Symplectic Geometry of Conical Symplectic Resolutions

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To my parents and brother, for their unending support and encouragement.

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Abstract

Ever since Kronheimer's celebrated hyperkähler construction of gravitational instantons thirty years ago, many similar constructions have appeared in the mathematical literature, eventually producing a vast class of holomorphic symplectic manifolds nowadays called Conical Symplectic Resolutions (CSRs). So far, they have been considered in representation theory, algebraic and differential geometry, and also in theoretical physics as they arise from certain supersymmetric gauge theories. However, much of their symplectic geometry is unexplored. This thesis aims to make the first steps in this direction. There are two essentially different natural real symplectic structures on CSRs; for this reason this thesis splits into two major parts.

The first part views CSRs as Liouville real symplectic manifolds (the symplectic form is exact). We investigate the presence of smooth closed exact Lagrangian submanifolds. The main theorem is that for any CSR there is a non-empty collection of non-isotopic such Lagrangians which arise from contracting $\mathbb{C}^*$-actions that act on the holomorphic symplectic structure by weight one. Lagrangians obtained from this method will be called *minimal components*. In particular, we use these to obtain (non-zero) lower bounds for the rank of symplectic cohomology. We will investigate two large subfamilies of CSRs where we are able to count minimal components and describe them. The family of *Nakajima Quiver Varieties*, in particular, we study in detail those of Dynkin type A; and the family of *resolutions of Slodowy varieties* that arise from the representation theory of semisimple Lie algebras and involve Springer theory. In the latter, we also obtain some further families of Lagrangians using Springer theoretic methods and certain *crystal operators.*

The second part studies CSRs with respect to non-exact symplectic structures arising from S^1-invariant Kähler structures on CSRs. We construct a family of symplectic cohomologies for them, labelled by different contracting $\mathbb{C}^*$-actions φ. Although we prove that these vanish, this vanishing result allows us to obtain a φ-dependent filtration by ideals on the ordinary cohomology of a CSR. Using Morse-Bott-Floer spectral sequences, we give many examples that explicitly describe these filtrations, in particular, we show an example where distinct filtrations arise from different φ actions.

Contents

Chapter 1

Introduction

1.1 Outline

The focus of this thesis lies in the interplay between Symplectic Topology and Geometric Representation Theory. Its main objects of interest are spaces called *Conical Symplectic Resolutions (CSRs)* [BPW16, BLPW16]. These form a broad family of holomorphic symplectic manifolds that are of interest to theoretical physicists [HaSp18, GrHa20, HaKa20], algebraic geometers [Ka06, Ka09, Nam08, BeSch16] and representation theorists [Nak94a, Nak98, Nak01, MO12, Nak15, BFN16]. Also, some of them have very interesting differential-geometric features [Kro89, KroNak90, Nak94a, BD00, Nak18]. This thesis will be concerned mostly with their symplectic geometry.

Examples of CSRs include many well-known families of spaces such as resolutions of Du Val singularities, Hilbert schemes of points on them, hyperpolygon spaces, quiver varieties, hypertoric varieties, cotangent bundles of flag varieties and nilpotent Slodowy varieties. All known examples of CSRs are complete hyperkähler manifolds,[1] which makes them a particularly nice geometric setting to work in.

Definition 1.1.1. A **Conical Symplectic Resolution (CSR)** is a projective resolution

$$\pi : \mathfrak{M} \to \mathfrak{M}_0$$

of a normal affine variety $\mathfrak{M}_0$, where $(\mathfrak{M}, \omega_\mathbb{C})$ is a holomorphic symplectic manifold and π is equivariant with respect to $\mathbb{C}^*$-actions on $\mathfrak{M}$ and $\mathfrak{M}_0$ (both denoted by φ). These actions satisfy two conditions:

[1] For compact Kähler manifolds, by Yau's theorem (together with a Bochner's formula and Berger's holonomy classification) the presence of a holomorphic symplectic form $\omega_\mathbb{C}$ guarantees the existence of a Kähler form ω such that $\omega, \mathrm{Re}(\omega_\mathbb{C}), \mathrm{Im}(\omega_\mathbb{C})$ is a hyperkähler structure. However, it is not known whether this also holds in our non-compact setting.

(1) the symplectic form $\omega_{\mathbb{C}}$ has a weight $k \in \mathbb{N}$, meaning $\varphi_t^* \omega_{\mathbb{C}} = t^k \omega_{\mathbb{C}}$, and

(2) the action φ contracts $\mathfrak{M}_0$ to a single fixed point x_0, so $\lim_{t \to 0} \varphi_t(x) = x_0$ for all $x \in \mathfrak{M}_0$.

Such $\mathbb{C}^*$-actions are called **conical** of **weight** k. In general, a CSR can have many conical actions, so we often denote a CSR by $(\mathfrak{M}, \varphi)$ to emphasise the choice of φ. The **core** of a CSR $\pi : \mathfrak{M} \to \mathfrak{M}_0$ is the central fibre

$$\mathfrak{L} := \pi^{-1}(x_0).$$

We denote by I the complex structure on $\mathfrak{M}$. The above definition implies that the core $\mathfrak{L} \subset \mathfrak{M}$ is an I-holomorphic projective subvariety, it is always $\omega_{\mathbb{C}}$-isotropic, and the $\mathbb{C}^*$-action contracts the whole manifold $\mathfrak{M}$ towards the core $\mathfrak{L}$, so $H^*(\mathfrak{M}) \cong H^*(\mathfrak{L})$. When the weight $k = 1$, the core $\mathfrak{L}$ is a (typically singular) complex Lagrangian subvariety of $(\mathfrak{M}, \omega_{\mathbb{C}})$.

We will denote[2] the real and imaginary parts of $\omega_{\mathbb{C}}$ by ω_J, ω_K,

$$\omega_{\mathbb{C}} = \omega_J + i\omega_K.$$

We abbreviate by $\omega_{J,K}$ any non-zero real linear combination of ω_J and ω_K. Then $\omega_{J,K}$ is a real exact symplectic form on $\mathfrak{M}$.[3] A consequence of condition (1) is that $(\mathfrak{M}, \omega_{J,K})$ admits a Liouville manifold structure such that the core $\mathfrak{L}$ is the Liouville skeleton. This structure is independent of the choice of $\omega_{J,K}$ up to Liouville isomorphism, so we call this the canonical Liouville manifold structure on $\mathfrak{M}$.

On the other hand, $\mathfrak{M}$ also admits a non-exact Kähler structure $(\mathfrak{M}, \omega_I)$ that is invariant under the S^1-part of the $\mathbb{C}^*$-action φ. We will prove that one can construct Floer cohomologies for $(\mathfrak{M}, \omega_I)$ which, since $\mathfrak{M}$ is Calabi-Yau,[4] are $\mathbb{Z}$-graded groups. Indeed, the gradings are canonical as $H^1(\mathfrak{M}) = 0$ holds for CSRs.

This thesis considers two independent projects that involve Conical Symplectic Resolutions. These are explained briefly now, for the convenience of the reader.

[2]We use the hyperkähler notation for these forms instead of the usual ω_{Re} and ω_{Im}, as in examples all known CSRs are indeed hyperkähler and these forms indeed represent the usual Kähler forms. Moreover, in general, we always have an *almost* hyperkähler structure (g, I, J, K) on $\mathfrak{M}$ such that $\omega_J = -g(\cdot, J\cdot)$ and similarly for K, as we explain later.

[3]Being the real part of the holomorphic form $re^{i\theta}\omega_{\mathbb{C}}$, for some $r > 0$, $\theta \in \mathbb{R}$. In particular, for any holomorphic two-form ω on $\mathfrak{M}$, letting $2n = \dim_{\mathbb{C}} \mathfrak{M}$, since $\omega^n \in \Lambda^{2n,0}T^*\mathfrak{M}$ is non-vanishing, also $(\omega + \overline{\omega})^{2n}$ and $(\omega - \overline{\omega})^{2n}$ are non-vanishing in $\Lambda^{2n,2n}T^*\mathfrak{M}$ (they are non-zero multiples of $\omega^n \wedge \overline{\omega}^n$), which implies that $\mathrm{Re}(\omega), \mathrm{Im}(\omega)$ are real symplectic forms.

[4]By Calabi-Yau we mean $c_1(\mathfrak{M}) = 0$, which holds since $\omega_{\mathbb{C}}^n$ trivialises the canonical bundle of $\mathfrak{M}$.

The first project, outlined in Sections 1.2-1.4, is focused on closed exact Lagrangian submanifolds in CSRs $(\mathfrak{M}, \varphi)$ endowed with the exact symplectic forms $\omega_{J,K}$. We find a collection of such Lagrangians that arise from weight-1 conical actions and we describe their Floer-theoretic invariants. In addition, we consider two families of CSRs, *Quiver Varieties of type A* (Section 1.3) and *Resolutions of Slodowy varieties of type A* (Section 1.4). In both cases we find families of weight-1 actions that produce closed exact Lagrangians. In the latter case, we also produce some further families of Lagrangians by generalising a certain construction from *Springer theory*, and by using the so-called *Crystal Operators.* In particular, as the obtained Lagrangians are smooth irreducible components of Springer fibres, they are interesting in their own right.

The second project (joint with A. Ritter) involves constructing and studying the symplectic cohomology $SH^*(\mathfrak{M}, \varphi, \omega_I)$ for CSRs $(\mathfrak{M}, \varphi)$ endowed with the highly non-exact symplectic structures ω_I mentioned above. This work is outlined in Section 1.5. We prove that, as we vary φ, these symplectic cohomologies vanish and therefore induce φ-dependent filtrations by ideals on the ordinary cohomology $H^*(\mathfrak{M})$. In addition, we construct filtrations on the Floer chain complexes of certain Hamiltonians that determine symplectic cohomology. As an application, we get a Floer-theoretic model for the ordinary cohomology of CSRs, and we describe Morse-Bott-Floer spectral sequences from which one can read off the aforementioned filtrations on $H^*(\mathfrak{M})$.

1.2 Exact Lagrangians in Conical Symplectic Resolutions

The content of this section summarises the results of Chapter 3.

A fundamental task in Symplectic Topology is finding Lagrangian submanifolds of a given symplectic manifold. Lagrangians are the objects of the Fukaya category – an increasingly important invariant studied in symplectic topology which constitutes one side of Homological Mirror Symmetry. This project focuses on discovering closed exact Lagrangian submanifolds of Conical Symplectic Resolutions.

All known examples of CSRs are complete hyperkähler manifolds, and in fact satisfy the following definition:

Definition 1.2.1. A **hyperkähler conical symplectic resolution (HKCSR)** is a CSR $(\mathfrak{M}, \omega_{\mathbb{C}})$ that is also a hyperkähler manifold $(\mathfrak{M}, \omega_I, \omega_J, \omega_K)$, such that

1. the complex structure of $\mathfrak{M}$ is I;
2. the holomorphic symplectic form $\omega_{\mathbb{C}}$ of $\mathfrak{M}$ is equal to $\omega_J + i\omega_K$;
3. the S^1-part of the $\mathbb{C}^*$-action acts by isometries, hence preserves ω_I.

On HKCSRs we will only consider $\mathbb{C}^*$-actions φ that are conical and satisfy condition (3), and we call these **HK conical actions**.

As mentioned above, the CSR $(\mathfrak{M}, \omega_{J,K})$ admits a canonical Liouville manifold structure for which the core $\mathfrak{L}$ is the Liouville skeleton. When the $\mathbb{C}^*$-action has weight $k = 1$, the skeleton $\mathfrak{L}$ of $\mathfrak{M}$ is in fact a complex Lagrangian subvariety. In examples, e.g. for certain quiver varieties, when a weight-1 action does not exist, the core $\mathfrak{L}$ turns out to have strictly smaller dimension than $\dim_{\mathbb{C}} \mathfrak{M}$, thus making $\mathfrak{M}$ a subcritical Stein manifold. In such a manifold exact Lagrangians **do not exist** due to the vanishing of symplectic cohomology [Cie02, p. 121] and Viterbo's criterion [Sei08, Prop. 5.1]. Therefore, we restrict observations to CSRs which admit weight-1 $\mathbb{C}^*$-actions, which we call **weight-1 CSRs**. This in particular includes all Nakajima Quiver Varieties [Nak94a] whose underlying graph does not have an edge-loop.

Given a weight-1 CSR $\mathfrak{M}$, its core $\mathfrak{L}$ is a complex projective variety and we denote its equidimensional irreducible components by $\mathfrak{L}_i$,

$$\mathfrak{L} = \bigcup_i \mathfrak{L}_i.$$

One can prove that if a component $\mathfrak{L}_i$ is smooth then it must be an exact Lagrangian submanifold. It follows that it is a non-trivial object of the compact Fukaya category $\mathcal{F}(\mathfrak{M})$. Also, different core components are non-isotopic, in particular they are essentially different objects in $\mathcal{F}(\mathfrak{M})$. So a natural question arises: does an arbitrary weight-1 CSR have smooth components in the core? The main theorem of the thesis, that summarizes Sections 3.1 and 3.2, answers this question as follows:

Theorem 1.2.2. *Given a weight-1 CSR $\mathfrak{M}$, there are at least $N \geq 1$ smooth irreducible components $\mathfrak{L}_i \subset \mathfrak{L}$ of the core, and these are non-isotopic smooth closed exact Lagrangians in $(\mathfrak{M}, \omega_{J,K})$.*
Moreover, $N = \max\{N_1, N_2\} \geq 1$ where

1. *N_1 = the maximal number of commuting weight-1 conical actions.*
2. *N_2 = the number of HK conical actions (if $\mathfrak{M}$ is also a HKCSR).*

Remark 1.2.3. This theorem also holds for any holomorphic symplectic manifold with a weight-1 action that contracts the whole manifold to a compact set. In particular, for the case of a moduli space of Higgs bundles [Hi87], the component of its skeleton that this method recovers is the well-known moduli space of stable vector bundles [AB83].

Given a weight-1 action φ, the theorem above finds the component $\mathfrak{F}_\varphi$ of its fixed locus that is the minimum of the moment map of the S^1-part of φ. Thus, we call **minimal components** the core components obtained by Theorem 1.2.2. We now describe Floer cohomologies of these Lagrangians, proved in Section 3.3.1.
We remark that in this section all cohomologies are assumed over $\mathbb{Z}/2$-**coefficients** instead of more general $\mathbb{Z}$-coefficients, due to the usual issues with the orientation signs in Lagrangian Floer cohomology, as the minimal components are not necessarily spin.[5]

Theorem 1.2.4. *Let $\mathfrak{M}$ be a weight-1 CSR. Then:*

1. *Its minimal components are exact Lagrangians, thus $HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi) \cong H^*(\mathfrak{F}_\varphi, \mathbb{Z}/2)$ for each minimal $\mathfrak{F}_\varphi$.*

2. *Any two minimal components $\mathfrak{F}_{\varphi^1}, \mathfrak{F}_{\varphi^2}$ intersect cleanly, thus we have $HF(\mathfrak{F}_{\varphi^1}, \mathfrak{F}_{\varphi^2}) \cong H(\mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}, \mathbb{Z}/2)$.*

3. *In particular, when $\mathfrak{M}$ is a HKCSR, we have a graded isomorphism $HF^*(\widetilde{\mathfrak{F}}_{\varphi^1}, \widetilde{\mathfrak{F}}_{\varphi^2}) \cong H^{*-\mu}(\mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}, \mathbb{Z}/2)$, where $\mu = \frac{1}{2}(\dim_{\mathbb{C}} \mathfrak{M} - \dim_{\mathbb{R}}(\mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}))$.*

Here, the gradings for minimal components can be canonically chosen as minimal components are special Lagrangians. At the end of Section 3.3.1, we prove that any holomorphic map from a Riemann surface to $\mathfrak{M}$ whose boundary lies on a core $\mathfrak{L}$ has to be constant. That tells us that the Floer product, and more generally, higher operations in the Fukaya category between smooth core components only involve constant solutions. However, saying something more precise about the Floer product between smooth, and in particular, minimal components, is currently of reach and we leave it for some future work.

In Section 3.3.2 we explain how the existence of minimal components yields lower bounds on the ranks of symplectic cohomology $SH^*(\mathfrak{M}, \omega_{J,K})$. Symplectic cohomology in general is notoriously hard to compute explicitly, so we usually have to make do with partial information. We will obtain the lower bounds of its **degree-wise** ranks,

[5] Take $\mathbb{C}P^2$ in $T^*\mathbb{C}P^2$ for instance.

hence it is important to notice that $SH^*(\mathfrak{M}, \omega_{J,K})$ has a canonical $\mathbb{Z}$-grading. Namely, recall that the construction of $SH^*(\mathfrak{M}, \omega_{J,K})$ involves the choice of an $\omega_{J,K}$-compatible almost complex structure S, and that it will be $\mathbb{Z}$-graded if $c_1(T\mathfrak{M}, S) = 0$. As complex symplectic structure $(\omega_{\mathbb{C}}, I)$ on $\mathfrak{M}$ can be upgraded to the almost hyperkähler structure (g, I, J, K), such that $\omega_{\mathbb{C}} = \omega_J + i\omega_K$,[6] we can choose the almost complex structure S in the S^2-family of obtained almost complex structures and then deform it to I (the complex structure of CSR $\mathfrak{M}$), and thus $c_1(T\mathfrak{M}, S) = c_1(T\mathfrak{M}, I) = 0$.[7] Thus, $SH^*(\mathfrak{M}, \omega_{J,K})$ has a $\mathbb{Z}$-grading. It is canonical due to the fact that $H^1(\mathfrak{M}) = 0$ for CSRs.[8]

Denote by $Con_1(\mathfrak{M})$ the set of all conical weight-1 actions on $\mathfrak{M}$. Each $\varphi \in Con_1(\mathfrak{M})$ gives rise to one minimal component $\mathfrak{F}_\varphi$ of the core $\mathfrak{L} \subset \mathfrak{M}$, and as we vary $\varphi \in Con_1(\mathfrak{M})$ we obtain the collection of all (possibly non-distinct) minimal components,

$$\mathrm{Min}(\mathfrak{M}) := \{\mathfrak{F}_\varphi \mid \varphi \in Con_1(\mathfrak{M})\}.$$

We can now state our estimate on the ranks of symplectic cohomology, in which $b_i(X)$ denotes the i-th Betti number of a topological space X and μ_j denotes the Morse-Bott index of a connected component $\mathfrak{F}_j$ of fixed locus for the $\mathbb{C}^*$-action.[9]

Proposition 1.2.5. *Let $(\mathfrak{M}, \varphi)$ be a weight-1 CSR, and let $\mathfrak{F} = \sqcup \mathfrak{F}_j$ be the fixed locus of φ decomposed into connected components $\mathfrak{F}_j$. Then*

$$rk\,(SH^k(\mathfrak{M}, \omega_{J,K})) \;\geq \sum_{\{\mathfrak{F}_j \subset L \,|\, L \in \mathrm{Min}(\mathfrak{M})\}} b_{k-\mu_j}(\mathfrak{F}_j)$$

for all $k \geq 0$. In particular, $rk\,(SH^{dim_{\mathbb{C}}\mathfrak{M}}(\mathfrak{M})) \geq |Min(\mathfrak{M})|$.

It is a general feature of weight-1 conical actions on CSRs that the fixed sets $\mathfrak{F}_j$ are bijectively distributed among components of the core, [Gi15, Prop. 4.6.1]. In the above sum we use precisely those that lie in *minimal* components. Thus, we get the following corollary:

Corollary 1.2.6. *When all core components of a CSR $\mathfrak{M}$ are minimal, the symplectic cohomology is degree-wise bounded from below by the singular cohomology,*

$$rk(SH^k(\mathfrak{M})) \geq rk(H^k(\mathfrak{M})),\ \forall k.$$

[6]Due to deformation retraction $Sp(2n, \mathbb{C})$ to its compact form $Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n)$.

[7]The vanishing $c_1(T\mathfrak{M}, I) = 0$ holds as I-holomorphic volume form $\omega_{\mathbb{C}}^n$ trivialises the canonical bundle $\mathcal{K}$, and $c_1(T\mathfrak{M}, I) = -c_1(\mathcal{K})$.

[8]Choices of $\mathbb{Z}$-gradings correspond to choices of trivialisation of the canonical bundle, and these are labelled by $H^1(\mathfrak{M})$.

[9]Which makes sense, as one can view the fixed locus as the critical locus of the moment map for the S^1-part of φ, and this moment map is a Morse-Bott function due to [AB83, Ki84].

This for example happens for the minimal resolution $X_{\mathbb{Z}/n} \to \mathbb{C}^2/(\mathbb{Z}/n)$ of a Du Val singularity of type A. As $X_{\mathbb{Z}/n}$ contracts to the wedge of $n-1$ (minimal) spheres, from the last corollary we get

$$\mathrm{rk}(SH^2(X_{\mathbb{Z}/n})) \geq \mathrm{rk}(H^2(X_{\mathbb{Z}/n})) = n - 1.$$

It is known that $\mathrm{rk}(SH^2(X_{\mathbb{Z}/n})) = n - 1$ by [EL17, Cor. 42], so in this case Proposition 1.2.5 gives the actual rank on the degree-2 part of symplectic cohomology.

Next, we proceed towards the examples, by considering two different families of CSRs where we count certain weight-1 conical actions. By Theorem 1.2.2 these yield minimal components, thus exact Lagrangians. In the second family of examples (Section 1.4) we use also some other methods to find and generate additional smooth core components, and, apart from being exact Lagrangian submanifolds, these are interesting in their own right.

1.3 Minimal components in Quiver varieties of type A

The content of this section summarises the results of Chapter 4.

We find families of minimal components in Nakajima quiver varieties of type A. Here we will not work through the construction of quiver varieties – it will be explained in Section 4.1. We just mention that from a graph $Q = (Q_0, Q_1)$, two integer-valued vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}_0^{Q_0}$, and a generic parameter $\zeta \in \mathbb{R}^{Q_0}$, one obtains the quiver variety $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$. Their main use in Geometric Representation Theory is that Borel-Moore homologies of their cores yield representations of Kac-Moody Lie algebras [Nak98, Thm. 10.2]. We consider the case of quiver varieties whose underlying graph Q is of Dynkin type A, which we will denote by $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.

In the construction of quiver varieties [Nak94a], Nakajima orients a graph Q, and then uses that same orientation in order to construct a conical weight-1 $\mathbb{C}^*$-action. We generalise this by fixing the orientation for the construction of a quiver variety, but then we use all possible orientations for constructing a family of conical weight-1 actions which we call **Nakajima actions** (Section 4.2). The number of different orientations on the A_n Dynkin graph is 2^{n-1}, but some of them may yield equivalent actions. In Section 4.3 we compute the exact number of non-equivalent Nakajima actions, and we now briefly explain this.

Given vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}_0^n$, there is the associated **dominant vector** $\mathbf{v}' = \mathbf{v}'(\mathbf{v}, \mathbf{w}) \in \mathbb{N}_0^n$ which comes from the representation theory of $\mathfrak{sl}_{n+1}$.[10] Instead of computing it here, we refer the reader to the end of Section 4.3. For vectors $\mathbf{v}$ and $\mathbf{w}$, we define a number $N(\mathbf{v}, \mathbf{w}) > 0$ that is easily computable combinatorially. Namely, it depends only on the support of the vectors $\mathbf{v}$ and $\mathbf{w}$, that is, positions where they non-vanish. In particular, if $\mathbf{v} > 0$ then $N(\mathbf{v}, \mathbf{w}) = N(\mathbf{w}) := \prod_{k=1}^{m-1}(s_{k+1} - s_k + 1)$, where $S = \{s_1, \ldots, s_m\}$ is the support of $\mathbf{w}$. We prove the following:

Theorem 1.3.1. *Given a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ of type A, there are exactly $N(\mathbf{v}', \mathbf{w})$ non-equivalent Nakajima actions, hence at least the same number of minimal components of its core,*

$$|\mathrm{Min}(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}))| \geq N(\mathbf{v}', \mathbf{w}).$$

For example, this theorem gives all components of the core in the case of the aforementioned Du Val singularities, seen as quiver varieties (Example 4.3.16). Given this theorem, the natural question to ask is if there are some other weight-1 conical actions, apart from Nakajima actions. In Section 4.4 we partially prove that there are none, where "partially" refers to "amongst certain weight-1 actions that one can construct on a quiver variety." Namely, on a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ there is:

- An $\omega_\mathbb{C}$-symplectic action by the group $GL(\mathbf{w}) := \prod_{i=1}^n GL(w_i)$.
- A natural weight-2 conical $\mathbb{C}^*$-action, which we call the **full quiver action**.

The $GL(\mathbf{w})$-action on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is rather natural because a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is constructed by $\omega_\mathbb{C}$-symplectic reduction for the $GL(\mathbf{v})$-action on the flat vector space $M(V, W)$ which includes vector spaces V_i and W_i of dimensions v_i and w_i, respectively. The full quiver action comes from the dilation action on $M(V, W)$. Since these two actions commute, by composing the latter with 1-parameter subgroups of $GL(\mathbf{w})$, one can construct families of weight-2 actions on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ that we call **twisted full actions**. Considering the even[11] and conical ones, we get the following:

Proposition 1.3.2. *Let $\mathbf{v}$ be a dominant vector. Then the twisted full actions on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ which are even and conical are precisely the squares of Nakajima actions.*

[10] More generally, these exist for any semisimple Lie algebra, but here we consider Dynkin type A graphs only.

[11] An action is "even" if it is the square of another action.

Thus, at least in the case of a dominant vector, we do not get more weight-1 actions other than Nakajima ones by using the twisted full actions. In addition, we suspect that, apart from twisted full actions, no other conical actions commuting with the full quiver action should exist. Thus, Nakajima actions should induce the only minimal components of the core given by such conical actions. However, this is currently out of reach – we speculate on this in Section 4.4.1.

1.4 Smooth components of Springer fibres

The content of this section summarises the results of Chapter 5. In addition to finding exact Lagrangians, its results are also of interest to Springer theory.

We construct two different families of smooth components of generalised Springer fibres of type A. The first family arises as minimal components in Resolutions of Slodowy varieties (thus, viewing Springer fibres as cores of those resolutions), whereas the other family comes as a generalisation of the so-called Richardson components of the *ordinary* Springer fibres. Motivated by the latter, we also define quasi-Richardson smooth components and prove some of their features. In the end, we show that one can generate many more smooth components of generalised Springer fibres using these smooth components and the so-called crystal operators.

Generalised Springer fibres of type A have been much investigated in Geometric Representation Theory. Their cohomologies provided irreducible representations of Weyl groups [Spr78, KaLu80, LuSpa85] and of $U(\mathfrak{sl}_n)$ [Gi91]. Moreover, they recover certain Parabolic Categories $\mathcal{O}$ [Str09] and Khovanov Arc Algebras [SW12, Sch12]. Their singular cohomology ring is well-known [BrOs11]. Still, little is known about their topology, and in particular, smoothness of their irreducible components. The state-of-the-art work [FrMe10] on this subject covers only the ordinary Springer fibres. In particular, it was not known whether an arbitrary generalised Springer fibre has a single smooth component. We answer this question affirmatively, using Theorem 1.2.2.

Let us first define Springer fibres and Slodowy varieties. Given a composition $p = (p_1, \ldots, p_n)$ of n,[12] denote by

$$\mathcal{B}_p := \{0 = F_0 \subset F_1 \subset \cdots F_{n-1} \subset F_n = \mathbb{C}^n \ \mid \dim F_i/F_{i-1} = p_i,\ i = 1, \ldots, n\}$$

[12]Meaning: $p_1 + \cdots + p_n = n, p_i \geq 0$.

the associated partial flag variety. The **Generalised Springer resolution** of type p is a resolution

$$\nu_p : T^*\mathcal{B}_p \to \overline{\mathcal{O}_{p_+^*}} \subset \mathfrak{sl}_n, \quad \{(F,e) \mid F \in \mathcal{B}_p, e \in \mathfrak{sl}_n, eF_i \subset F_{i-1}\} \mapsto e$$

of the closure of a nilpotent orbit $\mathcal{O}_{p_+^*}$ whose matrices have Jordan block partition p_+^*. Here p_+ is the weakly-descending permutation of p, and p_+^* denotes its dual partition (obtained via the dual Young diagram).

A **generalised Springer fibre** $\mathcal{B}_p^e = \nu_p^{-1}(e)$ is any fibre of a generalised Springer resolution. It can be seen as the core of the following CSR,

$$\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p},$$

where $\mathcal{S}_{e,p} := S_e \cap \mathcal{O}_{p_+^*}$ is a **Slodowy variety** and $\widetilde{\mathcal{S}}_{e,p}$ is its resolution. The set S_e, called the Slodowy slice, is an affine space centred at e that is transversal to all nilpotent orbits it meets. All these varieties do not depend, up to an isomorphism, on the choice of the nilpotent element $e \in \mathfrak{sl}_n$ within a given conjugacy class. We therefore denote them by $\mathcal{B}_p^\lambda, S_\lambda, \mathcal{S}_{\lambda,p}, \widetilde{\mathcal{S}}_{\lambda,p}$, where $\lambda = \lambda(e)$ is the Jordan partition of e.[13] In particular, when $p = (1 \dots 1)$ we denote $\mathcal{B}^\lambda := \mathcal{B}_{1\dots 1}^\lambda$, $\mathcal{S}_\lambda := \mathcal{S}_{\lambda,1\dots 1}$ and call them the **ordinary Springer fibre** and **ordinary Slodowy variety,** respectively.

Thus, by Theorem 1.2.2, weight-1 conical actions on $\widetilde{\mathcal{S}}_{\lambda,p}$ yield smooth components of $\mathcal{B}_p^\lambda$. We find a family of such actions similarly to the case of quiver varieties. Namely, it is well-known that on the Slodowy slice S_λ, and hence on $\widetilde{\mathcal{S}}_{\lambda,p}$, there is:

- an $\omega_\mathbb{C}$-symplectic action by a certain group Z_λ; and
- a natural weight-2 conical $\mathbb{C}^*$-action, called the **Kazhdan action**.

As these two actions commute, by composing the latter by 1-parameter subgroups of the former, we obtain a family of weight-2 actions that we call **twisted Kazhdan actions**. Among them, we search for the even and conical ones.

Given a partition λ of n, denote by $\mathbf{w}(\lambda) = (w_1, \dots, w_n)$ the vector such that $\lambda = 1^{w_1} 2^{w_2} \dots n^{w_n}$. The computations of Section 5.2 yield the following:

Theorem 1.4.1. *There is an explicit isomorphism $Z_\lambda \cong GL(\mathbf{w}(\lambda))$. Moreover, there are exactly $N(\mathbf{w}(\lambda))$ different even and conical twisted Kazhdan actions on the Slodowy slice S_λ. The same holds for the ordinary Slodowy variety $\mathcal{S}_\lambda$ and its resolution $\widetilde{\mathcal{S}}_\lambda$.*

[13] The partition of n by sizes of the Jordan blocks of e.

As a corollary, by Theorem 1.2.2, we obtain a family of $N(\mathbf{w}(\lambda))$ smooth components of the ordinary Springer fibre $\mathcal{B}^\lambda$, so

$$|\mathrm{Min}(\widetilde{\mathcal{S}}_\lambda)| \geq N(\mathbf{w}(\lambda)).$$

In Section 5.3 we describe that family explicitly, which we explain briefly now. Firstly, recall by [PaRe06, Sec. 7] that a component of $\mathcal{B}^\lambda$ that is invariant under the action of a parabolic subgroup $P \leq GL(n)$ is called a **Richardson component** of $\mathcal{B}^\lambda$. These are smooth components isomorphic to products of ordinary flag varieties, and are labelled bijectively by the set $\mathrm{Perm}(\lambda^*)$ of permutations of the dual partition of λ. We prove:

Proposition 1.4.2. *In a Springer fibre $\mathcal{B}^\lambda$, the set of $N(\mathbf{w}(\lambda))$ minimal components form an explicit subset of the set of Richardson components.*

By an "explicit subset" we mean that they correspond to an explicitly-computable subset $\mathrm{Good}(\lambda^*) \subset \mathrm{Perm}(\lambda^*)$. We prove Proposition 1.4.2 by noticing that a Richardson component corresponding to $\mu \in \mathrm{Perm}(\lambda^*)$ is fixed under a certain $\mathbb{C}^*$-action on $\mathcal{B}$, which in the case of $\mu \in \mathrm{Good}(\lambda^*)$ agrees with a twisted Kazhdan action on $\mathcal{B}^\lambda$, and that all twisted Kazhdan actions are obtained in this way.

Recall that components of a Springer fibre $\mathcal{B}^\lambda$ are bijectively labelled by the set $\mathbf{Std}^\lambda$ of standard Young tableaux of shape λ, by [Spa76]. We prove in last two subsections of Section 5.3 some further Springer-theoretic properties of the minimal components in $\mathcal{B}^\lambda$, namely that their tableaux are invariant under the so-called Schützenberger involution and that these components are among the type of smooth components of $\mathcal{B}^\lambda$ discovered in [BaZi08, GrZi11].

We believe that the appearance of the same number that counts even and conical actions in Theorems 1.3.1 and 1.4.1 should not be merely a coincidence, as there is an isomorphism:

$$\mathfrak{M}(\mathbf{v}(\lambda,p),\mathbf{w}(\lambda)) \xrightarrow{\cong} \widetilde{\mathcal{S}}_{\lambda,p},\ \ \mathfrak{L}(\mathbf{v},\mathbf{w}) \xrightarrow{\cong} \mathcal{B}^\lambda_p \tag{1.1}$$

between quiver varieties of type A and resolutions of Slodowy varieties, which sends core to core, by [Maf05]. Here $\mathbf{v} = \mathbf{v}(\lambda,p)$ is given by a linear relation. In particular, it is not hard to show that for $p = (1,\dots,1)$, one gets $N(\mathbf{v}(\lambda,p),\mathbf{w}(\lambda)) = N(\mathbf{w}(\lambda))$. So, in that case the count in Theorems 1.3.1 and 1.4.1 is indeed the same. The isomorphism (1.1) yields a corollary of Theorem 1.3.1:

Corollary 1.4.3. *There are at least $N(\mathbf{v}(\lambda,p)',\mathbf{w}(\lambda))$ smooth components in a Springer fibre $\mathcal{B}^\lambda_p$.*

Recall ([Spa76]) that the components of $\mathcal{B}_p^\lambda$ are labelled bijectively by the set $\mathbf{Std}_p^\lambda$ of p-semistandard Young tableaux of shape λ. It is not completely clear to us which tableaux the smooth components from Corollary 1.4.3 should correspond to. If the Maffei isomorphism (1.1) were equivariant with respect to the $\mathbb{C}^* \times GL(\mathbf{w})$ and $\mathbb{C}^* \times Z_\lambda$ actions on the left and right side of it, respectively,[14] then these components would arise as minimal components of twisted Kazhdan actions via Theorem 1.4.1, which would then give us better chances of computing their tableaux. However, this is out of the scope of this thesis.

In Section 5.4 we construct a generalisation of Richardson components, for all generalised Springer fibres. They are smooth components, isomorphic to products of generalised flag varieties. In the fibre $\mathcal{B}_p^\lambda$, they are labelled bijectively by the set $Coar(p) \cap \mathrm{Perm}(\lambda^*)$, where $Coar(p)$ is the set of all permutations μ of λ^* which are coarser than p, or in other words, such that p is a refinement of μ. In particular, as the composition $p = (1, \ldots, 1)$ is finer than any other one, this construction recovers the ordinary Richardson components in $\mathcal{B}^\lambda$. In addition, we find the tableaux in $\mathbf{Std}_p^\lambda$ to which these generalised Richardson components correspond. Motivated by this construction, we also define quasi-Richardson smooth components, which should comprise a much larger family of components. Unlike the case of ordinary Springer fibres (Proposition 1.4.2), minimal components in a generalised Springer fibre $\mathcal{B}_p^\lambda$ **need not** to lie among the Richardson or even quasi-Richardson components, as we see in examples. Thus, this construction yields some genuinely new smooth components.

In the end, in Section 5.5 we use the so-called **crystal operators** [Nak98, Sai02, Sa06] to generate more smooth components of generalised Springer fibres. These operators interchange between the irreducible components of different generalised Springer fibres. In general, they do not preserve smoothness of components, but we prove that under some further assumptions on components they do. Thus, as there are already existing families of minimal and Richardson smooth components, using the crystal operators one could generate, in principle, many other smooth components. Furthermore, as the crystal operators are also topologically well-described, we get that smooth components obtained in this way are iterated Grassmann bundles on the other smooth components. In particular, we believe that in this way one can recover the topological description of components of ordinary Springer fibres $\mathcal{B}^\lambda$ when λ is of hook type, given in [Fu03, Thm. 3.1].

[14] Recall by Theorem 1.4.1 that these are isomorphic groups.

1.5 Symplectic cohomology of Conical Symplectic Resolutions

This is joint work in progress with Alexander Ritter. The content of this section summarises the results of Chapter 6. Although this chapter is self-contained, our research on these topics is ongoing work. In particular, we highlight the technical Assumptions 1 and 2 above Corollary 1.5.10 which we hope to lift in future work.

Consider a conical symplectic resolution $(\mathfrak{M}, \varphi)$. We prove that there is an I-compatible Kähler form ω_I on $\mathfrak{M}$ such that the S^1-part of φ is a Hamiltonian action whose moment map $H : \mathfrak{M} \to \mathbb{R}$ is proper.

Recall the classical construction of symplectic cohomology:

Definition 1.5.1. We say that a symplectic manifold (M, ω) is **convex at infinity** if there exists a compact set K and a symplectomorphism $(M \setminus K, \omega_I) \cong (\Sigma \times [1, +\infty), d(R\alpha))$, where (Σ, α) is a contact manifold.

For such manifolds, the **symplectic cohomology** is defined as the direct limit of Floer cohomologies

$$SH^*(M) := \lim_{F \in \mathcal{H}(M)} HF^*(F), \tag{1.2}$$

over a class $\mathcal{H}(M)$ of Hamiltonians that are **linear at infinity** with respect to the radial coordinate R. A Hamiltonian F is linear at infinity when, outside of a compact set, $F = \lambda R$ for some generic[15] $\lambda > 0$. The morphisms between the Floer cohomologies in the direct limit are given by the continuation maps, and are directed towards a Hamiltonian with the larger slope λ. Thus, the limit lets $\lambda \to \infty$.

Following the ideas from Seidel's influential survey on symplectic cohomology [Sei08], one may ask the following.

Question 1.5.2. *Given a CSR* $(\mathfrak{M}, \omega_I)$*,*

(1) When is it convex at infinity?

(2) Can one always construct its symplectic cohomology $SH^*(\mathfrak{M})$*, and if yes, what is it?*

(3) Can one obtain a Morse-Bott spectral sequence that converges to $SH^*(\mathfrak{M})$*?*

[15] Here generic means not equal to a period of the Reeb vector field for (Σ, α).

It turns out that $(\mathfrak{M}, \omega_I)$ is **almost never** convex at infinity. Indeed, to be convex at infinity the singularity $0 \in \mathfrak{M}_0$ would have to be isolated, and the only such examples are $T^*\mathbb{C}P^n$, for $n \geq 2$ and the minimal resolutions of Du Val singularities $\mathbb{C}^2/\Gamma$, where $\Gamma \leq SU(2)$. The former cannot be convex at infinity for dimension reasons [R14, Rmk. in Sec. 11.1], whereas for the latter, the symplectic cohomology is known to vanish [R10, Thm. 48] and the answer to question (3) is affirmative by McLean-Ritter [McLR18, Sec. 7].

Thus, in general, $(\mathfrak{M}, \omega_I)$ is **highly non-convex:** there are usually many closed holomorphic curves at infinity. Therefore, constructing symplectic cohomology can become rather difficult. Our approach is to use a very natural class of Hamiltonian functions, namely those arising as functions of the moment map of the S^1-part of φ, and to prove that symplectic cohomology is well-defined So, in Section 6.2 we affirmatively answer to question (2) above:

Theorem 1.5.3. *For any CSR $(\mathfrak{M}, \varphi)$, one can construct*

$$SH^*(\mathfrak{M}, \omega_I, \varphi) := \varinjlim_{F \in \mathcal{H}(\mathfrak{M}, \varphi)} HF^*(F), \tag{1.3}$$

and it is a $\mathbb{Z}$-graded $\mathbb{K}$-algebra with respect to the pair-of-pants product.

Here, $\mathbb{K}$ is the Novikov field and $\mathcal{H}(\mathfrak{M}, \varphi)$ is the set of *φ-admissible Hamiltonians*, which outside of some compact set are linear functions $F = \lambda H$ of the moment map H for φ, of generic[16] slope $\lambda > 0$. In the above limit, we let $\lambda \to \infty$.

Remark 1.5.4. Another approach of constructing symplectic cohomology for CSRs could be to show that $(\mathfrak{M}, \omega_I)$ is geometrically bounded and apply Groman's [Gr15] machinery. However, apart from difficulties of proving geometrical boundedness, which is a necessary condition in order for Groman's symplectic cohomology to be well-defined, in order to connect his work to ours one would also need to prove that our Hamiltonians satisfy the dissipativity condition he imposes, which could be difficult.

In particular, when the slope of F is small, $HF^*(F) \cong H^*(\mathfrak{M})$[17] recovers the ordinary cohomology, and as a part of the direct limit we have maps

$$c_\lambda : H^*(\mathfrak{M}) \to HF^*(F_\lambda),$$

where F_λ has slope λ. Using the computations from Section 6.3, In Section 6.4 we prove:

[16] Here generic means not equal to a period of an S^1-orbit.
[17] NB All cohomologies are assumed to be in $\mathbb{K}$-coefficients in this section.

Proposition 1.5.5. *For any CSR* $(\mathfrak{M}, \varphi)$*, we have* $SH^*(\mathfrak{M}, \omega_I, \varphi) = 0$.

This vanishing result implies that each class $a \in H^*(\mathfrak{M})$ has a filtration value λ, defined to be the infimum of λ for which $c_\lambda(a) = 0$. Therefore, we have the following corollary:

Corollary 1.5.6. *Given a CSR* $(\mathfrak{M}, \varphi)$*, there is an associated filtration* $F^\varphi_\lambda H^*(\mathfrak{M})$ *by ideals of the ordinary cohomology ring* $H^*(\mathfrak{M})$*,*[18] *compatible with the cohomological grading.*

We remark here that there is an interest in filtrations on cohomology of CSRs in the representation-theoretic literature. Namely, in a recent paper [BeSch18], the authors construct filtrations on cohomologies of Springer fibres, which are, as we saw in Section 1.4, one of the principal examples of (cores of) CSRs. Their filtrations is compatible with the cohomological grading, just as ours, so one can ask e.g. what is the relation between these two filtrations on the top degree of cohomology. In particular, in the example of Springer fibre that corresponds to Du Val singularities of type A_n, we find a choice of the $\mathbb{C}^*$-action φ that yields (rank-wise) the same filtration as theirs. Apart from this, we remark also that, in the examples of resolutions of Du Val singularities (and possibly for other symplectic quotient singularities $\mathbb{C}^{2n}/\Gamma$) this Floer-theoretic method gives a refinement of McKay correspondence.[19]

Choosing different conical actions φ, this method yields a family $\{F^\varphi_\lambda\}_\varphi$ of filtrations on $H^*(\mathfrak{M})$. We have developed tools to compute these filtrations in examples, and we believe that, for different choices of φ, these should be different, as suggested by examples. Thus, although the symplectic cohomology $SH^*(\mathfrak{M}, \omega_I, \varphi)$ does not distinguish between the actions φ, the filtrations F^φ_λ should.

In Section 6.5 using the Morse-Bott argument, we compute explicitly the Floer cohomology for **pure Hamiltonians** λH :

Proposition 1.5.7. *For generic* $\lambda > 0$,

$$HF^*(\lambda H) \cong \bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)].$$

Here, $\mathfrak{M}^\varphi = \mathfrak{F} = \sqcup_\alpha \mathfrak{F}_\alpha$ is the decomposition of φ-fixed locus of into connected components, and $\mu_\lambda(\mathfrak{F}_\alpha)$ are certain degree shifts that are calculable only by knowing the weight-decomposition of the tangent space induced by the $\mathbb{C}^*$-action φ. For small

[18] In general, this should be quantum-cup ideals, but for CSRs quantum cohomology is known to be trivial, due to deformation of the complex structure I to an affine variety.

[19] As explained further in Remark 6.4.6.

λ, these shifts become the Morse-Bott indices μ_α of $\mathfrak{F}_\alpha$,[20] thus in this model, the continuation map $c_\lambda : H^*(\mathfrak{M}) \to HF^*(F_\lambda)$ becomes

$$\bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \to \bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)], \tag{1.4}$$

This presentation of the continuation map gives some hope in understanding the filtration $F^\varphi_\lambda H^*(\mathfrak{M})$ better. We discuss this in Section 6.10.1.

In Section 6.6 we construct a cofinal family of Hamiltonians $\{H_\lambda\}_\lambda$ of type $H_\lambda = c(H)$ with carefully chosen $c : \mathbb{R} \to \mathbb{R}$ that is convex and λ-linear at infinity, which admits a filtration on their Floer chain complexes:

Theorem 1.5.8. *Given a CSR $(\mathfrak{M}, \varphi)$, one can construct a cofinal family of Hamiltonians H_λ of slope λ, and a filtration on their Floer chain complexes $CF^*(H_\lambda)$, so that the fixed locus $\mathfrak{M}^\varphi$ yields a subcomplex and the other 1-orbits are filtered by the value of H.*

This was a tricky proof since we are in a highly non-exact setup: even at infinity we do not have the exactness needed to build a filtration as in [McLR18, Sec. 6]. Our new filtration allows us to quotient out by the subcomplex of fixed points of φ, thus obtaining the "positive symplectic cohomology" $SH^*_+(\mathfrak{M}, \omega_I, \varphi)$.

Corollary 1.5.9. *Given a CSR $\mathfrak{M}$ and an S^1-action φ, there is a canonical isomorphism*

$$SH^{*-1}_+(\mathfrak{M}, \omega_I, \varphi) \xrightarrow{\cong} H^*(\mathfrak{M}).$$

Thus, we get new cohomology models for the ordinary cohomology of a CSR $\mathfrak{M}$, labelled by its conical $\mathbb{C}^*$-actions φ, whose chain level generators involve certain Hamiltonian S^1-orbits. In the special case of resolutions of Du Val singularities this leads to the McKay correspondence via Floer theory [McLR18]. Corollary 1.5.9 is interesting especially as the singular cohomology of CSRs is not yet classically understood. Even for the case of quiver varieties, only recently a set of ring-generators for its singular cohomology was obtained by McGerty-Nevins [MN18].

As usual in homological algebra, the filtration on the chain complex $CF^*(H_\lambda)$ induces a spectral sequence E_r^{pq} that converges to its homology,

$$E_r^{pq} \Rightarrow HF^k(H_\lambda), \text{ where } E_1^{pq} = \begin{cases} H^q(\mathfrak{M}), & p = 0, \\ HF^k_{\text{loc}}(\mathcal{O}_p, H_\lambda), & p < 0, \\ 0, & \text{otherwise} \end{cases} \tag{1.5}$$

[20]Recall by [AB83, Ki84] that the moment map H of an S^1-action on a Kähler manifold is a Morse-Bott function, thus $\mathfrak{F}_\alpha$ are its critical submanifolds.

Here, the $\mathcal{O}_p$ are parametrised 1-periodic orbits of H_λ appearing on an energy level $\mathcal{S}_p = \{x \mid H = H_p\}$ and $HF^k_{\text{loc}}(\mathcal{O}_p, H_\lambda)$ is their local Floer cohomology, that counts only the Floer cylinders between solutions in $\mathcal{O}_p$. By $k = p+q$ we denote the **total degree.** Here we are using the **Morse-Bott model** for Floer cohomology of $HF^*(H_\lambda)$ instead of standard one (given by a time-dependent perturbation). The reason is that the perturbation in principle may ruin our filtration on the chain complex $CF^*(H_\lambda)$. Bourgeois-Oancea [BO09a, BO09b] shown that, in the case when manifolds of 1-orbits are circles, these two models compute the same Floer cohomology. We believe that their reasoning extends for more general Morse-Bott manifolds (i.e. not only circles), but proving that would be rather a substantial amount of work and is outside of scope of this thesis. Hence, we will make it as an assumption.

Assumption 1. Morse-Bott Floer complex for Hamiltonian H_λ computes the same Floer cohomology as the one obtained from a time-dependent perturbation.

Denote by $B^\varphi_p := \{x(0) \mid x \in \mathcal{O}_p\} \subset \mathcal{S}_p$, the manifold of 1-orbits of H_λ in the slice $\mathcal{S}_p$ and split it into connected components $B^\varphi_p = \sqcup B^\varphi_{p,c}$. The recent work [KwKo16] of Kwon-van Koert proves that, under certain assumptions for the manifolds of 1-orbits $B^\varphi_{p,c}$, the local Floer cohomology above is isomorphic to the Morse, hence ordinary cohomology, with a certain degree shift. We prove all those assumptions in our setup, except for the very restrictive *symplectic triviality* (ST) condition that asks for symplectic triviality of the tangent bundle of the ambient space restricted to manifolds $B^\varphi_{p,c}$. It is actually unlikely that this condition is satisfied for CSRs. Thus, one would have to bypass it somehow, which we leave for some future work. For now, we put the result [KwKo16, Prop. B.4] of Kwon-van Koert as an assumption.

Assumption 2. $HF^*_{\text{loc}}(\mathcal{O}_p, H_\lambda) \cong \bigoplus_c H^{*-\mu(B^\varphi_{p,c})}(B^\varphi_{p,c})$.

Now, letting λ to go to infinity in (1.5), we obtain the following in the direct limit:

Corollary 1.5.10. *Given a CSR $(\mathfrak{M}, \varphi)$, under Assumptions 1 and 2, there is a convergent spectral sequence*

$$E(\varphi)^{pq}_r \Rightarrow SH^k(\mathfrak{M}, \omega_I, \varphi), \text{ where } E(\varphi)^{pq}_1 = \begin{cases} H^q(\mathfrak{M}), & p = 0, \\ \bigoplus_c H^{*-\mu(B^\varphi_{p,c})}(B^\varphi_{p,c}), & p < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.6}$$

Thus, the answer to Question 1.5.2(3) is affirmative.

The importance of these spectral sequences is that from them one can read-off the filtrations $F^{\varphi}_{\lambda}H^*(\mathfrak{M})$ on ordinary cohomology obtained in Corollary 1.5.6. Denote by T_p the period of 1-orbits in B_p when seen as orbits of H (recall $H_\lambda = c(H)$).

Proposition 1.5.11. *The filtration $F^{\varphi}_{\lambda}H^*(\mathfrak{M})$ by slope λ is the subspace of the 0-th column of $E(\varphi)^{p,q}_r$ given as the image of the edge-differentials arising from the columns p that have $T_p \leq \lambda$.*

One can compute the cohomologies of the manifolds $B^{\varphi}_{p,c}$ in practice by viewing them as hypersurfaces at infinity of the submanifolds that consist of $\mathbb{Z}/m$-isotropic points under the φ-action. We also obtain the formula for degree shifts $\mu(B_{p,c})$ that one can use in practice, knowing only the S^1-weight decomposition of the tangent spaces of the fixed locus $\mathfrak{M}^{\varphi}$. We prove some further properties of the spectral sequences $E(\varphi)^{p,q}_r$, such as that they are periodic (with a downward shift) and also central-symmetric within the periodic blocks.

With the last paragraph in mind, one can compute the E_1-page of the spectral sequence in the examples. In particular, in Section 6.9 we compute the spectral sequences for generalised Springer resolutions of type A, minimal resolutions of Du Val singularities and resolutions of ordinary Slodowy varieties $\widetilde{\mathcal{S}}_\lambda$, for certain partitions λ.

In the end we mention that one can generalize this construction of symplectic cohomology beyond CSRs. We briefly explain it now and refer the reader to Section 6.10.2 for details. Consider a pseudoholomorphic S^1-equivariant proper map

$$\begin{array}{c} S^1 \overset{\varphi}{\curvearrowright} (\mathfrak{M}, I) \\ \pi \downarrow \\ S^1 \overset{\varphi}{\curvearrowright} (X, J) \end{array}$$

such that $(\mathfrak{M}, \omega_I, I)$ is a Kähler manifold with a holomorphic $\mathbb{C}^*$-action φ whose S^1-part is a Hamiltonian action with moment map H, and (X, ω, J), is a symplectic manifold that is convex at infinity, for which the S^1-action is J-holomorphic and agrees with the Reeb flow. There are a lot of interesting examples that fall in the realm of this definition, extending CSRs, like equivariant resolutions of singularities of affine varieties; in particular of weighted homogeneous singularities. Moreover, celebrated Moduli space of Higgs Bundles [Hi87] are also examples of this construction.

In this setup one can, analogously to Theorem 1.5.3,[21] define symplectic cohomology as a direct limit of Floer cohomologies

$$SH^*(\mathfrak{M}, \omega_I, \varphi) := \lim_{F \in \mathcal{H}(\mathfrak{M}, \varphi)} HF^*(F),$$

where $\mathcal{H}(\mathfrak{M}, \varphi)$ is the class of Hamiltonians $F : \mathfrak{M} \to \mathbb{R}$ which at infinity are linear functions of H.

Assuming further that $c_1(\mathfrak{M}) = 0$ and that the $\mathbb{C}^*$-action acts with a positive weight on the canonical bundle $\mathcal{K}$ of $\mathfrak{M}$, analogously to Proposition 1.5.5, one can get the vanishing result

$$SH^*(\mathfrak{M}, \omega_I, \varphi) = 0,$$

from which one gets a filtration of $QH^*(\mathfrak{M})$ by quantum-cup ideals.[22] Without the assumption about the $\mathbb{C}^*$-action on the canonical bundle, one could possibly get some interesting non-vanishing symplectic cohomologies $SH^*(\mathfrak{M}, \omega_I, \varphi)$.

The filtration constructed in Theorem 1.5.8 uses the filtration functional on $\mathbb{C}^N$ which is the special case of the functional defined in [McLR18, Sec. 6] for any symplectic manifold convex at infinity. Thus, by analogous arguments as for CSRs, one can obtain Hamiltonians H_λ on $\mathfrak{M}$ whose Floer chain complexes are filtered, hence induce the positive symplectic cohomology $SH^*_+(\mathfrak{M}, \omega_I, \varphi)$ and Morse-Bott Floer spectral sequences

$$E(\varphi)_r^{p,q} \Rightarrow SH^*(\mathfrak{M}, \omega_I, \varphi).$$

Still, in order to compute these spectral sequences in practice, one has to deal with the technical issues, which we have already encountered for CSRs (recall Assumption 1 and Assumption 2). For now, we hope to find an argument that proves or bypasses this assumptions for CSRs, but it could be possible to find one that also works for the general setup described in this section.

1.6 Statement of originality

The work written in this thesis is independent work of the author, except for (by the order in the thesis):

- Section 2.1, which is a brief review on Conical Symplectic Resolutions taken from the literature, except for Remark 2.1.2 that came out of discussion with Nicholas Proudfoot, and Definition 2.1.14 that is from the author.

[21] Where for the role of (X, ω) we use $(\mathbb{C}^N, \omega_{\mathbb{C}^N})$, as $\mathfrak{M}_0$ is affine, hence embeds $\mathbb{C}^*$-equivariantly into $\mathbb{C}^N$.

[22] Recall that for CSRs quantum product is just the cup product.

- Lemma 3.1.5, that was pointed out to the author by Alexander Ritter and Kevin McGerty.
- Proposition 3.3.2, that was pointed out to the author by Nigel Hitchin.
- Section 4.1 which is a review on Quiver Varieties taken from work of Hiraku Nakajima [Nak94a].
- Proposition 4.3.22, whose second part is due to Hiraku Nakajima.
- Section 5.1 which is a review on basics of Springer theory, taken from various sources.
- Sections 5.3.1, 5.3.2, 5.4.1, 5.4.2 which are reviews on some relevant topics in Springer theory, taken from the papers by Pagnon-Ressayre [PaRe06], Brundan-Ostrik [BrOs11] and Fresse [Fr09a].
- Section 5.5.1, which is a review on some basic facts about crystal operators, taken from [Sa06] and [Kas95].
- Proposition 5.5.7 which was proved by Kevin McGerty.
- Chapter 6 which is joint work with Alexander Ritter.

In addition, the author would like to thank (in the alphabetical order):

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  - Suggesting computing the shift μ in Theorem 3.3.5.
  - Pointing out that the previous version of Theorem 3.3.14 was false, along with the counterexample to it, given in Remark 3.3.15.
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  - Pointing out Proposition 2.2.6 to the author.
  - The idea that led to Proposition 6.1.2 and its proof.
  - Pointing out the filtration (6.13) along with Proposition 6.4.7 to the author.
- Nicholas Wilkins, for suggesting a generalisation outlined in Section 6.10.2.

Chapter 2

On Conical Symplectic Resolutions

In this chapter we define Conical Symplectic Resolutions, list their examples and prove some of their general features that we are going to use later in the text.

2.1 Basics

We will first give the definition and list examples of conical symplectic resolutions, taken from [BPW16, Sec. 2]. For any algebraic geometric language, we always assume that the ground field is $\mathbb{C}$.

Definition 2.1.1. A **conical symplectic resolution** (CSR) is a smooth complex-symplectic variety $(\mathfrak{M}, \omega_{\mathbb{C}})$ which is a $\mathbb{C}^*$-equivariant projective resolution of a normal singular affine variety $\mathfrak{M}_0$

$$\begin{array}{c} \mathbb{C}^* \overset{\varphi}{\curvearrowright} \mathfrak{M} \\ \pi \downarrow \\ \mathbb{C}^* \overset{\varphi}{\curvearrowright} \mathfrak{M}_0 \end{array} \tag{2.1}$$

such that under the $\mathbb{C}^*$-action φ the symplectic form $\omega_{\mathbb{C}}$ has a weight $k \in \mathbb{N}$

$$t \cdot \omega_{\mathbb{C}} = t^k \omega_{\mathbb{C}}$$

which we call the **weight of the action** φ, and the $\mathbb{C}^*$-action contracts $\mathfrak{M}_0$ to a single fixed point $x_0 \in \mathfrak{M}_0$:

$$\forall x \in \mathfrak{M}_0, \quad \lim_{t \to 0} t \cdot x = x_0.$$

We will call such $\mathbb{C}^*$-actions **conical**. Algebraically, it means that the coordinate ring of $\mathfrak{M}_0$ has a non-negative grading

$$\mathbb{C}[\mathfrak{M}_0] = \bigoplus_{n \geq 0} \mathbb{C}[\mathfrak{M}_0]^n,$$

such that $\mathbb{C}[\mathfrak{M}_0]^0 = \mathbb{C}$. Here, the **weight space** $\mathbb{C}[\mathfrak{M}_0]^n$ denotes the subspace of $\mathbb{C}[\mathfrak{M}_0]$ consisting of functions of weight n under the induced pull-back action.[1]
The fibre over the fixed point

$$\mathfrak{L} := \pi^{-1}(x_0)$$

is called the **core** of $\mathfrak{M}$.

Notice that our definition of CSR is slightly different from the original one from [BPW16, Sec. 2], as we require $\mathfrak{M}_0$ to be singular, in order to exclude the trivial example $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$,[2] which is non-interesting for the material written in this thesis. Namely, its core is just a point, so there are no Lagrangians in it (the content of Chapter 4). Also, its S^1-invariant Kähler structure is exact and Liouville indeed, so the content of Chapter 6 is already established for this example (see [Oan04, Sec. 3]).

Notice further difference from the original definition of CSR, that we do not require $\mathfrak{M}_0$ to be the affinisation $\mathrm{Aff}(\mathfrak{M}) := \mathit{Spec}(\Gamma(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}}))$ of $\mathfrak{M}$, although we do require $\mathfrak{M}_0$ to be a normal variety. However, one can prove that these two definitions are equivalent:

Lemma 2.1.2. *In Definition 2.1.1 of CSR, requiring $\mathfrak{M}_0$ to be normal is equivalent to requiring for $\mathfrak{M}_0$ to be the affinisation of $\mathfrak{M}$.*

Proof. Assume that $\mathfrak{M}_0$ is the affinisation on $\mathfrak{M}$. Then, the coordinate rings $\mathbb{C}[\mathfrak{M}] := \Gamma(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}})$ and $\mathbb{C}[\mathfrak{M}_0]$ are isomorphic. Being smooth, $\mathfrak{M}$ is normal, thus its local rings $\mathcal{O}_{\mathfrak{M},p}$ are integrally closed. As an intersection of integrally closed rings, the coordinate ring $\mathbb{C}[\mathfrak{M}] = \cap_{p\in\mathfrak{M}}\mathcal{O}_{\mathfrak{M},p}$ is also integrally closed. It follows that the coordinate ring of $\mathfrak{M}_0$ is integrally closed. Since $\mathfrak{M}_0$ is affine, this implies that it is normal.

Assume now that $\mathfrak{M}_0$ is normal. As $\mathfrak{M}_0$ is affine, we have a factorization of π through the affinisation map, $\mathfrak{M} \to \mathrm{Aff}(\mathfrak{M}) \to \mathfrak{M}_0$, thus it is enough to show that the second map is an isomorphism. One of the properties of the affinisation map is that complete connected subvarieties get shrunk to points. In particular, as the fibres of $\pi : \mathfrak{M} \to \mathfrak{M}_0$ are connected (Proposition 2.1.11) and projective (since π is projective), they shrink to points via the map $\mathfrak{M} \to \mathrm{Aff}(\mathfrak{M})$, which yields that the map $g : \mathrm{Aff}(\mathfrak{M}) \to \mathfrak{M}_0$ is a bijection, in particular, has finite fibres. Moreover, it is birational as $\mathfrak{M} \to \mathfrak{M}_0$ is. Thus, one can use a variant of Zariski's main theorem

[1] I.e. satisfying $(t \cdot f)(x) = f(t \cdot x) = t^n f(x)$.

[2] And indeed this condition only excludes $\mathbb{C}^{2n}$. Namely, having a smooth affine variety $\mathfrak{M}_0$ with a $\mathbb{C}^*$-action that contracts it to a single point, by the Białynicki-Birula decomposition (Theorem 2.3.3) applied to its $\mathbb{C}^*$-invariant compactification (which exist by [Su74, Thm. 1]), we get that it must be an affine space indeed.

which states that a birational morphism with finite fibres to a normal variety is an isomorphism onto an open subset. Hence, g is isomorphism to an open subset of $\mathfrak{M}_0$, thus being surjective, it is an isomorphism indeed. ■

Remark 2.1.3. We remark that although the definition of a conical symplectic resolution comes from [BPW16], these objects were already considered by various authors, most notably Kaledin [Ka06, Ka08, Ka09] and Namikawa [Nam08, Nam11].

Given a CSR $\mathfrak{M}$ there could be (and in general there are) more $\mathbb{C}^*$-actions φ that fit into the definition above. Fixing such an action φ, we will denote the CSR by $(\mathfrak{M}, \varphi)$. We define a special type of CSRs on which we will be focused in search for exact Lagrangians (Chapter 3).

Definition 2.1.4. A **weight-1 CSR** is a conical symplectic resolution $\mathfrak{M}$ which has a weight-1 conical action.

The known examples of CSRs to date are:

1. Minimal resolutions of Du Val singularities, i.e. $\widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$, where $\Gamma \leq SL(2, \mathbb{C})$. They are weight-1 CSRs.

2. Hilbert schemes of n-points $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma}) \to \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ on those minimal resolutions. They are weight-1 CSRs.

3. $\mathfrak{M} = T^*(G/P)$ for a semisimple[3] algebraic group G and a parabolic subgroup P, and $\mathfrak{M}_0$ is the affinisation of this variety. There is a finite morphism from $\mathfrak{M}_0$ to a nilpotent orbit closure; in type A this map is an isomorphism.[4] These are weight-1 CSRs.

4. $\mathfrak{M}$ is a hypertoric variety associated to a simple, unimodular hyperplane arrangement [BD00, Pro08]. They are weight-1 CSRs if the arrangement is coloop-free [HP04].

5. Nakajima quiver varieties. They are weight-1 CSRs when the underlying graph has no edge-loops [Nak94a].

[3]Considering more general reductive groups instead, one actually does not get more examples. Namely, given a reductive group G, one passes to the adjoint quotient $G_{ad} := G/Z(G)$, which is semisimple indeed (here $Z(G)$ is the centre of G). Now, denoting by P' the image of P in G_{ad} it is immediate that $P' \leq G_{ad}$ is parabolic and $G/P \cong G_{ad}/P'$.

[4]In detail: The image of the moment map $\mu : T^*(G/P) \to \mathfrak{g}^* \cong \mathfrak{g}$ of G-action on $T^*(G/P)$ is the closure $\overline{\mathcal{O}_P}$ of a nilpotent orbit $\mathcal{O}_P \subset \mathfrak{g}$ (here, isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ is provided by Killing form, as $\mathfrak{g}$ is semisimple). Then, it is known that the induced map $\mathfrak{M}_0 \to \overline{\mathcal{O}_P}$ is generically finite. If it is generically 1-1, it is the normalization map of $\overline{\mathcal{O}_P}$.

6. Resolution of a transverse slice to one Schubert variety Gr^μ in an affine Grassmannian inside another Gr^λ. When λ is a sum of minuscule coweights, this variety has a natural conical symplectic resolution constructed from a convolution variety.

7. Higgs/Coulomb branches of moduli spaces [Nak15, BFN16].[5]

Now we will give a couple of general topological facts about conical symplectic resolutions.

Firstly, given a CSR $\mathfrak{M}$ of complex dimension $2n$, we have that the top exterior power $\omega_{\mathbb{C}}^n$ of its complex-symplectic form $\omega_{\mathbb{C}}$ trivialises the canonical bundle $\mathcal{K} := \Lambda^{2n,0}T^*\mathfrak{M}$. Using the general fact that $c_1(\mathcal{K}) = c_1(T^*\mathfrak{M}) = -c_1(T\mathfrak{M})$, this implies:

Lemma 2.1.5. *Any CSR $\mathfrak{M}$ satisfies $c_1(T\mathfrak{M}, I) = 0$, where I is its complex structure.*

Moreover, we notice that the complex-symplectic structure on $\mathfrak{M}$ can be upgraded to an almost hyperkähler structure, which we define now:

Definition 2.1.6. Given a manifold M an **almost hyperkähler structure** on it is given by the quadruple (g, I, J, K), where g is a Riemannian metric and I, J, K are g-orthogonal almost complex structures satisfying $IJ = K$. This yields non-degenerate 2-forms $\omega_I, \omega_J, \omega_K$, defined by $\omega_S(\cdot,\cdot) := -g(\cdot, S\cdot)$, for $S = I, J, K$. Almost hyperkähler structure is called **hyperkähler** if we have $\nabla^g I = \nabla^g J = \nabla^g K = 0$, where ∇^g is the Levi-Civita connection of g. In particular, this implies that I, J, K are complex structures and $\omega_I, \omega_J, \omega_K$ are Kähler forms.

Lemma 2.1.7. *Any CSR $(\mathfrak{M}, I, \omega_{\mathbb{C}})$ can be enriched with an almost hyperkähler structure (g, I, J, K), such that $\omega_{\mathbb{C}} = \omega_J + i\omega_K$. In particular, we have that the almost complex structures $S_t := cost J + sint K$ satisfy $c_1(T\mathfrak{M}, S_t) = 0$.*

Proof. Briefly, a complex-symplectic structure on a complex manifold of dimension $2n$ corresponds to a (torsion free) $Sp(2n,\mathbb{C})$-structure on tangent bundle of $\mathfrak{M}$, and an almost hyperkähler structure to a $Sp(n)$-structure, and the key thing is that $Sp(2n,\mathbb{C})$ deformation retracts to $Sp(n) = Sp(2n,\mathbb{C}) \cap U(2n)$, hence a CSR can be enriched with an almost hyperkähler structure indeed. Now we will explain this in more details, for the convenience of the reader.

[5]Strictly speaking, the objects of a Coulomb branch are only $\mathfrak{M}_0$, affine Poisson varieties with conical actions. When such an object has a symplectic resolution, it is a CSR.

Firstly, having a complex symplectic structure $\omega_{\mathbb{C}}$ on $\mathfrak{M}$, we have local trivialisations of $\pi : T\mathfrak{M} \to \mathfrak{M}$, given by Darboux bases, i.e. vector fields $\{e_i, f_j\}_{i=1\dots n, j=1\dots,n}$ such that $\omega_{\mathbb{C}}(e_i, e_j) = \omega_{\mathbb{C}}(f_i, f_j) = 0, \omega_{\mathbb{C}}(e_i, f_j) = \delta_{ij}$. Given two such trivialisations

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^{2n}, \quad \varphi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{C}^{2n},$$

on their overlap we have

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^{2n} &\to (U_i \cap U_j) \times \mathbb{C}^{2n}, \\ (b, v) &\mapsto (b, g_{ij}(b)v), \end{aligned}$$

where the transition functions $g_{ij}(b) \in Sp(2n, \mathbb{C})$. Thus, one gets an associated $Sp(2n, \mathbb{C})$-principal bundle $P \to \mathfrak{M}$ using the transition functions g_{ij}. As $Sp(2n, \mathbb{C})$ deformation retracts to its compact form $Sp(n)$, the quotient $Sp(2n, \mathbb{C})/Sp(n)$ is contractible, hence by [Hu94, Cor 2.4, Ch. VI] there is a $Sp(n)$-reduction of P, that is, a principal $Sp(n)$-bundle Q such that $P \cong Q \times_{Sp(n)} Sp(2n, \mathbb{C})$. Then, by [Hu94, Thm. 4.1, Ch. VI],[6] that means that there are maps $r_i : U_i \to Sp(2n, \mathbb{C})$ such that $r_j(b)g_{ij}(b)r_i(b)^{-1} \in Sp(n)$, for every i, j. Next, denote

$$\widetilde{r}_i : U_i \times \mathbb{C}^{2n} \to U_i \times \mathbb{C}^{2n}, \ (b, v) \mapsto (b, r_i(b)v)$$

and define new local trivialisations $\widetilde{\varphi}_i := \widetilde{r}_i \circ \varphi_i$ of $T\mathfrak{M}$. Then, for all i, j we have

$$\begin{aligned} \widetilde{\varphi}_j \circ \widetilde{\varphi}_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^{2n} &\to (U_i \cap U_j) \times \mathbb{C}^{2n}, \\ (b, v) &\mapsto (b, r_j(b)g_{ij}(b)r_i(b)^{-1}v), \end{aligned}$$

thus, as $r_j(b)g_{ij}(b)r_i(b)^{-1} \in Sp(n)$, we get a $Sp(n)$-structure on $T\mathfrak{M}$. Now we will show that it yields an almost hyperkähler structure (g, I, J, K) on $\mathfrak{M}$, whose complex-symplectic part is the given one, $\omega_{\mathbb{C}} = \omega_J + i\omega_K$.

Denote by $(g^0, I^0, J^0, K^0, \omega_{\mathbb{C}}^0)$ the standard corresponding structures on $\mathbb{C}^{2n} \cong \mathbb{H}^n$. First, notice that $\varphi_i^* \omega_{\mathbb{C}}^0 = \omega_{\mathbb{C}}$, and $\varphi_i^* I^0 = I$ (where pull-back is understood fibre-wise, $\varphi_i : T_b\mathfrak{M} \to \{b\} \times \mathbb{C}^{2n}$), as we have used complex Darboux basis for defining φ_i. As $\widetilde{r}_i^* \omega_{\mathbb{C}}^0 = \omega_{\mathbb{C}}^0, \ \widetilde{r}_i^* I^0 = I$, we have $\widetilde{\varphi}_i^* \omega_{\mathbb{C}}^0 = \omega_{\mathbb{C}}, \ \widetilde{\varphi}_i^* I^0 = I$ as well.

Next, we define structures on $\mathfrak{M}$ using the pull-back from local trivialisations $\widetilde{\varphi}_i$:

$$g := \widetilde{\varphi}_i^* g^0, J := \widetilde{\varphi}_i^* J^0, K := \widetilde{\varphi}_i^* K^0.$$

They are well-defined (i.e. independent of i) as the transition functions $\widetilde{\varphi}_j \circ \widetilde{\varphi}_i^{-1}$ fibre-wise lie in $Sp(n)$, in particular, preserve g^0, J^0, K^0. Moreover, together with I and $\omega_{\mathbb{C}}$,

[6] Notice the slight difference between stated therein and here, as our transition function g_{ij} is their g_{ij}^{-1} and we take r_i to be their r_i^{-1}.

they form an almost hyperkähler structure on $\mathfrak{M}$, being a pull-back of the standard almost hyperkähler structure on $\mathbb{C}^{2n}$. This proves the first part of the lemma.

Now, notice that given a vector $u = (u_I, u_J, u_K) \in S^2$, a linear combination

$$S_u = u_I I + u_J J + u_K K$$

is an almost complex structure.[7] Thus, we get a smooth S^2-family of almost complex structures S_u, hence corresponding complex vector bundles $(T\mathfrak{M}, S_u)$ are all isomorphic,[8] and together with Lemma 2.1.5 we get $c_1(T\mathfrak{M}, S_u) = 0$. In particular, we have $c_1(T\mathfrak{M}, S_t) = 0$ for $S_t = cost J + sint K$. ■

Now we will consider the topology of fibres of CSRs. Fibres of projective resolutions are in general singular,[9] thus carry a mixed Hodge structure. However, in the case of *symplectic* resolutions their Hodge structure is pure and concentrated in (p, p)-classes, due to Kaledin. In particular, this applies to CSRs:

Theorem 2.1.8. [Ka06, Prop. 2.12] *Let F be a fibre of a CSR $\pi : \mathfrak{M} \to \mathfrak{M}_0$. For all odd k we have $H^k(F, \mathbb{C}) = 0$, while for even $k = 2p$ the Hodge structure on $H^k(F, \mathbb{C})$ is pure of weight k and Hodge type (p, p). In particular, this is true for $\pi^{-1}(0) = \mathfrak{L}$.*

As the $\mathbb{C}^*$-action contracts $\mathfrak{M}_0$ to a point, it also contracts $\mathfrak{M}$ to $\pi^{-1}(0) = \mathfrak{L}$. Although it is not a deformation retraction ($\mathbb{C}^*$ acts non-trivially on $\mathfrak{L}$), the argument from [BPW16][10] proves a homotopy equivalence, which then together with the previous theorem gives an information on cohomology of $\mathfrak{M}$ itself:

Corollary 2.1.9. [BPW16, Prop. 2.5] *Let $\mathfrak{M}$ be a CSR. The inclusion $\mathfrak{L} \subset \mathfrak{M}$ is a homotopy equivalence. Thus, for all odd k we have $H^k(\mathfrak{M}, \mathbb{C}) = 0$, while for even $k = 2p$ we have $H^{2p}(\mathfrak{M}, \mathbb{C}) = H^{p,p}(\mathfrak{M}, \mathbb{C})$.*

Remark 2.1.10. Corollary 2.1.9 is independently proved for quiver varieties by Nakajima, even using the integer coefficients for cohomology [Nak01, Sec. 7]. The argument therein uses an explicit decomposition of the diagonal of the fixed loci of the $\mathbb{C}^*$-action. Similar arguments for general CSRs are written in Kaledin's work [Ka08, Thm. 1.9, Cor. 1.10] and later in Ginzburg's work [Gi15, Sec. 9], though these arguments in general only apply to rational[11] coefficients.

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[7] Indeed, to prove it, one needs the relations $IJ = -JI$, $JK = -KJ$, $KI = -IK$ which follow from $I^2 = J^2 = K^2 = -\mathrm{Id}$ and $K = IJ$.

[8] As S^2 is path-connected.

[9] As they usually consist of few irreducible components, which may be singular as well.

[10] See also [Slo80, Prop. 1, Sec. 4.3].

[11] Thus, any characteristic 0 field.

It is not known, apart from quiver varieties, whether general CSRs indeed have free and even-only cohomology over integer coefficients. They certainly do, when there is another $\mathbb{C}^*$-action on $\mathfrak{M}$ that commutes with the given one and has finitely many fixed points,[12] as in that case the core $\mathfrak{L}$ would be preserved by this action, and paved by the affine spaces (e.g. by Białynicki-Birula decomposition applied to the projective completion of $\mathfrak{M}$ as in Corollary 2.3.4), so the claim about cohomology follows from e.g. [Ja04, Sec. 12.2].

It is also true that the fibres of a CSR π are connected. In order to prove it, we use normality of $\mathfrak{M}_0$.

Proposition 2.1.11. [Ka19] *Given a CSR $\pi : \mathfrak{M} \to \mathfrak{M}_0$, the fibres of π are connected.*

Proof. Take a Stein factorisation of π, that is: there is a variety $\mathfrak{M}_0'$ and morphisms $\pi' : \mathfrak{M} \to \mathfrak{M}_0'$ and $g : \mathfrak{M}_0' \to \mathfrak{M}_0$ such that $\pi = g \circ \pi'$, the fibres of π' are connected, and g is a finite morphism. As π is birational, so is g, but then (a variant of) Zariski's main theorem states that g, being a birational morphism with finite fibres to a normal variety, is an isomorphism onto an open subset of $\mathfrak{M}_0$. As π is surjective, the same holds for g. Hence, g is an isomorphism, thus all fibres of π are connected. ■

Thus, in particular the core $\mathfrak{L} = \pi^{-1}(0)$ is connected. As $\mathfrak{M}$ is homotopy equivalent to it (Corollary 2.1.9), we immediately have:

Corollary 2.1.12. *Any CSR $\mathfrak{M}$ is connected.*

It is known (due to Kaledin-Verbitsky [KaVe02, Thm. 1.1] and Namikawa [Nam08]) that, as a symplectic variety, any CSR $\mathfrak{M}$ has a (topologically trivial) deformation whose base is $H^2(\mathfrak{M}, \mathbb{C})$ and whose generic fibre is an affine algebraic variety. In such varieties, there are no I-holomorphic spheres so the quantum product is equal to the usual (cup) product. As the quantum product is preserved under deformations of the complex structure, we have the following corollary:

Proposition 2.1.13. *There is a ring isomorphism $QH^*(\mathfrak{M}) \cong H^*(\mathfrak{M}, \mathbb{K})$, where $\mathbb{K}$ is a Novikov field*[13] *over an arbitrary field.*

[12] This is not an unusual setup – the CSRs with this additional $\mathbb{C}^*$-action were considered in [BLPW16].

[13] See Definition A.1.1.

2.1.1 Hyperkähler conical symplectic resolutions

For compact Kähler manifolds, by Yau's theorem (together with a Bochner's formula and Berger's holonomy classification) the presence of a holomorphic symplectic form $\omega_{\mathbb{C}}$ guarantees the existence of a Kähler form ω such that $\omega, \mathrm{Re}(\omega_{\mathbb{C}}), \mathrm{Im}(\omega_{\mathbb{C}})$ is a hyperkähler structure. However, it is not known whether this also holds in our non-compact setting. That motivates the next definition:

Definition 2.1.14. Let $(\mathfrak{M}, \omega_{\mathbb{C}})$ be a CSR. We call it a **hyperkähler conical symplectic resolution** and abbreviate by HKCSR if there is a hyperkähler structure $(\mathfrak{M}, \omega_I, \omega_J, \omega_K)$ on it such that:

(1) The complex structure on $\mathfrak{M}$ is I

(2) The holomorphic symplectic form $\omega_{\mathbb{C}}$ of $\mathfrak{M}$ is equal to $\omega_J + i\omega_K$,

(3) The S^1-part of the $\mathbb{C}^*$-action acts by isometries, hence preserves ω_I.

On such CSRs we will only consider $\mathbb{C}^*$-actions satisfying condition (3). When such actions are also conical, we will call them **HK conical actions**.

In fact, all known CSRs ((1)-(7) from Section 2.1) are known to be HKCSR, and they have a HKCSR action. It seems that there is a deep reason why this holds, and why it may hold in general. Hopefully the future literature on CSRs will reveal this phenomenon.

2.2 Symplectic structures on conical symplectic resolutions

In this section we describe *real* symplectic structures on conical symplectic resolutions. Thus we will assume that symplectic forms are real, unless otherwise stated.

2.2.1 Canonical Liouville structure on a CSR

In this section we construct a canonical Calabi-Yau Liouville structure on a given CSR $(\mathfrak{M}, \varphi)$. Moreover, we prove that different commuting conical actions yield isomorphic Liouville structures. Let us start with a definition of Liouville manifolds:

Definition 2.2.1. A **Liouville manifold** (M, ω) is a non-compact exact symplectic manifold $\omega = d\theta$ which has a compact submanifold $K \subset M$ with boundary, such that there is a symplectomorphism

$$(M \setminus int(K), \omega) \cong (\Sigma \times [1, +\infty), d(R\alpha)) \tag{2.2}$$

where $\Sigma = \partial K,\ \alpha = \theta|_{\Sigma}$, R is the coordinate on $[1, \infty]$, and $R\alpha$ pulls back to θ via the symplectomorphism. The subset $M \setminus int(K) \cong (\Sigma \times [1, +\infty))$ we will call the **convex end**. The flow of the vector field Z defined by $i_Z\omega = \theta$ is called the **Liouville flow** of the Liouville manifold $(\mathfrak{M}, \omega)$. Its Lie derivative satisfies $L_Z\omega = \omega$. The **Liouville skeleton** is the set of points in M which do not escape every compact set under the Liouville flow.

Motivated by the last definition, any vector field Z on a symplectic manifold (M, ω) satisfying $L_Z\omega = \omega$ we call a **Liouville vector field.** We have the following useful lemma for finding a Liouville manifold structure on an arbitrary open symplectic manifold.

Lemma 2.2.2. *A symplectic manifold (M, ω) such that*

1. *There is a hypersurface $\Sigma = \partial K$ that bounds a compact submanifold K.*
2. *There is a Liouville vector field Z defined on M which is positively-integrable and non-zero outside of $int(K)$.*
3. *$Z \pitchfork T\Sigma$ with Z pointing outside of K.*

is a Liouville manifold.

In the setup of this lemma, the coordinate R appearing in Definition 2.2.1 is exactly $R = e^t$, where $t > 0$ is the time needed to flow via the vector field Z starting from Σ, in order to reach the given point. In particular, $\Sigma = \{R = 1\}$. Moreover, by the Cartan formula $L_Z\omega = d(i_Z\omega) + i_Z(d\omega) = d(i_Z\omega)$, we see that we can define $\theta = i_Z\omega$.

Next we define isomorphisms between two Liouville manifolds.

Definition 2.2.3. A **Liouville isomorphism** between Liouville manifolds $(M_1, d\theta_1)$ and $(M_2, d\theta_2)$ is a diffeomorphism $\psi : M_1 \to M_2$ satisfying $\psi^*\theta_2 = \theta_1 + df$ where f is compactly supported. It is immediate that any such ψ is symplectic, and compatible with the Liouville flow at infinity.

Let M be a manifold. Given a 1-form θ on it such that $d\theta =: \omega$ is symplectic, together with a compact hypersurface Σ, such that the vector field Z defined by $i_Z\omega = \theta$ satisfies the conditions from Lemma 2.2.2, $(M, d\theta)$ is a Liouville manifold. Thus, assuming those conditions are satisfied, we will call the pair (M, θ) a **Liouville structure** on M.

There is a deformation lemma of Moser type for Liouville structures which we will need:

Lemma 2.2.4. [SeiSm05, Lem. 5] *Let $(M, \theta_t)_{0\leq t\leq 1}$ be a smooth family of Liouville structures on M having the same hypersurface Σ. Then all the $(M, d\theta_t)$ are mutually Liouville isomorphic.*

Remark 2.2.5. We remark that, what we call Liouville manifold here, in [SeiSm05, Sec. 2] is called a *complete finite type convex symplectic manifold*, defined as a triple (M, θ, ϕ), where $\phi : M \to \mathbb{R}$ is a smooth exhausting function, such that the Liouville flow Z of θ is positively-integrable and there is a constant c_0 such that the level sets $\phi^{-1}(c)$ are transversal to Z for $c \geq c_0$. Given a Liouville manifold $(M, d\theta)$, one can turn it into this setup, by choosing $\phi := log(R)$ on the convex end (where R is the radial coordinate, recall Definition 2.2.1), and smooth it out on its complement by a cut-off function. Then, having the family of Liouville structures (M, θ_t), one immediately has that the corresponding triples (M, θ_t, ϕ_t) make a *complete finite type convex symplectic deformation*, which is the condition of cited [SeiSm05, Lem. 5] that ensures Liouville isomorphisms between $(M, d\theta_t)$.

Proposition 2.2.6. *Any CSR $(\mathfrak{M}, \varphi)$ has a canonical Calabi-Yau Liouville manifold structure $(\mathfrak{M}, \omega_{J,K})$, with symplectic form $\omega_{J,K}$ being any non-zero linear combination of the real ω_J and imaginary ω_K part of its holomorphic symplectic form $\omega_{\mathbb{C}} = \omega_J + i\omega_K$, and its Liouville vector field being the $1/k$-multiple of the vector field of the $\mathbb{R}_+$-action, where k is the weight of the $\mathbb{C}^*$-action φ. Moreover, the Liouville skeleton is exactly the core $\mathfrak{L}$.*

Proof. As φ is a weight-k action, we have that $t \cdot \omega_{\mathbb{C}} = t^k\omega_{\mathbb{C}}$ for every $t \in \mathbb{C}^*$. Using the real values of t and taking a derivation this gives us $L_{X_{\mathbb{R}_+}}\omega_{\mathbb{C}} = k\omega_{\mathbb{C}}$, where $X_{\mathbb{R}_+}$ is the vector field of the $\mathbb{R}_+$-part of the action. Therefore, the vector field $Z := \frac{1}{k}X_{\mathbb{R}_+}$ satisfies

$$L_Z\omega_{\mathbb{C}} = \omega_{\mathbb{C}},$$

which gives $L_Z\omega_J = \omega_J$ and $L_Z\omega_K = \omega_K$. The candidate for the hypersurface Σ can be found using the homogeneous generators f_i of the coordinate ring $\mathbb{C}[\mathfrak{M}_0]$. Let their weights be w_i respectively, and denote by w their least common multiple. Define the polynomial on $\mathfrak{M}$

$$\Phi := \pi^*(\sum_i |f_i|^{\frac{2w}{w_i}}). \tag{2.3}$$

As π is $\mathbb{C}^*$-equivariant and the polynomials f_i have weights w_i, denoting the $\mathbb{C}^*$-action on $\mathfrak{M}$ by φ_t we have $\varphi_t^*\Phi = |t|^{2w}\Phi$, thus $L_Z\Phi = 2w\Phi$. As $L_Z\Phi = d\Phi(Z)$, we see that 0 is the only singular value of Φ, and hence for any $\Phi_0 > 0$ the hypersurface

$$\Sigma := \Phi^{-1}(\Phi_0)$$

is smooth. As $d\Phi(Z) = 2w\Phi$, the pair (Σ, Z) satisfies the conditions from Lemma 2.2.2, with respect to an arbitrary linear combination $a\omega_J + b\omega_K$ of symplectic forms, thus yields Liouville structures for each of them. The unital linear combinations $\omega_t = \cos t\omega_J + \sin t\omega_K$ are all Liouville isomorphic due to Lemma 2.2.4. Moreover, scaling of the symplectic form does not change the Liouville structure, thus we conclude that there is a canonical Liouville structure $(\mathfrak{M}, \omega_{J,K})$ on a CSR $(\mathfrak{M}, \varphi)$. The Calabi-Yau property of forms $\omega_{J,K}$ is due to Lemma 2.1.7.

We comment that the choice of the homogeneous generators does not affect the obtained Liouville structure, as one uses the same vector-field, hence can get a symplectomorphism between the corresponding convex ends. Indeed, denote by Φ_0 and Φ_1 two polynomials obtained by the formula (2.3) by using two different sets of homogeneous coordinates. Now, choose two hypersurfaces $\Sigma_0 := \Phi_0^{-1}(c_0)$ and $\Sigma_1 := \Phi_0^{-1}(c_1)$. The flow of the Liouville vector field Z yields a contactomorphism

$$\Psi : (\Sigma_0, \alpha_0 := \theta_{J,K}|_{\Sigma_0}) \to (\Sigma_1, \alpha_1 := \theta_{J,K}|_{\Sigma_1}), \ \Psi^*\alpha_1 = e^f a_0, \tag{2.4}$$

where $f = f(y)$ is the time of the Liouville flow that takes from point $y \in \Sigma_0$ to hit Σ_1. The equality in (2.4) is due to $L_Z\theta_{J,K} = \theta_{J,K}$ and the fact that α_0 and α_1 are restrictions of $\theta_{J,K}$. Finally, Ψ induces a symplectomorphism ϕ of different convex ends $(\Sigma_0 \times [1, +\infty))$ and $(\Sigma_1 \times [1, +\infty))$, defined by

$$\phi : (e^r, y) \to (e^{r-f(y)}, \Psi(y)).$$

In the end, the statement that $\mathfrak{L}$ is the skeleton follows immediately from the proof of Corollary 2.3.4 (equation (2.8)). ■

In particular, as the family $(\mathfrak{M}, \omega_{J,K})$ of Liouville structures are mutually Liouville isomorphic, all corresponding symplectic cohomologies $SH^*(\mathfrak{M}, \varphi, \omega_{J,K})$ are mutually isomorphic (see Appendix A.2.1). In fact, the proof of Proposition 2.2.6 gives their description:

Corollary 2.2.7. *Given any CSR $(\mathfrak{M}, \varphi)$, its symplectic cohomology can be represented*

$$SH^*(\mathfrak{M}, \varphi, \omega_{J,K}) = HF^*(\mathfrak{M}, \Phi, \omega_{J,K})$$

as the Floer cohomology of the polynomial $\Phi = \sum_i |\pi^*(f_i)|^{\frac{2w}{w_i}}$ obtained by lifts $\pi^*(f_i)$ of a set f_i of φ-homogeneous generators of the coordinate ring $\mathbb{C}[\mathfrak{M}_0]$. Here w_i is the weight of f_i, and $w = lcm(w_i)$.

Proof. By the proof of Proposition 2.2.6 we have $L_Z\Phi = 2w\Phi$, and we claim that this implies that $\Phi = \Phi_0 R^{2w}$. Indeed, the given equation implies that $\partial_R \log \Phi = 2w/R$ (as $Z = R\partial_R$), therefore the Liouville flow maps the level set $\Sigma = \Phi^{-1}(\Phi_0)$ to another level set of Φ. This implies that in the $\Sigma \times [1, \infty)$ coordinates from (2.2), Φ is constant in the $T\Sigma$ directions. Thus the Z-directional derivative equation is enough to determine Φ, as claimed. Finally, $\Phi = h(R)$ is an admissible Hamiltonian such that $\lim_{R \to +\infty} h'(R) = +\infty$, so its Floer cohomology $HF^*(\mathfrak{M}, \Phi)$ gives the symplectic cohomology by definition (Appendix A.2.1). ■

Remark 2.2.8. In particular, computing the exact symplectic cohomology $SH^*(\mathfrak{M}, \varphi, \omega_{J,K})$ is an interesting question in itself. Apart from the case of cotangent bundles of generalised flag manifolds $T^*\mathcal{B}_p$ (which is computable due to Viterbo isomorphism [Vi96, Ab15]), amongst all CSRs the exact symplectic cohomology is computed only for resolutions of Du Val singularities of type A and D, due to Etgü-Lekili [EL17, Cor. 42 and Cor. 46], where they have used *Legendrian surgery techniques* [BEE12] as a means of computation.

Next, we show that the Liouville structure constructed in Proposition 2.2.6 does not depend on the choice of the conical action, amongst the commuting ones.

Proposition 2.2.9. *Given a CSR $\mathfrak{M}$, two different conical actions that commute yield isomorphic Liouville structures.*

Proof. Let us have two conical actions φ_0 and φ_1 on $\mathfrak{M}$ that commute and have weights k_1 and k_2 respectively. As in Proposition 2.2.6, $1/k_0$ and $1/k_1$-multiples of their $\mathbb{R}_+$-vector fields yield two Liouville vector fields Z_0 and Z_1, with respect to linear combinations $a\omega_J + b\omega_K$ of symplectic forms. As we proved in the same proposition that a choice of (a, b) does not impact on the Liouville structure, we will use the form ω_J for simplicity. Setting $Y := Z_1 - Z_0$, we define a family of Liouville vector fields

$$Z_t = Z_0 + tY$$

and corresponding primitive 1-forms θ_t of ω defined by $\theta_t = i_{Z_t}\omega_J$. By Lemma 2.2.4 it is enough to prove that the vector fields Z_t intersect the hypersurface $\Sigma_1 = \Phi_1^{-1}(c)$ transversely outwards for some $c > 0$, in other words $\mathrm{d}\Phi_1(Z_t) > 0$. We prove that by projecting to $\mathfrak{M}_0$.

Firstly, as the actions φ_0 and φ_1 commute, there is a set $\{f_k\}_{k=1}^N$ of generators of $\mathbb{C}[\mathfrak{M}_0]$ that are homogeneous under both actions. Thus, we have an embedding $j : \mathfrak{M}_0 \hookrightarrow \mathbb{C}^N,\ p \mapsto (f_1(p), \dots, f_N(p))$. Denote by $z_1, \dots, z_N$ the coordinates in $\mathbb{C}^N$. Denote the weights of f_k by w_k^0 and w_k^1 under the actions φ_0 and φ_1 respectively, and their corresponding least common multipliers by w^0, w^1. Then, the map $\nu := j \circ \pi : \mathfrak{M} \to \mathbb{C}^N$ is $\mathbb{C}^*$-equivariant with φ_0 and φ_1, where the corresponding actions on $\mathbb{C}^N$ are given by

$$t \cdot (z_1, \dots, z_N) = (t^{w_1^i} z_1, \dots, t^{w_N^i} z_N),$$

for $i = 0$ and $i = 1$ respectively. The vector fields of the $\mathbb{R}_+$-parts of these actions on $\mathbb{C}^N$ are given by the standard formulas $V_i(z) = \sum_{k=1}^N w_k^i (x_k \partial_{x_k} + y_k \partial_{y_k})$, where $z_k = x_k + i y_k$.

