

Analysis of the asymptotic response of complete contacts



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Abstract

Complete contacts are a rich area of study and are of significant interest. These contacts arise when one of the contacting bodies has a sharp corner, such as if machining burrs have not been removed, or if previous damage has resulted in material loss, leaving a sharp corner. The state of stress induced by the loading of a complete contact is singular in the neighbourhood of the contact edge. The inherent coupling in the problem and the inability to treat the problem, other than by asymptotic or numerical means, makes the study of these contact interactions more challenging.

This thesis aims to investigate the response of these contacts. The behaviour of these contacts must be investigated by asymptotic methods, using distributed dislocation techniques. This requires the development of existing and new asymptotic methods. As such, this thesis begins by defining and classifying contact properties. Asymptotic methods and phenomena such as crack growth and partial slip are reviewed.

The response of a general complete contact, emerges as a key research challenge, together with the understanding of how and when zones of partial slip form, or a crack grows. This is very important as partial slip will inevitably lead to wear, which not only damages the surfaces but can lead to further problems by debris ejection or by reducing the load bearing contact area. A growing crack is a more obvious concern, long cracks and short cracks behave differently, but if a short crack, such as those modelled here is allowed to grow without control it can lead to several different problems. A short crack can of course grow into a long crack,

which then result in total component failure, a short crack may instead branch, depending on material microstructure, and all cracks will weaken the material so it will not sustain as much load as intended.

This thesis presents predictive tools to indicate when a growing crack may form, or when a large partial slip zone will form at the edge of a complete contact. First of all we classify the response, to determine a priori whether a slip zone will exist, and whether or not intimate contact is maintained, a necessary condition for the use of the asymptotic method. Once the response type is known, we must also develop a dislocation method.

Crystalline dislocations induce stresses in bodies where they are present. However, the state of stress induced is described in closed form for an infinite plane, so using dislocations as strain nuclei in the presence of other features, such as free surfaces or the contact edge, requires a determination of the state of stress induced when interacting with the other features. This makes the stress kernel position dependent and more complex than a simple infinite plane kernel. These kernels can not be expressed in closed form, so a numerical method is described which allows the use of the distributed dislocation theory.

The asymptotic solution and the numerical solutions for the dislocation kernels are then used to enforce the conditions required to model a crack, and to model a zone of partial slip. The state of stress at the crack tip, characterised by the crack tip stress intensity factors is calibrated to remote loading to determine under what conditions the crack will grow. The size of the slip zone will also be found, for a given loading, and how this is affected by changes in the angle of the contact, and the coefficient of friction. This gives an estimate of the depth to which slip, and by extension wear, penetrates.

This work is original and no part of it has been submitted for a degree at this university or any other. Some of the work described here has been published or is under consideration for publication in the following journals:

1. **Riddoch, D. J.** and Hills, D. A., *Necessary conditions for near edge stick of Complete Contacts*, The Journal of Strain Analysis for Engineering Design 55: 172-180, March 2020
2. **Riddoch, D. J.** and Hills, D. A., *Dislocations in an arbitrary angle wedge. Part I: The dislocation kernel*, The Journal of Strain Analysis for Engineering Design: 03093247211042539, October 2021
3. **Riddoch, D. J.**, Cwiekala, N. and Hills, D. A., *Dislocations in an arbitrary angle wedge. Part II: Cracks in the wedge*, The Journal of Strain Analysis for Engineering Design: 03093247211047785, October 2021.
4. **Riddoch, D. J.** and Hills, D. A., *Slip at the edge of complete contacts*, The Proceedings of the iMechE, Part C: Journal of Mechanical Engineering Science, <https://doi.org/10.1177/09544062221142695>, January 2023
5. **Riddoch, D. J.** and Hills, D. A., *Distributed dislocation problems in wedge shaped domains*, European Journal of Mechanics/ A Solids, <https://doi.org/10.1016/j.euromechsol.2023.104941>, February 2023

In addition to these publications and the work detailed above, other work has been completed during the course of this D. Phil. This has resulted in the publication of :

Riddoch, D. J., Cicirello, A. and Hills, D. A., *Response of a mass-spring system subject to Coulomb damping and harmonic base excitation*, International Journal of Solids and Structures 193 (2020), 527-534.

This paper is included in appendix D, along with a brief outline of previous related work and an explanation of the developments found.

Much of the analysis contained within this thesis, including almost all numerical work was performed using MATLAB. The code used to produce these results and figures has been partially published in relation to certain papers. However, it has now been compiled completely and is made available along with instructions for its use at: <https://doi.org/10.5287/bodleian:dXRdVJEaA>

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Nomenclature

b_x	Burgers vector in the x-direction, normally referring to a glide edge dislocation
b_y	Burgers vector in the y-direction, normally referring to a climb edge dislocation
b_z	Burgers vector in the z-direction, normally referring to a screw dislocation
$B_x(x)$	Dislocation density
c	Slip stick boundary
c_0	Initial guess of slip zone size in a complete contact
d_0	Mode mixity parameter
f_c	Coefficient of friction
f_{ij}^k	Mode k eigenvector of the σ_{ij} component of the Williams solution
G_{ijk}	Dislocation kernels for dislocations with Burgers vector b_i in the σ_{jk} component
K_I	Mode I stress intensity factor
K_{II}	Mode II stress intensity factor
l	Crack length
M	Remote moment load
P	Remote normal force load
$p(x)$	Pointwise normal pressure along the contact interface
Q	Remote shear force load
$q(x)$	Pointwise shear stress along the contact interface
r	Radial distance polar coordinate
α	Wedge half-angle
β	Dundurs parameter
$\delta\sigma$	Differential bulk tension
κ	Kolosov's constant
λ_k	Mode k Williams eigenvalue

μ	Modulus of rigidity
ν	Poisson's ratio
θ	Measured angle polar coordinate
ρ	Dislocation core radial coordinate
ω	Fundamental function of Gauss-Chebyshev quadratures

Chapter 1

Contact mechanics

Contacts are used to transmit force between bodies. Indeed it is only through contact or body forces, that force may be transmitted. Contacts concentrate stress near the contact interface. Consequently, regions of the body local to the contact are often the first to fail. Hence stresses caused by contact are, in many cases, life limiting factors for components and structures. Therefore it is important to study and understand the effects of these contacts, whatever geometric form or state (stationary, sliding or rolling) the contact exhibits.

Some examples of contacts which are life limiting for critical structures are the dovetail roots of fan blades in a gas turbine [1, 2, 3] and locking segments used to connect pipelines to deep sea oil wells [4, 5].

Throughout this work we utilise several simplifications. We assume that both of the contacting bodies are made of linearly elastic material, which is capable of analysis within the framework of linear elasticity. Hence displacements are small and the first order definitions of strains apply. Further, we assume that these materials are the same, or at least have the same elastic properties for each body. Finally, we assume that, in the neighbourhood of the contact, the materials are

homogeneous and isotropic.

1.1 Definition and classification of contacts

We shall start this review by identifying and classifying engineering contacts into four distinct types as shown in Figure 1.1. This classification will be useful as each type of contact must be described differently [6, 7]. We introduce all of these types here. However we will concentrate on incomplete and complete contacts in this work.

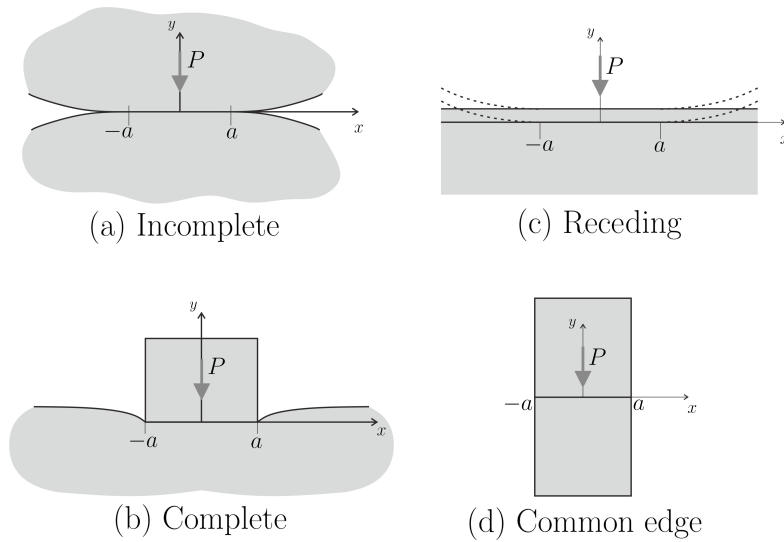


Figure 1.1: Types of contact

1.1.1 Incomplete contacts

Incomplete contacts are characterised by the fact that the contact area will increase with a higher normal load, P . Consequently, for each contact, this rule can be characterised by the equation known as the contact law, which relates contact size to the normal force. The contact law is, in many cases, nonlinear.

We may further subdivide the class of incomplete contacts into conformal and non-conformal contacts. A conformal contact is one in which the two contacting bodies have curvatures in the same direction, whereas a non-conformal contact is one where the two bodies have curvatures in the opposite direction. This is illustrated in Figure 1.2.

For non-conformal contacts, each of the bodies may be represented as a half-plane, and half-plane analysis may be used. Consequently, the geometry of the bodies remote from the contact is not important.

Incomplete contacts invariably display a contact pressure distribution which falls smoothly to zero at the contact edge [8]. If, as we assume throughout here, the materials have the same elastic properties, or alternatively the Dundurs parameter [9] $\beta = 0$ or the contact is lubricated so no shear traction arises, then the contact pressure falls to zero in a square root bounded manner.

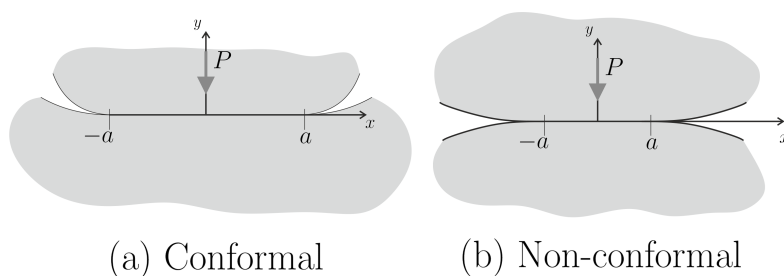


Figure 1.2: Types of incomplete contact

The seminal work in the study of incomplete contacts is Hertz's paper [10] from 1882, in which he was able to find closed-form solutions for the contact law and the contact pressure distribution for bodies whose profile is described by a second order polynomial. Subsequently studies by Cattaneo [11] and Mindlin [12, 13, 14] focussed on the development of solutions in the presence of partial slip. A generalisation of these results was found independently by Ciavarella [15] and

Jäger [16]. Further studies of incomplete contacts are beyond the scope of this review. Recent advances and a review of the response of these contacts can be found in Ramesh [17] and Andresen [18].

1.1.2 Complete contacts

Complete contacts are characterised by the fact that the size of the contact is predetermined by the geometry of the contacting bodies. The edges of the contact are defined by the location of a discontinuity in the gradient of the profile of the contacting bodies. They are exemplified by the square block on a half-plane shown in Figure 1.1(b).

Examples of complete contacts arise in mechanical joints rarely because in many practical instances edges are slightly rounded, either by burr removal or by design. However, such a contact can exhibit behaviour that is similar to that of a complete contact, as the effect of edge rounding may be small, especially at any distance removed from the contact edge.

True complete contacts still occur in incorrectly finished assemblies, and in cases where damage has occurred to one of the contacting bodies. It is important therefore to quantify the behaviour of such contacts, as the life of a damaged assembly is reduced, and its method of failure is important to safety considerations.

A feature of these contacts is that they cannot be tackled analytically, as it is not possible to idealise the bodies as half-planes. Instead, in order to evaluate the full contact interface, a finite element method or boundary element method must be employed to determine the contact pressure distribution [19]. This requires some calculations using the finite element method. This is complicated by the presence of singularities in both normal and shear traction at the edges of the

contact [20].

One of the reasons for the difficulties in analysing the problem is the presence of a coupling between normal and shear tractions, meaning that the application of a normal pressure induces both normal and shear tractions.

1.1.3 Receding contacts

Receding contacts are exemplified by an elastic layer resting on a half-plane, as shown in Figure 1.1(c). An interesting feature of some members of this class, for example a layer resting on a half plane subject to a line load, is that the application of an infinitesimally small normal load causes the layer to lift off and a contact of finite size to form. Any subsequent increase in the normal load will not increase or decrease the size of the contact, but merely increase the displacements, strain and stress in proportion [21, 22].

Receding contacts occur frequently in engineering design, being found for example at bolted joints such as flanged joints. However, further detailed study of these contacts is beyond the scope of this project. A detailed review of the properties of these contacts, especially in their axisymmetric form is provided by da Ponte Lopes [23].

1.1.4 Common edge contacts

A common edge contact is formed when two bodies with aligned edges are clamped together. The geometry of both bodies pre-determines the contact size. A similarity with complete contacts is the difficulty in analysing them generally because of their strong dependence on geometry. Uniquely among contacts, they have a finite contact pressure at the edge. Several studies have investigated the partial slip

problem in common edge contacts of specific geometries [24, 25, 26]. Further discussion of these contact problems is beyond the scope of this project.

1.2 Contact behaviour

Contacts are a rich area of study because they have a significant effect on the state of stress within the contacting bodies. Furthermore, this can then cause a behaviour or response which has further effects on the state of stress or condition of the contacting bodies. This can occur in several different ways.

1.2.1 Frictional phenomena

The relative movement of two bodies over one another will cause a net loss of energy because of the force of friction. Friction is a tangential reaction force which occurs across multiple scales and states. It can take many forms and can be both useful, such as in car tyres and brakes, and deleterious, such as in internal combustion engines and bearings. Friction is itself an extremely complex topic and the nature of friction is not well understood. There are many different models which may be used to describe frictional behaviour. A detailed analysis of all of these models is beyond the scope of this thesis, but we will study some of its effects.

Throughout this project we will use the Coulomb friction model, proposed by Amontons [27] and Coulomb [28], which relates the magnitude of the shear and normal forces required to effect sliding by a constant coefficient of friction. This is the most simple of the frictional models considered, but we also mention the static friction model, which has two distinct coefficients of friction, one describing the relationship needed for slip to start from a stationary position, and one describing

the relationship required for slip to persist, by observation the static coefficient of friction is larger than the dynamic one. Another model is the viscous friction model which proposes that the coefficient of friction depends linearly on the slip velocity, and the Stribeck model, which suggests a non-linear dependence which is not strictly increasing. A full description is given by Landes [29].

Sliding

Contacts transmit forces between contacting bodies, but, this is not always a lossless process. When external loads are resolved orthogonal to the contact interace, if the shear load is sufficiently large compared to the normal load the contact will slide. That is one body can move over the top of the other.

A contact is said to be sliding if all points on the contacting face of one body are moving relative to a point on the contacting face of the other body. In this case the two components of traction along the contact interface will be scaled versions of each other, as at every point $q(x) = f_c p(x)$, where $q(x), p(x)$ are the shear and normal tractions as functions of position respectively, and f_c is the coefficient of friction.

Partial slip

The phenomenon known as partial slip describes the process by which some particles along the contact interface slip but others remain stuck. Partial slip requires that the pointwise shear traction $q(x)$ along the interface satisfies the Coulomb friction condition for sliding [27, 28]. Hence we have that $q(x) = f_c p(x)$ for x in the slipping region. This localised slip will require elastic deformation of the material and is not therefore sustainable ad infinitum.

The shear traction distribution is changed from the sliding traction or adhered traction distributions, and if the contact is coupled, either because of the type of contact or an elastic mismatch, the contact pressure distribution will change too. One or more regions of partial slip may arise along a contact interface, and the contact is said to be in a state of partial slip if one or more points along the interface are slipping and one or more points along the interface are adhered (not slipping.)

Cyclic loading may lead to the formation of a zone of partial slip, and then these points becoming adhered again at load reversal. These points then may or may not slip in the other direction during the next loading phase. This is known as cyclic slip and is a common phenomenon, often found in incomplete contacts. Barber et al [30] showed that this will always occur during periodic loading of an incomplete contact if the loads are out of phase.

Wear

The presence of sliding, partial slip or cyclic slip over a sustained period will lead to wear. Wear is an overarching term for processes which lead to the damage or erosion of one or both of the contacting bodies. Wear is an extremely complex topic [31] and will not be discussed further in this thesis. However, the prospect of long term wear means that cyclic slip and sliding are processes which can ultimately lead to damage, and an understanding of these processes is important.

1.2.2 Edge cracks

Contacts affect the state of stress in contacting bodies, as has been stated before. This can lead to frictional effects, or, if one or both of the bodies contains a defect,

could also lead to the nucleation or propagation of cracks. Whilst it is possible for a contact to play some role in the nucleation or propagation of a buried crack, it is less likely and the existence of a contact is predicated on a state of normal compression on the contact interface, and so a large region of tension is unlikely to be caused by a contact under normal circumstances.

As such, attention is normally restricted to edge cracks. The growth of the cracks can be stopped, or accelerated by the nearby presence of a contact. Cwiekala and Hills [32] have shown that the relationship between the stress intensity factor for an open edge crack, and the loads sustained by an incomplete contact can be summarised by a calibration matrix, and Churchman and Hills [33] have shown similar for a crack at the edge of a square complete contact.

1.2.3 Plastic deformation

Plasticity, as opposed to elasticity, is a form of deformation which permanently changes the shape of a body. Plasticity also falls beyond the scope of this thesis, and an overview of the topic is given by Bertram [34]. Plasticity occurs when the stresses become very large, especially when the stress at a point exceeds the yield strength of the material. Material distortion results from the irreversible glide of dislocations rather than the reversible change change of interatomic distance.

1.3 Summary

This project is focussed on the asymptotic response of complete contact edges. Included in this is the modelling of slip problems. As outlined in section 1.2.1, the presence of friction phenomena will change the state of stress. So, if we wish

to model cracks, separations or slip, we must find a way of modifying the state of stress. This can be done by superposition, and to do this we must consider strain nuclei [35, 36].

We may choose any kind of strain nucleus, such as a dislocation, a dipole or a force-pair. For this project, we will choose dislocations, which are the topic of the next chapter. This choice has the advantage that closed form solutions may be found for the influence of a dislocation in an infinite plane and a semi-infinite plane, which will be useful later for verification [37, 38, 39, 40].

After this, the application of dislocations, and the distributed dislocation method is discussed and applied to complete contact problems. Chapter 3 discusses previous efforts to classify and describe the edge behaviour of complete contacts.

One of the difficulties encountered by previous studies was the need to classify the expected behaviour, then enforce conditions. In chapter 4, classification methods are developed which allow the expected behaviour to be identified without significant calculation, and before any conditions are enforced.

Another common problem comes from the use of the dislocations. Dislocation kernels are well known, but the domain in which the dislocation is placed will modify these kernels. For the purposes of complete contact problems, this domain will be a semi-infinite reentrant wedge. A method for calculating the kernels in wedges was devised by Hecker and Romanov, and in chapter 5 this is expanded and generalised. The method described in chapter 5 is used throughout, but an alternative method is discussed in chapter 8, and the advantages and disadvantages of the two methods are compared.

Finally, some examples of edge behaviour in general complete contacts are explored. Chapter 6 describes the effect of an edge crack at the edge of contact,

or a pair of cracks, originating from the same location. Chapter 7 describes the effect of a zone of partial slip, the slip displacements and the size of the slip zone formed when the contact is created.

Chapter 2

Dislocation theory

The study of dislocations initially comes from their application to modelling the state of stress in crystals [41]. However, we will utilise them here as a strain nuclei. They are useful for this purpose as their original interpretation was to induce a strain in a crystal and model the stress created. Therefore they produced a well-modelled and well understood stress field that we may superpose onto our bilateral solution.

The creation of an edge dislocation and subsequent induction of the associated state of stress is achieved by making a cut in an infinite solid from a given point, the core of the dislocation, to a free boundary, or infinity. Then the material either side of the cut, known as the path cut, is displaced by a fixed amount, known as the Burgers vector, in a given direction. Material is added where necessary and the solid welded together again.

There are three specific methods of creating dislocations that we will discuss here, and the resulting dislocations are known by the names glide, climb, (collectively called edge) and screw dislocations. An edge dislocation may be considered in a plane.

If we consider an infinite plane and make a cut, starting from the origin to an infinite point, displace all the material on one side of the cut tangentially by a fixed amount b_x , then weld the material together again, this is a glide edge dislocation. Similarly if we make the same cut but force the material on either side of the cut apart, and insert a semi-infinite strip of material of thickness b_y , then weld everything together, this is a climb edge dislocation.

Finally, to consider a screw dislocation we must consider the anti plane direction. We make a cut from the core in a direction perpendicular to the plane. Then we displace all the material on one side of the cut in the anti plane direction by the fixed amount b_z . We weld this back together to produce a screw dislocation. A sketch showing the creation of these three types of dislocations, including the direction of the Burgers vector, is shown in Figure 2.1.

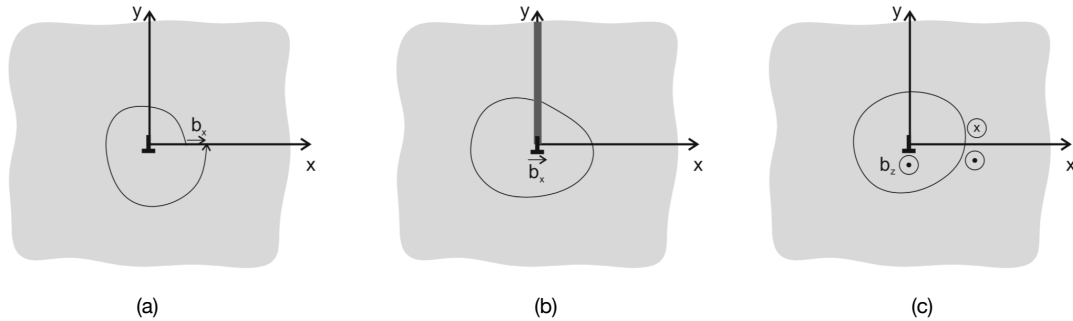


Figure 2.1: Sketch showing the formation of (a) a glide edge dislocation (b) a climb edge dislocation and (c) a screw dislocation

2.1 State of stress

With these dislocations now being created, we must determine the state of stress they induce in the body. To do this we require a stress function ϕ which satisfies the biharmonic equation:

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0. \quad (2.1)$$

The desired function is given by Dundurs [42]

$$\phi = -\frac{2\mu b_x \sqrt{x^2 + y^2}}{\pi(\kappa + 1)} \log(y), \quad (2.2)$$

where μ is the modulus of rigidity, κ is Kolosov's constant. Standard results [42, 43, 36] allows us to write

$$u_x = \frac{b_x}{2\pi(\kappa + 1)} \left[(\kappa + 1) \arctan\left(\frac{y}{x}\right) + \frac{2xy}{x^2 + y^2} \right], \quad (2.3)$$

$$u_y = -\frac{b_x}{2\pi(\kappa + 1)} \left[(\kappa - 1) \log(\sqrt{x^2 + y^2}) + \frac{2x^2}{x^2 + y^2} \right], \quad (2.4)$$

$$\sigma_{xx}(x, y) = \frac{2\mu}{\pi(\kappa + 1)} \left(b_x \left(-\frac{y(3x^2 + y^2)}{(x^2 + y^2)^2} \right) + b_y \left(\frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \right) \right), \quad (2.5)$$

$$\sigma_{xy}(x, y) = \frac{2\mu}{\pi(\kappa + 1)} \left(b_x \left(\frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \right) + b_y \left(\frac{y(x^2 - y^2)}{(x^2 + y^2)^2} \right) \right), \quad (2.6)$$

$$\sigma_{yy}(x, y) = \frac{2\mu}{\pi(\kappa + 1)} \left(b_x \left(\frac{y(x^2 - y^2)}{(x^2 + y^2)^2} \right) + b_y \left(\frac{x(x^2 + 3y^2)}{(x^2 + y^2)^2} \right) \right), \quad (2.7)$$

$$\sigma_{xz}(x, y) = \frac{2\mu b_z}{\pi(\kappa + 1)} \left(-\frac{y}{(x^2 + y^2)^2} \right), \quad (2.8)$$

$$\sigma_{yz}(x, y) = \frac{2\mu b_z}{\pi(\kappa + 1)} \left(\frac{x}{(x^2 + y^2)^2} \right). \quad (2.9)$$

2.1.1 The semi-infinite plane

Now that we are able to describe the state of stress in an infinite body, our attention inevitably turns to describing the stress in other kinds of body. Perhaps the most obvious choice when considering other bodies to study is the semi-infinite, or half-plane. Incomplete contacts use half-plane methods to solve problems generally [44, 6, 17, 18].

It is therefore natural to consider the state of stress induced by a dislocation in a half-plane. An Airy stress function approach may again be taken to calculate the stress induced by such a dislocation, as laid out by Dundurs and Sendeckyj [45, 36]. The resulting stress induced by an edge dislocation in a half-plane, where the dislocation core is at a distance c directly below the free surface is given by

$$\sigma_{xx} = \frac{\mu}{\pi(\kappa + 1)} [b_x G_{xxx}(x, y, c) + b_y G_{yxx}(x, y, c)], \quad (2.10)$$

$$\sigma_{xy} = \frac{\mu}{\pi(\kappa + 1)} [b_x G_{xxy}(x, y, c) + b_y G_{yyx}(x, y, c)], \quad (2.11)$$

$$\sigma_{yy} = \frac{\mu}{\pi(\kappa + 1)} [b_x G_{xyy}(x, y, c) + b_y G_{yyy}(x, y, c)], \quad (2.12)$$

where G_{ijk} are the half-plane kernel functions, given in appendix A.

2.1.2 The three-quarter plane

The final problem we will consider in this brief review is the influence of a dislocation in a three-quarter plane. This problem cannot be tackled in the same way as the half-plane problem and is indicative of a problem that will arise several times in this project. Whilst the problem of the state of stress induced by a dislocation in an infinite plane, and in certain other specific geometries, has a well known solution, geometries which place further bounds on the state of stress, such as

requiring free surfaces to be traction free, do not usually have a general solution.

A method for determining the state of stress in such geometries is therefore an important problem. Hecker and Romanov [46, 47] propose a method which involves distributing edge dislocations along trace lines in an infinite plane, where the free surfaces of the three quarter plane are to be, with the density of these distributions along them being held as a variable. An object dislocation is then placed at a given point in the infinite plane and the density of the distributions are found such that the stress induced on the free surfaces by the object dislocation vanishes. This setup is sketched in Figure 2.2.

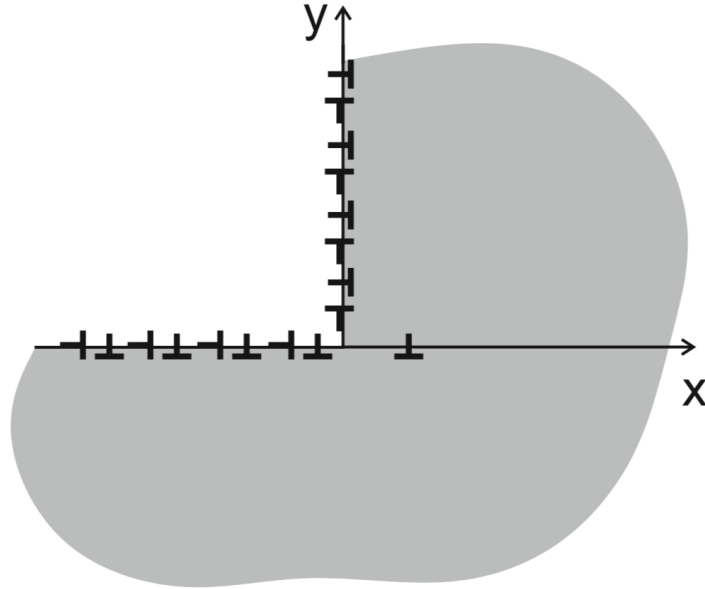


Figure 2.2: Sketch showing the distributions of dislocations used to find the influence of a dislocation in a three quarter plane

This approach has been taken to tackle the problem of dislocations in a three quarter plane by Churchman et al [48] and by Qiu et al [49]. They present a corrective function F_{ijk} which allows the stress kernels generally to be written as $G_{ijk} + F_{ijk}$ where G_{ijk} are the infinite plane kernels described in section 2.1. The

corrective functions are given in appendix B.

2.2 Stress superposition

We are now able to describe the state of stress induced by the insertion of dislocations in several different geometries. Let us now apply Bueckner's principle [35]. This states that we can superpose the stress fields produced by the introduction of a dislocation with other known stress fields, including those of other dislocations, in order to fulfil known geometric requirements, such as free surfaces being traction free.

Alternatively, the principle may be stated as [35, 36, 50, 23]:

The mode I stress intensity factor for a crack in a body subjected to external forces is identical to that for a similar crack, subjected to internal pressure in a similar body which has no external forces acting on it where the internal pressure acting on the crack is equal to the stress that would exist normal to the crack line in the uncracked body subjected to the external forces. An analogous procedure exists for shear stresses and the calculation of sliding mode stress intensity factors.

2.2.1 Buried crack

The easiest way to illustrate the principle is to apply it to some simple examples. We start here with the simplest possible example. We place a short, straight crack of known length $2a$ in an infinite solid, loaded by a tensile stress $\hat{\sigma}_{yy}(x)$, perpendicular to the line of the crack.

Applying Bueckner's principle, we must first determine the stress field induced in the uncracked body. The tractions along the line of the crack are the *stress that*

would exist normal to the crack line in the uncracked body subjected to the external forces. Let these stresses, both normal and shear, be denoted by $\hat{\sigma}_{xy}(x)$ and $\hat{\sigma}_{yy}(x)$, for $-a \leq x \leq a$.

Next, consider the state of stress induced by the addition of dislocations along the crack line. In this case, the material is an infinite solid, so the influence functions are given in section 2.1. It is obvious that if multiple dislocations are placed along the line, the superposition of their stress fields will lead to a new, more complicated, stress field.

When inserting these dislocations, we may choose the value of the Burgers vector b_x , or b_y . For now, let us hold these as variables. The placing of several such dislocations, with Burgers vector b_x , leads to the stress field described by

$$\bar{\sigma}_{yy}(x, 0) = \frac{2\mu}{\pi(\kappa + 1)} \sum_{\text{dislocations}} b_x(\xi) G_{xyy}(x, 0, \xi, 0),$$

where G_{xyy} , is the influence function of the dislocation. This sum of all the dislocations is inconvenient, so we introduce the continuous dislocation density $B_x(\xi)$, which is given by

$$B_x(\xi) = \frac{db_x(\xi)}{d\xi}.$$

This allows us to rewrite the stress as

$$\bar{\sigma}_{yy}(x, 0) = \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a B_x(\xi) G_{xyy}(x, 0, \xi, 0) d\xi, \quad (2.13)$$

where the dislocation density $B_x(\xi)$ is now our variable. A very similar operation may be undertaken for the shear stress.

It is required that the faces of the crack, which are free surfaces, are traction free. Therefore, we may superpose our solutions, giving rise to the pair of simultaneous

equations:

$$\begin{aligned} 0 &= \hat{\sigma}_{yy}(x, 0) + \bar{\sigma}_{yy}(x, 0), \\ 0 &= \hat{\sigma}_{xy}(x, 0) + \bar{\sigma}_{xy}(x, 0), \quad \text{for } -a \leq x \leq a \end{aligned}$$

which we now know can be written as a pair of simultaneous integral equations:

$$\begin{aligned} 0 &= \hat{\sigma}_{yy}(x, 0) + \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a B_x(\xi) G_{xyy}(x, 0, \xi, 0) d\xi, \\ 0 &= \hat{\sigma}_{xy}(x, 0) + \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a B_x(\xi) G_{xxy}(x, 0, \xi, 0) d\xi \quad \text{for } -a \leq x \leq a. \end{aligned}$$

Practically, both glide edge and climb edge dislocations will be required, so the equations become

$$0 = \hat{\sigma}_{yy}(x, 0) + \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a B_x(\xi) G_{xyy}(x, 0, \xi, 0) + B_y(\xi) G_{yxx}(x, 0, \xi, 0) d\xi, \quad (2.14)$$

$$0 = \hat{\sigma}_{xy}(x, 0) + \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a B_x(\xi) G_{xxy}(x, 0, \xi, 0) + B_y(\xi) G_{yxy}(x, 0, \xi, 0) d\xi, \quad (2.15)$$

$$\text{for } -a \leq x \leq a.$$

The solution of these equations allows us to find the dislocation densities and hence place the correct distributions of dislocations to model the presence of a short, straight crack in an infinite solid. The problem is complicated, however, by the fact that the functions G_{ijk} contain Cauchy singularities.

2.2.2 Singular integral equations

When applying Bueckner's principle and using a distribution of dislocations, it will often prove necessary to solve a set of simultaneous Cauchy singular integral equations. These problems require the inversion of the integrals, and in all but the simplest cases, this cannot be done analytically, requiring instead a numerical solution.

To do this, we will employ the Gauss-Chebyshev quadrature, using a method developed by Erdogan et al [51], based on the original Gaussian quadratures [52]. The idea of this method is to turn the integral equation into a set of simultaneous linear equations, and at the same time to consider the solution as a bounded function by separating the singularity.

A derivation and explanation of the method is beyond the scope of this thesis. Instead we will apply this method to the simplest case, with an example equation:

$$F(v) = \frac{1}{\pi} \int_{-1}^1 C(u) \frac{1}{v-u} du, \quad (2.16)$$

where $F(v)$ is a known function, representing in our case the desired stress along the line, and $C(u)$ is an unknown function, the dislocation density. This is a typical equation that we will come across with dislocation problems, and the first thing we must decide is the behaviour of the function $C(u)$. We have to decide the endpoint behaviour of the function, whether it is singular or bounded at $u = \pm 1$. Table 2.1 shows the choices we have.

Once we decide on the end point behaviour, we now separate our function $C(u)$. We define

$$C(u) = \omega(u)\phi(u),$$

ine -1\+1	Singular	Bounded
ine Singular	I	II
ine Bounded	III	IV
ine		

Table 2.1: The cases of end-point behaviour

where $\phi(u)$ is a bounded function whose value we will find later, and $\omega(u)$ is the fundamental function. The form of this function is different depending on the end point behaviour of $C(u)$, and the various options are set out in table 2.2. By doing this we may rewrite equation 2.16 as

$$F(v_j) = \sum_{i=1}^N W_i \frac{\phi(u_i)}{v_j - u_i} \quad \text{for } k = 1, \dots, N - n, \quad (2.17)$$

where N is some large natural number, and n is dependent on the end point behaviour, as outlined in table 2.2. This is a set of $N - n$, simultaneous equations, which may be solved using any chosen method. There are N discrete integration points, given by u_i , and $N - n$ discrete collocation points, given by v_j , these values are again given in table 2.2. It should be noted that the transition from equation 2.16 to equation 2.17 is not dependent on the end point behaviour.

2.3 Summary

This chapter has defined what a dislocation is, how they are created, their origins in crystalline lattice defect modelling and how they can be used to modify the state of stress. This includes a description of the kernel functions in the specific cases of an infinite and semi-infinite plane, and a description of how to calculate the kernel functions in more general geometries.

Having these kernel functions, Bueckner's principle can then be used. This

ine Case	$\omega(u)$	u_i	v_j	n	W_i
ine I	$\frac{1}{\sqrt{1-u^2}}$	$\cos\left(\pi \frac{2i-1}{2N}\right)$	$\cos\left(\pi \frac{j}{N}\right)$	1	$\frac{1}{N}$
ine II	$\sqrt{\frac{1-u}{1+u}}$	$\cos\left(\pi \frac{2i}{2N+1}\right)$	$\cos\left(\pi \frac{2j-1}{2N+1}\right)$	0	$\frac{2(1-u_i)}{2N+1}$
ine III	$\sqrt{\frac{1+u}{1-u}}$	$\cos\left(\pi \frac{2i-1}{2N+1}\right)$	$\cos\left(\pi \frac{2j}{2N+1}\right)$	0	$\frac{2(1+u_i)}{2N+1}$
ine IV	$\sqrt{1-u^2}$	$\cos\left(\pi \frac{i}{N+1}\right)$	$\cos\left(\pi \frac{2j-1}{2(N+1)}\right)$	-1	$\frac{(1-u_i^2)}{N+1}$
ine					

Table 2.2: The cases of end-point behaviour [36]

allows for the superpositions of stress fields caused by distributions of dislocations, with those created by other means. Practically, this means that distributions of dislocations may be inserted into a body to change the state of stress. This principle is used to tackle various problems, and is critical for the application of these methods to complete contact problems in the next chapter. In this chapter, this approach has been used to tackle the buried crack problem. This important illustrative problem demonstrates that the application of Bueckner's principle, will, in most cases, require the solution of a Cauchy singular integral equation(s).

The last part of this chapter has been dedicated to methods of solution for such equations. Specifically, the use of numerical quadratures to allow for numerical inversion; namely the Gauss-Chebyshev quadratures. These methods allow for a discretisation of the integral, and the conversion from an integral to a set of simultaneous finite sums. These simultaneous equations can then be solved using traditional methods, and the solution to these can then be converted back into a solution of the integrals.

Chapter 3

Edge behaviour of complete contacts

Complete contacts are those whose contact profiles are discontinuous at the contact edge. This definition was first appreciated by Boussinesq [20, 53] in 1885. The state of stress at the corner of this contact is singular and can take several different forms [54, 44] depending upon frictional conditions, indenter angle and elastic constants. In reality these contacts will have a plastic zone at the corner as all real materials yield plastically at finite values of stress.

The behaviour of complete contacts is the principle consideration of this thesis. There have been numerous previous attempts to classify and explain this behaviour, but the phenomena are complex and difficult to generalise. The main reason for this is that the edge of a complete contact cannot be approximated using half plane theory, as is done with incomplete contacts. This means that an analysis of this class of contacts must be done in one of three ways: an asymptotic approximation at the contact edge, a finite element model, or an elasticity description of both contacting bodies.

Each of these methods has its respective advantages and disadvantages. An asymptotic method will only describe the behaviour in the region of validity of the

asymptote, but is often the simplest way to focus on areas of interest. A finite element method will provide a description of the full contact interface - however the method is only applicable to a particular example. A full elasticity description will give the best results along the entire contact interface, however it is an extremely complex method and is not used in practice for this reason.

Asymptotic methods allow an analytical representation of the state of stress around a point of interest in the solution. This representation is limited in its nature. It is not always possible to represent accurately the state of stress analytically, and in other cases, it may be desirable to simplify the expression. Asymptotic methods are an approximation of the true solution, which is valid only for a specific region, known as the range of validity.

However asymptotic methods are highly applicable, the asymptote being able to be fitted into many different problems, irrespective of the remote geometry. Furthermore, an analytical solution gives us a greater understanding of the state of stress and how changing loading will affect it. Finally, we are able to use the asymptotic solution as a basis to tackle other more complex problems.

As stated in section 1.1.2 and given by Boussinesq [20], the state of stress at the edge of a complete contact is singular. Consequently, it is likely that life-limiting phenomena will begin at the edges of contact, so an asymptotic solution centred on the edge of contact should allow an analysis of these phenomena using an analytical approximation to the state of stress at the contact edge.

Considering instead the finite element method, this may appear to present some advantages. Unlike an asymptotic approach the finite element method is able to describe the tractions along the entire interface, rather than just in a certain region. Furthermore a finite element approach is much simpler than a full elasticity

description of all of the contacting bodies.

However, there are several major disadvantages to using the finite element method. As previously stated in section 1.1.2, the state of stress at the edge of contact will be singular. By the nature of the approach, the finite element method struggles to calculate accurately and represent singularities. Furthermore, the discretisation required to avoid this error requires a length dimension far smaller than that which can be practically achieved with the finite element method, as the distance must be small compared to the distance to the nearest feature, a term which can not be accurately defined.

3.1 The Williams solution

Centering the asymptotic solution for a complete contact at the edge of contact presents some difficulties. As discussed in section 1.2 the edge of contact can be a region in which slip or cracks occur. To begin with however, we will ignore this problem, as one of the benefits of an asymptotic solution is that it can be modified to consider behavioural changes at a later point. Considering then the adhered edge of a complete contact, it is known that the adhered contact will transmit all tractions perfectly. So we may consider the adhered edge of a complete contact as a single monolithic reentrant wedge.

The Williams solution was developed to tackle the problem of analysing the stress state around a reentrant wedge, where the stresses are singular. Initially this is done by defining a set of polar coordinates centred on the apex of the wedge, with the wedge bisector being the pole of the coordinate set. The stresses can then be expanded as an asymptotic series in powers of r .

We know that the tractions on the boundary vary in a power sense, $r^{\lambda-1}$:

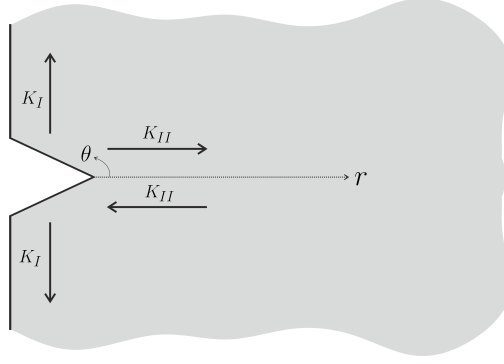


Figure 3.1: The stressed notch in a plate

we have used $\lambda - 1$ for convenience, as will become clear later. Hence the stress potential will have the form $\phi = r^{\lambda+1}f(\theta)$. The biharmonic equation may then be used to reduce this further to

$$\phi = r^{\lambda+1}[A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta]. \quad (3.1)$$

It is a condition that the boundaries be traction free, so we have the condition

$$\sigma_{r\theta} = \sigma_{\theta\theta} = 0 \quad \text{when} \quad \theta = \pm\alpha,$$

where α is half of the total internal angle of the wedge.

The stresses $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ are found by differentiation and the condition becomes

$$\begin{bmatrix} (\lambda_I + 1)\sin(\lambda_I + 1)\alpha & (\lambda_I - 1)\sin(\lambda_I - 1)\alpha \\ (\lambda_I + 1)\cos(\lambda_I + 1)\alpha & (\lambda_I - 1)\cos(\lambda_I - 1)\alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0 \quad (3.2)$$

and

$$\begin{bmatrix} (\lambda_{II} + 1)\cos(\lambda_{II} + 1)\alpha & (\lambda_{II} - 1)\cos(\lambda_{II} - 1)\alpha \\ (\lambda_{II} + 1)\sin(\lambda_{II} + 1)\alpha & (\lambda_{II} - 1)\sin(\lambda_{II} - 1)\alpha \end{bmatrix} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = 0. \quad (3.3)$$

These equations have non-trivial solutions if, and only if, the determinants of the coefficient matrices are non-zero. Hence we require

$$\lambda_I \sin(2\alpha) + \sin(2\lambda_I \alpha) = 0 \quad (3.4)$$

$$\lambda_{II} \sin(2\alpha) - \sin(2\lambda_{II} \alpha) = 0. \quad (3.5)$$

These are the characteristic equations of the Williams solution, and the values λ_I and λ_{II} are the eigenvectors, and characterise the nature of the singularity in the mode I and mode II terms of the expansion respectively.

Note that we require $\lambda_i > 0$ to satisfy the criterion that the strain energy, given by

$$U = \frac{1}{2} \int_0^{2\pi} \int_0^r \sigma_{ij} e_{ij} r dr d\theta = C \int_0^r r^{2a+1} dr,$$

where e_{ij} are the strains and C is a finite constant. The integral is bounded if, and only if, $a > -1$, hence we require $\lambda_i > 0$. We also require $\lambda_i \neq 1$ as in this case the stress potential becomes $\phi = A_4 \sin(0)$.

These eigenvalues are plotted with the variation of α in Figure 3.2. Notice that the mode II eigenvalue is only sufficient to cause a singularity for values of $2\alpha > 257.4^\circ$, so mode II excitation of a wedge of smaller angle does not give rise to singular stresses.

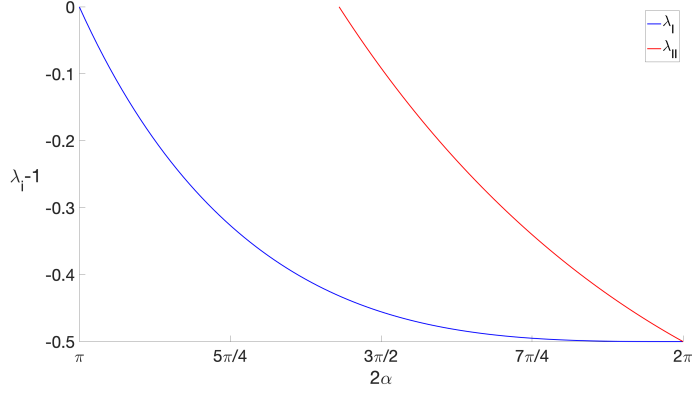


Figure 3.2: Variation of the eigenvalues of the Williams solution with wedge angle

3.1.1 Eigenvectors

We have the eigenvalues of the solution. We must now find the corresponding eigenvectors. We begin this process by considering equation (3.2). Expanding this condition we find expressions for A_1 and A_2 .

$$\begin{aligned} \frac{A_1}{K_I}(\lambda_I + 1)\sin(\lambda_I + 1)\alpha + \frac{A_2}{K_{II}}(\lambda_I - 1)\sin(\lambda_I - 1)\alpha &= 0 \\ \frac{A_1}{K_I}(\lambda_I + 1)\cos(\lambda_I + 1)\alpha + \frac{A_2}{K_{II}}(\lambda_I - 1)\cos(\lambda_I - 1)\alpha &= 0 \end{aligned}$$

where K_I and K_{II} are the mode I and mode II stress intensity factors, described in section 3.1.2, gives

$$A_1 = K_I \frac{\cos(\lambda_I - 1)\alpha}{\lambda_I(\lambda_I + 1)(\cos(\lambda_I - 1)\alpha - \cos(\lambda_I + 1)\alpha)} \quad (3.6)$$

$$A_2 = K_{II} \frac{\cos(\lambda_I + 1)\alpha}{\lambda_I(\lambda_I + 1)(\cos(\lambda_I + 1)\alpha - \cos(\lambda_I - 1)\alpha)}. \quad (3.7)$$

We repeat this process with equation (3.3) to find

$$A_3 = K_I \frac{\sin(\lambda_{II} - 1)\alpha}{\lambda_{II}((\lambda_{II} - 1)\sin(\lambda_{II} + 1)\alpha - (\lambda_{II} + 1)\sin(\lambda_{II} - 1)\alpha)} \quad (3.8)$$

$$A_4 = K_{II} \frac{\sin(\lambda_{II} + 1)\alpha}{\lambda_{II}((\lambda_{II} + 1)\sin(\lambda_{II} - 1)\alpha - (\lambda_{II} - 1)\sin(\lambda_{II} + 1)\alpha)}. \quad (3.9)$$

Differentiating equation (3.1) twice we find that the stresses are given by

$$\begin{aligned} \sigma_{rr} = r^{\lambda-1} & [-A_1\lambda(\lambda+1)\cos(\lambda+1)\theta - A_2\lambda(\lambda-3)\cos(\lambda-1)\theta \\ & - A_3\lambda(\lambda+1)\sin(\lambda+1)\theta - A_4\lambda(\lambda-3)\sin(\lambda-1)\theta] \end{aligned} \quad (3.10)$$

$$\begin{aligned} \sigma_{r\theta} = r^{\lambda-1} & [A_1\lambda(\lambda+1)\sin(\lambda+1)\theta + A_2\lambda(\lambda-1)\sin(\lambda-1)\theta \\ & - A_3\lambda(\lambda+1)\cos(\lambda+1)\theta - A_4\lambda(\lambda-1)\cos(\lambda-1)\theta] \end{aligned} \quad (3.11)$$

$$\begin{aligned} \sigma_{\theta\theta} = r^{\lambda-1} & [A_1\lambda(\lambda+1)\cos(\lambda+1)\theta + A_2\lambda(\lambda+1)\cos(\lambda-1)\theta \\ & - A_3\lambda(\lambda+1)\sin(\lambda+1)\theta - A_4\lambda(\lambda+1)\sin(\lambda-1)\theta]. \end{aligned} \quad (3.12)$$

We combine these equations with the solutions for A_i to find the eigenvectors, which are given by

$$f_{rr}^I(\theta) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\theta] - \frac{\lambda_I-3}{\lambda_I+1}\cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\theta]}{\cos[(\lambda_I + 1)\alpha] - \cos[(\lambda_I - 1)\alpha]} \quad (3.13)$$

$$f_{rr}^{II}(\theta) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\theta] - \frac{\lambda_{II}-3}{\lambda_{II}+1}\sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\theta]}{\sin[(\lambda_{II} - 1)\alpha] - \frac{\lambda_{II}-1}{\lambda_{II}+1}\sin[(\lambda_{II} + 1)\alpha]} \quad (3.14)$$

$$f_{r\theta}^I(\theta) = \frac{\sin[(\lambda_I - 1)\alpha]\sin[(\lambda_I + 1)\theta] - \sin[(\lambda_I + 1)\alpha]\sin[(\lambda_I - 1)\theta]}{\sin[(\lambda_I - 1)\alpha] - \frac{\lambda_I+1}{\lambda_I-1}\sin[(\lambda_I + 1)\alpha]} \quad (3.15)$$

$$f_{r\theta}^{II}(\theta) = \frac{\cos[(\lambda_{II} - 1)\alpha]\cos[(\lambda_{II} + 1)\theta] - \cos[(\lambda_{II} + 1)\alpha]\cos[(\lambda_{II} - 1)\theta]}{\cos[(\lambda_{II} - 1)\alpha] - \cos[(\lambda_{II} + 1)\alpha]} \quad (3.16)$$

$$f_{\theta\theta}^I(\theta) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\theta] - \cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\theta]}{\cos[(\lambda_I - 1)\alpha] - \cos[(\lambda_I + 1)\alpha]} \quad (3.17)$$

$$f_{\theta\theta}^{II}(\theta) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\theta] - \sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\theta]}{-\sin[(\lambda_{II} - 1)\alpha] + \frac{\lambda_{II}-1}{\lambda_{II}+1}\sin[(\lambda_{II} + 1)\alpha]}. \quad (3.18)$$

The notation used here is fairly standard and we will use it throughout this report. The subscripts refer to the corresponding components of the stress tensor, and the superscript refers to the mode I or mode II term as appropriate.

