

Elliptic Calabi-Yau fivefolds and 2d (0,2) F-theory landscape

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ABSTRACT: In this paper, we initiate the study of the 2d F-theory landscape based on compact elliptic Calabi-Yau fivefolds. In particular, we determine the boundary models of the landscape using Calabi-Yau fivefolds with the largest known Hodge numbers $h^{1,1}$ and $h^{4,1}$. The former gives rise to the largest geometric gauge group in the currently known 2d (0,2) supergravity landscape, which is $E_8^{482\,632\,421} \times F_4^{3\,224\,195\,728} \times G_2^{11\,927\,989\,964} \times \text{SU}(2)^{25\,625\,222\,180}$. Besides that, we systematically study the hypersurfaces in weighted projective spaces with small degrees, and check the gravitational anomaly cancellation. Moreover, we also initiate the study of singular bases in 2d F-theory. We find that orbifold singularities on the base fourfold have non-zero contributions to the gravitational anomaly.

KEYWORDS: Differential and Algebraic Geometry, F-Theory

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1 Introduction

In the pursuit of the global set of consistent quantum gravity theories, it is very important to identify the boundaries of the string theory landscape, in order to compare them with the swampland bounds [1]. For example, one can ask the following question:

In a given space-time dimension and amount of supersymmetry, what is the maximal number of fields of a given type in a string compactification model?

For non-chiral theories with 16 supercharges in $d > 3$ space-time dimensions, the maximal rank of gauge group is given by $r_G = 26 - d$, and it was matched with the swampland bounds [2].

For theories with eight supercharges, such as 6d $(1, 0)$ supergravity, the currently known maximal number of tensor multiplet, $T = 193$, and the maximal rank of the gauge group, $r_G = 296$, are both given by F-theory on the elliptic Calabi-Yau threefold X_3 with maximal $h^{1,1}$ [3–6]:¹

$$(h^{1,1}, h^{2,1}) = (491, 11). \quad (1.1)$$

For 5d $\mathcal{N} = 1$ supergravity, the maximal number of vector multiplets is also realized on the same geometry, from the M-theory starting point. These bounds have not been proven as a swampland condition, despite of the presence of worldsheet CFT techniques in these cases [8–11].

For theories with four supercharges, such as 4d $\mathcal{N} = 1$ supergravity, the maximal rank of gauge group $r_G = 121\,328$ is given by F-theory on the elliptic Calabi-Yau fourfold X_4 with maximal known $h^{1,1}$ [3, 12]:

$$(h^{1,1}, h^{2,1}, h^{3,1}) = (303\,148, 0, 252). \quad (1.2)$$

The same model also leads to the largest number of axions

$$N(\text{axion}) = 181\,820. \quad (1.3)$$

On the other hand, F-theory on the mirror Calabi-Yau fourfold with the largest $h^{3,1}$ would lead to the largest number of complex structure moduli and number of flux vacua on a single geometry [13].

As a general pattern, the F-theory landscape seems to always provide the answer to the above question in even space-time dimensions. In particular, the point of interest is always the elliptic Calabi-Yau manifold with the largest Hodge numbers.

In this paper, we will extend this logic to the case of 2d $(0, 2)$ supergravity with two supercharges, which comes from F-theory on a compact elliptic Calabi-Yau fivefold [14, 15]. As another motivation, the study of $(0, 2)$ gauge theories in two dimensions is a rich subject by itself, see e. g. [16–19], and it is interesting to investigate the coupling of a supergravity sector.

In particular, we will study the details of the elliptic Calabi-Yau fivefolds with maximal $h^{1,1}$ or $h^{4,1}$. For the case of maximal $h^{1,1}$:

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}, h^{2,2}) = (247\,538\,602\,581, 0, 0, 151\,701, 758\,522), \quad (1.4)$$

and the 2d $(0, 2)$ theory has a geometric gauge group

$$G = E_8^{482\,632\,421} \times F_4^{3\,224\,195\,728} \times G_2^{11\,927\,989\,964} \times \text{SU}(2)^{25\,625\,222\,180}. \quad (1.5)$$

The total rank of gauge group is

$$r_G = 66\,239\,044\,388, \quad (1.6)$$

which is conjectured to be the largest in the whole 2d $(0, 2)$ landscape.

¹By “maximal” we meant the extremal Hodge numbers of Calabi-Yau manifolds as a hypersurface of weighted projective spaces, which appeared in the sequence (3.3) of [7]. These numbers represent the records among all the known compact (elliptic) Calabi-Yau manifolds, which are also conjectured to be the rigorous bound in full generality, see [6] for the CY3 case. We will also use this notion of “maximal” later on.

The construction of the corresponding fourfold base with $h^{1,1}(B_4) = 181\,299\,558\,192$ is similar to the 4d case [12]. We tune E_8 gauge groups on the toric divisors of a starting point toric fourfold, and then blow up all the non-minimal loci in codimension-two, three and four.

Besides this particular geometric model, we also present the first attempt of studying the set of elliptic Calabi-Yau fivefolds and the 2d F-theory geometric landscape. The constructions of Calabi-Yau fivefolds were explored in [20–22], but the elliptic fibration structures have not been discussed in the literature. Namely, we study the Calabi-Yau hypersurfaces of reflexive weighted projective spaces up to degree $d \leq 150$ that have an elliptic fibration structure. For example, the generic fibration over a “generalized Hirzebruch fourfold” is given by a Calabi-Yau hypersurface inside $\mathbb{P}^{1,1,1,1,n,2n+8,3n+12}$. We also find Calabi-Yau fivefolds with non-zero Hodge numbers $h^{2,1}$ and $h^{3,1}$. The ones with non-zero $h^{3,1}$ describes 2d (0,2) supergravity coupled to 2d Fermi multiplets. The full table of these geometries is listed in appendix B.

Finally, we checked the 2d gravitational anomaly cancellation conditions [15, 23] in several cases with or without non-Abelian gauge groups. More interestingly, we also analyzed cases with a singular base, and we found that these orbifold singularities also have a non-zero contribution to the gravitational anomaly.

The structure of this paper is as follows: in section 2, we briefly recap the formulation of 2d F-theory and the gravitational anomaly computation. In section 3, we present the detailed construction of the elliptic Calabi-Yau fivefolds with either largest $h^{1,1}$ or $h^{4,1}$. In section 4, we study the geometric structure of a number of other elliptic Calabi-Yau fivefolds. In section 5, we check gravitational anomaly cancellation, including the models with a singular base.

2 Mathematics and physics of 2d F-theory compactifications

In this section, we introduce the basics of globally consistent compactification of F-theory to 1+1 dimensions on compact elliptic Calabi-Yau fivefolds, including the geometric tools and the gravitational anomaly computation of the low energy effective theory. In section 2.1 we introduce compactification of F-theory on elliptic Calabi-Yau fivefolds with an emphasis on the computation of the massless spectrum of the low energy effective theory. In section 2.2 we discuss the derivation of gravitational anomaly of the 2d effective theory. The materials in section 2.1 and 2.2 are not new and are all covered in [14, 15, 23]. In section 2.3 we review the basic toric geometry tools that we will make use of to construct examples of elliptic Calabi-Yau fivefolds.

2.1 Basic setup of 2d F-theory

We consider compactification of F-theory on an elliptic Calabi-Yau fivefold X_5 whose low energy effective theory is a 2d $\mathcal{N} = (0, 2)$ supersymmetric field theory coupled to gravity. In general, an elliptic Calabi-Yau $(n + 1)$ -fold has the following form:

$$\begin{array}{ccc} \pi : \mathbb{E}_\tau & \rightarrow & Y_{n+1} \\ & & \downarrow \\ & & B_n \end{array} \quad (2.1)$$

Codimension	Physical data
1	Gauge groups
2	Matters in $\mathbf{R} \oplus \overline{\mathbf{R}}$ Bulk-surface matter couplings
3 4	Holomorphic matter couplings

Table 1. Singularities and the corresponding physical data of the low energy 2d $\mathcal{N} = (0, 2)$ field theory.

and we will mainly focus on the $n = 4$ cases. We further assume that the fibration has a zero section therefore it can be described by a Weierstrass model:

$$y^2 = x^3 + fxz^4 + gz^6, \quad (2.2)$$

where $f \in \mathcal{O}(-4K_B)$ and $g \in \mathcal{O}(-6K_B)$. Here K_B is the canonical bundle of the base fourfold B_4 . We will mainly working in the local chart where we can set $z = 1$. Singularities of the elliptic fibration at different codimensions of the base B_4 correspond to different physical contents and we list such correspondences in table 1.

For our purpose it is sufficient to discuss the codimension-1 and 2 singularities on B_4 as we will focus only on the gauge groups and matters in this paper. Codimension-1 singularities are characterized by the vanishing of the discriminant locus:

$$\Delta = 4f^3 + 27g^2.$$

In the IIB physics, the locus $\Delta = 0$ is wrapped by 7-branes, and the gauge group G_S along the codimension-1 locus S is determined by the order of vanishing of (f, g, Δ) along S . The matters are localized at codimension-2 locus of B_4 where the order of vanishing of (f, g, Δ) along S enhances. The matter representations can be determined following Katz-Vafa [24]. There is also bulk matter that is not localized as we will discuss later. For us it is important to know that with gauge invariant G_4 flux, the bulk matter transforms in the adjoint representation of the gauge group G_S and it will contribute to the anomaly.

Besides the 7-branes wrapping codimension-1 loci of B_4 , there will also be D3-branes wrapping codimension-2 loci of B_4 due to tadpole cancellation. The interplay between D3-brane sector and 7-brane sector will also contribute to gravitational anomaly in 2d.

Another indispensable ingredient in the F-theory compactification is the G_4 flux which must satisfy the following condition:

$$G_4 + \frac{1}{2}c_2(X_5) \in H^4(X_5, \mathbb{Z}) \cap H^{2,2}(X_5),$$

in order for the M-theory compactification on Y_5 to preserve two supercharges [22]. We will see that G_4 flux contributes to the gravitational anomaly from the 3-7 sector.

We will summarize some properties of the supermultiplets in the 2d $\mathcal{N} = (0, 2)$ field theory. They include vector multiplets with one negative chirality complex fermion, chiral multiplets with one positive chirality Weyl fermion, Fermi multiplets with one negative

chirality complex fermion and a single gravity multiplet with one positive chirality complex dilatino and one negative chirality gravitino. In 2d there are also tensor multiplets containing real axionic scalar fields arising from KK reduction of the F-theory 4-form field C_4 . The tensor multiplets will play an important role in the Green-Schwarz mechanism of anomaly cancellation as will be discussed in the next section.

2.2 Gravitational anomaly cancellation

In 2d the gravitational and gauge anomaly can be described by a gauge invariant polynomial of degree 2 in gauge field strength F and the curvature 2-form R :

$$I_4 = \sum_{\mathbf{R},s} n_s(\mathbf{R}) I_s(\mathbf{R}), \quad (2.3)$$

where $I_s(\mathbf{R})$ is the anomaly polynomial of a single spin s matter field in representation \mathbf{R} and $n_s(\mathbf{R})$ is the multiplicity of that matter field.

In general I_4 does not have to vanish in a consistent quantum field theory. A gauge variant Green-Schwarz counter-term at tree level can cancel I_4 if I_4 factorizes suitably. This is possible in 2d because of the existence of an axionic scalar field c^α that gives rise to a self-dual one-form $H^\alpha = dc^\alpha + \Theta_i^\alpha A^i$, such that:

$$g_{\alpha\beta} * H^\beta = \Omega_{\alpha\beta} H^\beta.$$

The gauge variant pseudo-action that contains c^α and H^α is:

$$S_{\text{GS}} = -\frac{1}{4} \int g_{\alpha\beta} H^\alpha \wedge * H^\beta - \frac{1}{2} \int \Omega_{\alpha\beta} c^\alpha \wedge X^\beta, \quad (2.4)$$

where $dH^\alpha = X^\alpha$ and $X^\alpha = \Theta_i^\alpha F^i$, F^i is the field strength of the abelian gauge group factor $U(1)_i$. The axionic symmetry of c^α is gauged by A^i with the following transformation rule:

$$\begin{aligned} A^i &\rightarrow A^i + d\lambda^i, \\ c^\alpha &\rightarrow c^\alpha - \Theta_i^\alpha \lambda^i. \end{aligned}$$

It is then easy to obtain the gauge variation of S_{GS} is:

$$\delta S_{\text{GS}} = \frac{1}{2} \int \Omega_{\alpha\beta} \Theta_i^\alpha \lambda^i X^\beta := 2\pi \int I_{2,\text{GS}}^{(1)}(\lambda). \quad (2.5)$$

Using the descent equations:

$$I_{4,\text{GS}} = dI_{3,\text{GS}}, \quad \delta_\lambda I_{4,\text{GS}} = dI_{2,\text{GS}}^{(1)}(\lambda),$$

we have:

$$I_{4,\text{GS}} = \frac{1}{4\pi} \Omega_{\alpha\beta} X^\alpha X^\beta = \frac{1}{4\pi} \Omega_{\alpha\beta} \Theta_i^\alpha \Theta_j^\beta F^i F^j. \quad (2.6)$$

We require:

$$I_4 + I_{4,\text{GS}} = 0. \quad (2.7)$$

It is easy to see that since $I_{4,\text{GS}}$ contains only the field strengths of abelian gauge groups, the cancellation is possible only if the gravitational and non-abelian gauge anomalies vanish

2d multiplet	Multiplicity
Chiral	$h^{2,1}(X_5) + h^{4,1}(X_5) - (-h^{1,1}(B_4) + h^{2,1}(B_4) - h^{3,1}(B_4)) - 1$
Fermi	$h^{2,1}(B_4) - h^{3,1}(B_4) + h^{3,1}(X_5)$
Tensor	$\tau(B_4)$
Gravity	1

Table 2. The 2d supermultiplets in the moduli and gravitational sector of F-theory compactification on X_5 .

by themselves and the abelian gauge anomalies factorize suitably. In this paper, we will denote by I_4 the gravitational anomaly of the low energy effective theory from a 2d F-theory construction, and we will check if $I_4 = 0$ for a series of examples.

For simplicity we first consider the gravitational sector of F-theory compactification on a smooth Calabi-Yau fivefold X_5 . Using the duality between F-theory and IIB orientifold we have the following spectrum in the moduli and gravitational sector [15] in table 2.

Here the signature $\tau(B_4)$ is given by

$$\tau(B_4) = 48 + 2h^{1,1}(B_4) + 2h^{3,1}(B_4) - 2h^{2,1}(B_4). \quad (2.8)$$

Summing up the contributions of chiral, Fermi and tensor multiplets (+1 for chiral multiplets and (-1) for Fermi and tensor multiplets) to the 2d anomaly polynomial we have:

$$\begin{aligned} I_{4,\text{moduli}} &= \frac{1}{24}p_1(T)(-\tau(B_4) + \chi_1(X_5) - 2\chi_1(B_4)) \\ &\equiv \frac{1}{24}p_1(T)\mathcal{A}_{\text{grav|mod}}. \end{aligned} \quad (2.9)$$

where we have used the relation $h^{1,1}(X_5) = 1 + h^{1,1}(B_4)$ and the definition of arithmetic genus:

$$\chi_q(V) = \sum_{p=1}^{\dim V} (-1)^p h^{p,q}(V). \quad (2.10)$$

The gravitational anomaly from the gravity multiplet is:

$$\begin{aligned} I_{4,\text{grav}} &= \frac{1}{24}p_1(T) \times 24 \\ &\equiv \frac{1}{24}p_1(T)\mathcal{A}_{\text{grav|uni}}. \end{aligned} \quad (2.11)$$

We then consider the spectrum of 3-7 sector when a D3 brane wraps genus g curve C in B_4 . The spectrum is summarized in the table 3.