Now, fixing some $c > 0$, we want to prove $d\Phi_1(Z_t(p)) > 0$ for an arbitrary point $p \in \Phi_1^{-1}(c)$. As $\Phi_1 = \pi^*\alpha_1 = \nu^*\alpha_1$, we have $d\Phi_1(Z_t(p)) = d(\nu^*\alpha_1)(Z_t(p)) = \mathrm{d}\alpha_1(\nu_*(Z_t(p)))$, where $\alpha_1 := \sum_k |z_k|^{\frac{2w^1}{w_k^1}}$, and $\nu_*(Z_t) = \nu_*((1-t)Z_0 + tZ_1) = (1-t)V_0 + tV_1$, as ν is $\mathbb{C}^*$-equivariant. Thus, using $d\alpha_1 = \sum_{k=1}^N \frac{2w^1}{w_k^1}(x_k \mathrm{d}_{x_k} + y_k \mathrm{d}_{y_k})$ we have

$$\begin{aligned} \mathrm{d}\Phi_1(Z_t(p)) &= \mathrm{d}\alpha_1((1-t)V_0 + tV_1) = (1-t)\ \mathrm{d}\alpha_1(V_0) + t\ \mathrm{d}\alpha_1(V_1) \\ &= (1-t)\sum_{k=1}^N \frac{2w^1}{w_k^1} w_k^0 (x_k^2 + y_k^2) + t \sum_{k=1}^N \frac{2w^1}{w_k^1} w_k^1 (x_k^2 + y_k^2), \end{aligned}$$

thus it is positive whenever $\nu(p) = (x_1, y_1, \dots, x_N, y_N) \neq 0$, that is to say, for all $c > 0$.

Thus, by Lemma 2.2.4, $(\mathfrak{M}, d\theta_t)$ are Liouville isomorphic. In particular, $(\mathfrak{M}, d\theta_0)$ and $(\mathfrak{M}, d\theta_1)$ are Liouville isomorphic as well, hence the proposition is proved. ■

We will see examples of commuting conical actions in the following chapters. Namely, there are Nakajima actions on quiver varieties, defined in Section 4.2, and (even and conical) Twisted Kazhdan actions on Slodowy varieties, defined in Section 5.2 (commute due to Lemmas 5.2.22 and 5.2.23). However, we remark that in general, one should always expect that on a CSR there are non-commuting conical actions. Here we give an simple example of such occurrence.

Example 2.2.10. Let us consider an example of $\mathbb{C}^2$, with the standard complex symplectic form $\omega_{\mathbb{C}} = dz_1 \wedge dz_2$. Given an arbitrary polynomial $f \in \mathbb{C}[X]$, there are symplectomorphisms $\Phi_f \in Symp(\mathbb{C}^2, \omega_{\mathbb{C}})$ of type

$$\Phi_f : (z_1, z_2) \mapsto (z_1 + f(z_2), z_2),$$

which, when f is not of type $f(X) = aX$, do **not** commute with the standard conical action $t \cdot (z_1, z_2) = (tz_1, tz_2)$. Thus, by conjugation with Φ_f one obtains an action

$$\Psi_f(t) : (z_1, z_2) \mapsto (tz_1 - tf(z_2) + f(tz_2), tz_2)$$

that does not commute[14] with the standard action and is conical precisely when polynomial f does not have a free term.

Picking $f(X) = X^n$, and a primitive $n+1$-root of unity ε, one can see that the symplectomorphism Φ_{X^n} commutes with the $\mathbb{Z}/(n+1)$-action $\varepsilon \cdot (z_1, z_2) = (\varepsilon z_1, \varepsilon^{-1} z_2)$ on $\mathbb{C}^2$, hence it passes together with the action Ψ_{X^n} to the quotient $\mathbb{C}^2/(\mathbb{Z}/n+1)$, known as Du Val singularity.[15] This quotient has a symplectic resolution which is a CSR with respect to the standard action or the action induced by Ψ_{X^n}. However, these actions do not commute.[16]

Inspired by the previous example, we give the following definition.

Definition 2.2.11. Given a CSR $\mathfrak{M}$, a **conical structure** on it is a maximal set of mutually-commuting conical actions on it.

Hence, as we saw in the previous example, a CSR can possibly have more conical structures. In this new terminology, Propositions 2.2.6 and 2.2.9 show that a conical structure on CSR $\mathfrak{M}$ induces a canonical Liouville structure attached to it.

2.2.2 Non-exact symplectic structures on a CSR

We now show that an arbitrary CSR $\mathfrak{M}$ also admits a non-exact Calabi-Yau Kähler structure which is invariant under the S^1-part of its conical action.

Lemma 2.2.12. *Any CSR $\mathfrak{M}$ has an I-compatible S^1-invariant Calabi-Yau Kähler structure, where I is the complex structure of $\mathfrak{M}$. Moreover, when $\mathfrak{M}$ is not a point, the symplectic form of this structure is non-exact.*

[14] Indeed, this reduces to show that for $-stf(z_2) + sf(tz_2) = -tf(sz_2) + f(tsz_2)$ cannot hold for all s, t, z_2, unless $f(X) = aX$ for some $a \in \mathbb{C}$, which is a standard exercise in polynomial equations.

[15] These are explained in details in Section 6.9.2

[16] One can argue that although they not commute on $\mathbb{C}^2$, passing to the quotient they may commute. However, by explicitly writing down the formulas one sees that if they commute on the quotient, they do on $\mathbb{C}^2$ as well.

Proof. Being projective over an affine variety $\mathfrak{M}_0$, the variety $\mathfrak{M}$ is quasi-projective. Indeed, $\pi : \mathfrak{M} \to \mathfrak{M}_0$ being a projective morphism means that there is an embedding $\mathfrak{M} \hookrightarrow \mathfrak{M}_0 \times \mathbb{C}P^n$ for some n ([Ha77, p.103]). As $\mathfrak{M}_0$ is an affine variety there is an embedding $\mathfrak{M}_0 \hookrightarrow \mathbb{C}^m$ for some m, hence we have the sequence of embeddings

$$\mathfrak{M}_0 \times \mathbb{C}P^n \hookrightarrow \mathbb{C}^m \times \mathbb{C}P^n \hookrightarrow \mathbb{C}P^m \times \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{(m+1)(n+1)-1},$$

the last one being the Segre embedding. Altogether we get a holomorphic embedding $\mathfrak{M} \overset{\iota}{\hookrightarrow} \mathbb{C}P^N$, where $N := ((m+1)(n+1)-1)$. One can pull-back the Fubini-Study form ω_{FS} from $\mathbb{C}P^N$, and then average it over the S^1-part φ_t of the $\mathbb{C}^*$-action

$$\omega_I := \int_{S^1} \varphi_t^*(\iota^*\omega_{FS})dt,$$

getting an I-compatible S^1-invariant symplectic form ω_I on $\mathfrak{M}$. The I-compatibility exactly means that $g(\cdot,\cdot) := \omega_I(\cdot, I\cdot)$ is a Riemannian metric. As I is a complex structure, the triple (g, I, ω) forms an S^1-invariant Kähler structure. It is Calabi-Yau due to Lemma 2.1.5.

Let us explain why this symplectic structure is non-exact, in the case when $\mathfrak{M}$ is not a single point. Then, the core $\mathfrak{L}$ is not a single point, as otherwise $\mathfrak{M}$ would contract to it via the action and thus would be isomorphic to an affine space,[17] hence $\mathfrak{M}_0$ as well, and that does not fit with Definition 2.1.1 of a CSR ($\mathfrak{M}_0$ has to be singular). Thus the core is not a point. Assume first that it is fixed by the $\mathbb{C}^*$-action, i.e. $\mathfrak{M}^{\mathbb{C}^*} = \mathfrak{L}$. Then, $\mathfrak{L}$ is a smooth I-holomorphic, thus ω_I-symplectic submanifold of $\mathfrak{M}$.[18] If ω_I was exact on $\mathfrak{M}$ indeed, its pull-back on $\mathfrak{L}$ would yield an exact symplectic form on a connected closed manifold, which is impossible due to Stokes theorem. Otherwise, there is a point $x \in \mathfrak{L}$ which is non-fixed; its $\mathbb{C}^*$-orbit can be extended to a holomorphic map $u : \mathbb{C}P^1 \to \mathfrak{L}$, by [So75, Lem. II-A].[19] Hence, if $\omega_I = d\theta$ was closed on $\mathfrak{M}$ indeed, the Gromov lemma together with the Stokes formula makes a contradiction, $0 = \int_{\partial \mathbb{C}P^1} u^*\theta = \int_{\mathbb{C}P^1} u^*\omega_I = \frac{1}{2}\int_\Sigma ||du||^2 > 0$. ■

We add a simple lemma regarding these non-exact structures, that is going to be useful to us. Recall first that, given a function H on a symplectic manifold (M, ω), its **Hamiltonian vector field** X_H is defined by $dH(\cdot) = \omega(\cdot, X_H)$. The flow of this field preserves the symplectic form ω.

[17] Due e.g. to Białynicki-Birula decomposition applied to CSRs, Corollary 2.3.4.

[18] Due to Theorem 2.3.1(1a) and Lemma 2.3.2.

[19] Strictly speaking, one can use this result only for a compact Kähler manifold with a $\mathbb{C}^*$-action. However, being smooth, $\mathfrak{M}$ is a normal variety, so due to [Su74, Thm. 1] one can compactify it $\mathbb{C}^*$-equivariantly to a smooth projective variety X, which is then naturally Kähler.

Lemma 2.2.13. *Consider a CSR $(\mathfrak{M}, \varphi)$ with an S^1-invariant I-compatible Kähler structure (g, I, ω_I). The S^1-part of φ has a moment map H, defined with respect to ω_I. The vector field of the $\mathbb{R}_+$-part of φ is $X_{\mathbb{R}_+} = \nabla H$ and the vector field of the S^1-part is $X_{S^1} = X_H$. In particular, $X_H = I\nabla H$.*

Proof. By Corollary 2.1.9, $H^1(\mathfrak{M}) = 0$, hence there is a moment map $H : \mathfrak{M} \to \mathbb{R}$. The S^1-part ($X_{S^1} = X_H$) is immediate from the definition of the moment map. Then, as the $\mathbb{C}^*$-action is holomorphic, we have $X_{S^1} = IX_{\mathbb{R}_+}$. On the other hand, from

$$\omega_I(\xi, X_H) = dH(\xi) = g(\xi, \nabla H) = \omega_I(\xi, I\nabla H),$$

we get that $X_H = I\nabla H$, and thus the claim $X_{\mathbb{R}_+} = \nabla H$ follows immediately. ■

2.3 $\mathbb{C}^*$-geometry of conical symplectic resolutions

In this section we discuss some further geometric features of a conical $\mathbb{C}^*$-action on a CSR $\mathfrak{M}$. In particular, apart from the fixed locus, we consider the points that have cyclical isotropy subgroups under the action. These submanifolds will be used significantly as the components of the spectral sequences in Section 6.8.

Let us first recall a fundamental theorem about S^1-actions on compact Kähler manifolds. By $b_k(X)$ we denote the k-th Betti number of a space X.

Theorem 2.3.1. *Let M be a compact Kähler manifold M with an S^1-action by Kähler isometries. Then:*

(1) The moment map $H : M \to \mathbb{R}$ of the S^1-action is a perfect Morse-Bott function, thus

(a) The critical locus $Crit(H) = \mathfrak{F} = M^{S^1}$ is a smooth Kähler submanifold.

(b) It splits $\mathfrak{F} = \sqcup_{\alpha \in A} \mathfrak{F}_\alpha$ into connected components.

(c) $b_k(M) = \sum_{\alpha \in A} b_{k-\mu(F_\alpha)}(\mathfrak{F}_\alpha)$, where $\mu(\mathfrak{F}_\alpha)$ is the Morse-Bott index of $\mathfrak{F}_\alpha$.

(2) Under the linearised action, the tangent space of M at a fixed point $x \in \mathfrak{F}_\alpha$ splits as a complex S^1-representation

$$T_xM = \oplus_{k\in\mathbb{Z}} H_k(x),$$

where $H_k(x)$ is the weight-k subspace. In particular, $T_x\mathfrak{F}_\alpha = H_0(x)$.

(3) This yields the splitting

$$T_{\mathfrak{F}_\alpha} M = \oplus_{k\in\mathbb{Z}} H_k,$$

of the restriction of the tangent bundle to $\mathfrak{F}_\alpha$*. In particular,* $T\mathfrak{F}_\alpha = H_0$*.*

(4) The equality $\mu(\mathfrak{F}_\alpha) = \dim_{\mathbb{R}}(\oplus_{k<0} H_k)$ *holds, thus the Morse-Bott indices* $\mu(\mathfrak{F}_\alpha)$ *are even.*

Proof. Claim (1) is the standard statement from [AB83, Ki84]. Claim (4) follows from claim (2), as the subspace $\bigoplus_{k<0} H_k$ is the negative vector subspace, and Morse-Bott index $\mu(\mathfrak{F}_\alpha)$ is exactly its (real) dimension. It is even as all H_k are complex. Claim (2) is just the weight-decomposition of an S^1-representation. Finally, let us prove Claim (3). For $x \in \mathfrak{F}_\alpha$ denote by t_x the induced linearised action on $T_x M$ by $t \in S^1$. Now consider the characteristic polynomial $f(x)(X) = \det(X \cdot \mathrm{Id} - t_x)$ (recall this is independent of choices of local coordinates since t_x is an endomorphism of $T_x M$). As the action is Kähler and thus holomorphic, the coefficients of $f(x)(X)$ are holomorphic functions with respect to $x \in \mathfrak{F}_\alpha$. Since $\mathfrak{F}_\alpha$ is a closed connected complex manifold, global holomorphic functions on $\mathfrak{F}_\alpha$ are constant.[20] Thus the coefficients of $f(x)(X)$ are constant. Therefore, the eigenvalues of t_x and their multiplicities are independent of $x \in \mathfrak{F}_\alpha$. Thus, the weight subspace $H_k(x) = Ker(t_x - t^k Id \mid t \in S^1)$ varies smoothly with respect to x and is equidimensional, and thus forms a subbundle H_k of $T_{\mathfrak{F}_\alpha} M$. ∎

Although CSRs are not compact, this theorem holds for them as well, as we now show.

Lemma 2.3.2. *The statement of Theorem 2.3.1 holds for any CSR* $(\mathfrak{M}, \varphi)$*, using the* S^1*-part of* φ *and any* S^1*-invariant Kähler structure on* $\mathfrak{M}$*. Thus the moment map* $H : \mathfrak{M} \to \mathbb{R}$ *of the* S^1*-part of* φ *is a perfect Morse-Bott function. Moreover,*

$$\mathfrak{F} = \bigsqcup_{\alpha\in A} \mathfrak{F}_\alpha := \mathfrak{M}^{\mathbb{C}^*} = \mathfrak{M}^{S^1} = Crit(H) \subset \mathfrak{L}, \tag{2.5}$$

where $\mathfrak{F}_\alpha$ *denote the connected components of* $\mathfrak{F}$*. By perfectness of* H*, we have*

$$b_k(M) = \sum_{\alpha\in A} b_{k-\mu(F_\alpha)}(\mathfrak{F}_\alpha), \tag{2.6}$$

where $\mu(\mathfrak{F}_\alpha)$ *are the Morse-Bott indices of* $\mathfrak{F}_\alpha$*.*

[20]Using that non-constant holomorphic functions are open maps, and open compact subsets of $\mathbb{C}$ are points.

Proof. The proof of claim (1) of Theorem 2.3.1 also works for an open manifold if the negative gradient flow of the moment map of an arbitrary point stays in a compact set. This is true for an CSR $(\mathfrak{M}, \varphi)$ by the following. From Lemma 2.2.13, we have that the image of a negative gradient trajectory of a point x is $\{t \cdot x \mid t \in (0,1]\}$, which lies inside the set $\pi^{-1}(\{t \cdot \pi(x) \mid t \in [0,1]\})$. The last set is compact as π is a projective, hence proper map. The claims (4) and (2) of Theorem 2.3.1 follow immediately.

Observe that $Crit(H) = Zeros(X_H) = \mathfrak{M}^{S^1}$. The third equality in (2.5) is due to holomorphicity of the $\mathbb{C}^*$-action. That $\mathfrak{M}^{\mathbb{C}^*} \subset \mathfrak{L}$ is due to the fact that the only fixed point in $\mathfrak{M}_0$ is x_0, and $\mathfrak{L} = \pi^{-1}(x_0)$.

Thus, fixed locus $\mathfrak{F} = \mathfrak{M}^{\varphi}$ is a closed subvariety of the core $\mathfrak{L} = \pi^{-1}(0)$ which is compact, thus $\mathfrak{F}$ is compact as well. Hence, the argument for claim (3) of Theorem 2.3.1 follows. ■

Hence, given a CSR $(\mathfrak{M}, \varphi)$, the fixed locus $\mathfrak{F} := \mathfrak{M}^{\varphi}$ of the $\mathbb{C}^*$-action φ, and, equivalently, of its S^1-part, is a smooth subvariety of $\mathfrak{M}$. It breaks into connected components $\mathfrak{F} = \sqcup_{\alpha \in A} \mathfrak{F}_\alpha$. Focusing on the $\mathbb{C}^*$-convergence of the points in $\mathfrak{M}$ we get to the following spaces:

$$\mathfrak{L}_\alpha := \left\{ x \in \mathfrak{M} \mid \lim_{t \to \infty} t \cdot x \in \mathfrak{F}_\alpha \right\}, \; \mathfrak{D}_\alpha := \left\{ x \in \mathfrak{M} \mid \lim_{t \to 0} t \cdot x \in \mathfrak{F}_\alpha \right\}. \tag{2.7}$$

One can view these as the upward and downward Morse flow of the moment map H, as the vector field of the $\mathbb{R}_+$-action is the gradient flow ∇H (Lemma 2.2.13). Thus, these are smooth submanifolds in $\mathfrak{M}$. Moreover, on the algebraic side we have a version of the Białynicki-Birula decomposition for CSRs. Recall first the original result that follows from [BB73, Sec. 4].

Theorem 2.3.3 (Białynicki-Birula decomposition)**.** *Consider a smooth projective variety Y with an algebraic $\mathbb{C}^*$-action on it. Denoting the decomposition of the fixed locus $\mathfrak{F} = Y^{\mathbb{C}^*} = \sqcup_\alpha \mathfrak{F}_\alpha$, the upward and downward attracting sets $\mathfrak{L}_\alpha$ and $\mathfrak{D}_\alpha$ defined by (2.7) are locally closed smooth subvarieties that decompose Y, that is, $\sqcup_\alpha \mathfrak{L}_\alpha = \sqcup_\alpha \mathfrak{D}_\alpha = Y$. Moreover, the natural morphisms*

$$\mathfrak{L}_\alpha \to \mathfrak{F}_\alpha, \; x \mapsto \lim_{t \to \infty} t \cdot x,$$

$$\mathfrak{D}_\alpha \to \mathfrak{F}_\alpha, \; x \mapsto \lim_{t \to 0} t \cdot x,$$

are isomorphic to fibre bundles with affine fibres.

Corollary 2.3.4. *Given an CSR* $(\mathfrak{M}, \varphi)$, *its fixed locus* $\mathfrak{M}^\varphi = \mathfrak{F} = \sqcup_{\alpha \in A} \mathfrak{F}_\alpha$ *induces the decompositions* $\mathfrak{L} = \sqcup_\alpha \mathfrak{L}_\alpha, \mathfrak{M} = \sqcup_\alpha \mathfrak{D}_\alpha$ *into smooth locally closed subvarieties, defined by (2.7). The natural morphisms*

$$\mathfrak{L}_\alpha \to \mathfrak{F}_\alpha, \; x \mapsto \lim_{t \to \infty} t \cdot x,$$

$$\mathfrak{D}_\alpha \to \mathfrak{F}_\alpha, \; x \mapsto \lim_{t \to 0} t \cdot x,$$

are isomorphic to fibre bundles with affine fibres.

Proof. As every point $x \in \mathfrak{M}$ has a limit $\lim_{t \to 0} t \cdot x$ and it must be a fixed point, the decomposition $\mathfrak{M} = \sqcup_\alpha \mathfrak{D}_\alpha$ is immediate. To prove the other decomposition, it is enough to prove that

$$\mathfrak{L} = \{x \in \mathfrak{M} \mid \lim_{t \to \infty} t \cdot x \text{ exist}\}, \tag{2.8}$$

which we do as in [Gi15, Lem. 4.5.2]. Namely, having a point x such that $\lim_{t \to \infty} t \cdot x$ exist, the same will be true for its projection $y := \pi(x)$, and then as the limit $\lim_{t \to 0} t \cdot y = 0$ always exist, the map $\mathbb{C}^* \to \mathfrak{L}$, $t \mapsto t \cdot y$ extends to a regular map $f : \mathbb{P}^1 \to \mathfrak{M}_0$. As $\mathfrak{M}_0$ is affine, f must be a constant, hence $y = f(1) = f(0) = 0$, thus $x \in \pi^{-1}(0) = \mathfrak{L}$. Now, choose arbitrary $x \in \mathfrak{L}$. As $\mathfrak{L}$ is projective, hence complete, the map $\mathbb{C}^* \to \mathfrak{L}$, $t \mapsto t \cdot x$ extends to a regular map $\mathbb{P}^1$, thus the limit $\lim_{t \to \infty} t \cdot x$ exists.

Now, recall by the proof of Lemma 2.2.12 that $\mathfrak{M}$ is a quasi-projective variety. Being smooth, $\mathfrak{M}$ is normal, thus by [Su74, Thm. 1] one can extend the $\mathbb{C}^*$-action on it to a smooth projective variety Y which compactifies it, such that $\mathfrak{M} \subset Y$ is a $\mathbb{C}^*$-invariant Zariski open subvariety. As Y is smooth projective, it has the decompositions with the properties as in Theorem 2.3.3, thus it is enough to show that the decompositions $\mathfrak{L} = \sqcup_\alpha \mathfrak{L}_\alpha, \mathfrak{M} = \sqcup_\alpha \mathfrak{D}_\alpha$ are sub-decompositions of those in Y (i.e. every piece of decomposition in Y is either entirely contained in $\mathfrak{M}$ or in $Y \setminus \mathfrak{M}$). This is immediate: given any fixed locus component $\mathfrak{F}_\alpha$ in $\mathfrak{M}$, if there is a point $x \in Y$ such that $\lim_{t \to 0} t \cdot x \in \mathfrak{F}_\alpha / \lim_{t \to \infty} t \cdot x \in \mathfrak{F}_\alpha$, there is a neighbourhood U of $0/\infty$ such that $\{t \cdot x \mid t \in U\}$ is contained in $\mathfrak{M}$ (as $\mathfrak{M}$ is an open neighbourhood of $\mathfrak{F}_\alpha$ in Y). Thus, as $\mathfrak{M} \subset Y$ is $\mathbb{C}^*$-invariant, x also belongs to $\mathfrak{M}$. ∎

Recall the weight decomposition of the tangent bundle of $\mathfrak{M}$ restricted to $\mathfrak{F}_\alpha$ into subbundles,

$$T_{\mathfrak{F}_\alpha} \mathfrak{M} = \oplus_{k \in \mathbb{Z}} H_k. \tag{2.9}$$

Białynicki-Birula decomposition [BB73, Thm 4.3(iii)] also describes the tangent spaces of bundles $\mathfrak{L}_\alpha$ and $\mathfrak{D}_\alpha$ at $\mathfrak{F}_\alpha$,

$$T_{\mathfrak{F}_\alpha} \mathfrak{L}_\alpha = \oplus_{k \le 0} H_k, \qquad T_{\mathfrak{F}_\alpha} \mathfrak{D}_\alpha = \oplus_{k \ge 0} H_k. \tag{2.10}$$

In the literature it is often cited that by the Białynicki-Birula paper [BB73], the morphisms $\mathfrak{L}_\alpha \to \mathfrak{F}_\alpha$, $\mathfrak{D}_\alpha \to \mathfrak{F}_\alpha$, are isomorphic to complex[21] vector bundles on $\mathfrak{F}_\alpha$, but it is actually never proven. We fix that gap in the following proposition.

Proposition 2.3.5. *In the setup of Theorem 2.3.3, the morphisms $\mathfrak{L}_\alpha \to \mathfrak{F}_\alpha$ and $\mathfrak{D}_\alpha \to \mathfrak{F}_\alpha$ are isomorphic to $\mathbb{C}^*$-equivariant complex vector bundles over $\mathfrak{F}_\alpha$.*

Proof. Let us prove the statement for the morphism $f : \mathfrak{D}_\alpha \to \mathfrak{F}_\alpha$, the other one follows verbatim. Firstly, by [BB73],[22] we actually have a more precise information on the fibre bundle f than stated in Theorem 2.3.3. Namely, its local trivializations are $\mathbb{C}^*$-equivariant, where the $\mathbb{C}^*$-action on the fibre V is linear and isomorphic to the $\mathbb{C}^*$-action on the tangent space $T_{\mathfrak{F}_\alpha}\mathfrak{D}_\alpha = \oplus_{k\geq 0} H_k$ (this action is uniform throughout $\mathfrak{F}_\alpha$, recall proof of Theorem 2.3.1). Thus, given trivialisations over two open sets $Y_i, Y_j \subset \mathfrak{F}_\alpha$ that intersect, the transition functions

$$g_{ij} : Y_i \cap Y_j \to \mathrm{Aut}_{\mathbb{C}^*}(V), \tag{2.11}$$

land into the group $\mathrm{Aut}_{\mathbb{C}^*}(V)$ of $\mathbb{C}^*$-equivariant algebraic[23] automorphisms of the affine space V. Thus, switching to the coordinate ring of V and denoting $\dim V = n$, those are graded isomorphisms of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, where $deg(x_i) = a_i$ are the weights of the $\mathbb{C}^*$-action on V. In particular, when the weights on V are all equal, we are left with linear automorphisms only, thus the transition functions of bundle f land in $GL(V)$ indeed. Hence, in that case f has a structure of algebraic, thus holomorphic vector bundle, which is even stronger than what we need. Otherwise, one should expect non-linear transition functions in general, but the key point is that we can reduce the structure group $\mathrm{Aut}_{\mathbb{C}^*}(V)$ to its linear subgroup $\mathrm{GL}_{\mathbb{C}^*}(V)$, by the following lemma.

Lemma 2.3.6. *Given a complex vector space V with a linear $\mathbb{C}^*$-action with positive weights, the group of linear $\mathbb{C}^*$-invariant maps $GL_{\mathbb{C}^*}(V)$ is a deformation retract of the group $Aut_{\mathbb{C}^*}(V)$ of $\mathbb{C}^*$-invariant polynomial automorphisms[24] of V.*

Proof. Split $V = \oplus_{i=1}^r V_i$ into weight spaces $\{V_i\}_{i=1\ldots r}$ such that their with weights increase with $i = 1 \ldots r$. It is clear that each map in $\mathrm{Aut}_{\mathbb{C}^*}(V)/\mathrm{GL}_{\mathbb{C}^*}(V)$ has to

[21] Meaning: $\mathbb{R}$-smooth vector bundles with complex vector space as fibres and transition functions in $GL(n, \mathbb{C})$.

[22] More precisely: Proofs of [BB73, Thm. 2.5] and [BB73, Thm. 4.1].

[23] Recall that the setup of [BB73] is algebraic.

[24] Meaning: Polynomial maps whose inverses are also polynomial.

preserve V_i. Thus, denoting the coordinates in V_i by $x_i^1, \dots, x_i^{s_i}$ and the vectors $x_i = (x_i^1, \dots, x_i^{s_i})$, we have that an arbitrary map in $\mathrm{Aut}_{\mathbb{C}^*}(V)$ is of type

$$(x_1, \dots, x_r) \mapsto (L_1(x_1), L_2(x_2) + p_2(x_1), L_3(x_2) + p_3(x_1, x_2), \dots, L_r(x_r) + p_r(x_1, \dots, x_{r-1})) \tag{2.12}$$

where p_i are arbitrary polynomials and L_i are invertible linear maps. Indeed, passing to the map of coordinate ring $\mathbb{C}[V] = \mathbb{C}[(x_i^j)_{i,j}]$ we see inductively by i that L_i need to be invertible in order to get all monomials $\{x_i^j\}_{j=1,\dots,s_i}$ in the image. Furthermore, the map (2.12) is going to be invertible polynomial for arbitrary choices of p_i, as one can show inductively as well (the inverse is $(x_1, \dots, x_r) \mapsto (L_1^{-1}(x_1), L_2^{-1}(x_2 - p_2(L_1^{-1}(x_1))), \dots))$. Hence, the group $\mathrm{Aut}_{\mathbb{C}^*}(V)$ deformation retracts (by letting all coefficients of $p_2, \dots, p_r$ in (2.12) to go to zero) to its subgroup given by linear maps

$$(x_1, \dots, x_r) \mapsto (L_1(x_1), L_2(x_2), L_3(x_2), \dots, L_r(x_r)),$$

which is precisely the group $\mathrm{GL}_{\mathbb{C}^*}(V)$. Thus, the lemma is proved. ∎

Now the proposition follows by the following standard lemma from the theory of fibre bundles. Indeed, putting $G = \mathrm{Aut}_{\mathbb{C}^*}(V)$ and $H = \mathrm{GL}_{\mathbb{C}^*}(V)$ in it, we get that $f : \mathfrak{D}_\alpha \to \mathfrak{F}_\alpha$ is isomorphic to a V-fibre bundle with transition functions in $\mathrm{GL}_{\mathbb{C}^*}(V)$, hence it is a $\mathbb{C}^*$-invariant complex vector bundle.

Lemma 2.3.7. *Given a F-fibre bundle E whose transition functions land in Lie group $G \leq \mathit{Diff}(F)$ and a closed subgroup $H \leq G$ which is a deformation retract of G, there is a F-fibre bundle E' with transition functions landing in H, which is isomorphic to E as a F-fibre bundle.*

Proof. We can associate a principal G-bundle P to E, constructed using the transition functions of E. Then, the associated F-fibre bundle $P \times_G F$ is isomorphic to E as a F-fibre bundle. Now, as $H \leq G$ is a deformation retract, G/H is contractible, hence by [Hu94, Cor 2.4, Ch. VI] there is a H-reduction of P, that is, a principal H-bundle Q such that $P \cong Q \times_H G$. Thus, by [Hu94, Thm. 3.1, Ch. VI] we have isomorphism of F-fibre bundles $P \times_G F \cong Q \times_H F$. By definition, transition functions of the associated bundle $E' := Q \times_H F$ are the same as the transition functions of Q, hence they land in H. As $E' \cong E$, the lemma is proved. ∎

∎

We will now focus on points in a CSR $\mathfrak{M}$ that have finite isotropies under the $\mathbb{C}^*$-action on a CSR. Thus, we introduce the following definition:

Definition 2.3.8. A **torsion point** of a CSR $(\mathfrak{M}, \varphi)$ is a point that has non-trivial isotropy group under the $\mathbb{C}^*$-action φ. In particular, given an arbitrary integer $m \geq 2$, a $\mathbb{Z}/m$**-torsion point** is a point whose isotropy group contains $\mathbb{Z}/m$. In the same way we define a **torsion submanifold** and a $\mathbb{Z}/m$**-torsion submanifold** in $\mathfrak{M}$, respectively ($\mathbb{Z}/m$-)torsion points and ($\mathbb{Z}/m$-)**torsion subvarieties** in $\mathfrak{M}_0$. Given an arbitrary $m \in \mathbb{N}$, we introduce the notation:

$$\mathcal{R}_{\mathbb{Z}/m} := \{\mathbb{Z}/m\text{-torsion points in } \mathfrak{M}\} \qquad \mathcal{P}_{\mathbb{Z}/m} := \{\mathbb{Z}/m\text{-torsion points in } \mathfrak{M}_0\}.$$

Notice that $\mathcal{R}_{\mathbb{Z}/m} = \mathfrak{M}^{\mathbb{Z}/m}$ and $\mathcal{P}_{\mathbb{Z}/m} = \mathfrak{M}_0^{\mathbb{Z}/m}$, where the $\mathbb{Z}/m$-actions are the induced action from φ. As $\pi : \mathfrak{M} \to \mathfrak{M}_0$ is $\mathbb{C}^*$-equivariant, we have that $\pi(\mathcal{R}_{\mathbb{Z}/k}) = \mathcal{P}_{\mathbb{Z}/k}$.

Next, we define the torsion refinements of Białynicki-Birula bundles $\mathfrak{D}_a, \mathfrak{L}_\alpha$.

Definition 2.3.9. Given a connected component $\mathfrak{F}_\alpha$ of the fixed locus in a CSR $(\mathfrak{M}, \varphi)$, and an integer $m \neq 0$ that appears as a weight in (2.9), define its **homogeneous bundles**

$$\mathcal{H}_{m>0} := \{x \in \mathfrak{M} \mid \lim_{t\to 0} t \cdot x \in \mathfrak{F}_\alpha,\ x \text{ is } \mathbb{Z}/m\text{-torsion}\}.$$

$$\mathcal{H}_{m<0} := \{x \in \mathfrak{M} \mid \lim_{t\to \infty} t \cdot x \in \mathfrak{F}_\alpha,\ x \text{ is } \mathbb{Z}/m\text{-torsion}\}$$

The term "bundle" in the last definition makes sense due to the following lemma:

Lemma 2.3.10. *Any homogeneous bundle $\mathcal{H}_m$ is a complex vector bundle over the fixed locus component $\mathfrak{F}_\alpha$ to which it converges. Moreover, $T_{\mathfrak{F}_\alpha}\mathcal{H}_m = \oplus_{t\geq 0} H_{tm}$.*

Proof. This basically follows from the Białynicki-Birula decomposition and Proposition 2.3.5, applied on the connected component $\mathfrak{M}_\alpha^{\mathbb{Z}/m}$ of $\mathfrak{M}^{\mathbb{Z}/m}$ containing $\mathfrak{F}_\alpha$. Firstly, notice that $\mathfrak{M}^{\mathbb{Z}/m}$ is a smooth closed subvariety of $\mathfrak{M}$.[25] Thus, $\mathfrak{M}^{\mathbb{Z}/m}$ is a smooth quasi-projective variety,[26] and so is $\mathfrak{M}_\alpha^{\mathbb{Z}/m}$, hence we can apply the Białynicki-Birula decomposition theorem to it, the same way as for CSRs (Corollary 2.3.4). Together with Proposition 2.3.5, we have that the morphisms $\mathcal{H}_m \to \mathfrak{F}_\alpha,\ x \mapsto \lim_{t\to\infty} t \cdot x$, are isomorphic to complex vector bundles over $\mathfrak{F}_\alpha$. The equation $T_{\mathfrak{F}_\alpha}\mathcal{H}_m = \oplus_{t\geq 0} H_{tm}$ follows from the fact that

$$T_{\mathfrak{F}_\alpha}\mathfrak{M}_\alpha^{\mathbb{Z}/m} = \oplus_{t\in\mathbb{Z}} H_{tm}, \tag{2.13}$$

and the description of the tangent spaces of Białynicki-Birula pieces (2.10). We can show (2.13) using the $\mathbb{Z}/m$-equivariant tubular neighbourhood of $\mathfrak{F}_\alpha$. Namely, as the

[25] E.g. due to [CGP10, Prop. A.8.10], as $\mathbb{Z}/m$ is an affine algebraic group acting algebraically on the smooth variety $\mathfrak{M}$.

[26] As a closed subvariety of quasi-projective $\mathfrak{M}$.

$\mathbb{Z}/m$-action can be made isometric with respect to some Riemann metric, by [Bre72, Thm. 2.2, Ch. VI] there is a $\mathbb{Z}/m$-invariant tubular neighbourhood $\mathcal{N}\mathfrak{F}_\alpha \subset \mathfrak{M}$ of $\mathfrak{F}_\alpha$ given by the composition of $\mathbb{Z}/m$-equivariant exponential map and a contraction ψ of the normal bundle $N\mathfrak{F}_\alpha$ onto a neighbourhood of the zero section,

$$exp_\epsilon \circ \psi : N\mathfrak{F}_\alpha \xrightarrow{\cong} \mathcal{N}\mathfrak{F}_\alpha.$$

The claim (2.13) follows as $\mathbb{Z}/m$-fixed locus on $N\mathfrak{F}_\alpha$ is $\oplus_{t\in\mathbb{Z}} H_{tm}$, whereas on $\mathcal{N}\mathfrak{F}_\alpha$ it is an open subset of $\mathfrak{M}_\alpha^{\mathbb{Z}/m}$. ■

Definition 2.3.11. Given a homogeneous bundle $\mathcal{H}_m$ over the fixed connected component $\mathfrak{F}_\alpha$, depending on its embedding in the CSR $\mathfrak{M}$, we will call it:

- an **outer bundle,** if it lies outside of the core $\mathfrak{L}$, except for $\mathfrak{F}_\alpha$;
- an **inner bundle,** if it lies completely in the core;
- a **mixed bundle,** otherwise;
- a **torsion bundle,** if it is an outer bundle and $m \geq 2$. As it consists of $\mathbb{Z}/m$-torsion points, we will call it a $\mathbb{Z}/m$-torsion bundle as well.

The motivation for considering mixed and torsion bundles arises because the cohomologies of their hypersurfaces at infinity will be building blocks of the spectral sequence which we construct in Section 6.8. We have the following lemma that is immediate from the definition of $\mathcal{H}_m$.

Lemma 2.3.12. *Consider two fixed points $x_1 \in \mathfrak{F}_{\alpha_1}, x_2 \in \mathfrak{F}_{\alpha_2}$ in two different fixed components, such that there is a $\mathbb{C}^*$-orbit γ flowing from x_1 (when $t \to 0$) to x_2 (when $t \to \infty$). If $\gamma \subset \mathcal{H}_m$ with respect to $\mathfrak{F}_{\alpha_1}$, then $\gamma \subset \mathcal{H}_{-m}$ with respect to $\mathfrak{F}_{\alpha_2}$.*

Geometry of $\mathbb{C}^*$-orbits between different fixed locus components is depicted by the following object:

Definition 2.3.13. Given a CSR $(\mathfrak{M}, \varphi)$ its **attraction graph** $Q_\varphi = (Q_0, Q_1)$ is an oriented graph whose vertices Q_0 represent connected components $\mathfrak{F}_\alpha$ of the fixed set under φ, and edges represent the $\mathbb{C}^*$-flow between them: between vertices α_1 and α_2 we put N edges where N is the number of connected components of the manifold

$$\mathfrak{F}_{\alpha_1 \to \alpha_2} := \{x \in \mathfrak{M} | \lim_{t\to 0} t \cdot x \in \mathfrak{F}_{\alpha_1} \text{ and } \lim_{t\to\infty} t \cdot x \in \mathfrak{F}_{\alpha_2}\}.$$

The **leaves** of the graph Q are vertices that have no outgoing edges.

Lemma 2.3.14. *Given the attraction graph of a CSR $(\mathfrak{M}, \varphi)$, between any two of its vertices there cannot be edges going in both directions. Hence, there must exist a leaf vertex.*

Proof. Consider the moment map H of the S^1-action, with respect to a Kähler form from Lemma 2.2.12. The function H is constant on each component $\mathfrak{F}_\alpha$ because if there were a non-constant smooth path in $\mathfrak{F}_\alpha$ along which H varies, then that would imply that ∇H is non-zero at some point of $\mathfrak{F}_\alpha$, contradiction. The gradient trajectories of ∇H are $\mathbb{R}_+$-action orbits. As the value of the moment map is strictly increasing along a gradient trajectory, there cannot be two gradient trajectories (hence $\mathbb{R}_+$-action orbits) going in opposite directions between two fixed locus components. ■

Proposition 2.3.15. *Given a CSR $(\mathfrak{M}, \varphi)$ whose attraction graph has at least two vertices, each leaf has a torsion bundle attached to it. In particular, the action φ is not free outside of the fixed locus.*

Proof. We know that the attraction graph Q_φ has a leaf by Lemma 2.3.14. Pick one of them, and its corresponding fixed component $\mathfrak{F}_\alpha$. Then, as in Theorem 2.3.1, consider the S^1-representation

$$T_x\mathfrak{M} = \bigoplus_{k\in\mathbb{Z}} H_k$$

of the tangent space of a point $x \in \mathfrak{F}_\alpha$, that is induced by the S^1-part of φ. Here, H_k is the weight-k subspace. Denoting by l the weight of the symplectic form $\omega_\mathbb{C}$ under the $\mathbb{C}^*$-action, we claim that this induces a non-degenerate pairing

$$\omega_\mathbb{C} : H_k \oplus H_{l-k} \to \mathbb{C}.$$

Indeed for $v \in H_k$, $\omega_\mathbb{C}(v, w) = t^{-l}(\varphi_t^*\omega)(v, w) = t^{-l}\omega_\mathbb{C}(t^k v, \varphi_{t*}w) = \omega_\mathbb{C}(v, t^{k-l}\varphi_{t*}w)$. Then considering the weight-decomposition of w, and the leading terms when taking either the limit $t \to 0$ or $t \to \infty$, we deduce that $\varphi_{t*}w = t^{l-k}w$ whenever $\omega(v, w) \neq 0$, as required. As $\mathfrak{F}_\alpha$ is a leaf, the weight-spaces H_k that are tangent to the core-directions must have $k \leq 0$. In particular, $H_0 = T_x\mathfrak{F}_\alpha$.

Assume now that $\mathfrak{F}_\alpha$ does not have a weight space with negative k. Then the attraction graph consists only of the component $\mathfrak{F}_\alpha$, which contradicts with our assumption. Otherwise, chose an arbitrary weight space of H_k with $k < 0$. There must be an $\omega_\mathbb{C}$-dual tangent subspace H_{l-k}, with a corresponding homogenous bundle $\mathcal{H}_{l-k}$. As $l - k \geq 1 - (-1) = 2 > 0$, it does not belong to core directions (recall that $\mathfrak{F}_\alpha$ is a leaf). Hence, apart from its subset $\mathfrak{F}_\alpha$, the homogenous bundle $\mathcal{H}_{l-k}$ lies outside of the core, As $l - k \geq 2$, it is a torsion bundle indeed. ■

Having the previous proposition in mind, we decorate the attraction graph with this very last piece of information from the $\mathbb{C}^*$-data:

Definition 2.3.16. Given a CSR $(\mathfrak{M}, \varphi)$, its **extended attraction graph**[27] $\widetilde{Q_\varphi} = (Q_0, \widetilde{Q_1})$ is an extension of the attraction graph $Q_\varphi = (Q_0, Q_1)$ such that for each torsion bundle $\mathcal{H}_m \to \mathfrak{F}_\alpha$ we add an outward-pointing edge at a vertex that corresponds to the fixed component $\mathfrak{F}_\alpha$.

Example 2.3.17. We give an example of an extended attraction graph for the minimal resolution of the Du Val singularity $\pi : X_{\mathbb{Z}/5} \to \mathbb{C}^2/(\mathbb{Z}/5)$. The standard weight-2 $\mathbb{C}^*$-action on this CSR is induced from the weight-1 dilation action on $\mathbb{C}^2$. The core $\pi^{-1}(0)$ consists of a Dynkin A_4 tree of spheres that intersect transversely, and there are five $\mathbb{C}^*$-fixed points, three out of which are intersections of spheres. The other two are exactly the leaves of the attraction graph (Figure 2.1) and thus, according to Proposition 2.3.15, they have torsion bundles attached to them. Notice that on each arrow of the graph we label the weights that correspond to the weight space H_k from which and into which each arrow converges. By Lemma 2.3.12, on each arrow, the starting weight and ending weight should be the same number up to a sign. Notice that the sum of the numbers appearing at each vertex is equal to 2, which is due to the fact that the action in this example is weight-2.

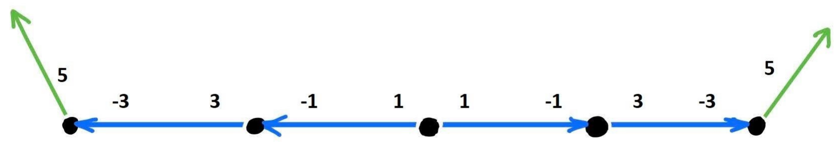


Figure 2.1: Extended attraction graph for the minimal resolution of $\mathbb{C}^2/(\mathbb{Z}/5)$.

We will come back to this example in Section 6.9, where we will compute its corresponding Morse-Bott Floer spectral sequence (Example 6.9.3 and Figure 6.4).

[27] We are slightly abusing the language, as this is not a graph in the usual sense (the last-added arrows have no inward vertices).

Chapter 3

Exact Lagrangians in Conical Symplectic Resolutions

In this chapter we construct a family of closed exact non-isotopic Lagrangian submanifolds of an arbitrary weight-1 CSR $(\mathfrak{M}, \omega_{J,K})$, in other words a family of distinct objects of its compact Fukaya category $\mathcal{F}(\mathfrak{M}, \omega_{J,K})$. Those are constructed as the minima of moment maps of the S^1-part of different weight-1 conical actions, thus we call them *minimal*. Moreover, these minimal components are irreducible components of the core of $\mathfrak{M}$, hence are also interesting outside the realm of symplectic topology.[1] We study their symplectic topology, prove that their Lagrangian Floer cohomologies are of strictly-topological nature, and also show that their existence yields some lower bounds on the rank of symplectic cohomology $SH^*(\mathfrak{M}, \omega_{J,K})$.

3.1 Existence of a smooth core component

We will start this section by recalling the main facts about the geometry of the core $\mathfrak{L}$ of a CSR $\mathfrak{M}$. The proof is due to V. Ginzburg for an arbitrary CSR, but the key ideas are from H. Nakajima's proof of the same theorem for quiver varieties [Nak94a, Thm. 5.8]. We denote by $t \cdot x$ the $\mathbb{C}^*$-action, both on $\mathfrak{M}$ and $\mathfrak{M}_0$.

Theorem 3.1.1. [Gi15] *Given an arbitrary CSR* $(\mathfrak{M}, \varphi)$*, its core* $\mathfrak{L}$ *is an* $\omega_{\mathbb{C}}$*-isotropic subvariety of* $\mathfrak{M}$ *equal to*

$$\mathfrak{L} = \left\{ x \in \mathfrak{M} \mid \lim_{t \to \infty} t \cdot x \ \textit{exists} \right\}. \tag{3.1}$$

[1] As we will see e.g. in Chapter 5.

Denoting the decomposition of fixed locus into connected components by $\mathfrak{M}^\varphi = \mathfrak{F} = \sqcup_\alpha \mathfrak{F}_\alpha$, *there is a partition* $\mathfrak{L} = \sqcup_\alpha \mathfrak{L}_\alpha$ *of the core by smooth locally closed subvarieties*

$$\mathfrak{L}_\alpha := \left\{ x \in \mathfrak{M} \mid \lim_{t\to\infty} t \cdot x \in \mathfrak{F}_\alpha \right\}.$$

Moreover, if φ *is a weight-1 action, each* $\mathfrak{L}_\alpha$ *is a* $\omega_\mathbb{C}$*-Lagrangian submanifold, thus the core* $\mathfrak{L}$ *has pure dimension* $\frac{1}{2}dim\mathfrak{M}$ *and its irreducible components are precisely the closures* $\overline{\mathfrak{L}_\alpha}$.

Proof. The core $\mathfrak{L}$ is $\omega_\mathbb{C}$-isotropic due to [Gi15, Thm. 4.2.1(2)]. We have already seen (3.1) and the partition $\mathfrak{L} = \sqcup_\alpha \mathfrak{L}_\alpha$ in the proof of Corollary 2.3.4. The last part regarding the weight-1 case is due to [Gi15, Sec. 4.5]. ∎

In particular, given a weight-1 CSR $\mathfrak{M}$, the irreducible components $\overline{\mathfrak{L}_\alpha}$ of its core $\mathfrak{L}$ are complex projective varieties, thus have well-defined fundamental classes in singular homology, which freely generate the top-dimensional homology of the core $H_{top}(\mathfrak{L})$.[2] Thus, together with Proposition 2.1.9 we have the following:

Lemma 3.1.2. *Given a weight-1 CSR* $\mathfrak{M}$, *the fundamental classes* $[\overline{\mathfrak{L}_\alpha}]_\alpha$ *make a basis of* $H_{top}(\mathfrak{L}, \mathbb{Z}) \cong H_{mid}(\mathfrak{M}, \mathbb{Z})$. *In particular,* $\overline{\mathfrak{L}_\alpha}$ *are non-isotopic in* $\mathfrak{M}$.

Given a weight-1 CSR, its irreducible core components $\overline{\mathfrak{L}_\alpha}$ are in general not smooth. However, we will prove that at least one of them is always smooth. Recall that by Lemma 2.2.12, there is an S^1-invariant I-compatible Kähler form ω_I on $\mathfrak{M}$, where I is the complex structure of $\mathfrak{M}$, and S^1 acts via φ. By Lemma 2.2.13, with respect to ω_I there is a moment map

$$H : \mathfrak{M} \to \mathbb{R}$$

for the S^1-action, and ∇H is the vector field of the $\mathbb{R}_+$-part of the $\mathbb{C}^*$-action φ.

Proposition 3.1.3. *Given a weight-1 CSR* $(\mathfrak{M}, \varphi)$ *there is a smooth irreducible component* $\overline{\mathfrak{L}_\alpha}$ *in* $\mathfrak{L}$, *therefore a holomorphic Lagrangian submanifold of* $(\mathfrak{M}, \omega_\mathbb{C})$.

Proof. We claim that the global minimum of the moment map H exists and is attained on the core $\mathfrak{L}$. Firstly, by Lemma 2.3.2, we have

$$\mathfrak{F} = \bigsqcup_{\alpha\in A} \mathfrak{F}_\alpha := \mathfrak{M}^{\mathbb{C}^*} = \mathfrak{M}^{S^1} = Crit(H) \subset \mathfrak{L},$$

[2] Indeed, an irreducible projective algebraic variety admits a triangulation, making it a closed oriented connected pseudomanifold, and these have fundamental classes by [SeTh80, Sec. 24]. The statement about the core $\mathfrak{L}$ follows from the Mayer-Vietoris sequence since intersections of irreducible components have real codimension bigger or equal to 2.

thus, if the minimum exists, it is in the core $\mathfrak{L}$. As $\pi : \mathfrak{M} \to \mathfrak{M}_0$ is a projective map, $\mathfrak{L} = \pi^{-1}(x_0)$ is projective, hence compact, so the minimum of $H|_{\mathfrak{L}}$ exists. It must be attained on a subset of $Crit(H)$, so by the equation above, the minimum is attained for a certain family of components $\sqcup_{\alpha \in A'} \mathfrak{F}_a$, call them minimal.

Picking an arbitrary minimal component $\mathfrak{F}_{\alpha_0}$, we want to show that it is an irreducible component of the core $\mathfrak{L}$. As H attains the minimum at $\mathfrak{F}_{\alpha_0}$, there is no gradient trajectory ∇H converging to it. Recalling that ∇H is the vector field of the $\mathbb{R}_+$-action, we have that

$$\mathfrak{L}_{\alpha_0} = \left\{ x \in \mathfrak{M} \mid \lim_{t \to \infty} t \cdot x \in \mathfrak{F}_{\alpha_0} \right\} = \mathfrak{F}_{\alpha_0}.$$

As $\mathfrak{F}_{\alpha_0}$ is already a closed subvariety,

$$\overline{\mathfrak{L}_{\alpha_0}} = \mathfrak{F}_{\alpha_0} = \mathfrak{L}_{\alpha_0}. \tag{3.2}$$

But $\mathfrak{L}_{\alpha_0}$ is smooth by Theorem 3.1.1. Hence the irreducible component $\overline{\mathfrak{L}_{\alpha_0}}$ of the core $\mathfrak{L}$ is also smooth.

In fact, there is always a **single** minimal component. As $\mathfrak{M}$ is connected by Corollary 2.1.12, equation (2.6) from Lemma 2.3.2,

$$b_k(M) = \sum_{\alpha \in A} b_{k-\mu(F_\alpha)}(\mathfrak{F}_\alpha),$$

tells us that precisely one component can contribute to $b_0(\mathfrak{M}) = 1$, and that component $\mathfrak{F}_\alpha$ must have Morse-Bott index zero, $\mu(\mathfrak{F}_\alpha) = 0$. By the definition of Morse-Bott index, this component is minimal. ■

Remark 3.1.4. The last proposition also holds for any holomorphic symplectic manifold with a weight-1 action that contracts the whole manifold to a compact set. In particular, for the case of a moduli space of Higgs bundles [Hi87], the component of its core that this method recovers is the well-known moduli space of stable vector bundles [AB83].

We also show a converse of Proposition 3.1.3, that is, if an action has a fixed component, it must be a power of a weight-1 action. We first need a lemma.

Lemma 3.1.5. *Consider a smooth irreducible complex algebraic variety X with an algebraic $\mathbb{C}^*$-action φ. If it has an m-th root ψ on a Zariski open subset U, then it extends to an holomorphic m-th root on the whole of X.*

Proof. Firstly, as an algebraic variety X is a separable scheme, the diagonal $\Delta \subset X \times X$ is a closed subvariety. Then, seeing the actions as functions $\varphi, \psi : \mathbb{C}^* \times X \to X$, from the condition that $\varphi = \psi^m$ holds on U, we have that $\varphi(\varepsilon t, x) = \varphi(t, x)$ holds for $x \in U$, for any m-th roof of unity ε. In other words, the image $F(\mathbb{C}^* \times U)$ of the morphism

$$F : \mathbb{C}^* \times X \to X \times X,\ (t, x) \mapsto (\varphi(t, x), \varphi(\varepsilon t, x))$$

lies in the diagonal Δ. As X is irreducible, $\mathbb{C}^* \times X = \overline{\mathbb{C}^* \times U}$. Thus, by continuity of F,

$$F(\mathbb{C}^* \times X) = F(\overline{\mathbb{C}^* \times U}) \subset \overline{F(\mathbb{C}^* \times U)} \subset \overline{\Delta} = \Delta.$$

Hence, $\varphi(\varepsilon t, x) = \varphi(t, x)$ holds everywhere on X. Thus, the map $\overline{\psi}(t, x) := \varphi(t^{1/m}, x)$ is well-defined for all $x \in X$. It is a holomorphic map, as this is a local condition, and one can always locally choose a holomorphic branch of the m-th root. Furthermore, it agrees with ψ on U, thus it is a holomorphic extension of ψ to X. ■

Proposition 3.1.6. *Consider a weight-k CSR* $(\mathfrak{M}, \varphi)$, *which has a Lagrangian core, and suppose the action* φ *fixes an irreducible component of the core. Then there is a weight-1 conical action* ψ *such that* $\varphi = \psi^k$.

Proof. Denote by $\mathfrak{F}$ the φ-fixed irreducible component of the core. Its weight-decomposition (Theorem 2.3.1 and Lemma 2.3.2) $T_{\mathfrak{F}}\mathfrak{M} = H_0 \oplus H_k$ consist just of spaces $H_0 = T\mathfrak{F}$ and its $\omega_{\mathbb{C}}$-dual H_k.[3] Thus, on the outer bundle[4] $\mathcal{H}_k \to \mathfrak{F}$ the action acts by weight k, hence one can define the k-th root of the action φ, call it ψ. The bundle $\mathcal{H}_k$ has maximal dimension in $\mathfrak{M}$, as its tangent space at $\mathfrak{F}$ is $H_0 \oplus H_k$. Thus, as $\mathcal{H}_k$ is a locally closed subset of maximal dimension in the irreducible $\mathfrak{M}$, its closure $\overline{\mathcal{H}_k}$ must be equal to $\mathfrak{M}$, and H_k is open in it.[5] Hence, one can use Lemma 3.1.5 to conclude that ψ is extendable to an action on the whole $\mathfrak{M}$, such that $\varphi = \psi^k$. Thus, ψ is a weight-1 action. Recall that the conical property for the action means that under the limit $t \to 0$ the $\mathbb{C}^*$-action contracts $\mathfrak{M}$ to the compact fixed locus. As the fixed loci of ψ and φ are equal, and $\varphi = \psi^k$ is conical, also ψ must be conical. ■

Remark 3.1.7. In particular, considering a weight-2 action that has a fixed component, from the last proposition we get that it **has** to be an even action (a square of an action). We will keep this in mind when searching for weight-2 actions that yield minimal components, in Chapters 4 and 5.

[3]Note that $\mathfrak{F}$ is Lagrangian, so $\dim_{\mathbb{C}} H_0 = \dim_{\mathbb{C}} \mathfrak{M}/2$, therefore also $\dim_{\mathbb{C}} H_k = \dim_{\mathbb{C}} \mathfrak{M}/2$, so $H_0 \oplus H_k$ already has full dimension

[4]Recall Definition 2.3.11.

[5]Recall that locally closed means open in its closure

3.2 Minimal components

The smooth core component from Proposition 3.1.3, obtained as the minimum locus of a certain moment map, is fixed under a conical weight-1 action. Moreover, by the Morse-Bott argument in the proof, any such action will have a single fixed component. That motivates the following definition.

Definition 3.2.1. Given a CSR $\mathfrak{M}$, a **minimal component** $\mathfrak{F}_\varphi$ of its core $\mathfrak{L}$ is the component that is fixed under a weight-1 conical action φ. We denote by $Con_1(\mathfrak{M}, \omega_{\mathbb{C}})$ the set of all weight-1 conical actions in $(\mathfrak{M}, \omega_{\mathbb{C}})$, and by

$$\mathrm{Min}(\mathfrak{M}) := \{\mathfrak{F}_\varphi \mid \varphi \in Con_1(\mathfrak{M})\}.$$

the collection of all minimal components in $\mathfrak{M}$.

In principle, given an arbitrary CSR, we can have many weight-1 conical actions. We show that the different actions yield different minimal components when they commute.

Proposition 3.2.2. *Let $\mathfrak{M}$ be a weight-1 CSR. Different commuting weight-1 conical actions on it induce different minimal components of its core.*

Proof. Firstly, from the proof of Proposition 3.1.3 we see that there is a Kähler metric g on $\mathfrak{M}$. Having two commuting conical actions φ and ψ, we can integrate over their S^1-actions to get a Kähler metric

$$\widetilde{g} = \int_{S^1 \times S^1} \varphi_t^* \psi_s^* g \, dt \, ds$$

that is S^1-invariant for both actions (since they commute). Now, let us assume that the actions φ and ψ are of weight-1 and that they have the same minimal component $\mathfrak{F}_{min}$. The tangent space of an arbitrary point $x \in \mathfrak{F}_{min}$ has the induced $\mathbb{C}^*$-action on it, hence splits

$$T_x\mathfrak{M} = \bigoplus_{k \in \mathbb{Z}} H_k \tag{3.3}$$

according to the weight decomposition of the action.

As the S^1-action preserves the Hermitian structure $\langle \cdot, \cdot \rangle = \widetilde{g}(\cdot, \cdot) - i\widetilde{g}(\cdot, I\cdot)$, the weight decomposition (3.3) is orthogonal. Now, as the symplectic form ω has weight-1, it induces a non-degenerate pairing

$$\omega : H_k \oplus H_{1-k} \to \mathbb{C}$$

between the weight spaces H_k. As $T_x\mathfrak{F}_{min} = H_0$ and its dimension (being a Lagrangian) is half of the dimension of $\mathfrak{M}$, we deduce that

$$T_x\mathfrak{M} = H_0 \oplus H_1.$$

As this is an orthogonal decomposition, we have that $H_1 = H_0^{\perp}$, so it is independent of the action. Hence, two actions φ_1 and φ_2 induce the same S^1-actions on the normal bundle $N\mathfrak{F}_{min}$ of $\mathfrak{F}_{min}$.

By the equivariant tubular neighbourhood theorem for isometric actions of compact Lie groups [Bre72, Thm. 2.2, Ch. VI], there is an S^1-invariant tubular neighbourhood $\mathcal{N}\mathfrak{F}_{\min}$ of $\mathfrak{F}_{\min}$ given by the composition of a contraction ψ of normal bundle to a neighbourhood of a zero section and S^1-equivariant exponential map

$$\varphi = exp_\epsilon \circ \psi : N\mathfrak{F}_{\min} \xrightarrow{\cong} \mathcal{N}\mathfrak{F}_{\min}.$$

Hence, the restrictions to S^1 of the two actions φ_1 and φ_2 agree on the open subset $\mathcal{N}\mathfrak{F}_{min}$ of $\mathfrak{M}$. As they act holomorphically, due to analytic continuation they need to agree on the whole of $\mathfrak{M}$. As $\mathbb{C}^*$ is the complexification of S^1, there is a unique holomorphic extension of a holomorphic S^1-action, hence the $\mathbb{C}^*$ actions φ_1 and φ_2 agree as well, and the proposition is proved. ■

In the setup when the CSR is hyperkähler, we can omit the commuting condition:

Proposition 3.2.3. *Let $\mathfrak{M}$ be a weight-1 HKCSR. Different weight-1 HK conical actions on it induce different minimal components of its core.*

Proof. By definition, the S^1-part of each HK conical action preserves the hyperkähler metric g on $\mathfrak{M}$. Hence, given two different HK conical actions, we already have the metric that is preserved by their S^1-parts, so the proof of Proposition 3.2.2 goes through. ■

Strictly speaking, Propositions 3.2.2 and 3.2.3 are independent of each other, but one should bear in mind that in the examples known to date, all CSR are HKCSR and all known conical actions on them are HK conical actions, so one can use (the stronger) Proposition 3.2.3 which is a finer one, as it does not need the commutativity between the two actions as a condition. Altogether, we have the following theorem:

Theorem 3.2.4. *Given a weight-1 CSR $\mathfrak{M}$, there are at least N smooth irreducible components of its core where $N = \max\{N_1, N_2\} \geq 1$ and*

1. *N_1 = the maximal number of commuting weight-1 conical actions.*
2. *N_2 = the number of HK conical actions (if $\mathfrak{M}$ is also a HKCSR).*

Proof. Follows by Propositions 3.1.3, 3.2.2 and 3.2.3. ■

3.2.1 Constructing weight-1 conical actions

Theorem 3.2.4 allows us to find essentially different exact Lagrangian submanifolds in CSRs using weight-1 conical actions that commute. Here we show how one could find a family of such actions in practice.

Firstly, CSRs usually come with a natural weight-2 action. Typical examples arise as the *Higgs branch of moduli spaces*, which are hyperkähler reductions of unitary actions on flat quaternionic spaces $\mathbb{H}^N$. The dilation action on $\mathbb{H}^N$ yields a weight-2 action on the reduced space. Well-known families of these spaces are quiver and hypertoric varieties. On the Springer-theoretic side, there is a natural Kazhdan action that acts with weight-2 on Slodowy varieties.

Moreover, having a weight-2 CSR $(\mathfrak{M}, \phi)$ we can construct a family of commuting weight-1 actions in the following way. Let us give a following definition first.

Definition 3.2.5. Having a weight-2 CSR $(\mathfrak{M}, \phi)$, we define its group of **conical symplectomorphisms** $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$ as the group of algebraic π-compatible[6] symplectomorphisms that commute with ϕ.

Lemma 3.2.6. *The group* $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$ *is finite-dimensional.*

Proof. Firstly, we will check that $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$ acts faithfully on $\Gamma(\mathfrak{M}, \mathcal{O}_\mathfrak{M}) \cong \mathbb{C}[\mathfrak{M}_0]$.[7] It is enough to prove that if an element φ of $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$ fixes $\Gamma(\mathfrak{M}, \mathcal{O}_\mathfrak{M}) \cong \mathbb{C}[\mathfrak{M}_0]$, then $\varphi = Id$. Now, if φ fixes $\Gamma(\mathfrak{M}, \mathcal{O}_\mathfrak{M}) \cong \mathbb{C}[\mathfrak{M}_0]$, that means that the induced map on $\mathfrak{M}_0$ is identity (recall $\mathfrak{M}_0$ is affine), hence the map on $\mathfrak{M}$ is the identity on the open dense set $\mathfrak{M}^{reg} = \pi^{-1}(\mathfrak{M}_0^{reg})$. Moreover, the set of points in $\mathfrak{M}$ that are fixed by φ is closed, so we conclude that it has to be the whole of $\mathfrak{M}$ (recall that $\mathfrak{M}$ is irreducible).

Hence, the induced action $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C}) \curvearrowright \mathbb{C}[\mathfrak{M}_0]$ is faithful. Further, as elements of $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$ commute with ϕ, they preserve the grading on $\mathbb{C}[\mathfrak{M}_0]$ induced by it. Thus, fixing some set $(f_i)_i$ of homogeneous generators of $\mathbb{C}[\mathfrak{M}_0]$ whose weights we denote by $(w_i)_i$, we have that the action $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C}) \curvearrowright \mathbb{C}[\mathfrak{M}_0]$ is determined by the induced actions on the weight-spaces $\mathbb{C}[\mathfrak{M}_0]^{w_i}$, on which this group acts linearly. Thus, we get the induced monomorphism $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C}) \hookrightarrow \prod_i GL(\mathbb{C}[\mathfrak{M}_0]^{w_i})$ into a finite-dimensional group, hence $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$ is finite-dimensional itself. ■

[6]Meaning: a map that preserves the fibres of π.
[7]This isomorphism holds due to Remark 2.1.2.

Now, construct a family of weight-2 actions by composing ϕ with 1-parameter subgroups

$$S_t \leq Z(Symp_\phi(\mathfrak{M}, \omega_\mathbb{C}))$$

of the centre of $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$. These subgroups all lie in the identity-component $Z(Symp_\phi(\mathfrak{M}, \omega_\mathbb{C}))^\circ$ of this centre, which is isomorphic to $(\mathbb{C}^*)^n$ for some $n \in \mathbb{N}$. Thus, these subgroups are labelled by some integer lattice $L_\phi(\mathfrak{M}, \omega_\mathbb{C}) \cong \mathbb{Z}^n$.

Then, the 1-parameter subgroups S_t for which the actions $\phi_t S_t$ are **conical** and **even**[8] correspond to some subset of the lattice $\Theta_\phi \subset L_\phi(\mathfrak{M}, \omega_\mathbb{C})$. By construction, these actions corresponding to Θ_ϕ commute, thus by Theorem 3.2.4 they induce different minimal components in the core. As the number of core-components is finite, so is the number of these actions, and the set Θ_ϕ itself.

The square roots of the actions that correspond to Θ_ϕ yield weight-1 conical actions, thus minimal components (Moreover, by Remark 3.1.7, these are the only ones in $L_\phi(\mathfrak{M}, \omega_\mathbb{C})$ that could yield minimal components). Thus, we have a corollary to Theorem 3.2.4.

Corollary 3.2.7. *Given a weight-2 CSR* $(\mathfrak{M}, \phi)$, *the number of minimal components of its core is at least* $|\Theta_\phi|$.

Passing to the realm of Nakajima quiver varieties (Chapter 4) and Slodowy varieties (Chapter 5), there exist canonical weight-2 actions ϕ, and explicit subgroups $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})'$ of $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})$. Thus, the calculation of the convex subsets Θ_ϕ of the lattices $L_\phi(\mathfrak{M}, \omega_\mathbb{C})'$ that correspond to $Z(Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})')^\circ$ becomes rather feasible, which we do in Section 4.4 and Section 5.2, respectively. In addition, we believe that, in the case of Nakajima quiver varieties of type A, these subgroups $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})'$ are actually equal to the whole group $Symp_\phi(\mathfrak{M}, \omega_\mathbb{C})^\circ$ (cf. Corollary 4.4.9).

3.3 Symplectic Topology of minimal components

In this section we will observe the symplectic-topological viewpoint of the earlier sections. Namely, we have proved that any CSR $\mathfrak{M}$, has a canonical structure of a Liouville manifold with symplectic structures $\omega_{J,K}$, such that the core $\mathfrak{L}$ is its Liouville skeleton. Moreover, in the case that $\mathfrak{M}$ is a weight-1 CSR, any smooth component of the core is a smooth exact Lagrangian submanifold, therefore a non-trivial object of the Fukaya category $\mathcal{F}(\mathfrak{M})$ of $\mathfrak{M}$. In particular, the minimal components are exact

[8]Meaning: being a square of an action.

Lagrangian submanifolds of $\mathfrak{M}$. Here we compute their mutual Floer cohomologies, provide some information on their closed-open string maps, and show that existence of minimal components yields lower bounds on the ranks of the symplectic cohomology $SH^*(\mathfrak{M}, \omega_{J,K})$.

NB As the symplectic structure we discuss here is exact, there is no need to use the Novikov field for the coefficients. Thus, throughout this section, **we assume that all cohomologies are defined over $\mathbb{Z}/2$ coefficients.** We do not use the more general $\mathbb{Z}$ coefficients, due to the usual issues with the orientation signs, as the minimal components are not necessarily spin[9] (take $\mathbb{C}P^2$ in $T^*\mathbb{C}P^2$ for instance).

3.3.1 Minimal components are exact Lagrangians

In this section we prove that minimal components are exact Lagrangians and compute their Lagrangian Floer cohomologies, as graded vector spaces.

Definition 3.3.1. Given an exact symplectic manifold $(M, \omega = d\theta)$, a **Lagrangian submanifold** is a half-dimensional submanifold $i : L \hookrightarrow M$ such that $i^*\omega = 0$. In other words, the pull back of the primitive 1-form θ is closed $d(i^*\theta) = 0$. When $i^*\theta = df$ is also exact, the Lagrangian is called **exact**. When $i^*\theta = df$ vanishes, the Lagrangian is called **Bohr–Sommerfeld**.

Proposition 3.3.2. *Given a CSR $\mathfrak{M}$ of weight-1, any smooth component L of its core $\mathfrak{L}$ is Bohr–Sommerfeld, in particular it is an exact Lagrangian submanifold of $(\mathfrak{M}, \omega_{J,K})$*

Proof. The condition on the Lagrangian to be exact/Bohr–Sommerfeld depends on the choice of the primitive. Here we consider the primitives coming from the canonical Liouville structure $\omega_{J,K} = d\theta_{J,K}$ from Proposition 2.2.6. They are defined by

$$\theta_{J,K} = i_Z \omega_{J,K},$$

where Z is the vector field of the $\mathbb{R}_+$-part of the $\mathbb{C}^*$-action. Now, from Theorem 3.1.1 we see that the open dense subsets $\mathfrak{L}_\alpha$ of the components $\overline{\mathfrak{L}_\alpha}$ are $\mathbb{C}^*$-invariant and Lagrangian. This in particular yields $Z \in T\mathfrak{L}_\alpha$, hence the form $\theta_{J,K}(\xi) = \omega_{J,K}(Z, \xi)$ must vanish for any $\xi \in T\mathfrak{L}_\alpha$. As vanishing is a closed condition, when $\overline{\mathfrak{L}_\alpha}$ is smooth we deduce that $\theta_{J,K}$ vanishes on it. So $\overline{\mathfrak{L}_\alpha}$ is a Bohr–Sommerfeld Lagrangian submanifold, and in particular it is exact. ■

[9] A manifold X is spin if its second Stiefel–Whitney class vanishes, $w_2(TX) = 0$.

Corollary 3.3.3. *Given a weight-1 CSR $\mathfrak{M}$, there are at least N smooth closed exact non-isotopic Lagrangian submanifolds of $(\mathfrak{M}, \omega_{J,K})$, where $N = \max\{N_1, N_2\} \geq 1$ and*

1. $N_1 =$ *the maximal number of commuting weight-1 conical actions.*

2. $N_2 =$ *the number of HK conical actions (if $\mathfrak{M}$ is also a HKCSR).*

Proof. By Theorem 3.2.4 and Proposition 3.3.2. The fact that they are non-isotopic follows from Lemma 3.1.2. ■

In the next proposition we will compute Lagrangian Floer cohomologies of minimal components. Recall that the Lagrangian Floer cohomology $HF(L_1, L_2)$ of two Lagrangians L_1 and L_2 is a homology of the chain complex $CF(L_1, L_2)$ whose generators are intersections of two Lagrangians, and the differential counts the pseudo-holomorphic strips with ends on these intersections. In particular, when Lagrangian L is exact, it recovers the ordinary cohomology $HF(L, L) \cong H(L)$. Moreover, following [Sei00, Sec. 2], when the ambient manifold (M, ω) satisfies $2c_1(M, \omega) = 0$ and $H^1(M, \mathbb{Z}) = 0$, we can canonically grade Lagrangians L satisfying $H^1(L, \mathbb{Z}) = 0$, and for each such Lagrangian there is a $\mathbb{Z}$-worth of choices of its gradings $\widetilde{L}[k], k \in \mathbb{Z}$.

Choosing a J-complex[10] volume form Ω these gradings can be defined in the following way. Denote by $\mathcal{L}$ the Lagrangian Grassmannian of M, i.e. the bundle over M whose fibre $\mathcal{L}_x$ above $x \in M$ consists of Lagrangian subspaces of $(T_x M, \omega)$. Now, define the *squared phase map*

$$\det{}^2_\Omega : \mathcal{L} \to S^1, \ \det{}^2_\Omega(\Lambda) = \frac{\Omega(\xi_1, \ldots, \xi_n)^2}{|\Omega(\xi_1, \ldots, \xi_n)|^2},$$

for any[11] basis $\xi_1, \ldots, \xi_n$ of Λ. Consider the pull-back of the universal cover $\mathbb{R} \to S^1$ via squared phase map

$$\widetilde{\mathcal{L}} = \{(\Lambda, t) \in \mathcal{L} \times \mathbb{R} \mid \det{}^2_\Omega(\Lambda) = e^{2\pi i t}\}. \tag{3.4}$$

The **grading** of L is a section $\widetilde{L} : L \to \widetilde{\mathcal{L}}$. Thus, for every $x \in L$ we have $\widetilde{L}(x) = (T_x L, t)$ and $\det^2_\Omega(T_x L) = e^{2\pi i t}$. Obviously, if $\widetilde{L}$ is a grading, so is

$$\widetilde{L}[k] : L \to \widetilde{\mathcal{L}}, \ \ \widetilde{L}[k](x) := (T_x L, \pi_{\mathbb{R}}(\widetilde{L}(x)) - k),$$

[10] Where J is an almost complex structure compatible with ω.

[11] As one can show by the pull-back formula for the top-degree forms, this does not depend on the choice of basis.

for every $k \in \mathbb{Z}$, and every grading is obtained in this way. Here, $\pi_{\mathbb{R}} : \widetilde{\mathcal{L}} \to \mathbb{R}$ is the projection. Thus, we get an $\mathbb{Z}$-worth of gradings for all Lagrangians L satisfying $H^1(L, \mathbb{Z}) = 0$. Moreover, this notion depends on the choice of the homotopy class of Ω in the space of J-complex volume forms. However, when $H^1(M, \mathbb{Z}) = 0$, these are all homotopic and the notion of a grading becomes canonical.

In addition, Lagrangians L satisfying $\mathrm{im}(\Omega)|L = 0$, usually called **special** Lagrangians, have a canonical choice of grading, defined by

$$\widetilde{L}(x) := (T_x L, 0), \ \forall x \in L, \tag{3.5}$$

which is well-defined as $\det^2_{\Omega}(T_x L) \equiv 1$.

Finally, given two graded Lagrangians $\widetilde{L}_1, \widetilde{L}_2$, one can canonically define the $\mathbb{Z}$-grading on the Floer cohomology $HF^*(\widetilde{L}_1, \widetilde{L}_2)$. Thus, as grading of each Lagrangian $\widetilde{L}_i$ is defined up to a $\mathbb{Z}$-shift, so is $HF^*(\widetilde{L}_1, \widetilde{L}_2)$. When $L_1 = L_2 = L$, these shifts cancel out[12] and $HF^*(\widetilde{L}_1, \widetilde{L}_1)$ is canonically graded.

Let us apply this in the setup of CSRs. As proved in Lemma 2.1.7 any CSR $(\mathfrak{M}, \omega_{\mathbb{C}})$ is an almost hyperkähler manifold $(\mathfrak{M}, g, I, J, K)$ such that $\omega_{\mathbb{C}} = \omega_J + i\omega_K$ and $\omega_S(\cdot, \cdot) := -g(\cdot, S\cdot)$, for $S = I, J, K$. Hence, for any complex structure $\Theta \in \{aJ + bK \mid a^2 + b^2 = 1\}$ it has a natural Θ-complex volume form $\Omega_\Theta := \frac{1}{(n/2)!}(\omega_I - i\omega_{I\Theta})^{n/2}$, which makes a notion of graded ω_Θ-Lagrangians. As $H^1(\mathfrak{M}, \mathbb{Z}) = 0$ (Corollary 2.1.9), this notion is canonical. In addition, minimal components have canonical choice of gradings, due to the following lemma:

Lemma 3.3.4. *Given a weight-1 CSR, its smooth core components are special Lagrangians with respect to all Ω_Θ. Thus, each smooth core component L satisfying $H^1(L, \mathbb{Z}) = 0$ has a canonical grading. In particular, this holds for minimal components.*

Proof. By Theorem 3.1.1, smooth core components are holomorphic Lagrangians with respect to $\omega_{\mathbb{C}}$, thus they are special with respect to all Ω_Θ, by [SoVe19, Lem. 3.3].[13] Such Lagrangians whose first cohomology vanishes have canonical grading (3.5). Let us show that a minimal component $\mathfrak{F}_\varphi$ satisfies $H^1(\mathfrak{F}_\varphi, \mathbb{Z}) = 0$. Denoting $\{\mathfrak{F}_\alpha\}_{\alpha \in A}$ the set of connected components of the fixed locus $\mathfrak{M}^\varphi$, by Lemma 2.3.2 we have

$$b_k(M) = \sum_{\alpha \in A} b_{k - \mu(F_\alpha)}(\mathfrak{F}_\alpha),$$

[12] As $HF^*(\widetilde{L}_1[k], \widetilde{L}_2[l]) = HF^{*-k+l}(\widetilde{L}_1, \widetilde{L}_2)$.
[13] Which goes back to [HaLa82, p. 154].

where $\mu(\mathfrak{F}_\alpha)$ are Morse-Bott indices of the moment map of the S^1-part of φ, and the index of $\mathfrak{F}_\varphi$ is equal to zero. Since $b_1(\mathfrak{M}) = 0$ by Corollary 2.1.9, that implies that $b_1(\mathfrak{F}_\varphi) = 0$, thus $H^1(\mathfrak{F}_\varphi, \mathbb{R}) = 0$. Moreover, since $H^1(\mathfrak{F}_\varphi, \mathbb{Z})$ has no torsion,[14] it vanishes as well. ■

Now we will compute the mutual Floer cohomologies of minimal components with respect to these canonical gradings.

Theorem 3.3.5. *Given a weight-1 CSR $\mathfrak{M}$, its minimal components are exact Lagrangian submanifolds of $(\mathfrak{M}, \omega_{J,K})$, hence $HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi) \cong H^*(\mathfrak{F}_\varphi, \mathbb{Z}/2)$ for each minimal $\mathfrak{F}_\varphi$. For each pair $\mathfrak{F}_{\varphi^1}, \mathfrak{F}_{\varphi^2}$ of minimal components whose actions φ^1, φ^2 commute we have*

$$HF^*(\widetilde{\mathfrak{F}}_{\varphi^1}, \widetilde{\mathfrak{F}}_{\varphi^2}) \cong H^{*-\mu}(\mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}, \mathbb{Z}/2), \tag{3.6}$$

for certain shift $\mu = \mu(\varphi^1, \varphi^2)$. Moreover, when the S^1-part of one of φ^1, φ^2 is isometric w.r.t almost hyperkähler metric from Lemma 2.1.7, we have that

$$\mu = \frac{1}{2}(\dim_{\mathbb{C}} \mathfrak{M} - \dim_{\mathbb{R}}(\mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2})). \tag{3.7}$$

In particular, that holds when $\mathfrak{M}$ is a HKCSR.[15] *In addition, if the composition $\varphi^1\varphi^2$ of two commuting actions φ^1, φ^2 is even, minimal components $\mathfrak{F}_{\varphi^1}, \mathfrak{F}_{\varphi^2}$ are disjoint.*

Proof. By Proposition 3.3.2, smooth core components, hence minimal components are exact Lagrangians. However, here we notice that one can prove this in the other way, using Lemma 3.3.4. It tells us that each minimal component $\mathfrak{F}_\varphi$ satisfies $H^1(\mathfrak{F}_\varphi, \mathbb{R}) = 0$, hence $\mathfrak{F}_\varphi$ is exact with respect to any primitive form of $\omega_{J,K}$. By the same vanishing, the Floer cohomology $HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi)$ is $\mathbb{Z}$-graded, and the exactness of $\mathfrak{F}_\varphi$ implies the graded isomorphism $HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi) \cong H^*(\mathfrak{F}_\varphi, \mathbb{Z}/2)$, as claimed.

Consider now two different commuting weight-1 conical actions φ^1, φ^2 and their corresponding minimal components $\mathfrak{F}_{\varphi^1}$, $\mathfrak{F}_{\varphi^2}$. The composition $\varphi^{12}_t := \varphi^1_t \circ \varphi^2_t$ is then a weight-2 conical action. If the intersection of $\mathfrak{F}_{\varphi^1}$ and $\mathfrak{F}_{\varphi^2}$ is empty, then (3.6) is trivially true as then $HF^*(\mathfrak{F}_{\varphi^1}, \mathfrak{F}_{\varphi^2}) = 0$. Otherwise, let us show that

$$\mathfrak{F}_{\varphi^{12}} = \mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}$$

holds, where $\mathfrak{F}_{\varphi^{12}}$ is the minimal component of φ^{12}. The action φ^{12} is not a weight-1 action, but it still has a well-defined unique minimal component that is connected, as

[14] Due to universal coefficients.

[15] As for them we consider only HK conical actions, whose S^1-parts are isometric w.r.t. hyperkähler metric on $\mathfrak{M}$ (recall Definition 2.1.14).

this follows from the analogous Morse-Bott argument as in the proof of Proposition 3.1.3. Like in the proof of Proposition 3.2.2, there is a Kähler metric g that is S^1-invariant for both φ^1 and φ^2. It is also S^1-invariant for φ^{12}, and as $H^1(\mathfrak{M}) = 0$, there exist corresponding moment maps H^1, H^2, H^{12} of those S^1-actions with respect to the Kähler form ω_I. From the general theory on symplectic reduction, we have $H^{12} = H^1 + H^2$, so the minimum of H^{12} is attained exactly at the intersection of minima of H^1 and H^2, hence $\mathfrak{F}_{\varphi^{12}} = \mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}$. Now, let us prove that this intersection is clean, that is,

$$T_x\mathfrak{F}_{\varphi^{12}} = T_x\mathfrak{F}_{\varphi^1} \cap T_x\mathfrak{F}_{\varphi^2}$$

The direction '$\subseteq$' is obvious as $\mathfrak{F}_{\varphi^{12}}$ is a submanifold of $\mathfrak{F}_{\varphi^1}$ and $\mathfrak{F}_{\varphi^2}$ so the inclusion between tangent spaces follows.
To prove the direction '$\supseteq$' notice that $v \in T_x\mathfrak{F}_{\varphi^1}$ means that $(\varphi^1_t)_* v = v$ and similarly for $v \in T_x\mathfrak{F}_{\varphi^2}$. Therefore for $v \in T_x\mathfrak{F}_{\varphi^1} \cap T_x\mathfrak{F}_{\varphi^2}$ we have $v \in T_x\mathfrak{F}_{\varphi^{12}}$ because

$$(\varphi^{12}_t)_* v = (\varphi^1_t)_*(\varphi^2_t)_* v = v.$$

As the two exact Lagrangians $\mathfrak{F}_{\varphi^1}$ and $\mathfrak{F}_{\varphi^2}$ intersect cleanly, and their intersection φ^{12} is connected, we have an isomorphism $HF(\mathfrak{F}_{\varphi^1}, \mathfrak{F}_{\varphi^2}) \cong H(\mathfrak{F}_{\varphi^{12}}, \mathbb{Z}/2)$. This is a well-known result that goes back to Poźniak [Po99, Thm. 3.4.11]. Moreover, as $H^1(\mathfrak{F}_{\varphi^1}) = H^1(\mathfrak{F}_{\varphi^2}) = 0$, these Lagrangians can be $\mathbb{Z}$-graded, $\widetilde{\mathfrak{F}}_{\varphi^1}, \widetilde{\mathfrak{F}}_{\varphi^2}$. Then, we have a graded isomorphism (see e.g. [KhSei02, Prop. 5.18]):

$$HF^*(\widetilde{\mathfrak{F}}_{\varphi^1}, \widetilde{\mathfrak{F}}_{\varphi^2}) \cong H^{*-\mu}(\mathfrak{F}_{\varphi^{12}}, \mathbb{Z}/2),$$

where $\mu = \mu_{\widetilde{\mathfrak{F}}_{\varphi^1}, \widetilde{\mathfrak{F}}_{\varphi^2}}$ is the shift which **depends** on the gradings,

$$\mu = \mu(\lambda_1, \lambda_2) + \frac{1}{2}(\dim_{\mathbb{C}} \mathfrak{M} - \dim_{\mathbb{R}} \mathfrak{F}_{\varphi^{12}}). \tag{3.8}$$

Here, $\mu(\lambda_1, \lambda_2)$ is the Maslov index for paths of Lagrangians ([RS93, Sec. 3]) $\lambda_1, \lambda_2 : [0,1] \to \mathcal{L}_x$ satisfying $\lambda_1(0) = T_x\mathfrak{F}_{\varphi^1},\ \lambda_1(1) = T_x\mathfrak{F}_{\varphi^2},\ \lambda_2(t) \equiv T_x\mathfrak{F}_{\varphi^2}$. The path λ_1 is chosen to be compatible with the choice of gradings $\widetilde{\mathfrak{F}}_{\varphi^1}, \widetilde{\mathfrak{F}}_{\varphi^2}$, i.e. it is the projection of a path $\widetilde{\lambda_1} : [0,1] \to \widetilde{\mathcal{L}}_x$ satisfying $\widetilde{\lambda_1}(0) = \widetilde{\mathfrak{F}}_{\varphi^1}(x),\ \widetilde{\lambda_1}(1) = \widetilde{\mathfrak{F}}_{\varphi^2}(x)$. It is unique, up to homotopy that fixes endpoints. Thus, the index $\mu(\lambda_1, \lambda_2)$ neither depends on the choice of λ_1, or on the chosen point $x \in \mathfrak{F}_{\varphi^{12}} = \mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}$, as this intersection is clean and connected.

Now, by Lemma 3.3.4, minimal components are special Lagrangians and they are canonically graded $\widetilde{\mathfrak{F}}_{\varphi^1}(x) = (T_x\mathfrak{F}_{\varphi^1}, 0),\ \widetilde{\mathfrak{F}}_{\varphi^2} = (T_x\mathfrak{F}_{\varphi^2}, 0)$. We are going to compute the index $\mu(\lambda_1, \lambda_2)$, hence shift μ, with respect to that grading. For brevity, we will use the form ω_J, thus complex volume form Ω_J, noticing that the same proof goes for any $\omega_\Theta, \Omega_\Theta$, where $\Theta \in \{aJ + bK \mid a^2 + b^2 = 1\}$.

Now, we assume that the S^1-part of one of the actions φ^1, φ^2, say of φ^1, is isometric with respect to the almost hyperkähler metric g from Lemma 2.1.7. This is automatically true for HKCSRs, as we consider only such conical actions on them.[16] Denote $V := T_x\mathfrak{M},\ N := T_x\mathfrak{F}_{\varphi^{12}},\ L_1 := T_x\mathfrak{F}_{\varphi^1},\ L_2 := T_x\mathfrak{F}_{\varphi^2}$, and $U_i := N^{\perp_g} \subset L_i$. The following lemma is crucial.

Lemma 3.3.6. $JU_1 = U_2$.

Proof. Firstly, for $i \in \{1, 2\}$ we have that $JL_i = L_i^{\perp_g}$, thus $V = L_i \oplus JL_i$ are g-orthogonal decompositions.[17] Thus, as $L_1 = N \oplus U_1$ is an orthogonal decomposition, so is $V = N \oplus U_1 \oplus JN \oplus JU_1$.[18] Thus, as $\dim JU_1 = \dim U_1 = \dim U_2$, it is enough to show that $(N \oplus U_1 \oplus JN) \perp_g U_2$. By definition, $U_2 \perp_g N$, and, as $JN \subset JL_2 = L_2^{\perp_g}$ is orthogonal to $L_2 \supset U_2$, it is orthogonal to U_2 as well. Thus, it is left to show that

$$U_2 \perp_g U_1. \tag{3.9}$$

To do this, we will use that $\mathfrak{F}_{\varphi^1}$ is a minimal component, hence a connected component of the fixed locus of a $\mathbb{C}^*$-action φ^1. Then, like in the proof of Proposition 3.2.2, we have the g-orthogonal weight composition with respect to φ^1 :

$$V = T_x\mathfrak{M} = H_0 \oplus H_1.$$

We have that $H_0 = T_x\mathfrak{F}_{\varphi^1} = L_1 \supset U_1$, thus in order to show (3.9), it is enough to show that $U_2 \subset H_1$. This follows from considering the restriction of the $\mathbb{C}^*$-action φ^1 to the component $\mathfrak{F}_{\varphi^2}$. Then, we again have the weight decomposition $T_x\mathfrak{F}_{\varphi^2} = H_0' \oplus H_1'$, where $H_0' = T_x\mathfrak{F}_{\varphi^{12}} = N$, and H_1' is its orthogonal complement, U_2. Thus, U_2 consists of weight-1 vectors, and we are done. ■

Now, pick a g-orthogonal basis $u_1, \ldots, u_k$ of N and extend it to a g-orthogonal basis $u_1, \ldots, u_n$ of L_1. Thus, $U_1 = \langle u_{k+1}, \ldots, u_n \rangle$. Denoting by $v_i := Ju_i$, previous lemma says that $L_2 = \langle u_1, \ldots, u_k, v_{k+1}, \ldots, v_n \rangle$. Now, consider a path of half-dimensional subspaces in V:

$$\gamma_1(t) = \langle u_1, \ldots, u_k, e^{i\frac{\pi}{2}t} u_{k+1}, \ldots, e^{i\frac{\pi}{2}t} u_{\frac{n+k}{2}}, e^{-i\frac{\pi}{2}t} u_{\frac{n+k}{2}+1}, \ldots, e^{-i\frac{\pi}{2}t} u_n \rangle, \tag{3.10}$$

[16] Recall Definition 2.1.14.
[17] As $L_i \subset V$ is a Lagrangian subspace, hence for $v, w \in L_i$ we have $g(v, Jw) = -\omega_J(v, w) = 0$.
[18] As J is g-orthogonal, $g(J\cdot, J\cdot) = g(\cdot, \cdot)$.

where $e^{i\theta}u_j := cos(\theta)u_j + sin(\theta)Ju_j$. The number $\frac{n+k}{2}$ is an integer, as both n and k are even.[19] From g-orthogonality of $(u_i)_{i=1}^n$, it follows that $\gamma_1 \subset \mathcal{L}_x$. Clearly, $\gamma_1(0) = L_1$ and $\gamma_1(1) = L_2$. The next lemma shows the importance of this path:

Lemma 3.3.7. *γ_1 is an eligible choice for a path λ_1.*

Proof. It is enough to show that

$$det^2_{\Omega_J}(\gamma_1(t)) \equiv 1, \tag{3.11}$$

as then $\widetilde{\lambda_1} := (\gamma_1(t), 0)$ is a path in $\widetilde{\mathcal{L}_x}$ satisfying $\widetilde{\lambda_1}(0) = \widetilde{\mathfrak{F}_{\varphi_1}}$, $\widetilde{\lambda_1}(1) = \widetilde{\mathfrak{F}_{\varphi_2}}$ and γ_1 is its projection to $\mathcal{L}$, hence an eligible choice for λ_1. In order to show (3.11), consider a J-complex volume form $\Omega := (u_1^* + iv_1^*) \wedge (u_2^* + iv_2^*) \wedge \cdots \wedge (u_n^* + iv_n^*)$ on V. Denoting the basis of $\gamma_1(t)$ in (3.10) by $(u_j'(t))_{j=1}^n$, it is immediate that

$$\Omega(u_1'(t), \ldots, u_n'(t)) = (e^{i\frac{\pi}{2}t})^{\frac{n-k}{2}}(e^{-i\frac{\pi}{2}t})^{\frac{n-k}{2}}\Omega(u_1, \ldots, u_n) = \Omega(u_1, \ldots, u_n) = 1,$$

thus, $det^2_{\Omega}(\gamma_1(t)) \equiv 1$.

On the other hand, Ω and Ω_J are both complex volume forms on V, hence $\Omega = \eta\Omega_J$ for some $\eta \in \mathbb{C}^*$. Then, $det^2_{\Omega} = \frac{\eta^2}{|\eta|^2} det^2_{\Omega_J}$. As $\mathfrak{F}_{\varphi^1}$ is special, we have $det^2_{\Omega_J}(T_x\mathfrak{F}_{\varphi^1}) = det^2_{\Omega_J}(L_1) = 1$. On the other hand, $det^2_{\Omega}(L_1) = det^2_{\Omega}(\gamma_1(0)) = 1$, thus $\frac{\eta^2}{|\eta|^2} = 1$ and $det^2_{\Omega} = det^2_{\Omega_J}$, which yields (3.11) indeed. ∎

Now we can compute the Maslov index $\mu(\lambda_1, \lambda_2)$, which, together with (3.8), finally gives the claimed shift $\mu = \frac{1}{2}(\dim_{\mathbb{C}} \mathfrak{M} - \dim_{\mathbb{R}} \mathfrak{F}_{\varphi^{12}})$.

