Supplementary Information: Interacting multi-channel topological boundary modes in a quantum Hall valley system

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THEORETICAL MODELING OF THE DOMAIN WALL

We first describe a Hartree-Fock mean-field ansatz that allows us to theoretically model the tunneling spectra at the domain wall. The analysis shows general qualitative and quantitative agreement with experimental data, capturing essential features including the magnitude of the spectral gap, the width of the domain wall, and the asymmetry in tunneling spectra along the axis transverse to the wall. This mean-field analysis however excludes interactions that become relevant for the $\tilde{\nu} = 2$ case, and can open up a gap in the tunneling spectra, as observed in the data. To describe this situation, we use a more sophisticated Luttinger liquid analysis, based on Ref. [1], to model the low energy excitations of the domain wall and estimate a gap that is in order-of-magnitude agreement with the experimental data. Our analysis shows the role of interactions constrained by two-dimensional momentum conservation in engendering new one-dimensional physics in these systems. We end with a discussion on the role of disorder in the system.

Hartree-Fock analysis

Wave-functions

The Bi(111) surface has 6 elongated hole pockets oriented along the Γ-M direction (as shown in Fig. 1c of the main text) that are probed in the tunneling data. First, we describe the single-particle wave-functions in this setting.

Following previous studies [2, 3], we assume a parabolic dispersion of hole pockets on the surface of bismuth and model them as approximate ellipses offset in momentum space. The mass along the extended direction (in momentum-space) is given by $m_\parallel$ and the mass in the perpendicular direction by $m_\perp$; together these correspond to an Fermi pocket aspect ratio $\sqrt{m_\parallel/m_\perp} \equiv \lambda \approx 5$. The momentum offset defining the center of the hole pocket has magnitude K, and is at an angle θ from the k_x-axis, where we choose the y-axis to denote the direction of the domain wall. In what follows, we will set the magnetic length $\ell_B = 1$, and $\hbar = 1$; these constants will be restored in final results. In the presence of a uniform magnetic field B, which corresponds to a vector potential $\boldsymbol{A} = (0, Bx)$, the Hamiltonian reads

$$H = \frac{1}{2m_\parallel}\left(p_x\cos\theta + p_y\sin\theta - K + x\sin\theta\right)^2 + \frac{1}{2m_\perp}\left(p_y\cos\theta - p_x\sin\theta + x\cos\theta\right)^2. \tag{1}$$

Since $[H, p_y] = 0$, eigenfunctions for this Hamiltonian can be chosen to be of the form $\psi_{n,X} = \frac{1}{\sqrt{L_y}} e^{i\boldsymbol{K}\cdot\boldsymbol{r}} e^{iXy} f_{n,X}(x)$, where $\boldsymbol{K} = (K\cos\theta, K\sin\theta)$, the subscript n denotes the Landau level index, X denotes the eigenvalue of the operator p_y and f_n is an eigenfunction of the Hamiltonian

$$H' = \frac{1}{2m_\parallel}\left(p_x\cos\theta + x'\sin\theta\right)^2 + \frac{1}{2m_\perp}\left(x'\cos\theta - p_x\sin\theta\right)^2 \tag{2}$$

where $x' = x + X$. H' can be solved in analogy to the usual quantum harmonic oscillator by defining conjugate operators $Q = p_x\cos\theta + x'\sin\theta$, and $P = -x'\cos\theta + p_x\sin\theta$, that satisfy $[Q, P] = i$. This yields the diagonalized form $H' = \frac{b^\dagger b}{\sqrt{m_\parallel m_\perp}}$, where $b = \frac{1}{\sqrt{2}}\left(Q/\lambda^{1/4} + i\lambda^{1/4}P\right)$.

