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Strong Stability of Bounded Evolution Families and Semigroups

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We prove several characterizations of strong stability of uniformly bounded evolution families $(U(t,s))_{t\geqslant s\geqslant 0}$ of bounded operators on a Banach space X, i.e. we characterize the property $\lim_{t\to\infty}\|U(t,s)x\| = 0$ for all $s\geqslant 0$ and all $x\in X$. These results are connected to the asymptotic stability of the well-posed linear nonautonomous Cauchy problem

$$\begin{cases} \dot u(t) = A(t)u(t), & t\geqslant s\geqslant 0,\\ u(s) = x, & x\in X. \end{cases}$$

In the autonomous case, i.e. when $U(t,s) = T(t-s)$ for some C_0-semigroup $(T(t))_{t\geqslant 0}$, we present, in addition, a range condition on the generator A of $(T(t))_{t\geqslant 0}$ which is sufficient for strong stability. This condition is more general than the condition in the ABLV-Theorem involving countability of the imaginary part of the spectrum of A. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

One difficult task in the study of a linear nonautonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = A(t)u(t), & t \geqslant s \geqslant 0, \\ u(s) = x, & x \in X \end{cases} \tag{1}$$

is the study of asymptotic behaviour of its solutions. Among the most interesting types of asymptotic behaviour we would like to mention stability in the sense that solutions vanish at infinity (see Definition 2.1).

In the case of the autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geqslant 0, \\ u(0) = x, & x \in X, \end{cases} \tag{2}$$

some results have already been obtained. For example, assuming that (2) is well-posed and that the operator A generates a bounded C_0-semigroup $(T(t))_{t \geqslant 0}$, there is the ABLV-Theorem and all its generalizations to bounded individual solutions or bounded uniformly continuous functions, [1–3, 5, 6, 9, 28]. One typical assumption in results of this type is the countability of some spectrum; for example countability of the boundary spectrum $\sigma(A) \cap i\mathbb{R}$.

Another group of results, based on resolvent estimates rather than simple spectral conditions, has developed recently, coming closer to a characterization of stable semigroups [7, 34]. By means of unitary dilations (in Hilbert spaces) and limit isometric groups there have been obtained growth conditions on the resolvent near the imaginary axis which are sufficient for stability, and which are close to being necessary (see especially the characterizations of strong stability for semigroups on Hilbert spaces in [34]).

In this paper, we give several characterizations of stability of evolution families such as arise from well-posed, nonautonomous Cauchy problems. The characterizations are in terms of bounded complete trajectories of the dual family, stability of associated evolution semigroups and spectral properties of the generator of an evolution semigroup. The results seem to be new even in the case of finite dimensional Banach spaces. In the autonomous case some of these are known characterizations of stability of a bounded C_0-semigroup, but our results include a new characterization analogous to Datko's characterization of exponential stability. Moreover, we also show that a sufficient condition for stability of a bounded semigroup on X generated by A is that the intersection of the ranges of $i\beta - A$ (as β

varies through $\mathbb{R}$) is dense in X. Finally, we present some applications of our results to abstract operator theory.

2. THE MAIN RESULTS

Throughout the paper we will denote by X a complex Banach space and by $\mathbf{L}(X)$ the space of bounded linear operators on X. A two-parameter family $\mathbf{U} = (U(t,s))_{t \geqslant s \geqslant 0} \subset \mathbf{L}(X)$ is called an *evolution family* if it satisfies the following three conditions:

(i) $U(t,t) = I$ for all $t \geqslant 0$.
(ii) $U(t,s)U(s,r) = U(t,r)$ for all $0 \leqslant r \leqslant s \leqslant t$.
(iii) $U(\cdot,\cdot)$ is strongly continuous from $\{(t,s) \in \mathbb{R}^2 \colon 0 \leqslant s \leqslant t\}$ into $\mathbf{L}(X)$.

Evolution families usually appear in the context of the nonautonomous Cauchy problem (1). Under suitable conditions on the operators $A(t)$, there is an evolution family such that for each $x \in X$ there is a unique solution of (1) (in an appropriate sense) given by $u(t) = U(t,s)x$ (see [8, Sect. 3.1; 30, Chap. 5; 32, Chap. 5]). In the autonomous case, i.e. when $A(t) = A$ is constant, the existence of a unique "mild" solution of (1) for each $x \in X$ is equivalent to the condition that A generates a C_0-semigroup $(T(t))_{t \geqslant 0}$ (see [3, Sect. 3.1]). In that case we have $U(t,s) = T(t-s)$ for all $t \geqslant s$.