Lemma 3.3.8. $\mu(\lambda_1, \lambda_2) = 0$.

Proof. By definition, Maslov index of paths of Lagrangians in a symplectic vector space V is computed using a symplectomorphism $\phi : (V, \omega_J) \to (\mathbb{R}^{2n}, \omega_{std})$, $\mu(\lambda_1, \lambda_2) := \mu(\phi_*\lambda_1, \phi_*\lambda_2)$. Here, ω_{std} is the standard symplectic structure on $\mathbb{R}^{2n}$. By the naturality of the Maslov index for paths, this does not depend on the choice of ϕ. We will define the most natural such symplectomorphism:

$$\phi : V \to \mathbb{R}^{2n}, \ \phi(u_i) = e_i, \ \phi(v_i) = f_i,$$

where $(e_i)_{i=1}^n, (f_i)_{i=1}^n$ are standard bases of $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$, respectively. Denote $\nu_i := \phi_*\lambda_i$, for $i \in \{1, 2\}$, and $K_i := \phi_*L_i$. Thus, $K_1 = \langle e_1, \ldots, e_n\rangle$ and $K_2 = \langle e_1, \ldots, e_k, f_{k+1}, \ldots, f_n\rangle$. It is immediate that $\nu_1(t) = \psi(t)K_1 = \psi(1-t)K_2$, where

$$\psi : [0, 1] \to U(n) \subset Sp(2n), \ \psi(t) = diag(\underbrace{1, \ldots, 1}_{k}, \underbrace{e^{i\frac{\pi}{2}t}, \ldots, e^{i\frac{\pi}{2}t}}_{\frac{1}{2}(n-k)}, \underbrace{e^{-i\frac{\pi}{2}t}, \ldots, e^{-i\frac{\pi}{2}t}}_{\frac{1}{2}(n-k)}).$$

[19] Being real dimensions of $I-$Kähler manifolds $\mathfrak{F}_{\varphi^1}$ and $\mathfrak{F}_{\varphi^{12}}$.

Here, the complex coordinates used on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ are standard, i.e. on the i-th place is the span $\langle e_i, f_i \rangle$. Using the elementary properties (given in [RS93]) of Maslov index for pairs of Lagrangian paths, and for its version $\mu(A) := \mu(A(0 \times \mathbb{R}^n), 0 \times \mathbb{R}^n)$ for a path $A = A(t)$ of symplectic matrices, we have a sequence of equalities:

$$\begin{aligned}
\mu(\lambda_1, \lambda_2) = \mu(\nu_1, \nu_2) &= \mu(\psi(1-t)K_2, K_2) = \mu(\psi(1-t)\varphi(0 \times \mathbb{R}^n), \varphi(0 \times \mathbb{R}^n)) \\
&= \mu(\varphi\psi(1-t)(0 \times \mathbb{R}^n), \varphi(0 \times \mathbb{R}^n)) = \mu(\psi(1-t)0 \times \mathbb{R}^n, 0 \times \mathbb{R}^n) \\
&= \mu(\psi(1-t)) = -\mu(\psi(t)) \\
&= -k\mu(1) - \frac{1}{2}(n-k)\mu((e^{i\frac{\pi}{2}t})_{t\in[0,1]}) - \frac{1}{2}(n-k)\mu((e^{-i\frac{\pi}{2}t})_{t\in[0,1]}) \\
&= -\frac{1}{2}(n-k)(\frac{1}{2}) - \frac{1}{2}(n-k)(-\frac{1}{2}) = 0,
\end{aligned}$$

where $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\varphi = diag(\underbrace{i, \ldots, i}_{k}, \underbrace{1, \ldots, 1}_{n-k})$ is a symplectomorphism which satisfies $\varphi(0 \times \mathbb{R}^n) = (\mathbb{R}^k \times 0) \times (0 \times \mathbb{R}^{n-k}) = L_2$, and clearly commutes with $\psi(t)$. The Maslov index $\mu((e^{\pm i\frac{\pi}{2}t})_{t\in[0,1]}) = \mu((e^{\pm i\frac{\pi}{2}t})_{t\in[0,\frac{1}{2}]}) + \mu((e^{\pm i\frac{\pi}{2}t})_{t\in[\frac{1}{2},1]}) = \mu((e^{\pm i\frac{\pi}{2}t})_{t\in[0,\frac{1}{2}]}) + 0 = \frac{1}{2}\text{sign}(tg(\pm\frac{\pi}{4})) - \frac{1}{2}\text{sign}(tg(0)) = \pm\frac{1}{2}$ is computed using the catenation, zero and localisation properties from [RS93, Thm. 2.3]. ■

Finally, let us prove the last claim of the Theorem. If the composition $\varphi^1\varphi^2$ is even, there is a minimal component attached to it (more precisely, to its square root). It cannot be the intersection of two different minimal components $\mathfrak{F}_{\varphi^1}, \mathfrak{F}_{\varphi^2}$, due to dimension reasons. Thus, the intersection $\mathfrak{F}_{\varphi^1} \cap \mathfrak{F}_{\varphi^2}$ is empty and they are disjoint. ■

Remark 3.3.9. The other direction of the last claim of the previous theorem is not true. As an example, consider the resolution of a Du Val singularity $X_{\mathbb{Z}/5} \to \mathbb{C}^2/\mathbb{Z}/5$ (Example 2.3.17), and compose the weight-1 actions whose minimal components are the "outer spheres" in the Dynkin A_4-chain of spheres (which constitutes the core). As one can prove by hand, the composition of those two actions is not even, but their minimal components are disjoint.

Following Theorem 3.3.5, an interesting avenue of further Floer-theoretic research on minimal components could be towards computing their Floer product, or more generally, computing the higher operations of the Fukaya category between them. At the moment this is far from reach, but let us mention an encouraging result which says that the operations in the Fukaya category of a CSR $\mathfrak{M}$ that involve smooth Lagrangians in the core come only from constant solutions. We say that a Riemann surface Σ is of **type-n** if its boundary $\partial\Sigma$ has punctures, which decompose it into n connected pieces, denoted by $\partial\Sigma_i, i = 1, \ldots, n$.

Proposition 3.3.10. ***(Holomorphic maps with boundary in the core are constant)*** *Consider a weight-1 CSR $\mathfrak{M}$ and pick any choice of symplectic form $\omega_{J,K}$, and its compatible almost complex structure S. Given any smooth core components $L_1, \ldots, L_n$, and picking a Riemann surface Σ of type-n, any S-holomorphic map $u : \Sigma \to \mathfrak{M}$, $u(\partial\Sigma_i) \subset \mathfrak{L}_i$ is constant.*

Proof. Let us assume that we have a non-constant S-holomorphic map

$$u : \Sigma \to \mathfrak{M},\ u(\partial\Sigma) \subset \mathfrak{L},$$

By Gromov Lemma we have $\int_\Sigma u^*\omega_{J,K} = \frac{1}{2}\int_\Sigma ||du||^2 > 0$. On the other hand, like in the proof of Proposition 2.2.6, we have $\omega_{J,K} = d\theta_{J,K}$ where $\theta_{J,K} = i_Z\omega_{J,K}$ and Z is vector field of the $\mathbb{R}_+$-action, up to a scale. Thus, by Stokes theorem $\int_\Sigma u^*\omega_{J,K} = \int_{\partial\Sigma} u^*\theta_{J,K}$, and the last integral vanishes due to $\theta_{J,K}(\xi) = 0$ for any vector $\xi \in T\mathfrak{L}$ (see the proof of Proposition 3.3.2). Thus we get a contradiction. ■

We remark that the result [SoVe19, Thm. 1.6] from a recent paper by Solomon-Verbitsky states the same[20] statement as Proposition 3.3.10 in the setup of arbitrary hyperkähler manifolds and holomorphic Lagrangians therein. However, this is the main result of their paper, whereas our proof for the setup of CSRs and the Lagrangians of the core goes rather easy, as we saw.

Remark 3.3.11. We have seen in this section that smooth components of the cores of CSRs are exact Lagrangians, hence well-defined objects in the closed Fukaya category of a CSR. From that viewpoint, of interest would also be the core components which are not necessarily smooth but immersed, as their Floer theory theory is well-defined as well [AkJo10]. However, it is hard to get the information whether a singular core component is immersed in general. Even the simplest singular core components, in the examples of cores being Springer fibres [Va79, Sec. 5], [FrMe10, Prop. 2.1], are rather defined by a set of equations than an (immersion) map.

3.3.2 Applications towards symplectic cohomology

In this section we use the existence of minimal components in order to give some information on the symplectic cohomology $SH^*(\mathfrak{M}) := SH^*(\mathfrak{M}, \omega_{J,K})$. Symplectic cohomology is notoriously hard to compute explicitly, so we usually have to make due with partial information. Recall that the symplectic structures $\omega_{J,K}$ are Calabi-Yau (due to Lemma 2.1.7), and also $H^1(\mathfrak{M}, \mathbb{Z}) = 0$ (due to Lemma 2.1.9), hence

[20] To be precise, our result is slightly stronger, as we omit the word "generic" which they have, for choosing the symplectic form among the forms $\omega_{J,K}$.

$SH^*(\mathfrak{M})$ is canonically $\mathbb{Z}$-graded (Corollary A.2.4). This is important, as we will get a degree-wise information on its ranks; getting information on its global rank would be not much of use as symplectic cohomology is usually infinite-dimensional (cf. Remark 3.3.22).

We will make a crucial use of the **closed-open string map**

$$\mathcal{CO}^0 : SH^*(M, \omega) \to HF^*(L, L)$$

for an exact Lagrangian submanifold $L \subset M$ of Liouville manifold M. As we will not use its definition but merely its properties, we refer to the original paper [Ab15] for a thorough exposition. Here we just mention that it counts half-cylinders satisfying the Floer equation and having a boundary on L.

Firstly, we show the non-vanishing of symplectic cohomology, due to the Viterbo-Seidel argument.

Corollary 3.3.12. *Given any weight-1 CSR* $\mathfrak{M}$, *its exact symplectic cohomology over* $\mathbb{Z}/2$*-coefficients is non-zero*

$$SH^*(\mathfrak{M}, \omega_{J,K}) \neq 0.$$

Proof. By the argument in [Sei08, 5a], given a closed exact Lagrangian submanifold $L \overset{i}{\hookrightarrow} \mathfrak{M}$, there is a commuting triangle

$$\begin{array}{ccc} H^*(\mathfrak{M}) & \xrightarrow{c^*} & SH^*(\mathfrak{M}) \\ & \searrow^{i^*} & \downarrow \mathcal{CO}^0 \\ & & HF^*(L, L) \end{array} \tag{3.12}$$

consisting of ring homomorphisms such that the diagonal map, under the isomorphism

$$HF^*(L, L) \cong H^*(L),$$

becomes the restriction map for singular cohomology. Hence, it sends the unit to the unit, therefore $c^*(1) \in SH^*(\mathfrak{M})$ cannot vanish. Now, by Proposition 3.1.3 there is at least one smooth core component of $\mathfrak{M}$, and by Proposition 3.3.2 it is a closed exact Lagrangian submanifold. The claim follows. ■

Remark 3.3.13. In addition, it should be noted that for CSRs which do not admit a weight-1 conical action, symplectic cohomology may in fact vanish, which in turn prevents the existence of exact Lagrangians (by the Viterbo-Seidel argument above). For example, quiver varieties and hypertoric varieties that do not admit weight-1 conical actions are all subcritical Stein manifolds, so by Cieliebak [Cie02, p. 121] the symplectic cohomology vanishes.

By studying the closed-open string map, we can also obtain lower bounds on the rank of $SH^*(\mathfrak{M})$. For that matter, we will need a result about the decomposition of singular homology of a variety in the presence of a $\mathbb{C}^*$-action, which we prove combining the work of Carrell–Goresky [CaGo83] and the Białynicki-Birula decomposition (Theorem 2.3.3). We remark that we will not be using the notation from that paper, rather we continue with the one we previously introduced in this thesis, for the reader's convenience.

Theorem 3.3.14 (Carell–Goresky, Białynicki-Birula)**.** *Let Y be a smooth projective variety with a holomorphic $\mathbb{C}^*$-action. Suppose $X \subset Y$ is a closed $\mathbb{C}^*$-invariant subvariety, and let $X^{\mathbb{C}^*} = \bigsqcup_i \mathfrak{F}_i$ be the decomposition of its fixed locus into connected components. Assume further that:*

(i) The fixed components $\mathfrak{F}_i$ are smooth.

(ii) Their attracting sets $\mathfrak{L}_i := \{x \in X \mid \lim_{t\to\infty} t \cdot x \in \mathfrak{F}_i\}$ in X are the same as their attracting sets in Y.

Then:

(1) The morphisms $p_i : \mathfrak{L}_i \to \mathfrak{F}_i,\ x \mapsto \lim_{t\to\infty} t \cdot x$ are fibre bundles with affine fibres.

(2) There is an isomorphism[21]

$$\Phi = \oplus_i \eta_i : \bigoplus_i H_*(\mathfrak{F}_i)[-\mu_i] \to H_*(X).$$

where $\eta_i : H_*(\mathfrak{F}_i)[-\mu_i] \to H_*(X),\ [C] \mapsto [\overline{p_i^{-1}(C)}]$, for a generic cycle C. Here the closure $\overline{p_i^{-1}(C)}$ is taken in X, and μ_i is the real dimension of the bundle p_i.

Proof. Consider a fixed component $\mathfrak{F}_i$ in X. It belongs to a fixed component $\mathfrak{F}'_i$ in Y. As $X \subset Y$ is a closed subvariety, $\mathfrak{F}_i \subset \mathfrak{F}'_i$ is so as well. Due to assumption (ii) we have that the map $p_i : \mathfrak{L}_i \to \mathfrak{F}_i$ is a restriction of the map $p'_i : \mathfrak{L}'_i \to \mathfrak{F}'_i,\ x \mapsto \lim_{t\to\infty} t \cdot x$ where $\mathfrak{L}'_i := \{x \in Y \mid \lim_{t\to\infty} t \cdot x \in \mathfrak{F}'_i\}$ is the attraction set of $\mathfrak{F}'_i$ in Y. Now, from Theorem 2.3.3 applied to Y, we know that the p'_i is a fibre bundle with affine fibres, so the same holds to its restriction p_i, which proves the claim (1). Now, we claim that one can apply the theorem [CaGo83, Thm. 1', Sec. 2] by Carrell-Goresky that

[21] Here we are using the standard shifting notation from homological algebra; Given a graded module A_*, the graded module $A_*[k]$ is obtained from it by shifting **down** by k. Also, notice that the result from [CaGo83] holds for integral homology. Nevertheless, their proof goes over the $\mathbb{Z}/2$ coefficients (which we are using here) as well.

claims the isomorphism (2) in this setup. Indeed, as stated in that theorem, one needs the $\mathbb{C}^*$-action on X has to be *good*, which means that the decomposition $X = \cup \mathfrak{L}_i$ needs to satisfy certain conditions (1a)-(1d), [CaGo83, p.368-369]. As mentioned in [CaGo83, Rmk, Sec. 2], the conditions (1b) and (1d) are always satisfied in this setup. The condition (1c) only makes sense if there are singular F_i, thus is vacuous due to our assumption (i). Finally, the condition (1a) is exactly the claim (1), hence the the $\mathbb{C}^*$-action on X is indeed good, and the isomorphism (2) holds. For its description given in the claim (2) we refer to [CaGo83, p.369]. ■

Remark 3.3.15. We remark that the assumptions in Theorem 3.3.14 are needed indeed, or at least assumption (ii), as the following example shows. Namely, consider a $\mathbb{C}^*$-action on $X = \mathbb{P}^2$ given by

$$t \cdot [z_0, z_1, z_2] = [z_0, tz_1, t^2 z_2].$$

Its fixed locus are points $\mathfrak{F}_0 = [1 : 0 : 0]$, $\mathfrak{F}_1 = [0 : 1 : 0]$, $\mathfrak{F}_2 = [0 : 0 : 1]$. Choose Y to be the union of three complex lines which these points form. We see that assumption (i) from the theorem holds, but notice that assumption (ii) does not. Namely, the attracting set $\mathfrak{L}_2' = \{x \in X \mid \lim_{t\to\infty} t \cdot x \in \mathfrak{F}_2\}$ attached to the point $\mathfrak{F}_2$ is equal to $[z_0, z_2, 1] \cong \mathbb{C}^2$ whereas the attracting set $\mathfrak{L}_2$ in Y attached to the same point is equal to the union of two lines, $[z_0 : 0 : 1]$, and $[0 : z_1 : 1]$, thus $L_2 \neq L_2'$ indeed. It is immediate that $\mathfrak{L}_2 \to \mathfrak{F}_2$ is not an affine bundle, and also that the homology isomorphism between fixed loci $\mathfrak{F}_i$ and Y cannot hold, as $H_*(Y) = \mathbb{K}[0] \oplus \mathbb{K}[-1] \oplus \mathbb{K}^3[-2]$, thus has a bigger rank than the total rank of all $H_*(\mathfrak{F}_i)$.

Proposition 3.3.16. *The closed-open string map*

$$\mathcal{CO}^0 : SH^*(\mathfrak{M}) \to HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi)$$

for a minimal component $\mathfrak{F}_\varphi$ *is surjective.*

Proof. Using diagram (3.12) for the minimal component $\mathfrak{F}_\varphi$

$$\begin{array}{ccc} H^*(\mathfrak{M}) & \xrightarrow{c^*} & SH^*(\mathfrak{M}) \\ & \searrow^{i^*} & \downarrow \mathcal{CO}^0 \\ & & HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi) \end{array} \tag{3.13}$$

we see that the surjectivity of $\mathcal{CO}^0$ would follow from the surjectivity of the map $H^*(\mathfrak{M}) \to HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi)$, which under the isomorphism $HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi) \cong H^*(\mathfrak{F}_\varphi)$ becomes the restriction map $H^*(\mathfrak{M}) \to H^*(\mathfrak{F}_\varphi)$. Recalling that the inclusion of the core

$\mathfrak{L} \subset \mathfrak{M}$ is a homotopy equivalence (Proposition 2.1.9), it is enough to show that the restriction map $H^*(\mathfrak{L}) \to H^*(\mathfrak{F}_\varphi)$ is surjective.

We will prove this by applying Theorem 3.3.14 to the core $\mathfrak{L}$ of $\mathfrak{M}$. Recall first that, as mentioned in the proof of Corollary 2.3.4, one can extend the $\mathbb{C}^*$-action on $\mathfrak{M}$ to a smooth projective variety Y which compactifies it, such that $\mathfrak{M} \subset Y$ is a $\mathbb{C}^*$-invariant Zariski open subvariety. Next, $\mathfrak{L} \subset Y$ is a closed $\mathbb{C}^*$-invariant subvariety which satisfies assumptions (i) and (ii) of Theorem 3.3.14. Indeed, the $\mathbb{C}^*$-fixed locus $\sqcup_i \mathfrak{F}_i$ in $\mathfrak{L}$ is the same as fixed locus in $\mathfrak{M}$, and is smooth (Lemma 2.3.2), thus assumption (i) holds. Moreover, by the characterisation of the core given by (2.8) in the proof of Corollary 2.3.4, we have that assumption (ii) holds as well, hence we have an isomorphism

$$\Phi_\varphi = \oplus_i \eta_i : \oplus_i H_*(\mathfrak{F}_i)[-\mu_i] \xrightarrow{\cong} H_*(\mathfrak{L}) \tag{3.14}$$

Notice that the minimal component $\mathfrak{F}_{i_0} := \mathfrak{F}_\varphi$ satisfies

$$\mathfrak{F}_{i_0} = \mathfrak{L}_{i_0} = \overline{\mathfrak{L}_{i_0}}$$

(equation (3.2) in the proof of Proposition 3.1.3). Thus, $\mu_{i_0} = 0$ and $\overline{p_{i_0}^{-1}(C)} = C$, for a cycle C, thus

$$\eta_{i_0} = (i_\varphi)_* : H_*(\mathfrak{F}_\varphi) \to H_*(\mathfrak{L}),$$

where $i_\varphi : \mathfrak{F}_\varphi \hookrightarrow \mathfrak{L}$ is the inclusion map. Hence by isomorphism (3.14), the map $(i_\varphi)_*$ is injective. As we are working over field coefficients, the restriction map on cohomology $(i_\varphi)^* : H^*(\mathfrak{L}) \to H^*(\mathfrak{F}_\varphi)$ is naturally isomorphic to the dual map $\mathrm{Hom}((i_\varphi)_*, \mathbb{Z}_2)$, hence is surjective. Thus by the argument above, the proposition is proved. ■

Hence, as a corollary, we obtain the lower bounds on the ranks of symplectic cohomology. Denote by $b_i(X)$ the i-th Betti number of a topological space X.

Corollary 3.3.17. *Given an arbitrary minimal component $\mathfrak{F}_\varphi$ of a weight-1 CSR $\mathfrak{M}$, for all $k \in \mathbb{N}_0$,*

$$rk(SH^k(\mathfrak{M})) \geq b_k(\mathfrak{F}_\varphi).$$

By looking carefully at the proof of Proposition 3.3.16, notice that we have shown that a block of $H^*(\mathfrak{M})$, that is isomorphic to $H^*(\mathfrak{F}_\varphi)$ via the restriction map, injects in $SH^*(\mathfrak{M})$ with the c^*-map. Thus, fixing an weight-1 action φ with the fixed locus $\mathfrak{F} = \sqcup \mathfrak{F}_i$, we can inject the cohomologies of all $\mathfrak{F}_i$ which lie in minimal components to $SH^*(\mathfrak{M})$ via the c^*-map. Notice that, in order to prove this, we have to use the isomorphism from Theorem 3.3.14 simultaneously for the action φ restricted to a minimal component and for the actions that produce minimal components, and to prove the compatibility of those isomorphisms. We do it in the following proposition.

Proposition 3.3.18. *Let $(\mathfrak{M}, \varphi)$ be a weight-1 CSR, and let $\mathfrak{F} = \sqcup_i \mathfrak{F}_i$ be the fixed locus of φ decomposed into connected components $\mathfrak{F}_i$, and μ_i their corresponding Morse-Bott indices. Then*

$$rk\,(SH^k(\mathfrak{M})) \;\geq \sum_{\{\mathfrak{F}_i \subset L \,|\, L \in \mathrm{Min}(\mathfrak{M})\}} b_{k-\mu_i}(\mathfrak{F}_i),$$

for all $k \geq 0$. In particular, $rk\,(SH^{dim_{\mathbb{C}}\mathfrak{M}}(\mathfrak{M})) \geq |Min(\mathfrak{M})|$.

Proof. As φ is a weight-1 conical action, by Theorem 3.1.1 the fixed sets $\mathfrak{F}_i$ are bijectively distributed in different irreducible components of the core. That is to say, $\overline{\mathfrak{L}_i}$ are irreducible components of $\mathfrak{L}$. We will focus on those that are minimal, $\overline{\mathfrak{L}_i} = \mathfrak{F}_\phi$, for some weight-1 action ϕ.

Given such a fixed set $\mathfrak{F}_i$, one can apply Theorem 3.3.14 for $X = \mathfrak{L}$ and Y a $\mathbb{C}^*$-equivariant compactification of $\mathfrak{M}$ (as in Proposition 3.3.16), and get an injective map

$$\eta_i : H_*(\mathfrak{F}_i)[-\mu_i] \to H_*(\mathfrak{L}),\; [C] \mapsto [\overline{p_i^{-1}(C)}],\; \text{for a generic cycle } C. \tag{3.15}$$

Moreover, as the action φ restricts to $\overline{\mathfrak{L}_i} = \mathfrak{F}_\phi$, Theorem 3.3.14 applies for $X = Y = \mathfrak{F}_\phi$ with the $\mathbb{C}^*$-action φ on it.[22] Thus, we get an injective map

$$\eta_i^\phi : H_*(\mathfrak{F}_i)[-\mu_i^\phi] \to H_*(\mathfrak{F}_\phi),\; [C] \mapsto [\overline{p_i^{-1}(C)}^\phi],\; \text{for a generic cycle } C. \tag{3.16}$$

Here the closure $\overline{p_i^{-1}(C)}^\phi$ is taken in $\mathfrak{F}_\phi$, and μ_i^ϕ is the dimension of the part of the bundle $\mathfrak{L}_i \to \mathfrak{F}_i$ that lies in $\mathfrak{F}_\phi$. But, since $\mathfrak{F}_\phi = \overline{\mathfrak{L}_i}$, the closures and shifts agree $\overline{p_i^{-1}(C)}^\phi = \overline{p_i^{-1}(C)}$, $\mu_i^\phi = \mu_i$. Thus, we have

$$(i_\phi)_* \eta_i^\phi = \eta_i$$

where $i_\phi : \mathfrak{F}_\phi \hookrightarrow \mathfrak{L}$ is the inclusion.

Thus, picking for arbitrary $\phi \in Con_1(\mathfrak{M})$ a φ-fixed connected component $\mathfrak{F}_{i(\phi)}$ defined by $\overline{\mathfrak{L}_{i(\phi)}} = \mathfrak{F}_\phi$, we get that the image of the map $H^*(\mathfrak{F}_\phi) \xrightarrow{(i_\phi)_*} H_*(\mathfrak{L})$ contains $\eta_{i(\phi)}(H_*(\mathfrak{F}_{i(\phi)})[-\mu_i])$. Thus, summing over all $\phi \in Con_1(\mathfrak{M})$ we get that the image of the map

$$\oplus_\phi\, H_*(\mathfrak{F}_\phi) \xrightarrow{\oplus (i_\phi)_*} H_*(\mathfrak{L}) \tag{3.17}$$

[22]In that case it reduces to the homology decomposition for closed Kähler manifolds with a $\mathbb{C}^*$-action, [CaSo79, Thm. 1].

contains the image of the map

$$\oplus_\phi \eta_{i(\phi)} : \oplus_\phi H_*(\mathfrak{F}_{i(\phi)})[-\mu_{i(\phi)}] \to H_*(\mathfrak{L}),$$

which is injective, by isomorphism (3.14). Denote by $U := Im(\oplus_\phi \eta_{i(\phi)})$ and by U' its arbitrary graded complement in $H_*(\mathfrak{L})$.

Applying the functor $\mathrm{Hom}(\cdot, \mathbb{Z}/2)$ on the map (3.17), we get the map

$$\mathrm{Hom}(H_*(\mathfrak{L}), \mathbb{Z}/2) \xrightarrow{\oplus_\phi Hom((i_\phi)_*, \mathbb{Z}/2)} \oplus_\phi \mathrm{Hom}(H_*(\mathfrak{F}_\phi), \mathbb{Z}/2) \tag{3.18}$$

which we show to be injective on the dual of U, by applying the following linear-algebraic lemma:

Lemma 3.3.19. *Given vector spaces V and $W = U \oplus U'$ and a linear map $L : V \to W$ such that $U \leq L(V)$, the dual map $L^* : W^* \to V^*$ is injective when restricted to $U^* := \{\xi \in W^* \mid \xi|_{U'} = 0\}$.*

Proof. We just have to show that if $L^*(\xi) = 0$ for $\xi \in U^*$ then $\xi = 0$. By assumption, $0 = L^*(\xi)(v) = \xi(L(v))$ for all $v \in V$, so $\xi|_{L(V)} = 0$, so $\xi|_U = 0$. As $\xi \in U^*$, we also have $\xi|_{U'} = 0$, hence $\xi = 0$. ∎

Thus, by this lemma the map $\oplus_\phi Hom((i_\phi)_*, \mathbb{Z}/2)$ injects U^* into $\oplus_\phi \mathrm{Hom}(H_*(\mathfrak{F}_\phi), \mathbb{Z}/2)$. As U' was chosen to be a graded complement, U^* is isomorphic to U as a graded vector space. Now, using the Kronecker isomorphisms $\kappa : H^*(\mathfrak{L}) \to Hom(H_*(\mathfrak{L}), \mathbb{Z}/2)$, $\kappa : H^*(\mathfrak{F}_\phi) \to \mathrm{Hom}(H_*(\mathfrak{F}_\phi), \mathbb{Z}/2)$ we pass from (3.18) to the map

$$H^*(\mathfrak{L}) \xrightarrow{\oplus_\phi i_\phi^*} \oplus_\phi H^*(\mathfrak{F}_\phi),$$

which then injects $\kappa^{-1}(U^*)$ into $\oplus_\phi H^*(\mathfrak{F}_\phi)$.

Finally, we connect this with symplectic cohomology. Considering the diagram (3.13) for a minimal component $\mathfrak{F}_\phi$, and using the isomorphism $r : H^*(\mathfrak{M}) \xrightarrow{\cong} H^*(\mathfrak{L})$ (given by the restriction map), and $S : HF^*(\mathfrak{F}_\varphi, \mathfrak{F}_\varphi) \xrightarrow{\cong} H^*(\mathfrak{F}_\varphi)$ we get the diagram

$$\begin{array}{ccc} H^*(\mathfrak{L}) & \xrightarrow{c} & SH^*(\mathfrak{M}) \\ & \searrow^{i_\phi^*} & \downarrow co_\phi \\ & & H^*(\mathfrak{F}_\varphi), \end{array} \tag{3.19}$$

where $c = c^* \circ r^{-1}$ and $co_\phi := S \circ \mathcal{CO}^0$. Summing over all $\phi \in Con_1(\mathfrak{M})$ we get the diagram

$$\begin{array}{ccc} H^*(\mathfrak{L}) & \xrightarrow{c} & SH^*(\mathfrak{M}) \\ & \searrow^{\oplus_\phi i_\phi^*} & \downarrow \oplus_\phi co_\phi \\ & & \oplus_\phi H^*(\mathfrak{F}_\phi) \end{array} \tag{3.20}$$

Observe that, as $\kappa^{-1}(U^*)$ injects by the diagonal map $\oplus_\phi i_\phi^*$ of this diagram, it injects by the horizontal map c as well. The claim of the proposition then follows as $\kappa^{-1}(U^*) \cong \oplus_\phi(H_*(\mathfrak{F}_{i(\phi)})[-\mu_{i(\phi)}])$ (graded isomorphism), and the summing goes through all $\phi \in Con_1(\mathfrak{M})$, thus $\overline{\mathfrak{L}_{i(\phi)}}$ goes through the set of all minimal components. ■

We get an immediate corollary of this proposition:

Corollary 3.3.20. *If all components of $\mathfrak{L}$ of a CSR $\mathfrak{M}$ are minimal, the singular cohomology $H^*(\mathfrak{M})$ embeds into $SH^*(\mathfrak{M})$ via the c^* map, hence for all $k \in \mathbb{N}_0$,*

$$rk(SH^k(\mathfrak{M})) \geq rk(H^k(\mathfrak{M})).$$

Let us see an instance of this corollary in the following example.

Example 3.3.21. Given the minimal resolution $\mathfrak{M} = \widetilde{\mathbb{C}^2/(\mathbb{Z}/(n+1))}$ of a Du Val singularity of type A_n, its core is topologically an n-wedge of 2-spheres, and these are all minimal components (Examples 4.3.16 and 5.3.16). Hence, in particular, we get that

$$\mathrm{rk}(SH^2(\mathfrak{M})) \geq H^2(\mathfrak{M}) = n.$$

It is known that $\mathrm{rk}(SH^2(\mathfrak{M})) = n$ by [EL17, Cor. 42], so in this case Proposition 3.3.18 gives the actual rank on the top degree of symplectic cohomology.

Remark 3.3.22. Note that, in principle, symplectic cohomology may be infinite-dimensional even degree-wise.[23] However, it is indeed finite-dimensional in each degree for certain examples of CSRs:

- Minimal resolutions of Du Val singularities $\widetilde{\mathbb{C}^2/\Gamma}$, due to [EL17].
- Cotangent bundles of generalised flag varieties $T^*(G/P)$, where G is complex semisimple group and P a parabolic subgroup. By Viterbo isomorphism [Vi96, Ab15], $SH^*(T^*(G/P))$ is isomorphic to the singular homology of the free loop space $\mathfrak{L}(G/P)$ of flag variety, so we can use a general result [Ser51, Prop. 9, Ch. IV] by Serre that the homology of the free loop space of a simply connected space is finite-dimensional degree-wise. The flag variety G/P is simply connected due to the decomposition into Schubert cells which are isomorphic to affine spaces [Bo54].

[23] The simplest example is T^*S^1, whose symplectic cohomology is supported in degrees 0 and 1 and has infinite rank in both.

Thus, we believe that for general CSR $\mathfrak{M}$ the ranks $rk(SH^k(\mathfrak{M})),\ k \in \mathbb{Z}$ should be finite-dimensional, so the lower bounds from previous statements actually give some non-trivial information.

Chapter 4

Minimal components in Quiver varieties

In the special case of Nakajima quiver varieties, we can describe the results of Chapter 3 more concretely. Namely, given a quiver variety $\mathfrak{M}(Q, \mathbf{v}, \mathbf{w})$ we can produce a number of weight-1 HK conical actions, which we call Nakajima actions, that yield a family of minimal components of the core $\mathfrak{L}(Q, \mathbf{v}, \mathbf{w})$. Then, in the special case when the graph Q is of Dynkin type A, we get an explicit number of these weight-1 actions, hence their corresponding minimal components. For convenience of the reader, we will firstly briefly review Nakajima's hyperkähler construction of quiver varieties from [Nak94a], to which we refer to for further details.

4.1 Review of quiver varieties

Let $Q = (Q_0, Q_1)$ be any undirected graph, with Q_0 the set of vertices, and Q_1 the set of edges. To each vertex k we assign two complex vector spaces V_k and W_k thus obtaining two Q_0-graded complex vector spaces

$$V := \bigoplus_{k \in Q_0} V_k, \quad W := \bigoplus_{k \in Q_0} W_k.$$

W is usually called *the framing.* Now define the *double quiver* $Q^{\#}$ as the quiver with the same vertices as Q, such that each edge $e \in Q_1$ connecting vertices i and j in Q gives rise to two oriented edges $h : i \to j, \bar{h} : j \to i$ in $Q^{\#}$. Call H the set of edges of $Q^{\#}$.

For an oriented edge $h \in H$ we denote by $s(h)$ and $t(h)$ its source and target respectively. We call h an *edge-loop* if $s(h) = t(h)$. Define the *space of framed representations* *of the double quiver* to be

$$M := M(Q, V, W) := R(Q^{\#}, V) \oplus L(W, V) \oplus L(V, W), \tag{4.1}$$

where

$$R(Q^{\#}, V) := \bigoplus_{h \in H} \mathrm{Hom}(V_{s(h)}, V_{t(h)}),$$
$$L(W, V) := \bigoplus_{k \in Q^0} \mathrm{Hom}(W_k, V_k),$$
$$L(V, W) := \bigoplus_{k \in Q^0} \mathrm{Hom}(V_k, W_k).$$

For an element of M, we denote by B_h, i_k, j_k its components in $\mathrm{Hom}(V_{s(h)}, V_{t(h)})$, $\mathrm{Hom}(W_k, V_k)$, $\mathrm{Hom}(V_k, W_k)$, respectively. We write

$$B = (B_h)_{h \in H}, \ i = (i_k)_{k \in Q^0}, \ j = (j_k)_{k \in Q^0}$$

For example, for the Dynkin A_n graph, a description of an element in M can be seen in Figure 4.1.

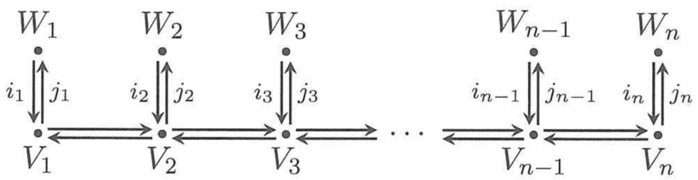


Figure 4.1: A framed representation of a double quiver

Equipping the spaces V_k with a Hermitian structure, the vector space M has a natural flat Kähler structure (M, g, I). We will explain below how to extend this to a flat hyperkähler space (M, g, I, J, K), by analogy with the assignment of the quaternionic structure on $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$. Recall that one identifies $(a + bi, c + di) \in \mathbb{C} \oplus \mathbb{C}$ with $a + bi + cj + dk \in \mathbb{H}$, so the quaternionic structure is determined by declaring that j acts on $\mathbb{C} \oplus \mathbb{C}$ by $j(z_1, z_2) = (-\overline{z_2}, \overline{z_1})$ (note i acts diagonally on each copy of $\mathbb{C}$, and $k = ij$).

Observe that the set H of edges of $Q^{\#}$ has a natural involution $h \to \bar{h}$ which reverses edge orientations. For a subset $\Omega \subset H$ we denote by $\overline{\Omega}$ the image under this involution. Therefore we can divide the oriented edges into two disjoint subsets,

$$H = \Omega_0 \sqcup \overline{\Omega_0}, \tag{4.2}$$

such that neither subset contains two mutually inverse edges in H. Such a division yields a function $\varepsilon_0 : H \to \{\pm 1\}$ defined by

$$\varepsilon_0 = \begin{cases} +1, & \text{if } h \in \Omega_0 \\ -1, & \text{if } h \in \overline{\Omega_0}. \end{cases}$$

This determines the map $J : M \to M$ given by

$$J(B_h, i_k, j_k) = ((-1)^{\varepsilon_0(\bar{h})} B^*_{\bar{h}}, -j^*_k, i^*_k).$$

Note in particular that for $(h_1, h_2) = (h_1, \overline{h_1}) \in \Omega_0 \sqcup \overline{\Omega_0}$, the pair (B_{h_1}, B_{h_2}) maps to $(-B^*_{h_2}, B^*_{h_1})$, analogously to the model for $\mathbb{H}$.

Defining $K := IJ$ one obtains a flat hyperkähler space (M, g, I, J, K). The complex-symplectic part of it is defined by $\omega_{\mathbb{C}} := \omega_J + i\omega_K$, where $\omega_J(\cdot,\cdot) := -g(\cdot, J\cdot)$, $\omega_K(\cdot,\cdot) = -g(\cdot, K\cdot)$. One can show that choosing different partitions (4.2) induces linear hyperkähler isometries between corresponding spaces, therefore the choice of partition is not essential. There is a natural $GL(V) = \prod_{k \in Q_0} GL(V_k)$-action on M by conjugation

$$g \curvearrowright (B, i, j) = (gBg^{-1}, gi, jg^{-1}), \tag{4.3}$$

The compact real form of this group $G := \prod_{k \in I} U(V_k)$ acts by hyperkähler isometries, hence one can do the hyperkähler reduction of [HKLR87], obtaining the *Nakajima quiver variety*

$$\mathfrak{M}_{\zeta_I, \zeta_J, \zeta_K}(Q, V, W) := \mu^{-1}(\zeta_I, \zeta_J, \zeta_K)/G. \tag{4.4}$$

Here $\mu = (\mu_I, \mu_J, \mu_K) : M(Q, V, W) \to \mathfrak{g}^* \otimes \mathbb{R}^3 = \oplus_{k \in Q^0} \mathfrak{u}(V_k)^* \otimes \mathbb{R}^3$ is the hyperkähler moment map, given as a direct product of moment maps for the three symplectic forms $\omega_I, \omega_J, \omega_K$. Explicit formulas for these are

$$\mu_I = \frac{1}{2} \left(\sum_{h \in H, k = t(h)} B_h B^*_h - B^*_{\bar{h}} B_{\bar{h}} + i_k i^*_k - j^*_k j_k \right)_k \tag{4.5}$$

$$\mu_{\mathbb{C}} := \mu_J + i\mu_K = \left(\sum_{h \in H, k = t(h)} \varepsilon_0(h) B_h B_{\bar{h}} + i_k j_k \right)_k \tag{4.6}$$

Let us denote by μ^k_I and $\mu^k_{\mathbb{C}}$ the k-th component of the moment maps $\mu_I, \mu_{\mathbb{C}}$ for $k \in Q_0$. Using the moment parameter of type $(\zeta_I, 0, 0)$, for a generic choice of parameter ζ_I the corresponding quiver variety $\mathfrak{M}_{\zeta_I, 0, 0}(Q, V, W)$ will be smooth. The "generic choice" here means that there is a finite number of hyperplanes of a parameter space $Z(\mathfrak{g}^*) \cong \mathbb{R}^{Q_0}$, whose complement consists of open chambers in which the corresponding quiver variety is smooth. We will consider two different choices of parameter ζ_I :

1. $\zeta_I = \zeta$ generic, which gives us a smooth quiver variety $\mathfrak{M}_\zeta(Q, V, W)$,

2. $\zeta_I = 0$, which gives us a singular *affine quiver variety* $\mathfrak{M}_0(Q, V, W)$.

From now on we will denote these quiver varieties simply by $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ where $\mathbf{v} = (\dim V_k)_{k\in Q_0} \in \mathbb{Z}^{Q_0}$ and $\mathbf{w} = (\dim W_k)_{k\in Q_0} \in \mathbb{Z}^{Q_0}$, because the different choices of Hermitian vector spaces V_k, W_k of the same dimension induce hyperkähler isomorphisms between the corresponding varieties.

Using the Kempf-Ness theorem [KeNe79], one can view the affine quiver variety as a GIT quotient instead

$$\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w}) = (\mu_I^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0))/G \cong \mu_{\mathbb{C}}^{-1}(0) /\!\!/ G_{\mathbb{C}}, \tag{4.7}$$

where $G_{\mathbb{C}} = \prod_{k\in Q_0} GL(V_k)$ is the complexification of G. Thus, the natural inclusion $\mu_I^{-1}(\zeta) \cap \mu_{\mathbb{C}}^{-1}(0) \hookrightarrow \mu_{\mathbb{C}}^{-1}(0)$, after the quotient and the Kempf-Ness isomorphism, induces a map

$$\pi : \mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w}) \tag{4.8}$$

which is a resolution onto its image $\mathfrak{M}^1(Q, \mathbf{v}, \mathbf{w}) := \pi(\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}))$. The last fact is due to Nakajima, in the case where the regular subset

$$\mathfrak{M}_0^{reg}(Q, \mathbf{v}, \mathbf{w}) := \{(B, i, j) \in \mu^{-1}(0, 0, 0) \mid \text{the stabiliser of } (B, i, j) \text{ is trivial}\}/G \tag{4.9}$$

is non-empty, as then it is an isomorphism over this set, which is open and dense in $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$. In the complete generality, it is only recently proven by McGerty-Nevins [MN19, Prop. 3.3]. More precisely, denoting by $\mathfrak{N}(Q, \mathbf{v}, \mathbf{w})$ the normalisation of $\mathfrak{M}^1(Q, \mathbf{v}, \mathbf{w})$, they prove that that the induced map[1]

$$\widetilde{\pi} : \mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}) \to \mathfrak{N}(Q, \mathbf{v}, \mathbf{w})$$

is a symplectic resolution. Thus, combined with the normalisation $\mathfrak{N}(Q, \mathbf{v}, \mathbf{w}) \to \mathfrak{M}^1(Q, \mathbf{v}, \mathbf{w})$, which is a birational map, it follows that π is a resolution as well, though not necessarily symplectic, as $\mathfrak{M}^1(Q, \mathbf{v}, \mathbf{w})$ may not be a normal variety.

In [Nak94a], Nakajima defines a $\mathbb{C}^*$-action on M given by

$$t \cdot (B_h, i_k, j_k) = (t^{(1-\varepsilon_0(h))/2} B_h, i_k, t j_k),\ t \in \mathbb{C}^*. \tag{4.10}$$

Note in particular that for $(h_1, h_2) = (h_1, \overline{h_1}) \in \Omega_0 \sqcup \overline{\Omega_0}$, the pair (B_{h_1}, B_{h_2}) maps to (B_{h_1}, tB_{h_2}), so this action is analogous to the action $t \cdot (z_1, z_2) = (z_1, tz_2)$ on $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} = T^*\mathbb{C}$.

[1] As $\mathfrak{M}(Q, \mathbf{v}, \mathbf{w})$ is smooth, π factors through the normalisation.

The action (4.10) is of weight-1 with respect to $\omega_{\mathbb{C}}$, hence induces weight-1 $\mathbb{C}^*$-actions on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ such that π is an equivariant map.[2]

Example 4.1.1. Given $Q = A_1$, dimension vectors $\mathbf{v} = 1, \mathbf{w} = n + 1$, and the stability condition $\zeta > 0$, the quiver variety $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ is $\omega_{\mathbb{C}}$-symplectomorphic to the cotangent bundle $T^*\mathbb{C}P^n$. Here the action (4.10) is the dilation on cotangent fibres with weight 1.

The following proposition tells us precisely when this $\mathbb{C}^*$-action is conical. By a **cycle** in Ω_0 we mean an oriented loop consisting of edges in Ω_0.

Proposition 4.1.2. *The action defined by (4.10) is a conical action on*

$$\pi : \mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$$

precisely when there are no cycles in Ω_0.

Proof. One side of this proposition (no cycles implies conical) is proved in [Kir16, Thm. 10.55(1),(2)], to which we refer the interested reader. We prove the other side here due to completeness, as it was not written in the existing literature (to the author's knowledge).

Hence, assume that there is a cycle in Ω_0, and let us prove that the action (4.10) is not conical indeed. Denote the edges in the cycle by $h_1, \ldots, h_n$. Looking at the description of the coordinate ring $\mathbb{C}[\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})]$ from Theorem 4.3.13, we see that the polynomial

$$f(B, i, j) = tr(B_{h_n} \ldots B_{h_2} B_{h_1})$$

is one of its generators ($\alpha := B_{h_n} \ldots B_{h_2} B_{h_1}$ is a closed B-path), and the action (4.10) acts on it with weight zero, which means that that it cannot be conical, as long as polynomial f is non-constant. We prove the last by explicitly finding a point $x = [(B_h, i_k, j_k)] \in \mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ such that $f(x) \neq 0$ (as then f is non-constant since $f(0) = 0$). Denoting by i the source vertex of edge h_i, we have a chain of linear maps $B_{h_i} : V_i \to V_{i+1}$. Let us choose orthonormal bases ${e_j}^i$ for Hermitian spaces V_i, and define the maps B_{h_i}, for $i = 1, \ldots, n$ by

$$B_{h_i}(e^i_j) = \begin{cases} e^{i+1}_1, \text{ when } j = 1 \\ 0, \text{ otherwise,} \end{cases}$$

[2] To be precise, the formula 4.10 makes sense on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ only for $t \in S^1$. However, one can extend the S^1-action to the $\mathbb{C}^*$-action on $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ using its GIT quotient interpretation (4.7), and to $\mathfrak{M}(Q, \mathbf{v}, \mathbf{w})$ using its holomorphic quotient interpretation. We do not write those details here due to brevity, rather we refer the reader to Nakajima's paper [Nak94a, Thm. 5.1(5)]

where $e_1^{n+1} := e_1^1$. Define all the other maps B_h for $h \notin \{h_1, \dots h_n\}$, and i_k, j_k to be equal to zero. We claim that $x := [(B_h, i_k, j_k)] \in \mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$. Indeed, it immediately follows that the complex moment map $\mu_{\mathbb{C}}$ given by formula (4.6) vanishes, as the terms $B_h B_{\bar{h}}$ in it involves maps B whose edges are in $\overline{\Omega_0}$, and the only non-zero B-maps are B_{h_i} which are in Ω_0, by assumption. The moment map μ_I given by formula (4.5) vanishes, due to Lemma 4.3.15. Altogether, we have $x = [(B_h, i_k, j_k)] \in (\mu_I^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0))/G = \mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$. indeed and $f(x) = tr(B_{h_n} \dots B_{h_2} B_{h_1}) = 1 \neq 0$, as by construction we have

$$B_{h_n} \dots B_{h_2} B_{h_1}(e_j^1) = \begin{cases} e_1^1, & \text{when } j = 1 \\ 0, & \text{otherwise.} \end{cases}$$

■

When the graph Q does not have edge-loops, one can always make a choice of edges Ω_0 such that Ω_0 does not contain a cycle. Indeed, pick any numbering $1, 2, \dots$ of the vertices in Q_0, now orient the edges in Ω_0 so that the numbering increases in the direction of the edge. Then there cannot be any cycle. Therefore we have a corollary of Proposition 4.1.2 and Proposition 3.1.3.

Corollary 4.1.3. *When the graph Q does not have loop-edges, $\pi : \mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}) \to \mathfrak{N}(Q, \mathbf{v}, \mathbf{w})$ is a CSR of weight-1, therefore there is a smooth component in the core $\mathfrak{L}_\zeta(Q, \mathbf{v}, \mathbf{w})$.*

Motivated by this corollary, we construct many more weight-1 actions on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ in the next section.

4.2 Nakajima actions

In the construction of quiver varieties $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ Nakajima uses a fixed partition $H = \Omega_0 \sqcup \overline{\Omega_0}$ of the edges H of the double quiver $Q^\#$, and then produces the $\mathbb{C}^*$-action (4.10) using *the same partition.* Here we construct other actions, varying all possible partitions of H, thus getting a family of weight-1 conical actions which we call Nakajima actions.

Definition 4.2.1. Given any partition $H = \Omega \sqcup \overline{\Omega}$ on the set of edges H of the double quiver $Q^\#$, such that neither subset contains two mutually inverse edges in H, we obtain again a function $\varepsilon : H \to \{\pm 1\}$,

$$\varepsilon = \begin{cases} +1, & \text{if } h \in \Omega \\ -1, & \text{if } h \in \overline{\Omega}, \end{cases}$$

and a corresponding $\mathbb{C}^*$-action on M,

$$t \cdot (B_h, i_k, j_k) = (t^{(1-\varepsilon(h))/2} B_h, i_k, t j_k), \ t \in \mathbb{C}^*. \tag{4.11}$$

This action induces weight-1 $\mathbb{C}^*$-actions on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ under which π is equivariant, which we call a **Nakajima action.**

From the definition we directly see that:

Lemma 4.2.2. *Any two Nakajima actions commute.* ■

Verbatim to the proof of Proposition 4.1.2, one can prove the following.[3]

Proposition 4.2.3. *The Nakajima action induced by a partition $H = \Omega \sqcup \overline{\Omega}$ is a conical action on $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ precisely when there are no cycles in Ω.*

Having this proposition in mind, due to simplicity we restrict our attention to the case when the graph Q is a tree, thus any partition induces a conical action.

4.2.1 Nakajima actions for tree graphs

Given a tree Q, one can construct exactly $2^{|Q^1|}$ partitions as in Definition 4.2.1,

$$H = \Omega \sqcup \overline{\Omega}.$$

Therefore, we have 2^{Q^1} Nakajima actions, and according to Proposition 4.2.3 they are all weight-1 conical actions, thus they yield minimal components. However, these actions are not all different in general. It is easy to see that:

Proposition 4.2.4. *The S^1-part of a Nakajima action is isometric on M. Being such, it preserves ω_I and its moment map is*

$$F(B, i, j) = \frac{1}{2}\left(\sum_{h \in \overline{\Omega}} \|B_h\|^2 + \sum_{k \in Q^0} \|j_k\|^2\right).$$

This all also holds when one passes to the quiver variety $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$.

Given this, one can compare different Nakajima actions using the moment map formula.

[3] Indeed, the proof of one direction, referred to [Kir16, Thm. 10.55(1)(2)] does not depend on the partition used for the action. The same holds for the proof of the other direction, as the only time we actually use the partition $H = \Omega_0 \sqcup \overline{\Omega_0}$ is when we refer to the formula (4.6) to prove that the vanishing of the complex moment map for point x. But then we only use the fact that the term $B_{\bar{h}} B_h$ vanishes as one of edges h and $\bar{h}$ is in $\overline{\Omega_0}$. However, the same is true for $\overline{\Omega}$ as well.

Definition 4.2.5. We will say that two Nakajima actions φ', φ'' are **equationally-equivalent** if their moment maps F' and F'' differ (formally) by a sum of traces of the components μ_I^k in equation (4.5) of the moment map μ_I,

$$F' - F'' = \sum_{k \in Q^0} n_k \operatorname{tr}(\mu_I^k), \qquad \text{for some } n_k \in \mathbb{Z}. \tag{4.12}$$

We will write $F' \sim F''$ to denote equational-equivalence. We call two Nakajima actions **equivalent** if they yield equal actions on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$. We will denote by $N(\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}))$ and $N_e(\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}))$ the number of non-equivalent, respectively, non-equationally equivalent Nakajima actions on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$.

We create this relation in order to build a passage for counting the number of non-equivalent Nakajima actions, hence the number of minimal components induced by them. Namely, restricting to quiver varieties of type[4] A $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, we first count the number $N_e(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}))$ of classes of equational-equivalence relation (Section 4.3.1), and then by comparing this relation and the ordinary equivalence-relation among Nakajima actions (Section 4.3.2) we finally get Theorem 4.3.21 that counts the classes $N(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}))$ of latter relation.

Remark 4.2.6. Definition 4.2.5 has a geometric description as well. Observe the maximal torus $T := \prod_{k \in Q^0} S_k^1$ of the group $G = \prod_{k \in Q^0} U(V_k)$, where $S_k^1 \leq U(V_k)$ are the maximal tori. Given an integer-vector $\mathbf{n} = (n_k)_{k \in Q^0} \in \mathbb{Z}^{Q_0}$, one has adjacent one-parameter subgroup $Z^{\mathbf{n}} \leq T$ having weights n_k with respect to S_k^1. The moment map of the action of this group on M is precisely given by formula $\sum_{k \in Q^0} n_k \operatorname{tr}(\mu_I^k)$. Thus, given two Nakajima actions φ', φ'', equation (4.12) means exactly that their difference[5] $\varphi'(\varphi'')^{-1} \leq T$ is a 1-parameter subgroup of the maximal torus T, when acted on M, or on the level set $\mu^{-1}(\zeta)$. Of course, passing to the quotient $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}) = \mu^{-1}(\zeta)/G$ this difference disappears and thus the actions become equivalent, as the next proposition shows.

Proposition 4.2.7. *Any two equationally-equivalent actions yield equivalent actions on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$, thus $N_e(\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})) \geq N(\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}))$.*

Proof. On $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w}) = \mu^{-1}(\zeta_I, 0, 0)/G$ we have that $\mu_I^k = \zeta_I^k$, thus the moment maps F' and F'' differ by a constant, so they yield the same S^1-action. Due to the unique extension of holomorphic S^1 actions to holomorphic $\mathbb{C}^*$-actions, the claim follows. ∎

[4] Meaning that the starting graph Q is of Dynkin type A.

[5] Recall that Nakajima actions commute (Lemma 4.2.2), thus their composition is a well-defined action.

The converse to this proposition is not true in general, as we are going to see in Example 4.3.18. Understanding the equational-equivalence relation between different Nakajima actions becomes a bit easier when one passes to Dynkin type A graphs, which we cover in the next section. We hope to understand this relation for general tree graphs in future work.

4.3 Minimal components of Nakajima actions in type A

Given a Dynkin graph A_n (see Figure (4.1)) and two non-negative integer vectors $\mathbf{v}, \mathbf{w}$, denote by $1, \ldots, n$ the vertices of the graph A_n, and let $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ denote the quiver variety $\mathfrak{M}_\zeta(A_n, \mathbf{v}, \mathbf{w})$. In this section we will compute explicitly the number of distinct minimal components in a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ corresponding to Nakajima actions.

4.3.1 Equational-equivalence classes

In this section we will give the formula for the number $N_e(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}))$ of equational-equivalence classes of Nakajima actions on the quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.

Let us assume firstly that $\mathbf{v} > 0$ (meaning $v_k > 0,\ \forall k$).

Definition 4.3.1. A **stop** is a vertex $k \in \{1, \ldots, n\}$ having $W_k \neq 0$.

On a quiver variety of type A_n, each Nakajima action φ can be represented by a string $\overline{a}(\varphi) = a_1 \ldots a_{n-1}$, where $a_k \in \{0, 1\}$, in the following way.

$$a_k = \begin{cases} 1, & \text{if the action acts with weight 1 on } B_{k,k+1} \\ 0, & \text{if the action acts with weight 1 on } B_{k+1,k} \end{cases}$$

Now, labelling stops $S = \{s_1, \ldots, s_m\}$ by vertical bars $|$, we add them to the string $\overline{a}(\varphi)$ at positions that correspond to their corresponding vertices in the graph, thus obtaining the string

$$a(\varphi) = a_1 \ldots a_{s_1-1} | a_{s_1} \ldots a_{s_2-1} | a_{s_2} \ldots a_{s_m-1} | a_{s_m} \ldots a_{n-1}$$

which we will call the **string** of the action φ. The **signature** of φ is a vector

$$r(\varphi) = (r_1, \ldots, r_{m-1}),$$

where r_i counts how many numbers labelled by 1 there are in the substring $a_{s_i}, \dots, a_{s_{i+1}}$ of $a(\varphi)$ between the stops s_i and s_{i+1}. We call an action **reduced** if its string satisfies the following:

- it starts with a (possibly empty) string of zeros $0\dots0|$,
- it finishes with a (possibly empty) string of zeros $|0\dots0$,
- and all substrings between two bars are of type $|\underbrace{1\dots1}_{k\geq0}\underbrace{0\dots0}_{l\geq0}|$.

Let us show the following two lemmas.

Lemma 4.3.2. *Any Nakajima action is equationally-equivalent to a reduced action.*

Proof. Observe that the move $\dots01\dots \to \dots10\dots$ on the places k and $k+1$ in the string a changes the moment map of the corresponding action by (after omitting the usual 1/2 factor):

$$\|B_{k,k+1}\|^2 + \|B_{k+2,k+1}\|^2 - \|B_{k+1,k}\|^2 - \|B_{k+1,k+2}\|^2 = \mathrm{tr}(\mu_I^{k+1}),$$

where we used (4.5), the fact that there is no stop, and the Frobenius norm of a matrix (so $\|A\|^2 = \mathrm{tr}(A^*A) = \sum_{i,j}|A_{ij}|^2$). Therefore the move does not change the equational-equivalence class of the corresponding action. Observe that the argument also works in the cases $\dots|01\dots \to \dots|10\dots$ and $\dots01|\dots \to \dots10|\dots$ since μ_I^{k+1} will only involve the B-matrices.

We also claim that the following move does not change the action's equational-equivalence class,

$$\underbrace{1\dots11}_{k}00\dots \to \underbrace{1\dots1}_{k-1}000\dots$$

Indeed the difference in the corresponding moment maps is

$$\begin{aligned}
\|B_{k,k+1}\|^2 - \|B_{k+1,k}\|^2 &= \|B_{k-1,k}\|^2 - \|B_{k,k-1}\|^2 - \mathrm{tr}(\mu_I^k)\\
&= \|B_{k-2,k-1}\|^2 - \|B_{k-1,k-2}\|^2 - tr(\mu_I^k) - tr(\mu_I^{k-1})\\
&\quad\dots\\
&= \|B_{1,2}\|^2 - \|B_{2,1}\|^2 - \sum_{l=2}^{k} tr(\mu_I^l)\\
&= -\sum_{l=1}^{k} tr(\mu_I^l) \sim 0.
\end{aligned}$$

Analogously, the move

$$\dots 00\underbrace{11\dots 1}_{k} \to \dots 000\underbrace{1\dots 1}_{k-1}$$

does not change the equational-equivalence class of the corresponding action. As any string a can be transformed, using a finite number of these moves, to a string of a reduced action, any action is equationally-equivalent to a reduced action. ■

Lemma 4.3.3. *Given two Nakajima actions with signatures $(r'_1, \dots, r'_{m-1})$ and $(r''_1, \dots, r''_{m-1})$, the difference between their moment maps is equationally-equivalent to*

$$\sum_{l=1}^{m-1} (k_l + \dots + k_{m-1})(\|i_{s_l}\|^2 - \|j_{s_l}\|^2), \tag{4.13}$$

where $k_l := r'_l - r''_l$.

Proof. By Lemma 4.3.2, without loss of generality we can assume that actions are reduced. We can divide the difference between the moment maps into the pieces that correspond to parts of graph between consecutive stops

$$F' - F'' = (F'_1 - F''_1) + \dots + (F'_{m-1} - F''_{m-1}).$$

Let us first calculate the last term $F'_{m-1} - F''_{m-1}$ From the discussion of the second move in the proof of Lemma 4.3.2, we see that if $k_{m-1} = 1$ then $F'_{m-1} - F''_{m-1} \sim \left\|B_{s_{m-1},s_{m-1}+1}\right\|^2 - \left\|B_{s_{m-1}+1,s_{m-1}}\right\|^2$. Using a similar computation, in general we get

$$F'_{m-1} - F''_{m-1} \sim k_{m-1}(\left\|B_{s_{m-1},s_{m-1}+1}\right\|^2 - \left\|B_{s_{m-1}+1,s_{m-1}}\right\|^2). \tag{4.14}$$

The μ_I-moment map equation at the vertex s_{m-1} gives

$$\left\|B_{s_{m-1},s_{m-1}+1}\right\|^2 - \left\|B_{s_{m-1}+1,s_{m-1}}\right\|^2 \sim \left\|B_{s_{m-1}-1,s_{m-1}}\right\|^2 - \left\|B_{s_{m-1},s_{m-1}-1}\right\|^2 + \left\|i_{s_{m-1}}\right\|^2 - \left\|j_{s_{m-1}}\right\|^2,$$

thus together with (4.14) we get

$$F'_{m-1} - F''_{m-1} \sim k_{m-1}(\left\|B_{s_{m-1}-1,s_{m-1}}\right\|^2 - \left\|B_{s_{m-1},s_{m-1}-1}\right\|^2) + k_{m-1}(\left\|i_{s_{m-1}}\right\|^2 - \left\|j_{s_{m-1}}\right\|^2).$$

Repeating this substitution (omitting the i, j maps whenever the vertex is not a stop), we inductively arrive at

$$F'_{m-1} - F''_{m-1} \sim k_{m-1}(\|B_{s_1,s_1+1}\|^2 - \|B_{s_1+1,s_1}\|^2) + \sum_{l=2}^{m-1} k_{m-1}(\|i_{s_l}\|^2 - \|j_{s_l}\|^2).$$

If $s_1 = 1$, using the moment map equation $\mu_I^{s_1}$ we get

$$F'_{m-1} - F''_{m-1} \sim \sum_{l=1}^{m-1} k_{m-1}(\|i_{s_l}\|^2 - \|j_{s_l}\|^2). \tag{4.15}$$

Otherwise, using the same equation, we get

$$F'_{m-1} - F''_{m-1} \sim k_{m-1}(\|B_{s_1-1,s_1}\|^2 - \|B_{s_1,s_1-1}\|^2) + \sum_{l=1}^{m-1} k_{m-1}(\|i_{s_l}\|^2 - \|j_{s_l}\|^2),$$

and then, using the second move from the proof of Lemma 4.3.2,

$$\|B_{s_1-1,s_1}\|^2 - \|B_{s_1,s_1-1}\|^2 \sim \|B_{s_1-2,s_1-1}\|^2 - \|B_{s_1-1,s_1-2}\|^2 \sim \cdots \sim 0,$$

hence we get (4.15) as well.

In the same way, we can get that $F'_{m-2} - F''_{m-2}$ is equationally-equivalent to the right hand side of equation (4.15) but with $m-1$ replaced by $m-2$, and so on. Then observe that $\sum_{p=1}^{m-1} k_p \sum_{l=1}^{p}(\|i_{s_l}\|^2 - \|j_{s_l}\|^2)$ can in fact be rewritten as in the claimed equation (4.13). ■

Reduced actions are labelled bijectively by their signatures. Moreover, we prove:

Proposition 4.3.4. *Reduced actions are equationally-equivalent iff they have the same signatures.*

Proof. Consider two reduced actions φ' and φ'' with different signatures $(r'_1, \ldots, r'_{m-1})$ and $(r''_1, \ldots, r''_{m-1})$. That means that there is a maximal number $t \leq m-1$ such that $r'_t \neq r''_t$. Assigning $k_l := r'_l - r''_l$, from Lemma 4.3.3, the difference between the moment maps of the actions φ', φ'' is equationally-equivalent to

$$\sum_{l=1}^{t}(k_l + \cdots + k_t)(\|i_{s_l}\|^2 - \|j_{s_l}\|^2).$$

Now, assume by contradiction that the actions φ' and φ'' are equationally-equivalent. Then

$$\sum_{l=1}^{t}(k_l + \cdots + k_t)(\|i_{s_l}\|^2 - \|j_{s_l}\|^2) = \sum_{k=1}^{n} p_k tr(\mu_I^k) \tag{4.16}$$

for some integers p_k. Recall that this equation is taking place on the space M from (4.1), therefore all variables B_h, i_k, j_k are independent. By comparing the coefficients of $\|i_{s_l}\|^2$ we deduce that $p_{s_l} = k_l + \cdots + k_t$, for all $l = 1, \ldots, t$, so in particular $p_{s_t} = k_t \neq 0$. Now, the $tr(\mu_I^{s_t})$ induces a term

$$\|B_{s_t+1,s_t}\|^2 - \|B_{s_t,s_t+1}\|^2$$

on the right hand side of (4.16), which does not exist on the left hand side. Therefore, it needs to cancel with terms on the right hand side, forcing $p_{s_t+1} \neq 0$. That in turn induces the term

$$\|B_{s_t+2,s_t+1}\|^2 - \|B_{s_t+1,s_t+2}\|^2$$

on the right side of (4.16), and so on – we inductively deduce that $p_{s_t+1}, \ldots, p_{s_{t+1}} \neq 0$. Note here that $t \leq m-1$, so the stop s_{t+1} exists. Finally, $p_{s_{t+1}} \neq 0$ would imply that the term

$$\left\|i_{s_{t+1}}\right\|^2 - \left\|j_{s_{t+1}}\right\|^2$$

appears on the right hand side of (4.16), contradicting the fact that it does not appear on the left. ■

Theorem 4.3.5. *Given a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ with $\mathbf{v} > 0$, two Nakajima actions on it are equationally-equivalent iff their signatures are same. Therefore, the number of equational-equivalence classes of Nakajima actions is*

$$N_e(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})) = N(\mathbf{w}) := \prod_{k=1}^{m-1} (s_{k+1} - s_k + 1),$$

where $S = \{s_1, \ldots, s_m\}$ is the set of stops.

Proof. The theorem follows from Lemma 4.3.2 and Proposition 4.3.4. ■

Notice that in the case $\mathbf{v} > 0$ the number $N(\mathbf{w})$ of equational-equivalence classes does not depend on $\mathbf{v}$, as it depends only on stops, which are defined only by $\mathbf{w}$. Now, we pass to the general case when v_i may be zero for some $i \in \{1 \ldots n\}$.

Definition 4.3.6. Given a non-negative integer vector $\mathbf{v} = (v_1, \ldots, v_n)$ of length n, a **gap** of $\mathbf{v}$ is any number $i \in \{1 \ldots n\}$ such that $v_i = 0$, and the **support** of $\mathbf{v}$ is the set

$$\mathit{supp}(\mathbf{v}) := \{i \mid v_i \neq 0\}.$$

Gaps of $\mathbf{v}$ divide its support into components

$$\mathit{supp}(\mathbf{v}) = \mathit{supp}(\mathbf{v})^1 \sqcup \mathit{supp}(\mathbf{v})^2 \sqcup \ldots \mathit{supp}(\mathbf{v})^k$$

such that each $\mathit{supp}(\mathbf{v})^i$ is a set of consecutive integers. Given an ordered pair of non-negative integer vectors $(\mathbf{v}, \mathbf{w})$ of length n, denote by

$$\mathbf{v}^i := (v_k | k \in \mathit{supp}(\mathbf{v})^i)$$

$$\mathbf{w}^i := (w_k | k \in \mathit{supp}(\mathbf{v})^i)$$

the parts of the vectors $\mathbf{v}$ and $\mathbf{w}$ that are supported on $\mathit{supp}(\mathbf{v})^i$.

Given two non-negative integer vectors $\mathbf{v}$ and $\mathbf{w}$, due to the construction of quiver varieties, it is easy to deduce that

$$\mathfrak{M}(\mathbf{v}, \mathbf{w}) = \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2) \times \cdots \times \mathfrak{M}(\mathbf{v}^k, \mathbf{w}^k),$$

so the following is an immediate corollary of Theorem 4.3.5.

Corollary 4.3.7. *Given a quiver variety* $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$*, the number of equational-equivalence classes of Nakajima actions is*

$$N_e(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})) = N(\mathbf{v}, \mathbf{w}) := \prod_{i=1}^{k} N(\mathbf{w}^i).$$

4.3.2 Equivalence versus equational-equivalence

In this section we will describe the relation between equivalence and equational-equivalence of Nakajima actions, deducing the number of minimal components induced by Nakajima actions on a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.

Definition 4.3.8. Given a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ of type A_n we call $\mu := \mathbf{w} - C\mathbf{v}$ its **weight vector**. Here $C = A - 2I$ is the **Cartan matrix** of A_n, where A is the adjacency matrix of the same graph. We say that the quiver variety has a **dominant weight** if $\mu \geq \mathbf{0}$.

A nice property for the quiver varieties with dominant weights is the following theorem which follows from Nakajima's work.

Theorem 4.3.9 (Nakajima)**.** *When a quiver variety* $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ *has a dominant weight, the morphism* $\pi : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ *is surjective.*

Proof. In [Nak94a, Thm. 4.1] Nakajima proves that the map π is a resolution, thus surjective, when the set of regular points $\mathfrak{M}_0^{reg}(Q, \mathbf{v}, \mathbf{w})$ (defined in (4.9)) is non-empty. Moreover, by [Nak98, Prop. 10.5 and Rmk. 10.9] we have that in particular for ADE quivers this set is non-empty when $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is non-empty and has a dominant weight, hence the theorem follows. ■

We will extend Theorem 4.3.9, thus characterising the image of morphism π for an arbitrary quiver variety of type A, at the end of this section (Proposition 4.3.22). For now, we remark that the surjectivity in the last theorem can hold **without** the assumption of dominant weight, as the next example shows.

Example 4.3.10. In the example of A_1 quiver $\mathbf{v} = v, \mathbf{w} = w$, the dominant weight condition becomes $w - 2v \geq 0$. Thus, we see that the quiver variety $\mathfrak{M}(3,5)$ does not satisfy it. By [Nak94a, Sec. 8] (see also [Kir16, Example 10.45]), we have that

$$\mathfrak{M}(n,r) \cong T^*Gr(n,r),^{6}$$

whereas

$$\mathfrak{M}_0(n,r) = \{y \in End(\mathbb{C}^r) | y^2 = 0, rank(y) \leq n\},$$

and the morphism $\pi : \mathfrak{M}(n,r) \to \mathfrak{M}_0(n,r)$ is isomorphic to the generalised Springer resolution, defined in Section 5.1.2. In particular, by Theorem 5.1.2(2) we have that the image of $\pi : \mathfrak{M}(3,5) \to \mathfrak{M}_0(3,5)$ is the closure of the nilpotent orbit $\mathcal{O}_{221}$ whereas we directly compute that

$$\mathfrak{M}_0(3,5) = \{y \in End(\mathbb{C}^5) | y^2 = 0, rank(y) \leq 3\} = \overline{\mathcal{O}_{221}}$$

as well. Thus, π is surjective although $\mathfrak{M}(3,5)$ does not have dominant weight.

Next, we give an example when surjectivity of Theorem 4.3.9 does **not** hold.

Example 4.3.11. Given a quiver variety of type A_1 with $\mathbf{v} = 4, \mathbf{w} = 6$, by the previous example and Theorem 5.1.2(2), we have that the image of $\pi : \mathfrak{M}(4,6) \to \mathfrak{M}_0(4,6)$ is the closure of nilpotent orbit $\mathcal{O}_{2211}$, whereas we directly compute that

$$\mathfrak{M}_0(4,6) = \{y \in End(\mathbb{C}^6) | y^2 = 0, rank(y) \leq 4\} = \overline{\mathcal{O}_{222}},$$

thus strictly bigger than the image of π.

We are going to use the explicit description of the coordinate ring $\mathbb{C}[\mathfrak{M}_0(\mathbf{v}, \mathbf{w})]$ by Lusztig-Maffei, which we recall now briefly.

Definition 4.3.12. We will define the set of **admissible strings** as

$$\mathcal{P} = \{j_q B_{q-1,q} \dots B_{r,r+1} B_{r+1,r} \dots B_{p,p-1} i_p \mid p,q,r = 1 \dots n, r \leq \min(p,q)\}.$$

Also we will call a **closed B-path** a string of type $\alpha = B_{h_n} B_{h_{n-1}} \dots B_{h_1}$ such that $s(h_1) = t(h_n)$.

[6]Here $Gr(n,r)$ denotes the Grassmann manifold of n-planes in $\mathbb{C}^r$.

Given a point $x = [(B_h, i_k, j_k)]$ in a quiver variety $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ and a string

$$\beta = j_q B_{q-1,q} \dots B_{r,r+1} B_{r+1,r} \dots B_{p,p-1} i_p$$

in $\mathcal{P}$, one obtains a natural evaluation $\beta(x) \in \mathrm{Hom}(W_{\beta_0}, W_{\beta_1})$. Here $\beta_0 = p$ and $\beta_1 = q$ are starting and ending vertices of the string β. Analogously, one can evaluate a closed B-path

$$\alpha = B_{h_n} B_{h_{n-1}} \dots B_{h_1}$$

to obtain $\alpha(x) \in \mathrm{Hom}(V_k, V_k)$, where $k = s(h_1) = t(h_n)$.

Theorem 4.3.13. [Lu98, Thm. 1.3], [Maf05, Lem. 7] *The coordinate ring of the affine quiver variety* $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ *is generated by the polynomials of type*

$$x \to f(\beta(x)), \text{ for } \beta \in \mathcal{P} \text{ and } f \in (Hom(W_{\beta_0}, W_{\beta_1}))^*$$

and

$$x \to Tr(\alpha(x)), \text{ for } \alpha \text{ a closed } B\text{-path.}$$

Corollary 4.3.14. *When* $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ *has a dominant weight and* $\mathbf{v} > \mathbf{0}$,

$$N(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})) = N_e(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})) = N(\mathbf{w}).$$

Proof. We will prove that non-equational-equivalence implies non-equivalence. That together with Proposition 4.2.7 and Theorem 4.3.5 gives a complete proof. Consider two Nakajima actions φ_1 and φ_2 that are non-equationally-equivalent. That means that there are two stops, s_1 and s_2, such that in the substrings of $a(\varphi_1)$ and $a(\varphi_2)$ between these stops there is a different number of numbers labelled by 1. But that would mean that the actions on the generators $f(\beta(x))$ of $\mathbb{C}[\mathfrak{M}_0(\mathbf{v}, \mathbf{w})]$ given by the string

$$\beta = j_{s_1} B_{s_1+1,s_1} \dots B_{s_2,s_2-1} i_{s_2}$$

are different for any non-trivial choice of form $f \in (\mathrm{Hom}(W_{\beta_0}, W_{\beta_1}))^*$. In fact, in order to prove that the actions are non-equivalent on $\mathfrak{M}_0$, we just have to prove that there is at least one form f such that $f(\beta(x))$ is a non-zero polynomial on $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, or equivalently, that $\beta(x)$ is a non-zero function on $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. We will do it explicitly by finding a point $x = [(B_h, i_k, j_k)] \in \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ such that $\beta(x) \neq 0$. Let us prove the following lemma first:

Lemma 4.3.15. *Let U, V, W be Hermitian vector spaces with orthonormal bases e_i, f_i, g_i, respectively. Define the linear maps $U \xrightarrow{L_1} V \xrightarrow{L_2} W$ by*

$$L_1(e_i) = \begin{cases} f_1, & \text{when } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$L_2(f_i) = \begin{cases} g_1, & \text{when } i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $L_1 L_1^* = L_2^* L_2$.

Proof. From the definition of L_1 and L_2 and orthonormality of bases, we have that $L_1^*(f_1) = e_1$ and $L_2^*(g_1) = f_1$ are the only non-zero values of L_1^* and L_2^* on bases f and g. Hence $L_1 L_1^* = L_2^* L_2$ follows immediately. ∎

Now, choosing orthonormal bases for the Hermitian spaces $W_{s_2}, V_{s_2}, \ldots, V_{s_1}, W_{s_1}$, we can make a string of non-zero maps

$$W_{s_2} \xrightarrow{i_{s_2}} V_{s_2} \xrightarrow{B_{s_2,s_2-1}} V_{s_2-1} \to \ldots \to V_{s_1+1} \xrightarrow{B_{s_1+1,s_1}} V_{s_1} \xrightarrow{j_{s_1}} W_{s_1}.$$

analogously to the maps L_1 and L_2 from the lemma. Let $x = (B_h, i_k, j_k) \in M(V, W)$ be the point having these maps as appropriate components and all other components being equal to zero. Then the moment map values μ_I^k for vertices $k = s_1, \ldots, s_2$ are exactly

$$\begin{aligned} \mu_I^{s_1}(x) &= -j_{s_1}^* j_{s_1} + B_{s_1+1,s_1} B_{s_1+1,s_1}^* \\ \mu_I^{s_1+1}(x) &= -B_{s_1+1,s_1}^* B_{s_1+1,s_1} + B_{s_1+2,s_1+1} B_{s_1+2,s_1+1}^* \\ &\ldots \\ \mu_I^{s_2}(x) &= -B_{s_2,s_2-1}^* B_{s_2,s_2-1} + i_{s_2} i_{s_2}^* \end{aligned}$$

hence vanish according to the lemma, whereas the values $\mu_I^k(x)$ for other vertices k are trivially equal to zero. The components $\mu_{\mathbb{C}}^k(x)$ of the other two moment maps are zero as well, as by construction of x we have $i_k j_k = 0$ for any $k \in Q^0$ and $B_h B_{\bar{h}} = 0$ for any $h \in H$. Hence, $x \in \mu^{-1}(0)$, so $0 \neq [x] \in \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, and

$$\beta(x) = j_{s_1} B_{s_1+1,s_1} \ldots B_{s_2,s_2-1} i_{s_2} \neq 0.$$

Hence, the two Nakajima actions φ_1 and φ_2 are non-equivalent on $\mathfrak{M}_0$. As the map π is surjective (by Theorem 4.3.9) and $\mathbb{C}^*$-equivariant, we have that the actions φ_1 and φ_2 are non-equivalent on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ as well. ∎

Example 4.3.16. A particular instance of Corollary 4.3.14 is the minimal resolution of a Du Val singularity of type A_n, which is a quiver variety $\mathfrak{M}(\underbrace{11\ldots11}_{n}, 1\underbrace{0\ldots0}_{n-2}1)$ of A_n type. Here the weight is dominant $\mu = \mathbf{w} - C\mathbf{v} = 0$, and the stops are $S = \{1, n\}$, so there are exactly

$$N(\mathbf{w}) = n - 1 + 1 = n$$

non-equivalent Nakajima actions, hence n induced minimal components. On the other hand, it is known that the core $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ is the Dynkin A_n tree of spheres. Thus in this case the minimal components *exhaust all* of the core components.

Remark 4.3.17. Of course, in the last example one could show that all components are minimal also by explicitly writing down different $\mathbb{C}^*$-actions "by hand". However, the point of this example was to show how one could easily count these actions using the developed technology, which extends far beyond the Du Val singularities case.

Now we give an example of a quiver variety which *does not* have a dominant weight, and the assertion of Corollary 4.3.14 is not true.

Example 4.3.18. A quiver variety $\mathfrak{M}(343, 210)$ of type A_3 corresponds to the lowest weight of the irreducible highest weight representation $L(210)$ of $\mathfrak{sl}_4$, so it is isomorphic to the quiver variety $\mathfrak{M}(000, 210) = \{pt\}$ under Nakajima reflection functors, described in [Nak03]. In other words, $\mathfrak{M}(343, 210)$ is a point, so all Nakajima actions on it are equivalent. On the other hand, the number of non-equationally equivalent Nakajima actions on it is

$$N(210) = 2 - 1 + 1 = 2$$

as the set of stops is $S = \{1, 2\}$.

We will now explain why this discrepancy between equivalence and equational-equivalence seen in this example is not unreasonable.

Given a general quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, the morphism $\pi : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ need not be surjective, so the proof of Corollary 4.3.14 does not go through. However, there are certain **Nakajima reflection functors** [Nak03] which are hyperkähler isometries between quiver varieties whose weights are related by the Weyl group action. In particular case of quiver varieties of type ADE, one can relate an arbitrary quiver variety with the one that has a dominant weight. That is, there is a hyperkähler isometry

$$\Phi_\sigma : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\sigma\cdot\zeta}(\sigma *_{\mathbf{w}} \mathbf{v}, \mathbf{w}), \tag{4.17}$$

such that $\mathfrak{M}_{\sigma\cdot\zeta}(\sigma *_{\mathbf{w}} \mathbf{v}, \mathbf{w})$ has a dominant weight. Here $\sigma \in W(A_n)$ is an element of the Weyl group $W(A_n) = S_{n+1}$ of the Lie algebra that corresponds to the Dynkin diagram A_n, and the actions of the Weyl group $\sigma \cdot \zeta$ and $\sigma *_{\mathbf{w}} \mathbf{v}$ are defined in a certain way that we will not describe here. The interested reader can confer [Nak03] (beginning of 2(i) and Definition 2.3, respectively). The argument in [BL13, Sec. 2.1.3] by Bezrukavnikov-Losev discusses the equivariance property of the reflection functor[7] Φ_σ with respect to the torus $T = (\mathbb{C}^*)^{Q_1} \times (\mathbb{C}^*)^{Q_0}$ action on both sides. Here, the $(\mathbb{C}^*)^{Q_0}$-part of the action on quiver varieties is induced from the $\mathbb{C}^*$ scalar actions on the vector spaces W_i and similarly the $(\mathbb{C}^*)^{Q_1}$-part is induced from the $\mathbb{C}^*$ scalar actions on the edges belonging to Ω_0 (recall the construction of a quiver variety). They claim that the reflection functor is twisted-equivariant, by a certain automorphism of T. When translated to the language of Nakajima actions, their claim becomes:

Proposition 4.3.19 (Bezrukavnikov-Losev)**.** *The Nakajima reflection functor Φ_σ interchange between Nakajima actions. This means that under it, a Nakajima action on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ becomes another Nakajima action on $\mathfrak{M}_{\sigma\cdot\zeta}(\sigma *_{\mathbf{w}} \mathbf{v}, \mathbf{w})$.*

Hence, two quiver varieties connected via the reflection functor have the same number of non-equivalent Nakajima actions. Therefore, given a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, it is enough to pass to the one $\mathfrak{M}_{\sigma\cdot\zeta}(\sigma *_{\mathbf{w}} \mathbf{v}, \mathbf{w})$, with dominant weight. Due to the general representation theory of semisimple Lie algebras, the vector $\mathbf{v}' = \sigma *_{\mathbf{w}} \mathbf{v}$ that induces the dominant weight is uniquely defined by $\mathbf{v}$. We will calculate it for completeness. Given $\mathbf{v}$ and $\mathbf{w}$, denoting by $\mu_k := (\mathbf{w} - C\mathbf{v})_k$ the weight vector components, consider the composition[8] $p = (p_1, \ldots, p_{n+1})$ of $\sum_i iw_i$ which satisfies $p_i - p_{i+1} = \mu_i$. Solving this system of equations, one comes up with a formula

$$p_i = \frac{1}{n+1}\left(\sum_{k=1}^{i-1} -k\mu_k + \sum_{k=i}^{n}(n+1-k)\mu_k + \sum_{k=1}^{n} iw_i\right).$$

Let ρ be the descending-ordered permutation of elements in p, and $\mu' = (\rho_1 - \rho_2, \ldots, \rho_n - \rho_{n+1})$ the corresponding weight vector. Finally define

$$\mathbf{v}' := C^{-1}(\mathbf{w} - \mu'), \tag{4.18}$$

thus $\mu' = \mathbf{w} - C\mathbf{v}' \geq 0$, as required.

[7] We remark that in that paper the authors call it a *LMN isomorphism* instead.

[8] Recall that a composition of n is a vector $(a_1, \ldots, a_k)$ of non-negative integers a_i that sum up to n.

The reason that $\mathbf{v}' \geq 0$ indeed holds is implicit and it is due to the classical representation theory of $\mathfrak{sl}_n$ and the correspondence between quiver varieties of type A and irreducible representations of $\mathfrak{sl}_n$, due to [Nak98]. For type A it simply states that for any weight space of weight μ of an irreducible representation $L(\mathbf{w})$ of $\mathfrak{sl}_n$ with highest weight $\mathbf{w}$, there is an associated quiver variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ of type A, such that $\mu = \mathbf{w} - C\mathbf{v}$, and vice-versa. In particular as we know that μ is a weight, so is μ', being in its Weyl-orbit (permuting p corresponds to action by $S_{n+1} = W(\mathfrak{sl}_n)$). Thus, **there is** a corresponding quiver variety with $\mu' = \mathbf{w} - C\mathbf{v}'$, hence $\mathbf{v}' \geq 0$.

Definition 4.3.20. Given a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, the vector $\mathbf{v}' = \mathbf{v}'(\mathbf{v}, \mathbf{w})$ obtained by this algorithm is called its **dominant vector**.

The importance of the dominant vector is that it allows us to count the number of non-equivalent Nakajima actions for an arbitrary quiver variety of type A, giving the main result of this chapter.

Theorem 4.3.21. *Given a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ of type A, denoting by $\mathbf{v}'$ its dominant vector, there are exactly $N(\mathbf{v}', \mathbf{w})$ non-equivalent Nakajima actions, and thus $N(\mathbf{v}', \mathbf{w})$ distinct minimal components induced by them.*

Proof. This follows by combining the previous results. Firstly, due to Proposition 4.3.19 and the paragraph after it, we have that $N(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})) = N(\mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w}))$ where $\mathbf{v}' = \mathbf{v}'(\mathbf{v}, \mathbf{w})$ is the associated dominant vector. Now, recall (Definition 4.3.6) the split of vectors $\mathbf{v}', \mathbf{w}$ into subvectors $\{\mathbf{v}'^i, \mathbf{w}^i\}_{i=1\ldots k}$, given by the support of $\mathbf{v}'$. This split induces the identification

$$\mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w}) \cong \prod_{i=1}^{k} \mathfrak{M}_{\zeta'^i}(\mathbf{v}'^i, \mathbf{w}^i)$$

(where ζ'^i are the corresponding subvectors of moment parameters) and the choices of picking a Nakajima action on $\mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w})$ correspond to collection of choices of picking Nakajima actions on each $\mathfrak{M}_{\zeta'^i}(\mathbf{v}'^i, \mathbf{w}^i)$. Thus, we have

$$N(\mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w})) = \prod_{i=1}^{k} N(\mathfrak{M}(\mathbf{v}'^i, \mathbf{w}^i)).$$

Then, notice that the quiver varieties $\mathfrak{M}_{\zeta'^i}(\mathbf{v}'^i, \mathbf{w}^i)$ have dominant weights. Picking an arbitrary i, let $\mathbf{v}'^i = (v_r, \ldots, v_{r+s})$, and $\mathbf{v} = (v_1, \ldots, v_n)$. We want to prove that $\mu'^i := \mathbf{w}^i - C^i\mathbf{v}'^i \geq 0$, where C^i is the Cartan matrix for the subgraph $(r, \ldots, r+s)$

of type A that is the support of vector $\mathbf{v}'^i$. As the quiver variety $\mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w})$ has dominant weight, we have $\mu' = \mathbf{w} - C\mathbf{v}' \geq 0$, and thus

$$0 \leq \mu'|_{supp(\mathbf{v}'^i)} = (\mathbf{w} - C\mathbf{v}')|_{supp(\mathbf{v}'^i)} = \mathbf{w}^i - C^{[r,r+s]\times[1,n]}\mathbf{v}', \tag{4.19}$$

where $C^{[r,r+s]\times[1,n]}$ is the submatrix of C consisting of rows $r, \ldots, r+s$. Now, notice that

$$C^{[r,r+s]\times[1,n]}\mathbf{v}' = C^i\mathbf{v}'^i, \tag{4.20}$$

since $C^i = C^{[r,r+s]\times[r,r+s]}$ and $C^{[r,r+s]\times[1,n]}$ is supported on it, with a possible exception of $C^{r,r-1} = -1$ and $C^{r+s,r+s+1} = -1$. But in those cases we have $v'_{r-1} = 0$ and $v'_{r+s+1} = 0$ accordingly, thus equality (4.20) holds. Together with (4.19), we get that the quiver varieties $\mathfrak{M}_{\zeta'^i}(\mathbf{v}'^i, \mathbf{w}^i)$ have dominant weights indeed.

As $\mathbf{v}'^i > 0$, according to Corollary 4.3.14, we have $N(\mathfrak{M}(\mathbf{v}'^i, \mathbf{w}^i)) = N(\mathbf{w}^i)$. Connecting with previously said, the number of non-equivalent Nakajima actions on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is

$$N(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})) = N(\mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w})) = \prod_{i=1}^{k} N(\mathfrak{M}(\mathbf{v}'^i, \mathbf{w}^i)) = \prod_{i=1}^{k} N(\mathbf{w}^i) = N(\mathbf{v}', \mathbf{w}).$$

The statement about minimal components follows due to Proposition 3.2.2 and Lemma 4.2.2. ■

In conclusion, given two vectors $\mathbf{v}$ and $\mathbf{w}$, notice that the number $N(\mathbf{v}', \mathbf{w})$ of minimal components in a quiver variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is fairly easy to compute.

At the end of this section, we characterise the image of the morphism $\pi : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ for an arbitrary quiver variety of type A, using the notion of the dominant vector.

Proposition 4.3.22. *Given a quiver variety $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ of type A, the image of the morphism $\pi : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ is isomorphic to $\mathfrak{M}_0(\mathbf{v}', \mathbf{w})$, where $\mathbf{v}'$ is the associated dominant vector.*

Proof. By [MN19, Prop. 3.9] (which goes back to [Nak94a, Prop. 6.7]), we have the stratification of the Poisson variety $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ by symplectic leaves

$$\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \bigsqcup_{\mathbf{u}\leq\mathbf{v},\ \mathbf{w}-C\mathbf{v}\geq 0} \mathfrak{M}_0^{reg}(\mathbf{u}, \mathbf{w}),$$

which is induced by natural inclusions $\iota_\mathbf{u} : \mathfrak{M}_0(\mathbf{u}, \mathbf{w}) \hookrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ given by considering subrepresentations of $U \subset V$. In this stratification, the closure of a leaf $\mathfrak{M}_0^{reg}(\mathbf{u}, \mathbf{w})$ is exactly the image of $\iota_\mathbf{u}$, isomorphic to $\mathfrak{M}_0(\mathbf{u}, \mathbf{w})$.

Thus, it is enough show that the image of π is the closure of the leaf $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$. Let us first show that this leaf is indeed in $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, i.e. that $\mathbf{v}' \leq \mathbf{v},\ \mathbf{w} - C\mathbf{v}' \geq 0$ holds. The second inequality is satisfied by definition of the dominant vector, so let us prove that the first one. Recall that we have defined the dominant vector using the composition $p = (p_1, \ldots, p_{p+1})$ of $\sum_i iw_i$ such that $p_i - p_{i+1} = \mu_i$, where $\mu = \mathbf{w} - C\mathbf{v}$ is the weight vector. Then $\mathbf{v}' := C^{-1}(\mathbf{w} - \mu')$, where μ' is the associated dominant weight, given as $\mu' = (\rho_1 - \rho_2, \ldots, \rho_n - \rho_{n+1})$ where ρ be the descending-ordered permutation of elements in p.

We will show $\mathbf{v}' \leq \mathbf{v}$ inductively, by considering their corresponding compositions. Namely, as one can get the composition ρ out of composition p just by doing swaps between adjacent numbers

$$\ldots a, b \ldots \to \ldots b, a \ldots$$

whenever $a < b$, it is enough to show that this move makes the corresponding dimension vector smaller or equal to the the previous one. Thus, let us consider the composition $q = (q_1, \ldots, q_{n+1})$ having $q_i < q_{i+1}$ and the composition q' that is obtained by their swap. Assuming that $i \notin \{1, n+1\}$, the corresponding weight vectors $\nu = (q_1 - q_2, \ldots, q_n - q_{n+1})$ and $\nu' = (q_1' - q_2', \ldots, q_n' - q_{n+1}')$ differ only on positions $i-1, i, i+1$,

$$\begin{aligned} \nu_{i-1}' &= \nu_{i-1} + q_i - q_{i+1} = \nu_{i-1} + \nu_i, \\ \nu_i' &= -\nu_i, \\ \nu_{i+1}' &= \nu_{i+1} + q_i - q_{i+1} = \nu_{i+1} + \nu_i. \end{aligned} \tag{4.21}$$

Now, we want to show the inequality $\mathbf{u} \leq \mathbf{u}'$ between the corresponding dimension vectors $\mathbf{u} = C^{-1}(\mathbf{w} - \nu), \mathbf{u}' = C^{-1}(\mathbf{w} - \nu')$. It reduces to show $C^{-1}\nu \leq C^{-1}\nu'$, which is equivalent to proving the inequality $\sum_{r=1}^n k_r\nu_r \leq \sum_{r=1}^n k_r\nu_r'$, for an arbitrary row $\mathbf{k} = (k_1, \ldots, k_n)$ of the matrix C^{-1}. Thus, using the formulae (4.21) this finally reduces to show

$$k_{i-1} + 2k_i - k_{k+1} \geq 0.$$

This is trivial now, as the left hand side of this inequality is exactly the product of the row $\mathbf{k} = (k_1, \ldots, k_n)$ with the i-th column[9] $(\underbrace{0, \ldots, 0}_{i-2}, -1, 2, -1, 0, \ldots, 0)^t$ of C, thus is either equal to 0 or 1. Assuming that $i = 1$ goes similarly, reducing to prove the inequality

$$2k_1 - k_2 \geq 0$$

[9]Recall that $C = 2I - A$, where A is the adjacency matrix of the graph A_n.

for an arbitrary row $\mathbf{k} = (k_1, \dots, k_n)$ of C^{-1}, which we do similarly by considering the product of $\mathbf{k}$ with the first column $(2, -1, 0, \dots, 0)^t$ of C. The $i = n+1$ case goes verbatim.