The eigenfunctions $f_n(x')$ can be found using the algebra of the boson operator b. In particular, since $b \sim \left(x'/x_0 + \partial_{x'/x_0}\right)$, and $bf_1(x') = 0$, we find $f_1(x') \sim e^{-x^2/2x_0^2}$. Note that x_0 is a complex number, and it will be useful to consider its decomposition as $1/x_0^2 = f_0 + ig_0$. The higher Landau-level wave-functions can found by repeated operations by $b^\dagger \sim \left(x'/x_0^{*2} - \partial_{x'}\right)$ on the Gaussian state corresponding to the lowest Landau level solution. Complete with normalization, the wave-functions read

$$\psi_{n,\kappa,X}(\boldsymbol{r}) = \frac{1}{\sqrt{L_y}} e^{i\boldsymbol{K}_\kappa\cdot\boldsymbol{r} + iXy/l_B^2} \frac{1}{\sqrt{l_B\sqrt{\pi/f_\kappa}\, 2^{n-1}(n-1)!}} H_{n-1}\left(\frac{\sqrt{f_\kappa}(x+X)}{l_B}\right) e^{-\frac{(x+X)^2}{2l_B^2}(f_\kappa + ig_\kappa)},$$
$$f_\kappa = \frac{\lambda}{\lambda^2\sin^2\theta_\kappa + \cos^2\theta_\kappa}, \;\; g_0 = \frac{\sin 2\theta_\kappa(1-\lambda^2)}{2(\lambda^2\sin^2\theta_\kappa + \cos^2\theta_\kappa)}. \tag{3}$$

where the additional index κ has been added to demarcate different electron valleys. Note that we have ignored the spin texture along the domain wall as well as the fact that the valleys are not precisely ellipsoidal; while these can have significant effects in the bulk, they are less germane to the present discussion of domain-wall properties.

In the experimental system, valleys A and $\bar{A}$ are populated on the 'left' side of the domain wall and valleys B and $\bar{B}$ lie on the 'right' side of the domain wall, as indicated in Fig. 1a of the main text. The remaining valleys (C and $\bar{C}$) are gapped out by strain and play no role in the symmetry breaking we observe. The valleys A and B, in particular, lie at angles $\theta = \pi/2$ and $\theta = -\pi/6$ with respect to the k_x-axis as per our conventions. It is not too difficult to see that while a generic uniaxial strain will likely have an effect that splits the three sets of valleys by comparable amounts, strains applied near certain high-symmetry directions can lead to a very large energy shift for of one of the three valley pairs while leaving the other two pairs much closer in energy, with a weak residual strain between them. Spatially modulating this strain then naturally gives rise to the domain wall scenario seen in the experiments.

Interactions

Coulomb interactions are captured by the Hamiltonian $H_{\text{e-e}} = \frac{1}{2A}\sum_{\boldsymbol{q}} V_{c,\text{sc}} \;:\; \rho(\boldsymbol{q})\rho(-\boldsymbol{q}) \;:$, where A is the two-dimensional surface area of the sample, $V_{c,\text{sc}}$ is the (screened) Coloumb interaction at momentum $\boldsymbol{q}$, and $: \ldots :$ denotes normal ordering. For the present purpose, we ignore inter-valley scattering interactions. These interactions are suppressed at least polynomially as $\sim \mathcal{O}\left[1/(K\ell_B)\right]$, which is valid for $K\ell_B \gg 1$. In this setup, $\ell_B = 6.8$ nm at 14 T and $K \approx 0.25$ Å^{-1}, and thus $K\ell_B \approx 17 \gg 1$. We will return to this assumption when considering the more complete Luttinger liquid analysis.

This density operator is then $\rho(\boldsymbol{q}) \approx \sum_\kappa \rho_{n=3,\kappa}(\boldsymbol{q})$, where κ parametrizes the different valleys, and $n = 3$ sets the Landau level studied in the experiment. We also neglect effects of Landau-level mixing, and therefore, only consider holes in the $n = 3$ Landau level. The interaction Hamiltonian reads

$$H_{\text{e-e}} = \frac{1}{2}\sum_{\kappa,\kappa',X,X',Y,Y'} \delta_{X+X',Y+Y'}\; V^{\kappa X,\kappa' X'}_{\kappa' Y',\kappa Y}\; c^\dagger_{\kappa X} c^\dagger_{\kappa' X'} c_{\kappa' Y'} c_{\kappa Y},$$
$$V^{\kappa X,\kappa' X'}_{\kappa' Y',\kappa Y} = \int \frac{dQ}{2\pi}\, V_{c,\text{sc}}\left[(Q, X-Y)\right]\; F_{n=3,\kappa}\left[\frac{X+Y}{2}, -(Q,X-Y)\right] F_{n=3,\kappa'}\left[\frac{X'+Y'}{2}, (Q,X-Y)\right],$$
$$F_{n,\kappa}(X,\boldsymbol{q}) = e^{iq_x X - \frac{f_\kappa}{4}q_y^2 - \frac{1}{4f_\kappa}(g_\kappa q_y + q_x)^2} L_{n-1}^{(0)}\left(\frac{f_\kappa}{2}q_y^2 + \frac{1}{2f_\kappa}(g_\kappa q_y + q_x)^2\right), \tag{4}$$

