State-feedback design for nonlinear saturating systems

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***Abstract*—This paper presents strategies for state-feedback control law design of non-linear control laws with saturating inputs. The input constraints are handled by considering a generalized local sector inequality allowing the study of non-symmetric saturation bounds. A numerical formulation is presented for polynomial systems and is based on the solution of Lyapunov inequalities with sum-of-squares programming.**

I. Introduction

The design of feedback control laws for nonlinear input-affine systems is a challenging problem for which different constructive solutions have been proposed, such as the back-stepping method [22] and nonlinear dynamic inversion [12]. An important subclass of nonlinear input-affine systems is that of polynomial systems. Polynomial vector fields can model biological systems (such as predator-prey dynamics [17]), DC-DC converters [18], and also describe, in the simple instance of quadratic systems, truncated models of infinite-dimensional systems with energy-preserving terms [7]. The interest in polynomial systems was also prompted by Sum-of-Squares programming (SOSP), whereby Lyapunov-based stability conditions can be efficiently solved with semi-definite programming. These polynomial Lyapunov inequalities establish conditions for both global and regional stability [5]. Unfortunately, polynomial Lyapunov functions (LF) for polynomial systems may not exist globally [1]. Also, the exact characterisation of the region of attraction of stable equilibria requires LFs that *blow up* on its boundary [27], suggesting the use of rational LFs instead of polynomial ones.

Methods based on SOSP for state-feedback design were proposed (see [4], [20]) and have attempted to generalise methods for linear state feedback design. To this aim, quadratic-like representations of the LF have been adopted [6], [11], [20], [30]. Due to the product between LF variables and state feedback gain coefficients, the computational solutions adopted by [4], [13], are based on iterations between the LF and the feedback gains. The LF structure of [20] was also studied in [32], where the inverse of the Lyapunov matrix is assumed to be polynomial, thus defining a rational LF and rational state-feedback laws. A different path was taken by [21] where density functions, a notion dual to LFs, were used. More recently, algebraic geometry has been used in [16] to solve Lyapunov equations associated with polynomial systems and parameterise polynomial feedback laws.

A desired feature of control design methods is the ability to handle input saturations, a ubiquitous nonlinearity. A standard approach is to handle the saturation using sector inequalities. Generalisations of sector inequalities have been proposed to study linear saturating systems in [25, Lemma 1.6], [9] and have been used in analysis and design problems [25], [31]. Importantly, these local sector inequalities are crucial for computing regions of attraction of stable equilibria. For polynomial systems, these sector inequalities were used in [28], [30]. In [10], a generalisation of the differential inclusion approach of [8] was considered.

A. Contribution

In this paper, we propose conditions for the stabilisation of input-affine saturating systems in Section IV. Using dissipation inequalities, we also present conditions to characterise reachable sets with bounded disturbances and to certify induced gains. These conditions are obtained thanks to a generalised sector condition for the saturation function with non-linear arguments, which is presented in Section III.

In Section V, the stabilisation inequalities are applied to polynomial systems. From their solution, we obtain polynomial and rational state feedback control laws. Importantly, no transformation of the polynomial vector field is required, which is in contrast to the approaches proposed in [6], [11], [20], [29], [30], [32], where a linear-like representation of the vector field needs to be computed first. No particular structure for the input matrices must be assumed, as required in [20].

The polynomial inequalities are then cast as Sum-of-Squares constraints of optimization problems. However, products between the LF and a set of multipliers are handled with an iterative procedure. These results are illustrated with examples taken from the literature in Section VI.

Notation. The Euclidean space of dimension n is denoted $\mathbb{R}^n$, $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers, $\mathbb{R}_{>0}$ denotes the set of positive real numbers. The set of symmetric matrices of dimension n with real entries is given by $\mathbb{S}^n$, the set of symmetric, positive semi-definite matrices of dimension n is denoted $\mathbb{S}^n_{\geq 0}$ and the set of diagonal matrices of dimension n is denoted $\mathbb{D}^n$. For a vector $x \in \mathbb{R}^n$ we denote $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. The ρ level set of a positive semi-definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\{x \in \mathbb{R}^n \mid V(x) \leq \rho\}$, is denoted $\mathcal{E}(V, \rho)$. The set of continuously differentiable functions with continuous derivatives is denoted $\mathcal{C}^1$. The gradient of a $\mathcal{C}^1$

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scalar function $\phi : \mathbb{R}^n \to \mathbb{R}$ is given by $\nabla\phi(x) := \left[\frac{\partial\phi}{\partial x_1}\ \frac{\partial\phi}{\partial x_2}\ \dots\ \frac{\partial\phi}{\partial x_n}\right]^\top$; the Jacobian matrix of a $\mathcal{C}^1$ vector function $\zeta : \mathbb{R}^n \to \mathbb{R}^d$ is given by $\nabla\zeta(x) = \left[\nabla\zeta_1(x)\ \nabla\zeta_2(x)\ \dots\ \nabla\zeta_d(x)\right]^\top$. We use $\mathcal{H}^c$ to denote the complement of a set $\mathcal{H} \subseteq \mathbb{R}^n$. The ring of the vector of polynomials of dimension n on variable x is denoted $\mathcal{P}^n[x]$, the ring of polynomial matrices of dimension $n\times m$ is denoted $\mathcal{P}^{n\times m}[x]$ and the set of vectors of sum-of-squares (SOS) polynomials of dimension n on variable x is denoted $\Sigma^n[x]$. The decentralized saturation $sat : \mathbb{R}^m \to \mathbb{R}^m$, $sat(u) = \left[sat_1(u_1)\ sat_2(u_2)\ \dots\ sat_m(u_m)\right]^\top$ has its entries $sat_i(u_i)$ defined by $sat_i(u_i) = \max(\min(u_i, \overline{u}_i), \underline{u}_i)$. We assume that the vectors of upper- and lower-bounds $\overline{u} \in \mathbb{R}^m$, $\underline{u} \in \mathbb{R}^m$ of the saturation nonlinearity satisfy $\overline{u}_i \geq 0$, $\underline{u}_i \leq 0$ and we define the deadzone function as $dz(u) := u - sat(u)$.

II. Problem statement

Consider the system $\dot{x} = f(x) + G(x)sat(u) + G_w(x)w$ where $x(0) = x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $G_w : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ where w is an exogenous signal and u is the control input. Using the definition of the deadzone nonlinearity, we obtain

$$\dot{x} = f(x) + G(x)u - G(x)dz(u) + G_w(x)w. \tag{1}$$

In this paper, we propose sufficient stabilization conditions and numerical methods to compute state-feedback control laws that solve the following problem for system (1).