Thus, we have $\mathbf{v}' \leq \mathbf{v}$, so $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$ indeed occurs as a stratum of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. We want to prove that its closure is the image $\mathfrak{M}^1(\mathbf{v}, \mathbf{w})$ of π, and we are done. It is enough to show that the $\mathfrak{M}^1(\mathbf{v}, \mathbf{w})$ contains $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$, as both are irreducible varieties of the same dimension. Indeed, $\mathfrak{M}^1(\mathbf{v}, \mathbf{w})$ is a closed irreducible subset of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, as an image of a connected variety under a projective morphism. Moreover, being smooth and connected,[10] $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$ is irreducible as well, and so is its closure $\mathfrak{M}_0(\mathbf{v}', \mathbf{w})$. Finally, these two irreducible varieties have the same dimension, due to following equalities

$$\dim \mathfrak{M}^1(\mathbf{v}, \mathbf{w}) = \dim(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})) = \dim(\mathfrak{M}_\zeta(\mathbf{v}', \mathbf{w})) = \dim(\mathfrak{M}_0(\mathbf{v}', \mathbf{w})).$$

The first equality is due to the fact that π is a resolution onto its image, the second is due to existence of a reflection functor $\Phi_\sigma : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w})$ (which is a diffeomorphism), and the third is due to surjectivity of the morphism $\pi' : \mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}', \mathbf{w})$ (Theorem 4.3.9).

Hence, let us prove that the image of π contains $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$, i.e. that the fibres of π above $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$ are non-empty. By [Nak01, Sec. 3.3], given a point $x \in \mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$, its fibre $\pi^{-1}(x)$ is isomorphic to the central fibre $\widetilde{\pi}^{-1}(0)$ of the morphism $\widetilde{\pi} : \mathfrak{M}(\mathbf{v} - \mathbf{v}', \mathbf{w} - C\mathbf{v}') \to \mathfrak{M}_0(\mathbf{v} - \mathbf{v}', \mathbf{w} - C\mathbf{v}')$. Now, notice that there is a reflection functor[11]

$$\mathfrak{M}(\mathbf{v} - \mathbf{v}', \mathbf{w} - C\mathbf{v}') \to \mathfrak{M}(0, \mathbf{w} - C\mathbf{v}'),$$

because the weights that correspond to those quiver varieties are $\mathbf{w} - C\mathbf{v}' - C(\mathbf{v} - \mathbf{v}') = \mathbf{w} - C\mathbf{v} = \mu$ and $\mathbf{w} - C\mathbf{v}' = \mu'$, hence lie in the same Weyl group orbit (as previously said, the Weyl group action on weights corresponds to permutation action on compositions p), which is needed for the existence of a reflection functor. Finally, we have $\mathfrak{M}_{\zeta'}(0, \mathbf{w} - C\mathbf{v}') \cong \{pt\}$, thus $\pi^{-1}(x) \cong \{pt\} \neq \emptyset$. ■

[10] As its isomorphic preimage $Z := \pi^{-1}(\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w}))$ is a subset in smooth connected $\mathfrak{M}(\mathbf{v}', \mathbf{w})$ with complement of real codimension at least 2, thus must be connected as well.

[11] We omit the moment parameters in the notation for quiver varieties, as they are not important for the argument.

4.4 Nakajima actions are the only weight-1 conical actions

Let $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ be a quiver variety of type A. We call **full quiver action** the $\mathbb{C}^*$-action which acts with weight-1 on all edges of the doubled graph $Q^\#$ (as opposed to the Nakajima action in (4.11) which only acts on half of the edges). Proceeding as in Section 3.2.1, we can compose the full quiver action with complex 1-parameter subgroups of the symplectic action $GL(\mathbf{w}) \curvearrowright \mathfrak{M}(\mathbf{v}, \mathbf{w})$ on the framing. In this section we prove that, in the case of quiver variety with dominant weight, squares of Nakajima actions are **the only** even[12] weight-2 conical actions amongst such composed actions. Therefore, the minimal components that we produce from Nakajima actions in Section 4.3 exhaust the set of all minimal components which one gets using $GL(\mathbf{w})$ (Having in mind Remark 3.1.7). We prove this by comparing these actions on $\mathfrak{M}_0$ using the Lusztig-Maffei description of the coordinate ring $\mathbb{C}[\mathfrak{M}_0]$ given in Theorem 4.3.13.

Definition 4.4.1. Given a graph Q, the space $M(Q, V, W)$ of framed representations of the double quiver $Q^\#$ has a natural $\mathbb{C}^*$-action by dilation

$$t \cdot (B_h, i_k, j_k) = (tB_h, ti_k, tj_k), \ t \in \mathbb{C}^* \tag{4.22}$$

which acts with weight-2 on $\omega_{\mathbb{C}}$, hence induces weight-2 $\mathbb{C}^*$-actions on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ such that π is a $\mathbb{C}^*$-equivariant map. We call this action the **full quiver action**, as it acts on all edges of the framed double quiver $Q^\#$.

Example 4.4.2. Observe again the example 4.1.1 of quiver variety $\mathfrak{M}_\zeta(A_1, 1, n+1) \cong T^*\mathbb{C}P^n$. Here the full quiver action contracts the fibres with weight-2, thus it is a square of a Nakajima action that contracts the fibres with weight-1. This is not true in general, as we will see below.

Further, notice that the group $GL(W) = \prod_{i \in Q_0} GL(W_i)$ acts by $\omega_{\mathbb{C}}$-symplectomorphisms on the space $M(Q, V, W)$ by conjugation

$$g \curvearrowright (B, i, j) = (B, ig^{-1}, gj). \tag{4.23}$$

This action induces a $GL(\mathbf{w})$ action on the quiver varieties $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$ such that π is a $GL(\mathbf{w})$-equivariant map.

Example 4.4.3. In our example $\mathfrak{M}_\zeta(A_1, 1, n+1) \cong T^*\mathbb{C}P^n$ we have that $GL(\mathbf{w}) = GL(n+1)$ acts $\omega_{\mathbb{C}}$-symplectomorphically on $T^*\mathbb{C}P^n$ by the action induced from the canonical action $GL(n+1) \curvearrowright \mathbb{C}P^n$.

[12] An action is called **even** if it is a square of another action.

As the $GL(\mathbf{w})$ action acts by $\omega_{\mathbb{C}}$-symplectomorphisms on $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$, and commutes with the full quiver action, the composition of the action of a 1-parameter subgroup of $GL(\mathbf{w})$ with the full quiver action gives a weight-2 $\mathbb{C}^*$-action on the quiver variety $\mathfrak{M}_\zeta(Q, \mathbf{v}, \mathbf{w})$. We will give these actions some special names.

Definition 4.4.4. Call G**-twisted full action** any composition of the action of a 1-parameter subgroup $G \leq GL(\mathbf{w})$ with the full quiver action. Varying the 1-parameter subgroup G, the set of all these actions are the **twisted full actions**. In particular, when a 1-parameter subgroup of $G \leq GL(\mathbf{w})$ consists of scalar operators, that is, $G = t^{m_1} Id_{W_1} \times \cdots \times t^{m_n} Id_{W_n}$, we call its twisted full action a **scalar action**.

Now, we will connect these actions with previously defined Nakajima actions. An action is called **even** if it is a square of an action.

Lemma 4.4.5. *Even conical scalar actions on* $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ *are exactly the squares of Nakajima actions. The same holds for* $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$.

Proof. We first consider the case $\mathbf{v} \geq 0$ for the affine quiver variety $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. Denote the set of stops by $S = \{s_1, \ldots, s_n\}$. Label each scalar action by the vector $\mathbf{m} = (m_1, \ldots, m_n)$ of weights m_i by which it acts on W_{s_i}. Now let us translate the condition of action being conical and even in terms of the vector $\mathbf{m}$. Firstly, let us consider the simplest strings that give generators of $\mathbb{C}[\mathfrak{M}_0(\mathbf{v}, \mathbf{w})]$

$$\beta_i' = j_{s_i} B_{s_i+1, s_i} \ldots B_{s_{i+1}, s_{i+1}-1} i_{s_{i+1}},$$

$$\beta_i'' = j_{s_{i+1}} B_{s_{i+1}-1, s_{i+1}} \ldots B_{s_i, s_i+1} i_{s_i}.$$

Their weights of the action are $2 + s_{i+1} - s_i + m_i - m_{i+1}$ and $2 + s_{i+1} - s_i + m_{i+1} - m_i$, respectively. These two being positive and even is equivalent to

$$|m_{i+1} - m_i| \leq s_{i+1} - s_i \tag{4.24}$$

and

$$m_{i+1} - m_i = s_{i+1} - s_i \pmod 2. \tag{4.25}$$

Altogether, that gives $s_{i+1} - s_i + 1$ possible choices for the value of $m_{i+1} - m_i$. Having the conditions (4.24) and (4.25) satisfied, all the other generators of the coordinate ring of $\mathbb{C}[\mathfrak{M}_0(\mathbf{v}, \mathbf{w})]$ will have positive and even weights. Indeed, the generators that come from a string of type

$$\beta = j_{s_i} B_{s_i-1, s_i} \ldots B_{r, r+1} B_{r+1, r} \ldots B_{s_j, s_j-1} i_{s_j}, \quad i < j$$

have the weight

$$\begin{aligned}2+s_j-r+s_i-r+m_i-m_j &= 2+s_j-s_i+m_i-m_j+2(s_i-r)\\ &\geq 2+s_j-s_i+m_i-m_j\\ &= 2+\sum_{k=i}^{j-1}(s_{k+1}-s_k+m_k-m_{k+1})\end{aligned}$$

and the last term is positive and even being a sum of such. The same proof works for strings of type

$$\beta = j_{s_i}B_{s_i-1,s_i}\dots B_{r,r+1}B_{r+1,r}\dots B_{s_j,s_j-1}i_{s_j}, \quad i>j.$$

Finally, the closed B-paths, that is, strings of type $\alpha = B_{h_n}B_{h_{n-1}}\dots B_{h_1}$ where $s(h_1)=t(h_n)$ will always have an even number of any edge showing up in the set $\{h_1,\dots,h_n\}$, (as a type A graph is a tree), hence will have a positive and even weight as well. Thus, we have shown that there are exactly

$$\prod_{k=1}^{m-1}(s_{k+1}-s_k+1) = N(\mathbf{w})$$

distinct choices for the vector $\mathbf{m}'=(m_2-m_1,\dots,m_n-m_{n-1})$, that yield distinct even conical scalar actions, whose number is the same as the number of all Nakajima actions. To finish the proof of lemma, let us now prove that a scalar action that is conical and even is the square of a Nakajima action. Given its vector $\mathbf{m}$, let us define

$$r_i := \frac{1}{2}(m_2-m_1+s_2-s_1),$$

which is a non-negative integer, due to (4.24) and (4.25). The claim is that the square of the Nakajima action with signature $(r_1,r_2,\dots r_{n-1})$ is the same as the given scalar action, which is a straightforward check that their weights on strings of type

$$\beta = j_{s_i}B_{s_i-1,s_i}\dots B_{r,r+1}B_{r+1,r}\dots B_{s_j,s_j-1}i_{s_j}$$

and

$$\alpha = B_{h_n}B_{h_{n-1}}\dots B_{h_1}, \text{ where } s(h_1)=t(h_n)$$

agree. This finishes the proof for $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$. Using the equivariant projection π we have that these actions agree on the open dense subset of $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$. Due to their continuity, they agree on the whole $\mathfrak{M}(\mathbf{v},\mathbf{w})$. For general $\mathbf{v}$, we split the vector $\mathbf{v}$ into disjoint $\mathbf{v}^i > 0$ vectors like in Definition 4.3.6 and run the same proof for $\mathfrak{M}(\mathbf{v}^i, \mathbf{w}^i)$, knowing that

$$\mathfrak{M}(\mathbf{v}, \mathbf{w}) = \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2) \times \cdots \times \mathfrak{M}(\mathbf{v}^k, \mathbf{w}^k)$$

and both the Nakajima and twisted full actions are products of those on the components $\mathfrak{M}(\mathbf{v}, \mathbf{w})$, hence the result holds in full generality. ■

So, all the minimal components coming from scalar actions are already obtained by Nakajima actions. The next step is to delve into the general twisted full actions.

Lemma 4.4.6. *A twisted full action on $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ that is even and conical must be a scalar action. The same holds for $\mathfrak{M}_\varsigma(\mathbf{v}, \mathbf{w})$, when its weight is dominant.*

Proof. First, we choose bases for W_k so the G-twisted full action becomes a diagonal one, that is, it consists of diagonal matrices. This is possible due to the fact that every 1-parameter subgroup lies in a maximal torus and the fact that all maximal tori are conjugate to the diagonal one. Now, pick an arbitrary vertex k. In the chosen basis, the map $j_k i_k : W_k \to W_k$ is represented as a matrix (a_{ij}). Denote the $GL(W_k)$-part of G by

$$g_k = \begin{bmatrix} t^{r_1} & & \\ & \ddots & \\ & & t^{r_{w_k}} \end{bmatrix}.$$

As the G-twisted full action acts on $j_k i_k$ by $t^2 g_k j_k i_k g_k^{-1}$ we get that it acts on the matrix entry a_{ij} with weight $t^{2+r_i-r_j}$. Let us assume that g_k is not a scalar matrix. We then have $r_i < r_j$ for some i, j hence the action acts on a_{ij} with weight $2+r_i-r_j < 2$.

So, let us prove that a_{ij} is indeed a non-zero polynomial. This we will do by constructing an explicit point $[x] \in \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ such that $a_{ij}(x) = 1$, similarly to the proof of Corollary 4.3.14. Firstly we can assume that the basis f_s of W_k under which the g_k is diagonal is orthonormal. Also, choose an orthonormal basis e_s of V_k. Now, define maps

$$i_k(f_s) = \begin{cases} e_1, & \text{when } s = j \\ 0, & \text{otherwise} \end{cases}$$

and

$$j_k(e_s) = \begin{cases} f_i, & \text{when } s = 1 \\ 0, & \text{otherwise}. \end{cases}$$

Then we have $i_k^*(e_1) = f_j$ and $j_k^*(f_i) = e_1$ and zero for other basis elements. Thus, we get that

$$i_k i_k^* - j_k^* j_k = 0, \quad i_k j_k = 0. \tag{4.26}$$

Now choosing the point $x = (B, i, j) \in M(V, W)$ with i_k and j_k as above, and all other components equal to zero, we see that equation (4.26) gives us $x \in \mu^{-1}(0)$ hence $0 \neq [x] \in \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. On the other hand, $j_k i_k(f_j) = f_i$ hence $a_{ij}(x) = 1$, that is, a_{ij} is not a zero polynomial, as required. Hence, g_k needs to be scalar for all vertices k, which is what we wanted to prove.

The statement for $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ when $\mathbf{v}$ is dominant follows immediately as then the projection $\pi : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ is surjective, hence a twisted full action that is even and conical on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is of the same type on $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, thus it is a scalar action. ■

The previous two lemmas yield the following proposition.

Proposition 4.4.7. *On a quiver variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ with dominant weight, twisted full actions that are even and conical are exactly the squares of Nakajima actions.*

Combining this with Theorem 4.3.21 we deduce the final statement about minimal components in quiver varieties of type A.

Corollary 4.4.8. *Given a quiver variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ with dominant weight, the distinct minimal components of its core $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ obtained from twisted full actions are exactly those arising from Nakajima actions, and there are exactly $N(\mathbf{v}, \mathbf{w})$ of them.*

4.4.1 Further remarks

We have seen that minimal components of Nakajima actions are the only minimal components that one can get from twisted full actions. In this section we give, in a slightly speculative way, some further remarks on why there should be no other minimal components, arising from the conical structure given by the full quiver action.[13] As in Definition 3.2.5, denote by $\mathrm{Symp}_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$ the group of **conical symplectomorphisms** (i.e. algebraic π-compatible[14] symplectomorphisms of $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ that commute with the full quiver action). We give a conjecture about this group, motivated by a comment given in [MN17, Sec. 3.4]. In their notation, let

$$\mathbb{K} := \mathrm{Im}(GL(\mathbf{w}) \to \mathrm{Symp}_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$$

[13] Recall (Definition 2.2.11) that a conical structure is a maximal set of mutually-commuting conical actions.

[14] Meaning: a map that preserves the fibres of π.

be the image of the $GL(\mathbf{w})$-group action on $\mathfrak{M}(\mathbf{v},\mathbf{w})$, which lands in $\mathrm{Symp}_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w}))$ as it commutes with the full quiver action. Even more precisely, $\mathbb{K}$ is in the connected component of the identity $\mathrm{Symp}_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w}))^o$ of this group.[15]

Conjecture 4.4.9. *Given a quiver variety $\mathfrak{M}(\mathbf{v},\mathbf{w})$ of type A, the action $GL(\mathbf{w})$ yields all conical symplectomorphisms of $\mathfrak{M}(\mathbf{v},\mathbf{w})$, that is*

$$\mathbb{K} = Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w}))^{\circ}. \tag{4.27}$$

Remark 4.4.10. We remark here that this conjecture is true for two families of examples of quiver varieties, giving a sketch of the proof for both.

- Du Val singularities of type A (Example 4.3.16). Using the Maffei-Lusztig description for the coordinate ring of the affine quiver variety (Theorem 4.3.13), one can pass to the usual description of the coordinate ring of Du Val singularity of type A_{n-1}, $\mathbb{C}[\mathfrak{M}_0] \cong \mathbb{C}[X,Y,Z]/V(XY - Z^n)$, together with its Poisson brackets, given by $\{X,Y\} = -nZ^{n-1}, \{Z,X\} = X, \{Y,Z\} = Y$. For $n \geq 3$,[16] we can directly check that the graded Poisson automorphism of the ring $\mathbb{C}[\mathfrak{M}_0]$ must be of type $(X,Y,Z) \to (tX, t^{-1}Y, Z)$ or $(X,Y,Z) \to (tY, (-1)^n t^{-1}X, -Z)$. Automorphisms of the second type swap the exceptional spheres in the resolution, thus act non-trivially on the homology, hence are not in the connected component of identity. Finally, one can compute that the group $GL(\mathbf{w})$ acts on $\mathfrak{M}_0$ in these coordinates by $t \cdot (X,Y,Z) = (t^n X, t^n Y, t^2 Z)$, i.e. exhausts all Poisson automorphisms of the first type. The equality (4.27) follows, as a conical symplectomorphism ϕ on $\mathfrak{M}$ projects to a Poisson automorphism ϕ_0 on $\mathfrak{M}_0$,[17] thus picking an element $g \in GL(\mathbf{w})$ whose action on $\mathfrak{M}_0$ is equal to ϕ_0, its action on $\mathfrak{M}$ is equal to ϕ on an open dense subset, hence by analytic continuity, on the whole $\mathfrak{M}$.

- Cotangent bundles of generalised flag varieties of type A, known to be quiver varieties by [Nak94a, Sec. 7]. Namely, we have the isomorphism of complex-symplectic manifolds

$$T^*\mathcal{F}(k_1,\ldots,k_n;r) \cong \mathfrak{M}(k_1,\ldots,k_n;r,0,\ldots,0) =: \mathfrak{M}, \tag{4.28}$$

[15]The group $\mathrm{Symp}_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w}))$ need not to be connected in general. Namely, it is known that quiver automorphisms can induce discrete group of automorphisms of the corresponding quiver varieties. In Example 4.3.16 of Du Val singularities, the A_n Dynkin graph can be reflected, which induces an $\mathbb{Z}/2$-automorphism group, whose generator acts non-trivially on the homology of the resolution (as it swaps the spheres in the core). Thus, this automorphism cannot belong in the connected component of the identity, thus the conical symplectomorphism group is disconnected.

[16]The case $n = 2$ is covered in the next example, as then $\mathfrak{M} = T^*\mathbb{C}P^1$.

[17]Recall that a conical symplectomorphism is π-compatible.

where $\mathcal{F} := \mathcal{F}(k_1, \ldots, k_n; r) = \{\mathbb{C}^r = F_0 \supset F_1 \supset \cdots \supset F_{n+1} = 0 \mid \dim F_i = k_i\}$ is a generalised flag variety, and on $T^*\mathcal{F}$ we have the standard complex-symplectic structure $\omega = d\alpha$. The pull back of the full quiver action to $T^*\mathcal{F}$ is precisely the square of the $\mathbb{C}^*$-action that contracts the fibres. Hence, a symplectomorphism of $(T^*\mathcal{F}, \omega)$ that commutes with it must preserve the 1-form α (as it is given by $\alpha = i_Z\omega$, where Z is the (complex) vector field of the contracting action). By [Can01, p. 20-21, Ex. 2,3], we have that such a symplectomorphism of a cotangent bundle must be induced from a automorphism of the base.[18] Now one uses the standard result [Akh95, Thm. 2, Sec. 3.3] which says that connected component of the identity of the group of the automorphisms of a generalised flag variety G/P is exactly G,[19] which for our flag variety $\mathcal{F}$ gives $GL(\mathbb{C}^r)$. Finally, we connect that with the fact (which one can conclude directly from the description of (4.28) from [Nak94a, Sec. 7]) that the action of $GL(\mathbf{w}) \cong GL(\mathbb{C}^r)$ seen on $T^*\mathcal{F}$ is exactly the induced action from the action $GL(\mathbb{C}^r)$ on $\mathcal{F}$.

Now, let us comment on why only the group $Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))^\circ$ is the relevant one for our purposes. Firstly, having a conical symplectomorphism $\psi \in Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w})) \setminus Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))^\circ$ we have that it does not yield any new actions, hence minimal components:

Corollary 4.4.11. *Assume that Conjecture 4.4.9 holds for a quiver variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$. Then conjugation by $\psi \in Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w})) \setminus Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))^\circ$ preserves Nakajima actions. That is, given a Nakajima action φ_t, the conjugation $\psi\varphi_t^1\psi^{-1}$ is another Nakajima action. In particular, if $\mathfrak{F}$ is a minimal component of a Nakajima action, $\psi(\mathfrak{F})$ is as well.*

Proof. As we have proved, the square of a Nakajima action is a twisted full action, thus φ_t^2 is the composition $\varphi_t^2 = \phi_t G_t$ of the full quiver action ϕ_t and a 1-parameter subgroup $G_t \leq GL(\mathbf{w})$. Then, $\psi\varphi_t^2\psi^{-1} = \psi\phi_t G_t\psi^{-1} = \phi_t\psi G_t\psi^{-1}$. Then, as G_t is in $Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))^\circ$, which is a normal subgroup of $Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$, the conjugation $G'_t := \psi G_t\psi^{-1}$ is in $Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))^\circ$. By Conjecture 4.4.9 it is the image of a 1-parameter subgroup of $GL(\mathbf{w})$. Thus

$$\psi\varphi_t^2\psi^{-1} = \phi_t G'_t$$

[18] The text in that book is on *real* symplectic structures, but the argument there goes through identically in the complex setup.

[19] This result does not for every simple G but certainly does type A, which we need.

is a twisted full action. It is obviously even: $\psi\varphi_t^2\psi^{-1} = (\psi\varphi_t\psi^{-1})^2$. Let us show that it is conical as well. Firstly, we have that ψ preserves the core

$$\mathfrak{L}(\mathbf{v},\mathbf{w}) = \{p \in \mathfrak{M}(\mathbf{v},\mathbf{w}) \mid \lim_{t\to\infty} \phi_t(p) \text{ exists}\}$$

as it is a continuous function and commutes with the full quiver action ϕ_t. Now, the conical property means precisely that for every $p \in \mathfrak{M}(\mathbf{v},\mathbf{w})$, we have $\lim_{t\to 0} \psi\varphi_t^2\psi^{-1}(p) \in \mathfrak{L}(\mathbf{v},\mathbf{w})$, which is true as ψ is continuous and preserves the core, and φ_t^2 is conical itself.

Thus the action $\psi\varphi_t^2\psi^{-1}$ is an even and conical twisted full action, hence by Proposition 4.4.7 it has to be square of a Nakajima action $\psi\varphi_t^2\psi^{-1} = (\varphi_t')^2$. If $\mathfrak{F}$ is the minimal component for φ_t, then $\psi(\mathfrak{F})$ is minimal component for $\varphi_t' = \psi\varphi_t\psi^{-1}$ as it is in its fixed locus and, as $\psi(\mathfrak{L}(\mathbf{v},\mathbf{w})) = (\mathfrak{L}(\mathbf{v},\mathbf{w}))$, it is an irreducible component of the core. ■

Thus, assuming that Conjecture 4.4.9 holds, the last corollary tells us that by using symplectomorphisms $\psi \in Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w})) \setminus Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w}))^\circ$ we **cannot** get more minimal components of the core $\mathfrak{L}(\mathbf{v},\mathbf{w})$ beyond those we already have. Call a minimal component **relevant** if is obtained from a conical action that commutes with the full quiver action. Thus, we deduce the following:

Corollary 4.4.12. *Assume that Conjecture 4.4.9 holds for a quiver variety $\mathfrak{M}(\mathbf{v},\mathbf{w})$ of dominant weight. Then all relevant minimal components of $\mathfrak{L}(\mathbf{v},\mathbf{w})$ are obtained through Corollary 4.4.8, thus are minimal components of Nakajima actions.*

Proof. Given a relevant minimal component obtained from a weight-1 action φ_t, we have that $G_t := \varphi_t^2\phi^{-1} \in \mathrm{Symp}_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v},\mathbf{w}))$, thus by Conjecture 4.4.9, G_t is equal to an image of a 1-parameter subgroup of $GL(\mathbf{w})$. Hence, φ_t^2 is an even and conical twisted full action and indeed a square of a Nakajima action (by Corollary 4.4.8). ■

Chapter 5

Smooth components of Springer fibres

Springer fibres are known to be the cores of those CSRs which in the literature are called *S3-varieties*[1] or resolutions of Slodowy varieties. We will use the latter term in this text. The theoretical framework of Chapter 3 can be applied to these spaces, hence obtaining exact Lagrangians in resolutions of Slodowy varieties, or equivalently, smooth components of Springer fibres.

Springer fibres are fibres of *Springer resolution*, which is basically the central object in Geometric Representation Theory. The geometry of these fibres is used to recover various representation-theoretic objects. In particular, their cohomologies provide irreducible representations of Weyl groups [Spr78, KaLu80, LuSpa85], and Hecke algebras [KaLu79]; this is an consequence of more general feature known as *Springer correspondence*. From it, one can deduce a substantial information about representation theory of reductive groups, even over p-adic and finite fields. Cohomologies of Springer fibres also yield geometric representations of $U(\mathfrak{sl}_n)$ [Gi91] (see also [BLM90]). Moreover, their geometry recovers certain Parabolic Categories $\mathcal{O}$ [Str09] and Khovanov Arc Algebras [SW12, Sch12].

In this chapter, we will be interested on finding smooth components of Springer fibres of type A, and understanding their topology. The question of (non-)smoothness of components of *ordinary Springer fibres* and their topology is a question within active research in Springer theory, with a lot of interesting work in the previous decade [Fu03, PaRe06, Fr09b, FrMe10, FrMe11, Fr11, FrMeS-O15, BaZi08, GrZi11]. Moreover, the (non-)smoothness of this components, at least in the two-column case,[2] has been translated to the representation-theoretic language, in terms of certain W-

[1]The name comes from Slodowy, Spaltenstein and Springer, as they have been studying these spaces independently.

[2]Meaning that the Springer fibre is attached to an nilpotent element of order 2.

graphs[3] attached to the corresponding representation of the symmetric group.[4] Basically speaking, components of a Springer fibre are vertices of a W-graph, and singular components are the vertices that have "too many" edges.[5]

Although a lot of work has been done to understand (non-)smoothness and topology of components of ordinary Springer fibres, no work has been done on understanding the same for components of *generalised Springer fibres*, with an exception of the two-row[6] case [Sch12]. In this chapter, we give some first results in that direction. First, we construct a family of weight-1 actions that yields a family of smooth components in any generalised Springer fibre. Second, we find another family of smooth components that generalises the well-known *Richardson components,* of ordinary Springer fibres and also define their generalisations which we call *quasi-Richardson.* Finally, we show that these families of components generate more smooth components via so-called *crystal operators.*

5.1 Review on Springer theory

In this section we will review the standard results of Springer theory which we will be using in the following sections. Standard literature on this matter, where these statements and their proofs can be found, are the books by Collingwood-McGovern [CoMcG93] and Chriss-Ginzburg [CGi97].

5.1.1 Springer resolution

Let $\mathfrak{g}$ be a semisimple Lie algebra. Recall the **nilpotent cone** $\mathcal{N} \subset \mathfrak{g}$ is the set of all ad-nilpotent elements in $\mathfrak{g}$. Denote by $\mathcal{B}$ the flag variety of $\mathfrak{g}$. Then the **Springer resolution** for $\mathfrak{g}$ is the resolution of $\mathcal{N}$ by the cotangent bundle of $\mathcal{B}$,

$$\begin{array}{c} T^*\mathcal{B} \\ \nu \downarrow \\ \mathcal{N} \end{array} \tag{5.1}$$

We will deal only with the Springer resolution for $\mathfrak{sl}_n$, so let us describe it now. In this case, the nilpotent cone becomes the set of nilpotent matrices,

$$\mathcal{N} = \{e \in \mathfrak{sl}_n \mid e \text{ is a nilpotent matrix}\},$$

[3]Defined in [KaLu79].
[4]I.e. the Weyl group in type A.
[5]For the precise statement, see [FrMe11, Thm. 1.5.] and the paragraph after it.
[6]Meaning that the fibre is over a nilpotent element that has two nilpotent blocks.

and the flag variety becomes the variety of **full flags** in $\mathbb{C}^n$,

$$\mathcal{B} := \{0 = F_0 \subset F_1 \subset \cdots F_{n-1} \subset F_n = \mathbb{C}^n \mid \dim F_i = i\}.$$

The space

$$\widetilde{\mathcal{N}} = \{(F, e) \mid F \in \mathcal{B}, e \in \mathfrak{sl}_n, eF_i \subset F_{i-1}\}$$

and the map

$$\nu : \widetilde{\mathcal{N}} \to \mathcal{N}, \ \nu(F, e) = e,$$

form the **Springer resolution** for $\mathfrak{sl}_n$, which follows from the following facts.

Theorem 5.1.1.

(1) The projection on the first coordinate $\pi : \widetilde{\mathcal{N}} \to \mathcal{B}$, $\pi(F, e) = F$ is isomorphic to the cotangent bundle $T^*\mathcal{B}_p \to \mathcal{B}_p$, thus giving the variety $\widetilde{\mathcal{N}}$ the structure of a holomorphic symplectic manifold.

(2) The map ν is a resolution of singularities.

(3) $\mathcal{N} \subset \mathfrak{sl}_n$ is a Poisson subvariety,[7] hence itself a Poisson variety.

(4) The map ν is a symplectic resolution.

(5) Under the dilation action on $\mathfrak{sl}_n$ and the contraction of cotangent fibres on $T^*\mathcal{B}$, the map ν becomes a CSR of weight-1. Its core is exactly the flag variety $\mathcal{B}$.

Hence, this Springer resolution is a CSR of weight-1.

5.1.2 Generalised Springer resolution

This is an extension of the previous example, which uses any generalised flag variety instead of the full one. By **composition** of n we will mean a vector $p = (p_1, \dots, p_n)$ of n non-negative integers p_i that sum up to n. Out of those, we call **partitions** those compositions that satisfy $p_1 \geq p_2 \geq \cdots \geq p_n$. When writing partitions, we will omit the zeroes at the end. Thus for example, $(2, 0, 2, 0)$ and $(3, 1)$ are composition and partition of 4, respectively. We call **nilpotent orbit** an orbit of a nilpotent element of $\mathfrak{sl}_n$ under the conjugation action, thus the nilpotent cone is a disjoint union of nilpotent orbits. Furthermore, the nilpotent orbits are labelled by partitions of the

[7]Here, we use the Killing form to pull-back the canonical Lie-Poisson structure on $\mathfrak{sl}_n^*$. Thus, the symplectic leaves on $\mathfrak{sl}_n$ of this Poisson structure are adjoint orbits.

integer n which are given by the lengths of the Jordan blocks of any element in the orbit. We denote by

$$\mathcal{O}_\lambda = \mathcal{O}_{\lambda(e)}$$

the nilpotent orbit labelled by the partition $\lambda = \lambda(e)$ corresponding to an element $e \in \mathcal{N}$, called the **Jordan partition** of e. Writing $\lambda \vdash n$ to denote that λ is a partition of n, we therefore have

$$\mathcal{N} = \bigsqcup_{\lambda \vdash n} \mathcal{O}_\lambda.$$

Define the partial **dominance order** $\lambda \preceq \eta$ on partitions $\lambda = (\lambda_1, \dots, \lambda_n)$, $\eta = (\eta_1, \dots, \eta_n)$, to mean $\lambda_1 + \cdots + \lambda_i \leq \eta_1 + \cdots + \eta_i, \forall i = 1, \dots, n$. It turns out that the closures of nilpotent orbits satisfy

$$\overline{\mathcal{O}}_\lambda = \bigsqcup_{\lambda' \preceq \lambda} \mathcal{O}_{\lambda'}.$$

To a composition p of n one associates a **generalised flag variety**[8]

$$\mathcal{B}_p := \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n \ \mid \dim F_i/F_{i-1} = p_i,\ 1 \leq i \leq n\},$$

consisting of p-partial flags, and, analogously to Section 5.1.1, a space

$$\widetilde{\mathcal{N}}_p = \{(F, e) \mid F \in \mathcal{B}_p, e \in \mathfrak{sl}_n, eF_i \subset F_{i-1}\}$$

with a map

$$\nu_p : \widetilde{\mathcal{N}}_p \to \mathcal{N},\ \mu(F, e) = e.$$

Now we give a generalisation of Theorem 5.1.1. Recall first that given a partition $p = (p_1, p_2, \dots, p_n)$ of a positive integer n, so that $p_1 \geq p_2 \geq \cdots \geq p_n$, one obtains a **Young diagram** $Y(p)$ with rows consisting of $p_1, p_2, \dots, p_n$ boxes. Given a Young

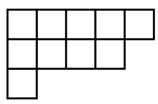


Figure 5.1: Young diagram Y(5,4,1).

diagram $Y = Y(p)$ we call the partition p its **shape.** Denoting the numbers of boxes in its columns by $p_1^*, p_2^*, \dots, p_n^*$, the *dual partition* of p is defined as $p^* = (p_1^*, \dots, p_n^*)$. For example, in Figure 5.1 the dual partition of $(5, 4, 1)$ is $(3, 2, 2, 2, 1)$, and $n = 10$.

[8]As $p = (p_1, \dots, p_n)$ is a composition of n, in general it will have zeroes, thus $F_{i+1} = F_i$ may occur within flags. In the examples we will omit the repeating flags for convenience.

Theorem 5.1.2.

(1) The projection on the first coordinate $\pi : \widetilde{\mathcal{N}}_p \to \mathcal{B}_p$, $\pi(F, e) = F$ is isomorphic to the cotangent bundle $T^*\mathcal{B}_p \to \mathcal{B}_p$, giving the variety $\widetilde{\mathcal{N}}_p$ the structure of a holomorphic symplectic manifold.

(2) The image of the map ν_p is the closure of the nilpotent orbit $\mathcal{O}_{p_+^*}$, where p_+ is the partition obtained from p by reordering it in descending order, and $p_+^* = (p_+)^*$ is its dual. Moreover, the map $\nu_p : \widetilde{\mathcal{N}}_p \to \overline{\mathcal{O}}_{p_+^*}$ is a resolution of singularities and an isomorphism over $\mathcal{O}_{p_+^*}$.

(3) Thus by (1) and (2), $T^*\mathcal{B}_p$ is a resolution of $\mathcal{O}_{p_+^*}$.

(4) $\overline{\mathcal{O}}_{p_+^*} \subset \mathfrak{sl}_n$ is a Poisson subvariety, hence itself a Poisson variety.

(5) The map ν_p is a symplectic resolution.

(6) Under the dilation action on $\mathfrak{sl}_n$ and the contraction of cotangent fibres on $T^*\mathcal{B}_p$, the map ν_p becomes a CSR of weight-1. Its core is exactly the generalised flag variety $\mathcal{B}_p$.

The map $\nu_p : \widetilde{\mathcal{N}}_p \to \overline{\mathcal{O}}_{p_+^*}$ is called a **generalised Springer resolution** for $\mathfrak{sl}_n$ for the partition p. Due to the theorem above, it is a CSR of weight-1. Its fibre $\mathcal{B}_p^e := \nu_p^{-1}(e)$ is called a **generalised Springer fibre**. Up to isomorphism of algebraic varieties, it depends only on the conjugacy class of the nilpotent element e. Thus, by abuse of notation, we label that fibre with:

$$\mathcal{B}_p^\lambda := \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n \mid \dim F_i/F_{i-1} = p_i,\ eF_i \subset F_{i-1},\ i = 1, \ldots, n\},$$

where $\lambda = \lambda(e)$ is the partition corresponding to the nilpotent orbit of e. When the composition p is equal to $(1, 1, \ldots, 1)$, these are the fibres of the ordinary Springer resolution, so we call them **(ordinary) Springer fibres** and denote them analogously by $\mathcal{B}^\lambda = \mathcal{B}^e := \nu^{-1}(e)$, hence

$$\mathcal{B}^\lambda = \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n \mid \dim F_i/F_{i-1} = 1,\ eF_i \subset F_{i-1},\ i = 1, \ldots, n\}.$$

5.1.3 Slodowy varieties

A way to get many more CSRs using (generalised) Springer resolutions is by using Slodowy slices. These are transverse slices to nilpotent orbits, constructed using only using the Lie-theoretic framework. As in the previous sections, let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{sl}_n$. For an arbitrary non-zero $e \in \mathcal{N}$, by the Jacobson-Morozov theorem we may pick a pair f, h of elements in $\mathfrak{sl}_n$ such that the triple (e, f, h) satisfies the sl_2-relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{5.2}$$

Moreover, any such pair f, h is determined uniquely up to conjugation by an element in the unipotent radical of the centralizer $C_{GL_n}(e)$. Now define its **Slodowy transversal slice** as the affine space

$$S_e = \{x \in \mathfrak{sl}_n \mid [x - e, f] = 0\} = e + ker(\mathrm{ad} f)$$

Up to isomorphism of algebraic varieties, $\mathcal{S}_e$ does not depend on the choice of f and h. For completeness, we also define $S_0 := \mathfrak{sl}_n$. As said before, the main property of the Slodowy slice is its transversal property.

Proposition 5.1.3. *The Slodowy slice S_e is transversal to all the nilpotent orbits in $\mathcal{N}$. It has a non-empty intersection exactly with orbits $\mathcal{O}_\eta$ such that $\lambda \preceq \eta$, where $\lambda = \lambda(e)$ is the Jordan partition of e. Moreover, $S_e \cap \mathcal{O}_\lambda = \{e\}$.*

Given a composition $p = (p_1, \ldots, p_n)$ of n with $p_i \geq 0$ we have the associated generalised Springer resolution $\nu_p : \widetilde{\mathcal{N}}_p \to \overline{\mathcal{O}_{p_+^*}}$. Let us denote by

$$\mathcal{S}_{e,p} := S_e \cap \overline{\mathcal{O}}_{p_+^*}, \quad \widetilde{\mathcal{S}}_{e,p} := \nu_p^{-1}(\mathcal{S}_{e,p}) \tag{5.3}$$

the **Slodowy variety** associated to e and p, and its resolution. For completeness, we also define $\mathcal{S}_{e,p} = \widetilde{\mathcal{S}}_{e,p} = \emptyset$ when $p_i < 0$ for some i. In particular, when $p = (1, 1, \ldots, 1)$, we call it the **ordinary Slodowy variety** associated to e, and denote it and its resolution by $\mathcal{S}_e$ and $\widetilde{\mathcal{S}}_e$. The important thing is that, up to isomorphism of algebraic varieties, all these varieties do not depend on the choice of nilpotent element e within the conjugacy class, nor on the choice of the other two elements of the sl_2 triple. Therefore, together with (5.3) we will use the notation $\mathcal{S}_{\lambda,p}$ and $\widetilde{\mathcal{S}}_{\lambda,p}$ as well, where $\lambda = \lambda(e)$ is the Jordan partition of e.

Theorem 5.1.4.

1. *The map $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ is a resolution of singularities and an isomorphism over $S_e \cap \mathcal{O}_{p_+^*}$*
2. *The variety $\mathcal{S}_{e,p} \subset \overline{\mathcal{O}}_{p_+^*}$ is a Poisson subvariety, hence itself a Poisson variety.*
3. *The map $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ is a symplectic resolution.*

Slodowy varieties are *not* invariant with respect to the weight-1 dilation action from previous sections. Therefore, we would need another action to make $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ into a CSR. Such an action exists, as we shall see in the next section.

5.1.4 Kazhdan action

Given a nilpotent element e and a chosen $\mathfrak{sl}_2$-triple (e, f, h), there is a natural $\mathbb{C}^*$-action that contracts the Slodowy slice S_e into the point $\{e\}$, usually called Kazhdan action in the literature [GaGi02, Sec. 4],[Sch17, Rmk. 11].

Definition 5.1.5. Given an sl_2-triple (e, f, h), where e is nilpotent, the $\mathbb{C}^*$-action on $\mathfrak{sl}_n$ given by

$$t \cdot x = t^2 \mathrm{Ad}(t^{-h})x \tag{5.4}$$

is called the **Kazhdan action** of the triple (e, f, h), and we explain the meaning of the matrix t^{-h} below.

Consider the Lie algebra homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_n$ defined by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h,$$

This Lie algebra homomorphism exponentiates to an algebraic group homomorphism $\tilde{\gamma} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n$. We define

$$\gamma : \mathbb{C}^* \to \mathrm{SL}_n \quad , \quad \gamma(t) = \tilde{\gamma} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad , \quad \forall t \in \mathbb{C}^*.$$

Now $t^{-h} := \gamma(t^{-1})$. For our purposes, h will always be a diagonal matrix $h = diag(h_1, \dots, h_n)$ in a given basis, when $t^{-h} = diag(t^{-h_1}, \dots, t^{-h_n})$. From the definition it is easy to prove

Proposition 5.1.6. *The Kazhdan action has the following properties:*

(1) It leaves $S_e = e + ker(\mathrm{ad} f)$ invariant.

(2) Moreover, it has only positive weights on S_e, hence contracts it to the unique fixed point $\{e\}$.

(3) It preserves the nilpotent orbits, hence leaves $S_{e,p}$ invariant.

(4) It acts by weight-2 on the Poisson brackets on $\mathfrak{sl}_n$.

(5) If lifts equivariantly from $\mathcal{N}$ to the Springer resolution $T^*\mathcal{B}$ by

$$t \cdot (x, F) = (t^2 \mathrm{Ad}(t^{-h})x, t^{-h}F).$$

Remark 5.1.7. Note above that the Kazhdan action $t^{-h}F$ on the flag $F \in B_{\lambda(e)}$ is induced by the action on $\mathbb{C}^n$ given by left-multiplication by t^{-h}. This action on $\mathbb{C}^n$ in turn determines a natural action on $\mathfrak{sl}_n$ by $x \mapsto t^{-h} \circ x \circ (t^{-h})^{-1} = \mathrm{Ad}(t^{-h})(x)$. Finally, the Kazhdan action is determined by the latter action by composing with the action of t^2. Thus, to recover the Kazhdan action it is enough to know how t^{-h} acts on a chosen basis for $\mathbb{C}^n$ in which e is in Jordan normal form. This observation will be used in Definition 5.2.34 to define the weight tableau of the Kazhdan action.

Hence we have the following useful corollary:

Corollary 5.1.8. *The map $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ together with the Kazhdan action is a weight-2 CSR. Its core is exactly the generalised Springer fibre $\mathcal{B}^e_p$.*

Example 5.1.9. We will describe the Kazhdan action in the case when e is the nilpotent element

$$e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{sl}_3$$

which has Jordan type $\lambda = (2, 1)$. A particular choice of its $\mathfrak{sl}_2$ triple are the following matrices

$$f = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and their corresponding Slodowy slice is

$$S_e = \left\{ \begin{bmatrix} a & 1 & 0 \\ b & a & d \\ c & 0 & -2a \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

As we have $t^h = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the Kazhdan action is

$$t \cdot \begin{bmatrix} a & 1 & 0 \\ b & a & d \\ c & 0 & -2a \end{bmatrix} = \begin{bmatrix} t^2a & 1 & 0 \\ t^4b & t^2a & t^3d \\ t^3c & 0 & -2t^2a \end{bmatrix}.$$

In the end of this section we mention a combinatorial characterisation on when the Kazhdan action is even (i.e. the square of an action). We will say that the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n is **even** if all λ_i are of the same parity, and say it is **odd** otherwise.

Proposition 5.1.10. *The Kazhdan action on S_e is even iff the Jordan partition of e is even.*

5.2 Twisted Kazhdan actions

Proceeding as in Section 3.2.1 (thus analogously to the strategy from Section 4.4), composing the Kazhdan action on $\widetilde{\mathcal{S}}_{e,p}$ with 1-parameter subgroups of a certain group that acts on $\widetilde{\mathcal{S}}_{e,p}$ by symplectomorphisms, we can get many more weight-2 conical actions that we call twisted Kazhdan actions. We will seek for even and conical ones, as their square roots will give us a family of weight-1 conical actions on $\widetilde{\mathcal{S}}_{e,p}$, which then induce minimal, thus smooth components of generalised Springer fibre $\mathcal{B}_p^e$. Having in mind Remark 3.1.7, in this way we exhaust *all* minimal, components arising from twisted Kazhdan actions.

Definition 5.2.1. Given a nilpotent e and its $\mathfrak{sl}_2$-triple (e, f, h), we denote by

$$Z_e := \{g \in GL(n) \mid ge = eg, gf = fg, gh = hg\}$$

the subgroup of $GL(n)$ that commutes with the triple, that is, the centraliser of the set $\{e, f, h\}$. This group, up to a conjugation by an element in the unipotent radical of $C_{GL_n}(e)$, does not depend on the choice of f and h, hence their omission in the notation.

We have the following proposition that is easy to prove.

Proposition 5.2.2. *The Slodowy Slice S_e is invariant under Z_e acting by conjugation. Hence, the same follows for the Slodowy varieties $\mathcal{S}_{e,p}$. Conjugation by Z_e naturally lifts to $T^*\mathcal{B}$ via the Springer resolution,*

$$g \cdot (x, F) = (gxg^{-1}, gF).$$

This induces an action on $\widetilde{\mathcal{S}}_{e,p}$, such that the map $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ is Z_e-equivariant. Moreover, the Z_e-action commutes with the Kazhdan action.

This is the Springer theoretic analogue of the group $GL(\mathbf{w})$ that acts equivariantly on the morphism $\pi : \mathfrak{M}(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ of quiver varieties, and that commutes with the full quiver action (Section 4.4). Hence, by analogy, we state the following definitions.

Definition 5.2.3. The composition of the action of a 1-parameter subgroup $G \leq Z_e$ on S_e with the Kazhdan action is a G**-twisted Kazhdan action**. Thus, denoting $G = (g_t)_{t \in \mathbb{C}^*}$, the twisted action on an arbitrary $\widetilde{\mathcal{S}}_{e,p}$ is given by

$$t \cdot (x, F) = (t^2 \mathrm{Ad}(g_t)\mathrm{Ad}(t^{-h})x, g_t t^{-h} F).$$

Varying the 1-parameter subgroup G, we call all these actions the **twisted Kazhdan actions**.

Remark 5.2.4. Similarly to Remark 5.1.7, the twisted Kazhdan action on flags arises from the action by left multiplication by $g_t t^{-h}$ on $\mathbb{C}^n$, and the twisted Kazhdan action is determined by this action on $\mathbb{C}^n$.

Now, in order to make an analogue of scalar actions from Section 4.4, we would need to view the group Z_e as a product of general linear groups first. With that in mind we will fix a choice of the triple (e, f, h) for a given nilpotent element e.

5.2.1 The standard $\mathfrak{sl}_2$-triple

Given a positive integer k we denote the following matrices of dimensions $k \times k$

$$e_k = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ & & & & 0 \end{bmatrix} \quad f_k = \begin{bmatrix} 0 & & & & \\ \alpha_1^k & 0 & & & \\ & \alpha_2^k & & & \\ & & \ddots & & \\ & & & \alpha_{k-1}^k & 0 \end{bmatrix} \quad h_k = \begin{bmatrix} h_1^k & & & \\ & h_2^k & & \\ & & \ddots & \\ & & & h_k^k \end{bmatrix}$$

where the numbers on all undeclared cells are equal to zero. Here $h_i^k := k-1-2i$, and $\alpha_i^k := \sum_{s=1}^{i} h_s^k$. An easy computation shows that

Proposition 5.2.5. *For any positive integer k, matrices e_k, f_k, h_k satisfy the $\mathfrak{sl}_2$-relations (5.2), thus form an sl_2-triple.* ■

Now, pick an element $e \in sl_n$ with Jordan partition $\lambda = (\lambda_1, \dots, \lambda_k)$. That means that its Jordan normal form is exactly the block-diagonal matrix

$$e_\lambda := \begin{bmatrix} e_{\lambda_1} & & & \\ & e_{\lambda_2} & & \\ & & \ddots & \\ & & & e_{\lambda_k} \end{bmatrix}.$$

Now, defining the block-diagonal matrices

$$f_\lambda := \begin{bmatrix} f_{\lambda_1} & & & \\ & f_{\lambda_2} & & \\ & & \ddots & \\ & & & f_{\lambda_k} \end{bmatrix} \quad h_\lambda := \begin{bmatrix} h_{\lambda_1} & & & \\ & h_{\lambda_2} & & \\ & & \ddots & \\ & & & h_{\lambda_k} \end{bmatrix}$$

we have the following generalisation of Proposition 5.2.5, whose proof directly follows from it.

Proposition 5.2.6. *Given a partition $\lambda \vdash n$, the matrices $e_\lambda, f_\lambda, h_\lambda$ satisfy the $\mathfrak{sl}_2$-relations (5.2), thus form an sl_2-triple.*

Definition 5.2.7. Given a partition λ, we call e_λ the **standard nilpotent element of type** λ and $(e_\lambda, f_\lambda, h_\lambda)$ the **standard $\mathfrak{sl}_2$-triple of type** λ. We will denote by S_λ and Z_λ their corresponding Slodowy slice and centraliser group. We will describe these two explicitly in the next two sections.

5.2.2 Slodowy slice of the standard triple

Let us first calculate the complex dimension of the Slodowy slice S_λ.

Proposition 5.2.8. $\dim(S_\lambda) = \sum_{i=1}^{k} (\lambda_i^*)^2 - 1$

Proof. As S_λ is transversal to the orbit $\mathcal{O}_\lambda$, its dimension is equal to $\dim \mathfrak{sl}_n - \dim \mathcal{O}_\lambda$. Recall that $\mathcal{O}_\lambda$ is resolved by $T^*\mathcal{B}_{\lambda^*}$, by Theorem 5.1.2(3). Thus $\dim \mathcal{O}_\lambda$ is twice the dimension of the generalised flag variety $\mathcal{B}_{\lambda^*}$, which is calculated easily by seeing it as a quotient $\mathcal{B}_{\lambda^*} = GL(n)/P_{\lambda^*}$ by the parabolic group P_{λ^*} that has diagonal blocks of sizes λ_i^*. Therefore, $\dim \mathcal{O}_\lambda = 2 \dim \mathcal{B}_{\lambda^*} = 2\frac{n^2 - \sum_{i=1}^{n} (\lambda_i^*)^2}{2} = n^2 - \sum_{i=1}^{n} (\lambda_i^*)^2$, and $\dim S_\lambda = \dim \mathfrak{sl}_n - \dim \mathcal{O}_\lambda = \sum_{i=1}^{n} (\lambda_i^*)^2 - 1$. ∎

Let us first describe the Slodowy slice S_n of the triple (e_n, f_n, h_n), that is we take $\lambda = (n)$ so $\lambda^* = (1, \ldots, 1)$ and $\dim S_n = n - 1$ by Lemma 5.2.8. We will call ***i*-subdiagonal** the lower subdiagonal that starts from the i-th row. Recall that f_n is a matrix with the 2-subdiagonal equal to $(\alpha_1^n, \ldots, \alpha_{n-1}^n)$, and all other entries equal to zero. We have the following lemma:

Lemma 5.2.9. *Any $n \times n$ matrix having zeroes outside the i-subdiagonal*

$$b_n = \begin{bmatrix} 0 & & & & & \\ \vdots & & & & & \\ \beta_1 & & & & & \\ & \beta_2 & & & & \\ & & \ddots & & & \\ & & & \beta_{n+1-i} & \ldots & 0 \end{bmatrix}$$

will commute with f_n if and only if $\alpha_{i-1+k}^n \beta_k = \beta_{k+1} \alpha_k^n$, $\forall k = 1, \ldots, n-i$.

Proof. This follows from computing $b_n f_n$ and $f_n b_n$, and comparing the non-zero values. ∎

Define $\alpha^n_k := 1$ for $k \in \mathbb{N}$ greater than $n-1$. For a non-empty finite subset $I \subset \mathbb{N}$, denote by $\alpha^n_I := \prod_{i \in I} \alpha^n_i$, and $\alpha_\emptyset := 1$. Also, for $a, b \in \mathbb{N}_0$ denote $[a, b] := \{a, a+1, \ldots, b\}$.

Given a complex n-dimensional vector $a = (a_0, \ldots, a_{n-1})$, define the matrix

$$S_n(a) = S_n(a_0, \ldots, a_{n-1}) := \begin{bmatrix} a_0 & & & & \\ \alpha^n_1 a_1 & a_0 & & & \\ \alpha^n_{[1,2]} a_2 & \alpha^n_2 a_1 & \ddots & & \\ \vdots & & & & \\ \alpha^n_{[1,n-1]} a_{n-1} & \cdots & \alpha^n_{[n-2,n-1]} a_2 & \alpha^n_{n-1} a_1 & a_0 \end{bmatrix}$$

hence

$$(S_n(a))_{ij} := \begin{cases} \alpha^m_{[j,i-1]} a_{i-j}, & \text{if } i \geq j \\ 0, & \text{otherwise} \end{cases}$$

Observe that the assignment $a \mapsto S_n(a)$ is linear in a.

Lemma 5.2.10. *Given a positive integer n and an n-dimensional vector a, the matrix $S_n(a)$ commutes with f_n.*

Proof. We see that the matrix $S_n(0, 0 \ldots, 0, a_{i-1}, 0, \ldots, 0)$ is of the type mentioned in Lemma 5.2.9, with $\beta_k = \alpha^n_{[k,i+k-2]} a_{i-1}$. By direct calculation, we see that this matrix satisfies the relation $\alpha^n_{i-1+k} \beta_k = \beta_{k+1} \alpha^n_k, \forall k = 1, \ldots, n-i$, hence by the lemma it commutes with f_n. That means that the sum $S_n(a) = \sum_{i=1}^n S_n(0, 0 \ldots, 0, a_{i-1}, 0, \ldots, 0)$ of such matrices also commutes with f_n. ■

Proposition 5.2.11.

$$S_n = \{e_n + S_n(0, a_1, \ldots, a_{n-1}) \mid a_1, \ldots, a_{n-1} \in \mathbb{C}\} \tag{5.5}$$

Proof. Firstly, a matrix $S_n(0, a_1, \ldots, a_{n-1})$ is trace-free, so the right-hand side of equation (5.5) is in $\mathfrak{sl}_n$. By Lemma 5.2.10, it is a subset of $e_n + ker(\mathrm{ad} f_n) = S_n$. Moreover, as $\dim S_n = n-1$ by Proposition 5.2.8, these two affine spaces have the same dimension, hence are indeed equal. ■

We have the rectangular generalisation of Lemma 5.2.9:

Lemma 5.2.12. *An $m \times n$ matrix having zeroes outside the i-subdiagonal*

$$b_{mn} = \begin{bmatrix} 0 & & & & & \\ \vdots & & & & & \\ \beta_1 & & & & & \\ & \beta_2 & & & & \\ & & \ddots & & & \\ & & & \beta_{m+1-i} & \cdots & 0 \end{bmatrix}$$

satisfies $f_m b_{mn} = b_{mn} f_n$ *if and only if* $\alpha^m_{i-1+k}\beta_k = \beta_{k+1}\alpha^n_k$, $\forall k = 1, \dots, m-i$.

Proof. This follows by computing $f_m b_{mn}$ and $b_{mn} f_n$, and comparing the non-zero values. ■

Now, given two positive integer numbers $n < m$ and an n-dimensional vector $a = (a_0, \dots, a_{n-1})$ we define an $n \times m$ matrix

$$(P_{nm}(a))_{ij} := \begin{cases} \alpha^m_{[j,i-1]} \dfrac{\alpha^n_{[i+1-j,i-1]}}{\alpha^m_{[i+1-j,i-1]}} a_{i-j}, & \text{if } i \geq j \\ 0, & \text{otherwise} \end{cases}$$

and a $m \times n$ matrix

$$(Q_{mn}(a))_{ij} := \begin{cases} \alpha^n_{[j,i-1]} \dfrac{\alpha^m_{[i+1-j,i-1]}}{\alpha^n_{[i+1-j,i-1]}} a_{i-j-m+n}, & \text{if } i \geq j+m-n \\ 0, & \text{otherwise.} \end{cases}$$

These are both generalisations of the matrix $S_n(a)$ in the sense that $P_{nn}(a) = Q_{nn}(a) = S_n(a)$. Moreover, we have the following lemma:

Lemma 5.2.13. *Given two positive integer numbers* $n < m$ *and an* n*-dimensional vector* $a = (a_0, \dots, a_{n-1})$, *the matrices* $P_{nm}(a)$ *and* $Q_{mn}(a)$ *satisfy*

$$f_n P_{nm}(a) = P_{nm}(a) f_m$$
$$f_m Q_{mn}(a) = Q_{mn}(a) f_n$$

Proof. Similarly to the proof of Lemma 5.2.10, by direct calculation we see that an i-subdiagonal $b_i := P_{nm}(0, 0 \dots, 0, a_{i-1}, 0, \dots, 0)$ satisfies the condition of Lemma 5.2.12, hence satisfies the property $f_n b_i = b_i f_m$. As $P_{nm}(a)$ is a sum of such subdiagonals, it satisfies the same property. The analogous proof works for the matrix $Q_{mn}(a)$. ■

Proposition 5.2.14. *The Slodowy slice of the standard* $\mathfrak{sl}_2$*-triple of type* $\lambda = (\lambda_1, \dots, \lambda_n)$ *has a block decomposition*

$$S_\lambda = \left\{ e_\lambda + \begin{bmatrix} S_{\lambda_1}(a^{11}) & Q_{\lambda_1\lambda_2}(a^{12}) & \dots & Q_{\lambda_1\lambda_n}(a^{1n}) \\ P_{\lambda_2\lambda_1}(a^{21}) & S_{\lambda_2}(a^{22}) & & \vdots \\ \vdots & & \ddots & \\ P_{\lambda_n\lambda_1}(a^{n1}) & \dots & & S_{\lambda_n}(a^{nn}) \end{bmatrix} \;\middle|\; a^{ij} \in \mathbb{C}^{\lambda_{\max\{i,j\}}}, \quad \sum_{i=1}^n \lambda_i a^{ii}_0 = 0 \right\} \tag{5.6}$$

Proof. First, let us prove that these two affine spaces have the same dimension. By Proposition 5.2.8,

$$\dim(S_\lambda) = \sum_{i=1}^{n} (\lambda_i^*)^2 - 1.$$

From the definition, $\dim S_{\lambda_i} = \lambda_i$ and $\dim P_{\lambda_i \lambda_j} = \dim Q_{\lambda_j \lambda_i} = \lambda_j$, for $i < j$, hence the matrix-space on the right-hand side of equation (5.6) all together has the dimension

$$2 \sum_{i=1}^{n} i\lambda_i - \sum_{i=1}^{n} \lambda_i - 1,$$

where the $(-1)-$term comes from the trace-vanishing equation $\sum_{i=1}^{n} \lambda_i a_0^{ii} = 0$. Thus the dimensions of these two spaces are the same by the following lemma:

Lemma 5.2.15. *Given a partition λ and its dual λ^*, the following holds:*

$$\sum_{i=1}^{n} (\lambda_i^*)^2 = 2 \sum_{i=1}^{n} i\lambda_i - \sum_{i=1}^{n} \lambda_i.$$

Proof. As $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i^*$, the equation $\sum_{i=1}^{n} (\lambda_i^*)^2 = 2\sum_{i=1}^{n} i\lambda_i - \sum_{i=1}^{n} \lambda_i$ is equivalent to

$$\sum_{i=1}^{n} \frac{\lambda_i^*(\lambda_i^* + 1)}{2} = \sum_{i=1}^{n} i\lambda_i. \tag{5.7}$$

The latter equation holds, which one can see by filling the boxes of the Young diagram that corresponds to the partition λ in a way that boxes in the i-th row are filled with the number i. Then both sides of equation (5.7) are the sum of all the numbers in the boxes, the left-hand side being the sum by columns first, whereas the right-hand side is the sum by rows first. ∎

Now, as these two affine spaces have the same dimension, it is left to prove that the right-hand side of equation (5.6) is a subset of $S_\lambda = e_\lambda + ker(\mathrm{ad} f_\lambda)$, or equivalently, whether a block-matrix written therein commutes with f_λ. By the block-multiplication of matrices, this is equivalent to

$$\begin{aligned} S_{\lambda_i}(a^{ii}) f_{\lambda_i} &= f_{\lambda_i} S_{\lambda_i}(a^{ii}), \\ P_{\lambda_i \lambda_j}(a^{ij}) f_{\lambda_j} &= f_{\lambda_i} P_{\lambda_i \lambda_j}(a^{ij}), \\ Q_{\lambda_j \lambda_i}(a^{ji}) f_{\lambda_i} &= f_{\lambda_j} Q_{\lambda_j \lambda_i}(a^{ji}), \end{aligned} \tag{5.8}$$

which is true by Lemmas 5.2.10 and 5.2.13. Thus, the proposition is proved. ∎

5.2.3 Centraliser of the standard triple

Given a partition $\lambda = (\underbrace{s_k, \ldots, s_k}_{w_k}, \ldots, \underbrace{s_2, \ldots, s_2}_{w_2}, \underbrace{s_1, \ldots, s_1}_{w_1}) = (s_k^{w_k} \ldots s_2^{w_2} s_1^{w_1})$ of n, we define a non-negative integer vector $\mathbf{w}(\lambda) := (0 \ldots 0, w_1, 0 \ldots 0, w_2, 0 \ldots 0, w_k)$, where w_i is in the s_i-th position.

Proposition 5.2.16. *The group Z_λ from Definition 5.2.7 is isomorphic to $GL(\mathbf{w}(\lambda)) := \prod_{i=1}^{k} GL(w_i)$.*

Proof. A matrix $g \in Z_\lambda$ commutes with f_λ hence, it is of type

$$g = \begin{bmatrix} S_{\lambda_1}(a^{11}) & Q_{\lambda_1\lambda_2}(a^{12}) & \ldots & Q_{\lambda_1\lambda_n}(a^{1n}) \\ P_{\lambda_2\lambda_1}(a^{21}) & S_{\lambda_2}(a^{22}) & & \vdots \\ \vdots & & \ddots & \\ P_{\lambda_n\lambda_1}(a^{n1}) & \ldots & & S_{\lambda_n}(a^{nn}) \end{bmatrix},$$

for some vectors $a^{nm} \in \mathbb{C}^{\lambda_{\max\{n,m\}}}$, by the proof of Proposition 5.2.14. The important fact is that the matrices $P_{\lambda_n\lambda_m}(a^{nm})$ and $Q_{\lambda_m\lambda_n}(a^{mn})$ are "lower triangular", meaning that their non-vanishing terms have $i - j \geq 0$ for $P_{\lambda_n\lambda_m}(a^{nm})$ and $i - j \geq \lambda_m - \lambda_n$ for $Q_{\lambda_m\lambda_n}(a^{mn})$. Also, we have that g commutes with e_λ, which analogously to the previous result would give us that these matrices are "upper-triangular", meaning the non-vanishing terms have $i-j \leq \lambda_n - \lambda_m$ for $P_{\lambda_n\lambda_m}(a^{nm})$ and $i-j \leq 0$ for $Q_{\lambda_m\lambda_n}(a^{mn})$. Altogether, for $\lambda_i \neq \lambda_j$ we have that $P_{\lambda_n\lambda_m}(a^{nm}) = 0$ and $Q_{\lambda_m\lambda_n}(a^{mn}) = 0$, and for $\lambda_m = \lambda_n$ the matrices $P_{\lambda_n\lambda_m}(a^{nm})$ and $Q_{\lambda_m\lambda_n}(a^{mn})$ are scalar, having only the diagonal terms non-vanishing. By the same reasoning, the matrices $S_{\lambda_i}(a^{ii})$ are scalar as well.

That is to say, we have proved that an arbitrary matrix $g \in Z_\lambda$ is quasi-diagonal,

$$g = \begin{bmatrix} D_k & & & \\ & D_{k-1} & & \\ & & \ddots & \\ & & & D_1 \end{bmatrix}$$

where square blocks D_i of lengths $s_i w_i$, are made by grouping λ_j's that are equal to s_i (recall $\lambda = (s_k^{w_k} \ldots s_2^{w_2} s_1^{w_1})$). Thus, each D_i is a block matrix

$$D_i = \begin{bmatrix} D_i^{11} & D_i^{12} & \ldots & D_i^{1w_i} \\ D_i^{21} & D_i^{22} & & \vdots \\ \vdots & & \ddots & \\ D_i^{w_i1} & \ldots & & D_i^{w_iw_i} \end{bmatrix},$$

whose blocks are scalar matrices $D_i^{kl} = d_i^{kl} I_{s_i}$ of length s_i. Thus, we have the mapping

$$Z_\lambda \to \prod_{i=1}^{n} GL(w_i), \quad g \mapsto \prod_{i=1}^{n} (d_i^{kl})_{k,l=1}^{w_i}$$

which is a monomorphism, and it remains to show that it is a surjection, that is, that for an arbitrary choice of scalars d_i^{kl} the matrix $g = g(d_i^{kl})$ commutes with h_λ as well. This is equivalent to equations analogous to (5.8), namely

$$D_i^{kl} h_{s_i} = h_{s_i} D_i^{kl},$$

which are trivially true as D_i^{kl} are scalar matrices. Thus, altogether, we get $Z_\lambda \cong \prod_{i=1}^{n} GL(w_i)$. ■

5.2.4 Scalar actions

Using the description of the group Z_λ from the last section, we are now in a position to define scalar actions, a special kind of twisted Kazhdan actions. Firstly, a 1-parameter subgroup G of Z_λ is contained in a maximal complex torus, and maximal tori are unique up to conjugation. That is to say, denoting by T_λ the **canonical maximal torus** in $Z_\lambda \cong \prod_{i=1}^{n} GL(w_i)$ that is a product of canonical (diagonal) maximal tori of $GL(w_i)$, any 1-parameter subgroup of Z_λ is conjugate to a 1-parameter subgroup of T_λ. The next lemma, which can be proved directly, shows that we can reduce our considerations to the case when $G \leq T_\lambda$.

Lemma 5.2.17. *Given a 1-parameter subgroup $G \leq Z_\lambda$ such that $G = gKg^{-1}$, the map*

$$S_\lambda \to S_\lambda, \quad e_\lambda + y \mapsto e_\lambda + g^{-1} y g$$

is an equivariant isomorphism of affine spaces, where on the left we have a G-action and on the right the K-action.

Thus, from now on, we will focus only on twisted actions by 1-parameter subgroup of Z_λ that are inside the canonical maximal torus T_λ. Given an integer vector $v = (v_1, \ldots, v_n)$ we denote by $t^v := diag(t^{v_1}, \ldots, t^{v_n})$. We have the following lemma whose proof is a direct matrix computation:

Lemma 5.2.18. *Given an arbitrary $n \times n$ matrix X and a vector $v = (v_1, \ldots, v_n)$, we have*

$$(t^v X t^{-v})_{i,j} = t^{v_i - v_j} X_{i,j}$$

This yields a corollary which will be important for us.

Corollary 5.2.19. *Given an integer vector $v = (v_1, \ldots, v_n)$, the Kazhdan action twisted by $G_t = t^v$ will act on $X \in S_\lambda$ by*

$$t \cdot (X_{ij}) = (t^{2+v_i-v_j+h_j-h_i} X_{ij}), \tag{5.9}$$

where $h_\lambda = diag(h_1, \ldots, h_n)$.

Definition 5.2.20. A twisted Kazhdan action, whose 1-parameter subgroup $G \leq T_\lambda$ consists of a product of scalar matrices $G_t = t^{a_k} I_{s_k w_k} \times \cdots \times t^{a_1} I_{s_1 w_1}$, is called a **scalar action** on S_λ. Collecting the weights into an integer vector $\vec{a} = (a_k, \ldots, a_1)$, we will denote with $G_{\vec{a}}$ such a subgroup, and the action by $\mathbb{C}^*_{\vec{a}}$. To depict the action on the basis of T_λ we will use the **extended vector**

$$\widetilde{a} := (\underbrace{a_k, \ldots, a_k}_{w_k s_k}, \ldots, \underbrace{a_1, \ldots, a_1}_{w_1 s_1}).$$

of vector $\vec{a}$ (thus e.g. in Corollary 5.2.19 one uses $v = \widetilde{a}$).

Example 5.2.21. Coming back to the example $\lambda = (21)$ (Example 5.1.9), let us describe a scalar action on its Slodowy slice. Firstly, recall from that example the description of S_{21} :

$$S_{21} = \left\{ \begin{bmatrix} a & 1 & 0 \\ b & a & d \\ c & 0 & -2a \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

Picking the vector $\vec{a} = (12)$, thus $\widetilde{a} = (122)$, its scalar action $\mathbb{C}^*_{\vec{a}}$ is by Corollary 5.2.19 equal to

$$t \cdot \begin{bmatrix} a & 1 & 0 \\ b & a & d \\ c & 0 & -2a \end{bmatrix} = \begin{bmatrix} t^2 a & t^{-1} & 0 \\ t^5 b & t^2 a & t^3 d \\ t^4 c & 0 & -2t^2 a \end{bmatrix}.$$

We now show that scalar actions commute, as we will need it later.

Lemma 5.2.22. *Any two scalar actions on S_λ commute.*

Proof. A scalar action on S_λ is defined as a composition of the Kazhdan $\mathbb{C}^*$-action and the conjugation action of a 1-parameter subgroup $G \leq T_\lambda$. The Kazhdan action commutes with conjugation by G (since $G \subset Z_e$ and by definition Z_e commutes with the Kazhdan action, see Proposition 5.2.2). Finally, any two subgroups $G_1, G_2 \leq T_\lambda$ commute (since T_λ is abelian). ■

The importance of scalar actions comes from the following lemma, which is the Springer-theoretic analogue to Lemma 4.4.6.

Lemma 5.2.23. *A twisted Kazhdan action on S_λ that is even and conical has to be a scalar action.*

Proof. Given a G-twisted Kazhdan action. By Lemma 5.2.17, we can assume that $G \leq T_\lambda$. Denoting $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$ as before, let us decompose matrices $G_t = (D_k(t), D_{k-1}(t), \dots, D_1(t))$ into $s_i w_i$ blocks. Thus, we want to show that each $D_i(t) = t^{a_i} I_{s_i w_i}$ is a scalar matrix. Let us assume the contrary, that there is an i such that $D_i(t)$ is not scalar. Divide $D_i(t) = (D_i^1(t), \dots, D_i^{w_i}(t))$ into blocks of size s_i. As $G_t \in T_\lambda \subset Z_\lambda$, we have that each $D_i^j(t)$ is a scalar matrix, so $D_i^j(t) = t^{m_j} I_{s_i}$, for some $m_1, \dots, m_{w_i} \in \mathbb{Z}$. Now, as $D_i(t)$ is not a scalar matrix, there are two indices, say j_1 and j_2, such that $m_{j_1} \neq m_{j_2}$. Then, let us split the matrices in S_λ into blocks S_λ^{ij} of size $s_i w_i \times s_j w_j$.

$$S_\lambda = \begin{bmatrix} S_\lambda^{kk} & S_\lambda^{kk-1} & \dots & S_\lambda^{k1} \\ S_\lambda^{k-1k} & S_\lambda^{k-1k-1} & & \vdots \\ \vdots & & \ddots & \\ S_\lambda^{1k} & \dots & & S_\lambda^{11} \end{bmatrix}$$

We can split the block S_λ^{ii} further into blocks of size $s_i \times s_i$. Then, according to Proposition 5.2.14, we have

$$S_\lambda^{ii} = \left\{ \begin{bmatrix} S_{s_i}(b^{11}) & S_{s_i}(b^{12}) & \dots & S_{s_i}(b^{1w_i}) \\ S_{s_i}(b^{21}) & S_{s_i}(b^{22}) & & \vdots \\ \vdots & & \ddots & \\ S_{s_i}(b^{w_i 1}) & \dots & & S_{s_i}(b^{w_i w_i}) \end{bmatrix} \middle| b^{rs} \in \mathbb{C}^{s_i} \right\} \tag{5.10}$$

Notice that the chunk of the element h_λ in the block S_λ^{ii}, call it h_λ^i, is equal to

$$h_\lambda^i = \begin{bmatrix} h_{s_i} & & & \\ & h_{s_i} & & \\ & & \ddots & \\ & & & h_{s_i} \end{bmatrix} \tag{5.11}$$

with w_i copies of the block h_{s_i}. Thus, according to Corollary 5.2.19, the weights of the diagonal entries in the block $S_{s_i}(b^{j_2 j_1})$ under the G-twisted action are equal to $2 + m_{j_2} - m_{j_1}$. Similarly, diagonal boxes of the block $S_{s_i}(b^{j_1 j_2})$ have weights $2 + m_{j_1} - m_{j_2}$. Since $m_{j_1} \neq m_{j_2}$, one of these two weights needs to be smaller than 2, hence cannot be even and positive, hence the G-twisted action is not even and conical, which is a contradiction. Hence, the action is indeed scalar. ■

The following is the Springer-theoretic analogue of Lemma 4.4.5, as it counts the scalar actions that are even and conical.

Lemma 5.2.24. *Given $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$, a scalar action $\mathbb{C}^*_{\vec{a}}$ on S_λ is even and conical exactly when $|a_i - a_{i+1}| \leq s_{i+1} - s_i$, for all $i = 1, 2, \dots k-1$.*

Proof. Let us pick a scalar action $\mathbb{C}^*_{\vec{a}}$, where $\vec{a} = (a_1, \dots, a_k)$. Thus it is G-twisted Kazhdan action where $G_t = t^{\widetilde{a}}$, and $\widetilde{a} := (\underbrace{a_k, \dots, a_k}_{w_k s_k}, \dots, \underbrace{a_1, \dots, a_1}_{w_1 s_1})$, where $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$. As in Lemma 5.2.23, consider the decomposition of S_λ into blocks S_λ^{ij} of sizes $s_i w_i \times s_j w_j$.

$$S_\lambda = \begin{bmatrix} S_\lambda^{kk} & S_\lambda^{kk-1} & \dots & S_\lambda^{k1} \\ S_\lambda^{k-1k} & S_\lambda^{k-1k-1} & & \vdots \\ \vdots & & \ddots & \\ S_\lambda^{1k} & \dots & & S_\lambda^{11} \end{bmatrix}$$

In order to use the fact that the action $\mathbb{C}^*_{\vec{a}}$ is even and conical, we have to compute the weights in each entry of this big matrix, using the formula (5.9). Firstly, we can see that the weights of entries in blocks S_λ^{ii} are not affected by twisting, as the chunk of the vector v in that block is constant. Having in mind the decomposition of h_λ in that block (5.11), and using the notation from equation (5.10), the weight of the (r_1, r_2)-entry in a matrix $S_{s_i}(b^{u_1 u_2})$ is exactly $2 + h_{r_2}^{s_i} - h_{r_1}^{s_i} = 2 + 2(r_1 - r_2)$. Hence it is even and positive, as $S_{s_i}(b^{u_1 u_2})$ is lower-triangular. Thus, weights of all entries in S_λ^{ii} are even and positive, regardless of $\vec{a}$.

Now, let us consider a block S_λ^{ji}, for $i > j$. According to Proposition 5.2.14 we have

$$S_\lambda^{ji} = \left\{ \left[\begin{matrix} P_{s_j,s_i}(b^{11}) & P_{s_j,s_i}(b^{12}) & \dots & P_{s_j,s_i}(b^{1w_i}) \\ P_{s_j,s_i}(b^{21}) & P_{s_j,s_i}(b^{22}) & & \vdots \\ \vdots & & \ddots & \\ P_{s_j,s_i}(b^{w_j 1}) & \dots & & P_{s_j,s_i}(b^{w_j w_i}) \end{matrix} \right] \middle| \, b^{rs} \in \mathbb{C}^{s_i} \right\} \tag{5.12}$$

Thus, since the action is scalar, the weights of it are the same in each of the $P_{s_j,s_i}(b^{u_1,u_2})$-blocks, so let us pick $P_{s_j,s_i}(b^{11})$ for instance. Its (r_1, r_2)-entry has weight $2 + a_j - a_i + h_{r_2}^{s_i} - h_{r_1}^{s_j} = 2 + a_j - a_i + s_i - s_j + 2(r_1 - r_2)$. So, we see that the smallest weights occur on the main diagonal ($r_1 = r_2$), hence the necessary and sufficient condition we need in order for the action to be even and conical on this block are $-(a_j - a_i) \leq s_i - s_j$ and $a_j - a_i = s_j - s_i \pmod 2$.

Similarly, considering the block S_λ^{ji} for $i < j$ we get that $a_j - a_i \leq s_i - s_j$, hence altogether, the necessary and sufficient conditions that the vector $\vec{a}$ has to satisfy in order for $\mathbb{C}^*_{\vec{a}}$ to be even and conical are

$$|a_j - a_i| \leq s_j - s_i \text{ and } a_j - a_i = s_j - s_i \pmod 2, \tag{5.13}$$

for all $i < j$ Actually, by

$$|a_j - a_i| \leq \sum_{t=1}^{j-1} |a_{t+1} - a_t| \leq \sum_{t=1}^{j-1} s_{t+1} - s_t \leq s_j - s_i$$
$$a_j - a_i = \sum_{t=1}^{j-1} a_{t+1} - a_t = \sum_{t=1}^{j-1} s_{t+1} - s_t = s_j - s_i \pmod 2$$

is suffices to have the conditions (5.13) satisfied only when $j = i + 1$. Denoting $n_i := \frac{1}{2}(a_i - a_{i+1} + s_{i+1} - s_i)$, these conditions are equivalent to $n_i \in \{0, \dots, s_{i+1} - s_i\}$, for all $i = 1, \dots, k-1$. By considering weights in the entries of S_λ, we see that a scalar action $\mathbb{C}^*_{\vec{a}}$ is completely determined by the values of $a_i - a_j$, hence by the numbers n_i. Thus, corresponding to different choices of numbers n_i, we get exactly $\prod_{i=1}^{k-1}(s_{i+1} - s_i + 1) = \prod_{i=1}^{r-1}(\lambda_{i+1} - \lambda_i + 1)$ scalar actions that are conical and even. Moreover, two actions having different n_i, for some i, have different $a_{i+1} - a_i$ and thus act by different weights on blocks S_λ^{i+1i} and S_λ^{ii+1}, hence are different actions on S_λ. Thus, this lemma is proved. ∎

Combining Lemmas 5.2.23 and 5.2.24, we get the following theorem which is analogous to Proposition 4.4.7.

Theorem 5.2.25. *Given $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$, the set of all twisted Kazhdan actions on S_λ that are even and conical is exactly the set of scalar actions $\mathbb{C}^*_{\vec{a}}$ having $|a_i - a_{i+1}| \leq s_{i+1} - s_i$. Thus, there are exactly $\prod_{i=1}^{k-1}(s_{i+1} - s_i + 1) = N(\mathbf{w}(\lambda))$ such actions.*

Hence, we give these vectors and their corresponding actions a special name.

Definition 5.2.26. Given $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$, a vector $\vec{a} = (a_k, \dots, a_1)$ is called **good** and the scalar action $\mathbb{C}^*_{\vec{a}}$ is called a **good action** if $|a_i - a_{i+1}| \leq s_{i+1} - s_i$, for all $i = 1, 2, \dots k-1$.

We will show that good actions are not only different on S_λ, but also on the ordinary Slodowy variety

$$\mathcal{S}_\lambda := \mathcal{S}_{\lambda,11..1} = S_\lambda \cap \mathcal{N}.$$

Lemma 5.2.27. *Different scalar actions on the Slodowy slice S_λ restricted to the ordinary Slodowy variety $\mathcal{S}_\lambda$ are different as well. In particular, the same holds for good actions.*

Proof. In the notation from the proof of Lemma 5.2.24, seeing the block S_λ^{ij} as a subspace of S_λ with other blocks vanishing, we just have to show that $S_\lambda^{ij} \cap \mathcal{N} \neq \emptyset$, for arbitrary $i \neq j, i, j = 1, \dots, k$. This is because given two different scalar actions $\mathbb{C}_{\vec{a'}}$ and $\mathbb{C}_{\vec{a''}}$, there are i and j such that $a'_i - a'_j \neq a''_i - a''_j$ and thus the weights on $S_\lambda^{ij} \cap \mathcal{N}$ will be different, hence producing different actions.

Now, $S_\lambda^{ij} \cap \mathcal{N} \neq \emptyset$ is true as S_λ^{ij} is a subset of $\mathcal{N}$, being strictly upper or strictly lower triangular. ■

The last lemma together with Theorem 5.2.25 implies that there are $N(\mathbf{w}(\lambda))$ different good actions on $\mathcal{S}_\lambda$, and hence on its resolution $\widetilde{\mathcal{S}}_\lambda = \nu^{-1}(\mathcal{S}_\lambda)$ as well. Since they commute (by Lemma 5.2.22), according to Proposition 3.2.2 their square roots yield different minimal components of the core of $\widetilde{\mathcal{S}}_\lambda$, which is the Springer fibre $\mathcal{B}^\lambda$. Thus, having in mind Remark 3.1.7, we obtain the following corollary.

Corollary 5.2.28. *There are precisely $N(\mathbf{w}(\lambda)) = \prod_{i=1}^{n-1}(\lambda_{i+1} - \lambda_i + 1)$ different minimal components of twisted Kazhdan actions in an arbitrary Springer fibre $\mathcal{B}^\lambda$.*

One would ideally like to have more information about these components, that is, which components in $\mathcal{B}^\lambda$ are the minimal ones, and what is their topology. We will completely answer this question in Section 5.3 (Proposition 5.3.12 and Corollary 5.3.13), after explaining the Spaltenstein labelling of irreducible components of Springer fibres. For now, in the following, rather technical section, we will describe the constructed even and conical actions on S_λ in a different way, from which it will be feasible to understand what the corresponding minimal components are, in Section 5.3.

5.2.5 Labelling scalar actions by permutations of λ^*

In this, somewhat technical section, we label the scalar actions $\mathbb{C}^*_{\vec{a}}$ by certain *good* permutations $\mu = \mu(\vec{a})$ of the dual partition λ^*, as these will be more convenient to use in Section 5.3.3 in order to get the explicit description of minimal components of Springer fibres $\mathcal{B}^\lambda$.

Given a partition $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$ and a scalar action $\mathbb{C}^*_{\vec{a}}$ on S_λ, as in the proof of Lemma 5.2.24 denote

$$n_i := \frac{1}{2}(a_i - a_{i+1} + s_{i+1} - s_i). \tag{5.14}$$

Recall from the same proof that the condition for $\mathbb{C}^*_{\vec{a}}$ to be even and scalar is precisely $n_i \in \{0, 1, \dots, s_{i+1} - s_i\}$, for all $i = 1, \dots, k-1$. Thus, denoting the dual partition by $\lambda^* = (q_k^{z_1} q_{k-1}^{z_2} \dots q_1^{z_k})$ (where $q_k > q_{k-1} > \dots > q_1$), we define its permutation

$$\mu = \mu(\vec{a}) := q_1^{n_{k-1}} \dots q_{k-1}^{n_1} q_k^{z_1} q_{k-1}^{z_2 - n_1} \dots q_1^{z_k - n_{k-1}}$$

Example 5.2.29. For $\lambda = (3, 2, 1) = (3^1 2^1 1^1)$ and $\vec{a} = (0, 1, 0)$, we have $\lambda^* = (3, 2, 1) = (3^1 2^1 1^1)$, $n_1 = 0$, $n_2 = 1$, $k = 3$ so $\mu(\vec{a}) = \mu(010) = (1^1 2^0 3^1 2^1 1^0) = (132)$.

Here we give a lemma that says how to calculate the dual partition λ^* of a given one, as we will need it later. Its proof goes directly by observing the Young diagram $Y(\lambda)$, hence we omit it.

Lemma 5.2.30. *Given a partition $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$, its dual partition is $\lambda^* = (q_k^{z_1} q_{k-1}^{z_2} \dots q_1^{z_k})$ where $q_i = \sum_{j=1}^{i} w_{k+1-j}$ and $z_1 = s_1, z_i = s_i - s_{i-1}$, for $i = 2, \dots k$.*

Definition 5.2.31. Denote by $\mathrm{Perm}(\lambda^*)$ the set of permutations of the dual partition λ. We say that a permutation $\mu = (\mu_1, \dots, \mu_r) \in \mathrm{Perm}(\lambda^*)$ is **good** if it is of type $\mu(\vec{a})$, for some good vector $\vec{a}$. Equivalently, μ is good if for no three indices $i < j < k$ we have both $\mu_i > \mu_j$ and $\mu_j < \mu_k$. We denote the set of good permutations of λ^* by $\mathrm{Good}(\lambda^*)$.

We will use good permutations to label minimal components in Springer fibres.

Definition 5.2.32. Given a permutation $\mu \in \mathrm{Good}(\lambda^*)$, denote by $\mathfrak{F}_p^\mu$ the minimal component in $\mathcal{B}_p^\lambda$ fixed by the good action $\mathbb{C}^*_{\vec{a}}$, where $\mu = \mu(\vec{a})$. In particular, we denote by $\mathfrak{F}^\mu := \mathfrak{F}_{1..1}^\mu$ the minimal components of the ordinary Springer fibre $\mathcal{B}^\lambda$.

Remark 5.2.33. This labelling of minimal components need not to be injective for general $\mathcal{B}_p^\lambda$, but it is at least for the ordinary Springer fibres $\mathcal{B}^\lambda$. Namely, two different good permutations μ', μ'' correspond to two different good vectors $\vec{a}', \vec{a}''$, thus two different good actions $\mathbb{C}_{\vec{a}'}, \mathbb{C}_{\vec{a}''}$, and we have seen that those induce different components on $\mathcal{B}^\lambda$ (see Lemma 5.2.27 and the argument below it).

Now, we will depict the way the action $\mathbb{C}^*_{\vec{a}}$ acts on $\mathcal{B}^\lambda$ by filling the Young diagram $Y(\lambda)$.

Definition 5.2.34. Let $e \in \mathfrak{sl}_n$ be a nilpotent element, and let $\lambda = \lambda(e)$. By **Jordan basis** of e we will mean a basis of $\mathbb{C}^n$,

$$\mathbf{v} = \{v_{i,j} \in \mathbb{C}^n \mid i = 1, \dots, k,\ j = 1, \dots, \lambda_i\},$$

in which e is in Jordan normal form. The Young diagram $Y(\lambda)$ filled with the Jordan basis will be denoted $Y(\mathbf{v})$, with $v_{i,j}$ in row i and column j. Since e is in Jordan normal form in this basis, e maps vectors leftwards in the tableau: $ev_{i,j} = v_{i,j-1}$ ($= 0$ if $j = 1$).
Let $\mathbb{C}^*_{\vec{a}}$ be a scalar action. Then, in that basis, T_λ and thus $G_t \leq T_\lambda$ will consist of diagonal matrices, so by Remarks 5.1.7 and 5.2.4 one can recover the (twisted) Kazhdan action from knowing how it acts on the Jordan basis. Replacing each entry $v_{i,j}$ of $Y(\mathbf{v})$ by the weight of the scalar action on $v_{i,j}$ defines the **weight tableau** $T(\vec{a})$.

Example 5.2.35. For example, consider $\lambda = (3, 2, 1)$. Then

$$Y(\mathbf{v}) = \begin{array}{|c|c|c|} \hline v_{1,1} & v_{1,2} & v_{1,3} \\ \hline v_{2,1} & v_{2,2} & \\ \cline{1-2} v_{3,1} & & \\ \cline{1-1} \end{array} \tag{5.15}$$

Using the symbol | to distinguish between different Jordan blocks of e_λ, we have $h_\lambda = \text{diag}(2\ 0\ \text{-}2 \mid 1\ \text{-}1 \mid 0)$ (notice these are the three h_k blocks from Section 5.2.1 for $k = 3$, $k = 2$ and $k = 1$ respectively). Since $h_\lambda = \text{diag}(h_1, h_2, \ldots, h_n)$ is diagonal, t^{-h_λ} acts on $\mathbb{C}^n$ with weights t^{-h_i}, which by Remark 5.1.7 is enough information to determine the Kazhdan action. We record these weights $-h_i$ in the weight tableau for the Kazhdan action $\mathbb{C}_{\vec{0}}$:

$$T(\vec{0}) = \begin{array}{|c|c|c|} \hline -2 & 0 & 2 \\ \hline -1 & 1 & \\ \cline{1-2} 0 & & \\ \cline{1-1} \end{array}$$

Suppose we now twist by $\vec{a} = (0, 1, 0)$. As $G_{\vec{a}}$ acts by diagonal matrices, the action of $g_t t^{-h_\lambda}$ on $\mathbb{C}^n$ is still diagonal, and we add the weights $-h_i$ above with the respective weight $\widetilde{a}_i$ in the extended vector $\widetilde{a} = (0, 1, 0)$ (in the sense of Definition 5.2.20, in our case $w_1 = w_2 = w_3 = 1$). Thus, twisting by $\vec{a} = (0, 1, 0)$ simply adds the i-th entry of $\widetilde{a}$ to each entry in the i-th row of the previous tableau:

$$T(0, 1, 0) = \begin{array}{|c|c|c|} \hline -2 & 0 & 2 \\ \hline 0 & 2 & \\ \cline{1-2} 0 & & \\ \cline{1-1} \end{array}$$

Now we construct another filling of the Young diagram $Y(\lambda)$, this time using a permutation $\mu \in \text{Perm}(\lambda^*)$.

Definition 5.2.36. Given $\mu = (\mu_1, \dots, \mu_r) \in \mathrm{Perm}(\lambda^*)$, we define the **$\mu$-weighted tableau** $T_{wt}(\mu)$ in the following way. Start with an array of r top-adjusted columns (meaning each column starts from row 1), with μ_i boxes in the i-th column filled in with the number $2(i-1)$. Then push all the boxes to the left, in order to obtain a Young diagram of shape $Y(\lambda)$.

Example 5.2.37. For example, given $\lambda = (3, 2, 1)$ and $\mu = (1, 3, 2)$ the Young tableau $T_{wt}(\mu)$ is given by

$$\begin{array}{|c|c|c|} \hline 0 & 2 & 4 \\ \hline & 2 & 4 \\ \cline{2-3} & 2 & \\ \cline{2-2} \end{array} \;\to\; \begin{array}{|c|c|c|} \hline 0 & 2 & 4 \\ \hline 2 & 4 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array} = T_{wt}(132).$$

Recall that $\mu(010) = (132)$ (see Example 5.2.29) and, from Examples 5.2.35 and 5.2.37, that the tableau $T_{wt}(132)$ is the same as the tableau $T(010)$ upon adding 2 to all the boxes. We prove that this is a general pattern:

Proposition 5.2.38. *Given a good vector $\vec{a}$, the tableaux $T(\vec{a})$ and $T_{wt}(\mu(\vec{a}))$ are equal up to a uniform shift in the entries of all boxes.*

Proof. We claim that the entries along each row of the tableau $T(\vec{a})$ increase by two. Firstly, this is true for the weight tableau for the Kazhdan action, as its rows are the inverted numbers of the matrices h_k, and the entries in h_k decrease by two. Then, twisting the action by $\vec{a}$ shifts the entries in a whole row by the same number, thus this property remains true in $T(\vec{a})$ as well. We now prove that this also holds for the tableau $T_{wt}(\mu(\vec{a}))$.

Lemma 5.2.39. *For a good vector $\vec{a}$, the entries along any row of the tableau $T_{wt}(\mu(\vec{a}))$ increase by two.*

Proof. Let us assume that there are two consecutive boxes in, say, the l-th row of $T_{wt}(\mu(\vec{a}))$, which are labelled by $2k$ and $2(k+t)$. That means that the sizes of the columns $k, k+1, \dots, k+t-2$ used in the construction of $T_{wt}(\mu(\vec{a}))$ are smaller than the sizes of the columns $k-1$ and $k+t-1$. Thus, we have $\mu_k < \mu_{k-1}$ and $\mu_k < \mu_{k+t-1}$, which violates that $\mu = \mu(\vec{a})$ is a good permutation. ■

Thus, it suffices to prove that first columns of the tableaux $T(\vec{a})$ and $T_{wt}(\mu(\vec{a}))$ are equal up to a uniform shift in all entries. Letting $\lambda = (s_k^{w_k} \dots s_2^{w_2} s_1^{w_1})$, the first column of $T(\vec{a})$ is exactly the string

$$(-(s_k - 1) + a_k)^{w_k} (-(s_{k-1} - 1) + a_{k-1})^{w_{k-1}} \dots (-(s_1 - 1) + a_1)^{w_1}, \tag{5.16}$$

where the a_i parts come from the twist, and the $-(s_i-1)$ parts come from the Kazhdan action (in Section 5.2.1, this is the opposite of the value h_i^k for $i = 0$, $k = s_i$).