We assume a screened Coulomb interaction given by $V_{sc}(\boldsymbol{q}) = \frac{e^2}{2\epsilon_0\epsilon(|\boldsymbol{q}|+q_0)}$. Here $q_0 = \frac{e^2}{2\epsilon_0\epsilon\nu_0}$ is the RPA screening length, $\epsilon \approx 45$ is the average of the dielectric constant of the vacuum and bulk bismuth, and ν_0 is the surface density of states. ν_0 can be estimated using the experimentally determined value of the Landau level spacing and an application of the semiclassical quantization rule. In particular, $2\pi l_B^2\nu_0 \approx \frac{1}{7.6\text{meV}} + \frac{6}{6.1\text{meV}}$, where 7.6 meV and 6.1 meV are the Landau level spacings for electrons and holes, respectively, and the factor of 6 in the numerator of the second term accounts for the degeneracy of the hole pockets [2].

Valley Zeeman field

The domain wall can be stabilized by the presence of a valley Zeeman field that provides an energy cost for the occupation of one valley over the other. Such a term can arise in the experimental setup due to the presence of a

uniaxial strain, as long as the direction of such a strain does not correspond to a mirror symmetry of the inequivalent valleys A and B. A variation of such a strain over a long wavelength would then support domain walls at locations corresponding to the zeros of this variation. Near the domain wall itself, we can model such a term as the energy cost $H_z = \Gamma \sum_X X \left(c^\dagger_{A,X} c_{A,X} + c^\dagger_{\bar{A},X} c_{\bar{A},X} - c^\dagger_{B,X} c_{B,X} - c^\dagger_{\bar{B},X} c_{\bar{B},X} \right)$. Since long-wavelength strain directly modifies the effective mass associated with the hole pockets, it leads to an energy of the order of the bare Landau level gap times the strain. This is particularly important in our experimental setting since the exchange gap ≈ 650 μeV is already about a tenth of the Landau level gap ≈ 6.1 meV.

Mean-field ansatz and Excitation Spectra

In what follows, we set $n = 3$ and suppress the index n in all functions. For the $\tilde{\nu} = 1$ case, we assume valley A is occupied (with holes) for $X < 0$, while valley B is occupied for $X > 0$. That is, the domain wall exists 'sharply' at $X = 0$. This assumption is well justified in the case of strongly anisotropic orbitals [3]. We will also assume that there is a fictitious uniform background of electronic charge that neutralize the net charge whose role will essentially be to take care of the singular contributions from the Hartree repulsion from other holes. This charge will assume to reside in valley A everywhere (a trick to yield uniform background charge). For $\tilde{\nu} = 2$, we additionally assume that valley $\bar{A}$ is occupied for $X < 0$ and $\bar{B}$ is occupied for $X > 0$.

Tunneling into such a system occurs via the creation of an extra hole in an unoccupied valley orbital, or the removal of a hole from an occupied orbital, at any position X. The tunneling energies are given by a combination of exchange and Hartree energies. Within the approximation of neglecting inter-valley scattering, the exchange energy comes only from exchanging holes residing in the same valley. On the other hand, the Hartree energy comes from electrostatic repulsion from other holes and attraction due to the background. The mean-field Hamiltonian is given by $H_{\mathrm{MF}} = \sum_{\kappa,X} E_{\kappa,X}\ c^\dagger_{\kappa,X} c_{\kappa,X}$. For $\tilde{\nu} = 1$, these energies are given by

