin{aligned} \|u_\varepsilon^\varepsilon \rho_\varphi^\varepsilon\|_{L^p} &\leq \|\varphi\|_{L^\infty} \|\rho^\varepsilon\|_{L^{p/(2-p)}}^{1/2} \|\sqrt{\rho^\varepsilon} u_\varepsilon^\varepsilon\|_{L^2} \\ &\leq \|\varphi\|_{L^\infty} \|\rho^\varepsilon\|_{L^{p/(2-p)}}^{1/2} \left(\int_{\Omega \times \mathbb{R}^d} |v|^2 f^\varepsilon dx dv \right)^{1/2}, \end{aligned}$$

which gives the uniform bound for $u_\varepsilon^\varepsilon \rho_\varphi^\varepsilon$ in $L^p(\Omega)$, since $p/(2-p) \in (1, (d+2)/d)$, and we already have the uniform bound for ρ^ε in L^q with $q \in (1, (d+2)/d)$ and $|v|^2 f^\varepsilon$ in L^1 . Thus, we can find m such that, up to a subsequence,

$$u_\varepsilon^\varepsilon \rho_\varphi^\varepsilon \rightharpoonup m \quad \text{in } L^\infty(0, T; L^p(\Omega)), \quad \forall p \in (1, (d+2)/(d+1)).$$

It remains to show that $m = u\rho_\varphi$. For this, we let $h_1, h_2 > 0$ and define

$$\mathcal{A}_{h_1}^{h_2} := \{(x, t) \in (B(0, h_1) \cap \Omega) \times (0, T) : \rho(x, t) > h_2\}.$$

For each h_1 and h_2 , we combine the pointwise convergence of ρ^ε to ρ with Egorov's theorem to deduce that for every $\delta > 0$, we may choose $A_\delta \subset \mathcal{A}_{h_1}^{h_2}$ satisfying

$$|A_{h_1}^{h_2} \setminus A_\delta| < \delta \quad \text{and} \quad \rho^\varepsilon \rightarrow \rho \quad \text{as } \varepsilon \rightarrow 0 \text{ uniformly on } A_\delta.$$

Then, for a sufficiently small ε , we have $\rho^\varepsilon > h_2/2$ on A_δ and thus we get

$$u_\varepsilon^\varepsilon \rho_\varphi^\varepsilon = \frac{\rho^\varepsilon u_\varepsilon^\varepsilon}{\rho^\varepsilon + \varepsilon} \rho_\varphi^\varepsilon \rightarrow m = u\rho_\varphi, \quad \text{on } A_\delta.$$

Since the choices of h_1 , h_2 and δ were arbitrary, we now obtain

$$m = u\rho_\varphi \quad \text{on } \{\rho > 0\},$$

and therefore, we have

$$\begin{aligned} \int_0^t \int_{\Omega \times \mathbb{R}^d} f^\varepsilon u_\varepsilon^\varepsilon \Psi dx dv ds &= \int_0^t \int_{\Omega} u_\varepsilon^\varepsilon \rho_\varphi^\varepsilon \psi dx ds \\ &\rightarrow \int_0^t \int_{\Omega} u \rho_\varphi \psi dx ds = \int_0^t \int_{\Omega \times \mathbb{R}^d} u f \Psi dx dv ds, \end{aligned}$$

for all test functions Ψ of the form $\Psi(x, v, t) = \psi(x, t)\varphi(v)$. Thus, we conclude that f is a weak solution to (1.1). It remains to show that the weak solution obtained above satisfies the entropy inequality (2.2). Note that the regularized solutions f^ε satisfies the entropy inequality (5.6) and the strong compactness of macroscopic fields $\rho^\varepsilon, \rho^\varepsilon u^\varepsilon$ are obtained in (5.9) via the velocity averaging lemma. Thus, we can use a similar argument as in the previous step together with Fatou's lemma to have the following entropy inequality:

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f \right) f dx dv + \frac{1}{2} \int_{\Omega} (W \star \rho) \rho dx \\ + \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f} |\sigma \nabla_v f - (v - u)f|^2 dx dv ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} |v|^2 f dx dv ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f(y, w) f(x, v) dx dy dv dw ds \\
 & \leq \int_{\Omega \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f_0 \right) f_0 dx dv + \frac{1}{2} \int_{\Omega} (W \star \rho_0) \rho_0 dx \\
 & \quad + \sigma dt \|f_0\|_{L^1} + \sigma d \int_0^t \int_{\Omega} (\phi \star \rho) \rho dx ds.
 \end{aligned}$$

5.4. Global-in-time existence of weak solutions for dimensions $d = 1, 2$

In this section, we investigate the global existence of weak solutions to (1.1) for the case $d = 1, 2$. In these cases, the positivity of the interaction energy is not guaranteed, and as a result, it makes some problems in obtaining the uniform upper bound estimates for solutions. Once we can find a way to get the uniform bound, then other parts of the proof for the existence of weak solutions can follow from almost the same analysis as in the previous sections. Since the results for $\Omega = \mathbb{T}^d$ is analogous to the case $\Omega = \mathbb{R}^d$, we only consider the case $\Omega = \mathbb{R}^d$. Let us introduce the regularized fundamental solutions of the Laplace's equation in $d = 1, 2$:

$$W^\varepsilon(x) := \begin{cases} -\frac{1}{2} \sqrt{\varepsilon + |x|^2} & \text{if } d = 1, \\ -\frac{1}{4\pi} \log(\varepsilon + |x|^2) & \text{if } d = 2. \end{cases}$$

Then, since we also have ∇W^ε is bounded and smooth, global-in-time existence of weak solutions to Eq. (5.2) is clear. Moreover, we can also deduce that the entropy inequality (5.4) and the following upper bound estimate hold:

$$\begin{aligned}
 & \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + \sigma |\log f^\eta| \right) f^\eta dx dv \\
 & \quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \\
 & \quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(u_\varepsilon^\eta)) f^\eta|^2 dx dv ds \\
 & \quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) dx dy dv dw ds \\
 & \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + \sigma |\log f_0^\eta| \right) f_0^\eta dx dv \\
 & \quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) dx dy + \sigma dt \|f_0^\eta\|_{L^1} + \sigma d \|\phi\|_{L^\infty} t \|f_0^\eta\|_{L^1}^2 \\
 & \quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (|v|^2 + |x|^2) f^\eta dx dv ds + C,
 \end{aligned}$$

where $C = C(T)$ is a positive constant independent of η . When $d = 1$, one uses Young's inequality to get

$$\begin{aligned} & -\frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) \, dx dy \\ &= \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} \sqrt{\varepsilon + |x-y|^2} \rho^\eta(x) \rho^\eta(y) \, dx dy \\ &\leq \frac{1}{4} \int_{\mathbb{R} \times \mathbb{R}} \left(1 + \frac{1}{4}(\varepsilon + |x-y|^2)\right) \rho^\eta(x) \rho^\eta(y) \, dx dy \\ &\leq \frac{1}{16}(4 + \varepsilon) + \frac{1}{16} \int_{\mathbb{R} \times \mathbb{R}} (|x|^2 + |y|^2) \rho^\eta(x) \rho^\eta(y) \, dx dy \\ &\leq C + \frac{1}{8} \int_{\mathbb{R}} |x|^2 \rho^\eta \, dx = C + \frac{1}{8} \int_{\mathbb{R} \times \mathbb{R}} |x|^2 f^\eta \, dx dv, \end{aligned}$$

and this gives

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + \sigma |\log f^\eta| \right) f^\eta \, dx dv \\ &+ \int_0^t \int_{\mathbb{R} \times \mathbb{R}} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(u_\varepsilon^\eta)) f^\eta|^2 \, dx dv ds \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) \, dx dy dv dw ds \\ &\leq \int_{\mathbb{R} \times \mathbb{R}} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} \sigma |\log f_0^\eta| \right) f_0^\eta \, dx dv \\ &+ \sigma dt \|f_0^\eta\|_{L^1} + \sigma d \|\phi\|_{L^\infty} t \|f_0^\eta\|_{L^1}^2 \\ &+ \int_0^t \int_{\mathbb{R} \times \mathbb{R}} (|v|^2 + |x|^2) f^\eta \, dx dv ds + C, \end{aligned}$$

where $C = C(T)$ is a constant independent of η , which implies the desired uniform upper bound estimate and this can be also used when $\varepsilon \rightarrow 0$. For $d = 2$, we note that the following inequality holds:

$$\begin{aligned} \varepsilon + |x-y|^2 &\leq (1+\varepsilon)(1+|x-y|^2) \\ &\leq (1+\varepsilon)(1+2|x|^2+2|y|^2) \\ &\leq 2(1+\varepsilon)(1+|x|^2)(1+|y|^2), \end{aligned}$$

which subsequently gives

$$\log(\varepsilon + |x-y|^2) \leq \log 2(1+\varepsilon) + \log(1+|x|^2) + \log(1+|y|^2).$$

We use the above inequality and $\log(1+x) \leq x$ for $x \geq 0$ to get

$$\begin{aligned}
 & \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \\
 &= -\frac{1}{8\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(\varepsilon + |x-y|^2) \rho^\eta(x) \rho^\eta(y) dx dy \\
 &\geq -\frac{1}{8\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} [\log 2(1+\varepsilon) + \log(1+|x|^2) + \log(1+|y|^2)] \rho^\eta(x) \rho^\eta(y) dx dy \\
 &\geq -\frac{1}{8\pi} \log 2(1+\varepsilon) - \frac{1}{8\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\log(1+|x|^2) + \log(1+|y|^2)) \rho^\eta(x) \rho^\eta(y) dx dy \\
 &\geq -\frac{1}{8\pi} \log 2(1+\varepsilon) - \frac{1}{4\pi} \int_{\mathbb{R}^2} \rho^\eta \log(1+|x|^2) dx \\
 &\geq -\frac{1}{8\pi} \log 2(1+\varepsilon) - \frac{1}{4\pi} \int_{\mathbb{R}^2} |x|^2 \rho^\eta dx.
 \end{aligned}$$