Summing up the contributions from chiral and Fermi multiplets (note again they have opposite contributions), we have:

$$\begin{aligned} I_{4,3-7} &= \frac{1}{24}p_1(T)(-6c_1(B_4) \cdot C) \\ &\equiv \frac{1}{24}p_1(T)\mathcal{A}_{\text{grav|3-7}}. \end{aligned} \quad (2.12)$$

Multiplet	Multiplicity
Chiral	$h^0(C, N_{C/B_4}) + g - 1 + c_1(B_4) \cdot C$
Fermi	$h^0(C, N_{C/B_4}) + g - 1 + 7c_1(B_4) \cdot C$

Table 3. The 2d supermultiplets in the 3-7 sector.

The various arithmetic genus above can be computed via index theorem and we have:

$$\chi_1(B_4) = \frac{1}{180} \int_{B_4} (-31c_4 - 11c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4), \quad (2.13)$$

$$\chi_1(X_5) = \int_{B_4} \left(90c_1^4 + 3c_1^2c_2 - \frac{1}{2}c_1c_3 \right), \quad (2.14)$$

$$\tau(B_4) = \frac{1}{180} \int_{B_4} (12c_2^2 - 56c_1c_3 + 56c_4 - 4c_1^4 + 16c_1^2c_2). \quad (2.15)$$

Here c_i is the i^{th} Chern class of the base B_4 . For a smooth Calabi-Yau fivefold we have:

$$[C] = \frac{1}{24} \pi_* c_4(X_5) = 15c_1^3 + \frac{1}{2}c_1c_2. \quad (2.16)$$

Here $\pi : X_5 \rightarrow B_4$ is the fibration map, and π_* is the push forward map from X_5 to B_4 .

Summing up all the contributions we have:

$$I_4 = I_{4,\text{moduli}} + I_{4,\text{grav}} + I_{4,3-7} = \frac{1}{24} p_1(T) (-24\chi_0(B_4) + 24) \quad (2.17)$$

where:

$$\chi_0(B_4) = \frac{1}{720} \int_{B_4} (-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4). \quad (2.18)$$

Recall that for a base B_4 to support a smooth elliptic fibration for a Calabi-Yau fivefold, we have $h^{0,0}(B_4) = 1$ and $h_{k,0}(B_4) = 0$ for $k \neq 0$. Therefore $\chi_0(B_4) = 1$ and the gravitational anomaly is cancelled for smooth elliptic Calabi-Yau fivefolds.

We now assume that the fibration contains non-abelian gauge groups from 7-branes and charged 7-7 matters. In addition we turn on G_4 flux. In this case the terms above needs slight modification and there will be a new term $I_{4,7-7}$ contributing to the gravitational anomaly from the 7-brane sector.

Suppose that the divisor $S \subset B_4$ is wrapped by 7-branes. The Kodaira fiber is singular over S and the Calabi-Yau fivefold X_5 is singular. We assume that the singular X_5 admits a crepant resolution $\tilde{f} : \tilde{X}_5 \rightarrow X_5$ and $G_4 \in H_{\text{vert}}^{2,2}(\tilde{X}_5)$. In this situation the $\chi(X_5)$ term in $I_{4,\text{moduli}}$ (2.9) is replaced by $\chi(\tilde{X}_5)$. The D3-brane class $[C]$ is corrected to:

$$[C] = \frac{1}{24} \pi_* c_4(\tilde{X}_5) - \frac{1}{2} \pi_*(G_4 \cdot G_4). \quad (2.19)$$

The anomaly polynomial from the non-trivial 7-brane sector is:

$$\begin{aligned} I_{4,7-7} &= \frac{1}{24} p_1(T) \left[\sum_{\mathbf{R}} \dim(\mathbf{R}) \chi(\mathbf{R}) - \text{rk}(G) \chi(\mathbf{adj}) \right] \\ &\cong \frac{1}{24} p_1(T) \mathcal{A}_{\text{grav}|7-7}. \end{aligned} \quad (2.20)$$

In section 5, we investigate cases with only non-Higgsable gauge groups and $\chi(\mathbf{adj})$ is purely geometric. To cancel the gravitational anomaly the following relation must hold:

$$\mathcal{A}_{\text{grav|mod}} + \mathcal{A}_{\text{grav|uni}} + \mathcal{A}_{\text{grav|3-7}} + \mathcal{A}_{\text{grav|7-7}} = 0. \quad (2.21)$$

The above equation puts a set of topological constraints that every crepant resolution $\tilde{X}_5 \rightarrow X_5$ with consistent background G_4 flux on \tilde{X}_5 must satisfy. It will be verified on a set of Calabi-Yau fivefolds \tilde{X}_5 in section 5.

2.3 Construction of Calabi-Yau fivefold hypersurfaces

In this section, we will review some basics tools of toric geometry that we will use to construct Calabi-Yau fivefolds as hypersurfaces in toric sixfolds. The techniques are standard and can be found in [25]. We will use Batyrev's construction [26] to construct Calabi-Yau hypersurfaces in a reflexive polytope. We will explain the details in a moment.

We will start with an $(n+1)$ -d reflexive polytope Δ in an $(n+1)$ -d lattice in $M_{\mathbb{R}}$. That is, $\Delta \subset M_{\mathbb{R}}$ contains $\mathbf{0}$ and both Δ and Δ^* are lattice polytopes where $\Delta^* \subset N_{\mathbb{R}}$ is defined as:

$$\Delta^* := \{v \in N_{\mathbb{R}} : \langle u, v \rangle \geq -1, \forall u \in \Delta\},$$

where $N_{\mathbb{R}}$ is the dual lattice of $M_{\mathbb{R}}$.

The polytope Δ^* defines a toric fan Σ and to each point v_i on the boundary of Δ^* one can associate a homogeneous coordinate z_i . We denote by Y_{n+1} the $(n+1)$ -d toric variety defined by Σ . To each point $u_i \in \Delta$ one can associate a monomial $m_i = \prod_j z_j^{\langle u_i, v_j \rangle + 1}$. The locus $\sum_i a_i m_i = 0$ (a_i are generic non-vanishing complex coefficients) defines a hypersurface $X_n \subset Y_{n+1}$ in the anticanonical class $-K_{Y_{n+1}}$ of Y_{n+1} . Therefore X_n is a Calabi-Yau n -fold. Note that there is no guarantee that X_n is smooth when $n > 3$.

For the Calabi-Yau n -fold hypersurface X_n defined from the reflexive pair (Δ^*, Δ) , the (stringy) Hodge numbers can be computed with the Batyrev formula [26, 27]:

$$h^{1,1}(X_n) = l(\Delta^*) - (n+2) - \sum_{\dim \Theta^* = n} l'(\Theta^*) + \sum_{\dim \Theta^* = n-1} l'(\Theta^*) l'(\Theta) \quad (2.22)$$

$$h^{m,1}(X_n) = \sum_{\dim \Theta^* = n-m} l'(\Theta^*) l'(\Theta) \quad (1 < m < n-1) \quad (2.23)$$

$$h^{n-1,1}(X_n) = l(\Delta) - (n+2) - \sum_{\dim \Theta = n} l'(\Theta) + \sum_{\dim \Theta = n-1} l'(\Theta) l'(\Theta^*) \quad (2.24)$$

Here Θ^* and Θ means the faces on Δ^* and Δ respectively. $l(\cdot)$ means the number of integral points in a polytope, and $l'(\cdot)$ means the number of interior points on a face.

For the cases we will discuss in this paper, they are all n -d hypersurfaces defined in some $(n+1)$ -d ambient toric varieties that are also elliptically fibered over some $(n-1)$ -d bases. Such a fibration structure can be easily read off by studying the toric fans of their ambient toric varieties. For all the examples in this paper, after a suitable $\text{SL}(6, \mathbb{Z})$ transformation, the vertices of Δ^* can be put into the following form:

$$\begin{aligned} \tilde{v}_1 &= (0, 0, 0, 0, 0, 1), & \tilde{v}_2 &= (0, 0, 0, 0, 1, 0), & \tilde{v}_3 &= (0, 0, 0, 1, -2, -3), \\ \tilde{v}_{i+3} &= (v_i, -2, -3). \end{aligned}$$

This is of the form introduced in [28] and is known to be a $\mathbb{P}^{2,3,1}$ fibration over a base toric variety B_4 . The fan of B_4 has toric rays v_i , and we denote the convex hull of it by the polytope Δ_{B_4} .

The Calabi-Yau hypersurface defined by the pair (Δ^*, Δ) is thus an elliptic fibration over B_4 . Note that to fully specify the toric variety corresponding to Δ^* , a triangulation is also required. We require the triangulation to be fine (uses all the points in Δ^*), regular (resulting variety is projective and Kähler) and star (the simplices define the cones of a toric fan). Though a triangulation of Δ_{B_4} is needed to compute some detailed geometrical data such as intersection numbers on B_4 , the computation of the Hodge numbers and the characteristic classes of B_4 depends only on the rays in the fan Σ_{B_4} associated with Δ_{B_4} . Therefore in later sections where we compute Hodge numbers and characteristic classes of B_4 and Y_5 , we will choose a convenient triangulation to facilitate our computations and the results are indeed independent from our choices.

The base varieties of the examples in section 5 are particularly easy in this sense since their triangulations are unique. In contrast, the triangulations of the bases of the examples in section 3 are far from being unique, but one does not need to worry about any specific choice of triangulation since we will be computing Hodge numbers only and the key data involved in this computation are the numbers of cones in various codimensions which are constants across all fine-star-regular triangulations (FRST).

For example, if the elliptic fibration does not have codimension-two $\text{ord}(f, g) \geq (4, 6)$, codimension-three $\text{ord}(f, g) \geq (8, 12)$ or codimension-four $\text{ord}(f, g) \geq (12, 18)$ non-minimal loci, then we expect the Shioda-Tate-Wazir formula to hold, independent of the triangulation of the base:

$$h^{1,1}(X_5) = h^{1,1}(B_4) + \text{rk}(G) + 1, \quad (2.25)$$

where G is the 2d geometric gauge group.

For most examples in our paper with E_8 geometric gauge groups, we will try to construct a smooth base B_4 that supports a flat fibration. To do that, we will first pick all the primitive rays ρ inside Δ_{B_4} and this we will denote by S this set of primitive rays. We will denote by B_{toric} the toric variety given by S (and a suitable triangulation of it). We then pick the subset $S_{E_8} \subset S$ whose elements are the rays that correspond to divisor supporting Kodaira II^* fiber, that is, carrying an E_8 gauge group. To find these rays we consider the following two polytopes:

$$\Delta_F = \{u \in \mathbb{Z}^4 \mid \langle u, v_i \rangle + 4 \geq 0, \forall v_i \in S\},$$

$$\Delta_G = \{u \in \mathbb{Z}^4 \mid \langle u, v_i \rangle + 6 \geq 0, \forall v_i \in S\}.$$

The points in Δ_F correspond to monomials in the class $-4K_{B_{\text{toric}}}$ and the points in Δ_G correspond to monomials in the class $-6K_{B_{\text{toric}}}$. The orders of vanishing of the polynomials $f \in O(-4K_{B_{\text{toric}}})$ and $g \in O(-6K_{B_{\text{toric}}})$ in the Weierstrass model along a divisor D_i corresponding to the primitive ray $u_i \in S$ are:

$$\begin{aligned} \text{ord}_{D_i}(f) &= \min_{u \in \Delta_F} (\langle u, v_i \rangle + 4), \\ \text{ord}_{D_i}(g) &= \min_{u \in \Delta_G} (\langle u, v_i \rangle + 6), \end{aligned}$$

We denote by S_{E_8} the set of v_i 's such that $\text{ord}_{D_i}(f) = 4$ and $\text{ord}_{D_i}(g) = 5$.

Usually the set S_{E_8} does not give rise to a compact base and we need to add several rays manually. After adding these rays by hand we arrive at a base we call B_{seed} . This base needs to be blown-up to be free from codimension-two $(4, 6)$ locus, codimension-three $(8, 12)$ and codimension-four $(12, 18)$ non-minimal loci. Focusing on S_{E_8} , we can compute the number of 4d cones in S_{E_8} , n_{4D} . By assigning a convenient triangulation to S_{E_8} we can then compute the number of 3d and 2d cones in S_{E_8} , n_{3D} and n_{2D} respectively and n_{1D} is simply the number of rays in S_{E_8} . Note that the n_{4D} , n_{3D} , n_{2D} and n_{1D} are all indeed independent of triangulation and our choice is simply to make the computation easier. There is the following correspondence between those numbers and the gauge web structure over S_{E_8} :

	Number of
n_{4D}	(E_8, E_8, E_8, E_8) point
n_{3D}	(E_8, E_8, E_8) curve
n_{2D}	(E_8, E_8) surface
n_{1D}	E_8 divisor

For each of the above intersecting E_8 structure there is a sequence of blow-ups one needs to perform over B_{seed} to finally arrive at a smooth base B_4 . We will present the process in the appendix [A](#).

3 The boundaries of 2d (0,2) F-theory landscape

3.1 Calabi-Yau d -fold with extremal Hodge numbers

We first compute the ambient reflexive polytope for Calabi-Yau d -fold with extremal Hodge numbers, which is a generalization of the sequence (3.3) in [\[7\]](#). We first define a sequence of integers m_k , with

$$m_0 = 1, \quad m_{k+1} = m_k(m_k + 1). \quad (3.1)$$

The first a few m_i are

$$m_1 = 2, \quad m_2 = 6, \quad m_3 = 42, \quad m_4 = 1\,806, \quad m_5 = 3\,263\,442. \quad (3.2)$$

Then the ambient reflexive polytope is a $(d+1)$ -dimensional weighted projective space $\mathbb{P}^{1,1,d_1,d_2,\dots,d_d}$. The weights are computed as:

$$d_1 = 2 \cdot m_{d-1}, \quad d_2 = (2 + d_1) \cdot m_{d-2}, \quad d_{k+1} = \left(2 + \sum_{i=1}^k d_i\right) \cdot m_{d-k-1}. \quad (3.3)$$

For the elliptic CY3 X_3 with $(h^{1,1}, h^{2,1}) = (11, 491)$, the ambient weighted projective space is $\mathbb{P}^{1,1,12,28,42}$.

For the elliptic CY4 X_4 with $(h^{1,1}, h^{2,1}, h^{3,1}) = (252, 0, 303\,148)$, the ambient weighted projective space is $\mathbb{P}^{1,1,84,516,1204,1806}$.

For the elliptic CY5 X_5 with the largest $h^{4,1}$, from the rules above, we expect the ambient weighted projective space to be $\mathbb{P}^{1,1,3612,151\,788,932\,412,2\,175\,628,3\,263\,442}$.

Using the terminologies in section 2.3, a weighted projective space $\mathbb{P}^{1,w_1,\dots,w_{d+1}}$ corresponds to an ambient polytope Δ^* with vertices:

$$\begin{aligned} v_1 &= (0, \dots, 0, 1) \\ v_2 &= (0, \dots, 1, 0) \\ &\vdots \\ v_{d+1} &= (1, 0, \dots, 0) \\ v_{d+2} &= (-w_1, -w_2, \dots, -w_{d+1}) \end{aligned} \tag{3.4}$$

As one can check, the pairs (Δ^*, Δ) above are all reflexive.