3.1.2 Stress intensity factors

This asymptotic solution is generally applicable to any problem with a suitable local geometry, ie. a wedge with total internal angle 2α . However, remote loads cannot be easily directly dealt with. Instead we use stress intensity factors. These allow us to write explicitly the asymptotic stresses in the form

$$\sigma_{ij}(r, \theta) = K_I f_{ij}^I(\theta) r^{\lambda_I-1} + K_{II} f_{ij}^{II}(\theta) r^{\lambda_{II}-1}, \quad (3.19)$$

where K_I and K_{II} are the mode I and mode II stress intensity factors. These are defined as follows

$$K_I = \lim_{r \rightarrow 0} \sigma_{\theta\theta}(r, 0) r^{1-\lambda_I} \quad f_{r\theta}^I(r, 0) = 0 \quad \& \quad f_{\theta\theta}^I(r, 0) = 1 \quad (3.20)$$

and

$$K_{II} = \lim_{r \rightarrow 0} \sigma_{r\theta}(r, 0) r^{1-\lambda_{II}} \quad f_{\theta\theta}^{II}(r, 0) = 0 \quad \& \quad f_{r\theta}^{II}(r, 0) = 1. \quad (3.21)$$

Clearly, this definition then allows us to write the stresses near the wedge apex in the form shown in equation (3.19). To find the stress intensity factors, we must perform finite element analysis of the geometry to determine a calibration between the external loads and the stress intensity factors.

This can be done individually for each of the proposed types of loading; by a normal force P , a shear force Q , a moment M , and a differential bulk tension $\Delta\sigma$.

$$\begin{bmatrix} K_I a^{\lambda_I-1} \\ K_{II} a^{\lambda_{II}-1} \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \begin{bmatrix} \frac{P}{2a} \\ \frac{Q}{2a} \\ \frac{M}{2a^2} \\ \frac{\Delta\sigma}{2} \end{bmatrix} \quad (3.22)$$

The problem is coupled as it is a complete contact so it is unlikely that there will be any zeros in the calibration matrix. One of the advantages of this type of problem is that it is linear, and so if multiple types of loading are applied, this calibration is still valid.

3.1.3 Validity of asymptotic solution

The state of stress at the edge of contact is now described asymptotically. However, as with any asymptotic solution, there are limits to the validity and usefulness of this solution. With this in mind, it is worth considering an alternative approach to the asymptotic method, which does not have some of the restrictions placed on the solution by the asymptotic approach, namely the finite element method.

The chief constraint on the asymptotic solution is that because of the assumption that we may treat the adhered contact as a wedge, intimate contact must be preserved along the length of the contact. This gives two constraints, firstly that the contact is preserved at the corner, this is ensured by $K_I f_{\theta\theta}^I(\phi) < 0$, where ϕ is the interface angle, and that the solution is valid if and only if $\sigma_{\theta\theta}(r) < 0$, meaning that the point where $\sigma_{\theta\theta}(r) = 0$, if it exists, is a bound on the range of asymptotic validity of the solution.

Another bound on the applicability of the asymptotic method, is that the edge of contact must be sufficiently remote from any other feature, such as another corner, contact or crack, that said feature will not affect the state of stress at the contact edge. Whilst not rigourously defined, this is nonetheless a constraint as the method should not be applied if other bounds would include or come close to the position of another feature.

Plasticity

It is known that a singular state of stress will always cause any real material to deform plastically, as all real materials have a finite yield stress. The problem of plastic deformation is not tackled in this thesis. We can make this assumption because of the work of Hills and Dini[55]. They compare the loading parameter G_0 , defined in by the same authors as $G_0 = K_I^{\frac{\lambda_{II}-1}{\lambda_{II}-\lambda_I}} K_{II}^{\frac{\lambda_I-1}{\lambda_I-\lambda_{II}}}$, normalised by the yield stress of the material k , to the distance of the plastic front from the contact edge, r_p , normalised by the mode mixity parameter d_0 , a length scale extracted from the Williams solution defined by $d_0 = \left| \frac{K_{II}}{K_I} \right|^{\frac{1}{\lambda_I-\lambda_{II}}}$.

They are able to show that the ratio $\frac{G_0}{k}$, is at least one order of magnitude greater than the ratio $\frac{r_p}{d_0}$, except possibly for extremely large loads observed near one of the free surfaces. For more reasonable loads, for example if $\frac{G_0}{k}$ has order 10^{-3} , the ratio $\frac{r_p}{d_0}$ is several orders of magnitude smaller. As such, we can proceed assuming that the plastic region will be very small compared to the region of asymptotic validity. Although this approximation does not account for the redistribution of stresses caused by plastic deformation, given the orders of magnitude, an elastic solution is a reasonable approximation.

3.2 Complete contact problems

The Williams solution gives us an asymptotic representation of the corner of an adhered complete contact. However, as laid out in sections 1.2.1 and 1.2.2, the edges of such contacts can behave in several different ways, within the confines of elasticity. If we wish to model the effects of such features we will have to modify the state of stress given by the Williams solution.

We have already seen how Bueckner's principle allows us to use dislocations to modify the state of stress. Therefore, if we insert a distribution of dislocations to either the adhered interface, or the intended crack line, we will be able to enforce the conditions necessary to model the state of stress created by a partial slip zone or an edge crack. As previously mentioned in relation to a three-quarter plane, it will first be necessary to use the method described by Hecker and Romanov [46, 47] to determine the kernel of the dislocations.

This problem has been tackled by Hecker and Romanov, and also by Churchman et al [48]. However, both of these come with limitations. Hecker and Romanov considered only a very small number of points, likely because of computational limitations. On the other hand, Churchman et al made a small mistake in their normalisation of the problem, which leads to a propagated error in the equations. Furthermore, they also failed to consider properly the scaling of the solution. Churchman asserted that for a dislocation placed at a point ρ and observed at a point r , we have that $F_{ijk}(r, \rho) = F_{ijk}(mr, m\rho)$, where F_{ijk} is the stress kernel, and m is some positive number. This is not true, whilst the state of stress scales, because the Burgers vector of the dislocation does not, the true relationship is given by $F_{ijk}(mr, m\rho) = \frac{1}{m}F_{ijk}(r, \rho)$.

This is based on the assumption that, because there is no inherent length scale

in the problem, the stress can be scaled, meaning that $\sigma_{jk}(r, \rho) = \sigma_{jk}(mr, m\rho)$, by scaling the entire system by the factor m . However, this would imply that $b_i F_{ijk}(r, \rho) = mb_i F_{ijk}(mr, m\rho)$, which means that the actual relationship must be given by $F_{ijk}(mr, m\rho) = \frac{1}{m} F_{ijk}(r, \rho)$. This problem is sketched in figure 3.3.

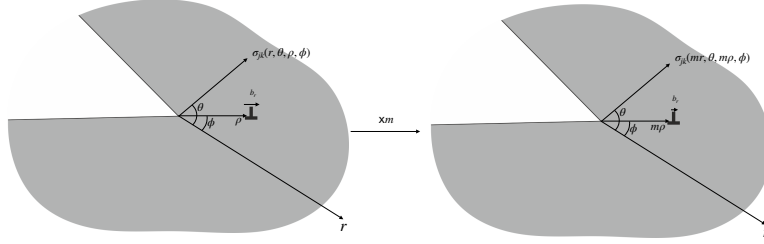


Figure 3.3: Sketch showing the effect of scaling the wedge without simultaneously scaling the Burgers vector

However, the biggest problem with both analyses is that they use Cartesian coordinate systems. Whilst this may not appear to be a problem initially, it restricts the considerations that either set of results can apply to, requiring problems which have orthogonal free surfaces and cracks/slip interfaces. This is very limiting, and restricts these approaches to the consideration of example problems only, as very little can be determined about the more general cases.

3.2.1 The crack problem

Let us begin by considering the problem of an open crack, whose root is at the edge of a complete contact. This problem is described by the Williams solution, however, we must use dislocations to ensure that the crack faces are traction free. To do this, we place a distribution of glide edge dislocations, having density B_r , and a distribution of climb edge dislocations, having density B_θ , along the crack faces, which we place at an angle ϕ to the bisector of the monolith representing

the adhered contact, which itself has internal angle 2α . The crack itself has length l .

This allows us to calculate the stresses on the crack faces, which are given by

$$\sigma_{r\theta}(r, \phi) = K_I f_{r\theta}^I(\phi) r^{\lambda_I-1} + K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II}-1} + \dots \quad (3.23)$$

$$\frac{2\mu}{\pi(\kappa+1)} \int_0^l B_r(\xi) F_{rr\theta}(r, \phi, \xi, \phi) + B_\theta(\xi) F_{\theta r\theta}(r, \phi, \xi, \phi) d\xi,$$

$$\sigma_{\theta\theta}(r, \phi) = K_I f_{\theta\theta}^I(\phi) r^{\lambda_I-1} + K_{II} f_{\theta\theta}^{II}(\phi) r^{\lambda_{II}-1} + \dots \quad (3.24)$$

$$\frac{2\mu}{\pi(\kappa+1)} \int_0^l B_r(\xi) F_{r\theta\theta}(r, \phi, \xi, \phi) + B_\theta(\xi) F_{\theta\theta\theta}(r, \phi, \xi, \phi) d\xi.$$

We know that the faces of an open crack must be traction free, so setting these equations equal to zero, we are able to use the inversion methods described in section 2.2.2 to solve for the dislocation densities. Using these we are able to describe fully the state of stress at the edge of a complete contact with a crack emanating from its edge. Crucially, this will allow a calibration of the crack tip stress intensity factors, which determine the behaviour of the crack itself, including whether or not the crack will grow.

This problem has been tackled by Churchman and Hills [33], but as alluded to previously, they have only been able to consider right angle problems, which has in practice limited consideration to the case of a square edged contact with a crack vertically down from the contact edge. This means that the effect of the crack and/or contact angle on the state of stress at the crack tip cannot be explored.

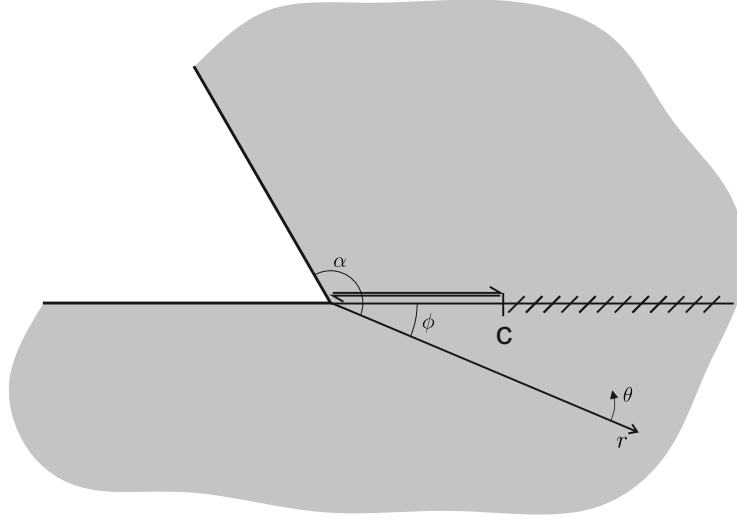


Figure 3.4: Sketch of the slip problem

3.2.2 The slip problem

Considering, instead, the problem of a zone of slip at the edge of the contact, we find that we first must find a way to determine if such a slip zone will exist. Hills and Dini [56] found an upper bound of the coefficient of friction which will permit the existence of such a slip zone which depends on the geometry. However, we are not able to verify that slip will always start from the edge of contact.

Previous simulations [19] have shown, however, that this is the case, and so we continue with the assumption that slip will always originate from the corner. Once again dislocations can be placed along the slip interface, having angle ϕ , from the edge of contact up to the slip-stick boundary, located at a point c . The system is sketched in figure3.4. This, then, allows the stresses to be written in the form;

$$\sigma_{r\theta}(r, \phi) = K_I f_{r\theta}^I(\phi) r^{\lambda_I - 1} + K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II} - 1} + \dots \quad (3.25)$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \int_0^c B_r(\xi) F_{rr\theta}(r, \phi, \xi, \phi) + B_\theta(\xi) F_{\theta r\theta}(r, \phi, \xi, \phi) d\xi, \\
\sigma_{\theta\theta}(r, \phi) = & K_I f_{\theta\theta}^I(\phi) r^{\lambda_I-1} + K_{II} f_{\theta\theta}^{II}(\phi) r^{\lambda_{II}-1} + \dots \\
& \frac{2\mu}{\pi(\kappa+1)} \int_0^c B_r(\xi) F_{r\theta\theta}(r, \phi, \xi, \phi) + B_\theta(\xi) F_{\theta\theta\theta}(r, \phi, \xi, \phi) d\xi,
\end{aligned} \tag{3.26}$$

as for the crack problem. These formulae are then applied to the Coulomb friction law, $\sigma_{r\theta}(r, \phi) = f_c \sigma_{\theta\theta}(r, \phi)$, which give rise to a single Cauchy singular integral equation, which may again be solved by the methods described in section 2.2.2. The problem which arises, here, is determining the position of the slip-stick boundary c .

Once again Churchman and Hills have previously tackled this problem [57], although again only for right-angle contacts. They were able to determine the size of the slip zone using two conditions, which may be summarised as; requiring there to be no slip implied in the slip region; and ensuring that the direction of slip is opposed by friction near the edge of the slip zone, but within it. These conditions have proved sufficient for problems considering finite contacts. Churchman and Hills also determined that the size of the slip zone, when normalised by an initial guess found from the Williams' solution, is invariant with changing coefficient of friction. These results were, however, found using the dislocation formula found by Churchman et al [48], which has been found to contain a mistake.

It was not possible for Churchman and Hills to investigate the effect of varying the contact angle on the size of the slip zone, and the question of the variance with the coefficient of friction remains open. Further experience has also shown that the conditions employed by Churchman are necessary but not sufficient, and this will be discussed further in chapter 7.

3.2.3 Separation

It is also possible that separation will occur at the edge of contact. It is trivially shown from the Williams solution, that given $\lambda_I < \lambda_{II}$, which holds for $\pi < 2\alpha \leq 2\pi$, there is a region from the contact edge to some point within the contact, where the mode I term is the dominant solution. Separation will occur if $K_I f_{\theta\theta}^I(\phi) > 0$, where ϕ is the contact angle. If this is the case then the mode I term will be positive(tensile) and as such the contact, which is unable to support tensile normal traction, will separate, at least in the region where the mode I stress is dominant. The size of this mode I dominant region is approximated by Hills and Dini [55], who denote the switchover point as d_0 , and call this the mode mixity parameter.

The effect of this separation is that the state of stress at the new edge of contact, which will be a point somewhat inside the original contact, will not be singular, as it is not defined by a gradient discontinuity. Paynter et al [58] conducted a study of possible slip events at the edge of this contact. In many cases, however, the degree of separation and the length of slip zone may extend beyond the limit of the asymptotic solution.

3.3 Summary

This chapter has discussed specifically the use of asymptotic methods and Bueckner's principle with distributions of dislocations to tackle the analysis of non-linear behaviour at the edges of complete contact problems. This requires the development of an asymptotic solution for the edge behaviour of complete contacts. Using the assumption that the contact is stuck, this is provided by the Williams solution, treating the two adhered contacting bodies as a monolith.

The Williams solution is derived by enforcing boundary conditions on a reentrant wedge, these are that the free surfaces of the wedge must be traction free. The state of stress is then assumed to vary in a power order way from the wedge apex. An Airy stress function used as a stress potential and the state of stress can be found by differentiation, and solving the resulting eigenvalue problem. The energy criterion is used to determine which non-trivial solutions to the eigenvalue problem are required. Finding the resulting eigenvectors then allows for a full description of the state of stress.

The resulting solution is an infinite expansion. However, there are two singular terms so, close to the wedge apex, an approximation may be used, which is found by considering only the singular terms. This asymptotic approximation may be collocated into a given problem by use of stress intensity factors, each term being descriptive of symmetric(mode I) and anti symmetric(mode II) stress field. The range of validity of the solution can be constrained by the first point at which the solution suggests that the normal contact stress is tensile. Contacts cannot support tensile normal traction, so this point is a bound on the solution.

The Williams solution is then used as a bilateral solution when applying Bueckner's principle. In this case, the true contact behaviour may violate the assumption that the contacting bodies are adhered, and so dislocations may be used to represent the change in the state of stress caused by the presence of an open crack, a slip zone, or a zone of separation. In the presence of a short open crack, dislocations are placed along the crack faces so they may be cleared of tractions. This allows for a description of the stress field around the crack tip using crack tip stress intensity factors. The solution of the problem, as provided in the case of a vertical crack in a three quarter plane by Churchman and Hills

[33], is a calibration which relates the remote loading(characterised by the wedge generalised stress intensity factors) to the crack tip stress intensity factors.

The solution of a slip problem requires the enforcement of a friction condition over the region of slip. This requires a choice of friction law - usually Coulomb's law is used - and a knowledge of the size of the slip zone. Previous studies have found the size of the slip zone by guessing and checking using two conditions. Churchman and Hills [57] found that, when normalised by an initial guess based on the Williams solution, the size of the slip zone is invariant with respect to the coefficient of friction. Finally, consideration was also given to problems of edge separation. Such problems result, however, in a change in the nature of the contact, and have previously fallen beyond the range of the asymptotic solution [58].

Chapter 4

Near edge stick in complete contacts

There are many different phenomena that can occur at the edge of complete contacts, and in this thesis we have already discussed the possibility of separation, slip and the growth of a crack. Each of these responses requires its own consideration and comes with its own conditions which must be enforced. However, before we can begin to consider enforcing these conditions, and to do this, we must know what type of response we are expecting. It will be useful therefore to have some classification of the behaviour we can expect, and some constraints on this behaviour. Ideally this will require as little calculation as possible.

In order to achieve this, we consider both singular terms of the Williams solution. For separation to occur at the contact edge, we recall that it is sufficient for $K_I f_{\theta\theta}^I(\phi) > 0$. However, we also require contact to be maintained throughout the asymptote, which requires the normal stress to be compressive throughout the range of asymptotic validity. As we have already ensured that the stress is compressive at the corner, the point where this sign changes can then be used as a bound on the range of asymptotic validity, provided there is no turning point in the normal traction, which might lead to a detached ‘island’ of separation.

We also must consider slip behaviour. We can easily determine that if we have corner lift-off, we will also get a region of local slip, and the previous work of Hills and Dini [56] has revealed the conditions required for the corner to slip. However, we must once again consider the possibility of a slip zone originating from inside the contact, and the potential for an ‘island’ of slip within the asymptote. If slip begins from inside the contact then this will fall beyond the range of the asymptotic solution, however an ‘island’ of slip could potentially be investigated. Such behaviour would require a turning point in the traction ratio.

As this is a consideration of a two term Williams solution, we are also able to investigate mode II dominated regions. We recall the existence of a mode I dominated region at the corner of the contact, but it is an open question whether or not a similar region will exist where the mode II terms are dominant, as any such theoretical region could be beyond the validity of the asymptote.

It will also be useful to consider how the asymptotic solution relates to real contacts. The solution is collocated using the two stress intensity factors, which are found from finite element analysis. This allows us to apply the conclusions reached above to contacts of given angles and consider how the behaviour changes with contact angle. Furthermore, the stress intensity factors are calibrated to external loading, so we can see how the external loading affects the response.

4.1 Publication - ‘Necessary conditions for near edge stick of Complete Contacts’

These investigations have all been carried out in a paper now attached as part of this thesis. This paper has been reproduced, as originally published:

Riddoch, D. J. and Hills, D. A., *Necessary conditions for near edge stick of Complete Contacts*, The Journal of Strain Analysis for Engineering Design 55: 172-180, March 2020

Necessary conditions for near edge stick of Complete Contacts

Daniel J. Riddoch¹, David A. Hills¹

Abstract—The conditions for slip within a contact and at the corner, together with separation conditions are presented in a general sense and applied to two example finite complete contact problems. These conditions can be found directly from the signs of the relevant stress intensity factors. Further, a new bound on the validity of the asymptotic solution is found, and then applied to find a solution relating to separation conditions. Finally this new bound is used to identify and classify cases in which a region of mode *II* dominance is seen from purely geometric considerations.

I. INTRODUCTION

Complete contacts are those in which the front (contacting) faces of the bodies being pressed together have the same shape in the unloaded state, that is they ‘conform’, and the extent of the contact is defined by a discontinuity in the gradient of the profile, as exemplified by the rectangular block-on-half plane arrangement shown in Figure 1. They have the property that the contact size is independent of the load carried and that a stress singularity will usually arise at the edges of the contact. Practically occurring contacts which are of this general form include splines, for example in certain car transmissions and in the split shafts of a gas turbine.

The object of the analysis presented here is to quantify the conditions for ensuring that the contact edges are stuck so that fretting damage is avoided, although this does leave the attendant risk of a notch type failure. We shall assume that the block and the half-plane are made from the same material, and also that the interface is, indeed, straight, as sketched, but note that the results to be found would apply equally well to a circular cylinder resting in a conforming groove, provided that the line of the interface were accounted for correctly.

A full analysis of the problem, including incorporating the presence of all features, calls for a finite element analysis and, indeed, finite element analysis must be used to calibrate the external loads to stress intensity factors to be defined, but our object is to learn as much as possible about the properties of the contact from an analysis of the half-plane and contact corner modelled together, as a monolith, and using Williams’ analysis of a wedge [1] as our starting point. The use of this classic asymptotic form, appropriate when the observation point is much nearer to the corner than any other features, adds considerable precision to our understanding of the local stress field.

This idea is one which we have used before. It enables the coefficient of friction needed to attain a fully stuck contact edge [2] to be determined. But in that analysis only the

dominant term in an eigenfunction expansion was used, and the question arises of what happens when we move slightly inwards from the edge; is the coefficient of friction needed to achieve stick at the edge also sufficient to attain stick in the rest of the neighbourhood? We answer that question, here, together with looking at further forms of excitation of the local stress field.

One of the main topics of practical analysis in this laboratory is the study of fretting, and fretting fatigue, and in pursuing these we note that there are normally four sets of external loads which are relevant in describing the local contact-edge stress field, (Figure 1), viz. the normal load, P , the shear force, Q , any moment carried by the contact-defining body, M , and underlying surface tension in the counterbody, $\Delta\sigma$. All of these quantities become ‘coupled’ in the context of a complete contact, in the sense that all four excite all eigenmodes. Here, we will concentrate on only the first two terms in the infinite Williams series, and investigate what these imply. This is a legitimate approximation to make as these are the only two terms in the series which are singular at the contact edge, all others terms being bounded. Hence, these terms dominate the solution.

II. FORMULATION

It is initially assumed that the two contacting bodies are in intimate contact everywhere, and fully adhered, giving a monolithic domain. We zoom in on the contact (wedge) corner. At the time of the initial consideration of this class of problem, we would write down the state of stress in the neighbourhood of the wedge (contact) corner [1] in the form

$$\sigma_{ij}(r, \theta) = K_I r^{\lambda_I - 1} f_{ij}^I(\theta) + K_{II} r^{\lambda_{II} - 1} f_{ij}^{II}(\theta) + \dots \quad (1)$$

where (r, θ) are polar coordinates measured from the notch bisector, Figure 1, (K_I, K_{II}) are the multipliers (generalised stress intensity factors) which match the eigensolution to the finite problem, $f_{ij}^K(\theta)$ are the eigenvectors (see Appendix A), and the eigenvalues $(\lambda_I, \lambda_{II})$ are given by the smallest, positive, non-trivial root of the following characteristic equations

$$\lambda_I \sin(2\alpha) + \sin(2\alpha\lambda_I) = 0 \quad \& \quad \lambda_{II} \sin(2\alpha) - \sin(2\alpha\lambda_{II}) = 0 \quad (2)$$

where 2α is the total wedge angle, and equal to $\pi + \phi$, where ϕ is the internal angle of the block, equal to $\pi/2$ in Figure 1 but to be generalised later. The angles are measured from the bisector as the eigensolutions uncouple along that line, and we also use this angular position to define the values of

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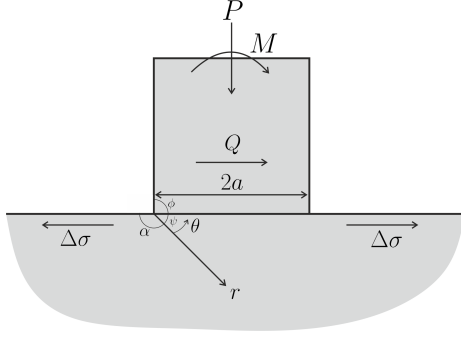


Fig. 1: Rectangular block resting on a half-plane subject to four external loads

the generalised stress intensity factors, ie.

$$K_I = \lim_{r \rightarrow 0} \sigma_{\theta\theta}(r, 0) r^{1-\lambda_I}$$

$$\text{with } f_{r\theta}^I(r, 0) = 0 \quad \& \quad f_{\theta\theta}^I(r, 0) = 1 \quad \text{and} \quad (3)$$

$$K_{II} = \lim_{r \rightarrow 0} \sigma_{r\theta}(r, 0) r^{1-\lambda_{II}}$$

$$\text{with } f_{\theta\theta}^{II}(r, 0) = 0 \quad \& \quad f_{r\theta}^{II}(r, 0) = 1. \quad (4)$$

Before introducing an alternative representation of this solution, we will consider the physical requirements for the solution to be developed to be appropriate when applied to contact problems. These are; (a) that the two bodies be in intimate contact, implied by $\sigma_{\theta\theta}(r, \psi) < 0$, where $\psi = (\alpha - \phi)$ is the angle to the interface, and (b) stick must prevail which requires $-\sigma_{r\theta}(r, \psi)/\sigma_{\theta\theta}(r, \psi) < f$, where f is the coefficient of friction, for all points ‘ r ’ on the interface. Some *local* violations of these inequalities may be tolerated without invalidating the whole solution, but first consider conditions where they *are* satisfied everywhere.

For the time being, focus attention wholly on the square block, Figure 1, so that $2\alpha = 3\pi/2$ and $\psi = \pi/4$. For this example the eigenvectors $(f_{ij}^I(\theta), f_{ij}^{II}(\theta))$ are given in Appendix A. We note that $f_{\theta\theta}^I(\pi/4) \simeq 0.7$ so that it follows, because we can always find points sufficiently close to the contact corner for the mode I contribution to dominate the solution, a requirement for the contact edge to remain closed is that $K_I < 0$. If we turn our attention, now, to the second term in the series, it has been shown in the appendix that $f_{\theta\theta}^{II}(\pi/4) \simeq -1.2$ so that if $K_{II} > 0$ we could be confident that the penetration of intimate contact inwards from the contact edge will last to at least the point where the contributions to the overall solution from modes I and II loading were equal. The question arises is ‘Physically, where is the point along the interface where mode I domination of the solution gives way to a contribution from the second solution which is of at least comparable magnitude?’ and to answer we may re-write the first two terms of equation (1), which brings out the internal length scale, d_0 , and introduce this quantity and a further quantity, G_0 , which has dimensions of stress, defined by [3],

$$d_0 = \left| \frac{K_{II}}{K_I} \right|^{\frac{1}{\lambda_I - \lambda_{II}}} \quad (5)$$

$$G_0 = |K_I|^{\frac{\lambda_{II}-1}{\lambda_I-\lambda_{II}}} |K_{II}|^{\frac{\lambda_I-1}{\lambda_{II}-\lambda_I}}, \quad (6)$$

so that equation (1) may be re-written as

$$\frac{\sigma_{ij}(r, \theta)}{G_0} = \left(\frac{r}{d_0} \right)^{\lambda_I-1} f_{ij}^I(\theta) - \left(\frac{r}{d_0} \right)^{\lambda_{II}-1} f_{ij}^{II}(\theta). \quad (7)$$

This form is extremely useful for studying notch-corner plasticity [3], and highlights that d_0 may be viewed as the notional radial distance which represents the partition between mode I domination ($r < d_0$) and higher order terms domination ($r > d_0$). It may also be used in our present study: although it is not possible to state unequivocally the distance from the contact corner where higher order terms in a series representation dominate the solution, it is possible to define the distance where the mode I term loses its significance, viz. d_0 . This quantity is defined wholly by the mode mixity (equation (5)), and in cases where this exceeds the distance to the nearest physically relevant feature (for example the contact half width, Figure 1), then we can be assured that the first term controls the state of stress until the observation point reaches this feature dominates.

III. A REMARK ON SEPARATION

It is well known that we may always find a point, sufficiently close to the corner, where the stress state is totally dominated by the mode I term. It’s sign is determined by the sign of the stress intensity factor K_I . Hence the lift-off or intimate contact of the corner itself is determined by the sign of this term alone, which is both loading and geometry dependent.

We now suppose that the edge of the contact is intimately closed but consider the possibility of lift-off at a point *within* the body, according to the asymptotic results. For this to occur, and to be observable within the asymptote, we require a turning point in the equation

$$\sigma_{\theta\theta} = K_I f_{\theta\theta}^I r^{\lambda_I-1} + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}-1}, \quad (8)$$

found by $\frac{d\sigma_{\theta\theta}}{dr} = 0$. This is shown in Figure 2, where in the central sketch we see a case where such a turning point exists, and how this may lead to an internal lift-off point.

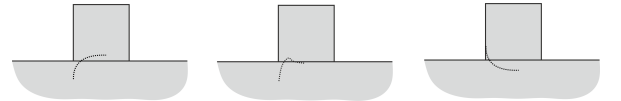


Fig. 2: Sketch of different possible shapes for $\sigma_{\theta\theta}$

The location of this point is given by

$$r = \frac{-K_I}{K_{II}}^{\frac{\lambda_I-2}{\lambda_I-2}} \left(\frac{f_{\theta\theta}^I(\lambda_I-1)}{f_{\theta\theta}^{II}(\lambda_{II}-1)} \right)^{\frac{\lambda_I-2}{\lambda_{II}-2}}. \quad (9)$$

Note that the quantity within brackets depends only on α , so the behaviour of the function is governed by the ratio K_I/K_{II} . Therefore for a given geometry, the behaviour and/or existence of such a turning point is purely loading dependent.

A. Application to the 90° block

Retaining a focus on the square block resting on a halfplane, we aim to improve on and refine the conditions set out by Churchman[4] for separation at the edge of the contact and within the body, as far as is possible with only asymptotic considerations.

To do this, we use a finite element analysis of the problem to calibrate the external loading with the stress intensity factors. This is done by modelling the problems then applying each of the four external loads individually. The stresses along the angle bisector are then found from FE, and the limits taken as described above in equations (3) and (4). The results of this for the square block are shown in equation 10. However, we will first concentrate on excitation by only a normal force P and a shear force applied along the line of the interface, Q .

$$\begin{bmatrix} K_I a^{\lambda_I-1} \\ K_{II} a^{\lambda_{II}-1} \end{bmatrix} = \begin{bmatrix} -0.2265 & 0.2443 & -0.3890 & 0.2462 \\ 0.1118 & 0.2536 & 0.3639 & 0.1483 \end{bmatrix} \begin{bmatrix} \frac{P}{2a} \\ \frac{Q}{2a} \\ \frac{M}{2a^2} \\ \frac{\Delta\sigma}{2} \end{bmatrix} \quad (10)$$

From the calibration, we see that the value of Q/P required to cause lift off at the corner is $Q/P = 0.927$. This differs slightly from the value found by Churchman[4], but this is purely due to differences in the calibrations from finite element analysis. The details of this analysis are discussed in appendix II.

We now intend to see how the location of the turning point (equation (9)) changes with changes in normal and shear load. We can see from Figure 3(a) the ratio K_I/K_{II} is negative for values of Q/P is in the range $-0.44 < Q/P < 0.927$. This implies an *internal* lift off point cannot be found once the loading ratio is within the range needed to maintain intimate contact at the corner, as the location of this turning point, as given by equation (9) is negative and hence outside the contact patch.

The positive value of the ratio K_I/K_{II} for negative values of Q/P would appear to imply that the corner would lift-off too for values of $Q/P < -0.44$. However, looking at Figure 3(b) we see that for values, roughly, $Q/P < 0.2$, the ratio $d_0/a > 1$, ie. the mode II term dominates the mode I term only at a point beyond the centre of the contact, beyond where the asymptotic solution can be assumed to apply. Therefore all points along the interface are mode I dominated, so the stress may be described by $\sigma_{\theta\theta} = K_I f_{\theta\theta}^I r^{\lambda_I-1}$ alone, which displays no turning point, and hence intimate contact is maintained within the asymptote for this load region as $K_I < 0$, giving $\sigma_{\theta\theta} < 0$ which is compressive.

So, for the 90° block, at least, we cannot improve on the statement presented by Churchman by the use of a two-term solution. We conclude that lift-off only occurs in the near edge when K_I is positive. Further inferences on the effect

of other methods of loading are made in section IV-B.1, together with observations on their effect on slip conditions.

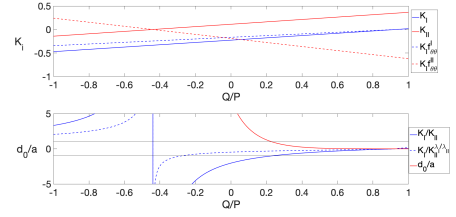


Fig. 3: The variation of stress intensity factors and resultant constants with varying Q/P

IV. SLIP

As laid out at the start of this paper, one of the objects of this analysis is to determine the conditions required for the corners of the contact to remain stuck, in order to avoid fretting damage. Equally important is ensuring that there is no region of slip *within* the body, which would equally lead to fretting damage.

Recall from Hills and Dini [3] that the corner will slip for a sufficiently low coefficient of friction. Hence, we must determine under what conditions slip will occur within the body, and within the asymptote, or at the corner. As already discussed in section III, there are conditions under which we get lift-off at the corners of the block. This will necessarily lead to some partial slip. We deduce this because the direct traction ($\sigma_{\theta\theta}$) changes sign, so must fall smoothly to zero in the region just interior to the true point of separation. Recall, further, that it was noted that a shear force had to be present to induce separation. The shear traction ($\sigma_{r\theta}$) will be non-zero local to the point of separation. So, there must exist a point where the equation $\sigma_{r\theta} = f\sigma_{\theta\theta}$ is satisfied. Hence, this point will be slipping.

A. The turning point

To determine whether an interior point is the point of first slip, let us first consider the ratio $\sigma_{r\theta}/\sigma_{\theta\theta}$. If this ratio is greater than the coefficient of friction over some distance, slip will occur there; if not, it will remain stuck. So, we will find the turning point of this ratio, as either the value near to this point, or the contact edge value, will give the coefficient of friction required to prevent all slip.

The turning point can be found by

$$\frac{d}{dr} \left(\frac{\sigma_{r\theta}}{\sigma_{\theta\theta}} \right) = \frac{(f_{r\theta}^{II} f_{\theta\theta}^I - f_{r\theta}^I f_{\theta\theta}^{II})(\lambda_{II} - \lambda_I) r^{\lambda_I + \lambda_{II} - 1} K_I K_{II}}{(f_{\theta\theta}^I K_I r^{\lambda_I} + f_{\theta\theta}^{II} K_{II} r^{\lambda_{II}})^2} = 0. \quad (11)$$

Provided that the denominator is non-zero, we may first deduce that we can in fact solve the equation

$$-(f_{r\theta}^{II} f_{\theta\theta}^I - f_{r\theta}^I f_{\theta\theta}^{II})(\lambda_I - \lambda_{II}) r^{\lambda_I + \lambda_{II} - 1} K_I K_{II} = 0. \quad (12)$$

Now set

$$A = f_{r\theta}^{II} f_{\theta\theta}^I - f_{r\theta}^I f_{\theta\theta}^{II} \quad B = -(\lambda_I - \lambda_{II}) \quad C = r^{\lambda_I + \lambda_{II} - 1} \quad (13)$$

So equation (12) becomes

$$ABCK_I K_{II} = 0.$$

From Williams solution we have that $\lambda_{II} > \lambda_I$ so we find that $B > 0$. Secondly, we consider positive r only, ie within the block, so $C > 0$. Finally we cannot easily see algebraically, but on plotting A we find that $A > 0$. Hence we have that the solutions to equation 11 are given by $K_I = 0$ or $K_{II} = 0$.

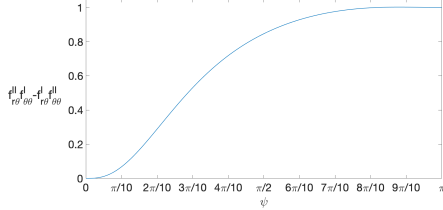


Fig. 4: Value of $f_{r\theta}^{II} f_{\theta\theta}^I - f_{r\theta}^I f_{\theta\theta}^{II}$ with ψ

So a turning point can exist only if $K_I = 0$ or $K_{II} = 0$. In these cases the ratio $\sigma_{r\theta}/\sigma_{\theta\theta}$ is equal to $f_{r\theta}^{II}/f_{\theta\theta}^I$ and $f_{r\theta}^I/f_{\theta\theta}^{II}$ respectively, and is independent of r . Hence, all we can say is that in the first case ($K_I = 0$) we will always see slip at the corner, and in the second, we will see slip at the corner if $f < f_{r\theta}^I/f_{\theta\theta}^{II}$, provided that $K_I < 0$. The independence of these ratios from r means that we cannot say what happens at any point other than the corner.

For all other cases, $K_I \neq 0$ and $K_{II} \neq 0$, we can say that there is no turning point within the asymptote, so, either slip will start at the corner, or within the body but beyond the asymptote. We can determine which of these cases applies by looking at the gradient of the ratio $\sigma_{r\theta}/\sigma_{\theta\theta}$. This could be found by taking the limit of the expression in equation 11 as $r \rightarrow 0$, but this limit does not exist. So instead, we find the sign of the expression in the same way the conditions for the existence of a turning point were found.

Recall that

$$\text{sign}(-(f_{r\theta}^{II} f_{\theta\theta}^I - f_{r\theta}^I f_{\theta\theta}^{II})(\lambda_I - \lambda_{II})r^{\lambda_I + \lambda_{II} - 1} K_I K_{II}) = \text{sign}(K_I K_{II}) \quad \text{and} \quad (14)$$

$$\text{sign}((f_{\theta\theta}^I K_I r^{\lambda_I} + f_{\theta\theta}^{II} K_{II} r^{\lambda_{II}})^2) \neq -1. \quad (15)$$

Hence,

$$\text{sign}\left(\frac{d}{dr} \frac{\sigma_{r\theta}}{\sigma_{\theta\theta}}\right) = \text{sign}(K_I K_{II}),$$

whenever the gradient is determinable.

The results here are interesting. We know that if $d(\sigma_{r\theta}/\sigma_{\theta\theta})/dr$ is positive, then slip will begin within the body, at some point beyond the validity of the asymptote. However, in the case where K_I is positive, the corner will also be slipping, giving a large region of slip. Whereas, if K_I is negative, slip will originate at a point within the body with $f > f_{r\theta}^I/f_{\theta\theta}^{II}$, but for some coefficients of friction, given

by Hills and Dini [3], stick will persist at the corner. The reverse is true if the gradient is negative, and slip will always originate at the corner provided $K_I < 0$, for $f < f_{r\theta}^I/f_{\theta\theta}^{II}$. If K_I is positive, then, as already outlined, slip will start at the corner but stick will persist within the body and, for sufficiently large coefficients of friction, within the asymptote. These results are summarised in Table I.

		K_I	
		Positive	Negative
K_{II}	Positive	Separation and entire asymptote slip(A)	Slip starts at corner(B)
	Negative	Separation and small slip region(C)	Slip from within(D)

TABLE I: Matrix of inferred result from stress intensity factor signs

B. Application to the 90° block

Returning focus to the 90° block, we may distinguish the regions of the load space where each of these forms of behaviour occurs. The results of this analysis are shown in Figure 5. The letters shown correspond to those defined in table I. Note that, under these loading conditions, there is no region in which $K_I > 0$ and $K_{II} < 0$.

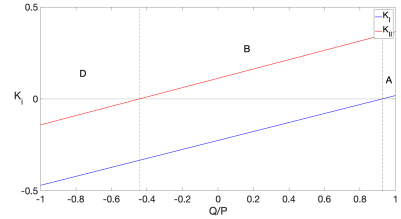


Fig. 5: Plot showing the regions of behaviour for the 90 degree block

1) *Further forms of loading:* So far we have considered excitation only by a normal and shearing force, and we have assumed that these are not sufficient to cause sliding, ie. $|Q|/fP < 1$. However, excitation by other means is possible and the appropriate calibrations for the 90° block were included in equation (10).

We can divide the load space to characterise behaviour under general loading as we did with just two loading types. Figure 6 shows the results of these where, in each, one of loading methods is not excited. Consequently we can see the load regions where each of these types of behaviour occur, again the letters refer to those in Table I.

V. A FURTHER BOUND ON ASYMPTOTIC VALIDITY

From section III, we can see that, provided that the corner is in intimate contact, the whole interface, within the region which can be approximated by the asymptotic solution, must remain in intimate contact. In other words, throughout the region of asymptotic validity, $\sigma_{\theta\theta} < 0$, provided that conditions are such that the corner itself is in intimate contact. A tidier way to locate the point where the interfacial

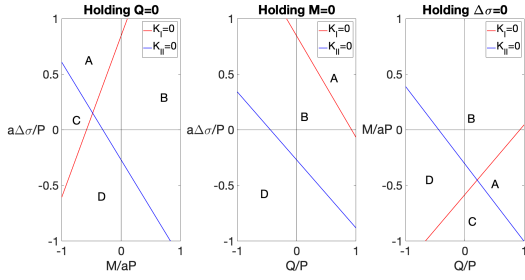


Fig. 6: Characterisation of regions of the four method load space(see Table I)

direct traction changes sign is to find a solution to $\sigma_{\theta\theta} = 0$. This point is itself beyond the limit of the validity of the asymptotic solution, but can provide an upper bound.

So, we solve the equation evaluated on $\theta = \psi$

$$K_I f_{\theta\theta}^I(\psi) r^{\lambda_I-1} + K_{II} f_{\theta\theta}^{II}(\psi) r^{\lambda_{II}-1} = 0 \quad (16)$$

to find the value of r which satisfies this. We denote this value by d_r , its value is given by

$$d_r = \left(\frac{-f_{\theta\theta}^{II}(\psi)}{f_{\theta\theta}^I(\psi)} \right)^{\frac{1}{\lambda_I - \lambda_{II}}} \left(\frac{K_{II}}{K_I} \right)^{\frac{1}{\lambda_I - \lambda_{II}}} \quad (17)$$

So we can state, using this bound, that for the stresses at a point, s along the interface to be approximated by the asymptotic solution, it is a necessary but not sufficient condition that $s/d_r < 1$.

VI. CONSIDERATION OF OTHER ANGLES

So far, throughout this paper our examples have been limited to a square block resting on half plane, however it is not necessary to do this and the general results presented are equally applicable to any complete contact. An obvious extension is to take the internal angle of the block as a parameter, ψ . In order to apply these results to a new example, it is necessary to perform calibrations by finite element analysis to find the stress intensity factors. Such a system is sketch in figure 7. Otherwise, however, we may proceed as for the 90 degree block. The details of this analysis are discussed in appendix II. Several such calibrations have been done and included here is such a calibration, whose internal angle is $\arctan(-3/4) \approx 143^\circ$. This is chosen so as to display a mode II dominant region, as explained in section VII.

$$\begin{bmatrix} K_I a^{\lambda_I-1} \\ K_{II} a^{\lambda_{II}-1} \end{bmatrix} = \begin{bmatrix} -0.2260 & 0.1070 & -0.4415 & 0.1516 \\ 0.0369 & 0.2711 & 0.0930 & 0.1094 \end{bmatrix} \begin{bmatrix} \frac{P}{2a} \\ \frac{Q}{2a} \\ \frac{M}{2a^2} \\ \frac{\Delta\sigma}{2} \end{bmatrix} \quad (18)$$

Use of this calibration leads us to conclude that under no combination of pure P , Q loading does this block lift-off at the corner, see Figure 9, but it can be made to do so by using

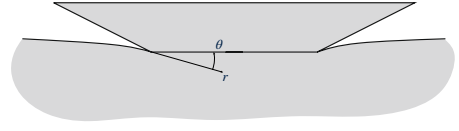


Fig. 7: Sketch of the large angle block

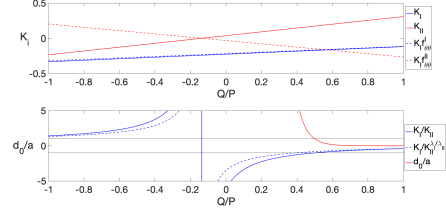


Fig. 8: The variation of stress intensity factors and resultant constants with varying Q/P for the 143° block

other methods of loading, as in Figure 10, however, only for large values of the ratios $a\Delta\sigma/P$ and M/aP . The argument against an interior lift-off point that applied to the 90° block applies again here, as can be seen in Figure 8.

Slip consideration can also be made, and the letters shown in figures 9 and 10 relate again to table I, which is still applicable here. Common throughout all of these, is that, for moderate loading, and largely for positive valued loading(as defined by convention in Figure 1), slip will originate from the corner, regions marked ‘B’, and hence the conditions necessary to prevent slip at the corner are sufficient to prevent interfacial slip throughout the interface.

Similar calibrations for other angles allow for similar partition of the load space. Four other angles have been calculated, the results are shown in appendix III.

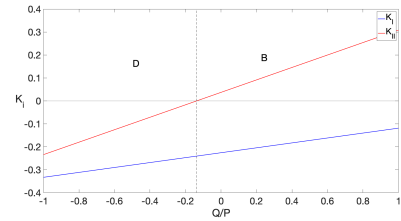


Fig. 9: Plot showing the regions of behaviour for the 143° block

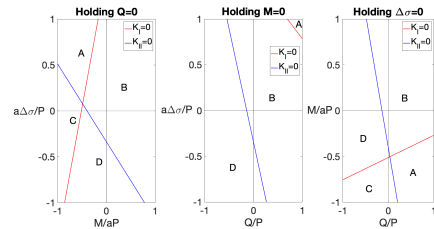


Fig. 10: Characterisation of slip behaviours of the 143° block in four load space

VII. A BOUND ON THE EXISTENCE OF MODE II DOMINATED REGIONS

Throughout this analysis we have utilised a two-term analysis. We have already seen that it is always possible to find a point where the state of stress is dominated by the first term in our expansion. However, we now ask whether the same can be said for the second term. We already know that these points, should they exist, would be at a point $r > d_0$. We also know that for them to be valid for consideration with asymptotic methods, we require $r < d_r$. So it is natural to conclude that we require the ratio $d_r/d_0 > 1$, for asymptotic consideration to be appropriate. A comparison of the forms of these values leads to the ratio

$$\frac{d_r}{d_0} = \left(\frac{-f_{\theta\theta}^{II}(\psi)}{f_{\theta\theta}^I(\psi)} \right)^{\frac{1}{\lambda_I - \lambda_{II}}} \quad (19)$$

Plotting this we find that, as shown in Figure 11, the value $d_r/d_0 = 1$ is achieved at approximately $\alpha = \alpha_0 = 0.89\pi$, or alternatively $\psi = \psi_0 = 139^\circ$. Consequently, we cannot find a point where the mode II term dominates the mode I and which is within the valid range of the asymptote, for values of $\alpha < 0.89\pi$. So we cannot see a region of mode II dominance for an indenter of internal angle less than 139° .

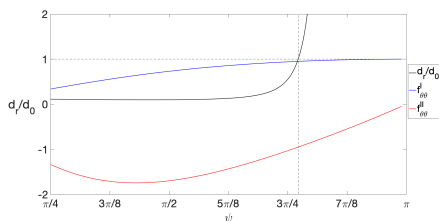


Fig. 11: The value of the ratio d_r/d_0 with varying total internal angle α

A. A new normalised form of the solution

Finally, we may use d_r to re-write the equation for σ_{ij} in a different way.

$$\frac{\sigma_{ij}(r, \psi)}{G_r} = \left(\frac{r}{d_r} \right)^{\lambda_I - 1} - \left(\frac{r}{d_r} \right)^{\lambda_{II} - 1} \quad (20)$$

where

$$G_r = \left(\frac{K_I}{f_{ij}^I(\psi)} \right)^{\frac{\lambda_{II} - 1}{\lambda_I - \lambda_{II}}} \left(-K_{II} f_{ij}^{II}(\psi) \right)^{\frac{\lambda_I - 1}{\lambda_I - \lambda_{II}}} \quad (21)$$

$$d_r = \left(\frac{-f_{ij}^{II}(\psi)}{f_{ij}^I(\psi)} \right)^{\frac{1}{\lambda_I - \lambda_{II}}} \left(\frac{K_{II}}{K_I} \right)^{\frac{1}{\lambda_I - \lambda_{II}}} \quad (22)$$

VIII. CONCLUSIONS

The object of this analysis was to gain extra information about the form of the solution from a two-term asymptotic consideration, compared to that found using a single-term approximation. This has proven to be a difficult task.

However, several general conclusions can be drawn from the work prevented.

A set of conditions which categorise slip behaviour solely from the signs of the relevant stress intensity factors has been devised. We find that, in cases with positive K_{II} , the coefficient of friction required to prevent slip at the corner is sufficient to prevent any slip within the asymptote. The same is not true when K_I is negative, with slip starting within the body. This allows, with appropriate finite element calibrations, for the categorisation of four load space into four regions of distinct behaviour covering the entire space.

Further, a bound on the validity of asymptotic solutions was established and from this, a new normalisation of the Williams solution is possible. Also resulting is a bound on indenter angle necessary for a mode II dominant region to be visible by asymptotic methods is established.

IX. ACKNOWLEDGEMENTS

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APPENDIX I WILLIAMS SOLUTION

A full description and derivation of the Williams solution can be found either from the original [1] or from Elasticity [5]. However, a simple outline of certain important points is given here.

Firstly, we consider the eigenvalues, which are found as the solutions to the characteristic equations

$$\lambda_I \sin(2\alpha) + \sin(2\alpha\lambda_I) = 0 \quad \& \quad \lambda_{II} \sin(2\alpha) - \sin(2\alpha\lambda_{II}) = 0. \quad (23)$$

We note that we have certain conditions placed on these solutions λ_I and λ_{II} . We require that both be positive, and that $\lambda_{II} > \lambda_I$, except in the special case $\psi = \pi$, in which case we find $\lambda_I = \lambda_{II} = 0.5$. Further more we require $\lambda_I \neq 1$ and $\lambda_{II} \neq 1$ as these would lead to trivial solutions.