Problem 1: Design a state-feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$ and obtain a set $\mathcal{D}$ such that the solutions to (1) satisfy

1) **(Asymptotic stability without disturbances)** If $w \equiv 0$ the origin is (locally) asymptotically stable and the set $\mathcal{D}$, $0 \in \mathcal{D}$, is included in its region of attraction. Namely $\forall x_0 \in \mathcal{D}$ and $w \equiv 0$, $\lim_{t\to\infty} x(t) = 0$.
2) **(Boundedness for zero initial conditions)** Given $\tilde{W}_w : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ and $\rho > 0$ defining $\mathcal{W} := \left\{w \in \mathbb{R}^p \middle| \int_0^\infty \tilde{W}_w(w(\tau))\mathrm{d}\tau \leq \rho\right\}$, if $x_0 = 0$ and $w \in \mathcal{W}$ then $x(t) \in \mathcal{D}\ \ \forall t \geq 0$.
3) **(Nonlinear gains)** Given $\tilde{W}_w : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$, $\tilde{W}_x : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, if $x_0 = 0$ and $w \in \mathcal{W}$ then $x(t) \in \mathcal{D}\ \ \forall t \geq 0$ and
$$\int_0^\infty \tilde{W}_x(x(\tau))\mathrm{d}\tau \leq \int_0^\infty \tilde{W}_w(w(\tau))\mathrm{d}\tau. \tag{2}$$

A solution to the above problem can be obtained from the solution to dissipation inequalities as in the following lemma

Lemma 1: Given $W(x, w)$ If there exists $V : \mathbb{R}^n \to \mathbb{R}_{>0}$, $V(0) = 0$, positive scalars β_ℓ, γ_ℓ, satisfying

$$\beta_\ell\|x\|^{\gamma_\ell} \leq V(x) \tag{3}$$

and a mapping $u : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$-\dot{V}(x, w) + W(x, w) > 0 \quad \forall x \in \mathcal{E}(V, \rho), \tag{4}$$

along the trajectories of (1), then $u(x)$ and $\mathcal{D} = \mathcal{E}(V, \rho)$, with $W(x, w) = 0$, solve Problem 1.1; with $W(x, w) = \tilde{W}_w(w)$, solve Problem 1.2, and, with $W(x, w) = \tilde{W}_w(w) - \tilde{W}_x(x)$, solve Problem 1.3.

Proof. We have: **1)** for $w \equiv 0$ and $W(x, w) = 0$, following the steps in [15, Theorem 4.9], the origin of (1) is uniformly asymptotically stable for all trajectories starting in $\mathcal{E}(V, \rho)$; **2)** for $W(x, w) = \tilde{W}_w(w)$, $\tilde{W}_w : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ integrate (4) over $[0, T]$ and use $x(0) = 0$, $V(0) = 0$ to obtain

$$V(x(T)) < \int_0^T \tilde{W}_w(w(\tau))\mathrm{d}\tau.$$

Since $\tilde{W}_w(w(t)) \geq 0\ \forall t \geq 0$, if $w \in \mathcal{W}$, we have $V(x(T)) < \int_0^T \tilde{W}_w(w(\tau))\mathrm{d}\tau \leq \rho$ thus $x(T) \in \mathcal{E}(V, \rho)\ \forall T \in \mathbb{R}_{\geq 0}$;
3) for $W(x, w) = \tilde{W}_w(w) - \tilde{W}_x(x)$, integrate (4) over $[0, T]$ and use $x(0) = 0$, $V(0) = 0$ to obtain

$$V(x(T)) + \int_0^T \tilde{W}_x(x(\tau))\mathrm{d}\tau < \int_0^T \tilde{W}_w(w(\tau))\mathrm{d}\tau.$$

which implies that $V(x(T)) < \int_0^T \tilde{W}_w(w(\tau))\mathrm{d}\tau$ for all T since $\tilde{W}_x(x) \geq 0$ for all x. We thus have $V(x(T)) < \rho$ for all w in $\mathcal{W}$. From (3), $V(x) > 0\ \forall x \in \mathbb{R}^n \setminus \{0\}$, we have $\int_0^T \tilde{W}_x(x(\tau))\mathrm{d}\tau < \int_0^T \tilde{W}_w(w(\tau))\mathrm{d}\tau$ for all $T > 0$, thus $T \to \infty$ gives (2). ■

Remark 1: The *supply rate* $W(x, w)$ defines the *dissipation inequality* used to assess properties of the closed-loop system. The reachable sets for bounded disturbances are characterized according to the function $\tilde{W}_w$, thus the set $\mathcal{W}$. Gain properties of the closed-loop can be specified by the functions $\tilde{W}_w$ and $\tilde{W}_x$. For instance, the induced input-to-state $\mathcal{L}_2$ norm is bounded by a positive scalar η provided the inequality (2) is defined by $\tilde{W}_w(w) = \eta^2 w^\top w$ and $\tilde{W}_x(x) = x^\top x$. ⋆

III. Generalised Sector inequalities

To cope with asymmetric input saturation in the *local* analysis (i.e. in a set containing the origin), we propose generalizations of local sector conditions for non-linear control laws in terms of inequalities. We also provide conditions for the inclusion of level sets of positive functions in the sets where sector inequalities hold.

Consider $u : \mathbb{R}^n \to \mathbb{R}^m$, a function $h : \mathbb{R}^n \to \mathbb{R}^m$ and the vectors of saturation bounds $\underline{u} \in \mathbb{R}^m_{<0}$ and $\overline{u} \in \mathbb{R}^m_{>0}$ defining

$$\begin{array}{lll}\mathcal{H}_{1j}(u, h) & := & \left\{x \mid \underline{u}_j \leq u_j(x) \leq \overline{u}_j\right\},\\ \mathcal{H}_{2j}(u, h) & := & \left\{x \mid (u_j(x) - \underline{u}_j) < 0, (h_j(x) - \underline{u}_j) \geq 0\right\},\\ \mathcal{H}_{3j}(u, h) & := & \left\{x \mid (u_j(x) - \overline{u}_j) > 0, (h_j(x) - \overline{u}_j) \leq 0\right\}.\end{array} \tag{5}$$

$j = 1, \dots, m$. Let us also define $\mathcal{H}_j(u, h) := \mathcal{H}_{1j}(u, h) \cup \mathcal{H}_{2j}(u, h) \cup \mathcal{H}_{3j}(u, h)$ and

$$\mathcal{H}(u, h) = \cap_{j=1}^m \mathcal{H}_j(u, h). \tag{6}$$

The lemma below is akin to the local sector condition of [9], [26], where u and h are linear functions.

Lemma 2 (Sector Inequalities): For every $T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{D}^m_{\geq 0}$, the inequality

$$-(dz(u(x)))^\top T(x, dz(u(x)))\,(dz(u(x)) - u(x) + h(x)) \geq 0 \tag{7}$$

holds in the set $\mathcal{H}(u, h)$.

Proof. Since $T(x, dz(u(x)))$ is a diagonal matrix for all x, we can rewrite (7) as (omitting the arguments of the matrix $T(x, dz(u(x)))$ and the deadzone function)

$$-\sum_{j=1}^m (dz)_j T_{jj}\left((dz)_j - u_j(x) + h_j(x)\right) \geq 0. \tag{8}$$

From the definition of the deadzone function, we have $(dz)_j T_{jj}\left((dz)_j - u_j(x) + h_j(x)\right) = (dz)_j T_{jj}\left(-sat_j(u(x)) + h_j(x)\right)$. We now show that

$$-(dz)_j T_{jj}\left(-sat_j(u(x)) + h_j(x)\right) \geq 0 \quad j = 1, \ldots, m \quad (9)$$

hold in (5) for any non-negative T_{jj}:

I) In $\mathcal{H}_{1j}(u,h)$, we have $\underline{u}_j \leq u_j(x) \leq \overline{u}_j$, which implies $(dz)_j = 0$, hence (9) holds;

II) For $u_j(x) - \underline{u}_j < 0$, we have $(dz)_j < 0$ and $sat_j(u) = \underline{u}_j$, thus $(dz)_j T_{jj}\left(-sat_j(u) + h_j(x)\right) = (dz)_j T_{jj}\left(-\underline{u}_j + h_j(x)\right)$. Hence, for $u_j(x) - \underline{u}_j < 0$, the inequality $-(dz)_j T_{jj}\left(-sat_j(u) + h_j(x)\right) \geq 0$ holds only if $\left(-\underline{u}_j + h_j(x)\right) \geq 0$, that is, if $x \in \mathcal{H}_{2j}(u,h)$.