In the following, we will always assume that an evolution family $\mathbf{U} = (U(t,s))_{t \geqslant s \geqslant 0}$ is given and, in addition, that $\mathbf{U}$ is uniformly bounded, i.e. $\sup_{t \geqslant s \geqslant 0} \|U(t,s)\| := M < \infty$.

Definition 2.1. We call an evolution family $(U(t,s))_{t \geqslant s \geqslant 0}$ on a Banach space X (*strongly*) *stable* if for all $s \geqslant 0$ and for all $x \in X$ one has $\lim_{t \to \infty} \|U(t,s)x\| = 0$.

As indicated in the Introduction we would like to characterize strong stability of an evolution family $\mathbf{U}$. In this context we will study *complete trajectories* of a dual evolution family, the asymptotic behaviour of associated *evolution semigroups*, *convolutions* and *range conditions* on the operator A in the autonomous case.

To define these concepts let us point out that the dual $(U(t,s)^*)_{t \geqslant s \geqslant 0}$ of an evolution family $\mathbf{U}$ is in general no longer an evolution family, but $(U(-s,-t)^*)_{s \leqslant t \leqslant 0}$ is (the definition of an evolution family is easily adapted to the case of the negative real line, and, moreover, strong continuity is replaced by weak* continuity). If $\mathbf{U}$ is associated with the well-posed

Cauchy problem (1) then the evolution family $(U(-s,-t)^*)_{s\leqslant t\leqslant 0}$ is, at least formally, associated with the nonautonomous backward Cauchy problem

$$\begin{cases} \dot v(t) = A(-t)^* v(t), & t \in (-\tau, 0], \\ v(0) = x^*, & x^* \in X^* \end{cases} \tag{3}$$

in which, in addition, the initial condition has been replaced by a condition at the end point of the time interval $(-\tau, 0]$ $(\tau > 0)$. In general, a solution v to the backward Cauchy problem (3) need not exist, but if it exists, then v satisfies the relation $v(t) = U(-s,-t)^* v(s)$ for all $\tau < s \leqslant t \leqslant 0$.

Now we call a function $g : \mathbb{R}_- \to X^*$ a *complete trajectory for* $(U(-s,-t)^*)_{s\leqslant t\leqslant 0}$ whenever it satisfies the condition $U(-s,-t)^* g(s) = g(t)$ for all $s \leqslant t \leqslant 0$. Note that this definition of a complete trajectory differs from that in the literature (see e.g. [4, 35]) in that g is only defined on the half-line $\mathbb{R}_-$. However, in the autonomous case when $U(t,s) = T(t-s)$, a complete trajectory in our sense can be uniquely extended to a complete trajectory on $\mathbb{R}$ by defining $g(t) = T(t)^* g(0)$ for $t \geqslant 0$. We call a complete trajectory $g : \mathbb{R}_- \to X^*$ *nontrivial* if g is not identically 0.

Let $1 \leqslant p \leqslant \infty$, and let $E_p := L^P(\mathbb{R}_+; X)$ if $1 \leqslant p < \infty$, and $E_\infty := C_{00}(\mathbb{R}_+; X)$ (the space of continuous functions vanishing at 0 and at infinity). The space E_p will be equipped with the norm $\|f\|_p := (\int_0^\infty \|f \times (s)\|^p\, ds)^{1/p}$ when $1 \leqslant p < \infty$ and with the sup-norm when $p = \infty$. It is well known (see e.g. [8, Sect. 3.3]) that the family $(\mathbf{T}_p(t))_{t\geqslant 0}$ defined by

$$(\mathbf{T}_p(t)f)(s) = \begin{cases} U(s, s-t)f(s-t), & s \geqslant t, \\ 0, & s < 1, \end{cases} \qquad t, s \geqslant 0,\ f \in E_p$$

is a C_0-semigroup on the Banach space E_p. We call $(\mathbf{T}_p(t))_{t\geqslant 0}$ the *evolution semigroup associated with* $(U(t,s))_{t\geqslant s\geqslant 0}$ *on the space* E_p, and we denote by $\mathbf{G}_p$ its generator.

Finally, we denote by $D(G)$, $\operatorname{Rg} G$ and $\operatorname{Ker} G$, the domain, the range and the kernel, respectively, of a closed linear operator G on a Banach space X, and $\sigma(G)$ (resp. $P\sigma(G)$) denotes the spectrum (resp. point spectrum) of G.

