Soft Computing

# Predictor-Corrector approach for numerical solution of fuzzy fractional differential equations and linear multi-term fuzzy fractional equations

--Manuscript Draft--

| | |
|---|---|
| Manuscript Number: | |
| Full Title: | Predictor-Corrector approach for numerical solution of fuzzy fractional differential equations and linear multi-term fuzzy fractional equations |
| Article Type: | Original Research |
| Keywords: | Volterra integral equation (VIE); Caputo Hukuhara differentiability; predictor-corrector method; fuzzy fractional multi-term methods; fuzzy fractional differential equations(FFDEs) |
| Corresponding Author: | Zhouchao Wei<br>China University of Geosciences<br>CHINA |
| Corresponding Author Secondary Information: | |
| Corresponding Author's Institution: | China University of Geosciences |
| First Author: | Wadhah Al-sadi |
| First Author Secondary Information: | |
| Order of Authors: | Wadhah Al-sadi |
| | Zhouchao Wei |
| | Irene Moroz |
| | Omar Al Balqa Applied University Abu Arqub |
| | Tariq Q. S. Abdullah |
| Funding Information: | National Natural Science Foundation of China (12172340) — Prof. Zhouchao Wei |
| | the Fundamental Research Funds for the Central Universities (G1323523061, G1323523041 and CUGDCJJ202216) — Prof. Zhouchao Wei |
| | Young Top-notch Talent Cultivation Program of Hubei Province — Prof. Zhouchao Wei |
| Abstract: | This article is devoted to establishing the existence and uniqueness of solution for fuzzy fractional differential equations(FFDEs) under sufficient assumptions and contraction principle. Our study is based on Caputo's generalized Hukuhara differentiability. After that, we present a numerical solution to the initial value problems for solving two families of fuzzy fractional-order problems: fuzzy fractional differential equations (FFDEs) and multi-term FFDEs by a predictor-corrector method. This method is considered a primary goal of the article. The suggested method has higher-order accuracy and reduces the error between exact and approximate solutions. Finally, our approach is demonstrated by solving some specific examples. |
| Section/Category: | Fuzzy systems and their mathematics |

# Predictor-Corrector approach for numerical solution of fuzzy fractional differential equations and linear multi-term fuzzy fractional equations

Wadhah Al-sadi[1], Zhouchao Wei[1*], Irene Moroz[2], Omar Abu Arqub[3], Tariq Q. S. Abdullah[1]

[1] School of Mathematics and Physics, China University of Geosciences, Wuhan, 430074, China

[2] Mathematical Institute, Oxford University, Oxford, OX2 6GG, United Kingdom

[3] Department of Mathematics, Faculty of Science, Al Balqa Applied University, Salt 19117, Jordan

**Abstract**

This article is devoted to establishing the existence and uniqueness of solution for fuzzy fractional differential equations(FFDEs) under sufficient assumptions and contraction principle. Our study is based on Caputo's generalized Hukuhara differentiability. After that, we present a numerical solution to the initial value problems for solving two families of fuzzy fractional-order problems: fuzzy fractional differential equations (FFDEs) and multi-term FFDEs by a predictor-corrector method. This method is considered a primary goal of the article. The suggested method has higher-order accuracy and reduces the error between exact and approximate solutions. Finally, our approach is demonstrated by solving some specific examples.

**Keywords:** Volterra integral equation (VIE), Caputo Hukuhara differentiability, predictor-corrector method, fuzzy fractional multi-term methods, fuzzy fractional differential equations(FFDEs).
MSC(2010) 26A33, 34B82, 45N05

## 1 Introduction

Fractional calculus and applications have acquired a lot of significance in various fields. So, fractional calculus has been used in several fields as a strong tool for more successful and accurate results in modeling many complicated phenomena due to its different applications in seemingly widespread and diverse areas of engineering and science, such as thermal engineering, signal processing, mechanics, image processing, electrical engineering, biophysics, automatic control and robotics. Moreover, the fractional differential equation has tremendous application potential in modeling various physical problems encountered in everyday life. Among these, we may include some models such as the measurement of material properties, the dynamic model, the modeling of earthquakes, etc. (For more, see [1, 2, 3, 4] and references therein).

Uncertainty in a model of physical phenomena is inescapable due to inexactitude or approximations. The essential tools for analyzing uncertainty are interval analysis [6], the probabilistic approach, and fuzzy

[1]**Corresponding author:** weizc@cug.edu.cn, weizhouchao@163.com.

concepts [5]. Consequently, fuzzy differential equations and fuzzy analysis theorems have been suggested and developed recently to deal with uncertainty brought on by insufficient information that may be seen in several mathematical models of some complex real-world processes. The study of FDEs forms an appropriate environment for the mathematical modeling of real-world problems that involve uncertainty or ambiguity pervades. Recent scientific and technical studies have shown the use of FDEs to represent many dynamic systems in recent decades. It has led to an increased interest in studying FDEs models. We observed that the idea of solutions for FDEs with uncertainty was recently put forward by Agarwal et al. [7]. In [8], it has been established that the solution for Fuzzy FDEs with fuzzy initial conditions exists and is unique under such a derivative. For further information, see [9 -16]. Recent concepts of fuzzy fractional derivatives for fuzzy functions are based either on the idea of the Hukuhara fractional derivative or the strongly generalized fractional derivative. However, the Hukuhara derivative concept is well-established and age-old. Hoa [19] investigated existence and uniqueness of solutions and studied the modified fractional Euler approach for FFDEs. The classes of FFDEs have been extended and studied using novel derivatives ideas in various articles by Bede et al such as [20]. Fard et al. [22] established many theories and definitions of fuzzy fractional calculus of variation. They presented some necessary conditions for the fuzzy-fractional Euler-Lagrange equations for both unconstrained and constrained fuzzy-fractional variational. Existence and uniqueness theories in studying initial value problems are established widely due to their importance in proving well-posed real-world problems. So, mathematical physics models must have the properties of existence, uniqueness, and continuing behavior that change the solution with the initial conditions. Singularity results play an essential role in the continuance of solutions and theories of independent systems. Even if the singularity results nearly always come at the expense of demanding circumstances, they are nonetheless helpful since it is hard to anticipate the behavior of physical systems without such unique results. The Banach fixed point theory was proposed in [17, 8] to demonstrate existence solution for fuzzy fractional IVP and its singularity.

Solving all fractional differential equations using a single approach would be impossible. We must use derivatives and solutions that correspond to the problem while resolving a practical problem. Based on the characteristics of linear transformations, Gasilov et al. [27] suggested a strategy to resolve a fuzzy IVP for the second order linear DEs. In 2013, S. Arshad [28] proved the EUS of the solution for FFDEs by Combining the Goetschel-Voxman derivative and Riemann-Liouville derivative. Numerous studies have been on FFDEs based on fuzzy fractional derivatives under particular circumstances mentioned above; for example, see [23],[24],[25],[26]. R. Abdollahi et al. [33] investigated the Caputo-Fabrizio derivative concept in combination with SGH-derivative. Granular fuzzy fractional derivatives emerged by M. Najariyan and Y. Zhao [34]by integrating Caputo and Riemann-Liouville derivatives with gr-derivative, i.e., granular-Caputo derivative and granular- Riemann-Liouville derivative. N. V. Hoa et al. [35] have demonstrated the EU of solution for FFDEs under the Caputo-Katugampola gH-derivative by using successive approximations under the generalized Lipschitz conditions. In 2020, T. Allahviranloo et al. [36] studied FFDEs under concept of Atangana-Baleanu gH-derivative. For more studies, see[29,30,31,32,37,38]. The motivation for developing numerical methods specifically designed to solve FDEs is due to the increasing interest in fractional calculus applications (FDEs). It is even more challenging to solve FDEs analytically than to solve standard ODEs. Most of the time, it is only possible to approximate the solution numerically. To our knowledge, no article exists devoted to studying solving Numerical FFDEs by a Predictor-Corrector Approach. This method has high accuracy results. So, we use this method to obtain accurate results, which is the primary goal of an article. In brief, this article aims at 1. studying the existence for fuzzy FDEs. 2. studying solving Numerical FFDEs by a Predictor-Corrector Approach. 3. studying solving Numerical multi-step methods of FFDEs by a Predictor-Corrector Approach.

Organize the article as follows: Section 2 briefly introduces some fundamentals of fuzzy function numbers, FFDEs, and fuzzy fractional integral equations. In Section 3, we investigate the existence and uniqueness of solutions for FFDEs under the Caputo gH-differentiability. Section 4.1 is devoted to illustrating a Predictor-Corrector approach for FFDEs and multi-step methods for FFDEs. In Section 4.2, we utilize a Predictor-Corrector approach to find the approximate solutions to the FFDEs and multi-step methods for FFDEs of fractional order under the Caputo fuzzy derivative. In Section 5, we offer some examples to show the accuracy of our results and, finally, conclusions of the contribution.

## 2 Preliminaries

Here, the fundamental definitions and lemmas for fuzzy calculus are provided, which are necessary and required in this article.

**Definition 2.1.** [18] . The class of fuzzy subset on $\mathbb{R}$ is indicate by $\mathbf{E} = \{\mu(\widetilde{\zeta}) : \mathbb{R} \longrightarrow [0,1]\}$ which achieves:

1. $\mu(\widetilde{\zeta})$ is upper semi-continuous;
2. $\mu(\widetilde{\zeta})$ is fuzzy convex, i.e., $\mu(\lambda\widetilde{\zeta}_1 + (1-\lambda)\widetilde{\zeta}_2) \geq \min\{\mu(\widetilde{\zeta}_1), \mu(\widetilde{\zeta}_2)\}\ \forall\widetilde{\zeta}_1, \widetilde{\zeta}_2 \in \mathbb{R}, \lambda \in [0,1]$;
3. $[\mu]^0 = \overline{\{\widetilde{\zeta}_0 \in \mathbb{R} | \mu(\widetilde{\zeta}_0 > 0)\}}$ is compact;
4. $\mu(\widetilde{\zeta})$ is normal, i.e., $\exists\widetilde{\zeta}_0 \in \mathbb{R}$ such that $\mu(\widetilde{\zeta}_0) = 1$.

**Definition 2.2. [18]** . The $\delta - cut$ set of $\mu \in \mathbf{E}$ is defined by

$$[\mu]^\delta = \begin{cases} \{\widetilde{\zeta}_0 \in \mathbb{R} | \mu(\widetilde{\zeta}_0 > \delta)\}, \\ \overline{\{\widetilde{\zeta}_0 \in \mathbb{R} | \mu(\widetilde{\zeta}_0 > 0)\}}, \end{cases} \qquad \forall 0 < \delta \leq 1,$$

which is a closed, bounded interval and
$[\mu]^\delta = [\underline{\mu}(\delta), \overline{\mu}(\delta)], \forall\delta \in [0,1]$.

**Definition 2.3** [17]. The parametric form of a fuzzy number $\mu$ is a pair $(\underline{\mu}^\delta, \overline{\mu}^\delta)$ of functions $\underline{\mu}^\delta(\widetilde{\zeta}), \overline{\mu}^\delta(\widetilde{\zeta}), \delta \in [0,1]$, which fulfill the following conditions:

1. $\underline{\mu}^\delta(\widetilde{\zeta})$ is a bounded non-decreasing left continuous function.
2. $\overline{\mu}^\delta(\widetilde{\zeta})$ is a monotonically non-increasing left continuous function.
3. $\underline{\mu}^\delta(\widetilde{\zeta}) \leq \overline{\mu}^\delta(\widetilde{\zeta}), 0 \leq \delta \leq 1$.

The Hausdorff distance between $\mu$ and $\nu$ is defined by $\mathbb{H} : E \times E \longrightarrow [0, +\infty]$,
$\mathbb{H}[\mu, \nu] = sup_{\eta\in[0,1]} max\{|\underline{\mu}(\widetilde{\zeta}) - \underline{\nu}(\widetilde{\zeta})|, |\overline{\mu}(\widetilde{\zeta}) - \overline{\nu}(\widetilde{\zeta})|\}$,
where $\mu = (\underline{\mu}(\widetilde{\zeta}), \overline{\mu}(\widetilde{\zeta})), \nu = (\underline{\nu}(\widetilde{\zeta}), \overline{\nu}(\widetilde{\zeta})) \subset \mathbb{R}$ is used in [5]. Then, it is simple to see that $\mathbb{H}$ is a metric in $E$, we offer some properties of the metric $\mathbb{H}$ :

1. $\mathbb{H}(\mu + \hbar, \nu + \hbar) = \mathbb{H}(\mu, \nu), \forall\mu, \nu, \hbar \in E$,
2. $\mathbb{H}(\rho\mu, \rho\nu) = |\rho|\mathbb{H}(\mu, \nu), \forall\rho \in \mathbb{R}, \mu, \nu \in E$,
3. $\mathbb{H}(\mu + \nu, \omega + \varrho) \leq \mathbb{H}(\mu, \varpi) + \mathbb{H}(\nu, \varrho), \forall\mu, \nu, \varpi, \varrho \in E$,
4. $(\mathbb{H}, E)$ is a complete metric space.

Noting that for $r \leq \varphi \leq h$, $r, \varphi, h \in \mathbb{R}$, a triangular fuzzy number $\mu = (r, \varphi, h)$ is given $\underline{\mu}^\delta = \varphi - (1-\delta)(\varphi - r)$ and $\overline{\mu}^\delta = +(1-\delta)(h-\varphi)$ are the end-point of the $\delta - cut$ set, $\forall\delta \in [0,1]$. In this article we use triangular fuzzy numbers. Subtraction $\mu - \nu$, addition $\mu + \nu$ and scalar multiplication by $\hbar$ are given as

$$[\mu+\nu]^{\delta} = [\underline{\mu}^{\delta}+\underline{\nu}^{\delta}, \overline{\mu}^{\delta}+\overline{\nu}^{\delta}],$$
$$[\mu-\nu]^{\delta} = [\underline{\mu}^{\delta}-\overline{\nu}^{\delta}, \overline{\mu}^{\delta}-\underline{\nu}^{\delta}].$$
$$[\hbar\mu]^{\delta} = \begin{cases} [\hbar\underline{\mu}^{\delta}, \hbar\overline{\mu}^{\delta}], & \hbar \geq 0, \\ [\hbar\overline{\mu}^{\delta}, \hbar\underline{\mu}^{\delta}], & \hbar < 0. \end{cases}$$

**Definition 2.4** [16] Let $\mu, \nu \in E$. If there exists $\varpi \in E$ such that $\mu = \nu + \varpi$ , then $\varpi$ is called (H-difference) of $\mu, \nu$, and it is indicated by $\mu \ominus \nu$. observe that $\mu \ominus \nu \neq \mu + (-1)\nu$.

**Definition 2.5 [19]**. The generalized Hukuhara difference of two fuzzy numbers $\mu, \nu \in E$ Hukuhara difference is described as follows:

$$\mu \ominus_{gH} \nu = \begin{cases} \mu = \nu + \varpi, \\ \nu = \mu - (-1)\varpi. \end{cases}$$

**Definition 2.6** [19] Let $t \in (r, h)$ and $z$ be such that $t + z \in (r, h)$, then (gH-differentiable) of a fuzzyvalued function $\eta : (r, h) \longrightarrow E$ at $t$ is described as

$$\mathcal{D}_{gH}\widetilde{\zeta}(t) = \lim_{z \longrightarrow 0} \frac{\widetilde{\zeta}(t+z) \ominus_{gH} \widetilde{\zeta}(t)}{z}$$

**Lemma 1.** [21] . For any $\vartheta_1, \vartheta_2, \vartheta_3 \in E$, we have

$$\mathcal{D}_0[\vartheta_1 \ominus (-1)\vartheta_2, \hat{0}] \leq \mathcal{D}[\vartheta_1, \hat{0}] + \mathcal{D}[\vartheta_2, \hat{0}],$$
$$\mathcal{D}_0[\vartheta_1 \ominus (-1)\vartheta_2, \vartheta_1 \ominus (-1)\vartheta_3] = \mathcal{D}_0[\vartheta_2, \vartheta_3].$$

**Theorem 2.1** [17]. Let $\mathbb{G}(\widetilde{\zeta}) : [r, \infty) \longrightarrow E$ be a fuzzy valued function on $[r, \infty)$ represented by $\underline{\mathbb{G}}(\widetilde{\zeta}; \delta), \overline{\mathbb{G}}(\widetilde{\zeta}; \delta)$. For any fixed $r \in [0, 1]$, suppose $\underline{\mathbb{G}}(\widetilde{\zeta}; \delta)$ and $\overline{\mathbb{G}}(\widetilde{\zeta}; \delta)$ are Riemann-integrable on $\mathbb{T} = [r, h] \subseteq \mathbb{R}$. Then $\mathbb{G}(\widetilde{\zeta})$ is fuzzy Riemann-integrable on $[r, \infty)$

$$\int_r^{\infty} |\mathbb{G}(\widetilde{\zeta}; \delta)| d\widetilde{\zeta} = [\int_r^{\infty} \underline{\mathbb{G}}(\widetilde{\zeta}; \delta) d\widetilde{\zeta}, \int_r^{\infty} \overline{\mathbb{G}}(\widetilde{\zeta}; \delta) d\widetilde{\zeta}].$$

We indicate by $\mathcal{C}^{\varepsilon}[r, h]$ the space of fuzzy function on an interval $[r, h]$. Also, we indicate by $\mathbb{L}^{\varepsilon}[r, h]$the metric space of fuzzy function on an interval $[r, h]$ and $AC^{\varepsilon}[r, h]$ implies the set fuzzy valued function which are absolute.