3.2 Maximal $h^{4,1}$

In this section, we construct the Calabi-Yau fivefold X_5 with the largest $h^{4,1}$ from the reflexive pair (Δ^*, Δ) , where Δ^* corresponds to $\mathbb{P}^{1,1,3\,612,151\,788,932\,412,2\,175\,628,3\,263\,442}$. We will explicitly construct the elliptic fibration structure and the base fourfold B_4 .

The weighted projective space $\mathbb{P}^{1,1,3\,612,151\,788,932\,412,2\,175\,628,3\,263\,442}$ has the following vertices:

$$\begin{aligned} \tilde{v}_1 &= (0, 0, 0, 0, 0, 1), & \tilde{v}_2 &= (0, 0, 0, 0, 1, 0), & \tilde{v}_3 &= (0, 0, 0, 1, 0, 0), \\ \tilde{v}_4 &= (0, 0, 1, 0, 0, 0), & \tilde{v}_5 &= (0, 1, 0, 0, 0, 0), & \tilde{v}_6 &= (1, 0, 0, 0, 0, 0), \\ \tilde{v}_7 &= (-1, -3\,612, -151\,788, -932\,412, -2\,175\,628, -3\,263\,442). \end{aligned} \tag{3.5}$$

Its dual polytope has the following vertices:

$$\begin{aligned} \tilde{u}_1 &= (-1, -1, -1, -1, -1, -1), & \tilde{u}_2 &= (-1, -1, -1, -1, -1, 1), \\ \tilde{u}_3 &= (-1, -1, -1, -1, 2, -1), & \tilde{u}_4 &= (-1, -1, -1, 6, -1, -1), \\ \tilde{u}_5 &= (-1, -1, 42, -1, -1, -1), & \tilde{u}_6 &= (-1, 1\,806, -1, -1, -1, -1), \\ \tilde{u}_7 &= (6\,526\,883, -1, -1, -1, -1, -1). \end{aligned} \tag{3.6}$$

From the Batyrev formula, one can compute $h^{1,1}(X_5) = 151701$. The last term in (2.22) vanishes. Similarly, $h^{2,1}(X_5)$ and $h^{3,1}(X_5)$ both vanishes as well.

The other Hodge numbers can be computed by Landau-Ginzburg methods [29]:

$$h^{4,1}(X_5) = 247\,538\,602\,581, \quad h^{2,3}(X_5) = 2\,722\,923\,718\,202, \quad h^{2,2}(X_5) = 758\,522. \tag{3.7}$$

They satisfy the relation [22]:

$$11h^{1,1} - 10h^{2,1} - h^{2,2} + h^{2,3} + 10h^{3,1} - 11h^{4,1} = 0 \tag{3.8}$$

We perform an $\text{SL}(6, \mathbb{Z})$ rotation on v_i :

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & -2 & -2 & -2 & 1 & 0 \\ -3 & -3 & -3 & -3 & 0 & 1 \end{pmatrix} \tag{3.9}$$

The resulting vertices are

$$\begin{aligned}\tilde{v}'_1 &= (0, 0, 0, 0, 0, 1), & \tilde{v}'_2 &= (0, 0, 0, 0, 1, 0), & \tilde{v}'_3 &= (0, 0, 0, 1, -2, -3), \\ \tilde{v}'_4 &= (0, 0, 1, 0, -2, -3), & \tilde{v}'_5 &= (0, 1, 0, 0, -2, -3), & \tilde{v}'_6 &= (1, 0, 0, 0, -2, -3), \\ \tilde{v}'_7 &= (-1, -3612, -151788, -932412, -2, -3).\end{aligned}\quad (3.10)$$

Hence it is in form of $\mathbb{P}^{1,2,3}$ bundle over a 4d base B_4 , whose 4d polytope Δ_{B_4} has the following vertices:

$$\Delta_{B_4} = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (-1, -3612, -151788, -932412)\}. \quad (3.11)$$

The base B_4 of X_5 is a B_3 fibration over \mathbb{P}^1 . B_3 is exactly the threefold base for the elliptic CY4 X_4 with $h^{1,1} = h^{3,1} = 151700$, as similar phenomenon is observed in the lower dimensional case [13]. Note that X_4 has an elliptic fibration with geometric gauge groups [3]

$$G_{4d} = E_8^{1285} \times F_4^{3792} \times G_2^{10092} \times \text{SU}(2)^{15108}. \quad (3.12)$$

To construct the rays and cones on B_4 and B_3 . We first compute the set of lattice points $\{x, y, z, w\}$ in the polytope (3.11), with the following condition:

$$\gcd(x, y, z, w) = 1. \quad (3.13)$$

Among these points, we select the ones that correspond to divisors with E_8 gauge group, which form the set S_{E_8} . Such a point v satisfy the following condition:

$$\min_{u \in \Delta_G} (\langle u, v \rangle + 6) = 5, \quad (3.14)$$

where the Δ_G polytope is the set of lattice points $u = (u_x, u_y, u_z, u_w)$ satisfying

$$u_x \geq -6, \quad u_y \geq -6, \quad u_z \geq -6, \quad u_w \geq -6, \quad -u_x - 3612u_y - 151788u_z - 932412u_w \geq -6. \quad (3.15)$$

It turns out that there are 1285 points satisfying the conditions, and they are all in the form of $(0, y, z, w)$. Then we can construct a non-compact toric threefold $B_{E_8}^{(3)}$ with the 3d rays (y, z, w) . After a triangulation, we find that there are 2508 (E_8, E_8, E_8) 3d cones and 3792 (E_8, E_8) 2d cones on $B_{E_8}^{(3)}$. Then we add three additional rays $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and a number of additional 3d cones into $B_{E_8}^{(3)}$, such that the resulting base is a compact one $B_{\text{seed}}^{(3)}$.

Finally, after we blow up the (E_8, E_8, E_8) 3d cones and (E_8, E_8) 2d cones according to [12] (also see appendix A), we get a base $B_{\text{toric}}^{(3)}$ with 90652 rays and $h^{1,1}(B_{\text{toric}}^{(3)}) = 90649$. The number of rays is computed as follows. We start with the $B_{\text{seed}}^{(3)}$ with 1288 rays. Then for each of the 2508 (E_8, E_8, E_8) 3d cones, we need to add 19 additional rays in the interior. For each of the 3792 (E_8, E_8) 2d cones, we need to add 11 additional rays on it. Thus these numbers add up to 90652. Finally, to get the base B_3 , we checked that there are 310 E_8 divisors p on B_{toric} with non-toric $(4, 6)$ -curves. This can be checked by the following criterion:

$$|\{u \in \Delta_G | \langle u, p \rangle + 6 = 5\}| > 1. \quad (3.16)$$

It turns out that all of these non-toric $(4, 6)$ -curves are irreducible. After these curves are blown up, we get the non-toric base B_3 with $h^{1,1}(B_3) = 90\,959$.

After adding up the rank of geometric gauge group, we get exactly the following Shioda-Tate-Wazir formula in CY4 case:

$$h^{1,1}(X_4) = h^{1,1}(B_3) + \text{rk}(G) + 1 = 151\,700. \quad (3.17)$$

Then the 4d base B_4 is constructed as B_3 fibered over \mathbb{P}^1 with the addition of two rays $(1, 0, 0, 0)$ and $(-1, -3\,612, -151\,788, -932\,412)$. The geometric gauge group on B_4 remains the same, and there is no additional base locus to be blow up. Thus the base B_4 has $h^{1,1}(B_4) = 90\,960$, and we have exactly

$$h^{1,1}(X_5) = h^{1,1}(B_4) + \text{rk}(G) + 1 = 151\,701. \quad (3.18)$$

For the 2d F-theory on X_5 , the geometric gauge group is also

$$G = E_8^{1\,285} \times F_4^{3\,792} \times G_2^{10\,092} \times \text{SU}(2)^{15\,108}. \quad (3.19)$$

3.3 Maximal $h^{1,1}$

In this section, we construct Calabi-Yau fivefold X_5 with the largest $h^{1,1}$, along with its elliptic fibration structure.

We take (3.6), and perform an $\text{SL}(6, \mathbb{Z})$ rotation:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad (3.20)$$

The resulting vertices are

$$\begin{aligned} \tilde{v}'_1 &= (-6, -6, -6, -6, -2, -3), & \tilde{v}'_2 &= (0, 0, 0, 0, 0, 1), & \tilde{v}'_3 &= (0, 0, 0, 0, 1, 0), \\ \tilde{v}'_4 &= (-6, -6, -6, 1, -2, -3), & \tilde{v}'_5 &= (-6, -6, 37, -6, -2, -3), \\ \tilde{v}'_6 &= (-6, 1\,801, -6, -6, -2, -3), & \tilde{v}'_7 &= (6\,526\,878, -6, -6, -6, -2, -3). \end{aligned} \quad (3.21)$$

Naively, it is in form of $\mathbb{P}^{1,2,3}$ bundle over a 4d base B_4 with vertices

$$\begin{aligned} \Delta_{B_4} = \{ & (-6, -6, -6, -6), (-6, -6, -6, 1), (-6, -6, 37, -6), \\ & (-6, 1\,801, -6, -6), (6\,526\,878, -6, -6, -6) \}. \end{aligned} \quad (3.22)$$

Nonetheless, the vertices such as $(-6, -6, -6, -6)$ cannot correspond to a ray on a smooth base B_4 , because all the coordinates are dividable by four. It should be interpreted as six times the ray $(-1, -1, -1, -1)$ on B_4 , which carries an E_8 gauge group.

Now we write down the set of rays S_{E_8} whose corresponding toric divisor supports E_8 gauge algebra:

$$\begin{aligned}
 S_{E_8} = & \left\{ (x, y, z, -1) \mid -1 \leq z \leq 6, -1 \leq y \leq \frac{932\,412 - 151\,788z}{3\,612}, \right. \\
 & \left. -1 \leq x \leq 932\,413 - 151\,788z - 3\,612y \right\} \\
 & \cup \{ (x, y, -1, 0) \mid -1 \leq y \leq 42, -1 \leq x \leq 151\,789 - 3\,612y \} \\
 & \cup \{ (x, -1, 0, 0) \mid -1 \leq x \leq 3\,613 \} \\
 & \cup \{ (1, 0, 0, 0), (-1, 0, 0, 0) \}.
 \end{aligned} \tag{3.23}$$

There are in total

$$n_{1D} = 482\,632\,421 \tag{3.24}$$

integral points in this set. Now we are going to construct the non-compact toric fourfold B_{E_8} with rays in the set S_{E_8} . We denote by Δ_{E_8} the convex hull polytope of S_{E_8} . Δ_{E_8} has a shape of hyper truncated pyramid, with the following 16 vertices, see figure 1:

$$\begin{aligned}
 v_1 &= (-1, -1, -1, -1), & v_2 &= (-1, 300, -1, -1), & v_3 &= (601, 300, -1, -1), \\
 v_4 &= (1\,087\,813, -1, -1, -1), & v_5 &= (-1, -1, 6, -1), & v_6 &= (-1, 6, 6, -1), \\
 v_7 &= (13, 6, 6, -1), & v_8 &= (25\,297, -1, 6, -1), & &
 \end{aligned} \tag{3.25}$$

$$\begin{aligned}
 v_9 &= (-1, -1, -1, 0), & v_{10} &= (155\,401, -1, -1, 0), & v_{11} &= (85, 42, -1, 0), \\
 v_{12} &= (-1, 42, -1, 0), & & & &
 \end{aligned} \tag{3.26}$$

$$v_{13} = (-1, -1, 0, 0), \quad v_{14} = (3\,613, -1, 0, 0), \tag{3.27}$$

$$v_{15} = (1, 0, 0, 0), \quad v_{16} = (-1, 0, 0, 0) \tag{3.28}$$

We observe that the vertices of Δ_{E_8} can be naturally organized in the following manner: the vertices in (3.25) are the 8 vertices of the first line of (3.23), the vertices in (3.26) are the 4 vertices of the second line of (3.23) and the vertices in (3.27) are the 2 ends of the third line of (3.23).

It is a fact that the number of simplicial 4d cones is independent of the choice of triangulation of the 4d fan given by the primitive rays in S_{E_8} . To compute the number of simplicial 4d cones, we only need to calculate the volume of the Δ_{E_8} , which turns out to be

$$\text{vol}(\Delta_{E_8}) = 114\,084\,800.$$

Therefore the number of simplicial 4d cones is:

$$n_{4D} = 4! \times \text{vol}(\Delta_{E_8}) = 2\,738\,035\,200. \tag{3.29}$$

To compute the total number of 3d cones on B_{E_8} , one can use the following trick. On a compact toric fourfold, each 4d cone contains four 3d cones, while each 3d cone is shared by two 4d cones. Hence the number of 3d cones on a compact toric fourfold should be the

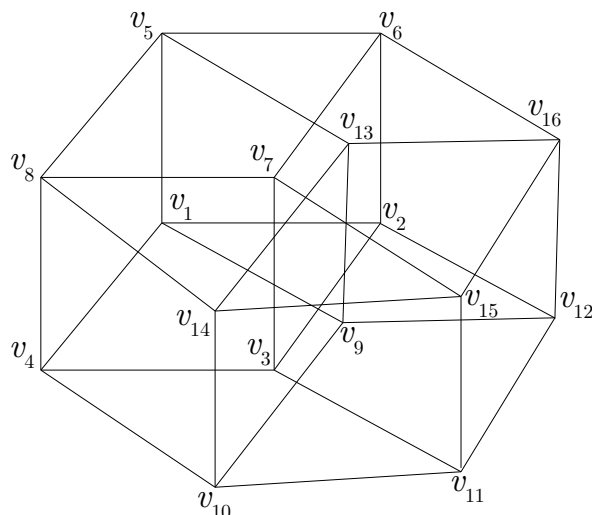


Figure 1. The vertices of hyper truncated pyramid Δ_{E_8} , for the elliptic Calabi-Yau fivefold with the largest $h^{1,1}$.

twice of the number of 4d cones. However, the base B_{E_8} is non-compact, with the following boundary 2d faces:

$$\begin{aligned}
 &v_2v_3v_6v_7, & v_5v_6v_7v_8, & v_9v_{10}v_{11}v_{12}, & v_{10}v_{11}v_{14}v_{15}, \\
 &v_9v_{10}v_{13}v_{14}, & v_9v_{12}v_{13}v_{16}, & v_2v_3v_{11}v_{12}, & v_2v_6v_{12}v_{16}, \\
 &v_3v_7v_{11}v_{15}, & v_5v_6v_{13}v_{16}, & v_5v_8v_{13}v_{14}, & v_7v_8v_{14}v_{15}.
 \end{aligned} \tag{3.30}$$

In the above list, we take the 2d faces inside a single 3d face with non-zero contribution to the 4d volume of Δ_{E_8} .