For an evolution family $\mathbf{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ and a function $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$, let

$$(U * f)(t) := \int_0^t U(t,s) f(s)\, ds \qquad \text{for all } t \geqslant 0.$$

When $U(t,s) = T(t-s)$ for a semigroup $\mathbf{T}$, $U * f$ is the *convolution* of $\mathbf{T}$ and f in the usual sense.

Our first main result is the following.

THEOREM 2.2. *Let $(U(t,s))_{t\geqslant s\geqslant 0}$ be a bounded evolution family on a Banach space X, and let $(\mathbf{T}_p(t))_{t\geqslant 0}$ be the evolution semigroup associated with $(U(t,s))_{t\geqslant s\geqslant 0}$ on E_p $(1\leqslant p\leqslant\infty)$. Then the following assertions are equivalent*:

(1) *The evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ is strongly stable.*

(2) *If B^* denotes the unit ball in X^*, then the set*

$$J^* := \bigcup_{s\geqslant 0} \bigcap_{t\geqslant s} U(t,s)^*(B^*) \tag{4}$$

is trivial, i.e. $J^* = \{0\}$.

(3) The *evolution family $(U(-s,-t)^*)_{s\leqslant t\leqslant 0}$ does not admit a bounded nontrivial complete trajectory.*

(4) *The semigroup $(\mathbf{T}_p(t))_{t\geqslant 0}$ is stable for some $1\leqslant p\leqslant\infty$.*

(5) *The semigroup $(\mathbf{T}_p(t))_{t\geqslant 0}$ is stable for all $1\leqslant p\leqslant\infty$.*

(6) $\operatorname{Rg}\mathbf{G}_1$ *is dense in $L^1(\mathbb{R}_+;X)$.*

(7) *The set*

$$F := \{f \in L^1(\mathbb{R}_+;X)\colon\ U*f \in L^1(\mathbb{R}_+;X)\} \tag{5}$$

is dense in $L^1(\mathbb{R}_+;X)$.

Let us point out that in the autonomous case the equivalence in (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) in Theorem 2.2 goes back to [13, Théorème 2; 27, Theorem 4.3]. The generalization from semigroups to evolution families is based on the same ideas, but one has to be careful in some steps. For our approach equivalence (1) $\Leftrightarrow$ (3) is important for the proof of the other results.

The Equivalences (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) should be compared to Datko's characterization of *uniformly exponentially* stable evolution families in which density of $\operatorname{Rg}\mathbf{G}_1$ has been replaced by surjectivity of $\mathbf{G}_p$ [3, Theorem 5.1.2; 12, Theorem 6; 29, Theorem 2.2].

THEOREM 2.3 (Datko). *Let $(U(t,s))_{t\geqslant s\geqslant 0}$ be an evolution family on a Banach space X, and let $1\leqslant p\leqslant\infty$. Then the following assertions are equivalent*:

(i) *The family $(U(t,s))_{t\geqslant s\geqslant 0}$ is exponentially stable.*

(ii) *The operator $\mathbf{G}_p$ is surjective.*

(iii) *For all $f\in E_p$ one has $U*f\in E_p$.*

However, whereas Datko's Theorem 2.3 is true for every $p\in[1,\infty]$, equivalence (1) $\Leftrightarrow$ (7) cannot be true for $p\in(1,\infty]$ (see Remark 5.3).

The second main result concerns the autonomous case, i.e. the case when $\mathbf{U}$ comes from a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$. In that case the Laplace transform of an orbit $T(\cdot)x$ is given by the local resolvent $R(\cdot, A)x$ on the right half-plane. Applying Laplace transform and Fourier transform techniques, we are able to find conditions on the resolvent of the generator A and range conditions on A leading to stability. In fact, as we show in Section 6, we can also prove local stability results (Theorem 6.3). The ABLV-Theorem mentioned in the Introduction is an easy corollary of Theorem 2.4 (Corollary 6.5).

Before stating the second result we define $A_+(\mathbb{R}; X)$ to be the image under Fourier transform of the space $L^1(\mathbb{R}_+; X)$, i.e. the space of all functions $f : \mathbb{R} \to X$ for which there exists $g \in L^1(\mathbb{R}_+; X)$ such that

$$f(\beta) = \int_0^\infty e^{-i\beta s} g(s)\, ds =: \mathbf{F}g(\beta)(\beta \in \mathbb{R}).$$

By injectivity of the Fourier transform the function g is uniquely determined and we put $\|f\|_{A_+} := \|g\|_1$.