**Definition 2.7** [17] Let $\mathbb{G} : (r, h) \longrightarrow E$ and $\widetilde{\zeta}_0 \in (r, h)$. We say that $\mathbb{G}$ is a strongly generalized differential at $\widetilde{\zeta}_0$, if there exists $\mathbb{G}'(\widetilde{\zeta}_0) \in E$ such that

$(i)$ for all $c > 0$, $\exists \mathbb{G}(\widetilde{\zeta}_0 + c) \ominus \mathbb{G}(\widetilde{\zeta}_0), \exists \mathbb{G}(\widetilde{\zeta}_0) \ominus c\mathbb{G}(\widetilde{\zeta}_0 - c)$ and

$$\lim_{c \searrow 0} \frac{\mathbb{G}(\widetilde{\zeta}_0 + c) \ominus \mathbb{G}(\widetilde{\zeta}_0)}{c} = \lim_{c \searrow 0} \frac{\mathbb{G}(\widetilde{\zeta}_0) \ominus \mathbb{G}(\widetilde{\zeta}_0 - c)}{c} = \mathbb{G}'(\widetilde{\zeta}_0),$$

or $(ii)$ for all $c > 0$, $\exists \mathbb{G}(\widetilde{\zeta}_0) \ominus \mathbb{G}(\widetilde{\zeta}_0 + c), \exists \mathbb{G}(\widetilde{\zeta}_0 - c) \ominus \mathbb{G}(\widetilde{\zeta}_0)$ and

$$\lim_{c \searrow 0} \frac{\mathbb{G}(\widetilde{\zeta}_0) \ominus \mathbb{G}(\widetilde{\zeta}_0 + c)}{-c} = \lim_{c \searrow 0} \frac{\mathbb{G}(\widetilde{\zeta}_0 - c) \ominus \mathbb{G}(\zeta_0)}{-c} = \mathbb{G}'(\widetilde{\zeta}_0),$$

or $(iii)$ for all $c > 0$, $\exists \mathbb{G}(\widetilde{\zeta}_0 + c) \ominus \mathbb{G}(\widetilde{\zeta}_0), \exists \mathbb{G}(\widetilde{\zeta}_0 - c) \ominus \mathbb{G}(\widetilde{\zeta}_0)$ and

$$\lim_{c \searrow 0} \frac{\mathbb{G}(\widetilde{\zeta}_0 + c) \ominus \mathbb{G}(\widetilde{\zeta}_0)}{c} = \lim_{c \searrow 0} \frac{\mathbb{G}(\widetilde{\zeta}_0 - c) \ominus \mathbb{G}(\zeta_0)}{-c} = \mathbb{G}'(\widetilde{\zeta}_0),$$

or $(iv)$ for all $c>0$, $\exists \mathbb{G}(\widetilde{\zeta}_0)\ominus \mathbb{G}(\widetilde{\zeta}_0+c), \exists \mathbb{G}(\widetilde{\zeta}_0)\ominus \mathbb{G}(\widetilde{\zeta}_0-c)$ and

$$\lim_{c\searrow 0}\frac{\mathbb{G}(\widetilde{\zeta}_0)\ominus \mathbb{G}(\widetilde{\zeta}_0+c)}{-c}=\lim_{c\searrow 0}\frac{\mathbb{G}(\widetilde{\zeta}_0)\ominus \mathbb{G}(\widetilde{\zeta}_0-c)}{c}=\mathbb{G}'(\widetilde{\zeta}_0).$$

**Definition 2.8** [17] Let $\mathbb{F}\in \mathcal{C}^{\varepsilon}[r,h]\cap \mathbb{L}^{\varepsilon}[r,h], \mathbb{G}$ is integrable, then the Caputo fractional derivative of $\mathbb{G}$ for $0<\gamma<1$ and $\widetilde{\zeta}\in[r,h]$ is denoted by $({}^{C}\mathcal{D}^{\gamma}_{r+}\mathbb{G})(\widetilde{\zeta})\in\mathbb{R}$ and defined by

$$({}^{C}\mathcal{D}^{\gamma}_{r+}\mathbb{G})(\widetilde{\zeta})=\frac{1}{\Gamma(1-\delta)}\int_r^{\widetilde{\zeta}}(\widetilde{\zeta}-t)^{-\gamma}\mathbb{G}'(t)dt. \tag{2.1}$$

Also, we say $\mathbb{G}$ is ${}^{CH}[(i)-\delta]$ -differentiable if Equation (2.1) holds while $\mathbb{G}$ is $(i)-$differentiable and $\mathbb{G}$ is ${}^{CH}[(ii)-\delta]$-differentiable if Eqution (2.1) holds while $\mathbb{G}$ is $(ii)-$differentiable.

**Theorem 2.2** [17] Let $0<\gamma<1$ and $\mathbb{G}\in AC^{\varepsilon}[r,h]$, then the fuzzy Caputo derivative exists on $(r,h)$ and $\forall\delta\in[0,1]$ we have

$$\begin{aligned}({}^{C}\mathcal{D}^{\gamma}_{r+}\mathbb{G})(\widetilde{\zeta};\delta)&=[\frac{1}{\Gamma(1-\gamma)}\int_r^{\widetilde{\zeta}}\frac{\underline{\mathbb{G}'}(t;\delta)dt}{(\widetilde{\zeta}-t)^{\gamma}},\frac{1}{\Gamma(1-\gamma)}\int_r^{\widetilde{\zeta}}\frac{\overline{\mathbb{G}'}(t;\delta)dt}{(\delta\widetilde{\zeta}-t)^{\gamma}}]\\&=[\mathcal{I}^{1-\gamma}_{r^+}(\mathcal{D}\underline{\mathbb{G}})(\widetilde{\zeta};\delta),\mathcal{I}^{1-\gamma}_{r^+}(\mathcal{D}\overline{\mathbb{G}})(\widetilde{\zeta};\delta)],\end{aligned}$$

when $\mathbb{G}$ is $(i)-$differentiable, and

$$\begin{aligned}({}^{C}\mathcal{D}^{\gamma}_{r+}\mathbb{G})(\widetilde{\zeta};\delta)&=[\frac{1}{\Gamma(1-\gamma)}\int_r^{\widetilde{\zeta}}\frac{\overline{\mathbb{G}'}(t;\delta)dt}{(\widetilde{\zeta}-t)^{\gamma}},\frac{1}{\Gamma(1-\gamma)}\int_r^{\widetilde{\zeta}}\frac{\underline{\mathbb{G}'}(t;\delta)dt}{(\widetilde{\zeta}-t)^{\gamma}}]\\&=[\mathcal{I}^{1-\gamma}_{r^+}(\mathcal{D}\underline{\mathbb{G}})(\widetilde{\zeta};\delta),\mathcal{I}^{1-\gamma}_{r^+}(\mathcal{D}\overline{\mathbb{G}})(\widetilde{\zeta};\delta)],\end{aligned}$$

when $\mathbb{G}$ is $(ii)-$differentiable.

**Theorem 2.3** [17] Let us consider $\mathbb{G}\in\mathcal{C}^{\varepsilon}[r,h]$, then we have the following:

$$(\mathcal{I}^{\gamma}_{r+}\mathcal{D}^{\gamma}_{r+}\mathbb{G})(\widetilde{\zeta})=\mathbb{G}(\widetilde{\zeta})\ominus\mathbb{G}(r),\qquad 0<\gamma<1,$$

when $\mathbb{G}$ is $[(i)-\gamma]-$differentiable, and

$$(\mathcal{I}^{\gamma}_{r+}\mathcal{D}^{\gamma}_{r+}\mathbb{G})(\widetilde{\zeta})=-\mathbb{G}(r)\ominus(-\mathbb{G}(\widetilde{\zeta})),\qquad 0<\gamma<1,$$

when $\mathbb{G}$ is $[(i)-\gamma]$-differentiable.

**Theorem 2.4 [16]**. Let $\mathbb{G}(\widetilde{\zeta})\in\mathcal{C}^{\varepsilon}[r,h]\cap\mathbb{L}^{\varepsilon}[r,h]$ be a fuzzyvalued function. The Riemann-Liouville integral of the $\mathbb{G}(\widetilde{\zeta})$, based on its $\delta-cut$ expression can be described by:

$$\mathcal{I}^{\gamma}\mathbb{G}(\widetilde{\zeta})]^{\delta}=[\mathcal{I}^{\gamma}\underline{\mathbb{G}}^{\delta}(\widetilde{\zeta}),\mathcal{I}^{\gamma}\overline{\mathbb{G}}^{\delta}(\widetilde{\zeta})],\qquad 0\le\delta\le 1,$$

where

$$\mathcal{I}^{\gamma}\underline{\mathbb{G}}(\widetilde{\zeta})=\frac{1}{\gamma}\int_r^{\widetilde{\zeta}}\frac{\underline{\mathbb{G}}(t)}{(\widetilde{\zeta}-t)^{1-\gamma}}dt,\qquad \widetilde{\zeta},\gamma\in\mathbb{R}_+,$$

$$\mathcal{I}^{\gamma}\overline{\mathbb{G}}(\widetilde{\zeta})=\frac{1}{\gamma}\int_r^{\widetilde{\zeta}}\frac{\overline{\mathbb{G}}(t)}{(\widetilde{\zeta}-t)^{1-\gamma}}dt,\qquad \widetilde{\zeta},\gamma\in\mathbb{R}_+.$$

**Lemma 2.** Let $0<\gamma<1$ and $\zeta>\delta\ge 0$ , we get $\zeta^{\gamma}-\delta^{\gamma}\le(\zeta-\delta)^{\gamma}$.

*Proof.* In fact
$\zeta^{\gamma}-\delta^{\gamma}=\gamma\int_{\delta}^{\gamma}y^{\gamma-1}dy$, and $(\zeta-\delta)^{\gamma}=\gamma\int_{\delta}^{\gamma}(y-\delta)^{\gamma-1}dy$.
Due to $\delta\le y\le\zeta, -1<\gamma-1<0$ , we have $y^{\gamma-1}\le(y-\delta)^{\gamma-1}$, then $\zeta^{\gamma}-\delta^{\gamma}\le(\zeta-\delta)^{\gamma}$. □

We need the following hypotheses to prove our results.

**Hypothesis 1** . For each constant $\delta \in [-\xi, \mathcal{T}], \hbar(\delta, \varpi, \varphi) \in E$ is a continuous function in $\varpi, \varphi$, and $\forall \varpi_1, \varpi_2, \varphi_1, \varphi_2 \in E$, the following inequality holds:
$\mathcal{D}_0[\hbar(\delta, \varpi_1, \varphi_1), \hbar(\delta, \varpi_2, \varphi_2)] \leq \lambda(\delta)\mathcal{M}(\mathcal{D}_0[\varpi_1, \varpi_2], \mathcal{D}_0[\varphi_1, \varphi_2])$,
where $\lambda \in y^{\sigma}([-\xi, \mathcal{T}], \mathbb{R}), \mathcal{M} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is increasing, concave and continuous function with $\mathcal{M}(0, 0) = 0$ and $\mathcal{M}(\rho, \rho) = \varsigma \mathcal{M}(\rho), \varsigma$ is a constant.

**Hypothesis 2**. For any $\varpi_1(\rho), \varpi_2(\rho) \in C([0, \mathcal{T}], E)$, we assume that there exists a function $\omega$ such that for $\mathcal{M}(\mathcal{D}_0[\varpi_1(\rho), \varpi_2(\rho)]) : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $\mathcal{M}(\mathcal{D}_0[\varpi_1(\rho), \varpi_2(\rho)]) \leq \omega(\rho)\mathcal{D}_0[\varpi_1(\rho), \varpi_2(\rho)]$ , where $\omega \in C([0, \mathcal{T}], \mathbb{R})$ .

# 3 Existence of the Approximate solutions

Here, we study the approximate solutions of FFDEs under Caputo's H-differentiability by the contraction principle with the help of the equivalent Volterra integral.

**Theorem 3.1** For $0 < \gamma < 1$ a function $\Upsilon^{\sigma}(\zeta) \in C[0, 1]$ satisfies Eq(3.1). Then solutions fuzzy FDEs initial condition:

$$\begin{cases} ({}^{C}_{gH}\mathcal{D}^{\gamma}_{r+}\Upsilon^{\sigma})(\xi) = \mathcal{Q}(\xi, \Upsilon^{\sigma}(\xi), \Upsilon^{\sigma}(\xi - \wp)), \\ \Upsilon^{\sigma}(\xi_0) = \Upsilon^{\sigma}_0 \in E, \end{cases} \tag{3.1}$$

is given by

$$\hat{\Upsilon}^{\sigma}_0(\zeta) = \hat{\Upsilon}^{\sigma}_0, \qquad \hat{\Upsilon}^{\sigma}(\zeta) = \hat{\Upsilon}^{\sigma}_0 + \frac{1}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \hat{\Upsilon}^{\sigma}(\xi), \hat{\Upsilon}^{\sigma}(\xi - \wp))}{(\zeta - \xi)^{1-\gamma}} d\xi, \qquad < \gamma \leq 1, \tag{3.2}$$

and

$$\Upsilon^{\sigma}{}_0(\zeta) = \Upsilon^{\sigma}_0, \qquad \Upsilon^{\sigma}(\zeta) = \Upsilon^{\sigma}_0 \ominus \frac{(-1)}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \Upsilon^{\sigma}(\xi)), \Upsilon^{\sigma}(\xi - \wp)}{(\zeta - \xi)^{1-\gamma}} d\xi, \qquad 0 < \gamma \leq 1. \tag{3.3}$$

where ${}^{C}_{gH}\mathcal{D}^{\gamma}_{r+}$ is the Caputo's $g$H-derivative, $\mathcal{Q} : \mathbb{R}_0 \to E$. In this article, we regard only ${}^{C}[(i) - gH]-$differentiable kind and ${}^{C}[(ii) - gH]-$differentiable kind solutions.

*Proof.* Without losing generality, we prove the case ${}^{C}[(ii) - \gamma]-$differentiability. The second case proof is omitted because it is exactly the same as the first case.

Let $\mathbf{Q} : C([0, \mathcal{T}], E) \longrightarrow C([0, \mathcal{T}], E)$ be a self operator defined by

$$\mathbf{Q}(\Upsilon^{\sigma}(\zeta)) = \Upsilon^{\sigma} \ominus \frac{(-1)}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \Upsilon^{\sigma}(\xi), \Upsilon^{\sigma}(\xi - \wp)}{(\zeta - \xi)^{1-\gamma}} d\xi.$$

For that, we divided the proof into the following steps.