Now, denoting $\lambda^* = (q_k^{z_1} q_{k-1}^{z_2} \dots q_1^{z_k})$, we have $\mu(\vec{a}) = q_1^{n_{k-1}} \dots q_{k-1}^{n_1} q_k^{z_1} q_{k-1}^{z_2-n_1} \dots q_1^{z_k-n_{k-1}}$. Recall that the tableau $T_{wt}(\mu(\vec{a}))$ is constructed out of an array of μ_i-sized top-adjusted columns. Thus, the first of the n_{k-1} q_1-sized columns yields the string 0^{q_1} at the beginning of the first column of $T_{wt}(\mu(\vec{a}))$. Then, the first among the next n_{k-2} q_2-sized columns yields the string $(2n_{k-1})^{q_2-q_1}$ on the following $q_2 - q_1$ places in the first column of $T_{wt}(\mu(\vec{a}))$, and so on. In the end we get that the string

$$0^{q_1}(2n_{k-1})^{q_2-q_1} \dots (2n_{k-1} + \dots + 2n_1)^{q_k-q_{k-1}} \tag{5.17}$$

is the first column of $T_{wt}(\mu(\vec{a}))$. Shifting down the string (5.16) by $-(s_k - 1) + a_k$, we get the string

$$0^{w_k}(a_{k-1} - a_k + s_k - s_{k-1})^{w_{k-1}} \dots (a_1 - a_k + s_k - s_1)^{w_1},$$

which is equal to the string (5.17), by formula (5.14) and Lemma 5.2.30. Thus, the proposition is proved. ■

5.2.6 Further work: Towards Maffei isomorphism

In this section we compare the results obtained in Section 5.2.4 with the results from Section 4.4. Namely, it is long known that there is a bijective correspondence between Slodowy varieties of type A and quiver varieties of type A. Nakajima conjectured this in his first paper on quiver varieties [Nak94a, Conj. 8.6] and A. Maffei proved it ten years later in [Maf05, Thm. 8].

Let us introduce some notation. Let $\mathbf{v} = (v_1, \dots, v_{n-1})$ and $\mathbf{w} = (w_1, \dots, w_{n-1})$ be two non-negative integer vectors of length $(n - 1)$. Define a partition

$$\lambda = (n-1)^{w_{n-1}} 2^{w_2} \cdots 1^{w_1}, \tag{5.18}$$

and a composition $p = (p_1, \dots, p_n)$,

$$\begin{aligned} p_1 &= w_1 + \dots + w_{n-1} - v_1 \\ p_i &= w_i + \dots + w_{n-1} - v_i + v_{i-1}, \text{ for } i = 2, \dots, n-1 \\ p_n &= v_{n-1}. \end{aligned} \tag{5.19}$$

of $r = \sum_{i=1}^{n-1} i w_i$. Denote by $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ the quiver variety of type A_{n-1} whose moment parameter $\zeta_I = (\zeta_i)_{i=1}^{n-1}$ is chosen such that all $\zeta_i < 0$. All such choices lie in the same contractible chamber of moment parameters, hence there is no monodromy with respect to the I-holomorphic and $\omega_{\mathbb{C}}$-symplectic structures. Moreover, they are also equivariant under the full quiver and $GL(\mathbf{w})$ actions. Thus, there is no ambiguity in using the notation $\mathfrak{M}(\mathbf{v},\mathbf{w})$ for our purpose (to study the twisted full quiver actions). Denote as before by $\mathfrak{M}^1(\mathbf{v},\mathbf{w})$ the image of $\pi : \mathfrak{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w})$.

Theorem 5.2.40. [Maf05, Thm. 8] *Let* $\mathbf{v}$, $\mathbf{w}, p = p(\mathbf{v},\mathbf{w}), \lambda = \lambda(\mathbf{v},\mathbf{w})$ *be as above. Then:*

(1) There are isomorphisms of algebraic varieties $\widetilde{\varphi} : \mathfrak{M}(\mathbf{v},\mathbf{w}) \to \widetilde{\mathcal{S}}_{\lambda,p}$, *and* $\varphi_1 : \mathfrak{M}^1(\mathbf{v},\mathbf{w}) \to \mathcal{S}_{\lambda,p}$, *such that the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{M}(\mathbf{v},\mathbf{w}) & \xrightarrow{\widetilde{\varphi}} & \widetilde{\mathcal{S}}_{\lambda,p} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\nu_p} \\ \mathfrak{M}^1(\mathbf{v},\mathbf{w}) & \xrightarrow{\varphi_1} & \mathcal{S}_{\lambda,p} \end{array}$$

(2) Moreover $\varphi_1(0) = e$, *so the core* $\mathfrak{L}(\mathbf{v},\mathbf{w}) = \pi^{-1}(0) \subset \mathfrak{M}(\mathbf{v},\mathbf{w})$ *corresponds via* $\widetilde{\varphi}$ *to the generalised Springer fibre* $\mathcal{B}_p^\lambda = \nu_p^{-1}(e_\lambda)$.

Thus, the theorem above combined with Theorem 4.3.21 (which gives the number of minimal components in a quiver variety) give a lower bound on smooth components in a generalised Springer fibre $\mathcal{B}_p^\lambda$:

Corollary 5.2.41. *There are at least* $N(\mathbf{v}(\lambda,p)',\mathbf{w}(\lambda))$ *smooth components in a Springer fibre* $\mathcal{B}_p^\lambda$.

Here, $\mathbf{v} = \mathbf{v}(\lambda,p)$ and $\mathbf{w} = \mathbf{w}(\lambda)$ are obtained through the correspondence from Theorem 5.2.40, and $\mathbf{v}' = \mathbf{v}(\lambda,p)'$ is the dominant vector of the pair $(\mathbf{v},\mathbf{w})$, which is easily-calculable (see equation (4.18) and the calculations above it). Although having the family of smooth components in $\mathcal{B}_p^\lambda$ given in Corollary 5.2.41, without some further knowledge on the Maffei isomorphism $\widetilde{\varphi}$, we cannot tell something more about them.

Thus, let us describe how the work written here speculates some further property of the Maffei isomorphism, which could reveal components from Corollary 5.2.41. By "the quiver side" we will refer to a quiver variety, and by "the Springer side" we will refer to the corresponding Slodowy variety.

- We have seen that on both sides there are natural full weight-2 conical $\mathbb{C}^*$-actions: on the quiver side it is the full quiver action, whereas on the Springer side it is the Kazhdan action.

- On the quiver side there is a symplectomorphism group $GL(\mathbf{w})$ that commutes with the full quiver action, whereas on the Springer side there is the triple centraliser group Z_λ that commutes with the Kazhdan action.
- Thus on the quiver side we defined twisted full quiver actions, whereas on the Springer side we defined twisted Kazhdan actions.
- We have proved that for corresponding varieties $GL(\mathbf{w}) \cong Z_\lambda$ (Proposition 5.2.16).
- Moreover, by Corollary 4.3.14, and Theorem 5.2.25, there is the same number $N(\mathbf{w})$ of even conical actions on both sides. To be precise, on the quiver side this is true only when $\mathbf{v} > 0$ and the weight is dominant, but on the Springer side the actions may coincide in general, thus reducing the number to the smaller value $N(\mathbf{v}', \mathbf{w})$ of different actions on the quiver side.

So, with all these facts in mind, we expect the following:

Conjecture 5.2.42. *Maffei's commutative diagram in Theorem 5.2.40 is equivariant with respect to the action of $\mathbb{C}^* \times GL(\mathbf{w})$. Here $\mathbb{C}^*$ acts by the full quiver action and the Kazhdan action respectively, and $GL(\mathbf{w})$ acts on the right side of the diagram after applying the isomorphism $GL(\mathbf{w}) \cong Z_\lambda$ from Proposition 5.2.16.*

Maffei isomorphism is not explicit as in general it is given by an implicit solution of a system of equations in terms of matrices. However, the recent preprint [ILW20] constructs an explicit solution to these equations, which then makes it feasible to check Conjecture 5.2.42, which we leave for some future work. It would be interesting to prove it, as it would refine the "quiver-Springer correspondence", thus allowing some results on one side to transfer to the other. In particular, if the conjecture held, then we would be able to deduce the number of different minimal components of twisted Kazhdan actions on generalised Springer fibres $\mathcal{B}_p^\lambda$, as follows.

Corollary 5.2.43. *Assuming that Conjecture 5.2.42 holds, the number of different even and conical twisted Kazhdan actions on $\widetilde{\mathcal{S}}_{\lambda,p}$ is exactly $N(\mathbf{v}', \mathbf{w})$, where $\mathbf{v}$ and $\mathbf{w}$ are determined by (5.18) and (5.19). Thus, the number of minimal components in $\mathcal{B}_p^\lambda$ is precisely $N(\mathbf{v}', \mathbf{w})$, and they are precisely the ones obtained in Corollary 5.2.41.*

5.3 Minimal components of ordinary Springer fibres

In this section we describe the minimal components $\{\mathfrak{F}^\mu \mid \mu \in \mathrm{Good}(\lambda^*)\}$ of ordinary Springer fibres $\mathcal{B}^\lambda$ obtained by the twisted Kazhdan actions in the last section. In particular, we show that they lie in the intersection of the set of well-known *Richardson components* [Kr78, He78, PaRe06] and the set of Barchini-Graham-Zierau components [BaZi08, GrZi11], which shows that they are of interest also from a Springer-theoretic viewpoint. In particular we get that they are isomorphic to products of flag manifolds, and are invariant under the torus action on the flag variety $\mathcal{B}$. Also, we show that their corresponding standard Young tableaux (combinatorial objects that label components of $\mathcal{B}^\lambda$) are invariant under the so-called Schützenberger involution.

As it is going to be the central object of this section, recall that the **(ordinary) Springer fibre** $\mathcal{B}^e$ of a nilpotent element $e \in \mathfrak{sl}_n$ is defined as

$$\mathcal{B}^e := \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n \ \mid \dim F_i/F_{i-1} = 1,\ eF_i \subset F_{i-1},\ i = 1, \dots, n\}$$

Denoting by $\lambda = \lambda(e)$ the Jordan partition of e, recall that **we denote by** $\mathcal{B}^\lambda$ the Springer fibre $\mathcal{B}^e$, as it does not depend (up to an isomorphism) on the choice of particular nilpotent element e of Jordan type λ.

In the first two subsections (5.3.1 and 5.3.2) of this section we recall the classical facts from the Springer-theoretic literature that we will use in this section.

5.3.1 Spaltenstein map

In this section we recall Spaltenstein's result [Spa76] that the irreducible components of an ordinary Springer fibre are labelled by standard Young tableaux.

Definition 5.3.1. Given a Young diagram of shape $\lambda \vdash n$, its **standard tableau** is a filling of it with the numbers $1, \dots, n$, such that in each row and in each column these numbers increase. Given a tableau T, we will denote by $Sh(T)$ its Young diagram. Denote by $\mathbf{Std}^\lambda$ the collection of all standard Young tableaux of shape λ.

Example 5.3.2. The standard Young tableaux of shape $(2, 1, 1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\ .$$

Consider a nilpotent element $e \in \mathfrak{sl}_n$ whose Jordan partition is $\lambda = \lambda(e)$. To a flag

$$F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n) \in \mathcal{B}^\lambda$$

we attach its **shape,** an increasing (i.e. nested) sequence

$$sh(F) := (Y(e|_{F_0}), Y(e|_{F_1}), \ldots, Y(e|_{F_n}))$$

of Young diagrams, where $Y(u)$ denotes the Young diagram whose shape is the Jordan partition of a nilpotent matrix u. To an increasing sequence $Y = (Y_0, Y_1, \ldots, Y_n)$ of Young diagrams we attach the Young tableaux $Tab(Y)$ of shape Y_n by filling the boxes with numbers from 1 to n in the order in which they appear in the sequence Y_i.

Remark 5.3.3. In the examples, we will simplify our notation by labelling the vectors in the Jordan basis of e_λ by $v_1, \ldots, v_n$ in the natural order given by the matrix e (instead of using the notation $v_{i,j}$ from (5.15)). For example, for $\lambda = (2, 1, 1)$ the basis $v_1, \ldots, v_4$ constitutes the λ-shaped tableau

$$\begin{ytableau} v_1 & v_2 \\ v_3 \\ v_4 \end{ytableau}\,.$$

Example 5.3.4. Consider a nilpotent element $e \in \mathfrak{sl}_3$ such that $\lambda(e) = (21)$. As mentioned in Remark 5.3.3, we denote by $(v_1, v_2|v_3)$ its Jordan basis (thus $ev_2 = v_1$ and $ev_3 = 0$). Consider the flag $F_1 = (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \mathbb{C}^3)$.

As $e|\langle v_1 \rangle = 0$ and $\lambda(e|\langle v_1, v_2 \rangle) = (2)$ we see that the flag F_1 has shape

$$sh(F_1) = \left(\ydiagram{1}, \ydiagram{2}, \ydiagram{2,1} \right),$$

hence

$$Tab(sh(F_1)) = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,.$$

We denote the set of flags of given tableau type T by

$$\mathcal{F}_{e,T} := \{F \in \mathcal{B}^\lambda \mid Tab(sh(F)) = T\}. \tag{5.20}$$

We can now state Spaltenstein's result from [Spa76].

Theorem 5.3.5 (Spaltenstein)**.** *Given a Springer fibre $\mathcal{B}^\lambda$, for $\lambda = \lambda(e)$, the following is true:*

(1) For each $T \in \mathbf{Std}^\lambda$, $\mathcal{F}_{e,T}$ is a non-empty smooth open and irreducible subvariety of $\mathcal{B}^\lambda$.

(2) The varieties $\{\mathcal{F}_{e,T}\}_{T\in\mathbf{Std}^\lambda}$ are equidimensional, $\dim \mathcal{F}_{e,T} = \frac{1}{2}\sum_{i=1}^{\lambda_1} \lambda_i^*(\lambda_i^* - 1)$.

(3) $\mathcal{B}^\lambda = \bigsqcup_{T\in\mathbf{Std}^\lambda} \mathcal{F}_{e,T}$,

(4) Thus the closures $\mathcal{K}_T := \overline{\mathcal{F}_{e,T}}$ form the set of (equidimensional) irreducible components of $\mathcal{B}^\lambda$.

Hence, denoting by $\mathcal{B}(\lambda)$ the set of irreducible components of $\mathcal{B}^\lambda$, the map

$$\mathrm{Spal} : \mathbf{Std}^\lambda \to \mathcal{B}(\lambda), \quad T \mapsto \mathcal{K}_T,$$

is a bijection. We call it called the **Spaltenstein map**.

5.3.2 Richardson components

In this section we recall the known facts about a set of smooth components in $\mathcal{B}^\lambda$ called *Richardson components.* These are components that are homogeneous under a parabolic subgroup of GL_n, and were classified by Kraft [Kr78] and Hesselink [He78]. In [PaRe06, Sec. 7.1], the authors recall the standard Young tableaux that correspond to these components (under the Spaltenstein map), and we use this paper as a reference for the statements below.

Definition 5.3.6. A **Richardson component** is an irreducible component of $\mathcal{B}^\lambda$ which is homogeneous under the action of a parabolic subgroup of GL_n.[9]

To get the set of standard tableaux that correspond to these components (via the Spaltenstein map), we give the following:

Definition 5.3.7. Given a partition $\lambda \vdash n$, we denote by $\mathrm{Perm}(\lambda^*)$ the set of permutations of its dual partition λ^*. For an element $\mu = (\mu_1, \dots, \mu_r) \in \mathrm{Perm}(\lambda^*)$ we define the standard Young tableau $T^\mu \in \mathbf{Std}^\lambda$ associated to it in the following way. Take the μ-weighted tableau $T_{wt}(\mu)$ from Definition 5.2.36 and replace the boxes labelled '0' by $1, \dots, \mu_1$ in order from top to bottom, then replace the boxes labelled '2' by $\mu_1 + 1, \dots, \mu_1 + \mu_2$ from top to bottom, etc. For example, given $\lambda = (3, 2, 1)$, so $\lambda^* = (3, 2, 1)$, for $\mu = (3, 1, 2)$ the Young tableau T^μ is

$$T^{312} = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & \\ \cline{1-2} 3 & & \\ \cline{1-1}\end{array}$$

[9]I.e. the action of the parabolic subgroup on the component is transitive.

The sets of $\mu_1, \mu_2, \ldots, \mu_r$ boxes that we fill with this procedure we will call **μ-lines**. Thus, the k-th μ-line consists precisely of the boxes labelled $2(k-1)$ in the μ-weighted tableau $T_{wt}(\mu)$ from Definition 5.2.36. We enumerate μ-lines from left to right, thus the i-th μ-line starts from the box $(1, i)$. Filling the Young diagram $Y(\lambda)$ with numbers $1, 2, \ldots, r$ corresponding to μ-lines we get the tableau that we denote by $Y(\mu)$. In the last example, we have

$$Y(312) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array}$$

Let $e \in \mathfrak{sl}_n$ be an arbitrary nilpotent element with Jordan partition $\lambda = \lambda(e) = (\lambda_1, \ldots, \lambda_k)$. Pick a Jordan basis $\mathbf{v}$ for e (Definition 5.2.34).

Then to a permutation $\mu \in \text{Perm}(\lambda^*)$ we associate a partial flag

$$\overline{W_\mu} := (0 = W_\mu^0 \subset \overline{W_\mu^1} \subset \cdots \subset \overline{W_\mu^r} = \mathbb{C}^n), \tag{5.21}$$

where $\overline{W_\mu^k}$ is spanned by the $v_{i,j}$ from μ-lines $1, \ldots, k$, so $\overline{W_\mu^i} := \oplus_{k=1}^i W_\mu^k$ with $W_\mu^k := \langle v_{i,j} \mid Y(\mu)_{i,j} = k \rangle$.

We could also define $\overline{W_\mu}$ invariantly as the unique partial flag in $\mathcal{B}_\mu^\lambda \cong \{pt\}$.

Given $\mu \in \text{Perm}(\lambda^*)$, we **define the subvariety $\mathcal{R}^\mu \subset \mathcal{B}$** as

$$\mathcal{R}^\mu := \{F = (F_0, \ldots, F_n) \mid F_{d_i} = \overline{W_\mu^i}\}, \tag{5.22}$$

where $d_i = \sum_{j=1}^i \mu_j = \dim \overline{W_\mu^i}$.

Define the parabolic subgroup $P^\mu \leq GL_n$ as the set of matrices that preserve the partial flag $\overline{W_\mu}$.

Theorem 5.3.8. *Given an arbitrary $\mu \in \text{Perm}(\lambda^*)$, the subvariety $\mathcal{R}^\mu \subset \mathcal{B}$ satisfies the following:*

(1) It is a closed subset of $\mathcal{B}^\lambda$.

(2) It is isomorphic to the product of flag varieties $\mathcal{B}^{\mu_1} \times \cdots \times \mathcal{B}^{\mu_r}$.

(3) It is an irreducible component of $\mathcal{B}^\lambda$.

(4) Its corresponding Young tableau is the standard Young tableau T^μ.

(5) It is fixed under the parabolic subgroup P^μ.

(6) Hence, it is a Richardson component of $\mathcal{B}^\lambda$.

(7) Moreover, the set $\{\mathcal{R}^\mu \mid \mu \in \mathrm{Perm}(\lambda^*)\}$ *contains all Richardson components of* $\mathcal{B}^\lambda$.

Remark 5.3.9. The fact (4) from the theorem above explains the name *Richardson components*, as the tableau T^μ is obtained through the Robinson insertion algorithm from a Richardson element, that is, the longest element of the parabolic subgroup W_{P^μ} of the Weyl group $W = S_n$.

5.3.3 Minimal components are Richardson

In this section we prove that the set of minimal components in $\mathcal{B}^\lambda$ forms an explicitly-defined subset of the set of Richardson components, hence due to Theorem 5.3.8 we get the set of standard Young tableaux that correspond to minimal components in the arbitrary ordinary Springer fibre $\mathcal{B}^\lambda$.

We first define a family of $\mathbb{C}^*$-actions on flag varieties.

Definition 5.3.10. Let $e \in \mathfrak{sl}_n$ be an arbitrary nilpotent element with Jordan partition $\lambda = \lambda(e)$. Pick its Jordan basis $\mathbf{v} = (v_{i,j})$. Given a permutation $\mu \in \mathrm{Perm}(\lambda^*)$, we have the associated subspaces $W_\mu^k := \langle v_{i,j} \mid Y(\mu)_{i,j} = k \rangle$. The $\mathbb{C}_\mu^*$**-action on** $\mathbb{C}^n$ is a $\mathbb{C}^*$-action that acts on $v_{i,j}$ homogeneously

$$t \cdot v_{i,j} = (T_{wt}(\mu))_{i,j} v_{i,j}$$

by weights that correspond to the numbers of μ-weighted tableau $T_{wt}(\mu)$. Equivalently, its acts with weight $2(k-1)$ on W_μ^k. This action induces the $\mathbb{C}_\mu^*$**-actions on flag varieties** $\mathcal{B}$ **and** $\mathcal{B}_p$.

The importance of these actions lies in the following corollary of Proposition 5.2.38.

Corollary 5.3.11. *Given a good permutation* $\mu = \mu(\vec{a}) \in \mathrm{Perm}(\lambda^*)$ *(see Definition 5.2.31), the* $\mathbb{C}_\mu^*$*-action restricts to Springer fibres* $\mathcal{B}^\lambda$ *and* $\mathcal{B}_p^\lambda$*, agreeing with the scalar* $\mathbb{C}_{\vec{a}}^*$ *action.*

Proof. As μ is a good permutation, $\mu = \mu(\vec{a})$ for a good vector $\vec{a}$. By Proposition 5.2.38, the weights of the Jordan base vectors $v_{i,j}$ of e for the $\mathbb{C}_\mu^*$-action and $\mathbb{C}_{\vec{a}}^*$-action are the same, up to a constant shift. Thus the image of a vector subspace of $\mathbb{C}^n$ will be the same under either action. Thus the $\mathbb{C}_{\vec{a}}^*$ and $\mathbb{C}_\mu^*$ actions agree on flags, thus on $\mathcal{B}^\lambda$ and $\mathcal{B}_p^\lambda$ as well. In particular, as $\mathbb{C}_{\vec{a}}^*$ keeps those Springer fibres invariant, so does $\mathbb{C}_\mu^*$. ■

Finally, we get the description of minimal components in $\mathcal{B}^\lambda$.

Proposition 5.3.12. *The set of minimal components of twisted Kazhdan actions in $\mathcal{B}^\lambda$ is precisely*

$$\{\mathcal{R}^\mu \mid \mu \in \mathrm{Good}(\lambda^*)\},$$

hence it is a subset of the set of Richardson components. In particular, these minimal components are isomorphic to products of flag varieties.

Proof. As the sets W_μ^i are preserved by the $\mathbb{C}_\mu^*$-action, so is the Richardson component $\mathcal{R}^\mu$. But then, due to Corollary 5.3.11, it is fixed under the $\mathbb{C}_{\vec{a}}^*$-action, for a vector $\vec{a}$ such that $\mu = \mu(\vec{a})$, hence it is exactly the minimal component $\mathfrak{F}^\mu$. By Theorem 5.2.25, the set $\{\mathbb{C}_{\vec{a}}^* \mid \vec{a} \text{ is good}\}$ is the set of all twisted Kazhdan actions that are even and conical, hence their minimal components are indeed $\{\mathcal{R}^\mu \mid \mu \text{ is good}\}$. ∎

As a corollary, we describe the set of standard Young tableaux that correspond to minimal components:

Corollary 5.3.13. *The set of standard Young tableaux in* $\mathbf{Std}^\lambda$ *that correspond to minimal components in $\mathcal{B}^\lambda$ is exactly $\{T^\mu \mid \mu \in \mathrm{Good}(\lambda^*)\}$.*

Another corollary is that minimal components are torus-invariant. We call **diagonal maximal torus** the maximal torus of GL_n consisting of diagonal matrices with respect to the basis $v_{i,j}$.

Corollary 5.3.14. *Minimal components in $\mathcal{B}^\lambda$ are invariant under the action of the diagonal maximal torus.*

Proof. Minimal components are Richardson, and Richardson components are invariant under the action of a parabolic subgroup, hence under the diagonal maximal torus as well, as it is a subgroup of each of them. ∎

In the end, we describe exactly when the set of minimal components is equal to the set of Richardson components. Recall a partition is of **hook-type** when its Young diagram has the shape of a hook, so $\lambda = (k, \underbrace{11\ldots1}_{n})$, for arbitrary $k, n \in \mathbb{N}$. These form a subset of the partitions that arise in the following.

Corollary 5.3.15. *The set of minimal components agrees with the set of Richardson components in the Springer fibre $\mathcal{B}^\lambda$ exactly when the partition is of type $\lambda = (\underbrace{kk,\ldots,k}_{m\geq 0}, \underbrace{11\ldots1}_{n\geq 0})$.*

Proof. According to Corollary 5.3.13 we just have to find the set

$$\Lambda = \{\lambda \vdash N, \text{ for some } N \mid \text{Good}(\lambda^*) = \text{Perm}(\lambda^*)\}.$$

The dual λ^* of a partition $\lambda \in \Lambda$ cannot contain three numbers such that two of them are bigger than the third one, as then one could make a non-good permutation $\mu \in \text{Perm}(\lambda^*)$ that has the smallest one in between the other two. Thus, for some $m, n \geq 0, k \geq 1$ we have $\lambda^* = (m + n \ \underbrace{mm \dots m}_{k-1})$, hence $\lambda = (\underbrace{kk, \dots, k}_{m}, \underbrace{11 \dots 1}_{n})$, as claimed. ∎

Now we give a family of examples when minimal components constitute all components of a Springer fibre.

Example 5.3.16. In Springer fibres of type $\lambda = (n, 1)$, all components are minimal. Namely, the set of standard Young tableaux $\mathbf{Std}^{n,1}$ has exactly n elements (one picks a number between 2 and $n+1$ to place in the second row, and the remaining numbers are in the first row in increasing order). The dual partition is $\lambda^* = (2, \underbrace{1, \dots, 1}_{n-1})$ hence

$$\text{Perm}(\lambda^*) = \{2\underbrace{1 \dots 1}_{n-1}, 12\underbrace{1 \dots 1}_{n-2}, \dots, \underbrace{11 \dots 1}_{n-1}2\},$$

that is, there are n Richardson components. Together with Corollary 5.3.15 we get that in Springer fibres $\mathcal{B}^{n,1}$ all components are minimal. We remark here that this is somewhat foreseen, as the Slodowy variety $\mathcal{S}_{n,1}$ and its resolution form the Springer-theoretic model for the minimal resolution of the Du Val singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$, thus due to the quiver model of Du Val singularities (Example 4.3.16), and the expectancy that Maffei isomorphism is equivariant (Conjecture 5.2.42), we would indeed expect that minimal components of twisted Kazhdan actions exhaust the core $\mathcal{B}^{n,1}$.

Remark 5.3.17. We remark that the previous example covers all examples of ordinary Springer fibres $\mathcal{B}^\lambda$ in which all components are minimal. The proof goes directly: given a partition λ of type from Corollary 5.3.15, which is not of type $\lambda = (n, 1)$, one can show that there is a tableau that does not belongs to Richardson ones, hence its corresponding component is not minimal. For other partitions, there are non-minimal Richardson components.

5.3.4 Minimal components are Schützenberger-invariant

In this section we prove an interesting fact about minimal components, from the Springer-theoretic viewpoint. Namely, the set of Young tableaux that correspond to minimal components inside $\mathcal{B}^\lambda$ are invariant under the so-called *Schützenberger involution,* an involution on $\mathbf{Std}^\lambda$ that is combinatorially defined.

Given a standard Young tableau T with n boxes, we define $\pi_{1,i}(T)$ as the tableau obtained from T by removing the boxes labelled with numbers $i+1$ to n. Moreover, we define $\pi_{i,n}(T)$ as the tableau obtained from T by removing boxes labelled with numbers 1 to $i-1$ by the so-called procedure of **jeu de taquin**. First we remove the box with number 1. In its place, we move whichever adjacent box to the right or beneath it contained the smaller number. Thus, we have caused another empty space. Continue the procedure until we obtain a standard tableau. Continue the procedure until the numbers 1 to $i-1$ are removed and then subtract $i-1$ from all numbers to get a standard tableau, called $\pi_{i,n}(T)$.

Example 5.3.18. Let us show that e.g. $\pi_{2,3}(T^{221}) = T^{121}$, in the following sequence of tableau-operations:

$$T^{221} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \to \begin{array}{c|c|c|}\cline{2-3} & 3 & 5 \\ \hline \multicolumn{1}{|c|}{2} & 4 \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 \\ \cline{2-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} = T^{121}.$$

Recall that $Sh(S)$ deotes the Young diagram of a tableau S, The **Schützenberger involution** of a standard tableau T of size n is defined as the standard tableau

$$Sch(T) := Tab(\emptyset, Sh(\pi_{n,n}(T)), \dots, Sh(\pi_{2,n}(T)), Sh(\pi_{1,n}(T))),$$

Equivalently, the tableau $Sch(T)$ is defined by equalities

$$Sh(\pi_{1,i}(Sch(T))) = Sh(\pi_{n+1-i,n}(T)), \quad i = 1, 2, \dots, n.$$

Example 5.3.19. Given the tableau $T^{21} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$, by jeu de taquin we get

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \to \begin{array}{c|c|}\cline{2-2} & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} = \pi_{2,3}(T^{21}),$$

and $\pi_{3,3}(T^{21}) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$ trivially. Thus

$$Sch(T^{21}) = Tab\left(\begin{array}{|c|}\hline \\ \hline\end{array}, \begin{array}{|c|c|}\hline & \\ \hline\end{array}, \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = T^{12}.$$

Recall in Section 5.3.2 we defined the standard tableaux $T^{\mu} = T^{\mu_1,\dots,\mu_r}$. Then:

Lemma 5.3.20. *$\pi_{2,n}(T^{\mu_1,\dots,\mu_r}) = T^{\mu_1-1,\dots,\mu_r}$ and $\pi_{1,n-1}(T^{\mu}) = T^{\mu_1,\dots,\mu_r-1}$ where $n = \mu_1 + \dots, +\mu_r$.*

Proof. $\pi_{2,n}(T^{\mu_1,\dots,\mu_k})$ is the jeu de taquin performed on tableau $T^{\mu_1,\dots,\mu_k}$. Thus, we remove the box with number 1 and then the box with number 2, which is beneath it, moves to its place. Then the box that was occupied by 2 is empty, and gets filled by the number 3 which is in the box below, and so on until all the numbers $2, 3, \dots, \mu_1$ move one box up. Subtracting 1 from each box, yields exactly the tableau $T^{\mu_1-1,\dots,\mu_r}$. Tableau $\pi_{1,n-1}(T^{\mu})$ is obtained by deleting the box n from the tableau $T^{\mu_1,\dots,\mu_k}$, hence it is trivially equal to the tableau $T^{\mu_1,\dots,\mu_r-1}$. ■

Given an element $\mu = (\mu_1, \dots, \mu_k) \in \mathrm{Perm}(\lambda^*)$, define **reflection** $\bar{\mu}$ is defined as $\bar{\mu} = (\mu_k, \dots, \mu_1)$.

Lemma 5.3.21. *For an arbitrary $\mu \in \mathrm{Perm}(\lambda^*)$ we have $Sch(T^{\mu}) = T^{\bar{\mu}}$.*

Proof. We will prove this by induction on the number $n(\mu) := \mu_1 + \dots + \mu_r$, where $\mu = (\mu_1, \dots, \mu_r)$. The base $n(\mu) = 1$ is trivially true. Now let us assume that the statement $Sch(T^{\mu}) = T^{\bar{\mu}}$ is true for all μ s.t. $n(\mu) = n-1$, and choose $\mu = (\mu_1, \dots, \mu_r)$ with $n(\mu) = n$. Then, $\mu' = (\mu_1 - 1, \mu_2, \dots, \mu_r)$ has $n(\mu') = n - 1$, hence it is true that

$$Sch(T^{\mu'}) = T^{\bar{\mu}'}$$

by the inductive hypothesis. That means that we have

$$Sh(\pi_{1,i}(T^{\bar{\mu}'})) = Sh(\pi_{n-i,n-1}(T^{\mu'})), \quad i = 1, 2, \dots, n-1.$$

Now using Lemma 5.3.20 this becomes

$$Sh(\pi_{1,i}(\pi_{1,n-1}(T^{\bar{\mu}}))) = Sh(\pi_{n-i,n-1}(\pi_{2,n}(T^{\mu}))), \quad i = 1, 2, \dots, n-1.$$

and thus

$$Sh(\pi_{1,i}(T^{\bar{\mu}})) = Sh(\pi_{n+1-i,n}(T^{\mu}))), \quad i = 1, 2, \dots, n-1.$$

Together with the trivial equality $Sh(\pi_{1,n}(T^{\bar{\mu}})) = Sh(\pi_{1,n}(T^{\mu})))$ we get $Sch(T^{\mu}) = T^{\bar{\mu}}$. ■

Thus, from Corollary 5.3.13, Lemma 5.3.21, and fact that the set $\mathrm{Good}(\lambda^*)$ is invariant under the reflection, we get the claimed result:

Proposition 5.3.22. *The set of Young tableaux that correspond to minimal components in $\mathcal{B}^{\lambda}$ are invariant under the Schützenberger involution.*

5.3.5 Minimal components are of Barchini-Graham-Zierau type

In [BaZi08, GrZi11], the authors define a family of smooth components of ordinary Springer fibres and show they are isomorphic to iterated bundles of generalised flag varieties. Abbreviating surnames of the authors, let us call these BGZ components. In this section we show that minimal components of the ordinary Springer fibres lie in the intersection of the sets of BGZ components and Richardson components. Firstly, let us briefly recall their construction.

Given any subgroup $K = GL(p) \times GL(q)$ of $G = GL(n)$ with $p+q = n$, K acts on the flag variety $\mathcal{B}$, with finitely many orbits. Fix a closed K-orbit Q. Then, denote the restriction of the Springer resolution $\nu : T^*\mathcal{B} \to \mathcal{N}$ to the conormal bundle $T^*_Q\mathcal{B}$ by

$$\gamma_Q := \nu|_{T^*_Q\mathcal{B}} \to Im(\gamma_Q) \subset \mathcal{N}.$$

Its image is the closure of a K-adjoint orbit, $Im(\gamma_Q) = \overline{K \cdot f}$, where $f \in \mathcal{N}$ is a generic element in $Im(\gamma_Q)$. We have the following:

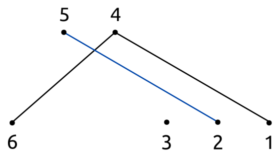
Theorem 5.3.23. [BaZi08, GrZi11, Thm. 2.7] $\gamma_Q^{-1}(f) \subset \mathcal{B}^f$ *is an irreducible component. Moreover, it is isomorphic to an iterated bundle of generalised flag varieties.*

Definition 5.3.24. The components of the Springer fibre obtained from the last theorem we will call **BGZ components.**

As these components depend only on the choice of the subgroup K, and the closed K-orbit Q in $\mathcal{B}$, roots systems theory yields their combinatorial labelling by arrays. An **array** consists of two parallel lines of numbered dots labelled by $1, 2, \ldots, n$ from right to left, e.g. see the first picture in Example 5.3.25 for the case $n = 6$. A **block** is a subset of dots in a line that are labelled by consecutive numbers. E.g. in Example 5.3.25, there are three blocks: '6', '5,4', and '3,2,1'. The **string** of an array is the zig-zag line joining the rightmost dots in each block such that it alternates between the two rows, e.g. in Example 5.3.25 the black zig-zag line connecting '6,4,1'. Given an array A we obtain the array A' by deleting the dots connected by the string of A, then joining blocks if necessary in the obvious way, and finally relabelling as appropriate. This process of joining blocks is called **collapsing**.

Now, given an array A, we construct a tableau $T(A)$ of the corresponding BGZ component. Its first row consists of the numbers in the string of A, written from right to left. The next row consists of the numbers in the string of A' using the original labelling from A, and so on.

Example 5.3.25. Here we give an example of an array and its corresponding tableau. Given an array on the picture



we see that its string consists of numbers (1,4,6). By deleting these numbers we get the next string (2,5) (the string of the collapsed array, but using the original labelling), and finally the last string is (3). Thus, the tableau of the given array is

1	4	6
2	5	
3		

Let us summarize what we have said in the previous paragraphs:

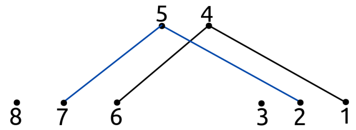
Theorem 5.3.26. [GrZi11, App. A] *BGZ components are labelled bijectively by arrays, such that the tableau of a component that corresponds to array A is exactly $T(A)$.*

In a remark [GrZi11, Rmk. A.6], the authors state a condition for a BGZ component to be Richardson as well. As this remark has not been proved in the paper, we prove it here for completeness.

Proposition 5.3.27. *A BGZ component of $\mathcal{B}^\lambda$ is Richardson exactly when the block sizes of its array form a permutation of the dual partition λ^*, and no collapsing of blocks occurs in the construction of its tableau.*

Proof. Given a BGZ component which is Richardson, say $\mathcal{R}^\mu$, for some permutation $\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathrm{Perm}(\lambda^*)$. Then, by Proposition 5.3.8 its Young tableau is T^μ. So the first row of its Young tableau is $(1, 1+\mu_1, 1+\mu_1+\mu_2, \dots, 1+\mu_1+\dots+\mu_{r-1})$, thus the string of the array for $\mathcal{R}^\mu$ has exactly these numbers.

That immediately yields the claim about the block sizes. Now, the collapsing in the construction of the tableau occurs exactly when there is an odd number $2k+1$ of consecutive blocks of the same size, say b, that is surrounded by two blocks of strictly bigger sizes than b, which hence collapse to one block after b iterations (in the array below: $b = 2$).



Let us assume that such a collapsing occurs, and let the collapsing blocks start at numbers $1 + d_{i+1}, \ldots, 1 + d_{i+2k+1}$, where $d_i = \sum_{j=1}^{i} \mu_j$ as before. Their common size is $b = \mu_{i+2} = \cdots = \mu_{i+2k+2}$. Then, the blocks on the right and left start at numbers $1 + d_i$, and $1 + d_{i+2k+2}$ respectively. Thus, the μ-line that starts with the number $1 + d_{i+2k+2}$ has the numbers $s + d_{i+2k+2}$ in the rows $s = 1, 2, \ldots, b$ but the number $b+1+d_{i+2k+2}$ does not occur in the $(b+1)$-th row, as it becomes a non-rightmost point of the joint block after b iterations (in the above array: $b + 1 + d_{i+2k+2} = 8$). Thus, the tableau that we get from this array (see the tableau below for the given array above) will not be the tableau T^{μ}, hence the component obtained is not Richardson, which is a contradiction. Hence, no collapsing occurs.

1	4	6
2	5	7
3		
8		

Now, let us prove that a non-collapsing array whose blocks are of sizes $\mu = (\mu_1, \ldots, \mu_r)$ (from right to left) indeed gives the Richardson component $R^{\overline{\mu}}$, by induction on the length of the vector μ. When $\mu = \mu_1$, then the tableau indeed has one column labelled by numbers $1, \ldots, \mu_1$ hence it is the tableau of $\mathcal{R}^{\mu}$. Now let us assume we have an array of n blocks with sizes $\mu = (\mu_1, \ldots, \mu_r)$. Then, by the inductive hypothesis, if we neglect the leftmost block, the tableau obtained would be $T^{\mu'}$ where $\mu' = (\mu_1, \ldots, \mu_{r-1})$ Now, notice that without neglecting the leftmost block, at the i-th step of the process we just add the number $i + \mu_{r-1}$ to the end of the i-th row. Hence, the obtained tableau is $T^{\mu'}$ with an added μ-line with numbers $(1 + d_{r-1}, \ldots, d_r)$ on the right, which is exactly the tableau T^{μ}. ■

Corollary 5.3.28. *Minimal components are of BGZ type.*

Proof. Every minimal component is Richardson $\mathcal{R}^{\mu}$ with $\mu \in \mathrm{Good}(\lambda)$. Recall that the set $\mathrm{Good}(\lambda)$ consists of permutations μ which do not contain three indices $i < j < k$ such that $\mu_i > \mu_j$ and $\mu_j < \mu_k$. Thus, no collapsing would occur in creating a tableau from the array of type μ, hence by Proposition 5.3.27, we see that the minimal component $\mathcal{R}^{\mu}$ is a BGZ component as well. ■

It would be interesting to see whether the BGZ construction can be generalised to the case of generalised Springer fibres. In the case that it is possible, assuming that some analogue of Proposition 5.3.28 exists as well, that would yield a family of Young tableaux that correspond to some family of special minimal components that are products of generalised flag manifolds. The author has discussed this topic with the authors of [GrZi11], and leaves it for some future work.

In the end of this section, we briefly comment on smooth components in the ordinary Springer fibres. Apart from the Richardson and BGZ constructions, there is also a family of so-called *generalised Richardson components* constructed by L. Fresse in [Fr11], which in particular contains Richardson components. So far, it is unclear how this family relates to BGZ. We illustrate all these families in Figure 5.2.

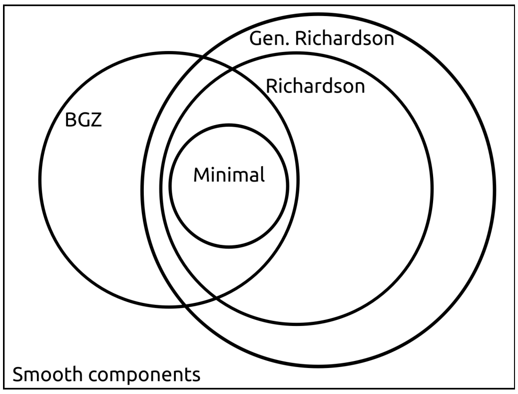


Figure 5.2: Families of smooth components in $\mathcal{B}^\lambda$

5.4 Minimal and Richardson components of generalised Springer fibres

In this section we construct a novel family of smooth components of the generalised Springer fibres $\mathcal{B}_p^\lambda$. We prove that these families generalise Richardson components, hence we use the same name for them. Although, the essential difference from the case of ordinary Springer fibres is that the set of minimal components of a generalised Springer fibre $\mathcal{B}_p^\lambda$ is not a subset of the set of Richardson components (see Example 5.4.40), hence in a search for smooth components in $\mathcal{B}_p^\lambda$, both families of smooth components (minimal and Richardson) that we obtain are equally interesting to consider. In addition, we define quasi-Richardson components as a generalisation of these Richardson components, and find their corresponding semistandard tableaux.

As it is going to be the central object of this section, recall that the **(generalised) Springer fibre** $\mathcal{B}_p^e$ of a nilpotent element $e \in \mathfrak{sl}_n$ is defined as

$$\mathcal{B}_p^e := \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n \mid \dim F_i/F_{i-1} = p_i,\ eF_i \subset F_{i-1},\ i = 1, \ldots, n\}$$

Denoting by $\lambda = \lambda(e)$ the Jordan partition of e, recall that **we denote by** $\mathcal{B}_p^\lambda$ the generalised Springer fibre $\mathcal{B}_p^e$, as it does not depend (up to an isomorphism) on the choice of particular nilpotent element e of Jordan type λ.

In the first two subsections (5.4.1 and 5.4.2) we recall the classical facts from the Springer-theoretic literature that we will use in this section.

5.4.1 Brundan-Ostrik construction

In this section we will recall the Brundan-Ostrik generalisation [BrOs11, Sec. 2] of the Spaltenstein construction (Section 5.3.1) to generalised Springer fibres. This shows that the irreducible components of a generalised Springer fibre $\mathcal{B}_p^\lambda$ are labelled bijectively by the *semistandard* Young p-tableaux of shape λ.[10]

Definition 5.4.1. Given a partition $\lambda \vdash n$, and a composition $p = (p_1, \ldots, p_n)$ of n, a **Young p-tableau of shape** λ is a filling of the Young diagram of shape λ with numbers, such that the number i fills exactly p_i boxes. We call such a tableau **row-standard** when the numbers in each row strictly increase, and **semistandard** when the numbers in each row strictly increase and in each column weakly increase

[10] NB for the reader: With a difference from the Brundan-Ostrik notation, we will use the row-standard instead of column-standard tableaux. Also, whilst they use the notation x for the nilpotent element, we prefer to keep e instead, in order to have consistent notation in this chapter.

(i.e. may be stationary). The set of all ordinary, row standard and semistandard Young p-tableaux of shape λ will be denoted by $\mathbf{Tab}_p^\lambda$, $\mathbf{Row}_\mathbf{p}^\lambda$ and $\mathbf{Std}_p^\lambda$, respectively.

Consider a nilpotent element $e \in \mathfrak{sl}_n$ whose Jordan partition is $\lambda = \lambda(e)$, and a composition $p = (p_1, \dots, p_n)$ of n. Given a p-partial flag

$$F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n \mid \dim F_i/F_{i-1} = p_i) \in \mathcal{B}_p^\lambda$$

its **shape** is a weakly-increasing sequence

$$sh(F) := (Y(e|_{F_0}), Y(e|_{F_1}), \dots, Y(e|_{F_n}))$$

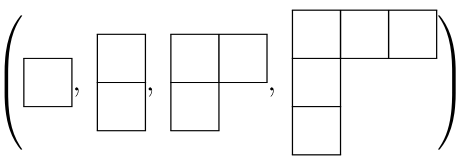
of Young diagrams. To a weakly-increasing sequence $Y = (Y_0, Y_1, \dots, Y_n)$ of Young diagrams we attach the semistandard p-tableaux $Tab(Y)$ of shape Y_n by filling the boxes in Y_i that are not in Y_{i-1} with numbers i.

Example 5.4.2. Consider a nilpotent element e whose Jordan partition is $\lambda(e) = (3, 1, 1)$. Let us denote by $(v_1, v_2, v_3|v_3|v_4)$ its Jordan basis (thus $ev_3 = v_2, ev_2 = v_1, ev_1 = 0$ and $ev_4 = ev_3 = 0$). Consider the flag

$$F = \{0 \subset \langle \lambda v_1 + \mu(\alpha v_4 + \beta v_5) \rangle \subset \langle v_1, \alpha v_4 + \beta v_5 \rangle \subset \langle v_1, v_2, \alpha v_4 + \beta v_5 \rangle \subset \mathbb{C}^5\},$$

for some $[\alpha : \beta], [\lambda : \mu] \in \mathbb{P}^1 \times \mathbb{P}^1$.

We see that $e|\langle v_1 + \mu(\alpha v_4 + \beta v_5)\rangle = 0$, $e|\langle v_1, \alpha v_4 + \beta v_5 \rangle = 0$, and $\lambda(e|\langle v_1, v_2, \alpha v_4 + \beta v_5 \rangle) = (21)$, as $ev_2 = v_1,\ ev_1 = 0$ and $e(\alpha v_4 + \beta v_5) = 0$. Thus, altogether, we have

$sh(F) =$ 

hence

$$Tab(sh(F_1)) = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} .$$

Analogously to Section 5.3.1, we define the set

$$\mathcal{F}_{e,T} := \{F \in \mathcal{B}_p^\lambda \mid Tab(sh(F)) = T\}.$$

Now we have a generalisation of Spaltenstein's Theorem 5.3.5.

Theorem 5.4.3. *Given a generalised Springer fibre $\mathcal{B}_p^\lambda$, the following is true:*

(1) For each $T \in \mathbf{Std}_p^\lambda$, $\mathcal{F}_{e,T}$ *is a non-empty smooth open and irreducible subvariety of* $\mathcal{B}_p^\lambda$.

(2) Varieties $\{\mathcal{F}_{e,T}\}_{T\in\mathbf{Std}_p^\lambda}$ *are equidimensional,* $\dim \mathcal{F}_{e,T} = \frac{1}{2}\sum_{i=1}^{\lambda_1} \lambda_i^*(\lambda_i^* - 1) - \frac{1}{2}\sum_{i=1}^{n} p_i(p_i - 1)$.

(3) $\mathcal{B}_p^\lambda = \sqcup_{T\in\mathbf{Std}_p^\lambda}\mathcal{F}_{e,T}$.

(4) Hence, the closures $\mathcal{K}_T := \overline{\mathcal{F}_{e,T}}$ *form the set of (equidimensional) irreducible components of* $\mathcal{B}_p^\lambda$.

Hence, denoting by $\mathcal{B}(p,\lambda)$ the set of irreducible components of $\mathcal{B}_p^\lambda$, the map

$$\mathrm{Spal}_p : \mathbf{Std}_p^\lambda \to \mathcal{B}(p,\lambda), \quad T \mapsto \mathcal{K}_T,$$

is a bijection. We call it the **generalised Spaltenstein map** for the composition p, following the attribution given in the Brundan-Ostrik paper.

Moreover, in the same paper they describe the closures of the cells $\mathcal{F}_{e,T}$. We first require some definitions.

Definition 5.4.4. Given a composition $p = (p_1, \dots, p_n)$ and a tableaux $T \in \mathbf{Std}_p^\lambda$, we define the vector $l(T) = (l_1, \dots, l_k)$ and a tableau $\overline{T} \in \mathbf{Std}_p^\lambda$ in the following way:

- $1 \le l_1 < \cdots < l_k$ index the rows of T containing the entry n;
- $\overline{T}$ is obtained from T by deleting all the boxes that contain the number n.

We define the partial order $\preceq$ on $\mathbf{Std}_p^\lambda$ inductively: Given two tableaux T_1, T_2 we denote $T_1 \preceq T_2$ if either $l(T_1) \le l(T_2)$ or $l(T_1) = l(T_2)$ and $\overline{T_1} \preceq \overline{T_2}$. Here for two vectors η, ν by $\eta \le \nu$ we mean $\eta_i \le \nu_i$ for all i.

Hence, we have defined a partial order on the set of semistandard tableaux $\mathbf{Std}_p^\lambda$. This order is essential in the topology of the closure of Spaltenstein cells:

Theorem 5.4.5. *Given a semistandard tableau* $T \in \mathbf{Std}_p^\lambda$,

$$\mathcal{K}_T = \overline{\mathcal{F}_{e,T}} \subset \bigcup_{S \preceq T} \mathcal{F}_{e,S}.$$

5.4.2 Torus fixed points

In this section we briefly recall the standard folklore in Springer theory (look e.g. in [Fr09a, Sec. 3.9]) about the fixed points of the diagonal torus action on $\mathcal{B}_p$, which lie on $\mathcal{B}_p^\lambda$. Let e be a λ-nilpotent matrix and $v_{i,j}$ its Jordan base, as before. Define $H := H_e \leq GL_n$ to be **the diagonal torus with respect to the base** $v_{i,j}$.

Definition 5.4.6. Given a p-tableau T, we attach to it the p-partial flag $F_T := (0 \subset F_{\overline{p_1}} \ldots, F_{\overline{p_n}} = \mathbb{C}^n)$ defined by

$$F_{\overline{p_k}} := \langle v_{i,j} \mid \text{ the number in the position } (i,j) \text{ in } T \text{ is } \leq k \rangle.$$

Recall, by Section 5.4.1, that when T is semistandard, it corresponds to the component $\mathcal{K}_T = \overline{\mathcal{F}_{e,T}}$. It follows immediately from the definition that $F_T \in \mathcal{F}_{e,T} \subset \mathcal{K}_T$.

A standard fact from Springer theory is that these flags are exactly the fixed points under the action of the torus H.

Proposition 5.4.7. $(\mathcal{B}_p)^H = \{F_T \mid T \in \boldsymbol{Tab}_p^\lambda\}$ *and* $(\mathcal{B}_p^\lambda)^H = \{F_T \mid T \in \mathbf{Row}_\mathbf{p}^\lambda\}$

Remark 5.4.8. The torus H **does not** act on $\mathcal{B}_p^\lambda$. However, we can still discuss the fixed points $(\mathcal{B}_p^\lambda)^H$ of the H-action on $\mathcal{B}_p$ which lie in $\mathcal{B}_p^\lambda$.

Moreover, there is a refinement of the second part of the last proposition, according to the Spaltenstein decomposition $\mathcal{B}_p^\lambda = \sqcup_{T \in \mathbf{Std}_p^\lambda} \mathcal{F}_T$.

Definition 5.4.9. Given a row-standard tableau $T \in \mathbf{Row}_\mathbf{p}^\lambda$, its **standardisation** is the unique semistandard tableau $Std(T) \in \mathbf{Std}_p^\lambda$ such that it has the same numbers in each column as T, thus it is obtained from T by shuffling the numbers within columns.

Proposition 5.4.10. *Given a semistandard tableau S, the torus fixed points in the cell $\mathcal{F}_S$ are*

$$\mathcal{F}_S^H = \{F_T \mid T \in \mathbf{Row}_\mathbf{p}^\lambda, \quad Std(T) = S\}.$$

5.4.3 Richardson components in generalised Springer fibres

In this section we generalize the construction of the Richardson components given in Section 5.3.2. More precisely, we obtain a family of smooth components in generalised Springer fibres that are isomorphic to products of generalised flag manifolds and are invariant under the parabolic subgroups.

Definition 5.4.11. Given two compositions $p = (p_1, \dots, p_n), \mu = (\mu_1, \dots, \mu_r)$ of n, we say that p is **finer** than μ, and μ is **coarser** than p, if for some increasing indices $0 = k_0, k_1, \dots, k_r = n$

$$\begin{aligned} \mu_1 &= p_{k_0+1} + \dots + p_{k_1} \\ \mu_2 &= p_{k_1+1} + \dots + p_{k_2} \\ \dots & \\ \mu_r &= p_{k_{r-1}+1} + \dots + p_{k_r} \end{aligned} \tag{5.23}$$

holds. Denote by $Coar(p)$ the set of all compositions of n that are coarser than p. For the composition $p = (1, \dots, 1)$, the set $Coar(p)$ is the set of all compositions μ such that $\mu_i \geq 1$.

Definition 5.4.12. Given an element $\mu = (\mu_1, \dots, \mu_r) \in \mathrm{Perm}(\lambda^*)$ we define the Young p-tableau $T_p^\mu \in \mathbf{Std}_p^\lambda$ associated to it in the following way. Consider the string $q_p := \underbrace{1, \dots, 1}_{p_1}, \underbrace{2, \dots, 2}_{p_2}, \dots, \underbrace{n, \dots, n}_{p_n}$. Take the μ-weighted tableau $T_{wt}(\mu)$ from Definition 5.2.36 and replace the boxes labelled '0' by first μ_1 numbers of the string q_p in order from top to bottom, then replace the boxes labelled '2' by next μ_2 numbers of the string q_p in order from top to bottom, etc. It is easy to see that when $\mu \in Coar(p)$, the p-tableau T_p^μ is semistandard.

Notice that this is a generalisation of Definition 5.3.7 for the ordinary case $p = (1 \dots 1)$. As there, the sets of $\mu_1, \mu_2, \dots, \mu_r$ boxes that we fill with this procedure we will call **μ-lines**.

Example 5.4.13. Let $\lambda = (321)$, $p = (211110)$, so $q_p = (112345)$, $\lambda^* = (321)$, then for $\mu = (312) \in \mathrm{Perm}(\lambda^*)$ the Young p-tableau is the Young diagram of type $\lambda = (321)$ filled with numbers as follows:

$$T_{211110}^{312} = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 1 & 5 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array}$$

where for the ease of the reader we highlighted in different colours the three substrings of q_p of length μ_i. As a second example, suppose we want to find the tableau $T_p^\mu = T_{2121}^{42}$. As $p = (2121)$ we have $q_p = 112334$, and since $\mu = (42) \in \mathrm{Perm}(\lambda^*)$ we have $\lambda^* = (42)$ so $\lambda = (2211)$. Thus we want a tableau of Young type (2211), we put the substring 1123 of q_p into the μ_1-line, and 34 in the μ_2-line:

$$T^{42}_{2121} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Recall that, given a partition λ, the set of Richardson components in the Springer fibre $\mathcal{B}^\lambda$ are labelled by the set $\mathrm{Perm}(\lambda^*)$ of permutations of the dual partition. Here we will extend this to the generalised Springer fibre, by constructing a family of components of $\mathcal{B}^\lambda_p$ labelled by $Coar(p) \cap \mathrm{Perm}(\lambda^*)$. Recall from Section 5.3.2 that, given a partition $\lambda \vdash n$, and a permutation $\mu = (\mu_1, \ldots, \mu_r) \in \mathrm{Perm}(\lambda^*)$, we have the partial flag $\overline{W_\mu} := (0 = W^0_\mu \subset \overline{W^1_\mu} \subset \cdots \subset \overline{W^r_\mu} = \mathbb{C}^n)$ defined as the unique partial flag in $\mathcal{B}^\lambda_\mu \cong \{pt\}$.

Definition 5.4.14. Given a partition λ, a composition p, and a composition $\mu \in Coar(\lambda, p) := Coar(p) \cap \mathrm{Perm}(\lambda^*)$, we define the **Richardson component** $\mathcal{R}^\mu_p \subset \mathcal{B}_p$ by

$$\mathcal{R}^\mu_p := \{0 = F_0 \subset F_{\overline{p_1}} \subset \cdots \subset F_{\overline{p_n}} = \mathbb{C}^n \mid F_{d_i} = \overline{W^i_\mu}\},$$

where $\overline{p_i} = \sum_{j=1}^i p_j$ and $d_i = \sum_{j=1}^i \mu_j = \dim \overline{W^i_\mu}$.

We see that this definition generalises the ordinary Richardson component $\mathcal{R}^\mu$. Namely, $Coar(1, \ldots, 1) \cap \mathrm{Perm}(\lambda^*) = \mathrm{Perm}(\lambda^*)$ and $\mathcal{R}^\mu_{1\ldots1} = \mathcal{R}^\mu$.

Moreover, we prove an analogue of Proposition 5.3.8. Recall that the parabolic subgroup P^μ denotes the the set of elements in GL_n that preserve the partial flag $\overline{W_\mu}$.

Theorem 5.4.15. *Given an arbitrary $\mu \in Coar(p) \cap \mathrm{Perm}(\lambda^*)$, the Richardson component $\mathcal{R}^\mu_p \subset \mathcal{B}_p$ satisfies the following:*

(1) It is a closed subset of $\mathcal{B}^\lambda_p$.

(2) It is isomorphic to a product of generalised flag varieties

$$\mathcal{R}^\mu_p \cong \mathcal{B}_{p^1} \times \mathcal{B}_{p^2} \times \cdots \times \mathcal{B}_{p^r}$$

where $p^i := (p_{k_{i-1}+1}, \ldots, p_{k_i})$ and k_i are defined by (5.23).

(3) It is an irreducible component of $\mathcal{B}^\lambda_p$.

(4) Its corresponding Young tableau is exactly T^μ_p.

(5) It is fixed under the parabolic subgroup P^μ.

Proof. As $\overline{W_\mu^i}$ is defined as the unique partial flag in $\mathcal{B}_\mu^\lambda \cong \{pt\}$, we have that $e_\lambda \overline{W_\mu^k} \subset \overline{W_\mu^{k-1}}$ for every k. Thus, having an arbitrary $\overline{p_i}$, as μ is coarser than p, there is an index j such that

$$d_{j-1} = \sum_{k=0}^{j-1} \mu_k < \overline{p_i} \leq \sum_{k=0}^{j} \mu_k = d_j$$

hence

$$e_\lambda F_{\overline{p_i}} \subset e_\lambda F_{d_j} = e_\lambda \overline{W_\mu^j} \subset \overline{W_\mu^{j-1}} = F_{d_{j-1}} \subset F_{\overline{p_{i-1}}},$$

where the last inclusion follows from $d_{j-1} \leq p_{i-1}$ which is true as $d_{j-1} < p_i$ and μ is coarser than p. Thus $\mathcal{R}_p^\mu \subset \mathcal{B}_p^\lambda$, hence we have proved claim (1).

The claim (2) is easy to see, simply by constructing an isomorphism

$$\mathcal{R}_p^\mu \to \mathcal{B}_{p^1} \times \mathcal{B}_{p^2} \times \cdots \times \mathcal{B}_{p^r}, \quad F \mapsto \widetilde{F}_1 \times \cdots \times \widetilde{F}_r$$

where

$$\widetilde{F}_i := 0 \subset F_{\overline{p_{k_{i-1}+1}}} / F_{\overline{p_{k_{i-1}}}} \subset \cdots \subset F_{\overline{p_{k_i}}} / F_{\overline{p_{k_{i-1}}}}$$

is a p^i-partial flag where $p^i = (p_{k_{i-1}+1}, \ldots, p_{k_i})$.

The variety $\mathcal{R}_p^\mu$ is closed and irreducible, hence to show that it is an irreducible component, it remains to show that its dimension is equal to $\dim \mathcal{B}_p^\lambda = \frac{1}{2}\sum_{i=1}^{\lambda_1} \lambda_i^*(\lambda_i^* - 1) - \frac{1}{2}\sum_{i=1}^{n} p_i(p_i - 1)$.

By (2), we have that

$$\begin{aligned}
\dim \mathcal{R}_p^\mu = \sum_{i=1}^{r} \dim \mathcal{B}_{p^i} &= \sum_{i=1}^{r} \left(\frac{1}{2}\mu_i(\mu_i - 1) - \frac{1}{2} \sum_{t=k_{i-1}+1}^{k_i} p_t(p_t - 1) \right) \\
&= \sum_{i=1}^{r} \frac{1}{2}\mu_i(\mu_i - 1) - \frac{1}{2}\sum_{t=1}^{n} p_i(p_i - 1) \\
&= \frac{1}{2}\sum_{i=1}^{\lambda_1} \lambda_i^*(\lambda_i^* - 1) - \frac{1}{2}\sum_{i=1}^{n} p_i(p_i - 1) = \dim \mathcal{B}_p^\lambda
\end{aligned}$$

where we have used the fact that μ is a permutation of λ^*. Hence, claim (3) is proved.

The claim (4) follows immediately from a more general claim in Theorem 5.4.30, hence we will give a proof later (Corollary 5.4.33)

Finally, as the condition $F_{d_i} = \overline{W_\mu^i}$ is P^μ-invariant, $\mathcal{R}_p^\mu$ itself is P^μ-invariant. ■

We expect these components to generalise the Richardson components of ordinary Springer fibres also in sense of Definition 5.3.6, thus:

Conjecture 5.4.16. *The components* $\{\mathcal{R}_p^\mu \mid \mu \in Coar(p,\lambda)\}$ *are the only components of* $\mathcal{B}_p^\lambda$ *invariant under a parabolic subgroup of* GL_n.

Notice that, unlike in the ordinary case, different Richardson components of the same generalised Springer fibre $\mathcal{B}_p^\lambda$ need not be isomorphic. Here is an example:

Example 5.4.17. Let $\lambda = (321)$ and $p = (211110)$. Then we have

$$\mathrm{Perm}(\lambda^*) = \{123, 132, 213, 231, 312, 321\}.$$

and furthermore

$$Coar(\lambda, p) = \{213, 231, 312, 321\},$$

hence we have 4 Richardson components in $\mathcal{B}^{321}_{211110}$. Consider $\mathcal{R}^{213}_{211110}$ first. As μ=(213), its components μ_i split into p_j as follows:

$$\mu_1 = 2 = p_1,\ \ \mu_2 = 1 = p_2,\ \ \mu_3 = 1+1+1 = p_3+p_4+p_5.$$

Thus, according to Theorem 5.4.15, we get $\mathcal{R}^{213}_{211110} \cong \mathcal{B}_2 \times \mathcal{B}_1 \times \mathcal{B}_{1,1,1} \cong \mathcal{B}_{1,1,1}$, where the last isomorphism holds because a partial flag $\mathcal{B}_p$ is just a point when $p = (p_1)$ (its only flag is $0 \subset \mathbb{C}^{p_1}$). Similarly, we get:

$$\begin{aligned} R^{231}_{211110} &\cong \mathcal{B}_{1,1,1}, \\ R^{312}_{211110} &\cong \mathcal{B}_{2,1} \times \mathcal{B}_{1,1}, \\ R^{321}_{211110} &\cong \mathcal{B}_{2,1} \times \mathcal{B}_{1,1}. \end{aligned}$$

5.4.4 Quasi-Richardson components

In this section we define an enlargement of the set of Richardson components, which we will call quasi-Richardson components. They are smooth irreducible components of generalised Springer fibres, and we will find their corresponding tableaux. They will not satisfy, in general, all the properties of Richardson components (such as torus-invariance, being products of generalised flag manifolds,etc.), hence the difference in names.

Definition 5.4.18. Consider a composition $\mu = (\mu_1, \dots, \mu_r)$ of n and a splitting $\mathbb{C}^n = \oplus_{i=1}^r W_\mu^i$, where subspaces W_μ^i have dimensions μ_i. define the string

$$q_\mu := \underbrace{1\dots1}_{\mu_1}\underbrace{2\dots2}_{\mu_2}\dots\underbrace{r\dots r}_{\mu_r}$$

of length n, and denote by $q_\mu[1,k]$ the substring of first k characters of q_μ.

We will say that the k-dimensional subset $V_k \subset \mathbb{C}^n$ is **of** $q_\mu[1,k]$**-type** if it is a direct sum of subspaces $V_k = V_k^1 \oplus \cdots \oplus V_k^r$ such that $V_k^i \subset W_\mu^i$ and $\dim V_k^i = \#\{\text{occurrences of } i \text{ in } q_\mu[1,k]\}$. Equivalently, there is a basis $(u_i)_{i=1}^k$ of V_k made of homogeneous vectors under the splitting $\mathbb{C}^n = \oplus_{i=1}^r W_\mu^i$, which we can label by numbers $n_i \in \{1,\dots,r\}$ according to which $W_\mu^{n_i}$ they belong to and get the string $q_\mu[1,k]$ as a result.

Now, given a splitting $\mathbb{C}^n = \oplus_{i=1}^r W_\mu^i$, define the closed P_μ-orbit $C_p^\mu \subset \mathcal{B}_p$ in the following manner.

$$C_p^\mu := \{F = (F_0, F_{\overline{p_1}}, \dots, F_{\overline{p_n}}) \mid F_{\overline{p_i}} \text{ is of } q_\mu[1,\overline{p_i}]\text{-type}\} \tag{5.24}$$

where $\overline{p_i} = \sum_{j=1}^i p_j$ as before.

We give an example of this somewhat cumbersome-defined variety, for the convenience of the reader.

Example 5.4.19. Choose $\mu = (13)$ and $p = (2110)$. Consider $\mathbb{C}^4 = \langle v_1, v_2, v_3, v_4\rangle = W_\mu^1 \oplus W_\mu^2$, where

$$W_\mu^1 = \langle v_1\rangle, \quad W_\mu^2 = \langle v_2, v_3, v_4\rangle.$$

Noticing that $\overline{p_1} = 2, \overline{p_2} = 3, \overline{p_3} = 4, \overline{p_4} = 4$ and $q_\mu = 1222$, observe an arbitrary flag

$$(0 \subset F_2 \subset F_3 \subset F_4 = \mathbb{C}^4) \in C_{2110}^{13}.$$

By definition of C_{2110}^{13}, we have that F_2 has a 1-dimensional summand from $W_\mu^1 = \langle v_1\rangle$ and a 1-dimensional summand V_1 from W_μ^2. Similarly for F_3 but now gets a 2-dimensional summand V_2 from $W_\mu^2 = \langle v_2, v_3, v_4\rangle$. Thus we get

$$C_{2110}^{13} = \{0 \subset \langle v_1, V_1\rangle \subset \langle v_1, V_2\rangle \subset \mathbb{C}^4 \mid 0 \subset V_1 \subset V_2 \subset \langle v_2, v_3, v_4\rangle \text{ is a full flag}\}.$$

Now, let us connect the variety C_p^μ with Definition 5.4.14 of a Richardson component. Recall first the explicit definition of the partial flag $\overline{W_\mu}$ attached to the nilpotent element e from Section 5.3.2. Let $e \in \mathfrak{sl}_n$ be an arbitrary nilpotent element with Jordan partition $\lambda = \lambda(e)$, with a Jordan basis $\mathbf{v} = v_{i,j}$ (Definition 5.2.34). Then, to a permutation $\mu \in \mathrm{Perm}(\lambda^*)$ we associate $\overline{W_\mu^i} := \oplus_{k=1}^i W_\mu^k$ where

$$W_\mu^k := \langle v_{i,j} \mid Y(\mu)_{i,j} = k\rangle, \tag{5.25}$$

and $Y(\mu)$ is the tableau that labels μ-lines (Definition 5.3.7).

Now, **choosing the split** $\mathbb{C}^n = \oplus_{i=1}^r W_\mu^i$, for $\mu \in Coar(\lambda^*)$ we get

$$C_p^\mu = \mathcal{R}_p^\mu,$$

as the conditions in (5.24) that F_{p_i} are of $q_\mu[1,p_i]$-type become equivalent to $F_{d_i} = \overline{W^i_\mu}$, which is precisely the definition of $\mathcal{R}^\mu_p$. For general $\mu \in \mathrm{Perm}(\lambda^*)$, the variety C^μ_p is not a subset of $\mathcal{B}^\lambda_p$. Thus, in order to obtain a component of it, we have to intersect it with $\mathcal{B}^\lambda_p$. This motivates the next definition:

Definition 5.4.20. Given an arbitrary $\mu \in \mathrm{Perm}(\lambda^*)$, if the set

$$\mathcal{R}^\mu_p := C^\mu_p \cap \mathcal{B}^\lambda_p \subset \mathcal{B}^\lambda_p$$

is a smooth component, we call it a **quasi-Richardson (QR)** component of $\mathcal{B}^\lambda_p$. The set of all such μ we denote by $QR(\lambda, p)$. It is clear that $Coar(\lambda, p) \subset QR(\lambda, p)$.

Remark 5.4.21. Given a nilpotent element e, we will always assume that C^μ_p is defined by the split $\mathbb{C}^n = \oplus_{i=1}^r W^i_\mu$ whose blocks W^i_μ come from a Jordan basis of e (Equation (5.25)). Such C^μ_p we will call e-**adapted.**

Now we give an example of a quasi-Richardson component. As in earlier examples, we will simplify our notation by labelling the vectors in the Jordan basis of a nilpotent e by $v_1, \ldots, v_n$ in the natural order given by the matrix e (instead of using the cumbersome notation $v_{i,j}$ from (5.15)).

Example 5.4.22. Continuing with Example 5.4.19, consider a nilpotent element e with Jordan partition $\lambda(e) = (211)$. Picking $\mu = (13)$ as before, we get the corresponding tableau $Y(\mathbf{v})$

$$Y(\mathbf{v}) = \begin{array}{|c|c|} \hline v_1 & v_2 \\ \hline v_3 & \\ \cline{1-1} v_4 & \\ \cline{1-1} \end{array}$$

where we distinct different μ-lines with different colours. Hence, we have

$$W^1_\mu = \langle v_1 \rangle, \quad W^2_\mu = \langle v_2, v_3, v_4 \rangle.$$

Thus, the variety C^{13}_{2110} associated to the split $\mathbb{C}^4 = W^1_\mu \oplus W^2_\mu$ is by Example 5.4.19 equal to:

$$C^{13}_{2110} = \{0 \subset \langle v_1, V_1 \rangle \subset \langle v_1, V_2 \rangle \subset \mathbb{C}^4 \mid 0 \subset V_1 \subset V_2 \subset \langle v_2, v_3, v_4 \rangle \text{ is a full flag}\}.$$

Now, intersecting C^{13}_{2110} with $\mathcal{B}^{211}_{2110}$ amounts to use the conditions of the Springer fibre $\mathcal{B}^{211}_{2110}$, Thus using the fact that $ev_2 = v_1$ (indeed, e acts by shifting vectors to the left along each row in the tableau $Y(\mathbf{v})$), and forcing $eF_2 = 0$ and $eF_3 \subset F_2$, we get

$$\mathcal{R}^{13}_{2110} = \{0 \subset \langle v_1, w_2 \rangle \subset \langle v_1, w_2, w_3 \rangle \subset \mathbb{C}^4 \mid w_2 \in \langle v_3, v_4 \rangle,\ w_2, w_3 \in W^2_\mu \text{ lin. ind.}\}.$$ [11]

[11] Here lin. ind. stands for linearly independent.

This is an irreducible projective and smooth subvariety of dimension two, thus of maximal dimension in $\mathcal{B}^{211}_{2110}$, (recall the dimension formula in Theorem 5.4.3), hence is a quasi-Richardson component.

Now we give an analogue of Proposition 5.3.12 for generalised Springer fibres, which gives a criterion for when a quasi-Richardson is a minimal component as well.

Proposition 5.4.23. *When* $\mu \in \mathrm{Good}(\lambda^*) \cap QR(\lambda, p)$, *its quasi-Richardson and minimal components coincide,*

$$\mathcal{R}^\mu_p = \mathfrak{F}^\mu_p.$$

In particular, the tableau of such a minimal component is $\widetilde{T^\mu_p}$ *(Definition 5.4.28).*

Proof. From the definition of C^μ_p, we immediately see that it is fixed under the $\mathbb{C}^*_\mu$-action on $\mathcal{B}_p$ (Definition 5.3.10). When μ is good, the $\mathbb{C}^*_\mu$-action restricts to $\mathcal{B}^\lambda_p$, and is the same as the scalar $\mathbb{C}^*_{\vec{a}}$ action for $\mu = \mu(\vec{a})$, by Corollary 5.3.11. Thus $\mathcal{R}^\mu_p$ is the minimal component of $\mathbb{C}^*_{\vec{a}}$, hence it equals $\mathfrak{F}^\mu_p$. The last statement follows from Theorem 5.4.30. ∎

The important difference from the case of ordinary Springer fibres is that here minimal components can be non-Richardson, and even non quasi-Richardson, as we will see in Example 5.4.41.

5.4.5 Semistandard tableaux of quasi-Richardson components

In this section we obtain the semistandard tableaux that correspond to quasi-Richardson components, by calculating the torus fixed points on C^μ_p and by using Theorem 5.4.5 which gives a geometric feature of Brundan-Ostrik order on tableaux. Firstly, we introduce the following definition:

Definition 5.4.24. Given a partition λ and a permutation $\mu \in \mathrm{Perm}(\lambda^*)$, the set $\mathrm{Shuffle}(T^\mu_p)$ is the set of all Young tableaux that one obtains from the tableau T^μ_p by shuffling the numbers along the μ-lines.

Example 5.4.25. Consider tableaux

$$T^{123}_{211110} = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad T^{132}_{211110} = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

After a shuffle along μ-lines (labelled in distinct colour, for a better picture) we the following tableaux:

$$\widetilde{T^{123}_{211110}} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array} \quad \widetilde{T^{132}_{211110}} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 1 & 5 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array}$$

Now, we justify the purpose of introducing the last definition. Pick a nilpotent element e with Jordan partition λ and permutation $\mu \in \mathrm{Perm}(\lambda^*)$. The maximal torus $H := H_e \leq GL_n$ diagonal with respect to Jordan basis of e preserves the e-adapted C^{μ}_{p} (Remark 5.4.21). We calculate its fixed points (recall the notation from Section 5.4.2):

Proposition 5.4.26. *Given a nilpotent element e, the e-adapted variety C^{μ}_{p} is invariant under the action of the diagonal torus H. Its torus fixed points are exactly*

$$(C^{\mu}_{p})^H = \{F_T \mid T \in \mathrm{Shuffle}(T^{\mu}_{p})\}.$$

Proof. Being P^{μ}-invariant, C^{μ}_{p} is also H invariant, as $H \leq P^{\mu}$. Now that we have a smooth algebraic torus action on a smooth projective variety, with isolated fixed points, a classical result [Iv72] asserts that the number of fixed points is equal to the rank of the cohomology of the variety itself. Hence, it suffices to prove two statements:

(1) $\{F_T \mid T \in \mathrm{Shuffle}(T^{\mu}_{p})\} \subset C^{\mu}_{p}$.

(2) $rk(H^*(C^{\mu}_{p}))$=$|\{F_T \mid T \in \mathrm{Shuffle}(T^{\mu}_{p})\}|$

Proof of (1). Let $F_{T^{\mu}_{p}} = (F_0, F_{\overline{p_1}}, \dots, F_{\overline{p_n}})$. Recall that $F_{\overline{p_k}}$ has a basis formed by all $v_{i,j}$ such that number in T^{μ}_{p} in the position (i,j) is less than or equal to k. By the construction of the tableau T^{μ}_{p}, if we label those basis vectors $v_{i,j}$ by numbers $l \in \{1, \dots, r\}$ according to which W^{l}_{μ} they belong to, we will get exactly the substring $q_{\mu}[1, \overline{p_k}]$. Thus $F_{\overline{p_k}}$ is of $q_{\mu}[1, \overline{p_k}]$-type, hence $F_{T^{\mu}_{p}} \in C^{\mu}_{p}$.

Now, notice that any shuffling of the entries within the l-th μ-line preserves the count of boxes labelled by the numbers smaller than or equal to k in the l-th μ-line, hence it preserves the number of vectors $v_{i,j}$ in the flag $F_{\overline{p_k}}$ that belongs to W^{l}_{μ}. Thus, given any $T \in \mathrm{Shuffle}(T^{\mu}_{p})$, denoting $F_T := (F_0, F_{\overline{p_1}}, \dots, F_{\overline{p_n}})$, space $F_{\overline{p_k}}$ remains of $q_{\mu}[1, \overline{p_k}]$-type.

Proof of (2). Firstly, given $\mu = (\mu_1, \dots, \mu_r)$ and $p = (p_1, \dots, p_n)$, we divide the string $q_{\mu} = q^1_{\mu} q^2_{\mu} \dots q^n_{\mu}$ into substrings q^j_{μ} of length p_j (see (5.26) for $\mu = (123)$ and $p = (2121)$).

$$q_{123} = 12|2|33|3 \tag{5.26}$$

Denoting by t^i_j the number of occurrences of the number i in q^j_μ, we see that the map

$$C^\mu_p \to \mathcal{B}_{t^1_1,t^1_2,\dots,t^1_n} \times \cdots \times \mathcal{B}_{t^r_1,t^r_2,\dots,t^r_n},\ (F_0, F_{\overline{p_1}}, \dots, F_{\overline{p_n}}) \mapsto ((V^1_0, V^1_1, \dots, V^1_n), \dots, (V^r_0, V^r_1, \dots, V^r_n))$$

is an isomorphism, where $F_{\overline{p_j}} = \oplus_i V^i_j$ and $V^i_j \subset W^i_\mu$. Hence, the rank of the cohomology of C^μ_p is equal to

$$rk(H^*(C^\mu_p)) = \frac{\mu_1!}{t^1_1!t^1_2!\cdots t^1_n!} \cdots \frac{\mu_r!}{t^r_1!t^r_2!\cdots t^r_n!},$$

where we have used the standard formula for the rank of the cohomology of a generalised flag variety $rk(H^*(\mathcal{B}_{p_1,\dots,p_n})) = \frac{(p_1+\cdots+p_n)!}{p_1!\dots p_n!}$. On the other hand, let us divide the string

$$q_p = \underbrace{1\dots1}_{p_1}\underbrace{2\dots2}_{p_2}\dots\underbrace{n\dots n}_{p_n} = q^1_p q^2_p \dots q^r_p$$

into substrings q^i_p of length μ_i, and denote by p^i_j the number of occurrences of the number j in q^i_p. (see (5.27) for $p = (2121)$ and $\mu = (123)$)

$$q_{2121} = 1|12|334 \tag{5.27}$$

Notice then that the number of non-equivalent shuffles of the tableau T^μ_p is exactly

$$|\text{Shuffle}(T^\mu_p)| = \frac{\mu_1!}{p^1_1!p^1_2!\cdots p^1_n!} \cdots \frac{\mu_r!}{p^r_1!p^r_2!\cdots p^r_n!},$$

as the i-th μ-line contains exactly the numbers from q^i_p, hence the number of its shuffles is $\frac{\mu_i!}{p^i_1!p^i_2!\cdots p^i_n}$. Thus, in order to prove claim (2), we just have to prove that $T = \{(t^i_j)^{i=1\dots r}_{j=1\dots n}\}$ and $P = \{(p^i_j)^{i=1\dots r}_{j=1\dots n}\}$ are the same sets.

Let us now divide the strings q_μ and q_p, simultaneously by μ_i-blocks and p_j-blocks (see (5.28) for $p = (2121)$ and $\mu = (123)$, red bars denote μ_i-blocks and blue bar denote p_j-blocks).

$$\begin{aligned} q_{123} &= 1|2|2||33|3 \\ q_{2121} &= 1|1|2||33|4 \end{aligned} \tag{5.28}$$

Thus we obtain divisions of q_μ and q_p into sets of the same cardinalities. It is easy to see that these divisions correspond to the sets T and P respectively. Namely, in the string q_μ we count t^i_j until we reach an end of a chunk divided by either p_j or μ_i (in (5.28): red or blue bar). If it is a p_j block end (in (5.28): blue bar), we start counting t^i_{j+1} in the next chunk, if it is the μ_i block end (in (5.28): red bar) we start counting t^{i+1}_j, and if both p_j and μ_i blocks are ending, we pass to count t^{i+1}_{j+1}. The same applies to the division of q_p. Hence, the sets T and P are the same and thus the claim (2) is proved. ■

Corollary 5.4.27. $(\mathcal{R}_p^\mu)^H = \{F_T \mid T \in \text{Shuffle}(T_p^\mu) \cap \mathbf{Row_p^\lambda}\}$

This corollary will help us in finding the tableau that corresponds to a quasi-Richardson component $\mathcal{R}_p^\mu$.

Definition 5.4.28. Given the tableau T_p^μ, define its **maximisation** $\widetilde{T_p^\mu}$ as

$$\widetilde{T_p^\mu} := \max(\text{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda),$$

under the Brundan-Ostrik order $\preceq$ (Definition 5.4.4) on $\mathbf{Std}_p^\lambda$.

Lemma 5.4.29. *Given $p = (p_1, \ldots, p_m)$, the number m occurs in a tableau $T \in$ Shuffle$(T_p^\mu) \cap \mathbf{Std}_p^\lambda$ only in the last two μ-lines. Moreover, all the occurrences of m in the penultimate μ-line lie strictly below (in terms of row-numbers) the occurrences of m from the last μ-line.*

Proof. Firstly, if m occurs in the last three μ-lines of a tableau $T \in \text{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda$, that means that the whole penultimate μ-line is filled with the number m, but then the box $(1, \lambda_1 - 1)$ is filled with m, thus the box $(1, \lambda_1)$ has to be labelled by a number bigger than m since $T \in \mathbf{Std}_p^\lambda$, and that is impossible.

Thus, m occurs in the last two μ-lines only. If it appears only in the last μ-line we are done. Otherwise, the whole last μ-line is filled with the number m. Assume that there is a box b_p of the penultimate μ-line containing m such that its row number r_p is smaller than the row number r_u of a box b_u of the last μ-line containing m. Then, there is a box in the last μ-line whose row number is r_p, which then is impossible as we would have two boxes filled with m in the same row of T, which contradicts that tableaux in $\mathbf{Std}_p^\lambda$ have strictly increasing rows. ■

Finally, we prove that the tableau $\widetilde{T_p^\mu}$ corresponds to a quasi Richardson component $\mathcal{R}_p^\mu$.

Theorem 5.4.30. *Given a quasi-Richardson component $\mathcal{R}_p^\mu$, its corresponding tableau is exactly $\widetilde{T_p^\mu}$.*

Proof. Firstly, as $\mathcal{R}_p^\mu$ is a component, denoting by $T = \text{Spal}_p^{-1}(\mathcal{R}_p^\mu) \in \mathbf{Std}_p^\lambda$, we have $F_T \in \mathcal{R}_p^\mu$. Due to Corollary 5.4.27, we have $T \in \text{Shuffle}(T_p^\mu)$, thus $T \in \text{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda$. As

$$\{F_S \mid S \in \text{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda\} \subset \mathcal{R}_p^\mu = \mathcal{K}_T,$$

and

$$\mathcal{K}_T \subset \cup_{S \preceq T} \mathcal{F}_S$$

by Theorem 5.4.5, that implies that T is indeed bigger than any other tableau in $\mathrm{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda$, thus it is equal to $\widetilde{T_p^\mu}$. In fact, this proves that the maximum of the set $\mathrm{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda$ exists (recall that $\preceq$ is a partial order only). ■

Remark 5.4.31. Here we remark on how one should get the tableau $\widetilde{T_p^\mu}$ in practice. It should be by the following inductive algorithm:

- Fill the first μ-line in the same way as T_p^μ.
- Fill the second μ-line such that the obtained tableau is the maximal possible amongst the semistandard ones.
- Continue the process in the same way until all μ-lines are filled.

In each step we are maximising the obtained tableau, so this tableau should hopefully be exactly $\widetilde{T_p^\mu}$. We say "should" as the author has not yet proved (but eventually intends to prove) that this tableau is indeed the maximal one in $\mathrm{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda$.

Lemma 5.4.32. *When the tableau T_p^μ is standard, it is equal to $\widetilde{T_p^\mu}$.*

Proof. We will prove this by total induction on the number of boxes in the tableau T_p^μ. Let us take $\mu = (\mu_1, \dots, \mu_r)$ and $p = (p_1, \dots, p_m)$ and consider T_p^μ and an arbitrary tableau $S \in \mathrm{Shuffle}(T_p^\mu) \cap \mathbf{Std}_p^\lambda$. According to Lemma 5.4.29, boxes labelled by the number m only exist in the last two μ-lines of T_p^μ and S. Recall that by $l(T)$ we label the rows of the boxes containing m, and by $\overline{T}$ we label the tableau T without the boxes labelled m. We split the proof into three cases:

1) The boxes labelled by m are in the last μ-line only and they fill it completely. Thus, $l(T_p^\mu) = l(S)$, and the tableaux $\overline{T_p^\mu}$ and $\overline{S}$ are exactly obtained by deleting the last μ-line. Thus, denoting by $\mu' = (\mu_1, \dots, \mu_{r-1})$ and $p' = (p_1, \dots, p_{m-1})$, we have $\overline{T_p^\mu} = T_{p'}^{\mu'}$, hence, by inductive hypothesis, $\overline{S} \preceq \overline{T_p^\mu}$, and so $S \preceq T_p^\mu$.

2) The boxes labelled by m are in the last and in the penultimate μ-line. That implies that the whole last line is labelled by m. Hence, if $l(T_p^\mu) = l(S)$, we can delete all the boxes filled with m and get the tableau $\overline{T_p^\mu} = T_{p'}^{\mu"}$, for $\mu" = (\mu_1, \dots, \mu_{r-1} - N)$ and $p' = (p_1, \dots, p_{m-1})$, where N is the number of boxes labelled by m in the penultimate line. Thus again by inductive hypothesis $\overline{S} \preceq \overline{T_p^\mu}$, and so $S \preceq T_p^\mu$. Otherwise, let us assume that $l(T_p^\mu) < l(S)$. Then, by Lemma 5.4.29 we get that $l(T_{p'}^{\mu'}) < l(S')$ where S' is the tableau S without the last μ-line. Thus, $T_{p'}^{\mu'} \preceq S'$ which is impossible due to the inductive hypothesis. Thus, $l(T_p^\mu) > l(S)$ and so $S \preceq T_p^\mu$.

3) The boxes labelled with m are in the last μ-line only and there are some other numbers in it. Then, since in T_p^μ boxes filled by m are in the last rows, $l(T_p^\mu) \geq l(S)$.

If it is a strict inequality, we are done, otherwise define $p' = (p_1, \ldots, p_{m-1})$ and $\mu' = (\mu_1, \ldots, \mu_r - p_m)$, and by the inductive hypothesis we have $\overline{S} \preceq T^{\mu'}_{p'} = \overline{T^{\mu}_{p}}$, hence $S \preceq T^{\mu}_{p}$. ■

As promised, we now prove claim (4) of Theorem 5.4.15, which we restate here for the convenience of the reader:

Corollary 5.4.33. *Given a Richardson component $\mathcal{R}^{\mu}_{p}$, its corresponding Young tableau is exactly T^{μ}_{p}.*

Proof. As a Richardson component is quasi-Richardson, according to Theorem 5.4.30, its corresponding tableau is $\widetilde{T^{\mu}_{p}}$. However, as $\mu \in Coar(\lambda, p)$ the tableau T^{μ}_{p} is standard, hence by Lemma 5.4.32 is equal to $\widetilde{T^{\mu}_{p}}$. ■

Theorem 5.4.30 motivates the next definitions.

Definition 5.4.34. We say that $\mu \in \mathrm{Perm}(\lambda^*)$ is **p-tableau-shuffle-allowable** if

$$\mathrm{Shuffle}(T^{\mu}_{p}) \cap \mathbf{Std}^{\lambda}_{p} \neq \emptyset,$$

and **p-tableau-allowable** if the tableau T^{μ}_{p} is semistandard. The sets of such permutations we denote by $TSA(\lambda, p)$ and $TA(\lambda, p)$, respectively. Obviously, $Coar(\lambda, p) \subset TA(\lambda, p) \subset TSA(\lambda, p)$.

Example 5.4.35. Let $p = (211110)$ As seen in Example 5.4.25, permutation $\mu = (123)$ is not p-tableau allowable as the tableau

$$T^{123}_{211110} = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}$$

is not semistandard. However, this permutation is p-tableau-shuffle-allowable as the shuffle

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}$$

of T^{123}_{211110} is semistandard.

Thus, in this new notation, Theorem 5.4.30 implies:

Corollary 5.4.36. $QR(\lambda, p) \subset TSA(\lambda, p)$.

We conjecture the contrary, that is

Conjecture 5.4.37. $TSA(\lambda, p) \subset QR(\lambda, p)$.

which would, together with Corollary 5.4.36, mean that quasi-Richardson components correspond **exactly** to the combinatorially-calculable set $\{\widetilde{T_p^\mu} \mid \mu \in TSA(\lambda, p)\}$ of semistandard tableaux.

5.4.6 Further examples

In this section we give some further examples of quasi Richardson and minimal components in generalised Springer fibres.

Firstly, let us give an example of a quasi-Richardson component which is not tableau-allowable.

Example 5.4.38. Continue with Example 5.4.22, thus let $\lambda = (211)$, $p = (2110)$, and $\mu = (13)$. As the tableau

$$T^{13}_{2110} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

is not semistandard, $\mu \notin TA(\lambda, p)$, however as we have seen, μ yields a quasi-Richardson component:

$$\mathcal{R}^{13}_{2110} = \{0 \subset \langle v_1, w_2\rangle \subset \langle v_1, w_2, w_3\rangle \subset \mathbb{C}^4 \mid w_2 \in \langle v_3, v_4\rangle,\ w_2, w_3 \in W^2_\mu \text{ lin. ind.}\}^{12}.$$

By Theorem 5.4.30, we know that its corresponding semistandard tableau is the maximal[13] tableau amongst the shuffled ones:

$$\mathrm{Spal}_p^{-1}(\mathcal{R}^{13}_{2110}) = \widetilde{T^{13}_{2110}} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{5.29}$$

Let us see directly why this equality is true. The condition $eF_2 = 0$ explains the shape $(1, 1)$ of the subtableau of two boxes labelled by 1. Then, as $w_3 \in W^2_\mu = \langle v_2, v_3, v_4\rangle$, we have $ew_3 = \alpha w_1$. Thus, **generically** $\alpha \neq 0$, hence $\langle w_3, w_1\rangle$ form a Jordan 2-block in F_3. As $ew_2 = 0$, this altogether gives $\lambda(e|F_3) = (2, 1)$ which explains the shape $(2, 1)$ of the subtableau of 3 boxes labelled by 1 and 2. Finally, we know that the whole tableau is of shape $\lambda = (2, 1, 1)$, which gives the unique way to add the box 3.

[12] Here lin. ind. stands for linearly independent.
[13] Under the Brundan-Ostrik order $\preceq$ (Definition 5.4.4).

We have seen that Richardson components, being invariant under a parabolic subgroup of GL_n, are invariant under the diagonal maximal torus action (for the basis $v_{i,j}$). In the next example we will see that this is not the case for quasi-Richardson components in general. That explains the distinction between these two sets of components.

Example 5.4.39. Let $\lambda = (32)$, $p = (12110)$ and $\mu = (221)$. Then $\mu \in \text{Good}(\lambda^*)$, hence induces a minimal component. Let us prove that it is quasi-Richardson.

Namely, we have

$$W_\mu^1 = \langle v_1, v_4 \rangle,\ W_\mu^2 = \langle v_2, v_5 \rangle,\ W_\mu^3 = \langle v_3 \rangle$$

and $q_\mu = 11223$ hence similarly to the last example we get

$$C_{12110}^{221} = \{0 \subset \langle \alpha v_1 + \beta v_4 \rangle \subset \langle v_1, v_4, \gamma v_2 + \delta v_5 \rangle \subset \langle v_1, v_2, v_4, v_5 \rangle \subset \mathbb{C}^5\} \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

thus intersecting with $\mathcal{B}_{12110}^{32}$ we get

$$\mathcal{R}_{12110}^{221} = \{0 \subset \langle \alpha v_1 + \beta v_4 \rangle \subset \langle v_1, v_4, \alpha v_2 + \beta v_5 \rangle \subset \langle v_1, v_2, v_4, v_5 \rangle \subset \mathbb{C}^5\} \cong \mathbb{P}^1$$

(since we force $e(\gamma v_2 + \delta v_5) \in \langle \alpha v_1 + \beta v_4 \rangle$, and $ev_2 = v_1$, $ev_5 = v_4$).
Thus $\mathcal{R}_{12110}^{221}$ is a component in $\mathcal{B}_{12110}^{32}$ as it is irreducible and of highest dimension. Hence it is quasi-Richardson, but it is not invariant under the action

$$(t_1, \ldots, t_5) \curvearrowright (v_1, \ldots v_5) = (t_1 v_1, \ldots, t_5 v_5)$$

of the diagonal maximal torus, hence it is not Richardson. One can argue that this is immediately true since $\mu = (221)$ is not coarse, but as we will see in the Example 5.4.40, there might be some other μ' that is coarse, such that $\mathcal{R}_{12110}^{\mu'} = \mathcal{R}_{12110}^{221}$.