III) For $u_j(x) - \overline{u}_j > 0$, we have $(dz)_j > 0$ and $sat_j(u) = \overline{u}_j$, thus $(dz)_j T_{jj}\left(-sat_j(u) + h_j(x)\right) = (dz)_j T_{jj}\left(-\overline{u}_j + h_j(x)\right)$. Hence, for $u_j(x) - \overline{u}_j > 0$, the inequality $-(dz)_j T_{jj}\left(-sat_j(u) + h_j(x)\right) \geq 0$ holds only if $\left(-\overline{u}_j + h_j(x)\right) \leq 0$, that is, if $x \in \mathcal{H}_{3j}(u,h)$.

Therefore the inequalities in (9) hold in the union $\mathcal{H}_j(u,h)$ of the disjoint sets (5). Since each term in (9) holds in $\mathcal{H}_j(u,h)$ they also hold in the intersection of the sets $\mathcal{H}_j(u,h)$, thus we have that (8) holds in the set $\mathcal{H}(u,h)$. ■

From (5), (6) we clearly have that $h(x) = 0$ yields $\mathcal{H}(u,h) = \mathbb{R}^n$, and we retrieve the global sector inequality [15, Sec. 6.1] as a particular case of Lemma 2.

Remark 2: In [28] and [30], the inequality (8) is used by considering it holds in the set $\mathcal{H}_b(u,h) := \left\{x \mid \underline{u}_j \leq h_j(x) \leq \overline{u}_j, j = 1, \ldots, m\right\}$. Figure 1 illustrates both sets $\mathcal{H}$ and $\mathcal{H}_b$ for $x \in \mathbb{R}$, the identity function $u(x) = x$ and a nonlinear function $h(x)$.

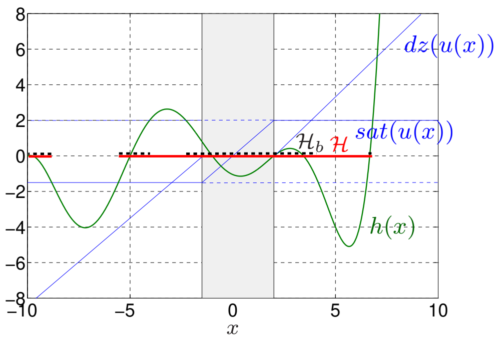

Fig. 1: Sets $\mathcal{H}(u,h)$ (red solid line on top of the x axis), and $\mathcal{H}_b(u,h)$ (black dashed line on top of the x axis) obtained with $u(x) = x$, a nonlinear function $h(x)$ and saturation bounds $\underline{u} = -1.7$ and $\overline{u} = 2$. The connected subset of $\mathcal{H}(u,h)$ containing the origin clearly contains the connected subset of $\mathcal{H}_b(u,h)$ containing the origin. The grey shaded area contains points with x coordinate on the set $\mathcal{H}_{1j}(u,h)$ of (5).

With symmetric saturation bounds $\overline{u} = -\underline{u} = u_0$ we have $(\overline{u}_j - h_j(x))(-\underline{u}_j + h_j(x)) = -(h_j^2(x) - u_{0j}^2)$ and $u_{0j}^2 - h_j^2(x) \geq 0 \Leftrightarrow \frac{h_j^2(x)}{u_{0j}^2} \leq 1$. We can thus express $\mathcal{H}_b(u,h) = \left\{x \mid \frac{h_j^2(x)}{u_{0j}^2} \leq 1\right\}$ as in [9], [26]. ⋆

The following lemma presents a condition to check whether level sets of radially unbounded functions are contained in the set $\mathcal{H}(u,h)$.

Lemma 3 (Inclusion Conditions): For $\omega : \mathbb{R}^n \to \mathbb{R}$, $\omega(x) \geq \beta\|x\|^\gamma - C$, $C > 0$, $\rho > 0$, $u : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$, if there exists $s_{u\underline{u}} : \mathbb{R}^n \to \mathbb{R}^m_{\geq 0}$, $s_{h\underline{u}} : \mathbb{R}^n \to \mathbb{R}^m_{\geq 0}$, $s_{u\overline{u}} : \mathbb{R}^n \to \mathbb{R}^m_{\geq 0}$, $s_{h\overline{u}} : \mathbb{R}^n \to \mathbb{R}^m_{\geq 0}$ such that

$$\begin{aligned}
&\psi_{\underline{u}j}(\omega, \rho, u, h, s_{u\underline{u}j}, s_{h\underline{u}j}) := \\
&(\omega(x) - \rho) + s_{u\underline{u}j}(x)(u_j(x) - \underline{u}_j) + s_{h\underline{u}j}(x)(h_j(x) - \underline{u}_j) \geq 0 \\
&\psi_{\overline{u}j}(\omega, \rho, u, h, s_{u\overline{u}j}, s_{h\overline{u}j}) := \\
&(\omega(x) - \rho) + s_{u\overline{u}j}(x)(\overline{u}_j - u_j(x)) + s_{h\overline{u}j}(x)(\overline{u}_j - h_j(x)) \geq 0 \\
&\quad \forall x \in \mathbb{R}^n, \ j = 1, \ldots, m
\end{aligned} \quad (10)$$

then $\mathcal{E}(\omega(x), \rho) \subset \mathcal{H}(u,h)$.

Proof. We show that (10) implies $\mathcal{E}(\omega(x), \rho) \subset \mathcal{H}(u,h)$ by showing that $\mathcal{E} \cap \mathcal{H}^c = \varnothing$. From the definition of $\mathcal{H}$ in (6) we have that $\mathcal{H}^c = \cup_{j=1}^m \mathcal{H}_j^c$, hence

$$\mathcal{E} \cap \mathcal{H}^c = \mathcal{E} \cap \left(\cup_{j=1}^m \mathcal{H}_j^c\right) = \cup_{j=1}^m \left(\mathcal{E} \cap \mathcal{H}_j^c\right). \quad (11)$$

For $\cup_{j=1}^m \left(\mathcal{E} \cap \mathcal{H}_j^c\right) = \varnothing \Leftrightarrow \mathcal{E} \cap \mathcal{H}_j^c = \varnothing \ j = 1, \ldots, m$, it suffices to show that $\mathcal{E} \cap \mathcal{H}_j^c = \varnothing \ j = 1, \ldots, m$. Following the definitions in (5), let us write the sets $\mathcal{H}_j^c(u,h)$,

$$\begin{aligned}
\mathcal{H}_j^c(u,h) = &\left\{x | (u_j(x) - \underline{u}_j) < 0, (h_j(x) - \underline{u}_j) < 0\right\} \\
&\cup \left\{x | (\overline{u}_j - u_j(x)) < 0, (\overline{u}_j - h_j(x)) < 0\right\}.
\end{aligned}$$

Since $s_{u\underline{u}j} \geq 0$, $s_{h\underline{u}j} \geq 0$, if (10) holds then

$$\begin{aligned}
(\omega(x) - \rho) \geq\ &(\omega(x) - \rho) \\
&+ s_{u\underline{u}j}(x)(u_j(x) - \underline{u}_j) + s_{h\overline{u}j}(x)(h_j(x) - \underline{u}_j) \geq 0
\end{aligned}$$

for all $x \in \left\{x | (u_j(x) - \underline{u}_j) < 0, (h_j(x) - \underline{u}_j) < 0\right\}$. Similarly, since $s_{u\overline{u}j} \geq 0$, $s_{h\overline{u}j} \geq 0$ we have

$$\begin{aligned}
(\omega(x) - \rho) \geq\ &(\omega(x) - \rho) \\
&+ s_{u\overline{u}j}(x)(\overline{u}_j - u_j(x)) + s_{h\overline{u}j}(x)(\overline{u}_j - h_j(x)) \geq 0
\end{aligned}$$

for all $x \in \{x | (\overline{u}_j - u_j(x)) < 0, (\overline{u}_j - h_j(x)) < 0\}$. Thus $x \in \{x \mid \omega(x) \geq \rho\}$ for every $x \in \mathcal{H}_j^c(u,h)$, $j = 1, 2$, and hence for all $x \in \mathcal{H}^c(u,h)$ that is $\mathcal{E}(\omega(x), \rho) \cap \mathcal{H}^c(u,h) = \varnothing$, which implies $\mathcal{E}(\omega(x), \rho) \subset \mathcal{H}(u,h)$. ■

IV. Stability and Stabilization Conditions

Following Lemma 1, the solution to (3)-(4) yields a solution to Problem 1. The results in this section use Lemma 2 to formulate inequalities (3)-(4), where the dz function appears since $\dot{V}$ is taken along the trajectories of system (1). We present a result for the stability analysis of saturating systems. Its proof relies on Lemma 3 and on the lemma below, which gives sufficient conditions for the non-negativity of a mapping in sets defined by inequalities and equalities.