**Step 1**. In the first step, we prove that $\mathbf{Q}$ is completely continuous. For that, we need to establish that

**1)** $\mathbf{Q}$ is continuous. by Hypothesis 1 and Lemma 1, we get

$$\begin{aligned}
\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^\sigma_n(\zeta)), \mathbf{Q}(\mathtt{Y}^\sigma(\zeta))] &= \mathcal{D}_0[\mathtt{Y}^\sigma_0 \ominus \frac{(-1)}{\Gamma(\gamma)}\int_0^\zeta \frac{\mathcal{Q}(\xi, \mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}d\xi, \mathtt{Y}^\sigma_0 \ominus \frac{(-1)}{\Gamma(\gamma)}\int_0^\zeta \frac{\mathcal{Q}(\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}d\xi] \\
&= \mathcal{D}_0[\frac{(1)}{\Gamma(\gamma)}\int_0^\zeta \frac{\mathcal{Q}(\xi, \mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}d\xi, \frac{(1)}{\Gamma(\gamma)}\int_0^\zeta \frac{\mathcal{Q}(\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}d\xi] \\
&\leq \frac{(1)}{\Gamma(\gamma)}\int_0^\zeta \frac{1}{(\zeta-\xi)^{1-\gamma}}\mathcal{D}_0[\mathcal{Q}(\xi, \mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi-\wp)), \mathcal{Q}(\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma(\xi-\wp))]d\xi \\
&\leq \frac{\mathcal{T}^\gamma}{\Gamma(\gamma+1)}\mathcal{D}_0[\mathcal{Q}(\xi, \mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi-\wp)), \mathcal{Q}(\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma(\xi-\wp))] \\
&\leq \frac{\mathcal{T}^\gamma\lambda(\xi)}{\Gamma(\gamma+1)}\mathcal{M}(\mathcal{D}_0[\mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi)], \mathcal{D}_0[\mathtt{Y}^\sigma_n(\xi-\wp), \mathtt{Y}^\sigma(\xi-\wp)]).
\end{aligned} \tag{3.4}$$

With the help of the definition of $\mathcal{D}_0$ , we get

$$\begin{aligned}
&\mathcal{D}_0[\mathtt{Y}^\sigma_n(\xi-\wp), \mathtt{Y}^\sigma(\xi-\wp)] \\
&= sup_{0\leq\mathcal{J}\leq1} max_{0\leq\xi\leq\zeta}\{|\underline{\mathtt{Y}^\sigma}_n(\xi-\wp,\mathcal{J}) - \underline{\mathtt{Y}^\sigma}(\xi-\wp,\mathcal{J})|, |\overline{\mathtt{Y}^\sigma}_n(\xi-\wp,\mathcal{J}) - \overline{\mathtt{Y}^\sigma}(\xi-\wp,\mathcal{J})|\} \\
&\leq sup_{0\leq\mathcal{J}\leq1} max_{-\wp\leq\xi_1\leq\zeta}\{|\underline{X^\sigma}_n(\xi_1,\mathcal{J}) - \underline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|, |\overline{\mathtt{Y}^\sigma}_n(\xi_1,\mathcal{J}) - \overline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|\} \\
&\leq sup_{0\leq\mathcal{J}\leq1} max_{-\wp\leq\xi_1\leq0}\{|\underline{\mathtt{Y}^\sigma}_n(\xi_1,\mathcal{J}) - \underline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|, |\overline{\mathtt{Y}^\sigma}_n(\xi_1,\mathcal{J}) - \overline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|\} \\
&+ sup_{0\leq\mathcal{J}\leq1} max_{-\wp\leq\xi_1\leq\zeta}\{|\underline{\mathtt{Y}^\sigma}_n(\xi_1,\mathcal{J}) - \underline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|, |\overline{\mathtt{Y}^\sigma}_n(\xi_1,\mathcal{J}) - \overline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|\} \\
&= sup_{0\leq\mathcal{J}\leq1} max_{0\leq\xi_1\leq\zeta}\{|\underline{\mathtt{Y}^\sigma}_n(\xi_1,\mathcal{J}) - \underline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|, |\overline{\mathtt{Y}^\sigma}_n(\xi_1,\mathcal{J}) - \overline{\mathtt{Y}^\sigma}(\xi_1,\mathcal{J})|\} \\
&= sup_{0\leq\mathcal{J}\leq1} max_{0\leq\xi\leq\zeta}\{|\underline{\mathtt{Y}^\sigma}_n(\xi,\mathcal{J}) - \underline{\mathtt{Y}^\sigma}(\xi,\mathcal{J})|, |\overline{\mathtt{Y}^\sigma}_n(\xi,\mathcal{J}) - \overline{\mathtt{Y}^\sigma}(\xi,\mathcal{J})|\} \\
&= \mathcal{D}_0[\mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi)].
\end{aligned}$$

Then

$$\begin{aligned}
&\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^\sigma_n(\zeta)), \mathbf{Q}(\mathtt{Y}^\sigma(\zeta))] \\
&\qquad \leq \frac{\mathcal{T}^\gamma\lambda(\xi)}{\Gamma(\gamma+1)}\mathcal{M}(\mathcal{D}_0[\mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi)], \mathcal{D}_0[\mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi)]) \\
&\qquad = \frac{\mathcal{T}^\gamma\lambda(\xi)\varsigma}{\Gamma(\gamma+1)}\mathcal{M}(\mathcal{D}_0[\mathtt{Y}^\sigma_n(\xi), \mathtt{Y}^\sigma(\xi)]).
\end{aligned}$$

□

Since $\mathcal{M}$ is continuous and $\lambda(\xi) \in U$ is bounded, we get $\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^\sigma_n(\zeta)), \mathbf{Q}(\mathtt{Y}^\sigma(\zeta))] \longrightarrow 0$ when $n \longrightarrow \infty$ . So $\mathbf{Q}$ is continuous. **2)** $\mathbf{Q}$ maps bounded set in $C([0,\mathcal{T}], E)$ . We establish that there exists a real positive constant $\eta$, and for $\forall\alpha > 0$ satisfying $\forall\mathtt{Y}^\sigma(\zeta) \in \mathcal{B}_\alpha = \{\mathtt{Y}^\sigma(\zeta) \in C([0,\mathcal{T}], E) : \mathcal{D}_0[\mathtt{Y}^\sigma(\zeta), \hat{0}] \leq \alpha\}$, one has $\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^\sigma(\zeta)), \hat{0}] \leq \eta$. Now $\forall\zeta \in [0,\mathcal{T}]$ and $\forall\mathtt{Y}^\sigma(\zeta) \in \mathcal{B}_{\alpha_1}$, we acquire

$$\begin{aligned}
\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^\sigma(\xi)), \hat{0}] &= \mathcal{D}_0[\mathtt{Y}^\sigma_0 \ominus \frac{-1}{\Gamma(\gamma)}\int_0^\zeta \frac{\mathcal{Q}(\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}d\xi, \hat{0}] \\
&\leq \mathcal{D}_0[\mathtt{Y}^\sigma, \hat{0}] + \mathcal{D}_0[\frac{1}{\Gamma(\gamma)}\int_0^\zeta \frac{\mathcal{Q}(\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}d\xi, \hat{0}] \\
&\leq \mathcal{D}_0[\mathtt{Y}^\sigma, \hat{0}] + \mathcal{D}_0[\frac{1}{\Gamma(\gamma)}\int_0^\zeta \mathcal{D}_0[\mathcal{Q}((\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma)(\xi-\wp)), \hat{0}]d\xi \\
&\leq \mathcal{D}_0[\mathtt{Y}^\sigma, \hat{0}] + \frac{\mathcal{T}^\gamma}{\Gamma(\gamma+1)}\mathcal{D}_0[\mathcal{Q}(\xi, \mathtt{Y}^\sigma(\xi), \mathtt{Y}^\sigma(\xi-\wp)), \hat{0}].
\end{aligned}$$

By the conditions of $\mathcal{Q} \in C([0,\mathcal{T}],E)$ , there is a constant $C_{\mathcal{Q}} > 0$ such that $\mathcal{D}_0[\mathcal{Q}(\zeta,\nu,\mu),\hat{0}] \leq C_{\mathcal{Q}}, \forall(\zeta,\nu,\mu) \in [0,\mathcal{T}] \times E \times E$. Therefore $\mathcal{D}_0[\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp)),\hat{0}] \leq C_{\mathcal{Q}}$. Then we get $\mathcal{D}_0[\mathcal{Q}(\mathtt{Y}^\sigma(\xi)),\hat{0}] \leq \mathcal{D}_0[\mathtt{Y}^\sigma,\hat{0}] + \dfrac{\mathcal{T}^\gamma C_{\mathcal{Q}}}{\Gamma(\gamma+1)} := \eta_1$.

Hence, for every $\mathtt{Y}^\sigma(\zeta) \in \mathcal{B}_{\alpha_1}$, we have $\mathcal{D}_0[\mathcal{Q}(\mathtt{Y}^\sigma(\zeta)),\hat{0}] \leq \eta_1$ , which denotes that $\mathbf{Q}\mathcal{B}_{\alpha_1} \subset \mathcal{B}_{\eta_1}$. **3) Q** maps bounded set into equicontinuous set. For each $\mathtt{Y}^\sigma(\zeta) \in \mathcal{B}_{\alpha_2}, \zeta_1, \zeta_2 \in [0,\mathcal{T}]$ with $\zeta_1 < \zeta_2$ , with help of the Lemma 1 and 2 to yield

$$\begin{aligned}
\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^\sigma(\zeta_1)),\mathbf{Q}(\mathtt{Y}^\sigma(\zeta_2))] &= \mathcal{D}_0[\mathtt{Y}_0^\sigma \ominus \frac{-1}{\Gamma(\gamma)}\int_0^{\zeta_1}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_1-\xi)^{1-\gamma}}d\xi, \mathtt{Y}_0^\sigma \ominus \frac{-1}{\Gamma(\gamma)}\int_0^{\zeta_2}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_2-\xi)^{1-\gamma}}d\xi] \\
&= \mathcal{D}_0[\frac{1}{\Gamma(\gamma)}\int_0^{\zeta_1}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_1-\xi)^{1-\gamma}}d\xi, \frac{1}{\Gamma(\gamma)}\int_0^{\zeta_2}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_2-\xi)^{1-\gamma}}d\xi] \\
&= \mathcal{D}_0[\frac{1}{\Gamma(\gamma)}\int_0^{\zeta_1}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_1-\xi)^{1-\gamma}}d\xi, \frac{1}{\Gamma(\gamma)}\int_0^{\zeta_1}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\tau))}{(\zeta_2-\xi)^{1-\gamma}}d\xi \\
&+ \frac{1}{\Gamma(\gamma)}\int_{\zeta_1}^{\zeta_2}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\tau))}{(\zeta_2-\xi)^{1-\gamma}}d\xi] \\
&\leq \mathcal{D}_0[\frac{1}{\Gamma(\gamma)}\int_0^{\zeta_1}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_1-\xi)^{1-\gamma}}d\xi, \frac{1}{\Gamma(\gamma)}\int_0^{\zeta_1}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_2-\xi)^{1-\gamma}}d\xi] \\
&+ \mathcal{D}_0[\frac{1}{\Gamma(\gamma)}\int_{\zeta_1}^{\zeta_2}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_2-\xi)^{1-\gamma}}d\xi,\hat{0}] \\
&\leq \int_0^{\zeta_1}\mathcal{D}_0[\frac{1}{\Gamma(\gamma)}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_1-\xi)^{1-\gamma}}, \frac{1}{\Gamma(\gamma)}\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp))}{(\zeta_2-\xi)^{1-\gamma}}]d\xi \\
&+ \frac{1}{\Gamma(\gamma)}\int_{\zeta_1}^{\zeta_2}\mathcal{D}_0[\frac{\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\tau))}{(\zeta_2-\xi)^{1-\gamma}},\hat{0}]d\xi \\
&= \int_0^{\zeta_2}|\frac{(\zeta_1-\xi)^{\gamma-1}-(\zeta_2-\xi)^{\gamma-1}}{\Gamma(\gamma)}|\mathcal{D}_0[\mathcal{Q}(\xi,X^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp)),\hat{0}]d\xi \\
&+ \int_{\zeta_1}^{\zeta_2}\frac{(\zeta_2-\xi)^{\gamma-1}}{\Gamma(\gamma)}\mathcal{D}_0[\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp)),\hat{0}]d\xi \\
&= \frac{(\zeta_2-\zeta_1)^\gamma + (\zeta_2^\gamma - \zeta_1^\gamma)}{\Gamma(\gamma+1)}\mathcal{D}_0[\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp)),\hat{0}] \\
&+ \frac{(\zeta_2-\zeta_1)^\gamma}{\Gamma(\gamma)+1}.\mathcal{D}_0[\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp)),\hat{0}] \\
&\leq 3\frac{(\zeta_2-\zeta_1)^\gamma}{\Gamma(\gamma+1)}.\mathcal{D}_0[\mathcal{Q}(\xi,\mathtt{Y}^\sigma(\xi),\mathtt{Y}^\sigma(\xi-\wp)),\hat{0}] \\
&\leq 3C_{\mathcal{Q}}\frac{(\zeta_2-\zeta_1)^\gamma}{\Gamma(\gamma+1)}.
\end{aligned}$$

We can see that $3C_{\mathcal{Q}}\dfrac{(\zeta_2-\zeta_1)^\gamma}{\Gamma(\gamma+1)}$ is independent of $\mathtt{Y}^\sigma(\zeta) \in \mathcal{B}_{\alpha_2}$ and $3C_{\mathcal{Q}}\dfrac{(\zeta_2-\zeta_1)^\gamma}{\Gamma(\gamma+1)} \longrightarrow 0$ when $\zeta_2 \longrightarrow \zeta_1$. It means that $\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^\sigma(\zeta_1)),\mathbf{Q}(\mathtt{Y}^\sigma(\zeta_2))] \longrightarrow 0$. Therefore, the set $\mathbf{Q}\mathcal{B}_{\alpha_2}$ is equicontinuous. With the help of the Arzela- Ascoli Theory, it's easy to know that **Q** is completely continuous.

**Step 2**. We will establish that there is a closed, convex, and bounded subset $\mathcal{B}_{\alpha'} = \{\mathtt{Y}^\sigma(\zeta) \in C([0,T],E) : \mathcal{D}_0[\mathtt{Y}^\sigma(\zeta),\hat{0}] \leq \alpha'\}$ such that $\mathbf{Q}\mathcal{B}_{\alpha'} \subseteq \mathcal{B}_{\alpha'}$ . We know that $\mathcal{B}_{\alpha'}$ is a closed, convex, and bounded subset of $C([0,\mathcal{T}],E)$ for $\forall\alpha' \in \mathbb{R}^+$. If for $\forall\alpha' \in \mathbb{R}^+$ , there exists $\mathtt{Y}_{\alpha'}^\sigma(\zeta) \in \mathcal{B}_{\alpha'}$ such that $(\mathbf{Q}(\mathtt{Y}_{\alpha'}^\sigma)) \notin \mathcal{B}_{\alpha'}$ that is

$\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^{\sigma}_{\alpha'}), \hat{0}] > \alpha'$, then

$$\begin{aligned}
\alpha' < \mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^{\sigma}_{\alpha'}), \hat{0}] &= \mathcal{D}_0[\mathtt{Y}^{\sigma}_0 \ominus \frac{-1}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_{\alpha'}(\xi), \mathtt{Y}^{\sigma}_{\alpha'}(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}, \hat{0}]d\xi \\
&\leq \mathcal{D}_0[\mathtt{Y}^{\sigma}_0, \hat{0}] + \frac{1}{\Gamma(\gamma)} \int_0^{\zeta} \mathcal{D}_0[\frac{\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_{\alpha'}(\xi), \mathtt{Y}^{\sigma}_{\alpha'}(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}}, \hat{0}]d\xi \\
&\leq \mathcal{D}_0[\mathtt{Y}^{\sigma}_0, \hat{0}] + \frac{\mathcal{T}^{\gamma} C_{\mathcal{Q}}}{\Gamma(\gamma+1)}.
\end{aligned}$$

Taking limit as $\alpha' \longrightarrow +\infty$ , it's immediate to get $\mathcal{D}_0[\mathtt{Y}^{\sigma}_0, \hat{0}] + \frac{\mathcal{T}^{\gamma} C_{\mathcal{Q}}}{\Gamma(\gamma+1)} \longrightarrow +\infty$, which is conflict to $\mathcal{D}_0[\mathtt{Y}^{\sigma}_0, \hat{0}] + \frac{\mathcal{T}^{\gamma} C_{\mathcal{Q}}}{\Gamma(\gamma+1)}$ bounded. Thus, for every positive constant $\alpha'$ , we get $\mathbf{Q}\mathcal{B}_{\alpha'} \subseteq \mathcal{B}_{\alpha'}$ . Using Schauder's fixed point Theorem there exist at least one solution of the Eq. (3.1) . The proof is completed.

**Theorem 3.2.** Presume that Hypotheses 1-2 hold. If $sup_{0\leq t\leq \mathcal{T}}[\lambda(\xi)\omega(\xi)] < \frac{\Gamma(\gamma+1)}{\varsigma \mathcal{T}^{\gamma}}$, then the solution to the fuzzy system (3.1) is unique.