The eigenvectors of the solution are given by the equations

$$f_{rr}^I(\psi) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\psi] - \frac{\lambda_I - 3}{\lambda_I + 1}\cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\psi]}{\cos[(\lambda_I + 1)\alpha] - \cos[(\lambda_I - 1)\alpha]} \quad (24)$$

$$f_{rr}^{II}(\psi) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\psi] - \frac{\lambda_{II} - 3}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\psi]}{\sin[(\lambda_{II} - 1)\alpha] - \frac{\lambda_{II} - 1}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]} \quad (25)$$

$$f_{r\theta}^I(\psi) = \frac{\sin[(\lambda_I - 1)\alpha]\sin[(\lambda_I + 1)\psi] - \sin[(\lambda_I + 1)\alpha]\sin[(\lambda_I - 1)\psi]}{\sin[(\lambda_I - 1)\alpha] - \frac{\lambda_I + 1}{\lambda_I - 1}\sin[(\lambda_I + 1)\alpha]} \quad (26)$$

$$f_{r\theta}^{II}(\psi) = \frac{\cos[(\lambda_{II} - 1)\alpha]\cos[(\lambda_{II} + 1)\psi] - \cos[(\lambda_{II} + 1)\alpha]\cos[(\lambda_{II} - 1)\psi]}{\cos[(\lambda_{II} - 1)\alpha] - \cos[(\lambda_{II} + 1)\alpha]} \quad (27)$$

$$f_{\theta\theta}^I(\psi) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\psi] - \cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\psi]}{\cos[(\lambda_I - 1)\alpha] - \cos[(\lambda_I + 1)\alpha]} \quad (28)$$

$$f_{\theta\theta}^{II}(\psi) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\psi] - \sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\psi]}{-\sin[(\lambda_{II} - 1)\alpha] + \frac{\lambda_{II} - 1}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]} \quad (29)$$

APPENDIX II FINITE ELEMENT MODELLING

Mentioned throughout this paper is the finite element calibration used in examples. We now briefly discuss how this modelling was done, some of the precautions taken to ensure an accurate and reliable calibration. The modelling technique was the same for both examples, but here we discuss the 90° block. All of the modelling was done using the commercial FE code ABAQUS.

In this case, we have a square block, of side length $2a$ resting on a half-plane. We assumed intimate contact so we may model this system as a single monolith. The halfplane was represented by a square block of side length $100a$. The combined shape of the half-plane and the smaller block make up the monolith. This means that no assumptions need to be made about the contact, such as coefficient of friction.

A mesh of standard linear plane strain elements was used, with 100 elements along the bisector. This mesh was settled on after mesh dependency analysis was performed and this found to be sufficient to capture the effects of the singularity. This was done by repeating an experimental case with various numbers of elements along the bisector. The singularity effect is measured remotely along the bisector, and 100 elements was sufficiently large that an increase to 250 elements did not improve the results noticeably.

The half-plane was then constrained on its bottom edge to with a pinned boundary condition. A uniform distribution of pressure on the top face served to apply the normal force. A concentrated shear force was applied to the top of the block, with a corresponding moment to cancel the moment created by this, being equivalent to applying the load along the interface. This moment, and the moment applied directly were applied by a two forces at the corners of the block, acting as a couple. Finally a negative pressure was applied to the half-plane to act as the bulk tension. In this last case, the boundary conditions were changed to restrict a single central point on the bottom of the half-plane, not the entire surface.

APPENDIX III CLAIBRATIONS FOR OTHER ANGLES

As has already been shown for the 90° block, and the 143° block, finite element analysis allows us to calibrate remote loads into stress intensity factors local to the contact edge. These calibrations are dependent on the angle of the contact and must be calibrated individually. In addition to the previously mentioned examples, some other calibrations were completed, using the same method described in appendix II. The results are reproduced here as calibration matrices.

4.2 Conclusions

As a result of the investigations detailed in the preceeding paper, we are now able to answer some of the questions posed earlier in this chapter. We are able to use the stress intensity factors to classify the behaviour of the contact. We also wish to introduce a bound on the existence of a separation ‘island,’ which we do using the stress intensity factors and the contact and interface angles, which give rise to the eigenvectors and eigenvalues. Consequently, we are able to determine, before any use of dislocations to modify the state of stress, whether or not intimate contact is maintained along the contact interface.

Furthermore, we have found similar results for slip. This time using only the stress intensity factors, and assuming that intimate contact is maintained throughout, we are able to determine whether or not slip will originate from the edge of the contact, which is the type of behaviour we can model using the asymptotic solution. We have also seen that we can not have an ‘island’ of slip. The required turning point exists if and only if one of the stress intensity factors is zero, which would make the traction ratio invariant with r .

Chapter 5

Dislocations in an arbitrary angle wedge

Now having the understanding of when slip or separation will occur, under what combinations of loading and contact angles, we can either seek out or avoid such cases. It is worth noting that we haven't predicted when cracks will occur, because crack nucleation is highly dependent on the presence and location of material defects. As such, we consider only crack propagation, which means that the crack already exists.

Before we can model different behaviours however, we must return to the problem of strain nuclei. In chapter 2 we introduced the concept of using crystalline dislocations as strain nuclei to modify the state of stress using Bueckner's principle. In chapter 3 we examined the method devised by Hecker and Romanov [46, 47] of determining the stress kernels for dislocations in wedge shaped domains. It should be remembered that an adhered complete contact is equivalent to a wedge, so these kernels may be placed in an adhered complete contact.

However, both the solution provided by Hecker and Romanov, and the later

expansion on that by Churchman et al [48], are limited, as discussed in section 3.2. As we are using an asymptotic representation centred on the contact edge, and the stress is then described using polar coordinates, it will prove useful to represent the dislocation stresses using polar coordinates.

Consequently, it has been necessary to transform the infinite plane kernels, which are well known [36], into polar coordinates, and use the polar stress components, rather than their cartesian forms. Then the same approach as Hecker and Romanov has to be applied, taking care not to repeat the mistake made by Churchman et al in normalising the problem.

Obviously, correcting this mistake will lead to a different result from that found by Churchman et al, so it will also be useful to verify this somehow. For this purpose, we will use the half-plane kernels calculated analytically for comparison with our numerical solution. Furthermore, it will be useful to compare the results at other angles, so a finite element solution is again used for comparison purposes for specific examples. This solution is limited by being a finite element solution, but it is broadly indicative of the expected state of stress for particular cases.

5.1 Publication - ‘Dislocations in an arbitrary angle wedge. Part I: The dislocation kernel’

These investigations have all been carried out in a paper now attached as part of this thesis. This paper has been reproduced as originally published:

Riddoch, D. J. and Hills, D. A., *Dislocations in an arbitrary angle wedge. Part I: The dislocation kernel*, The Journal of Strain Analysis for Engineering Design: 03093247211042539, October 2021

Dislocations in an arbitrary angle wedge.

Part I: The dislocation kernel

Daniel J. Riddoch¹, David A. Hills¹

Abstract—We consider the state of stress created by the presence of an edge dislocation at an arbitrary position, in a wedge of arbitrary internal angle. A method for determining the state of stress in the wedge is demonstrated and verified against finite element method simulations. Furthermore, a Mellin transform is employed to ensure that the free surfaces of the wedge remain traction free along their length.

I. INTRODUCTION

In applied mechanics, dislocations are often used as kernels to model pressure of plastic or frictional slip or crack opening, as well as lattice defects in a crystalline structure. In this paper, we will be discussing the former. In this case, the stress caused by the dislocation, known as the kernel, is used to modify the stress present in a body under load to fulfil conditions imposed by defects, such as cracks or regions of localised slip. The stress caused by an edge dislocation in an infinite plane is well understood and concise formulae can be used to describe this stress[1], [2], [3]. However, no such clean formula exists for the stress resultant from a dislocation present in a semi-infinite wedge of arbitrary internal angle.

We must consider a method to determine the influence of this dislocation, starting from the solution for a dislocation in an infinite plane. The method we will use is not a new one[4]. We begin by reminding ourselves that the free surfaces of the wedge must be free from tractions. Now, we place a single dislocation(the object dislocation) at a given location. The stresses resulting from this dislocation can be found using the formulae for a dislocation in an infinite plane. Next, along the half-lines corresponding to the positions of the free surfaces, we place distributions of dislocations, the density of which is chosen to cancel the tractions produced by the object dislocation and ensure that the free surfaces remain traction free. This set up is shown in figure 1.

This approach was first set out by Hecker and Romanov[4], [5] and was further considered and applied by Churchman et al[6], [7], [8]. All of these authors considered specific cases of the problem, and a more general form is sought. Furthermore, in order to make the analysis more practical and useable, Churchman et al truncated the interval over which the distributions of dislocations were placed, and we would like to avoid this approximation.

The object of this analysis therefore is to generalise the method in order to produce a set of solutions for any given internal wedge angle. A secondary consideration is the desire to reintroduce the Mellin transform, as described by Hecker and Romanov[4], in order to increase the accuracy

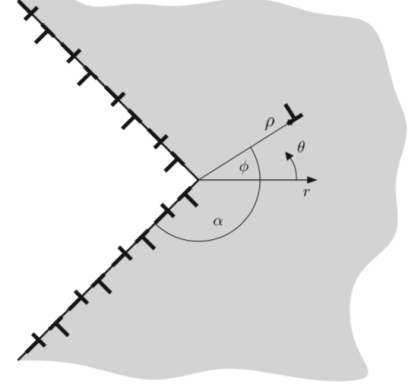


Fig. 1: Sketch showing the method of distributing dislocations on free surfaces

of the solution. It should also be noted that a small error was found in the results presented by Churchman et al[6], this is explained further in section V-A and a correction is presented.

II. BILATERAL SOLUTION

We begin by considering the form of the state of stress in a wedge of arbitrary angle, without any dislocations present. The modelling for these stresses relies on the long established method of Williams[9]. The solution is an asymptotic representation of the stresses near the apex of a semi-infinite wedge. The stresses resulting are described in a polar coordinate system whose r -axis(the line $\theta = 0$) is defined by the bisector line through the apex of the wedge.

A full derivation of the representation can be found in Barber[1], the upshot of which is that the stresses may generally be described by the formula

$$\sigma_{ij}(r, \theta) = K_I f_{ij}^I(\theta) r^{\lambda_I - 1} + K_{II} f_{ij}^{II}(\theta) r^{\lambda_{II} - 1} \dots + \text{higher order terms.} \quad (1)$$

In this formulation K_I and K_{II} are the mode I and mode II stress intensity factors respectively, λ_I and λ_{II} are the mode 1 and mode 2 eigenvalues respectively, and f_{ij}^I and f_{ij}^{II} are the mode 1 and mode 2 eigenvectors. The method for calculating the eigenvalues and eigenvectors is detailed in appendix I.

It is known that for wedges whose total internal angle is greater than approximately 257° , we have that $\lambda_I < \lambda_{II} < 1$, and hence the contributions from both the leading

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eigensolutions are singular. Furthermore $\lambda_I < \lambda_{II}$, and for all angles greater than 180° , we have that $\lambda_I < 1$. This means that the first term in the solution will be dominant near the apex of the wedge, and is singular.

III. FORMULATION

A. Infinite plane kernels

We start with the formulae for the stress resulting from an edge dislocation. These are well known and can be found in many sources[10], given in terms of Cartesian stress components and Cartesian coordinates. For a dislocation whose core is located at (x_d, y_d) , where, the resulting stress from the dislocation having Burgers vector components (b_x, b_y) is then given by

$$\sigma_{jk}(x, y, x_d, y_d) = \frac{2\mu}{\pi(\kappa + 1)} (G_{xjk}(x, y, x_d, y_d)b_x(x_d, y_d) \dots + G_{yjk}(x, y, x_d, y_d)b_y(x_d, y_d)), \quad (2)$$

where b_x and b_y are the Burgers vector components in the x and y directions respectively, μ is the modulus of rigidity and κ is Kolosov's constant. The kernel functions G_{ijk} observed at the point (x, y) are given by

$$G_{xxx}(x, y, x_d, y_d) = -\frac{(y - y_d)(3(x - x_d)^2 + (y - y_d)^2)}{((x - x_d)^2 + (y - y_d)^2)^2} \quad (3)$$

$$G_{xxy}(x, y, x_d, y_d) = \frac{(x - x_d)((x - x_d)^2 - (y - y_d)^2)}{((x - x_d)^2 + (y - y_d)^2)^2} \quad (4)$$

$$G_{xyy}(x, y, x_d, y_d) = \frac{(y - y_d)((x - x_d)^2 - (y - y_d)^2)}{((x - x_d)^2 + (y - y_d)^2)^2} \quad (5)$$

$$G_{yxx}(x, y, x_d, y_d) = -\frac{(x - x_d)((x - x_d)^2 - (y - y_d)^2)}{((x - x_d)^2 + (y - y_d)^2)^2} \quad (6)$$

$$G_{yyx}(x, y, x_d, y_d) = \frac{(y - y_d)((x - x_d)^2 - (y - y_d)^2)}{((x - x_d)^2 + (y - y_d)^2)^2} \quad (7)$$

$$G_{yyy}(x, y, x_d, y_d) = \frac{(x - x_d)((x - x_d)^2 + 3(y - y_d)^2)}{((x - x_d)^2 + (y - y_d)^2)^2}. \quad (8)$$

We wish to use polar coordinates to formulate the problem in a robust and adaptable way. It is therefore necessary to convert these formulae to give us the stress in polar components at location defined by polar coordinates.

This can be done in one of two ways. The most obvious method may be to rotate and transform the stress components using the coordinate transforms. Alternatively, the kernels may be derived using the Muskhelishvili potential methods. For the sake of brevity, we will discuss the first of these here only.

Firstly, let us recall the coordinate transforms. These can be derived from the definition of the coordinate systems, and

are given by

$$\begin{aligned} x &= r \cos(\theta), \\ y &= r \sin(\theta). \end{aligned}$$

Next we consider the tensor transforms needed to transform the Cartesian components into polar forms. These are given by

$$\sigma_{rr} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos(2\gamma) + \sigma_{xy} \sin(2\gamma) \quad (9)$$

$$\sigma_{r\theta} = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin(2\gamma) + \sigma_{xy} \cos(2\gamma) \quad (10)$$

$$\sigma_{\theta\theta} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos(2\gamma) - \sigma_{xy} \sin(2\gamma) \quad (11)$$

where $\gamma = \theta - \phi$. A representation of the rotation of the state of stress is shown in figure 2.

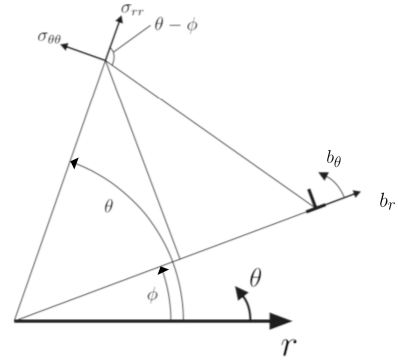


Fig. 2: A sketch showing how the state of stress is rotated during transformation

Combining these two sets of transformations, we can now generate the kernels for the stress resulting from a dislocation

in an infinite plane in polar coordinates. These are given by

$$G_{rrr}(r, \theta, \rho, \phi) = \sin(\phi - \theta) \dots \frac{r^3 - 2r\rho^2 - r\rho^2\cos(\phi - \theta) + 2\rho^3\cos(\phi - \theta)}{(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2}, \quad (12)$$

$$G_{rr\theta}(r, \theta, \rho, \phi) = \frac{r(2r^2 + 5\rho^2)\cos(\phi - \theta)}{2(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2} + \dots \frac{r\rho(\rho\cos(3(\phi - \theta)) - 4r) - 2\rho(r^2 + \rho^2)\cos(2(\phi - \theta))}{2(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2}, \quad (13)$$

$$G_{r\theta\theta}(r, \theta, \rho, \phi) = \sin(\phi - \theta) \dots \frac{r^3 + 4r\rho^2 + r\rho^2\cos(2(\phi - \theta)) - 2\rho(2r^2 + \rho^2)\cos(\phi - \theta)}{(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2}, \quad (14)$$

$$G_{\theta rr}(r, \theta, \rho, \phi) = \frac{-\rho(6r^2 + 2\rho^2 + r\rho\cos(3(\phi - \theta)))}{2(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2} + \dots \frac{r(2r^2 + 7\rho^2)\cos(\phi - \theta)}{2(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2} \quad (15)$$

$$G_{\theta r\theta}(r, \theta, \rho, \phi) = -r\sin(\phi - \theta) \dots \frac{\rho^2\cos(2(\phi - \theta)) + r(r - 2\rho\cos(\phi - \theta))}{(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2}, \quad (16)$$

$$G_{\theta\theta\theta}(r, \theta, \rho, \phi) = \frac{r(2r^2 + 5\rho^2)\cos(\phi - \theta)}{2(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2} + \dots \frac{r\rho^2\cos(3(\phi - \theta)) - 2\rho(r^2 + \rho^2 + 2r^2\cos(2(\phi - \theta)))}{2(r^2 + \rho^2 - 2r\rho\cos(\phi - \theta))^2}, \quad (17)$$

where the dislocation's core is placed at (ρ, ϕ) and the stress is observed at (r, θ) .

With these formulae now to hand, we can proceed to use them to determine the stress caused by placing a dislocation in a semi-infinite wedge of arbitrary internal angle.

B. Wedge kernels formulation

Now we have the kernels for the stress in polar coordinates, we employ the same methodology described by Churchman et al.[6] and previously by Hecker and Romanov[4], [5]. The approach they describe, and the one we will be using, takes the dislocation formulae just described. A single dislocation(hereafter called the object dislocation) is placed at a prescribed point in an infinite plane, and distributions of dislocations are placed along half-lines from the origin, corresponding to the free surfaces of the wedge we wish to model. The density of these dislocation distributions is then found such that the free surfaces are traction free, as required, and the combined effect of all the dislocations is now the state of stress of the object dislocation in the wedge. This concept is illustrated in Figure 1, which shows a single dislocation placed at (ρ, ϕ) in a wedge of total internal angle 2α .

Let us now apply this method to finding the influence of a dislocation in a wedge of total internal angle 2α . Using the axes described by the Williams solution, we define θ from on the angle bisector, with the origin at the wedge apex. The free surfaces then fall on the two half-lines described by

$\theta = \pm\alpha$, and $r > 0$. We place the dislocation at an arbitrary point with coordinates (ρ, ϕ) , having that $-\alpha < \phi < \alpha$.

We now place continuous distributions of both glide edge and climb edge dislocations along these free surfaces, and our object dislocation at (ρ, ϕ) . This results in the formation of four simultaneous integral equations. These are detailed fully in appendix II, however, they all have a similar form, so we will write explicitly only one here, viz

$$\int_0^\infty G_{rr\theta}(r, \alpha, \xi, \alpha) B_r^\oplus(\xi) + G_{\theta r\theta}(r, \alpha, \xi, \alpha) B_\theta^\oplus(\xi) + \dots G_{rr\theta}(r, \alpha, \xi, -\alpha) B_r^\ominus(\xi) + G_{\theta r\theta}(r, \alpha, \xi, -\alpha) B_\theta^\ominus(\xi) d\xi = [b_r G_{rr\theta}(r, \alpha, \rho, \phi) + b_\theta G_{\theta r\theta}(r, \alpha, \rho, \phi)], \quad (18)$$

where b_r , and b_θ are the r direction and θ direction components of the Burgers vector of the object dislocation, and B_i are the dislocation densities for the distributions along the free surfaces, found by

$$B_i(\xi) = \frac{db_i}{d\xi}.$$

The densities labelled $\oplus$ correspond to distributions on the line $\theta = \alpha$, and those labelled $\ominus$ with distributions along the line $\theta = -\alpha$. Equation 18 therefore clears the positive half-line of shear, or $r\theta$, tractions. The other three integral equations similarly clear both half-lines of both shear and normal tractions.

IV. SOLVING THE INTEGRAL EQUATIONS

We now need to consider how we solve these equations. Previous studies have suggested using a Mellin transform[4], [5], but more recent studies have deemed a finite truncation of the integral range to be sufficient[6]. We will now use a standard inversion method of the Gauss-Chebyshev type. The method behind this inversion is described by Hills et al[10], and we will now describe our application of this method.

First, we must ensure that we are right to apply this method. As described in section II, the stresses in the wedge without the dislocation present are singular at the wedge apex, of the order r^{λ_I-1} . For the sake of convenience, we may simply treat this as square root singular. Previous studies[6], [11] considering similar problems indicate that this does not affect convergence. Away from the apex, the stress decreases monotonically, and so we must choose a fundamental function that is singular at one end, and bounded at the other.

We can therefore transform the radial coordinates in order to change the integration range to the interval $[-1, 1]$ in order to use the quadrature. We introduce a transform for the collocation(observation) points

$$r = \frac{l}{2} \frac{\log\left(\frac{2}{1+t}\right)}{\log 2}, \quad (19)$$

and a similar transform for the integration points

$$\xi = \frac{l}{2} \frac{\log\left(\frac{2}{1+s}\right)}{\log 2}, \quad (20)$$

where l is an arbitrary multiplier. Hence, we choose as the fundamental function the function $\omega(s_j)$ having the form

$$\omega(s) = \sqrt{\frac{1-s}{1+s}}, \quad (21)$$

and so we now rewrite the dislocation densities in the form

$$B_i(\xi) = \omega(\xi) \varphi_i(\xi). \quad (22)$$

Now, the numerical quadrature, as described by Erdogan et al[12] for the choice of fundamental function, as in equation 21, leads to the following formulations for the integration and collocation points

$$t = \cos\left(\frac{\pi(2i-1)}{2n+1}\right) \quad (23)$$

$$s = \cos\left(\frac{2i\pi}{2n+1}\right), \quad (24)$$

for some predetermined number of points n , where $i = 1, \dots, n$. Finally, we introduce the weighting function

$$W(s) = \frac{2(1-s)}{2n+1}, \quad (25)$$

which allows us to discretise the integral equation.

Equation 18 therefore can first be rewritten as

$$\begin{aligned} & \int_{-1}^1 \left(G_{rr\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \frac{l \log \frac{2}{1+s}}{2 \log 2}, \alpha \right) B_r^\oplus(s) + \dots \right. \\ & G_{\theta r\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \frac{l \log \frac{2}{1+s}}{2 \log 2}, \alpha \right) B_\theta^\oplus(s) + \dots \\ & G_{rr\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \frac{l \log \frac{2}{1+s}}{2 \log 2}, -\alpha \right) B_r^\ominus(s) + \dots \\ & \left. G_{\theta r\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \frac{l \log \frac{2}{1+s}}{2 \log 2}, -\alpha \right) B_\theta^\ominus(s) \right) \frac{l}{2(s+1)\log 2} ds \\ & = \left[b_r G_{rr\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \rho, \phi \right) + \dots \right. \\ & \left. b_\theta G_{\theta r\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \rho, \phi \right) \right], \quad (26) \end{aligned}$$

by applying the transforms. Notice that generally we find

$G_{ijk}(\gamma r, \alpha, \gamma \rho, \phi) = \frac{1}{\gamma} G_{ijk}(r, \alpha, \rho, \phi)$, so the IE becomes

$$\begin{aligned} & \int_{-1}^1 \left(G_{rr\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s}, \alpha \right) B_r^\oplus(s) + \dots \right. \\ & G_{\theta r\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s}, \alpha \right) B_\theta^\oplus(s) + \dots \\ & G_{rr\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s}, -\alpha \right) B_r^\ominus(s) + \dots \\ & \left. G_{\theta r\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s}, -\alpha \right) B_\theta^\ominus(s) \right) \frac{1}{s+1} ds \\ & = \left[b_r G_{rr\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \rho, \phi \right) + \dots \right. \\ & \left. b_\theta G_{\theta r\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \rho, \phi \right) \right]. \quad (27) \end{aligned}$$

Finally, rewriting the dislocation densities, we find,

$$\begin{aligned} & \sum_{i=1}^n \left(G_{rr\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s(i)}, \alpha \right) \varphi_r^\oplus(s(i)) W(s(i)) + \dots \right. \\ & G_{\theta r\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s(i)}, \alpha \right) \varphi_\theta^\oplus(s(i)) W(s(i)) + \dots \\ & G_{rr\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s(i)}, -\alpha \right) \varphi_r^\ominus(s(i)) W(s(i)) + \dots \\ & \left. G_{\theta r\theta} \left(\log \frac{2}{1+t}, \alpha, \log \frac{2}{1+s(i)}, -\alpha \right) \varphi_\theta^\ominus(s(i)) W(s(i)) \right) \\ & \frac{1}{s(i)+1} = \left[b_r G_{rr\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \rho, \phi \right) + \dots \right. \\ & \left. b_\theta G_{\theta r\theta} \left(\frac{l \log \frac{2}{1+t}}{2 \log 2}, \alpha, \rho, \phi \right) \right]. \quad (28) \end{aligned}$$

We have shown how to discretise one of the equations, and the other three follow suit. These four equations then produce $4n$ equations, in $4n$ unknowns. These are the values of the functions denoted

$$\begin{aligned} & \varphi_r^\oplus(s_1), \dots, \varphi_r^\oplus(s_n), \varphi_r^\ominus(s_1), \dots, \varphi_r^\ominus(s_n), \varphi_\theta^\oplus(s_1), \dots \\ & \varphi_\theta^\oplus(s_n), \varphi_\theta^\ominus(s_1), \dots, \varphi_\theta^\ominus(s_n). \quad (29) \end{aligned}$$

V. RESULTS AND VERIFICATION

The solutions to the equations above allow us to distribute dislocations along the free surfaces, to ensure that they remain traction free while the influence of the object dislocation is also felt throughout the wedge. It is important to note that for $|r - \rho| \ll 1$, ie close to the core of the object dislocation, the singularity induced by the dislocation will be dominant, but otherwise, the free surfaces will have a greater impact on the local state of stress. As such, when representing the solution, it makes sense to split the solution for the kernel into two parts. The influence of the object dislocation is given by G_{ijk} , and the influence of the free surfaces is denoted by F_{ijk} .

Thus far, in the formulation and the method we have, very deliberately, maintained total generality of the wedge

angle, and the dislocation location. We have therefore, the ability to find the kernel for any given position of the object dislocation, and any given wedge angle. Now, however, we must specify these angles and locations in order to display the results.

A. The three-quarter plane

Firstly, let us consider the three-quarter plane, or 270 degree wedge. The case of the three-quarter plane, with the object dislocation at the angle $\phi = \frac{\pi}{4}$ has been extensively considered before by Churchman et al[6].

Whilst their method has generally been followed, a Mellin transform has been employed. This should improve the accuracy. It should also be noted that we have taken pains not to normalise the distances along the free surfaces from the wedge apex by the distance of the object dislocation from the apex. One of the assumptions of this normalisation is that $F_{ijk}(r, \theta, \rho, \phi) = F_{ijk}(\gamma r, \theta, \gamma \rho, \phi)$, where F_{ijk} are the influence functions defined above. This argument does not strictly hold. It is accurate to say that $\bar{\sigma}_{jk}(r, \theta, \rho, \phi) = \bar{\sigma}_{jk}(\gamma r, \theta, \gamma \rho, \phi)$, where γ is some real number and $\bar{\sigma}_{jk}(\gamma r, \theta, \gamma \rho, \phi) = \gamma b_i F_{ijk}(\gamma r, \theta, \gamma \rho, \phi)$. Therefore

$$\begin{aligned} \gamma b_i F_{ijk}(\gamma r, \theta, \gamma \rho, \phi) &= b_i F_{ijk}(r, \theta, \rho, \phi), \\ F_{ijk}(\gamma r, \theta, \gamma \rho, \phi) &= \frac{1}{\gamma} F_{ijk}(r, \theta, \rho, \phi). \end{aligned} \quad (30)$$

Now we may consider and compare the results from our method, to those found by Churchman et al. Figure 3 shows plot of all four relevant F_{ijk} functions, with our results shown in blue(solid line), and Churchman's results overlaid in red(dotted line). We can see from this figure that our results match almost perfectly.

As discussed by Churchman, it is possible to fit a function to these influence functions. However, we will not replicate that here. The reasons for not doing so are two-fold. Firstly, we will display many different new results for many different angles, reproducing sufficient coefficients for all the relevant fits would impractical. Secondly, as part of this publication, we are publishing the MATLAB code used to produce these curves. As such, the reader may use this code to reproduce these curves exactly; the added error which will inevitably be introduced by a fit can then be avoided.

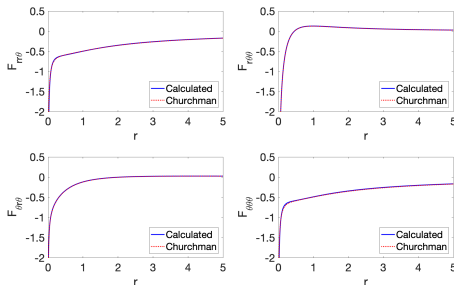


Fig. 3: Overlay plots comparing our results to those of Churchman[6].

B. The half-plane

The problem of the influence of a dislocation in a half-plane has a closed form solution. It is described by Hills et al[10]. Our solution may again be repurposed to compare these kernels. Figure 4 shows an overlay of our results for the kernels of a dislocation at a position of unit distance perpendicular to the free surface, observed along a line, also perpendicular to the surface, running through the dislocation core.

As before, our results are shown in blue(solid line) and the results of Hills et al are shown overlaid in red(dotted line). Once again, inspection of the figure shows that our results are near identical to those calculated analytically.

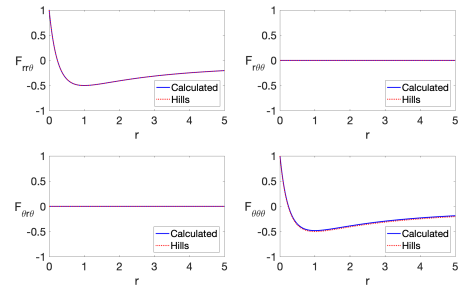


Fig. 4: Overlay plots comparing our results to those calculated from Hills et al[10].

C. The continuation line

There are many more combinations of angles that may be considered. Our results may, for example, be compared to those of Philipps et al[13]. However, we will content ourselves here by considering the case of a dislocation placed on the continuation line of one of the free surfaces in a wedge of varying angle, the continuation line being a half-line collinear to one of the free surfaces originating from the notch apex.

This time, a finite element analysis has been conducted to verify our findings. Again the results are plotted with our results shown in blue(solid line) and the FE data shown in red(dotted line). Several different wedge angles are considered, and these results are plotted in Figures 5, 6, 7, for total internal wedge angles of 240° , 280° and 340° . Comparisons have been made with 44 different angle values using FEM, and the FEM data is published along with the code generating the curves, so the reader may inspect all of these comparisons. For the sake of brevity, we only reproduce three such figures here.

An inspection of these figures once again shows that our results almost exactly match the FE solution. It is worth noting in some cases numerical problems related to singularities may cause errors in the two or three points closest to the wedge apex. This is simply a numerical feature caused by the interaction of many singularities very close to the wedge apex, and does not reflect any inaccuracies within the method or the results outside of this very small area.

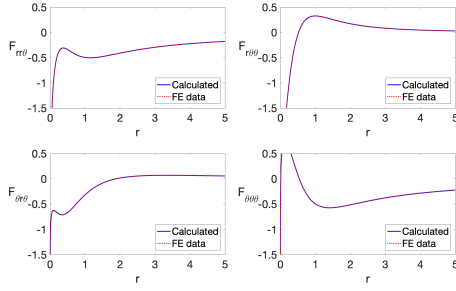


Fig. 5: Overlay plots comparing our results to and FEM analysis at an internal wedge angle of 240° .

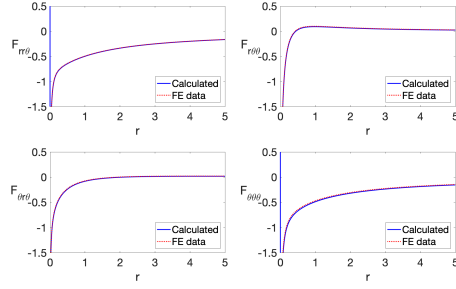


Fig. 6: Overlay plots comparing our results to and FEM analysis at an internal wedge angle of 280° .

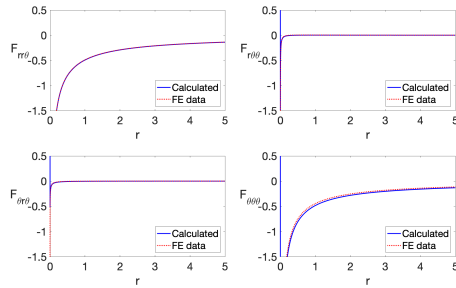


Fig. 7: Overlay plots comparing our results to and FEM analysis at an internal wedge angle of 340° .

VI. CONCLUSION

This analysis has derived a general method for finding the influence of an edge dislocation in a semi-infinite wedge of arbitrary internal angle. This has been done using the more accurate Mellin transform method, and the use of polar coordinates has allowed the problem to be generalised for an arbitrary internal angle.

The solution has been verified by comparison with known results and finite element analysis. In all cases the singularity caused by the dislocation has been considered separately to the influence of the free surfaces. The influence functions produced can now be used to consider the behaviour of cracks and/or slip regions near the edge of a complete contact. MATLAB code providing a transportable, verified form of the solutions for incorporation into numerical analysis of cracks, slip zones etc is available with this

document as supplementary files.

VII. ACKNOWLEDGEMENTS

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APPENDIX I WILLIAMS SOLUTION

A brief description of the Williams solution is given in section II. In this appendix we outline the method of finding the eigenvalues and eigenvectors.

Firstly, let us calculate the eigenvalues, these are the solution of the characteristic equations

$$\lambda_I \sin(2\alpha) + \sin(2\alpha\lambda_I) = 0 \quad \& \quad \lambda_{II} \sin(2\alpha) - \sin(2\alpha\lambda_{II}) = 0. \quad (31)$$

In order to avoid trivial solutions and ensure the appropriate singular behaviour, we must enforce constraints on the eigenvalues. Both eigenvalues must be positive, and $\lambda_{II} > \lambda_I$, except in the special case $\phi = \pi$, in which case we find $\lambda_I = \lambda_{II} = 0.5$. Finally $\lambda_I \neq 1$ and $\lambda_{II} \neq 1$ as these are trivial solutions.

The corresponding eigenvectors are found by

$$f_{rr}^I(\psi) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\psi] - \frac{\lambda_I - 3}{\lambda_I + 1}\cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\psi]}{\cos[(\lambda_I + 1)\alpha] - \cos[(\lambda_I - 1)\alpha]} \quad (32)$$

$$f_{rr}^{II}(\psi) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\psi] - \frac{\lambda_{II} - 3}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\psi]}{\sin[(\lambda_{II} - 1)\alpha] - \frac{\lambda_{II} - 1}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]} \quad (33)$$

$$f_{r\theta}^I(\psi) = \frac{\sin[(\lambda_I - 1)\alpha]\sin[(\lambda_I + 1)\psi] - \sin[(\lambda_I + 1)\alpha]\sin[(\lambda_I - 1)\psi]}{\sin[(\lambda_I - 1)\alpha] - \frac{\lambda_I + 1}{\lambda_I - 1}\sin[(\lambda_I + 1)\alpha]} \quad (34)$$

$$f_{r\theta}^{II}(\psi) = \frac{\cos[(\lambda_{II} - 1)\alpha]\cos[(\lambda_{II} + 1)\psi] - \cos[(\lambda_{II} + 1)\alpha]\cos[(\lambda_{II} - 1)\psi]}{\cos[(\lambda_{II} - 1)\alpha] - \cos[(\lambda_{II} + 1)\alpha]} \quad (35)$$

$$f_{\theta\theta}^I(\psi) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\psi] - \cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\psi]}{\cos[(\lambda_I - 1)\alpha] - \cos[(\lambda_I + 1)\alpha]} \quad (36)$$

$$f_{\theta\theta}^{II}(\psi) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\psi] - \sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\psi]}{-\sin[(\lambda_{II} - 1)\alpha] + \frac{\lambda_{II} - 1}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]} \quad (37)$$

APPENDIX II INTEGRAL EQUATIONS

$$\begin{aligned} \int_0^\infty G_{rr\theta}(r, \alpha, \xi, \alpha) B_r^\oplus(\xi) + G_{\theta r\theta}(r, \alpha, \xi, \alpha) B_\theta^\oplus(\xi) + G_{rr\theta}(r, \alpha, \xi, -\alpha) B_r^\ominus(\xi) + G_{\theta r\theta}(r, \alpha, \xi, -\alpha) B_\theta^\ominus(\xi) d\xi \\ = [b_r G_{rr\theta}(r, \alpha, \rho, \phi) + b_\theta G_{\theta r\theta}(r, \alpha, \rho, \phi)] \end{aligned} \quad (38)$$

$$\begin{aligned} \int_0^\infty G_{rr\theta}(r, -\alpha, \xi, \alpha) B_r^\oplus(\xi) + G_{\theta r\theta}(r, -\alpha, \xi, \alpha) B_\theta^\oplus(\xi) + G_{rr\theta}(r, -\alpha, \xi, -\alpha) B_r^\ominus(\xi) + G_{\theta r\theta}(r, -\alpha, \xi, -\alpha) B_\theta^\ominus(\xi) d\xi \\ = [b_r G_{rr\theta}(r, -\alpha, \rho, \phi) + b_\theta G_{\theta r\theta}(r, -\alpha, \rho, \phi)] \end{aligned} \quad (39)$$

$$\begin{aligned} \int_0^\infty G_{r\theta\theta}(r, \alpha, \xi, \alpha) B_r^\oplus(\xi) + G_{\theta\theta\theta}(r, \alpha, \xi, \alpha) B_\theta^\oplus(\xi) + G_{r\theta\theta}(r, \alpha, \xi, -\alpha) B_r^\ominus(\xi) + G_{\theta\theta\theta}(r, \alpha, \xi, -\alpha) B_\theta^\ominus(\xi) d\xi \\ = [b_r G_{r\theta\theta}(r, \alpha, \rho, \phi) + b_\theta G_{\theta\theta\theta}(r, \alpha, \rho, \phi)] \end{aligned} \quad (40)$$

$$\begin{aligned} \int_0^\infty G_{r\theta\theta}(r, -\alpha, \xi, \alpha) B_r^\oplus(\xi) + G_{\theta\theta\theta}(r, -\alpha, \xi, \alpha) B_\theta^\oplus(\xi) + G_{r\theta\theta}(r, -\alpha, \xi, -\alpha) B_r^\ominus(\xi) + G_{\theta\theta\theta}(r, -\alpha, \xi, -\alpha) B_\theta^\ominus(\xi) d\xi \\ = [b_r G_{r\theta\theta}(r, -\alpha, \rho, \phi) + b_\theta G_{\theta\theta\theta}(r, -\alpha, \rho, \phi)] \end{aligned} \quad (41)$$

5.2 Conclusions

As a result of these investigations we now have a method for introducing dislocations along the contact interface that is both versatile and reliable. The use of polar coordinates means that many different phenomena may now be investigated, including the possibility of varying the contact, crack (where appropriate) and wedge angles, and observing the effect of these changes on the behaviour of the contact.

Furthermore, because of the verification using the half plane kernels, and the use of finite element analysis, we can be confident that we have not repeated the mistake made by Churchman et al, or made a similar mistake ourselves.

Chapter 6

Cracks in a wedge of arbitrary angle

In chapter 4 we found a method of modelling the behaviour in a general complete contact with bilateral boundary conditions. In chapter 5 we found a way to modify the state of stress in such a geometry to accomodate unilateral boundary conditions. So, now we may use both of these methods to tackle the first of our real world problems, the presence of a crack at the edge of a complete contact. Crack problems present slightly easier conditions to enforce than slip problems, as we assume the crack is open, so we will discuss these first. The faces of open cracks must be free of traction. These crack faces are free surfaces and cannot sustain any normal or shear traction.

The location, length and angle of the crack must all be predetermined to allow a proper modelling of the resultant stresses. For this thesis we restrict our attention to cracks whose root is at the edge of contact to simplify the algebra. Previous studies, most notably that by Churchman and Hills [33], have restricted consideration of the crack angle to perpendicular cracks and free surfaces. Because of the developments to the dislocation kernels discussed in chapter 5, this restriction is no longer necessary and we can explore a variety of crack angles.

The solution is linear, and each remote load (characterised by the stress intensity factors K_I and K_{II}) may be applied individually. This means that in each load case, crack tip stress intensity factors may be found as a proportion of the remote loading. Furthermore, the crack length can be isolated and the dependence based on the eigenvalues of the wedge solution found. This allows us to write a calibration matrix which directly connects the crack tip stress intensity factors to the remote loading for a crack of any length, whose mouth is at the edge of contact, the angle of the crack being known, at a contact of given angle.

This is already a significant improvement on the previous studies as the angle of a crack arising cannot be easily controlled, so it will be beneficial in actual problems to be able to calculate the effect of skew cracks. Another common problem which arises is the growth of secondary cracks, which can occur either due to the same defect that caused the initiation of the original crack, or by crack branching. It is also possible for a crack to kink as it grows, which can itself lead to crack branching.

As we are only considering short cracks, rather than tackling the kinked crack problem, let us consider the problem of two similar cracks, both having their mouths at the edge of the contact. This second crack will have its own angle and length, and the resulting changes in the stress field from its presence will affect the first crack and vice versa. The magnitude and type of effect is dependent on the ratio of the two crack lengths and their angles relative to each other and the free surfaces.

Determining the crack tip stress intensity factors is an easy way to determine the long term behavioural trend of the crack; a crack having $K_I^c > 0$, will remain open but is likely to continue to grow, for example. Having this knowledge could

allow loading to be altered so as to arrest crack growth (by forcing K_I^c to become negative) without needing to replace parts within an assembly.

6.1 Publication - ‘Dislocations in an arbitrary angle wedge. Part II: Cracks in the wedge’

These investigations have all been carried out in a paper now attached as part of this thesis. This paper has been reproduced, as originally published:

Riddoch, D. J., Cwiekala, N. and Hills, D. A., *Dislocations in an arbitrary angle wedge. Part II: Cracks in the wedge*, The Journal of Strain Analysis for Engineering Design: 03093247211047785, October 2021.

The second author of this paper is Nils Cwiekala, another DPhil student in our group. His contribution to this paper was mainly consisting of technical help with coding, including utilising some code written for another problem. This work forms no part of his thesis.

Dislocations in an arbitrary angle wedge.

Part II: Cracks in the wedge

Daniel J. Riddoch¹, Nils Cwiekala¹, David A. Hills¹

Abstract—We describe a method for calculating the crack tip stress intensity factors for the problem of one or two cracks at the apex of an arbitrary angle wedge. The kernels for a dislocation in an arbitrary angle wedge described in part I of this paper are used extensively. Consideration is given to variations of crack length, crack angle and wedge angle.

I. INTRODUCTION

A sharp-angled, semi-infinite wedge may be used to represent, the edge of a complete contact between elastically similar bodies under conditions of full adhesion, and produces a singular state of stress at the wedge's apex, whenever the contact interface remains at least locally closed[1], [2], [3]. Such a stress state is unsustainable, as practical materials having a finite yield strength and some plastic flow must occur to relieve the state of stress and give rise to rounding and hence a relaxation of the singularity[4]. The singularity may also be relaxed by frictional slip[5], [6], or by the development of a crack whose mouth is precisely at the edge of the contact[7].

In this analysis, we will concentrate on the last of these phenomena. Cracks have long been an area of interest in applied mechanics, and many studies have examined their behaviour[8], [9]. A cracked body behaves very differently from an uncracked body under loading. Furthermore, if left unchecked a crack in a body may continue to grow until failure occurs.

In part I of this paper, we discussed and developed a method for calculating the influence of a dislocation in such a system as that which we are discussing here. We will now use these results to quantify the behaviour of the tip of a crack in a semi-infinite sharp angled wedge. We recall that the faces of a crack must be traction free, and therefore, by distributing edge dislocations along the faces of a crack, we may modify the state of stress in the wedge, and determine its new value around the tip of the crack, as characterised by the crack-tip stress intensity factors. Such a system has been sketched for two particular cases in figures 1 and 5.

This method, and indeed this problem, in a very specific case, have been tackled before. Churchman and Hills[7] looked at the specific case of a crack in one particular location in a three quarter plane. Their paper has been found to contain a small error, and corrections are presented here along with many extensions of their work. The object of this analysis is to present calibrations for the crack tip stress intensity factors for one or two cracks in arbitrary angle wedges, the cracks themselves also being at various angles.

II. ASYMPTOTIC FORMULATION

Before we tackle the modelling of the crack, we must first consider the state of stress in the uncracked wedge. To do this, we will use the long established Williams solution[10], which represents the state of stress in an asymptotic manner near the apex of the wedge. A full derivation is set out by Barber[11].

The resulting solution is dominated by two eigensolutions, the mode I, or symmetric, and the mode II, or anti-symmetric, term. The stress characterised by the mode I term is singular for wedges of total internal angle greater than 180° . The stress characterised by the mode II term is singular for wedges of total internal angle greater than approximately 257° . The stress is described by the formula

$$\sigma_{ij}(r, \theta) = K_I f_{ij}^I(\theta) r^{\lambda_I - 1} + K_{II} f_{ij}^{II}(\theta) r^{\lambda_{II} - 1} + \dots$$

higher order terms. (1)

where K_I and K_{II} are the mode I and mode II stress intensity factors respectively, λ_I and λ_{II} are the mode I and mode II eigenvalues respectively, and f_{ij}^I and f_{ij}^{II} are the mode I and mode II eigenvectors.

III. THE SINGLE CRACK PROBLEM

Let us now concentrate on the simplest of the many cases described above. In this section, we will consider a wedge, of total internal angle 2α , containing a single, straight, crack of length l , emanating from the apex of the wedge at an angle ϕ from the bisector line through the wedge apex. This cracked wedge is sketched in Figure 1. The far-field load is filtered through the medium of Williams' solution, and we will assume that the cracks are sufficiently short for them to experience a state of stress controlled by a hinterland given by the first two terms in a series expansion, as laid out in equation 1.

A. Formulation

An open crack in any circumstance will always have the property that the 'faces' of the crack will be free of both normal and shear traction. We will use this feature here in order to model the stresses around the crack. In order to do this, we will employ dislocations. The influence functions for these dislocations and the method of finding them is set out in part I of this paper.

With these functions to hand, we may proceed to distribute dislocations along the free surfaces of the crack. The state of stress induced by them must cause the tractions produced along the line of the crack in the uncracked body to vanish.

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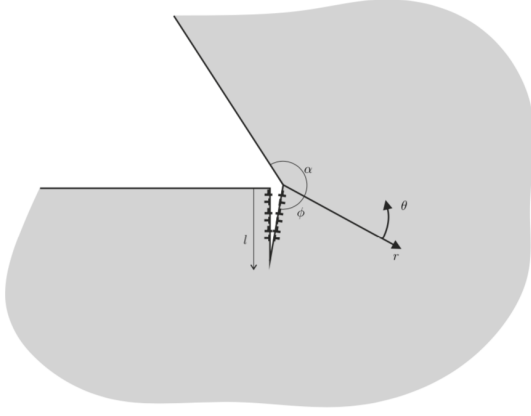


Fig. 1: Sketch of distribution of dislocations along crack faces

The stress along this line is determined using the Williams solution, as outlined above. We can formulate the problem using integral equations.

The problem is characterised by two integral equations, and these are set out in full in appendix III-A. For the sake of brevity, we will display and use one only here, viz:

$$\begin{aligned}
 -K_I f_{r\theta}^I(\phi) r^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II}-1} &= \frac{2\mu}{\pi(\kappa+1)} \dots \\
 \int_0^l B_r(\xi) [F_{rr\theta}(r, \phi, \rho, \phi) + G_{rr\theta}(r, \phi, \rho, \phi)] &+ \dots \\
 B_\theta(\xi) [F_{\theta r\theta}(r, \phi, \rho, \phi) + G_{\theta r\theta}(r, \phi, \rho, \phi)] d\xi, & \\
 \text{for } 0 \leq r \leq l. &
 \end{aligned} \quad (2)$$

In this formulation the influence functions are denoted by $F_{ijk}(r, \phi, \rho, \phi)$, and the single dislocation kernels by $G_{ijk}(r, \phi, \rho, \phi)$, as outlined in paper 1. K_I and K_{II} are the mode I and mode II stress intensity factors respectively, and λ_I, λ_{II} are the Williams eigenvalues, and $f_{r\theta}^I, f_{r\theta}^{II}$ are the Williams eigenvectors, l is the length of the crack. Finally, μ is the modulus of rigidity and κ is Kolosov's constant, equal to $3 - 4\nu$, where ν is Poisson's ratio, in plane strain.

We must now solve a pair of simultaneous singular integral equations, as we did for the kernel of a dislocation in a wedge. Also as before, we will employ the Gauss-Chebyshev quadrature, described by Erdogan et al[12], with appropriate transformations. Again we wish to use the interval $[-1, 1]$ as the integration range.