Lemma 4: Given $s_0 : \mathbb{R}^{n_\xi} \to \mathbb{R}$, $s : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_i}$, $r : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_e}$, if there exist mappings $s : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_i}_{\geq 0}$, and $r : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_e}$ such that $s_0(\xi) - s^\top(\xi)p(\xi) + r^\top(\xi)q(\xi) > 0 \quad \forall \xi \in \mathbb{R}^{n_\xi}$ then $s_0(\xi) > 0 \quad \forall \xi \in \{\xi \in \mathbb{R}^{n_\xi} \mid p_i(\xi) \geq 0, q_j(\xi) = 0, i = 1 \ldots n_i, j = 1 \ldots n_j\}$.

A. Stability Conditions

To simplify the notation, we drop the arguments of the deadzone function.

Theorem 1 (Stability Analysis): Given a state feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$, and $W : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ if there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\beta_\ell > 0$, $\gamma_\ell > 0$ satisfying (3), a function $h : \mathbb{R}^n \to \mathbb{R}^m$, a non-negative matrix function $T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{D}^m_{\geq 0}$, and a non-negative functions s, suct that

$$-\nabla V\left(f(x)+G(x)u(x)-G(x)dz+G_w(x)w\right)+W(x,w) +s(x)\left(\|x\|^{\gamma_\ell}-\beta_\ell^{-1}\rho\right)+dz^\top T(x,dz)(dz-u(x)+h(x))>0 \quad \forall x\in\mathbb{R}^n,\ \forall dz\in\mathbb{R}^m,\ \forall w\in\mathbb{R}^p \tag{12}$$

and non-negative functions $s_{u\underline{u}}$, $s_{h\underline{u}}$, $s_{u\overline{u}}$, $s_{h\overline{u}}$

$$\psi_{\underline{u}j}(V,\rho,u,h,s_{u\underline{u}},s_{h\underline{u}})\geq 0 \quad \psi_{\overline{u}j}(V,\rho,u,h,s_{u\overline{u}},s_{h\overline{u}})\geq 0 \quad \forall\ x\in\mathbb{R}^n,\ j=1,\ldots,m \tag{13}$$

then $u(x)$ and $\mathcal{D} = \mathcal{E}(V,\rho)$ satisfy 1)-3) in Problem 1. Moreover if (12) holds with $s = 0$ and $h = 0$ then $u(x)$ and $\mathcal{D} = \mathbb{R}^n$ satisfy 1)-3) in Problem 1.

Proof. Since $\dot V = \nabla V(f(x)+G(x)u-G(x)dz+G_w(x)w)$ and since, from Lemma 2, we have $-dz^\top T(x,dz)(dz-u(x)+h(x))\geq 0\ \forall x\in\mathcal{H}(u,h)$, (12) implies

$$-\dot V(x,w)+W(x,w)+s(x)\left(\|x\|^{\gamma_\ell}-\beta_\ell^{-1}\rho\right)>0 \quad \forall x\in\mathcal{H}(u,h),\forall w\in\mathbb{R}^p.$$

Provided (3) holds and since $s(x) \geq 0\ \forall x$, we have $\beta_\ell^{-1}s(x)V(x)\geq s(x)\|x\|^{\gamma_\ell}$ and we obtain

$$-\dot V(x,w)+W(x,w)+\beta_\ell^{-1}s(x)\left(V(x)-\rho\right)>0 \quad \forall x\in\mathcal{H}(u,h),\forall w\in\mathbb{R}^p.$$

Using Lemma 4 we have

$$-\dot V(x,w)+W(x,w)>0\ \forall x\in\mathcal{H}(u,h)\cap\mathcal{E}(V,\rho),\forall w\in\mathbb{R}^p.$$

From Lemma 3, (13) implies $\mathcal{H}(u,h)\cap\mathcal{E}(V,\rho)=\mathcal{E}(V,\rho)$ we have $-\dot V(x,w)+W(x,w)>0\quad\forall x\in\mathcal{E}(V,\rho)$. Following Lemma 1, u and $\mathcal{D}=\mathcal{E}(V,\rho)$ satisfy 1)-3) in Problem 1. Moreover if $h = 0$, we have $\mathcal{H}(u,0)=\mathbb{R}^n$ (the sector inequality holds globally) and if $s = 0$ we obtain

$$-\dot V(x)+W(x,w)>0\quad\forall x\in\mathbb{R}^n,\forall w\in\mathbb{R}^p$$

hence u and $\mathcal{D}=\mathbb{R}^n$ satisfy 1)-3) in Problem 1. ∎

B. Stabilization Conditions

This section presents three conditions for the stabilization of system (1) by considering different structures for the LF and the feedback law.

Theorem 2 (Stabilization with structured LF): Given a function $\zeta:\mathbb{R}^n\to\mathbb{R}^{n_\zeta}$, $\zeta\in\mathcal{C}^1$, $\zeta(0)=0$, and $W:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$, if there exist $P\in\mathbb{S}^{n_\zeta}_{\geq 0}$, $\beta_\ell>0$, $\gamma_\ell>0$, satisfying (3) with

$$V(x)=\alpha^{-1}\zeta^\top(x)P\zeta(x), \tag{14}$$

$\kappa:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^m$, $h:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^m$, $s:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}_{\geq 0}$, $\alpha>0$ and $N:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^{n_\zeta}$, satisfying

$$-2y^\top\nabla\zeta(x)(f(x)+G(x)\kappa(x,y)-G(x)dz+G_w(x)w)+W(x,w) +s(x,y)\left(\|x\|^{\gamma_\ell}-\beta_\ell^{-1}\rho\right)+dz^\top T(x,dz)(dz-\kappa(x,y)-h(x,y)) + N(x,y)(\alpha y-P\zeta(x))>0 \quad \forall\ x\in\mathbb{R}^n,\forall\ y\in\mathbb{R}^{n_\zeta},\forall\ dz\in\mathbb{R}^m,\forall\ w\in\mathbb{R}^p \tag{15}$$

and $N_{\overline{u}j}:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^{n_\zeta}$, $N_{\underline{u}j}:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^{n_\zeta}$ and non-negative functions $s_{u\underline{u}}$, $s_{h\underline{u}}$, $s_{u\overline{u}}$, $s_{h\overline{u}}$ satisfying

$$\begin{aligned}&\psi_{\underline{u}j}(V,\alpha\rho,\kappa,h,s_{u\underline{u}},s_{h\underline{u}})+N_{\underline{u}j}^\top(x,y)(\alpha y-P\zeta(x))\geq 0,\\ &\psi_{\overline{u}j}(V,\alpha\rho,\kappa,h,s_{u\overline{u}},s_{h\overline{u}})+N_{\overline{u}j}^\top(x,y)(\alpha y-P\zeta(x))\geq 0,\\ &\forall x\in\mathbb{R}^n,\ \forall\ y\in\mathbb{R}^{n_\zeta}\ j=1,\ldots,m,\end{aligned} \tag{16}$$

then, the feedback

$$u(x)=\kappa(x,\alpha^{-1}P\zeta(x)), \tag{17}$$

and $\mathcal{D}=\mathcal{E}(V,\rho)$, solve Problem 1.