*Proof.* The function $\mathtt{Y}^{\sigma}(\zeta)$ is a solution to equation (3.1) if

$$\mathbf{Q}(\mathtt{Y}^{\sigma}(\zeta)) = \mathtt{Y}^{\sigma} \ominus \frac{(-1)}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}(\xi), \mathtt{Y}^{\sigma}(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}} d\xi.$$

hold. If $\mathtt{Y}^{\sigma}(\zeta) \in C([0,\mathcal{T}], E)$ is a fixed point of $\mathbf{Q}$ which define as in the Theorem 3.1 , then $\mathtt{Y}^{\sigma}(\zeta)$ is the solution of the equation (3.1) . Let $\mathtt{Y}^{\sigma}_1(\zeta), \mathtt{Y}^{\sigma}_2(\zeta) \in C([0,\mathcal{T}], E)$ and $\zeta \in [-\xi, 0], \mathtt{Y}^{\sigma}_1(\zeta) = \mathtt{Y}^{\sigma}_2(\zeta) = \mathtt{Y}^{\sigma}_0(\zeta)$ . For $\forall \zeta \in [0, \mathcal{T}]$, we can get

$$\begin{aligned}
\mathcal{D}_0[\mathbf{Q}(\mathtt{Y}^{\sigma}_1)), \mathbf{Q}(\mathtt{Y}^{\sigma}_2))] =& \mathcal{D}_0[\mathtt{Y}^{\sigma}_0 \ominus \frac{-1}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_1(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}} d\xi, \\
& \mathtt{Y}^{\sigma}_0 \ominus \frac{-1}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_2(\xi), \mathtt{Y}^{\sigma}_2(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}} d\xi] \\
=& \mathcal{D}_0[\frac{1}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_1(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}} d\xi, \\
& \frac{1}{\Gamma(\gamma)} \int_0^{\zeta} \frac{\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_2(\xi), \mathtt{Y}^{\sigma}_2(\xi-\wp))}{(\zeta-\xi)^{1-\gamma}} d\xi] \\
\leq& \int_0^{\zeta} \frac{(\zeta-\xi)^{\gamma-1}}{\Gamma(\gamma)} \mathcal{D}_0[\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_1(\xi-\wp)), \mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_2(\xi), \mathtt{Y}^{\sigma}_2(\xi-\wp))]d\xi \\
\leq& \frac{\mathcal{T}^{\gamma}}{\Gamma(\gamma+1)} \mathcal{D}_0[\mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_1(\xi-\wp)), \mathcal{Q}(\xi, \mathtt{Y}^{\sigma}_2(\xi), \mathtt{Y}^{\sigma}_2(\xi-\wp))] \\
\leq& \frac{\mathcal{T}^{\gamma}\lambda(\xi)}{\Gamma(\gamma+1)} \mathcal{M}(\mathcal{D}_0[\mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_2(\xi)], \mathcal{D}_0[\mathtt{Y}^{\sigma}_1(\xi-\wp), \mathtt{Y}^{\sigma}_2(\xi-\wp)]) \\
\leq& \frac{\mathcal{T}^{\gamma}\lambda(\xi)}{\Gamma(\gamma+1)} \mathcal{M}(\mathcal{D}_0[\mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_2(\xi)], \mathcal{D}_0[\mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_2(\xi)]) \\
\leq& \frac{\mathcal{T}^{\gamma}\lambda(\xi)}{\Gamma(\gamma+1)} \mathcal{M}(\mathcal{D}_0[\mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_2(\xi)]) \\
\leq& \frac{\mathcal{T}^{\gamma}\lambda(\xi)}{\Gamma(\gamma+1)} \mathcal{M}(\zeta)(\mathcal{D}_0[\mathtt{Y}^{\sigma}_1(\xi), \mathtt{Y}^{\sigma}_2(\xi)]).
\end{aligned}$$

Hence, $\mathbf{Q}$ has a unique fixed point $\mathtt{Y}^{\sigma}(\zeta) \in C([0,\mathcal{T}], E)$ based on the Banach contraction principle. Therefore, the proof of the Theorem is completed □

## 4 The Predictor-Corrector Method

It is far more complex than the standard integer order to solve FFDEs accurately, reliably, and efficiently. Most computer tools do not supply built-in functions for these types of problems. This article investigates the Predictor-Corrector method as the most effective numerical approach for fractional-order problems, to find the approximate solutions for FFDEs. For clarity and without losing generality, we consider the following FFDEs with the initial condition defined by

$$\begin{cases} ({}^{C}\mathcal{D}^{\gamma}_{0+}\mathtt{Y})(\widetilde{t}) = \mathcal{G}(\widetilde{t}, \mathtt{Y}(\widetilde{t})), & \gamma > 0, \\ \mathtt{Y}(0) = \mathtt{Y}_0, \quad \widetilde{t} \in [0,1], \end{cases} \tag{4.1}$$

the solution to Eq.(4.1) is define on $[0, \mathcal{T}]$, where $\mathcal{T}$ is a non-negative number. It is common knowledge that a fractional differential operator can be defined in various ways. The special operator ${}^{C}\mathcal{D}^{\gamma}_{0+}$ that we are utilizing in (4.1) is defined by

$${}^{C}\mathcal{D}^{\gamma}_{0+} = \mathcal{J}^{n-\gamma}\mathtt{Y}^{(n)}(\widetilde{t}). \tag{4.2}$$

We will resolve it using two kinds of Caputo fractional gH-derivative. Thus, based on theorem 2.1, we have two cases.

**Case 1**. If $\mathtt{Y}(\widetilde{t})$ is $[(i)-gH]^{C}_{\gamma}-$ differentiable then $[({}^{c}\mathcal{D}^{\gamma}_{0+}\mathtt{Y})(\widetilde{t})]^{\varsigma} = [{}^{c}\mathcal{D}^{\gamma}_{0+}\underline{\mathtt{Y}}(\widetilde{t},\varsigma), {}^{c}\mathcal{D}^{\gamma}_{0+}\overline{\mathtt{Y}}(\widetilde{t},\varsigma)]$ and (4.1) is summarized into the following FFDEs system:

$$\begin{cases} {}^{c}\mathcal{D}^{\gamma}_{0+}\underline{\mathtt{Y}}(\widetilde{t},\varsigma) = \underline{\mathcal{G}}(\widetilde{t},\varsigma,\underline{\mathtt{Y}}(\widetilde{t},\varsigma),\overline{\mathtt{Y}}(\widetilde{t},\varsigma)), & \widetilde{t} \in [0,1], \\ {}^{c}\mathcal{D}^{\gamma}_{0+}\overline{\mathtt{Y}}(\widetilde{t},\varsigma) = \overline{\mathcal{G}}(\widetilde{t},\varsigma,\overline{\mathtt{Y}}(\widetilde{t},\varsigma),\underline{\mathtt{Y}}(\widetilde{t},\varsigma)), & \widetilde{t} \in [0,1], \\ \underline{\mathtt{Y}}(\widetilde{t},\varsigma) = \underline{\phi}(\widetilde{t},\varsigma), \quad \overline{\mathtt{Y}}(\widetilde{t},\varsigma) = \overline{\phi}(\widetilde{t},\varsigma). \end{cases} \tag{4.3}$$

**Case 2**. If $\mathtt{Y}(\widetilde{t})$ is $[(ii)-gH]^{C}_{\gamma}-$ differentiable then $[({}^{c}\mathcal{D}^{\gamma}_{0+}\mathtt{Y})(\widetilde{t})]^{\varsigma} = [{}^{c}\mathcal{D}^{\gamma}_{0+}\overline{\mathtt{Y}}(\widetilde{t},\varsigma), {}^{c}\mathcal{D}^{\gamma}_{0+}\underline{\mathtt{Y}}(\widetilde{t},\varsigma)]$ and (4.1) is summarized into the following FFDEs system:

$$\begin{cases} {}^{c}\mathcal{D}^{\gamma}_{0+}\overline{\mathtt{Y}}(\widetilde{t},\varsigma) = \underline{\mathcal{G}}(\widetilde{t},\varsigma,\underline{\mathtt{Y}}(\widetilde{t},\varsigma),\overline{\mathtt{Y}}(\widetilde{t},\varsigma)), & \widetilde{t} \in [0,1], \\ {}^{c}\mathcal{D}^{\gamma}_{0+}\underline{\mathtt{Y}}(\widetilde{t},\varsigma) = \overline{\mathcal{G}}(\widetilde{t},\varsigma,\overline{\mathtt{Y}}(\widetilde{t},\varsigma),\underline{\mathtt{Y}}(\widetilde{t},\varsigma)), & \widetilde{t} \in [0,1], \\ \underline{\mathtt{Y}}(\widetilde{t},\varsigma) = \underline{\phi}(\widetilde{t},\varsigma), \quad \overline{\mathtt{Y}}(\widetilde{t},\varsigma) = \overline{\phi}(\widetilde{t},\varsigma). \end{cases} \tag{4.4}$$

Remark 4.1. If we make sure that the solutions $(\underline{\mathtt{Y}}(\widetilde{t},\varsigma), \overline{\mathtt{Y}}(\widetilde{t},\varsigma))$ of our systems of FFDEs (4.3) and (4.4) are valid sets of fractional differential functions and if the derivatives $({}^{C}\mathcal{D}^{\gamma}_{0+}\underline{\mathtt{Y}}(\widetilde{t},\varsigma), {}^{C}\mathcal{D}^{\gamma}_{0+}\overline{\mathtt{Y}}(\widetilde{t},\varsigma)$ are valid sets of fractional differential functions with two types differentiability, we can create the solutions of the FFDEs (4.1).

In many of the articles, the numerical solutions of the FFDEs are investigated. For example, Mazandarani et al. [16] suggested the modified fractional Euler approach (MFEP) for solving FFDEs under Caputo fuzzy derivatives. The MFEP based on a modified trapezoidal rule and generalized Taylor's formula is used for solving FFDEs of order $\gamma \in (0,1)$. Van Hoa et al. studied numerical solutions for FFDEs by a modified Adams-Bashforth-Moulton approach in [38].

In this article, we use Predictor-Corrector approach to find the numerical solutions to FFDEs under the Caputo gH-differentiability.

## 4.1 The Predictor-Corrector for FFDEs

This subsection suggests the Predictor-Corrector approach for solving FFDEs under the Caputo fractional derivative. To motivate our numerical method, before presenting this method, we will briefly recall the notion behind Predictor-Corrector approach for ODEs.
Our method is based on the analytical condition that the IVP (4.1) is equivalent to the Volterra-integral equation.

$$\mathtt{Y}(\widetilde{t}) = \sum_{i=1}^{\lceil\gamma\rceil-1} \mathtt{Y}_0^{(i)} \frac{\widetilde{t}^i}{i!} + \frac{1}{\Gamma(\gamma)} \int_0^{\widetilde{t}} (\widetilde{t} - \widetilde{\mu})^{\gamma-1} \mathcal{G}(\widetilde{\mu}, \mathtt{Y}(\widetilde{\mu})) d\widetilde{\mu}, \tag{4.5}$$

where $n := \lceil\gamma\rceil$ is only the value rounded to the closest integer In this Eq (4.1).
For a start, we present a view of the IVP for the first order differential equations

$$\begin{cases} (\mathcal{D}\mathtt{Y})(\widetilde{t}) = \mathcal{G}(\widetilde{t}, \mathtt{Y}(\widetilde{t})), \\ \mathtt{Y} = \mathtt{Y}_0, \end{cases} \tag{4.6}$$

we suppose the function $\mathcal{G}$ is such that a unique solution on $[0, \mathcal{T}]$, suppose that we are using a uniform grid
$\widetilde{t}_m = mh : m = 0, 1, ..., M$ and $h = \dfrac{\mathcal{T}}{M}$ be the time step. The primary notion is, supposing that we have actually calculated approximations $\mathtt{Y}_h(\widetilde{t}_k) \approx \mathtt{Y}(\widetilde{t}_k)(k = 1, 2, ..., m)$, that we are striving to get the approximation $\mathtt{Y}_h(\widetilde{t}_{m+1})$ by this equation

$$\mathtt{Y}(\widetilde{t}_{m+1}) = \mathtt{Y}(\widetilde{t}_m) + \int_{\widetilde{t}_m}^{\widetilde{t}_{m+1}} \mathcal{G}(\zeta, \mathtt{Y}(\zeta)) d\zeta. \tag{4.7}$$

This equation follows upon integrating (4.6) on $[\widetilde{t}_m, \widetilde{t}_{m+1}]$. The two-point trapezoidal quadrature formula then replaces the integral on the righthand side of (4.7)

$$\int_{\widetilde{t}_m}^{\widetilde{t}_{m+1}} \mathcal{G}(\zeta, \mathtt{Y}(\zeta)) d\zeta \approx \frac{h}{2}(\mathcal{G}(\widetilde{t}_m, \mathtt{Y}(\widetilde{t}_m)) + \mathcal{G}(\widetilde{t}_{m+1}, \mathtt{Y}(\widetilde{t}_{m+1}))). \tag{4.8}$$

Therefore,
providing an equation for the unknown approximation $\mathtt{Y}_h(\widetilde{t}_{m+1})$, as follows:

$$\mathtt{Y}_h(\widetilde{t}_{m+1}) = \mathtt{Y}_h(\widetilde{t}_m) + \frac{h}{2}[\mathcal{G}(\widetilde{t}_m, \mathtt{Y}(\widetilde{t}_m)) + \mathcal{G}(\widetilde{t}_{m+1}, \mathtt{Y}(\widetilde{t}_{m+1}))]. \tag{4.9}$$

In the implicit equation for the Adams-Moulton method, we have to replace $\mathtt{Y}(\widetilde{t}_m)$ and $\mathtt{Y}(\widetilde{t}_{m+1})$ with their approximations $\mathtt{Y}_h(\widetilde{t}_m)$ and $\mathtt{Y}_h(\widetilde{t}_{m+1})$, as following

$$\mathtt{Y}_h(\widetilde{t}_{m+1}) = \mathtt{Y}_h(\widetilde{t}_m) + \frac{h}{2}[\mathcal{G}(\widetilde{t}_m, \mathtt{Y}_h(\widetilde{t}_m)) + \mathcal{G}(\widetilde{t}_{m+1}, \mathtt{Y}_h(\widetilde{t}_{m+1}))]. \tag{4.10}$$

The unknown quantity $\mathtt{Y}_h(\widetilde{t}_{m+1})$ on both sides appears. That is the problem in this equation, and due to the nature of the non-linear function $\mathcal{G}$. In general, we can't directly resolve for $\mathtt{Y}_h(\widetilde{t}_{m+1})$. So, we may utilize Eq (4.10) in an iterative process, by placing an initial approximation for $\mathtt{Y}_h^p(\widetilde{t}_{m+1})$ on both sides, we can then utilize that to determine a better approximation. The preliminary approximation t is called a predictor and can be obtained in another way, only putting the rectangle rule instead of the trapezoidal quadrature formula

$$\int_{\widetilde{t}_m}^{\widetilde{t}_{m+1}} \mathcal{G}(\zeta) d\zeta \approx (\widetilde{t}_{m+1} - \widetilde{t}_m)\mathcal{G}(\widetilde{t}_m), \tag{4.11}$$

giving the explicit method

$$\mathtt{Y}_h^p(\widetilde{t}_{m+1}) = \mathtt{Y}_h(\widetilde{t}_m) + h\mathcal{G}(\widetilde{t}_m, \mathtt{Y}_h(\widetilde{t}_m)). \tag{4.12}$$

Known as the one-step Adams-Bashforth method. It's well known that the process described by Eq (4.12) and

$$\mathtt{Y}_h(\widetilde{t}_{m+1}) = \mathtt{Y}_h(\widetilde{t}_m) + \frac{h}{2}[\mathcal{G}(\widetilde{t}_m, \mathtt{Y}_h(\widetilde{t}_m)) + \mathcal{G}(\widetilde{t}_{m+1}, \mathtt{Y}_h^p(\widetilde{t}_{m+1}))]. \tag{4.13}$$