We begin by transforming the collocation points

$$r = \frac{l}{2}(v+1), \quad (3)$$

and a similar transform for the integration points

$$\xi = \frac{l}{2}(u+1). \quad (4)$$

The integration range is thus transformed. It will also prove useful to have

$$\frac{d\xi}{du} = \frac{l}{2}. \quad (5)$$

Considering the quadrature, we require the distribution of dislocations to be singular at $u = 1$, which represents the crack tip, but bounded at $u = -1$, which represents the wedge apex. Hence, we choose as the fundamental function the function $\omega(u_j)$ having the form

$$\omega(u) = \sqrt{\frac{1+u}{1-u}}, \quad (6)$$

and so we may rewrite the dislocation densities, using the bounded function $\varphi_i(\xi)$, in the form

$$B_i(\xi) = \omega(\xi) \varphi_i(\xi). \quad (7)$$

We should also note the weight function for the numerical quadrature, given by

$$W(u) = \frac{2(1+u)}{2n+1}, \quad (8)$$

where n is the number of integration and collocation points in the quadrature. Having these substitutions, using the steps outlined in appendix IV, the integral equation can be written as

$$\begin{aligned}
 -K_I f_{r\theta}^I(\phi) \left(\frac{l}{2}(v+1)\right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi) \left(\frac{l}{2}(v+1)\right)^{\lambda_{II}-1} &\dots \\
 = \frac{2\mu}{\pi(\kappa+1)} \int_{-1}^1 \omega_r(u) \varphi_r(u) [F_{rr\theta}(v+1, \phi, u+1, \phi) + \dots & \\
 G_{rr\theta}(v+1, \phi, u+1, \phi)] + \omega_\theta(u) \varphi_\theta(u) \dots & \\
 [F_{\theta r\theta}(v+1, \phi, u+1, \phi) + G_{\theta r\theta}(v+1, \phi, u+1, \phi)] du. &
 \end{aligned} \quad (9)$$

The application of the quadrature allows us to discretise the equation as follows

$$\begin{aligned}
 -K_I f_{r\theta}^I(\phi) \left(\frac{l}{2}(v+1)\right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi) \left(\frac{l}{2}(v+1)\right)^{\lambda_{II}-1} &\dots \\
 = \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(u_i) \varphi_r(u_i) [F_{rr\theta}(v+1, \phi, u_i+1, \phi) + \dots & \\
 G_{rr\theta}(v+1, \phi, u_i+1, \phi)] + \sum_{i=1}^n W(u_i) \varphi_\theta(u_i) \dots & \\
 [F_{\theta r\theta}(v+1, \phi, u_i+1, \phi) + G_{\theta r\theta}(v+1, \phi, u_i+1, \phi)] \right]. &
 \end{aligned} \quad (10)$$

By similar operations, the other integral equation can be similarly discretised, and as a result, we produce a system of $2n$ simultaneous equations in $2n$ unknowns, denoted by $\varphi_r(u_1), \dots, \varphi_r(u_n), \varphi_\theta(u_1), \dots, \varphi_\theta(u_n)$.

Finally, in order to determine the stress intensity factors at the crack tip which will characterise the state of stress in the neighbourhood of the crack tip, we use Krenk's interpolation[13], to calculate the value of φ_r and φ_θ at the crack tip. This then allows us to immediately calculate the

stress intensity factors. The following formulae result

$$K_I^c = \frac{4\mu\sqrt{2\pi l}}{(2n+1)(\kappa+1)} \cdots \sum_{i=1}^n \cot\left(\frac{\pi(2i-1)}{2(2n+1)}\right) \sin\left(\frac{n\pi(2i-1)}{2n+1}\right) \varphi_\theta(u_i), \quad (11)$$

$$K_{II}^c = \frac{4\mu\sqrt{2\pi l}}{(2n+1)(\kappa+1)} \cdots \sum_{i=1}^n \cot\left(\frac{\pi(2i-1)}{2(2n+1)}\right) \sin\left(\frac{n\pi(2i-1)}{2n+1}\right) \varphi_r(u_i). \quad (12)$$

Now that the stress intensity factors may be determined, we are able to produce the following relationship,

$$\frac{1}{\sqrt{\pi l}} \begin{bmatrix} K_I^c \\ K_{II}^c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} K_I l^{\lambda_I-1} \\ K_{II} l^{\lambda_{II}-1} \end{bmatrix}, \quad (13)$$

where a, b, c, d are dimensionless numbers to be determined, which form a calibration between crack tip stress intensity factors and those at the wedge apex.

B. The Churchman results

This problem has been previously studied by Churchman and Hills[7]. They considered the specific case of a crack in a three quarter plane, placed at a 90° angle to one of the free surfaces. That paper and several related papers formed a basis for this analysis, but it is important to note that during the course of this analysis, a small error was discovered in the original Churchman paper.

This emerges in the normalisation they perform between equations 11 and 12, and equations 14 and 15, where they replace $\frac{1}{x-\xi}$ with $\frac{1}{\bar{x}-\bar{\xi}}$, with $\bar{x} := \frac{x}{d_0}$ and $\bar{\xi} := \frac{\xi}{d_0}$, where they define $d_0 = \left(\frac{K_I f_{\theta\theta}^I(\phi)}{K_{II} f_{\theta\theta}^{II}(\phi)} \right)^{\frac{1}{\lambda_{II}-\lambda_I}}$. In reality we have that

$$\frac{1}{x-\xi} = \frac{d_0}{\bar{x}-\bar{\xi}}.$$

A similar mistake is again made when discretising the integral in equations 27 and 28. Both these mistakes are minor in themselves, but nonetheless do produce a small change in the results. Churchman's results were detailed in equation 35 of their paper[7], but we have recalculated them using our method, without the errors mentioned, and the results are

$$\frac{1}{\sqrt{\pi l}} \begin{bmatrix} K_I^c \\ K_{II}^c \end{bmatrix} = \begin{bmatrix} 1.5060 & 1.0927 \\ 0.6083 & -0.3695 \end{bmatrix} \begin{bmatrix} K_I l^{\lambda_I-1} \\ K_{II} l^{\lambda_{II}-1} \end{bmatrix}. \quad (14)$$

C. Generalising the angle

One of the priorities of this new analysis was to generalise the angle of the wedge and of the crack within the wedge. Previous analyses of similar problems[7], [14] have focused on specific combinations of wedge angle and crack angle. However, our formulation is written in terms of a general angle, and the method of finding the kernels has also been done generally. So we are in a position to calculate

calibration matrices such as that given in equation 14 for any combination of crack and wedge angle.

We can plot the resultant stress intensity factor ratios against the angle of the crack or the plane. Figure 2 shows how the calibration matrix entries, as defined in equation 13, vary as a crack, which is a continuation of a plane face and has length 1, is placed in a wedge of various half-angles, from $\frac{3\pi}{4} \leq \alpha \leq \frac{9\pi}{10}$, see figure 1 for a definition of α , in this case we choose $\phi = \frac{\pi}{2} - \alpha$. Similarly, figure 3 shows the calibration matrix entries as the angle of the crack, set within a three quarter plane and of length 1, varies between $0 \leq \phi \leq \frac{4\pi}{10}$, see figure 1 for a definition of ϕ .

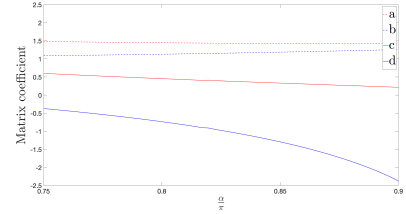


Fig. 2: Plot of variation of stress intensity factors with changing wedge angle, 2α , and $\phi = \frac{\pi}{2} - \phi$.

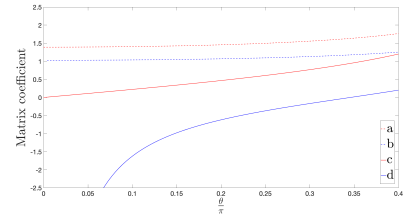


Fig. 3: Plot of variation of stress intensity factors with changing crack angle for a three-quarter plane, $2\alpha = \frac{3\pi}{2}$.

It is useful to fit a polynomial to these curves to allow for a more accurate use of these results. Fits for all eight curves shown in figures 2 and 3 can be found in appendix II.

D. Verification

It is reasonable to want to verify these results, especially since Churchman et al compared their results to known stress intensity factors for long cracks. We present an example problem with a well known solution.

1) *The cracked half-plane:* We consider the problem of a crack in a half-plane, subjected to remote tensile stress parallel to the free surface. This specific problem was tackled by Nowell and Hills[9]. They produced a clean formulation for the stress intensity factors at the tip of a crack using distributions of dislocations along the crack face with kernels specifically calculated for a dislocation in a half-plane, and we will use their results here. Figure 4 shows an overlay of their stress intensity factor ratios(in black), with our calibrations for the same stress intensity factors overlaid(dashed, green and red).

An examination of figure 4 shows a good match between the calibrations. The results for angles greater than 75° are

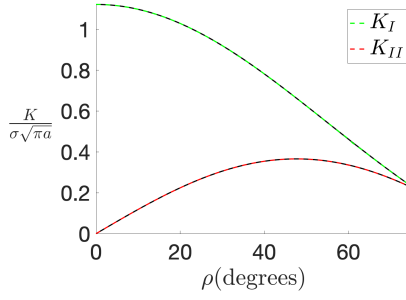


Fig. 4: Overlay of our stress intensity factor calibrations with the results of Nowell and Hills[9]

omitted, as significant numerical problems occur here with both methods. This is inevitable when using the method of distributed dislocations for producing the dislocation kernels, as lots of dislocations will be close together and have nearly parallel Burgers vectors. This will inevitably lead to numerical errors, but the fidelity of the match for angles of slant less than approximately 75° shows that the method is valid.

IV. THE TWO CRACK PROBLEM

The problem of the single crack in the semi-infinite wedge has been tackled before and we are now able to generalise the problem in a much more comprehensive way. However, it would seem obvious to consider next how the presence of a second crack may complicate the solution. The system is sketched in figure 5

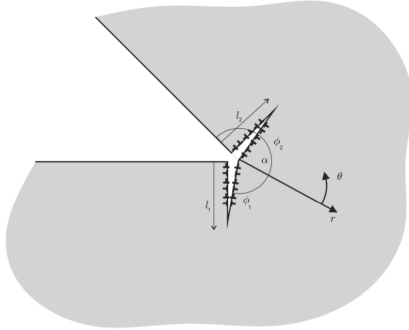


Fig. 5: Sketch of distribution of dislocation along the faces of both cracks

Crack tips produce a singular state of stress, but we will not consider here the possibility of branching or splitting cracks, an introduction to these topics is given by Hills et al.[8]. Instead, we consider the possibility of another crack emanating from the notch apex. The state of stress here is still extremely high, and may result in the formation of another crack.

A. Formulation

The basic approach to the two crack problem is identical to that taken to the single crack problem. Once again we will distribute dislocations along the faces of the cracks, this time

we must place four distributions of dislocations, two along each crack. We will distinguish between variables relating to the cracks by the means of the subscript/superscript numerals 1 and 2.

As such we may proceed in much the same way, by writing the integral equations which characterise the problem. On this occasion, four integral equations are required, all of which are displayed in appendix III-B. For the sake of brevity we will display only one, viz:

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_1} B_r^1(\rho) [F_{rr\theta}(r, \phi_1, \rho, \phi_1) + G_{rr\theta}(r, \phi_1, \rho, \phi_1)] + \dots \\ & B_\theta^1(\rho) [F_{\theta r\theta}(r, \phi_1, \rho, \phi_1) + G_{\theta r\theta}(r, \phi_1, \rho, \phi_1)] d\rho + \dots \\ & \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_2} B_r^2(\rho) [F_{rr\theta}(r, \phi_1, \rho, \phi_2) + G_{rr\theta}(r, \phi_1, \rho, \phi_2)] + \dots \\ & B_\theta^2(\rho) [F_{\theta r\theta}(r, \phi_1, \rho, \phi_2) + G_{\theta r\theta}(r, \phi_1, \rho, \phi_2)] d\rho \dots \\ & = -K_I f_{r\theta}^I(\phi_1) r^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1) r^{\lambda_{II}-1} \dots \\ & \text{for } 0 \leq r \leq l_1 \dots \end{aligned} \quad (15)$$

Very similar transforms are applied to those that were applied to the single crack integral equations, and these are outlined in appendix IV. The resulting equations are also outlined there and are characterised by equation 16, resulting from transforms of equation 15.

$$\begin{aligned} & \int_{-1}^1 \hat{B}_r^1(u) [F_{rr\theta}(v+1, \phi_1, u+1, \phi_1) + \dots \\ & G_{rr\theta}(v+1, \phi_1, u+1, \phi_1)] + \dots \\ & \hat{B}_\theta^1(u) [F_{\theta r\theta}(v+1, \phi_1, u+1, \phi_1) + \dots \\ & G_{\theta r\theta}(v+1, \phi_1, u+1, \phi_1)] du + \dots \\ & l_R \int_{-1}^1 \hat{B}_r^2(u) [F_{rr\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + \dots \\ & G_{rr\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] + \dots \\ & \hat{B}_\theta^2(u) [F_{\theta r\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + \dots \\ & G_{\theta r\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] du \dots \\ & = -K_I f_{r\theta}^I(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - \dots \\ & K_{II} f_{r\theta}^{II}(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}, \end{aligned} \quad (16)$$

where $l_r = \frac{l_2}{l_1}$, and $\hat{B}_i^j(u) = \frac{2\mu}{\pi(\kappa+1)} B_i^j(u)$.

We use the Gauss-Chebyshev quadrature as before. Again we require the distribution to be singular at $u = 1$, and bounded at $u = -1$. So we write

$$\omega(u) = \sqrt{\frac{1+u}{1-u}}, \quad (17)$$

$$\hat{B}_i(\xi) = \omega(\xi) \hat{\varphi}_i(\xi) \quad (18)$$

$$W(u) = \frac{2(1+u)}{2n+1}, \quad (19)$$

notice that we are using the same number of points n on

both cracks. So, equation 16 can be discretised to become

$$\begin{aligned}
& \sum_{i=1}^n \hat{\varphi}_r^1(u_i) W(u_i) [F_{rr\theta}(v+1, \phi_1, u_i+1, \phi_1) + \dots \\
& \quad G_{rr\theta}(v+1, \phi_1, u_i+1, \phi_1)] + \hat{\varphi}_\theta^1(u_i) W(u_i) \dots \\
& \quad [F_{\theta r\theta}(v+1, \phi_1, u_i+1, \phi_1) + \dots \\
& \quad G_{\theta r\theta}(v+1, \phi_1, u_i+1, \phi_1)] + \dots \\
& l_R \sum_{i=1}^n \hat{\varphi}_r^2(u_i) W(u_i) [F_{rr\theta}(v+1, \phi_1, l_R(u_i+1), \phi_2) + \dots \\
& \quad G_{rr\theta}(v+1, \phi_1, l_R(u_i+1), \phi_2)] + \dots \\
& \quad \hat{\varphi}_\theta^2(u_i) W(u_i) [F_{\theta r\theta}(v+1, \phi_1, l_R(u_i+1), \phi_2) + \dots \\
& \quad G_{\theta r\theta}(v+1, \phi_1, l_R(u_i+1), \phi_2)] \dots \\
& = -K_I f_{r\theta}^I(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - \dots \\
& \quad K_{II} f_{r\theta}^{II}(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}. \quad (20)
\end{aligned}$$

Krenk interpolation is once more used to determine the crack tip stress intensity factors from the solutions to the arising simultaneous equations

$$K_I^{1c} = \frac{4\mu\sqrt{2\pi l_1}}{(2n+1)(\kappa+1)l_1^{\lambda_I-1}} \dots \sum_{i=1}^n \cot\left(\frac{\pi(2i-1)}{2(2n+1)}\right) \sin\left(\frac{n\pi(2i-1)}{2n+1}\right) \varphi_\theta(u_i), \quad (21)$$

$$K_{II}^{1c} = \frac{4\mu\sqrt{2\pi l_1}}{(2n+1)(\kappa+1)l_1^{\lambda_{II}-1}} \dots \sum_{i=1}^n \cot\left(\frac{\pi(2i-1)}{2(2n+1)}\right) \sin\left(\frac{n\pi(2i-1)}{2n+1}\right) \varphi_r(u_i), \quad (22)$$

$$K_I^{2c} = \frac{4\mu\sqrt{2\pi l_2}}{(2n+1)(\kappa+1)l_2^{\lambda_I-1}} \dots \sum_{i=1}^n \cot\left(\frac{\pi(2i-1)}{2(2n+1)}\right) \sin\left(\frac{n\pi(2i-1)}{2n+1}\right) \varphi_\theta(u_i), \quad (23)$$

$$K_{II}^{2c} = \frac{4\mu\sqrt{2\pi l_2}}{(2n+1)(\kappa+1)l_2^{\lambda_{II}-1}} \dots \sum_{i=1}^n \cot\left(\frac{\pi(2i-1)}{2(2n+1)}\right) \sin\left(\frac{n\pi(2i-1)}{2n+1}\right) \varphi_r(u_i), \quad (24)$$

Now that the stress intensity factors may be determined, we are able to produce the following relationship,

$$\frac{1}{\sqrt{\pi}} \begin{bmatrix} K_I^1 l_1^{\frac{\lambda_I-1}{2}} \\ K_{II}^1 l_1^{\frac{\lambda_{II}-1}{2}} \\ K_I^2 l_2^{\frac{\lambda_I-1}{2}} \\ K_{II}^2 l_2^{\frac{\lambda_{II}-1}{2}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}, \quad (25)$$

where a, b, c, d, e, f, g, h are dimensionless numbers to be determined.

B. The three-quarter plane

The first example we address is the three quarter plane, with two cracks, at angles $\pm\frac{\pi}{4}$ from the bisector. This problem can be tackled by setting $\alpha = \frac{3\pi}{4}, \phi_1 = \frac{\pi}{4}, \phi_2 = -\frac{\pi}{4}$. Then, using the method set out above the calibration for cracks of equal length, we get the following relationship:

$$\frac{1}{\sqrt{\pi}} \begin{bmatrix} K_I^1 l_1^{\frac{\lambda_I-1}{2}} \\ K_{II}^1 l_1^{\frac{\lambda_{II}-1}{2}} \\ K_I^2 l_2^{\frac{\lambda_I-1}{2}} \\ K_{II}^2 l_2^{\frac{\lambda_{II}-1}{2}} \end{bmatrix} = \begin{bmatrix} 1.0558 & -2.1207 \\ 0.7439 & 0.3416 \\ 1.0558 & 2.1207 \\ -0.7439 & 0.3416 \end{bmatrix} \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}. \quad (26)$$

We may also consider the effect of varying the ratio of the length of the cracks. Figure 6 plots the variation of the coefficients of the matrix against the logarithm of the variation of the crack length ratio, note that the entries b, c, f and g are not constant, just weak functions of $\frac{l_2}{l_1}$.

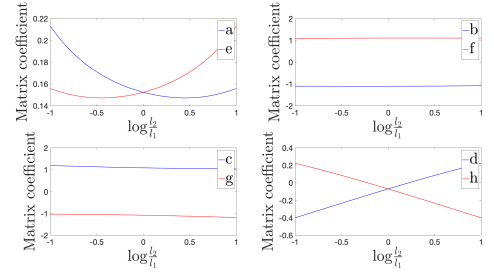


Fig. 6: Plot of the coefficients of the matrix given in equation 25

C. Generalising the wedge angle

Next, we consider variations of the angle of the wedge. There are infinitely many possible combinations of wedge and crack angles to consider. First, we will consider what happens if we place two cracks, one of unit length at angle $\phi_1 = \frac{\pi}{4}$, and a second of half-unit length at angle $\phi_2 = -\frac{\pi}{4}$ in a three-quarter plane. Fixing these cracks, figure 7 shows how the matrix coefficients change as we vary the angle α , the half-angle of the wedge, again note that the entries b, c, f and g are not constant, just weak functions of $\frac{\alpha}{\pi}$.

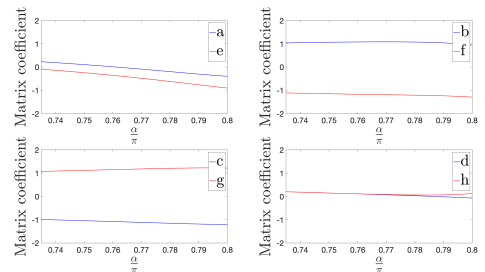


Fig. 7: Plot of the coefficients of the matrix given in equation 25 as the internal wedge angle varies

D. Generalising the crack angle

Alternatively, we may consider what happens if we only change the angle of one of the cracks. Figure 8 shows the variation of the matrix coefficients as the angle of one crack is varied. Specifically, we considered a three quarter plane, with one crack of unit length fixed at an angle $\phi_1 = \frac{\pi}{4}$, and another crack of half-unit length, whose angle is given by ϕ_2 and varies from $-\frac{\pi}{2}$ to 0, or along the bisector.

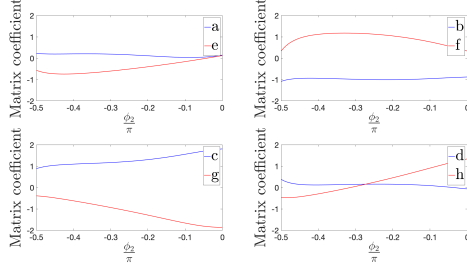


Fig. 8: Plot of the coefficients of the matrix given in equation 25 as the angle of one of the cracks varies

V. CONCLUSIONS

We have described in this paper a method for evaluating the crack tip stress intensity factors for one or two cracks of arbitrary length, placed at arbitrary angles in a sharp-cornered wedge, itself of arbitrary internal angle. The method used has utilised the previously discussed kernels for a dislocation placed in such a wedge.

The results presented are verified and compared to known results for specific cases, and in the particular case discussed by Churchman and Hills[15]. A correction to their results is made. Furthermore, as is consistent with the objectives of this analysis, calibrations, or at least the method of finding such, are presented for cases of either one or two cracks in arbitrary angle wedges. Owing to the many possible configuration that readers may wish to investigate specifically, the MATLAB code we used to generate these results is published alongside this article.

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APPENDIX I WILLIAMS SOLUTION

The solution of Williams[10] is well known, and produces characteristic eigenvalues and eigenvectors. In this appendix, the formulae to find the eigenvalues and eigenvectors are given, more detail is given by Barber[1]. The eigenvalues are the solutions of the equations

$$\lambda_I \sin(2\alpha) + \sin(2\alpha\lambda_I) = 0 \quad \& \quad \lambda_{II} \sin(2\alpha) - \sin(2\alpha\lambda_{II}) = 0. \quad (27)$$

The eigenvectors are given by

$$f_{rr}^I(\psi) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\psi] - \frac{\lambda_I - 3}{\lambda_I + 1}\cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\psi]}{\cos[(\lambda_I + 1)\alpha] - \cos[(\lambda_I - 1)\alpha]} \quad (28)$$

$$f_{rr}^{II}(\psi) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\psi] - \frac{\lambda_{II} - 3}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\psi]}{\sin[(\lambda_{II} - 1)\alpha] - \frac{\lambda_{II} - 1}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]} \quad (29)$$

$$f_{r\theta}^I(\psi) = \frac{\sin[(\lambda_I - 1)\alpha]\sin[(\lambda_I + 1)\psi] - \sin[(\lambda_I + 1)\alpha]\sin[(\lambda_I - 1)\psi]}{\sin[(\lambda_I - 1)\alpha] - \frac{\lambda_I + 1}{\lambda_I - 1}\sin[(\lambda_I + 1)\alpha]} \quad (30)$$

$$f_{r\theta}^{II}(\psi) = \frac{\cos[(\lambda_{II} - 1)\alpha]\cos[(\lambda_{II} + 1)\psi] - \cos[(\lambda_{II} + 1)\alpha]\cos[(\lambda_{II} - 1)\psi]}{\cos[(\lambda_{II} - 1)\alpha] - \cos[(\lambda_{II} + 1)\alpha]} \quad (31)$$

$$f_{\theta\theta}^I(\psi) = \frac{\cos[(\lambda_I - 1)\alpha]\cos[(\lambda_I + 1)\psi] - \cos[(\lambda_I + 1)\alpha]\cos[(\lambda_I - 1)\psi]}{\cos[(\lambda_I - 1)\alpha] - \cos[(\lambda_I + 1)\alpha]} \quad (32)$$

$$f_{\theta\theta}^{II}(\psi) = \frac{\sin[(\lambda_{II} - 1)\alpha]\sin[(\lambda_{II} + 1)\psi] - \sin[(\lambda_{II} + 1)\alpha]\sin[(\lambda_{II} - 1)\psi]}{-\sin[(\lambda_{II} - 1)\alpha] + \frac{\lambda_{II} - 1}{\lambda_{II} + 1}\sin[(\lambda_{II} + 1)\alpha]} \quad (33)$$

APPENDIX II CURVE FITTING

This appendix contains the coefficients of quadratic polynomials which have been fitted to the curves displayed in figures 2 and 3. These are summarised in tables I and II.

Matrix Coefficient	Coefficient of $\left(\frac{\alpha}{\pi}\right)^2$	Coefficient of $\frac{\alpha}{\pi}$	Constant
a	5.344	-9.227	5.412
b	5.446	-7.798	3.881
c	4.647	-10.26	5.692
d	-75.88	112.7	-42.27

TABLE I: Table showing polynomial fit coefficients for curves shown in figure 2

Matrix Coefficient	Coefficient of $\left(\frac{\theta}{\pi}\right)^2$	Coefficient of $\frac{\theta}{\pi}$	Constant
a	2.84	-0.3164	1.39
b	1.781	-0.1986	1.03
c	3.038	1.588	0.02141
d	-181.2	95.58	-11.55

TABLE II: Table showing polynomial fit coefficients for curves shown in figure 3

APPENDIX III INTEGRAL EQUATIONS

A. Single crack problem

$$-K_I f_{r\theta}^I(\phi) r^{\lambda_I - 1} - K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II} - 1} = \dots$$

$$\frac{2\mu}{\pi(\kappa + 1)} \int_0^l B_r(\rho) \left[F_{rr\theta}(r, \phi, \rho, \phi) + G_{rr\theta}(r, \phi, \rho, \phi) \right] + B_\theta(\rho) \left[F_{\theta r\theta}(r, \phi, \rho, \phi) + G_{\theta r\theta}(r, \phi, \rho, \phi) \right] d\rho \quad (34)$$

$$-K_I f_{\theta\theta}^I(\phi) r^{\lambda_I - 1} - K_{II} f_{\theta\theta}^{II}(\phi) r^{\lambda_{II} - 1} = \dots$$

$$\frac{2\mu}{\pi(\kappa + 1)} \int_0^l B_r(\rho) \left[F_{r\theta\theta}(r, \phi, \rho, \phi) + G_{r\theta\theta}(r, \phi, \rho, \phi) \right] + B_\theta(\rho) \left[F_{\theta\theta\theta}(r, \phi, \rho, \phi) + G_{\theta\theta\theta}(r, \phi, \rho, \phi) \right] d\rho \quad (35)$$

B. Two crack problem

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_1} B_r^1(\rho) [F_{rr\theta}(r, \phi_1, \rho, \phi_1) + G_{rr\theta}(r, \phi_1, \rho, \phi_1)] + B_\theta^1(\rho) [F_{\theta r\theta}(r, \phi_1, \rho, \phi_1) + G_{\theta r\theta}(r, \phi_1, \rho, \phi_1)] d\rho + \dots \\
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_2} B_r^2(\rho) [F_{rr\theta}(r, \phi_1, \rho, \phi_2) + G_{rr\theta}(r, \phi_1, \rho, \phi_2)] + B_\theta^2(\rho) [F_{\theta r\theta}(r, \phi_1, \rho, \phi_2) + G_{\theta r\theta}(r, \phi_1, \rho, \phi_2)] d\rho \dots \\
& = -K_I f_{r\theta}^I(\phi_1) r^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1) r^{\lambda_{II}-1}
\end{aligned} \tag{36}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_1} B_r^1(\rho) [F_{r\theta\theta}(r, \phi_1, \rho, \phi_1) + G_{r\theta\theta}(r, \phi_1, \rho, \phi_1)] + B_\theta^1(\rho) [F_{\theta\theta\theta}(r, \phi_1, \rho, \phi_1) + G_{\theta\theta\theta}(r, \phi_1, \rho, \phi_1)] d\rho + \dots \\
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_2} B_r^2(\rho) [F_{r\theta\theta}(r, \phi_1, \rho, \phi_2) + G_{r\theta\theta}(r, \phi_1, \rho, \phi_2)] + B_\theta^2(\rho) [F_{\theta\theta\theta}(r, \phi_1, \rho, \phi_2) + G_{\theta\theta\theta}(r, \phi_1, \rho, \phi_2)] d\rho \dots \\
& = -K_I f_{\theta\theta}^I(\phi_1) r^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_1) r^{\lambda_{II}-1}
\end{aligned} \tag{37}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_1} B_r^1(\rho) [F_{rr\theta}(r, \phi_2, \rho, \phi_1) + G_{rr\theta}(r, \phi_2, \rho, \phi_1)] + B_\theta^1(\rho) [F_{\theta r\theta}(r, \phi_2, \rho, \phi_1) + G_{\theta r\theta}(r, \phi_2, \rho, \phi_1)] d\rho + \dots \\
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_2} B_r^2(\rho) [F_{rr\theta}(r, \phi_2, \rho, \phi_2) + G_{rr\theta}(r, \phi_2, \rho, \phi_2)] + B_\theta^2(\rho) [F_{\theta r\theta}(r, \phi_2, \rho, \phi_2) + G_{\theta r\theta}(r, \phi_2, \rho, \phi_2)] d\rho \dots \\
& = -K_I f_{r\theta}^I(\phi_2) r^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_2) r^{\lambda_{II}-1}
\end{aligned} \tag{38}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_1} B_r^1(\rho) [F_{r\theta\theta}(r, \phi_2, \rho, \phi_1) + G_{r\theta\theta}(r, \phi_2, \rho, \phi_1)] + B_\theta^1(\rho) [F_{\theta\theta\theta}(r, \phi_2, \rho, \phi_1) + G_{\theta\theta\theta}(r, \phi_2, \rho, \phi_1)] d\rho + \dots \\
& \frac{2\mu}{\pi(\kappa+1)} \int_0^{l_2} B_r^2(\rho) [F_{r\theta\theta}(r, \phi_2, \rho, \phi_2) + G_{r\theta\theta}(r, \phi_2, \rho, \phi_2)] + B_\theta^2(\rho) [F_{\theta\theta\theta}(r, \phi_2, \rho, \phi_2) + G_{\theta\theta\theta}(r, \phi_2, \rho, \phi_2)] d\rho \dots \\
& = -K_I f_{\theta\theta}^I(\phi_2) r^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_2) r^{\lambda_{II}-1}
\end{aligned} \tag{39}$$

APPENDIX IV TWO CRACK TRANSFORMS

Firstly, let us assume without loss of generality that $l_1 \geq l_2$. Let us define the following transforms

$$\begin{aligned}
r &= \frac{l_1}{2}(v+1) & \rho &= \frac{l_1}{2}(u+1) & \text{alternatively} \\
v &= \frac{2r}{l_1} - 1 & u &= \frac{2\rho}{l_1} - 1.
\end{aligned} \tag{40}$$

Further note that

$$\frac{d\rho}{du} = \frac{l_1}{2} \quad \text{therefore} \quad d\rho = du \frac{l_1}{2}. \tag{41}$$

Finally we define

$$l_R = \frac{l_2}{l_1}$$

and then define similar transformations relating to crack 2

$$\rho = l_R \frac{l_1}{2}(u+1) \quad \text{alternatively} \quad u = \frac{1}{l_R} \frac{2\rho}{l_1} - 1. \tag{42}$$

Further note that

$$\frac{d\rho}{du} = l_R \frac{l_1}{2} \quad \text{therefore} \quad d\rho = l_R \frac{l_1}{2} du. \tag{43}$$

If we now consider just equation 36 and apply these transformations, we get

$$\begin{aligned}
& \int_{-1}^1 \bar{B}_r^1(u) \left[F_{rr\theta} \left(\frac{l_1}{2}(v+1), \phi_1, \frac{l_1}{2}(u+1), \phi_1 \right) + G_{rr\theta} \left(\frac{l_1}{2}(v+1), \phi_1, \frac{l_1}{2}(u+1), \phi_1 \right) \right] + \dots \\
& \bar{B}_\theta^1(u) \left[F_{\theta r\theta} \left(\frac{l_1}{2}(v+1), \phi_1, \frac{l_1}{2}(u+1), \phi_1 \right) + G_{\theta r\theta} \left(\frac{l_1}{2}(v+1), \phi_1, \frac{l_1}{2}(u+1), \phi_1 \right) \right] du + \dots \\
& l_R \int_{-1}^1 \bar{B}_r^2(u) \left[F_{rr\theta} \left(\frac{l_1}{2}(v+1), \phi_1, l_R \frac{l_1}{2}(u+1), \phi_2 \right) + G_{rr\theta} \left(\frac{l_1}{2}(v+1), \phi_1, l_R \frac{l_1}{2}(u+1), \phi_2 \right) \right] + \dots \\
& \bar{B}_\theta^2(u) \left[F_{\theta r\theta} \left(\frac{l_1}{2}(v+1), \phi_1, l_R \frac{l_1}{2}(u+1), \phi_2 \right) + G_{\theta r\theta} \left(\frac{l_1}{2}(v+1), \phi_1, l_R \frac{l_1}{2}(u+1), \phi_2 \right) \right] du \dots \\
& = -K_I f_{r\theta}^I(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}
\end{aligned} \tag{44}$$

where

$$\bar{B}_i^j(s) = \frac{l_1 \mu}{\pi(\kappa+1)} B_i^j(s).$$

It is important to note that equation 36 is referring to clearing tractions from the interface of crack number 1, and as such, the observation points are defined in relation to crack number 1.

We have previously shown by Burgers vector argument that

$$F_{ijk}(r, \theta, \rho, \phi) = m F_{ijk}(mr, \theta, m\rho, \phi). \tag{45}$$

So making use of this in equation 44

$$\begin{aligned}
& \int_{-1}^1 \hat{B}_r^1(u) [F_{rr\theta}(v+1, \phi_1, u+1, \phi_1) + G_{rr\theta}(v+1, \phi_1, u+1, \phi_1)] + \dots \\
& \hat{B}_\theta^1(u) [F_{\theta r\theta}(v+1, \phi_1, u+1, \phi_1) + G_{\theta r\theta}(v+1, \phi_1, u+1, \phi_1)] du + \dots \\
& l_R \int_{-1}^1 \hat{B}_r^2(u) [F_{rr\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + G_{rr\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] + \dots \\
& \hat{B}_\theta^2(u) [F_{\theta r\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + G_{\theta r\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] du \dots \\
& = -K_I f_{r\theta}^I(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}
\end{aligned}$$

where

$$\hat{B}_i^j(u) = \frac{2\mu}{\pi(\kappa+1)} B_i^j(u).$$

So the formulation of equations 36 and 37 becomes

$$\begin{aligned}
& \int_{-1}^1 \hat{B}_r^1(u) [F_{rr\theta}(v+1, \phi_1, u+1, \phi_1) + G_{rr\theta}(v+1, \phi_1, u+1, \phi_1)] + \dots \\
& \hat{B}_\theta^1(u) [F_{\theta r\theta}(v+1, \phi_1, u+1, \phi_1) + G_{\theta r\theta}(v+1, \phi_1, u+1, \phi_1)] du + \dots \\
& l_R \int_{-1}^1 \hat{B}_r^2(u) [F_{rr\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + G_{rr\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] + \dots \\
& \hat{B}_\theta^2(u) [F_{\theta r\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + G_{\theta r\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] du \dots \\
& = -K_I f_{r\theta}^I(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}
\end{aligned} \tag{46}$$

$$\begin{aligned}
& \int_{-1}^1 \hat{B}_r^1(u) [F_{r\theta\theta}(v+1, \phi_1, u+1, \phi_1) + G_{r\theta\theta}(v+1, \phi_1, u+1, \phi_1)] + \dots \\
& \hat{B}_\theta^1(u) [F_{\theta\theta\theta}(v+1, \phi_1, u+1, \phi_1) + G_{\theta\theta\theta}(v+1, \phi_1, u+1, \phi_1)] du + \dots \\
& l_R \int_{-1}^1 \hat{B}_r^2(u) [F_{r\theta\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + G_{r\theta\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] + \dots \\
& \hat{B}_\theta^2(u) [F_{\theta\theta\theta}(v+1, \phi_1, l_R(u+1), \phi_2) + G_{\theta\theta\theta}(v+1, \phi_1, l_R(u+1), \phi_2)] du \dots \\
& = -K_I f_{\theta\theta}^I(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_1) \left(\frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}
\end{aligned} \tag{47}$$

The other two equations refer to clearing tractions along crack 2. So we will use a new transform

$$r = l_R \frac{l_1}{2}(v+1),$$

thus substitution into equation 38 yields

$$\begin{aligned}
& \int_{-1}^1 \bar{B}_r^1(u) \left[F_{rr\theta} \left(l_R \frac{l_1}{2}(v+1), \phi_2, \frac{l_1}{2}(u+1), \phi_1 \right) + G_{rr\theta} \left(l_R \frac{l_1}{2}(v+1), \phi_2, \frac{l_1}{2}(u+1), \phi_1 \right) \right] + \dots \\
& \bar{B}_\theta^1(u) \left[F_{\theta r\theta} \left(l_R \frac{l_1}{2}(v+1), \phi_2, \frac{l_1}{2}(u+1), \phi_1 \right) + G_{\theta r\theta} \left(l_R \frac{l_1}{2}(v+1), \phi_2, \frac{l_1}{2}(u+1), \phi_1 \right) \right] du + \dots \\
& l_R \int_{-1}^1 \bar{B}_r^2(u) \left[F_{rr\theta} \left(l_R \frac{l_1}{2}v+1, \phi_2, l_R \frac{l_1}{2}(u+1), \phi_2 \right) + G_{rr\theta} \left(l_R \frac{l_1}{2}v+1, \phi_2, l_R \frac{l_1}{2}(u+1), \phi_2 \right) \right] + \dots \\
& \bar{B}_\theta^2(u) \left[F_{\theta r\theta} \left(l_R \frac{l_1}{2}v+1, \phi_2, l_R \frac{l_1}{2}(u+1), \phi_2 \right) + G_{\theta r\theta} \left(l_R \frac{l_1}{2}v+1, \phi_2, l_R \frac{l_1}{2}(u+1), \phi_2 \right) \right] du \dots \\
& = -K_I f_{r\theta}^I(\phi_2) \left(l_R \frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_2) \left(l_R \frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}
\end{aligned} \tag{48}$$

where still

$$\bar{B}_i^j(u) = \frac{l_1 \mu}{\pi(\kappa+1)} B_i^j(u).$$

Further kernel reduction as before gives

$$\begin{aligned}
& \int_{-1}^1 \hat{B}_r^1(u) [F_{rr\theta}(l_R(v+1), \phi_2, u+1, \phi_1) + G_{rr\theta}(l_R(v+1), \phi_2, u+1, \phi_1)] + \dots \\
& \hat{B}_\theta^1(u) [F_{\theta r\theta}(l_R(v+1), \phi_2, u+1, \phi_1) + G_{\theta r\theta}(l_R(v+1), \phi_2, u+1, \phi_1)] du + \dots \\
& \int_{-1}^1 \hat{B}_r^2(u) [F_{rr\theta}(v+1, \phi_2, u+1, \phi_2) + G_{rr\theta}(v+1, \phi_2, u+1, \phi_2)] + \dots \\
& \hat{B}_\theta^2(u) [F_{\theta r\theta}(v+1, \phi_2, u+1, \phi_2) + G_{\theta r\theta}(v+1, \phi_2, u+1, \phi_2)] du \dots \\
& = -K_I f_{r\theta}^I(\phi_2) \left(l_R \frac{l_1}{2}(v+1) \right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_2) \left(l_R \frac{l_1}{2}(v+1) \right)^{\lambda_{II}-1}
\end{aligned}$$

where still

$$\hat{B}_i^j(u) = \frac{2\mu}{\pi(\kappa+1)} B_i^j(u).$$

So the formulation of equations 38 and 39 becomes equations

$$\begin{aligned} & \int_{-1}^1 \hat{B}_r^1(u) [F_{rr\theta}(l_R(v+1), \phi_2, u+1, \phi_1) + G_{rr\theta}(l_R(v+1), \phi_2, u+1, \phi_1)] + \dots \\ & \hat{B}_\theta^1(u) [F_{\theta r\theta}(l_R(v+1), \phi_2, u+1, \phi_1) + G_{\theta r\theta}(l_R(v+1), \phi_2, u+1, \phi_1)] du + \dots \\ & \int_{-1}^1 \hat{B}_r^2(u) [F_{rr\theta}(v+1, \phi_2, u+1, \phi_2) + G_{rr\theta}(v+1, \phi_2, u+1, \phi_2)] + \dots \\ & \hat{B}_\theta^2(u) [F_{\theta r\theta}(v+1, \phi_2, u+1, \phi_2) + G_{\theta r\theta}(v+1, \phi_2, u+1, \phi_2)] du \dots \\ & = -K_I f_{r\theta}^I(\phi_2) \left(l_R \frac{l_1}{2} (v+1) \right)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_2) \left(l_R \frac{l_1}{2} (v+1) \right)^{\lambda_{II}-1} \end{aligned} \quad (49)$$

$$\begin{aligned} & \int_{-1}^1 \hat{B}_r^1(u) [F_{r\theta\theta}(l_R(v+1), \phi_2, u+1, \phi_1) + G_{r\theta\theta}(l_R(v+1), \phi_2, u+1, \phi_1)] + \dots \\ & \hat{B}_\theta^1(u) [F_{\theta\theta\theta}(l_R(v+1), \phi_2, u+1, \phi_1) + G_{\theta\theta\theta}(l_R(v+1), \phi_2, u+1, \phi_1)] du + \dots \\ & \int_{-1}^1 \hat{B}_r^2(u) [F_{r\theta\theta}(v+1, \phi_2, u+1, \phi_2) + G_{r\theta\theta}(v+1, \phi_2, u+1, \phi_2)] + \dots \\ & \hat{B}_\theta^2(u) [F_{\theta\theta\theta}(v+1, \phi_2, u+1, \phi_2) + G_{\theta\theta\theta}(v+1, \phi_2, u+1, \phi_2)] du \dots \\ & = -K_I f_{\theta\theta}^I(\phi_2) \left(l_R \frac{l_1}{2} (v+1) \right)^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_2) \left(l_R \frac{l_1}{2} (v+1) \right)^{\lambda_{II}-1} \end{aligned} \quad (50)$$

6.2 Conclusions

As has been shown in the above paper, we are now able to calculate the crack tip stress intensity factors in a number of different cases. We have examined, first, the case of a single crack, having length l , originating from the edge of an adhered complete contact, which remains adhered. For the most simple case of a perpendicular crack at the edge of a square edged contact, we can observe from equation 14 of the above that the dominant influence on the mode I crack tip stress intensity factor, which controls crack growth, is the remote mode I loading. A negative K_I , which implies that the contact is preserved, also implies a closed crack, which will not grow. This dominance is reduced as the crack grows ($l > d_0$) at which point the antisymmetric mode II remote loading has a growing influence.

Utilising the ability of the dislocation formulation discussed in chapter 5, we have also been able to vary the contact and crack angles. Figure 2 shows that the tendency of the crack to open as discussed above does not change greatly as the contact angle varies; instead the antisymmetric mode II behaviour at the crack tip will become stronger as the angle of the contact increases. The reverse is true as we vary the crack angle, as seen in Figure 3. In this case as we increase the crack angle (moving away from a notch bisecting crack) the tendency to open is increasingly controlled by the remote K_I load. The remote K_{II} load has a diminishing influence, especially on the antisymmetric crack tip stress field, although its influence on the symmetric stress field grows.

We have also been able to tackle the two crack problem, and begin to understand how one crack can affect another. Figure 6 shows the system to be symmetric or antisymmetric. It is expected that the ‘ a ’ and ‘ e ’ entries are symmetric as we vary the ratio of crack lengths as these both relate to the symmetric

mode I loading and the symmetric stress field. Similarly, the ‘ d ’ and ‘ h ’ entries both refer to the antisymmetric mode II loading and the antisymmetric stress field. These cancel out so the result is a symmetric relationship to crack length ratio. The ‘ b ,’ ‘ c ,’ ‘ f ,’ and ‘ g ’ entries on the other hand either relate the antisymmetric loading to the symmetric stress field or vice versa, so these are antisymmetric as crack length ratio is varied.

Considering the possibility of crack opening, we refer to equation 26 to consider the symmetrical problem of two mutually orthogonal cracks placed in a three-quarter plane, which is simply a complete contact whose total internal angle is $\frac{3\pi}{2}$. This shows that, unlike in the single crack case, it is the antisymmetric loading which now has the most effect on whether or not the crack will tend to open. The antisymmetric nature of this loading means that one crack will tend to open and the other will not, unless the mode I loading is of such magnitude as to ‘drown out’ the influence of the mode II loading. The symmetric mode I loading will have the same effect on both cracks, so that whatever the ratio of mode I to mode II loading, the cracks will not exhibit the same behaviour; even if the loading is sufficient to make both cracks grow, one will naturally grow faster than the other.

Once again it is possible to vary the crack and wedge angles. Figure 7 shows that the effect of varying the wedge angle is surprisingly small on all but the ‘ a ’ and ‘ e ’ entries, where the increasing wedge angle suggests that the cracks are more likely to open because of a compressive mode I remote load, this effect being felt on both cracks. Figure 8, which illustrates moving one of the crack angles, gives a good indication of the complexity of the problem once any symmetry is removed and is much harder to interpret. As the second crack moves closer to being parallel

to the free surface, it is more likely to open under compressive mode I loading. At the same time the effect of the antisymmetric mode II loading on the opening mode of both cracks, the ' c ' and ' g ' entries, is greatly reduced when compared to the crack approaching the wedge bisector.

Chapter 7

Slip at the edge of complete contacts

The problem of cracks at the edge of arbitrary complete contacts has been thoroughly investigated, and it is now that we turn to the other behaviour that we wish to investigate in depth, the problem of a zone of frictional partial slip, originating from the edge of the contact. We restrict our considerations to these cases, rather than considering ‘islands’ of slip, because, as shown in chapter 4, such islands only occur when one end of the slip zone is beyond the range of the asymptote, and as such these cases are not suitable for treatment using the asymptotic methods. The other condition which we must satisfy is that the corner itself slips. This condition is laid out by Hills and Dini [56], and is briefly summarised as $f_c < \frac{f_{r\theta}^I(\phi)}{f_{\theta\theta}^I(\phi)}$.

Having satisfied ourselves that these conditions are met, it is then necessary to consider the outputs of the problem that will be useful. The reason we want to quantify this type of behaviour is to understand how much damage this could cause, and across what region. The two major outputs from the problem are the slip zone size, and the slip displacements. The size of the slip zone is important as it characterises the potential depth of wear or fretting. A larger slip zone will lead to the penetration of damage to a greater depth within the contact. Slip

displacements of greater magnitude will lead to more energy dissipation, which in turn will lead to more fretting wear and more damage to the contacting bodies. Understanding these outputs can allow preventative action like load alterations and potentially even lubrication to reduce the potential damage.

Once the corrected tractions are found, it is relatively simple to determine the slip displacements. Using the distributed dislocation method, the slip displacements are simply found by integrating the dislocation density, from the slip stick boundary to the point of interest. This is predicated on having the corrected forms of the tractions, found using Bueckner's principle and the dislocation kernels found in chapter 5.

One of the requirements for solving these equations is knowing the size of the slip zone, which is used as an integration limit. This is necessary to know the area over which you are distributing dislocations, and choosing the incorrect slip zone size will artificially force the slip zone to be the chosen size. Unfortunately, there is no closed form solution for the slip zone size which can be found a priori. An estimate must first be made, based on the size of the slip zone implied by the point where the Williams solution indicates that $f_c \sigma_{\theta\theta}(r, \phi) = \sigma_{r\theta}(r, \phi)$, which we denote c_0 .

This guess must then be refined by some checks on the solution. In practice, this means imposing some constraints on the solution; the full set of constraints should only be satisfied if the guess was correct. A similar problem was faced by Churchman and Hills [57], and they used what we will refer to as the two constraints method. These constraints may be summarised as preventing slip in the stick region and ensuring that the slip displacements occur in the same direction as implied by the stresses in the slip zone. The former of these *should* prevent the

slip zone being too small, and the latter *should* prevent the slip zone being too large. It will also be useful to consider the solution provided by Conminou [59] which we can use to describe the state of stress in a *sliding* complete contact using an asymptotic representation at the edge of contact.

Once these issues are resolved it will again prove interesting to consider two variables. The first is familiar. We wish to vary the contact angle; it is perfectly possible in theory also to vary the interface angle, meaning that two bodies with gradient discontinuities contact with both of these corners meeting precisely. But, due to the relative rarity of such occurrences in practice, and the fact that even a small slip displacement will destroy this phenomenon, we restrict our attention to a sharp cornered body indenting a half-plane, thus fixing one of the angles. In any case, it is interesting to consider what effect, if any, the angle of the contacting body has on the tendency of the contact to slip.

The other variable is one that has been previously investigated by Churchman and Hills [57], and that is the coefficient of friction. Churchman and Hills state that, when normalised by c_0 , the initial guess found from the Williams solution, the size of the slip zone is invariant with changing coefficient of friction. However, this solution did use their dislocation formulation which contains a scaling error, and we must re-examine this problem.

7.1 Publication - ‘Slip at the edge of complete contacts’

These investigations have all been carried out in a paper now attached as part of this thesis. This paper is currently under review:

Riddoch, D. J. and Hills, D. A., *Slip at the edge of complete contacts*, The Proceedings of the iMechE, Part C: Journal of Mechanical Engineering Science,

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Slip at the edge of complete contacts

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Abstract— We describe an asymptotic method for calculating the size of a zone of partial slip at the edge of a complete contact of arbitrary angle and coefficient of friction, bounds are set on both of these values. A distributed dislocation method is used and the use of different types of numerical quadrature is explored.

I. INTRODUCTION

A complete contact is one whose extent is defined by a discontinuity in the gradient profile of one of the contacting surfaces. In layman's terms, this means that the edge of the contact is located at the corner of either of the contacting bodies. This gives rise to a stress singularity at the edge of the contact[1], [2], [3] provided that the contact remains intimate throughout the initial contact region, ie. the loading is such that the corner does not lift off. The singularity has to be relieved, and this can be done by plasticity[4], by the nucleation and propagation of cracks[5], or by local frictional slip along the contact interface. This contact may persist across the entire interface (ie. two separate bodies in contact), or it may be a closed pair of crack faces,[6] we consider the former in this case.

It is the last of these effects which we will consider here. The creation of a zone of partial slip is dependent on the loading, as described by Riddoch and Hills[7], the intimate contact must be maintained by the correct signs of the stress intensity factors, and as laid out by Hills and Dini [8], the coefficient of friction must be lower than the value of the traction ratio implied within the dominant eigenvector of the Williams solution[9].