Moreover, the integral of $|\log(\varepsilon + |x|)|^p$ on $|x| \leq 1$ can be bounded uniformly in ε for every $p \in [1, \infty)$ and thus we have

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) dx dy \right| \\
 &\leq C (\|W^\varepsilon(\cdot) \mathbf{1}_{\{|\cdot| \leq 1\}}\|_{L^q} \|\rho_0^\eta\|_{L^p}^2 + \|W^\varepsilon(\cdot) \mathbf{1}_{\{|\cdot| > 1\}}\|_{L^\infty} \|\rho_0^\eta\|_{L^1}^2) \\
 &\leq C,
 \end{aligned} \tag{5.11}$$

where C is a constant independent of η and $1/q + 2/p = 2$ with $p \in (1, (d+2)/d)$. Hence, for $d = 2$, we can obtain

$$\begin{aligned}
 & \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + \sigma |\log f^\eta| \right) f^\eta dx dv \\
 &+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{f^\eta} |\sigma \nabla_v f^\eta - (v - \chi_\zeta(u_\varepsilon^\eta)) f^\eta|^2 dx dv ds \\
 &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^4 \times \mathbb{R}^4} \phi(x-y) |w-v|^2 f^\eta(y, w) f^\eta(x, v) dx dy dv dw ds \\
 &\leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2} + \sigma |\log f_0^\eta| \right) f_0^\eta dx dv \\
 &+ \sigma dt \|f_0^\eta\|_{L^1} + \sigma d \|\phi\|_{L^\infty} t \|f_0^\eta\|_{L^1}^2 \\
 &+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} (|v|^2 + |x|^2) f^\eta dx dv ds + C,
 \end{aligned}$$

which gives the desired uniform upper bound estimate.

For the free energy inequality, we first note that the following inequality still holds for $d = 1, 2$:

$$\begin{aligned}
& \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f^\varepsilon \right) f^\varepsilon dx dv \\
& + \frac{1}{2} \liminf_{R \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} W^\varepsilon(x-y) \rho^\eta(x) \rho^\eta(y) dx dy \\
& + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\sigma \nabla_v f^\varepsilon - (v - u_\varepsilon) f^\varepsilon|^2 dx dv ds \\
& + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f^\varepsilon dx dv ds \\
& + \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \phi(x-y) |w-v|^2 f^\varepsilon(y, w) f^\varepsilon(x, v) dx dy dv dw ds \\
& \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + V + \sigma \log f_0^\varepsilon \right) f_0^\varepsilon dx dv \\
& + \frac{1}{2} \limsup_{R \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} W^\varepsilon(x-y) \rho_0^\eta(x) \rho_0^\eta(y) dx dy + \sigma dt \|f_0\|_{L^1} \\
& + \sigma d \int_0^t \int_{\mathbb{R}^d} (\phi \star \rho^\varepsilon) \rho^\varepsilon dx ds.
\end{aligned}$$

When $d \geq 3$, the interaction potential W is positive, thus we used Fatou's Lemma to obtain the desired inequality. Although it is no longer possible to use Fatou's Lemma when $d = 1, 2$, we use Theorem 5.2 instead to show that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} (W^\varepsilon \star \rho^\eta) \rho^\eta dx = \int_{\mathbb{R}^d} (W^\varepsilon \star \rho^\varepsilon) \rho^\varepsilon dx$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} (W^\varepsilon \star \rho^\varepsilon) \rho^\varepsilon dx = \int_{\mathbb{R}^d} (W \star \rho) \rho dx$$

for each $t \in [0, T]$. Since the proof for the $\varepsilon \rightarrow 0$ case is similar, we only consider the case $R \rightarrow \infty$.

First, we show that $W^\varepsilon \star \rho^\eta$ converges to $W^\varepsilon \star \rho^\varepsilon$ pointwise. Indeed, the pointwise convergence $\rho^\eta \rightarrow \rho^\varepsilon$ implies $W^\varepsilon(x - \cdot) \rho^\eta(\cdot)$ converges to $W^\varepsilon(x - \cdot) \rho^\varepsilon(\cdot)$ for each x . If $d = 1$, for each x .

$$\begin{aligned}
\int_{|y| \geq L} W^\varepsilon(x-y) \rho^\eta(y) dy & \leq \int_{|y| \geq L} \sqrt{\varepsilon + |x-y|^2} \rho^\eta(y) dy \\
& \leq (|x| + \sqrt{\varepsilon}) \int_{|y| \geq L} \rho^\eta(y) dy + \int_{|y| \geq L} |y| \rho^\eta(y) dy \\
& \leq \left(\frac{|x|}{L^2} + \frac{1}{L} \right) \int_{\mathbb{R}^d} |y|^2 \rho^\eta dy \rightarrow 0, \quad \text{as } L \rightarrow \infty.
\end{aligned}$$

When $d = 2$, one uses $|\log x| \leq \max\{x, x^{-1}\}$, $x > 0$, and chooses L sufficiently large so that $L \gg |x|$ to get

$$\begin{aligned} & \int_{|y| \geq L} W^\varepsilon(x-y) \rho^\eta(y) dy \\ & \leq \frac{1}{2\pi} \int_{|y| \geq L} \log \sqrt{\varepsilon + |x-y|^2} \rho^\eta(y) dy \\ & \leq \frac{1}{2\pi} \int_{|y| \geq L} \max\{\sqrt{\varepsilon + |x-y|^2}, (\varepsilon + |x-y|^2)^{-1/2}\} \rho^\eta(y) dy \\ & = \frac{1}{2\pi} \int_{|y| \geq L} \sqrt{\varepsilon + |x-y|^2} \rho^\eta(y) dy, \end{aligned}$$

which also gives the desired estimate Theorem 5.2(ii). For the last condition (iii) in Theorem 5.2, we choose a measurable set E . Then for $d = 1$, we use Hölder's inequality and the uniform bounds for ρ^η in L^p with $p \in (1, 2)$ to get

$$\begin{aligned} & \int_E W^\varepsilon(x-y) \rho^\eta(y) dy \\ & \leq (|x| + \sqrt{\varepsilon}) \int_E \rho^\eta(y) dy + \int_E |y| \rho^\eta(y) dy \\ & \leq (|x| + \sqrt{\varepsilon}) \|\rho^\eta\|_{L^p} m(E)^{1/p'} + \left(\int_E |y|^2 \rho^\eta(y) dy \right)^{1/2} \left(\int_E \rho^\eta(y) dy \right)^{1/2} \\ & \leq C(|x| + \sqrt{\varepsilon}) m(E)^{1/(2p')} \rightarrow 0 \quad \text{as } m(E) \rightarrow 0, \end{aligned}$$

where $p' = p/(p-1)$ is the Hölder conjugate of p . When $d = 2$, we obtain

$$\begin{aligned} & \int_E W^\varepsilon(x-y) \rho^\eta(y) dy \\ & \leq \frac{1}{\pi} \int_E \max\{(\varepsilon + |x-y|^2)^{1/4}, (\varepsilon + |x-y|^2)^{-1/4}\} \rho^\eta(y) dy \\ & \leq \int_{E \cap \{|x-y| \geq 1-\varepsilon\}} \sqrt{\varepsilon + |x-y|^2} \rho^\eta(y) dy \\ & \quad + \int_{E \cap \{|x-y| < 1-\varepsilon\}} (\varepsilon + |x-y|^2)^{-1/4} \rho^\eta(y) dy \\ & \leq C(|x| + \sqrt{\varepsilon}) m(E)^{1/(2p')} + \|(\varepsilon + |x|^2)^{-1/4} \mathbf{1}_{\{|x| \leq 1\}}\|_{L^{7/2}} \|\rho^\eta\|_{L^{7/4}} m(E)^{1/7}, \end{aligned}$$

which guarantees the condition (iii). Thus, we have the pointwise convergence of $W^\varepsilon \star \rho^\eta$ to $W^\varepsilon \star \rho^\varepsilon$ and hence $(W^\varepsilon \star \rho^\eta) \rho^\eta$ converges to $(W^\varepsilon \star \rho^\varepsilon) \rho^\varepsilon$ almost everywhere. To prove the desired convergence, we note that

$$|W^\varepsilon \star \rho^\eta| \leq C(1 + |x|),$$

where C is a constant independent of η . More precisely, we find

$$|W^\varepsilon \star \rho^\eta| \leq (1 + \varepsilon) \int_{\mathbb{R}^d} (1 + |x| + |y|) \rho^\eta(y) dy \leq C(1 + |x|)$$

for $d = 1$ and

$$\begin{aligned} & \int_{\mathbb{R}^d} W^\varepsilon(x - y) \rho^\eta(y) dy \\ & \leq \frac{1}{\pi} \int_{\mathbb{R}^d} \max\{(\varepsilon + |x - y|^2)^{1/4}, (\varepsilon + |x - y|^2)^{-1/4}\} \rho^\eta(y) dy \\ & \leq \int_{\{|x-y| \geq 1-\varepsilon\}} \sqrt{\varepsilon + |x - y|^2} \rho^\eta(y) dy \\ & \quad + \int_{\{|x-y| < 1-\varepsilon\}} (\varepsilon + |x - y|^2)^{-1/4} \rho^\eta(y) dy \\ & \leq C(1 + |x|) + \int_{\{|x-y| < 1\}} |x - y|^{-1/2} \rho^\eta(y) dy \\ & \leq C(1 + |x|) + \| |\cdot|^{-1/2} \mathbf{1}_{\{|\cdot| < 1\}} \|_{L^{7/2}} \|\rho^\eta\|_{L^{7/5}} \leq C(1 + |x|) \end{aligned}$$

for $d = 2$. Thus, we use the above estimate to validate the conditions in Theorem 5.2.