Now one takes two times the number of 4d cones (3.29), plus additional 3d cones from the boundary set (3.30) divided by two. We get

$$\begin{aligned}
 n_{3D} &= 2n_{4D} + \frac{1}{2} \times 7\,056\,216 \\
 &= 5\,479\,598\,508.
 \end{aligned} \tag{3.31}$$

Then to compute the number of 2d cones on B_{E_8} , one needs to carefully add up all the contributions from each faces of Δ_{E_8} . The result is

$$n_{2D} = 3\,224\,195\,728. \tag{3.32}$$

With the number of 4d, 3d and 2d cones, we now construct the base B_4 by blowing up the (E_8, E_8, E_8, E_8) , (E_8, E_8, E_8) and (E_8, E_8) collisions, according to section A. For each 4d cone, there are in total 15 exceptional divisor in the interior after blowing up the (E_8, E_8, E_8, E_8) collision. For each 3d cone and 2d cone, there are in total 19 and 11 exceptional divisors, respectively. Finally, there are a number of non-toric blow ups on the divisors on B_{E_8} . They can be checked by the criterion (3.16) in this case as well, and there are in total

$$N_{\text{non-toric}} = 167\,873\,112 \tag{3.33}$$

of these divisors (which are all irreducible). In this whole process, we are only blowing up loci where $(4, 6) \leq \text{ord}(f, g) < (8, 12)$ at codimension-two, $(8, 12) \leq \text{ord}(f, g) < (12, 18)$ at codimension-three and $(12, 18) \leq \text{ord}(f, g) < (16, 24)$ at codimension-four. Hence the number of complex structure moduli of X_5 is unchanged and it is still within a finite distance of the moduli space.

Finally, we need to add the rays $(-6, -6, -6, 1)$, $(-6, -6, 37, -6)$ and $(-6, 1801, -6, -6)$ back into the base, to make B_4 compact. The total $h^{1,1}(B_4)$ is then

$$\begin{aligned} h^{1,1}(B_4) &= n_{1D} + 15n_{4D} + 19n_{3D} + 11n_{2D} + N_{\text{non-toric}} + 3 - 4 \\ &= 181\,299\,558\,192. \end{aligned} \quad (3.34)$$

To compute the $h^{1,1}(X_5)$ of this elliptic Calabi-Yau fivefold. We add the rank of non-Higgsable gauge groups: for each 4d cone, there is a single $\text{SU}(2)$; for each 3d cone, the additional gauge group is $G_2 \times \text{SU}(2)^3$; for each 2d cone, the additional gauge group is $F_4 \times G_2^2 \times \text{SU}(2)^2$; for each E_8 ray, the gauge rank is 8.

Thus we have (2.25)

$$\begin{aligned} h^{1,1}(X_5) &= h^{1,1}(B_4) + 8n_{1D} + n_{4D} + 5n_{3D} + 10n_{2D} + 1 \\ &= 247\,538\,602\,581. \end{aligned} \quad (3.35)$$

This number is exactly the same as the $h^{4,1}$ of its mirror in section 3.2. Hence the elliptic fibration structure is completely correct.

The numbers of each type of gauge groups are

$$\begin{aligned} n(E_8) &= n_{1D} \\ &= 482\,632\,421, \\ n(F_4) &= n_{2D} \\ &= 322\,419\,5728, \\ n(G_2) &= n_{3D} + 2n_{2D} \\ &= 11\,927\,989\,964, \\ n(\text{SU}(2)) &= n_{4D} + 3n_{3D} + 2n_{2D} \\ &= 25\,625\,222\,180. \end{aligned} \quad (3.36)$$

The total 2d geometric gauge group is

$$G = E_8^{482\,632\,421} \times F_4^{3\,224\,195\,728} \times G_2^{11\,927\,989\,964} \times \text{SU}(2)^{25\,625\,222\,180}. \quad (3.37)$$

4 Various elliptic Calabi-Yau fivefolds

In this section, we explicitly study a number of elliptic Calabi-Yau fivefolds as hypersurfaces of $\mathbb{P}^{1,w_1,w_2,w_3,w_4,w_5,w_6}$, which constructed from a reflexive polytope. While the full list for $\sum_{i=1}^6 w_i < 150$ is presented in appendix B, we will discuss a few examples in full detail and explain the origin of the non-vanishing Hodge numbers $h^{2,1}(X_5)$ and $h^{3,1}(X_5)$.

n	Gauge group	$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}, h^{2,3})$
1	None	(2, 0, 0, 56 977, 626 727)
2	None	(2, 0, 0, 59 054, 649 574)
3	None	(2, 0, 0, 72 888, 801 751)
4	None	(3, 1, 0, 93 190, 1 025 070)
6	SU(3)	(5, 0, 0, 151 471, 1 666 132)
8	SO(8)	(7, 0, 0, 235 299, 2 588 220)
12	E_6	(9, 0, 0, 494 933, 5 444 174)
24	E_8	(11, 0, 0, 2 314 879, 25 463 560)

Table 4. The Hodge numbers of the generic elliptic CY5 over a smooth base B_n . The non-Higgsable gauge group is also listed.

4.1 Hypersurface of $\mathbb{P}^{1,1,1,1,n,2n+8,3n+12}$

In these section, we consider ambient spaces in form of $\mathbb{P}^{1,1,1,1,n,2n+8,3n+12}$, $n \in \mathbb{Z}_+$. For $n \geq 4$, the toric base fourfold is a “generalized Hirzebruch fourfold” $B_{n,4}$. In general, it is a toric fourfold with $h^{1,1}(B_{n,4}) = 2$ and it has the structure of a \mathbb{P}^1 fibration over \mathbb{P}^3 . The fan of $B_{n,4}$ has the following rays

$$\begin{aligned} v_1 &= (1, 0, 0, 0), & v_2 &= (0, 1, 0, 0), & v_3 &= (0, 0, 1, 0), & v_4 &= (0, 0, 0, 1), \\ v_5 &= (-1, -1, -1, -n), & v_6 &= (0, 0, 0, -1). \end{aligned}$$

The list of 4d cones of the toric variety is complete:

$$\begin{aligned} &(1, 2, 3, 4), (1, 2, 3, 6), (1, 2, 4, 5), (1, 2, 5, 6), \\ &(1, 3, 4, 5), (1, 3, 5, 6), (2, 3, 4, 5), (2, 3, 5, 6), \end{aligned}$$

where (i, j, k, l) denotes the 4D cone whose rays are v_i, v_j, v_k and v_l . These bases have

$$\chi(B_{n,4}) = 8. \quad (4.1)$$

For $1 \leq n \leq 3$, the base fourfold is a weighted projective space $\mathbb{P}^{1,1,1,1,n}$. The rays are

$$\begin{aligned} v_1 &= (1, 0, 0, 0), & v_2 &= (0, 1, 0, 0), & v_3 &= (0, 0, 1, 0), & v_4 &= (0, 0, 0, 1), \\ v_5 &= (-1, -1, -1, -n). \end{aligned}$$

The list of 4d cones is

$$(1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 4, 5), (1, 3, 4, 5), (2, 3, 4, 5). \quad (4.2)$$

For $\mathbb{P}^{1,1,1,1,n,2n+8,3n+12}$ to be reflexive, n can only take the following values:

$$n = 1, 2, 3, 4, 6, 8, 12, 24. \quad (4.3)$$

The data of X_5 for these cases are summarized in the table 4.

For $n = 1$, the Calabi-Yau fivefold X_5 is a generic elliptic fibration over \mathbb{P}^4 . The fibration is smooth, and the Hodge numbers are

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (2, 0, 0, 56\,977). \quad (4.4)$$

For $n = 2$, X_5 is a generic fibration over the weighted projective space $\mathbb{P}^{1,1,1,1,2}$. The base $\mathbb{P}^{1,1,1,1,2}$ has a codimension-four $\mathbb{C}^4/\mathbb{Z}_2$ orbifold singularity at the intersection point $v_1 v_2 v_3 v_5 = D_1 \cdot D_2 \cdot D_3 \cdot D_5$. Similarly, the Calabi-Yau fivefold also has a codimension-four terminal singularity over this point. From the Batyrev formula, the Hodge numbers are different from the generic fibration over \mathbb{P}^4 :

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (2, 0, 0, 59\,054). \quad (4.5)$$

For $n = 3$, similarly X_5 is a generic fibration over the weighted projective space $\mathbb{P}^{1,1,1,1,3}$, with a $\mathbb{C}^4/\mathbb{Z}_3$ orbifold singularity at the intersection point $v_1 v_2 v_3 v_5 = D_1 \cdot D_2 \cdot D_3 \cdot D_5$. The Hodge numbers are:

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (2, 0, 0, 72\,888). \quad (4.6)$$

For $n = 4$, X_5 is a generic fibration over a generalized Hirzebruch fourfold $B_{4,4}$. There is no gauge group on X_5 , and the Hodge numbers are

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (3, 1, 0, 93\,190). \quad (4.7)$$

$h^{1,1}(X_5)$ exactly matches (2.25), and there is a non-zero $h^{2,1}(X_5) = 1$. The harmonic $(2, 1)$ -form is constructed as follows. The normal bundle and canonical bundle of the divisor D_6 corresponding to $v_6 = (0, 0, 0, -1)$ satisfies

$$N_{D_6} = K_{D_6}. \quad (4.8)$$

Hence the base is locally Calabi-Yau near the divisor D_6 . Then the elliptic fiber over D_6 is a smooth toric T^2 with a constant modulus τ . Now we take the $(1, 0)$ form of this T^2 and wedge it with the Poincaré dual of D_6 (a $(1, 1)$ -form). Thus we get an a contribution to $h^{2,1}$. This divisor is similar to a single (-2) -curve on the base in the cases of elliptic CY3, which also has an additional contribution to $h^{2,1}$ of the CY3 [5].²

For $n = 6$, X_5 is a generic fibration over the generalized Hirzebruch fourfold $B_{6,4}$. There is a type IV_s singular fiber on D_6 with an $SU(3)$ gauge group, the Hodge numbers are

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (5, 0, 0, 151\,471). \quad (4.9)$$

We can check that $h^{1,1}(X_5) = h^{1,1}(B_{6,4}) + \text{rk}(SU(3)) + 1$.

For $n = 8$, X_5 is a generic fibration over the generalized Hirzebruch threefold $B_{8,4}$. There is a type $I_{0,s}^*$ singular fiber on D_6 with an $SO(8)$ gauge group, the Hodge numbers are

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (7, 0, 0, 235\,299). \quad (4.10)$$

Hence we have $h^{1,1}(X_5) = h^{1,1}(B_{8,4}) + \text{rk}(SO(8)) + 1$.

²We thank Andreas Braun and Washington Taylor for the discussions here, in an unfinished project before.

For $n = 12$, X_5 is a generic fibration over the generalized Hirzebruch threefold $B_{12,4}$. There is a type IV_s^* singular fiber on D_6 with an E_6 gauge group, the Hodge numbers are

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (9, 0, 0, 494\,933). \quad (4.11)$$

Hence $h^{1,1}(X_5) = h^{1,1}(B_{12,4}) + \text{rk}(E_6) + 1$.

For $n = 24$, X_5 is a generic fibration over the generalized Hirzebruch threefold $B_{24,4}$. There is a type II^* singular fiber on D_6 with an E_8 gauge group. The Hodge numbers are

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (11, 0, 0, 2\,314\,879). \quad (4.12)$$

Hence $h^{1,1}(X_5) = h^{1,1}(B_{24,4}) + \text{rk}(E_8) + 1$.

In other dimensions, there also exists a similar series of elliptically fibered Calabi-Yau $(d+1)$ -dimensional hypersurfaces X_{d+1} in $(d+2)$ -dimensional ambient weighted projective spaces. Consider a $(d+2)$ -dimensional weighted projective space $W_{n,d} = \mathbb{P}^{1,1,\dots,1,n,2(n+d),3(n+d)}$, for its corresponding polyhedron to be reflexive, n can only take the values $6d$ and its divisors. For $n \geq d$, the Calabi-Yau hypersurface X_{d+1} in $W_{n,d}$ is elliptically fibered over a toric base $\mathbb{F}_n^{(d)}$ with the following rays:

$$\begin{aligned} v_1 &= (1, 0, \dots, 0), \\ v_2 &= (0, 1, \dots, 0), \\ &\vdots \\ v_d &= (0, 0, \dots, 1), \\ v_e &= (-1, -1, \dots, -n), \\ v_g &= (0, 0, \dots, -1). \end{aligned}$$

The triangulation of $\Delta_{B_{n,d}}$ is:

$$\begin{aligned} &(1, 2, \dots, d), (1, 2, \dots, d-1, g), \\ &(1, 2, \dots, d-2, d, e), (1, 2, \dots, d-2, e, g), \\ &(1, 2, \dots, d-3, d-1, d, e), (1, 2, \dots, d-3, d-1, e, g), \\ &\vdots \\ &(1, 3, 4, \dots, d-1, d, e), (1, 3, 4, \dots, d-1, e, g), \\ &(2, 3, \dots, d-1, d, e), (2, 3, \dots, d-1, e, g). \end{aligned}$$

and we have $\chi(B_{n,d}) = 2d$.

Note that when d is even the largest four divisors of $n_{\max} = 6d$ are $6d, 3d, 2d$ and $\frac{3}{2}d$ and when d is odd the largest four divisors of $n_{\max} = 6d$ are $6d, 3d, 2d$ and d (or $\frac{6}{5}d$ depends on whether $5|d$). For $n = 6d$, there is E_8 gauge group along D_g . For $n = 3d$, there is E_6 group along D_g . For $n = 2d$, there is $\text{SO}(8)$ group along D_g . When d is even, for $n = \frac{3}{2}d$, there is $\text{SU}(3)$ along D_g and for $n < \frac{3}{2}d$ there is no gauge group on $B_{n,d}$. When d is odd, for $n \leq d$ (or $n \leq \frac{6}{5}d$), there is no gauge group on $B_{n,d}$.

v_1	$(1,0,0,0)$
v_2	$(0,1,0,0)$
v_3	$(0,0,1,0)$
v_4	$(0,0,0,1)$
v_5	$(0,0,0,-1)$
v_6	$(-1,-1,-1,-3)$
v_7	$(-1,-1,-1,-4)$
v_8	$(-2,-2,-2,-7)$
v_9	$(-3,-3,-3,-10)$

Table 5. The rays on the toric fourfold base B_4 of the elliptic Calabi-Yau fourfold in $\mathbb{P}^{1,3,3,3,10,40,60}$, with Hodge numbers $(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (7, 3, 171, 53\,192)$.

The most well-known case of this series is when $d = 2$. The Hirzebruch surfaces \mathbb{F}_3 , \mathbb{F}_4 , \mathbb{F}_6 and \mathbb{F}_{12} carry $SU(3)$, $SO(8)$, E_6 and E_8 non-Higgsable gauge groups respectively. The series in 3d, known as generalized Hirzebruch threefolds, has also been explored in literatures [30, 31]. Note that here $d = 3$ is odd. As $n = 3$ is the fourth largest divisor of $n_{\max} = 18$, there is no gauge group on the base $B_{3,3}$ which is the generalized Hirzebruch threefold $\tilde{\mathbb{F}}_3$.

4.2 An example with non-zero $h^{2,1}$ and $h^{3,1}$: (7, 3, 171, 53 192)

Here the Calabi-Yau fivefold X_5 is the degree 120 hypersurface in $\mathbb{P}^{1,3,3,3,10,40,60}$. X_5 is a $\mathbb{P}^{2,3,1}$ fibration over the base given by the FRST of the polytope Δ_{B_4} whose rays are listed in table 5.