In which (4.13) is called a corrector, it's well known that this approach is convergent of order 2, i.e., $\max_{m=0,1,...,M}|\mathtt{Y}(\widetilde{t}_m) - \mathtt{Y}_h(\widetilde{t}_m)|= O(h^2)$.
To begin, we would compute the predictor in Eq (4.12), then evaluate $\mathcal{G}(\widetilde{t}_{m+1}, \mathtt{Y}_h^p(\widetilde{t}_{m+1}))$, utilize this to calculate the corrector in Eq(4.13), and finally evaluate $\mathcal{G}(\widetilde{t}_{m+1}, \mathtt{Y}_h(\widetilde{t}_{m+1}))$. After introducing this approach, we attempt to apply the essential concepts to the fractional-order problem while making some unavoidable changes.
We utilize the integral to replace the product trapezoidal quadrature formula, where nodes $\widetilde{t}_k(k = 0, 1, ..., m+1)$ are taken concerning the weight function $(\widetilde{t}_{m+1})^{\gamma-1}$. In other form, we apply the following approximation

$$\int_0^{\widetilde{t}_{m+1}} (\widetilde{t}_{m+1} - \zeta)^{\gamma-1}\mathcal{G}(\zeta)d\zeta \approx \int_0^{\widetilde{t}_{m+1}} (\widetilde{t}_{m+1} - \zeta)^{\gamma-1}\widetilde{\mathcal{G}}_{m+1}(\zeta)d\zeta, \tag{4.14}$$

where $\widetilde{\mathcal{G}}_{m+1}$ is the piecewise linear-interpolant for $\mathcal{G}$ with nodes the $\widetilde{t}_k, k = 0, 1, 2, ..., m+1$. Using quadrature theory, we see that we can express the integral of (4.14) as

$$\int_0^{\widetilde{t}_{m+1}} (\widetilde{t}_{m+1} - \zeta)^{\gamma-1}\widetilde{\mathcal{G}}_{m+1}(\zeta)d\zeta = \frac{h^\gamma}{\gamma(\gamma+1)} \sum_{k=0}^{m+1} a_{k,m+1}^{(\gamma)}\mathcal{G}(\widetilde{t}_k),$$

where

$$a_{k,m+1}^{(\gamma)} = \begin{cases} m^{\gamma+1} - (m-\gamma)(m+1)^\gamma, & \text{if} \quad k = 0, \\ (m-k+2)^{\gamma+1} + (m-k)^{m+1} - 2(m-k+1)^{\gamma+1}, & \text{if} \quad 1 \le k \le m, \\ 1, & \text{if} \quad k = m+1. \end{cases} \tag{4.15}$$

For example, it's necessary and sufficient to approximate each interval to extend to FDEs (implicit) backward Euler methods and the (explicit) forward $[\widetilde{t}_k, \widetilde{t}_{k+1}]$, one gets a generalization (of the implicit kind) of the standard trapezoidal rule:

$$\text{implicit. Trapezoidal :} \qquad \mathtt{Y}_{m+1} = \sum_{i=1}^{\lceil\gamma\rceil-1} \frac{\widetilde{t}_{m+1}^i}{i!}\mathtt{Y}_0^{(i)} + \frac{h^\gamma}{\Gamma(\gamma+2)} \sum_{k=0}^{m} a_{k,m+1}^{(\gamma)}\mathcal{G}(\widetilde{t}_k, \mathtt{Y}_k) \tag{4.16}$$

As a result, we are given the corrector formula, which is the fractional version of the one-step Adams-Moulton approach

$$\mathtt{Y}_{m+1} = \sum_{i=1}^{\lceil\gamma\rceil-1} \frac{\widetilde{t}_{m+1}^i}{i!}\mathtt{Y}_0^{(i)} + \frac{h^\gamma}{\Gamma(\gamma+2)}\mathcal{G}(\widetilde{t}_{m+1}, \mathtt{Y}_{m+1}^p) + \frac{h^\gamma}{\Gamma(\gamma+2)} \sum_{k=0}^{m} a_{k,m+1}^{(\gamma)}\mathcal{G}(\widetilde{t}_k, \mathtt{Y}_k), \tag{4.17}$$

The residual problem is determining the predictor formula required to calculate the value $\mathtt{Y}_h^p(\widetilde{t}_{m+1})$. The notion we utilize to generalize the one-step ABM is identical to that previously defined for the ABM approach: we replace the integral of Eq (4.5) by the rectangle rule

$$\int_0^{\widetilde{t}_{m+1}} (\widetilde{t}_{m+1} - \zeta)^{\gamma-1}\mathcal{G}(\zeta)d\zeta \approx \sum_{k=0}^{m} b_{k,m+1}^{(\gamma)}\mathcal{G}(\widetilde{t}_k),$$

where now

$$b_{k,m+1}^{(\gamma)}=\frac{h^{\gamma}}{\gamma}((m+1-k)^{\gamma}-(m-k)^{\gamma}). \tag{4.18}$$

Consequently, the predicted value $\mathtt{Y}_{m+1}^{p}$ is determined by the fractional ABM

$$\text{explicit. Rectangular :}\qquad \mathtt{Y}_{m+1}^{p}=\sum_{i=1}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\mathtt{Y}_0^{(i)}+\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{m-1}b_{k,m-1}^{(\gamma)}\mathcal{G}(\widetilde{t}_k,\mathtt{Y}_k), \tag{4.19}$$

$$\text{implicit. Rectangular :}\qquad \mathtt{Y}_{m+1}^{p}=\sum_{i=0}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\mathtt{Y}_0^{(i)}+\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{m}b_{k,m+1}^{(\gamma)}\mathcal{G}(\widetilde{t}_k,\mathtt{Y}_{m+1}). \tag{4.20}$$

The fractional ABM is fully represented now by Eq (4.17) and (4.19), with the weights $a_{k,m+1}^{(\gamma)}$ and $b_{k,m+1}^{(\gamma)}$ being determined according to (4.15) and (4.18), respectively. After completing our numerical algorithm for the fractional differential. We may anticipate the error to behave as

$$\max_{m=0,1,\ldots,M}|\mathtt{Y}(\widetilde{t}_m)-\mathtt{Y}_h(\widetilde{t}_m)|=O(h^{P}). \tag{4.21}$$

Where $P=min(2,1+\gamma)$.

To follow our work, a Predictor-Corrector method for solving the fuzzy IVP of FFDEs (4.1) will be offered. Using the modified trapezoidal rule, the numerical scheme in (4.3) and (4.4) to evaluate the Corrector can be described as:

$$\begin{cases}\underline{\mathtt{Y}}_{m+1}=\sum_{i=1}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\underline{\mathtt{Y}}_0^{(i)}+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\underline{\mathcal{G}}(\widetilde{t}_{m+1},\underline{\mathtt{Y}}_{m+1}^{p},\overline{\mathtt{Y}}_{m+1}^{p})\\ \quad+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\sum_{k=0}^{m}a_{k,m+1}^{(\gamma)}\underline{\mathcal{G}}(\widetilde{t}_k,\underline{\mathtt{Y}}_k,\overline{\mathtt{Y}}_k),\\ \overline{\mathtt{Y}}_{m+1}=\sum_{i=1}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\overline{\mathtt{Y}}_0^{(i)}+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\overline{\mathcal{G}}(\widetilde{t}_{m+1},\underline{\mathtt{Y}}_{m+1}^{p},\overline{\mathtt{Y}}_{m+1}^{p})\\ \quad+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\sum_{k=0}^{m}a_{k,m+1}^{(\gamma)}\overline{\mathcal{G}}(\widetilde{t}_k,\underline{\mathtt{Y}}_k,\overline{\mathtt{Y}}_k),\\ \widetilde{t}\in[0,\mathcal{T}],\gamma\in(0,1),\end{cases} \tag{4.22}$$

for Case 1, and

$$\begin{cases}\underline{\mathtt{Y}}_{m+1}=\sum_{i=1}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\underline{\mathtt{Y}}_0^{(i)}+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\overline{\mathcal{G}}(\widetilde{t}_{m+1},\underline{\mathtt{Y}}_{m+1}^{p},\overline{\mathtt{Y}}_{m+1}^{p})\\ \quad+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\sum_{k=0}^{m}a_{k,m+1}^{(\gamma)}\overline{\mathcal{G}}(\widetilde{t}_k,\underline{\mathtt{Y}}_k,\overline{\mathtt{Y}}_k),\\ \overline{\mathtt{Y}}_{m+1}=\sum_{i=1}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\overline{\mathtt{Y}}_0^{(i)}+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\underline{\mathcal{G}}(\widetilde{t}_{m+1},\underline{\mathtt{Y}}_{m+1}^{p},\overline{\mathtt{Y}}_{m+1}^{p})\\ \quad+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\sum_{k=0}^{m}a_{k,m+1}^{(\gamma)}\underline{\mathcal{G}}(\widetilde{t}_k,\underline{\mathtt{Y}}_k,\overline{\mathtt{Y}}_k),\\ \widetilde{t}\in[0,\mathcal{T}],\gamma\in(0,1),\end{cases} \tag{4.23}$$

for Case 2.

The approximations $\underline{\mathtt{Y}}_{m+1}^{p}$ and $\overline{\mathtt{Y}}_{m+1}^{p}$ are utilized in (4.23) and (4.24) to evaluate predictor terms

$$\text{implicit. Rectangular :}\begin{cases}\underline{\mathtt{Y}}_{m+1}^{p}=\sum_{i=0}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\underline{\mathtt{Y}}_0^{(i)}+\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{m}b_{k,m+1}^{(\gamma)}\underline{\mathcal{G}}(\widetilde{t}_k,\underline{\mathtt{Y}}_k,\overline{\mathtt{Y}}_k),\\ \overline{\mathtt{Y}}_{m+1}^{p}=\sum_{i=0}^{\lceil\gamma\rceil-1}\frac{\widetilde{t}_{m+1}^{i}}{i!}\overline{\mathtt{Y}}_0^{(i)}+\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{m}b_{k,m+1}^{(\gamma)}\overline{\mathcal{G}}(\widetilde{t}_k,\underline{\mathtt{Y}}_k,\overline{\mathtt{Y}}_k),\end{cases} \tag{4.24}$$

for Case 1, and

$$\text{implicit. Rectangular}: \begin{cases} \underline{\mathtt{Y}}^{p}_{m+1} = \sum_{i=0}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\underline{\mathtt{Y}}_0^{(i)} + \dfrac{1}{\Gamma(\gamma)} \sum_{k=0}^{m} b^{(\gamma)}_{k,m+1}\overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}^{p}_{m+1} = \sum_{i=0}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\overline{\mathtt{Y}}_0^{(i)} + \dfrac{1}{\Gamma(\gamma)} \sum_{k=0}^{m} b^{(\gamma)}_{k,m+1}\underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.25}$$

for Case 2.

To extend our work to FFDEs, the (explicit. Rectangular ) and (implicit. Trapezoidal ) Euler methods it is suitable to approximate in $[\widetilde{t}_m, \widetilde{t}_{m+1}]$

$$\text{explicit. Rectangular}: \begin{cases} \underline{\mathtt{Y}}^{p}_{m+1} = \sum_{i=0}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\underline{\mathtt{Y}}_0^{(i)} + \dfrac{1}{\Gamma(\gamma)} \sum_{k=0}^{m} b^{(\gamma)}_{k,m+1}\underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}^{p}_{m+1} = \sum_{i=0}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\overline{\mathcal{Y}}_0^{(i)} + \dfrac{1}{\Gamma(\gamma)} \sum_{k=0}^{m} b^{(\gamma)}_{k,m+1}\overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.26}$$

for Case 1, and

$$\text{explicit. Rectangular}: \begin{cases} \underline{\mathtt{Y}}^{p}_{m+1} = \sum_{i=0}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\underline{\mathtt{Y}}_0^{(i)} + \dfrac{1}{\Gamma(\gamma)} \sum_{k=0}^{m-1} b^{(\gamma)}_{k,m-1}\overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}^{p}_{m+1} = \sum_{i=0}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\overline{\mathtt{Y}}_0^{(i)} + \dfrac{1}{\Gamma(\gamma)} \sum_{k=0}^{m-1} b^{(\gamma)}_{k,m-1}\underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.27}$$

for Case 2.

$$\text{implicit. Trapezoidal}: \begin{cases} \underline{\mathtt{Y}}_{m+1} = \sum_{i=1}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\underline{\mathtt{Y}}_0^{(i)} + \dfrac{h^{\gamma}}{\Gamma(\gamma+2)} \sum_{k=0}^{m} a^{(\gamma)}_{k,m+1}\underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}_{m+1} = \sum_{i=1}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\overline{\mathtt{Y}}_0^{(i)} + \dfrac{h^{\gamma}}{\Gamma(\gamma+2)} \sum_{k=0}^{m} a^{(\gamma)}_{k,m+1}\overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.28}$$

for Case 1.

$$\text{implicit. Trapezoidal}: \begin{cases} \underline{\mathtt{Y}}_{m+1} = \sum_{i=1}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\underline{\mathtt{Y}}_0^{(i)} + \dfrac{h^{\gamma}}{\Gamma(\gamma+2)} \sum_{k=0}^{m} a^{(\gamma)}_{k,m+1}\overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}_{m+1} = \sum_{i=1}^{\lceil\gamma\rceil-1} \dfrac{\widetilde{t}^{i}_{m+1}}{i!}\overline{\mathtt{Y}}_0^{(i)} + \dfrac{h^{\gamma}}{\Gamma(\gamma+2)} \sum_{k=0}^{m} a^{(\gamma)}_{k,m+1}\underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.29}$$

for Case 2.

## 4.2 The Predictor-Corrector Method for Linear Multi-Term fuzzy FDEs

A further particular case of FDEs is when more than one fractional-derivative appears in a single equation. These kinds of equations are called multiterm FDEs, and the linear form, they are defined as:

$$\eta_\sigma \mathcal{D}^{\gamma_\sigma}_{\widetilde{t}_0}\mathtt{Y}(\widetilde{t}) + \eta_{\sigma-1}\mathcal{D}^{\gamma_{\sigma-1}}_{\widetilde{t}_0}\mathtt{Y}(\widetilde{t}) + ... + \eta_2\mathcal{D}^{\gamma_2}_{\widetilde{t}_0}\mathtt{Y}(\widetilde{t}) + \eta_1\mathcal{D}^{\gamma_1}_{\widetilde{t}_0}\mathtt{Y}(\widetilde{t}) = \mathcal{G}(\widetilde{t}, \mathtt{Y}(\widetilde{t})), \tag{4.30}$$

where $\eta_1, \eta_2, ..., \eta_{\sigma-1}, \eta_\sigma$ are some coefficients and the orders $\gamma_1, \gamma_2, ..., \gamma_{\sigma-1}, \gamma_\sigma$ of the fractional derivatives are supposed to be sorted in ascending order, $i.e., 0 < \gamma_1 < \gamma_2 < ... < \gamma_{\sigma-1} < \gamma_\sigma$, with $\eta_\sigma \neq 0$. Here, we focus on multiterm FDEs, which are linear concerning the fractional derivatives but may be nonlinear of $\mathcal{G}(\widetilde{t}, \mathcal{Z})$. Some of the initial conditions are defined by $m_\sigma$, where $m_i = \lceil\gamma_i\rceil$, $i = 1, ..., \sigma$ and they are expressed as derivatives of the solution:

$$\mathtt{Y}(\widetilde{t}_0) = \mathtt{Y}_0, \qquad \frac{d}{d\widetilde{t}}\mathtt{Y}(\widetilde{t}_0) = \mathtt{Y}_0^{(1)}, ..., \frac{d^m\sigma^{-1}}{d\widetilde{t}^m\sigma^{-1}}\mathtt{Y}(\widetilde{t}_0) = \mathtt{Y}_0^{(m_{\sigma-1})}.$$

Multiterm FDEs are more challenging to resolve than FDEs, as suggested in [53,54]. It's possible to rewrite Eq (4.31) in a way that some of the methods for FDEs can be smoothly adapted. In fact, through Equation (4.5) and by applying $\mathcal{J}_{\widetilde{t}_0}^{\gamma_\sigma}$ to Eq (4.31), we can reformulate the multiterm FDEs as:

$$\mathtt{Y}(\widetilde{t}) = \sum_{i=0}^{\lceil\gamma\rceil_\sigma - 1} \frac{\widetilde{t}^i}{i!}\mathtt{Y}_0^{(i)} - \sum_{j=1}^{\sigma-1} \frac{\eta_j}{\eta_\sigma} \mathcal{J}_{\widetilde{t}_0}^{\gamma_\sigma - \gamma_j} [\mathtt{Y}(\widetilde{t}) - \sum_{i=0}^{\lceil\gamma\rceil_j - 1} \frac{\widetilde{t}^i}{i!}\mathtt{Y}_0^{(i)}] + \frac{1}{\eta_\sigma} \mathcal{J}_{\widetilde{t}_0}^{\gamma_\sigma} \mathcal{G}(\widetilde{t}, \mathtt{Y}(\widetilde{t})). \tag{4.31}$$