For the condition (ii) in Theorem 5.2, we choose $L > 0$ to get

$$\begin{aligned} \int_{|x| \geq L} (W^\varepsilon \star \rho^\eta) \rho^\eta dx & \leq C \int_{|x| \geq L} (1 + |x|) \rho^\eta dx \\ & \leq C \left(\frac{1}{L} + \frac{1}{L^2} \right) \int_{|x| \geq L} |x|^2 \rho^\eta dx \rightarrow 0 \quad \text{as } L \rightarrow \infty. \end{aligned}$$

For the condition (iii) in Theorem 5.2, we have

$$\begin{aligned} \int_E (W \star \rho^\eta) \rho^\eta dx & \leq C \int_E (1 + |x|) \rho^\eta dx \\ & \leq C \|\rho^\eta\|_{L^p m(E)}^{1/p'} + C \left(\int_E |x|^2 \rho^\eta dx \right)^{1/2} \left(\int_E \rho^\eta dx \right)^{1/2} \\ & \leq C (\|\rho^\eta\|_{L^p m(E)}^{1/p'})^{1/2} (1 + (\|\rho^\eta\|_{L^p m(E)}^{1/p'})^{1/2}). \end{aligned}$$

Hence, we can obtain the entropy inequality similarly as in the case $d \geq 3$.

6. Local-in-Time Existence of Strong Solutions to the Systems (1.3) and (1.5)

In this section, we study the local-in-time existence and uniqueness of strong solutions to (1.3) and (1.5) in the periodic domain $\Omega = \mathbb{T}^d$. Since the proof for the system (1.5) is similar to that for (1.3), we only provide the details of the proof for the system (1.3), see Sec. 6.4 for the brief idea of the proof for the pressureless case.

For the case with smooth interaction potential W , we briefly mention the existence result in Remark 6.1 below.

To be more specific, we are mainly interested in the local-in-time solvability of the following isothermal Euler–Poisson system with nonlocal forces:

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho u) &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho &= -\rho u - \rho(\nabla V + \nabla W \star \rho) \\ &\quad - \rho \int_{\mathbb{T}^d} \phi(x-y)(u(x) - u(y))\rho(y) dy.\end{aligned}\quad (6.1)$$

Here we set $\gamma = \lambda = \alpha = 1$ for the sake of simplicity. We then reformulate the above system by setting $g := \log \rho$ and rewrite it as follows:

$$\begin{aligned}\partial_t g + \nabla g \cdot u + \nabla \cdot u &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\ \partial_t u + (u \cdot \nabla)u + \nabla g &= -u - (\nabla V + \nabla W \star e^g) - (\phi \star e^g)u + \phi \star (e^g u),\end{aligned}\quad (6.2)$$

subject to initial data:

$$(g(x, 0), u(x, 0)) = (g_0(x), u_0(x)), \quad x \in \mathbb{T}^d. \quad (6.3)$$

We now state the result on the well-posedness of the system (6.2)–(6.3).

Theorem 6.1. *Let $s > d/2 + 1$. Suppose that the confinement potential and communication weight satisfy $(\nabla V, \phi) \in H^s(\mathbb{T}^d) \times \mathcal{W}^{s, \infty}(\mathbb{T}^d)$, and the initial data $(g_0, u_0) \in H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ with $e^{g_0} > 0$. Then for any positive constants $\epsilon_0 < M_0$, there exists a positive constant T^* such that if $\|g_0\|_{H^s} + \|u_0\|_{H^s} < \epsilon_0$, then the system (6.2)–(6.3) admits a unique solution $(g, u) \in \mathcal{C}([0, T^*]; H^s(\mathbb{T}^d)) \times \mathcal{C}([0, T^*]; H^s(\mathbb{T}^d))$ satisfying*

$$\sup_{0 \leq t \leq T^*} (\|g(\cdot, t)\|_{H^s} + \|u(\cdot, t)\|_{H^s}) \leq M_0.$$

Note that the solution (g, u) obtained above has \mathcal{C}^1 -regularity and in particular g is bounded, we can easily show that (ρ, u) with $\rho := e^g$ is a strong solution to (6.1). More precisely, we can have the equivalence relation between the classical solutions to the systems (6.1) and (6.2).

Proposition 6.1. *For any fixed $T > 0$, $(\rho, u) \in \mathcal{C}^1(\mathbb{T}^d \times [0, T]) \times \mathcal{C}^1(\mathbb{T}^d \times [0, T])$ solves the system (6.1) with $\rho > 0$ if and only if $(g, u) \in \mathcal{C}^1(\mathbb{T}^d \times [0, T]) \times \mathcal{C}^1(\mathbb{T}^d \times [0, T])$ solves the system (6.2) with $e^g > 0$.*

The well-posedness theory for Eq. (6.1) has not been developed so far to the best of our knowledge. On the other hand, if the velocity alignment forces, the last term on the right-hand side of the momentum equations in (6.1) and the confinement forces are ignored, the system (6.1) reduces to the damped isothermal Euler–Poisson system. For that system, the global existence of weak/strong solutions is studied in Refs. 23, 24, 29, 30 and 44. We refer to Ref. 16 for a general survey on the Euler equations and related conservation laws. Critical thresholds phenomena leading to

a finite-time blow-up or a global regularity of strong solutions for the Euler–Poisson system are also investigated in Refs. 12 and 27.

6.1. Solvability for the linearized system

In this section, we linearize the system (6.2) and discuss the local-in-time estimates of solutions to that system. More precisely, for a given

$$(\tilde{g}, \tilde{u}) \in \mathcal{C}([0, T]; H^s(\mathbb{T}^d)) \times \mathcal{C}([0, T]; H^s(\mathbb{T}^d)),$$

we consider the associated linear system:

$$\begin{aligned} \partial_t g + \tilde{u} \cdot \nabla g + \nabla \cdot u &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\ \partial_t u + (\tilde{u} \cdot \nabla)u + \nabla g &= -u - (\nabla V + \nabla W \star e^{\tilde{g}}) - (\phi \star e^{\tilde{g}})\tilde{u} + \phi \star (e^{\tilde{g}}\tilde{u}), \end{aligned} \quad (6.4)$$

with the initial data $(g_0, u_0) \in H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$.

Lemma 6.1. *Let $T > 0$ and $s > d/2 + 1$. For any positive constants $N < M$, if*

$$\|g_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 < N \quad (6.5)$$

and

$$\sup_{0 \leq t \leq T} (\|\tilde{g}(\cdot, t)\|_{H^s}^2 + \|\tilde{u}(\cdot, t)\|_{H^s}^2) < M,$$

then the Cauchy problem (6.4) has a unique classical solution $(g, u) \in \mathcal{C}([0, T]; H^s(\mathbb{T}^d)) \times \mathcal{C}([0, T]; H^s(\mathbb{T}^d))$ satisfying

$$\sup_{0 \leq t \leq T^*} (\|g(\cdot, t)\|_{H^s}^2 + \|u(\cdot, t)\|_{H^s}^2) < M$$

for some $T^* \leq T$.

Proof. We first easily obtain the existence and uniqueness of solutions to (6.4) by a standard linear theory of PDEs. Thus, it suffices to provide bound estimates for g and u . A straightforward computation gives

$$\frac{1}{2} \frac{d}{dt} \|g\|_{L^2}^2 \leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|g\|_{L^2}^2 - \int_{\mathbb{T}^d} g \nabla \cdot u \, dx$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 &\leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|u\|_{L^2}^2 - \int_{\mathbb{T}^d} \nabla g \cdot u \, dx + \|\nabla V\|_{L^2} \|u\|_{L^2} \\ &\quad + \|\nabla W \star e^{\tilde{g}}\|_{L^2} \|u\|_{L^2} + C\|\phi\|_{L^\infty} (1 + e^{\|\tilde{g}\|_{L^\infty}}) \|u\|_{L^2} \|\tilde{u}\|_{L^2} \\ &\leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|u\|_{L^2}^2 - \int_{\mathbb{T}^d} \nabla g \cdot u \, dx + \|\nabla V\|_{L^2} \|u\|_{L^2} + \|\nabla W\|_{L^1} e^{\|\tilde{g}\|_{L^\infty}} \|u\|_{L^2} \\ &\quad + C\|\phi\|_{L^\infty} (1 + e^{\|\tilde{g}\|_{L^\infty}}) \|u\|_{L^2} \|\tilde{u}\|_{L^2}. \end{aligned}$$

Then, we use Sobolev's inequality and Young's inequality to get

$$\frac{d}{dt} (\|g\|_{L^2}^2 + \|u\|_{L^2}^2) \leq Ce^{CM}(1+M) (\|g\|_{L^2}^2 + \|u\|_{L^2}^2) + Ce^{CM}(1+M), \quad (6.6)$$

where $C > 0$ only depends on $s, d, \nabla V$ and ϕ .