Proof. Set $U(x,y,w) = 2\zeta^\top(x)\alpha^{-1}P\nabla\zeta(x)\,(f(x) +G(x)\kappa(x,y)+G(x)dz+G_w(x)w)$. Following Lemma 3 we have that if (16) holds then $\mathcal{E}(V,\rho)\subset\mathcal{H}(u,h)$ holds. Thus we use (7) in (15) and use (3) to obtain

$$-U(x,y,w)+W(x,w)+\beta_\ell^{-1}s(x,y)\left(V(x)-\rho\right) + N(x,y)(\alpha y-P\zeta(x))>0$$

$\forall\ x\in\mathcal{E}(V,\rho),\forall\ y\in\mathbb{R}^{n_\zeta},\forall\ w\in\mathbb{R}^p$.

Since $s(x,y) \geq 0$, following Lemma 4, we have $-U(x,y,w)+W(x,w)>0$, $\forall(x,y)\in\left\{x\in\mathbb{R}^n,y\in\mathbb{R}^{n_\zeta}\mid x\in\mathcal{E}(V,\rho),y=\alpha^{-1}P\zeta(x)\right\},\forall\ w\in\mathbb{R}^p$.

Replacing $y=\alpha^{-1}P\zeta(x)$ in the above inequality, we obtain $\dot V(x,w)=U(x,\alpha^{-1}P\zeta(x),w)$ to obtain

$$-\dot V(x,w)+W(x,w)>0,\ \forall x\in\mathcal{E}(V,\rho),\forall\ w\in\mathbb{R}^p.$$

Hence, according to Lemma 1 we solve Problem 1 with u as in (20) and $\mathcal{D}=\mathcal{E}(V,\rho)$. ∎

The corollary below is the particular case of the above theorem using (18) with $y=\alpha^{-1}P\zeta(x)$. Its proof is obtained by replacing

$$\kappa(x,y)=-\frac{1}{2}S(x,y)G^\top(x)\nabla\zeta^\top(x)y, \tag{18}$$

$S:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{S}^m_{>0}$ in (15) and (16).

Corollary 1: Given a vector function $\zeta:\mathbb{R}^n\to\mathbb{R}^{n_\zeta}$, $\zeta\in\mathcal{C}^1$, and $W:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$, if there exist $P\in\mathbb{S}^{n_\zeta}_{\geq 0}$, $\beta_\ell>0$, $\gamma_\ell>0$, satisfying (3) with V as in (14), a matrix function $S:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^{m\times m}$, $h:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^m$, $s:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}_{\geq 0}$, $\alpha>0$ and $N:\mathbb{R}^n\times\mathbb{R}^{n_\zeta}\to\mathbb{R}^{n_\zeta}$, satisfying

$$\begin{aligned}&-2y^\top\nabla\zeta(x)\left(f(x)-G(x)dz+G_w(x)w\right)\\ &+y^\top\nabla\zeta(x)G(x)S(x,y)G^\top(x)\nabla\zeta^\top(x)y+W(x,w)\\ &+s(x,y)\left(\|x\|^{\gamma_\ell}-\beta_\ell^{-1}\rho\right)\\ &+dz^\top T(x,dz)(dz-S(x,y)G^\top(x)\nabla\zeta^\top(x)y+h(x,y))\\ &+N^\top(x,y)(\alpha y-P\zeta(x))>0\\ &\forall\ x\in\mathbb{R}^n,\forall\ y\in\mathbb{R}^{n_\zeta},\forall\ dz\in\mathbb{R}^m,\forall\ w\in\mathbb{R}^p\end{aligned} \tag{19}$$

and $N_{\overline{u}j} : \mathbb{R}^n \times \mathbb{R}^{n_\zeta} \to \mathbb{R}^{n_\zeta}$, $N_{\underline{u}j} : \mathbb{R}^n \times \mathbb{R}^{n_\zeta} \to \mathbb{R}^{n_\zeta}$ and non-negative functions $s_{u\underline{u}}$, $s_{h\underline{u}}$, $s_{u\overline{u}}$, $s_{h\overline{u}}$ satisfying (16) then the feedback $u(x) = \kappa(x, \alpha^{-1}P\zeta(x))$, with κ as in (18), and $\mathcal{D} = \mathcal{E}(V, \rho)$, solve Problem 1.

The theorem below provides stabilization conditions with a generic function $V(x)$. Inspired by control laws from solutions to the Hamilton-Jacobi-Bellman equation [14], [22, Sec. 3.5], we consider the state-feedback

$$u(x) = -R^{-1}(x)G^\top(x)\nabla V(x). \tag{20}$$

with $R : \mathbb{R}^n \to \mathbb{D}^m_{>0}$ and we impose $T(x) = R(x)$ in the sector inequality (7).

Theorem 3 (Stabilization): Given $W : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, if there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $V(0) = 0$, positive scalars β_ℓ, γ_ℓ satisfying (3), a matrix function $R : \mathbb{R}^n \to \mathbb{D}^m$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, $N : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_\zeta}$ satisfying

$$\begin{aligned} &-\nabla V(x)(f(x)-G(x)dz+G_w(x)w)+u^\top R(x)u+W(x,w)\\ &+s(x,u)\big(\|x\|^{\gamma_\ell}-\beta_\ell^{-1}\rho\big)+dz^\top\big(R(x)dz-G^\top(x)\nabla V(x)+h(x)\big)\\ &\qquad +N^\top(x,u)(R(x)u+G^\top(x)\nabla V(x))>0\\ &\qquad \forall\, x\in\mathbb{R}^n, \forall\, u\in\mathbb{R}^m, \forall\, dz\in\mathbb{R}^m, \forall\, w\in\mathbb{R}^p \end{aligned} \tag{21}$$

and

$$\begin{aligned} &(V(x)-\rho)+(G^\top(x))_j\nabla V(x)+h_j(x)-2R_{jj}(x)\underline{u}_j\geq 0,\\ &(V(x)-\rho)-(G^\top(x))_j\nabla V(x)-h_j(x)+2R_{jj}(x)\overline{u}_j\geq 0,\\ &\qquad \forall\, x\in\mathbb{R}^n, j=1,\ldots,m, \end{aligned} \tag{22}$$

then, provided $R^{-1}(x)$ is well-defined in $\{x \in \mathbb{R}^n \mid \|x\|^{\gamma_\ell} \leq \beta_\ell^{-1}\rho\}$ the feedback (20) solves Problem 1 with $\mathcal{D} = \mathcal{E}(V, \rho)$. Moreover if (21) holds with $s(x) \equiv 0$ and $h(x) \equiv 0$ then $u(x)$ solves Problem 1 with $\mathcal{D} = \mathbb{R}^n$.

The proof follows the same steps as the proof of Theorem 2.

Remark 3: It is straightforward to modify the inequalities in the above theorems to address the case of systems without saturation, namely system (1) with $dz \equiv 0$. Indeed, it suffices to drop the inequalities (13), (16) and (22) related to the inclusion conditions from Lemma 3 and to remove the terms containing dz in (12), (15), (19), and (21). ⋆

V. Computation of feedback laws for polynomial systems

In this section we assume that f, G, G_w, S, R and V in (1), (14), (18) and (20) are polynomials, yielding a polynomial state-feedback for (18) and a rational state-feedback for (20). The propositions below give, respectively, particular cases of conditions in Corollary 1 and Theorem 3.