Theorem 2.4. *Assume that* $U(t,s) = T(t-s)$ *for some bounded* C_0-*semigroup* $(T(t))_{t\geqslant 0}$ *on* X, *and denote by* A *the generator of* $(T(t))_{t\geqslant 0}$. *Then conditions* (1)–(7) *are equivalent to*:

(8) $P\sigma(A) \cap i\mathbb{R} = \emptyset$ *and the multiplication operator* $\mathbf{M}$ *defined by*

$$D(\mathbf{M}) := \{f \in A_+(\mathbb{R}; X):\ f(\beta) \in \mathrm{Rg}(i\beta - A) \text{ and}$$

$$\beta \mapsto (i\beta - A)^{-1} f(\beta) \in A_+(\mathbb{R}; X)\},$$

$$\mathbf{M}f(\beta) := (i\beta - A)^{-1} f(\beta)$$

is densely defined.

The condition

(9) *The space* $\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}(i\beta - A)$ *is dense in* X

implies (1)–(8).

Since the completion of this paper, the second and third authors, and also M. Lin and P. Wojtaszczyk, have shown that implication (1) $\Rightarrow$ (9) does not hold in the context of Theorem 2.4; in particular, (9) is not satisfied for some of the stable semigroups considered in [5, Example 4.1]. This and other information will appear in [9a].

The rest of the paper is devoted to the proof of Theorems 2.2 and 2.4 as well as of their local counterparts. Since we will frequently make use of the

Mean Ergodic Theorem for bounded C_0-semigroups, and since this theorem shows, in particular, how range conditions and (Abel) ergodicity are connected, we state that theorem explicitly (see [3, Proposition 4.3.1; 17, Lemma V.4.4, Theorem V.4.5, Example V.4.7]).

THEOREM 2.5. *Let A be the generator of a bounded C_0-semigroup on a Banach space X. For every $x \in X$ the following assertions are equivalent*:

(i) $\lim_{\lambda \to 0} \lambda R(\lambda, A)x$ *exists.*
(ii) $x \in \operatorname{Ker} A \oplus \overline{\operatorname{Rg} A}$.

If (i) *or* (ii) *holds, then* $\lim_{\lambda \to 0} \lambda R(\lambda, A)x = x_1$, *where* $x = x_1 + x_2$ *with* $x_1 \in \operatorname{Ker} A$ *and* $x_2 \in \overline{\operatorname{Rg} A}$.

If X is reflexive, then (i) *and* (ii) *are always true.*

3. PROOF OF (1) ⇔ (2) ⇔ (3)

In this section, we study the connection of the existence or nonexistence of bounded nontrivial complete trajectories to stability of the evolution family **U**. This connection is important for the subsequent results because we will use it both in the proof of (6) ⇒ (1) and in the proof of (9) ⇒ (1).

The proof of (1) ⇔ (2) ⇔ (3) is based on the next proposition which is an individual quantitative version of these equivalences. In the autonomous (and discrete) case a proof of Proposition 3.1 can be found in [4, Theorem 3.1; 13, Théorème 2; 27, Theorem 4.3; 35].

PROPOSITION 3.1. *Let $(U(t,s))_{t \geqslant s \geqslant 0}$ be a bounded evolution family on a Banach space X. Let B^* be the closed unit ball in the dual space X^* and define for fixed $s \geqslant 0$*

$$J_s^* := \bigcap_{t \geqslant s} U(t,s)^*(B^*). \tag{6}$$

Then:

(i) *For every $x \in X$ we have*

$$\frac{1}{M} \limsup_{t \to \infty} \|U(t,s)x\| \leqslant \sup_{x^* \in J_s^*} |\langle x, x^* \rangle| \leqslant \liminf_{t \to \infty} \|U(t,s)x\|, \tag{7}$$

where $M = \sup_{t \geqslant s \geqslant 0} \|U(t,s)\|$.

(ii) $\lim_{t \to \infty} \|U(t,s)x\| = 0$, *if and only if x annihilates J_s^*.*

(iii) *For every $x^* \in J_0^*$ there exists a bounded complete trajectory g for the evolution family $(U(-s,-t)^*)_{s \leqslant t \leqslant 0}$ such that $g(0) = x^*$.*

Proof. (i) Let $x \in X$, and let J_s^* be defined as in (6). For every $x^* \in J_s^*$ and every $t \geqslant s$ there exists $x_t^* \in B^*$ such that $U(t,s)^* x_t^* = x^*$. From that we obtain

$$|\langle x, x^* \rangle| = |\langle x, U(t,s)^* x_t^* \rangle| = |\langle U(t,s)x, x_t^* \rangle| \leqslant \|U(t,s)x\|,$$

which implies the second inequality in (7).