We consider the generalization of the implicit and explicit first-order (4.19) and (4.20). For this goal, we first notice that:

$$\mathcal{J}_{t_0}^{\gamma} \sum_{i=0}^{\lceil\gamma\rceil-1} \frac{\widetilde{t}^i}{i!}\mathtt{Y}_0^{(i)} = \sum_{i=0}^{\lceil\gamma\rceil-1} \mathtt{Y}_0^{(i)} \mathcal{J}_{\widetilde{t}_0}^{\gamma} \frac{\widetilde{t}^i}{i!} = \sum_{i=0}^{\lceil\gamma\rceil-1} \frac{\widetilde{t}^{i+\gamma}}{\Gamma(i+\gamma)}\mathtt{Y}_0^{(i)},$$

and therefore, after designating:

$$\mathbb{Q}(\widetilde{t}) := \sum_{i=0}^{\lceil\gamma\rceil_\sigma - 1} \frac{\widetilde{t}^i}{i!}\mathtt{Y}_0^{(i)} + \sum_{j=1}^{\sigma-1} \frac{\eta_j}{\eta_\sigma} \sum_{i=1}^{m_j - 1} \frac{\widetilde{t}^{i+\gamma_\sigma-\gamma_j}}{\Gamma(i+\gamma_\sigma-\gamma_j+1)}\mathtt{Y}^{(i)},$$

the related methods for the multiterm FDEs (4.31) are respectively:

$$\text{explicit. Rectangular}: \qquad \mathtt{Y}_m = \mathbb{Q}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \frac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=0}^{m-1} b_{m-k-1}^{\gamma_\sigma-\gamma_j}\mathtt{Y}_k + \frac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=0}^{m-1} b_{m-k-1}^{(\gamma_\sigma)} \mathcal{G}(\widetilde{t}_k, \mathtt{Y}_k), \tag{4.32}$$

and

$$\text{implicit. Rectangular}: \qquad \mathtt{Y}_m^p = \mathbb{Q}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \frac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=1}^{m} b_{m-k}^{\gamma_\sigma-\gamma_j}\mathtt{Y}_k + \frac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=1}^{m} b_{m-k}^{(\gamma_\sigma)} \mathcal{G}(\widetilde{t}_k, \mathtt{Y}_k), \tag{4.33}$$

where $b_m^{(\gamma)} = \dfrac{((m+1)^\gamma - m^\gamma)}{\Gamma(\gamma+1)}$. The implicit trapezoidal (4.16) can be expanded to multiterm FDEs as well.

$$\begin{aligned} \text{implicit. Trapezoidal}: \qquad \mathtt{Y}_m = \mathbb{Q}(\widetilde{t}) &- \sum_{j=1}^{\sigma-1} \frac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)}\mathtt{Y}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)}\mathtt{Y}_k) \\ &+ \frac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \mathcal{G}(\widetilde{t}_0, \mathtt{Y}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \mathcal{G}(\widetilde{t}_k, \mathtt{Y}_k)), \end{aligned} \tag{4.34}$$

where

$$\tilde{a}_m^{(\gamma)} = \frac{(m-1)^{\gamma+1} - m^\gamma(m-\gamma-1)}{\Gamma(\gamma+2)}, \qquad a_m^{(\gamma)} = \begin{cases} \dfrac{1}{\Gamma(\gamma+2)}, & m = 0, \\ \dfrac{(m-1)^{\gamma+1} - 2m^{\gamma+1} + (m+1)^{\gamma+1}}{\Gamma(\gamma+2)}, & m = 1, 2.... \end{cases}$$

To avert the solution of the nonlinear equations for the evaluation of $\mathtt{Y}_m$. Sometimes, a predictor-corrector method is preferred, in which the initial approximation of $\mathtt{Y}_m$ is predicted utilizing the explicit rectangular rule (4.33) and hence corrected by the implicit trapezoidal quadrature (4.35) according to:

$$\begin{cases} \mathtt{Y}_m^p = \mathbb{Q}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \frac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=1}^{m} b_{m-k}^{\gamma_\sigma-\gamma_j}\mathtt{Y}_k + \frac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=1}^{m} b_{m-k}^{(\gamma_\sigma)} \mathcal{G}(\widetilde{t}_k, \mathtt{Y}_k), \\ \mathtt{Y}_m = \mathbb{Q}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \frac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)}\mathtt{Y}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)}\mathtt{Y}_k) \\ \qquad + \frac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \mathcal{G}(\widetilde{t}_0, \mathtt{Y}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \mathcal{G}(\widetilde{t}_k, \mathtt{Y}_k) + a_0^{(\gamma)} \mathcal{G}(\widetilde{t}_m, \mathtt{Y}_m^p)). \end{cases} \tag{4.35}$$

In the sequel of our work, a Predictor-Corrector Approach for solving the fuzzy IVP of Linear Multi-Term FDEs (4.31) will be offered. Using the modified trapezoidal rule, the numerical scheme to evaluate the Corrector can be described as:

$$\begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_k) \\ & + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_0, \underline{\mathtt{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k) + a_0^{(\gamma)} \underline{\mathcal{G}}(\widetilde{t}_m, \underline{\mathtt{Y}}_m^p, \overline{\mathtt{Y}}_m^p)), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_k) \\ & + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_0, \underline{\mathtt{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k) + a_0^{(\gamma)} \overline{\mathcal{G}}(\widetilde{t}_m, \underline{\mathtt{Y}}_m^p, \overline{\mathtt{Y}}_m^p)), \end{cases} \tag{4.36}$$

for case 1, and

$$\begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_k) \\ & + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \overline{\mathtt{Y}}(\widetilde{t}_0, \underline{\mathcal{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k) + a_0^{(\gamma)} \overline{\mathcal{G}}(\widetilde{t}_m, \underline{\mathtt{Y}}_m^p, \overline{\mathtt{Y}}_m^p)), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_k) \\ & + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_0, \underline{\mathtt{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathcal{Y}}_k, \overline{\mathtt{Y}}_k) + a_0^{(\gamma)} \underline{\mathcal{G}}(\widetilde{t}_m, \underline{\mathtt{Y}}_m^p, \overline{\mathtt{Y}}_m^p)), \end{cases} \tag{4.37}$$

for Case 2.

The approximations solution $\underline{\mathtt{Y}}_m^p$ and $\overline{\mathtt{Y}}_m^p$ are utilized in (4.33) and (4.34) to evaluate predictor terms

$$\text{explicit. Rectangular}: \begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=0}^{m-1} b_{m-k-1}^{\gamma_\sigma-\gamma_j} \underline{\mathtt{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=0}^{m-1} b_{m-k-1}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=0}^{m-1} b_{m-k-1}^{\gamma_\sigma-\gamma_j} \overline{\mathtt{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=0}^{m-1} b_{m-k-1}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.38}$$

for case 1, and

$$\text{explicit. Rectangular}: \begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=0}^{m-1} b_{m-k-1}^{\gamma_\sigma-\gamma_j} \underline{\mathtt{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=0}^{m-1} b_{m-k-1}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=0}^{m-1} b_{m-k-1}^{\gamma_\sigma-\gamma_j} \overline{\mathtt{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=0}^{m-1} b_{m-k-1}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.39}$$

for case 2

$$\text{implicit. Rectangular}: \begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=1}^{m} b_{m-k}^{\gamma_\sigma-\gamma_j} \underline{\mathtt{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=1}^{m} b_{m-k}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=1}^{m} b_{m-k}^{\gamma_\sigma-\gamma_j} \overline{\mathtt{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=1}^{m} b_{m-k}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.40}$$

for case 1, and

$$\text{implicit. Rectangular}: \begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=1}^{m} b_{m-k}^{\gamma_\sigma-\gamma_j} \underline{\mathcal{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=1}^{m} b_{m-k}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) - \sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} \sum_{k=1}^{m} b_{m-k}^{\gamma_\sigma-\gamma_j} \overline{\mathtt{Y}}_k + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} \sum_{k=1}^{m} b_{m-k}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k), \end{cases} \tag{4.41}$$

for case 2

$$\text{implicit. Trapezoidal}: \begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_k) \\ & + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_0, \underline{\mathtt{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k)), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_k) \\ & + \dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_0, \underline{\mathtt{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k)), \end{cases} \tag{4.42}$$

for case 1, and

$$
\text{implicit. Trapezoidal}: \begin{cases} \underline{\mathtt{Y}}_m = \underline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \underline{\mathtt{Y}}_k) \\ & +\dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_0, \underline{\mathtt{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \overline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k)), \\ \overline{\mathtt{Y}}_m = \overline{\mathbb{Q}}(\widetilde{t}) & -\sum_{j=1}^{\sigma-1} \dfrac{\eta_j}{\eta_\sigma} h^{\gamma_\sigma-\gamma_j} (\tilde{a}_m^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_0 + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma-\gamma_j)} \overline{\mathtt{Y}}_k) \\ & +\dfrac{1}{\eta_\sigma} h^{\gamma_\sigma} (\tilde{a}_m^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_0, \underline{\mathtt{Y}}_0, \overline{\mathtt{Y}}_0) + \sum_{k=1}^{m} a_{m-k}^{(\gamma_\sigma)} \underline{\mathcal{G}}(\widetilde{t}_k, \underline{\mathtt{Y}}_k, \overline{\mathtt{Y}}_k)), \end{cases} \tag{4.43}
$$

for case 2.

A predictor-corrector method might be helpful in some practical help in numerical solutions.

Even though many different methods have been suggested to resolve linear multiterm FDEs, we believe the method discussed in this section may be preferable due to its high accuracy.

# 5 Applicative Numerical Examples

Here, We present four examples to demonstrate the procedures described in the previous sections. As we'll see, it typically demonstrates high accuracy. All the experiments are conducted in MATLAB Ver. 9.10.0.1602886 (R2021a).

**Example 1**. This example aims to demonstrate the implicit approach's superiority in terms of stability. For this goal, we consider this first example of the simple fuzzy fractional linear equation:

$$
\begin{cases} \underline{\mathtt{Y}}'(\widetilde{t}, \widetilde{r}) = (0.99 + 0.01 * \widetilde{r})\eta \underline{\mathtt{Y}}(\widetilde{t}, \widetilde{r}), \\ \overline{\mathtt{Y}}'(\widetilde{t}, \widetilde{r}) = (1.05 - 0.05 * \widetilde{r})\eta \overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r}), \\ \mathtt{Y}(\widetilde{t_0}) = \mathtt{Y}_0, \end{cases} \tag{5.1}
$$

whose exact solution is

$$
\begin{cases} \underline{\mathtt{Y}}(\widetilde{t}, \widetilde{r}) = (0.99 + 0.01 * \widetilde{r}) \mathbb{E}_\gamma(\eta(\widetilde{t} - \widetilde{t}_0)), \\ \overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r}) = (1.05 - 0.05 * \widetilde{r}) \mathbb{E}_\gamma(\eta(\widetilde{t} - \widetilde{t}_0)), \end{cases}
$$

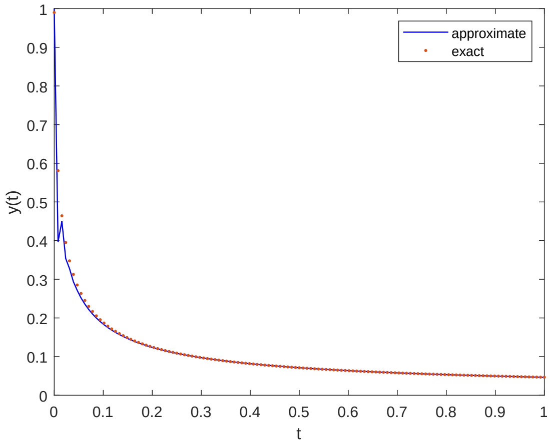
with

$$
\mathbb{E}_\gamma(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\gamma j + 1)},
$$

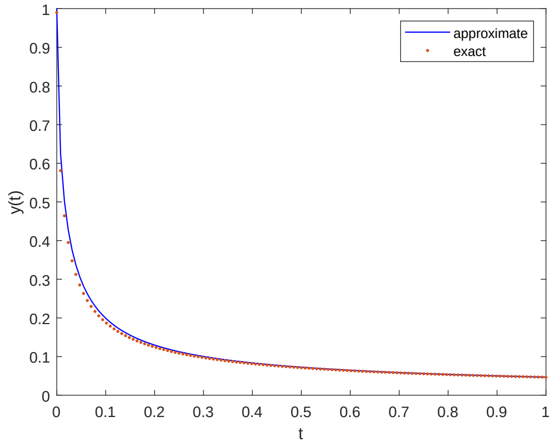
the Mittag-Leffler function of order $\gamma$ can be solved based on the algorithm depicted in [39,40]. We have solved this problem on $[0, 1]$ for $\gamma = 0.6$ and $\eta = -10$ whose numerical results are shown in Table 1. From Table 1 - 8, we studied the numerical results of each rectangular explicit and implicit rule and the trapezoidal implicit rule. After that, we introduced the main results of the predictor-corrector method.

Table 1: Solutions Errors at $\mathcal{T} = 1$ and $\widetilde{r} = 1$ for the FFDE (5.1) $\underline{Y}(\widetilde{t}, \widetilde{r})$ with $\gamma = 0.6$ and $\eta = -10.0$.

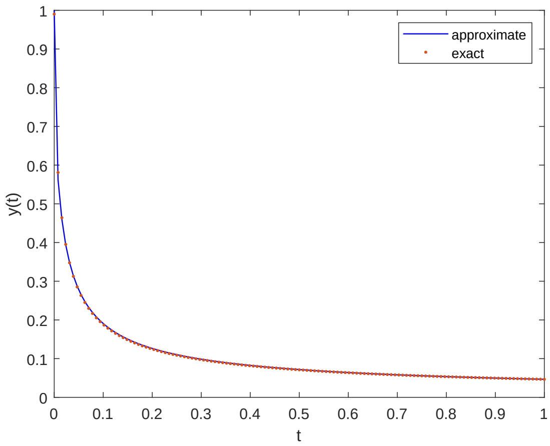
| .. h | Exp. Rectangular Error | Impl. Rectangular Error | Impl. Trapezoidal Error | Predictor-Corrector Error |
|---|---|---|---|---|
| $2^{-2}$ | 3.233(2) | 8.759(-3) | 3.496(-3) | 1.878(-3) |
| $2^{-3}$ | 2.588(3) | 4.108(-3) | 2.118(-3) | 2.363(-3) |
| $2^{-4}$ | 2.675(3) | 1.986(-3) | 6.516(-4) | 1.679(-4) |
| $2^{-5}$ | 0.378 | 9.918(-4) | 1.797(-4) | 1.059(-4) |
| $2^{-6}$ | 4.033(-4) | 5.128(-4) | 2.696(-5) | 6.377(-5) |
| $2^{-7}$ | 1.799(-4) | 2.781(-4) | 2.274(-5) | 1.690(-5) |



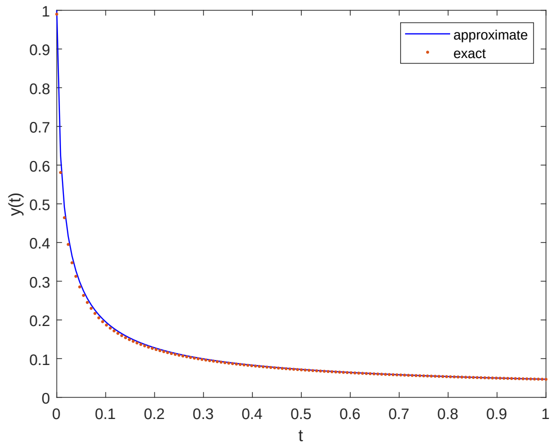

(a) Solutions Errors to (5.1) $\underline{Y}(\widetilde{t}, \widetilde{r})$ by explicit. Rectangular at $h^{-7}$




(b) Solutions Errors to (5.1) $\underline{Y}(\widetilde{t}, \widetilde{r})$ by implicit. Rectangular at $h^{-7}$



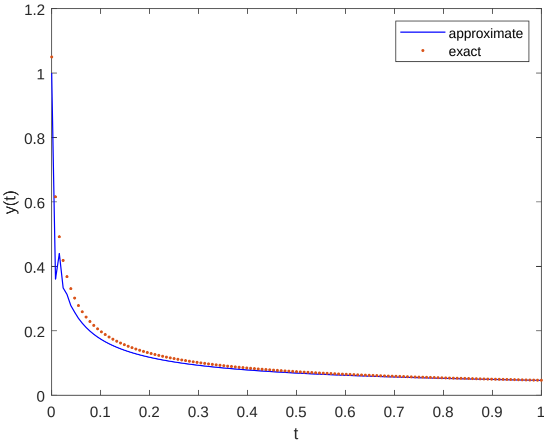

(a) Solutions Errors to (5.1) $\underline{Y}(\widetilde{t}, \widetilde{r})$ by implicit. Trapezoidal at $h^{-7}$



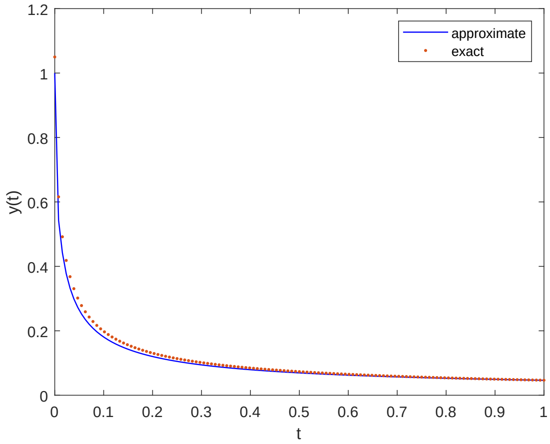

(b) Solutions Errors to (5.1) $\underline{Y}(\widetilde{t}, \widetilde{r})$ by Predictor-Corrector method at $h^{-7}$

Table 2: Solutions Errors at $\mathcal{T} = 1$ and $\widetilde{r}= 1$ for the FFDE (5.1) $\overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ with $\gamma = 0.6$ and $\eta = -10.0$.