$$
\begin{aligned}
E_{A,X} &= \Gamma X - \sum_{X'<0} V^{A\,X,A\,X'}_{A\,X,A\,X'} + \sum_{X'>0} \left(V^{A\,X,B\,X'}_{B\,X',A\,X} - V^{A\,X,A\,X'}_{A\,X',A\,X} \right) - \mu \\
E_{B,X} &= -\Gamma X + \sum_{X'>0} \left(-V^{B\,X,B\,X'}_{B\,X,B\,X'} + V^{B\,X,B\,X'}_{B\,X',B\,X} - V^{B\,X,A\,X'}_{A\,X',B\,X} \right) - \mu \\
E_{\bar{A},X} &= \Gamma X + \sum_{X'>0} \left(V^{\bar{A}\,X,B\,X'}_{B\,X',\bar{A}\,X} - V^{\bar{A}\,X,A\,X'}_{A\,X',\bar{A}\,X} \right) - \mu \\
E_{\bar{B},X} &= -\Gamma X + \sum_{X'>0} \left(V^{\bar{B}\,X,B\,X'}_{B\,X',\bar{B}\,X} - V^{\bar{B}\,X,A\,X'}_{A\,X',\bar{B}\,X} \right) - \mu.
\end{aligned} \tag{5}
$$

For $\tilde{\nu} = 2$, the energies are given by

$$
\begin{aligned}
E_{A,X} &= \Gamma X - \sum_{X'<0} V^{A\,X,A\,X'}_{A\,X,A\,X'} + 2 \sum_{X'>0} \left(V^{A\,X,B\,X'}_{B\,X',A\,X} - V^{A\,X,A\,X'}_{A\,X',A\,X} \right) - \mu \\
E_{B,X} &= -\Gamma X + \sum_{X'>0} \left(-V^{B\,X,B\,X'}_{B\,X,B\,X'} + 2V^{B\,X,B\,X'}_{B\,X',B\,X} - 2V^{B\,X,A\,X'}_{A\,X',B\,X} \right) - \mu \\
E_{\bar{A},X} &= E_{A,X} \\
E_{\bar{B},X} &= E_{B,X}.
\end{aligned} \tag{6}
$$

We note that the Hartree contributions are essentially doubled to the case before, and the symmetry between valley anisotropy pairs (such as A, and $\bar{A}$) is restored. Finally, note that the operators $c_{\kappa,X}$ and $c^\dagger_{\kappa,X}$ are the same annihilation and creation operators, respectively, associated with the single-particle orbitals $\psi_{\kappa,X}(\boldsymbol{r})$; that is, the Hartree-Fock orbitals are the usual single-particle orbitals within this ansatz.

Tunneling Spectra

The STM probe measures the differential conductance dI/dV as the bias voltage is changed. Here we assume a constant density of states for the STM probe, and a tip that is able to resolve tunneling at length scales much shorter

than the magnetic length ℓ_B. Such a probe then effectively measures the energy-resolved local density of states in the Quantum Hall system. Up to overall constants specific to the properties of the tunneling tip, the STM measures the differential conductance $G(\boldsymbol{r}, V)$ at position $\boldsymbol{r}$ and bias V,

$$G(\boldsymbol{r}, V) \propto \int_{-\infty}^{\infty} d\omega \left(-\frac{\partial n_F(\omega + eV)}{\partial \omega} \right) A(\boldsymbol{r}, \omega), \tag{7}$$

where $A(\boldsymbol{r}, \omega)$ is the local single-electron spectral function. Within the simple Hartree-Fock theory, the local spectral density is evaluated as follows

$$A(\boldsymbol{r}, \omega) = \sum_{\kappa, X} |\psi_{\kappa,X}(\boldsymbol{r})|^2 \, \delta(E - E_{\kappa,X}), \tag{8}$$

where we have used that the fact that the energy of the Hartree-Fock orbital, $E_{\kappa,X}$, is also the energy difference between many-body Hartree-Fock states with one more (or less, if $E_{\kappa,X} < 0$) electron. Combining Eqs. (7) and (8), we find

$$G(\boldsymbol{r}, V) \propto \sum_{\kappa, X} n_F'(E_{\kappa,X} + eV) \, |\psi_{\kappa,X}(\boldsymbol{r})|^2 \, . \tag{9}$$

Thus, the temperature of the tip results in broadening of the tunneling spectra. The majority of the experimental measurements are performed at a temperature $T \approx 250$ mK with AC lockin oscillation of $V_{rms} = 74$ μV, corresponding to an expected total energy resolution of FWHM ≈ 200 μeV. A Gaussian broadening of width $\sigma \approx 100$ μeV, or equivalently FWHM ≈ 235 μeV best captures the data. With these assumptions, the theoretical estimate of the tunneling spectrum is in good qualitative and quantitative agreement with the experimental data.