For higher-order estimates, we first recall the Moser-type inequality:

$$\|\nabla^k(fg) - f\nabla^k g\|_{L^2} \leq C (\|\nabla f\|_{L^\infty} \|\nabla^{k-1} g\|_{L^2} + \|\nabla^k f\|_{L^2} \|g\|_{L^\infty}).$$

For $1 \leq k \leq s$, we estimate $\nabla^k g$ as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla^k g\|_{L^2}^2 \\ &= - \int_{\mathbb{T}^d} \nabla(\nabla^k g) \cdot \tilde{u} \nabla^k g \, dx - \int_{\mathbb{T}^d} (\nabla^k(\nabla g \cdot \tilde{u}) - \nabla(\nabla^k g) \cdot \tilde{u}) \nabla^k g \, dx \\ & \quad - \int_{\mathbb{T}^d} (\nabla \cdot (\nabla^k u)) \nabla^k g \, dx \\ &\leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|\nabla^k g\|_{L^2}^2 + C \|\nabla^k g\|_{L^2} (\|\nabla \tilde{u}\|_{L^\infty} \|\nabla(\nabla^{k-1} g)\|_{L^2} \\ & \quad + \|\nabla g\|_{L^\infty} \|\nabla^k \tilde{u}\|_{L^2}) - \int_{\mathbb{T}^d} (\nabla \cdot (\nabla^k u)) \nabla^k g \, dx \\ &\leq CM \|\nabla^k g\|_{L^2} + CM \|\nabla^k g\|_{L^2} \|g\|_{H^s} - \int_{\mathbb{T}^d} (\nabla \cdot (\nabla^k u)) \nabla^k g \, dx, \end{aligned}$$

where $C > 0$ depends only on d and s . Similarly, we estimate $\nabla^k u$ as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla^k u\|_{L^2}^2 = - \int_{\mathbb{T}^d} (\tilde{u} \cdot \nabla(\nabla^k u)) \cdot \nabla^k u \, dx \\ & \quad - \int_{\mathbb{T}^d} (\nabla^k(\tilde{u} \cdot \nabla u) - \tilde{u} \cdot \nabla(\nabla^k u)) \cdot \nabla^k u \, dx \\ & \quad - \|\nabla^k u\|_{L^2}^2 - \int_{\mathbb{T}^d} (\nabla(\nabla^k V) + \nabla^k(\nabla W \star e^{\tilde{g}})) \cdot \nabla^k u \, dx \\ & \quad - \int_{\mathbb{T}^d} \nabla(\nabla^k g) \cdot \nabla^k u \, dx \\ & \quad + \int_{\mathbb{T}^d \times \mathbb{T}^d} \nabla_x^k \phi(x-y) \tilde{u}(y) e^{\tilde{g}(y)} \nabla^k u(x) \, dy \, dx \\ & \quad - \int_{\mathbb{T}^d \times \mathbb{T}^d} \nabla_x^k (\phi(x-y) \tilde{u}(x)) e^{\tilde{g}(y)} \nabla^k u(x) \, dy \, dx \\ &\leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|\nabla^k u\|_{L^2} + C \|\nabla^k u\|_{L^2} \left(\|\nabla \tilde{u}\|_{L^\infty} \|\nabla(\nabla^{k-1} u)\|_{L^2} \right. \\ & \quad \left. + \|\nabla u\|_{L^\infty} \|\nabla^k \tilde{u}\|_{L^2} \right) \\ & \quad + \|\nabla V\|_{H^k} \|\nabla^k u\|_{L^2} + \|\nabla W \star (\nabla^k e^{\tilde{g}})\|_{L^2} \|\nabla^k u\|_{L^2} \end{aligned}$$

$$\begin{aligned}
& - \int_{\mathbb{T}^d} \nabla(\nabla^k g) \cdot \nabla^k u \, dx + C \|\phi\|_{\mathcal{W}^{k,\infty}} \|\tilde{u}\|_{H^k} e^{\|\tilde{g}\|_{L^\infty}} \|\nabla^k u\|_{L^2} \\
& \leq CM \|\nabla^k u\|_{L^2}^2 + CM \|\nabla^k u\|_{L^2} \|u\|_{H^s} + Ce^{CM} (1+M) \|\nabla^k u\|_{L^2} \\
& \quad + C \|\nabla^k(e^{\tilde{g}})\|_{L^2} \|\nabla^k u\|_{L^2} - \int_{\mathbb{T}^d} \nabla(\nabla^k g) \cdot \nabla^k u \, dx,
\end{aligned}$$

where $C > 0$ only depends on $d, s, \nabla V, \|\nabla W\|_{L^1}$ and ϕ . To estimate the Poisson interaction term, we let $a_k := \|\nabla^k(e^{\tilde{g}})\|_{L^2}$. As shown previously, we have $a_0 \leq e^{\|\tilde{g}\|_{L^\infty}} \leq Ce^{CM}$. Then, we use the Moser-type inequality and Sobolev inequality to obtain

$$\begin{aligned}
a_k &= \|\nabla^{k-1}(e^{\tilde{g}} \nabla \tilde{g})\|_{L^2} \\
&\leq \|e^{\tilde{g}} \nabla^k \tilde{g}\|_{L^2} + \|\nabla^{k-1}(e^{\tilde{g}} \nabla \tilde{g}) - e^{\tilde{g}} \nabla^k \tilde{g}\|_{L^2} \\
&\leq Me^{CM} + C(\|\nabla e^{\tilde{g}}\|_{L^\infty} \|\nabla^{k-1} \tilde{g}\|_{L^2} + \|\nabla^{k-1}(e^{\tilde{g}})\|_{L^2} \|\nabla \tilde{g}\|_{L^\infty}) \\
&\leq CM a_{k-1} + CM e^{CM} (1+M),
\end{aligned}$$

where $C > 0$ only depends on d and k , and inductively, we get

$$a_k \leq CM^k a_0 + CM^{k-1} e^{CM} (1+M) \leq Ce^{CM}.$$

Here $C > 0$ only depends on d and k . Now, we combine the estimates for $\nabla^k g$ and $\nabla^k u$ to yield

$$\begin{aligned}
& \frac{d}{dt} (\|\nabla^k g\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2) \\
& \leq CM (\|\nabla^k g\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2) + CM \left(\|\nabla^k g\|_{L^2} \|g\|_{H^s} + \|\nabla^k u\|_{L^2} \|u\|_{H^s} \right) \\
& \quad + Ce^{CM} (1+M) \|\nabla^k u\|_{L^2}. \tag{6.7}
\end{aligned}$$

We sum the relation (6.7) over $1 \leq k \leq s$ and combine this with (6.6) to get

$$\frac{d}{dt} (\|g\|_{H^s}^2 + \|u\|_{H^s}^2) \leq Ce^{CM} (1+M) (\|g\|_{H^s}^2 + \|u\|_{H^s}^2) + Ce^{CM} (1+M).$$

We write $h(M) := Ce^{CM} (1+M)$ and use Grönwall's lemma to obtain

$$\begin{aligned}
\|g\|_{H^s}^2 + \|u\|_{H^s}^2 &\leq (\|g_0\|_{H^s}^2 + \|u_0\|_{H^s}^2) e^{h(M)t} + e^{h(M)t} (1 - e^{-h(M)t}) \\
&\leq N e^{h(M)t} + e^{h(M)t} (1 - e^{-h(M)t}) \\
&= N + (N+1) (e^{h(M)t} - 1).
\end{aligned}$$

Note that $N < M$ and $e^{h(M)t} - 1$ can be arbitrary small if $t \ll 1$. This allows us to find $T^* > 0$ such that

$$N + (N+1) (e^{h(M)T^*} - 1) < M.$$

This concludes the desired result. \square

6.2. Construction of approximate solutions

Now, we construct a sequence that approximates a (unique) solution to (1.3). More precisely, we consider a sequence (g^n, u^n) which is a solution to the following system:

$$\begin{aligned} \partial_t g^{n+1} + \nabla g^{n+1} \cdot u^n + \nabla \cdot u^{n+1} &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\ \partial_t u^{n+1} + (u^n \cdot \nabla) u^{n+1} + \nabla g^{n+1} &= -u^{n+1} - (\nabla V + \nabla W \star e^{g^n}) - \phi \star (e^{g^n}) u^n + \phi \star (e^{g^n} u^n), \end{aligned} \quad (6.8)$$

with the initial step and initial data defined by

$$(g^0(x, t), u^0(x, t)) = (g_0(x), u_0(x)) \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+$$

and

$$(g^n(x, 0), u^n(x, 0)) = (g_0(x), u_0(x)) \quad \forall n \in \mathbb{N}, x \in \mathbb{T}^d,$$

respectively. We note that the approximation sequence (g^n, u^n) is well-defined due to Lemma 6.1. Moreover, by Lemma 6.1, we have the following uniform-in- n bound estimates for the approximation sequence.

Corollary 6.1. *Let $s > d/2 + 1$. For any $M > N$, there exists $T^* > 0$ such that if the initial data (g_0, u_0) satisfy (6.5), then for each $n \in \mathbb{N}$*

$$(g^n, u^n) \in \mathcal{C}([0, T^*]; H^s(\mathbb{T}^d)) \times \mathcal{C}([0, T^*]; H^s(\mathbb{T}^d))$$

and

$$\sup_{0 \leq t \leq T^*} (\|g^n(\cdot, t)\|_{H^s}^2 + \|u^n(\cdot, t)\|_{H^s}^2) < M, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Proof. For the proof, we use the inductive argument. Since the initial step ($n = 0$) is obvious, it suffices to consider the induction step. We recall from Lemma 6.1 that

$$Ne^{h(M)T^*} + e^{h(M)T^*} (1 - e^{-h(M)T^*}) < M$$

for some $T^* > 0$. Then, by the induction hypothesis, we get

$$\sup_{0 \leq t \leq T^*} (\|g^n(\cdot, t)\|_{H^s}^2 + \|u^n(\cdot, t)\|_{H^s}^2) < M.$$

This together with the same analysis in Lemma 6.1, we have

$$\begin{aligned} \|g^{n+1}\|_{H^s}^2 + \|u^{n+1}\|_{H^s}^2 &\leq (\|g_0\|_{H^s}^2 + \|u_0\|_{H^s}^2) e^{h(M)t} + e^{h(M)t} (1 - e^{-h(M)t}) \\ &\leq Ne^{h(M)t} + e^{h(M)t} (1 - e^{-h(M)t}) < M \end{aligned}$$

for $0 \leq t \leq T^*$. This completes the proof. \square

In the lemma below, we show that the approximation sequence (g^n, u^n) is a Cauchy sequence in $\mathcal{C}([0, T^*]; L^2(\mathbb{T}^d)) \times \mathcal{C}([0, T^*]; L^2(\mathbb{T}^d))$.