In the next example, which is a continuation of the Example 5.4.17, we will see that two quasi-Richardson components can coincide.

Example 5.4.40. Let $\lambda = (321)$ and $p = (211110)$. We have already seen that

$$Coar(321, 211110) = \{213, 231, 312, 321\}$$

hence there are four Richardson components. Since $q_p = 112345$, by Theorem 5.4.15 they respectively correspond to the tableaux

<table>
<tr><td>1</td><td>2</td><td>3</td><td></td><td>1</td><td>2</td><td>5</td><td></td><td>1</td><td>3</td><td>4</td><td></td><td>1</td><td>3</td><td>5</td></tr>
<tr><td>1</td><td>4</td><td></td><td></td><td>1</td><td>3</td><td></td><td></td><td>1</td><td>5</td><td></td><td></td><td>1</td><td>4</td><td></td></tr>
<tr><td>5</td><td></td><td></td><td></td><td>4</td><td></td><td></td><td></td><td>2</td><td></td><td></td><td></td><td>2</td><td></td><td></td></tr>
</table>

Calculating as in the earlier examples, we get that the remaining two permutations $\mu' = (123)$ and $\mu'' = (132)$ yield quasi-Richardson components

$$\mathcal{R}^{123}_{211110} = \{0 \subset \langle v_1, v_4\rangle \subset \langle v_1, v_2, v_4\rangle \subset \langle v_1, v_2, v_4, V_1\rangle \subset \langle v_1, v_2, v_4, V_2\rangle \subset \mathbb{C}^6 \mid \\ \text{where } 0 \subset V_1 \subset V_2 \subset \langle v_3, v_5, v_6\rangle \text{ is a full flag}\}$$

$$\mathcal{R}^{132}_{211110} = \{0 \subset \langle v_1, \alpha v_4 + \beta v_6\rangle \subset \langle v_1, \alpha v_4 + \beta v_6, \mu v_2 + \lambda(-\bar{b} v_4 + \bar{\alpha} v_6)\rangle \subset \\ \subset \langle v_1, v_2, v_4, v_6\rangle \subset \langle v_1, v_2, v_4, v_6, \alpha v_3 + \beta v_5\rangle \subset \mathbb{C}^6\}$$

thus, either directly or by Theorem 5.4.30 and Example 5.4.25 (and observing that there shuffled tableaux therein are indeed the maximal ones), one gets that their corresponding tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 1 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

respectively. Hence, we see that the two quasi-Richardson components $\mathcal{R}^{213}_{211110} = \mathcal{R}^{123}_{211110}$ coincide.

Also, we see that here $QR(321, 211110) = \text{Perm}(321^*)$, hence by Proposition 5.4.23 all minimal components, labelled by

$$\text{Good}(321^*) = \{123, 132, 231, 321\},$$

are quasi-Richardson as well. Hence, in total, we get that 5 out of 9 components of $\mathcal{B}^{321}_{211110}$ are quasi-Richardson, out of which two different quadruples are Richardson and minimal.

All examples so far suggest that minimal components are quasi-Richardson as well. That is however not true, as we see in the following last example.

Example 5.4.41. Let $\lambda = (311)$, $p = (11120)$ and $\mu = (311)$. Then $\mu \in \text{Good}(\lambda^*)$, thus there is a minimal component attached to it.

We have

$$W^1_\mu = \langle v_1, v_4, v_5\rangle, W^2_\mu = \langle v_2\rangle, W^3_\mu = \langle v_3\rangle$$

and $q_\mu = (11123)$ hence

$$C^{311}_{11120} = \{0 \subset V_1 \subset V_2 \subset \langle v_1, v_4, v_5\rangle \subset \mathbb{C}^5 \mid 0 \subset V_1 \subset V_2 \subset \langle v_1, v_4, v_5\rangle \text{ is a complete flag}\} \cong \mathcal{B}_{111}.$$

Intersecting with $\mathcal{B}^{311}_{11120}$ we get an empty set since $e_\lambda\mathbb{C}^5 = \langle v_1, v_2\rangle \not\subset \langle v_1, v_4, v_5\rangle$, thus

$$\mathcal{R}^{311}_{11120} = \emptyset.$$

On the other hand, $\mu = (311)$ yields the minimal component, which we compute directly. Recall by Corollary 5.3.11 that the action $\mathbb{C}^*_\mu$ on $\mathcal{B}^\lambda_p$ that fixes $\mathfrak{F}^\mu_p$ has the weights on the Jordan basis $\langle v_1, \ldots, v_5\rangle$ of e given by the weight tableau

$$T_{wt}(311) = \begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 0 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}.$$

In particular, we see that triple v_1, v_4, v_5 have the same weights.

Now, consider the variety of flags

$$C^{11213}_{11120} = \{0 \subset F_1 \subset F_2 \subset F_3 \subset \mathbb{C}^5\}$$

defined in the same way as C^{311}_{11120}, with a difference that in it, space $F_{\overline{p_i}}$ is of $\alpha'[1, \overline{p_i}]$-type, for $\alpha' = 11213$. That means that F_1 and F_2 lie in $W^1_\mu = \langle v_1, v_4, v_5$, whereas F_3 has, in addition, one generator in W^2_μ. That gives us the following presentation

$$C^{11213}_{11120} = \{0 \subset V_1 \subset V_2 \subset V_2, v_2 \subset \mathbb{C}^5 \mid 0 \subset V_1 \subset V_2 \subset \langle v_1, v_4, v_5\rangle \text{ is a complete flag}\}$$

This variety is fixed by the $\mathbb{C}^*_\mu$-action on $\mathcal{B}_p$, for the same reasons as variety C^{311}_{11120}. Thus, intersecting it with $\mathcal{B}^\lambda_p$ we get a $\mathbb{C}^*_\mu$-fixed locus. Given a flag $0 \subset F_1 \subset F_2 \subset F_3 \subset \mathbb{C}^5$ in the intersection, the conditions of Springer fibre give $\langle v_1, v_2\rangle = e\mathbb{C}^5 \subset F_3$, thus $v_1 = ev_2 \subset eF_3 \subset F_2$, hence we get

$$C^{11213}_{11120} \cap \mathcal{B}^\lambda_p = \{0 \subset \langle \lambda v_1 + \mu(\alpha v_4 + \beta v_5)\rangle \subset \langle v_1, \alpha v_4 + \beta v_5\rangle \subset \langle v_1, v_2, \alpha v_4 + \beta v_5\rangle \subset \mathbb{C}^5\}.$$

We see that this is a smooth irreducible 2-dimensional, hence maximal-dimensional subvariety in $\mathcal{B}^\lambda_p$. As it is also fixed by the $\mathbb{C}^*_\mu$-action, it is indeed the minimal component $C^{11213}_{11120} \cap \mathcal{B}^\lambda_p = \mathfrak{F}^{311}_{11120}$. Thus, according to Example 5.4.2 this component corresponds to the tableau

$$\mathrm{Spal}_p^{-1}(\mathfrak{F}^{311}_{11120}) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

One can check that the other two permutations $\mu = (113)$ and $\mu = (131)$ yield two different quasi-Richardson components that correspond to the tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

respectively. Thus we see that the component $\mathfrak{F}^{311}_{11120}$ is minimal but not quasi-Richardson.

5.4.7 Further remarks

In this section we give some further remarks on the topic of smooth components in generalised Springer fibres, including some possible future work on it.

Firstly, let us delve further on the way we have obtained the non-quasi-Richardson minimal component in Example 5.4.41. Given a $\mathbb{C}^*_\mu$-action on $\mathcal{B}_p$, the set $\mathcal{B}_p^{\mathbb{C}^*_\mu}$ of its fixed points is a disjoint union

$$\mathcal{B}_p^{\mathbb{C}^*_\mu} = \bigsqcup_{\alpha \in \mathrm{Perm}(q_\mu)/\mathrm{Shuffle}(p)} C_p^\alpha$$

of connected components labelled by permutations $\mathrm{Perm}(q_\mu)$ of the string q_μ, up to shuffling in $[p_i, p_{i+1}]$ substrings. The set C_p^μ is one of them, for $\alpha = q_\mu$. When $\mu \in \mathrm{Good}(\lambda^*)$, fixed points of the $\mathbb{C}^*_\mu$-action on $\mathcal{B}_p^\lambda$ are exactly the intersections of the sets C_p^α with $\mathcal{B}_p^\lambda$. As we have seen in the last example, $\mathcal{R}_p^\mu = C_p^\mu \cap \mathcal{B}_p^\lambda$ can be empty, but then some other $\alpha' = \alpha'(\mu) \in \mathrm{Perm}(q_\mu)$ **has** to yield the minimal component $\mathfrak{F}_p^\mu = C_p^{\alpha'} \cap \mathcal{B}_p^\lambda$. In Example 5.4.41, that was $\alpha' = 11213$. Using an analogue of Proposition 5.4.26, one can prove that the corresponding semistandard tableau of the smooth component $C_p^\alpha \cap \mathcal{B}_p^\lambda$ is the tableau obtained from $\widetilde{T_p^\mu}$ upon permutation of boxes in the same way that α permutes q_μ. For instance, in Example 5.4.41, we see that

$$T^{311}_{11120} = \widetilde{T^{311}_{11120}} = \begin{array}{|c|c|c|}\hline 1^{1} & 4^{2} & 4^{3} \\ \hline 2^{1} \\ \cline{1-1} 3^{1} \\ \cline{1-1} \end{array} \quad \text{and } \mathrm{Spal}^{-1}(C^{11213}_{11120} \cap \mathcal{B}_p^\lambda) = \begin{array}{|c|c|c|}\hline 1^{1} & 3^{1} & 4^{3} \\ \hline 2^{1} \\ \cline{1-1} 4^{2} \\ \cline{1-1} \end{array} \tag{5.30}$$

(with the green numbers we label the strings q_μ and α', respectively).

In fact, it could be true that for every $\mu \in \mathrm{Perm}(\lambda^*)$ there is a smooth component in $\mathcal{B}_p^\lambda$ obtained as an intersection $C_p^\alpha \cap \mathcal{B}_p^\lambda$ for some $\alpha = \alpha(\mu) \in \mathrm{Perm}(q_\mu)$. In that case these components would generalise both quasi-Richardson and minimal ones, hence they would be of interest to us, but we will leave this for future work.

So, to sum up, in an arbitrary generalised Springer fibre $\mathcal{B}_p^\lambda$ we have two families of smooth components, minimal components that are labelled by the set Good(λ^*) and quasi-Richardson that are labelled by the set $QR(\lambda, p)$. We have that $QR(\lambda, p) \subset TSA(\lambda, p)$ and conjecturally these sets are equal. Also, there is a set $Coar(\lambda, p)$ that parametrises Richardson components. In Figure 5.3 we illustrate the relations between these sets in the general and in the ordinary case. The set $QR(\lambda, p) \subset TSA(\lambda, p)$ is illustrated as a dotted circle due to Conjecture 5.4.37.

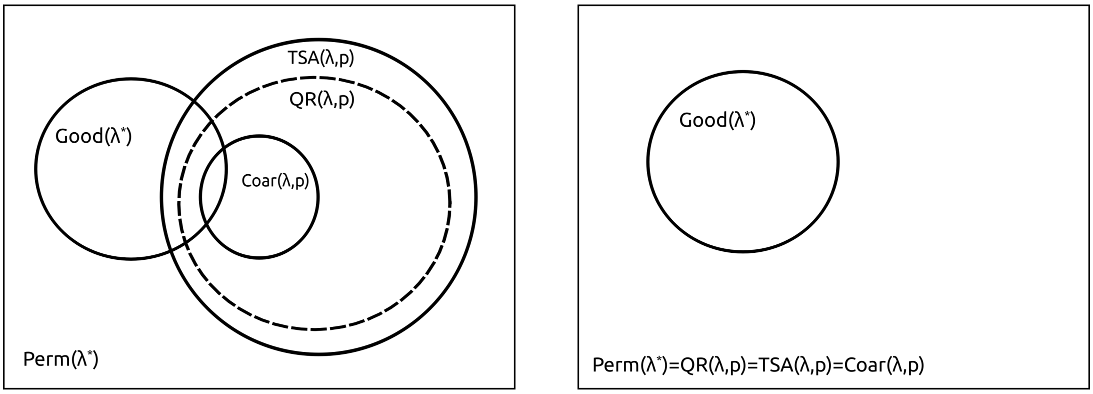


Figure 5.3: Different labels of smooth components in generalised (left) and in ordinary Springer fibres (right)

From these families we can get many more smooth components by *crystal operators*, as we shall see in the next section. On the other hand, one can try the approach of *adjacent tableaux* as in [PaRe06]. Namely, given a standard tableau T one can define its k-adjacent tableau $T(k)$ by swapping the numbers k and $k-1$. In [PaRe06], the authors prove that, given the tableau T^μ of a Richardson component, its k-adjacent tableau $T^\mu(k)$ corresponds to a smooth component when $T^\mu \preceq T^\mu(k)$.[14] The author plans to generalise this approach for Richardson components in generalised Springer fibres in some future work.

5.5 More smooth components from crystal operators

In this section we enhance the results from last two sections using the so-called *crystal operators* between irreducible components of different Springer fibres. By this method

[14]Where $\preceq$ denotes the Brundan-Ostrik order (Section 5.4.1).

we generate more smooth components of generalised Springer fibres $\mathcal{B}_p^\lambda$ out of existing ones (minimal, quasi-Richardson), as we prove that these operators often preserve non-singularity.

In Section 5.5.1 we recall the definition and facts about crystal operators that we will use in this section. Next, in Section 5.5.2 we find a couple of conditions under which crystal maps preserve non-singularity of components, by understanding that topologically these maps are Grassmann bundles. Finally, in Section 5.5.3 we give some remarks on where one could possibly go by using the statements proved in Section 5.5.2.

5.5.1 Background on crystal operators

Crystals were discovered by Kashiwara [Kas90, Kas91, Kas94], by looking at the representations of Drinfeld-Jimbo's quantized universal enveloping algebra $U_q(\mathfrak{g})$, also known as quantum group [Dr85, Ji85]. Namely, by in the limit $q \to 0$, a representation becomes convenient, admitting a base (called a crystal base) such that such that generators of quantum group have a particularly simple action on it. This construction was motivate by physics, namely the quantum group was introduced in order to explain trigonometric R-matrices in 2-dimensional solvable models in statistic mechanics, where the parameter q is the temperature parameter. Now, one would expect that at the absolute zero things get simplified, which is exactly what crystal base gives. The $U_q(\mathfrak{g})$-module structure induces a combinatorial structure on the crystal bases called the crystal graph. This combinatorial gadget allows one to reduce problems in representation theory to the combinatorial problems. By abstracting the combinatorial aspects of crystal bases, one comes up with a notion of a crystal.

Let us now explain how crystals connect to Springer fibres, which is our main interest. In [Nak98] Nakajima constructed certain operators that interchange between core-components of different quiver varieties, by generalising certain Lagrangian correspondences. Saito later proved [Sai02] that these operators together with irreducible components of cores of quiver varieties constitute crystals of the highest weight modules. In later years, Savage in [Sa06] gave the description of these crystal operators on the Springer side, using the correspondence between Springer fibres and (cores of) quiver varieties of type A given by Maffei [Maf05]. We briefly summarise this description.

As in the earlier sections, let $e_\lambda \in \mathfrak{sl}_n$ be a nilpotent element with Jordan partition λ, and $p = (p_1, \ldots, p_n)$ a composition of n. Recall that $F \in \mathcal{B}_p^\lambda$ is a partial flag, satisfying $e_\lambda F_k \subset F_{k-1}$. We now consider the possible vector subspaces $S \subset F_{k+1}$

obtained by enlarging F_k (for a fixed k) but which still satisfy $e_\lambda S \subset F_{k-1}$. We also fix the difference c between the dimensions of S and F_k. The variety that parametrises these objects is

$$\mathcal{B}_p^\lambda(k,c) := \{(F,S) \mid F \in \mathcal{B}_p^\lambda,\ F_k \subset S \subset F_{k+1} \cap e_\lambda^{-1}(F_{k-1}),\ \dim S/F_k = c\}. \quad (5.31)$$

This variety comes with two forgetful maps π_1, π_2 which respectively forget F_k and S :

$$\mathcal{B}_{p^{k,c}}^\lambda \xleftarrow{\pi_1} \mathcal{B}_p^\lambda(k,c) \xrightarrow{\pi_2} \mathcal{B}_p^\lambda,$$
$$(F_1,..,F_{k-1},S,F_{k+1},\dots,F_n) \overset{\pi_1}{\hookleftarrow} (F_1,\dots,F_n,S) \overset{\pi_2}{\mapsto} (F_1,\dots,F_n).$$

Here, $p^{k,c} := (p_1,\dots,p_{k-1},p_k+c,p_{k+1}-c,p_{k+2},\dots,p_n)$.

In order to understand these projections better, we make a stratification of $\mathcal{B}_p^\lambda$ in the following way.

Definition 5.5.1. Given a partial flag $F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n) \in \mathcal{B}_p^\lambda$, let

$$\varepsilon_k(F) := \dim(F_{k+1} \cap e_\lambda^{-1}(F_{k-1})) - \dim F_k.$$

Then, define

$$(\mathcal{B}_p^\lambda)_{k,c} := \{F \in \mathcal{B}_p^\lambda \mid \varepsilon_k(F) = c\}.$$

Denote by $Gr(k,n)$ the Grassmann manifold of k-planes in $\mathbb{C}^n$. By **Grassmann bundle** we mean any fibre bundle whose fibre is isomorphic to a Grassmann manifold. We have the following:

Theorem 5.5.2. *(Nakajima-Saito-Savage)*

(1) For an arbitrary $m \in \mathbb{N}_0$, $\bigcup_{k\le m}(\mathcal{B}_p^\lambda)_{k,c}$ is an open subset of $\mathcal{B}_p^\lambda$. Hence, $(\mathcal{B}_p^\lambda)_{k,0} \subset \mathcal{B}_p^\lambda$ is open.

(2) $\varepsilon_k(\pi_1(\pi_2^{-1}(F))) = \varepsilon_k(F) - c$.

(3) $\pi_1^{-1}((\mathcal{B}_{p^{k,c}}^\lambda)_{k,0}) = \pi_2^{-1}((\mathcal{B}_p^\lambda)_{k,c}) := Z_p^\lambda(k,c)$.

(4) Restricted to $Z_p^\lambda(k,c)$, the map π_2 is an isomorphism, and the map π_1 is a Grassmann bundle with fibre isomorphic to $Gr(c, p_k - p_{k+1} + 2c)$.

(5) Thus, the composition $\pi_1 \circ \pi_2^{-1} : (\mathcal{B}_p^\lambda)_{k,c} \to (\mathcal{B}_{p^{k,c}}^\lambda)_{k,0}$ is a $Gr(c, p_k - p_{k+1} + 2c)$-fibre bundle.

(6) Suppose $(\mathcal{B}_p^\lambda)_{k,c} \neq \emptyset$. Then there is a 1-1 correspondence between the set of irreducible components of $(\mathcal{B}_{p^{k,c}}^\lambda)_{k,0}$ and the set of irreducible components of $(\mathcal{B}_p^\lambda)_{k,c}$.

Denoting by $\mathrm{Irr}(p,\lambda)$[15] the set of irreducible components in $\mathcal{B}_p^\lambda$, let $\mathrm{Irr}(\lambda) := \bigsqcup_p \mathrm{Irr}(p,\lambda)$. We will be using the following notation often, so we emphasize it via a separate definition:

Definition 5.5.3. for $X \in \mathrm{Irr}(p,\lambda)$ define $\varepsilon_k(X) := \varepsilon_k(F)$ for a generic flag $F \in X$.

Hence, $\varepsilon_k(X) \leq p_{k+1}$. Then for $c \in \mathbb{Z}_{\geq 0}$ define

$$\mathrm{Irr}(p,\lambda)_{k,c} := \{X \in \mathrm{Irr}(p,\lambda) \mid \varepsilon_k(X) = c\}.$$

Thus, by Proposition 5.5.2, we have

$$\mathrm{Irr}(p^{k,c},\lambda)_{k,0} \cong \mathrm{Irr}(p,\lambda)_{k,c}.$$

Let us explain this in more details, for the better understanding of the following text in this section.

Let $X \in \mathrm{Irr}(p,\lambda)_{k,c}$. Hence, there is an open subset $Y_{k,c} \subset X$ whose each flag satisfies $\varepsilon_k(F) = c$. This means by definition that $\dim(F_{k+1} \cap e_\lambda^{-1}(F_{k-1})) - \dim F_k = c$, thus the only S satisfying (5.31) is

$$S = F_{k+1} \cap e_\lambda^{-1}(F_{k-1}),$$

and so the map π_2 is an isomorphism on $Y_{k,c}$. Notice that the flag

$$F' = \pi_1(\pi_2^{-1}(F)) = (F_1, .., F_{k-1}, F_{k+1} \cap e_\lambda^{-1}(F_{k-1}), F_{k+1}, \dots, F_n)$$

satisfies $\varepsilon_k(F') = 0$ by definition.

By part (5) of the theorem above, the composition

$$\pi_1 \circ \pi_2^{-1} : Y_{k,c} \to \pi_1(\pi_2^{-1}(Y_{k,c}))$$

is a $Gr(c, p_k - p_{k+1} + 2c)$-fibre bundle. Using the dimension formula for generalised Springer fibres given in Theorem 5.4.3(2), one gets that the dimension of irreducible variety $\pi_1(\pi_2^{-1}(Y_{k,c}))$ is maximal in $\mathcal{B}_{p^{k,c}}^\lambda$, thus its closure is an irreducible component. We denote it by

$$\widetilde{E_k}^c(X) := \overline{\pi_1(\pi_2^{-1}(Y_{k,c}))} \in \mathrm{Irr}(p^{k,c},\lambda).$$

[15] NB In the original paper [Sa06], the notation for this, and the analogous sets is $\mathcal{B}(p,\lambda)$, but we use this one in order to make it more convenient for the reader.

In fact, by the argument above, $\varepsilon_k(\pi_1(\pi_2^{-1}(Y_{k,c}))) = 0$, hence $\widetilde{E_k}^c(X) \in \mathrm{Irr}(p^{k,c}, \lambda)_{k,0}$. This algorithm is reversible, hence we get mutually-inverse maps:

$$\widetilde{F_k}^c : \mathrm{Irr}(p^{k,c}, \lambda)_{k,0} \to \mathrm{Irr}(p, \lambda)_{k,c}, \quad \widetilde{F_k}^c(\bar{X}) = X,$$
$$\widetilde{E_k}^c : \mathrm{Irr}(p, \lambda)_{k,c} \to \mathrm{Irr}(p^{k,c}, \lambda)_{k,0}, \quad \widetilde{E_k}^c(X) = \bar{X}.$$

Definition 5.5.4. We define the **crystal operators** $\widetilde{E_k}, \widetilde{F_k} : \mathrm{Irr}(\lambda) \to \mathrm{Irr}(\lambda) \sqcup \{0\}$ by

$$\widetilde{E_k} : \mathrm{Irr}(p, \lambda)_{k,c} \xrightarrow{\widetilde{E_k}^c} \mathrm{Irr}(p^{k,c}, \lambda)_{k,0} \xrightarrow{\widetilde{F_k}^{c-1}} \mathrm{Irr}(p^{k,1}, \lambda)_{k,c-1},$$
$$\widetilde{F_k} : \mathrm{Irr}(p, \lambda)_{k,c} \xrightarrow{\widetilde{E_k}^c} \mathrm{Irr}(p^{k,c}, \lambda)_{k,0} \xrightarrow{\widetilde{F_k}^{c+1}} \mathrm{Irr}(p^{k,-1}, \lambda)_{k,c+1}.$$

We set $\widetilde{E_k}(X) = 0$ for $X \in \mathrm{Irr}(p, \lambda)_{k,0}$ and $\widetilde{F_k}(X) = 0$ for $X \in \mathrm{Irr}(p, \lambda)_{k,c}$ with $\mathrm{Irr}(p, \lambda)_{k,c+1} = \emptyset$. It is clear from their definition that $\widetilde{E_k}$ and $\widetilde{F_k}$ are inverse to each other, and that

$$\varepsilon_k(\widetilde{E_k}X) = \varepsilon_k(X) - 1, \quad \varepsilon_k(\widetilde{F_k}X) = \varepsilon_k(X) + 1. \tag{5.32}$$

whenever $\widetilde{E_k}X$ and $\widetilde{F_k}X$ are non-zero.

The following lemma, which follows easily from the definitions, justifies the notation $\widetilde{E_k}^c$, since for low values of c it agrees with the composition of c copies of the operator $\widetilde{E_k}$.

Lemma 5.5.5. *Given an irreducible component $Y \in \mathcal{B}_p^\lambda$ with $\varepsilon_k(Y) = r$, the components $Y, \widetilde{E_k}Y, \widetilde{E_k}^2Y, \dots, \widetilde{E_k}^rY$ are all non-zero and $\widetilde{E_k}^j = \underbrace{\widetilde{E_k} \circ \dots \circ \widetilde{E_k}}_{j}$, for all $j = 1, \dots, r$.*

In order to compute crystal operators in practice, we use their description via Young tableaux, given by Kashiwara in [Kas95, p.18][16].

Definition 5.5.6. Given a semistandard tableau $T \in \mathbf{Std}_p^\lambda$, the crystal operators $\widetilde{E_k}, \widetilde{F_k}$ on it are combinatorially described as follows. Let us make a string of entries of T, written row by row, starting from the last row upwards. For example

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \mapsto 3 \otimes 4 \otimes 1 \otimes 2$$

Now, having such a string $a_1 \otimes a_2 \otimes \dots \otimes a_N$, we do the following procedure:

[16]NB for the reader: With a difference from that paper, we use the row-standard instead from column-standard notation for tableaux.

- Take out all numbers a_i whose value is not k or $k+1$
- Take out all appearances of $k \otimes (k+1)$
- at the end of the first two steps, we are left with a string of the form

$$(k+1) \otimes (k+1) \otimes \cdots \otimes (k+1) \otimes k \otimes k \cdots \otimes k$$

  Operator $\widetilde{E_k}$ changes the rightmost $k+1$ to k while $\widetilde{F_k}$ changes the leftmost k to $k+1$. Then the tableaux $\widetilde{E_k}T$ and $\widetilde{F_k}T$ are defined by changing the value in the corresponding boxes in the same way. If such $k+1$ or k do not exist, we define $\widetilde{E_k}T = 0$ or $\widetilde{F_k}T = 0$, respectively.

Thus, in the above example, the operator $\widetilde{E_2}$ does

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \mapsto 3 \otimes 4 \otimes 1 \otimes 2 \mapsto 3 \otimes 2 \mapsto 2 \otimes 2.$$

Hence, the obtained tableau is $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \stackrel{\widetilde{E_2}}{\longmapsto} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array}$.

Thus, denoting $\mathbf{Std}(\lambda) := \sqcup_p \mathbf{Std}_p^\lambda$, we obtain the **tableaux crystal operators** $\widetilde{E_k}, \widetilde{F_k} : \mathbf{Std}(\lambda) \to \mathbf{Std}(\lambda) \sqcup \{0\}$.

Now, in order to use these combinatorial operators to calculate the geometrical ones, we need the following proposition:

Proposition 5.5.7. *Given a semistandard Young tableau $T \in \mathcal{B}_p^\lambda$, the tableaux $\widetilde{E_k}T$ and $\widetilde{F_k}T$ correspond to the components $\widetilde{E_k}\mathcal{K}_T$ and $\widetilde{F_k}\mathcal{K}_T$, respectively. In particular, when one of them is equal to zero, the other is as well.*

Proof. The crystal structure on components of Spaltenstein varieties recovered by Savage in [Sa06] was first explained by Malkin in [Mal02]. To see that the map $\mathcal{K}_T \mapsto T$ is compatible with the crystal operators, it is convenient to use the description of the crystal operators as given in [Mal02]: There the operators are induced from the rank one case (or $\mathfrak{gl}_2$-case in the terminology of that paper). Since the crystal structure on tableaux is similarly deduced from the rank two case (i.e. one throws away all labels except k and $k+1$ when defining the action of $\widetilde{E_k}$ and $\widetilde{F_k}$. But then the description of the operators in the $\mathfrak{gl}_2$-case in [Mal02, Sec. 2.2] immediately completes the proof. ∎

From now on, we will use the same letters $\widetilde{E_k}$ and $\widetilde{F_k}$ for maps between irreducible components and for maps between their corresponding tableaux.

5.5.2 Equisingularity via crystal operators

In this section we obtain results on when different components of Springer fibres related by crystal operators are **equisingular**, meaning: either both are singular, or both are smooth. Thus the smooth components from our previous sections will yield new smooth components.

Definition 5.5.8. We say that two irreducible components of (possibly different) generalised Springer fibres are **equisingular**[17] if they are either both smooth or both singular. Moreover, we say that two semistandard tableaux are equisingular if their corresponding components are so.

Definition 5.5.9. We say that the component $X \in \mathcal{B}(p, \lambda)$ is ε_k**-homogeneous** if $\varepsilon_k(F)$ is constant for $F \in X$.

Proposition 5.5.10. *Let X be an ε_k-homogeneous component with $\varepsilon_k(X) = r$. Then the components*

$$X, \widetilde{E_k}X, \widetilde{E_k}^2 X, \ldots, \widetilde{E_k}^r X, \widetilde{F_k}X, \widetilde{F_k}^2 X, \ldots$$

are all equisingular.

Proof. By Proposition 5.5.2(5), we have a Grassmann bundle

$$X_{k,r} \to (\widetilde{E_k}^r X)_{k,0},$$

where for an irreducible component Y we denote $Y_{k,c} := \{F \in X \mid \varepsilon_k(F) = c\}$. Since X is ε_k-homogeneous, $X = X_{k,r}$. By Proposition 5.5.2(1), the set $(\widetilde{E_k}^r X)_{k,0}$ is open in $\widetilde{E_k}^r X$. But also, it is compact, being an image of a projective variety X. Hence, $(\widetilde{E_k}^r X)_{k,0} \subset \widetilde{E_k}^r X$ is closed and thus being open and closed, needs to be equal to $\widetilde{E_k}^r X$. Thus, we get a Grassmann bundle $X \to \widetilde{E_k}^r X$, and hence X and $\widetilde{E_k}^r X$ are equisingular.

Now, given an arbitrary $j \in \{1, \ldots, r\}$, by Lemma 5.5.5 the component $\widetilde{E_k}^{r-j} X$ has $\varepsilon_k = j$ and $\widetilde{E_k}^j (\widetilde{E_k}^{r-j} X) = \widetilde{E_k}^r X$. Thus, there is a Grassmann bundle $(\widetilde{E_k}^{r-j} X)_{k,j} \to (\widetilde{E_k}^r X)_{k,0}$, and since $(\widetilde{E_k}^r X)_{k,0} = \widetilde{E_k}^r X$ is projective, we get that $(\widetilde{E_k}^{r-j} X)_{k,j}$ is projective as well. Thus, it is a closed and dense (as $\varepsilon_k(\widetilde{E_k}^{r-j} X) = j$) subset of $\widetilde{E_k}^{r-j} X$, hence must be equal to it. Thus the Grassmann bundle becomes $\widetilde{E_k}^{r-j} X \to \widetilde{E_k}^r X$, and hence $\widetilde{E_k}^{r-j} X$ and $\widetilde{E_k}^r X$ are equisingular.

The same proof works for a component $\widetilde{F_k}^s X$, when it is non-zero. Namely, then it has $\varepsilon_k = r + s$ and $\widetilde{E_k}^{r+s} (\widetilde{F_k}^s X) = \widetilde{E_k}^r$ hence observe the Grassmann bundle

[17]Not to be confused with other meanings of the word 'equisingular'

$(\widetilde{F_k}^s X)_{k,r+s} \to (\widetilde{E_k}^r X)_{k,0} = \widetilde{E_k}^r X$ and conclude as above that $(\widetilde{F_k}^s X)_{k,r+s} = \widetilde{F_k}^s X$, thus by Grassmann bundle it is equisingular to $\widetilde{E_k}^r X$. ■

Corollary 5.5.11. *Assume that $X \in \mathcal{B}(p,\lambda)$ has $\varepsilon_k(X) = p_{k+1}$. Then the components*

$$X, \widetilde{E_k}X, \widetilde{E_k}^2 X, \ldots, \widetilde{E_k}^{p_{k+1}} X, \widetilde{F_k}X, \widetilde{F_k}^2 X, \ldots$$

are all equisingular.

Proof. As $\varepsilon_k(F) \leq p_{k+1}$ for any p-partial flag F, p_{k+1} is the maximum value for ε_k on $X \in \mathcal{B}(p,\lambda)$. Hence, by Proposition 5.5.2(1), we get that the set $X_{k,p_{k+1}}$ is closed in X. As we have $\varepsilon_k(X) = p_{k+1}$, that means that this $X_{k,p_{k+1}}$ is a dense subset of X, hence altogether, it needs to be the whole component X. Thus, X is ε_k-homogeneous, and the statement follows from Proposition 5.5.10. ■

The next proposition shows that this condition is satisfied in a large family of examples.

Proposition 5.5.12. *Let $m \in \mathbb{N}$, and suppose X is a component in whose tableau all appearances of $1, \ldots, m$ occur in the first column. Then $\varepsilon_k(X) = p_{k+1}$ for $k = 1, \ldots, m-1$.*

Proof. We have $\varepsilon_k(F) = \dim(F_{k+1} \cap e_\lambda^{-1}(F_{k-1})) - \dim F_k$. As in the tableau of X all the numbers $1, \ldots, m$ appear in the first column, for a generic flag $F \in X$ we have that $e_\lambda(F_k) = 0$ for $k = 1, \ldots, m$. Thus, for $k = 1, \ldots, m-1$, we have $F_{k+1} \subset e_\lambda^{-1}(0) \subset e_\lambda^{-1}(F_{k-1})$, hence

$$\varepsilon_k(F) = \dim(F_{k+1} \cap e_\lambda^{-1}(F_{k-1})) - \dim F_k = \dim F_{k+1} - \dim F_k = p_{k+1}.$$

As this holds for generic F, we get that $\varepsilon_k(X) = p_{k+1}$. ■

In the next proposition we show that minimal components induce further minimal components via the $\widetilde{E_k}^c$ maps.

Proposition 5.5.13. *Given a minimal component $X \in \mathcal{B}_p^\lambda$ that has $\varepsilon_k(X) = r$, the component $\widetilde{E_k}^r X$ is also minimal.*

Proof. Basically the idea is to prove that the induced component $\widetilde{E_k}^r X$ is fixed under the same action that fixes the component X. Given a minimal $X \in \mathcal{B}_p^\lambda$, we know that there is $\mu \in \mathrm{Perm}(\lambda^*)$ such that the action $\mathbb{C}_\mu^*$ on $\mathcal{B}_p^\lambda$ fixes X. Hence, any flag

$F = (F_1, \dots, F_n) \in X$ is fixed under this action, thus the subspaces F_i are $\mathbb{C}^*_\mu$-invariant. Now, we have the Grassmann bundle

$$X_{k,r} \to (\widetilde{E_k}^r X)_{k,0}, \quad (F_1, \dots, F_n) \mapsto (F_1, \dots, F_{k-1}, F_{k+1} \cap e_\lambda^{-1}(F_{k-1}), F_{k+1}, \dots, F_n),$$

hence it suffices to prove that the subspace $F_{k+1} \cap e_\lambda^{-1}(F_{k-1})$ is $\mathbb{C}^*_\mu$-invariant, as then the set $(\widetilde{E_k}^r X)_{k,0}$ is fixed under $\mathbb{C}^*_\mu$-action and so is its closure $\widetilde{E_k}^r X$. Thus, it is a minimal component.

Now, the subspace $F_{k+1} \cap e_\lambda^{-1}(F_{k-1})$ is $\mathbb{C}^*_\mu$-invariant as the subspace $e_\lambda^{-1}(F_{k-1})$ itself is $\mathbb{C}^*_\mu$-invariant. Namely, let $v \in e_\lambda^{-1}(F_{k-1})$. Then $e_\lambda v \in F_{k-1}$ and thus $t \cdot e_\lambda v \in F_{k-1}$, where

$$t \cdot u = Q_\mu t^{-h_\lambda} u,$$

denotes the $\mathbb{C}^*_\mu$-action, and $Q_\mu \leq Z_\lambda$ is a 1-parameter subgroup that twists the Kazhdan action. As $[h_\lambda, e_\lambda] = 2e_\lambda$, we get $t^{h_\lambda} e_\lambda t^{-h_\lambda} = t^2 e_\lambda$. Therefore,

$$\begin{aligned} t \cdot e_\lambda v =& Q_\mu t^{-h_\lambda} e_\lambda v = Q_\mu t^{-2} e_\lambda t^{-h_\lambda} v = t^{-2} Q_\mu e_\lambda t^{-h_\lambda} v \\ =& t^{-2} e_\lambda Q_\mu t^{-h_\lambda} v = t^{-2} e_\lambda t \cdot v. \end{aligned}$$

By assumption, this is in F_{k-1}, so $e_\lambda t \cdot v \in F_{k-1}$ as well, which gives $t \cdot v \in e_\lambda^{-1}(F_{k-1})$. Thus, $e_\lambda^{-1}(F_{k-1})$ is $\mathbb{C}^*_\mu$-invariant and the proposition is proved. ■

Remark 5.5.14. The same proof almost works for quasi-Richardson components. Given a QR component $\mathcal{R}^\mu_p$, by the same argument we would get that $\widetilde{E_k}^r \mathcal{R}^\mu_p$ is fixed under the $\mathbb{C}^*_\mu$-action but there is no guarantee that it is smooth, hence a quasi-Richardson component. At least this could be guaranteed if the starting QR component $\mathcal{R}^\mu_p$ was ε_k-homogeneous, due to Proposition 5.5.10.

Despite the last proposition, the next two examples show that in general the crystal maps $\widetilde{E_k}$ and $\widetilde{F_k}$ **may not** preserve minimal or quasi-Richardson components.

Example 5.5.15. As we have seen in Definition 5.5.6, the crystal map $\widetilde{E_2} : \mathbf{Std}^{22}_{1111} \to \mathbf{Std}^{22}_{1201}$ sends

$$T_1 := \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \overset{\widetilde{E_2}}{\longmapsto} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array} = T_2.$$

Now, let us prove that $\varepsilon_2(K_{T_1}) = 1$. As, $\varepsilon_2(F) = \dim(F_3 \cap e_\lambda^{-1}(F_1)) - \dim F_2$, it is equivalent to prove that $e^{-1}(F_1) = F_3$ for (generic) flags in K_{T_1}. Now, from the Spaltenstein description of the component that correspond to tableau T_1,[18] given a

[18]Equation (5.20) in Section 5.3.1.

generic flag $F = (0 \subset F_1 \subset F_2 \subset F_3 \subset \mathbb{C}^4) \in K_{T_1}$ we have that $\lambda(e|F_2) = (2)$, thus $F_2 = \langle u_2, u_1 \rangle$ where $eu_2 = u_1$, and $u_1 = 0$. Next, as $eF_1 = 0$, we have $F_1 = \langle u_1 \rangle$. Again by Spaltenstein description, $\lambda(e|F_3) = (2,1)$, hence F_3 contains the whole 2-dimensional kernel of e, (e is of type (2,2) in this example, hence has a 2-dimensional kernel). Thus, $F_3 = \langle u_2, u_1, u_3 \rangle$, where $eu_3 = 0$, and finally $e^{-1}(F_1) = e^{-1}(u_1) = \langle u_2, u_1, u_3 \rangle = F_3$.

Thus, having $\varepsilon_2(K_{T_1}) = 1$, and $p = (1,1,1,1)$, by Corollary 5.5.11 the components $\mathcal{K}_{T_1}$ and $\mathcal{K}_{T_2}$ are equisingular. As the component is the only one in $\mathcal{K}_{T_2} \in \mathrm{Irr}(1201, 22)$, it is minimal. By easy computations as in previous examples we can see that it is QR as well. Altogether, it is smooth. Thus, we get that the component $\mathcal{K}_{T_1}$ is smooth as well. As this component is not minimal or quasi-Richardson (the only eligible permutation $\mu = (2,2)$ yields is the other component in $\mathrm{Irr}(1111, 22)$), we did not know this beforehand.[19] Moreover, as $\widetilde{F_2}(\mathcal{K}_{T_2}) = \mathcal{K}_{T_1}$, we see that minimal/QR components are not preserved via the $\widetilde{F_k}$ crystal operators.

Example 5.5.16. The crystal map $\widetilde{E_2} : \mathbf{Std}^{2211}_{2031} \to \mathbf{Std}^{2211}_{2121}$ sends

$$T_1 := \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \overset{\widetilde{E_2}}{\longmapsto} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = T_2;$$

Analogously to examples in Section 5.4.6, we get that the component $\mathcal{K}_{T_1}$ is quasi-Richardson and minimal, whereas component $\mathcal{K}_{T_2}$ is neither. Indeed, due to Theorem 5.4.30, if they exist, components $\mathcal{R}^{42}_{2121}$ and $\mathcal{R}^{24}_{2121}$ correspond respectively to the tableaux

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

(note the tableau on the left was explained in the second example in Example 5.4.13), neither of which is the tableau T_2. Moreover, these quasi-Richardson components do exist, thus are minimal as well. Namely, as $\mu = (24)$ is coarse, $\mathfrak{F}^{24}_{2121} = \mathcal{R}^{24}_{2121}$ is Richardson. Regarding the permutation $\mu = (42)$, one can (as in the examples in Section 5.4.6) explicitly compute the variety $C^{42}_{2121} = \{0 \subset F_2 \subset F_3 \subset v_1, v_3, v_5, v_6, \alpha v_2 +$

[19] Remark: In the Springer theory literature it has been proved that 2-row ordinary Springer fibres have only smooth components. Thus, saying "we did not know this beforehand" refers to using methods previously written in this thesis.

$\beta v_4 \subset \mathbb{C}^6\}$ and, intersecting it with $\mathcal{B}^{2211}_{2121}$, obtain the quasi-Richardson, hence minimal component

$$\mathfrak{F}^{42}_{2121} = \{0 \subset F_2 \subset F_2, \alpha v_1 + \beta v_3 \subset v_1, v_3, v_5, v_6, \alpha v_2 + \beta v_4 \subset \mathbb{C}^6 \mid F_2 \subset \langle v_1, v_3, v_5, v_6 \rangle\}$$

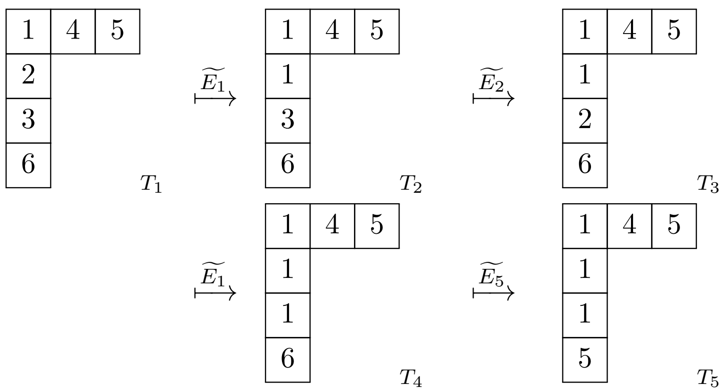
Thus, minimal/quasi-Richardson components are not preserved via the $\widetilde{E_k}$ crystal operators.

Still, it is possible that minimal components do *preserve smoothness* under the crystal maps. It could be true if they were ε_k-homogeneous, due to Proposition 5.5.10. Thus, we leave it as an open question:

Question 5.5.17. *Given a minimal/QR component X with $\varepsilon_k(X) \neq 0$, is it true that X is ε_k-homogeneous?*

We conclude with a less straightforward example of how we can get more smooth non-minimal components out of a minimal one via crystal operators.

Example 5.5.18. There is a sequence of crystal maps



We start with $p(T_1) = (111111)$, then recall that $\widetilde{E_k}$ adds one to p_k and take away one from p_{k+1}. Thus $p(T_2) = (201111)$, $p(T_3) = (210111)$, $p(T_4) = (300111)$ and $p(T_5) = (300120)$. By Proposition 5.5.12, we get $\varepsilon_1(T_1) = p(T_1)_2 = 1, \varepsilon_2(T_2) = p(T_2)_3 = 1$ and $\varepsilon_1(T_3) = p(T_3)_2 = 1$.

Also, one can immediately see from definition that $\varepsilon_5(T_4) = 1$. Thus, these crystal maps induce Grassmann bundles, whose fibres we calculate via Theorem 5.5.2(5): the first one is $Gr(1,2) = \mathbb{P}^1$, the second is a point, the third is $Gr(1,3)$, and the fourth is $\mathbb{P}^1$. The tableau T_5, being the only one in $\mathcal{B}^{3111}_{300120}$, represents a minimal component. Calculating the partial flags that describe this component, we directly get that it is isomorphic to $Gr(2,3)$. Altogether, we see that the non-minimal component representing the standard tableau

1	4	5
2		
3		
6		

is an iterated Grassmann bundle.

5.5.3 Further remarks

One could possibly use this inductive "down and up" approach as in Example 5.5.18 in order to prove the that all the components of Springer fibres $\mathcal{B}^\lambda$ of hook-type[20] are iterated Grassmann bundles, and to compare it with the similar result due to Fung which says that they are isomorphic to iterated Grassmann/Flag manifold bundles ([Fu03, Thm. 3.1]). Moreover, as we see in the same Example, we get the same topological description (iterated Grassmann bundles) also for components of generalised Springer fibres that lie in the path between the ordinary Springer fibre and the last one (components that correspond to tableaux T_2, T_3, T_4 in the Example). Thus, using this approach one could possibly prove a more general statement, which then would be an essentially new result:

Conjecture 5.5.19. *All components of the generalised Springer fibres of hook-type are iterated Grassmann bundles.*

A further long term goal would be to prove that the answer for the following question is affirmative (as verified by all known examples):

Question 5.5.20. *Are all smooth components of generalised Springer fibres $\mathcal{B}^\lambda_p$, or equivalently, of cores $\mathfrak{L}(\mathbf{v},\mathbf{w})$ of quiver varieties of type A, iterated Grassmann bundles?*

Remark 5.5.21. On the quiver variety side, this is not true outside of type A. Namely, there is an example [Nak06] of a quiver variety of type E_6 with a core component that is a blow-up of $\mathbb{P}^2$ at three points, hence not an iterated bundle of generalised flag manifolds. Also, in [HS02, Sec. 9] there is an example of a quiver variety of type $K_{2,3}$ (complete bipartite graph) that has a core consisting of 6 $\mathbb{P}^2$'s and one blow-up of $\mathbb{P}^2$ at 3 points.

[20]Meaning that the Young diagram $Y(\lambda)$ looks like a hook, that is, all the rows except for the first have one box.

Chapter 6

Symplectic cohomology of conical symplectic resolutions

This is joint work with Alexander Ritter. In this chapter we will assume that the reader has some familiarity with Hamiltonian Floer theory and symplectic cohomology. For a brief overview of those, we refer the reader to Appendix A, and for more extensive treatment to [Sa97, Sei08, R10].

Given an arbitrary CSR $(\mathfrak{M}, \varphi)$ we construct symplectic cohomologies for the highly non-exact Calabi-Yau symplectic structures on CSRs described in Section 2.2.2. Moreover, we show that these symplectic cohomologies always vanish, hence yield families of filtrations of the singular cohomology ring $H^*(\mathfrak{M})$. In addition, we construct a novel filtration on the Floer chain complex that yields both *positive symplectic cohomology* and a Morse-Bott-Floer spectral sequence whose E_1-page consists of cohomologies of torsion submanifolds of the $\mathbb{C}^*$-action φ. Although this chapter is self-contained, our research on these topics is ongoing work. At the end of the chapter we will list the conjectures that we aim to prove in future work. In particular, as mentioned in the Introduction, we highlight the technical Assumption 1 (above Proposition 6.8.7) and Assumption 2 (above Remark 6.8.8), which we hope to lift in future work.

6.1 Preliminaries

Consider an arbitrary CSR $(\mathfrak{M}, \varphi)$ as in Definition 2.1.1, where φ is the given $\mathbb{C}^*$-action. By Lemma 2.2.12, we can fix a choice of Kähler structure $(\mathfrak{M}, \omega_I, g)$ such that ω_I and g are invariant under the φ-action by $S^1 \subset \mathbb{C}^*$. We denote by

$$H : \mathfrak{M} \to \mathbb{R} \tag{6.1}$$

the moment map, i.e. the Hamiltonian that generates the S^1-action on $(\mathfrak{M}, \omega_I)$.

Recall the definition of symplectic manifold that is convex at infinity:

Definition 6.1.1. We say that a symplectic manifold (M, ω) is **convex at infinity** if there exists a compact set K and a symplectomorphism

$$(M \setminus K, \omega) \cong (\Sigma \times [1, +\infty), d(R\alpha)),$$

where (Σ, α) is a contact manifold.

We now explain why the manifold $(\mathfrak{M}, \omega_I)$ is rarely ever convex at infinity.

Proposition 6.1.2. *Suppose that $0 \in \mathfrak{M}_0$ is not an isolated singularity. Then* any *choice of I-compatible (real) symplectic form ω on $\mathfrak{M}$ is non-exact at infinity.*

Proof. As $0 \in \mathfrak{M}_0$ is a symplectic singularity, by [Ka06, Thm. 2.3] there is a finite stratification $\mathfrak{M}_0 = \sqcup_{a \in A} \mathfrak{M}_0^a$ by locally closed smooth strata, where $\mathfrak{M}_0^0 = 0$. Let us assume that we have a CSR with a non-isolated singularity. Thus, there is at least another non-generic stratum $\mathfrak{M}_0^1$. As the $\mathbb{C}^*$-action on $\mathfrak{M}_0$ is algebraic, it leaves the strata invariant. As the points in $\mathfrak{M}_0^1$ have finite isotropy subgroups, an arbitrary $\mathbb{C}^*$-orbit makes the stratum $\mathfrak{M}_0^1$ non-compact. Thus, there is a sequence of points $(x_i)_{i \in \mathbb{N}} \in \mathfrak{M}_0^1$ that goes to infinity. Their fibres $\pi^{-1}(x_i)$ are I-holomorphic, hence ω-symplectic, projective subvarieties in $\mathfrak{M}$. Thus, integrating their irreducible components with ω gives a positive value.

Now assume by contradiction that ω is exact outside of a compact set K, say $\omega = d\theta$ for some 1-form θ on $\mathfrak{M} \setminus K$. Picking a fibre $\pi^{-1}(x_i)$ that is outside of K, its irreducible components have well defined homology classes in $H_*(\mathfrak{M})$ [GrHa78, p.61], indeed in $H_*(\mathfrak{M} \setminus K)$. Hence, pairing them with ω would yield zero values, which is a contradiction. ■

Symplectic resolutions $\mathfrak{M} \to \mathfrak{M}_0$ for which $0 \in \mathfrak{M}_0$ is an isolated singularity are completely classified (hence CSRs are as well). In the (lowest) complex dimension 2, they are the minimal resolutions of Du Val singularities (due to [Be00, Prop. 1.3] and [Ish97, Thm. 7.5.1]), whereas in the higher dimensions, they are the cotangent bundles $T^*\mathbb{C}P^n$, due to [CMS-B02, Thm. 8.3].

For the former, it is known that they are convex at infinity due to Ritter [R10, Lem. 42] and the vanishing of their non-exact symplectic cohomology was shown in the same paper. For the latter, as in the argument in [R14, Rmk. in Sec. 11.1], one can show that for $n \geq 2$ they are not convex at infinity.

Thus, apart from those two families of examples, $0 \in \mathfrak{M}_0$ is not an isolated singularity. This is problematic as the usual definition of symplectic cohomology $SH^*(M,\omega)$ presumes that (M,ω) is convex at infinity (Appendix A.2), which fails by Proposition 6.1.2. One of the reasons for desiring convexity at infinity is because the induced radial coordinate R determines a natural class of Hamiltonians. Indeed for Hamiltonians $F : M \to \mathbb{R}$ which are linear in R at infinity, the Floer cohomologies $HF^*(F)$ can be constructed, as well as their direct limit $SH^*(M)$ as we let the slope of F at infinity diverge. A key reason to pick these Hamiltonians is that they prevent the non-compactness in the moduli spaces of Floer solutions, caused by solutions escaping to infinity.

In our setup, there is a natural class of Hamiltonians: in the role of the radial coordinate R, we instead use the moment map H. In this chapter we will prove that we can then ensure compactness of moduli spaces of Floer solutions. Thus we can define symplectic cohomology as a direct limit of Floer cohomologies,

$$SH^*(\mathfrak{M},\omega_I,\varphi) := \lim_{F\in\mathcal{H}(\mathfrak{M},\varphi)} HF^*(F),$$

where $\mathcal{H}(\mathfrak{M},\varphi)$ is the class of Hamiltonians $F : \mathfrak{M} \to \mathbb{R}$ which at infinity are linear functions of H.

As we fixed the choice of ω_I, we will simplify the notation to $SH^*(\mathfrak{M},\varphi)$ in the rest of this chapter. Although we will prove that $SH^*(\mathfrak{M},\varphi)$ vanishes, and thus emphasizing the dependence on φ may appear superfluous at first, what we are actually interested in is the directed system of Floer cohomologies $HF^*(F)$ which depends on the choice φ. We will show in examples that this will induce φ-dependent filtrations on $H^*(\mathfrak{M})$.

We begin by constructing a map which will be very useful for our purposes.

Proposition 6.1.3. *The CSR $(\mathfrak{M},\varphi)$ admits a proper $\mathbb{C}^*$-equivariant holomorphic map*

$$\Psi = \Theta \circ j \circ \pi : \mathfrak{M} \to \mathbb{C}^N,$$

with $\Psi^{-1}(0) = \mathfrak{L}$, where $\mathbb{C}^*$ acts diagonally on $\mathbb{C}^N$ by some integer weight w. This map arises from composing $\pi : \mathfrak{M} \to \mathfrak{M}_0$ with a $\mathbb{C}^*$-equivariant holomorphic map $\Theta \circ j : \mathfrak{M}_0 \to \mathbb{C}^N$ which is a local embedding except at $0 \in \mathfrak{M}_0$.

Proof. From the definition of the conical symplectic resolution $\mathfrak{M}$, there is a $\mathbb{C}^*$-equivariant projection $\mathfrak{M} \xrightarrow{\pi} \mathfrak{M}_0$ to the affine variety $\mathfrak{M}_0$. Recall that the coordinate ring of the affine variety $\mathfrak{M}_0$ with an algebraic $\mathbb{C}^*$-action is a graded ring

$$\mathbb{C}[\mathfrak{M}_0] = \bigoplus_{n \geq 0} \mathbb{C}[\mathfrak{M}_0]^n,$$

whose grading prescribes the weight of the action. Fix a choice of homogeneous polynomials $f_1, \ldots, f_N \in \mathbb{C}[\mathfrak{M}_0]$ which generate $\mathbb{C}[\mathfrak{M}_0]$. Since $\mathbb{C}[\mathfrak{M}_0]^0 = \mathbb{C}$ (as $\mathbb{C}^*$ contracts $\mathfrak{M}_0$ to a unique point), we may assume that all f_i are non-constant, so their weights w_i are positive. Thus the generators f_i determine an embedding $j : \mathfrak{M}_0 \to \mathbb{C}^N$, $p \mapsto (f_1(p), \ldots, f_N(p))$, with $j(0) = 0$. Denote by w the least common multiple of the weights w_i of the f_i. Then $j \circ \pi : \mathfrak{M} \to \mathbb{C}^N$ is a $\mathbb{C}^*$-equivariant holomorphic map if we make $\mathbb{C}^*$ act on $\mathbb{C}^N$ by

$$t \cdot (z_1, \ldots, z_N) = (t^{w_1} z_1, .., t^{w_N} z_N).$$

The claim now follows by defining Θ to be the following holomorphic map,

$$\mathbb{C}^N \xrightarrow{\Theta} \mathbb{C}^N, \quad \Theta(z_1, .., z_N) = (z_1^{w/w_1}, \ldots, z_N^{w/w_N}),$$

in particular Ψ is proper since π, j, Θ are all proper. ■

Remark 6.1.4. Observe from the above construction that Ψ is explicitly

$$\Psi = (\pi^*(f_1)^{w/w_1}, \ldots, \pi^*(f_N)^{w/w_N}).$$

Therefore the function $\Phi = \sum \pi^*(|f_i|^{2w/w_i}) : \mathfrak{M} \to \mathbb{R}$, which also arose in the proof of Proposition 2.2.6, satisfies $\Phi^{-1}(0) = \mathfrak{L}$ and corresponds to

$$\Phi = \Psi^*(|z_1|^2 + \cdots + |z_N|^2).$$

Proposition 6.1.5. *The moment map H from* (6.1) *is an exhausting function, i.e. it is proper and bounded from below.*

Proof. As $\mathfrak{L}$ is compact, the restriction $H|_{\mathfrak{L}}$ has a minimum, and we claim that this is the global minimum of H, thus H is bounded below. Recall that the Hamiltonian vector field X_H satisfies $\omega_I(\cdot, X_H) = dH = g(\cdot, \nabla H) = \omega_I(\cdot, I\nabla H)$, so $-\nabla H = IX_H$. Thus, picking any point $p \in \mathfrak{M}$, its negative gradient flowline is geometrically the same as the $t \leq 1$ part of its $\mathbb{R}_+$-orbit via φ_t, which converges as $t \to 0$ to a fixed point $p' \in \mathfrak{L}$. Hence $H(p) \geq H(p')$, as required.

To show that H is proper it suffices to show that any given sublevel set of H, say

$$S := \{H \leq c\}$$

for $c \in \mathbb{R}$, lies in some compact subset $K \subset \mathfrak{M}$. Using Ψ, Φ from Remark 6.1.4, the preimages

$$A = \Phi^{-1}[\tfrac{1}{2}, 1] = \Psi^{-1}\{z \in \mathbb{C}^N : \tfrac{1}{2} \leq |z| \leq 1\} \quad B = \Phi^{-1}[0,1] = \Psi^{-1}\{z \in \mathbb{C}^N : |z| \leq 1\}$$

are compact since Ψ is proper. The flow ϕ_V^s of $V = \nabla H / \|\nabla H\|^2$ for time $s \in \mathbb{R}$ is defined on $\mathfrak{M} \setminus \mathfrak{L}$ as $\mathrm{Crit}(H) \subset \mathfrak{L}$ and as H is a proper function (hence the flow of V does not diverge in finite time). We claim that the desired compact set $K \subset \mathfrak{M}$ can be taken to be

$$K = B \cup \bigcup_{s \in [0,\tau]} \phi_V^s(A)$$

for $\tau = c - \min H(A)$. Indeed, let $x \in S \setminus B$, so $\Phi(x) > 1$. The $\mathbb{R}_+$-action φ_t for $t < 1$ contracts $\mathfrak{M}$ to $\mathfrak{L} = \Phi^{-1}(0)$, so $\Phi(\varphi_t(x)) \to 0$ as $t \to 0$. Thus $\Phi(\varphi_{t_0}(x)) = 1$ for some $t_0 \in (0,1)$, so $\varphi_{t_0}(x) \in A$. We showed above that the $-\nabla H$ flow and the $\mathbb{R}_+$-action φ_t agree geometrically, so $\phi_V^{-s}(x) \in \Sigma$ for some $s \in \mathbb{R}$. Since $H(\phi_V^{-s}(x)) = H(x) - s \geq \min H(A)$, we deduce that $s \leq c - \min H(A)$, as required. ■

Corollary 6.1.6. *The fibres of the moment map H are connected.*

Proof. The moment map of a Hamiltonian S^1-action on a compact symplectic manifold has connected fibres, by F. Kirwan [Ki84b]. Although our $\mathfrak{M}$ is not compact, having a proper moment map H leads to the same conclusion. ■

6.2 C^0-bounds for Floer trajectories

In this section we construct the symplectic cohomology $SH^*(\mathfrak{M}, \varphi)$ and prove that it is a $\mathbb{Z}$-graded $\mathbb{K}$-algebra with respect to the pair-of-pants product, where the Novikov field $\mathbb{K}$ in a formal variable T, working over a base field $\mathcal{K}$ is

$$\mathbb{K} = \left\{ \sum_{i=1}^{\infty} a_i T^{n_i} : a_i \in \mathcal{K}, n_i \in \mathbb{R}, n_i \to \infty \right\}. \tag{6.2}$$

Remark 6.2.1. We remark here that, as we are in a non-exact setup, there are no *a-priori* energy estimates for the Floer solutions that occur for the class of Hamiltonians that we consider in constructing the symplectic cohomology. Thus, in principle it may happen that the energies of the Floer solutions between two periodic orbits are not bounded, hence Novikov field is necessary indeed. However, we remark that the outcomes of our construction, for example, filtration on the ordinary cohomology $H^*(\mathfrak{M}, \mathbb{K})$ (Corollary 6.4.7), pass immediately to ordinary coefficients, as the Novikov field $\mathbb{K}$ is flat over the base field $\mathcal{K}$.

Let us define first a class of admissible Hamiltonians, that we are going to use in the construction of symplectic cohomology. Recall that H was the moment map of the S^1-part of φ.

Definition 6.2.2. We call a Hamiltonian $F : \mathfrak{M} \to \mathbb{R}$ **admissible** if, for some compact subset $K \subset \mathfrak{M}$,

$$F = \lambda H \quad \text{on} \quad \mathfrak{M} \setminus K$$

for some $\lambda > 0$. We call λ the **slope** of F. We will denote the set of admissible Hamiltonians by $\mathcal{H}(\mathfrak{M}, \varphi)$.

Remark 6.2.3. We will always assume that λ is not a period of any Hamiltonian orbit of H.[1] We call such slopes **φ-generic**, or just **generic** if φ is understood. **We are using the convention that the Hamiltonian flow of H is 2π-periodic.** Thus non-generic slopes λ lie in $2\pi\mathbb{Q}$. Further, we will always assume (by making K larger if necessary) that $K = \Psi^{-1}(B)$ for some compact ball $B = \{z \in \mathbb{C}^N : \frac{1}{2}|z|^2 \leq R_0\}$. These assumptions imply that there are no 1-periodic orbits for F outside of K. Indeed, let $p \in \mathfrak{M} \setminus K$, then the S^1-orbit of $\Psi(p)$ stays within B. By equivariance of Ψ the S^1-orbit of p lies in $\mathfrak{M} \setminus K$. But in $\mathfrak{M} \setminus K$ the flow for F is just the λ-accelerated S^1-action, so the flow for F of p stays in $\mathfrak{M} \setminus K$.

We now show that the Floer trajectories for F project via $\Psi = \Theta \circ j \circ \pi : \mathfrak{M} \to \mathbb{C}^N$ (Proposition 6.1.3) to cylinders in $\mathbb{C}^N$ which at infinity are Floer for a constant multiple of the standard Hamiltonian. This allows us to apply a maximum principle in $\mathbb{C}^N$ to ensure that Floer trajectories cannot escape to infinity.

Proposition 6.2.4. *Let F be any admissible Hamiltonian, and let $x_\pm$ be any 1-periodic Hamiltonian orbits for F. Then there is a compact subset $C \subset \mathfrak{M}$, depending only on $x_\pm$, such that all Floer trajectories for F converging to $x_\pm$ are contained in C. In particular, the Floer cohomology $HF^*(F)$ is well-defined.*

Proof. Recall that the first part of the statement suffices to ensure that Floer cohomology is well-defined, by Hofer-Salamon [HS95], since in our setup $c_1(\mathfrak{M}) = 0$ by Lemma 2.1.5.

Recall a Floer trajectory as in the claim is a map $u : \Sigma := \mathbb{R} \times S^1 \to \mathfrak{M}$ with $u(-\infty, t) = x_-(t)$, $u(+\infty, t) = x_+(t)$, satisfying the Floer equation

$$\partial_s u + I(\partial_t u - X_F) = 0. \tag{6.3}$$

[1] Or equivalently, a period of an S^1-orbit of the action φ.

Since Ψ is $\mathbb{C}^*$-equivariant, in particular S^1-equivariant, by Remark 6.1.4 we deduce

$$\Psi_* X_H = w X_{\mathbb{C}^N} \tag{6.4}$$

where $X_{\mathbb{C}^N}$ is the Hamiltonian vector field of the standard Hamiltonian $R := \frac{1}{2}|z|^2$ in $\mathbb{C}^N$.

As F is admissible, pick K as in Remark 6.2.3, so $F = \lambda H$ and $X_F = \lambda X_H$ on $\mathfrak{M} \setminus K$ for a generic λ. Thus the part of the projected map $v := \Psi \circ u : \Sigma \to \mathbb{C}^N$ lying in $\Psi(\mathfrak{M} \setminus K)$ satisfies the equation

$$\partial_s v + i(\partial_t v - \lambda w X_{\mathbb{C}^N}) = 0. \tag{6.5}$$

For solutions v of such equations, the usual maximum principle for Floer solutions from [Vi99, Sei08] holds. Namely, the function R cannot attain a maximum at an interior point of v (unless R is locally constant). In our case, we only consider the part of v lying outside of the ball B from Remark 6.2.3 (in particular, $x_\pm \subset K$ so $\Psi(x_\pm) \subset B$). Thus, in our situation, the maximum of R on that part of v is forced to lie in B. So all of v lies in B. Since Ψ is proper, we obtain the required compact set $C = \Psi^{-1}(B)$. ■

Remark 6.2.5. One might ask if one can apply the above argument to Hamiltonians that at infinity are functions $F = h(H)$ of H (by analogy with Liouville manifolds where one uses functions $h(R)$ of the radial coordinate). Notice $X_{h(H)} = h'(H)\, X_H$, and projecting we have $\Psi_* X_F = h'(H(u(s,t))) \cdot w X_{\mathbb{C}^N}$, whose coefficient function is domain-dependent. In this case the maximum principle can fail. Indeed, consider the example of minimal resolution $\mathfrak{M} = X_{\mathbb{Z}/5}$ of Du Val singularity $\mathfrak{M}_0 = \mathbb{C}^2/(\mathbb{Z}/5) = V(XY - Z^5) \subset \mathbb{C}^3$, Examples 2.3.17 and 6.9.3. As we show in the latter example, non-fixed Hamiltonian orbits of the Hamiltonian $H_\lambda = c(H)$ of convex type[2] lie above the two lines $(X, 0, 0)$ and $(0, Y, 0)$ in $\mathfrak{M}_0$, and the fixed ones lie in the core. By construction, the latter are precisely the $\mathbb{C}^*$-fixed locus, so Example 2.3.17 show us that they are isolated, there is exactly 5 of them, 2 of which are "outer" in the chain of spheres, and the other 3 are "inner" (black dots on Figure 2.1). Thus, the non-fixed orbits converge via the $\mathbb{C}^*$-action towards the outer points, and one could think that they cannot reach inner points by a Floer trajectory. Indeed, as in the proof of Proposition 6.2.4, we can project a Floer trajectory u in $\mathfrak{M}$ connecting a non-fixed orbit with a fixed one in to "almost" Floer trajectories v_X, v_Y, v_Z that lie in the coordinate planes $\mathbb{C}_X, \mathbb{C}_Y, \mathbb{C}_Z$. We say "almost" as they satisfy Floer equations

[2]Meaning that c is a convex function.

$$\partial_s v_X + i(\partial_t v_X - k(s,t)X_{\mathbb{C}}) = 0, \tag{6.6}$$

where $k(s,t) = \frac{w}{w_X} h'(H(u(s,t)))$ depends on the domain.[3] Thus, picking a non-fixed orbit u living above the line $(X,0,0)$ we see that its Y-projected almost Floer trajectory v_Y starts and ends in the origin. If it was a honest Floer trajectory of a radial Hamiltonian in $\mathbb{C}_Y$, i.e.

$$\partial_s v_Y + i(\partial_t v_Y - h'(R)X_{\mathbb{C}}) = 0$$

(where R is the radial coordinate $R = |Y|^2$), the maximum principle would **not** allow it to move from the origin, thus u would need to lie completely above the line $(X,0,0)$, hence eventually hitting only the outer point indeed. However, equation (6.6) is not of that type, we cannot even see $k(s,t)$ as a function on $\mathbb{C}_X$. It is only a Floer solution at infinity where $k(s,t) = \frac{w}{w_X}\lambda$ becomes a constant, hence why we can apply maximum principle to bound it. Thus, the maximum principle in the coordinate planes can fail, and indeed it must, as we know that all fixed orbits need to be killed since the symplectic cohomology vanishes, by Proposition 6.4.3.[4] Spectral sequence for this example depicts this nicely (Figure 6.4): The top classes in the first column, corresponding to fixed orbits, are getting killed by the classes of the later columns, corresponding to non-fixed orbits.

Definition 6.2.6. An **admissible homotopy** between two admissible Hamiltonians F_{λ_1} and F_{λ_2}, with respective slopes λ_1 and λ_2, is a function $F : \mathbb{R} \times \mathfrak{M} \to \mathbb{R}$ such that there is a compact $K \subset \mathfrak{M}$ with:

1. $F_s = \lambda_s H$ on $\mathfrak{M} \setminus K$, for some $\lambda_s > 0$,
2. $\frac{d}{ds}\lambda_s \leq 0$,
3. $F_s = F_{\lambda_2}$ for $s \ll 0$,
4. $F_s = F_{\lambda_1}$ for $s \gg 0$.

As in Remark 6.2.3, we may as well assume that $K = \Psi^{-1}(B)$ for some ball $B = \{z \in \mathbb{C}^N : \frac{1}{2}\|z\|^2 \leq R_0\}$.

Proposition 6.2.7. *Given two admissible Hamiltonians F_{λ_1} and F_{λ_2}, with slopes $\lambda_1 \leq \lambda_2$, the Floer continuation map $\Phi_{\lambda_1,\lambda_2} : HF^*(F_{\lambda_1}) \to HF^*(F_{\lambda_2})$ is well-defined.*

[3] Here $w_X = 5$ is the weight of the coordinate X and $w = 10$ is the least common multiple of weights 5,5,2 of the coordinates.

[4] And for this example previously proved in [R10, Th. 48].

Proof. The proof is essentially the same as for Proposition 6.2.4. In this case, x_- is a 1-periodic orbit of F_{λ_2}, x_+ is a 1-periodic orbit of F_{λ_1}, and we replace X_F in equation (6.3) by X_{F_s}, and in (6.5) we replace λ by λ_s. The maximum principle still applies by [Vi99, Sei08] thanks to the condition $\frac{d}{ds}\lambda_s \leq 0$. ■

Thus, combining the last two propositions, we get the following theorem:

Theorem 6.2.8. *Given a CSR* $(\mathfrak{M}, \varphi)$*, the direct limit over the maps* $\Phi_{\lambda_1,\lambda_2}$ *from Proposition 6.2.7,*

$$SH^*(\mathfrak{M}, \varphi) := \lim_{F \in \mathcal{H}(\mathfrak{M},\varphi)} HF^*(F),$$

is a well defined $\mathbb{Z}$*-graded vector space over the Novikov field* $\mathbb{K}$*.*

Proof. This now follows by Propositions 6.1.5, 6.2.4 and 6.2.7. The groups are canonically $\mathbb{Z}$-graded by the Robbin-Salamon index (Lemma A.1.5), because $c_1(\mathfrak{M}) = 0$ due to Lemma 2.1.5 and $H^1(\mathfrak{M}) = 0$, due to Corollary 2.1.9. ■

Proposition 6.2.9. *Let* $(\mathfrak{M}, \varphi)$ *be a CSR. Then* $SH^*(\mathfrak{M}, \varphi)$ *is a* $\mathbb{Z}$*-graded* $\mathbb{K}$*-algebra with respect to the pair-of-pants product, which arises as the direct limit of the pair-of-pants products*

$$HF^*(F_{\lambda_1}) \otimes HF^*(F_{\lambda_2}) \to HF^*(F_{\lambda_1+\lambda_2}).$$

Proof. For the detailed construction of the pair-of-pants product on symplectic cohomology we refer to [R13]. The new ingredient here is to explain why pair-of-pants solutions $u : S \to \mathfrak{M}$ do not escape to infinity. At infinity, $F = \lambda H$, so the projection argument from Proposition 6.2.4 can be carried out, by applying the maximum principle for pair-of-pants solutions in $\mathbb{C}^N$, see [R13, App. D.3].

That Appendix shows that in a complex coordinate $z = s + \sqrt{-1}t$ on the pair-of-pants surface S the Floer equations are: $\partial_t u = X_F \beta_t + I\partial_s u - IX_F \beta_s$ and $\partial_s u = X_F \beta_s - I\partial_t u + IX_F \beta_t$, where β is a certain auxiliary one-form on S. At infinity, the projected solution $v = \Psi \circ u$ will therefore satisfy the same equations with X_F and I replaced respectively by $\lambda w X_{\mathbb{C}^N}$ and i, so the maximum principle applies. ■

6.3 Robbin-Salamon indices of Morse-Bott manifolds of orbits

To prove the vanishing of symplectic cohomology in the next section, we will first need a precise calculation of Robbin-Salamon indices of the 1-periodic orbits of Hamiltonians of the form λH on $(\mathfrak{M}, \omega_I)$, which we carry out in this section. As λ is generic, the only 1-periodic orbits of λH are constant orbits in the core lying in the fixed point set of the S^1-action. As in Lemma 2.3.2, denote by $\mathfrak{F} := \mathfrak{M}^\varphi$ the fixed locus of φ whose connected components $\mathfrak{F}_\alpha$ are the Morse-Bott manifolds that define the critical locus of H. Since λ is generic, the $\mathfrak{F}_\alpha$ are therefore the Morse-Bott manifolds of 1-periodic orbits of λH in $\mathfrak{M}$.

We denote by $RS(x, F)$ the *Robbin-Salamon* index (see Appendix A.1.1) of a periodic orbit x under the flow for a Hamiltonian F. As we will need it in the following text, recall the function

$$W : \mathbb{R} \to \mathbb{Z}, \quad W(t) := \begin{cases} 2\lfloor t/2\pi \rfloor + 1 & \text{if } t \notin 2\pi\mathbb{Z} \\ t/\pi & \text{if } t \in 2\pi\mathbb{Z}. \end{cases} \tag{6.7}$$

from Theorem A.1.4 in the Appendix.

Remark 6.3.1. **(Grading conventions)** We remark that we will use the grading convention for Floer cohomologies as in [McLR18], i.e. the grading of an orbit x of Hamiltonian F is given by

$$|x| := dim_{\mathbb{C}}\mathfrak{M} - RS(x, F).$$

Given a Hamiltonian λH, the **grading** of a fixed component $\mathfrak{F}_\alpha$ we define as

$$\mu_\lambda(\mathfrak{F}_\alpha) = dim_{\mathbb{C}}\,\mathfrak{M} - RS(x, \lambda H) - dim_{\mathbb{C}}\,\mathfrak{F}_\alpha,$$

for an arbitrary $x \in \mathfrak{F}_\alpha$. The motivation for this definition is that $\mu_\lambda(\mathfrak{F}_\alpha)$ is going to be the Floer grading of $\mathfrak{F}_\alpha$ seen as manifold of Hamiltonian 1-orbits of Hamiltonian λH (Corollary 6.5.6). In particular, for λ small enough, $\mu_\lambda(\mathfrak{F}_\alpha)$ becomes the Morse Bott index of $\mathfrak{F}_\alpha$.

Proposition 6.3.2. *The grading $\mu_\lambda(\mathfrak{F}_\alpha)$ is an even integer.*

Proof. By definition, $RS(x, \lambda H)$ is the Robbin-Salamon index of the linearised flow of λH along the fixed orbit x. Recall from Theorem 2.3.1(2) that there is a split

$$T_x\mathfrak{M} = \bigoplus_k H_k \tag{6.8}$$

by weight-spaces of the S^1-action. As the S^1-action is Kähler , the weight spaces H_k are Hermitian-orthogonal, with respect to the Hermitian form $\langle \cdot, \cdot \rangle = g(\cdot, \cdot) + i\omega_I(\cdot, \cdot)$. Thus, picking orthogonal bases $(z_k^j)_k$, we see that the linearisation of the flow ϕ_t of λH is

$$(\phi_t)_* z_k^j = e^{i\lambda k t} z_k^j.$$

Thus, by Theorem A.1.4(3),(5) its Robbin-Salamon index $RS(x, \lambda H)$ is equal to

$$RS(x, \lambda H) = \sum_k dim(H_k)W(\lambda k). \tag{6.9}$$

As λ is generic, for $k \neq 0$ we have $\lambda k \notin 2\pi\mathbb{Z}$ thus each $W(\lambda k)$ in the sum is odd, or zero (for $k = 0$). Therefore,

$$\begin{aligned}\mu_\lambda(\mathfrak{F}_\alpha) &\equiv dim_{\mathbb{C}}\mathfrak{M} - dim_{\mathbb{C}}\mathfrak{F}_\alpha - \sum_k dim_{\mathbb{C}}(H_k)W(\lambda k) \\ &\equiv dim_{\mathbb{C}}\mathfrak{M} - dim_{\mathbb{C}}\mathfrak{F}_\alpha - \sum_{k\neq 0} dim_{\mathbb{C}}(H_k) = dim_{\mathbb{C}}\mathfrak{M} - \sum_k dim_{\mathbb{C}}(H_k) = 0 \pmod 2,\end{aligned}$$

where in the penultimate equality we have used that H_0 is the tangent space of $\mathfrak{F}_\alpha$. ■

We state the formula obtained in the last proposition as it will be used in the following text.

$$\mu_\lambda(\mathfrak{F}_\alpha) = dim_{\mathbb{C}}\mathfrak{M} - dim_{\mathbb{C}}\mathfrak{F}_\alpha - \sum_k dim_{\mathbb{C}}(H_k)W(\lambda k) \tag{6.10}$$

Proposition 6.3.3. *For any point $x \in \mathfrak{F}_\alpha$,*

$$\lim_{\lambda\to+\infty} RS(x, \lambda H) = +\infty. \tag{6.11}$$

Proof. Refining the splitting (6.8) in Proposition 6.3.2 by picking unitary bases in each weight space H_k, we have an Hermitian-orthogonal splitting

$$T_x\mathfrak{M} = \bigoplus_{m_i} \mathbb{C}_{m_i}$$

of the tangent space, where $\mathbb{C}_{m_i}$ denotes a copy of $\mathbb{C}$ on which the S^1-action has weight m_i. Thus, rewriting the formula (6.9) in this splitting, we get,

$$RS(x, \lambda H) = \sum_{i=1}^d W(\lambda m_i) = \sum_{i=1}^d (2\lfloor \lambda m_i \rfloor + 1) > \sum_{i=1}^d (2(\lambda m_i - 1) + 1) = 2\lambda \sum_{i=1}^d m_i - d$$

(in the second equality we have used that λ is generic, thus $\lambda m_i \notin 2\pi\mathbb{Z}$).

Now, denoting by k the $\omega_{\mathbb{C}}$-weight of φ, recall that there is a duality between the s-weight space and the $(k-s)$-weight space in $T_x\mathfrak{M}$. Hence, every weight m_i has a paired weight $k - m_i$, thus $\sum_{i=1}^d m_i = kd/2$, which gives $RS(x, \lambda H) = d(\lambda k - 1)$. Thus, letting $\lambda \to +\infty$, (6.11) is satisfied and the proposition is proved. ■

Proposition 6.3.4. *Let $f_\alpha : \mathfrak{F}_\alpha \to \mathbb{R}$ be any Morse function on $\mathfrak{F}_\alpha$. One can construct an autonomous perturbation $\widetilde{F}$ of $F = \lambda H$, supported in disjoint neighbourhoods of the $\mathfrak{F}_\alpha$, such that the following hold.*

(1) $\widetilde{F}$ *is Morse and its critical points are precisely the critical points* p *of the* f_α.

(2) *The* 1*-periodic orbits of* $\widetilde{F}$ *are the (isolated) constant orbits* $x_{\alpha,p}$ *at the* $p \in \mathrm{Crit}(f_\alpha)$.

(3) *Their gradings in* $HF^*(\tilde{F})$ *are* $|x_{\alpha,p}| = \mu_{f_\alpha}(p) + dim_{\mathbb{C}}\,\mathfrak{M} - RS(p,F) - \frac{1}{2}\dim \mathfrak{F}_\alpha$, *where* $\mu_{f_\alpha}(p)$ *is the Morse index of* p.

(4) $|x_{\alpha,p}| \to -\infty$ *as* $\lambda \to +\infty$.

Proof. This is a standard perturbation argument. Since F has only constant 1-periodic orbits (as λ is generic), the Morse-Bott property of F ensures that the Floer action functional of any sufficiently small autonomous perturbation $\widetilde{F}$ of F is still Morse-Bott and its 1-periodic orbits are still constant orbits at the critical points of $\widetilde{F}$. Thus (2) will follow from (1). Following [BH13], we pick bump functions ρ_α supported near the $\mathfrak{F}_a$ and we define

$$\widetilde{F} = F + \sum \varepsilon_\alpha \rho_\alpha \widetilde{f}_\alpha, \tag{6.12}$$

where $\widetilde{f}_\alpha$ is an extension of f_α that is constant in normal directions to $\mathfrak{F}_\alpha$ (after parametrising a tubular neighbourhood of $\mathfrak{F}_\alpha$ by the normal bundle of $\mathfrak{F}_\alpha \subset \mathfrak{M}$ via the exponential map). Then Claim (1) follows for sufficiently small constants $\varepsilon_\alpha > 0$. Now, we have that $RS(x_{\alpha,p}, \widetilde{F}) = RS(p,F) + \frac{1}{2}\dim \mathfrak{F}_\alpha - \mu_{f_\alpha}(p)$, by [Oan04, Sec. 3.3], hence by our grading conventions (Remark 6.3.1), we get (3). Finally, (4) follows by Proposition 6.3.3. ■

6.4 Vanishing of $SH^*(\mathfrak{M},\varphi)$ and filtrations on the cohomology of CSRs

NB In this and following sections, ordinary cohomology will be **with coefficients in the Novikov field** $\mathbb{K}$.

As the continuation maps between Hamiltonians with the same slope are isomorphisms, we can choose the specific Hamiltonians λH from Section 6.3 to calculate $SH^*(\mathfrak{M},\varphi)$:

Corollary 6.4.1. $SH^*(\mathfrak{M},\varphi) = \lim\limits_{\lambda\to+\infty} HF^*(\lambda H)$.

Remark 6.4.2. As the 1-periodic orbits of λH are not isolated, by convention $HF^*(\lambda H)$ actually means that λH is first perturbed (if desired, in a time-dependent way)[5] to make the 1-periodic orbits isolated. By a standard continuation argument, the choice of perturbation does not matter up to isomorphism. Note that in our case the perturbation is compactly supported, as there are no 1-periodic orbits at infinity, and a continuation map for a compactly supported homotopy of Hamiltonians is always an isomorphism.

Thanks to the index calculations in Section 6.3, the proof of the vanishing of $SH^*(\mathfrak{M}, \varphi)$ now follows by the same trick as in [R10, Thm. 48], [McLR18, Thm. 2.10].

Proposition 6.4.3. *Let* $(\mathfrak{M}, \varphi)$ *be any CSR, then* $SH^*(\mathfrak{M}, \varphi, \omega_I) = 0$.

Proof. By Corollary 6.4.1, we need to compute the direct limit of the $HF^*(\lambda H)$. The maps in this direct limit are grading-preserving, so it is enough to show that $HF^*(\lambda H)$ is supported in arbitrarily large negative degrees, as $\lambda \to \infty$. We described an explicit perturbation of λH in Proposition 6.3.4. Thus by Proposition 6.3.4(4) the claim follows. ∎

By [PSS96] (see also [R13]), we have that:

Proposition 6.4.4. *When* $\delta > 0$ *is smaller than any period of* X_H*, and* H_δ *is an admissible Hamiltonian of slope* δ*, we have an isomorphism of graded rings*

$$HF^*(H_\delta) \cong QH^*(\mathfrak{M}).$$

In particular, we have an isomorphism of graded vector spaces $HF^*(H_\delta) \cong H^*(\mathfrak{M}; \mathbb{K})$.

As a consequence of Propositions 6.4.3 and 6.4.4, every class in $H^*(\mathfrak{M}) \cong H^*(H_\delta)$ must eventually map to zero in some $HF^*(\lambda H)$ since the direct limit $SH^*(\mathfrak{M}) = 0$. Thus, we obtain a filtration on $H^*(\mathfrak{M})$:

Definition 6.4.5. The $\varphi-$**filtration** $F^\varphi H^*(\mathfrak{M})$ on $H^*(\mathfrak{M})$ is defined by

$$F^\varphi_\lambda H^*(\mathfrak{M}) := \bigcap_{\mu > \lambda} \ker(H^*(\mathfrak{M}) \to HF^*(H_\mu)), \tag{6.13}$$

where H_μ denotes any admissible Hamiltonian of slope μ, and $H^*(\mathfrak{M}) \cong H^*(H_\delta) \to HF^*(H_\mu)$ is the Floer continuation map.

[5]In general, one has to do it in a time-dependent way indeed. However, in our case the Hamiltonian orbits are all constant, thus autonomous perturbation is also possible, as we saw in Proposition 6.3.4.

Remark 6.4.6. We remark here that there is an interest in filtrations on cohomology of CSRs in the representation-theoretic literature. Namely, the recent work of Bellamy–Schedler [BeSch18] constructs filtrations on cohomologies of Springer fibres, which are, as we saw in Chapter 5, one of the principal examples of (cores of) CSRs. Their filtrations is compatible with the cohomological grading, just as ours, so one can ask e.g. what is the relation between these two filtrations on the top degree of cohomology. In particular, in the example of Springer fibre that corresponds to Du Val singularities of type A_n, ([BeSch18, Ex. 1.5] in their paper) there is a choice of the $\mathbb{C}^*$-action φ that yields (rank-wise) the same filtration as theirs. For $n = 2$ it is given in Example 6.9.5. Apart from this comparison, an interesting question would be whether our filtration relates to filtrations that one would obtain by using algebraic-geometric techniques, such as perverse filtrations; However, we have not investigated into this so far.

Furthermore, we remark that in the case of resolution of Du Val singularities (and possibly of other holomorphic-symplectic quotient singularities $\mathbb{C}^{2n}/\Gamma$), this filtration yields a refinement of the McKay correspondence [Re92] which states that the cohomology class of the resolution are in graded bijection with the conjugacy classes, where the grading on the conjugacy classes is given by certain *age* grading. An example of this for the resolution $\mathfrak{M}$ of $\mathfrak{M}_0 := \mathbb{C}^2/(\mathbb{Z}/5)$ can be seen on Figure 6.4. Namely, we see that in that example the top cohomology has two filtration levels, which correspond to two pair of orbits, which all have the same age grading equal to 1 (orbits are loops in $\mathfrak{M} \setminus \mathfrak{L}$, and those are labelled naturally by their free homotopy classes, as explained e.g. in [McLR18, Eqn. (1.4)] for the case of isolated quotient singularity, and the footnote afterwards for the general case). Thus, our filtration makes a distinction between the orbits lying above $[e^{2\pi i t/5}, 0], [0, e^{2\pi i t/5}]$ and $[e^{4\pi i t/5}, 0], [0, e^{4\pi i t/5}]$, corresponding to conjugacy classes of $\varepsilon^1, \varepsilon^{-1}$ and $\varepsilon^2, \varepsilon^{-2}$ in the group $\mathbb{Z}/5$ (here ε is a primitive root of unity).

In the end, we expect that the filtrations (6.13) do not depend on the choice of the symplectic form ω_I, hence, they are labelled by the set of different conical actions φ, and we expect this parametrisation to be injective, following Example 6.9.5.

As two continuation maps compose to a continuation map, and the continuation map between two Hamiltonians of the same slopes is an isomorphism, $\ker(H^*(\mathfrak{M}) \to HF^*(H_\mu))$ is defined regardless of the choice of H_μ, hence so is the filtration $F^\varphi_\lambda H^*(\mathfrak{M})$.

From the definition we see that $F^\varphi_\lambda H^*(\mathfrak{M}) = 0$, for $\lambda \leq 0$. Also, $F^\varphi_\lambda H^*(\mathfrak{M})$ is monotone-increasing; it strictly increases only when λ crosses a period of an S^1-orbit.

Proposition 6.4.7. *$F^\varphi_\lambda H^*(\mathfrak{M})$ is a filtration on the singular cohomology $H^*(\mathfrak{M})$ by ideals with respect to the cup product.*

Proof. By Proposition 6.4.4, $QH^*(\mathfrak{M}) \cong HF^*(H_\delta)$ is a ring isomorphism. Using that fact and the compatibility of continuation maps with the pair-of-pants products [R13], we obtain the following commutative diagram

$$\begin{array}{ccc} QH^*(\mathfrak{M}) \otimes QH^*(\mathfrak{M}) & \longrightarrow & HF^*(H_\varepsilon) \otimes HF^*(H_k) \\ \downarrow & & \downarrow \\ QH^*(\mathfrak{M}) & \longrightarrow & HF^*(H_{\varepsilon+k}) \end{array} \tag{6.14}$$

where the left and right vertical maps are the quantum product and the pair-of-pants product in Floer cohomology (see Proposition 6.2.9), respectively, and the horizontal maps are the continuation maps. Given $x \in F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ and an arbitrary class $q \in H^*(\mathfrak{M})$, there are two situations:

1. λ is generic. In that case, $x \in F^{\varphi}_{\lambda-\delta}$ for sufficiently small δ, so by putting $k = \lambda - \delta$ and $\varepsilon = \delta$ in diagram (6.14) we get that $q * x \in F^{\phi}_{\lambda}$
2. λ is not generic. In that case $x \in F^{\varphi}_{\lambda+\delta/2}$ so by putting $k = \lambda + \delta/2$ and $\varepsilon = \delta/2$ in diagram (6.14) we get that $q * x \in F^{\varphi}_{\lambda+\delta}$ for arbitrarily small $\delta > 0$, hence $q * x \in F^{\varphi}_{\lambda}$.

Thus we conclude that $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ is a filtration by ideals with respect to the quantum product. By Proposition 2.1.13, $QH^*(\mathfrak{M}) \cong H^*(\mathfrak{M}, \mathbb{K})$ are isomorphic rings, so quantum product is cup product. ■

Proposition 6.4.8. *The filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ respects the cohomological grading on $H^*(\mathfrak{M})$.*

Proof. The maps $H^*(\mathfrak{M}) \to HF^*(H_\mu)$ are grading-preserving so their kernels are graded ideals, hence $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ are graded ideals as well. ■

A possible way of finding more precise information about the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ could be by using the specific Hamiltonians λH and the presentation of their Floer cohomologies given in next section (Corollary 6.5.6). In this presentation, the continuation map

$$H^*(\mathfrak{M}) \to HF^*(H_\lambda)$$

becomes

$$\bigoplus_{\alpha} H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \to \bigoplus_{\alpha} H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)], \tag{6.15}$$

thus $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ is the kernel of it (here μ_α are Morse-Bott indices of $\mathfrak{F}_\alpha$). Another way of getting some more information on the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ is by the Morse-Bott Floer spectral sequence (Proposition 6.8.14). We discuss these two approaches and their connection through a Gysin long sequence in Section 6.10.1.

6.5 The Morse-Bott computation of $HF^*(\lambda H)$

One can in fact precisely calculate $HF^*(\lambda H)$, by a Morse-Bott argument as follows. Recall the perturbation of $F = \lambda H$,

$$\widetilde{F} = F + \sum \varepsilon_\alpha \rho_\alpha \widetilde{f}_\alpha \tag{6.16}$$

from Proposition 6.3.4. As before, denote by $\mathfrak{F}_\alpha$ the connected components of the fixed locus $\mathfrak{F} := \mathfrak{M}^\varphi$ of φ.

Lemma 6.5.1. *Let $u : \mathbb{R} \times S^1 \to \mathfrak{M}$ be a Floer trajectory for any Hamiltonian function $G : \mathfrak{M} \to \mathbb{R}$ converging to critical points p_-, p_+ of G respectively, and let $A \in \pi_2(\mathfrak{M})$ denote the homotopy class of the sphere arising as the image of u (extending u continuously at the ends via $u(\pm\infty) = p_\pm$). Then the energy of u is*

$$E(u) := \int_{\mathbb{R}\times S^1} \|\partial_s u\|^2 \, ds \, dt = \omega_I[A] + G(p_-) - G(p_+).$$

Proof. By the Floer equation $I\partial_s = \partial_t u - X_G$ we obtain

$$\|\partial_s u\|^2 = g(\partial_s u, \partial_s u) = \omega_I(\partial_s u, I\partial_s u) = \omega_I(\partial_s u, \partial_t u) - dG(\partial_s u).$$

Thus $E(u) = \int u^*\omega_I - \int \partial_s G(u) \, dt \, ds = \omega_I[A] - \int (G(p_+) - G(p_-)) \, dt$ and the claim follows. ∎

Remark 6.5.2. We remark that in the $\omega_I(\pi_2) = 0$ case[6] there is a well-defined action functional (defined on contractible loops), and so Floer trajectories lying in the U_α defined below would have equal action values at the ends, thus (trivially) have energy zero.