Δ_{B_4} is small enough such that a concrete triangulation can be easily found. We triangulate Δ_{B_4} by giving the 4D cones as follows:

$$\begin{aligned} &(1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 4, 6), (1, 2, 5, 7), (1, 2, 6, 9), (1, 2, 8, 9), (1, 2, 7, 8), \\ &(1, 3, 4, 6), (1, 3, 5, 7), (1, 3, 6, 9), (1, 3, 8, 9), (1, 3, 7, 8), (2, 3, 4, 6), (2, 3, 5, 7), \\ &(2, 3, 6, 9), (2, 3, 8, 9), (2, 3, 7, 8), \end{aligned}$$

where (i, j, k, l) denotes the 4d cone whose rays are v_i , v_j , v_k and v_l . There is an $SU(2)$ gauge group on the divisor D_6 corresponding to the ray v_6 . We can compute

$$\chi(B_4) = \int_{B_4} c_4 = 17. \quad (4.13)$$

In this case, the non-zero $h^{2,1}(X_5)$ can be explained similar to the case of generic fibration on $\mathbb{F}_4^{(4)}$. The divisor D_5 has normal bundle $N_{D_5} = K_{D_5}$, which can be checked from

$$v_5 = \frac{1}{4}(v_1 + v_2 + v_3 + v_7). \quad (4.14)$$

v_1 , v_2 , v_3 and v_7 are neighbors of v_5 . Hence there is a harmonic $(2,1)$ -form, which is constructed from wedging the Poincaré dual $(1,1)$ -form of D_5 with the $(1,0)$ -form on the constant torus over D_5 .

Similar thing happens for D_7 and D_8 , as the rays satisfy

$$\begin{aligned} v_7 &= \frac{1}{2}(v_5 + v_8), \\ v_8 &= \frac{1}{2}(v_7 + v_9). \end{aligned} \quad (4.15)$$

In total, there are three harmonic $(2,1)$ -form of X_5 constructed in this way, which matches $h^{2,1}(X_5) = 3$.

The non-zero $h^{3,1}(X_5)$ is explained in another way. Denote the base coordinates of B_4 by z_1, z_2, \dots, z_9 . The local Tate model near the divisor D_6 with $SU(2)$ is [32]

$$y^2 + b_3 z_6 y + b_6 z_6^2 = x^3 + b_1 z_6 x y + b_2 z_6 x^2 + b_4 z_6^2 x. \quad (4.16)$$

We have

$$b_3 = F_{20}(z_1, z_2, z_3), \quad b_6 = F_{40}(z_1, z_2, z_3). \quad (4.17)$$

Here F_i is generic homogeneous polynomial of degree i . Note that the coefficients b_3 and b_6 of Tate model do not depend on z_4 and z_9 , although D_4 and D_9 intersect D_6 . Thus b_3 and b_6 can be thought as sections of line bundles on $\mathbb{P}^2 \times \mathbb{P}^1$. The coordinates of \mathbb{P}^2 are z_1, z_2, z_3 and the coordinates of \mathbb{P}^1 are z_4 and z_9 .

After the resolution $(x, y, z_6; \delta_1)$ [33],³ the equation is transformed into

$$y^2 + F_{20}(x_1, x_2, x_3) z_6 y + F_{40}(z_1, z_2, z_3) z_6^2 = (x^3 + b_1 z_6 x y + b_2 z_6 x^2 + b_4 z_6^2 x) \delta_1. \quad (4.18)$$

The exceptional divisor $\delta_1 = 0$ has equation

$$y^2 + F_{20}(z_1, z_2, z_3) z_6 y + F_{40}(z_1, z_2, z_3) z_6^2 = 0. \quad (4.19)$$

Note that if one set $z_6 = 1$, then the equation

$$y^2 + F_{20}(z_1, z_2, z_3) y + F_{40}(z_1, z_2, z_3) = 0 \quad (4.20)$$

is a complex surface S with the following Newton polytope:

$$\Delta_3 = \{(0, 0, 2), (0, 0, 0), (40, 0, 0), (0, 40, 0)\}. \quad (4.21)$$

This Newton polytope has 171 interior points, hence S has $h^{2,0}(S) = 171$. Taking into account the coordinate z_6, z_4 and z_9 , the whole topology of the exceptional divisor $\delta_1 = 0$ should be $S \times \mathbb{P}^1 \times \mathbb{P}^1$. Wedging the non-trivial $(2,0)$ -form of S with the Poincaré dual $(1,1)$ -form of $\delta_1 = 0$, we get 171 $(3,1)$ -forms in X_5 , which exactly matches $h^{3,1}(X_5)$.

³It means the replacement $(x, y, z_6) \rightarrow (x\delta_1, y\delta_1, z_6\delta_1)$ followed by the dividing the equation by δ_1^2 . The exceptional divisor is given by the equation $\delta_1 = 0$.

4.3 An example with a large $h^{3,1}(X_5)$: (11, 0, 2 024, 28 575)

Here we study an elliptic Calabi-Yau fivefold with a large $h^{3,1}(X_5)$. We take the toric ambient space to be the weighted projective space $\mathbb{P}^{1,6,6,6,6,50,75}$. The Hodge numbers are

$$(h^{1,1}(X_5), h^{2,1}(X_5), h^{3,1}(X_5), h^{4,1}(X_5)) = (11, 0, 2\,024, 28\,575). \quad (4.22)$$

After the $\mathrm{SL}(6, \mathbb{Z})$ rotation (3.9), the vertices of the 6d reflexive polytope Δ^* are

$$\begin{aligned} \tilde{v}_1 &= (0, 0, 0, 0, 0, 1) \\ \tilde{v}_2 &= (0, 0, 0, 0, 1, 0) \\ \tilde{v}_3 &= (0, 0, 0, 1, -2, -3) \\ \tilde{v}_4 &= (0, 0, 1, 0, -2, -3) \\ \tilde{v}_5 &= (0, 1, 0, 0, -2, -3) \\ \tilde{v}_6 &= (1, 0, 0, 0, -2, -3) \\ \tilde{v}_7 &= (-6, -6, -6, -6, -2, -3) \end{aligned} \quad (4.23)$$

The vertices of the 4d base polytope Δ_{B_4} are:

$$\begin{aligned} v_1 &= (1, 0, 0, 0) \\ v_2 &= (0, 1, 0, 0) \\ v_3 &= (0, 0, 1, 0) \\ v_4 &= (0, 0, 0, 1) \\ v_5 &= (-6, -6, -6, -6) \end{aligned} \quad (4.24)$$

The vertex v_5 is a multiple of six. Hence one can speculate that the elliptic Calabi-Yau fivefold is an elliptic fibration over \mathbb{P}^4 , with type II^* Kodaira fiber on the ray $(-1, -1, -1, -1)$ (tuned E_8 gauge group). We label the corresponding divisor of the rays of \mathbb{P}^4 as follows:

$$\begin{aligned} (1, 0, 0, 0) : z_1 = 0, & \quad (0, 1, 0, 0) : z_2 = 0, & \quad (0, 0, 1, 0) : z_3 = 0, \\ (0, 0, 0, 1) : z_4 = 0, & \quad (-1, -1, -1, -1) : z_5 = 0. \end{aligned} \quad (4.25)$$

The Calabi-Yau hypersurface equation can be read off from the lattice points in the polytope Δ , which is the dual polytope of Δ^* . The vertices are:

$$\begin{aligned} \tilde{u}_1 &= (-6, -6, -6, -6, -1, -1) \\ \tilde{u}_2 &= (19, -6, -6, -6, -1, -1) \\ \tilde{u}_3 &= (-6, 19, -6, -6, -1, -1) \\ \tilde{u}_4 &= (-6, -6, 19, -6, -1, -1) \\ \tilde{u}_5 &= (-6, -6, -6, 19, -1, -1) \\ \tilde{u}_6 &= (0, 0, 0, 0, 2, -1) \\ \tilde{u}_7 &= (0, 0, 0, 0, -1, 1). \end{aligned} \quad (4.26)$$

The Tate model of X_5 can be written as:

$$\begin{aligned} y^2 + F_4(z_1, z_2, z_3, z_4, z_5)z_5xy + F_{12}(z_1, z_2, z_3, z_4, z_5)z_5^3y \\ = x^3 + F_8(z_1, z_2, z_3, z_4, z_5)z_5^2x^2 + F_{16}(z_1, z_2, z_3, z_4, z_5)z_5^4x + F_{25}(z_1, z_2, z_3, z_4, z_5)z_5^5, \end{aligned} \quad (4.27)$$

and the Weierstrass model can be written as:

$$y^2 = x^3 + F_{16}(z_1, z_2, z_3, z_4, z_5)z_5^4x + F_{25}(z_1, z_2, z_3, z_4, z_5)z_5^5. \quad (4.28)$$

Here $F_i(z_1, z_2, z_3, z_4, z_5)$ are generic homogeneous polynomials of degree i in the variables z_1, z_2, z_3, z_4, z_5 . As one can see, the Weierstrass g polynomial has the following expansion around $u = 0$:

$$g = F_{25}(z_1, z_2, z_3, z_4)z_5^5 + \mathcal{O}(u^6). \quad (4.29)$$

We need to blow up the non-minimal codimension-two $(4, 6)$ locus at $z_5 = F_{25}(z_1, z_2, z_3, z_4) = 0$, which describes a Fermat surface with degree 25.

As a consequence, the new non-toric base fourfold B_4 has $h^{3,1} = 2024$. The reason is that the Fermat surface $F_{25}(v, w, s, t) = 0$ has the following Hodge numbers (see e. g. [34]):

$$h^{i,j} = \begin{pmatrix} 1 & 0 & 2024 \\ 0 & 9225 & 0 \\ 2024 & 0 & 1 \end{pmatrix} \quad (4.30)$$

Especially, the Hodge number $h^{2,0} = 2024$. The harmonic $(2, 0)$ -forms on the Fermat surface, wedged with the $(1, 1)$ -form which is the Poincaré dual of the E_8 divisor, give rise to $(3, 1)$ -forms on the base B_4 . Hence we have

$$h^{3,1}(B_4) = 2024, \quad (4.31)$$

which can be again uplifted to the $h^{3,1}(X_5) = 2024$.

On the other hand, the value of $h^{1,1}(X_5)$ matches (2.25):

$$h^{1,1}(X_5) = h^{1,1}(B_4) + \text{rank}(G) + 1 = 11. \quad (4.32)$$

Here $h^{1,1}(B_4) = 2$ after the single blow up along the non-toric Fermat surface, and the gauge group rank is 8.

4.4 An example with a large $h^{2,1}(X_5)$: (28 575, 2 024, 0, 11)

To construct the mirror Calabi-Yau fivefold of the X_5 in the last section, we take the vertices (4.26) and perform the $\text{SL}(6, \mathbb{Z})$ rotation (3.20)

We get the vertices

$$\begin{aligned} \tilde{v}_1 &= (-6, -6, -6, -6, -2, -3) \\ \tilde{v}_2 &= (19, -6, -6, -6, -2, -3) \\ \tilde{v}_3 &= (-6, 19, -6, -6, -2, -3) \\ \tilde{v}_4 &= (-6, -6, 19, -6, -2, -3) \\ \tilde{v}_5 &= (-6, -6, -6, 19, -2, -3) \\ \tilde{v}_6 &= (0, 0, 0, 0, 1, 0) \\ \tilde{v}_7 &= (0, 0, 0, 0, 0, 1). \end{aligned} \quad (4.33)$$

Therefore X_5 is the Calabi-Yau hypersurface in a $\mathbb{P}^{2,3,1}$ bundle fibered over the base variety associated with the 4d polytope Δ_{B_4} with the vertices:

$$\Delta_{B_4} = \{(-6, -6, -6, -6), (19, -6, -6, -6), (-6, 19, -6, -6), (-6, -6, 19, -6), (-6, -6, -6, 19)\}. \quad (4.34)$$

Δ_{B_4} is generated by 21437 primitive vectors. The rays in the set S_{E_8} are given by the non-zero vectors in the polytope Δ_{E_8} whose vertices are:

$$\Delta_{E_8} = \{(-1, -1, -1, -1), (3, -1, -1, -1), (-1, 3, -1, -1), (-1, -1, 3, -1), (-1, -1, -1, 3)\}. \quad (4.35)$$

We have $|S_{E_8}| = 69$ and $\text{vol}(\Delta_{E_8}) = \frac{32}{3}$. The numbers of n -dimensional cones in $\Delta_{S_{E_8}}$ with an arbitrary FRST are:

$$\begin{aligned} n_{4D} &= 4! \times \text{vol}(\Delta_{E_8}) = 256, \\ n_{3D} &= 2n_{4D} + \frac{1}{2} \times 64 = 544, \\ n_{2D} &= 356. \end{aligned} \quad (4.36)$$

In this case, the numbers of gauge groups can be explicitly worked out due to the relatively small number of rays in Δ_4 . The numbers of rays that support different gauge groups are:

$$n(E_8) = 69, \quad n(F_4) = 356, \quad n(G_2) = 1\,256, \quad n(\text{SU}(2)) = 2\,600. \quad (4.37)$$

In addition, there are also 5712 primitive rays carry type II fiber. With the extra 53 non-toric blow-ups, we can compute h^{11} to be:

$$\begin{aligned} h^{11} &= 21437 - 4 + 69 \times \text{rank}(E_8) + 356 \times \text{rank}(F_4) \\ &\quad + 1\,256 \times \text{rank}(G_2) + 2\,600 \times \text{rank}(\text{SU}(2)) + 53 + 1 \\ &= 28\,575. \end{aligned} \quad (4.38)$$

The numbers of gauge groups can also be used to derive the numbers of n -dimensional cones of the polytope Δ_{E_8} . Using the results of blow-ups of Δ_{E_8} at different codimensions that were worked out in section A, we have:

$$n(E_8) = n_{\text{rays}}, \quad n(F_4) = n_{2D}, \quad n(G_2) = 2n_{2D} + n_{3D}, \quad n(\text{SU}(2)) = 2n_{2D} + 3n_{3D} + n_{4D}. \quad (4.39)$$

Therefore we obtain $n_{\text{rays}} = 69$, $n_{4D} = 256$, $n_{3D} = 544$ and $n_{2D} = 356$ which match our computation using only the combinatorial data of the polytope Δ_{E_8} .

For the non-trivial $h^{2,1}(X_5)$, it can be explained as following. Consider the 3d face in the base polytope Δ_4 with vertices $(19, -6, -6, -6)$, $(-6, 19, -6, -6)$, $(-6, -6, 19, -6)$ and $(-6, -6, -6, 19)$. This face has 2024 interior points, which corresponds to 2024 base divisor with a locally trivial elliptic fibration (constant τ). Then for each base divisor D ,

we can construct a $(2,1)$ -form in terms of the Poincaré dual $(1,1)$ -form of D wedge the $(1,0)$ -form on the constant torus. There are in total 2024 of them.

This is also consistent with the computation of $h^{2,1}(X_5)$ using Batyrev formula. The dual face of (v_2, v_3, v_4, v_5) in (4.33) is a 2d face with vertices $(-6, -6, -6, -6, -1, -1)$, $(0, 0, 0, 0, -1, 1)$, $(0, 0, 0, 0, 2, -1)$, which exactly has one interior point $(-1, -1, -1, -1, 0, 0)$. Thus we can compute $h^{2,1}(X_5) = 2024$ from (2.23).