Calibrations of external loads to the stress intensity factors have shown that the contact will tend to spread outwards under purely normal loading[7], [10], [11], so initial investigations focus on outward slip.

Applying Coulomb's friction law throughout, this analysis will endeavour to model the slip and determine the size of the slip zone. Furthermore, previous work has considered only the three-quarter plane problem[12], [13], which applies, for example to the problem of a square block indenting a half-space. In this analysis, this will be extended to consider a variety of indenter angles. Additionally, previously it has been stated that the size of the slip zone, when normalised by a value calculated from the Williams solution, is independent of the coefficient of friction, which has now been found to be incorrect. Figure 1 shows a sketch of the resultant system, with the slip zone present at the edge of contact.

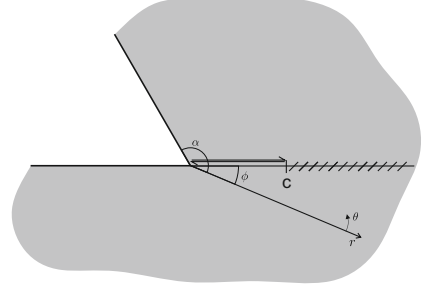


Fig. 1: Sketch of a slip zone at the edge of a complete contact of arbitrary internal angle

II. ASYMPTOTIC FORMULATION

It has already been mentioned that the Williams solution[9], [14] will be used extensively. This provides an asymptotic solution to determine the state of stress in the neighbourhood of the apex of a wedge. This solution places the origin of a polar coordinate system at the edge, a feature which is preserved throughout this analysis.

The state of stress is dominated by two terms, which are the symmetric (here after referred to as mode I) and anti-symmetric (hereafter referred to as mode II) eigensolutions. For a wedge having total internal angle greater than 180° , the mode I term is singular, and for a wedge having total internal angle greater than 257.4° , the mode II term is also singular.

The stress is described by the formula

$$\sigma_{ij}(r, \theta) = K_I f_{ij}^I(\theta) r^{\lambda_I - 1} + K_{II} f_{ij}^{II}(\theta) r^{\lambda_{II} - 1} \dots + \text{higher order terms.} \quad (1)$$

where K_I and K_{II} are the mode I and mode II stress intensity factors respectively, f_{ij}^I and f_{ij}^{II} are the mode I and mode II eigenvectors, and λ_I and λ_{II} are the mode I and mode II eigenvalues respectively.

This solution may be used as a bilateral solution, as an adhered complete contact is equivalent to a wedge whose internal angle is the sum of a half-plane and the internal angle of the contacting body. Therefore, this solution can be used as the basis for this analysis. At this stage, we will define two constants, that we find from the Williams solution and which will be useful later.

We begin by recalling the work of Hills and Dini[4], and using this to extract the characteristic length dimension of the problem. Before doing this, however, it will be useful to align the eigensolution of the problem to the slip line

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with new basis vectors along the interface, as this will allow easier normalisation later on. Quite simply this is done by applying multipliers to the vectors; $K_I^o = K_I f_{\theta\theta}^I(\phi)$, and $K_{II}^o = K_{II} f_{\theta\theta}^{II}(\phi)$, where K_I, K_{II} are the two stress intensity factors, ϕ is the angle of the interface line to the bisector of the combined body, and $f_{\theta\theta}^I, f_{\theta\theta}^{II}$ are the mode I and mode II parts of the $\sigma_{\theta\theta}$ component of the Williams eigenvector, respectively. So, we can now extract our length dimension, denoted by d_0 and defined by

$$d_0 = \left| \frac{K_I^o}{K_{II}^o} \right|^{\frac{1}{\lambda_{II} - \lambda_I}}.$$

The second important quantity we wish to define is the implied size of the slip zone, as indicated by the ratio of the tractions given by the Williams solution. This is the point where $-f_c \sigma_{\theta\theta} = \sigma_{r\theta}$, which is the Coulomb friction law, where f_c is the coefficient of friction. The Williams solution provides this point, which we denote as c_0 , given by

$$c_0 = \left(-\frac{(f_c - g_{r\theta}^I) K_I^o}{(f_c - g_{r\theta}^{II}) K_{II}^o} \right)^{\frac{1}{\lambda_{II} - \lambda_I}}. \quad \text{Therefore} \quad (2)$$

$$\frac{c_0}{d_0} = \left(-\frac{f_c - g_{r\theta}^I}{f_c - g_{r\theta}^{II}} \right)^{\frac{1}{\lambda_{II} - \lambda_I}}, \quad (3)$$

where $g_{r\theta}^I = \frac{f_{r\theta}^I(\phi)}{f_{\theta\theta}^I(\phi)}$, and $g_{r\theta}^{II} = \frac{f_{r\theta}^{II}(\phi)}{f_{\theta\theta}^{II}(\phi)}$, are the eigenvector ratios.

Finally, we will use our inherent length scale(d_0) to normalise the problem. This will allow us to relate problems under different loads or different conditions. Drawing again on the work of Churchman and Hills[10], and Hills and Dini[4], we see that we may write the stresses as

$$\frac{\sigma_{\theta\theta}(r)}{G_0} = \left(\frac{r}{d_0} \right)^{\lambda_I - 1} + \left(\frac{r}{d_0} \right)^{\lambda_{II} - 1}, \quad (4)$$

$$\frac{\sigma_{r\theta}(r)}{G_0} = g_{r\theta}^I \left(\frac{r}{d_0} \right)^{\lambda_I - 1} + g_{r\theta}^{II} \left(\frac{r}{d_0} \right)^{\lambda_{II} - 1}, \quad (5)$$

where $G_0 = K_I^o \frac{\lambda_{II} - 1}{\lambda_{II} - \lambda_I} K_{II}^o \frac{\lambda_I - 1}{\lambda_{II} - \lambda_I}$.

III. DISLOCATION FORMULATION

Turning now to the problem of modelling the actual extent of slip, we require a knowledge of the size of the slip zone, and a method of modifying the stresses. The former will be something we will return to later, but for the latter, we will use dislocations. Crystalline dislocations are used in material science to represent discontinuities in lattice structures, although that is not how we will use them here.

Instead we are interested in the stress and displacement fields that the dislocations create. The form of these is well known and understood[15], [16], [17], [18] and will be used extensively. These influence functions are known for a dislocation in an infinite plane, and so a method must be found to determine the influence of such dislocations in a semi-infinite wedge, a system which, as we have said, is

analogous to an adhered complete contact. This has been done previously by considering a single dislocation and using distributions of dislocations to clear the free surfaces of induced traction[19], [20], [21], [22], and in this analysis we will use the results calculated by Riddoch and Hills[22].

Denoting the wedge influence of a dislocation by $\bar{G}_{ijk}(r, \theta, \xi, \rho)$, where i corresponds to the Burgers vector of the dislocation, jk is the stress component under consideration, r is the observation radius from the origin (placed at the edge of contact, or apex of the wedge), θ is the observation angle from the wedge bisector, ξ is the radius of the dislocation core from the origin, and ρ is the angle of the dislocation core. We place a distribution of dislocations along the contact interface, from the edge of contact to the actual slip stick boundary, which we denote by c . The density of this distribution is denoted by B_r . This distribution of dislocations is sketched in Figure 2 which shows the dislocations placed along the slip interface of the problem, as shown in Figure 1.

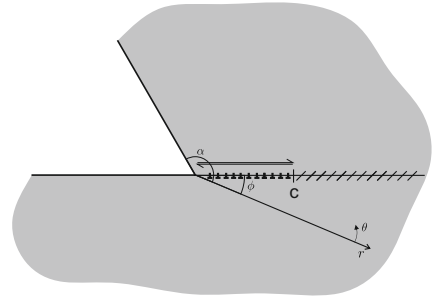


Fig. 2: Sketch showing the distribution of dislocations along the slip interface

Consequently, the corrected stresses along the contact interface can be written as

$$\frac{\sigma_{r\theta}(r, \phi)}{G_0} = g_{r\theta}^I \left(\frac{r}{d_0} \right)^{\lambda_I - 1} + g_{r\theta}^{II} \left(\frac{r}{d_0} \right)^{\lambda_{II} - 1} + \dots \quad (6)$$

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa + 1)} \int_0^c B_r \left(\frac{r}{d_0} \right) \bar{G}_{rr\theta} \left(\frac{r}{d_0}, \phi, \xi, \phi \right) d\xi, \\ & \text{for } 0 \leq r \leq c, \\ \frac{\sigma_{\theta\theta}(r, \phi)}{G_0} &= \left(\frac{r}{d_0} \right)^{\lambda_I - 1} + \left(\frac{r}{d_0} \right)^{\lambda_{II} - 1} + \frac{2\mu}{\pi(\kappa + 1)} \dots \\ & \int_0^c B_r \left(\frac{r}{d_0} \right) \bar{G}_{r\theta\theta} \left(\frac{r}{d_0}, \phi, \xi, \phi \right) d\xi, \\ & \text{for } 0 \leq r \leq c. \end{aligned} \quad (7)$$

Within the slip region, the Coulomb friction law must be satisfied, meaning

$$\begin{aligned} 0 &= (g_{r\theta}^I - f_c) \tilde{r}^{\lambda_I - 1} + (g_{r\theta}^{II} - f_c) \tilde{r}^{\lambda_{II} - 1} + \dots \\ & \int_0^c \hat{B}_r(\xi) (\bar{G}_{rr\theta}(\tilde{r}, \phi, \xi, \phi) - f_c \bar{G}_{r\theta\theta}(\tilde{r}, \phi, \xi, \phi)) d\xi, \\ & \text{for } 0 \leq r \leq c, \end{aligned} \quad (8)$$

where $\tilde{r} = \frac{r}{d_0}$ and $\hat{B} = \frac{2\mu}{\pi(\kappa+1)}B$, for values of $0 \leq r \leq c$. Equation 8 is the governing equation, and the solution to this equation will solve the problem fully. There are, however, two unknowns in the problem. The first of these is the dislocation density, and the second is the size of the slip zone c . The first may be found by inverting the integral equation, but the second must be found by some other method.

IV. A NUMERICAL INVERSION

Considering, first, the inversion of the integral equation, we note that we will require a numerical method. This is because the dislocation kernels, as found by Riddoch and Hills[22], do not have a closed-form expression. The kernels are, however, singular in nature, so equation 8 is a Cauchy singular integral equation.

A. Numerical quadrature

Consequently, a Gauss-Chebyshev quadrature will be employed as described by Erdogan et al[23]. This quadrature deals with the singularity by separating the behaviour into a well-behaved unknown function, denoted by φ , and a fundamental function, denoted ω , which encapsulates the power order singularity, but is well known and can be removed from the equation by use of the quadrature.

The choice of this fundamental function determines the type of behaviour that can be expected of the solution at the end points. However, there are only two types of behaviour which are accounted for, square root singular, and square root bounded, and so with either possible at both ends, giving a total of four different cases, as laid out in Table 2.2 by Hills et al[24].

We must consider the state of stress at the edge of contact and the slip stick boundary. Considering, first, the latter, in the adhered case, the stress at this point is bounded, with each component's value being a finite number, this will still be true as the contact slips. So here the corrective term must be bounded.

The question is not so easily resolved at the contact edge end of the interval however. The stresses in the bilateral solution are singular at the edge of contact, specifically, they are power order singular, and the power is given by $\lambda_I - 1$. When the interface is slipping the form may not immediately be so obvious. However, at a very local level, for $r \ll c$, the tractions will be similar in form to those which arise when the contact is in full sliding. The nature of the tractions when the contact is sliding is known, and they are asymptotically described by a power order singular term, with order $\lambda_s - 1$, where λ_s is an eigenvalue solution, as described by Conminou[25]. We have generally that $\lambda_I \neq \lambda_s$, and $0 < \lambda_I < 1, 0 < \lambda_s < 1$ so we can say that, for any $L > 0$, there exists $\epsilon > 0$ such that $|\epsilon^{\lambda_I-1} - \epsilon^{\lambda_s-1}| > L$. This is fairly trivially proven, and hence no proof is provided here. The implication of this however, is that the difference between the adhered and sliding solutions grows without bound as r approaches 0, the edge of contact. Therefore a singular behaviour at the edge of contact is expected.

B. Applying singular-bounded quadrature

Consequently, a singular-bounded quadrature is employed. It should be noted that this is the same quadrature as was applied by Churchman and Hills[12]. Table 2.2(case II) in Hills et al[24] then gives us a method for choosing our collocation and integration points.

Before we can apply the quadrature, we must first transform the integral. The quadrature requires the integration range to be $[-1, 1]$, so we must find a transformation from the current range $[0, c]$. This is achieved fairly easily by the applying the transform $\tilde{r} = \frac{c}{2}(s+1)$, for the collocation points, and $\xi = \frac{c}{2}(t+1)$ for the integration points. It is useful at this stage to note that $\frac{d\xi}{dt} = \frac{c}{2}$.

Applying these substitutions gives us the integral equation

$$\begin{aligned} & (g_{r\theta}^I - f_c) \left(\frac{c}{2}(s+1) \right)^{\lambda_I-1} + (g_{r\theta}^{II} - f_c) \left(\frac{c}{2}(s+1) \right)^{\lambda_{II}-1} \\ & + \dots \int_{-1}^1 \hat{B}_r(t) \left(\bar{G}_{rr\theta} \left(\frac{c}{2}(s+1), \phi, \frac{c}{2}(t+1), \phi \right) \right. \\ & \left. - \dots f_c \bar{G}_{r\theta\theta} \left(\frac{c}{2}(s+1), \phi, \frac{c}{2}(t+1), \phi \right) \right) \frac{c}{2} dt = 0, \\ & \text{for } -1 \leq s \leq 1. \end{aligned} \quad (9)$$

At this point it is worth noting that generally $G_{ijk}(r, \theta, \xi, \rho) = \frac{1}{\gamma} G_{ijk}(\gamma r, \theta, \gamma \xi, \rho)$, so we may rearrange the equation to find

$$\begin{aligned} & (f_c - g_{r\theta}^I) \left(\frac{c}{2}(s+1) \right)^{\lambda_I-1} + (f_c - g_{r\theta}^{II}) \left(\frac{c}{2}(s+1) \right)^{\lambda_{II}-1} \dots \\ & = \int_{-1}^1 \hat{B}_r(t) (\bar{G}_{rr\theta}(s+1, \phi, t+1, \phi) - \dots \\ & f_c \bar{G}_{rr\theta}(s+1, \phi, t+1, \phi)) dt, \quad \text{for } -1 \leq s \leq 1. \end{aligned} \quad (10)$$

This equation is now in a form where the quadrature may be applied. This leads us to a set of N simultaneous equations in N unknowns, which are described by

$$\begin{aligned} & (f_c - g_{r\theta}^I) \left(\frac{c}{2}(s_k+1) \right)^{\lambda_I-1} + (f_c - g_{r\theta}^{II}) \left(\frac{c}{2}(s_k+1) \right)^{\lambda_{II}-1} \dots \\ & = \sum_{i=1}^N W_i \varphi(t_i) (\bar{G}_{rr\theta}(s_k+1, \phi, t_i+1, \phi) - \dots \\ & f_c \bar{G}_{rr\theta}(s_k+1, \phi, t_i+1, \phi)) \end{aligned} \quad (11)$$

where W_i is the weight function described by Hills et al[24] and φ is such that $\omega(s)\varphi(s) = B_r(s)$, with ω the fundamental function also described by Hills et al. The values $\varphi(t_i)$ are the N unknowns which we solve the equations for.

C. The size of the slip zone

Having in hand this inversion method, we now come to solve the set of equations. By use of the transforms, this requires knowledge of the size of the slip zone c . No condition has been explicitly set out for this, and the problem is fully specified.

1) *Two constraints:* Previous studies[12], [13] have utilised a pair of conditions to calculate the size of the slip zone. This is done by guessing the size of the slip zone, and then checking these conditions to ensure that the estimated slip zone size satisfies both.

These conditions are derived from two basic phenomena. The first of these is that Coulomb's law is satisfied at a given point if and only if the point is in the slip region. For a point within the slip region, as specified by equation 8, this condition will be automatically satisfied by any solution to equation 8. Points outside of the slip zone satisfy this condition:

$$|\sigma_{r\theta}(r)| < f_c \sigma_{\theta\theta}(r), \quad \text{for } r > c, \quad \text{with } \sigma_{\theta\theta}(r) < 0. \quad (12)$$

The second condition within the slip region is derived from the orthogonality condition. This requires the velocity of a slipping point to have the same sign as the shear stress at that point. As this analysis is quasi-static in nature, we do not have a rigorous definition of velocity. However, it is a sufficient approximation to consider instead the slip displacement. Slip displacement is defined as the integral of the dislocation density, since the dislocations themselves represent the displacement discontinuity. We may write this condition as

$$\text{sign}(h(r)) := \text{sign}\left(\int_0^r B_r(\xi) d\xi\right) = \text{sign}(\sigma_{r\theta}(r)) \quad \text{for } 0 \leq r \leq c. \quad (13)$$

We now have four conditions, summarised in table I. The first of these, condition A, is automatically enforced by the solution of the integral equations. Similarly, the last of these, condition D, is automatically true; no dislocations are present along the interface in the stick region and so no displacement is caused. We can, however, use the remaining two conditions, B and C, to constrain the solution. These place restrictions on the solution in the slip region and the stick region, and it may appear that these conditions are sufficient.

	Slip region	Stick region
Tractions	$ \sigma_{r\theta}(r) = f_c \sigma_{\theta\theta}(r)$ (A)	$ \sigma_{r\theta}(r) < f_c \sigma_{\theta\theta}(r)$ (B)
Displacements	$\text{sgn}(h(r)) = \text{sgn}(\sigma_{r\theta}(r))$ (C)	$h(r) = 0$ (D)

TABLE I: Table of conditions in the slip and stick regions

Experience has shown that in this case, the conditions, whilst necessary, are not sufficient to define a unique size for the slip zone. Figure 3 shows the stresses implied by a slip zone which is too small, according to our findings later in the 'Results' section. It can be observed that both conditions described here are satisfied, the left hand plot shows (solid black line) that no slip is implied in the slip region. Furthermore, it can easily be observed that both the shear traction (solid red line) and the slip displacements (right hand plot) are positive throughout the slip zone. This would therefore appear to be a valid solution.

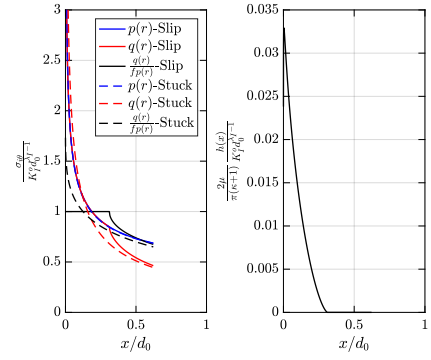


Fig. 3: Plot of the stresses in a slipping square edge indenter with a slip zone of size $c = 2.5c_0$.

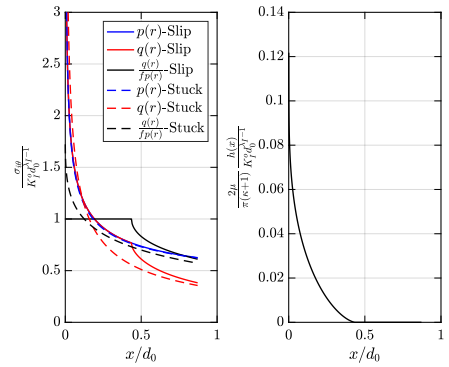


Fig. 4: Plot of the stresses in a slipping square edge indenter with a slip zone of size $c = 3.5c_0$.

Figure 4 shows the stress in the same contact, but in this case we specify a slip zone size that we believe to be too large. Once again, it can be observed that there is no violation of the slip condition in the stick region (solid black line, left hand plot) and that slip displacement and shear stress have the same sign (right hand plot and solid red line, left hand plot respectively.) So this too would appear to be a valid solution.

This is obviously problematic, the constraints are not only providing multiple solutions, but rather a range. Experience shows for example that for $c = 100c_0$, both conditions are still satisfied. So, given the correct solution is not given by the minimum viable solution, nor the maximum, but rather some point in the middle, we must utilise other constraints to determine the solution. This is made more difficult by the fact that the state of stress is qualitatively similar for both cases, as is highlighted in Figure 5 which overlays the stresses found for each of these cases.

2) *The sliding solution:* As previously mentioned by Churchman and Hills[12], it may be useful to consider the solution for a sliding block, as developed by Conminou[25]. This gives rise to an eigenvalue, denoted λ_s , calculated by

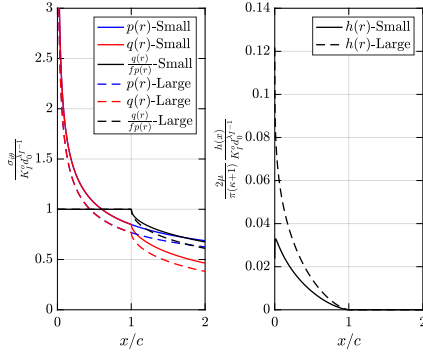


Fig. 5: Overlay plot showing the stresses produced for two different slip zone sizes, small being $c = 2.5c_0$, and large being $c = 3.5c_0$.

finding the solution to

$$0 = \cos(\pi\lambda_s)(\sin(\lambda_s\phi)^2 - \lambda_s^2\sin(\phi)^2) + \dots \quad (14)$$

$$\frac{\sin(\pi\lambda_s)(\sin(2\lambda_s\phi) + \lambda_s\sin(2\phi))}{2} + \dots$$

$$f_c\sin(\pi\lambda_s)(\lambda_s(1 + \lambda_s)\sin(\phi)^2). \quad (15)$$

However, the sliding asymptote is valid only for a very small region of the slip zone. We do not have to hand, a bound of the range of validity of this solution, so we cannot say with any confidence that this approximation will be accurate at any point a finite distance from the contact edge. The solution is exact only when $r \ll c$, or $\frac{r}{c} \rightarrow 0$, neither of which gives a quantifiable value of r , below which the solution is always valid, or above which the solution is never valid. It is therefore difficult to enforce conditions based on this approximation.

D. Slip displacements

The orthogonality condition requires the calculation of the slip displacements, which must be of the same sign as the shear stress. It is tempting to find other conditions. Perhaps the most obvious of these to consider is the prevention of a turning point in the slip displacements. Intuitively it may seem that the slip displacements must be strictly increasing from the slip stick boundary to the edge of contact.

However, this statement has been made without justification, and no immediate justification is apparent. Slip displacements are calculated from dislocation densities, and hence reference discontinuities, so no relation between the stress and the strain may be drawn. So, whilst it may appear at first sight to be obvious, we cannot say with any certainty that this will be the case, and so we can not enforce a condition based on this assumption.

Similarly, it could be argued that the slip displacement should not show singular behaviour at the edge of contact either. It is tempting therefore to enforce the condition that the minimum value of the second derivative of slip displacement with respect to position is bounded at the edge of contact. In practical terms this would mean that the slip length at which this value is minimised would be the true

slip length. However, this is once again making assumptions about the form of the slip displacements, which cannot be rigorously justified. It is, therefore, not possible to strictly enforce any constraints on the slip displacements, beyond the use of the orthogonality condition.

V. AN ALTERNATIVE NUMERICAL APPROACH

All of the problems described above present a serious obstacle to accurately determining the size of the slip zone. Let us consider a different approach. Earlier, we described at length, how and why we chose the singular bounded quadrature. Let us now consider an alternative conclusion.

The choice of end point behaviour forces a choice between square root singular and square root bounded. The reason that we are forced to make this choice is so that any singular behaviour can be separated. This means that we have one function whose form is known precisely (the fundamental function) and the other part of the function, φ , which is unknown, well behaved, and allows us to sample at chosen points without worrying about singular behaviour being missed. However, if the power order behaviour of the solution is not close to $\pm\frac{1}{2}$, then φ will display behaviour that is either near singular or near bounded.

For this reason, when we find that the power order behaviour of the solution is not close to either square root bounded or square root singular, but rather lies somewhere in between, particularly if the behaviour is weakly singular, (i.e. of power order much smaller than square root) we can investigate the behaviour as if the problem is bounded.

We may employ the bounded-bounded quadrature when investigating the slip problem. The power order behaviour of the corrective term will be of order $\lambda_s - \lambda_I$, but this is not close to square root singular. So we apply the same transformations as we did for the singular-bounded case, albeit with the selection of collocation, integration and weights being from the bounded-bounded type. This gives us $N + 1$ equations in N unknowns, given by

$$(f_c - g_{r\theta}^I) \left(\frac{c}{2}(s_k + 1) \right)^{\lambda_I - 1} + (f_c - g_{r\theta}^{II}) \left(\frac{c}{2}(s_k + 1) \right)^{\lambda_{II} - 1} \dots$$

$$= \sum_{i=1}^N W_i \varphi(t_i) (\bar{G}_{rr\theta}(s_k + 1, \phi, t_i + 1, \phi) - \dots$$

$$f_c \bar{G}_{r\theta\theta}(s_k + 1, \phi, t_i + 1, \phi)). \quad (16)$$

This is a fact that we will use to our advantage later; for now though it presents a problem. The method of solution for these equations is to invert a coefficient matrix. This is only possible if the matrix to be inverted is square. So, the way we tackle this problem is to select one of the equations arbitrarily and remove it. This leaves us N equations, and the system is invertible once again. This of course is an approximation, but provided that N is sufficiently large (in practice anything larger than about 30 appears sufficient, based on experience of trying various values for N) this is an appropriate approximation to make.

A. Determining the size of the slip zone

Now having the inversion method in hand, we must now determine the true size of the slip zone. Once again this cannot be done using pre-existing conditions, and so an estimate must be used and refined. We still use the friction law (equation 12) and the orthogonality condition (equation 13) as these conditions must still be fulfilled. Once again, though, these conditions prove necessary but not sufficient.

However, we have another condition that we can utilise. The equation which we discarded when inverting the system must still be satisfied. This is a far stricter condition, and enforces the friction law at the point described by the equation. In practice, this expression is written as

$$\begin{aligned}
 & - (f_c - g_{r\theta}^I) \left(\frac{c}{2} (s_k + 1) \right)^{\lambda_I - 1} - \dots \\
 & (f_c - g_{r\theta}^{II}) \left(\frac{c}{2} (s_k + 1) \right)^{\lambda_{II} - 1} + \dots \\
 & \sum_{i=1}^N W_i \varphi(t_i) (\bar{G}_{rr\theta}(s_k + 1, \phi, t_i + 1, \phi) - \dots \\
 & f_c \bar{G}_{rr\theta}(s_k + 1, \phi, t_i + 1, \phi)),
 \end{aligned}$$

and the value of this expression is minimised. This gives us the slip zone size which satisfies the friction law at this point, which is therefore, the true size of the slip zone.

VI. RESULTS

Throughout these results, the primary output will be the size of the slip zone c . For ease of comparison, this will always be normalised, either by the initial guess for the size of the slip zone determined from the ‘full stick’ solution and violation of friction law, c_0 , or by the problem’s inherent length dimension, d_0 .

Let us first examine the problem of a square block indenting a half plane. This problem has previously been solved by Churchman and Hills[12] who suggested that the ratio $\frac{c}{c_0} = 2.4$, and does not vary with the coefficient of friction. However, several issues have been found with this work, as discussed by Riddoch and Hills[22], and here in ‘Two Constraints’ section. Consequently, we do not believe this result to be accurate, and we have found that for a coefficient of friction of $f_c = 0.3$, the correct value is $\frac{c}{c_0} = 2.79$, to three significant figures. Figure 6 shows the stresses that this solution implies and the slip displacements. It can clearly be seen that both constraints are satisfied.

The next question we then ask is, “Was Churchman right to say that the ratio $\frac{c}{c_0}$ is invariant with friction?” Again, our results would suggest that this is not true. However, the dependence is weak. Figure 7 shows how that ratio $\frac{c}{c_0}$ varies for the above problem when we vary the coefficient of friction between $0.2 \leq f_c \leq 0.45$. These values are chosen carefully: for very small coefficients of friction, the slip zone will be very large, and thus it risks moving out of the range of validity of the asymptotic solution. For example, in the case of the 90° block, the ratio $\frac{c_0}{d_0} > 1$ for $f_c < 0.2$, which

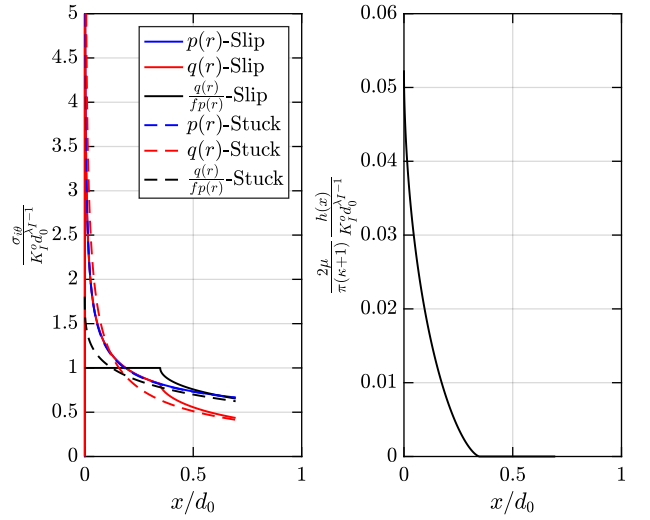


Fig. 6: Plot showing the tractions along the slip interface before (dotted) and after (solid) slip, and the slip displacements for a square block indenting a half-space with the coefficient of friction being 0.3

means that the slip zone extends into the mode II dominated region, the existence of which cannot be proved[7]. On the other hand, for large coefficients of friction there will be a very small slip zone. Hills and Dini[8] found a bound on this, and as we approach it the size of the slip zone becomes sufficiently small for problems to arise, as discussed later in this section, as figure 10 shows the ratio $\frac{c_0}{d_0}$ has order 10^{-2} , meaning that the size of the slip zone is orders of magnitude smaller than the mode I dominated region. As mentioned the dependence is weak, and the variation is almost linear. For ease of use, a quadratic may be fitted to the curve in figure 7, which is given by $\frac{c}{c_0} \approx 0.06026 f_c^2 + 0.1297 f_c + 2.748$.

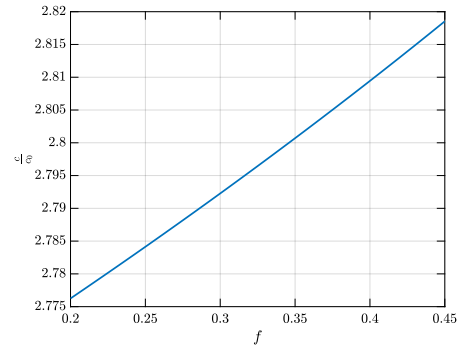


Fig. 7: Plot showing the variation of slip zone size for a square block indenting a half-space for varying coefficients of friction

Next we come to the issue of varying the angles of the contact defining block involved. In practice it is unlikely that two sharp corners would meet perfectly, thus meaning that it makes most sense to consider a block of differing angle indenting a half-space. Furthermore, any slip displacements will destroy the nature of the contact. The method described

can tackle such problems, although they are of little practical interest. It is worth noting, however, that in the case where the contacting bodies have the same geometry so the contact interface is colinear with the wedge bisector, then no slip will occur, as by definition only mode I stresses are present, in which case the ratio of shear to normal stress is invariant with distance from the origin.

Returning to the problem of a block indenting a half-plane, turning now to the result when we vary the angle of the indenter between 0.45π and 0.55π , Figure 8 shows how the ratios $\frac{c}{c_0}$ and $\frac{c}{d_0}$ vary. We have continued to use α to quantify the variation of the angle, although in fact what we vary is the internal angle of the indenter. This is because several bounds exist on the viable solution region, and these are most easily expressed using α . As for the results found for varying the coefficient of friction, a quadratic may be fitted to give easier use of the results, this is given by $\frac{c}{c_0} \approx 0.4797\left(\frac{\alpha}{\pi}\right)^2 + 5.762\frac{\alpha}{\pi} - 1.815$.

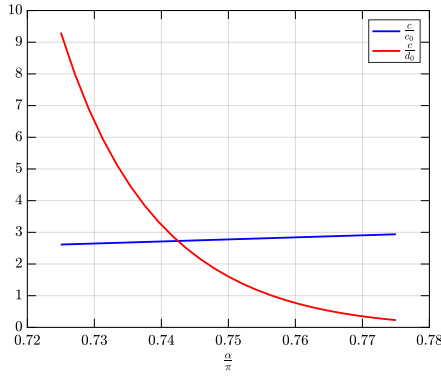


Fig. 8: Plot showing the variation of slip zone size for block of various internal angles indenting a half-space

The first of these bounds is that the mode II eigensolution remains singular in nature. This is only the case provided that $\lambda_{II} < 1$, this is true only for $2\alpha > \approx 257^\circ = 1.4278\pi^c$. The second constraint is related to the value of c_0 , whose value depends on the coefficient of friction and the wedge angle, through the properties of the dominant eigenvector. As can be seen in Figure 9 as we let α approach 0.8π , the values of both c_0 and $\frac{c_0}{d_0}$ approach 0. This means two things; firstly it means that any small numerical error becomes very significant; and secondly it means that the gradient of the stress ratio is small.

When the ratio $\frac{c_0}{d_0}$ is small, the Williams solution is dominated by mode I terms. This means, as a first order approximation, that we may consider the stresses to be given by just the mode I term,

$$\sigma_{ij}(r) = K_I f_{ij}^I r^{\lambda_I - 1},$$

meaning that the stress ratio is given by

$$\frac{\sigma_{r\theta}(r)}{\sigma_{\theta\theta}(r)} = \frac{K_I f_{r\theta}^I r^{\lambda_I - 1}}{K_I f_{\theta\theta}^I r^{\lambda_I - 1}} = \frac{f_{r\theta}^I}{f_{\theta\theta}^I},$$

and is thus invariant with r , over the range where the second term really can be neglected. In practice, of course, there

will be an influence from terms in the expansion other than the mode I terms, but that term will dominate, and hence the stress ratio will be nearly constant. This means that any small perturbation, in either the coefficient of friction or the angle, will lead to a large change in the apparent value of c_0 .

The variation of d_0 , c_0 , and $\frac{c_0}{d_0}$, with the angle is shown in Figure 9, and their variation with the coefficient of friction is shown in Figure 10, although d_0 does not vary with coefficient of friction.

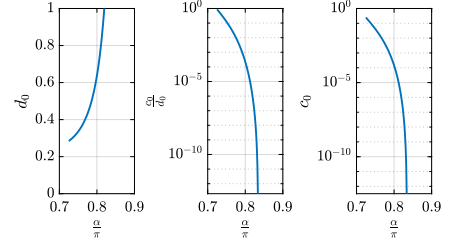


Fig. 9: Plot showing the variation of d_0 , $\frac{c_0}{d_0}$ and c_0 with angle α .

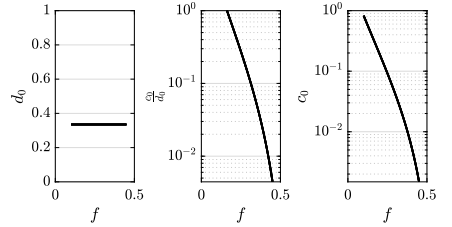


Fig. 10: Plot showing the variation of d_0 , $\frac{c_0}{d_0}$ and c_0 with coefficient of friction

A. Comparison of inversion methods

As we progress with the bounded-bounded quadrature, it is perhaps wise to check that the results that we get using this method are not dissimilar to those found using the singular-bounded quadrature, when supplying that method with the correct size for the slip zone. Figure 11 shows an overlay plot of the stresses and slip displacements for the two methods for a square indenter with coefficient of friction $f_c = 0.3$. As can be seen, the results are almost identical for the stresses, and the offset is only very marginally different for the slip displacements. This gives us confidence in the correctness and value of the method introduced, as the choice appears to have very little influence on the calculated result.

VII. DISCUSSION

These results are satisfactory and make sense on an intuitive level. However, so far several problems have been left open in this analysis. The first of these, that we will discuss is, “Why are the two conditions which previously were sufficient, now only necessary, these being the friction law and orthogonality conditions?”

In attempting to answer this question, let us consider briefly a different problem. Consider the problem of an

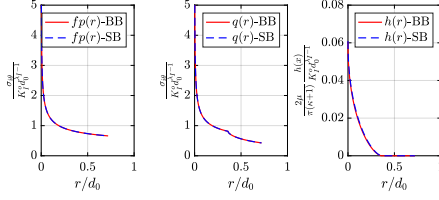


Fig. 11: Plot showing an overlay of the stresses calculated using the bounded-bounded(red) and singular-bounded(blue, dashed) quadratures

semi-infinite incomplete contact, loaded so as to produce a zone of partial slip at the contact edge. These problems are not the same: the incomplete contact is uncoupled, that is changes in the normal traction or normal load will not affect the shear traction and vice versa. Secondly, the contact pressure is square root bounded at the contact edge. The adhered shear traction is square root singular, and we may use a distribution of dislocations to apply a corrective term to model slip.

Happily, on this occasion, a simpler dislocation kernel may be used, and Moore et al[26] were able to find a closed form solution by inverting the integral equation. This also gives us a true size for the slip zone. However, using a dislocation formulation, we can observe the state of stress generated if the modelled slip zone size is forced to be either too small or too large.

We denote the size of the slip zone calculated by Moore et al as d_M . Figure 12 shows the surface tractions and slip displacements calculated for a slip zone 90% of the true slip zone size. It can readily be seen that both the friction law and orthogonality conditions are fulfilled here. Similarly Figure 13 shows the same tractions and displacements for a slip zone 110% the true size, once again both conditions are fulfilled. Whilst in neither case does the form of the slip displacement calculated(dashed red line) match that which is found analytically(solid black line), neither case violates any of the conditions we are able to set out. The slip displacements in these plots being calculated by integrating equation 29 in the paper by Moore et al[26].

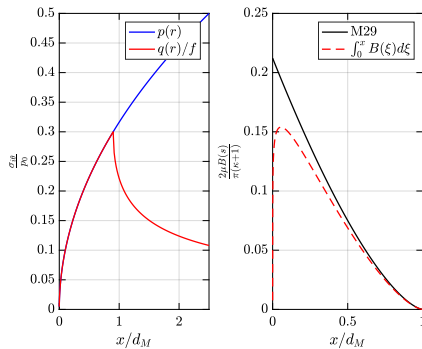


Fig. 12: Plot showing the surface tractions(left hand plot) and slip displacement(right hand plot) for a slipping incomplete contact with a slip zone 90% of the true size

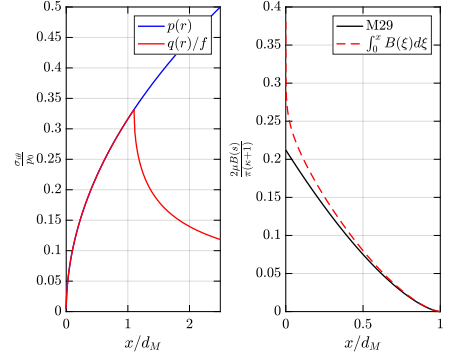


Fig. 13: Plot showing the surface tractions(left hand plot) and slip displacement(right hand plot) for a slipping incomplete contact with a slip zone 110% of the true size

For all the reasons previously stated, these problems are not the same, and the behaviour of the stresses is very different, most notably the corrected solution is not singular. However, it is noteworthy that the same problems arise when trying to determine slip zone size in another semi-infinite contact, using the same friction law and orthogonality conditions.

A. Sliding asymptote

We have already stated that the sliding solution may not be used to constrain the slip solution with any confidence as we are not able to say over what range the asymptotic representation is correct. Even once we have a solution that we are happy with this is a difficult problem, as we do not have values for the sliding stress intensity factor.

However, the sliding asymptote may be fitted to our slip traction at the edge of contact, where an accurate representation is expected. Doing this will obviously result in a very good match at the points which we use for the fit, but it is instructive nevertheless, as it gives an indication of how good an approximation the sliding asymptote is throughout the slip region.

Figure 14 shows a comparison of the tractions found using our method described above and the sliding asymptote, fitted at the edge of contact. The ratio between the two(Figure 14, dashed black line) is particularly instructive as it shows that the asymptotic approximation remains close to the calculated traction for a large portion of the slip region.

B. Slip displacements

When discussing the slip displacements calculated using the singular bounded quadrature, we stated that we are unable to exclude the possibility of a turning point in the slip displacement, but in all cases examined so far no turning point has been present. So the question may be asked, does a turning point arise, or, as expected, is no turning point present?

Figure 15 shows a normalised overlay of the slip displacements found for five different cases for the coefficient of friction(left hand plot) and for five different

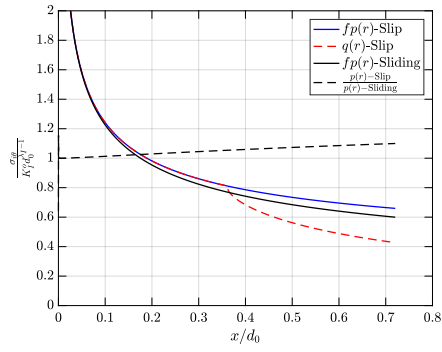


Fig. 14: Plot showing a comparison of the slip and sliding tractions

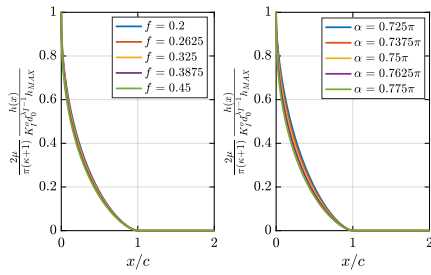


Fig. 15: Plot showing a normalised overlay of the slip displacements for different coefficients of friction(left hand plot) and indenter angles(right hand plot)

angles(right hand plot). It can be seen that the form of each of these is broadly similar, and none displays a turning point behaviour. This is by no means a rigorous proof, or an exhaustive examination, but it is indicative that it seems extremely unlikely that a turning point in slip displacement might be observed.

VIII. CONCLUSION

In this analysis, we have determined the size of a zone of partial slip at the edge of a complete contact. We have shown how the size of this slip zone varies with changes in the coefficient of friction, and the contact angle. This includes a correction of previously published results. Furthermore, bounds have been placed on the range of validity of the asymptotic solution, and/or the existence of a slip region.

Additionally, we have discussed and explored two different numerical methods, and the reasons for choosing either method. Previous conditions used to determine the size of the slip zone have been investigated, and have proved insufficient. Alternative constraints have been explored and ruled out, although these are shown to also be true, but not enforceable as constraints. Finally, a method is arrived at using a feature of one of the numerical methods, although the choice of numerical quadratures has been shown not to affect the solution.

IX. ACKNOWLEDGEMENTS

Both authors thank Rolls-Royce plc and the EPSRC for the support under the Prosperity Partnership Grant 'Cornerstone:

7.2 Conclusions

These investigations have revealed several new effects. Perhaps the most significant of these is the determination of the slip zone size. It was previously thought sufficient to use the two constraint method. However, as has been discussed at length above, these conditions have proved to be necessary but not sufficient. Even using the sliding asymptote as found by Conminou, we still do not have enough conditions to constrain the solution.

It has been necessary to consider a new inversion method, using a bounded-bounded quadrature, which gives us another condition that we can use to help find the true size of the slip zone. The reasons for doing this have been explained at length so do not bear repeating, but significant care has been taken to ensure that the choice of quadrature does not effect the result. Consequently we can say with confidence that the calculated slip zone size is correct, and that the slip displacements are correct, as is highlighted by figure 11.

It has also been found that the previous assertion by Churchman and Hills that the normalised slip zone size is invariant with coefficient of friction is not quite true. Figure 7 shows the variation in normalised slip zone size that can be expected for a square edged contact. The correlation is clear, although the variation is quite small. The variation is very slightly non-linear, but this effect is small enough over a comparatively large range of values as to be almost unnoticed. If the value of the coefficient of friction is further increased the corner will not slip, and if it is further reduced the slip will extend beyond the asymptote.

Similarly, we have investigated the variation of slip zone size with varying the contact angle. Figure 8 shows that increasing the contact angle will lead to a larger normalised slip zone. When normalised by the Williams solution estimate

c_0 this variation is also fairly small, and almost linear. Only a small range of possible angles could be calculated however, as for larger angles the mode mixity parameter d_0 becomes complex, and for shallower angles the mode II stress ceases to be singular.

Finally, it has been shown that the slip displacements calculated are all maximised at the contact edge, as might be expected, so the most damage will occur here. We have also seen that the problem of using the two conditions is not unique to the complete contact problem. Lastly, figure 14 shows the validity of the sliding asymptote compared to the shear and normal tractions.

Chapter 8

Distributed dislocation problems in wedge shaped domains

The principal properties of the complete contact have been examined, and this has been done by the use of Bueckener's principle and the dislocation method, described by Hecker and Romanov, and improved on by Riddoch and Hills [60]. This method has proved to be a very useful way of investigating multiple problems reusing the same kernels. However, it has not been without issues. The method has been computationally expensive when using larger numbers of points, as we require for an accurate analysis, and the mixture of kernel and problem coordinates can be confusing.

For this reason, it may be beneficial to consider a new approach to tackling these problems. The Hecker and Romanov method is a sequential consideration - first the dislocation kernels are found, and then these kernels are used to solve a specific problem. Consequently, two sets of simultaneous singular integral equations must be solved sequentially. Instead of this, let us consider combining the two problems into one, larger, simultaneous problem.

In both the crack problems and the slip problem we must enforce conditions along either the crack faces or the slip interface. Similarly, to find the dislocation kernels we enforce conditions on the free surfaces. If we enforce all of these conditions simultaneously, we will be able to use the infinite plane kernels on a larger set of simultaneous singular integral equations.

By reducing the number of stages of the calculation from two to one, the computational speed should increase, allowing us to more efficiently calculate the dislocation densities required in many different cases of the crack or slip problems. Similarly, the use of a single set of equations removes the possibility of transcription errors or normalisation mistakes by unifying the coordinate system.

Consequently, this method is very appealing, but care must be taken. The Hecker and Romanov method was known to converge quickly for kernel calculations, and while some difficulties were found during calculations for the slip problem generally the method converged quickly when applied to crack problems. This convergence must be checked using the simultaneous method described above. However, it should be possible to verify the new method using the solution found with the Hecker and Romanov method.

8.1 Publication - ‘Distributed dislocation problems in wedge shaped domains’

These investigations have all been carried out in a paper now attached as part of this thesis. This paper is currently under review:

Riddoch, D. J. and Hills, D. A., *Distributed dislocation problems in wedge shaped domains*, European Journal of Mechanics/ A Solids,

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Distributed dislocation problems in wedge shaped domains

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Abstract—We describe a method to tackle problems such as the presence of cracks or a region of partial slip, in wedge shaped domains, using distributed dislocations, alternatively considered as the edges of complete contacts. An asymptotic solution is sought, which is then applicable to many different contacts. Efficiencies are sought and described in comparison to previous approaches, both in terms of computational cost and kernel complexity. Three example problems, a crack, a slip zone, and a pair of cracks, are discussed.

I. INTRODUCTION

Wedge shaped domains arise in practical engineering problems, either in their own right, or when associated with the edge of a complete contact problem, where the contact in its adhered state is considered to be similar to a wedge, whose internal angle is the sum of the angle of contact within the contact defining body and a half plane. The surface tractions are identical on the adhered surfaces, and the interface transmits contact tractions. For this reason it is often of practical importance to determine the state of stress in wedges, and to be able to account for limited violations of continuity, such as the presence of a short crack, or the attainment of limiting friction, in the case of an adhered complete, or sharp-edged, contact.

A common approach to this problem is to use the well known solution of Williams[1] to describe the state of stress in the wedge. The Williams solution indicates that the stresses at the apex of a reentrant wedge will be singular, and composed of a symmetric(mode I) and anti-symmetric(mode II) component, with other bounded terms in the infinite expansion that is found being dominated by these two terms at the edge of contact. The mode I term will be singular for all reentrant wedges, that is all wedges having total internal angle greater than 180° , where as the dominant mode II term is singular for wedges having total internal angle greater than approximately 257.4° , so one where the contact defining body has an internal angle less than 77.4° .

In practical applications a singular state of stress cannot be sustained, as all realistic materials have a finite yield stress. The state of stress must therefore be relieved. In this analysis we consider methods of relieving this stress by permitting slip. Plasticity might also be present, as has been discussed by previous studies, including by Hills and Dini[2], but we will not discuss it any further here. We can ignore this effect because Hills and Dini show that the dimensionless load parameter G_0 has to be of the same order of magnitude as the yield stress of the material, k , for the plastic front to be of the same order of magnitude as the mode mixity parameter d_0 . This is an extremely high load case, and as

the order of magnitude of the load parameter is reduced, the order of magnitude of the plastic front relative to the mode mixity parameter decreases more rapidly; for example if $\frac{G_0}{k}$ is of order 10^{-2} , then the plastic front normalised by d_0 will have order 10^{-3} .

In the absence of plasticity, the singular state of stress may be relieved by either the presence of a crack, or by a zone of partial slip. These phenomena themselves introduce constraints upon the problem. The faces of an open crack must be traction free, and a zone of partial slip must obey a friction law. A method is therefore required which enforces these conditions, and preserves the conditions of the wedge, chiefly that the free surfaces of the wedge be free of tractions, and allows us to calculate the resulting state of stress in the wedge.

Using the Williams solution as a basis, previous work has used a series of distributed dislocations to modify the state of stress to account for the friction law, or the requirement for the faces of an open crack to be traction free[3], [4], [5]. However, there is a problem with this approach, which arises from the need to know the state of stress caused by the dislocations themselves in the domain under consideration.

The kernels, functions which describe the stress components induced by a dislocation, are well understood for a dislocation placed in an infinite plane. However, no closed form solution exists for dislocations in a wedge, except in special cases, such as a half-plane, which is simply a wedge with total internal angle of 180° . Some method must be devised based on the infinite plane kernels and which does not induce tractions on the free surfaces.

Hecker and Romanov[6], [7] proposed a method whereby an object dislocation is placed in an infinite plane, in the same position as the dislocation for which the state of stress is sought. This is referred to as the object dislocation. Then, distributions of dislocations are placed along trace lines which coincide with the free surfaces of the wedge. The density of these distributions is initially held as a variable, and then found using the condition that these lines must be free of traction. That is the influence of the object dislocation, and the distributions of glide and climb edge dislocations on both free surfaces must cancel out exactly on the free surface trace lines.