Proposition 6.5.3. *Let U_α be any closed neighbourhood of $\mathfrak{F}_\alpha$ disjoint from the other fixed components, and let $V_\alpha \subset U_\alpha$ be an arbitrarily small open neighbourhood of $\mathfrak{F}_\alpha$. Let*

$$0 < \hbar < (\textit{the smallest positive value of } \omega_I \textit{ on } \pi_2(\mathfrak{M})) > 0. \tag{6.17}$$

Then, for all sufficiently small $\varepsilon_\alpha > 0$ in (6.16)*, any Floer trajectory in $(\mathfrak{M}, \omega_I)$ of $\widetilde{F}$ lying in U_α of energy smaller than $\hbar$ must in fact lie in V_α.*

[6]Actually, we believe that this case never occurs, but cannot neglect it. Namely, one could in principle have $\mathfrak{M} = T^*X$, where X is a projective variety whose homology is generated by algebraic cycles (cf. Theorem 2.1.8) and still has $\pi_2 = 0$ (e.g. fake projective plane, fake quadric, etc). Now one would need to prove that such T^*X are not CSRs, and that might be hard – there is a conjecture [Ka09, Conj. 1.3] claiming that the only T^*X that are symplectic resolutions are for $X = G/P$ being a generalized flag variety (whose $\pi_2 \neq 0$ indeed).

Proof. The following proof adapts the argument of Cieliebak-Floer-Hofer-Wysocki [CFHW96, Lem. 2.1] to our setup. By Proposition 6.3.4 we may assume that the only 1-periodic orbits of $\widetilde{F}$ are the critical points of the f_α. Suppose by contradiction that we have a sequence u_n of Floer trajectories for $\widetilde{F}_n$ in U_α converging to critical points $p_\pm \in \mathrm{Crit}(f_\alpha)$, which do not entirely lie in V_α, where $\widetilde{F}_n$ is a perturbation of the form (6.16) for which the choices of ε_α converge to zero as $n \to \infty$. By $\mathbb{R}$-reparametrization, we may assume $u_n(0,\cdot)$ intersects $U_\alpha \setminus V_\alpha$, say at $u_n(0,t_n)$. By passing to a subsequence, we may assume that $t_n \to \tau$ and $u_n(0,t_n) \to p$ for some $p \notin V_\alpha$. By passing to a further subsequence, using a standard Gromov compactness argument we may assume u_n converges in C^∞_{loc} to some u, and $E(u) \leq \lim E(u_n)$.

Then $u(0,\tau) = p$ and u is a Floer trajectory for F. In particular, since u lies in U_α, and it has bounded energy, it converges at the ends to critical points in $\mathfrak{F}_\alpha$ (as these are the only 1-periodic orbits in U_α by assumption). Since F is constant on $\mathfrak{F}_\alpha$, by Lemma 6.5.1 for $G = F$ we deduce $E(u) = \omega_I(A)$. Since $0 \leq E(u) \leq E(u_n) < \hbar$, it follows that $\omega_I(A) = 0$. Thus $E(u) = 0$, so $\partial_s u = 0$. But then the Floer equation $\partial_t u = X_F$ implies that u lies at a critical point in $\mathfrak{F}_\alpha$, in particular u lies in V_α, contradicting that $u(0,\tau) \notin V_\alpha$. ∎

Corollary 6.5.4. *For each $\mathfrak{F}_\alpha$ define the local Floer chain complex $CF^*_{loc,\hbar}(U_\alpha)$ to be generated by the critical points of $\widetilde{F}$, such that among the Floer trajectories normally counted by the Floer differential we only count those which lie entirely in U_α and have energy less than $\hbar$. Then for small enough ε_α in* (6.16)*, $CF^*_{loc,\hbar}(U_\alpha)$ is a chain complex.*

Proof. Again we are adapting [CFHW96, Sec. 2] to our setup. By Proposition 6.5.3, we may assume that all Floer trajectories in U_α actually lie in V_α. If a 1-family of Floer solutions of energy less than $\hbar$ contained a Floer trajectory in U_α or limited to one, then some Floer trajectory in the family would lie U_α but not entirely in V_α, which is not allowed. This ensures that the proof that the Floer differential squares to zero also holds for our local Floer complex (in particular noting that in 1-families of Floer trajectories the energy is fixed, by Lemma 6.5.1, and equals the total energy of the broken Floer trajectories it may converge to). ∎

We call the cohomology of $CF^*_{loc,\hbar}(U_\alpha)$ the **low-energy local Floer cohomology** $HF^*_{loc,\hbar}(\mathfrak{F}_\alpha)$.

Proposition 6.5.5. *For sufficiently small ε_α in* (6.16) *(without varying ρ_α), $CF^*_{loc,\hbar}(\mathfrak{F}_\alpha)$ is the Morse complex of f_α on $\mathfrak{F}_\alpha$, up to a grading shift by*

$$\mu_\lambda(\mathfrak{F}_\alpha) = dim_{\mathbb{C}}\,\mathfrak{M} - RS(p,\lambda H) - dim_{\mathbb{C}}\,\mathfrak{F}_\alpha, \tag{6.18}$$

where $p \in \mathfrak{F}_\alpha$ (the choice does not matter). Thus

$$HF^*_{loc,\hbar}(\mathfrak{F}_\alpha) \cong H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)].$$

Proof. First observe that there cannot be a Floer trajectory for $\widetilde{F}$ of energy smaller than $\hbar$ whose ends are the same critical point, by Lemma 6.5.1. Suppose by contradiction that we had a family of time-dependent Floer trajectories u_n of $\widetilde{F}_n$ as in the proof of Proposition 6.5.3 with ends $p_- \neq p_+$. Here we fix ρ_α, and we let $\varepsilon^n_\alpha \to 0$ as $n \to \infty$. By Proposition 6.5.3 we may choose the V_α also to depend on n, so that u_n lies entirely in $V_{\alpha,n}$ and $V_{\alpha,n}$ converges to $\mathfrak{F}_\alpha$ (so $\mathrm{dist}(V_{\alpha,n},\mathfrak{F}_a) \to 0$ as $n \to \infty$). Since the bump function ρ_α are fixed and equal 1 near $\mathfrak{F}_\alpha$, we may assume $\rho_\alpha = 1$ on $V_{\alpha,n}$, thus $\widetilde{F}_n = F + \varepsilon^n_\alpha \widetilde{f}_\alpha$ on $V_{\alpha,n}$.

Since the Kähler metric g is S^1-invariant, there is an S^1-equivariant tubular neighbourhood of $\mathfrak{F}_\alpha$ ([Bre72, Thm. 2.2, Ch. VI]), thus our extension $\widetilde{f}_\alpha$ in (6.16) can be made S^1-equivariant. Recall also that I is S^1-equivariant, and that ∇H is S^1-equivariant.[7] Thus the composite $\varphi_\theta \circ u_n$ is also a Floer trajectory, for all $\theta \in S^1$. Hence, the u_n come in an $\mathbb{R} \times S^1$ family rather than just in an $\mathbb{R}$-family, which would contradict that the index of u_n is 1 unless the S^1-action does not yield genuinely new solutions: $\varphi_\theta \circ u_n = u_n$. The latter implies that u_n is fixed by the S^1-action, so u_n lies in $\mathfrak{F}_\alpha$. Since $\nabla F = 0$ on $\mathfrak{F}_\alpha$ it follows that u_n is a Floer solution for $\varepsilon^n_\alpha \widetilde{f}_\alpha$ lying in $\mathfrak{F}_\alpha \subset V_{\alpha,n}$.

Then, as the $\varepsilon^n_\alpha \widetilde{f}_\alpha$ are C^2-small Hamiltonians in $V_{\alpha,n}$, a standard argument due to Salamon-Zehnder [SZ92, Sec. 7] shows that u_n has to be time-independent, hence we get a contradiction.[8]

The claim follows, in particular the difference in grading between the Morse index of $p \in \mathrm{Crit}(f_\alpha)$ and the Conley-Zehnder index of the constant 1-periodic orbit $x_{\alpha,p}$ at p was explained in Proposition 6.3.4. ■

Corollary 6.5.6. *Then for generic $\lambda > 0$,*

$$HF^*(\lambda H) \cong \bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)],$$

[7] Since ∇H is the $\mathbb{R}_+$-action for the $\mathbb{C}^*$-action φ, and $\mathbb{C}^*$ is an abelian group.

[8] We note that [SZ92] relies on the symplectic form vanishing on π_2 to rule out holomorphic sphere bubbling, and in our case we have ruled sphere bubbling out artificially by demanding that the energy of the Floer solutions we consider is less than $\hbar$, see (6.17).

where $\mu_\lambda(\mathfrak{F}_\alpha) = dim_{\mathbb{C}}\,\mathfrak{M} - RS(x,\lambda H) - dim_{\mathbb{C}}\,\mathfrak{F}_\alpha$, *for an arbitrary* $x \in \mathfrak{F}_\alpha$. *In particular* $HF^*(\lambda H)$ *is supported in even degrees.*

Proof. We can filter the Floer cochain differential as $\partial = \partial_0 + \partial_1$, where ∂_0 is the differential defined for the local Floer chain complexes in Corollary 6.5.4. This yields a spectral sequence converging to $HF^*(\lambda H)$ such that the E_1-page is $\oplus_\alpha HF^*_{loc,\hbar}(\mathfrak{F}_\alpha)$. By Proposition 6.5.5, the E_1-page is therefore concentrated in even total degrees because $H^*(\mathfrak{F}_\alpha)$ is generated in even degrees (by Corollary 2.1.9 and equation (2.6) from Lemma 2.3.2) and the fact that $\mu_\lambda(\mathfrak{F}_\alpha)$ is even (by Lemma 6.3.2). Since the differentials in the spectral sequence increase the total grading by one, we deduce that all differentials vanish after the E_1-page. Thus the spectral sequence has already converged at the E_1-page, and by Proposition 6.5.5 it is the desired cohomology. ■

Remark 6.5.7. In particular, one can easily check that for small δ grading $\mu_\delta(\mathfrak{F}_\alpha)$ agrees with the Morse-Bott index $\mu(\mathfrak{F}_\alpha)$, hence the last corollary together with Proposition 6.4.4 recovers the Morse-Bott theorem for the perfect Morse-Bott function δH,

$$H^*(\mathfrak{M}) \cong H^*(\delta H) \cong \bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_a].$$

6.6 Filtration on Floer chain complex

In this section we construct carefully chosen Hamiltonians H_λ, for each generic slope λ, such that the Floer chain complex of H_λ has an exhausting filtration given by the value of the moment map H. This will yield both a construction of positive symplectic cohomology and of Morse-Bott-Floer spectral sequence in the following sections.

6.6.1 Construction of the Hamiltonian H_λ on $\mathfrak{M}$

The Hamiltonian H_λ of generic slope λ will be constructed as in Figure 6.1 in terms of a function c of H,

$$H_\lambda := c \circ H$$

The construction of $c : [\min H, +\infty) \to \mathbb{R}$ involves a choice of values $\{R'_0, R''_0, \ldots, R'_r, R''_r\}$ for Φ and $\{H'_0, H''_0, \ldots, H'_r, H''_r\}$ for H, which we describe later, as well as the values[9] $0 = T_0 < T_{-1} < \cdots < T_{-r}$ of the periods of the S^1-action that are smaller than λ, and a choice of generic slopes λ_i in between,

$$0 = \tau_0 < \lambda_0 < T_{-1} < \lambda_1 < T_{-2} < \cdots < T_{-r} < \lambda_r := \lambda.$$

[9]The indices are negative due to their later use in the spectral sequence, for which the negative-grading notation is more convenient.

This data then determines $c : [\min H, +\infty) \to \mathbb{R}$ so that

(1) $c' \geq 0$.

(2) $c'' \geq 0$.

(3) $c''(H) > 0$ whenever $c'(H) = T_{-i}$, for all $i = 1, \ldots, r$.

(4) $c(H) = \lambda_i H$, when $H \in [H_i'', H_{i+1}']$, for all $i = 0, \ldots, r-1$.

(5) $c(H) = \lambda_r H$ for $H \in [H_r'', +\infty)$.

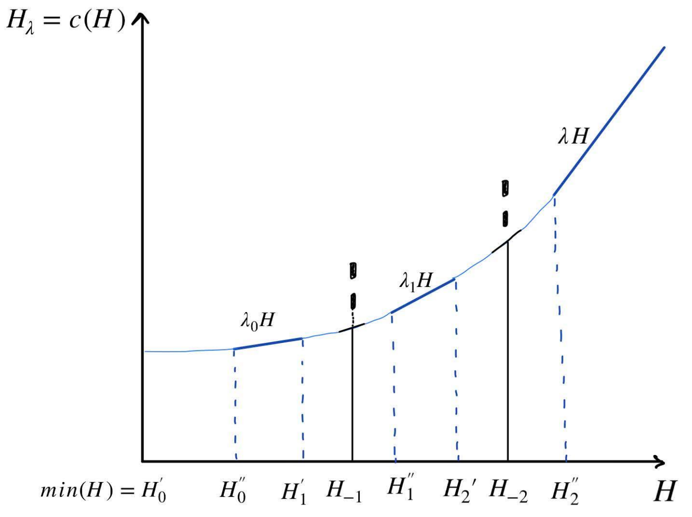


Figure 6.1: Graph of H_λ for $r = 2$

To build those values, we use from Remark 6.1.4 the map $\Psi : \mathfrak{M} \to \mathbb{C}^N$, and the slightly redefined map Φ,

$$\Phi := \Psi^*(\tfrac{1}{2}|z|^2).$$

Let Z denote the vector field of the $\mathbb{R}_+$ part of the $\mathbb{C}^*$-action on $\mathfrak{M}$. Recall from the proof of Proposition 2.2.6 that all sets $\Phi^{-1}(c)$ are smooth for $c > 0$ (whereas zero is a singular value as $\mathfrak{L} = \Phi^{-1}(0)$).

Before plunging into details, the idea is to pick the above values to get a strict chain of inclusions in $\mathfrak{M}$,

$$\Phi^{-1}(R_i') \subset H^{-1}(H_i') \subset H^{-1}(H_i'') \subset \Phi^{-1}(R_i'') \subset \Phi^{-1}(R_{i+1}'),$$

so that we have a uniform interval of space in the Φ (respectively H) values in between the inclusions.

Firstly, let $R'_0 = 0$ and $H'_0 = \min H$. Choose an arbitrary $\varepsilon > 0$. Then let $H''_0 := \max H(\Phi^{-1}(0)) + \varepsilon$, and $R''_0 := \max \Phi(H^{-1}(H''_0)) + \varepsilon$. Now, for the rest $i = 1, \dots, r$ we similarly define:

$$\begin{aligned}
R'_i :=& R''_{i-1} + \varepsilon \\
H'_i :=& \max H(\Phi^{-1}(R'_i)) + \varepsilon \\
H''_i :=& H_i + \varepsilon \\
R''_i :=& \max \Phi(H^{-1}(H''_i)) + \varepsilon.
\end{aligned}$$

In Figure 6.1 we indicated H-values H_{-1}, H_{-2} where 1-periodic orbits of H_λ arise, thus defined by $c'(H_i) = T_i$.

6.6.2 Filtration functional on $\mathbb{C}^N$

Here we construct the filtration functional on $\mathbb{C}^N$ that is a special case of the one constructed in [McLR18, Sec. 6], for the specific Hamiltonian $H_{\mathbb{C}^N}$ that we define below.

Let $R = \frac{1}{2}|z|^2$ on $\mathbb{C}^N$. The smooth cut-off function $\phi : [0, +\infty) \to \mathbb{R}$ is chosen to satisfy

(1) $\phi' \geq 0$.

(2) $\phi = 0$ on $[0, R''_0]$.

(3) $\phi = 1$ on $[R'_1, +\infty)$.

Notice that $\phi' = 0$ outside of $[R''_0, R'_1]$.

Let $H_{\mathbb{C}^N} = h(R)$ be a radial Hamiltonian where $h : \mathbb{R} \to [0, +\infty)$ satisfies

(1) $h' \geq 0$.

(2) $h'' \geq 0$.

(3) $h'(R) = w\lambda_i$ on $[R''_i, R'_{i+1}]$ for all $i = 0, \dots, r-1$.

(4) $h'(R) = w\lambda$ on $[R''_r, +\infty]$.

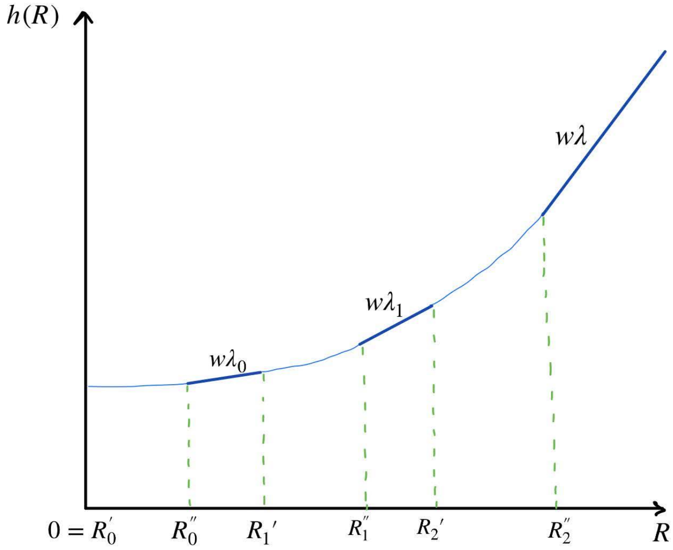


Figure 6.2: Graph of $h(R)$ for $r = 2$

Denoting by ω the standard Liouville form in $\mathbb{C}^N$, we have $\omega = d(R\alpha)$ where α is the contact form on the contact hypersurface $\{R = 1\}$. The cut-off function determines an exact 2-form on $\mathbb{C}^N$,

$$\eta = d(\phi(R)\alpha) = \phi(R)\, d\alpha + \phi'(R)\, dR \wedge \alpha,$$

and an associated 1-form Ω_η on the free loop space $\mathcal{L}\mathbb{C}^N = C^\infty(S^1, \mathbb{C}^N)$ given by

$$\Omega_\eta : T_x\mathcal{L}\mathbb{C}^N = C^\infty(S^1, x^*T\mathbb{C}^N), \;\; \xi \mapsto -\textstyle\int \eta(\xi, \partial_t x - X_{H_{\mathbb{C}^N}})\, dt. \tag{6.19}$$

Further, let $f : \mathbb{R} \to [0, \infty)$ be the smooth function defined by

$$f(R) := \textstyle\int_0^R \phi'(\tau)\, h'(\tau)\, d\tau.$$

Define the **filtration functional** $F : \mathcal{L}\mathbb{C}^N \to \mathbb{R}$ on the free loop space by

$$F(x) := -\int_{S^1} x^*(\phi\alpha) + \int_{S^1} f(R(x(t)))\, dt.$$

Lemma 6.6.1. [McLR18, Thm. 6.2(1)] *F is a primitive of Ω_η. That is, $dF(x)(\xi) = \Omega_\eta(x)(\xi)$.* ∎

6.6.3 Filtration on $CF^*(H_\lambda)$

By Proposition 6.1.3, $\Psi : \mathfrak{M} \to \mathbb{C}^N$ is $\mathbb{C}^*$-equivariant, with weight w action on $\mathbb{C}^N$. For a 1-orbit x of H_λ, we define by abuse of notation:

$$F(x) := F(\Psi(x)).$$

Theorem 6.6.2. *The Floer chain complex $CF^*(H_\lambda)$ has a filtration given by the value of F. That is, given two 1-periodic orbits x_-, x_+ of H_λ and a Floer cylinder for H_λ from x_- to x_+,*

$$F(x_-) \geq F(x_+). \tag{6.20}$$

Proof. The Floer cylinder $u : \mathbb{R} \times S^1 \to \mathfrak{M}$ for H_λ satisfies

$$\partial_s u + I(\partial_t u - X_{H_\lambda}) = 0.$$

Let $X_{\mathbb{C}^N}$ be the vector field of the S^1-rotation action on $\mathbb{C}^N$, so the Hamiltonian vector field of $R = \frac{1}{2}|z|^2$. By the same trick as in (6.4),

$$\Psi_*(X_{H_\lambda}) = \Psi_*(c'(H)X_H) = c'(H)\Psi_*(X_H) = c'(H)wX_{\mathbb{C}^N}, \tag{6.21}$$

noting that $c'(H)$ depends on the original coordinates in $\mathfrak{M}$. Composing Ψ with u gives us a mapping

$$v := \Psi \circ u : \mathbb{R} \times S^1 \to \mathbb{C}^N, \quad \partial_s v + i(\partial_t v - k(s,t)X_{\mathbb{C}^N}) = 0 \tag{6.22}$$

that converges to $y_- = \Psi(x_-)$, $y_+ = \Psi(x_+)$ at $s = -\infty, +\infty$, respectively, where

$$k(s,t) := w\, c'(H(u(s,t)))$$

is a domain-dependent function. Now define the **s-parametrised 1-form**

$$\Omega^v_\eta(s) : T_x\mathcal{L}\mathbb{C}^N = C^\infty(S^1, x^*T\mathbb{C}^N), \;\; \xi \mapsto -\textstyle\int \eta(\xi, \partial_t x - k(s,t)X_{\mathbb{C}^N})\, dt. \tag{6.23}$$

Using $\eta = d(\phi(R)\alpha) = \phi(R)\, d\alpha + \phi'(R)\, dR \wedge \alpha$, $d\alpha(\cdot, X_{\mathbb{C}^N}) = 0$, $dR(X_{\mathbb{C}^N}) = 0$ and $\alpha(X_{\mathbb{C}^N}) = 1$:

$$\begin{aligned}
\Omega^v_\eta(s)(x)(\xi) &= -\int \eta(\xi, \partial_t x)\; dt + \int k(s,t)\, \eta(\xi, X_{\mathbb{C}^N}) dt \\
&= -\int \eta(\xi, \partial_t x)\; dt + \int k(s,t)\, \phi(R) d\alpha(\xi, X_{\mathbb{C}^N})\; dt \\
&\quad + \int k(s,t)\, \phi'(R)(dR \wedge \alpha)(\xi, X_{\mathbb{C}^N})\; dt \\
&= -\int \eta(\xi, \partial_t x)\; dt + \int \phi'(R) dR(\xi) k(s,t)\; dt.
\end{aligned} \tag{6.24}$$

Similarly,

$$\Omega_\eta(x)(\xi) = -\int \eta(\xi, \partial_t x)\; dt + \int \phi'(R) dR(\xi) h'(R(x(t)))\; dt. \tag{6.25}$$

From the way we have chosen the Hamiltonian H_λ and the function $h(R)$, for $R(v(s,t)) \in [R_0'', R_1']$ we have

$$k(s,t) = w\ c'(H(u(s,t))) = w\lambda_0 = h'(R(v(s,t))), \tag{6.26}$$

whereas outside that region we have $\phi'(R) = 0$. All together we have

$$\phi'(R(v(s,t)))k(s,t) = \phi'(R(v(s,t)))h'(R(v(s,t))), \tag{6.27}$$

which from (6.24) and (6.25) yields

$$\Omega_\eta(v(s,t))(\partial_s v) = \Omega_\eta^v(s)(v(s,t))(\partial_s v). \tag{6.28}$$

Thus, together with Lemma 6.6.1 we have

$$\begin{aligned} F(x_-) - F(x_+) = F(y_-) - F(y_+) &= -\int_{-\infty}^{+\infty} dF(v(s,t))(\partial_s v)ds \\ &= -\int_{-\infty}^{+\infty} \Omega_\eta(v(s,t))(\partial_s v)ds \\ &= -\int_{-\infty}^{+\infty} \Omega_\eta^v(s)(v(s,t))(\partial_s v)ds. \end{aligned} \tag{6.29}$$

Hence, it is sufficient to prove that

$$\Omega_\eta^v(s)(v(s,t))(\partial_s v) \leq 0$$

holds for v satisfying equation (6.22). This we do verbatim to the proof of [McLR18, Lem. 6.1]. Hence, substituting the Floer equation $\partial_t v - k(s,t)X_{\mathbb{C}^N} = i\partial_s v$, and abbreviating $\rho = R \circ v$,

$$\begin{aligned} \eta(\partial_s v, \partial_t v - k(s,t)X_{\mathbb{C}^N}) &= \eta(\partial_s v, i\partial_s v) \\ &= \phi(\rho) \cdot d\alpha(\partial_s v, i\partial_s v) + \phi'(\rho) \cdot (dR \wedge \alpha)(\partial_s v, i\partial_s v) \\ &= \text{positive} \cdot \text{positive} + \text{positive} \cdot (dR \wedge \alpha)(\partial_s v, i\partial_s v), \end{aligned} \tag{6.30}$$

where "positive" here means "non-negative". To estimate the last term, we may assume that $R \geq R_0''$ since $\phi' = 0$ otherwise. Thus, we decompose $\partial_s v$ according to an orthogonal decomposition of $T\mathbb{C}^n$:

$$\partial_s v = C \oplus yX_{\mathbb{C}^N} \oplus zZ \in \xi \oplus \mathbb{R}X_{\mathbb{C}^N} \oplus \mathbb{R}Z,$$

where $Z = -iX_{\mathbb{C}^N} = R\partial_R$ is the Liouville vector field (as before: $R = \frac{1}{2}|z|^2$) and $\xi = \ker \alpha|_{R=1}$. Notice that $\ker \alpha = \xi \oplus \mathbb{R}Z$. Thus: $dR(\partial_s v) = Rz$ and $\alpha(i\partial_s v) = \alpha(izZ) = \alpha(zX_{\mathbb{C}^N}) = z$. So,

$$(dR \wedge \alpha)(\partial_s v, i\partial_s v) = dR(\partial_s v)\alpha(i\partial_s v) - \alpha(\partial_s v)dR(i\partial_s v) = Rz^2 + Ry^2 \geq 0.$$

Hence $\Omega_\eta^v(s)(v(s,t))(\partial_s v) \leq 0$, thus $F(\Psi(x_-)) - F(\Psi(x_+)) \geq 0$. ■

Corollary 6.6.3. *$CF^*(H_\lambda)$ has a filtration given by the value of H (the "radial coordinate" in $\mathfrak{M}$).*

Proof. Let us calculate the value of the functional $F(y)$ explicitly for the projection $y := \Psi(x(t))$ of a 1-periodic orbit x of H_λ. If x is a fixed point, $y = 0$ so $F(y) = 0$. Otherwise, $c'(H(x)) = T_{-k}$ for some $0 < k < r$. Then its projection is a 1-periodic orbit of the Hamiltonian $\frac{1}{2}wT_{-k}|z|^2$, so

$$\begin{aligned} F(y(t)) &= -\int_{S^1} y^*(\phi\alpha) + \int_{S^1} f(R(y(t)))dt \\ &= -\phi(y)wT_{-k} + \int_{R_0''}^{R_1'} \phi'(R)h'(R)dR \\ &= -wT_{-k} + \lambda_0 w = w(\lambda_0 - T_{-k}), \end{aligned}$$

where $\phi(y) = 1$ as 1-orbits of h in $\mathbb{C}^N$ with slopes $\geq T_{-1}$ lie in the region $R > R_1'$ (as $T_{-1} > \lambda_0$), and there $\phi = 1$. Thus $F(x(t))$ has a negative value which strictly decreases when T_{-k}, hence $H(x)$, increases. ■

6.7 Positive symplectic cohomology and models for ordinary cohomology

By Corollary 6.6.3 we have that the fixed loci $\mathfrak{F} = \mathfrak{M}^\varphi$ (which are constant 1-orbits for H_λ) determine a subcomplex $CF_0^*(H_\lambda) \subset CF^*(H_\lambda)$. Therefore, passing to the quotient $CF_+(H_\lambda) := CF^*(H_\lambda)/CF_0^*(H_\lambda)$ we obtain positive Floer cohomology $HF_+^*(H_\lambda) = H_*(CF_+(H_\lambda))$. The direct limit of those defines the **positive symplectic cohomology**,

$$SH_+(\mathfrak{M}, \varphi, \omega_I) := \lim_{\lambda\to\infty} HF_+^*(H_\lambda).$$

Proposition 6.7.1. *There is a long exact sequence*

$$\cdots \to QH^*(\mathfrak{M}, I) \to SH^*(\mathfrak{M}, \varphi, \omega_I) \to SH_+^*(\mathfrak{M}, \varphi, \omega_I) \to QH^{*+1}(\mathfrak{M}, I) \to \cdots$$

Proof. Firstly, let us prove that the subcomplex $CF_0^*(H_\lambda)$ is equal to $CF^*(H_{\lambda_0})$, and recall that λ_0 was chosen to be smaller than any non-zero S^1-period. The generators of $CF_0^*(H_\lambda)$ are the same as the generators of $CF^*(H_\delta)$. Moreover, the differential in $CF_0^*(H_\lambda)$ counts Floer solutions that do not escape the region $R \leq R_0''$, due to the filtration. Namely, assuming the contrary, the value of ϕ on a Floer solution will be positive in a small neighbourhood, hence, by (6.30), Ω_η^v will be strictly negative in such a neighbourhood, hence by (6.29) the filtration difference will be strictly positive, which is a contradiction. Hence, the Floer differential in $CF_0^*(H_\lambda)$ counts the same solutions as $CF^*(H_{\lambda_0})$, and thus their homologies are isomorphic, and so it is isomorphic to $QH^*(\mathfrak{M})$ by Proposition 6.4.4.

Now, the short exact sequence of complexes

$$0 \to CF_0^*(H_\lambda) \to CF^*(H_\lambda) \to CF^*(H_\lambda)/CF_0^*(H_\lambda) \to 0$$

yields the long exact sequence

$$QH^*(\mathfrak{M}) \to HF^*(H_\lambda) \to HF_+^*(H_\lambda) \to QH^{*+1}(\mathfrak{M})$$

which, using the compatibility of Hamiltonians H_λ, yields the claim by taking the direct limit over continuation maps. ■

Therefore, as a consequence of the vanishing in Proposition 6.4.3, and Proposition 2.1.13, we obtain:

Corollary 6.7.2. ***(Models for ordinary cohomology)*** *Given a CSR $\mathfrak{M}$ and a conical action φ on it, we have a Hamiltonian model for its ordinary cohomology:*

$$H^*(\mathfrak{M}) \cong SH_+^{*-1}(\mathfrak{M}, \varphi, \omega_I).$$

Hence, varying the actions φ we in principle get different models for the ordinary cohomology $H^*(\mathfrak{M})$ of a CSR $\mathfrak{M}$. We will see this in the examples in Section 6.9.

6.8 Morse-Bott-Floer spectral sequences

In this section we construct Morse-Bott-Floer spectral sequences that converge to $SH^*(\mathfrak{M}, \varphi)$ and $SH_+^*(\mathfrak{M}, \varphi)$, arising from the filtration of the Floer chain complex $CF^*(H_\lambda)$ from Section 6.6. Further, we list main properties of these spectral sequences, and list some techniques that allow one to compute them in practice.

6.8.1 Hamiltonian 1-orbits of H_λ

Let us see what the 1-periodic orbits of the Hamiltonian H_λ are (from now on, we will call them 1-orbits). As $X_{H_\lambda} = c'(H)X_H$, the 1-orbits of H_λ are either critical points in $\mathfrak{F} = Crit(H_\lambda) = Crit(H) = \mathfrak{M}^\varphi$, or, for some $p = -1, \ldots, -r$, lie on a **slice**

$$\mathcal{S}_p := \{x \in \mathfrak{M} \mid H(x) = H_p\} \subset \mathfrak{M}, \tag{6.31}$$

where the values H_p are defined by $c'(H_p) = T_p$. Let us denote by $\mathcal{O}_p := \mathcal{O}_{p,H_\lambda}$ the moduli space of parametrised 1-orbits of H_λ in $\mathcal{S}_p$, and by B_p the set of their initial points,

$$B_p := \{x(0) \mid x \in \mathcal{O}_p\} \subset \mathcal{S}_p.$$

Recall (Definition 2.3.8) that the fixed locus under the time $2\pi/m$ flow of the S^1-action is denoted by

$$\mathcal{R}_{\mathbb{Z}/m} := \{\mathbb{Z}/m\text{-torsion points of } \varphi \text{ in } \mathfrak{M}\}.$$

Now, in order to suppress the annoying 2π-factors in the following text, we define:

Definition 6.8.1. Given an 1-orbit of H_λ that lies in $\mathcal{S}_p$, its **normalised period** is $\tau_p := T_p/2\pi$. This is precisely the period of this orbit in the normalised time-1 S^1-flow.

Lemma 6.8.2. *$B_p = \mathcal{S}_p \cap \mathcal{R}_{\mathbb{Z}/m}$ where $\tau_p = k/m$ for coprime $k, m \in \mathbb{N}$.*

Proof. The points in B_p are fixed under the time-1 flow of H_λ. As $X_{H_\lambda} = c'(H)X_H$ and on $c'(H) = 2\pi\tau_p$ on $\mathcal{S}_p$, we see that the points in B_p are fixed under the time-$2\pi\tau_p$ flow of the S^1-action. As all closed subgroups of S^1 are discrete, τ_p has to be a rational number, $\tau_p = k/m$ for some coprime $k, m \in \mathbb{N}$.

As $(k, m) = 1$, there are $\alpha, \beta \in \mathbb{Z}$ such that $\alpha k + \beta m = 1$. Thus, points in B_p are fixed under the time $2\pi\alpha k/m = (2\pi/m) - 2\pi\beta$ flow, and hence by the time $2\pi/m$ flow, so they lie in $\mathcal{R}_{\mathbb{Z}/m}$. Thus, $B_p \subset \mathcal{S}_p \cap \mathcal{R}_{\mathbb{Z}/m}$. Conversely, a point in $\mathcal{S}_p \cap \mathcal{R}_{\mathbb{Z}/m}$ is fixed under the time-$2\pi/m$ and thus time-$2\pi\tau_p$ flow of S^1, hence is an 1-orbit of H_λ, thus lies in B_p. ■

In particular, this lemma shows that B_p is the fixed locus of the $\mathbb{Z}/m$-part of action φ on $\mathcal{S}_p$. Thus, being a fixed locus of a compact Lie group action on a smooth manifold, it is smooth itself [DK00, p. 108]. We denote the connected components of B_p by $B_{p,c}$ where the labelling by c depends on p,

$$B_p = \bigsqcup_c B_{p,c}.$$

Now, denote by $\xi := (\langle X_{\mathbb{R}_+}\rangle \oplus \langle X_{H_\lambda}\rangle)^\perp$ the complex codimension-1 distribution on $\mathfrak{M} \setminus \mathfrak{F}$. Here $X_{\mathbb{R}_+}$ is the vector field of the $\mathbb{R}_+$-action, and the orthogonal $\perp$ is calculated with respect to the Kähler metric $g(\cdot,\cdot) = \omega_I(\cdot, I\cdot)$. Thus, for any point $x \in \mathfrak{M} \setminus \mathfrak{F}$, we have g-orthogonal splittings of tangent spaces

$$T_x\mathfrak{M} = \langle X_{\mathbb{R}_+}\rangle \oplus \langle X_{H_\lambda}\rangle \oplus \xi, \quad T_x\mathcal{S}_p = \langle X_{H_\lambda}\rangle \oplus \xi. \tag{6.32}$$

Definition 6.8.3. Denote by ϕ_t^X the time-t flow of a vector field X. In particular, when $X = X_F$ is a Hamiltonian vector-field, we will sometimes also denote this flow by ϕ_t^F.

Lemma 6.8.4. $(\phi_\tau^{H_\lambda})_* X_{\mathbb{R}_+} = X_{\mathbb{R}_+} + \tau c''(H) \|\nabla(H)\|^2 X_H$.

Proof. Let us fix a point $x \in \mathfrak{M}$. Letting $\gamma(s) = \phi_s^{X_{\mathbb{R}_+}}(x)$, we have that

$$(\phi_\tau^{H_\lambda})_*(x) X_{\mathbb{R}_+} = \phi_\tau^{H_\lambda}{}_* \left(\frac{d}{ds}\bigg|_{s=0} \gamma(s) \right) = \frac{d}{ds}\bigg|_{s=0} (\phi_\tau^{H_\lambda}(\gamma(s))) = \frac{d}{ds}\bigg|_{s=0} (\phi^H_{\tau c'(H(s))}(\gamma(s))).$$

where we used that $X_{H_\lambda} = c'(H) X_H$. Abbreviate $u(t,s) = \phi_t^{X_H}(\gamma(s))$, then the last term is equal to

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} u(\tau c'(H(s)), s) &= \left(\frac{d}{ds}\bigg|_{s=0} \tau c'(H(s)) \right) \frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} \\ &= (\tau c''(H(\gamma(s))) d_{\gamma(s)} H \cdot \gamma'(s))_{s=0} X_H + X_{\mathbb{R}_+} \\ &= (\tau c''(H(\gamma(s))) \|\nabla H(\gamma(s))\|^2)_{s=0} X_H + X_{\mathbb{R}_+} \\ &= \tau c''(H(x)) \|\nabla H(x)\|^2 X_H + X_{\mathbb{R}_+}, \end{aligned}$$

where in the penultimate equality we have used that $X_{\mathbb{R}_+} = \nabla H$, and in the second equality we used that $\partial_s u|_{s=0} = d\varphi_{it} \cdot X_{\mathbb{R}_+}|_x = d\varphi_{it} \cdot \partial_r|_{r=1} \varphi_r(x) = \partial_r|_{r=1} \varphi_r(\varphi_{it}(x)) = \partial_r|_{r=1} \varphi_r(u) = X_{\mathbb{R}_+}|_u$. ■

Now we prove the following non-degeneracy property of submanifolds $B_{p,c}$, which is analogous to the Morse-Bott non-degeneracy condition for critical submanifolds.[10]

Proposition 6.8.5. *Given any point $x \in B_{p,c}$, the 1-eigenspace of the linearised return map of the flow for H_λ is precisely the tangent space of $B_{p,c}$,*

$$Ker\left((\phi_1^{H_\lambda})_* x - Id\right) = T_x B_{p,c}.$$

Proof. As in Lemma 6.8.2, let $\tau_p = k/m$, for coprime $k, m \in \mathbb{N}$. Then, on $\mathcal{S}_p$,

$$\phi_1^{H_\lambda} = \phi^H_{2\pi\tau_p} = (\phi^H_{2\pi/m})^k.$$

Hence given a vector $v \in T_x \mathcal{S}_p$, the condition $(\phi_1^{H_\lambda})_*(v) = v$ is equivalent to $(\phi^H_{2\pi/m})^k_*(v) = v$, and thus to $(\phi^H_{2\pi/m})_*(v) = v$, due to $(k,m) = 1$. We see that $\phi^H_{2\pi/m}$ generates a $\mathbb{Z}/m$-action on $\mathcal{S}_p$ by g-isometries. Its fixed locus is precisely B_p, by Lemma 6.8.2. Thus, picking an arbitrary connected component $B_{p,c}$, by the equivariant tubular

[10] As we can think of $B_{p,c}$ being the critical submanifolds for the action functional $d\mathcal{A}_{H_\lambda}(x)(\xi) = \int \omega_I(\dot{x} - X_{H_\lambda}, \xi)$.

neighbourhood theorem for isometric actions of compact Lie groups [Bre72, Thm. 2.2, Ch. VI], there is a $\mathbb{Z}/m$-invariant tubular neighbourhood $\mathcal{N}B_{p,c} \subset \mathcal{S}_p$ of $B_{p,c}$ given by the composition of $\mathbb{Z}/m$-equivariant exponential map and a contraction ψ of the normal bundle onto a neighbourhood of the zero section,

$$exp_\epsilon \circ \psi : NB_{p,c} \xrightarrow{\cong} \mathcal{N}B_{p,c}$$

(where the $\mathbb{Z}/m$-action on $NB_{p,c}$ is generated by $(\phi^H_{2\pi/m})_*$). Hence, if a vector $v \in T_x\mathcal{S}_p$ satisfies $(\phi^H_{2\pi/m})_*(v) = v$, its class $[v] \in (NB_{p,c})_x$ is fixed by the $\mathbb{Z}/m$-action and so is the curve $exp(vt) \subset \mathcal{N}B_{p,c}$. As $B_{p,c}$ is the isolated fixed locus in $\mathcal{N}B_{p,c}$, this curve has to lie in $B_{p,c}$, thus $v \in B_{p,c}$.

To conclude, we have proved that $\mathrm{Ker}(((\phi_1^{H_\lambda})_* - Id)|_{T_x\mathcal{S}_p}) = T_xB_{p,c}$. As the complement of $T_x\mathcal{S}_p$ in $T_x\mathfrak{M}$ is generated by $X_{\mathbb{R}_+}$, together with Lemma 6.8.4 and the fact that $c''(H) > 0$ when $c'(H) = \tau_p$,[11] we immediately get the desired claim. ■

Lemma 6.8.6. *The linearisation of the S^1-flow is complex linear with respect to a unitary trivialisation of ξ along every 1-orbit in $B_{p,c}$.*

Proof. Follows immediately as the S^1-flow is I-linear and a unitary trivialisation is complex-linear. ■

6.8.2 Constructing the spectral sequence for $HF^*(H_\lambda)$

In this section we construct Morse-Bott-Floer spectral sequences that converge to $HF^*(H_\lambda)$ and $HF^*_+(H_\lambda)$. We use arguments that are analogous to those written in [McLR18, Sec. 7]. The difference from their setup is that their ambient space is a symplectic manifold convex at infinity. In particular, it has a contact hypersurface, so there is a Reeb flow. Manifolds in our setup usually are not convex at infinity, thus there is no contact hypersurface. The role of it is taken by an energy level of the moment map H, and the role of the Reeb flow is taken by the flow of the S^1-action.

As the Hamiltonian H_λ is autonomous, its 1-orbits are not isolated. Thus, in order to compute $HF^*(H_\lambda)$, one would typically do a C^2-small time-dependent perturbation of H_λ localised near the manifolds $B_{p,c}$ and $\mathfrak{F}_\alpha$ and compute the homology of the perturbed complex. However, doing so would ruin[12] our filtration on the complex $CF^*(H_\lambda)$ obtained in Section 6.6.3, which we need in order to construct the

[11] Condition (3) in the definition of c.

[12] One could argue that for sufficiently small perturbations, the Floer trajectories converge to cascades, and since the filtration holds for the cascades, the filtration must also holds for the Floer trajectories obtained for sufficiently small perturbations. However, we decided to avoid that approach.

spectral sequence. The issue occurs as (6.21) fails if we perturb H_λ, thus we get the non-vanishing term $d\alpha(\xi, Y)$ in equation (6.24), where $Y = \Psi_*(X_{\widetilde{H_\lambda}} - X_{H_\lambda})$ is the projection of the difference in Hamiltonian vector fields in $\mathfrak{M}$ that occurred by perturbation. We cannot say how big is the change in the difference of the filtration values (6.29), caused by integration $\int_{-\infty}^{+\infty} d\alpha(\partial_s v, Y)ds$ of this term.

Thus, we will instead use the **Morse-Bott Floer complex** which introduces auxiliary Morse functions f_i on each $B_{p,c}$ and on each $\mathfrak{F}_\alpha$. The union $\bigsqcup_i Crit(f_i)$ of all critical points are the generators of the complex, after appropriate degree shifts. The differential counts **cascades** i.e. alternately following the flows of the negative gradients $-\nabla f_i$ or following Floer solutions that join two 1-orbits. This is the natural complex that would arise from a limit, as one undoes small time-dependent perturbations of H_λ localised near the manifolds $B_{p,c}$ and $\mathfrak{F}_\alpha$.

Bourgeois-Oancea [BO09a, BO09b] showed, in the case when manifolds of 1-orbits are circles, that the Morse-Bott Floer complex for an autonomous Hamiltonian computes the same Floer cohomology as when using a time-dependent perturbation. They require transversal non-degeneracy of 1-orbits, which translated to the general case becomes the condition that we proved in Lemma 6.8.5. We believe that their reasoning extends for more general Morse-Bott manifolds (i.e. not only circles), but proving that would be rather a substantial amount of work and is outside of scope of this thesis. Hence, we will make it as an assumption.

Assumption 1. Morse-Bott Floer complex for $HF^*(H_\lambda)$ computes the same Floer cohomology as the one obtained from a time-dependent perturbation.

The Morse-Bott Floer complex is filtered: a critical point $x \in Crit(f_i)$ has filtration value $F(x) := F(y)$, where y is an arbitrary 1-orbit lying in the manifold on which f_i is introduced. It does not depend on the choice of a 1-orbit due to Corollary 6.6.3. From the filtration we then run the standard argument in homological algebra to construct a spectral sequence E_r^{pq}. Abbreviate by $\mathbf{k} = \mathbf{p} + \mathbf{q}$ the **total degree**. Let C^* denote the Floer complex $CF^*_+(H_\lambda)$ or $CF^*(H_\lambda)$. The filtration is defined by letting

$$F^p(C^k) := \{x \in C^k \mid F(x) \geq F_p\},$$

where

$$F_p := \begin{cases} F(y) & \text{for } p < 0 \text{ and } y \in B_{p,c} \\ p & \text{for } p \geq 0. \end{cases}$$

In particular, $F^p(C^k) = 0$ for $p > 0$ since $F \leq 0$ on all 1-orbits. Recall the spectral sequence for this filtration has $E_0^{pq} = F^p(C^k)/F^{p+1}(C^k)$. As the filtration is exhaustive and bounded below, it yields:

Proposition 6.8.7. *There are convergent spectral sequences,*

$$E_r^{pq} \Rightarrow HF_+^*(H_\lambda) \quad \text{where} \quad E_1^{pq} = HF_{\text{loc}}^k(\mathcal{O}_p, H_\lambda) \text{ for } p<0, \text{ and } 0 \text{ otherwise}$$
$$E_r^{pq} \Rightarrow HF^*(H_\lambda) \quad \text{as above, except} \quad E_1^{0q} = H^q(\mathfrak{M}).$$

There are finitely many non-zero columns, as there are only finitely many $\mathcal{O}_p$ for every Hamiltonian H_λ. Above, $HF_{\text{loc}}^*(\mathcal{O}_p, H_\lambda)$ refers to the cohomology of the local Morse-Bott Floer complex generated by $\mathcal{O}_p$. The chain level generators of $CF_{loc}^*(\mathcal{O}_p, H_\lambda)$ are the critical points of the auxiliary Morse function on $B_{p,c}$. By construction, its differential only counts cascades which do not change the filtration value, so the Floer solutions stay trapped in the slice $\mathcal{S}_p$. If one were to make a very small time-dependent perturbation of H_λ supported near $\mathcal{S}_p$, the argument in [CFHW96, Prop.2.2] and [Oan04, Sec.3.3] would show that this is quasi-isomorphic to the local Floer complex for that slice, where one only considers Floer solutions whose filtration value stays bounded within a small neighbourhood of the value $F = F_p$. The state-of-the-art work on the local Floer complex is the recent paper by Kwon-van Koert, which proves that the local Floer cohomology of manifolds of 1-orbits is isomorphic to the Morse cohomology with a shift [KwKo16, Prop.B.4]. Their result asks for three conditions to be satisfied for the manifolds $B_{p,c}$, two out of which we have proved, Proposition 6.8.5 and Lemma 6.8.6. However, the third one is a very restrictive **symplectic triviality** (ST) condition for submanifolds:

(ST) The restriction $T_{B_{p,c}}\mathfrak{M}$ of the tangent bundle of $\mathfrak{M}$ on an arbitrary $B_{p,c}$ is symplectically trivial. Moreover, each 1-orbit in $B_{p,c}$ has a capping disk to which the trivialisation extends.

The ambient manifolds which they consider in the paper are Brieskorn varieties, whose tangent bundles are symplectically trivial, thus the (ST) is satisfied immediately. A similar situation appears in the work of Ritter-McLean [McLR18], as they resolve $\mathbb{C}^n/G$, so they have (outside of a compact set) global coordinates, as the resolution is biholomorphic to $\mathbb{C}^n/G$ away from the core. In the setup of an arbitrary CSR it is not clear why the (ST) condition should be true - we actually believe that it is unlikely to hold. Nevertheless, we believe that the result [KwKo16, Prop.B.4] is still true in our setup. At the moment we have not yet developed the tools to prove it, so we state it as an assumption:

Assumption 2. $HF_{\text{loc}}^*(\mathcal{O}_p, H_\lambda) \cong \bigoplus_c H^{*-\mu(B_{p,c})}(B_{p,c})$.

Remark 6.8.8. We are not entirely certain that we can exclude the possibility of a Floer trajectory u lying entirely in the preimage of an S^1-orbit in $\mathfrak{M}_0$ under π, causing cancellations between two classes respectively in $H^*(B_{p,c})$, $H^*(B_{p,c'})$, for two different connected components of B_p. This issue only arises if

$$\pi(B_{p,c}) \cap \pi(B_{p,c'}) \neq \emptyset \subset \mathfrak{M}_0.$$

Although the filtration values at the ends of u are equal, (6.30) could vanish if $\partial_s v = 0$, i.e. $v(s,t) = v(t)$.

In the statement of the above assumption, observe that $\mu(B_{p,c})$ is the grading of the 1-orbit that corresponds to the minimum of the Morse function used to perturb $B_{p,c}$ in the construction of the local Floer complex $CF^*_{loc}(\mathcal{O}_p, H_\lambda)$. We will compute $\mu(B_{p,c})$ explicitly in Proposition 6.8.17 as we will need it in the examples. The above discussion is analogous to the local Floer cohomology discussed in Section 6.5, where we also discussed these grading shifts, except above we do not impose low-energy conditions like we did in Section 6.5.

Now, in order to obtain the spectral sequence that converges to symplectic cohomology out of those constructed in Proposition 6.8.7, we have to make a more clever choice of Hamiltonians H_λ such that their spectral sequences become more compatible, in particular, allowing the direct limit when $\lambda \to +\infty$.

6.8.3 Tweaking the construction of H_λ and spectral sequence for $SH^*(\mathfrak{M}, \varphi)$

Before we embark on defining a more complicated version of H_λ, we will motivate why it is required.

In Floer theory it is often convenient to ensure that $CF^*(H_{\lambda_1})$ is a subcomplex of $CF^*(H_{\lambda_2})$ whenever $\lambda_1 < \lambda_2$. The usual approach is to construct the H_λ inductively so that $H_{\lambda_2} = H_{\lambda_1}$ except in the region T at infinity where H_{λ_1} has slope $< \lambda_1$, and to build H_{λ_2} by increasing the slope in T up to the new slope λ_2. So $H_{\lambda_1} = H_{\lambda_2}$ on $\mathfrak{M} \setminus T$. If a maximum principle held for H_{λ_2} in T, then Floer solutions for H_{λ_2} with ends in $\mathfrak{M} \setminus T$ would stay there and therefore would agree with the Floer solutions of H_{λ_1}. By Theorem 6.6.2, the filtration prohibits Floer solutions from going from an end in T to an end in $\mathfrak{M} \setminus T$, and therefore the Floer differential on 1-orbits in $\mathfrak{M} \setminus T$ agrees for both H_{λ_1} and H_{λ_2}. Thus $CF^*(H_{\lambda_1}) \subset CF^*(H_{\lambda_2})$ is a subcomplex.

Similarly, it is often convenient if one can construct the continuation map

$$CF^*(H_{\lambda_1}) \to CF^*(H_{\lambda_2})$$

to be the inclusion of the subcomplex. The natural homotopy to use is one that decreases the slope of H_{λ_2} in T back to λ_1, whilst staying constantly equal to H_{λ_1} on $\mathfrak{M} \setminus T$. As above, by an analogue of Theorem 6.6.2, the filtration prohibits Floer continuation solutions that go from T to $\mathfrak{M} \setminus T$. If the ends are both in $\mathfrak{M} \setminus T$, then provided the maximum principle applies in T we deduce that the Floer continuation solution is trapped in $\mathfrak{M} \setminus T$. But there the homotopy is s-independent, so the continuation solution must be constant as otherwise it would arise in a $\mathbb{R}$-family, due to the reparametrization action (Floer continuation solutions have index zero, unlike Floer trajectories which have index 1 and always admit an $\mathbb{R}$-reparametrization action).

In our setup, the maximum principle only holds in the regions where c' is locally constant so that $c'(H)$ in (6.4) is not domain-dependent. Therefore, in both situations above, there may be an "evil Floer solution" that has ends in $\mathfrak{M} \setminus T$, and enters T where it has a maximum in a region where c' is not locally constant. The implication for the preceding spectral sequence is that, as we increase the slope from λ_1 to λ_2, new arrows may appear in the old spectral sequence for $HF^*(H_{\lambda_1})$ caused by evil Floer solutions that only H_{λ_2} sees, even though it only involves the "old" columns of the spectral sequence. This also makes it difficult to read off the filtration for $H^*(\mathfrak{M})$ from the spectral sequence, because a generator of $H^*(\mathfrak{M})$ killed by such an arrow would have filtration value λ_2, not λ_1, even though the arrow only involves generators that already existed for H_{λ_1}.

To solve this issue, we will adjust the construction of H_λ so that a substantial drop in filtration value arises whenever a Floer solution passes through a region where c is linear (see Figure 6.1).

Firstly, in the notation from Section 6.6.1, we do not require the same ε to be used throughout. Instead, we let the width

$$R'_{i+1} - R''_i = \varepsilon_i > 0$$

of the interval $[R''_i, R'_{i+1}]$ on which $h' = w\lambda_i$ be arbitrarily large (we will choose it later).

Secondly, we allow the ϕ from Section 6.6.2 to have $\phi' > 0$ on the intervals (R''_i, R'_{i+1}) where $h' = w\lambda_i$:

(1) $\phi' \geq 0$.

(2) $\phi = 0$ on $[0, R''_0]$.

(3) $\phi' \geq 1$ on a subinterval of $[R_i'', R_{i+1}']$ of length $\geq \varepsilon_i/2$, for $i = 0, \ldots, r-1$.

(4) $\phi = \phi_0 + \phi_1 + \cdots + \phi_{i-1}$ is constant on $[R_i', R_i'']$ for $i = 1, \ldots, r-1$, and on $[R_r', \infty)$ for $i = r$.

The above constants ϕ_i will be chosen later (notice ϕ_i is the amount ϕ has increased after crossing the interval $[R_i'', R_{i+1}']$ where $h' = w\lambda_i$).

Our previous results all continue to hold. Indeed, in (6.26) we obtain

$$k(s,t) = w\lambda_i = h'(R(v(s,t)) \text{ for } R(v(s,t)) \in [R_i'', R_{i+1}'],$$

so (6.27) still holds. So the filtration is defined. In the proof of Corollary 6.6.3, $\phi(y) = \phi_0 + \cdots + \phi_{k-1}$ and

$$\begin{aligned} F(y(t)) &= -\phi(y)wT_{-k} + w\lambda_0\phi_0 + \cdots + w\lambda_{k-1}\phi_{k-1}, \\ &= (\lambda_0 - T_{-k})w\phi_0 + \cdots + (\lambda_{k-1} - T_{-k})w\phi_{k-1}, \end{aligned}$$

which is negative as $\lambda_i < T_{-k}$ for $i \leq k-1$.

Lemma 6.8.9. *Given $\delta_i > 0$, for suitable choices of the above data any Floer trajectory $u : \mathbb{R} \times S^1 \to \mathfrak{M}$ for H_λ which crosses the region $\Phi^{-1}[R_i'', R_{i+1}']$ satisfies the filtration estimate*

$$F(u(-\infty, \cdot)) - F(u(+\infty, \cdot)) > \delta_i.$$

Proof. Recall that (6.30) is pointwise non-negative. It suffices therefore that we bound the integral $\int \eta(\partial_s v, \partial_t v - w\lambda_i X_{\mathbb{C}^N})$ from below by δ_i the over the subset of $(s,t) \in \mathbb{R} \times S^1$ for which $v(s,t)$ lies in the region where $h' = w\lambda_i$ and $\phi' \geq 1$. We choose the constants $\phi_0, \phi_1, \ldots$ so that ϕ is larger than the function R on the region $R \geq R_1'$ (we can do this by picking ϕ' very large on a subinterval of $[R_i'', R_{i+1}']$). Then, for the above (s,t), in (6.30) we obtain:

$$\eta(\partial_s v, \partial_t v - w\lambda_i X_{\mathbb{C}^N}) \geq R \cdot d\alpha(\partial_s v, i\partial_s v) + (dR \wedge \alpha)(\partial_s v, i\partial_s v) = d(R\alpha)(\partial_s v, \partial_t v - w\lambda_i X_{\mathbb{C}^N}).$$

The latter expression is precisely the integrand of the energy $\int \|\partial_s v\|^2 \, ds \, dt$ for a Floer solution v in $\mathbb{C}^N$ for the Hamiltonian $w\lambda_i \frac{1}{2}|z|^2$. So the problem reduces to showing that a Floer solution in $\mathbb{C}^n$ for a radial Hamiltonian consumes a lot of energy if it crosses a long radial stretch in which the slope of the Hamiltonian is constant. This is now a standard "monotonicity lemma argument". Namely, one uses Gromov's trick of turning a Floer solution in a symplectic manifold M into a pseudo-holomorphic section of a bundle with fibre M (e.g. see [R14, Sec.5.3]). One can then apply the monotonicity lemma to obtain lower bounds on the energy (e.g. see [R14, Lem. 33]).

The key observation in our case, is to explain why the constants in the monotonicity lemma do not decay to zero as the radial coordinate increases. This follows because $\mathbb{C}^n$ is geometrically bounded at infinity and the auxiliary almost complex structure in Gromov's trick (see [R14, Def. 26]) is radially invariant, because both $w\lambda_i X_{\mathbb{C}^N}$ and the standard complex structure of $\mathbb{C}^N$ are radially invariant. ■

The same argument can be applied to Floer continuation solutions for the monotone homotopy from H_{λ_2} to H_{λ_1} described at the start of this section ([R14, Sec.5.3] discusses also the case of monotone homotopies). Thus we deduce the following.

Corollary 6.8.10. *For suitable choices of the above data, we can inductively build a sequence of Hamiltonians $H_{\lambda_1}, H_{\lambda_2}, \ldots$, where $H_{\lambda_{i+1}}$ is obtained from H_{λ_i} by increasing its (generic) slope at infinity to $\lambda_{i+1} > \lambda_i$, so that $CF^*(H_{\lambda_i})$ is a subcomplex of $CF^*(H_{\lambda_{i+1}})$, and the continuation map $CF^*(H_{\lambda_i}) \to CF^*(H_{\lambda_{i+1}})$ is the inclusion.*

Proof. We choose the interval $[R_i'', R_{i+1}']$ (where $h' = w\lambda_i$) to have sufficiently large length ε_i so that we can pick δ_i in Lemma 6.8.9 to be larger than the difference of filtration values of any two 1-periodic orbits of H_{λ_i}. Then, the comments at the beginning of this section will rule out any "evil Floer solutions" for $H_{\lambda_{i+1}}$ (and indeed any H_{λ_j} for $j \geq i+1$) because they would require a filtration-value difference at the ends larger than δ_i to escape the region $\mathfrak{M} \setminus T$. Indeed the maximum principle applies in the region $\Psi^{-1}[R_i'', R_{i+1}']$ (as $c' = \lambda_i$ is constant there), so the Floer solution (with ends in $\mathfrak{M} \setminus T$) would have to cross the entirety of $\Psi^{-1}[R_i'', R_{i+1}']$ to go from $\mathfrak{M} \setminus T$ to T, but Lemma 6.8.9 prohibits it. ■

It is only thanks to the last corollary that we know that the differential of the E_1 page of the spectral sequences for obtained in Proposition 6.8.7 stops changing below the column corresponding to slope λ as soon as the Hamiltonian H_μ has slope $\mu \geq \lambda$. Thus, one can make sense of the direct limit of those spectral sequences when λ goes to infinity, obtaining the following corollary.

Corollary 6.8.11. *Given a CSR $(\mathfrak{M}, \varphi)$, there are convergent spectral sequences*

$$E_+(\varphi)_r^{pq} \Rightarrow SH_+^*(\mathfrak{M}, \varphi), \text{ where } E(\varphi)_1^{pq} = \begin{cases} \bigoplus_c H^{*-\mu(B_{p,c})}(B_{p,c}), & p < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{6.33}$$

$$E(\varphi)_r^{pq} \Rightarrow SH^*(\mathfrak{M}, \varphi), \text{ where } E(\varphi)_1^{pq} = \begin{cases} H^q(\mathfrak{M}), & p = 0, \\ \bigoplus_c H^{*-\mu(B_{p,c})}(B_{p,c}), & p < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{6.34}$$

Notice that, due to the vanishing of $SH^*(\mathfrak{M},\varphi)$ from Proposition 6.4.3, the spectral sequence $E(\varphi)_r^{p,q}$ converges to zero. Moreover, by the following Proposition 6.8.20, it actually becomes empty by page E_{1+2N}, where N is the number of different S^1-periods of φ smaller than or equal to 1.

To appreciate the following corollary, it may help to peek at the figures of spectral sequences in Section 6.9, where green arrows indicate what edge-differentials there must be on later pages to ensure $SH^*(\mathfrak{M},\varphi) = 0$.

Definition 6.8.12. Given a p-th column of the spectral sequences from Proposition 6.8.7, we will call the number $T_p = 2\pi\tau_p$ its **slope**.

Corollary 6.8.13. *The E_1-page for the spectral sequence for $HF^*(H_\lambda)$ (Proposition 6.8.7) agrees with the part of the E_1-page of the spectral sequence for $SH^*(\mathfrak{M})$ (Corollary 6.8.11) given by the columns below slope λ. Thus $HF^*(H_\lambda)$ arises as the cohomology of that part of the E_1-page for $SH^*(\mathfrak{M})$ for some suitable differential, in particular it must agree with the computation of $HF^*(\lambda H)$ from Corollary 6.5.6, which is supported in even degrees.*

Proof. This follows by Corollary 6.8.10, and the fact that Floer cohomology only depends on the choice of slope at infinity for the Hamiltonian, not on the specific construction of the Hamiltonian. ■

In the end of this section, we explain the connection between the filtration $F_\lambda^\varphi H^*(\mathfrak{M})$ on ordinary cohomology constructed in Section 6.4, and the spectral sequence $E(\varphi)_r^{p,q}$.

Proposition 6.8.14. *The filtration $F_\lambda^\varphi H^*(\mathfrak{M})$ by slope λ is the subspace of the 0-th column of $E(\varphi)_r^{p,q}$ that gets killed by images of the edge-differentials arising from the columns p whose slopes are less than or equal to λ.*

Proof. Firstly, as the spectral sequence was constructed from the filtration on $CF^*(H_\lambda)$, the image of the differentials that hit the 0-th column starting from the columns p with $\tau_p \leq \lambda$ is equal to the image of the differential $d : CF^*(H_\lambda) \to CF^{*+1}(H_\lambda)$ intersected with the subcomplex $CF_0^*(H_\lambda)$ generated by the fixed points. More precisely, it is the image of that intersection on the cohomology level since on the E_1-page the 0-th column is $H_*(CF_0^*(H_\lambda), d)$. Then, observe that this is the kernel of the inclusion map $CF_0^*(H_\lambda) \hookrightarrow CF^*(H_\lambda)$ on the cohomology level. Now, recall from the proof of 6.7.1 that $(CF_0^*(H_\lambda), d) = (CF^*(H_{\lambda_0}), d)$. Thus, the last chain map is equal to $CF^*(H_{\lambda_0}) \to CF^*(H_\lambda)$, which on the cohomology level gives the continuation map $HF^*(H_{\lambda_0}) \to HF^*(H_\lambda)$. As the filtered subspace $F_\lambda^\varphi H^*(\mathfrak{M})$ is defined by the kernel of the last map, the proposition follows. ■

We will calculate (rank-wise) the filtrations $F^{\varphi}_{\lambda}H^*(\mathfrak{M})$ in this way in examples in Section 6.9.

6.8.4 General properties of spectral sequences

In this section we give some general properties of spectral sequences $E(\varphi)^{p,q}_r$ and $E_+(\varphi)^{p,q}_r$, that allow us to compute their E_1-pages in practice, only by having topological information on torsion submanifolds in $\mathfrak{M}$ and information about their convergence (under the $\mathbb{C}^*$-action) to the fixed locus $\mathfrak{F} = \mathfrak{M}^{\varphi}$, i.e. the extended attraction graph of a CSR $(\mathfrak{M}, \varphi)$ (Definition 2.3.16). We will always work with the spectral sequence $E(\varphi)^{p,q}_r$, but the same results will also apply to the spectral sequence $E_+(\varphi)^{p,q}_r$, due to it having the same data as $E(\varphi)^{p,q}_r$ except with the 0-th column erased.

Notation 6.8.15. By **Morse-Bott submanifolds** we will call the connected manifolds $B_{p,c}$ of 1-orbits of H_{λ}. When discussing the spectral sequences, we will use the conventions that agree with the **time-1 normalised flow of the S^1-action.** In particular, the column p of the spectral sequence will be called **the time-τ_p column**. Thus, the time-1 column consists of the cohomology $H^*(\mathcal{S}_p)$ of the whole slice with an appropriate shift. We will **enumerate the rows of the spectral sequence by the total degree** $p+q$, instead of the usual numbering by q (e.g. Figure 6.3).

Given a point $x \in \mathfrak{M}$, define its **convergence point** x_{∞} as

$$x_{\infty} := \lim_{\mathbb{C}^* \ni t \to 0} t \cdot x.$$

Recall that x_{∞} exists for any x, as the action is conical. We see immediately that the convergence point is a fixed point of the $\mathbb{C}^*$-action.

Before computing the gradings $\mu(B_{p,c})$ of the Morse-Bott submanifolds $B_{p,c}$, we mention a useful lemma that yields canonical capping discs for our 1-orbits.

Lemma 6.8.16. *For any $x \in \mathfrak{M}$, the $\mathbb{C}^*$-action determines a holomorphic map on the punctured disc,*

$$\psi_x : \{z \in \mathbb{C} : 0 < |z| \leq 1\} \to \mathfrak{M}, \quad \psi_x(z) = \varphi_z(x).$$

This map extends to a holomorphic map over the disc, $\psi_x : \{z \in \mathbb{C} : |z| \leq 1\} \to \mathfrak{M}$ with $\psi_x(0) = x_{\infty}$. Moreover, given a choice of unitary basis for $T_{x_{\infty}}\mathfrak{M}$, there is a canonical unitary (hence symplectic) trivialisation of $\psi_x^* T\mathfrak{M}$.

Proof. Observe that ψ_x extends continuously over 0 via $\psi_x(0) = x_\infty$. Pick complex coordinates for $\mathfrak{M}$ in a neighbourhood of x_∞. Then each coordinate of ψ_x for small z is a holomorphic function $\{z \in \mathbb{C} : 0 < |z| \leq \varepsilon\} \to \mathbb{C}$ which is bounded at $z = 0$, and therefore the singularity at 0 is removable.

The bundle $\psi_x^* T\mathfrak{M}$ is a complex vector bundle over the unit disc in $\mathbb{C}$, and we can pick a trivialisation by a unitary basis with respect to the Kähler metric g, by parallel transporting radially outwards from the centre of the disc (recall that parallel transport for a Kähler manifold $(\mathfrak{M}, g)$ is unitary). So the unitary trivialisation for $\psi_x^* T\mathfrak{M}$ is canonical up to a choice of unitary basis at x_∞ for $T_{x_\infty}\mathfrak{M}$. The trivialisation is also symplectic with respect to ω_I since $(\mathfrak{M}, g, \omega_I)$ is Kähler. ■

Proposition 6.8.17. *The grading of a Morse-Bott submanifold $B_{p,c}$ is*

$$\mu(B_{p,c}) = \dim_{\mathbb{C}} \mathfrak{M} - \frac{1}{2} \dim_{\mathbb{R}} B_{p,c} - \frac{1}{2} - \sum_i W(T_p m_i), \tag{6.35}$$

where $T_{x_\infty}\mathfrak{M} = \oplus_i \mathbb{C}_{m_i}$ is the 1-dimensional weight-decomposition of the tangent space of the convergence point x_∞ of an arbitrary point $x \in B_{p,c}$, and $\mathbb{C}_{m_i}$ denotes a copy of $\mathbb{C}$ whose $\mathbb{C}^*$-action has weight m_i. In particular, $\mu(B_{p,c})$ does not depend on the choice of $x \in B_{p,c}$.

Proof. The proof uses an idea similar to the computation in [Oan04, Sec.3.3]. Briefly: first we recall that the difference between the Hamiltonian flows of H_λ and $T_p H$ causes a symplectic shear which gives the $-\frac{1}{2}$ term in (6.35). Then, we compute the Robbin-Salamon index of $B_{p,c}$ with respect to the Hamiltonian $T_p H$, which yields the $\sum_i W(T_p m_i)$-part.

Recall from (6.32) the orthogonal splittings $T_x\mathfrak{M} = \langle X_{\mathbb{R}_+} \rangle \oplus \langle X_{H_\lambda} \rangle \oplus \xi$, $T_x \mathcal{S}_p = X_{H_\lambda} \oplus \xi$. The matrix of the linearised flow of H_λ using this splitting has the form

$$(\phi_t^{H_\lambda})_* = \begin{bmatrix} 1 & 0 & \\ \tau c''(H) \left\|\nabla H\right\|^2 & 1 & \\ & & (\phi_t^{H_\lambda})_*|_\xi \end{bmatrix}.$$

The first column is due to Lemma 6.8.4. The second column is immediate, and the rest is due to the fact that the flow $\phi_t^{H_\lambda}$ restricted to $\mathcal{S}_p$ is equal to $\phi_t^{T_p H}$, hence preserves the metric and thus the orthogonals. As it preserves X_{H_λ}, it preserves its orthogonal ξ.

Thus, we have that $(\phi_t^{H_\lambda})_* = \chi(t) \circ (\phi_t^{T_p H})_*$, where

$$\chi(t) = \begin{bmatrix} 1 & 0 & \\ \tau c''(H) \left\|\nabla H\right\|^2 & 1 & \\ & & Id_{2d-2} \end{bmatrix}.$$

Letting $\Psi(t) = (\phi_t^{T_pH})_*$ as in the proof of [CFHW96, Prop.2.2], there is a homotopy

$$K(s,\, t) = \begin{cases} \chi(st)\Psi(\frac{2t}{s+1}), & t \leq \frac{s+1}{2} \\ \chi\big((s+2)t - (s+1)\big)\Psi(1), & \frac{s+1}{2} \leq t \end{cases}$$

of paths with fixed ends between the path $(\phi_t^{H_\lambda})_*$ and the concatenation of $(\phi_t^{T_pH})_*$ and $\chi(t) \circ (\phi_1^{T_pH})_*$.

Hence, by Theorem A.1.4.(2),(6) we have $RS(x, H_\lambda) = RS(x, T_pH) + \frac{1}{2}$. Thus, it remains to compute the index $RS(x, T_pH)$. We have to pick a symplectic trivialisation of $x(t)^*T\mathfrak{M}$ first.

A choice of unitary basis z_i that generates the weight spaces $\mathbb{C}_{m_i} = \mathbb{C}z_i$ of the weight decomposition $T_{x_\infty}\mathfrak{M} = \mathbb{C}_{m_i}$ yield, by Lemma 6.8.16, a canonical holomorphic capping disc ψ_x and a canonical symplectic trivialisation of $\psi_x^*T\mathfrak{M}$. As $x(t)$ is a reparametrization of an S^1-orbit, all of its points yield the same capping disc up to S^1-reparametrization, and the same unitary and symplectic trivialisation of $T\mathfrak{M}$ over the capping disc. We remark that if the S^1-orbit of x is an m-fold cover of $x(t)$, then the capping disc ψ_x is $\mathbb{Z}/m$ equivariant (acting by m-th roots of unity ζ). Moreover, by the construction in the proof of Lemma 6.8.16, the unitary basis chosen for $\psi_x(z)^*T\mathfrak{M}$ at $\psi_x(z)$ is the same as the chosen basis at $\psi_x(\zeta z)^*T\mathfrak{M}$ (since $\psi_x(z) = \psi_x(\zeta z)$). So the choice of unitary basis z_i at x_∞ also determines a canonical unitary and symplectic trivialisation for $x(t)^*T\mathfrak{M}$.

By the construction from Lemma 6.8.16, the linearised S^1-flow along the S^1-orbit of x in the chosen trivialisation of $\psi_x^*T\mathfrak{M}$ is equal to the linearised S^1-flow at x_∞ in the trivialisation of $T_{x_\infty}\mathfrak{M}$ given by the basis z_i. The same argument applies to $x(t)^*T\mathfrak{M}$ using the time $2\pi/m$ flow of the S^1-action. By the continuity property for RS indices (Theorem A.1.4), we reduce to computing that $RS(x_\infty, T_pH) = \sum_i W(T_pm_i)$, which follows by the same argument as in the proofs of Propositions 6.3.2 and 6.3.3.

Now, recall that the grading of the Morse-Bott manifold $B_{p,c}$ corresponds to the grading of the orbit that represents the minimum of the Morse function $f_{p,c}$ that perturbs $B_{p,c}$. By the argument in [Oan04, Sec. 3.3], the Morse term in the Robbin-Salamon index formula for an orbit is equal to half of the signature of the Hessian $\nabla f_{p,c}$ at the corresponding critical point; thus at a minimum it is equal to $\frac{1}{2}\dim_{\mathbb{R}} B_{p,c}$. Finally, recalling that the grading in our conventions is equal to $\dim_{\mathbb{C}} \mathfrak{M}$ - RS, the formula (6.35) is proved. ■

We will now deduce the consequences of this proposition, with regards to the spectral sequence. Recall from Lemma 6.8.2 that $B_p = \mathcal{S}_p \cap \mathcal{R}_{\mathbb{Z}/m}$, where $\tau_p = k/m$ for coprime $k, m \in \mathbb{N}$. As $\mathcal{R}_{\mathbb{Z}/m}$ is an S^1-invariant Kähler submanifold of $(\mathfrak{M}, \omega_I)$, by the same reasoning as in Corollary 6.1.6, we get that each connected component of $\mathcal{R}_{\mathbb{Z}/m}$ that leaves the core has a connected intersection with $\mathcal{S}_p = H^{-1}(H_p)$. Thus, denoting by $\mathcal{R}^{out}_{\mathbb{Z}/m}$ the submanifold of $\mathcal{R}_{\mathbb{Z}/m}$ that leaves the core, one has its decomposition into connected components $\mathcal{R}^{out}_{\mathbb{Z}/m} = \sqcup_c \mathcal{R}_{\mathbb{Z}/m,c}$ which yield the connected components of B_p,

$$B_{p,c} = \mathcal{S}_p \cap \mathcal{R}_{\mathbb{Z}/m,c}. \tag{6.36}$$

Lemma 6.8.18. *Suppose $\tau_{p_1} = k_1/m$ and $\tau_{p_2} = k_2/m$ have the same denominators, where $(k_1, m) = (k_2, m) = 1$. Then B_{p_1} and B_{p_2} are diffeomorphic, indeed diffeomorphic to the B_p with $\tau_p = 1/m$.*

Proof. As $B_{p_1,c} = \mathcal{S}_{p_1} \cap \mathcal{R}_{\mathbb{Z}/m,c}$ and $B_{p_2,c} = \mathcal{S}_{p_2} \cap \mathcal{R}_{\mathbb{Z}/m,c}$, we have

$$B_{p_1,c} \cong B_{p_2,c}$$

via the normalised gradient flow $\nabla H / \|\nabla H\|^2$ that flows from one to the other. This holds since H is constant on $\mathcal{S}_p$, and $\mathcal{R}_{\mathbb{Z}/m,c}$ is preserved by the gradient flow of H (being the $\mathbb{R}_+$-part of $\mathbb{C}^*$-action). ■

We now deduce that the columns of the E_1-page are 1-periodic,[13] up to a constant shift in the q-coordinate. The shift depends only on the dimension of $\mathfrak{M}$ and the weight of the action φ on $\omega_{\mathbb{C}}$.

Corollary 6.8.19. *Given $p > p'$ such that $\tau_{p'} - \tau_p = 1$, we have that*

$$\mu(B_{p',c}) = \mu(B_{p,c}) + kd,$$

where k is the weight of φ, and $d = \dim_{\mathbb{C}} \mathfrak{M}$.

Proof. By formula (6.35), we have

$$\begin{aligned}\mu(B_{p',c}) - \mu(B_{p,c}) &= \sum_i W(T_{p'} m_i) - \sum_i W(T_p m_i) = \sum_i [W(T_{p'} m_i) - W(T_p m_i)] \\ &= \sum_i [W((T_p + 1) m_i) - W(T_p m_i)] = \sum_i 2m_i,\end{aligned}$$

where we have used $W(a + 2\pi n) = W(a) + 2n$, for $n \in \mathbb{Z}$ (which follows immediately from the definition (6.7) of function W). Recall that the m_i come from the weight-decomposition $T_{x_\infty}\mathfrak{M} = \oplus_i \mathbb{C}_{m_i}$. Also recall that as the action φ acts by weight k on the symplectic form $\omega_{\mathbb{C}}$, there is a duality between the weight spaces for m and $k - m$. Thus $\sum_i 2m_i = 2k \cdot \frac{1}{2} \dim_{\mathbb{C}} M = kd$. ■

[13] In the sense of Notation 6.8.15.

Corollary 6.8.20. *(Periodicity) The E_1-page of the spectral sequence $E(\varphi)_r^{p,q}$ is time-1 periodic, up to a downward shift by kd rows, where $d = \dim_{\mathbb{C}} \mathfrak{M}$ and k is the weight of φ.*

Thus, in order to compute the E_1-page of $E(\varphi)_r^{p,q}$, it is enough to compute the columns p having the normalised period τ_p smaller or equal to 1. We show now that it is actually enough to compute the columns whose period is smaller or equal to $\frac{1}{2}$. By **point "$(\frac{1}{2}, n - \frac{1}{2})$"** of the spectral sequence we will mean a point that lies on the column with time $\tau_p = \frac{1}{2}$, on the line between rows n and $n-1$. If a column with $\tau_p = \frac{1}{2}$ does not exist, then it is the point on the line between the two columns that have periods τ_p nearest to $\frac{1}{2}$. For example, the 4-star symbol in Figure 6.4 is the point $(\frac{1}{2}, -\frac{1}{2})$, and the following corollary confirms that this is a point of central symmetry for the generators in the spectral sequence.

Corollary 6.8.21. *(Local central symmetry) Given a CSR $(\mathfrak{M}, \varphi)$, there is a central symmetry of the sub-block of the E_1-page of $E(\varphi)_r^{p,q}$ consisting of columns p with $0 < \tau_p < 1$, about the point*

$$\left(\frac{1}{2}, 2r - kr - \frac{1}{2}\right),$$

where $\dim_{\mathbb{R}} \mathfrak{M} = 4r$ is the real dimension of $\mathfrak{M}$ and k is the weight of φ. In particular, when the action φ is weight-2, it is the point $(\frac{1}{2}, -\frac{1}{2})$, and when the action φ is weight-1 it is the point $(\frac{1}{2}, r - \frac{1}{2})$.

Proof. Consider a Morse-Bott submanifold $B_{p,c}$ that lies in the p-th column, so consisting of S^1-orbits with normalised period τ_p. By (6.35), the top-dimensional class (so $H^{top}(B_{p,c})$) lies on the row

$$r_1 := \dim_{\mathbb{R}} B_{p,c} + \dim_{\mathbb{C}} \mathfrak{M} - \frac{1}{2}\dim_{\mathbb{R}} B_{p,c} - \frac{1}{2} - \sum_i W(T_p m_i).$$

Now, in the column p' such that $\tau_{p'} = 1 - \tau_p$, we have the manifold $B_{p',c}$ that is diffeomorphic to $B_{p,c}$, by Lemma 6.8.18. Again by (6.35), its bottom-dimensional class (so $H^0(B_{p',c})$) lies on the row

$$r_2 := \dim_{\mathbb{C}} \mathfrak{M} - \frac{1}{2}\dim_{\mathbb{R}} B_{p',c} - \frac{1}{2} - \sum_i W(T'_p m_i).$$

Thus, by Corollary 6.8.19, we get $\frac{r_1+r_2}{2} = \frac{4r-1-2kr}{2} = 2r - kr - \frac{1}{2}$. So $H^{top}(B_{p,c})$ and $H^0(B_{p',c})$ match-up via the central symmetry about the point $(\frac{1}{2}, 2r - kr - \frac{1}{2})$. In the other degrees, the ranks match up due to Poincaré duality and the fact that $B_{p,c} \cong B_{p',c}$. ■

We also mention the following pattern, which manifests itself in the examples in Section 6.9.

Proposition 6.8.22. *After the E_1-page, the spectral sequence $E(\varphi)_r^{p,q}$ splits, horizontally, into a direct sum of two-row spectral sequences. In other words, the non-zero edge-homomorphisms (from the E_1-page onwards) all go from odd rows to even rows.*

Proof. By Corollary 6.8.13, every new odd-degree generator that appears in a new column of the E_1-page spectral sequence for H_λ (as we increase the slope λ), must have an edge-differential (on E_1 or later pages) whose image was not generated by previous generators. Otherwise, for that particular slope λ, an odd-degree class would survive and cannot be killed in later pages of the spectral sequence for $HF^*(H_\lambda)$. But that spectral sequence converges to $HF^*(H_\lambda) \cong HF^*(\lambda H)$, which is supported in even degrees by Corollary 6.5.6. This is a contradiction. Thus either the above case applies, or that generator is killed by an even-degree class in the same column (so it does not survive to $HF^*(\lambda H)$). ■

6.8.5 Techniques for computing the E_1-page

Having obtained the grading shifts of the Morse-Bott manifolds $B_{p,c}$ in Proposition 6.8.17, we now comment on how to compute their cohomologies $H^*(B_{p,c})$. With these two pieces of information, one obtains the graded generators for the E_1-page of the spectral sequence.

From the proof of Lemma 6.8.18, it suffices to compute the cohomologies of $B_{p,c}$ for normalised periods τ_p, so of the form $1/m$ for $m \in \mathbb{N}$. Let us first consider the case $m \geq 2$. By (6.36), it suffices to know the cohomology of a hypersurface section of $\mathcal{R}_{\mathbb{Z}/m,c}$, where the hypersurface in $\mathcal{R}_{\mathbb{Z}/m,c}$ is a level set of the Hamiltonian. The following lemma tells us that the choice of level set is immaterial.

Lemma 6.8.23. *Let M be an open manifold, and $H : M \to \mathbb{R}$ an exhausting function whose critical locus is compact. Let $\Sigma_c = H^{-1}(c)$ denote the level set. Then, for sufficiently large $c \in \mathbb{R}$, the cohomology $H^*(\Sigma_c)$ is independent of c and is independent of the choice of H. Thus, if M is a vector bundle and $S(M)$ its unit bundle, $H^*(\Sigma_c) \cong H^*(S(M))$ for large c.*

Proof. For large c, the critical locus of H is contained in $H < c$ (as H is exhausting). Thus, for large c, c' we have a diffeomorphism $\Sigma_c \cong \Sigma_{c'}$ defined by the flow of $\nabla H / \|\nabla H\|^2$.

Denote by $\Sigma_{c,\infty} = H^{-1}[c,\infty) \subset \mathfrak{M}$ the subset at infinity diffeomorphic to $\Sigma_c \times [0,\infty)$ obtained by flowing for positive time with $\nabla H/\|\nabla H\|^2$. Thus $H^*(\Sigma_{c,\infty}) \cong H^*(\Sigma_c)$ (for large c). Given two exhausting functions H_1 and H_2, respectively with level sets Σ'_c, Σ''_c, we can choose c-values to get nested sets

$$\Sigma'_{c_1,\infty} \subset \Sigma''_{c_2,\infty} \subset \Sigma'_{C_1,\infty} \subset \Sigma''_{C_2,\infty},$$

where the c-values are large so that the critical loci of H_1, H_2 lie in lower sublevel sets. The composite of the first two inclusions is isotopic to a diffeomorphism (by using the flow by $\nabla H_1/\|\nabla H_1\|^2$). Similarly, for the composite of the last two inclusions (using H_2). Thus on cohomology those two composite maps induce isomorphisms. Thus $H^*(\Sigma''_{c_2,\infty}) \leftarrow H^*(\Sigma'_{C_1,\infty})$ is both injective and surjective, so it is an isomorphism. By the first part, up to isomorphism the choices of large c_2, C_1 do not matter.

For the second claim, for a vector bundle, $M \to B$, we pick a Riemannian metric g on its fibres and obtain the exhausting function $H : M \to \mathbb{R}$ given by $H(b,\xi) = g(\xi,\xi)$. Then $S(M) = H^{-1}(1)$. ∎

Recall from (6.36) and (6.31) that

$$B_{p,c} = \mathcal{S}_p \cap \mathcal{R}_{\mathbb{Z}/m,c} \qquad \text{and} \qquad \mathcal{S}_p = \{x \in \mathfrak{M} \mid H(x) = H_p\} \subset \mathfrak{M},$$

where H_p are the values of H where the slope of $H_\lambda = c(H)$ satisfies $c'(H_p) = T_p$. Then by Lemma 6.8.23:

(1) If $\mathcal{R}_{\mathbb{Z}/m,c} = \mathcal{H}_m$ is a torsion bundle[14] over a component of the fixed locus $\mathfrak{F}_\alpha$, then the cohomology of its corresponding Morse-Bott submanifold $B_{p,c}$ is

$$H^*(B_{p,c}) \cong H^*(S(\mathcal{H}_m)),$$

which is calculable by considering the Gysin sequence for the sphere bundle $S(\mathcal{H}_m) \to \mathfrak{F}_\alpha$.

(2) More generally, if $\mathcal{R}_{\mathbb{Z}/m,c}$ is a bundle over a submanifold D_m of the core which is also in $\mathcal{R}_{\mathbb{Z}/m,c}$, but which is not necessarily fixed under the $\mathbb{C}^*$-action (unlike $\mathfrak{F}_\alpha$ above), then

$$H^*(B_{p,c}) \cong H^*(S(\mathcal{R}_{\mathbb{Z}/m,c})),$$

which is calculable by considering the Gysin sequence for the sphere bundle $S(\mathcal{R}_{\mathbb{Z}/m,c}) \to D_m$.

[14]Recall Definitions 2.3.9 and 2.3.11.

(3) We have so far observed only the situations (1) and (2) in the examples (Section 6.9), but we do not claim that these are necessarily the only possible ones.

Now we show how to calculate the time-1 column of the E_1-page. For such a column, we have $B_p = \mathcal{S}_p$, thus it is equal to $H^*(\mathcal{S}_p)$ with a shift calculated by Corollary 6.8.20. The ranks of the cohomology of the hypersurface $\mathcal{S}_p$ are easily calculable (up to the two middle degrees) using classical algebraic topology:

Proposition 6.8.24. *Assume that the Novikov field $\mathbb{K}$ is defined over a characteristic zero field. Abbreviate $4n = \dim_{\mathbb{R}} \mathfrak{M}$. Then for some $r \leq \dim_{\mathbb{K}} H^{2n}(\mathfrak{M})$ we have:*

$$H^k(\mathcal{S}_p) = \begin{cases} H^{4n-1-k}(\mathfrak{M}) & k \geq 2n+1, \\ \mathbb{K}^r & k = 2n-1, 2n, \\ H^k(\mathfrak{M}) & k \leq 2n-2. \end{cases}$$

Explicitly, r is the nullity of the natural map for the pair $(\mathfrak{M}, \mathfrak{M}_\infty)$ in degree $2n$,

$$H^{2n}(\mathfrak{M}) \cong H^{2n}(\mathfrak{M}, \mathfrak{M}_\infty) \to H^{2n}(\mathfrak{M}),$$

where $\mathfrak{M}_\infty = H^{-1}[H_p, \infty)$ is the region at infinity in $\mathfrak{M}$ outside of $\mathcal{S}_p$.

Proof. Recall by Corollary 2.1.9 that $H^*(\mathfrak{M}) \cong H^*(\mathfrak{L})$ is supported in even degrees,[15] which are less or equal to $2n$ (due to $\mathfrak{L}$ being $\omega_{\mathbb{C}}$-isotropic, Theorem 3.1.1). Denote $M := \{H \leq H_p\}$, so $\partial M = \mathcal{S}_p$. Thus $\mathfrak{M}$ is homotopy equivalent to M, as it deformation retracts onto M via the (rescaled) $\mathbb{R}_+$-action. By Lefschetz duality and universal coefficients (working over a field $\mathbb{K}$), $H^k(M, \partial M) \cong H_{4n-k}(M) \cong H^{4n-k}(M)$. Using the latter isomorphisms, the long exact sequence for the pair $(M, \partial M)$ becomes

$$H^{4n-k-1}(M) \leftarrow H^k(\partial M) \leftarrow H^k(M) \leftarrow H^{4n-k}(M).$$

For $k \leq 2n-2$ we have $4n-k \geq 2n+2$ and $4n-k-1 \geq 2n+1$, so the outside terms above vanish. Thus $H^k(\partial M) \cong H^k(M)$ for $k \leq 2n-2$. Finally, by Poincaré duality for the closed orientable $(4n-1)$-manifold $\partial M = \mathcal{S}_p$ and universal coefficients, we have $H^k(\mathcal{S}_p) \cong H_{4n-1-k}(\mathcal{S}_p) \cong H^{4n-1-k}(\mathcal{S}_p)$. The final claim follows from the above exact sequence for $k = 2n$ and $k = 2n-1$, and using that $H^{\text{odd}}(M) = 0$. ■

[15] This is why we need the characteristic zero assumption for the base field of the Novikov field.

6.9 Examples of spectral sequences

In this section we give a variety of examples of Morse-Bott Floer spectral sequences $E(\varphi)_r^{p,q}$ constructed in Section 6.8. As a consequence, by Proposition 6.8.14, we get the (rank-wise) filtrations of the ordinary cohomology rings for these CSRs.

We use the following notation for gradings: $A[m]$ means a graded group A is shifted down by m, so $A[m]_d = A_{m+d}$. In the pictures of spectral sequences we label B_p by B_{τ_p} instead, as τ_p tells the torsion subgroup of the Morse-Bott submanifold to which it corresponds.

NB In this section we will assume that the Novikov field is over a **characteristic zero** field due to easier computability of our spectral sequences (e.g. recall that Corollary 2.1.9 and Proposition 6.8.24 give us cohomological information over characteristic zero only). We remark that working over non-characteristic zero fields could in principle give us some further information, but we stay in the characteristic zero for the purposes of this section.

6.9.1 Generalised Springer resolutions of type A

Recall from Section 5.1.2 that the generalised Springer resolution of type A is a CSR

$$T^*\mathcal{B}_p \to \overline{\mathcal{O}_{p_+^*}},$$

where $p = (p_1, \ldots, p_n)$ is a composition of n and $\mathcal{B}_p$ is a variety of p-partial flags, whereas $\mathcal{O}_{p_+^*}$ is the nilpotent orbit whose matrices have Jordan partition p_+^*.[16] This is a weight-1 CSR, where the action acts on $T^*\mathcal{B}_p$ by contracting the fibres and on $\overline{\mathcal{O}_{p_+^*}}$ by dilation.

Thus, the action on $T^*\mathcal{B}_p$ is free outside the fixed locus $\mathcal{B}_p$, hence there are no torsion points, which implies that the spectral sequence $E(\varphi)_r^{p,q}$ will be very simple. The 0-th column is the cohomology $H^*(\mathcal{B}_p)$ without a shift, whereas the 1st and 2nd columns are the cohomologies of the sphere bundle $H^*(S(T^*\mathcal{B}_p))$ with shifts of $d := \dim_{\mathbb{R}} \mathcal{B}_p$ and $2d$ rows downward (using Lemma 6.8.23 and Corollary 6.8.20).

The cohomology $H^*(S(T^*\mathcal{B}_p))$ can be computed by Proposition 6.8.24, except for the middle ranks. We see that they vanish by looking at the Leray-Serre spectral sequence for $S(T^*\mathcal{B}_p) \to \mathcal{B}_p$. Namely, the edge-differential $H^{top}(S^{2d-1}) \to H^{top}(\mathcal{B}_p)$ from the top class in the 0-th column to the rightmost class in the 0-th row is multiplication by the Euler number of the bundle, which is strictly negative as the Euler

[16] Here, p_+ is a weakly-descending permutation of p, and p_+^* is its dual.

characteristic of $\mathcal{B}_p$ is positive (it has only even Betti numbers). Hence, this differential is an isomorphism and that makes the two middle classes vanish. We start with the simplest example:

Example 6.9.1. $T^*\mathbb{C}P^1 \to \overline{\mathcal{O}_{11}}$.