Comparison to Experiment

Numerical calculations using the above ansatz preduct a bulk exchange gap $\Delta_{\text{ex}} \approx 535$ μeV, in remarkably good agreement with the experiment, and without adjusting any free parameters.

The theoretical plots correspond to the choice $\Gamma = 0.009 \Delta_{\text{ex}} \ell_B$, which lead to a one percent change (on the order of the exchange gap) of the local orbital energy upon moving a distance ℓ_B, and of about 7% across the domain wall. All other parameters are as explained above.

A dipole moment is expected at the domain wall due to the sudden change in the spatial distribution of charges across the domain wall. Its magnitude is stronger for $\tilde{\nu} = 2$ compared to $\tilde{\nu} = 1$, due to the charge contribution from the pair of valleys with the same orientation.

Symmetry discussion and Luttinger liquid analysis

A detailed discussion of symmetry based arguments ensuring the presence of Luttinger liquid modes at the domain wall and a concomitant microscopic derivation of the properties of such a Luttinger liquid is presented in Ref. [1]. For completeness, we present here arguments pertinent to the present experimental study from that work. We further provide an estimate of the charge gap adapting the analysis of Ref. [1] to our experimental setup.

Symmetries

There are 4 $U(1)$ symmetries at the level of the single-particle Hamiltonian describing the electrons in each of the 4 valleys of interest. (Recall that 2 valleys of the six-fold spectrum are energetically separated in our setup by strain.) These 4 $U(1)$ generators are the densities N_κ, with $\kappa \in \{A, \bar{A}, B, \bar{B}\}$ of electrons in the corresponding valley. The generators can be rearranged into an equivalent set of 4 $U(1)$ charges $\mathcal{N} = N_A + N_B + N_{\bar{A}} + N_{\bar{B}}$, $\mathcal{P}^z = \frac{1}{2}(N_A + N_B - N_{\bar{A}} - N_{\bar{B}})$, $\mathcal{I}^z = \frac{1}{2}(N_A - N_B + N_{\bar{A}} - N_{\bar{B}})$, and $\mathcal{Q}^z = \frac{1}{2}(N_A - N_B - N_{\bar{A}} + N_{\bar{B}})$. In the case of $\nu = 1$, $N_{\bar{A}} = N_{\bar{B}} = 0$, and these reduce to two charges $\mathcal{N}, \mathcal{I}^z$.

Interactions generically break the charge conservation symmetry associated with the valleys since electrons can now hop between valleys. However, the interactions in our system are constrained by 2D momentum conservation—interactions that involve rearranging electrons with a net momentum change are suppressed exponentially $\sim e^{-K\ell_B}$

and can be safely neglected. The remaining interactions that respect 2D momentum conservation only break select valley symmetries and lead to novel one-dimensional physics at the domain wall.

At $\nu = 1$, in particular, the constrained interactions do not break the two relevant charges $\mathcal{N}$ and $\mathcal{I}^z$. It is worth noting that because of the locking of momentum and the position of the Landau orbits, the Fermi wave-vector $k_F = 0$ in our system. In the usual one-dimensional setting, this would allow for a process in which two left-movers are simultaneously scattered into two right-moving states, which would lead to a gap. Such processes are however forbidden in our setting because they involve a large momentum transfer (from valley A to B). Thus, $\mathcal{N}$ and $\mathcal{I}^z$ are conserved. Next, because of the anomalous (quantum Hall) bulk response associated with these charges, they must lead to chiral edge modes along the domain wall owing to the Callan-Harvey mechanism [4]. One way to understand the presence of chiral edge modes is to note that an application of an electric field in the direction of the domain wall would lead to an accumulation of charge $\mathcal{I}^z$ at the domain wall; this accumulation must be circumvented by the presence of chiral zero modes that remove the charge. Alternatively, a 'spin'-gradient (a potential gradient for $\mathcal{I}^z$) would lead to charge accumulation which must be ferried away by a chiral mode along the domain wall. These chiral modes interact and form a Luttinger liquid. Also note that at $\tilde{\nu} = 1$ one only expects a *soft gap* associated with a tunneling amplitude for Luttinger parameter K given by $\sim \omega^{\frac{1}{2}(K+1/K)-1}$, that is zero only at zero frequency, and is unobservable at finite temperature.