Proposition 1: Given $\zeta \in \mathcal{P}^n[x]$, $\zeta(0) = 0$, $W \in \mathcal{P}[x, w]$, and $\rho > 0$, if there exist $P \in \mathbb{S}^{n_\zeta}_{\geq 0}$, $\beta_\ell > 0$, $\gamma_\ell > 0$, such that

$$V(x) - \beta_\ell\|x\|^{\gamma_\ell} \in \Sigma[x] \tag{23}$$

with V as in (14), and there exist $\alpha > 0$, $S \in \mathcal{P}^{m\times m}[x, y]$ and $s \in \Sigma[x, y]$, $T \in \Sigma^m[x, dz]$, a positive scalar ϵ_A, and a positive integer δ_x, satisfying

$$\begin{aligned} &-2y^\top\nabla\zeta(x)\,(f(x)-G(x)dz+G_w(x)w)\\ &\quad+y^\top\nabla\zeta(x)G(x)S(x,y)G^\top(x)\nabla\zeta^\top(x)y+W(x,w)\\ &\quad+N^\top(x,y)(\alpha y-P\zeta(x))+s(x,y)\left(\|x\|^{\gamma_\ell}-\beta_\ell^{-1}\rho\right)\\ &+dz^\top T(x,dz)(dz-S(x,y)G^\top(x)\nabla\zeta^\top(x)y+h(x,y))\\ &\qquad\qquad+\epsilon_A(\|x\|^{\delta_x})\in\Sigma[x,y,dz,w] \end{aligned} \tag{24}$$

and $s_{u\underline{u}}, s_{h\underline{u}}, s_{u\overline{u}}, s_{h\overline{u}} \in \Sigma[x, y]$, $N \in \mathcal{P}^{n_\zeta}[x, y]$, satisfying

$$\begin{aligned} &\psi_{\underline{u}j}(V,\alpha\rho,\kappa,h,s_{u\underline{u}},s_{h\underline{u}})+N_{\underline{u}j}^\top(x,y)(\alpha y-P\zeta(x))\in\Sigma[x,y],\\ &\psi_{\underline{u}j}(V,\alpha\rho,\kappa,h,s_{u\overline{u}},s_{h\overline{u}})+N_{\overline{u}j}^\top(x,y)(\alpha y-P\zeta(x))\in\Sigma[x,y],\\ &j=1,\ldots,m, \end{aligned} \tag{25}$$

then the feedback $u(x) = \kappa(x, \alpha^{-1}P\zeta(x))$, with κ as in (18), and $\mathcal{D} = \mathcal{E}(V, \rho)$, solve Problem 1.

Remark 4: The use of quadratic-like representations as $V(x) := \zeta^\top(x)P\zeta(x)$ for polynomial LFs has been proposed for the synthesis of state feedback laws for polynomial systems *without saturating inputs* in [6], [11], [20], [30], [32]. Instead of establishing conditions in terms of polynomial scalar constraints as in the above propositions, the synthesis conditions in [6], [11], [20], [30] are formulated in terms of polynomial matrix inequalities. These stabilization conditions are obtained from linear-like representations of (1) with $dz \equiv 0$ as

$$\dot{x} = (A(x) + E(x) + G(x)K(x))\,\zeta(x) \tag{26}$$

where $A(x)\zeta(x) = f(x)$, the controller structure as $u(x) = K(x)\zeta(x)$ is imposed, and $E \in \mathcal{P}[x]$ satisfies $E(x)\zeta(x) = 0$. With the LF (14) the time-derivative along the trajectories of system (1) is given by $\dot{V}(x) = 2\zeta^\top(x)P(x)(\nabla\zeta(x))^\top(A(x)+E(x)+G(x)K(x))\zeta(x)$. Introducing variable $y(x) = P\zeta(x)$, and defining $Q := P^{-1}$ which gives $\zeta(x) = Qy(x)$ and $L(x) := K(x)Q$, one obtains

$$\dot{V}(x, y) = 2y^\top(x)(\nabla\zeta(x))^\top(A(x)Q+G(x)L(x))y(x). \tag{27}$$

Clearly, for given $A(x)$ and $E(x)$, if the matrix inequality

$$2(\nabla\zeta(x))^\top(A(x)Q + E(x)Q + G(x)L(x)) < 0 \tag{28}$$

holds for all $x \in \mathbb{R}^n$ then (27) holds for all $x \in \mathbb{R}^n$. However, the satisfaction of (28) is only a sufficient condition for (27) since, for a matrix $M(x)$, $y^\top M(x)y < 0\ \forall x, \forall y$ implies $\zeta(x)^\top M(x)\zeta(x) < 0\ \forall x$ but the converse does not necessarily hold. To mitigate the conservativeness introduced by solving the inequalities with fixed $A(x)$ and $E(x)$, [30] proposes a strategy to solve a sequence of semi-definite programs where either Q or $E(x)$ are taken as a decision variables and a representation of the $\dot{\zeta}(x)$, instead of (26) is considered. At each iteration a bound for the closed-loop $\mathcal{L}_2$ performance is optimized. The non-uniqueness of state-dependent linear-like representation as (26) has also been studied in the context of solutions to state-dependent Riccati Equations [23]. ⋆

Proposition 2: Given $W \in \mathcal{P}[x, w]$ and $\rho > 0$ if there exist $V \in \mathcal{P}[x]$, $V(0) = 0$, $\beta_\ell > 0$, $\gamma_\ell > 0$, satisfying (23), $N \in$

$\mathcal{P}^m[x,u]$, $s \in \Sigma[x,u]$, a diagonal polynomial matrix $R \in \mathcal{P}^{m\times m}[x]$, $\epsilon_A > 0$, and $\delta_x > 0$ satisfying

$$\begin{aligned}&-\nabla V(x)(f(x) - G(x)dz + G_w(x)w) + W(x,w)\\ &+s(x,u)\big(\|x\|^{\gamma_\ell} - \beta_\ell^{-1}\rho\big) + dz^\top\big(R(x)dz - G^\top(x)\nabla V(x) + h(x)\big)\\ &+u^\top R(x)u + N^\top(x,u)(R(x)u + G^\top(x)\nabla V(x)) + \epsilon_A(\|x\|^{\delta_x})\\ &\in \Sigma[x,u,dz,w] \quad (29)\end{aligned}$$

and

$$\begin{aligned}&(V(x)-\rho) + (G^\top(x))_j\nabla V(x) + h_j(x) - 2R_{jj}(x)\underline{u}_j \in \Sigma[x],\\ &(V(x)-\rho) - (G^\top(x))_j\nabla V(x) - h_j(x) + 2R_{jj}(x)\overline{u}_j \in \Sigma[x],\\ &j = 1,\ldots,m, \quad (30)\end{aligned}$$

then, provided that $R^{-1}(x)$ exists in $\{x \in \mathbb{R}^n \mid \|x\|^{\gamma_\ell} \le \beta_\ell^{-1}\rho\}$, the feedback law (20) solves Problem 1 with $\mathcal{D} = \mathcal{E}(V,\rho)$.

An advantage of the inclusion conditions (30) of Proposition 2 over (25) of Proposition 1 is that they do not present products of unknowns.

Algorithm 1 below applies to the conditions of Propositions 1 and 2 (the square brackets within the steps indicate the steps for Proposition 1). The proposed iterative method is required since (29) in Proposition 2 contains a product between N and both V and R. On the other hand, Proposition 1 presents products between all the multipliers $\{N, T, s_{u\underline{u}}, s_{h\underline{u}}, s_{u\overline{u}}, s_{h\overline{u}}\}$ and the control law parameters α, S and P of (17)-(18). At each step of the algorithm we take $W(x,w) = \eta w^\top w - x^\top x$ and we minimize η subject to the SOS constraints of Propositions 1 and 2. This objective function provides induced $\mathcal{L}_2$ gain bounds $\frac{\|x\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}} \le \sqrt{\eta}$.