5 Gravitational anomaly cancellation in 2d

5.1 Smooth base with non-Higgsable gauge groups

As introduced in section 2.2, to cancel the gravitational anomaly, the following expression needs to vanish:

$$I_{\text{grav}} = \frac{1}{24} p_1(T) (\mathcal{A}_{\text{grav}|7-7} + \mathcal{A}_{\text{grav}|\text{mod}} + \mathcal{A}_{\text{grav}|\text{uni}} + \mathcal{A}_{\text{grav}|3-7}) \quad (5.1)$$

where

$$\mathcal{A}_{\text{grav}|7-7} = \sum_{\mathbf{R}} \dim(\mathbf{R}) \chi(\mathbf{R}) - \text{rk}(G) \chi(\mathbf{adj}), \quad (5.2)$$

$$\mathcal{A}_{\text{grav}|\text{mod}} = -\tau(B_4) + \chi_1(\tilde{X}_5) - 2\chi_1(B_4), \quad (5.3)$$

$$\mathcal{A}_{\text{grav}|\text{uni}} = 24, \quad (5.4)$$

$$\mathcal{A}_{\text{grav}|3-7} = -6c_1(B_4) \cdot \left(\frac{1}{24} \pi_*(c_4(\tilde{X}_5)) - \frac{1}{2} \pi_*(G_4 \cdot G_4) \right). \quad (5.5)$$

For a smooth base, the Hirzebruch signature $\tau(B_4)$ and Chern character $\chi_1(B_4)$ can be computed by [7]

$$\tau(B_4) = \frac{1}{180} (12c_2^2 - 56c_1c_3 + 56c_4 - 4c_1^4 + 16c_1^2c_2) \quad (5.6)$$

$$\chi_1(B_4) = \frac{1}{180} (-31c_4 - 14c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4). \quad (5.7)$$

We compute $\pi_*(c_4(\tilde{X}_5))$ with at most a single non-abelian gauge group supported on a base divisor whose class is S and list them in table 6. Part of them were already computed in [14, 23], and these formula can also be found in an analogous computation for the elliptic Calabi-Yau fourfolds in [35, 36].

We test the gravitational anomaly cancellation using the series of CY fivefolds with non-abelian gauge groups we constructed in section 4.1. The base manifolds are generalized Hirzebruch fourfolds $B_{n,4}$. The CY5s in this series all have non-Higgsable non-abelian gauge groups and have matter only in the adjoint representation of the gauge group. The anomaly can be cancelled when $c_1(B_4) \cdot \pi_*(G_4 \cdot G_4)$ vanishes. For all the bases $B_{n,4}$, the divisor carrying non-Abelian gauge group is $S = \mathbb{P}^3$, and we have the purely geometric $\chi(\mathbf{adj})$:

$$\begin{aligned} \chi(\mathbf{adj}) &= -\frac{1}{24} \int_S c_1(S) c_2(S) \\ &= -1. \end{aligned} \quad (5.8)$$

Gauge group	$\pi_*(c_4(\hat{X}_5))$
—	C_s
SU(2)	$C_s - 294c_1^2S + 84c_1S^2 - 6S^3$
SU(3)	$C_s - 456c_1^2S + 192c_1S^2 - 24S^3$
SU(4)	$C_s - 600c_1^2S + 336c_1S^2 - 60S^3$
SU(5)	$C_s - 750c_1^2S + 525c_1S^2 - 160S^3$
SU(6)	$C_s - 894c_1^2S + 753c_1S^2 - 210S^3$
SO(8)	$C_s - 648c_1^2S + 384c_1S^2 - 72S^3$
SO(10)	$C_s - 756c_1^2S + 528c_1S^2 - 120S^3$
G_2	$C_s - 456c_1^2S + 192c_1S^2 - 24S^3$
F_4	$C_s - 648c_1^2S + 384c_1S^2 - 72S^3$
E_6	$C_s - 774c_1^2S + 549c_1S^2 - 126S^3$
E_7	$C_s - 810c_1^2S + 600c_1S^2 - 144S^3$
E_8	$C_s - 960c_1^2S + 840c_1S^2 - 240S^3$

Table 6. The values of $\pi_*(c_4(\tilde{X}_5))$ in the gravitational anomaly formula. c_k denotes the k^{th} Chern class of the base B_4 . We set $C_s = 360c_1^3 + 12c_1c_2$ for convenience.

Base	G	$\chi_1(\tilde{X}_5)$	$\mathcal{A}_{\text{grav} 3-7}$	$\mathcal{A}_{\text{grav} 7-7}$
$B_{4,4}$	—	93 188	−93 216	0
$B_{6,4}$	SU(3)	151 466	−151 488	−6
$B_{8,4}$	SO(8)	235 292	−235 296	−24
$B_{12,4}$	E_6	494 924	−494 880	−72
$B_{24,4}$	E_8	2 314 868	−2 314 656	−240

Table 7. Gravitational anomaly coefficients for the generic elliptic fivefold over the generalized Hirzebruch fourfolds $B_{n,4}$.

We summarize the topological numbers that are involved in the computation of gravitational anomaly cancellation in table 7. For completeness we also include in the table the case $B_{4,4}$ where there is no non-abelian gauge group. The bases in the table all have $\tau(B_{n,4}) = 0$ and $\chi_1(B_{n,4}) = -2$.

Since the gravitational anomaly has already been cancelled, according to (5.5) any consistent G_4 -flux on these bases must satisfy the condition

$$c_1(B_4) \cdot \pi_*(G_4 \cdot G_4) = 0. \quad (5.9)$$

For example, we consider $\tilde{X}_5 \xrightarrow{\pi} B_{6,4}$ for which $h^{1,1}(\tilde{X}_5) = 5$. The five generators of $H^{1,1}(\tilde{X}_5)$ are two vertical divisors D_1 and D_2 , two exceptional divisors E_1 and E_2 and the zero section σ of the elliptic fibration. For simplicity we set:

$$H_1 = D_1, \quad H_2 = D_2, \quad H_3 = E_1, \quad H_4 = E_2, \quad H_5 = \sigma. \quad (5.10)$$

We consider the G_4 -fluxes in the vertical cohomology group $H_V^{2,2}(X_5)$ therefore we have:

$$G_4 = \sum_{i,j} n_{ij} H_i \cdot H_j. \quad (5.11)$$

In order for G_4 to uplift to fluxes in F-theory, it must satisfy the transversality conditions:

$$\int G_4 \wedge S_0 \wedge \omega_4 = 0, \quad \int G_4 \wedge \omega_6 = 0, \quad \forall \omega_4 \in H^4(B_4), \quad \omega_6 \in H^6(B_4). \quad (5.12)$$

We also require that G_4 does not break non-abelian gauge groups, therefore it satisfies:

$$\int G_4 \wedge E_i \wedge \omega_4 = 0, \quad \forall \omega_4 \in H^4(B_4). \quad (5.13)$$

Applying (5.12) and (5.13) on G_4 using ω_4 's generated by $H_i \cdot H_j$ and ω_6 's generated by $H_i \cdot H_j \cdot H_k$, and making use of the intersection numbers on $B_{6,4}$, we have:

$$\begin{aligned} n_{11} &= 6n_{13} + 6n_{31} - 36n_{33}, \\ n_{22} &= -\frac{1}{6}n_{12} - \frac{1}{6}n_{13} - \frac{1}{6}n_{21} - \frac{1}{6}n_{31} + n_{33}, \\ n_{43} &= -\frac{1}{2}n_{13} + \frac{1}{3}n_{14} - \frac{1}{2}n_{31} + 5n_{33} - n_{34} + \frac{1}{3}n_{41}, \\ n_{44} &= \frac{1}{2}n_{13} - \frac{1}{6}n_{14} + \frac{1}{2}n_{31} - 5n_{33} - \frac{1}{6}n_{41}, \\ n_{52} &= -\frac{5}{6}n_{13} - n_{15} - n_{25} - \frac{5}{6}n_{31} + 5n_{33} - n_{51}, \\ n_{55} &= -\frac{1}{6}n_{13} - \frac{1}{6}n_{15} - \frac{1}{6}n_{31} + n_{33} - \frac{1}{6}n_{51}. \end{aligned} \quad (5.14)$$

It is then easy to show that any $G_4 = \sum_{i,j} n_{ij} H_i \cdot H_j$ such that the n_{ij} 's satisfy the above restrictions also satisfies $c_1(B_{6,4}) \cdot \pi_*(G_4 \cdot G_4) = 0$.

Here we also prove that for an X_5 over a smooth B_4 with no gauge group (codimension-one singular fiber), we always have

$$c_1(B_4) \cdot \pi_*(G_4 \cdot G_4) = 0. \quad (5.15)$$

We prove the uplift of such equality in X_5 :

$$\pi^*(c_1(B_4)) \cdot G_4 \cdot G_4 = 0. \quad (5.16)$$

Denote the zero section by S_0 and vertical divisors in X_5 by D_i , the general form of vertical G_4 -flux is

$$G_4 = aS_0 \cdot S_0 + \sum_i b_i S_0 \cdot D_i + \sum_{i,j} c_{ij} D_i \cdot D_j. \quad (5.17)$$

We write the anticanonical divisor of B_4 as

$$-K(B_4) = c_1(B_4) = \sum_i m_i \pi_*(D_i), \quad (5.18)$$

where $m_i \in \mathbb{Z}$ are coefficients associated to B_4 and the choice of basis D_i .

Then we have the following intersection number relations in X_5 :

$$\begin{aligned} S_0^2 \cdot D_i \cdot D_j \cdot D_k &= -S_0 \cdot \pi^*(c_1(B_4)) \cdot D_i \cdot D_j \cdot D_k \\ &= -\sum_l m_l S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l. \end{aligned} \quad (5.19)$$

$$\begin{aligned} S_0^3 \cdot D_i \cdot D_j &= -S_0^2 \cdot \pi^*(c_1(B_4)) \cdot D_i \cdot D_j \\ &= \sum_{k,l} m_k m_l S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l. \end{aligned} \quad (5.20)$$

$$\begin{aligned} S_0^4 \cdot D_i &= -S_0^3 \cdot \pi^*(c_1(B_4)) \cdot D_i \\ &= -\sum_{j,k,l} m_j m_k m_l S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l. \end{aligned} \quad (5.21)$$

Besides these relations, we have obviously $D_i \cdot D_j \cdot D_k \cdot D_l \cdot D_m = 0$.

The transversality conditions (5.12) on G_4 become:

$$G_4 \cdot S_0 \cdot D_i \cdot D_j = 0, \quad G_4 \cdot D_i \cdot D_j \cdot D_k = 0 \quad (5.22)$$

for any i, j, k . Plug in (5.17) and using (5.19), (5.20), (5.21), they are further reduced to

$$\sum_{k,l} (am_k m_l - b_k m_l + c_{k,l}) S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l = 0, \quad (5.23)$$

$$\sum_l (am_l - b_l) S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l = 0. \quad (5.24)$$

Note that these equations are equivalent to the following equations:

$$\begin{aligned} \sum_l (am_l - b_l) S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l &= 0 \\ \sum_{k,l} c_{k,l} S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l &= 0. \end{aligned} \quad (5.25)$$

Now we can rewrite (5.16):

$$\begin{aligned} \pi^*(c_1(B_4)) \cdot G_4 \cdot G_4 &= \sum_l m_l D_l \cdot G_4 \cdot G_4 \\ &= \sum_{i,j,k,l} m_l \left(-a^2 m_i m_j m_k + 2ab_i m_j m_k - b_i b_j m_k \right. \\ &\quad \left. - 2ac_{ij} m_k + 2b_i c_{jk} \right) S_0 \cdot D_i \cdot D_j \cdot D_k \cdot D_l. \end{aligned} \quad (5.26)$$

Due to the second equation of (5.25), the terms with c_{ij} all vanishes. The rest of terms vanish as well from the first equation of (5.25).

Thus we have proved (5.16) and equivalently

$$c_1(B_4) \cdot \pi_*(G_4 \cdot G_4) = 0. \quad (5.27)$$

For example, we can simply check that the generic fibration X_5 over \mathbb{P}^4 , with Hodge numbers $(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (2, 0, 0, 56\,977)$ satisfies the gravitational anomaly cancellation with G_4 flux. This also holds for $B_{4,4}$.

5.2 Orbifold singularity and anomaly

In this section, we consider a number of bases with orbifold singularity, and check the gravitational anomaly cancellation in these cases. As a result, we found that there need to be finite contributions from the orbifold singularities to cancel the anomaly.

The bases we considered are the weighted projective space $\mathbb{P}^{1,1,1,1,n}$, where n takes the values in table 4. The rays of the base are

$$\begin{aligned} v_1 &= (1, 0, 0, 0), & v_2 &= (0, 1, 0, 0), & v_3 &= (0, 0, 1, 0), & v_4 &= (0, 0, 0, 1), \\ v_5 &= (-1, -1, -1, -n). \end{aligned} \quad (5.28)$$

The 4d cones are

$$(1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 4, 5), (1, 3, 4, 5), (2, 3, 4, 5). \quad (5.29)$$

As the volume of the 4d cone $\text{Vol}(v_1 v_2 v_3 v_5) = n$, there is a $\mathbb{C}^4/\mathbb{Z}_n$ orbifold singularity at $z_1 = z_2 = z_3 = z_5 = 0$. Unlike B_n , there is no toric divisor carrying non-Higgsable gauge group on $\mathbb{P}^{1,1,1,1,n}$.

Now we compute the topological quantities involved in the gravitational anomaly cancellation (5.5). For the divisors D_i corresponds to the ray v_i , we have the linear equivalence relation:

$$D_1 = D_2 = D_3 = D_5 = H, \quad D_4 = nH, \quad (5.30)$$

and intersection number

$$H^4 = \frac{1}{n}. \quad (5.31)$$

The various Chern classes of $\mathbb{P}^{1,1,1,1,n}$ are

$$\begin{aligned} c_1 &= (n+4)H \\ c_2 &= (4n+6)H^2 \\ c_3 &= (6n+4)H^3 \\ c_4 &= 4 + \frac{1}{n}. \end{aligned} \quad (5.32)$$

However, in this case the formula (5.6) and (5.7) will no longer hold, as they give rise to fractional numbers for a general n . For a singular toric variety, the topological numbers τ and χ_1 are computed in a combinatoric way instead [37]. In particular, these numbers for $\mathbb{P}^{1,1,1,1,n}$ are exactly the same as the ones of \mathbb{P}^4 :

$$\begin{aligned} \tau(\mathbb{P}^{1,1,1,1,n}) &= 1 \\ \chi_1(\mathbb{P}^{1,1,1,1,n}) &= -1. \end{aligned} \quad (5.33)$$

Adding up the contributions in (5.5), we get the total gravitational anomaly:

$$\begin{aligned} \mathcal{A}_{\text{grav}} &= -\tau(\mathbb{P}^{1,1,1,1,n}) + \chi_1(X_5) - 2\chi_1(\mathbb{P}^{1,1,1,1,n}) + 24 - \frac{1}{4}c_1 \cdot (360c_1^3 + 12c_1c_2) \\ &= \chi_1(X_5) - 90n^3 - 1452n^2 - 8754n - 23351 - \frac{23328}{n}. \end{aligned} \quad (5.34)$$

\mathbb{Z}_n	$\mathcal{A}_{\text{orbifold}}$
\mathbb{Z}_2	-1
\mathbb{Z}_3	1
\mathbb{Z}_4	3
\mathbb{Z}_6	9
\mathbb{Z}_8	15
\mathbb{Z}_{12}	27
\mathbb{Z}_{24}	63

Table 8. The additional gravitational anomaly contribution from $\mathbb{C}^4/\mathbb{Z}_n$ orbifold singularity on a compact singular base.