This method was refined by Churchman et al[8] and applied to a three quarter plane, and successfully used to solve the problem of a short crack at the edge of a square contact. Later, Riddoch and Hills[9] would generalise this approach using polar coordinates to calculate the kernels at any wedge or interface angle, and use this method to tackle the problem of a crack or pair of cracks at the apex of a wedge[5], and

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the problem of a zone of slip at the edge of a contact of arbitrary angle[10].

II. AN ALTERNATIVE APPROACH

In all the examples mentioned so far, the basic approach has been the same as that originally described by Hecker and Romanov, where a kernel is found for an individual dislocation, and then these kernels are used to solve a distributed dislocation problem by superposition. However, in this analysis, we will consider another way. Given that the resultant dislocations are used to solve a distributed dislocation problem, and are themselves found by solving a distributed dislocation problem, and the conditions that both enforce must be simultaneously true for the problem to be solved, we can consider the two problems simultaneously.

As such, consider distributing dislocation along the trace lines representing the free surfaces of the wedge in an infinite plane, and at the same time distributing dislocations along the crack faces or slip region whose effect on the state of stress of the wedge is to be calculated. We once again enforce the conditions that the crack faces be traction free, or that the slip zone follows a friction law, but now we must also consider simultaneously the effect of the dislocations on the free surfaces. We also enforce the conditions that the free surfaces be traction free, considering the effect of the other free surface, and the crack faces or slip interface.

III. A CRACK AT THE WEDGE APEX

Let us now consider an example problem. We have already discussed the example of a crack at the edge of a complete contact, or the apex of a wedge. This problem has been solved by Churchman and Hills[3] for a three quarter plane, and generalised by Riddoch et al[5]. Both previous studies used the method laid out by Hecker and Romanov, using what we shall now call the sequential method.

Using our new method, we consider the problem of a single crack of length l , placed at an angle ϕ from the bisector of a wedge of total internal angle 2α . We place dislocations along the crack face, and along the free surface trace lines, using $\oplus$ to designate the line with $\theta = \alpha$, and $\ominus$ to denote the line having $\theta = -\alpha$. Figure 1 shows the layout of this system, which we now refer to as the simultaneous method.

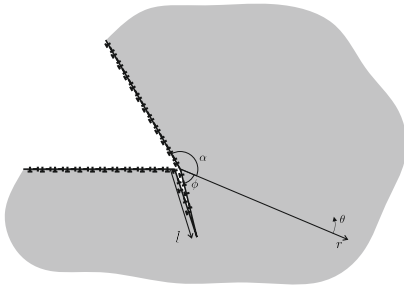


Fig. 1: Sketch of distributed dislocations on free surfaces and crack faces

The influence of a dislocation on the state of stress at a remote point is calculated by the product of the Burgers vector and the relevant dislocation kernel function. The influence of a distribution is then calculated by a sum, which can, in practice, be turned into an integral, by converting the Burgers vector's into a dislocation density. This is a contour integral along the line which the dislocations are distributed along. In this example, these contours are straight lines, which in polar coordinates can all be written as $\theta = \psi$, with ψ being some value of angle measured from the bisector of the wedge.

One of the disadvantages of the work of Churchman et al[8], is that the dislocation kernels are calculated only along perpendicular lines. One of the consequences of this assumption is that not all distributions of dislocations arise, as glide edge dislocations do not induce any normal stress along the line parallel to their Burgers vector, and climb edge dislocations do not induce any shear stress on the line parallel to their Burgers vector. However, as part of the motivation for this analysis is the generalisation of the crack and/or contact angles, it is necessary to consider all distributions of dislocations here.

This problem requires six distributions of dislocations along three lines. The free surfaces of the wedge are unable to support tractions. Therefore it is necessary to ensure that no shear or normal traction is induced on these free surfaces, so a distribution of glide edge dislocations is placed on each of the free surfaces lines to remove any induced shear tractions, and a distribution of climb edge dislocations placed similarly to remove any induced normal traction.

In this analysis, a glide edge dislocation will have a Burgers vector b_r , so a distribution of glide edge dislocations is said to have density $B_r(\rho)$, where ρ is the position along the line the dislocations are distributed along, running from the origin of the coordinate system, whose origin is placed at the apex of the wedge. The free surfaces of the wedge correspond to the lines $\theta = \pm\alpha$ in this coordinate system. Therefore, the symbol $\oplus$ is used to denote dislocation densities for those distributions along the line $\theta = \alpha$, and similarly $\ominus$ is used when considering those distributions along the line $\theta = -\alpha$. Hence the density of the two distributions of glide edge dislocations along the free surface lines are represented by $B_r^\oplus(\rho)$ and $B_r^\ominus(\rho)$ for the lines $\theta = \alpha$ and $\theta = -\alpha$ respectively. Similarly, climb edge dislocations have a Burgers vector b_θ , and so the distributions of climb edge dislocations along the free surface lines are represented by $B_\theta^\oplus(\rho)$ and $B_\theta^\ominus(\rho)$ for the lines $\theta = \alpha$ and $\theta = -\alpha$ respectively.

So, the influence on the stress component at a point (r, ψ) , for $r > 0$, and $-\alpha < \psi < \alpha$, of the distributions of dislocations along the free surface line is given by

$$\begin{aligned} \sigma_{r\theta}(r, \psi) = & \frac{2\mu}{\pi(\kappa + 1)} \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \psi, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \psi, \rho, \alpha) + \dots \\ & B_r^\ominus(\rho) G_{rr\theta}(r, \psi, \rho, -\alpha) + \dots \\ & B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \psi, \rho, -\alpha) d\rho. \end{aligned} \quad (1)$$

The other line along which dislocations must be distributed is the crack line. As we are modelling open cracks, the faces of the crack are free surfaces, and so cannot support any shear or normal traction. The two faces of the crack are treated as a single line, as the opening width of the crack is assumed to be small compared to the crack length. In the same way as we generate the free surfaces, a distribution of glide edge dislocations is placed along the crack line to remove any induced shear traction, the density of this distribution is denoted $B_r(\xi)$, where ξ is the location along the crack line from the origin, note that $\xi \leq l$, the crack length. No superscript is used to specifically identify the crack line. Similarly, a distribution of climb edge dislocations is used to clear any induced normal traction, the density of this distribution is then denoted $B_\theta(\xi)$.

The influence of these distributions of dislocations on a stress component at a point (r, ψ) , for $r > 0$, and $-\alpha < \psi < \alpha$, is then found by

$$\sigma_{r\theta}(r, \psi) = \frac{2\mu}{\pi(\kappa + 1)} \int_0^l B_r(\xi) G_{rr\theta}(r, \psi, \xi, \phi) + \dots B_\theta(\xi) G_{\theta r\theta}(r, \psi, \xi, \phi) d\xi, \quad (2)$$

where l is the crack length, and ϕ is the crack angle.

The combined influence of all of these distributions of dislocations on the state of stress at a point (r, ψ) , for $r > 0$, and $-\alpha < \psi < \alpha$, is then given by

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa + 1)} \left(\int_0^l B_r(\xi) G_{rr\theta}(r, \psi, \xi, \phi) + \dots \right. \\ & B_\theta(\xi) G_{\theta r\theta}(r, \psi, \xi, \phi) d\xi + \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \psi, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \psi, \rho, \alpha) + \dots \\ & \left. B_r^\ominus(\rho) G_{rr\theta}(r, \psi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \psi, \rho, -\alpha) d\rho \right). \end{aligned} \quad (3)$$

It should be noted that the shear stress component, $\sigma_{r\theta}$ has been used here as an example, and the same procedure can be followed with all stress components, most notably the direct stress $\sigma_{\theta\theta}$.

This influence can then be used to enforce boundary conditions on the free surface lines and the crack line. These have already been mentioned, but are explicitly stated in Table I. These conditions ensure that the free surfaces are cleared of both shear and normal traction, which they cannot support.

Line	Normal traction	Shear traction	Range
$\theta = \alpha$	$\sigma_{\theta\theta}(r, \alpha) = 0$	$\sigma_{r\theta}(r, \alpha) = 0$	$r \geq 0$
$\theta = -\alpha$	$\sigma_{\theta\theta}(r, -\alpha) = 0$	$\sigma_{r\theta}(r, -\alpha) = 0$	$r \geq 0$
$\theta = \phi$	$\sigma_{\theta\theta}(r, \phi) = 0$	$\sigma_{r\theta}(r, \phi) = 0$	$l \geq r \geq 0$

TABLE I: Table of boundary conditions

In addition to the stress induced by the placement of dislocations, the enforcement of these boundary conditions also requires a consideration of the tractions induced on these lines by the Williams solution. For the lines $\theta = \pm\alpha$,

this is trivial. The boundary conditions of the Williams solution itself requires that the lines be free of tractions, so the tractions implied along them is zero. For the crack line however, this is slightly more complicated, as this line will support stress according to the Williams solution. The tractions along this line are given by

$$\sigma_{r\theta}(r, \phi) = K_I f_{r\theta}^I(\phi) r^{\lambda_I - 1} + K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II} - 1}, \quad (4)$$

$$\sigma_{\theta\theta}(r, \phi) = K_I f_{\theta\theta}^I(\phi) r^{\lambda_I - 1} + K_{II} f_{\theta\theta}^{II}(\phi) r^{\lambda_{II} - 1}, \quad (5)$$

and these stresses must be added to the influence of the dislocations along the crack line.

The six boundary conditions described in Table I can now be enforced. In each case, the sum of the influence of all distributions of dislocations, given for an arbitrary point in equation 3, is added to the stress induced by the Williams solution. The conditions state that these six expressions must all equal zero. For the sake of clarity, these equations are reproduced in full in appendix I, and will not be stated here. However, for illustrative purposes, the equation which enforces the last boundary condition in Table I, that $\sigma_{r\theta}(r, \phi) = 0$ for $l \geq r \geq 0$, is given by

$$\begin{aligned} & K_I f_{r\theta}^I(\phi) r^{\lambda_I - 1} + K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II} - 1} + \dots \\ & \frac{2\mu}{\pi(\kappa + 1)} \int_0^l B_r(\xi) G_{rr\theta}(r, \phi, \xi, \phi) + \dots \\ & B_\theta(\xi) G_{\theta r\theta}(r, \phi, \xi, \phi) d\xi + \dots \\ & \frac{2\mu}{\pi(\kappa + 1)} \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \phi, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \phi, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, \phi, \rho, -\alpha) + \dots \\ & B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \phi, \rho, -\alpha) d\rho = 0, \quad \text{for } 0 \leq r \leq l. \end{aligned} \quad (6)$$

It can be observed that equation 6 contains six unknown functions. These are the six dislocation densities; $B_r, B_\theta, B_r^\oplus, B_\theta^\oplus, B_r^\ominus$, and $B_\theta^\ominus$. It should also be noted that the functions G_{ijk} contain singular behaviour, so equation 6 is a singular integral equation. In order to solve singular integral equations like this, it is necessary to discretise the integrals. In this case, the number of points used to discretise the equations will be decided later, but we can denote the number of points used to discretise the integral along the crack line as n , and the number of points used to discretise the integral along the free surface lines as N .

This discretisation is done using the Gauss-Chebyshev quadratures, described by Erdogan et al[11]. This is a long winded process, and is once again described in full in appendix I. One of the requirements of this quadrature is that the end point behaviour is known *a priori*, and it is then isolated and treated separately. This is done by splitting the unknown function, for example B_r , into a pointwise product of two functions, a known function which captures the end point behaviour and singularity and is known as the fundamental function, and an unknown function, denoted φ , which is well behaved.

The choice of end point behavior is either square root singular or square root bounded. This is necessarily an approximation, as the end point behaviour of the unknown

functions will not perfectly match this. Previous studies[8], [9] have shown that a sufficient simplification is to think of these options as simply singular or bounded. The choice of end point behaviour must be made with careful consideration of the form of the expected state of stress along each line.

Considering, first, the four distributions along the free surfaces, at the wedge apex, the induced state of stress will be singular in nature. Therefore the density of dislocations required to clear this stress will also be singular. Similarly, at the remote end, the state of stress will be bounded, so the density of dislocations must be bounded. The quadrature also requires the integral to be transformed to the interval $[-1, 1]$, as the integral for the free surface distribution is semi infinite, this requires a Mellin transform, which is laid out in appendix I. This transform translates the wedge apex to the -1 end of the interval, and the choice of quadrature is described as singular-bounded.

Then considering the distributions along the crack line, these dislocations are treated similarly, so the distribution of dislocations at the wedge apex must be singular, and bounded at the crack tip. As this is a finite interval, a linear transform is used, again set out in appendix I. This transform translates the crack tip to the -1 end of the interval, so the choice of quadrature is described as bounded-singular. The fundamental functions, weight functions and formula for distribution of points are in all cases given by Hills et al[12].

The result of this discretisation is that equation 6 becomes a set of n equations in $2n + 4N$ unknowns. Similar discretisation of the equations resulting from the other boundary conditions gives a set of $2n + 4N$ equations in $2n + 4N$ unknowns, which can then be solved simultaneously. The solution to this set of simultaneous equations can then be used to solve the integral equations, and so to find the six unknown functions; $B_r, B_\theta, B_r^\oplus, B_\theta^\oplus, B_r^\ominus, B_\theta^\ominus$.

A. Comparison of results

Turning now to a comparison of the results found with either method, firstly, let us compare the calibration matrix which relates the crack tip stress intensity factors K_I^c, K_{II}^c , to the wedge stress intensity factors K_I, K_{II} . This is for a single crack placed at an angle $\frac{\pi}{4}$, in a wedge of half angle $\alpha = \frac{3\pi}{4}$.

Using the sequential method, Riddoch et al[5] found that

$$\frac{1}{\sqrt{\pi}l} \begin{bmatrix} K_I^c \\ K_{II}^c \end{bmatrix} = \begin{bmatrix} 1.4928 & 1.0894 \\ 0.6070 & -0.3709 \end{bmatrix} \begin{bmatrix} K_I l^{\lambda_I-1} \\ K_{II} l^{\lambda_{II}-1} \end{bmatrix}. \quad (7)$$

Whereas using the simultaneous method we find that

$$\frac{1}{\sqrt{\pi}l} \begin{bmatrix} K_I^c \\ K_{II}^c \end{bmatrix} = \begin{bmatrix} 1.4913 & 1.0892 \\ 0.6122 & -0.3713 \end{bmatrix} \begin{bmatrix} K_I l^{\lambda_I-1} \\ K_{II} l^{\lambda_{II}-1} \end{bmatrix}. \quad (8)$$

These results are very slightly different. However, no element is more than 0.16% different from its previous value, which is well within the margin of error that may be attributed to rounding errors. This is also highlighted in Figure 2 which shows a comparison of all entries in the calibration matrix for a varying value of α between the three

quarter plane $\alpha = 0.75\pi$, and nearer to a full plane $\alpha = 0.9\pi$. It can easily be seen that the values of these coefficients are very similar and well within the margin of error that may be expected when comparing two numerically calculated results.

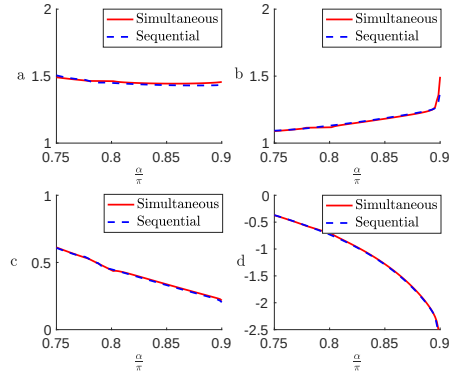


Fig. 2: Plot comparing the calibration matrix coefficients found by both methods for $\frac{3\pi}{4} \leq \alpha \leq \frac{9\pi}{10}$

IV. A SLIP ZONE AT THE CONTACT EDGE

Now we turn our attention to the other problem which we have been discussing. We consider the presence of a zone of partial slip originating from the edge of contact. Previous studies[13], [14] have indicated that the contact is likely to slip outwards under most loading scenarios. Hills and Dini[14] have also found a bound on the existence of a slip zone, based on the eigenvector ratio at the contact edge.

The Williams solution can also imply the existence of a slip zone, and give an initial guess at to the size of the slip zone. This is the point where $-f_c \sigma_{\theta\theta} = \sigma_{r\theta}$, from the Coulomb friction law, where f_c is the coefficient of friction. The Williams solution indicates that this point, which we denote as c_0 is given by

$$c_0 = \left(- \frac{(f_c - g_{r\theta}^I) K_I^o}{(f_c - g_{r\theta}^{II}) K_{II}^o} \right)^{\frac{1}{\lambda_{II} - \lambda_I}}, \quad \text{therefore} \quad (9)$$

$$\frac{c_0}{d_0} = \left(- \frac{f_c - g_{r\theta}^I}{f_c - g_{r\theta}^{II}} \right)^{\frac{1}{\lambda_{II} - \lambda_I}}, \quad (10)$$

where $g_{r\theta}^I = \frac{f_{r\theta}^I(\phi)}{f_{\theta\theta}^I(\phi)}$, and $g_{r\theta}^{II} = \frac{f_{r\theta}^{II}(\phi)}{f_{\theta\theta}^{II}(\phi)}$, are the eigenvector ratios, and ϕ is the angle of the contact interface relative to the bisector of the wedge created by combining the two contacting bodies. Note that this is not the true size of the slip, merely an estimate, and we denote the true size of the slip zone c .

As was done for the crack problem, the state of stress can be modified using dislocations to enforce the friction law. This must be done in a way that does not introduce tractions on the free surfaces. Once again, this can be written as a set of boundary conditions, in this case there are five given by

BC1 The positive free surface line is free of shear stress, $\sigma_{r\theta}(r, \alpha) = 0$, for $r \geq 0$.

- BC2 The positive free surface line is free of normal stress, $\sigma_{\theta\theta}(r, \alpha) = 0$, for $r \geq 0$.
- BC3 The negative free surface line is free of shear stress, $\sigma_{r\theta}(r, -\alpha) = 0$, for $r \geq 0$.
- BC4 The negative free surface line is free of normal stress, $\sigma_{\theta\theta}(r, -\alpha) = 0$, for $r \geq 0$.
- BC5 The friction law is satisfied in the slip region along the contact interface, $-f_c \sigma_{\theta\theta}(r, \phi) = \sigma_{r\theta}(r, \phi)$, for $c \geq r \geq 0$.

In order to enforce these boundary conditions, it is once again necessary to place distributions of glide and climb edge dislocations along the free surfaces. This is done in exactly the same way as for the crack problem described above, and the same notation is used throughout, so the explanation will not be repeated here.

However, along the slip interface, it is only necessary to distribute glide edge dislocations, as the slipping interface can support normal(compressive) traction. This distribution is placed in the slip region, from the corner of the contact, to the slip stick boundary, denoted c . The density of this distribution of glide dislocations is denoted by $\bar{B}$, in order to avoid confusion with the crack problem. The influence of this distribution of dislocations is found in exactly the same way as was done for the other distributions of dislocations.

The state of stress induced at a point (r, ψ) , for $r > 0$, $-\alpha < \psi < \alpha$, by these dislocations is then found by

$$\frac{2\mu}{\pi(\kappa + 1)} \left(\int_0^c \bar{B}(\xi) G_{rr\theta}(r, \psi, \xi, \phi) d\xi + \dots \right. \quad (11)$$

$$\left. \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \psi, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \psi, \rho, \alpha) + \dots \right.$$

$$\left. B_r^\ominus(\rho) G_{rr\theta}(r, \psi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \psi, \rho, -\alpha) d\rho \right).$$

Considering also the stress described by the Williams solution, and the state of stress along the contact interface can be described by

$$\sigma_{r\theta}(r, \phi) = K_I f_{r\theta}^I(\phi) + K_{II} f_{r\theta}^{II}(\phi) + \dots \quad (12)$$

$$\frac{2\mu}{\pi(\kappa + 1)} \left(\int_0^c \bar{B}(\xi) G_{rr\theta}(r, \phi, \xi, \phi) d\xi + \dots \right.$$

$$\left. \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \phi, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \phi, \rho, \alpha) + \dots \right.$$

$$\left. B_r^\ominus(\rho) G_{rr\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \phi, \rho, -\alpha) d\rho \right),$$

$$\sigma_{\theta\theta}(r, \phi) = K_I f_{\theta\theta}^I(\phi) + K_{II} f_{\theta\theta}^{II}(\phi) + \dots \quad (13)$$

$$\frac{2\mu}{\pi(\kappa + 1)} \left(\int_0^c \bar{B}(\xi) G_{\theta\theta}(r, \phi, \xi, \phi) d\xi + \dots \right.$$

$$\left. \int_0^\infty B_r^\oplus(\rho) G_{\theta\theta}(r, \phi, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta\theta}(r, \phi, \rho, \alpha) + \dots \right.$$

$$\left. B_r^\ominus(\rho) G_{\theta\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta}(r, \phi, \rho, -\alpha) d\rho \right).$$

Using BC5, the enforcement of the friction law, the formulae given in equations 12 and 13 are then substituted to give the equation

$$0 = (f_c f_{\theta\theta}^I(\theta) + f_{r\theta}^I(\theta)) K_I r^{\lambda_I - 1} + \dots$$

$$(f_c f_{\theta\theta}^{II}(\theta) + f_{r\theta}^{II}(\theta)) K_{II} r^{\lambda_{II} - 1} + \dots$$

$$\frac{2\mu}{\pi(\kappa + 1)} \left(\int_0^c \bar{B}(\xi) [f_c G_{r\theta\theta}(r, \phi, \xi, \phi) + \dots \right.$$

$$G_{rr\theta}(r, \phi, \xi, \phi)] d\xi + \int_0^\infty f_c [B_r^\oplus(\rho) G_{r\theta\theta}(r, \phi, \rho, \alpha) + \dots$$

$$B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \phi, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, \phi, \rho, \alpha) + \dots$$

$$B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \phi, \rho, \alpha) + B_r^\oplus(\rho) G_{rr\theta}(r, \phi, \rho, \alpha) + \dots$$

$$B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \phi, \rho, \alpha) d\rho \quad \text{for } 0 \leq r \leq c, \quad (14)$$

which is a singular integral equation.

The enforcement of the other four boundary conditions is done in the same way and leads to four more singular integral equations. For the sake of clarity, these are reproduced in full in appendix II, and are not reproduced here.

Once again, the similarity to the crack problem described above is clear. Equation 14 is a singular integral equation with five unknown functions; $\bar{B}$, $B_r^\oplus$, $B_\theta^\oplus$, $B_r^\ominus$, and $B_\theta^\ominus$. This equation is again discretised, using n points within the slip region, and N points along the free surfaces. These distributions of dislocations are sketched in Figure 3.

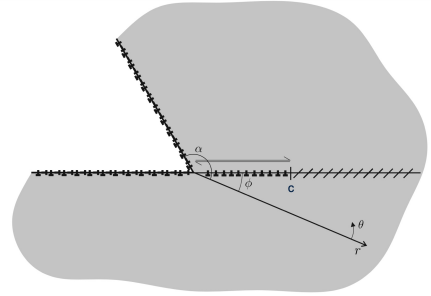


Fig. 3: Sketch of distributed dislocations on free surfaces and slip interface

Also, this discretisation is done using the Gauss-Chebyshev quadrature, which requires a choice of fundamental function based on end point behaviour, and a transformation of the integral range to the interval $[-1, 1]$. Once again, a Mellin transform is used for the free surface dislocations, and the singular-bounded quadrature is chosen, the reasons for this are explained above. The choice of quadrature for the slip problem is a little less clear, the behaviour at the slip stick boundary is fairly clear, this must be bounded, as the corrective term created by the influence of the dislocations is correcting a finite value to another finite value.

The behaviour at the contact corner is less clear, as the corrective term must change the state of stress from one singular behaviour to another. This choice was investigated thoroughly by Riddoch and Hills[10], and their conclusion

was that the bounded-bounded quadrature is the correct choice, although choosing singular behaviour was shown to not significantly affect the results. The slip region is finite, and so a finite transformation is used in both cases.

This discretisation results in equation 14 becoming a set of $n+1$ equations in $n+4N$ unknowns. Similar discretisation of the integral equations arising from enforcing the other four boundary conditions results in a set of $n+4N+1$ equations in $n+4N$ unknowns, which can be solved simultaneously and the results used to calculate the value of the unknown functions $\bar{B}$, $B_r^\oplus$, $B_\theta^\oplus$, $B_r^\ominus$, and $B_\theta^\ominus$. This requires us to discard one of the equations. The solution of this being used later to check the size of the slip zone.

A complication presented by this problem is the choice of the size of the slip zone. This must be chosen before the equations can be discretised and solved, but cannot be directly calculated as it relies on the solution of the same equations. The solution is to guess the size, based on the estimate c_0 , and then check this guess using conditions imposed on the final solution. The process for doing this is explained in more detail, including an explanation and derivation of the conditions used, by Riddoch and Hills[10].

A. Convergence

We now consider the problems of convergence. Using either the sequential or simultaneous methods, both the slip interface and the free surfaces are represented by a finite number of discrete points. For the slip interface, n points are used, and for the free surfaces N points are used. Furthermore, the quadrature requires the choice of two sets of points, known as integration and collocation points. The two sets of points are distinct from each other, meaning that there is always a finite, non-zero distance between any integration and any collocation point. This means that any singular, or near singular, behaviour can be described without providing a numerical error, such as requiring division by zero.

According to the choice of end point behaviour, the quadrature describes the distribution of integration and collocation points along the interval $[-1, 1]$. Along the slip interface, a finite transform is used to transform the integration range from $[0, c]$ to $[-1, 1]$, and each of the integration and collocation points can be transformed individually. Along the free surfaces, a Mellin transform is used to transform the range from $[0, \infty)$ to $[-1, 1]$. A log based transform was chosen for this, to simplify the differentiation required during discretisation, but other forms of transform, such as $\tan$ or reciprocal transforms could be used.

The discretisations are necessary for the solution method, but as they represent a discretisation of a truly continuous function, a larger number of points should always give a slightly more accurate result. However, for practical reasons, the greater the number of points the more demanding and therefore the slower the calculation.

So we should consider how many points are necessary. To do this we perform a convergence test, and we will do this for both the sequential and simultaneous methods. We

choose two values, N and n , but as these are related, we can infer a ratio between these two. Practical experience has shown that if the number of points on the free surface is five times that on the slip interface, ensuring a similar physical density of points along the slip interface with the choice of transforms, described in appendix II.

Figure 4 shows the calculated size of the slip zone changes as the number of points used to calculate changes. It can be seen that a very large number of points is required to guarantee convergence with either method. Computational limitations prevented us from exploring values beyond $N = 2500$, and whilst this proved sufficient to achieve a converged solution using the sequential method, this was not the case for the simultaneous method. This can be determined by the gradient of the lines in Figure 4.

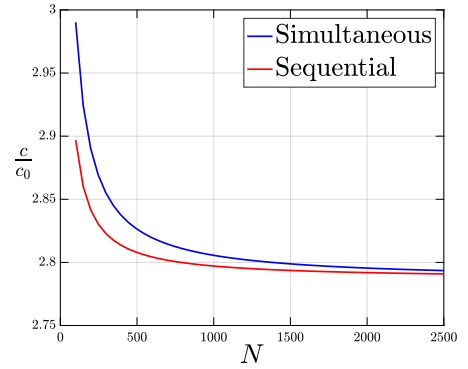


Fig. 4: Variation of calculated slip zone size with number of points used, using both dislocation methods, for a square edged contact with coefficient of friction $f = 0.3$, and remote loading $K_I = -1, K_{II} = 1$.

The distinction made here is quite fine, but is an important one. Figure 5 shows that the calculated solution changes with coefficient of friction, and that this also affects the convergence of the solution. This highlights the distinction that we have made between a converged and non-converged solution.

It is noteworthy that for this problem, the simultaneous method has not yet converged. This is disappointing and

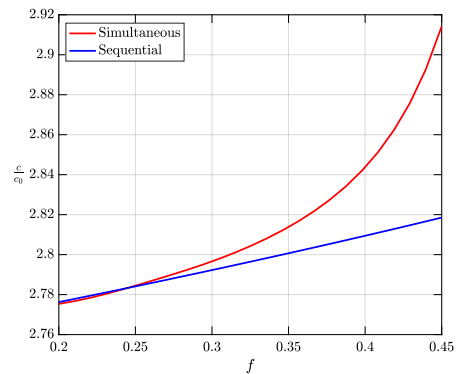


Fig. 5: Overlay plot showing the variation with coefficient of friction of the slip zone size using both dislocation methods

somewhat confusing. However, it is clear that the solution is converging, and it may reasonably be expected to converge for some larger value of N . Unfortunately, it is not possible to verify this at this point in time.

V. TWO CRACKS AT A WEDGE APEX

Considering again the modelling of crack problems, let us consider the problem of two short straight cracks, whose roots are both at the contact edge/wedge apex. This is a simple development of the single crack problem tackled above, and the same method may be employed. Some modifications will be required to the method and the notation to account for the influence of the second crack, and the effect that may be seen as a result of the two cracks having different lengths.

As before, distributions of glide and climb edge dislocations are placed along the free surfaces, these having densities denoted by; $B_r^\oplus$ for the glide edge dislocations along the line $\theta = \alpha$; $B_\theta^\oplus$ for the climb edge dislocations along the line $\theta = \alpha$; $B_r^\ominus$ for the glide edge dislocations along the line $\theta = -\alpha$, and; $B_\theta^\ominus$ for the climb edge dislocations along the line $\theta = -\alpha$.

In this case, distributions of dislocations must be placed along both crack faces. To differentiate between the cracks, we number them 1 and 2; the choice of number is not important as long as it is consistent. The number assigned to the crack is used as a subscript or superscript. Distributions of glide and climb edge dislocations, are distributed along the crack faces, their density is denoted; B_r^1 , for the glide dislocations on crack 1, B_θ^1 , for the climb dislocations on crack 1, B_r^2 , for the glide dislocations on crack 2, B_θ^2 , for the climb dislocations on crack 2. This system is sketched in Figure 6.

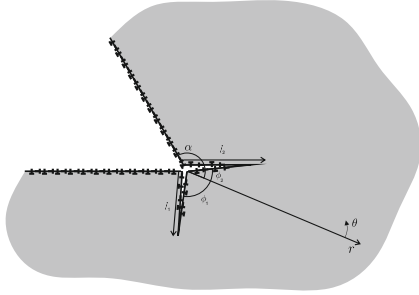


Fig. 6: Sketch of distributed dislocations on free surfaces and crack faces

The influence of these dislocations can then be calculated in the same way as before, and these can be used to enforce boundary conditions. In this case, as in all cases, the free surfaces cannot support any traction. As in the single crack case, we are modelling the cracks as open cracks, whose faces also cannot support any traction. As such, eight boundary conditions may be enforced. These are laid out in Table II.

Using the influence functions of the distributions of dislocations, the enforcement of these boundary conditions

Line	Normal traction	Shear traction	Range
$\theta = \alpha$	$\sigma_{\theta\theta}(r, \alpha) = 0$	$\sigma_{r\theta}(r, \alpha) = 0$	$r \geq 0$
$\theta = -\alpha$	$\sigma_{\theta\theta}(r, -\alpha) = 0$	$\sigma_{r\theta}(r, -\alpha) = 0$	$r \geq 0$
$\theta = \phi_1$	$\sigma_{\theta\theta}(r, \phi_1) = 0$	$\sigma_{r\theta}(r, \phi_1) = 0$	$l_1 \geq r \geq 0$
$\theta = \phi_2$	$\sigma_{\theta\theta}(r, \phi_2) = 0$	$\sigma_{r\theta}(r, \phi_2) = 0$	$l_2 \geq r \geq 0$

TABLE II: Table of boundary conditions

will give rise to eight simultaneous singular integral equations. As before, these may be discretised using the Gauss-Chebyshev quadratures. Once again a Mellin transform is used to transform the integral range for the distributions along the free surfaces, and a linear transforms is used for the cracks. As for the single crack problem, the singular-bounded quadrature is chosen for the free surfaces and bounded-singular for the cracks.

Again as before, this problem has been previously solved using the sequential method by Riddoch et al[5]. Using the simultaneous dislocation method, we are now able to enforce eight conditions, clearing the crack faces and free surfaces of both shear and normal traction. This is detailed in appendix III. Having these equations, and solving them using MATLAB, we can find the crack tip stress intensity factors related to the wedge stress intensity factors.

A. Comparison of results

The stress intensity factors are the primary outputs of the problem, and it is these which we may compare to the results found by Riddoch et al[5] using the sequential dislocation method. Firstly, let us compare two cracks of equal length. Whilst reviewing the method and results published, an error was found. The correct calibration for a pair of cracks of equal length at right angles to each other and the free surfaces in a three quarter plane, is shown below. Calibrations found with both the simultaneous and sequential methods are shown in Table III.

Method	Calibration matrix	
Sequential	$\frac{1}{\sqrt{\pi}} \begin{bmatrix} K_I^1 l_1^{\frac{1}{\lambda_I - \frac{1}{2}}} \\ K_{II}^1 l_1^{\frac{1}{\lambda_{II} - \frac{1}{2}}} \\ K_I^2 l_2^{\frac{1}{\lambda_I - \frac{1}{2}}} \\ K_{II}^2 l_2^{\frac{1}{\lambda_{II} - \frac{1}{2}}} \end{bmatrix}$	$= \begin{bmatrix} 0.9483 & -1.3765 \\ 0.2459 & 0.1160 \\ 0.9483 & 1.3765 \\ -0.2459 & 0.1160 \end{bmatrix} \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}$
Simultaneous	$\frac{1}{\sqrt{\pi}} \begin{bmatrix} K_I^1 l_1^{\frac{1}{\lambda_I - \frac{1}{2}}} \\ K_{II}^1 l_1^{\frac{1}{\lambda_{II} - \frac{1}{2}}} \\ K_I^2 l_2^{\frac{1}{\lambda_I - \frac{1}{2}}} \\ K_{II}^2 l_2^{\frac{1}{\lambda_{II} - \frac{1}{2}}} \end{bmatrix}$	$= \begin{bmatrix} 0.9519 & -1.3768 \\ 0.2502 & 0.1179 \\ 0.9519 & 1.3768 \\ -0.2502 & 0.1179 \end{bmatrix} \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}$

TABLE III: Table comparing the calibration matrices for symmetric cracks in a three quarter plane using different dislocation methods

This comparison is encouraging, as although the precise values are not identical, once again the results are well within the margin of error that is expected from numerical

calculations. No entry is more than 1.75% different between the two methods.

Alternatively, we may consider the ‘single crack’ problem. This is calculated by making the length of one of the cracks significantly larger than the other, in practice at least 1000 times larger. The effect of the shorter crack is then negligible. The results will not be exactly the same as the results calculated using the single crack model, as the presence of the other crack will have a small effect on the state of stress at the crack root and create a singularity near, but not at the crack root. However, by bringing this singularity, which arises from the tip of the very short crack, nearly to the wedge apex, the effect is reduced. The calibration generated for the shorter crack is useless, as the crack is so short as to be drowned out by noise. Table IV shows the calibration matrices calculated using both dislocation methods.

Method	Calibration matrix	
Sequential	$\frac{1}{\sqrt{\pi}} \begin{bmatrix} K_{II}^1 l_1^{\lambda_I - \frac{1}{2}} \\ K_{II}^1 l_1^{\lambda_{II} - \frac{1}{2}} \\ K_{II}^2 l_2^{\lambda_I - \frac{1}{2}} \\ K_{II}^2 l_2^{\lambda_{II} - \frac{1}{2}} \end{bmatrix}$	$= \begin{bmatrix} 1.0614 & -1.1759 \\ 0.4941 & 0.3733 \\ N/A & N/A \\ N/A & N/A \end{bmatrix} \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}$
Simultaneous	$\frac{1}{\sqrt{\pi}} \begin{bmatrix} K_{II}^1 l_1^{\lambda_I - \frac{1}{2}} \\ K_{II}^1 l_1^{\lambda_{II} - \frac{1}{2}} \\ K_{II}^2 l_2^{\lambda_I - \frac{1}{2}} \\ K_{II}^2 l_2^{\lambda_{II} - \frac{1}{2}} \end{bmatrix}$	$= \begin{bmatrix} 1.0938 & -1.1901 \\ 0.4416 & 0.4021 \\ N/A & N/A \\ N/A & N/A \end{bmatrix} \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}$

TABLE IV: Table comparing the calibration matrices for a ‘single crack’ in a three quarter plane using different dislocation methods

Once again these results are encouraging as the calculated results are within the margin of error when compared to each other.

Finally we can now consider the problem of varying the wedge angle. We vary the angle between the three-quarter plane and $\alpha = 0.8\pi$, and Figure 7 shows how the entries of the calibration matrix compare for cracks of unit and three-quarter unit length respectively, placed at angles $\pi - \alpha$, and $\alpha - \pi$, respectively, so that at the three quarter plane the cracks are perpendicular to each other and the free surfaces and remain perpendicular to each other and the negative free surface throughout.

VI. DISCUSSION

Having now considered three example problems, we can consider the efficacy of our new method, as well as discuss some of the advantages and disadvantages presented by the method.

Firstly, let us consider the crack problems. With both the single and two crack problems, close agreement was found between the two dislocation methods. However, there were problems considering the slip problem, which appear to be

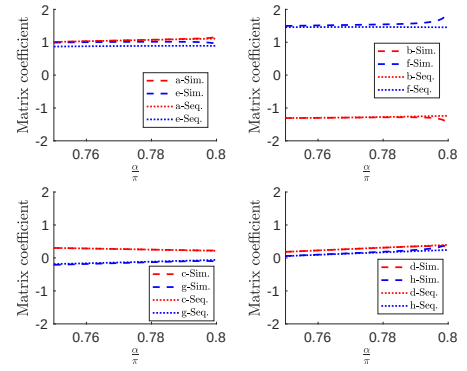


Fig. 7: Plot comparing the calibration matrix coefficients found by both methods for $\frac{3\pi}{4} \leq \alpha \leq \frac{8\pi}{10}$

caused by convergence problems, which are unfortunately unavoidable due to technical limitations of the software used.

This is all well and good, but in all three cases the sequential method developed by Hecker and Romanov[6], [7], and applied by Churchman et al[8], [3] and Riddoch et al[9], [5] produced equally good results, and in the slip problem, achieved faster convergence. So we must consider what advantages the simultaneous method described here gives. First among these is computational speed. Experience has shown that this method is significantly faster, by around 20%. This figure is arrived at by comparing the time taken to produce a single graph, like that shown in Figure 2, the sequential method taking 77.927 seconds, compared to 63.051 seconds taken by the simultaneous method. The precise time will obviously vary depending upon the computer but this is indicative of experience, and the more complex or extensive the problem, the faster the simultaneous method becomes, in relation to the sequential method. This is especially relevant when considering very high resolution problems, using large numbers of points and considering very similar angles each time for example.

The second major advantage is simplicity. This may seem counter-intuitive as the simultaneous method requires fewer integral equations to describe the problem. However, before tackling an example problem the calculation of the stress kernels must be undertaken correctly. This has, of course, been done previously, but to take advantage of this would require the creation and use of a look up table. Furthermore, the only method of verification that the kernel calculations have undergone has been done either by applying the kernels, which may introduce other errors, or by comparison with finite element analysis. By its nature finite element analysis struggles to accurately simulate discontinuities, such as those found at a dislocation core, and the calculation of singularities can lead to errors. The sequential method avoids this problem by using well known and well understood infinite plane kernels, which have been verified over many years of research[12], [15].

It should be considered that the simultaneous method has shown problems with convergence for the slip problem, so care must be taken to avoid this issue when considering other

problems using this method. Also, the method does generate more integral equations than the simultaneous method, increasing the chance of an error during transcription or discretisation.

VII. CONCLUSION

In this analysis we have developed and explained an alternative dislocation based approach to tackling wedge problems. We have demonstrated that this method has significant advantages in terms of speed and complexity, and does not compromise the fidelity of the results, provided that convergence is guaranteed. We have also explored the convergence of both the sequential and simultaneous methods for the slip and demonstrated why this problem is difficult to solve and prone to errors.

VIII. ACKNOWLEDGEMENTS

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APPENDIX I
SINGLE CRACK INTEGRAL EQUATIONS

The single crack problem requires us to enforce six conditions. Clearing the crack face of shear traction, clearing the crack face of normal traction, clearing the positive free surface of shear traction, clearing the positive free surface of normal traction, clearing the negative free surface of shear traction, and finally clearing the negative free surface of normal traction. This gives rise to six integral equations, as described in section III. These are given by:

$$\begin{aligned}
 -K_I f_{r\theta}^I(\phi)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II}-1} &= \frac{2\mu}{\pi(\kappa+1)} \int_0^l B_r(\xi) G_{rr\theta}(r, \phi, \xi, \phi) + \dots \\
 B_\theta(\xi) G_{\theta r\theta}(r, \phi, \xi, \phi) d\xi + \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \phi, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \phi, \rho, \alpha) + \dots \\
 B_r^\ominus(\rho) G_{rr\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \phi, \rho, -\alpha) d\rho, \quad \text{for } 0 \leq r \leq l,
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 -K_I f_{\theta\theta}^I(\phi) r^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi) r^{\lambda_{II}-1} &= \frac{2\mu}{\pi(\kappa+1)} \int_0^l B_r(\xi) G_{r\theta\theta}(r, \phi, \xi, \phi) + \dots \\
 B_\theta(\xi) G_{\theta\theta\theta}(r, \phi, \xi, \phi) d\xi + \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, \phi, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \phi, \rho, \alpha) + \dots \\
 B_r^\ominus(\rho) G_{r\theta\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, \phi, \rho, -\alpha) d\rho, \quad \text{for } 0 \leq r \leq l,
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 0 &= \frac{2\mu}{\pi(\kappa+1)} \int_0^c B_r(\xi) G_{rr\theta}(r, \alpha, \xi, \phi) + B_\theta(\xi) G_{\theta r\theta}(r, \alpha, \xi, \phi) d\xi + \dots \\
 \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \alpha, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \alpha, \rho, \alpha) + \dots \\
 B_r^\ominus(\rho) G_{rr\theta}(r, \alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty,
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 0 &= \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r(\xi) G_{r\theta\theta}(r, \alpha, \xi, \phi) + B_\theta(\xi) G_{\theta\theta\theta}(r, \alpha, \xi, \phi) d\xi + \dots \\
 \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, \alpha, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, \alpha, \rho, -\alpha) + \dots \\
 B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty,
 \end{aligned} \tag{18}$$

$$\begin{aligned}
 0 &= \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r(\xi) G_{rr\theta}(r, -\alpha, \xi, \phi) + B_\theta(\xi) G_{\theta r\theta}(r, -\alpha, \xi, \phi) d\xi + \dots \\
 \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, -\alpha, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta r\theta}(r, -\alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, -\alpha, \rho, -\alpha) + \dots \\
 B_\theta^\ominus(\rho) G_{\theta r\theta}(r, -\alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty,
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 0 &= \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r(\xi) G_{r\theta\theta}(r, -\alpha, \xi, \phi) + B_\theta(\xi) G_{\theta\theta\theta}(r, -\alpha, \xi, \phi) d\xi + \dots \\
 \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, -\alpha, \rho, \alpha) + B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, -\alpha, \rho, -\alpha) + \dots \\
 B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty.
 \end{aligned} \tag{20}$$

These equations can then be discretised using the Gauss-Chebyshev quadratures, as described by Hills et al[12]. We require the free surface dislocations to have a singular-bounded quadrature(denoted v for collocation points, u for integration points and X for weights), and the crack dislocations to have a bounded-singular quadrature(denoted t for collocation points, s for integration points and W for weights).

These points must be transformed in two different ways. The free surface points use a Mellin transform, given by $r = l \log \frac{2}{1-t}$, and $\xi = l \log \frac{2}{1-s}$. Whereas the crack points use a finite transform given by $r = \frac{2v}{l} - 1$, and $\rho = \frac{2u}{l} - 1$.

Finally, we can now write the discretised forms of the integral equations as:

$$\begin{aligned}
 -K_I f_{r\theta}^I(\phi) t^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi) t^{\lambda_{II}-1} &= \frac{2\mu}{\pi(\kappa+1)} \dots \\
 \left[\sum_{i=1}^n W(s_i) \varphi_r(s_i) G_{rr\theta}(t, \phi, s_i, \phi) + W(s_i) \varphi_\theta(s_i) G_{\theta r\theta}(t, \phi, s_i, \phi) + \dots \right.
 \end{aligned}$$

$$\sum_{i=1}^N X(u_i) \varphi_r^{\oplus}(u_i) G_{rr\theta}(t, \phi, u_i, \alpha) + X(u_i) \varphi_{\theta}^{\oplus}(u_i) G_{\theta r\theta}(t, \phi, u_i, \alpha) + \dots$$

$$\left. X(u_i) \varphi_r^{\ominus}(u_i) G_{rr\theta}(t, \phi, u_i, -\alpha) + X(u_i) \varphi_{\theta}^{\ominus}(u_i) G_{\theta r\theta}(t, \phi, u_i, -\alpha) \right], \quad (21)$$

$$- K_I f_{\theta\theta}^I(\phi) t^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi) t^{\lambda_{II}-1} = \frac{2\mu}{\pi(\kappa+1)} \dots$$

$$\left[\sum_{i=1}^n W(s_i) \varphi_r(s_i) G_{r\theta\theta}(t, \phi, s_i, \phi) + W(s_i) \varphi_{\theta}(s_i) G_{\theta\theta\theta}(t, \phi, s_i, \phi) + \dots \right.$$

$$\sum_{i=1}^N X(u_i) \varphi_r^{\oplus}(u_i) G_{r\theta\theta}(t, \phi, u_i, \alpha) + X(u_I) \varphi_{\theta}^{\oplus}(u_i) G_{\theta\theta\theta}(t, \phi, u_i, \alpha) + \dots$$

$$\left. X(u_i) \varphi_r^{\ominus}(u_i) G_{r\theta\theta}(t, \phi, u_i, -\alpha) + X(u_i) \varphi_{\theta}^{\ominus}(u_i) G_{\theta\theta\theta}(t, \phi, u_i, -\alpha) \right], \quad (22)$$

$$0 = \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r(s_i) G_{rr\theta}(v, \alpha, s_i, \phi) + W(s_i) \varphi_{\theta}(s_i) G_{\theta r\theta}(v, \alpha, s_i, \phi) + \dots \right.$$

$$\sum_{i=1}^N X(u_i) \varphi_r^{\oplus}(u_i) G_{rr\theta}(v, \alpha, u_i, \alpha) + X(u_I) \varphi_{\theta}^{\oplus}(u_i) G_{\theta r\theta}(v, \alpha, u_i, \alpha) + \dots$$

$$\left. X(u_i) \varphi_r^{\ominus}(u_i) G_{rr\theta}(v, \alpha, u_i, -\alpha) + X(u_i) \varphi_{\theta}^{\ominus}(u_i) G_{\theta r\theta}(v, \alpha, u_i, -\alpha) \right], \quad (23)$$

$$0 = \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r(s_i) G_{r\theta\theta}(v, \alpha, s_i, \phi) + W(s_i) \varphi_{\theta}(s_i) G_{\theta\theta\theta}(v, \alpha, s_i, \phi) + \dots \right.$$

$$\sum_{i=1}^N X(u_i) \varphi_r^{\oplus}(u_i) G_{r\theta\theta}(v, \alpha, u_i, \alpha) + X(u_I) \varphi_{\theta}^{\oplus}(u_i) G_{\theta\theta\theta}(v, \alpha, u_i, \alpha) + \dots$$

$$\left. X(u_i) \varphi_r^{\ominus}(u_i) G_{r\theta\theta}(v, \alpha, u_i, -\alpha) + X(u_i) \varphi_{\theta}^{\ominus}(u_i) G_{\theta\theta\theta}(v, \alpha, u_i, -\alpha) \right], \quad (24)$$

$$0 = \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r(s_i) G_{rr\theta}(v, -\alpha, s_i, \phi) + W(s_i) \varphi_{\theta}(s_i) G_{\theta r\theta}(v, -\alpha, s_i, \phi) + \dots \right.$$

$$\sum_{i=1}^N X(u_i) \varphi_r^{\oplus}(u_i) G_{rr\theta}(v, -\alpha, u_i, \alpha) + X(u_I) \varphi_{\theta}^{\oplus}(u_i) G_{\theta r\theta}(v, -\alpha, u_i, \alpha) + \dots$$

$$\left. X(u_i) \varphi_r^{\ominus}(u_i) G_{rr\theta}(v, -\alpha, u_i, -\alpha) + X(u_i) \varphi_{\theta}^{\ominus}(u_i) G_{\theta r\theta}(v, -\alpha, u_i, -\alpha) \right], \quad (25)$$

$$0 = \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r(s_i) G_{r\theta\theta}(v, -\alpha, s_i, \phi) + W(s_i) \varphi_{\theta}(s_i) G_{\theta\theta\theta}(v, -\alpha, s_i, \phi) + \dots \right.$$

$$\sum_{i=1}^N X(u_i) \varphi_r^{\oplus}(u_i) G_{r\theta\theta}(v, -\alpha, u_i, \alpha) + X(u_i) \varphi_{\theta}^{\oplus}(u_i) G_{\theta\theta\theta}(v, -\alpha, u_i, \alpha) + \dots$$

$$\left. X(u_i) \varphi_r^{\ominus}(u_i) G_{r\theta\theta}(v, -\alpha, u_i, -\alpha) + X(u_i) \varphi_{\theta}^{\ominus}(u_i) G_{\theta\theta\theta}(v, -\alpha, u_i, -\alpha) \right]. \quad (26)$$

The solution of this set of simultaneous equations will allow us to determine the dislocation densities along the crack faces and the free surfaces. This allows us to determine the state of stress in the wedge, and the crack tip stress intensity factors, in relationship to the wedge stress intensity factors.