At $\nu = 2$, interactions break $\mathcal{I}^z$ from a $U(1)$ symmetry down to Z_2—electrons from valleys A and $\bar{A}$ can scatter to electrons in B and $\bar{B}$, and vice-versa. The momentum-constrained interactions preserve other symmetries. Concomitantly, we do not expect gapless modes associated with $\mathcal{I}^z$ and its conjugate $\mathcal{N}$, which results in a gapping of the charge mode as observed experimentally in discussed in detail below. The remaining symmetries $\mathcal{P}^z$ and $\mathcal{Q}^z$ give rise to a neutral valley mode that is a gapless Luttinger mode by the Callan-Harvey mechanism.

Estimation of the charge gap for $\nu = 2$

As noted above, for $\nu = 2$, the domain wall excitations associated with the total charge $\mathcal{N}$ are gapped. Here we estimate this gap using the Luttinger analysis of [1], with three key modifications:

- we adjust the angular coordinates of the centers of the four valleys to properly model the dispersion in Bi
- we assume screened Coulomb interactions $V_{sc}(\boldsymbol{q})$.
- we work in the $n = 3$ Landau level.

Making these changes and using the methods and notation of [1] and its Supplementary Information, we find

$$\begin{aligned}
v_F^0 &\approx \frac{1}{2\pi}\int \frac{dq_x}{2\pi} V_{sc}(\boldsymbol{q})\,|F_{AA}(\boldsymbol{q})|^2\,\big|_{q_y=0},\\
v_F^1(1+\chi) &\approx \frac{1}{2\pi}\int \frac{dq_x}{2\pi} V_{sc}(\boldsymbol{q}+\boldsymbol{K}_A-\boldsymbol{K}_B)\,|F_{AB}(\boldsymbol{q})|^2\,\big|_{q_y=0},\\
v_F^1(1-\chi) &\approx \frac{1}{2\pi}\int \frac{dq_x}{2\pi} V_{sc}(\boldsymbol{q}+\boldsymbol{K}_A+\boldsymbol{K}_B)\,|F_{A\bar{B}}(\boldsymbol{q})|^2\,\big|_{q_y=0},\\
g &\approx \int \frac{dq_x}{2\pi}\left[V_{sc}(\boldsymbol{q}+\boldsymbol{K}_A-\boldsymbol{K}_B)-V_{sc}(\boldsymbol{q}+\boldsymbol{K}_A+\boldsymbol{K}_B)\right]\left[F_{AB}(\boldsymbol{q})\right]^2\big|_{q_y=0},
\end{aligned} \tag{10}$$

where we now have for the form factors

$$\begin{aligned}
F_{\alpha\beta}(\boldsymbol{q},X) = \frac{(f_\alpha f_\beta)^{1/4}}{48\sqrt{\pi}}\, e^{iq_xX}\int dx\; H_3\left(\sqrt{f_\alpha}\left(x-\frac{q_y}{2}\right)\right) H_3\left(\sqrt{f_\beta}\left(x+\frac{q_y}{2}\right)\right) e^{-iq_xx}\\
\times\, e^{-\frac{f_\alpha-ig_\alpha}{2}\left(x-\frac{q_y}{2}\right)^2} e^{-\frac{f_\beta+ig_\beta}{2}\left(x+\frac{q_y}{2}\right)^2}.
\end{aligned} \tag{11}$$

The Hamiltonian for the $U(1)$ phase ϕ_ρ, conjugate to the charge current density Π_ρ is given by

$$H_\rho = \frac{u_\rho}{2\pi}\int dy\;\left[\frac{1}{K_\rho}\left(\nabla\phi_\rho\right)^2 + K_\rho\left(\pi\Pi_\rho\right)^2\right] + \frac{2}{(2\pi\alpha)^2}\int dy\;\mathrm{Re}\left[ge^{i\sqrt{8}\phi_\rho}\right] \tag{12}$$

where u_ρ is the effective Luttinger velocity and K_ρ is the Luttinger parameter. Note that due to the repulsive Coulomb interactions between the modes, we generically expect $K_\rho < 1$. More concretely, the Luttinger analysis [1]

reveals that $K_\rho = \sqrt{\frac{\Gamma}{\Gamma+4v_F^0-v_F^1}}$; here v_F^0 is positive and larger than v_F^1 which arises due to inter-valley exchange interactions. Consequently, $K_\rho < 1$. In this case, the cosine potential $\sim \cos(\sqrt{8}\phi_\rho)$ is always relevant at the domain wall, and will lead to the opening of a tunneling gap in the charge sector. We now estimate the size of this gap, employing Feynman's variational ansatz.