Remark 5: Initialization of Algorithm 1. It should be noted that there is no guarantee of convergence to global optima in iterative algorithms as Algorithm 1. Also, the proposed procedure relies on the existence of initial feasible values for the LF and R. In the general case, these initial feasible solutions may be difficult to obtain. For the case of systems *without input saturation*, with stabilizable linear approximations, it is possible to obtain quadratic LFs and linear gains as the starting solution. Powers of quadratic LFs such as $V(x) = (x^\top Px)^{2r}$ can be used as initial polynomial LF as suggested by the converse results in [2]. ⋆

Algorithm 1 Control Design for saturating systems

Inputs: positive scalar δ, parameterised W, degrees of the polynomial variables
STEP 0:
  $i \leftarrow 0$.
  Fix ρ
  $V \leftarrow V_0$, $R \leftarrow R_0$ [$P \leftarrow P_0$, $\alpha \leftarrow \alpha_0$, $h \leftarrow h_0$, $S \leftarrow S_0$].
  $\gamma_0 \leftarrow \infty$.
STEP 1) Computation of multipliers:
  Fix V and R [Fix P, α, S, h]
  $\underset{N}{\text{Minimize}}$ η subject to (29), (30) [$\underset{N,T,s_{u\underline{u}},s_{h\underline{u}},s_{u\overline{u}},s_{h\overline{u}}}{\text{Minimize}}$ η subject to (24), (25)]
  $N \leftarrow N^*$; [$N \leftarrow N^*$; $T_i \leftarrow T^*$; $s_{u\underline{u}} \leftarrow s^*_{u\underline{u}}$; $s_{h\underline{u}} \leftarrow s^*_{h\underline{u}}$; $s_{u\overline{u}} \leftarrow s^*_{u\overline{u}}$; $s_{h\overline{u}} \leftarrow s^*_{u\underline{u}}$].
STEP 2) Computation of LF:
  Fix N [Fix $N, T, s_{u\underline{u}}, s_{h\underline{u}}, s_{u\overline{u}}, s_{h\overline{u}}$]
  $\underset{V,R}{\text{Minimize}}$ η subject to (23), (29), (30), [$\underset{P,S,h,\alpha}{\text{Minimize}}$ η subject to (23), (24), (25),]
  $i \leftarrow i+1$.
  $V \leftarrow V^*$, $R \leftarrow R^*$ [$P \leftarrow P^*$, $\alpha \leftarrow \alpha^*$, $S \leftarrow S^*$, $h \leftarrow h^*$].
  $\gamma_i \leftarrow \sqrt{\eta^*}$.
if $\gamma_{i-1} - \gamma_i < \delta$, **STOP**
else goto *STEP 1.*

VI. EXAMPLES

This section presents three examples that illustrate the results in Section V. Solutions to the proposed SOS constraints are obtained with SOSTOOLS [19].

Example 1 In this example we illustrate the impact of saturations in the closed loop system with a rational state feedback. Consider the saturating system borrowed from [30, Example 3]

$$\begin{cases}\dot{x}_1 &= 1.5x_1^2 + x_2^2 - 0.5x_1^3 + 0.3w - sat(u)\\ \dot{x}_2 &= -x_2 + x_1x_2 + 0.3w - 0.5sat(u)\end{cases} \quad (31)$$

The saturation limits are $[\underline{u}, \overline{u}] = [-1, 2]$. We take P, the solution to a Lyapunov equation for the linearized system with a linear state-feedback and take the function $V_0(x) = x^\top Px + (x^\top Px)^2 + (x^\top Px)^3$, as the starting LF for the Algorithm 1, applied to Proposition 2. As a solution to Algorithm 1 we obtain a rational state-feedback law as in (20) that guarantees a local induced $\mathcal{L}_2$ gain of $\gamma = 0.5$. That is, the control law guarantees that the trajectories satisfy $\|x\|_2 \le 0.5\rho$ for $x(0) = 0$ and for all $w \in \mathcal{L}_2$ such that $\|w\|_2 \le \rho$, with $\rho \le 0.4$. We first show trajectories of the closed loop without saturation in Figure 2 (left). Figure 2 (right) depicts trajectories of the saturated closed-loop and the sets related to the estimation of the region of attraction of the origin of the saturating system, that was obtained with the solution to the SOS relaxation of inequalities of Theorem 1. The gray curve corresponds to the boundary of the ERA of the origin. The dashed black curves delimit the set $\mathcal{H}_{11} = \{x \in \mathbb{R}^n | \underline{u} \le u(x) \le \overline{u}\}$ and the set $\mathcal{H}_b = \{x \in \mathbb{R}^n | \underline{u} \le h(x) \le \overline{u}\}$ is delimited by the dash-dotted green curves. Note that the obtained ERA is included in the set $\mathcal{H}(u,h)$ as in (6) ($m = 1$), which contains the union of the sets $\mathcal{H}_{11}$ and $\mathcal{H}_b$ and where the sector inequality (7) holds. Note also that the ERA is not fully included in $\mathcal{H}_b$, thus exploiting the generalization of the local sector condition of Lemma 2 (see Remark 2).

Example 2 Consider the system defined by $f(x) = \big[2x_1^3 + x_1^2x_2 - 6x_1x_2^2 + 5x(2)^3 \quad 0\big]^\top$ and $G(x) = [0 \quad 1]^\top$. Without saturation, [16, Example 1] shows that the control law $u = -(d_1^2+4)x_1^3 + (2d_1^2+4)x_1^2x_2 + (3-2\sqrt{15}d_1 + 4d_1^2)x_1x_2^2 - (10-4\sqrt{15}d_1 + 8d_1^2)x_2^3$, with $d_1 = \sqrt{15}/10$ globally stabilizes the origin of the system. With saturation limits $\underline{u} = -1$, $\overline{u} = 1$ the closed-loop system presents an unstable limit cycle. To estimate the region of attraction of the origin, we solve the inequalities of Theorem 1 with SOS relaxations, similar to the relaxations carried out in Propositions 1 and 2. With

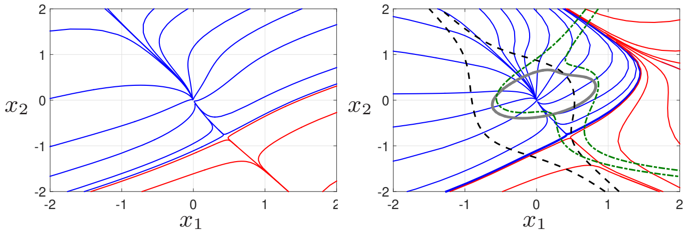

Fig. 2: *Closed-loop trajectories for Example 1. Trajectories converging to the origin are depicted in blue and diverging trajectories in red. On the left we have the trajectories for the unsaturated system with a rational state feedback that locally optimize the $\mathcal{L}_2$ gain. On the right the trajectories of the saturated feedback. The dashed black lines delimit the set $\mathcal{H}_{11}$, that is, the set where the input lies within the saturation limits. The dash-dotted green lines delimit the set $\mathcal{H}_b$, where the function $h(x)$ is within the saturation bounds.*