Lemma 6.2. *Let (g^n, u^n) be a sequence of the approximated solutions with the initial data (g_0, u_0) satisfying (6.5). Then we have*

$$\begin{aligned} & \| (g^{n+1} - g^n)(\cdot, t) \|_{L^2}^2 + \| (u^{n+1} - u^n)(\cdot, t) \|_{L^2}^2 \\ & \leq C \int_0^t (\| (g^n - g^{n-1})(\cdot, s) \|_{L^2}^2 + \| (u^n - u^{n-1})(\cdot, s) \|_{L^2}^2) ds \end{aligned}$$

for $0 \leq t \leq T^*$ and $n \in \mathbb{N}$, where $C > 0$ is independent of n .

Proof. First, it follows from (6.8) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \| g^{n+1} - g^n \|_{L^2}^2 \\ & = - \int_{\mathbb{T}^d} u^n \cdot \nabla (g^{n+1} - g^n) (g^{n+1} - g^n) dx \\ & \quad - \int_{\mathbb{T}^d} (u^n - u^{n-1}) \cdot \nabla g^n (g^{n+1} - g^n) dx \\ & \quad - \int_{\mathbb{T}^d} \nabla \cdot (u^{n+1} - u^n) (g^{n+1} - g^n) dx \\ & \leq \frac{\| \nabla \cdot u^n \|_{L^\infty}}{2} \| g^{n+1} - g^n \|_{L^2}^2 + \| \nabla g^n \|_{L^\infty} \| u^n - u^{n-1} \|_{L^2} \| g^{n+1} - g^n \|_{L^2} \\ & \quad - \int_{\mathbb{T}^d} \nabla \cdot (u^{n+1} - u^n) (g^{n+1} - g^n) dx \\ & \leq C (\| g^{n+1} - g^n \|_{L^2}^2 + \| u^n - u^{n-1} \|_{L^2}^2) - \int_{\mathbb{T}^d} \nabla \cdot (u^{n+1} - u^n) (g^{n+1} - g^n) dx. \end{aligned}$$

Next, we estimate

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \| u^{n+1} - u^n \|_{L^2}^2 \\ & = - \int_{\mathbb{T}^d} u^n \cdot \nabla (u^{n+1} - u^n) \cdot (u^{n+1} - u^n) dx \\ & \quad - \int_{\mathbb{T}^d} (u^n - u^{n-1}) \cdot \nabla u^n \cdot (u^{n+1} - u^n) dx \\ & \quad - \int_{\mathbb{T}^d} \nabla (g^{n+1} - g^n) \cdot (u^{n+1} - u^n) dx - \| u^{n+1} - u^n \|_{L^2}^2 \\ & \quad - \int_{\mathbb{T}^d} \nabla W \star (e^{g^n} - e^{g^{n-1}}) \cdot (u^{n+1} - u^n) dx \\ & \quad - \int_{\mathbb{T}^d \times \mathbb{T}^d} \phi(x - y) ((u^n - u^{n-1})(x) - (u^n - u^{n-1})(y)) e^{g^n(y)} \end{aligned}$$

$$\begin{aligned}
& \cdot (u^{n+1} - u^n)(x) dy dx \\
& - \int_{\mathbb{T}^d \times \mathbb{T}^d} \phi(x-y)(u^{n-1}(x) - u^{n-1}(y)) \left(e^{g^n(y)} - e^{g^{n-1}(y)} \right) \\
& \cdot (u^{n+1} - u^n)(x) dy dx \\
& \leq \frac{\|\nabla \cdot u^n\|_{L^\infty}}{2} \|u^{n+1} - u^n\|_{L^2}^2 + \|\nabla u^n\|_{L^\infty} \|u^{n+1} - u^n\|_{L^2} \|u^n - u^{n-1}\|_{L^2} \\
& - \int_{\mathbb{T}^d} \nabla(g^{n+1} - g^n) \cdot (u^{n+1} - u^n) dx \\
& + \|\nabla W\|_{L^1} \|e^{g^n} - e^{g^{n-1}}\|_{L^2} \|u^{n+1} - u^n\|_{L^2} \\
& + 2\|\phi\|_{L^\infty} e^{\|g^n\|_{L^\infty}} \|u^n - u^{n-1}\|_{L^2} \|u^{n+1} - u^n\|_{L^2} \\
& + 2\|\phi\|_{L^\infty} \|u^{n-1}\|_{L^2} \left\| e^{g^n} - e^{g^{n-1}} \right\|_{L^2} \|u^{n+1} - u^n\|_{L^2} \\
& \leq C (\|u^{n+1} - u^n\|_{L^2}^2 + \|u^n - u^{n-1}\|_{L^2}^2 + \|g^n - g^{n-1}\|_{L^2}^2) \\
& - \int_{\mathbb{T}^d} \nabla(g^{n+1} - g^n) \cdot (u^{n+1} - u^n) dx,
\end{aligned}$$

where we used the mean value theorem to get

$$\begin{aligned}
\|e^{g^n} - e^{g^{n-1}}\|_{L^2} & \leq \exp(\max\{\|g^n\|_{L^\infty}, \|g^{n-1}\|_{L^\infty}\}) \|g^n - g^{n-1}\|_{L^2} \\
& \leq C \|g^n - g^{n-1}\|_{L^2}.
\end{aligned}$$

Combining all of the above estimates yields

$$\begin{aligned}
& \frac{d}{dt} (\|(g^{n+1} - g^n)(\cdot, t)\|_{L^2}^2 + \|(u^{n+1} - u^n)(\cdot, t)\|_{L^2}^2) \\
& \leq C (\|(g^{n+1} - g^n)(\cdot, t)\|_{L^2}^2 + \|(u^{n+1} - u^n)(\cdot, t)\|_{L^2}^2 \\
& \quad + \|(g^n - g^{n-1})(\cdot, t)\|_{L^2}^2 + \|(u^n - u^{n-1})(\cdot, t)\|_{L^2}^2)
\end{aligned}$$

for $0 \leq t \leq T^*$, where $C > 0$ is independent of n . We finally apply Grönwall's lemma to conclude the desired result. \square

6.3. Proof of Theorem 6.1

Now, we prove the well-posedness of strong solutions to (1.3). First, Lemma 6.2 implies that

$$g^n \rightarrow g \quad \text{in } \mathcal{C}([0, T]; L^2(\mathbb{T}^d)) \quad \text{and} \quad u^n \rightarrow u \quad \text{in } \mathcal{C}([0, T]; L^2(\mathbb{T}^d))$$

as $n \rightarrow \infty$. Moreover, we can extend the convergence in $\mathcal{C}([0, T]; L^2(\mathbb{T}^d))$ to the one in $\mathcal{C}([0, T]; H^{s-1}(\mathbb{T}^d))$ by interpolating this with the uniform bound in $\mathcal{C}([0, T]; H^s(\mathbb{T}^d))$ from Corollary 6.1:

$$g^n \rightarrow g \quad \text{in } \mathcal{C}([0, T]; H^{s-1}(\mathbb{T}^d)) \quad \text{and} \quad u^n \rightarrow u \quad \text{in } \mathcal{C}([0, T]; H^{s-1}(\mathbb{T}^d)).$$

To obtain the H^s -regularity of (g, u) , we can use a standard argument from functional analysis. For detail, we refer to Ref. 12.

For the uniqueness, we consider two solutions (g, u) and (\hat{g}, \hat{u}) with the same initial data (g_0, u_0) . Then, the Cauchy estimate in Lemma 6.2 gives

$$\begin{aligned} & \| (g - \hat{g})(\cdot, t) \|_{L^2}^2 + \| (u - \hat{u})(\cdot, t) \|_{L^2}^2 \\ & \leq C \int_0^t (\| (g - \hat{g})(\cdot, s) \|_{L^2}^2 + \| (u - \hat{u})(\cdot, s) \|_{L^2}^2) ds \end{aligned}$$

for $t \leq T^*$. Applying Grönwall's lemma to the above concludes the uniqueness of solutions.

6.4. Pressureless Euler–Poisson system

For the pressureless case, by setting $g := \rho - \rho_c$, we can reformulate the system (1.5) as

$$\begin{aligned} \partial_t g + \nabla \cdot ((1 + g)u) &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\ \partial_t u + (u \cdot \nabla)u &= -u - (\nabla V + \nabla W \star (1 + g)) - (\phi \star (1 + g))u + \phi \star ((1 + g)u), \end{aligned} \quad (6.9)$$

where we simply let $\rho_c = 1$, since ρ_c is preserved in time. Then, similarly as before, we construct an approximation sequence to the reformulated system:

$$\begin{aligned} \partial_t g^{n+1} + \nabla \cdot ((1 + g^{n+1})u^n) &= 0, \\ \partial_t u^{n+1} + (u^n \cdot \nabla)u^{n+1} &= -u^{n+1} - (\nabla V + \nabla W \star (1 + g^n)) \\ &\quad - (\phi \star (1 + g^n))u^n + \phi \star ((1 + g^n)u^n). \end{aligned}$$

In this case, we use the following estimate for the Poisson interaction term:

$$\begin{aligned} \int_{\mathbb{T}^d} \nabla(\nabla W \star (1 + g^n)) : \nabla u^{n+1} dx &= \sum_{i,j=1}^d \int_{\mathbb{T}^d} \partial_{x_j} (\partial_{x_i} (W \star (1 + g^n))) \partial_{x_j} u_i^{n+1} dx \\ &= \sum_{i,j=1}^d \int_{\mathbb{T}^d} \partial_{x_j} \partial_{x_j} (W \star (1 + g^n)) \partial_{x_i} u_i^{n+1} dx \\ &= \int_{\mathbb{T}^d} \Delta W \star (1 + g^n) \nabla \cdot u^{n+1} dx \\ &= - \int_{\mathbb{T}^d} (1 + g^n) \nabla \cdot u^{n+1} dx. \end{aligned}$$

Thus, under suitable assumptions on the confinement potential ∇V and the communication weight ϕ , we can use the above estimate to get H^{s+1} -estimates for u , i.e. for any $M > N$, if

$$\|g_0\|_{H^s}^2 + \|u_0\|_{H^{s+1}}^2 < N,$$

then there exists $T^* > 0$ such that

$$\sup_{0 \leq t \leq T^*} (\|g^n(\cdot, t)\|_{H^s}^2 + \|u^n(\cdot, t)\|_{H^{s+1}}) < M, \quad \forall n \in \mathbb{N}.$$

Then, the similar argument as the above provides the local-in-time existence and uniqueness of strong solutions to the pressureless Euler system (6.9).