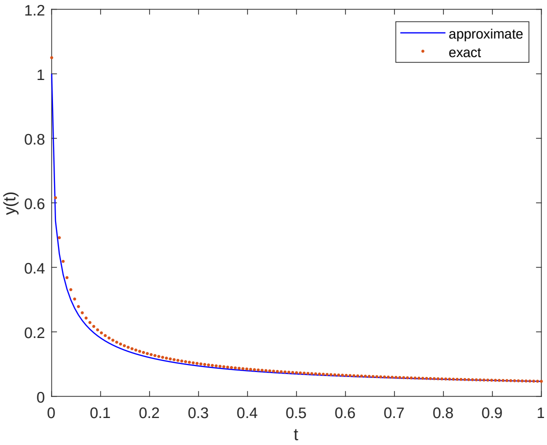
| ... | Exp. Rectangular | Impl. Rectangular | Impl. Trapezoidal | Predictor-Corrector |
|---|---|---|---|---|
| h | Error | Error | Error | Error |
| $2^{-2}$ | 3.830(2) | 8.498(-3) | 3.692(-3) | 2.426(-3) |
| $2^{-3}$ | 3.551(3) | 3.832(-3) | 2.517(-3) | 3.976(-4) |
| $2^{-4}$ | 5.021(3) | 1.709(-3) | 9.664(-4) | 4.653(-4) |
| $2^{-5}$ | 1.394 | 7.161(-4) | 4.670(-4) | 2.179(-4) |
| $2^{-6}$ | 6.764(-4) | 2.378(-4) | 3.054(-4) | 4.913(-5) |
| $2^{-7}$ | 4.536(-4) | 3.502(-5) | 2.528(-4) | 2.352(-5) |



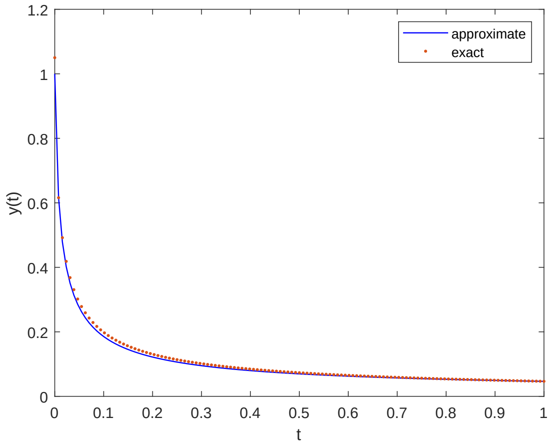

(a) Solutions Errors to (5.1) $\overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ by explicit. Rectangular at $h^{-7}$




(b) Solutions Errors to (5.1) $\overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ by implicit. Rectangular at $h^{-7}$




(a) Solutions Errors to (5.1) $\overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ by implicit. Trapezoidal at $h^{-7}$




(b) Solutions Errors to (5.1) $\overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ by Predictor-Corrector at $h^{-7}$

**Example 5.2** In this example, we consider the approximate solution $\mathtt{Y}$ with a smooth derivative of

Table 3: Solutions Errors at $\mathcal{T} = 1$ and $\widetilde{r} = 1$ for the FFDE (5.2) $\underline{Y}(\widetilde{t}, \widetilde{r})$ with $\gamma = 0.25$ and $\eta = 0.25$.

| ... <br> h | Exp. Rectangular <br> Error | Impl. Rectangular <br> Error | Impl. Trapezoidal <br> Error | Predictor-Corrector <br> Error |
|---|---|---|---|---|
| $2^{-4}$ | 8.835(-2) | 5.248(-2) | 4.149(-2) | 1.893(-3) |
| $2^{-5}$ | 4.245(-2) | 2.769(-2) | 5.334(-3) | 5.839(-4) |
| $2^{-6}$ | 2.054(-2) | 1.446(-2) | 1.906(-3) | 1.608(-4) |
| $2^{-7}$ | 1.000(-2) | 7.487(-3) | 8.583(-3) | 3.637(-5) |
| $2^{-8}$ | 4.888(-3) | 3.852(-3) | 3.954(-4) | 1.558(-5) |
| $2^{-9}$ | 2.394(-3) | 1.974(-3) | 1.823(-4) | 7.914(-6) |
| $2^{-10}$ | 1.473(-3) | 1.010(-3) | 8.609(-5) | 1.046(-6) |

Table 4: Solutions Errors at $\mathcal{T} = 1$ and $\widetilde{r}= 1$ for the FFDE (5.2) $\overline{Y}(\widetilde{t}, \widetilde{r})$ with $\gamma = 0.25$ and $\eta = 0.25$.

| ... <br> h | Exp. Rectangular <br> Error | Impl. Rectangular <br> Error | Impl. Trapezoidal <br> Error | Predictor-Corrector <br> Error |
|---|---|---|---|---|
| $2^{-4}$ | 8.840(-2) | 5.247(-2) | 4.150(-2) | 1.914(-3) |
| $2^{-5}$ | 4.249(-2) | 2.768(-2) | 5.320(-3) | 6.058(-4) |
| $2^{-6}$ | 2.058(-2) | 1.444(-2) | 1.886(-3) | 1.832(-4) |
| $2^{-7}$ | 1.002(-2) | 7.466(-3) | 8.370(-4) | 5.894(-5) |
| $2^{-8}$ | 4.912(-3) | 3.830(-3) | 3.733(-4) | 2.416(-5) |
| $2^{-9}$ | 2.418(-3) | 1.952(-3) | 1.599(-4) | 1.470(-5) |
| $2^{-10}$ | 1.196(-3) | 9.884(-4) | 6.354(-5) | 1.215(-6) |

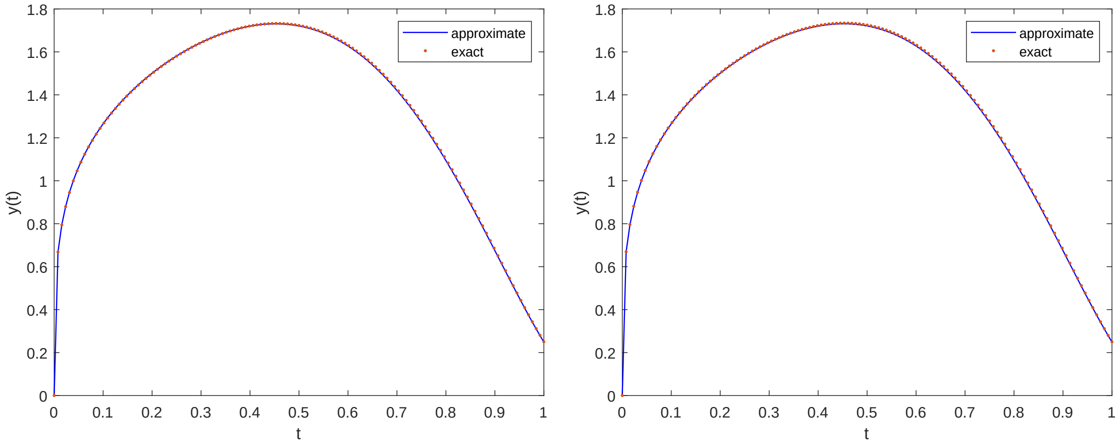
order $\gamma$. by applying this method to the following fuzzy linear equation

$$\begin{cases} \mathcal{D}_0^{\gamma}\underline{Y}(\widetilde{t},\widetilde{r}) = (0.999 + 0.001 * \widetilde{r})[\frac{40320}{\Gamma(9-\gamma)}\widetilde{t}^{8-\gamma} - 3\frac{\Gamma(5+\frac{\gamma}{2})}{\Gamma(5-\frac{\gamma}{2})}\widetilde{t}^{4-\frac{\gamma}{2}} + \frac{9}{4}\Gamma(\gamma+1) + (\frac{3}{2}\widetilde{t}^{\frac{\gamma}{2}} - \widetilde{t}^{4})^3 - [\underline{Y}(\widetilde{t},\widetilde{r})]^{(\frac{3}{2})}], \\ \mathcal{D}_0^{\gamma}\overline{Y}(\widetilde{t},\widetilde{r}) = (1.001 - 0.001 * \widetilde{r})[\frac{40320}{\Gamma(9-\gamma)}\widetilde{t}^{8-\gamma} - 3\frac{\Gamma(5+\frac{\gamma}{2})}{\Gamma(5-\frac{\gamma}{2})}\widetilde{t}^{4-\frac{\gamma}{2}} + \frac{9}{4}\Gamma(\gamma+1) + (\frac{3}{2}\widetilde{t}^{\frac{\gamma}{2}} - \widetilde{t}^{4})^3 - [\overline{Y}(\widetilde{t},\widetilde{r})]^{(\frac{3}{2})}], \\ \underline{Y}(0,\widetilde{r}) = 0, \underline{Y}'(0,\widetilde{r}) = 0, \qquad \overline{Y}(0,\widetilde{r}) = 0, \overline{Y}'(0,\widetilde{r}) = 0. \end{cases} \tag{5.2}$$

This equation has been selected because it shows a difficult(nonsmooth and nonlinear) right-hand side, and yet we can find its exact solution, Therefore we can compare the numerical solution with the exact solution. The exact result of this problem is. In fact, the exact solution is

$$\begin{cases} \underline{Y}(\widetilde{t},\widetilde{r}) = (0.999 + 0.001 * \widetilde{r})(t^8 - 3\widetilde{t}^{4-\frac{\gamma}{2}} + \frac{9}{4}\widetilde{t}^{\gamma}), \\ \overline{Y}(\widetilde{t},\widetilde{r}) = (1.001 - 0.001 * \widetilde{r})(\widetilde{t}^8 - 3\widetilde{t}^{4-\frac{\gamma}{2}} + \frac{9}{4}\widetilde{t}^{\gamma}). \end{cases}$$

Indeed, this problem is interesting because it does not give an artificially smooth solution, which is unrealistic for most fractional-order applications, unlike many other problems frequently suggested in the article.




(a) Solutions Errors to (5.2) $\underline{Y}(\widetilde{t},\widetilde{r})$ by Predictor-Corrector at $h^{-7}$

(b) Solutions Errors to (5.2) $\overline{Y}(\widetilde{t},\widetilde{r})$ by Predictor-Corrector at $h^{-7}$

Starting from the second example, we are sufficient to draw the numerical solution for the predictor-corrector method, which clarifies the accuracy of the relationship between the exact and approximate solutions.

**Example 5.3** The example is the benchmark Problem, which is another interesting example given in[41], and we will study it for the multi-term FFDE:

$$\begin{cases} \underline{Y}'''(\widetilde{t},\widetilde{r}) + \mathcal{D}_0^{\frac{5}{2}}\underline{Y}(\widetilde{t},\widetilde{r}) + \underline{Y}''(\widetilde{t},\widetilde{r}) + 4\underline{Y}'(\widetilde{t},\widetilde{r}) + \mathcal{D}_0^{\frac{1}{2}}\underline{Y}(\widetilde{t},\widetilde{r}) + 4\underline{Y}(\widetilde{t},\widetilde{r}) = (0.99 + 0.0001 * \widetilde{r})6cos(\widetilde{t},\widetilde{r}), \\ \overline{Y}'''(\widetilde{t},\widetilde{r}) + \mathcal{D}_0^{\frac{5}{2}}\overline{Y}(\widetilde{t},\widetilde{r}) + \overline{Y}''(\widetilde{t},\widetilde{r}) + 4\overline{Y}'(\widetilde{t},\widetilde{r}) + \mathcal{D}_0^{\frac{1}{2}}\overline{Y}(\widetilde{t},\widetilde{r}) + 4\overline{Y}(\widetilde{t},\widetilde{r}) = (1.01 - 0.0001 * \widetilde{r})6cos(\widetilde{t},\widetilde{r}), \\ \underline{Y}(0) = 1, \underline{Y}'(0) = 1, \underline{Y}''(0) = -1, \qquad \overline{Y}(0) = 1, \overline{Y}'(0) = 1, \overline{Y}''(0) = -1, \end{cases} \tag{5.3}$$

whose exact solution is

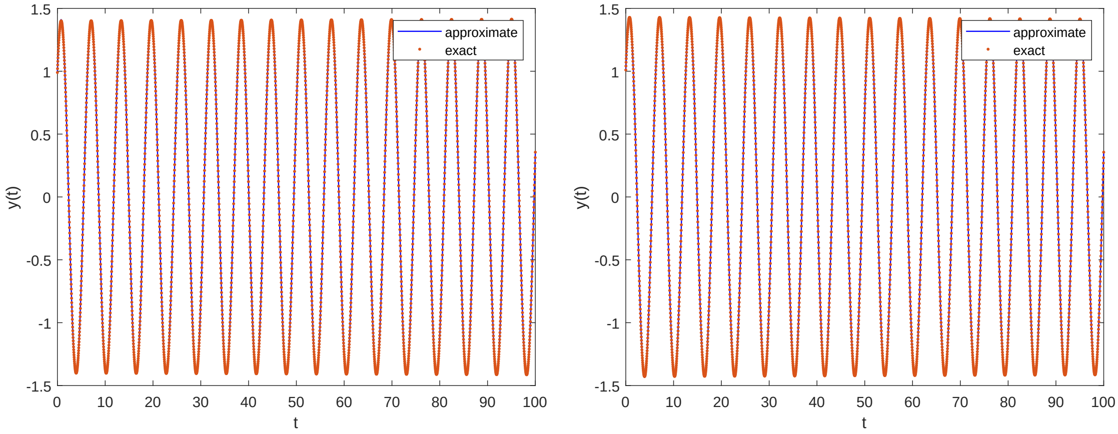
$$\begin{cases} \underline{Y}(\widetilde{t},\widetilde{r}) = (0.99 + 0.0001 * \widetilde{r})2^{\frac{1}{2}}sin(\widetilde{t},\pi/4), \\ \overline{Y}(\widetilde{t},\widetilde{r}) = (1.01 - 0.0001 * \widetilde{r})2^{\frac{1}{2}}sin(\widetilde{t},\pi/4). \end{cases}$$

Table 5: Solutions Errors at $\mathcal{T} = 100$ and $\widetilde{r} = 1$ for multi-term FFDE (benchmark Problem)(5.3) $\underline{Y}(\widetilde{t},\widetilde{r})$ with $\gamma = [3, 2.5, 2, 1, 0.5, 0]$ and $\eta = [1, 1, 1, 4, 1, 4]$.

| ... | Exp. Rectangular | Impl. Rectangular | Impl. Trapezoidal | Predictor-Corrector |
|---|---|---|---|---|
| h | Error | Error | Error | Error |
| $2^{-2}$ | 2.704(-2) | 3.084(-2) | 2.208(-2) | 1.521(-3) |
| $2^{-3}$ | 1.089(-2) | 1.349(-2) | 4.461(-3) | 2.508(-4) |
| $2^{-4}$ | 4.192(-3) | 6.266(-3) | 1.373(-3) | 5.074(-5) |
| $2^{-5}$ | 2.115(-3) | 3.032(-3) | 5.402(-4) | 1.240(-5) |
| $2^{-6}$ | 1.034(-3) | 1.532(-3) | 2.796(-4) | 5.420(-6) |
| $2^{-7}$ | 4.600(-4) | 8.224(-4) | 1.931(-4) | 2.417(-6) |