On the other hand, the Hodge numbers of the smooth X_5 over $\mathbb{P}^{1,1,1,1,n}$ is the same as the generic elliptic CY5 over B_n , given in table 4. The reason is that the 6d reflexive polytopes for X_5 are exactly the same in the two cases, and the Batyrev formula (2.22), (2.23), (2.24) hold. Plug in the $\chi(X_5)$ from table 4 into (5.34), we found that $\mathcal{A}_{\text{grav}}$ is always non-zero. To compensate this, we propose a new 2d sector from the orbifold singularity, which has the contribution $\mathcal{A}_{\text{orbifold}}$ to $\mathcal{A}_{\text{grav}}$ in table 8. Notably, the case of \mathbb{Z}_2 and \mathbb{Z}_3 has a contribution (-1) and $(+1)$, respectively. Hence a $\mathbb{C}^4/\mathbb{Z}_2$ singularity would effectively act as a Fermi or tensor multiplet, while a $\mathbb{C}^4/\mathbb{Z}_3$ singularity effectively acts as a chiral multiplet.

Finally we make more comments on the physics of singular bases in F-theory. In the case of singular base surface in 6d F-theory, such as the \mathbb{Z}_3 orbifold in [38], there is a localized SCFT sector coupled to gravity. The gravitational anomaly will cancel after the contribution of the SCFT is included. In the case of 2d F-theory here, we expect a similar story. Nonetheless, for the case of \mathbb{Z}_2 and \mathbb{Z}_3 , one cannot blow up the singular loci and still get a X_5 with the same Hodge numbers. One can also check this from the Hodge numbers of X_5 in table 4, where the $h^{1,1}(X_5)$ over B_2 and B_3 are the same as the ones over \mathbb{P}^4 . The SCFT sector in these cases will not have a Coulomb branch after the dimensional reduction to 1d.

6 Discussions

In this paper, we constructed the elliptic fibration structure for a variety of elliptic Calabi-Yau fivefolds. Especially, we studied the elliptic Calabi-Yau fivefolds with the largest $h^{1,1}$ or $h^{4,1}$, as well as hypersurfaces of weighted projective spaces with small degrees. The non-vanishing $h^{2,1}$ and $h^{3,1}$ in some examples are explained as well. Nonetheless, we have not studied the detailed condition on G_4 flux in many of these geometries. For example, for the Calabi-Yau fivefold with the largest $h^{1,1}$, one needs to know whether a non-vanishing G_4 is required, and if a generic G_4 flux choice would break any gauge symmetry. This would be a question for the future work.

Moreover, one can also study the set of smooth compact fourfold bases in a more systematic way, applying the methods in 4d F-theory, such as \mathbb{P}^1 fibrations [39], Monte Carlo and random walk methods [31, 40], and systematic blow ups from weak-Fano bases [41].

Besides the cases with a smooth fourfold base, we have also initiated the study of singular base fourfold in 2d F-theory. This question is especially interesting in 2d, because of the presence of pure gravitational anomaly and one can study the correction term from base singularities. In this paper, we have studied the contribution of an orbifold singularity $\mathbb{C}^4/\mathbb{Z}_n$ with $n = 2, 3, 4, 6, 8, 12, 24$ in table 8. For more general types of base singularities, we will study them in the future. Of course, it is also crucial to explain the physical origin of these effects.

Finally, one can ask what are the details of the 2d (0,2) SCFT constructed from either a base singularity or a non-minimal loci. It would be curious to relate them with the existing 2d (0,2) literature [17–19, 42–59].

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A Blow up of 4 – E_8 collision

Consider four rays v_1, v_2, v_3, v_4 of the fan corresponding to the toric base B_4 , and for each of them we tune an E_8 gauge algebra along the toric divisor $D_i := \{z_i = 0\}$ corresponding to it hence the order of vanishing of (f, g) along z_i is $(4, 5)$ and each D_i carries Kodaira type II^* fiber. To remove the non-minimal loci, we first blow up the intersection point $z_1 = z_2 = z_3 = z_4 = 0$ by adding a new ray $v_E = v_1 + v_2 + v_3 + v_4$ that corresponds to the exceptional divisor D_E of the blow-up. Using the linearity of the inner product it is easy to show that the order of vanishing of (f, g) along D_E is $(4, 2)$. Therefore D_E carries Kodaira type IV fiber and supports an $SU(2)$ gauge group. The configuration of the base structure is plotted in figure 2 after the blow-up.

Then there are six new 3d cones with $(E_8, E_8, SU(2))$ Collisions:

$$\begin{aligned} & (v_1, v_2, v_1 + v_2 + v_3 + v_4), \quad (v_1, v_3, v_1 + v_2 + v_3 + v_4), \quad (v_1, v_4, v_1 + v_2 + v_3 + v_4), \\ & (v_2, v_3, v_1 + v_2 + v_3 + v_4), \quad (v_2, v_4, v_1 + v_2 + v_3 + v_4), \quad (v_3, v_4, v_1 + v_2 + v_3 + v_4). \end{aligned} \tag{A.1}$$

Then we can blow up these codimension-three loci, according to figure 3. Note that the order of vanishing of g on the new exceptional divisor is zero. The whole tetrahedron will look like figure 4 after this step (we did not draw out all the subdivision cones).

After these blow ups, there are four $(E_8, SU(2))$ collisions in the middle of the tetrahedron, which need to be blown up twice for each. Finally, we can just blow up the four (E_8, E_8, E_8) collisions on the faces, according to [12]. For convenience purpose, we plot the blow up of (E_8, E_8, E_8) collisions in figure 5. We also show the fully subdivided $(E_8, E_8, SU(2))$ collision in figure 6. The final geometric configuration will be absent of non-minimal loci, and the elliptic fibration is flat.

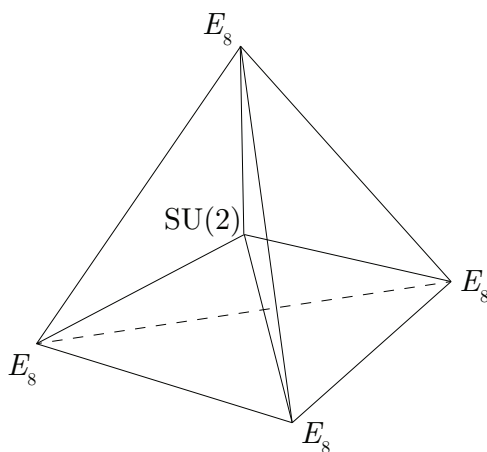


Figure 2. The blow up of the intersection point of four divisors with E_8 .

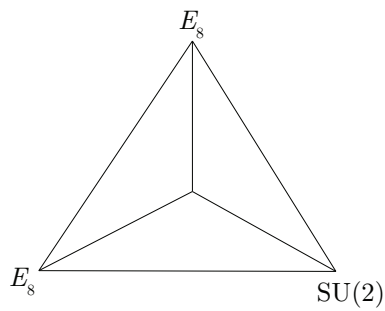


Figure 3. The $(E_8, E_8, \text{SU}(2))$ collision and a single blow up at the intersection point.

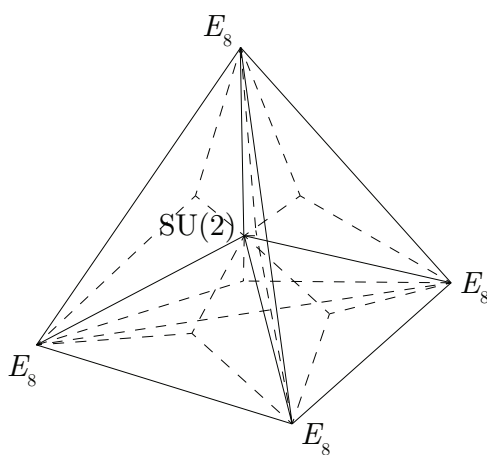


Figure 4. The tetrahedron after the blow up of the (E_8, E_8, E_8, E_8) and $(E_8, E_8, \text{SU}(2))$ collisions.

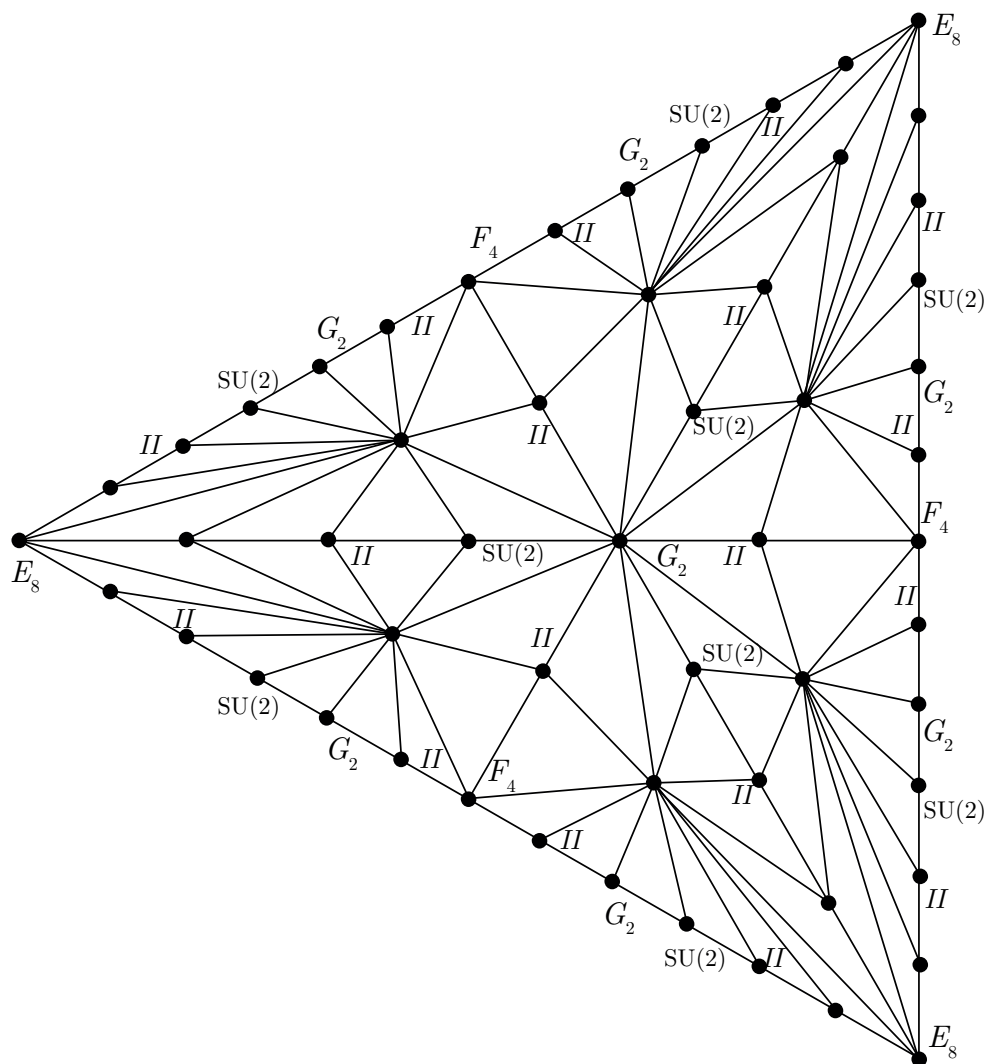


Figure 5. The fully blown up (E_8, E_8, E_8) collision.

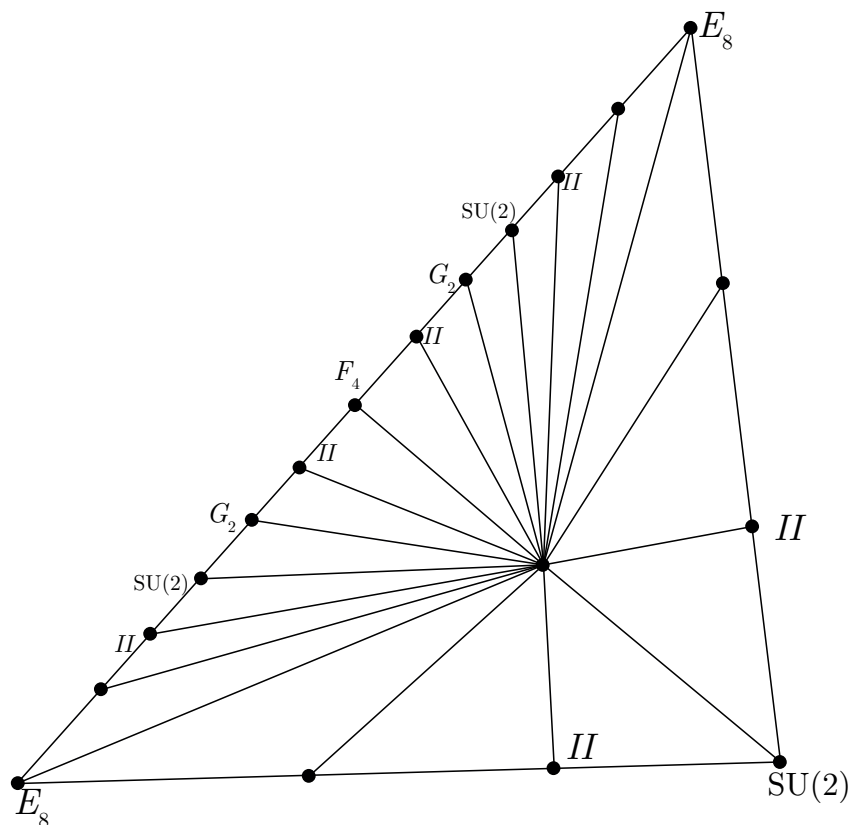


Figure 6. The fully blown up $(E_8, E_8, SU(2))$ collision.

B A list of elliptic Calabi-Yau fivefolds

In this appendix, we list the elliptic CY5 as hypersurfaces of weighted projective spaces $\mathbb{P}^{1,w_1,w_2,w_3,w_4,w_5,w_6}$. We impose the following conditions:

1. The lattice polytope associated to $\mathbb{P}^{1,w_1,w_2,w_3,w_4,w_5,w_6}$ is reflexive.
2. The weights satisfy $3w_5 = 2w_6$ and $1 + w_1 + w_2 + w_3 + w_4 + w_5 = w_6$, such that the 6d rays can be rotated by the matrix 3.9 to get a $\mathbb{P}^{2,3,1}$ fibration structure (the “naive” piling).

To get a finite list, we require that the degree⁴

$$d \equiv 1 + w_1 + w_2 + w_3 + w_4 + w_5 + w_6 \leq 150. \quad (\text{B.1})$$

We list these models along with the Hodge numbers of CY5 and the 2d F-theory geometric gauge group in table 9–12. Note that for many cases, the base fourfold has to be singular. We do not list all the possible base topologies in detail.

⁴The complete list of weights giving rise to reflexive polytopes in this category was already worked out in [20].