Theorem 6.2. *Let $s > d/2 + 1$. Suppose that the confinement potential and communication weight satisfy $(\nabla V, \phi) \in H^{s+1}(\mathbb{T}^d) \times \mathcal{W}^{s+1, \infty}(\mathbb{T}^d)$, and the initial data $(g_0, u_0) \in H^s(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d)$ with $g_0 + 1 > 0$. Then for any positive constants $\epsilon_0 < M_0$, there exists a positive constant T^* such that if $\|g_0\|_{H^s} + \|u_0\|_{H^{s+1}} < \epsilon_0$, then the system (6.9)–(6.3) admits a unique solution $(g, u) \in \mathcal{C}([0, T^*]; H^s(\mathbb{T}^d)) \times \mathcal{C}([0, T^*]; H^{s+1}(\mathbb{T}^d))$ satisfying*

$$\sup_{0 \leq t \leq T^*} (\|g(\cdot, t)\|_{H^s} + \|u(\cdot, t)\|_{H^{s+1}}) \leq M_0.$$

Remark 6.1. Our analysis can be naturally extended to the case when ∇W is sufficiently smooth. More precisely, if $\nabla W \in H^s(\mathbb{T}^d)$, then the same result can be obtained for the isothermal Euler system (1.3) and $\nabla W \in H^{s+1}(\mathbb{T}^d)$ for the pressureless Euler system (1.5). Furthermore, when $\nabla W \in H^s(\mathbb{T}^d)$, we can also extend the well-posedness result to (1.3) to the case when the domain is \mathbb{R}^d . For this, we refer to Refs. 12 and 20.

Appendix A. Proof of Corollary 2.2: Convergence Towards the Local Maxwellian

In this part, we provide the details on the proof of Corollary 2.2. For this, we recall the definition of relative entropy:

$$\begin{aligned} \mathcal{H}(f^\varepsilon | M_{\rho, u}) &:= f^\varepsilon \log f^\varepsilon - M_{\rho, u} \log M_{\rho, u} - (\log M_{\rho, u} + 1)(f^\varepsilon - M_{\rho, u}) \\ &= f^\varepsilon \log \left(\frac{f^\varepsilon}{M_{\rho, u}} \right) + M_{\rho, u} - f^\varepsilon \\ &\geq \frac{1}{2} \min \left\{ \frac{1}{f^\varepsilon}, \frac{1}{M_{\rho, u}} \right\} |f^\varepsilon - M_{\rho, u}|^2. \end{aligned}$$

Then, we use Cauchy–Schwartz inequality and

$$1 \leq (x + y) \min \left\{ \frac{1}{x}, \frac{1}{y} \right\} \quad \text{for } x, y > 0$$

to get

$$\begin{aligned} \left(\int_{\Omega \times \mathbb{R}^d} |f^\varepsilon - M_{\rho, u}| dx dv \right)^2 &\leq \left(\int_{\Omega \times \mathbb{R}^d} (M_{\rho, u} + f^\varepsilon) dx dv \right) \\ &\quad \times \left(\int_{\Omega \times \mathbb{R}^d} \min \left\{ \frac{1}{f^\varepsilon}, \frac{1}{M_{\rho, u}} \right\} |f^\varepsilon - M_{\rho, u}|^2 dx dv \right) \\ &\leq 4 \int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho, u}) dx dv. \end{aligned}$$

Thus, for the desired estimate, we investigate the relative entropy $\mathcal{H}(f^\varepsilon|M_{\rho,u})$. We first note that

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon|M_{\rho,u}) dx dv &= \int_{\Omega} \left(\int_{\mathbb{R}^d} \left(\log f^\varepsilon + \frac{|u-v|^2}{2} \right) f^\varepsilon dv \right) \\ &\quad - \rho^\varepsilon \left(\log \rho - \log(2\pi)^{d/2} \right) dx. \end{aligned}$$

We then estimate

$$\begin{aligned} \frac{d}{dt} \int_{\Omega \times \mathbb{R}^d} f^\varepsilon \log f^\varepsilon dx dv &= \int_{\Omega \times \mathbb{R}^d} \partial_t f^\varepsilon \log f^\varepsilon dx dv \\ &= d + d \int_{\Omega} (\phi \star \rho^\varepsilon) \rho^\varepsilon dx \\ &\quad - \frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^d} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) \frac{\nabla_v f^\varepsilon}{f^\varepsilon} dx dv \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \rho^\varepsilon \log \rho dx &= \int_{\Omega} \partial_t \rho^\varepsilon \log \rho dx + \int_{\Omega} \rho^\varepsilon \frac{\partial_t \rho}{\rho} dx \\ &= \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot \frac{\nabla \rho}{\rho} dx + \int_{\Omega} \rho u \cdot \nabla \left(\frac{\rho^\varepsilon}{\rho} \right) dx \\ &= \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \frac{\nabla \rho}{\rho} dx + \int_{\Omega} u \cdot \nabla \rho^\varepsilon dx. \end{aligned}$$

We also find

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}^d} \frac{|u-v|^2}{2} f^\varepsilon dx dv - \int_{\Omega \times \mathbb{R}^d} \frac{|u_0-v|^2}{2} f_0^\varepsilon dx dv \\ &= \int_0^t \frac{d}{ds} \left(\int_{\Omega \times \mathbb{R}^d} \frac{|u-v|^2}{2} f^\varepsilon dx dv \right) ds \\ &= \int_0^t \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot \partial_s u dx ds + \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{|u-v|^2}{2} \partial_s f^\varepsilon dx dv ds \\ &=: I + J, \end{aligned}$$

where I can be estimated as

$$\begin{aligned} I &= \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot (u \cdot \nabla u) dx ds + \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \frac{\nabla \rho}{\rho} dx ds \\ &\quad + \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot (\nabla V + u) dx ds + \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla W \star \rho dx ds \\ &\quad + \int_0^t \int_{\Omega \times \Omega} \phi(x-y) \rho(y) (u(x) - u(y)) \cdot [\rho^\varepsilon (u^\varepsilon - u)](x) dx dy ds \end{aligned}$$

$$\begin{aligned} &\leq \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \frac{\nabla \rho}{\rho} dx ds + C \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u| dx ds \\ &\quad + \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla W \star \rho dx ds. \end{aligned}$$

For J , we have

$$\begin{aligned} J &= - \int_0^t \int_{\Omega \times \mathbb{R}^d} \nabla_x \cdot (v f^\varepsilon) \frac{|u - v|^2}{2} dx dv ds \\ &\quad - \int_0^t \int_{\Omega \times \mathbb{R}^d} (v + \nabla V + \nabla W \star \rho^\varepsilon) \cdot (v - u) f^\varepsilon dx dv ds \\ &\quad + \int_0^t \int_{\Omega \times \mathbb{R}^d} F[f^\varepsilon] \cdot (v - u) f^\varepsilon dx dv ds \\ &\quad - \frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) \cdot (v - u) dx dv ds \\ &= \int_0^t \int_{\Omega \times \mathbb{R}^d} v f^\varepsilon \otimes (u - v) : \nabla u dx dv ds - \int_0^t \int_{\Omega \times \mathbb{R}^d} v \cdot (v - u) f^\varepsilon dx dv ds \\ &\quad - \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot (\nabla V + \nabla W \star \rho^\varepsilon) dx ds \\ &\quad + \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x - y) (w - v) f^\varepsilon(y, w) \cdot (v - u(x)) f^\varepsilon(x, v) dx dy dv dw ds \\ &\quad - \frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) \cdot (v - u) dx dv ds \\ &= - \int_0^t \int_{\Omega \times \mathbb{R}^d} (u - v) \otimes (u - v) f^\varepsilon : \nabla u dx dv ds \\ &\quad + \int_0^t \int_{\Omega} u \otimes (u^\varepsilon - u) \rho^\varepsilon : \nabla u dx ds \\ &\quad - \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds - \int_0^t \int_{\Omega} u^\varepsilon \cdot (u^\varepsilon - u) \rho^\varepsilon dx ds \\ &\quad - \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot (\nabla V + \nabla W \star \rho^\varepsilon) dx ds \\ &\quad - \frac{1}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x - y) |v - w|^2 f^\varepsilon(x, v) f^\varepsilon(y, w) dx dy dv dw ds \\ &\quad - \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x - y) (w - v) f^\varepsilon(y, w) \cdot u(x) f^\varepsilon(x, v) dx dy dv dw ds \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) \cdot (v - u) dx dv ds \\
& = - \int_0^t \int_{\Omega \times \mathbb{R}^d} (u - v) \otimes (u - v) f^\varepsilon : \nabla u dx dv ds \\
& \quad + \int_0^t \int_{\Omega} u \otimes (u^\varepsilon - u) \rho^\varepsilon : \nabla u dx ds \\
& \quad - \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds - \int_0^t \int_{\Omega} u^\varepsilon \cdot (u^\varepsilon - u) \rho^\varepsilon dx ds \\
& \quad - \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot (\nabla V + \nabla W \star \rho^\varepsilon) dx ds \\
& \quad - \frac{1}{2} \int_0^t \int_{\Omega^2 \times \mathbb{R}^{2d}} \phi(x - y) |v - w|^2 f^\varepsilon(x, v) f^\varepsilon(y, w) dx dy dv dw ds \\
& \quad + \frac{1}{2} \int_0^t \int_{\Omega \times \Omega} \phi(x - y) \rho^\varepsilon(x) \rho^\varepsilon(y) |u^\varepsilon(x) - u^\varepsilon(y)|^2 dx dy ds \\
& \quad - \frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) \cdot (v - u) dx dv ds =: \sum_{i=1}^8 J_i.
\end{aligned}$$