This ansatz assumes that an effective mass term $\sim \Delta\phi_\rho^2$ can approximate the effect of the cosine potential. The mass Δ is then determined self-consistently by minimizing the variational upper bound on the free energy $F \leq F_{\rm var} = F_0 + T\langle S - S_0\rangle_0$, with respect to the mass Δ. Here S is the complete sine-Gordon action, while S_0 is the Gaussian approximation to it, and $\langle\rangle_0$ represents averaging with respect to the Gaussian action S_0. The mass Δ also sets the spectral gap in this approximation. Applied to the problem at hand, this procedure yields a gap Δ_ρ for fluctuations in ϕ_ρ given by

$$\Delta_\rho = u_\rho\Lambda\left(\frac{4K_\rho|g|}{(\alpha\Lambda)^2}\pi u_\rho\right)^{\frac{1}{2-2K_\rho}} \approx \Delta_{\rm ex}\left(4K_\rho\frac{|g|}{\pi u_\rho}\right)^{\frac{1}{2-2K_\rho}}, \tag{13}$$

where we used $\alpha \approx \Lambda^{-1}$ and $u_\rho\Lambda \approx \Delta_{\rm ex}$, the exchange splitting in the bulk. In the limit of weak Zeeman coupling, and using values for the anisotropy and screening length as mentioned previously, we find

$$\Delta_\rho(\Gamma \to 0) \approx 0.2\Delta_{\rm ex} \approx 120\ \mu{\rm eV}. \tag{14}$$

The central result of this analysis is that the gap is likely a sizable fraction of the exchange splitting for the experimental parameters, which is in order-of-magnitude agreement with the observation of a gap $\Delta_{\rm charge} \approx 350\ \mu$eV as seen in the experiments, and which only shows up for the $\tilde{\nu} = 2$ case. This gap in the ϕ_ρ fluctuations also controls the gap in tunneling properties of the Luttinger liquid because an electron must fractionalize into both ϕ_ρ and ϕ_σ ($U(1)$ phase conjugate to the valley-spin current density which does remain gapless) modes to tunnel into the system [1]. This charge-valley separation results in the gapping of the charge mode from the ϕ_ρ field, which shows up in the tunneling spectra, whereas the valley mode is neutral and remains gapless, but is not probed by STM.

Although the orthogonality catastrophe of tunneling into a Luttinger liquid predicts a suppression of the local density of states at E=0 in both cases, this effect is challenging to measure experimentally and not observed in the current setup.

Ruling out Disorder Effects as an Alternative Explanation

Throughout, we have ignored the role of point-like impurities that scatter the holes from one valley to another. These will generically lead to localization along the wall; we may then expect a soft gap at a scale $\sim e^2/\xi_{\rm loc}$, corresponding to Coulomb blockade effects on the scale of a localization length $\xi_{\rm loc}$. (Note that even including interactions, disorder will be relevant for the expected value of the Luttinger parameters, $K \ll 1$). However, we would expect that a disorder-driven gap arises for *both* the $\tilde{\nu} = 1$ and $\tilde{\nu} = 2$ case, and furthermore that it will show significant spatial variation along the domain wall. A purely localization driven explanation of the gap at the domain wall given by $\Delta_{\rm loc} = u_\rho\frac{2\pi}{\xi_{\rm loc}} \implies \xi_{\rm loc} \approx 3\ell_B$—requires an extremely small localization length. These facts are inconsistent with the experimental observation of a gap, which only appears in the $\tilde{\nu} = 2$ case. Further evidence that disorder plays a subsidiary role is the fact that both the $\tilde{\nu} = 1, 2$ domain walls are realized at the *same* physical location of the sample and therefore are likely to see a similar disorder potential, and thus cannot account for the striking differences in spectra (gapless vs. gapped) between the two cases. Therefore, we conclude that the charge gap for $\tilde{\nu} = 2$ is due to interactions rather than disorder.

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