The first three columns[17] are $H^*(S(T^*\mathcal{B}_p)) = \mathbb{K}[0] \oplus \mathbb{K}[-3]$, with a shift down by 2, 4 and 6. We remark that in this case we know explicitly that $S(T^*\mathbb{C}P^1) = \mathbb{R}P^3$, whose cohomology over a field $\mathbb{K}$ of characteristic zero agrees with that of S^3. Now, by looking at the spectral sequence (see the figure 6.3), the filtration $F^{\varphi}_{\lambda} H^*(T^*\mathbb{C}P^1)$ of the singular cohomology that one obtains is rank-wise:

$$0 \subset \mathbb{K}[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}[-2] = H^*(T^*\mathbb{C}P^1).$$

As $H^*(T^*\mathbb{C}P^1) = \mathbb{K}[a]/a^2,\ |a| = 2$, in this example we get the complete information about the filtration,

$$0 \subset \langle a \rangle \subset H^*(T^*\mathbb{C}P^1).$$

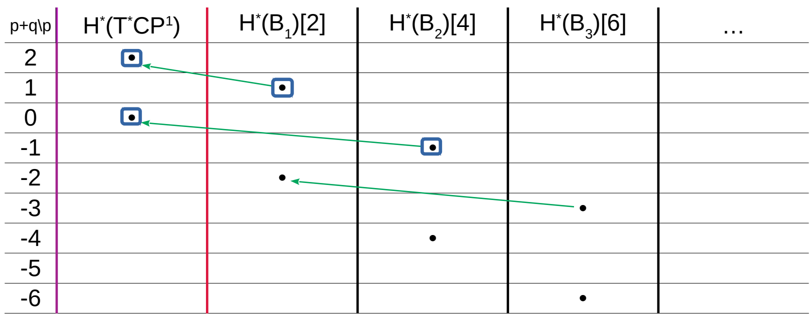


Figure 6.3: Spectral sequence for $T^*\mathbb{C}P^1$

Example 6.9.2. $T^*\mathcal{B}_p \to \overline{\mathcal{O}_{p^*_+}}$.

Analogously to the previous example, we get that the first three columns are $H^*(S(T^*\mathcal{B}_p))$ shifted down by d, $2d$ and $3d$ (where $d = \dim_{\mathbb{R}} \mathcal{B}_p$). Thus, only the first two columns can hit the 0-th column via the differential, and moreover, the second column can hit only $1 \in H^0(T^*\mathcal{B}_p)$, due to the shift. Recall from above that $H^*(S(T^*\mathcal{B}_p))$ has no middle degree classes, so rows 0 and -1 of the first column are zero, so there is no class in the first column that can hit $1 \in H^0(T^*\mathcal{B}_p)$. Thus, the obtained filtration is again

$$0 \subset H^{\geq 2}(T^*\mathcal{B}_p) \subset H^*(T^*\mathcal{B}_p),$$

[17] Recall that we numerate columns from 0-th column, thus here we mean "the first three columns after the 0-th column."

where $H^{\geq 2}(T^*\mathcal{B}_p)$ is the maximal ideal generated by all cohomological classes in degrees ≥ 2.

6.9.2 Resolutions of Du Val singularities

The simplest conical symplectic resolutions are resolutions of Du Val singularities, also known as simple surface singularities, ADE singularities, Kleinian singularities or rational double points. These are the minimal resolutions of quotient singularities

$$\pi_\Gamma : X_\Gamma \to \mathbb{C}^2/\Gamma$$

for all finite subgroups $\Gamma \leq SU(2)$. Any such group Γ (up to conjugation) can be labelled bijectively via an ADE Dynkin graph Q_Γ which is called its *McKay graph*.[18]

The Dynkin graph A_n corresponds to the cyclic group $\mathbb{Z}/(n+1)$, the graph D_n corresponds to the binary dihedral group $BD_{4(n-2)}$, whereas the graphs E_6, E_7 and E_8 correspond to the binary tetrahedral, octahedral and icosahedral groups, respectively. It is known that the core $\pi_\Gamma^{-1}(0)$ consists of a union of 2-spheres, whose dual graph of intersections is exactly the McKay graph Q_Γ.

Klein showed that the coordinate ring $\mathbb{C}[\mathbb{C}^2/\Gamma]$ is generated by three polynomials, hence the variety $\mathbb{C}^2/\Gamma$ embeds into $\mathbb{C}^3$ via a polynomial map $(z_1, z_2) \mapsto (f, g, h) \subset \mathbb{C}^3$. These embeddings for types A and D are as follows:

$$\begin{aligned} &\text{For } \mathbb{Z}/n : \quad (z_1, z_2) \mapsto (z_1^n, z_2^n, z_1 z_2), \text{ the image is } XY - Z^n = 0, \\ &\text{For } BD_{4n} : \ (z_1, z_2) \mapsto ((z_1^{2n} - z_2^{2n}) z_1 z_2, z_1^{2n} + z_2^{2n}, z_1^2 z_2^2), \\ &\qquad\qquad \text{the image is } X^2 - Y^2 Z + 4Z^{n+1} = 0. \end{aligned} \tag{6.37}$$

For the types E_6, E_7, E_8, the invariant polynomials are rather cumbersome so we will not write them here (see [Dol07, Sec. 1.2]).

The natural conical weight-2 action on these spaces comes from the dilation action

$$\mathbb{C}^* \curvearrowright \mathbb{C}^2, \ t \cdot (z_1, z_2) = (tz_1, tz_2) \tag{6.38}$$

We call the action that it yields on $X_\Gamma \to \mathbb{C}^2/\Gamma$ the **standard action**.

We discuss type A_n singularities first, splitting into two cases by parity of n. Type A_1 is the case $T^*\mathbb{C}P^1$ already covered in Section 6.9.1, so we omit it.

Example 6.9.3. $\mathfrak{M}_0 = \mathbb{C}^2/(\mathbb{Z}/k)$**, for odd** $k \geq 3$**.**

In the coordinates (6.37), the action (6.38) becomes

$$t \cdot X = t^k X, \quad t \cdot Y = t^k Y, \quad t \cdot Z = t^2 Z,$$

[18]Which comes from representation theory of the group Γ.

thus the only torsion subvarieties are the lines $l_1 = (0, Y, 0)$ and $l_2 = (X, 0, 0)$, which have $\mathbb{Z}/k$-isotropies.

Thus, the only torsion submanifolds in the resolution $\mathfrak{M} = X_{\mathbb{Z}/k}$ are the two $\mathbb{Z}/k$-torsion lines, $\widetilde{l_1}$ and $\widetilde{l_2}$, that converge (by the $\mathbb{C}^*$-action, when $t \to 0$) to the two different fixed points in the core. Their intersections with slices S_p, for $p = -1, \dots, -(k-1)$ yield two Morse-Bott submanifolds

$$B_{p,1} \cong B_{p,2} \cong S^1.$$ [19]

By the same argument as in Lemma 6.8.23, the hypersurface B_k has cohomology isomorphic to the cohomology of $S^3/(\mathbb{Z}/k)$, thus, as we are working over fields of characteristic zero, $H^*(B_k) = \mathbb{K}[0] \oplus \mathbb{K}[-3]$. As the core of $\mathfrak{M}$ is an A_{k-1}-tree of 2-spheres, the cohomology of $\mathfrak{M}$ is $H^*(\mathfrak{M}) = \mathbb{K}[0] \oplus \mathbb{K}^{k-1}[-2]$. Thus, in order to construct the E_1-page of the spectral sequence $E(\varphi)_r^{p,q}$ we only have to know the shifts $\mu(B_{p,1})$ and $\mu(B_{p,2})$. According to formula (6.35), it suffices to know the weights of the converging points of lines $\widetilde{l_1}$ and $\widetilde{l_2}$. But this is straightforward as their weight decompositions are $\mathbb{C}_k \oplus \mathbb{C}_{2-k}$, due to the duality $H_k \overset{\omega_{\mathbb{C}}}{\longleftrightarrow} H_{2-k}$ induced by the weight-2 action. Thus, we get that

$$\begin{aligned} \mu(B_{p,1}) = \mu(B_{p,2}) &= 2 - \tfrac{1}{2} - \tfrac{1}{2} - W(2\pi \tfrac{-p}{k} k) - W(2\pi \tfrac{-p}{k}(2-k)) \\ &= W(2\pi \tfrac{-2p}{k}) = \begin{cases} -2, & p < -\frac{k-1}{2}, \\ 0, & p \geq -\frac{k-1}{2}. \end{cases} \end{aligned}$$

Hence, the first $\frac{k-1}{2}$ columns of the spectral sequence consist of two copies of the cohomology of S^1 and the next $\frac{k-1}{2}$ columns consist of the same, shifted down by 2, and then followed by the torsion-free[20] cohomology of $S^3/(\mathbb{Z}/k)$, shifted down by $2 \cdot 2 = 4$ (both the weight and $\dim_{\mathbb{C}} \mathfrak{M}$ are equal to 2). Thus, knowing that all the classes in the 0-th column $H^*(\mathfrak{M})$ have to vanish eventually, we see that the only possibility is that they vanish two by two in each page, finishing with the class $1 \in H^0(\mathfrak{M})$ that vanishes last at some page E_r, where $r > k + \frac{1}{2}$. Hence, by Proposition 6.8.14 we get that the filtration $F_\lambda^\varphi H^*(\mathfrak{M})$ that one obtains on the cohomology is rank-wise:

$$0 \subset \mathbb{K}^2[-2] \subset \cdots \subset \mathbb{K}^{k-1}[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^{k-1}[-2] = H^*(X_{\mathbb{Z}/k}).$$

We give a picture of the first few columns of the spectral sequence in the case $k = 5$ (Figure 6.4). We match by blue boxes the sets of classes that are killed one from the

[19] As explained in the paragraph after Proposition 6.8.17, these intersections are connected, thus being connected 1-dimensional manifolds, they are circles.

[20] Recall that we work over a characteristic zero coefficient field.

other by the green edge-differentials (as in general we do not know whether these maps are diagonalizable). With this labelling, one can read off the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ from the spectral sequence more easily. Also, we denote by a **4-star symbol** the point of local central symmetry in the spectral sequence, which exists due to Corollary 6.8.21.

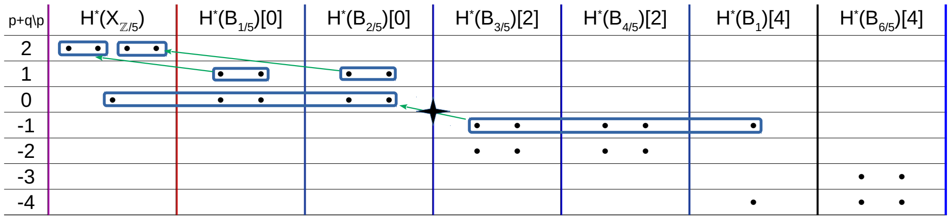


Figure 6.4: Spectral sequence for $X_{\mathbb{Z}/5}$

Example 6.9.4. $\mathfrak{M}_0 = \mathbb{C}^2/\mathbb{Z}/k$**, for even** $k \geq 2$**.**

Similarly to the previous case, we get that there are two $\mathbb{Z}/k$-torsion lines converging to two points, thus they yield two Morse-Bott submanifolds $B_{p,1} \cong B_{p,2} \cong S^1$. The difference is that here the action is even. Thus, by considering the square root of the action, we conclude that the time-$\frac{1}{2}$ column of the E_1-page is equal to $H^*(\mathcal{S}_p)[2]$, and after that its columns repeat with the shift $2 \cdot 1 = 2$ (the square root of the action has weight 1). Thus, we see that the classes in $H^2(\mathfrak{M})$ vanish two by two until the last one is taken by the top class of the time-$\frac{1}{2}$ column, and then the class $1 \in H^0(\mathfrak{M})$ gets killed at some later page.

Thus, rank-wise the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ is:

$$0 \subset \mathbb{K}^2[-2] \subset \cdots \subset \mathbb{K}^{k-2}[-2] \subset \mathbb{K}^{k-1}[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^{k-1}[-2] = H^*(X_{\mathbb{Z}/k}).$$

We give a picture of the first few columns of the spectral sequence in the case $k = 6$ (Figure 6.5).

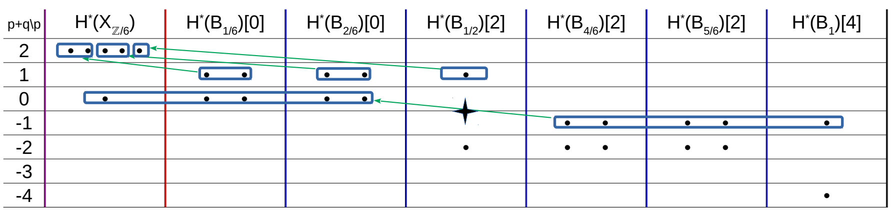


Figure 6.5: Spectral sequence for $X_{\mathbb{Z}/6}$

Next, let us consider a different conical action on $\mathbb{C}^2/(\mathbb{Z}/3)$, and observe that the filtration obtained is different from the previous.

Example 6.9.5. $\mathfrak{M}_0 = \mathbb{C}^2/(\mathbb{Z}/3)$ **with a non-standard action.**

As seen in Example 6.9.3, the standard action on $\mathbb{C}^2/(\mathbb{Z}/3)$ is

$$t \cdot X = t^3 X,\ t \cdot Y = t^3 Y,\ t \cdot Z = t^2 Z.$$

Here, we consider the action

$$t \cdot X = t^2 X,\ t \cdot Y = t^4 Y,\ t \cdot Z = t^2 Z.$$

We immediately see that all points are $\mathbb{Z}/2$-torsion and in addition there is a $\mathbb{Z}/4$-torsion line $l_1 = (0, Y, 0)$. Its resolution $\widetilde{l_1}$ is a $\mathbb{Z}/4$-torsion line that converges to a fixed point that has weight decomposition $\mathbb{C}_4 \oplus \mathbb{C}_{-2}$. Thus according to formula (6.35) we get that the shift of the corresponding Morse-Bott submanifold $B_1 = \widetilde{l_1} \cap \mathcal{S}_1$ is $\mu(B_1) = 0$. As all the points are $\mathbb{Z}/2$-torsion, we get that $B_2 = \mathcal{S}_2$, and the shift, that we calculate analogously, is $\mu(B_2) = -2$. Now, we can use the property of central symmetry (Corollary 6.8.21) and obtain the 3rd and 4th column. For degree reasons, the other columns cannot hit the classes in the ordinary cohomology $H^*(\mathfrak{M})$. See Figure 6.6.

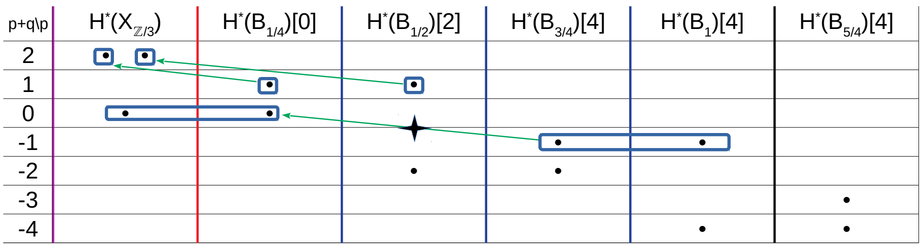


Figure 6.6: Spectral sequence for $X_{\mathbb{Z}/3}$ with a non-standard action.

Thus, rank-wise the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ is:

$$0 \subset \mathbb{K}[-2] \subset \mathbb{K}^2[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^2[-2] = H^*(X_{\mathbb{Z}/3}),$$

which is different from the filtration obtained by the standard action in Example 6.9.3:

$$0 \subset \mathbb{K}^2[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^2[-2] = H^*(X_{\mathbb{Z}/3}).$$

Hence, this example and Example 6.9.3 for $k = 3$ yield two different Hamiltonian models for the ordinary cohomology $H^*(X_{\mathbb{Z}/3})$ of a CSR $X_{\mathbb{Z}/3}$, given by the classes in spectral sequence which kill the cohomology classes of $X_{\mathbb{Z}/3}$ (the 0-th column).

Now we continue with the Du Val singularities of type D. Firstly, observe that in the coordinates (6.37), the action (6.38) on a D_{n+2}-singularity becomes

$$t \cdot X = t^{2n+2}X, \quad t \cdot Y = t^{2n}Y, \quad t \cdot Z = t^4 Z,$$

thus all the points are $\mathbb{Z}/2$-torsion. Hence, in order to simplify the action we take a square-root of it:

$$t \cdot X = t^{n+1}X, \quad t \cdot Y = t^{n}Y, \quad t \cdot Z = t^2 Z. \tag{6.39}$$

As in the type A case, we distinguish between two cases, by parity of n.

Example 6.9.6. $\mathfrak{M}_0 = \mathbb{C}^2/BD_{8k}$ **(D_{2k+2}-singularity),** $k \geq 1$

Here, the square-root of the standard action (6.39) becomes

$$t \cdot X = t^{2k+1}X, \quad t \cdot Y = t^{2k}Y, \quad t \cdot Z = t^2 Z.$$

Hence, it is easy to see that there are exactly three torsion lines

$$\alpha = (0, Y, 0), \;\; \beta_1 = (0, 2Z^k, Z), \;\; \beta_2 = (0, -2Z^k, Z),$$

which have torsion groups $\mathbb{Z}/2k, \mathbb{Z}/2, \mathbb{Z}/2$, respectively. As the action is a square-root of a weight-2 action, it is a weight-1 action, thus has a minimal component. It must be the sphere that corresponds to the unique trivalent vertex of the graph D_{2k+2} (that sphere has three intersection points fixed by the action, hence has to be fixed). The resolutions $\tilde{\alpha}, \tilde{\beta}_1, \tilde{\beta}_2$ of the torsion lines α, β_1, β_2 are lines again, and they converge to the fixed points of the spheres that correspond to the leaves of the graph D_{2k+2}. This is true because these are the leaves of the attraction graph, and by Proposition 2.3.15 we know that each leaf of an attraction graph has a torsion bundle that converges to it.

Using formula (6.35), as in the previous examples, one can calculate the degree shifts $\mu(B_{p,c})$ of the Morse Bott submanifolds $B_{p,c}$ which are intersections of the lines $\tilde{\alpha}, \tilde{\beta}_1, \tilde{\beta}_2$ with the slices $\mathcal{S}_p$, hence are circles. One deduces that they all have zero shift, for $\tau_p < 1$. Then, $B_1 = \mathcal{S}_1$ has a shift down by 2 (as the action has weight 1), and thus the other columns are determined, by Corollary 6.8.20.

To sum up, the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ is (rank-wise):

$$\text{for } k = 1: \;\; 0 \subset \mathbb{K}^3[-2] \subset \mathbb{K}^4[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^4[-2] = H^*(\mathfrak{M}),$$
$$\text{for } k \geq 2: \;\; 0 \subset \mathbb{K}[-2] \subset \cdots \subset \mathbb{K}^{k-1}[-2] \subset \mathbb{K}^{k+2}[-2] \subset \mathbb{K}^{k+3}[-2]$$
$$\cdots \subset \mathbb{K}^{2k+1}[-2] \subset \mathbb{K}^{2k+2}[-2] \subset H^*(\mathfrak{M}),$$

where in both cases the jump in the rank by 3 occurs at the time-$\frac{1}{2}$ column, as all three lines are $\mathbb{Z}/2$-isotropic.

We give a picture (Figure 6.7) of the spectral sequence in the case $k = 2$ (thus, D_6-singularity):

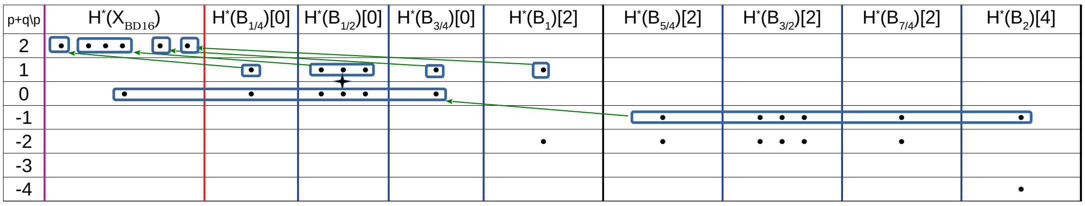


Figure 6.7: Spectral sequence for D_6.

Example 6.9.7. $\mathfrak{M}_0 = \mathbb{C}^2/BD_{8k+4}$ **(D_{2k+3}-singularity),** $k \geq 1$

Here, the square-root of the standard action (6.39) becomes

$$t \cdot X = t^{2k+2}X, \ t \cdot Y = t^{2k+1}Y, \ t \cdot Z = t^2 Z.$$

Thus, similarly to the previous example, we get exactly three torsion lines

$$\alpha = (0, Y, 0), \ \beta_1 = (2iZ^k, 0, Z), \ \beta_2 = (-2iZ^k, 0, Z),$$

which have isotropies $\mathbb{Z}/(2k+1), \mathbb{Z}/2, \mathbb{Z}/2$, respectively. These yield circles as Morse-Bott submanifolds, and as in the previous example, they all have zero shifts. The only difference from the previous example is that here in the time $\frac{1}{2}$-column there are 2 Morse-Bott manifolds, thus the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ at $\lambda = \frac{1}{2}$ jumps by rank 2 in the degree 2.

Thus, rank-wise the filtration $F^{\varphi}_{\lambda} H^*(\mathfrak{M})$ is:

$$0 \subset \mathbb{K}[-2] \subset \cdots \subset \mathbb{K}^k[-2] \subset \mathbb{K}^{k+2}[-2] \subset \mathbb{K}^{k+3}[-2] \subset \cdots \subset \mathbb{K}^{2k+3}[-2] \subset H^*(\mathfrak{M}).$$

We give a picture (Figure 6.8) of the spectral sequence in the case $k = 1$ (thus, D_5-singularity).

We remark that for Du Val singularities of type D, in order to get the information on Morse-Bott submanifolds and thus on the spectral sequence, we did not have to know the invariant polynomials and coordinates of the singularity in $\mathbb{C}^3$. Instead, we could have just observed the extended attraction graph in the corresponding resolution, which we now explain.

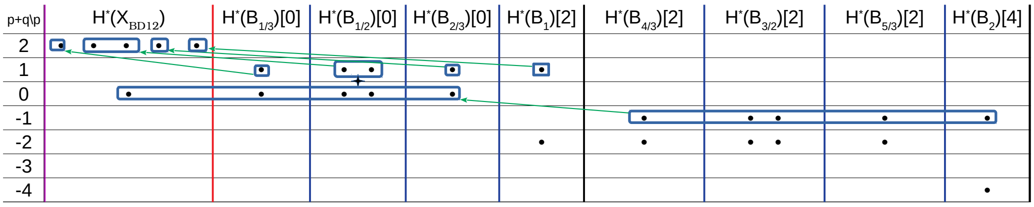


Figure 6.8: Spectral sequence for D_5.

Example 6.9.8. ***D_n*-singularity via the extended attraction graph**
Consider the minimal resolution of a D_n singularity. As already mentioned, the core is topologically a Dynkin tree D_n of spheres. The sphere C corresponding to the trivalent vertex has to be fixed by the action, as it has three fixed points of intersection with the other spheres. The other spheres cannot be fixed, due to a homological Morse-Bott argument (like in the last part of the proof of Proposition 3.1.3). Thus, all other spheres have two fixed points, and hence the attraction graph is a Dynkin graph D_n as well. As there is a fixed component of the core C, by Proposition 3.1.6, the standard action has a square-root, which is a weight-1 conical action. Thus, the tangent space weight decomposition on C is $H_0 \oplus H_1$, where H_0 and H_1 are 1-dimensional. In particular, this is true for the intersection points p_1, p_2, p_3 with the neighbouring spheres C_1, C_2, C_3 to the sphere C. Thus, there are $\mathbb{C}^*$-orbits flowing out of p_i towards the other fixed points q_i in the spheres C_i, respectively. They will converge to H_{-1} weight spaces, according to Lemma 2.3.12. Thus, by $\omega_{\mathbb{C}}$-pairing duality of the weight-1 action, the points q_i also have a H_2 weight-space. Two of them are leaf nodes in the attraction graph, thus these weight-spaces will give rise to $\mathbb{Z}/2$-torsion line bundles $\mathcal{H}_2$, hence will indeed yield circles when intersected with slices. In the D_4 case, the third point will also have a $\mathbb{Z}/2$-torsion line bundle, whereas for $n > 4$ we continue further along the attraction graph using the same duality principle, getting that the weight decompositions for fixed points are $H_{-2} \oplus H_3, H_{-3} \oplus H_4, \ldots, H_{-(n-3)} \oplus H_{n-2}$, thus ultimately getting the $\mathbb{Z}/(n-2)$-torsion line bundle, like we have obtained in Examples 6.9.7 and 6.9.6.

The same method as in the last example can be used to calculate the Morse-Bott manifolds in type E. Here this method helps **substantially** as the invariant polynomials for type E are quite cumbersome, hence the method of finding the torsion subsets in $\mathfrak{M}_0$ and then finding their resolutions on $\mathfrak{M}$ becomes much harder.

Example 6.9.9. ***E_6*-Singularity**

Analogously to the previous example, as the core is the E_6-Dynkin tree of spheres, the sphere C that corresponds to the trivalent node of E_6 is fixed. Thus again, by Proposition 3.1.6, the standard action has a square-root, which is a weight-1 conical action. Thus, the weight decomposition of C is $H_0 \oplus H_1$. Analogously to the type D case, here the attraction graph is of E_6 type, thus yields one $\mathbb{Z}/2$-torsion and two $\mathbb{Z}/3$-torsion line bundles over fixed points that are leaves of the attraction graph (Figure 6.9. Except for the attraction graph, which drawn in black dots and blue arrows, on the figure we also depict the topology of the core by drawing the spheres as black circles. The central sphere is completely black, being fixed.)

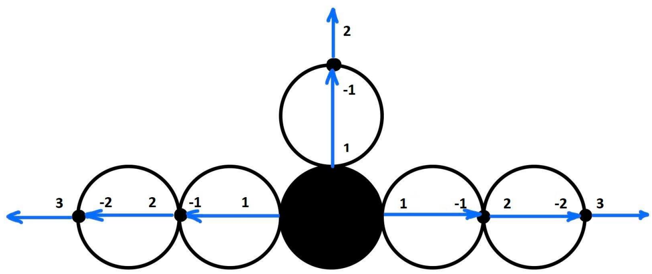


Figure 6.9: Extended attraction graph of minimal resolution of E_6-singularity.

Thus, the Morse-Bott submanifolds are circles, and their shifts are calculated in the standard way, knowing the weight decompositions $H_{-1} \oplus H_2, H_{-2} \oplus H_3, H_{-2} \oplus H_3$ of their converging points. We give a picture of the obtained spectral sequence (Figure 6.10).

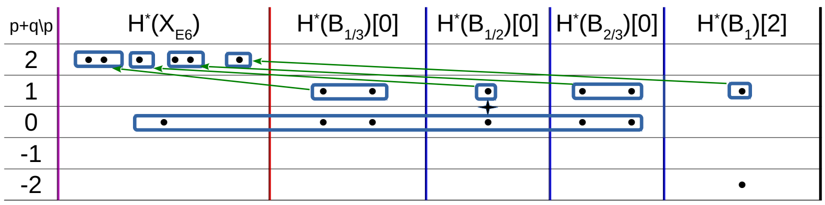


Figure 6.10: Spectral sequence for E_6.

From it, one reads-off the filtration on $H^*(\mathfrak{M})$ (rank-wise):

$$0 \subset \mathbb{K}^2[-2] \subset \mathbb{K}^3[-2] \subset \mathbb{K}^5[-2] \subset \mathbb{K}^6[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^6[-2] = H^*(\mathfrak{M}).$$

Example 6.9.10. *E_7*-**Singularity**

Analogously, we get three torsion line bundles over fixed points that are leaves of the attraction graph, with torsion groups $\mathbb{Z}/2, \mathbb{Z}/3$, and $\mathbb{Z}/4$. We give a picture of the obtained spectral sequence (Figure 6.11). We do not put the edge-differentials on this one as it would make the picture unreadable.

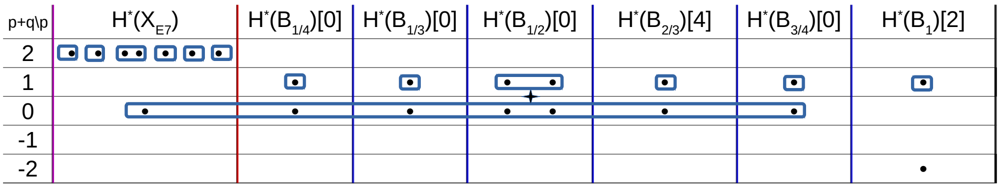


Figure 6.11: Spectral sequence for E_7.

From it, one reads-off the filtration on $H^*(\mathfrak{M})$ (rank-wise):

$$0 \subset \mathbb{K}^1[-2] \subset \mathbb{K}^2[-2] \subset \mathbb{K}^4[-2] \subset \mathbb{K}^5[-2] \subset \mathbb{K}^6[-2] \subset \mathbb{K}^7[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^7[-2] = H^*(\mathfrak{M}).$$

Example 6.9.11. E_8**-Singularity**

Analogously, we get three torsion line bundles over fixed points that are leaves of the attraction graph, with torsion groups $\mathbb{Z}/2, \mathbb{Z}/3$, and $\mathbb{Z}/5$. We give a picture of the obtained spectral sequence (Figure 6.12). We do not put blue boxes and edge-differentials on this one as there is just one possibility for them in this case.

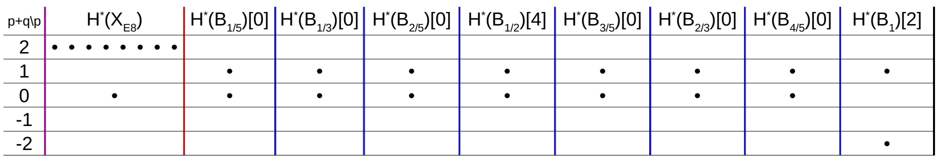


Figure 6.12: Spectral sequence for E_8.

From it, one reads-off the filtration on $H^*(\mathfrak{M})$ (rank-wise):

$$0 \subset \mathbb{K}^1[-2] \subset \mathbb{K}^2[-2] \subset \mathbb{K}^3[-2] \subset \cdots \subset \mathbb{K}^7[-2] \subset \mathbb{K}^8[-2] \subset \mathbb{K}[0] \oplus \mathbb{K}^8[-2] = H^*(\mathfrak{M}).$$

(in every step the rank increases by one).

6.9.3 Slodowy varieties of type A

In this section we consider resolutions of ordinary Slodowy varieties of type A (Section 5.1.3) and the Kazhdan action on them (Section 5.1.4). Let us recall them briefly for the convenience of the reader. Given an $\mathfrak{sl}_2$ triple (e, f, h) of nilpotent elements, one constructs the Slodowy slice

$$S_e = \{x \in \mathfrak{sl}_n \mid [x - e, f] = 0\} = e + ker(\mathrm{ad} f).$$

The ordinary Slodowy variety and its resolution are

$$\mathcal{S}_e := S_e \cap \mathcal{N}, \quad \widetilde{\mathcal{S}}_e := \nu^{-1}(\mathcal{S}_e). \tag{6.40}$$

On the Slodowy slice, and hence on the Slodowy variety, the Kazhdan action is defined by

$$t \cdot x = t^2 \mathrm{Ad}(t^{-h})x. \tag{6.41}$$

In the matrix notation $x = (X_{ij})$, this becomes

$$t \cdot (X_{ij}) = (t^{2+h_j-h_i} X_{ij}), \tag{6.42}$$

which we will use in the examples. The Kazhdan action on $\mathcal{S}_e$ pulls back via the Springer resolution to the Kazhdan action on $\widetilde{\mathcal{S}}_e$ by

$$t \cdot (x, F) = (t^2 \mathrm{Ad}(t^{-h})x, t^{-h}F). \tag{6.43}$$

Picking a Jordan basis (v_i) for the triple (e, f, h), the action on flags is induced by the action on $\mathbb{C}^n$ given by

$$t \cdot v_i = t^{-h_i} v_i, \tag{6.44}$$

where $h = diag(h_1, \ldots, h_n)$ in that basis. We abbreviate $\mathcal{S}_\lambda := \mathcal{S}_e$, $\widetilde{\mathcal{S}}_\lambda := \widetilde{\mathcal{S}}_e$ where $\lambda = \lambda(e)$ is the associated Jordan partition to the nilpotent element e (i.e. the sizes of the Jordan blocks of e).

The basic algorithm that we are going to use here is that we first find the torsion points $\{\mathcal{T}_{\mathbb{Z}/k}\}_{k\geq 2}$ in the Slodowy slice S_λ, then we restrict to the nilpotent cone $\mathcal{N}$ in order to get the torsion points

$$\mathcal{P}_{\mathbb{Z}/k} = \mathcal{T}_{\mathbb{Z}/k} \cap \mathcal{N}$$

in $\mathcal{S}_\lambda$. Then, using the Springer resolution ν, we find the torsion points $\mathcal{R}_{\mathbb{Z}/k}$ in $\widetilde{\mathcal{S}}_\lambda = \nu^{-1}(\mathcal{S}_\lambda)$. As the fibres of the resolution ν depend on the nilpotent orbit $\mathcal{O}_\lambda$ where they are taken, we will keep track of λ in the intersections

$$\mathcal{P}_{\mathbb{Z}/k,\lambda} := \mathcal{T}_{\mathbb{Z}/k} \cap \mathcal{O}_\lambda$$

obtained for $\mathcal{T}_{\mathbb{Z}/k}$ with other nilpotent orbits.

In addition, when the Kazhdan action is even, we will rather consider a square-root of it, in order to avoid all points being $\mathbb{Z}/2$-torsion.

In the following examples we will use the notation B_{τ_p} instead of B_p for the Morse-Bott submanifolds, so that it agrees with the spectral sequence labelling.

Example 6.9.12. Slodowy variety $\mathcal{S}_{22}$

From Proposition 5.2.14 we get the description of the affine Slodowy slice S_{22}

$$S_{22} = \left\{ \begin{bmatrix} a & 1 & c & 0 \\ m & a & p & c \\ e & 0 & -a & 1 \\ x & e & w & -a \end{bmatrix} \middle| \, a, c, e, m, p, x, w \in \mathbb{C} \right\}.$$

As we have $t^{h_{22}} = diag(t\ t^{-1}\ t\ t^{-1})$, by formula (6.42) the Kazhdan action on S_{22} is equal to:

$$t \cdot \begin{bmatrix} a & 1 & c & 0 \\ m & a & p & c \\ e & 0 & -a & 1 \\ x & e & w & -a \end{bmatrix} = \begin{bmatrix} t^2a & 1 & t^2c & 0 \\ t^4m & t^2a & t^4p & t^2c \\ t^2e & 0 & -t^2a & 1 \\ t^4x & t^2e & t^4w & -t^2a \end{bmatrix}. \tag{6.45}$$

Thus, we see that the action is even, so we take a square root of it. Then, we only have powers t^1 and t^2 of t in equation (6.45), hence we get only $\mathbb{Z}/2$-torsion points in S_{22}.

$$\mathcal{T}_{\mathbb{Z}/2} = \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ m & 0 & p & 0 \\ 0 & 0 & 0 & 1 \\ x & 0 & w & 0 \end{bmatrix} \middle| m, p, x \in \mathbb{C} \right\}.$$

Hence intersecting it with the nilpotent cone $\mathcal{N} = \{A \mid A^4 = 0\}$ after some amount of computations one gets the set of $\mathbb{Z}/2$-torsion points in $\mathcal{S}_{22}$:

$$\mathcal{P}_{\mathbb{Z}/2} = \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ m & 0 & p & 0 \\ 0 & 0 & 0 & 1 \\ x & 0 & w & 0 \end{bmatrix} \middle| \, m^2 + px = 0, \; m, p, x \in \mathbb{C} \right\} \cong \mathbb{C}^2/(\mathbb{Z}/2) \tag{6.46}$$

One can check that these matrices, except for the zero matrix, all lie in the regular orbit $\mathcal{O}_4$, hence are bijectively lifted to $\widetilde{\mathcal{S}}_{22}$ under the Springer resolution. Denoting by A_{mpx} the matrix in equation (6.46), for $(m, p, x) \neq (0, 0, 0)$, its lift is the Springer fibre $(A_{mpx}, \mathcal{F}_{mpx})$. Thus the flag

$$\mathcal{F}_{mpx} = \{0 \subset F_1 \subset F_2 \subset F_3 \subset \mathbb{C}^4\}$$

is defined by $A_{mpx}F_i \subset F_{i-1}$. Denoting by (v_1, v_2, v_3, v_4) the basis in which all these matrices are given, after some calculations we get

$$\mathcal{F}_{mpx} = (0 \subset \langle pv_1 - mv_3 \rangle \subset \langle pv_1 - mv_3, pv_2 - mv_4 \rangle \subset \langle v_1, v_3, pv_2 - mv_4 \rangle \subset \mathbb{C}^4),$$

when $(p,m) \neq (0,0)$, and

$$\mathcal{F}_{mpx} = (0 \subset \langle pv_1 - mv_3 \rangle \subset \langle pv_1 - mv_3, pv_2 - mv_4 \rangle \subset \langle v_1, v_3, pv_2 - mv_4 \rangle \subset \mathbb{C}^4),$$

when $(m,x) \neq (0,0)$.

Now, using the formula (6.44) we get that the Kazhdan action acts on $\mathbb{C}^n$ by

$$t \cdot (v_1, v_2, v_3, v_4) = (t^{-1}v_1, tv_2, t^{-1}v_3, tv_4),$$

thus the induced action on the flag $\mathcal{F}_{mpx}$ fixes it. Hence, the fixed locus to which $\mathcal{R}_{\mathbb{Z}/2}$ converges (when $t \to 0$) is a $\mathbb{P}^1$ in the core:

$$\mathfrak{F}_{mpx} := (0 \subset \langle \alpha v_1 + \beta v_3 \rangle \subset \langle \alpha v_1 + \beta v_3, \alpha v_2 + \beta v_4 \rangle \subset \langle v_1, v_3, \alpha v_2 + \beta v_4 \rangle \subset \mathbb{C}^4 \mid [\alpha : \beta] \in \mathbb{P}^1).$$

Thus, $\mathcal{R}_{\mathbb{Z}/2}$ has a single connected component, which is a torsion bundle $\mathcal{R}_{\mathbb{Z}/2} = \mathcal{H}_2 \to \mathfrak{F}_{mpx}$. Being a resolution of the cone $\mathcal{P}_{\mathbb{Z}/2} \cong \mathbb{C}^2/(\mathbb{Z}/2)$, it is a Du Val singularity resolution, hence $\mathcal{H}_2 \cong T^*\mathbb{C}P^1$. So, the cohomology of its hypersurface $B_{1/2} = \mathcal{S}_{1/2} \cap \mathcal{H}_2$ is isomorphic to the cohomology of the unit bundle $S(T^*\mathbb{C}P^1) \cong \mathbb{R}P^3$, by Lemma 6.8.23. As we work over the field $\mathbb{K}$ of characteristic zero,

$$H^*(B_{1/2}) \cong H^*(\mathbb{R}P^3) = \mathbb{K}[0] \oplus \mathbb{K}[-3].$$

The cohomology of the space $\widetilde{\mathcal{S}}_{22}$ is isomorphic to the cohomology of its core, the Springer fibre $\mathcal{B}^{22}$. Vector space generators of $H^*(\mathcal{B}^\lambda)$ are labelled by row-standard tableaux of shape λ, and the degrees of the generators can be easily computed by some tableaux combinatorics. We will not delve into this, instead we refer the interested reader to [Fr09a, Sec. 1.3]. Using the prescribed method therein, one obtains

$$H^*(\widetilde{\mathcal{S}}_{22}) \cong H^*(\mathcal{B}^{22}) \cong \mathbb{K}[0] \oplus \mathbb{K}^3[-2] \oplus \mathbb{K}^2[-4].$$

Thus, from Proposition 6.8.24 one obtains the cohomology of the slice. Hence, we have all the elements of the spectral sequence, except for the degree shifts.

As the weight decomposition of $\mathfrak{F}_{mpx}$ has $H_0 \oplus H_2$ as a summand, by the $\omega_\mathbb{C}$-duality it has also $H_{1-2} = H_{-1}$ and $H_{1-0} = H_1$ (recall that we have square-rooted the action so now $\omega_\mathbb{C}$ has weight 1). Thus we have $T_{\mathfrak{F}_{mpx}}\widetilde{\mathcal{S}}_{22} = H_{-1} \oplus H_0 \oplus H_1 \oplus H_2$, and then by formula (6.35) one gets that $\mu(B_{1/2}) = 0$. Having in mind time-1 periodicity of the spectral sequence, that is all that we need. We give a picture of the obtained spectral sequence (Figure 6.13). In it, **the asterisks **** label the unknown ranks in the middle cohomologies of the slice $\mathcal{S}_p$. We will always put $\mathrm{rk}(H^{mid}(\mathfrak{M}))$ asterisks

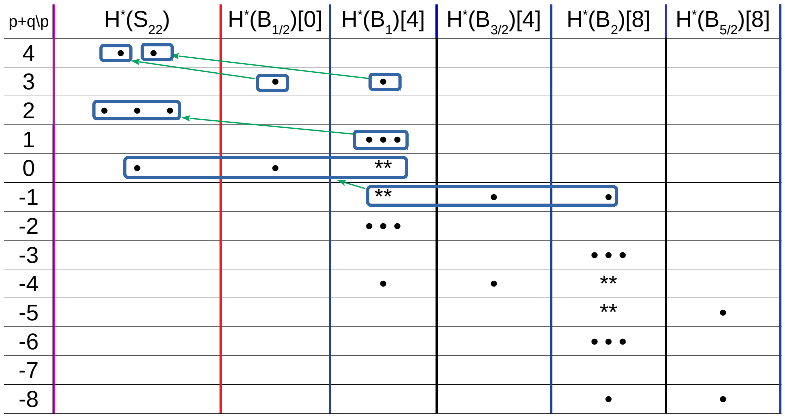


Figure 6.13: Spectral sequence for $\mathcal{S}_{22}$

in the pictures, as that is the maximum possible rank of these middle cohomologies (recall Proposition 6.8.24).

Assuming that there are no asterisks, from the spectral sequence one gets the (rank-wise) filtration on $H^*(\widetilde{\mathcal{S}}_{22})$:

$$0 \subset \mathbb{K}^1[-4] \subset \mathbb{K}^3[-2] \oplus \mathbb{K}^2[-4] \subset \mathbb{K}[0] \oplus \mathbb{K}^3[-2] \oplus \mathbb{K}^2[-4] = H^*(\widetilde{\mathcal{S}}_{22}).$$

However, if if a degree -1 asterisk in the B_1-column kills the $H^0(\widetilde{\mathcal{S}}_{22})$ generator, the filtration is:

$$0 \subset \mathbb{K}^1[-4] \subset \mathbb{K}[0] \oplus \mathbb{K}^3[-2] \oplus \mathbb{K}^2[-4] = H^*(\widetilde{\mathcal{S}}_{22}).$$

At the moment it is not clear to us which of these two situations happens.

Example 6.9.13. Slodowy variety $\mathcal{S}_{32}$

From Proposition 5.2.14 we get the description of the affine Slodowy slice S_{32}

$$S_{32} = \left\{ \begin{bmatrix} 2a & 1 & 0 & 0 & 0 \\ f & 2a & 1 & j & 0 \\ l & f & 2a & p & 2j \\ 2y & 0 & 0 & -3a & 1 \\ w & y & 0 & x & -3a \end{bmatrix} \middle| \; a, f, j, l, p, x, y, w \in \mathbb{C} \right\}.$$

As $t^{h_{32}} = diag(t^2\ t^0\ t^{-2}\ t^1\ t^{-1})$, the Kazhdan action on S_{32} is, by formula (6.42), equal to:

$$t \cdot \begin{bmatrix} 2a & 1 & 0 & 0 & 0 \\ f & 2a & 1 & j & 0 \\ l & f & 2a & p & 2j \\ 2y & 0 & 0 & -3a & 1 \\ w & y & 0 & x & -3a \end{bmatrix} = \begin{bmatrix} 2t^2a & 1 & 0 & 0 & 0 \\ t^4f & 2t^2a & 1 & t^3j & 0 \\ t^6l & t^4f & 2t^2a & t^5p & 2t^3j \\ 2t^3y & 0 & 0 & -3t^2a & 1 \\ t^5w & t^3y & 0 & t^4x & -3t^2a \end{bmatrix}.$$

Thus, we see that S_{32} has $\mathbb{Z}/k$-torsion points for all $k = 2, 3, 4, 5, 6$. Restricting to $\mathcal{S}_{32} = \{A \in S_{32} \mid A^5 = 0\}$, after some amount of matrix calculation via a computer program, we get only $\mathbb{Z}/3$ and $\mathbb{Z}/5$-torsion points on it.

$$\mathcal{P}_{\mathbb{Z}/3} = \left\{ \left. \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & j & 0 \\ -5jy & 0 & 0 & 0 & 2j \\ 2y & 0 & 0 & 0 & 1 \\ 0 & y & 0 & 0 & 0 \end{bmatrix} \right| j, y \in \mathbb{C} \right\} \cong \mathbb{C}^2(j, y), \tag{6.47}$$

$$\mathcal{P}_{\mathbb{Z}/5} = \left\{ \left. \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 1 \\ w & 0 & 0 & 0 & 0 \end{bmatrix} \right| pw = 0 \right\} \cong \mathbb{C}(p) \cup \mathbb{C}(w), \tag{6.48}$$

where in round brackets we mention the names of the coordinates used on $\mathbb{C}^2$ and $\mathbb{C}$.

$\mathbb{Z}/5$ **torsion points and their convergence to the core.** Apart from the zero matrix, $\mathcal{P}_{\mathbb{Z}/5}$ lies in the regular orbit $\mathcal{O}_5$. Therefore is bijectively lifted to $\widetilde{\mathcal{S}}_{32}$ under the Springer resolution. Denoting the matrix in equation (6.48) by A_{pw}, its lift is $(A_{p0}, \mathcal{F}_p)$ and $(A_{0w}, \mathcal{F}_w)$ where

$$\mathcal{F}_p := (0 \leq \langle v_1 \rangle \leq \langle v_1, v_2 \rangle \leq \langle v_1, v_2, v_3 \rangle \leq \langle v_1, v_2, v_3, v_4 \rangle \leq \mathbb{C}^5)$$

$$\mathcal{F}_w := (0 \leq \langle v_4 \rangle \leq \langle v_4, v_5 \rangle \leq \langle v_1, v_4, v_5 \rangle \leq \langle v_1, v_2, v_4, v_5 \rangle \leq \mathbb{C}^5)$$

as one can easily calculate from the Springer fibre conditions $A_{0w}\mathcal{F}_w \subset \mathcal{F}_w$, $A_{p0}\mathcal{F}_p \subset \mathcal{F}_p$. Here, (v_1, v_2, v_3, v_4) is the basis in which all the matrices are given. Thus, in the limit $t \to 0$ of the Kazhdan action,

$$\lim_{t \to 0} t \cdot (A_{p0}, \mathcal{F}_p) = (e_{32}, \mathcal{F}_p), \quad \lim_{t \to 0} t \cdot (A_{0w}, \mathcal{F}_w) = (e_{32}, \mathcal{F}_w)$$

hence, these two $\mathbb{Z}/5$-torsion lines converge to two points $\mathcal{F}_p$ and $\mathcal{F}_w$ in the Springer fibre $\mathcal{B}^{32}$. They yield two Morse-Bott submanifolds

$$B_{1/5} = B_{1/5,1} \sqcup B_{1/5,2} \cong S^1 \sqcup S^1,$$

however for now we cannot compute their shifts $\mu(B_{1/5,i})$ as we do not have complete information on the weight decomposition of fixed points $\mathcal{F}_p$ and $\mathcal{F}_w$ in the core $\mathcal{B}^{32}$. We will be able to do it after considering the $\mathbb{Z}/3$-torsion points as well.

$\mathbb{Z}/3$ **torsion points and their convergence to the core.** Denote the matrix in equation (6.47) by A_{jy}. By checking for which $j, y \in \mathbb{C}$ we have $A_{jy}^4 = 0$, we get that in $\mathcal{P}_{\mathbb{Z}/3}$ two lines $j = 0$ and $y = 0$ lie on the subregular orbit $\mathcal{O}_{41}$, whereas the rest lie in $\mathcal{O}_5$.

Thus, for $j, y \neq 0$ the fibre above A_{jy} is $(A_{jy}, \mathcal{F}_{jy})$, where after some calculation one gets

$$\mathcal{F}_{jy} = (0 \subset \langle jv_3 - v_4 \rangle \subset \langle jv_3 - v_4, v_1 + yv_5 \rangle \subset \langle jv_3 - v_4, v_1 + yv_5, v_2 \rangle \subset \langle v_2, v_3, v_4, v_1 + yv_5 \rangle \subset \mathbb{C}^5).$$

This flag also makes sense for $j = 0$ or $y = 0$, and due to continuity, we get a slice

$$S_{jy} = \{(A_{jy}, \mathcal{F}_{jy}) \mid j, y \in \mathbb{C}\} \cong \mathbb{C}^2.$$

By formula (6.44), the Kazhdan action acts on $\mathbb{C}^n$ by

$$t \cdot (v_1, v_2, v_3, v_4, v_5) = (t^{-2} v_1, v_2, t^2 v_3, t^{-1} v_4, t v_5),$$

thus one immediately gets $\lim_{t \to 0} t \cdot (A_{jy}, \mathcal{F}_{jy}) = (e_{32}, \mathcal{F}_{big})$, where

$$\mathcal{F}_{big} := (0 \subset \langle v_4 \rangle \subset \langle v_1, v_4 \rangle \subset \langle v_1, v_2, v_4 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5).$$

Thus the whole slice S_{jy} converges to the flag $\mathcal{F}_{big}$, hence yields a trivial 2-dimensional torsion bundle $\mathcal{H}_3 \to \mathcal{F}_{big}$. The corresponding Morse-Bott submanifold $B_{1/3,big}$ is cohomologically isomorphic to the sphere bundle

$$H^*(B_{1/3,big}) \cong H^*(S(\mathcal{H}_3)) \cong H^*(S^3) = \mathbb{K}[0] \oplus \mathbb{K}[-3], \tag{6.49}$$

and knowing the weight decomposition $H_3 \oplus H_3 \oplus H_{-1} \oplus H_{-1}$ of $\mathcal{F}_{big}$ we also compute $\mu(B_{1/3,big}) = 0$.

Now let us consider the $\mathbb{Z}/3$-torsion points in the fibre over A_{j0}, where $j \neq 0$. The Springer fibre over A_{j0} is a Dynkin A_4-tree of spheres $\mathbb{P}^1_1 \cup \mathbb{P}^1_2 \cup \mathbb{P}^1_3 \cup \mathbb{P}^1_4$ that intersect transversally. Standard computations from Springer theory yield their explicit flag description:

$$\begin{aligned}
\mathbb{P}^1_1[\alpha : \beta] =& (0 \subset \langle \alpha v_1 + \beta(\tfrac{1}{j} v_4 - v_3) \rangle \subset \langle v_1, \tfrac{1}{j} v_4 - v_3 \rangle \subset \langle v_1, v_2, \tfrac{1}{j} v_4 - v_3 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5) \\
\mathbb{P}^1_2[\alpha : \beta] =& (0 \subset \langle v_1 \rangle \subset \langle v_1, \alpha v_2 + \beta(\tfrac{1}{j} v_4 - v_3) \rangle \subset \langle v_1, v_2, \tfrac{1}{j} v_4 - v_3 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5) \\
\mathbb{P}^1_3[\alpha : \beta] =& (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, \alpha v_3 + \beta \tfrac{1}{j} v_4 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5) \\
\mathbb{P}^1_4[\alpha : \beta] =& (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, 2v_3 + \tfrac{1}{j} v_4 \rangle \\
& \subset \langle v_1, v_2, 2v_3 + \tfrac{1}{j} v_4 \rangle \subset \langle v_1, v_2, 2v_3 + \tfrac{1}{j} v_4, \alpha v_3 + \beta \tfrac{1}{j} v_5 \rangle \subset \mathbb{C}^5)
\end{aligned}$$

In short: One modifies the base v_i such that it becomes a Jordan base for A_{j0} and then one uses the description of the Springer fibre of the standard matrix e_{41}.

Recall that the Kazhdan action is $t \cdot (v_1, v_2, v_3, v_4, v_5) = (t^{-2}v_1, v_2, t^2 v_3, t^{-1}v_4, tv_5)$, which for the primitive third root of unity ε gives

$$\varepsilon \cdot (v_1, v_2, v_3, v_4, v_5) = (\varepsilon v_1, v_2, \varepsilon^{-1}v_3, \varepsilon^{-1}v_4, \varepsilon v_5).$$

Notice that the $\mathbb{Z}/3$-torsion points in the fibre are exactly the flags fixed by the ε action. Thus, we get that the $\mathbb{Z}/3$-torsion points in the Springer fibre above A_{j0} are

$$\mathbb{P}^1_1[0:1], \quad \mathbb{P}^1_1[1:0], \quad \mathbb{P}^1_3[\alpha:\beta], \quad \mathbb{P}^1_4[0:1],$$

thus three points and a sphere. Notice that the point $\mathbb{P}^1_1[0:1] = \mathcal{F}_{j0}$ is already contained in the slice S_{jy}. One can calculate directly that $\lim_{t\to 0} t \cdot (A_{j0}, \mathbb{P}^1_1[1:0]) = (e_{32}, \mathcal{F}^3_j)$, and $\lim_{t\to 0} t \cdot (A_{j0}, \mathbb{P}^1_4[0:1]) = (e_{32}, \mathcal{F}^1_j)$, where

$$\mathcal{F}^3_j := (0 \subset \langle v_1 \rangle \subset \langle v_1, v_4 \rangle \subset \langle v_1, v_2, v_4 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5)$$

$$\mathcal{F}^1_j := (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_4 \rangle \subset \langle v_1, v_2, v_4, v_5 \rangle \subset \mathbb{C}^5)$$

Thus, these two yield $\mathbb{Z}/3$-torsion line bundles over the points $\mathcal{F}^3_j$ and $\mathcal{F}^1_j$. Hence, their Morse-Bott submanifolds

$$B_{1/3,j^3} \cong B_{1/3,j^1} \cong S^1$$

are circles.

The convergence (under Kazhdan action and $t \to 0$) of the spheres

$$\mathbb{P}^1_3[\alpha:\beta]_j = (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, \alpha v_3 + \beta \tfrac{1}{j} v_4 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5 \mid [\alpha:\beta] \in \mathbb{P}^1) \quad (6.50)$$

is a bit more involved. Namely, for $\beta = 0$, (6.50) converges to the flag

$$\mathcal{F}_p := (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_3 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5)$$

and otherwise (thus, generically) it converges to the flag

$$\mathcal{F}'_j := (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_4 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5).$$

Notice also that there is a $\mathbb{Z}/3$-torsion sphere in the core between these two points:

$$C_{jp} = (0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, \alpha v_3 + \beta v_4 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle \subset \mathbb{C}^5).$$

Thus, altogether with $\mathbb{P}^1_3[\alpha:\beta]$ we get a $\mathbb{Z}/3$-torsion line bundle

$$\{(A_{j0}, C_{jp}) \mid j \in \mathbb{C}\} \to C_{jp}$$

over the (non fixed!) sphere C_{jp}. This bundle is then trivial, and by Lemma 6.8.23 the corresponding Morse-Bott submanifold $B_{1/3,jp}$ is cohomologically isomorphic to its sphere bundle,

$$H^*(B_{1/3,jp}) \cong H^*(S^2 \times S^1) \cong \mathbb{K}[0] \oplus \mathbb{K}[-1] \oplus \mathbb{K}[-2] \oplus \mathbb{K}[-3]. \tag{6.51}$$

The weight-decomposition of $\mathcal{F}_j'$ is $H_3 \oplus H_3 \oplus H_{-1} \oplus H_{-1}$, thus by formula (6.35) one computes the grading $\mu(B_{1/3,jp}) = 0$. Now we compute the gradings of the Morse-Bott submanifolds $B_{1/3,j^3}$ and $B_{1/3,j^1}$. We claim that the weight-decompositions of their corresponding fixed loci $\mathcal{F}_j^3$ and $\mathcal{F}_j^1$ are both

$$H_3 \oplus H_{-1} \oplus H_1 \oplus H_1, \tag{6.52}$$

due to the following **accumulative** argument. Assuming the contrary and that there is a weight-space H_2, it then yields an $\mathbb{C}^*$-flowline in the core (as there are no $\mathbb{Z}/2$-torsion points outside of the core) which converges to another fixed set to weight-space H_{-2} (Lemma 2.3.12). Thus, by duality, that fixed set has an H_4 weight-space, which again yields a $\mathbb{C}^*$-orbit etc. As the number of components of the fixed locus is finite, this process must stop and we get a contradiction. The same would happen if we had assumed that there were a weight-space $H_{k\geq 4}$ in $\mathcal{F}_j^3$ and $\mathcal{F}_j^1$. Otherwise, assume that H_3 is 2-dimensional. That would mean that there are more outer[21] $\mathbb{Z}/3$-torsion points converging to $\mathcal{F}_j^3$ and $\mathcal{F}_j^1$, so again, a contradiction. Thus the weight-decompositions (6.52) are indeed true, and by formula (6.35) we get

$$\mu(B_{1/3,j^3}) = \mu(B_{1/3,j^1}) = 0. \tag{6.53}$$

Completely analogously, considering A_{0y} and its fibres we get (cohomologically) the same Morse-Bott submanifolds together with the same gradings.

Now, similarly as in the last example, by counting the row-standard Young tableaux of shape $(3,2)$ with grading defined in [Fr09a, Sec. 1.3] we get the ordinary cohomology

$$H^*(\widetilde{\mathcal{S}}_{32}) \cong H^*(\mathcal{B}^{32}) \cong \mathbb{K}[0] \oplus \mathbb{K}^4[-2] \oplus \mathbb{K}^5[-4].$$

We give a picture of the obtained spectral sequence (Figure 6.14). In the $B_{2/3}$ column we put the **asterisk sign * in the shift** as the shifts of different connected components in it **are different.** As can be seen on the picture, for the four circles the shift is equal to 2 whereas for S^3 and the two $S^2 \times S^1$ it is is equal to 4. We will use the asterisk notation for mixed shifts in the following examples as well.

[21] I.e., points that are outside of the core.

From this picture onward, we will sometimes denote the rank of the cohomology of the slice $\mathcal{S}_p$ in some entries of the picture by a number rather than using black dots, for convenience.

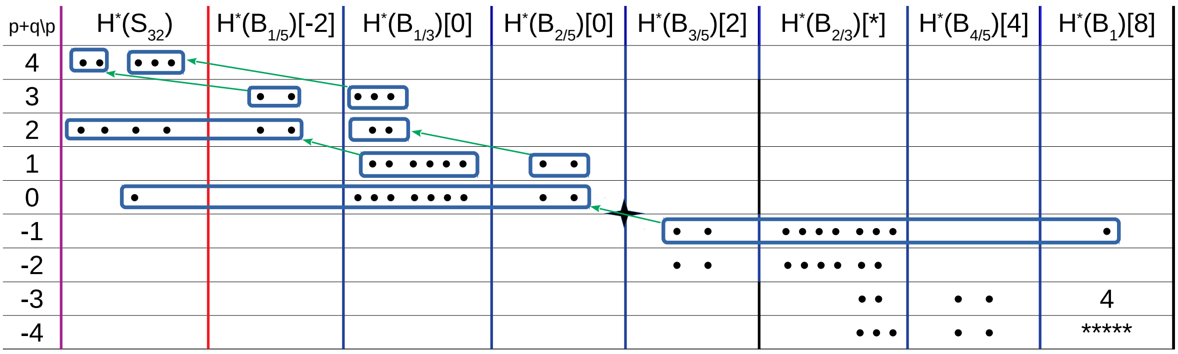


Figure 6.14: Spectral sequence for $\mathcal{S}_{32}$

From the spectral sequence, one easily reads-off the (rank-wise) filtration on $H^*(\widetilde{\mathcal{S}}_{32})$:

$$0 \subset \mathbb{K}^2[-4] \subset \mathbb{K}^4[-2] \oplus \mathbb{K}^5[-4] \subset \mathbb{K}[0] \oplus \mathbb{K}^4[-2] \oplus \mathbb{K}^5[-4] = H^*(\widetilde{\mathcal{S}}_{32}).$$

Remark 6.9.14. Corollaries 6.5.6 and 6.8.13 can also be used to read off information about the cohomology of the fixed locus components $\mathfrak{F}_\alpha$ from the spectral sequence. Namely, we see that the 0-th column has total rank 10, and this rank must persist even as we add new columns each time provided we take into account all the green arrows. Thus, we see that if we consider the block given by all columns p whose periods τ_p are below a certain slope, we must get total rank 10. By Corollary 6.5.6 the degrees in which these ranks live will shift in complicated ways, as we increase the slope. Observing how the gradings shift can reveal information about the cohomology of individual fixed components $\mathfrak{F}_\alpha$. In Figure 6.14 an extreme example occurs: if we consider all columns below $B_{2/5}$, we see that all of the rank 10 is supported in degree 0. By Corollary 6.5.6, that means each $\mathfrak{F}_\alpha$ has cohomology supported in just one degree, so each $\mathfrak{F}_\alpha$ is a point. So $\mathfrak{F} = \mathfrak{M}^{\mathbb{C}^*}$ consists of 10 points. From the representation theory point of view, this follows from the fact that the Kazhdan action's weights $(-2, 0, 2, -1, 1)$ on basis vectors are all different, so the fixed subspaces are spanned just by a single basis vector, and thus the fixed flags arise as isolated points.

Example 6.9.15. Slodowy varieties $\mathcal{S}_{211}, \mathcal{S}_{311}, \mathcal{S}_{33}, \mathcal{S}_{42}$

In addition, we give the pictures of the spectral sequences $E(\varphi)_r^{p,q}$ for $\mathcal{S}_{211}$ (Figure 6.15), $\mathcal{S}_{311}$ (Figure 6.16), $\mathcal{S}_{33}$ (Figure 6.17) and $\mathcal{S}_{42}$ (Figure 6.18), that we calculate analogously to previous examples. As before, from them one can read-off the filtrations on ordinary cohomology (rank-wise).

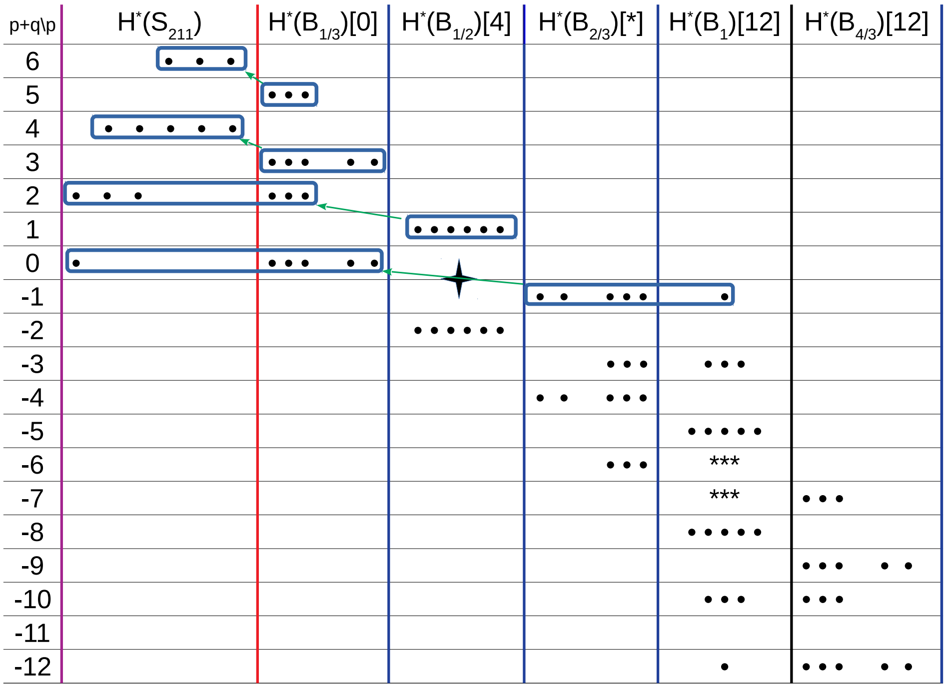


Figure 6.15: Spectral sequence for $\mathcal{S}_{211}$

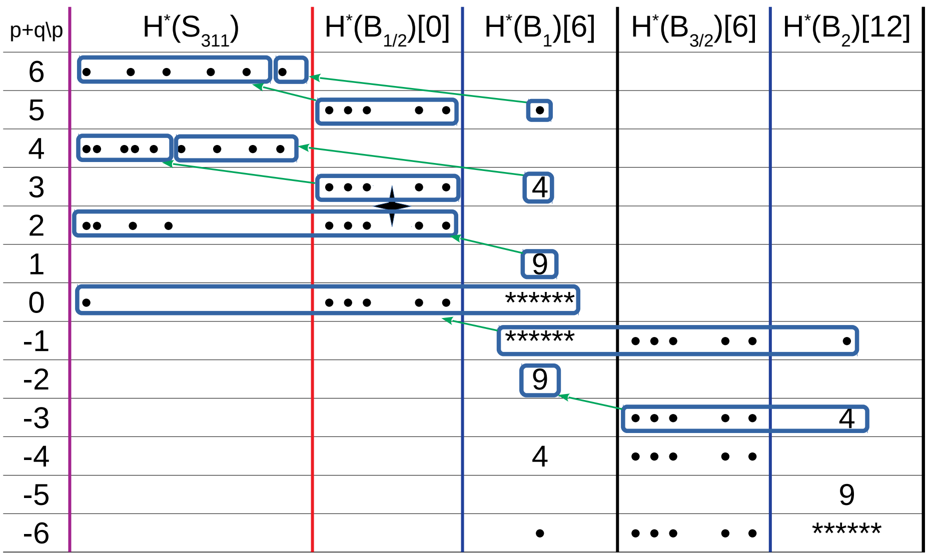


Figure 6.16: Spectral sequence for $\mathcal{S}_{311}$

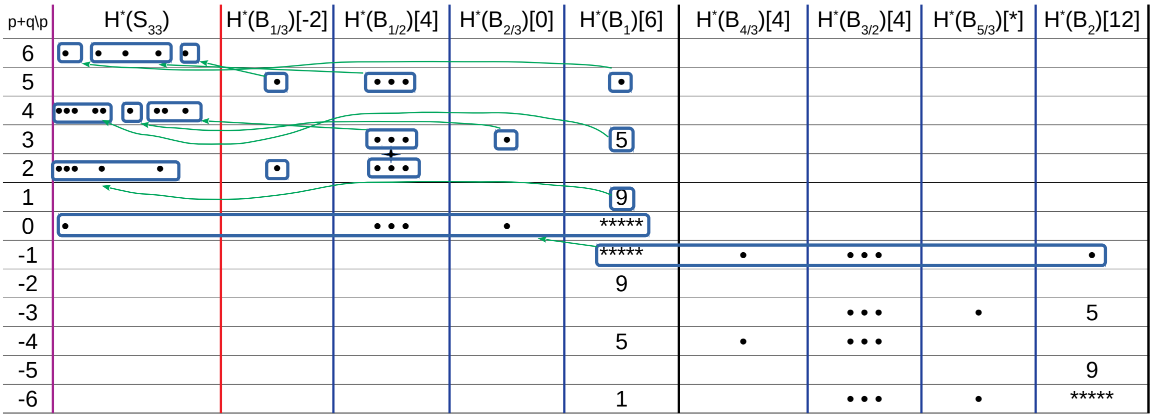


Figure 6.17: Spectral sequence for $\mathcal{S}_{33}$

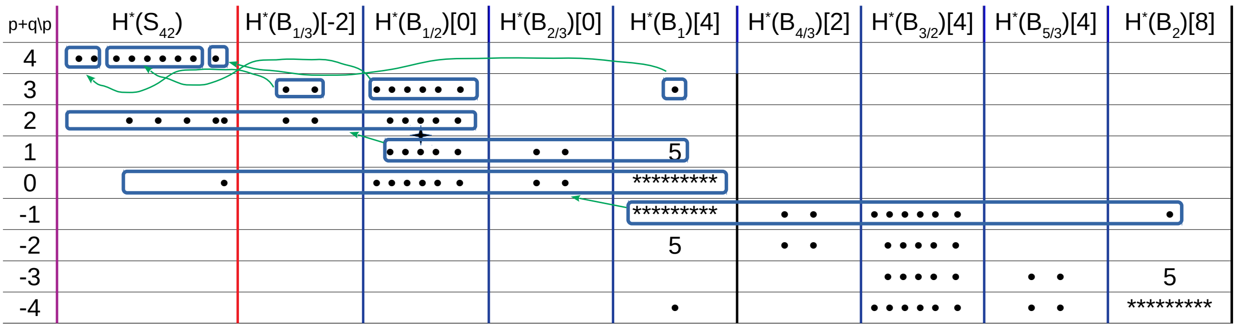


Figure 6.18: Spectral sequence for $\mathcal{S}_{42}$

6.10 Final remarks

In this section we explain some further ideas of our ongoing work on this topic, jointly with A. Ritter.

6.10.1 Calculating the filtrations $F_\lambda^\varphi H^*(\mathfrak{M})$

We have seen that one can obtain the rank-wise information for the filtration $F_\lambda^\varphi H^*(\mathfrak{M})$ from the spectral sequence $E(\varphi)_r^{p,q}$. Here we discuss how one could get some more precise information about these filtrations.

Consider an arbitrary CSR $(\mathfrak{M}, \varphi)$ and denote as usual $\mathfrak{M}^\varphi = \mathfrak{F} = \sqcup_\alpha \mathfrak{F}_\alpha$ its fixed locus decomposed into connected components. As already mentioned at the end of Section 6.4, using the Hamiltonian $F_\lambda = \lambda H$, the continuation map $H^*(\mathfrak{M}) \to$

$HF^*(F_\lambda)$ becomes

$$\bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \to \bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)], \tag{6.54}$$

by the presentation from Corollary 6.5.6 and Remark 6.5.7 after it. Here, μ_α are the Morse-Bott indices of $\mathfrak{F}_\alpha$. We are led to ask the following:

Question 6.10.1. *What is the continuation map (6.54) for arbitrary λ? In particular:*

1. *Is the mapping (6.15) $\oplus$-preserving?*
2. *If yes, what are the induced maps*

$$K_{\alpha,\lambda} : H^*(\mathfrak{F}_\alpha) \to H^{*+\mu_\alpha-\mu_\lambda(\mathfrak{F}_\alpha)}(\mathfrak{F}_\alpha) \ ?$$

If the answer to question (1) is affirmative, then the knowledge of the index $\mu_\lambda(\mathfrak{F}_\alpha)$ and of $\dim \mathfrak{F}_\alpha$ already yield some consequences about which part of $H^*(\mathfrak{F}_\alpha)$ can lie in $F^\varphi_\lambda H^*(\mathfrak{M})$, as the continuation map preserves the total grading (namely, we obtain the above maps $K_{\alpha,\lambda}$). We have a conjecture about the answer for question (2) in a special case. Firstly, recall the notions of the homogeneous/outer bundle (Definitions 2.3.9 and 2.3.11, respectively). Consider the biggest m such that there are $\mathbb{Z}/m$-torsion points. Then there exist a homogeneous bundle $\mathcal{H}_m$ converging to a fixed component $\mathfrak{F}_\alpha$. Due to Lemma 2.3.10 and the maximality m, we have $T_{\mathfrak{F}_\alpha}\mathcal{H}_m = H_0 \oplus H_m$. Moreover, $\mathcal{H}_m$ must be an outer bundle, as otherwise there would be a $\mathbb{C}^*$-flowline from $\mathfrak{F}_\alpha$ which would create a weight space H_{-m} on the other component of the fixed locus, yielding $\omega_{\mathbb{C}}$-dual weight space H_{l+m}, where l is the $\mathbb{C}^*$-weight of $\omega_{\mathbb{C}}$. However, this contradicts with maximality of m. Thus $\mathcal{H}_m$ is an outer bundle. By easy computations using the formula (6.10) we see that the restriction maps $K_{\alpha,\lambda}$ do not have a degree-shift for $\lambda < 2\pi/m$, hence we expect those to be isomorphisms. As soon as λ passes the "critical value" $2\pi/m$ we expect to get an interesting phenomenon:

Conjecture 6.10.2. *The restriction of the continuation map*

$$K_{\alpha,2\pi/m} : H^*(\mathfrak{F}_\alpha) \to H^{*+\mu_\alpha-\mu_{2\pi/m}(\mathfrak{F}_\alpha)}(\mathfrak{F}_\alpha) \tag{6.55}$$

is the cup product

$$\cup e(\mathcal{H}_m)$$

with the Euler class of the bundle $\mathcal{H}_m$, up to a scaling in the Novikov field $\mathbb{K}$.

The fact that the continuation maps are described through cupping with the Euler class has been proved in Ritter's work [R14] for the case when the total space is a negative vector bundle.[22] So, a natural guess would be that cupping with the Euler class could appear here. Also, the grading shift between these two maps agree, according to the following lemma, which is provable by a direct computation using formula (6.10).

Lemma 6.10.3. *In the setup of Conjecture 6.10.2, $rk_{\mathbb{R}}(\mathcal{H}_m) = \mu_\alpha - \mu_{2\pi/m}(\mathfrak{F}_\alpha)$.*

In order to reinforce Conjecture 6.10.2, observe the **Gysin long exact sequence**

$$\ldots \to H^{*-rk_{\mathbb{R}}\mathcal{H}_m+1}(S(\mathcal{H}_m)) \xrightarrow{\pi_*} H^*(\mathfrak{F}_\alpha) \xrightarrow{\cup e(\mathcal{H}_m)} H^{*+\mathrm{rk}_{\mathbb{R}}\mathcal{H}_m}(\mathfrak{F}_\alpha) \to \ldots \qquad (6.56)$$

for the associated sphere bundle $\pi : S(\mathcal{H}_m) \to \mathfrak{F}_\alpha$. The image of the first map is precisely the kernel of the second. Moreover, we know by Proposition 6.8.14 that the kernel of the restriction of the continuation map (6.55) has to get killed by the edge-differentials of columns of the spectral sequence $E(\varphi)_r^{p,q}$ whose slope is less or equal to $2\pi/m$, or equivalently, whose time τ_p is less or equal to $1/m$. Also, as mentioned, we expect the maps $K_{\lambda,\alpha}$ to be isomorphisms for $\lambda < 2\pi/m$, hence the kernel of the map $K_{\alpha,2\pi/m}$ should get killed in the spectral sequence exactly by time $1/m$-columns. Having that in mind, notice the following grading-compatibility with (6.56) that can be checked directly using the formula (6.10).

Lemma 6.10.4. *In the setup of Conjecture 6.10.2, let $B_{p,c} = \mathcal{H}_m \cap \mathcal{S}_p$, where $\tau_p = \frac{1}{m}$. In the spectral sequence $E(\varphi)_r^{p,q}$, the class $H^{*-rk_{\mathbb{R}}\mathcal{H}_m+1}(B_{p,c})$ is shifted one row below the class $H^*(\mathfrak{F}_\alpha)$.*

Thus, having in mind the isomorphism $\Phi : H^*(B_{p,c}) \to H^*(S(\mathcal{H}_m))$ from Lemma 6.8.23, in $E(\varphi)_r^{p,q}$ the blocks $H^*(B_{p,c})$ and $H^*(\mathfrak{F}_\alpha)$ are shifted in a way such that the potential edge-differentials

$$H^{*-rk_{\mathbb{R}}\mathcal{H}_m+1}(B_{p,c}) \xrightarrow{d_r} H^*(\mathfrak{F}_\alpha)$$

have the same degree-shifts as the map $H^{*-rk_{\mathbb{R}}\mathcal{H}_m+1}(S(\mathcal{H}_m)) \xrightarrow{\pi_*} H^*(\mathfrak{F}_\alpha)$ from the Gysin sequence. That supports Conjecture 6.10.2, as according to it and what is written above, these two maps should have the same images.

[22] More precisely, that paper proves it is a quantum cup product with the Euler class. However, in our CSR $\mathfrak{M}$ the quantum cup product is the usual cup product (Proposition 2.1.13).

Thus, putting all this together, one obtains the following conjectural diagram, involving the topological (on top) and the Floer-theoretic (on the bottom) long exact sequences.

$$\begin{array}{ccccccccc} \ldots \to H^*(S(\mathcal{H}_m)) & \xrightarrow{\pi_*} & H^{*-rk_{\mathbb{R}}\mathcal{H}_m+1}(\mathfrak{F}_\alpha) & \xrightarrow{\cup e(\mathcal{H}_m)} & H^{*+1}(\mathfrak{F}_\alpha) & \xrightarrow{\pi^*} & H^{*+1}(S(\mathcal{H}_m)) \to \ldots \\ \downarrow \Phi & & \downarrow \mathrm{id} & & \downarrow \mathbb{K}\cdot\mathrm{id} & & \downarrow \Phi \\ \ldots \to H^*(B_{p,c}) & \xrightarrow{d_r} & H^{*-rk_{\mathbb{R}}\mathcal{H}_m+1}(\mathfrak{F}_\alpha) & \xrightarrow{K_{\alpha,\frac{1}{m}}} & H^{*+1}(\mathfrak{F}_\alpha) & \xrightarrow{\Theta} & H^{*+1}(B_{p,c}) \to \ldots \end{array}$$

At the moment, we are not sure what the mysterious map Θ should be.

6.10.2 Possible generalizations

At the very end of this chapter and of the thesis itself, we remark that the constructions and results of this chapter could be generalised to a much bigger class of spaces than conical symplectic resolutions. By carefully looking at the key facts about a CSR $\pi : \mathfrak{M} \to \mathfrak{M}_0$ that we have used in our arguments, we get to the following "minimal" setup.

Consider a pseudoholomorphic S^1-equivariant proper map

$$\begin{array}{c} S^1 \overset{\varphi}{\curvearrowright} (\mathfrak{M}, I) \\ \pi \downarrow \\ S^1 \overset{\varphi}{\curvearrowright} (X, J) \end{array}$$

such that:

(1) $(\mathfrak{M}, \omega_I, I)$ is a Kähler manifold with a holomorphic $\mathbb{C}^*$-action φ whose S^1-part is a Hamiltonian action. Thus, there is a moment map $H : \mathfrak{M} \to \mathbb{R}$.

(2) (X, ω, J), is a symplectic manifold that is convex at infinity (see Appendix A.2), for which the S^1-action is J-holomorphic and agrees with the Reeb flow.

One can then, analogously to Sections 6.1 and 6.2 (where in the role of X, ω we had $\mathbb{C}^N, \omega_{\mathbb{C}^N}$), define symplectic cohomology as a direct limit of Floer cohomologies:

$$SH^*(\mathfrak{M}, \omega_I, \varphi) := \varinjlim_{F \in \mathcal{H}(\mathfrak{M}, \varphi)} HF^*(F), \tag{6.57}$$

where $\mathcal{H}(\mathfrak{M}, \varphi)$ is the class of Hamiltonians $F : \mathfrak{M} \to \mathbb{R}$ which at infinity are linear functions of H. Basically, all the compactness arguments needed to define symplectic cohomology for CSRs used the maximum principles for Floer solutions of linear

Hamiltonians in $\mathbb{C}^N$. As we know that those maximum principles also exist for any symplectic manifolds convex at infinity [R10, Lem. 1], the symplectic cohomology (6.57) is indeed well-defined.

The Liouville backward flow on X contracts it to a compact subset. As the map π is S^1-invariant and pseudoholomorphic, and the map π is proper, the $\mathbb{C}^*$-flow contracts $\mathfrak{M}$ to a compact subset as well. Thus, the fixed locus $\mathfrak{F} := \mathfrak{M}^{\mathbb{C}^*}$ is contained in a compact set. Hence, assuming further that:

(3) $c_1(\mathfrak{M}) = 0$ and the $\mathbb{C}^*$-action acts with a positive weight on the canonical bundle $\mathcal{K}$ of $\mathfrak{M}$
$$t \cdot \mathcal{K} = t^{k>0}\mathcal{K},$$

analogously to the arguments in Sections 6.3 and 6.4 one can get the vanishing result

$$SH^*(\mathfrak{M}, \omega_I, \varphi) = 0,$$

from which one gets a filtration of $QH^*(\mathfrak{M})$ by quantum-cup ideals.[23] Without the second assumption in (3), one could possibly get some interesting non-vanishing symplectic cohomologies $SH^*(\mathfrak{M}, \omega_I, \varphi)$.

For the same reason as for CSRs, the moment map $H : \mathfrak{M} \to \mathbb{R}$ is a Morse-Bott function, and thus so is λH. Its critical locus is the compact set of fixed points $Crit(\lambda H) = \mathfrak{M}^{\varphi} = \sqcup_{\alpha} \mathfrak{F}_{\alpha}$, where $\mathfrak{F}_{\alpha}$ are the connected components. Thus, by the same analytic arguments as in Section 6.5, one gets the isomorphism

$$HF^*(\lambda H) \cong \bigoplus_{\alpha} H^*(\mathfrak{F}_{\alpha})[-\mu_{\lambda}(\mathfrak{F}_{\alpha})],$$

where one can compute the gradings $\mu_{\lambda}(\mathfrak{F}_{\alpha})$ by using only the weight decomposition of the fixed loci.

The filtration on Floer chain complexes constructed in Section 6.6 uses the filtration functional on $\mathbb{C}^N$ which is the special case of the functional defined in [McLR18, Sec. 6] for any symplectic manifold convex at infinity. Thus, by analogous arguments as in Section 6.6, one can obtain Hamiltonians H_{λ} on $\mathfrak{M}$ whose Floer chain complexes are filtered, hence induce the positive symplectic cohomology $SH^*_+(\mathfrak{M}, \omega_I, \varphi)$ and Morse-Bott Floer spectral sequences

$$E(\varphi)_r^{p,q} \Rightarrow SH^*(\mathfrak{M}, \omega_I, \varphi).$$

[23] Recall that for CSRs quantum product is just the cup product.

Still, in order to compute these spectral sequences in practice, one has to deal with the technical issues, which we have already encountered for CSRs (recall Assumption 1 and Assumption 2). For now, we hope to find an argument that proves or bypasses this assumptions for CSRs, but it could be possible to find one that also works for the general setup described in this section.

Appendix A

Floer theory basics

In this section we give a **very brief** overview of basics of Floer theory, for the convenience of the reader. For more detailed discussion, we refer the reader to classical literature on this subject:

1. Hamiltonian Floer cohomology [Sa97]
2. Robbin-Salamon index [RS95, Gu14]
3. Symplectic cohomology in the exact (Liouville) setting [Sei08],[R13]
4. Symplectic cohomology in the non-exact setting [R10]

A.1 Hamiltonian Floer cohomology

Hamiltonian Floer cohomology is the homology of a chain complex associated to the Hamiltonian flow on a symplectic manifold. It is defined over a specific choice of a field, which we define now.

Definition A.1.1. Given an arbitrary field $\mathcal{K}$, its corresponding **Novikov field** is

$$\mathbb{K} = \big\{ \textstyle\sum_{j=0}^{\infty} n_j T^{a_j} : a_j \in \mathbb{R}, a_j \to \infty, n_j \in \mathcal{K} \big\}.$$

Here, T is a formal variable in grading zero.

Now, let (M, ω) be a closed symplectic manifold. By a Hamiltonian we call an arbitrary smooth function $H : M \to \mathbb{R}$ on it. Hamiltonian vector field of H is given by equation $\omega(\cdot, X_H) = dH(\cdot)$. Its flow we call **Hamiltonian flow.** Denoting by $\mathcal{L}M = C^\infty(\mathbb{S}^1, M)$ the space of free loops in M, define the **Floer chain complex** of Hamiltonian H by

$$CF^*(H) := \bigoplus \mathbb{K}\big\{x \mid x \in \mathcal{L}M, \dot{x}(t) = X_H(x(t)),\ x(0) = x(1)\big\}.$$

Thus, it is a free $\mathbb{K}-$module generated by the set of 1-periodic orbits of X_H. It is always supposed that one first makes a generic C^2-small time-perturbation H_t of H, so that there are only finitely many 1-periodic orbits of X_{H_t}, and therefore $CF^*(H)$ is a finitely generated $\mathbb{K}$ module.

The **Floer differential** on this chain complex counts solutions of **Floer equations**

$$u : S^1 \times \mathbb{R} \to M,\ \partial_s u + J(\partial_t u - X_H) = 0.$$

Lemma A.1.2. *Denoting by δ the Floer differential $(CF^*(H), d)$, is a chain complex, i.e. $\delta \circ \delta = 0$.*

Definition A.1.3. The homology $HF^*(H) := H_*(CF^*(H), d)$ is called the **Hamiltonian Floer cohomology** of Hamiltonian H.

The grading on Hamiltonian Floer cohomology is given by Robbin-Salamon indices, and will be discussed in the next section.

A.1.1 Robbin-Salamon indices

Gradings of non-degenerate 1-periodic Hamiltonian orbits[1] are defined using the Conley-Zehnder index. This is a $\mathbb{Z}$-valued index defined for certain non-degenerate paths of symplectic matrices. However, when we want to work over Hamiltonians that are degenerate (e.g. autonomous Hamiltonians), we have to introduce the more general **Robbin-Salamon** index, that is a $\frac{1}{2}\mathbb{Z}$-valued index defined for **any** continuous path of symplectic matrices.

For the definition of the Robbin-Salamon index by $\mu_{RS} : [0,1] \to Sp(\mathbb{C}^n, \Omega_0)$ for a path of symplectic matrices in $\mathbb{C}^n$, we refer to [RS95]. Here, we will only list its main properties, that we will need in the thesis. Denote by Ω_0 the standard symplectic structure on $\mathbb{C}^n$ and by $\psi' \diamond \psi'' : \mathbb{C}^{n'} \oplus \mathbb{C}^{n''} \to \mathbb{C}^{n'} \oplus \mathbb{C}^{n''}$ the direct sum of two symplectic matrices $\psi' : \mathbb{C}^{n'} \to \mathbb{C}^{n'}$ and $\psi'' : \mathbb{C}^{n''} \to \mathbb{C}^{n''}$.

Theorem A.1.4. *The Robbin-Salamon index satisfies the following properties:*

(1) μ_{RS} is invariant under homotopies with fixed endpoints,

(2) μ_{RS} is additive under concatenation of paths,

(3) μ_{RS} has the product property $\mu_{RS}(\psi' \diamond \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$.

[1] I.e. 1-periodic orbits $x(t)$ of X_H satisfying $ker((\phi_1^H)_* - Id)_{x(0)} = 0$, where ϕ_t^H is the Hamiltonian flow of H.

(4) Consider two continuous paths of symplectic matrices $\psi, \phi : [0,1] \to Sp(\mathbb{R}^{2n}, \Omega_0)$. Then

$$\mu_{RS}(\phi\psi\phi^{-1}) = \mu_{RS}(\psi).$$

(5) $\mu_{RS}((e^{is})_{s\in[0,t]}) = W(t)$, *where*

$$W : \mathbb{R} \to \mathbb{Z}, \quad W(t) := \begin{cases} 2\lfloor t/2\pi \rfloor + 1 & \text{if } t \notin 2\pi\mathbb{Z} \\ t/\pi & \text{if } t \in 2\pi\mathbb{Z}. \end{cases} \tag{A.1}$$

(6) The Robbin-Salamon index of the symplectic shear $\begin{bmatrix} 1 & 0 \\ b(t) & 1 \end{bmatrix}$ *is equal to* $\frac{1}{2}(sign(b(1)) - sign(b(0)))$.

Now, consider an 1-orbit x of a Hamiltonian H in a symplectic manifold M of dimension $2n$. Denoting by ϕ_t the Hamiltonian flow for H, we consider its linearisation $(\phi_t)_* : T_{x(0)}M \to T_{x(t)}M$. We pick a symplectic trivialisation

$$\Phi : x^*TM \to \mathbb{C}^n \times S^1,$$

of the tangent bundle above the orbit x to get a path of symplectic matrices $\Phi(\phi_t)_*\Phi^{-1} : \mathbb{C}^n \to \mathbb{C}^n$. Define

$$RS(x, H) := \mu_{RS}(\Phi(\phi_t)_*\Phi^{-1}) \tag{A.2}$$

This obviously in general depends on the choice of trivialisation Φ. However, if we assume that $c_1(M) = 0$, one can trivialise the canonical bundle $\mathcal{K} = \Lambda^{n,0}(T^*M)$, and then choose the trivialisation Φ that is compatible with it.

In the case when $H^1(M) = 0$, all choices of trivialisations of $\mathcal{K}$ are equivalent, thus the values of Robbin-Salamon indices of orbits (A.2) are uniquely determined. Define the **grading** of an 1-orbit x of a Hamiltonian $H : M \to \mathbb{R}$ by

$$|x| := \dim_{\mathbb{C}} M - RS(x, H). \tag{A.3}$$

By previously said, we have the following:

Lemma A.1.5. *Consider a symplectic manifold (M, ω) with a ω-compatible almost complex structure I satisfying $c_1(TM, I) = 0$ and $H^1(M) = 0$. Then, the Floer cohomologies $HF^*(H)$ are canonically $\mathbb{Z}$-graded by* (A.3).

A.2 Symplectic cohomology

Symplectic cohomology is a version of Hamiltonian Floer cohomology for open manifolds. Here, we give its brief overview.

Definition A.2.1. A **contact manifold** (Σ, α) is a manifold with 1-form α that is $\alpha \wedge (\mathrm{d}\alpha)^k \neq 0$. Its **contact distribution** is $\xi := ker(\alpha)$. The Reeb vector field on Σ is defined by $d\alpha(, \mathcal{R}) = 0$ and $\alpha(\mathcal{R}) = 1$. The flow of this vector field is called **Reeb flow** on Σ.

Definition A.2.2. A symplectic manifold (M, ω) is **convex at infinity** if there exists a compact set K and a symplectomorphism $\varphi : (M \setminus K, \omega_I) \xrightarrow{\cong} (\Sigma \times [1, +\infty), d(R\alpha))$, where (Σ, α) is a contact manifold, and R is a coordinate on $[1, +\infty)$.

Given (M, ω) that is convex at infinity, on the **convex end** $M \setminus K \cong \Sigma \times [1, +\infty)$ one has a well-defined Reeb vector field $\mathcal{R}$ defined as a pull-back of the Reeb flow on (Σ, α). Also, denoting by $\theta = \varphi^*(R\alpha)$, on the convex end there is **Liouville vector field** Z, defined by $\omega(Z, \cdot) = \theta$. In the R coordinate, we have $Z = R\partial R$. In order to define Floer cohomologies on these open symplectic manifolds, we consider only ω-compatible almost complex structures J that are **contact at infinity,** meaning $JZ = \mathcal{R}$, or equivalently, $\theta = -dR \circ J$. Thus, saying that (M, ω, J) is convex at infinity we will always assume that J is contact at infinity. For such manifolds, the **symplectic cohomology** is defined as the direct limit of Floer cohomologies

$$SH^*(M) := \lim_{F \in \mathcal{H}(M)} HF^*(F), \tag{A.4}$$

over a class $\mathcal{H}(M)$ of Hamiltonians that are **linear at infinity** with respect to the radial coordinate R. A Hamiltonian F is linear at infinity when, outside of a compact set, $F = \lambda R$ for some generic[2] $\lambda > 0$. The morphisms between the Floer cohomologies in the direct limit are given by the continuation maps, and are directed towards a Hamiltonian with the larger slope λ. Thus, the limit lets $\lambda \to \infty$.

Remark A.2.3. We remark that the ambient manifold M is open, hence we could in principle have a sequence of Floer solutions for arbitrary Hamiltonian $F : M \to \mathbb{R}$ that go off to infinity. However, the choice of the class of Hamiltonians $\mathcal{H}(M)$ together with our assumption on the almost complex structure, we ensure that the Floer solutions must satisfy certain maximum principle, which does not allow them to go beyond their boundaries, in the R coordinate. An analogous maximum principle exist for

[2] I.e. not equal to a Reeb period for (Σ, α).

continuation solutions, thus the continuation maps are well defined, hence so is their direct limit $SH^*(M)$.

We remark that one can grade the symplectic cohomology using the grading of Floer Cohomologies, as the continuation maps in the direct limit (A.4) preserve the gradings. Thus, we get the corollary of Lemma A.1.5.

Corollary A.2.4. *Consider a symplectic manifold (M, ω) convex at infinity with a ω-compatible almost complex structure I satisfying $c_1(TM, I) = 0$ and $H^1(M) = 0$. Then, the symplectic cohomology $SH^*(M)$ is canonically $\mathbb{Z}$-graded.*

For Hamiltonians $F = \delta H$ with small slope $\delta > 0$, Hamiltonian Floer cohomology is isomorphic to the ordinary cohomology $HF^*(F) \cong H^*(M)$, hence the direct limit (A.4) yields **the c^* map**

$$c^* : H^*(M) \to SH^*(M).$$

This map gives information on closed Reeb orbits. In particular, having $c^* = 0$ yields that there exist a closed Reeb orbit on Σ.

A.2.1 Liouville manifolds

We mention here the special case of symplectic manifolds convex at infinity that are globally exact, called the Liouville manifolds.

Definition A.2.5. Liouville manifold (M, ω) is a symplectic manifold convex at infinity,

$$\varphi : (M \setminus K, \omega) \xrightarrow{\cong} (\Sigma \times [1, +\infty), d(R\alpha)),$$

such that the 1-form $\varphi^*(R\alpha)$ extends **globally** to a primitive θ of ω. In particular, M is exact $\omega = d\theta$.

The symplectic cohomology (A.4) for such manifolds can also be defined as the Floer cohomology of arbitrary Hamiltonian $F = h(R)$ with slope going to infinity $\lim_{R \to +\infty} h'(R) = +\infty$,

$$SH^*(M) := HF^*(F). \tag{A.5}$$

One can show (e.g. see [Sei08, 3e]) that this definition is equivalent to (A.4).

Now we define an isomorphism between Liouville manifolds.

Definition A.2.6. A **Liouville isomorphism** between Liouville manifolds $(M_1, d\theta_1)$ and $(M_2, d\theta_2)$ is a diffeomorphism $\psi : M_1 \to M_2$ satisfying $\psi^*\theta_2 = \theta_1 + df$ where f is compactly supported function.

An importance of such isomorphisms is that they preserve the symplectic cohomology.

Lemma A.2.7. *Symplectic cohomology $SH^*(M)$ is invariant under the Liouville isomorphisms.*

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