For J_1 , we obtain

$$\begin{aligned}
J_1 & = - \int_0^t \int_{\Omega \times \mathbb{R}^d} [(u - u^\varepsilon) \otimes (u - u^\varepsilon) + (u^\varepsilon - v) \otimes (u^\varepsilon - v)] f^\varepsilon : \nabla u dx dv ds \\
& \leq \|\nabla u\|_{L^\infty} \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \\
& \quad - \int_0^t \int_{\Omega \times \mathbb{R}^d} ((u^\varepsilon - v) \sqrt{f^\varepsilon} - 2 \nabla_v \sqrt{f^\varepsilon}) \otimes (u^\varepsilon - v) \sqrt{f^\varepsilon} : \nabla u dx dv ds \\
& \quad - \int_0^t \int_{\Omega \times \mathbb{R}^d} 2 \nabla_v \sqrt{f^\varepsilon} \otimes (u^\varepsilon - v) \sqrt{f^\varepsilon} : \nabla u dx dv ds \\
& \leq C \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \\
& \quad + \|\nabla u\|_{L^\infty} \left(\int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds \right)^{1/2} \left(\int_0^t \mathcal{D}_1(f^\varepsilon)(s) ds \right)^{1/2} \\
& \quad - \int_0^t \int_{\Omega \times \mathbb{R}^d} \nabla_v f^\varepsilon \otimes (u^\varepsilon - v) : \nabla u dx dv ds \\
& \leq C \left(\sqrt{\varepsilon} + \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \right) + \int_0^t \int_{\Omega} \nabla \rho^\varepsilon \cdot u dx ds.
\end{aligned}$$

For J_2 and J_4 , it is easy to get

$$\begin{aligned} J_2 + J_4 &\leq \| |u| |\nabla u| \|_{L^\infty} \int_0^t \int_\Omega \rho^\varepsilon |u^\varepsilon - u| \, dx ds \\ &\quad + \left(\int_0^t \int_\Omega \rho^\varepsilon |u^\varepsilon|^2 \, dx ds \right)^{1/2} \left(\int_0^t \int_\Omega \rho^\varepsilon |u^\varepsilon - u|^2 \, dx ds \right)^{1/2} \\ &\leq C \left(\int_0^t \int_\Omega \rho^\varepsilon |u^\varepsilon - u|^2 \, dx ds \right)^{1/2}. \end{aligned}$$

Moreover, we use the relation from Proposition 2.1 in Ref. 35 or Lemma 7.3 in Ref. 34 to get

$$\begin{aligned} J_6 + J_7 &\leq C\varepsilon + \frac{1}{2\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon|^2 \, dx dv ds \\ &\quad - d \int_0^t \int_\Omega (\phi \star \rho^\varepsilon) \rho^\varepsilon \, dx ds. \end{aligned}$$

Thus, we combine all the previous estimates to get

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho, u}) \, dx dv - \int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f_0^\varepsilon | M_{\rho_0, u_0}) \, dx dv \\ &\leq C \left(\sqrt{\varepsilon} + \left(\int_0^t \int_\Omega \rho^\varepsilon |u^\varepsilon - u|^2 \, dx ds \right)^{1/2} + \int_0^t \int_\Omega \rho^\varepsilon |u^\varepsilon - u|^2 \, dx ds \right) \\ &\quad - \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon \, dx dv ds + dt \\ &\quad - \frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) (\nabla_v f^\varepsilon - (u - v) f^\varepsilon) \, dx dv ds \\ &\quad + \frac{1}{2\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon|^2 \, dx dv ds \\ &\quad + \int_0^t \int_\Omega \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla W \star (\rho - \rho^\varepsilon) \, dx ds. \end{aligned}$$

On the other hand, we find

$$\begin{aligned} dt &= - \int_0^t \int_{\Omega \times \mathbb{R}^d} (v - u^\varepsilon) \cdot \nabla_v f^\varepsilon \, dx dv ds \\ &= - \int_0^t \int_{\Omega \times \mathbb{R}^d} (v - u^\varepsilon) \cdot (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) \, dx dv ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds \\
& \leq \left(\int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds \right)^{1/2} \\
& \quad \times \left(\int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon|^2 dx dv ds \right)^{1/2} \\
& \quad + \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds \\
& \leq C\sqrt{\varepsilon} + \int_0^t \int_{\Omega \times \mathbb{R}^d} |u^\varepsilon - v|^2 f^\varepsilon dx dv ds
\end{aligned}$$

and

$$\begin{aligned}
& -\frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) dx dv ds \\
& = -\frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon|^2 dx dv ds \\
& \quad - \frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} (\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon) \cdot (u^\varepsilon - u) dx dv ds \\
& = -\frac{1}{\varepsilon} \int_0^t \int_{\Omega \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla_v f^\varepsilon - (u^\varepsilon - v) f^\varepsilon|^2 dx dv ds.
\end{aligned}$$

If the interaction potential W is given by the Coulomb, then by Lemma 3.2, we obtain

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla W \star (\rho - \rho^\varepsilon) dx ds \\
& = -\frac{1}{2} \int_0^t \frac{d}{ds} \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx ds \\
& \quad + \int_0^t \int_{\Omega} \nabla W \star (\rho^\varepsilon - \rho) \cdot u(\rho^\varepsilon - \rho) dx ds,
\end{aligned}$$

and integrating by parts the second term on the right-hand side of above yields

$$\begin{aligned}
& \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla W \star (\rho - \rho^\varepsilon) dx ds \\
& \leq -\frac{1}{2} \int_0^t \frac{d}{ds} \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx ds + C \int_0^t \int_{\Omega} |\nabla W \star (\rho^\varepsilon - \rho)|^2 dx ds
\end{aligned}$$

for some $C > 0$ which depends only on $\|\nabla u\|_{L^\infty}$. Therefore, we use Theorem 2.1 and apply Grönwall's lemma to have

$$\begin{aligned}
 & \int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho, u}) dx dv - \int_{\Omega \times \mathbb{R}^d} \mathcal{H}(f_0^\varepsilon | M_{\rho_0, u_0}) dx dv + \int_{\Omega} |\nabla W \star (\rho - \rho^\varepsilon)|^2 dx \\
 & \leq C\sqrt{\varepsilon} + C \int_{\Omega} |\nabla W \star (\rho_0 - \rho_0^\varepsilon)|^2 dx \\
 & \quad + C \left(\left(\int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \right)^{1/2} + \int_0^t \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \right) \\
 & \quad + C \int_0^t \int_{\Omega} |\nabla W \star (\rho^\varepsilon - \rho)|^2 dx ds \\
 & \leq C\varepsilon^{1/4} + C \left(\left(\int_{\Omega} |\nabla W \star (\rho_0^\varepsilon - \rho_0)|^2 dx \right)^{1/2} \right. \\
 & \quad \left. + \int_{\Omega} |\nabla W \star (\rho_0^\varepsilon - \rho_0)|^2 dx \right). \tag{A.1}
 \end{aligned}$$

On the other hand, if the interaction potential W is regular such that $\nabla W \in L^\infty(\Omega)$, then

$$\begin{aligned}
 & \int_0^t \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla W \star (\rho - \rho^\varepsilon) dx ds \\
 & \leq \|\nabla W\|_{L^\infty} \int_0^t \|(\rho - \rho^\varepsilon)(\cdot, s)\|_{L^1} \left(\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} ds \\
 & \leq C \int_0^t \int_{\Omega} \mathcal{E}(U^\varepsilon | U) dx ds,
 \end{aligned}$$

and this gives (A.1) without the terms with W . This completes the proof.

Appendix B. Proof of Theorem 5.2

In this part, we present the proof for Theorem 5.2. Here, we only present the proof for the case $\Omega = \mathbb{R}^d$, since the case $\Omega = \mathbb{T}^d$ is analogous. First, the condition (ii) and the integrability of h imply that for every $\varepsilon_0 > 0$, there exists $L > 0$ such that

$$\left(\sup_{n \in \mathbb{N}} \int_{\{|x| > L\}} |h_n| dx \right) + \int_{\{|x| > L\}} |h| dx < \frac{\varepsilon_0}{2}.$$

Moreover, we can also choose $\delta > 0$ such that

$$\left(\sup_{n \in \mathbb{N}} \int_E |h_n| dx \right) + \int_E |h| dx < \frac{\varepsilon_0}{2}, \quad \text{whenever } m(E) < \delta.$$

For those choices of L and δ , we use Egoroff's theorem to get a set A_δ such that $m(\{|x| \leq L\} \setminus A_\delta) < \delta$ and

$$h_n \rightarrow h \quad \text{uniformly on } A_\delta.$$

Thus, we have

$$\begin{aligned}
 & \int_{\mathbb{R}^d} |h_n - h| dx \\
 &= \int_{\{|x| \leq L\}} |h_n - h| dx + \int_{\{|x| > L\}} |h_n - h| dx \\
 &\leq \int_{A_\delta} |h_n - h| dx + \int_{\{|x| \leq L\} \setminus A_\delta} (|h_n| + |h|) dx + \int_{\{|x| > L\}} (|h_n| + |h|) dx \\
 &\leq \int_{A_\delta} |h_n - h| dx + \varepsilon_0,
 \end{aligned}$$

which implies

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} |h_n - h| dx \leq \varepsilon_0.$$

Since the choice of ε_0 was arbitrary, we conclude the proof.