$(w_1, w_2, w_3, w_4, w_5, w_6)$	$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}, h^{2,3})$	Gauge group
(1, 1, 1, 1, 10, 15)	(2, 0, 0, 56 977, 626 727)	—
(1, 1, 1, 2, 12, 18)	(2, 0, 0, 59 054, 649 574)	—
(1, 1, 1, 3, 14, 21)	(2, 0, 0, 72 888, 801 751)	—
(1, 1, 2, 2, 14, 21)	(2, 0, 0, 54 703, 601 696)	—
(1, 1, 1, 4, 16, 24)	(3, 1, 0, 93 190, 1 025 070)	—
(1, 1, 2, 3, 16, 24)	(2, 0, 0, 62 187, 684 040)	—
(1, 2, 2, 2, 16, 24)	(3, 0, 0, 46 727, 513 968)	—
(1, 1, 3, 3, 18, 27)	(3, 0, 2, 66 398, 730 358)	—
(1, 2, 2, 3, 18, 27)	(2, 0, 0, 49 821, 547 988)	—
(2, 2, 2, 2, 18, 27)	(4, 0, 0, 39 495, 434 405)	G_2
(1, 1, 1, 6, 20, 30)	(5, 0, 0, 151 471, 1 666 132)	SU(3)
(1, 1, 2, 5, 20, 30)	(3, 1, 0, 90 996, 1 000 936)	—
(1, 1, 3, 4, 20, 30)	(2, 0, 0, 75 877, 834 635)	—
(1, 2, 2, 4, 20, 30)	(3, 0, 0, 56 975, 626 700)	—
(1, 2, 3, 3, 20, 30)	(2, 0, 0, 50 618, 556 783)	—
(2, 2, 2, 3, 20, 30)	(3, 0, 0, 38 076, 418 810)	—
(1, 1, 2, 6, 22, 33)	(3, 0, 0, 110 963, 1 220 566)	—
(1, 3, 3, 3, 22, 33)	(4, 0, 45, 49 405, 543 298)	SU(2)
(2, 2, 3, 3, 22, 33)	(2, 0, 0, 37 069, 407 745)	—
(1, 1, 1, 8, 24, 36)	(7, 0, 0, 235 299, 2 588 220)	SO(8)
(1, 1, 3, 6, 24, 36)	(4, 1, 1, 104 806, 1 152 847)	—
(1, 2, 2, 6, 24, 36)	(4, 1, 0, 78 669, 865 332)	—
(1, 2, 4, 4, 24, 36)	(4, 0, 2, 59 052, 649 543)	—
(1, 3, 3, 4, 24, 36)	(3, 0, 3, 52 470, 577 153)	—
(2, 2, 3, 4, 24, 36)	(3, 0, 0, 39 435, 433 760)	—
(2, 3, 3, 3, 24, 36)	(4, 0, 0, 35 110, 386 173)	—
(1, 2, 3, 6, 26, 39)	(2, 0, 0, 72 216, 794 363)	—
(2, 2, 2, 6, 26, 39)	(4, 0, 0, 56 282, 619 065)	G_2
(3, 3, 3, 3, 26, 39)	(6, 0, 0, 33 809, 371 839)	F_4
(1, 1, 4, 7, 28, 42)	(4, 2, 0, 124 796, 1 372 743)	—
(1, 2, 3, 7, 28, 42)	(3, 1, 0, 83 223, 915 441)	—
(1, 2, 4, 6, 28, 42)	(3, 0, 0, 72 887, 801 738)	—
(1, 3, 3, 6, 28, 42)	(4, 0, 36, 64 770, 712 343)	SU(2)
(1, 4, 4, 4, 28, 42)	(5, 0, 93, 54 700, 601 299)	G_2
(2, 2, 2, 7, 28, 42)	(5, 2, 0, 62 633, 688 936)	—
(2, 2, 3, 6, 28, 42)	(3, 0, 0, 48 652, 535 150)	—
(2, 3, 4, 4, 28, 42)	(3, 0, 0, 36 516, 401 654)	—
(3, 3, 3, 4, 28, 42)	(4, 0, 78, 32 417, 356 321)	—
(1, 1, 2, 10, 30, 45)	(7, 0, 0, 229 774, 2 527 448)	SO(8)
(1, 1, 3, 9, 30, 45)	(6, 0, 1, 170 371, 1 874 032)	SU(2)
(1, 1, 6, 6, 30, 45)	(5, 13, 0, 127 933, 1 407 347)	G_2

Table 9. A list of elliptic CY5 hypersurfaces in weighted projective spaces $\mathbb{P}^{1,w_1,w_2,w_3,w_4,w_5,w_6}$.

$(w_1, w_2, w_3, w_4, w_5, w_6)$	$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}, h^{2,3})$	Gauge group
(1, 2, 2, 9, 30, 45)	(5, 0, 0, 127 801, 1 405 768)	SU(3)
(1, 2, 5, 6, 30, 45)	(3, 1, 0, 76 807, 844 867)	—
(1, 3, 5, 5, 30, 45)	(4, 0, 4, 61 469, 676 141)	—
(2, 2, 5, 5, 30, 45)	(4, 0, 4, 46 113, 507 225)	—
(2, 3, 3, 6, 30, 45)	(4, 0, 0, 42 783, 470 581)	—
(3, 3, 3, 5, 30, 45)	(5, 1, 0, 34 351, 377 825)	—
(1, 1, 1, 12, 32, 48)	(9, 0, 0, 494 933, 5 444 174)	E_6
(1, 2, 4, 8, 32, 48)	(5, 1, 1, 93 189, 1 025 052)	—
(1, 2, 6, 6, 32, 48)	(5, 7, 0, 82 874, 911 639)	SU(2)
(1, 3, 3, 8, 32, 48)	(4, 2, 0, 82 820, 911 007)	—
(1, 4, 4, 6, 32, 48)	(4, 0, 3, 62 185, 684 012)	—
(2, 2, 3, 8, 32, 48)	(4, 1, 0, 62 184, 684 002)	—
(2, 3, 4, 6, 32, 48)	(3, 0, 0, 41 498, 456 458)	—
(3, 3, 3, 6, 32, 48)	(6, 0, 0, 38 407, 422 421)	F_4
(3, 4, 4, 4, 32, 48)	(5, 0, 0, 31 266, 343 882)	SU(2)
(1, 3, 6, 6, 34, 51)	(5, 0, 28, 70 409, 774 390)	—
(2, 2, 6, 6, 34, 51)	(5, 0, 0, 54 381, 598 162)	G_2
(1, 1, 3, 12, 36, 54)	(8, 0, 1, 264 675, 2 911 356)	SO(8)
(1, 1, 6, 9, 36, 54)	(6, 3, 0, 176 763, 1 944 376)	—
(1, 2, 2, 12, 36, 54)	(8, 0, 0, 198 578, 2 184 285)	SO(8)
(1, 3, 4, 9, 36, 54)	(4, 1, 1, 88 430, 972 719)	—
(1, 4, 6, 6, 36, 54)	(5, 0, 4, 66 397, 730 342)	—
(2, 2, 4, 9, 36, 54)	(5, 2, 0, 66 499, 731 468)	—
(2, 3, 3, 9, 36, 54)	(5, 1, 0, 59 050, 649 517)	—
(2, 3, 6, 6, 36, 54)	(6, 0, 2, 44 360, 487 913)	—
(2, 5, 5, 5, 36, 54)	(7, 0, 0, 38 221, 421 070)	G_2
(3, 4, 4, 6, 36, 54)	(4, 0, 2, 33 242, 365 639)	—
(3, 3, 6, 6, 38, 57)	(6, 0, 0, 37 938, 417 281)	F_4
(1, 1, 2, 15, 40, 60)	(9, 0, 0, 483 320, 5 316 436)	E_6
(1, 1, 5, 12, 40, 60)	(7, 1, 0, 242 246, 2 664 655)	SU(3)
(1, 2, 4, 12, 40, 60)	(7, 0, 1, 151 470, 1 666 115)	SU(3)
(1, 2, 6, 10, 40, 60)	(5, 2, 0, 121 291, 1 334 182)	—
(1, 2, 8, 8, 40, 60)	(6, 13, 0, 113 741, 1 251 228)	G_2
(1, 3, 3, 12, 40, 60)	(7, 0, 36, 134 621, 1 480 680)	SU(3)
(1, 3, 5, 10, 40, 60)	(5, 1, 2, 97 033, 1 067 349)	—
(1, 4, 4, 10, 40, 60)	(6, 2, 4, 90 994, 1 000 909)	—
(1, 4, 6, 8, 40, 60)	(4, 0, 2, 75 876, 834 617)	—
(1, 5, 5, 8, 40, 60)	(5, 0, 6, 72 831, 801 120)	—
(1, 6, 6, 6, 40, 60)	(7, 0, 72, 67 467, 741 894)	F_4

Table 10. A list of elliptic CY5 hypersurfaces in weighted projective spaces $\mathbb{P}^{1,w_1,w_2,w_3,w_4,w_5,w_6}$ (cont.).

$(w_1, w_2, w_3, w_4, w_5, w_6)$	$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}, h^{2,3})$	Gauge group
(2, 2, 3, 12, 40, 60)	(6, 0, 0, 101 041, 1 111 401)	SU(3)
(2, 2, 5, 10, 40, 60)	(6, 1, 2, 72 861, 801 444)	—
(2, 3, 4, 10, 40, 60)	(4, 1, 0, 60 700, 667 682)	—
(2, 3, 6, 8, 40, 60)	(3, 0, 0, 50 616, 556 764)	—
(2, 4, 5, 8, 40, 60)	(4, 0, 2, 45 580, 501 359)	—
(2, 5, 6, 6, 40, 60)	(4, 1, 0, 40 533, 445 844)	—
(3, 3, 3, 10, 40, 60)	(7, 3, 171, 53 932, 592 648)	SU(2)
(3, 3, 5, 8, 40, 60)	(3, 1, 0, 40 478, 445 255)	—
(3, 4, 4, 8, 40, 60)	(5, 0, 0, 38 072, 418 756)	—
(3, 4, 6, 6, 40, 60)	(5, 0, 36, 33 766, 371 296)	SU(2)
(3, 5, 5, 6, 40, 60)	(4, 0, 6, 32 395, 356 333)	—
(4, 4, 5, 6, 40, 60)	(4, 0, 4, 30 404, 334 425)	—
(4, 5, 5, 5, 40, 60)	(7, 1, 0, 29 366, 322 971)	—
(1, 1, 9, 9, 42, 63)	(7, 12, 0, 218 302, 2 401 378)	F_4
(1, 2, 3, 14, 42, 63)	(7, 0, 0, 210 158, 2 311 680)	SO(8)
(1, 3, 7, 9, 42, 63)	(4, 1, 2, 93 637, 1 029 994)	—
(1, 6, 6, 7, 42, 63)	(5, 19, 0, 70 261, 773 021)	G_2
(2, 2, 2, 14, 42, 63)	(10, 0, 0, 161 415, 1 775 471)	$G_2, \text{SO}(8)$
(2, 2, 7, 9, 42, 63)	(4, 2, 0, 70 240, 772 635)	—
(2, 3, 6, 9, 42, 63)	(4, 0, 0, 54 700, 601 677)	—
(2, 6, 6, 6, 42, 63)	(8, 0, 93, 42 004, 461 614)	G_2, G_2
(3, 3, 7, 7, 42, 63)	(6, 0, 12, 40 161, 441 751)	—
(1, 2, 6, 12, 44, 66)	(6, 5, 0, 147 911, 1 627 018)	SU(2)
(1, 3, 6, 11, 44, 66)	(5, 3, 0, 107 617, 1 183 779)	—
(1, 4, 4, 12, 44, 66)	(6, 0, 75, 110 961, 1 220 238)	G_2
(2, 2, 6, 11, 44, 66)	(6, 3, 0, 80 885, 889 716)	—
(2, 3, 4, 12, 44, 66)	(4, 0, 0, 74 014, 814 125)	—
(2, 4, 4, 11, 44, 66)	(6, 3, 0, 60 687, 667 536)	—
(3, 3, 3, 12, 44, 66)	(7, 0, 0, 67 767, 745 372)	F_4
(3, 3, 4, 11, 44, 66)	(4, 2, 0, 53 832, 592 147)	—
(3, 6, 6, 6, 44, 66)	(9, 0, 45, 33 939, 373 123)	$F_4, \text{SU}(2)$
(1, 1, 3, 18, 48, 72)	(10, 0, 1, 556 757, 6 124 238)	E_6
(1, 1, 9, 12, 48, 72)	(9, 3, 0, 279 225, 3 071 429)	SU(3)
(1, 2, 2, 18, 48, 72)	(10, 0, 0, 417 652, 4 594 081)	E_6
(1, 2, 4, 16, 48, 72)	(9, 0, 1, 235 298, 2 588 205)	SO(8)
(1, 2, 8, 12, 48, 72)	(7, 3, 0, 157 144, 1 728 561)	—
(1, 3, 3, 16, 48, 72)	(9, 0, 7, 209 132, 2 300 383)	SO(8)
(1, 4, 6, 12, 48, 72)	(7, 1, 3, 104 805, 1 152 828)	—
(1, 4, 9, 9, 48, 72)	(6, 8, 0, 93 192, 1 025 143)	SU(2)
(1, 6, 8, 8, 48, 72)	(7, 1, 6, 78 672, 865 365)	—

Table 11. A list of elliptic CY5 hypersurfaces in weighted projective spaces $\mathbb{P}^{1,w_1,w_2,w_3,w_4,w_5,w_6}$ (cont.).

$(w_1, w_2, w_3, w_4, w_5, w_6)$	$(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}, h^{2,3})$	Gauge group
(2, 2, 3, 16, 48, 72)	(8, 0, 0, 156 930, 1 726 165)	SO(8)
(2, 3, 6, 12, 48, 72)	(7, 1, 1, 69 955, 769 462)	SU(2)
(2, 3, 9, 9, 48, 72)	(6, 7, 0, 62 185, 684 056)	SU(2)
(2, 4, 8, 9, 48, 72)	(5, 1, 2, 52 513, 577 627)	—
(2, 6, 6, 9, 48, 72)	(6, 0, 3, 46 724, 513 926)	—
(3, 3, 8, 9, 48, 72)	(6, 2, 0, 46 771, 514 455)	—
(3, 4, 4, 12, 48, 72)	(7, 1, 1, 52 531, 577 798)	—
(3, 4, 8, 8, 48, 72)	(7, 0, 4, 39 431, 433 696)	—
(3, 6, 6, 8, 48, 72)	(7, 1, 3, 35 109, 386 156)	—
(4, 4, 6, 9, 48, 72)	(5, 0, 6, 35 022, 385 218)	—
(5, 5, 5, 8, 48, 72)	(9, 2, 0, 30 163, 333 037)	G_2 , SU(2)
(1, 2, 6, 15, 50, 75)	(6, 0, 0, 197 152, 2 168 627)	SU(3)
(1, 3, 5, 15, 50, 75)	(7, 0, 2, 157 742, 1 735 118)	SU(3)
(1, 3, 10, 10, 50, 75)	(6, 13, 0, 118 451, 1 303 044)	G_2
(2, 2, 5, 15, 50, 75)	(7, 0, 2, 118 321, 1 301 488)	SU(3)
(2, 2, 10, 10, 50, 75)	(8, 13, 0, 90 632, 997 010)	G_2 , G_2
(2, 6, 6, 10, 50, 75)	(5, 0, 0, 50 391, 554 292)	G_2
(3, 3, 3, 15, 50, 75)	(10, 0, 0, 90 009, 990 007)	F_4 , SU(3)
(3, 5, 6, 10, 50, 75)	(4, 0, 4, 39 526, 434 773)	—
(6, 6, 6, 6, 50, 75)	(11, 0, 2 024, 28 575, 303 200)	E_8

Table 12. A list of elliptic CY5 hypersurfaces in weighted projective spaces $\mathbb{P}^{1,w_1,w_2,w_3,w_4,w_5,w_6}$ (cont.).

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