APPENDIX II

SLIP PROBLEM INTEGRAL EQUATIONS

The normal and shear tractions along the slip interface are given by

$$\sigma_{r\theta} = K_{II} f_{r\theta}^I(\theta) r^{\lambda_I-1} + K_{II} f_{r\theta}^{II}(\theta) r^{\lambda_{II}-1} + \dots \quad (27)$$

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa+1)} \int_0^c B_r(\xi) G_{rr\theta}(r, \phi, \xi, \phi) d\xi + \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \phi, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \phi, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \phi, \rho, -\alpha) d\rho, \\ \sigma_{\theta\theta} &= K_I f_{\theta\theta}^I(\theta) r^{\lambda_I-1} + K_{II} f_{\theta\theta}^{II}(\theta) r^{\lambda_{II}-1} + \dots \quad (28) \\ & \frac{2\mu}{\pi(\kappa+1)} \int_0^c B_r(\xi) G_{r\theta\theta}(r, \phi, \xi, \phi) d\xi + \frac{2\mu}{\pi(\kappa+1)} \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, \phi, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \phi, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, \phi, \rho, -\alpha) d\rho, \end{aligned}$$

throughout these the superscript $\oplus$ or $\ominus$ refers to distributions of dislocation along the positive and negative angle free surfaces respectively.

Now having these formulae for the tractions we can proceed exactly as described by Riddoch and Hills[10] to find the integral equation governing slip behaviour. However, unlike previously where this is the only integral equation for us to solve, we now have a further four integral equations associated with the free surfaces, and ensuring that these are free of traction. The five resulting integral equations are given by

$$0 = (f_c f_{\theta\theta}^I(\theta) + f_{r\theta}^I(\theta)) K_I r^{\lambda_I-1} + (f_c f_{\theta\theta}^{II}(\theta) + f_{r\theta}^{II}(\theta)) K_{II} r^{\lambda_{II}-1} + \dots \quad (29)$$

$$\begin{aligned} & \int_0^c \bar{B}(\xi) [f_c G_{r\theta\theta}(r, \phi, \xi, \phi) + G_{rr\theta}(r, \phi, \xi, \phi)] d\xi + \int_0^\infty f_c [B_r^\oplus(\rho) G_{r\theta\theta}(r, \phi, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \phi, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, \phi, \rho, -\alpha)] + B_r^\oplus(\rho) G_{rr\theta}(r, \phi, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \phi, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, \phi, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \phi, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq c, \\ 0 &= \int_0^c \bar{B}(\xi) G_{rr\theta}(r, \alpha, \xi, \phi) d\xi + \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \alpha, \rho, \alpha) + \dots \quad (30) \\ & B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, \alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty. \end{aligned}$$

$$\begin{aligned} 0 &= \int_0^c \bar{B}(\xi) G_{r\theta\theta}(r, \alpha, \xi, \phi) d\xi + \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, \alpha, \rho, \alpha) + \dots \quad (31) \\ & B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, \alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty. \end{aligned}$$

$$\begin{aligned} 0 &= \int_0^c \bar{B}(\xi) G_{rr\theta}(r, -\alpha, \xi, \phi) d\xi + \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, -\alpha, \rho, \alpha) + \dots \quad (32) \\ & B_\theta^\oplus(\rho) G_{\theta r\theta}(r, -\alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, -\alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, -\alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty. \end{aligned}$$

$$\begin{aligned} 0 &= \int_0^c \bar{B}(\xi) G_{r\theta\theta}(r, -\alpha, \xi, \phi) d\xi + \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, -\alpha, \rho, \alpha) + \dots \quad (33) \\ & B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, -\alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, -\alpha) d\rho \quad \text{for } 0 \leq r \leq \infty. \end{aligned}$$

These integral equations now must be discretised. We will use exactly the same Mellin transform for the free surface points as we did for the single crack problem, given by $r = c \log \frac{2}{1-t}$, for the collocation points and $\xi = c \log \frac{2}{1-s}$, for the integration points. However, the points on the slip interface use a different quadrature to that used by those on the crack faces. Whereas the crack face tractions used the bounded-singular quadrature, the slip points will use a bounded-bounded quadrature as discussed by Riddoch and Hills[10].

For the sake of clarity we will denote the collocation points by p , the integration points by q , and the weights by Y . The transformation is then the same linear transformation given by $r = \frac{2p}{c} - 1$, and $\xi = \frac{2q}{c} - 1$. Therefore we can discretise these equations by

$$\begin{aligned} 0 &= (f_c f_{\theta\theta}^I(\theta) + f_{r\theta}^I(\theta)) K_I p^{\lambda_I-1} + (f_c f_{\theta\theta}^{II}(\theta) + f_{r\theta}^{II}(\theta)) K_{II} p^{\lambda_{II}-1} + \dots \\ & \sum_{i=1}^n Y(q_i) \bar{\varphi}(q_i) [f_c G_{r\theta\theta}(p, \phi, q_i, \phi) + G_{rr\theta}(p, \phi, q_i, \phi)] \dots \\ & + \sum_{j=1}^N f_c [W(s_i) \varphi_r^\oplus(s_i) G_{r\theta\theta}(p, \phi, s_i, \alpha) + W(s_i) \varphi_\theta^\oplus(s_i) G_{\theta\theta\theta}(p, \phi, s_i, \alpha) + \dots \\ & W(s_i) \varphi_r^\ominus(s_i) G_{r\theta\theta}(p, \phi, s_i, \alpha) + W(s_i) \varphi_\theta^\ominus(s_i) G_{\theta\theta\theta}(p, \phi, s_i, \alpha)] + W(s_i) \varphi_r^\oplus(s_i) G_{rr\theta}(p, \phi, s_i, \alpha) + \dots \quad (34) \end{aligned}$$

$$\begin{aligned}
& W(s_i)\varphi_\theta^\oplus(s_i)G_{\theta r\theta}(p, \phi, s_i, \alpha) + W(s_i)\varphi_r^\ominus(s_i)G_{rr\theta}(p, \phi, s_i, \alpha) + W(s_i)\varphi_\theta^\ominus(s_i)G_{\theta r\theta}(p, \phi, s_i, \alpha), \\
0 = & \sum_{i=1}^n Y(q_i)\bar{\varphi}(q_i)G_{rr\theta}(t, \alpha, s_i, \phi) + \sum_{i=1}^N W(s_i)\varphi_r^\oplus(s_i)G_{rr\theta}(t, \alpha, s_i, \alpha) + \dots
\end{aligned} \tag{35}$$

$$\begin{aligned}
& W(s_i)\varphi_\theta^\oplus(s_i)G_{\theta r\theta}(t, \alpha, s_i, \alpha) + W(s_i)\varphi_r^\ominus(s_i)G_{rr\theta}(t, \alpha, s_i, -\alpha) + W(s_i)\varphi_\theta^\ominus(s_i)G_{\theta r\theta}(t, \alpha, s_i, -\alpha), \\
0 = & \sum_{i=1}^n Y(q_i)\bar{\varphi}(q_i)G_{r\theta\theta}(t, \alpha, q_i, \phi) + \sum_{i=1}^N W(s_i)\varphi_r^\oplus(s_i)G_{r\theta\theta}(t, \alpha, s_i, \alpha) + \dots
\end{aligned} \tag{36}$$

$$\begin{aligned}
& W(s_i)\varphi_\theta^\oplus(s_i)G_{\theta\theta\theta}(t, \alpha, s_i, \alpha) + W(s_i)\varphi_r^\ominus(s_i)G_{r\theta\theta}(t, \alpha, s_i, -\alpha) + W(s_i)\varphi_\theta^\ominus(s_i)G_{\theta\theta\theta}(t, \alpha, s_i, -\alpha), \\
0 = & \sum_{i=1}^n Y(q_i)\bar{\varphi}(q_i)G_{rr\theta}(t, -\alpha, q_i, \phi) + \sum_{i=1}^N W(s_i)\varphi_r^\oplus(s_i)G_{rr\theta}(t, -\alpha, s_i, \alpha) + \dots
\end{aligned} \tag{37}$$

$$\begin{aligned}
& W(s_i)\varphi_\theta^\oplus(s_i)G_{\theta r\theta}(t, -\alpha, s_i, \alpha) + W(s_i)\varphi_r^\ominus(s_i)G_{rr\theta}(t, -\alpha, s_i, -\alpha) + W(s_i)\varphi_\theta^\ominus(s_i)G_{\theta r\theta}(t, -\alpha, s_i, -\alpha), \\
0 = & \sum_{i=1}^n Y(q_i)\bar{\varphi}(q_i)G_{r\theta\theta}(t, -\alpha, q_i, \phi) + \sum_{i=1}^N W(s_i)\varphi_r^\oplus(s_i)G_{r\theta\theta}(t, -\alpha, s_i, \alpha) + \dots \\
& W(s_i)\varphi_\theta^\oplus(s_i)G_{\theta\theta\theta}(t, -\alpha, s_i, \alpha) + W(s_i)\varphi_r^\ominus(s_i)G_{r\theta\theta}(t, -\alpha, s_i, -\alpha) + W(s_i)\varphi_\theta^\ominus(s_i)G_{\theta\theta\theta}(t, -\alpha, s_i, -\alpha).
\end{aligned} \tag{38}$$

The solution of this set of equations, and the attendant determination of the true size of the slip zone will allow us to determine the state of stress in a slipping complete contact.

APPENDIX III TWO CRACK INTEGRAL EQUATIONS

As with the single crack problem these two pairs of distributions will have different integration ranges, and again we denote the free surface dislocations using the superscripts $\oplus, \ominus$. This leads us to a set of eight simultaneous singular integral equations, given by:

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho)G_{rr\theta}(r, \phi_1, \rho, \phi_1) + B_\theta^1(\rho)G_{\theta r\theta}(r, \phi_1, \rho, \phi_1)d\rho + \dots \right. \\
& \int_0^{l_2} B_r^2(\rho)G_{rr\theta}(r, \phi_1, \rho, \phi_2) + B_\theta^2(\rho)G_{\theta r\theta}(r, \phi_1, \rho, \phi_2)d\rho + \int_0^\infty B_r^\oplus(\rho)G_{rr\theta}(r, \phi_1, \rho, \alpha) + \dots \\
& \left. B_\theta^\oplus(\rho)G_{\theta r\theta}(r, \phi_1, \rho, \alpha) + B_r^\ominus(\rho)G_{rr\theta}(r, \phi_1, \rho, -\alpha) + B_\theta^\ominus(\rho)G_{\theta r\theta}(r, \phi_1, \rho, -\alpha)d\rho \right] = \dots \\
& - K_I f_{r\theta}^I(\phi_1)r^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1)r^{\lambda_{II}-1}.
\end{aligned} \tag{39}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho)G_{r\theta\theta}(r, \phi_1, \rho, \phi_1) + B_\theta^1(\rho)G_{\theta\theta\theta}(r, \phi_1, \rho, \phi_1)d\rho + \dots \right. \\
& \int_0^{l_2} B_r^2(\rho)G_{r\theta\theta}(r, \phi_1, \rho, \phi_2) + B_\theta^2(\rho)G_{\theta\theta\theta}(r, \phi_1, \rho, \phi_2)d\rho + \int_0^\infty B_r^\oplus(\rho)G_{r\theta\theta}(r, \phi_1, \rho, \alpha) + \dots \\
& \left. B_\theta^\oplus(\rho)G_{\theta\theta\theta}(r, \phi_1, \rho, \alpha) + B_r^\ominus(\rho)G_{r\theta\theta}(r, \phi_1, \rho, -\alpha) + B_\theta^\ominus(\rho)G_{\theta\theta\theta}(r, \phi_1, \rho, -\alpha)d\rho \right] = \dots \\
& - K_I f_{\theta\theta}^I(\phi_1)r^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_1)r^{\lambda_{II}-1}.
\end{aligned} \tag{40}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho)G_{rr\theta}(r, \phi_2, \rho, \phi_1) + B_\theta^1(\rho)G_{\theta r\theta}(r, \phi_2, \rho, \phi_1)d\rho + \dots \right. \\
& \int_0^{l_2} B_r^2(\rho)G_{rr\theta}(r, \phi_2, \rho, \phi_2) + B_\theta^2(\rho)G_{\theta r\theta}(r, \phi_2, \rho, \phi_2)d\rho + \int_0^\infty B_r^\oplus(\rho)G_{rr\theta}(r, \phi_2, \rho, \alpha) + \dots \\
& \left. B_\theta^\oplus(\rho)G_{\theta r\theta}(r, \phi_2, \rho, \alpha) + B_r^\ominus(\rho)G_{rr\theta}(r, \phi_2, \rho, -\alpha) + B_\theta^\ominus(\rho)G_{\theta r\theta}(r, \phi_2, \rho, -\alpha)d\rho \right] = \dots \\
& - K_I f_{r\theta}^I(\phi_2)r^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_2)r^{\lambda_{II}-1}.
\end{aligned} \tag{41}$$

$$\frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho)G_{r\theta\theta}(r, \phi_2, \rho, \phi_1) + B_\theta^1(\rho)G_{\theta\theta\theta}(r, \phi_2, \rho, \phi_1)d\rho + \dots \right.$$

$$\begin{aligned} & \int_0^{l_2} B_r^2(\rho) G_{r\theta\theta}(r, \phi_2, \rho, \phi_2) + B_\theta^2(\rho) G_{\theta\theta\theta}(r, \phi_2, \rho, \phi_2) d\rho + \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, \phi_2, \rho, \alpha) + \dots \\ & B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \phi_2, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, \phi_2, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, \phi_2, \rho, -\alpha) d\rho \Big] = \dots \\ & - K_I f_{\theta\theta}^I(\phi_2) r^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_2) r^{\lambda_{II}-1}. \end{aligned} \quad (42)$$

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho) G_{rr\theta}(r, \alpha, \rho, \phi_1) + B_\theta^1(\rho) G_{\theta r\theta}(r, \alpha, \rho, \phi_1) d\rho + \dots \right. \\ & \int_0^{l_2} B_r^2(\rho) G_{rr\theta}(r, \alpha, \rho, \phi_2) + B_\theta^2(\rho) G_{\theta r\theta}(r, \alpha, \rho, \phi_2) d\rho + \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, \alpha, \rho, \alpha) + \dots \\ & \left. B_\theta^\oplus(\rho) G_{\theta r\theta}(r, \alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, \alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, \alpha, \rho, -\alpha) d\rho \right] = 0. \end{aligned} \quad (43)$$

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho) G_{r\theta\theta}(r, \alpha, \rho, \phi_1) + B_\theta^1(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, \phi_1) d\rho + \dots \right. \\ & \int_0^{l_2} B_r^2(\rho) G_{r\theta\theta}(r, \alpha, \rho, \phi_2) + B_\theta^2(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, \phi_2) d\rho + \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, \alpha, \rho, \alpha) + \dots \\ & \left. B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, \alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, \alpha, \rho, -\alpha) d\rho \right] = 0. \end{aligned} \quad (44)$$

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho) G_{rr\theta}(r, -\alpha, \rho, \phi_1) + B_\theta^1(\rho) G_{\theta r\theta}(r, -\alpha, \rho, \phi_1) d\rho + \dots \right. \\ & \int_0^{l_2} B_r^2(\rho) G_{rr\theta}(r, -\alpha, \rho, \phi_2) + B_\theta^2(\rho) G_{\theta r\theta}(r, -\alpha, \rho, \phi_2) d\rho + \int_0^\infty B_r^\oplus(\rho) G_{rr\theta}(r, -\alpha, \rho, \alpha) + \dots \\ & \left. B_\theta^\oplus(\rho) G_{\theta r\theta}(r, -\alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{rr\theta}(r, -\alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta r\theta}(r, -\alpha, \rho, -\alpha) d\rho \right] = 0, \end{aligned} \quad (45)$$

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa+1)} \left[\int_0^{l_1} B_r^1(\rho) G_{r\theta\theta}(r, -\alpha, \rho, \phi_1) + B_\theta^1(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, \phi_1) d\rho + \dots \right. \\ & \int_0^{l_2} B_r^2(\rho) G_{r\theta\theta}(r, -\alpha, \rho, \phi_2) + B_\theta^2(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, \phi_2) d\rho + \int_0^\infty B_r^\oplus(\rho) G_{r\theta\theta}(r, -\alpha, \rho, \alpha) + \dots \\ & \left. B_\theta^\oplus(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, \alpha) + B_r^\ominus(\rho) G_{r\theta\theta}(r, -\alpha, \rho, -\alpha) + B_\theta^\ominus(\rho) G_{\theta\theta\theta}(r, -\alpha, \rho, -\alpha) d\rho \right] = 0. \end{aligned} \quad (46)$$

Once again we will discretise these equations using the Gauss-Chebyshev quadratures. This is done in exactly the same way as for the single crack problem, the only difference being the introduction of the factor $l_R := \frac{l_1}{l_2}$. So, as before the points must be transformed in two different ways. The free surface points use a Mellin transform, given by $r = l \log \frac{2}{1-t}$, and $\xi = l \log \frac{2}{1-s}$. Whereas the crack points use a finite transform given by $r = \frac{2v}{l_1} - 1$, and $\rho = \frac{2u}{l_1} - 1$. This therefore allows us to discretise our eight integral equations into eight sums, given by:

$$\begin{aligned} & \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{rr\theta}(t, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta r\theta}(t, \phi_1, s_i, \phi_1) + \dots \right. \\ & W(s_i) \varphi_r^2(s_i) G_{rr\theta}(t, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta r\theta}(t, \phi_1, l_R s_i, \phi_2) + \dots \\ & \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{rr\theta}(t, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta r\theta}(t, \phi_1, u_i, \alpha) + \dots \\ & \left. X(u_i) \varphi_r^\ominus(u_i) G_{rr\theta}(t, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta r\theta}(t, \phi_1, u_i, -\alpha) \right] = \dots \\ & - K_I f_{r\theta}^I(\phi_1) t^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1) t^{\lambda_{II}-1}, \end{aligned} \quad (47)$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{r\theta\theta}(t, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta\theta\theta}(t, \phi_1, s_i, \phi_1) + \dots \right. \\
& W(s_i) \varphi_r^2(s_i) G_{r\theta\theta}(t, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta\theta\theta}(t, \phi_1, l_R s_i, \phi_2) + \dots \\
& \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{r\theta\theta}(t, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta\theta\theta}(t, \phi_1, u_i, \alpha) + \dots \\
& \left. X(u_i) \varphi_r^\ominus(u_i) G_{r\theta\theta}(t, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta\theta\theta}(t, \phi_1, u_i, -\alpha) \right] = \dots \\
& - K_I f_{\theta\theta}^I(\phi_1) t^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_1) t^{\lambda_{II}-1}, \tag{48}
\end{aligned}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{rr\theta}(l_R t, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta r\theta}(l_R t, \phi_1, s_i, \phi_1) + \dots \right. \\
& W(s_i) \varphi_r^2(s_i) G_{rr\theta}(l_R t, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta r\theta}(l_R t, \phi_1, l_R s_i, \phi_2) + \dots \\
& \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{rr\theta}(l_R t, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta r\theta}(l_R t, \phi_1, u_i, \alpha) + \dots \\
& \left. X(u_i) \varphi_r^\ominus(u_i) G_{rr\theta}(l_R t, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta r\theta}(l_R t, \phi_1, u_i, -\alpha) \right] = \dots \\
& - K_I f_{r\theta}^I(\phi_1) (l_R t)^{\lambda_I-1} - K_{II} f_{r\theta}^{II}(\phi_1) (l_R t)^{\lambda_{II}-1}, \tag{49}
\end{aligned}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{r\theta\theta}(l_R t, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta\theta\theta}(l_R t, \phi_1, s_i, \phi_1) + \dots \right. \\
& W(s_i) \varphi_r^2(s_i) G_{r\theta\theta}(l_R t, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta\theta\theta}(l_R t, \phi_1, l_R s_i, \phi_2) + \dots \\
& \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{r\theta\theta}(l_R t, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta\theta\theta}(l_R t, \phi_1, u_i, \alpha) + \dots \\
& \left. X(u_i) \varphi_r^\ominus(u_i) G_{r\theta\theta}(l_R t, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta\theta\theta}(l_R t, \phi_1, u_i, -\alpha) \right] = \dots \\
& - K_I f_{\theta\theta}^I(\phi_1) (l_R t)^{\lambda_I-1} - K_{II} f_{\theta\theta}^{II}(\phi_1) (l_R t)^{\lambda_{II}-1}, \tag{50}
\end{aligned}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{rr\theta}(v, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta r\theta}(v, \phi_1, s_i, \phi_1) + \dots \right. \\
& W(s_i) \varphi_r^2(s_i) G_{rr\theta}(v, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta r\theta}(v, \phi_1, l_R s_i, \phi_2) + \dots \\
& \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{rr\theta}(v, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta r\theta}(v, \phi_1, u_i, \alpha) + \dots \\
& \left. X(u_i) \varphi_r^\ominus(u_i) G_{rr\theta}(v, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta r\theta}(v, \phi_1, u_i, -\alpha) \right] = 0, \tag{51}
\end{aligned}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{r\theta\theta}(v, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta\theta\theta}(v, \phi_1, s_i, \phi_1) + \dots \right. \\
& W(s_i) \varphi_r^2(s_i) G_{r\theta\theta}(v, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta\theta\theta}(v, \phi_1, l_R s_i, \phi_2) + \dots \\
& \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{r\theta\theta}(v, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta\theta\theta}(v, \phi_1, u_i, \alpha) + \dots \\
& \left. X(u_i) \varphi_r^\ominus(u_i) G_{r\theta\theta}(v, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta\theta\theta}(v, \phi_1, u_i, -\alpha) \right] = 0, \tag{52}
\end{aligned}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{rr\theta}(v, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta r\theta}(v, \phi_1, s_i, \phi_1) + \dots \right. \\
& W(s_i) \varphi_r^2(s_i) G_{rr\theta}(v, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta r\theta}(v, \phi_1, l_R s_i, \phi_2) + \dots
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{rr\theta}(v, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta r\theta}(v, \phi_1, u_i, \alpha) + \dots \\
& \left. X(u_i) \varphi_r^\ominus(u_i) G_{rr\theta}(v, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta r\theta}(v, \phi_1, u_i, -\alpha) \right] = 0, \tag{53}
\end{aligned}$$

$$\begin{aligned}
& \frac{2\mu}{\pi(\kappa+1)} \left[\sum_{i=1}^n W(s_i) \varphi_r^1(s_i) G_{r\theta\theta}(v, \phi_1, s_i, \phi_1) + W(s_i) \varphi_\theta^1(s_i) G_{\theta\theta\theta}(v, \phi_1, s_i, \phi_1) + \dots \right. \\
& W(s_i) \varphi_r^2(s_i) G_{r\theta\theta}(v, \phi_1, l_R s_i, \phi_2) + W(s_i) \varphi_\theta^2(s_i) G_{\theta\theta\theta}(v, \phi_1, l_R s_i, \phi_2) + \dots \\
& \sum_{i=1}^N X(u_i) \varphi_r^\oplus(u_i) G_{r\theta\theta}(v, \phi_1, u_i, \alpha) + X(u_i) \varphi_\theta^\oplus(u_i) G_{\theta\theta\theta}(v, \phi_1, u_i, \alpha) + \dots \\
& \left. X(u_i) \varphi_r^\ominus(u_i) G_{r\theta\theta}(v, \phi_1, u_i, -\alpha) + X(u_i) \varphi_\theta^\ominus(u_i) G_{\theta\theta\theta}(v, \phi_1, u_i, -\alpha) \right] = 0.
\end{aligned}$$

8.2 Conclusions

The paper has shown that this new method is an efficient way of solving the crack problems, reducing the computational cost and converging very well. The results closely match those found with the Hecker and Romanov method, so we can be confident, both in the original results, and that the new method is working as expected.

The results found for the slip problem are not a surprise; given the convergence issues found with the Hecker and Romanov method, described in section IV-A, it is not a surprise that this new method experiences similar problems. However, despite not being able to verify this because of computational limitations, it is clear that the solution is converging, if not yet converged.

The simplicity which this method buys by using the infinite plane kernels may not be obvious, as the number of equations involved in any single problem increases, and there are multiple quadratures to consider, but the ability to use only one set of kernels has obvious advantages, and the set of equations now fully specifies the problem, rather than requiring further calculation.

Chapter 9

Conclusions

In this thesis, the behaviour of complete contacts has been investigated by considering their edge behaviour using asymptotic methods. These contacts are very difficult to model in full, and any such models are impossible to generalise, hence the use of asymptotic methods. As the state of stress at the edge of contact is singular, the edges of complete contacts are often the areas of most interest. The use of asymptotic methods allows us to investigate these areas, but caution must be exercised and the limitations of the asymptotic approach understood. These include, but are not limited to, the approximation inherent within an asymptote, and the bound that exists on the validity of this approximation.

The motivation for this work arises from the fact that complete contacts are often found in assemblies, when burrs have not been removed or if some kind of damage has occurred. This means that the damage caused by the formation of such a contact, and its behaviour under remote loading, could be important in determining the life of a component or assembly. To this end, in this thesis, the development of slip zones and edge cracks at the edge of complete contacts, both of which can lead to significant damage or fracture, have been examined.

Chapter 1 discusses different types of contacts, and the corresponding behaviour. Chapter 2 explains how the state of stress is modelled, and how Bueckner's principle can be used to model more accurately the state of stress. Chapter 3 applies Bueckner's principle to complete contacts: limitations of different approaches to modelling complete contacts are investigated. Finally, focus is shifted to the edge behaviour of this class of contacts and the results found are discussed.

Chapter 4 focuses on behaviour at the contact edge. This is done considering the remote loading and the contact geometry. Cases in which separation will occur are easily identified, and conditions are laid out to prevent separation. Conditions are also laid out which can determine whether or not slip will occur, and where the slip will originate from, at the same time ruling out the possibility of slip 'islands.' Finally, a bound on the asymptotic solution is quantified and precludes certain behaviour from consideration which occurs beyond the range of validity of the asymptote.

Being able to predict the expected behaviour of the contact, methods to model the actual contact behaviour must now be considered. Chapters 5 and 8 describe two different approaches to the problem of modifying the state of stress using dislocations. Chapter 5 is based on the Hecker and Romanov approach, and builds on the work of Churchman to describe the stress caused by a dislocation in a semi-infinite reentrant wedge. Chapter 8 describes a new approach, which aims to tackle the problem at hand directly, enforcing free surface effects as conditions in the same way as the conditions of the problem. Both approaches have been used to tackle various problems, and both appear to work well. Each method has its own merits, and each presents issues; the sequential method of chapter 5 is more reliably converges, whereas the simultaneous method of chapter 8 is

computationally cheaper.

Having these dislocation methods and the expected behaviour in hand, the true behaviour of the contact can then be modelled. Chapter 6 considers the presence of edge cracks. The crack tip stress intensity factors have been calibrated to the wedge stress intensity factors using a single matrix, with contact length dependence incorporated. Each such matrix is dependent on the contact and crack angles. The result of this work is that the problem may be generalised to consider any angle combination. Consequently, the effect of the form and combination of loading on the behaviour of the crack and the contact can be explored

Similarly, chapter 6 also discussed the two crack problem, in which a pair of cracks is present at the contact edge. As with the single crack problem the problem has been generalised to consider multiple different angles, and the calibration matrix once again includes a dependence on crack lengths. The interaction of the cracks complicates the problem somewhat, but the effects of different combinations of loading on each of the respective cracks has been considered. Furthermore, in some cases, loading which causes one crack to open will close the other and vice versa.

Finally, chapter 7 examined the presence of a region of partial slip at the edge of contact. This required an entirely new inversion method, using different types of Gauss Chebyshev quadrature in order to determine the slip zone size. Consequently extensive work was undertaken to ensure the validity of the new method and verify the size of slip zone and the slip displacements found. Using this new inversion method, it has also been possible to correct the previous assertion that the normalised slip zone size is invariant with coefficient of friction, and a dependence has been found, a larger coefficient of friction leading to slightly larger

slip zones once normalised with an initial guess, which does itself depend on the coefficient of friction, although this dependence is weak. Similarly the effect of varying the contact angle has been explored and it is found that a larger contact angle also tends to lead towards a larger normalised slip zone size. In all cases, the peak slip displacements are at the edge of the contact.

In summary, this thesis set out to understand better and explain the edge behaviour of complete contacts, using asymptotic methods. This thesis has classified and predicted the type of behaviour and then modelled, explained and understood the effects of the presence of (multiple) edge cracks and a region of partial slip. Finally, let us consider some other projects which this work leads to.

9.1 Future work

Having understood the behaviour of a crack and of a slip region, it may seem logical to combine these, and examine the problem of a zone of partial slip in the presence of an edge crack. To investigate this problem, consider the formation of a slip zone in the presence of an existing crack, to avoid considerations of crack nucleation. As such, it is useful to consider the state of stress along the intended contact interface in the presence of a crack.

This problem has been considered for the incomplete contact, which was discussed earlier when considering the inversion method in chapter 7. Cwiekala and Hills [32] found an asymptotic description of the contact edge was sufficient to approximate the value of the crack tip stress intensity factors for cracks beyond the contact edge, and the same authors later found a similar result [61] for cracks whose root is within the slip zone.

Beginning by modelling the presence of a crack and observing the stress

along the contact interface, figure 9.1 shows the state of stress along the contact interface for four different contact angles. These results present some immediate challenges to modelling this class of problem. The first two cases, of $\alpha = 0.75\pi$ and $\alpha = 0.8\pi$ both have an apparent lift off point either within or very close to the edge of the apparent slip region. Given the multiplier found in chapter 7 both of these are likely to experience lift off within the true slip region. The third case is most promising as it does not exhibit lift off, it does however imply inwards slip, and the normal traction has at least two turning points, both of these factors being contrary to assumptions made when modelling slip previously. The last case shows corner lift off, which requires a different approach to deal with separation.

Considering only frictional slip, it should be noted that all the loading described has occurred purely ‘in plane’, that is that no load is applied in and out of the two dimensional representation used throughout. Within the plane framework it is possible to apply such loads, and the asymptotic representation of the Williams solution can be extended to consider such loads.

Another problem which arises is the question of quadrature choice. In chapter 7 it was shown that the choice of end point behaviour does not significantly affect the results obtained. This has been shown by simply trying multiple options and comparing the results. However, it may be worthwhile investigating the underlying quadratures and determining a rigorous method for showing the effect of the choice of end point behaviour.

Another problem worthy of consideration is the so called convection problem. This arises when slip occurs in a three dimensional contact. The Poisson effect means that a component of slip will occur in the orthogonal direction. This effectively changes the slip direction, and the magnitude of the effect is dependent

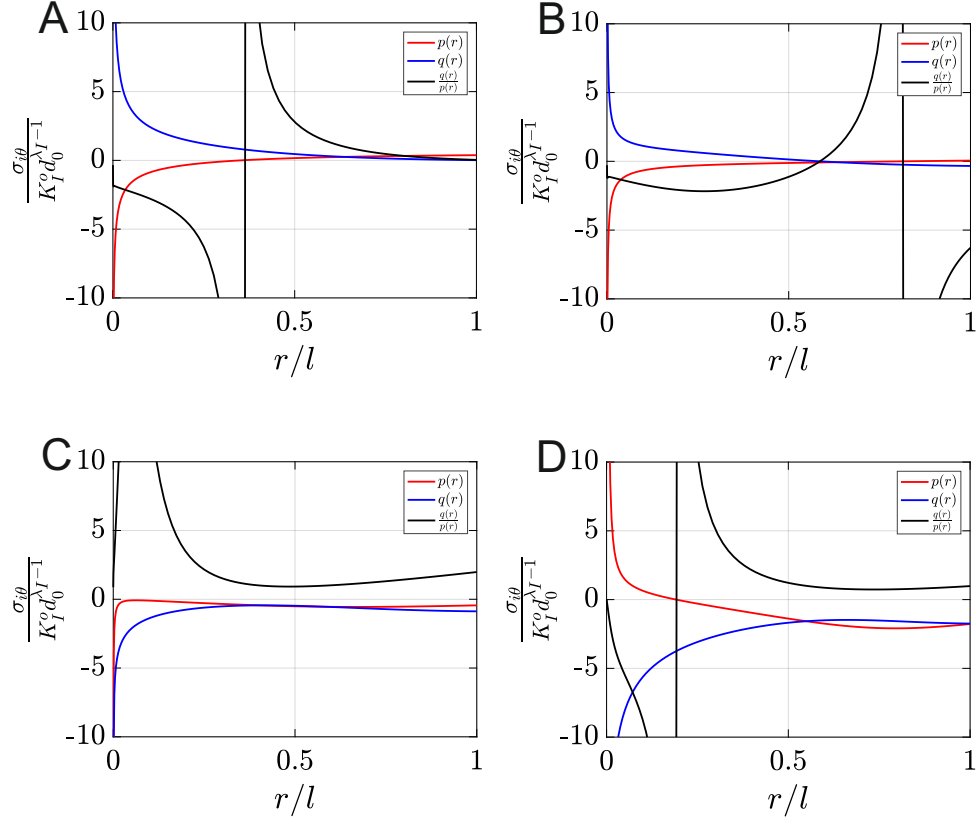


Figure 9.1: Plot of the normal and shear tractions, and the traction ratio on the contact interface of sharp edged contacts with a crack of length l present into the half-plane, the total internal angles being $A = 0.75\pi$, $B = 0.8\pi$, $C = 0.85\pi$, $D = 0.9\pi$.

on the position within the slip zone. The effect is quite small, but could be significant, for example, it may cause a slight local misalignment of features within experimental or practical applications. The size of the effect and the effect on slip displacement is an open question, both for the incomplete and complete contacts. Some preparatory work on these problems, and the anti plane slip problem described above has been done and is described in appendix C.

A useful generalisation of the crack problem discussed in chapter 6 would be the development of a solution for offset cracks. These are short, straight cracks

which do not develop at the contact edge, but at a finite distance from the edge of contact. Such cracks are of interest as one of the key constraints of the existing solution is that the crack must occur in a specific location. One of the difficulties faced when tackling such a problem is to ensure that the crack remains within the range of validity of the asymptote. This problem has been tackled in the case of an incomplete contact by Cwiekala and Hills [32].

Another extension of the crack problem would be to consider different crack paths, such as kinked or branched cracks. These can occur due to the microstructure of the materials or locations of defects. Such cracks are more likely to occur than a pair of cracks, although again it will be necessary to ensure that the analysis occurs within the range of the asymptotic method.

Another possible extension is to consider a three dimensional problem. Barnett proposed a solution [62, 63] which allows for a surface within a three dimensional body to be cleared of tractions, using triangular loops of dislocations. In combination with a development of the Williams solution which allows for conditions other than plane stress or plane strain, this would allow three dimensional problems of complete contacts of finite depths to be considered, and for slip and/or cracks to be modelled.

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Appendix A

Half-plane kernel functions

The stress kernel functions for a dislocation are presented here, having been set out by Hills et al [36, 40].

$$G_{xxx}(x, y, c) = y \left(-\frac{2}{r_1^2} - \frac{4(x-c)^2}{r_1^4} + \frac{2}{r_2^2} + \frac{4(x+c)^2}{r_2^4} - \frac{8c(x+c)}{r_2^4} + \dots \right. \\ \left. \frac{32c(x+c)^3}{r_2^6} + \frac{8c^2}{r_2^4} - \frac{32c^2(x+c)^2}{r_2^6} \right) \quad (\text{A.1})$$

$$G_{xxy}(x, y, c) = -\frac{2(x-c)}{r_1^2} + \frac{4(x-c)^3}{r_1^4} + \frac{2(x+c)}{r_2^2} - \frac{4(x+c)^3}{r_2^4} - \frac{4c}{r_2^2} + \dots \\ \frac{32c(x+c)^2}{r_2^4} - \frac{32c(x+c)^4}{r_2^6} - \frac{24c^2(x+c)}{r_2^4} + \frac{32c^2(x+c)^3}{r_2^6} \quad (\text{A.2})$$

$$G_{xyy}(x, y, c) = y \left(-\frac{2}{r_1^2} + \frac{4(x-c)^2}{r_1^4} + \frac{2}{r_2^2} - \frac{4(x+c)^2}{r_2^4} + \frac{24c(x+c)}{r_2^4} - \dots \right. \\ \left. \frac{32c(x+c)^3}{r_2^6} - \frac{8c^2}{r_2^4} + \frac{32c^2(x+c)^2}{r_2^6} \right) \quad (\text{A.3})$$

$$G_{yxx}(x, y, c) = -\frac{2(x-c)}{r_1^2} + \frac{4(x-c)^3}{r_1^4} + \frac{2(x+c)}{r_2^2} - \frac{4(x+c)^3}{r_2^4} - \frac{4c}{r_2^2} - \dots \\ \frac{16c(x+c)^2}{r_2^4} + \frac{32c(x+c)^4}{r_2^6} + \frac{24c^2(x+c)}{r_2^4} - \frac{32c^2(x+c)^3}{r_2^6} \quad (\text{A.4})$$

$$G_{yxy}(x, y, c) = y \left(-\frac{2}{r_1^2} + \frac{4(x-c)^2}{r_1^4} + \frac{2}{r_2^2} - \frac{4(x+c)^2}{r_2^4} - \frac{8c(x+c)}{r_2^4} + \dots \right. \\ \left. \frac{32c(x+c)^3}{r_2^6} + \frac{8c^2}{r_2^4} - \frac{32c^2(x+c)^2}{r_2^6} \right) \quad (\text{A.5})$$

$$G_{yxx}(x, y, c) = \frac{6(x-c)}{r_1^2} - \frac{4(x-c)^3}{r_1^4} - \frac{6(x+c)}{r_2^2} + \frac{4(x+c)^3}{r_2^4} - \frac{4c}{r_2^2} + \dots \\ \frac{32c(x+c)^2}{r_2^4} - \frac{32c(x+c)^4}{r_2^6} - \frac{24c^2(x+c)}{r_2^4} + \frac{32c^2(x+c)^3}{r_2^6}, \quad (\text{A.6})$$

where

$$r_1^2 = (x-c)^2 + y^2 \quad (\text{A.7})$$

$$r_2^2 = (x+c)^2 + y^2. \quad (\text{A.8})$$

Appendix B

Three quarter plane kernels

These are the fitted corrective functions for the influence of a dislocation in a three quarter plane, as found by Churchman et al [48] and Qiu et al [49].

$$F_{xxy}(x, 0, \xi, 0) = \frac{1}{\left(\frac{x}{\xi}\right)^{1-\lambda_I} + \left(\frac{x}{\xi}\right)} \left(-0.3650 - 0.1416 \left(\frac{x}{x+\xi} \right) - \dots \right. \\ \left. 3.942 \left(\frac{x}{x+\xi} \right)^2 + 3.450 \left(\frac{x}{x+\xi} \right)^3 \right) \quad (\text{B.1})$$

$$F_{xyy}(x, 0, \xi, 0) = \frac{1}{\left(\frac{x}{\xi}\right)^{1-\lambda_I} + \left(\frac{x}{\xi}\right)^{1.45}} \left(-0.8416 + 3.602 \left(\frac{x}{x+\xi} \right) - \dots \right. \\ \left. 3.146 \left(\frac{x}{x+\xi} \right)^2 + 0.6288 \left(\frac{x}{x+\xi} \right)^3 \right) \quad (\text{B.2})$$

$$F_{yxy}(x, 0, \xi, 0) = \frac{1}{\left(\frac{x}{\xi}\right)^{1-\lambda_I} + \left(\frac{x}{\xi}\right)^{1.45}} \left(-0.2726 - 1.016 \left(\frac{x}{x+\xi} \right) + \dots \right. \\ \left. 2.333 \left(\frac{x}{x+\xi} \right)^2 - 0.3687 \left(\frac{x}{x+\xi} \right)^3 \right) \quad (\text{B.3})$$

$$F_{yyy}(x, 0, \xi, 0) = \frac{1}{\left(\frac{x}{\xi}\right)^{1-\lambda_I} + \left(\frac{x}{\xi}\right)} \left(-0.3650 - 0.1416 \left(\frac{x}{x+\xi} \right) - \dots \right)$$

$$3.942\left(\frac{x}{x+\xi}\right)^2 + 3.450\left(\frac{x}{x+\xi}\right)^3 \Bigg) \quad (\text{B.4})$$

$$F_{zyz}(x, 0, \xi, 0) = \frac{1}{\left(\frac{x}{\xi}\right)^{\frac{1}{3}} + \left(\frac{x}{\xi}\right)} \left(-0.9106 + 0.46\left(\frac{x}{x+\xi}\right) - \dots \right. \\ \left. 2.055\left(\frac{x}{x+\xi}\right)^2 + 1.545\left(\frac{x}{x+\xi}\right)^3 \right) \quad (\text{B.5})$$

Appendix C

Anti plane slip and convection

Thus far, the problem of a complete contact of arbitrary angle has been considered, under loading that causes the formation of a slip zone or the development of a crack. All of this loading however, has been ‘in plane.’ This means that the loads are applied parallel to the plane that the contact is viewed in. Now consider what happens if the contact is not loaded in precisely this way.

Such loading is immediately split into the in plane and anti plane components. This will allow us to consider the effects of the in plane and anti plane loadings separately. Plenty of previous consideration has been given to the in plane loading in chapters 6 and 7 and by Qiu [64], so consider instead the anti plane loading.

Begin by noting several key factors: the first is that the Williams solution prohibits gradients of stress in the z -direction, so the anti plane stresses must be constant through the material. Hence, this analysis is invalid in the locality of a vertex. Second, the Williams solution also supposes that in plane loading only affects in plane stresses and vice versa. Hence, the anti plane stresses are only influenced by the anti plane loading, and the anti plane loading only affects anti plane stresses. In finite bodies this will not be the case because of the Poisson

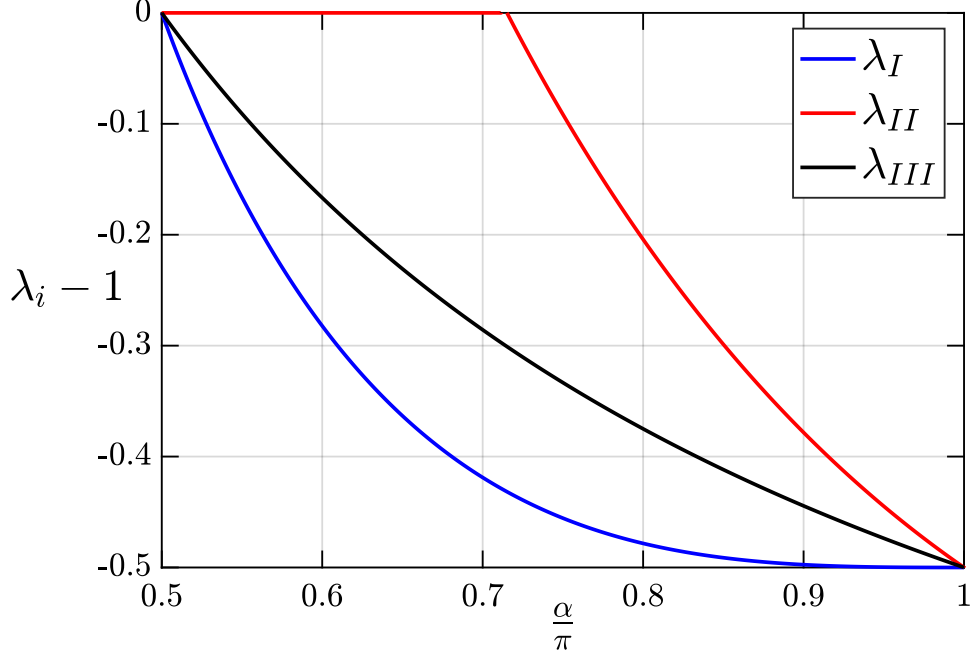


Figure C.1: Variation of the eigenvalues of the Williams solution

effect, but we will only consider infinite bodies.

C.1 Williams solution

Section 3.1 has already covered the classical Williams solution for in plane loading. However, the same formulation may be employed, and is detailed in [65, 43, 66]. The method is not recounted here, but the results are fairly well known.

The mode III eigenvalue is trivial compared to the mode I and mode II, and is given by

$$\lambda_{III} = \frac{\pi}{2\alpha}, \quad (\text{C.1})$$

where α is again the wedge half-angle. Figure C.1 shows how the three eigenvalues of the Williams solution vary as the wedge angle α changes.

The eigenvectors are similarly straightforward, with the two anti plane eigenvectors being given by:

$$f_{zr}^{III}(\phi) = -\sin(\lambda_{III}\phi), \quad (\text{C.2})$$

$$f_{z\theta}^{III}(\phi) = \cos(\lambda_{III}\phi). \quad (\text{C.3})$$

So, the stresses are given by

$$\sigma_{zr}(\rho, \phi) = K_{III} f_{zr}^{III}(\phi) \rho^{\lambda_{III}-1}, \quad (\text{C.4})$$

$$\sigma_{z\theta}(\rho, \phi) = K_{III} f_{z\theta}^{III}(\phi) \rho^{\lambda_{III}-1}. \quad (\text{C.5})$$

C.2 Corner slip

Considering the problem of slip at the corner of the contact; for the in plane loading only problem, based on the geometry of the contact alone, it can be determined whether or not the corner of the contact will slip [56].

Recall that, near this corner, the mode I term dominates the mode II term, and so it is appropriate to take a single term expansion for $\sigma_{\theta\theta}$, therefore the ratio is given by

$$\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}} = \frac{K_{III} f_{z\theta}^{III}(\phi)}{K_I f_{\theta\theta}^I(\phi)} r^{\lambda_{III}-\lambda_I}, \quad (\text{C.6})$$

in its simplest form. Note that the r terms do not cancel, so at the corner, as $r \rightarrow 0$, we find

$$\lim_{r \rightarrow 0} \frac{\sigma_{z\theta}}{\sigma_{\theta\theta}} = 0,$$

as $\lambda_I > \lambda_{III}$, for $\frac{\pi}{2} < \alpha < \pi$. Hence, under anti plane only loading, ie. with no in plane shear $\sigma_{r\theta}(0, \psi) = 0$, the corner will never slip. In practice, it is not possible

to have $\sigma_{r\theta}(0, \phi) = 0$ and $\sigma_{\theta\theta}(0, \phi) \neq 0$, as the ratio

$$\frac{\sigma_{r\theta}(0, \phi)}{\sigma_{\theta\theta}(0, \phi)} = \frac{f_{r\theta}^I(\phi)}{f_{\theta\theta}^I(\phi)} \neq 0.$$

So when anti plane loading is applied, the corner will slip provided that the corner slips in plane, ie. provided that $\frac{f_{r\theta}^I(\phi)}{f_{\theta\theta}^I(\phi)} < -f_c$, the coefficient of friction. Conversely, if this is not the case and the in plane shear traction does not cause slip at the corner, the corner will stick under any anti plane loading. This is a result of the equivalent shear traction under mixed loading being given by $\sqrt{\sigma_{r\theta}(r, \phi)^2 + \sigma_{z\theta}(r, \phi)^2} = \sqrt{(K_I f_{r\theta}^I(\phi) r^{\lambda_I-1})^2 + (K_{III} f_{z\theta}^{III}(\phi) r^{\lambda_{III}-1})^2}$, as $r \rightarrow 0$, this reduces simply to $K_I f_{r\theta}^I(\phi) r^{\lambda_I-1} = \sigma_{r\theta}$, the in plane shear, as $\lambda_I < \lambda_{III}$, for $\frac{\pi}{2} < \alpha < \pi$. So the in plane shear dominates at the corner.

C.3 Turning point

As for the purely in plane loading, it is again necessary to consider the possibility of islands of anti plane slip, in from the corner, but possibly entirely within the asymptote. For now, let us work with the assumption that $\sigma_{r\theta} = 0$, so the shear traction is just given by $\sigma_{z\theta}$.

For such an island to exist, a turning point in the ratio $\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}}$, must exist. This would be indicated by a zero in the gradient, which is given by

$$\frac{d}{dr} \left(\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}} \right) = \frac{K_{III} f_{z\theta}^{III} r^{\lambda_{III}-1} (K_I f_{\theta\theta}^I r^{\lambda_I} (\lambda_{III} - \lambda_I) + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}} (\lambda_{III} - \lambda_{II}))}{(K_I f_{\theta\theta}^I r^{\lambda_I} + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}})^2}. \quad (\text{C.7})$$

The zeros of equation C.7 are given by either

$$K_{III}f_{z\theta}^{III}r^{\lambda_{III}-1} = \sigma_{z\theta} = 0 \quad \text{or} \quad (C.8)$$

$$K_I f_{\theta\theta}^I r^{\lambda_I} (\lambda_{III} - \lambda_I) + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}} (\lambda_{III} - \lambda_{II}) = 0.$$

A further restriction on the existence of such an island is that it be a true turning point, not an inflexion, hence a further constraint is

$$\begin{aligned} \frac{d^2}{dr^2} \left(\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}} \right) &= \sigma_{z\theta} \left(\frac{2(K_I f_{\theta\theta}^I r^{\lambda_I} (\lambda_I - \lambda_{II}))^2}{K_I f_{\theta\theta}^I r^{\lambda_I} + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}}} \dots \right. \\ &\quad \left. - K_I f_{\theta\theta}^I r^{\lambda_I} (\lambda_I^2 + 2\lambda_{II}^2 + \lambda_{III} - \lambda_{III}^2 + \lambda_I(2\lambda_{III} - 4\lambda_{II} - 1)) + \dots \right. \\ &\quad \left. K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}} (\lambda_{II} - \lambda_{III})(1 + \lambda_{II} - \lambda_{III}) \right) \neq 0. \end{aligned} \quad (C.9)$$

The solution $\sigma_{z\theta} = 0$, is therefore not a true turning point. For such an island to appear, it is necessary that

$$\begin{aligned} K_I f_{\theta\theta}^I r^{\lambda_I} (\lambda_{III} - \lambda_I) + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}} (\lambda_{III} - \lambda_{II}) &= 0 \implies \\ \lambda_{III} &= \frac{K_I f_{\theta\theta}^I \lambda_I + K_{II} f_{\theta\theta}^{II} \lambda_{II} r^{\lambda_{II} - \lambda_I}}{K_I f_{\theta\theta}^I + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II} - \lambda_I}}; \quad (C.10) \\ \left(\frac{2(K_I f_{\theta\theta}^I r^{\lambda_I} (\lambda_I - \lambda_{II}))^2}{K_I f_{\theta\theta}^I r^{\lambda_I} + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}}} + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}} (\lambda_{II} - \lambda_{III})(1 + \lambda_{II} - \lambda_{III}) \dots \right. \\ &\quad \left. - K_I f_{\theta\theta}^I r^{\lambda_I} (\lambda_I^2 + 2\lambda_{II}^2 + \lambda_{III} - \lambda_{III}^2 + \lambda_I(2\lambda_{III} - 4\lambda_{II} - 1)) \right) \neq 0. \end{aligned}$$

When the equation for λ_{III} is substituted into condition C.10, the left-hand side collapses to zero, meaning that these conditions can never be simultaneously satisfied. Hence, no islands of purely anti plane slip can exist.