Table 6: Solutions Errors at $\mathcal{T}=100$ and $\widetilde{r}=1$ for multi-term FFDE (benchmark Problem)(5.3) $\overline{\mathrm{Y}}(\widetilde{t},\widetilde{r})$ with $\gamma=[3,2.5,2,1,0.5,0]$ and $\eta=[1,1,1,4,1,4]$.

| ... | Exp. Rectangular | Impl. Rectangular | Impl. Trapezoidal | Predictor-Corrector |
|---|---|---|---|---|
| h | Error | Error | Error | Error |
| $2^{-2}$ | 1.758(-2) | 3.058(-2) | 2.184(-2) | 1.851(-3) |
| $2^{-3}$ | 9.650(-3) | 1.326(-2) | 4.245(-3) | 5.566(-4) |
| $2^{-4}$ | 4.478(-3) | 6.052(-3) | 1.113(-3) | 2.476(-4) |
| $2^{-5}$ | 2.474(-3) | 2.800(-3) | 2.563(-4) | 1.724(-4) |
| $2^{-6}$ | 1.368(-3) | 1.274(-3) | 1.540(-4) | 1.222(-5) |
| $2^{-7}$ | 7.752(-4) | 5.462(-4) | 1.495(-4) | 1.014(-6) |




(a) Solutions Errors to (5.3) $\underline{\mathrm{Y}}(\widetilde{t},\widetilde{r})$ by Predictor-Corrector at $h^{-5}$

(b) Solutions Errors to (5.3) $\overline{\mathrm{Y}}(\widetilde{t},\widetilde{r})$ by Predictor-Corrector at $h^{-5}$

**Example 5.4**, We conclude our work by studying the BagleyTorvik equation as a test equation for the multi-term FFDE:

$$\begin{cases} \underline{\mathrm{Y}}''(\widetilde{t},\widetilde{r})+A\mathcal{D}_{t_0}^{\frac{3}{2}}\underline{\mathrm{Y}}(\widetilde{t},\widetilde{r})+B\underline{\mathrm{Y}}(\widetilde{t},\widetilde{r})=(0.9999+0.0001*\widetilde{r})(\widetilde{t}+1), \\ \overline{\mathrm{Y}}''(\widetilde{t},\widetilde{r})+A\mathcal{D}_{t_0}^{\frac{3}{2}}\overline{\mathrm{Y}}(\widetilde{t},\widetilde{r})+B\overline{\mathrm{Y}}(\widetilde{t},\widetilde{r})=(1.00001-0.00001*\widetilde{r})(\widetilde{t}+1), \\ \underline{\mathrm{Y}}(0)=1,\underline{\mathrm{Y}}'(0)=1, \\ \overline{\mathrm{Y}}(0)=1,\overline{\mathrm{Y}}'(0)=1, \end{cases} \tag{5.4}$$

whose exact solution is

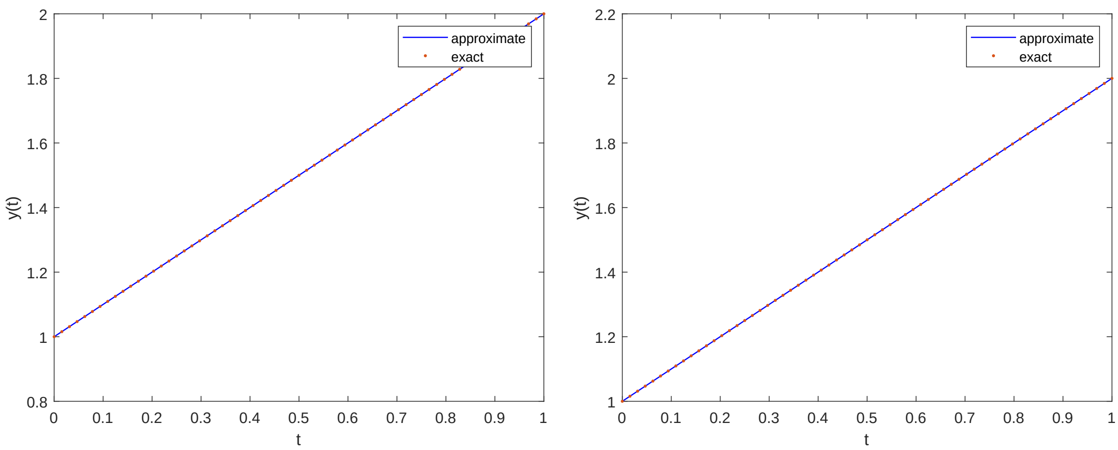
$$\begin{cases} \underline{\mathrm{Y}}(\widetilde{t},\widetilde{r})=(0.9999+0.0001*\widetilde{r})(\widetilde{t}+1), \\ \overline{\mathrm{Y}}(\widetilde{t},\widetilde{r})=(1.00001-0.00001*\widetilde{r})(\widetilde{t}+1). \end{cases}$$

Table 7: Solutions Errors at $\widetilde{r} = 1$ and $\mathcal{T} = 1$ for multi-term FFDE (Bagley-Torvik equation)(5.4) $\underline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ with $\gamma = [2, 3/2, 0]$ and $\eta = [1, 1, 1]$.

| ... <br> h | Exp. Rectangular <br> Error | Impl. Rectangular <br> Error | Impl. Trapezoidal <br> Error | Predictor-Corrector <br> Error |
|---|---|---|---|---|
| $2^{-2}$ | 6.788(-2) | 2.710(-2) | 4.172(-2) | 2.305(-2) |
| $2^{-3}$ | 3.214(-2) | 2.526(-2) | 1.160(-2) | 2.331(-3) |
| $2^{-4}$ | 1.542(-2) | 1.320(-2) | 3.581(-3) | 2.338(-4) |
| $2^{-5}$ | 7.507(-3) | 6.797(-3) | 1.174(-3) | 2.339(-4) |
| $2^{-6}$ | 3.680(-3) | 3.470(-3) | 4.070(-4) | 2.340(-5) |
| $2^{-7}$ | 1.808(-3) | 1.767(-3) | 1.537(-4) | 2.340(-6) |

Table 8: Solutions Errors at $\mathcal{T} = 1$ and $\widetilde{r} = 1$ for multi-term FFDE (Bagley-Torvik equation)(5.4) $\overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ with $\gamma = [2, 3/2, 0]$ and $\eta = [1, 1, 1]$.

| ... <br> h | Exp. Rectangular <br> Error | Impl. Rectangular <br> Error | Impl. Trapezoidal <br> Error | Predictor-Corrector <br> Error |
|---|---|---|---|---|
| $2^{-2}$ | 6.790(-2) | 4.708(-2) | 4.170(-2) | 2.350(-2) |
| $2^{-3}$ | 3.216(-2) | 2.524(-2) | 1.158(-2) | 2.145(-3) |
| $2^{-4}$ | 1.545(-2) | 1.318(-2) | 3.555(-3) | 2.333(-4) |
| $2^{-5}$ | 7.533(-3) | 6.772(-3) | 1.148(-3) | 2.339(-4) |
| $2^{-6}$ | 3.706(-3) | 3.444(-3) | 3.813(-4) | 2.502(-5) |
| $2^{-7}$ | 1.834(-3) | 1.741(-3) | 1.280(-4) | 2.360(-6) |




(a) Solutions Errors to (5.4) $\underline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ by Predictor-Corrector at $h^{-6}$

(b) Solutions Errors to (5.4) $\overline{\mathtt{Y}}(\widetilde{t}, \widetilde{r})$ by Predictor-Corrector at $h^{-6}$

All the FFDEs in this work investigated the exact solutions. However, numerical solutions are the only way to use them in real applications. Naturally, the validation technique in this study cannot be employed if the analytical solutions are unknown. From previous examples, we notice that the error decreases with increasing step size $h$.

# 6 Conclusion of the paper

In this article, we have studied existence and uniqueness of FFDEs under Caputo's H-differentiability by the contraction principle. The exact solutions to FFDEs are typically challenging to derive. Therefore, it's vital to develope reliable and efficient techniques to solve FFDEs. So, in this article, we used the Predictor-Corrector method for numerically solving systems of FFDEs. We compared the exact and approximate solutions for two families of fuzzy fractional-order problems: fuzzy fractional differential equations (FFDEs) and linear multi-term FFDEs. This method has shown us high accuracy and reduces errors between the exact and approximate solutions. Moreover, Their application was illustrated in detail using some examples. For future studies, we will use the Predictor-Corrector method to solve a class of FFDEs under Riemann-Liouville H-differentiability and the fuzzy Laplace transforms technique with an initial value problem.

**ACKNOWLEDGEMENTS**

This work was supported by the National Natural Science Foundation of China (NNSFC) (No. 12172340), the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan) (Nos. G1323523061, G1323523041 and CUGDCJJ202216), and the Young Top-notch Talent Cultivation Program of Hubei Province.

# Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

## Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

## Availability of data and material

Not applicable.

# References

[1] T. Allahviranloo, T. Allahviranloo, . (2021). Fuzzy fractional differential equations (pp. 127-192). Springer International Publishing..

[2] T. Allahviranloo, L. Avazpour, MJ. Ebadi , D. Baleanu , Salahshour S. Fuzzy fractional Ostrowski inequality with Caputo differentiability. J Inequal Appl 2013;2013:50.

[3] S. Salahshour , A. Ahmadian , N. Senu , D. Baleanu, P. Agarwal . On analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem. Entropy 2015;17:885–902.

[4] S. Aadhithiyan, R. Raja, Zhu Q , J. Alzabut, M. Niezabitowski , Lim CP . Modified projective synchronization of distributive fractional order complex dynamic networks with model uncertainty via adaptive control. Chaos Soliton Fract 2021(147):110853 .

[5] B. Bede . Mathematics of fuzzy sets and fuzzy logic. London: Springer; 2013 .

[6] A. Ahmadian, M. Suleiman, S. Salahshour, D. Baleanu, A Jacobi operational matrix for solving a fuzzy linear fractional differential equation, Advances in Difference Equations 2013 (2013) 1-29 .

[7] R.P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, Fuzzy fractional integral equations under compactness type condition, Fractional Calculus and Applied Analysis 15 (2012) 572-590.

[8] S. Arshad, V. Lupulescu. Fractional differential equation with the fuzzy initial conditions. Elect J Differ Equat 2011;34:1–8.

[9] Abu Arqub, O. (2017). Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm–Volterra integrodifferential equations. Neural Computing and Applications, 28, 1591-1610.

[10] Z. Alijani, D. Baleanu, B. Wu Shiri, G. C. (2020). Spline collocation methods for systems of fuzzy fractional differential equations. Chaos, Solitons and Fractals, 131, 109510.

[11] Arqub, O. A. Al-Smadi, M. (2020). Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions. Soft Computing, 24(16), 12501-12522.

[12] Vu H , Hoa NV . Non-instantaneous impulses interval-valued fractional differ- ential equations with caputo-katugampola fractional derivative concept. Fuzzy Set Syst 2021;404:111–40 .

[13] Hoa NV , Ho V . A survey on the initial value problems of fuzzy implicit frac- tional differential equations. Fuzzy Set Syst 2020;400:90–133 .

[14] Vu, H., Ghanbari, B. Van Hoa, N. (2022). Fuzzy fractional differential equations with the generalized Atangana-Baleanu fractional derivative. Fuzzy Sets and Systems, 429, 1-27.

[15] M. Akram, T. Ihsan, T. Allahviranloo, (2023). Solving Pythagorean fuzzy fractional differential equations using Laplace transform. Granular Computing, 8(3), 551-575.

[16] M. Mazandarani, A. V. Kamyad (2013). Modified fractional Euler method for solving fuzzy fractional initial value problem. Communications in Nonlinear Science and Numerical Simulation,18(1), 12-21.

[17] S. Salahshour, T. Allahviranloo, S. Abbasbandy and D. Baleanu, Existence and uniqueness results for fractional differential equations with uncertainty, Advances in Difference Equations 2012, 2012:112.

[18] Hoa NV , Vu H , Duc TM . Fuzzy fractional differential equations un- der caputo-katugampola fractional derivative approach. Fuzzy Set Syst 2019;375:70–99.

[19] N. V. Hoa, Fuzzy fractional functional integral and differential equations, Fuzzy Sets and Systems 280 (2015) 58-90.

[20] B. Bede and L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems, 230 (2013) 119-141.

[21] Vu, Ho, and Ngo Van Hoa. "Hyers-Ulam stability of fuzzy fractional Volterra integral equations with the kernel function via successive approximation method." Fuzzy Sets and Systems 419 (2021): 67-98.

[22] O.S. Fard, M. Salehi, A survey on fuzzy fractional variational problems, Journal of Computational and Applied Mathematics 271 (2014) 71-82

[23] Osman, M.Xia, Y. (2023). Solving fuzzy fractional differential equations with applications. Alexandria Engineering Journal, 69, 529-559.

[24] R. Alikhani, F. Bahrami, Global solutions for nonlinear fuzzy fractional integral and integro-differential equations,Communications in Nonlinear Science and Numerical Simulation 18 (2013) 2007-2017.

[25] A. Khastan, J. J. Nieto, R. Rodr ́ıguez-L ́opez, Schauder fixed-point theorem in semilinear spaces and its application to fractional differential equations with uncertainty, Fixed Point Theory and Applications (2014) 2014: 21 .

[26] F. Santo Pedro, M. M. Lopes, V. F. Wasques, E.de Esmi, L. C. Barros, (2023). Fuzzy fractional differential equations with interactive derivative. Fuzzy Sets and Systems, 108488.

[27] N. A. Gasilov, A. G. Fatullayev, S. E. Amrahov, A. Khastan, A new approach to fuzzy initial value problem, Soft Computing 18 (2014) 217–225.

[28] S. Arshad, existence and uniqueness of solution of fuzzy fractional differential equations," Iranian J. Fuzzy Syst., vol. 10, no. 6, pp. 137-151,2013.

[29] N. V. Hoa (2020). On the initial value problem for fuzzy differential equations of non-integer order $\alpha \in (1, 2)$ . Soft Computing, 24(2), 935-954.

[30] A. Ahmadian, S. Salahshour, M. Ali-Akbari, F. Ismail, & D. Baleanu (2017). A novel approach to approximate fractional derivative with uncertain conditions. Chaos, Solitons & Fractals, 104, 68-76.

[31] H. Van Ngo, V. Lupulescu, & D. O'Regan (2018). A note on initial value problems for fractional fuzzy differential equations. Fuzzy Sets and Systems, 347, 54-69.

[32] M. Najariyan, M. Mazandarani, & V. E. Balas(2016, August). Fuzzy fractional derivative: A new definition. In International Workshop Soft Computing Applications (pp. 314-329). Springer, Cham.

[33] R. Abdollahi, A. Khastan, J. J. Nieto, & R. Rodríguez-López, (2018). On the linear fuzzy model associated with Caputo–Fabrizio operator. Boundary Value Problems, 2018(1), 1-18.

[34] M. Najariyan,& Y. Zhao (2017). Fuzzy fractional quadratic regulator problem under granular fuzzy fractional derivatives. IEEE Transactions on Fuzzy Systems, 26(4), 2273-2288.

[35] N. Van Hoa,, H. Vu, & T. M. Duc (2019). Fuzzy fractional differential equations under Caputo–Katugampola fractional derivative approach. Fuzzy Sets and Systems, 375, 70-99.

[36] T. Allahviranloo, & B. Ghanbari (2020). On the fuzzy fractional differential equation with interval Atangana–Baleanu fractional derivative approach. Chaos, Solitons & Fractals, 130, 109397.

[37] H. Vu, B. Ghanbari, & N. Van Hoa (2020). Fuzzy fractional differential equations with the generalized Atangana-Baleanu fractional derivative. Fuzzy Sets and Systems.

[38] N. Van Hoa, V. Lupulescu,& D. O'Regan (2017). Solving interval-valued fractional initial value problems under Caputo gH-fractional differentiability. Fuzzy Sets and Systems, 309, 1

[39] R. Garrappa Numerical evaluation of two and three parameter Mittag-Leffler functions. SIAM J. Numer. Anal. 2015, 53, 1350–1369.

[40] R. Garrappa On linear stability of predictor-corrector algorithms for fractional differential equations. Int. J. Comput. Math. 2010, 87, 2281–2290.

[41] D. Xue, L.Bai problems for Caputo fractional-order ordinary differential equations. Fract. Calc. Appl. Anal. 2017, 20, 1305–1312.