Acknowledgments

JAC was partially supported by EPSRC grant number EP/P031587/1 and the Advanced Grant Nonlocal-CPD (Nonlocal PDEs for Complex Particle Dynamics: Phase Transitions, Patterns and Synchronization) of the European Research Council Executive Agency (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 883363). YPC was supported by NRF grant (No. 2017R1C1B2012918), POSCO Science Fellowship of POSCO TJ Park Foundation, and Yonsei University Research Fund of 2019-22-021. JJ was supported by NRF grant (No. 2019R1A6A1A10073437).

References

1. L. Ambrosio, N. Gigli and G. Savaré, *Gradient Flows: In Metric Spaces and in the Space of Probability Measures* (Springer Science & Business Media, 2008).
2. B. Aylajm, N. Bellomo, L. Gibelli and A. Reali, A unified multiscale vision of behavioral crowds, *Math. Models Methods Appl. Sci.* **30** (2020) 1–22.
3. N. Bellomo and L. Gibelli, Toward a mathematical theory of behavioral-social dynamics for pedestrian crowds, *Math. Models Methods Appl. Sci.* **25** (2015) 2417–2437.
4. N. Bellomo and J. Soler, On the mathematical theory of the dynamics of swarms viewed as complex systems, *Math. Models Methods Appl. Sci.* **22** (2012) 1140006.
5. J. A. Cañizo, J. A. Carrillo and J. Rosado, A well-posedness theory in measures for some kinetic models of collective motion, *Math. Models Methods Appl. Sci.* **21** (2011) 515–539.
6. J. A. Carrillo and Y.-P. Choi, Quantitative error estimates for the large friction limit of Vlasov equation with nonlocal forces, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **37** (2020) 925–954.
7. J. A. Carrillo, Y.-P. Choi and M. Hauray, The derivation of swarming models: Mean-field limit and Wasserstein distances, in *Collective Dynamics from Bacteria to Crowds, An Excursion Through Modeling, Analysis and Simulation Series*, CISM International Centre for Mechanical Sciences, Vol. 553 (Springer, 2014), pp. 1–46.

8. J. A. Carrillo, Y.-P. Choi, M. Hauray and S. Salem, Mean-field limit for collective behavior models with sharp sensitivity regions, *J. Eur. Math. Soc.* **21** (2019) 121–161.
9. J. A. Carrillo, Y.-P. Choi and T. K. Karper, On the analysis of a coupled kinetic-fluid model with local alignment forces, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016) 273–307.
10. J. A. Carrillo, Y.-P. Choi and S. Pérez, A review on attractive–repulsive hydrodynamics for consensus in collective behavior, *Active Particles, Advances in Theory, Models, Applications, Modeling and Simulation in Science Engineering and Technology*, Vol. I (Birkhäuser/Springer, 2017), pp. 259–298.
11. J. A. Carrillo, Y.-P. Choi, E. Tadmor and C. Tan, Critical thresholds in 1D Euler equations with nonlocal forces, *Math. Models Methods Appl. Sci.* **26** (2016) 185–206.
12. J. A. Carrillo, Y.-P. Choi and E. Zatorska, On the pressureless damped Euler–Poisson equations with quadratic confinement: Critical thresholds and large-time behavior, *Math. Models Methods Appl. Sci.* **26** (2016) 2311–2340.
13. J. A. Carrillo, E. Feireisl, P. Gwiazda and A. Świerczewska-Gwiazda, Weak solutions for Euler systems with non-local interactions, *J. London Math. Soc.* **95** (2017) 705–724.
14. J. A. Carrillo, Y. Peng and A. Wróblewska-Kamińska, Relative entropy method for the relaxation limit of hydrodynamic models, *Netw. Heterog. Media* **15** (2020) 369–387.
15. C. Cercignani, R. Illner and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Applied Mathematical Sciences, Vol. 106 (Springer-Verlag, 1994).
16. G.-Q. Chen, Euler Equations and Related Hyperbolic Conservation Laws, in *Evolutionary Equations Handbook of Differential Equations*, Vol. II (Elsevier, 2005), pp. 1–104.
17. Y.-P. Choi, Large friction limit of pressureless Euler equations with nonlocal forces, preprint, arXiv:2002.01691.
18. Y.-P. Choi, The global Cauchy problem for compressible Euler equations with a non-local dissipation, *Math. Models Methods Appl. Sci.* **29** (2019) 185–207.
19. Y.-P. Choi, S.-Y. Ha and Z. Li, Emergent Dynamics of the Cucker–Smale Flocking Model and its Variants, *Active Particles, Advances in Theory, Models, Applications, Modeling and Simulation in Science, Engineering and Technology*, Vol. I (Birkhäuser/Springer, 2017), pp. 299–331.
20. Y.-P. Choi and J. Haskovec, Hydrodynamic Cucker–Smale model with normalized communication weights and time delay, *SIAM J. Math. Anal.* **51** (2019) 2660–2685.
21. Y.-P. Choi and J. Jung, Asymptotic analysis for a Vlasov–Fokker–Planck/Navier–Stokes system in a bounded domain, preprint, arXiv:1912.13134.
22. Y.-P. Choi and S. Salem, Propagation of chaos for aggregation equations with no-flux boundary conditions and sharp sensing zones, *Math. Models Methods Appl. Sci.* **28** (2018) 223–258.
23. G.-Q. Chen and D. Wang, Convergence of shock capturing schemes for the compressible Euler–Poisson equations, *Comm. Math. Phys.* **179** (1996) 333–364.
24. S. Cordier, Global solutions to the isothermal Euler–Poisson plasma model, *Appl. Math. Lett.* **8** (1995) 19–24.
25. F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Automat. Control* **52** (2007) 852–862.
26. C. M. Dafermos, The second law of thermodynamics and stability, *Arch. Ration. Mech. Anal.* **70** (1979) 167–179.
27. S. Engelberg, H. Liu and E. Tadmor, Critical thresholds in Euler–Poisson equations, *Indiana Univ. Math. J.* **50** (2001) 109–157.

28. A. Figalli and M.-J. Kang, A rigorous derivation from the kinetic Cucker–Smale model to the pressureless Euler system with nonlocal alignment, *Anal. PDE* **12** (2019) 843–866.
29. Y. Guo, Smooth irrotational flows in the large to the Euler–Poisson system in \mathbb{R}^{3+1} , *Comm. Math. Phys.* **195** (1998) 249–265.
30. Y. Guo and B. Pausader, Global Smooth Ion Dynamics in the Euler–Poisson System, *Comm. Math. Phys.* **303** (2011) 89–125.
31. S.-Y. Ha and J.-G. Liu, A simple proof of the Cucker–Smale flocking dynamics and mean-field limit, *Commun. Math. Sci.* **7** (2009) 297–325.
32. P.-E. Jabin and Z. Wang, Mean-field limit for stochastic particle systems, in *Active Particles – Advances in Theory, Models, Applications*, Series: Modeling and Simulation in Science and Technology, Vol. I (Birkhäuser, 2017), pp. 379–402.
33. M.-J. Kang, From the Vlasov–Poisson equation with strong local alignment to the pressureless Euler–Poisson system, *Appl. Math. Lett.* **79** (2018) 85–91.
34. T. Karper, A. Mellet and K. Trivisa, Existence of weak solutions to kinetic flocking models, *SIAM Math. Anal.* **45** (2013) 215–243.
35. T. K. Karper, A. Mellet and K. Trivisa, Hydrodynamic limit of the kinetic Cucker–Smale flocking model, *Math. Models Methods Appl. Sci.* **25** (2015) 131–163.
36. Y. Katz, K. Tunstrom, C. C. Ioannou, C. Huepe and I. D. Couzin, Inferring the structure and dynamics of interactions in schooling fish, *Proc. Natl. Acad. Sci.* **108** (2011) 18720–18725.
37. C. Lattanzio and A. E. Tzavaras, Relative entropy in diffusive relaxation, *SIAM J. Math. Anal.* **45** (2013) 1563–1584.
38. C. Lattanzio and A. E. Tzavaras, From gas dynamics with large friction to gradient flows describing diffusion theories, *Comm. Partial Differential Equations* **42** (2017) 261–290.
39. N. E. Leonard, D. A. Paley, F. Lekien, R. Sepulchre, D. M. Fratantoni and R. E. Davis, Collective motion, sensor networks and ocean sampling, *Proc. IEEE* **95** (2007) 48–74.
40. R. Lukeman, Y.-X. Li and L. Edelstein-Keschet, Inferring individual rules from collective behavior, *Proc. Natl. Acad. Sci.* **107** (2010) 12576–12580.
41. A. Mellet and A. Vasseur, Global weak solutions for a Vlasov–Fokker–Planck/Navier–Stokes system of equations, *Math. Models Methods Appl. Sci.* **17** (2007) 1039–1063.
42. S. Motsch and E. Tadmor, A new model for self-organized dynamics and its flocking behavior, *J. Stat. Phys.* **144** (2011) 923–947.
43. B. Perthame and P. E. Souganidis, A limiting case for velocity averaging, *Ann. Sci. École Norm. Sup.* **31** (1998) 591–598.
44. F. Poupaud, M. Rasle and J. P. Vila, Global solutions to the isothermal Euler–Poisson system with arbitrarily large data, *J. Differential Equations* **123** (1995) 93–121.
45. C. M. Topaz, A. L. Bertozzi and M. A. Lewis, A nonlocal continuum model for biological aggregation, *Bull. Math. Biol.* **68** (2006) 1601–1623.
46. C. Villani, *A Review of Mathematical Topics in Collisional Kinetic Theory, Handbook of Mathematical Fluid Dynamics*, Vol. I (Elsevier, 2002), pp. 71–305.