C.4 Classification of slip behaviour

The possibility of slip islands now being discounted, it is necessary to determine where slip will originate from. It has been shown that slip will never occur at the corner unless there is also some in plane slip. Hence, in all cases, if $\sigma_{r\theta} = 0$, anti plane slip must begin within the body. The absence of a turning point suggests that this originates beyond the range of the asymptotic solution.

Now suppose that a very small amount of in plane stress is present, just at the corner. Consequently, consider the equivalent shear traction, however, letting $\sigma_{r\theta}$ be negligible in size everywhere compared to $\sigma_{z\theta}$, this can be thought of as pure anti plane loading for the moment. These assumptions mean that the value of the ratio $\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}}$ has to be finite at the corner, so the important factor to determine where slip begins as the coefficient of friction is reduced is the sign of the gradient of $\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}}$.

Considering $\text{sign}\left(\frac{d}{dr}\left(\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}}\right)\right)$, it can be seen that

$$\text{sign}\left(\frac{d}{dr}\left(\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}}\right)\right) = \text{sign}(\sigma_{z\theta})\text{sign}(K_I f_{\theta\theta}^I r^{\lambda_I}(\lambda_{III} - \lambda_I) + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}}(\lambda_{III} - \lambda_{II})).$$

By plotting $f_{z\theta}^{III}$, it can be seen that $\text{sign}(\sigma_{z\theta}) = \text{sign}(K_{III})$.

Recall that $\lambda_I < \lambda_{II}$, so taking the limit $r \rightarrow 0$, there is always a point where the mode I term dominates. Assuming that near the corner $\text{sign}(K_I f_{\theta\theta}^I r^{\lambda_I}(\lambda_{III} - \lambda_I) + K_{II} f_{\theta\theta}^{II} r^{\lambda_{II}}(\lambda_{III} - \lambda_{II})) = \text{sign}(K_I f_{\theta\theta}^I r^{\lambda_I}(\lambda_{III} - \lambda_I))$. Note that $\text{sign}(f_{\theta\theta}^I) = \text{sign}(r^{\lambda_I}) = \text{sign}(\lambda_{III} - \lambda_I) = 1$, so $\text{sign}\left(\frac{d}{dr}\left(\frac{\sigma_{z\theta}}{\sigma_{\theta\theta}}\right)\right) = \text{sign}(K_I)\text{sign}(K_{III})$.

Hence for this psuedo anti plane loading, an equivalent table of behaviour to that found in chapter 4 for the purely in plane case, may be found.

			K_I	
ine			Positive	Negative
ine	Sign of K_i			
ine K_{III}	Positive	Separation and entire asymptote slip		Slip starts at corner
ine	Negative	Separation and small slip region		Slip from within
ine				

Table C.1: Matrix of inferred result from stress intensity factor signs

C.5 Dislocations in the wedge

These are the conditions required for anti plane slip to occur, but in order to model this phenomenon, the dislocations already developed in chapter 5 to model the required in plane slip are required. Furthermore a similar formulation for screw dislocations to model the anti plane slip will also be required.

To start with, it will be necessary to calculate the kernel for a screw dislocation in polar coordinates. The kernels for the screw dislocation in Cartesian coordinates are well known and are given by:

$$G_{zz}(x, y, x_d, y_d) = -\frac{(y - y_d)}{(x - x_d)^2 + (y - y_d)^2},$$

$$G_{yz}(x, y, x_d, y_d) = \frac{(x - x_d)}{(x - x_d)^2 + (y - y_d)^2}.$$

The kernels for the $z\theta$ and zr stress components are needed, so a rotation of the stress components must be performed. The stress is rotated through the angle $\theta - \phi$, using the substitutions

$$\begin{aligned} x &= r \cos(\theta - \phi) - \rho, \\ y &= r \sin(\theta - \phi). \end{aligned} \tag{C.11}$$

Therefore the kernels are given by:

$$\begin{aligned}
G_{zz\theta}(r, \theta, \rho, \phi) &= G_{zyz}(r, \theta, \rho, \phi)\cos(\theta - \phi) - G_{zxx}(r, \theta, \rho, \phi)\sin(\theta - \phi), \\
G_{zz\theta}(r, \theta, \rho, \phi) &= \frac{(r\cos(\theta - \phi) - \rho)}{(r\cos(\theta - \phi) - \rho)^2 + (r\sin(\theta - \phi))^2}\cos(\theta - \phi) - \dots \\
&\quad \frac{-r\sin(\theta - \phi)}{(r\cos(\theta - \phi) - \rho)^2 + (r\sin(\theta - \phi))^2}\sin(\theta - \phi) \quad \text{so,} \\
G_{zz\theta}(r, \theta, \rho, \phi) &= \frac{r - \rho\cos(\theta - \phi)}{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}, \tag{C.12}
\end{aligned}$$

$$\begin{aligned}
G_{zzr}(r, \theta, \rho, \phi) &= G_{zxx}(r, \theta, \rho, \phi)\cos(\theta - \phi) + G_{zyz}(r, \theta, \rho, \phi)\sin(\theta - \phi), \\
G_{zzr}(r, \theta, \rho, \phi) &= \frac{-r\sin(\theta - \phi)}{(r\cos(\theta - \phi) - \rho)^2 + (r\sin(\theta - \phi))^2}\cos(\theta - \phi) + \dots \\
&\quad \frac{(r\cos(\theta - \phi) - \rho)}{(r\cos(\theta - \phi) - \rho)^2 + (r\sin(\theta - \phi))^2}\sin(\theta - \phi) \quad \text{so,} \\
G_{zzr}(r, \theta, \rho, \phi) &= \frac{-\rho\sin(\theta - \phi)}{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}. \tag{C.13}
\end{aligned}$$

So, as with the in plane loading, place an object screw dislocation in an infinite prism of a semi-infinite wedge of plane angle 2α , place this dislocation at the point (ρ, ϕ) , with Burgers vector b_z . Then it is necessary to place distributions of screw dislocations along the free surfaces at angles $\phi = \pm\alpha$. Note it is not necessary to employ any glide or climb dislocations in this formulation as the screw dislocations do not induce any in plane stresses.

Let the distribution of the dislocations on the free surface with angle $\phi = \alpha$ have distribution denoted by $B_z^\oplus(\xi)$, and similarly the distribution on the other free surface be denoted by $B_z^\ominus(\xi)$. Then the integral equations governing the problem

are given by:

$$\frac{\mu}{2\pi} \int_0^\infty G_{zz\theta}(r, \alpha, \xi, \alpha) B_z^\oplus(\xi) + G_{zz\theta}(r, \alpha, \xi, -\alpha) B_z^\ominus(\xi) d\xi = b_z G_{zz\theta}(r, \alpha, \rho, \phi) \quad (\text{C.14})$$

$$\frac{\mu}{2\pi} \int_0^\infty G_{zzr}(r, \alpha, \xi, \alpha) B_z^\oplus(\xi) + G_{zzr}(r, \alpha, \xi, -\alpha) B_z^\ominus(\xi) d\xi = b_z G_{zzr}(r, \alpha, \rho, \phi) \quad (\text{C.15})$$

$$\frac{\mu}{2\pi} \int_0^\infty G_{zz\theta}(r, -\alpha, \xi, \alpha) B_z^\oplus(\xi) + G_{zz\theta}(r, -\alpha, \xi, -\alpha) B_z^\ominus(\xi) d\xi = b_z G_{zz\theta}(r, -\alpha, \rho, \phi) \quad (\text{C.16})$$

$$\frac{\mu}{2\pi} \int_0^\infty G_{zzr}(r, -\alpha, \xi, \alpha) B_z^\oplus(\xi) + G_{zzr}(r, -\alpha, \xi, -\alpha) B_z^\ominus(\xi) d\xi = b_z G_{zzr}(r, -\alpha, \rho, \phi). \quad (\text{C.17})$$

These integral equations may be solved using the same method as described in chapter 5. The resulting influence functions can be used to describe the corrected tractions along the interface. The corrected anti plane tractions along the interface are given by:

$$\sigma_{zr}(\rho, \phi) = K_{III} f_{zr}^{III}(\phi) \rho^{\lambda_{III}-1} + \frac{\mu}{2\pi} \int_0^d [F_{zzr}(\rho, \phi, \xi, \phi) + G_{zzr}(\rho, \phi, \xi, \phi)] B_z(\xi) d\xi, \quad (\text{C.18})$$

$$\sigma_{z\theta}(\rho, \phi) = K_{III} f_{z\theta}^{III}(\phi) \rho^{\lambda_{III}-1} + \frac{\mu}{2\pi} \int_0^d [F_{zzr}(\rho, \phi, \xi, \phi) + G_{zzr}(\rho, \phi, \xi, \phi)] B_z(\xi) d\xi, \quad (\text{C.19})$$

where B_z is the distribution of dislocation along the slip interface and d is the slip stick boundary.

C.6 The slip problem

Using the corrected tractions given above, the problem of a block, which must first be brought into contact with a non-zero pressure P , and is then loaded by shear in any direction, which we reduce to plane(Q_x) and anti plane(Q_z) components, may be considered. To do this, two equations must be satisfied simultaneously

$$f_c^2 \sigma_{\theta\theta}^2(r) = \sigma_{r\theta}^2(r) + \sigma_{z\theta}^2(r) \quad (\text{C.20})$$

$$\frac{\sigma_{z\theta}(r)}{\sigma_{r\theta}(r)} = -\frac{\dot{u}_z(r)}{\dot{u}_r(r)}, \quad (\text{C.21})$$

where u_i is the i direction of the surface displacement vector, and the dot refers to a time derivative of this. The relevant displacements are given by

$$u_r(r) = \int_r^d B_r(\xi) d\xi \quad (\text{C.22})$$

$$u_z(r) = \int_r^d B_z(\xi) d\xi. \quad (\text{C.23})$$

Now define

$$\gamma(r) = \frac{\dot{u}_z(r)}{\dot{u}_r(r)}. \quad (\text{C.24})$$

Combining and rearranging equations C.24 and C.20 gives

$$\sigma_{r\theta}(r) = \mp \frac{f_c \sigma_{\theta\theta}(r)}{\sqrt{1 + \gamma^2(r)}}, \quad (\text{C.25})$$

$$\sigma_{z\theta}(r) = \mp \frac{f \sigma_{\theta\theta}(r) \gamma(r)}{\sqrt{1 + \gamma^2(r)}}, \quad (\text{C.26})$$

$$\sigma_{z\theta}(r) + \gamma(r) \sigma_{r\theta}(r) = 0. \quad (\text{C.27})$$

Finally, using the corrected tractions, as defined by equation C.19 and in chapter 6, the problem becomes

$$\begin{aligned}
& -K_I r^{\lambda_I-1} \left(f_{r\theta}^I(\phi) \mp \frac{f_c f_{\theta\theta}^I(\phi)}{\sqrt{1+\gamma^2(r)}} \right) - K_{II} r^{\lambda_{II}-1} \left(f_{r\theta}^{II}(\phi) \mp \frac{f_c f_{\theta\theta}^{II}(\phi)}{\sqrt{1+\gamma^2(r)}} \right) = \dots \\
& \frac{2\mu}{\pi(\kappa+1)} \int_0^d B_r(\xi) \left[F_{rr\theta}(r, \phi, \rho, \phi) + G_{rr\theta}(r, \phi, \rho, \phi) \dots \right. \\
& \quad \mp \frac{f_c(F_{r\theta\theta}(r, \phi, \rho, \phi) + G_{r\theta\theta}(r, \phi, \rho, \phi))}{\sqrt{1+\gamma^2(r)}} \left. \right] + B_\theta(\xi) \left[F_{\theta r\theta}(r, \phi, \rho, \phi) + \dots \right. \\
& \quad \left. G_{\theta r\theta}(r, \phi, \rho, \phi) + \frac{f_c(F_{\theta\theta\theta}(r, \phi, \rho, \phi) + G_{\theta\theta\theta}(r, \phi, \rho, \phi))}{\sqrt{1+\gamma^2(r)}} \right] d\xi, \tag{C.28}
\end{aligned}$$

$$\begin{aligned}
& -K_{III} f_{z\theta}^{III}(\phi) \pm \frac{K_I r^{\lambda_I-1} f_c f_{\theta\theta}^I(\phi)}{\sqrt{1+\gamma^2(r)}} \pm \frac{K_{II} r^{\lambda_{II}-1} f_c f_{\theta\theta}^{II}(\phi)}{\sqrt{1+\gamma^2(r)}} = \frac{2\mu}{\pi(\kappa+1)} \dots \\
& \int_0^d B_z(\xi) [F_{zz\theta}(r, \phi, \rho, \phi) + G_{zz\theta}(r, \phi, \rho, \phi)] \mp \dots \\
& B_r(\xi) \left[\frac{f_c(F_{r\theta\theta}(r, \phi, \rho, \phi) + G_{r\theta\theta}(r, \phi, \rho, \phi))}{\sqrt{1+\gamma^2(r)}} \right] \mp \dots \\
& B_\theta(\xi) \left[\frac{f_c(F_{\theta\theta\theta}(r, \phi, \rho, \phi) + G_{\theta\theta\theta}(r, \phi, \rho, \phi))}{\sqrt{1+\gamma^2(r)}} \right] d\xi, \tag{C.29}
\end{aligned}$$

$$\begin{aligned}
& K_{III} f_{zr}^{III}(\phi) \rho^{\lambda_{III}-1} + \gamma(r) K_I f_{r\theta}^I(\phi) r^{\lambda_I-1} + \gamma(r) K_{II} f_{r\theta}^{II}(\phi) r^{\lambda_{II}-1} + \dots \\
& \frac{\mu}{2\pi} \int_0^d B_z(\xi) [F_{zzr}(\rho, \phi, \xi, \phi) + G_{zzr}(\rho, \phi, \xi, \phi)] + \frac{(\kappa+1)\gamma(r)}{4} \dots \\
& \left[B_r(\xi) F_{rr\theta}(r, \phi, \rho, \phi) + G_{rr\theta}(r, \phi, \rho, \phi) + \dots \right. \\
& \quad \left. B_\theta(\xi) F_{\theta r\theta}(r, \phi, \rho, \phi) + G_{\theta r\theta}(r, \phi, \rho, \phi) \right] d\xi = 0. \tag{C.30}
\end{aligned}$$

Once again, the solution of these equations requires a priori knowledge of the size of the slip zone d and the convection effect $\gamma(r)$. As such, initial guesses for these values are needed and then the results must be checked using known conditions.

Consider first the slip stick boundary d . The in plane and anti plane slip stick boundaries are given by finding d_r and d_z such that $\frac{\bar{\sigma}_{r\theta}(d_r, \phi)}{\bar{\sigma}_{\theta\theta}(d_r, \rho)} = -f_c$, and $\frac{\bar{\sigma}_{z\theta}(d_z, \phi)}{\bar{\sigma}_{\theta\theta}(d_z, \rho)} = -f_c$, where $\bar{\sigma}_{ij}$ is the Williams solution tractions only. The initial guess can then be found by $c_0 = \sqrt{d_r^2 + d_z^2}$. The choice of initial guess for the convection effect may change based on experience, but one example might be $\gamma(r) = 1.01 * (r - d) * \arctan\left(\frac{K_{II}}{K_{III}}\right)$.

These integral equations can then be solved, using the inversion methods described in chapter 7. This is complicated by the minus plus signs, which originate from taking the square root of equation C.20. Choosing the correct branch must be done with care to ensure the accuracy of the solution.

Appendix D

Response of a mass-spring system subject to Coulomb damping

This appendix further extends and considers the work of Den Hartog [67]. The dynamic properties of this deceptively simple system are complex and interesting and the interrupted motion regime set out by Hong and Liu [68], and Papangelo and Ciavarella [69] warrants further discussion and investigation.

The continuous motion regimen of the solution is further investigated by Marino et al [70], but this appendix primarily considers the interrupted motion solution. A new method expanding on the ideas of Hundal [71] is presented which allows rapid analysis of the steady state solution to the problem.

Additionally, a branch of different solution behaviour has been identified in the parameter space. The reason for the existence of this branch is explained and what causes the creation or destruction of finite stops as we move in the parameter space.

This work has been published, and a pre print of the paper is reproduced below.

Riddoch, D. J., Cicirello, A. and Hills, D. A., *Response of a mass-spring system subject to Coulomb damping and harmonic base excitation*, International Journal of Solids and Structures 193 (2020), 527-534

Response of a mass-spring system subject to Coulomb damping and harmonic base excitation

Daniel J. Riddoch¹, Alice Cicirello¹, David A. Hills¹

Abstract—The periodic response of a mass-spring system is studied and the behaviour described, including the creation of finite dwell periods. A parameter space investigation is conducted and a branch of behaviour with the space is explained. A new solution method is proposed for direct calculation of the steady-state response without calculating a transient phase. A new approach to the display of the steady-state response is given. Finally, the system behaviour near its resonance is explored and new asymptotic solutions are proposed for the response amplitude near resonance. The accuracy of these solutions is then determined by comparison to the true solution.

I. INTRODUCTION

Mechanical assemblies include a large number of joints, many of them bolted. When external vibrational forces are applied these joints may slip and work is then done against friction, so that they provide damping. The primary immediate application we have in mind, here, is the frictional contact between a fan blade in a gas turbine located in a dovetail socket in the rotor, and where we know that, during run up of the turbine, the centrifugal force generated pulling the blade into its socket causes sliding, i.e. limiting friction is achieved at all points along the dovetail flanks; but the results to be discovered are very general. Classical viscous damping has been exhaustively studied and is fully understood. By contrast, the effect of frictional damping has been studied very little until recently, and the number of papers which have been published on the subject is very small.

We will consider, here, a simple system with a rigid mass, m , resting on a frictional surface and where the limiting frictional force is of magnitude F_f , which will often be the product μmg where μ is the coefficient of friction and g the acceleration due to gravity, but we maintain this notation, for the time being, as in some cases the frictional force may be externally imposed. We distinguish between three similar configurations, Figure 1. In Figure 1(a) we show the simplest arrangement where an external force $F(t)$, where t is time, is applied directly to the mass. Related problems in which the excitation is by imposed displacement, $Y(t)$, at the end of the spring are shown in Figure 1(b,c). In the first of these the surface on which the mass is resting is also referred to the position $Y(t)$, and in the second the frictional force is referred to stationary ground. In this paper the problem shown in Figure 1(c) is studied, and harmonic excitation is used throughout.

The seminal paper in this area is that of Den Hartog [1] from 1930. The key results of his study are establishing the

boundary, in the steady-state, between continuous motion and one where stick-slip motion occurs, and the magnitude of the response in the former case. More recently, a detailed numerical investigation was conducted by Hong and Liu [2] who considered and categorised the multi-stop behaviour of the system. Papangelo and Ciavarella [3] concentrated on low forcing frequency cases with a different number of stops. All three of these authors analysed the imposed force problem, as shown in Figure 1(a). Others have distinguished between the dynamic and static coefficients of friction [4], and lastly, in [5], a further configuration was looked at where the mass rests on a continuously moving belt, connected by a spring to a fixed wall. The case of a joined base-wall system, as shown in figure 1(b), has been investigated by [6] and an extension to Den Hartog's work is presented.

The paper is laid out as follows; in section II-A the results of the Den Hartog paper are discussed further and a new graphical procedure is introduced. In section III, the response of the system during stick slip motion is discussed and explained, this is extended to multiple stop cases in section III-A. A feature of the parameter space response is identified and discussed in section IV for the first time and the behaviour of the system near resonance is examined thoroughly in section VI. Some comments are made about energy dissipation and work done in section VII.

II. FORMULATION

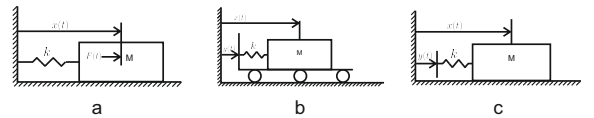


Fig. 1: Coulomb friction damper subject to a base excitation

Refer to Figure 1(c). The force in the spring, F_s , is given by

$$F_s = k(y - x),$$

and, if both the magnitude of this force is less than the limiting frictional force ($|F_s| < F_f$) and the mass is stationary ($\dot{x} = 0$) it will continue to be at rest. It will be in equilibrium and the spring force will be precisely opposed by the frictional force.

Figure 2 displays behaviour of the system in dimensionless displacement space & figure 2(a) shows the quasi static response. If the mass is positioned anywhere within the red lines it will remain stationary. If the amplitude of forcing is such that the limiting frictional force is reached, and it

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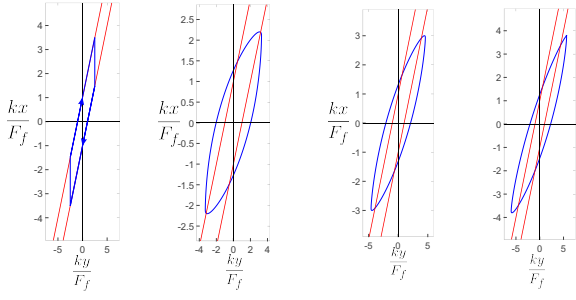


Fig. 2: Dimensionless plots of base input against response showing quasi-static(a) and dynamic response(b-d), including interrupted motion(b), out of phase continuous motion(c) and in phase continuous motion(d)

varies harmonically in time but at a very low frequency so that inertia forces are very small, the mass will track out a loop of the form shown in blue.

If either the spring force exceeds the limiting frictional force ($|F_s| > F_f$) or the mass is in motion ($|\dot{x}| > 0$), or, indeed, if both of these inequalities hold, the mass has a net force on it causing it to accelerate, and application of Newton's second Law of Motion gives

$$F_s - F_f \text{sgn}(\dot{x}) = m\ddot{x}$$

or

$$\ddot{x} + \omega_n^2(x(t) - y(t)) + \frac{F_f}{m} \text{sgn}(\dot{x}) = 0 \quad (1)$$

where $\text{sgn}(\cdot)$ is the signum function, and we can now see that the physical problem is characterised by two quantities, the natural frequency of the problem, $\omega_n^2 = \frac{k}{m}$, and the ratio of the frictional force to the inertial mass, $F_f/m \equiv p$, and which would be given by μg when the frictional force originates in the weight of the mass.

We now restrict our attention to harmonic forcing, and choose $y(t) = Y \cos(\omega_b t + \phi)$ where the 'phase shift', ϕ , is included here to simplify the ensuing algebra. This also sets the start of the measured time period as the maximum response amplitude of the system, when the system is in continuous motion. The general solution to differential equation (1) is given by

$$\frac{x(t)}{Y} = \begin{cases} \frac{A \cos(\omega_n t) + B \sin(\omega_n t)}{Y} + \frac{p}{Y \omega_n^2} + \frac{\cos(\omega_b t + \phi)}{1 - r^2} & \dot{x} > 0 \\ \frac{A \cos(\omega_n t) + B \sin(\omega_n t)}{Y} - \frac{p}{Y \omega_n^2} + \frac{\cos(\omega_b t + \phi)}{1 - r^2} & \dot{x} < 0 \end{cases} \quad (2)$$

where r is the dimensionless forcing frequency $r = \frac{\omega_b}{\omega_n}$, and $\frac{Y \omega_n^2}{p}$ represents the dimensionless magnitude of forcing. Also at this point, we define $T = \frac{2\pi}{\omega_b}$, the period of the steady-state motion.

At this point, it should be noted that equation 2 is the solution to equation 1, which is an equation of motion, and hence it is valid only when the mass is in motion. In section III we describe responses with finite periods of arrest, when the mass is stationary and we cannot apply these equations throughout. Hence, when the solution incorporates periods

of dwell, their beginning and end points, together with the preserved displacement of the mass, must be found.

A. Existing solution

The motion of the mass may be expected to display a transient phase, and then to settle down into a steady-state periodic motion, and it is this which is of primary interest here. The dimensionless amplitude of forcing is given by the group $\frac{Y \omega_n^2}{p}$ and we note that, if the mass is already stationary, it will remain so if this quantity is less than unity. We also state, without proof, that even if the mass is initially moving it will become stuck within a finite time when this holds true.

Although Den Hartog [1] studied a slightly different problem, figure 1(a), some of the results he found are relevant to the problem at hand. He used boundary conditions, described in appendix A to show that, when the amplitude of forcing is high enough the mass will be in continuous motion, the amplitude of which, X , is given by

$$\left(\frac{X}{Y}\right)^2 = \left(\frac{1}{1 - r^2}\right)^2 - \left(\frac{p \sin\left(\frac{\pi}{r}\right)}{Y r \omega_n^2 \left(1 + \cos\left(\frac{\pi}{r}\right)\right)}\right)^2. \quad (3)$$

The resulting motion is very complicated. It is periodic, and the corresponding angular frequency is ω_b , but within an individual cycle there is content at the natural angular frequency, ω_n , and it is not possible to provide a description of the waveform in closed form.

If the amplitude of the forcing function is gradually reduced, there comes a point at which the mass halts at some point during the cycle. The boundary between continuous and interrupted motion is found by enforcing the condition that the sign of the velocity must remain the same at all instants within any half cycle (and must change sign between consecutive half cycles) and applying symmetric, continuous boundary displacement conditions. The requirement that the velocity not change sign in any half cycle becomes

$$\left(\frac{kY}{F_f}\right)^2 \geq (1 - r^2)^2 \left(\frac{S^2}{r^2} + \frac{\sin^2\left(\frac{\pi}{r}\right)}{r^2 \left(1 + \cos\left(\frac{\pi}{r}\right)\right)^2} \right). \quad (4)$$

where S is the maximum value in the time interval $0 < t < \pi/\omega_b$ of

$$\frac{r \sin\left(\frac{\pi}{r}\right) + \frac{\sin\left(\frac{\pi}{r}\right)}{1 + \cos\left(\frac{\pi}{r}\right)} (\cos(\omega_b t) - \cos(\omega_n t))}{\sin(\omega_b t)}. \quad (5)$$

For most values of the forcing frequency it is not possible to simplify the condition embodied in equation (4,5), but, for the particular case when forcing is at the resonant frequency ($r = 1$), the right hand side of equation (4) becomes equal to $16/\pi^2$.

These results allow us to introduce figure 3, which we will refer back to several times throughout this paper. The results shown in figure 3 are applicable only for the specific value of μ of 0.3875. Figure 3(a) shows the number of stops per cycle of the steady state, and we have already defined the

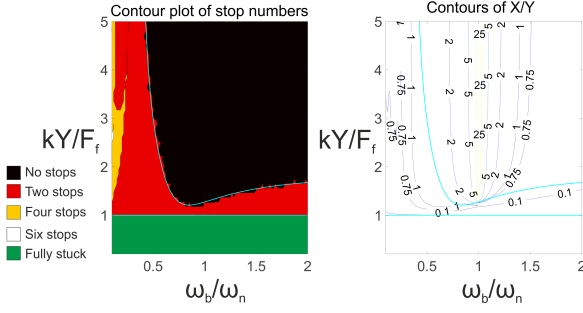


Fig. 3: The parameter space response of the system at the value $\mu = 0.3875$

boundaries between (i) permanent stick and motion, and (ii) the boundary between continuous motion and motion with at least one period of dwell. Figure 3(b) shows the amplitude of response in the steady state, normalised by the magnitude of excitation, $\frac{X}{Y}$. Its value has already been stated (equation 3).

The boundaries between fully stuck and interrupted motion solutions, and between interrupted and continuous motion which are shown here can be calculated directly for any value of the coefficient of friction. The former is given by $kY = pm$ and the latter by equation 4.

III. THE STOPPING PROBLEM

Recall that, in section II, we state that the equation of motion (equation 1) does not apply during dwell periods. Hence, a marching in time solution is initially sought to determine the system response in cases of finite dwell periods. Using this solution a transient phase is calculated and the steady-state response is reached subsequently. In cases away from the system resonance, this is achieved quite quickly. We have reviewed the existing steady-state solution, appropriate when the mass comes to rest only instantaneously at the extremes of displacement. We now describe the solution of the problem when there are finite periods of dwell. The principles adduced will generally apply when there is interrupted motion, but the method becomes unwieldy as the number of stops increases and it is, in any case, most instructive to look at the ‘two-stop’ case first. This applies at all frequencies beyond the natural frequency, and intermediate levels of forcing, see figure 3, and also at a wide range of amplitudes and lower frequencies, as shown in the red shaded region of the figure 3(a), where the mass exhibits two stops of finite duration per steady state cycle of the base motion. Starting from the point of a case with no stops of finite duration and now gradually increasing the coefficient of friction, we observe that two finite stops develop at the extremes of the base motion. At the beginning of these stop periods, the velocity reaches zero and the value of the force in the spring is not sufficient to cause sliding. So the mass becomes stuck, and the frictional force balances the force in the spring. At the end of these stop periods, the force in the spring reaches the limiting value of the friction as the spring is stretched or compressed further

by the continuing base motion. Note that, the reference point in time is changed here, from the peak of the response to the peak of the base motion, this is now carried throughout the paper from here. This has been done as there is no distinct peak of the response, as this is where the dwell periods are to be found.

The development of these two periods of finite dwell is best seen in cases near to the switching condition, which figure 4 is an example of. The top plot shows the force in the spring in pink, with the limiting value of the friction in black. The subsequent plots show the displacement, velocity and acceleration of the mass, normalised by the amplitude of the base displacement, velocity and acceleration respectively. Note the discontinuity in acceleration at the points where the velocity changes sign (and hence the sense of the spring force reverses).

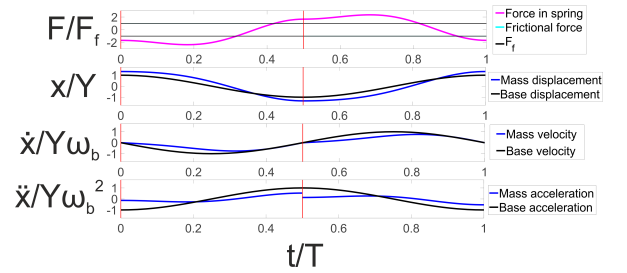


Fig. 4: A typical system in continuous motion

By contrast with this, we can see in figure 5 a typical two stop response. The two finite stop regions can be seen between the red vertical lines. The two finite dwell periods are where the extremes of motion were.

A set of boundary conditions defining the periodic

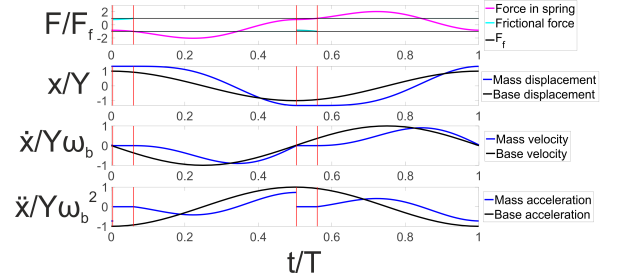


Fig. 5: A typical response of a two stop system

behaviour was developed. This is described in full and algebraically in appendix B. This method is much faster, for individual cases, than traditional time marching solutions. However, as the problem is underspecified by the equation of motion alone, the solvers are not reliable around the boundaries between continuous and interrupted motion, and motion with different numbers of finite dwell periods.

A. The multiple stop problem

As previously stated, the principles above can be extended to cases with multiple stops. In the ‘four-stop’ system there

are four finite dwell periods per cycle of the base motion in the steady state. Intermediate stops develop when the ratio between the driven and natural frequencies is low, which gives rise to multiple peaks of the force in the spring. These intermediate peaks and their corresponding troughs become larger, with either a reduction of the frequency ratio or an increase in the input amplitude. An example of one of these cases can be seen in figure 6.

The boundary conditions and solution method can be extended to cover problems with more than two stops, as described in appendix C, but the system remains underspecified and the method becomes unwieldy. This presents a problem exploring regions with a very low excitation frequency compared to the natural frequency, as these cases have lots of sub-resonances, giving rises to many stops per cycle. This region was explored in more detail by Papangelo and Ciavarella [3].

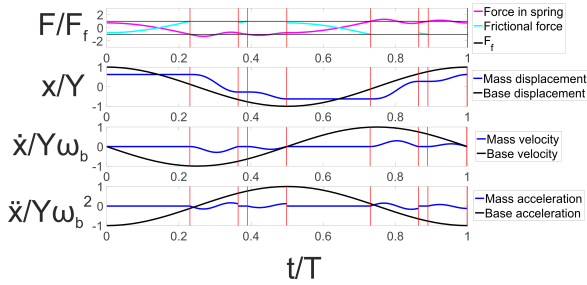


Fig. 6: A typical four stop system

IV. THE STOP BRANCH

Referring back to figure 3(a), it can be seen that there is an isolated ‘branch’ of input parameters which lead to more finite dwell periods than other cases around them. These solutions have $0.15 < \frac{\omega_b}{\omega_n} < 0.25$ and $2 < \frac{kY}{F_f} < 4.5$. We shall refer to this as the stop branch. We seek to explain the reasons that this branch exists and to do this, we use the four stop branch, shown in yellow on figure 3(a), as an illustrative case. A set of example plots is shown in figure 7, where (F) is a zoomed on part of figure 3(a), and figure 7 A-E represent various cases around the stop branch.

To simplify the explanation, we separate the changes into frequency ratio and input amplitude variations, and consider them separately. Initially, consider a constant frequency ratio, and different cases of input amplitude. It can be observed that increasing the amplitude directly increases the response amplitude. As such, in figure 7(D), the amplitude of the second (positive) peak in the force in the spring is not sufficient to cause sliding. This peak becomes large enough to cause sliding in (B), hence we get a second period of sliding, whereas the trough between the peaks is not small enough to cause stick in (A), so we do not get a period of stick.

Now consider, alternatively, the case of changing the frequency of excitation only. Increasing the driving frequency changes the ratio between the magnitude of the peaks described above. In the same way that changing the

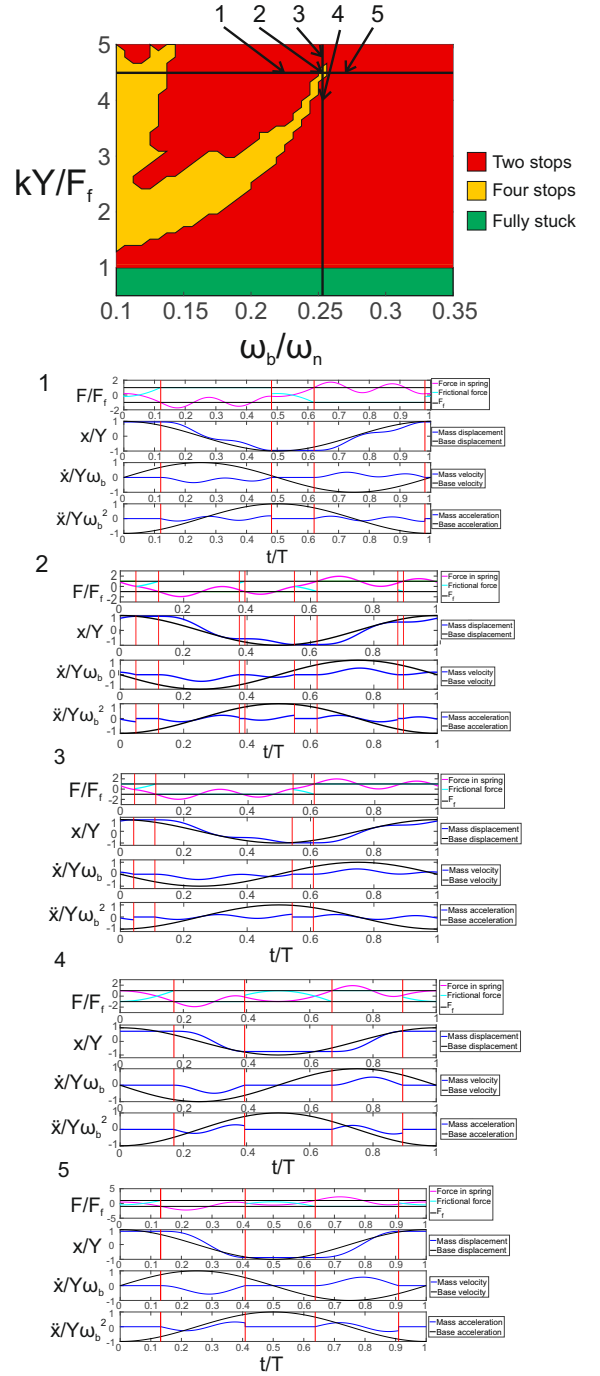


Fig. 7: Illustrative cases around the stop branch

magnitude of the peaks causes the restart to occur and then the trough to become sufficiently large as to prevent slip, changing the magnitude of the second peak has the same effect. As can be seen by comparing figure 7 (C), (B) and (E).

V. DISPLAY OF BEHAVIOUR OVER ONE CYCLE

We introduced a method of displaying the steady state behaviour with figure 2, for the quasi-static solution and cases of continuous motion. Applying this method to

dynamic cases in which the steady state solution has multiple periods of finite dwell per half-cycle, this display regime shows more information.

We compare the graphical procedure for cases with no finite stops, two and four finite stops per cycle in figure 8. It can be seen that the number of distinct sections in which the response is shown as a vertical corresponds to the number of finite stops per cycle of the steady state. This must be the case, as a vertical line corresponds to no change in x , in other words the mass is not moving, but a change in y . Further, the rounded corners of the case in figure 8(b) indicate that the mass was in motion at the extreme of the base motion, in other words, a phase shift was present. This is the case because the extreme values of x and y do not align; otherwise a ‘corner’ would be present.

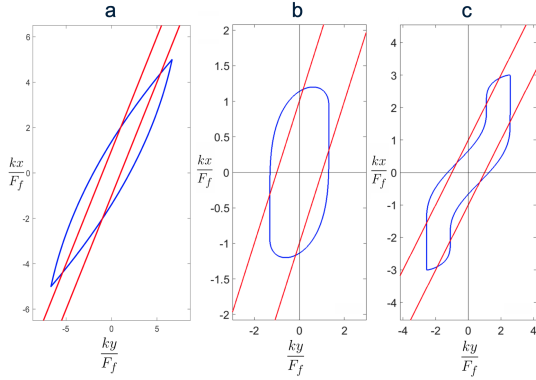


Fig. 8: Dimensionless plots of normalised input displacement against response in cases of continuous and interrupted motion

VI. BEHAVIOUR NEAR RESONANCE

When $\frac{\omega_b}{\omega_n} = 1$, the system is excited at its resonant frequency. Figure 3 shows that there are only three possible responses. Clearly, if $\frac{kY}{F_f} < 1$, the system is fully stuck in the steady state. If $\frac{kY}{F_f} > \frac{4}{\pi}$, then the simplification of equation 4 tells us that the system is a case of continuous motion. Finally, if $1 < \frac{kY}{F_f} < \frac{4}{\pi}$, then the mass must display stick-slip motion, which is, in fact, two-stop motion. The amplitude of this oscillation is shown in figure 9. Its limit at the onset of continuous motion being equal to the amplitude of the base motion. If the motion is continuous, it does not reach a limit cycle, so the amplitude grows without bound.

Using the equations set out in section II, an asymptotic solution is sought for the steady amplitude of the system near resonance, as the ratio of the driven to the natural frequency approaches unity:

$$X = \frac{Y \cos\left(\frac{T \arcsin\left(\frac{4F_f}{kY\pi}\right)}{2\pi}\right)}{1 - r^2} \quad (6)$$

The accuracy of the solution can be seen by means of a set of contours, shown in figure 10, which shows the absolute error of the asymptotic solution compared the full solution presented for no stop cases by Den Hartog.

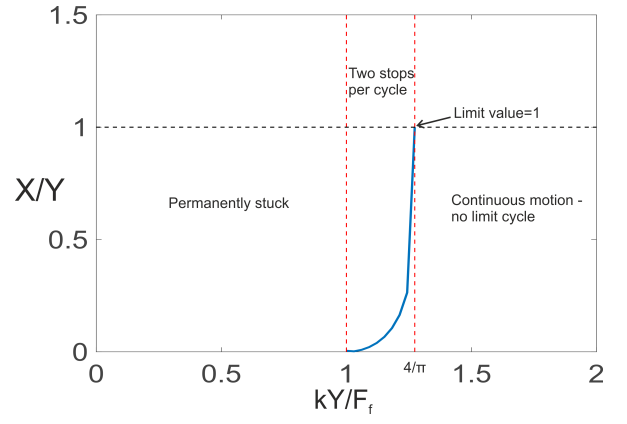


Fig. 9: Plot of the resonant amplitude of the system

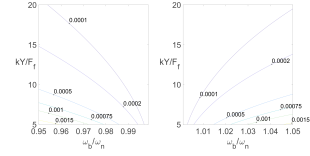


Fig. 10: Error contours in the asymptotic solution

VII. ENERGY BALANCE AND DAMPING

As the magnitude of the frictional force is constant when the block is sliding, it is clear that the work done per cycle, W , is directly proportional to amplitude of the response and is given by $W = 4F_f X$. Therefore the damping is dependent only on the amplitude of the motion. Neither the work done, nor the amount of damping is dependent on the number of stops in the system.

VIII. CONCLUDING REMARKS

We have described the behaviour of the system modelled throughout the parameter space. A new approach has been proposed to solve directly for the steady state behaviour of the system in regions of stick-slip motion. An explanation has been offered for the stop branch phenomena present in figure 3.

A new graphical procedure has been developed to easily identify features of the solution, including the number of stops, phase shift and relative velocity of the mass. The behaviour of the system at resonance has been explored and results of this investigation were presented, along with asymptotic solutions for the amplitude of the system near resonance.

Further, comments were made about the energy balance and damping properties of the system and their dependence on features of the response. Finally, an example of the frequency components of the inertial force was presented showing the disparity between this and the apparent period of the response.

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APPENDIX

The notation x_1 is used to denote motion with a negative velocity, x_2 has positive velocity. $x_{1,1}$ represents the first phase of motion with a negative velocity and so on. The unknowns used in section C are defined in figure 11. In each case the response is referred to a base motion cycle starting from its positive extreme.

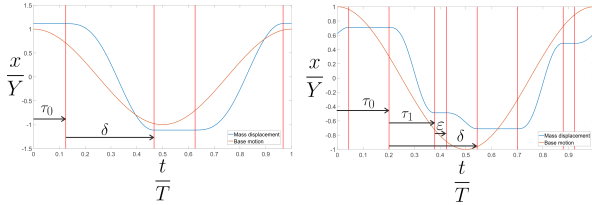


Fig. 11: Definitions of time unknowns in different stop regimes

A. No stops

Here we lay out algebraically the conditions determined by Den Hartog, applied to both half-cycles of the steady state (denoted by x_1 and x_2) which allow us to solve directly for the steady state behaviour

- 1) $x_1(\tau_0) = x_2(\phi)$
- 2) $x_1(\phi + \frac{T}{2}) = x_2(\phi + \frac{T}{2})$
- 3) $x_1(\phi) + x_1(\phi + \frac{T}{2}) = 0$
- 4) $x_2(\phi) + x_2(\phi + \frac{T}{2}) = 0$
- 5) $\dot{x}_1(\phi) = 0$
- 6) $\dot{x}_2(\phi) = 0$
- 7) $\dot{x}_1(\phi + \frac{T}{2}) = 0$
- 8) $\dot{x}_2(\phi + \frac{T}{2}) = 0$

B. Two stops

From the description of two stop systems, we are able to introduce boundary conditions constraining the two stop system. First we define two new unknowns. These allow us to replace ϕ from the original formulation as its meaning is no longer clear, as explained in section III. These two new unknowns τ_0 and δ are shown in figure 11 and formally defined as follows:

- τ_0 - the time from the peak of the base motion until the mass first slides, with negative velocity
- δ - the time across each half-cycle that the mass is in motion.

With these new unknowns and the equations of motion, which are still valid for each period where the mass is sliding, symmetry and continuity are used to determine conditions across the cycle.

Consider first displacement. This has to be continuous and across a single cycle of the steady-state, is assumed to be symmetrical with respect to the half-period. In other words, the amplitude of response is equal in both the positive and negative directions. Moving to velocity, we know that at the start and end of each motion period, the velocity is zero. Finally the force in the spring must be precisely equal to the limiting value of the friction at the start of the motion period, and hence the acceleration at this point must be zero. These statements are set out algebraically now.

- 1) $x_1(\tau_0) = x_2(\tau_0 + \delta + \frac{T}{2})$
- 2) $x_1(\tau_0 + \delta) = x_2(\tau_0 + \frac{T}{2})$
- 3) $x_1(\tau_0) + x_2(\tau_0 + \frac{T}{2}) = 0$
- 4) $x_1(\tau_0 + \delta) + x_2(\tau_0 + \delta + \frac{T}{2}) = 0$
- 5) $\dot{x}_1(\tau_0) = 0$
- 6) $\dot{x}_1(\tau_0 + \delta) = 0$
- 7) $\dot{x}_2(\tau_0 + \frac{T}{2}) = 0$
- 8) $\dot{x}_2(\tau_0 + \delta + \frac{T}{2}) = 0$
- 9) $\dot{x}_1(\tau_0) = 0$
- 10) $\ddot{x}_2(\tau_0 + \frac{T}{2}) = 0$
- 11) $F_{spring}(\tau_0) = -\mu W$
- 12) $F_{spring}(\tau_0 + \frac{T}{2}) = \mu W$

It should be noted, at this point, that these boundary conditions are insufficient to solve the problem analytically as Den Hartog did. A set of numerically stable solvers has been developed but these do require prior knowledge of the number of stops. This approach allows us to jump straight to the steady-state behaviour of the system.

C. Four stops

We can extend the two stop description to cover four stop cases. Again we must define more unknowns in time. We introduce τ_1 and ε :

- τ_1 is the time taken for the mass to stick again having first started to slide;
- ε is the duration of the intermediate stop.

Now we again apply continuity and symmetry as for the two-stop solution, and the following algebraic conditions result:

- 1) $x_{1,1}(\tau_0) = x_{2,2}(\tau_0 + \delta + \frac{T}{2})$
- 2) $x_{1,2}(\tau_0 + \delta) = x_{2,1}(\tau_0 + \frac{T}{2})$
- 3) $x_{1,1}(\tau_0 + \tau_1) = x_{1,2}(\tau_0 + \tau_1 + \varepsilon)$
- 4) $x_{2,1}(\tau_0 + \tau_1 + \frac{T}{2}) =$
 $x_{2,2}(\tau_0 + \tau_1 + \varepsilon + \frac{T}{2})$
- 5) $x_{1,1}(\tau_0) + x_{2,1}(\tau_0 + \frac{T}{2}) = 0$
- 6) $x_{1,2}(\tau_0 + \delta) +$
 $x_{2,2}(\tau_0 + \delta + \frac{T}{2}) = 0$
- 7) $x_{1,1}(\tau_0 + \tau_1) +$
 $x_{2,1}(\tau_0 + \tau_1 + \frac{T}{2}) = 0$
- 8) $\dot{x}_{1,1}(\tau_0) = 0$
- 9) $\dot{x}_{1,1}(\tau_0 + \tau_1) = 0$
- 10) $\dot{x}_{1,2}(\tau_0 + \tau_1 + \varepsilon) = 0$
- 11) $\dot{x}_{1,2}(\tau_0 + \delta) = 0$
- 12) $\dot{x}_{2,1}(\tau_0 + \frac{T}{2}) = 0$
- 13) $\dot{x}_{2,1}(\tau_0 + \tau_1 + \frac{T}{2}) = 0$
- 14) $\dot{x}_{2,2}(\tau_0 + \tau_1 + \varepsilon + \frac{T}{2}) = 0$
- 15) $\dot{x}_{2,2}(\tau_0 + \delta + \frac{T}{2}) = 0$
- 16) $\ddot{x}_{1,1}(\tau_0) = 0$
- 17) $\ddot{x}_{1,2}(\tau_0 + \tau_1 + \varepsilon) = 0$
- 18) $\ddot{x}_{2,1}(\tau_0 + \frac{T}{2}) = 0$
- 19) $\ddot{x}_{2,2}(\tau_0 + \tau_1 + \varepsilon + \frac{T}{2}) = 0$
- 20) $F_{spring}(\tau_0) = -\mu W$
- 21) $F_{spring}(\tau_0 + \tau_1 + \varepsilon) = -\mu W$
- 22) $F_{spring}(\tau_0 + \frac{T}{2}) = \mu W$
- 23) $F_{spring}(\tau_0 + \tau_1 + \varepsilon + \frac{T}{2}) = \mu W$

D. Six stop region

Figure 3 and the description of it mention a six stop region. Whilst present, this is a very small region and difficult to see. For that reason a zoomed in portion of that figure is reproduced here, which allows a clearer view of the six stop region.

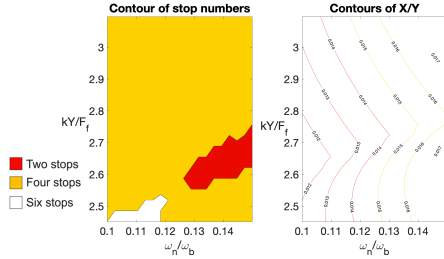


Fig. 12: The parameter space response of the system at the value $\mu = 0.3875$, focussed on the six stop region.

The kind of branching discussed for the four stop system can be seen here too, with a branch of six stop solutions, surrounded by four stop solutions. The inverse is also true, with a branch of two stop solutions being visible for similar reasons.

In addition to the objectives described previously, which are primarily focussed on the method for determining the steady state solution and the existence of the stop branch, several further conclusions have been drawn. Most prominently we have devised a new method for visualising the single cycle behaviour of the system in the steady state which allows for an easy visual recognition of the type of behaviour and the system behaviour at each end of the load cycle.

We have also explored the resonant and near resonant behaviour of the system. We have been able to show that, provided the magnitude of excitation is small, excitation at resonant frequencies will not necessarily cause a runaway situation, but may be controlled for a period of time. We have also developed an asymptotic solution for the amplitude of the motion near resonance in the steady state.