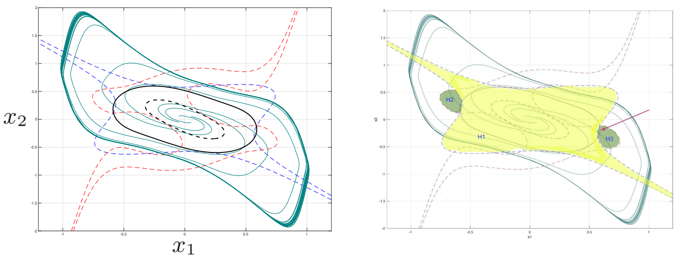

Fig. 3: *Trajectories for Example 2 are indicated in green. The dashed blue lines indicate the boundary of the set $\mathcal{H}_{11} = \{x \mid \underline{u} \leq u(x) \leq \overline{u}\}$, namely the set of control values within the saturation bounds. The dashed red lines indicate the boundary of the set $\mathcal{H}_b(u,h) := \{x \mid \underline{u} \leq h(x) \leq \overline{u}\}$ The figure on the right highlights the sets (5) in $\mathcal{H}_1 = \mathcal{H}_{11} \cup \mathcal{H}_{21} \cup \mathcal{H}_{31}$ with $\mathcal{H}_{11}$ in yellow and $\mathcal{H}_{21}$, $\mathcal{H}_{31}$ in green. The red arrow in this figure indicates that the largest level set of the LF within $\mathcal{H}_1$ includes points in the green areas.*

the computed V and h, Figure 3 illustrates the obtained RA (solid black line) and the set $\mathcal{H}$. For the obtained function h, the set $\mathcal{H}_b$ (see Remark 2) corresponds to the area between the dashed red lines. The largest level set of the computed Lyapunov function within the set $\mathcal{H}_b$ is given by the black dashed line.

Example 3 [Cart and Pendulum] We consider the cart and pendulum system given in [3, eq. (1)-(2), (5)]. We consider $u = V_{in}$, $w = -T_{fric}$ and $F_{fric} = 0$ and set state variables as $x_1 = x$; $x_2 = \theta$, $x_3 = \dot{x}$, and $x_4 = \dot{\theta}$ (x is the cart position and θ the pendulum position). By using the approximations $\sin(x_2) \approx x_2$ and $\cos(x_2) \approx 1$ we obtain model (1) with

$$f(x)=\begin{bmatrix} x_3 \\ x_4 \\ \bar{M}^{-1}\begin{bmatrix} mLx_2x_4^2 - c_2x_3 \\ mgLx_2 \end{bmatrix} \end{bmatrix}; G=\begin{bmatrix} 0 \\ 0 \\ \bar{M}^{-1}\begin{bmatrix} c_1 \\ 0 \end{bmatrix} \end{bmatrix};$$

$$G_w = \begin{bmatrix} 0 \\ 0 \\ \bar{M}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \text{ with } \bar{M} = \begin{bmatrix} M_t & mL \\ mL & J_m \end{bmatrix}.$$

the parameters are $L = 0.32$, $m = 0.231$ $M_t = 1.142$, $J_m = 0.03153$, $c_1 = 1.3290$, $c_2 = 5.561$. The input saturation limits are $-12V$, $+12V$. We have designed a control law

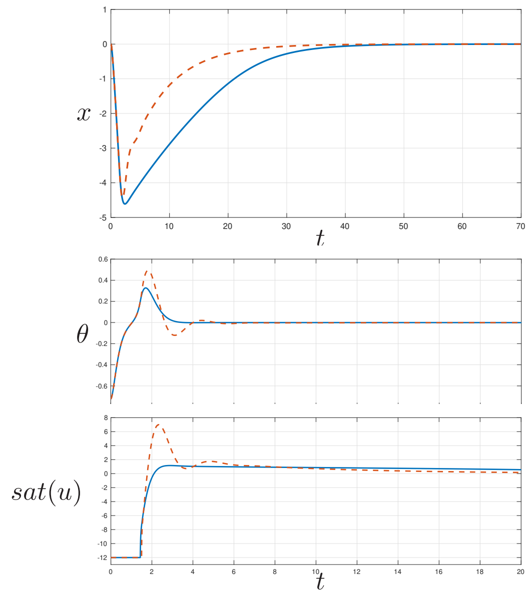

Fig. 4: *Trajectories of the positions, angular positions and the corresponding control signals for Example 3. Two simulations starting in the same initial condition are depicted: for a state-feedback computed using standard linear saturating systems design (dashed red line) and a rational state feedback law obtained using Proposition 2 (solid blue line).*

following Algorithm 1 applied to Proposition 2 and another control law for the linearized system with input saturations using the local sector conditions of [25, Lemma 1.6]. We simulate trajectories starting from the initial condition $x = \begin{bmatrix} 0 & -0.7225 & 0 & 0 \end{bmatrix}$ (only the initial angular positions is not zero) and the trajectory for variable θ as well as the control laws are presented in Figure 4. The rational control obtained with Proposition 2 gives a faster convergence and smaller overshoot for variable θ for very similar time interval in saturation. Note that the control does not converge to zero during the simulation time as the other variables (x in particular) present dynamics slower than for θ.

Example 4 [Attitude Control of a Rigid Body] Consider the model for the attitude of a rigid body [24, eqs. (330), (358)] studied in [20, Section V]

$$\begin{cases} \dot{\psi} &= \frac{1}{2}\left(\omega - \omega \times \psi + (\omega^\top \psi)\psi\right) \\ J\dot{\omega} + \omega \times J\omega &= sat(u) \end{cases} \tag{32}$$

where $\omega \in \mathbb{R}^3$ is the angular velocity described a body frame, $\rho \in \mathbb{R}^3$ is the Rodrigues parameter vector and $u \in \mathbb{R}^3$ is the control torque. The matrix $J \in \mathbb{R}^{3\times 3}$ represents the inertia matrix described in the body frame. We use $J = \text{diag}(\begin{bmatrix} 4 & 2 & 1 \end{bmatrix})$. In this example we consider that the three control inputs signals with bounds $\underline{u}_i = -10$, $\overline{u}_i = 10$.

Using an initial quadratic LF obtained for the linearized system, we increase the value of parameter ρ for the constraints of Propositions 1 and 2 by alternating the search between the LF and control parameters, and the multipliers. A quadratic function and parameters S and R in the control functions of degree zero result in a linear control law, obtained for the nonlinear system while taking into account the saturations.

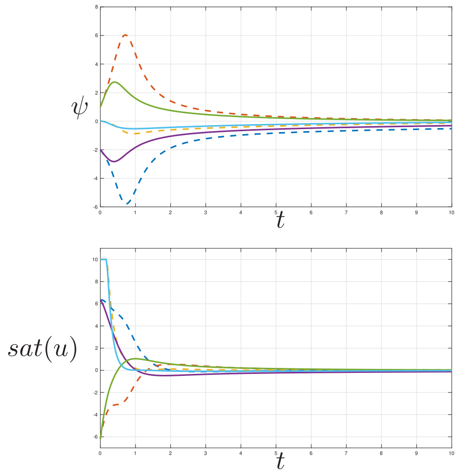

Fig. 5: *Angular positions and the control signals for Example 4. Two simulations starting in the same initial condition are depicted Proposition 1 (solid lines) Proposition 2 (dashed lines). The variables correspond to the lines starting at $\phi_1(0) = -2$, $\phi_2(0) = 1$, $\phi_3(0) = 0$.*

The simulation results for the three angular position as well as the three input signals are illustrated in Figure 5 for an initial condition starting at $x_0 = \begin{bmatrix} -2 & 1 & 0 & -1 & 2 & -3 \end{bmatrix}$. Even if the saturation occurs for a short period, this initial condition could not be scaled by a factor larger than 1.3 preserving convergence to the origin. This example shows that the SOSP strategies presented in the paper can handle a system of degree 3 with 6 state variables.

VII. Conclusion

We have presented conditions for the stabilization of saturating nonlinear input-affine systems. To cope with (possibly asymmetric) magnitude saturation, we have introduced the nonlinear extension of the generalised sector condition presented in [9], [26]. For the class of polynomial nonlinear systems, we obtain SOS programs allowing for the computation of polynomial or rational control laws. Four numerical examples illustrate the proposed analysis and feedback strategies and the numerical method to compute the controller parameters.

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