Conversely, by Hahn–Banach, there exists for every $t \geqslant s$ an element $x_t^* \in B^*$ such that

$$\begin{aligned}\|U(t,s)x\| &\leqslant |\langle U(t,s)x, x_t^* \rangle| + \frac{1}{t+1} \\ &= M\left|\left\langle x, \frac{1}{M} U(t,s)^* x_t^* \right\rangle\right| + \frac{1}{t+1}.\end{aligned}$$

Let $N_t^* := \bigcap_{t \geqslant r \geqslant s} U(r,s)^*(B^*)$. Then $(N_t^*)_{t \nearrow \infty}$ is a decreasing net of weak* compact sets. Due to the equality $U(r,s)^* U(t,r)^* = U(t,s)^*$ we have $\frac{1}{M} U(t,s)^* x_t^* \in N_t^*$. Hence, every weak* accumulation point x^* of the net $(\frac{1}{M} U(t,s)^* x_t^*)_{t \geqslant s}$ belongs to the intersection $\bigcap_{t \geqslant s} N_t^* = J_s^*$. This implies

$$\limsup_{t \to \infty} \|U(t,s)x\| \leqslant M \sup_{x^* \in J_s^*} |\langle x, x^* \rangle|$$

which is just the first inequality in (7).

Statement (ii) is an obvious consequence of (i).

Choose any $x^* \in J_0^*$. By definition we find for every $n \in \mathbb{N}$ an element $x_n^* \in B^*$ such that $U(n,0)^* x_n^* = x^*$. Define functions $g_n : \mathbb{R}_- \to X^*$ by

$$g_n(t) := \begin{cases} U(n,-t)^* x_n^*, & t \in (-n, 0], \\ 0, & t \in (-\infty, -n). \end{cases}$$

Then for all $-n \leqslant s \leqslant t \leqslant 0$ we have

$$U(-s,-t)^* g_n(s) = U(-s,-t)^* U(n,-s)^* x_n^* = U(n,-t)^* x_n^* = g_n(t) \quad (8)$$

and, by definition, $g_n(0) = x^*$ and $\|g_n\|_\infty \leqslant M$ for every $n \in \mathbb{N}$.

Let $B_M^* := \{x^* \in X^* : \|x^*\| \leqslant M\}$ with the weak* topology, and consider the product space $(B_M^*)^{\mathbb{R}_-}$ equipped with the topology of pointwise weak* convergence. By the theorems of Banach–Alaoglu and Tikhonov this product space is compact, and thus the set of accumulation points of the sequence $(g_n)_{n \in \mathbb{N}}$ in that space is nonempty. Let g be an accumulation point of $(g_n)_{n \in \mathbb{N}}$.

Let $s \leqslant t \leqslant 0$ be arbitrary. Then equation (8) (for $n \geqslant -s$) and the weak* continuity of $U(-s,-t)^*$ imply that

$$U(-s,-t)^* g(s) = g(t).$$

Hence, the function g is a bounded complete trajectory satisfying $g(0) = x^*$ and we have proved (iii). ∎

Proof of (1) ⇔ (2) ⇔ (3). Equivalence (1) ⇔ (2) follows immediately from Proposition 3.1(ii).

Next, if g is a bounded nontrivial complete trajectory for the evolution family $(U(-s,-t)^*)_{s\leqslant t\leqslant 0}$, then there exists $s_0 \leqslant 0$ such that $g(s_0) \neq 0$. From this and the definition of a complete trajectory it follows that the set $\bigcap_{t \geqslant s_0} U(t,s_0)^*(B^*)$ is nontrivial. In particular, the set J^* is nontrivial, i.e. (2) implies (3).

Assume that J^* is nontrivial. Then there exists $s_0 \geqslant 0$ such that $J^*_{s_0}$ is nontrivial. Choosing $x^* \in J^*_{s_0} \setminus \{0\}$, Proposition 3.1(iii) says that there exists a bounded and necessarily nontrivial complete trajectory g for $(U(s_0 - s, s_0 - t)^*)_{s\leqslant t\leqslant 0}$ with $g(0) = x^*$. Let $h(s) := g(s + s_0)$ when $s \leqslant -s_0$ and $h(s) := U(s_0, -s)^* g(0)$ when $-s_0 < s \leqslant 0$. One easily verifies that h is a bounded nontrivial complete trajectory for $(U(-s,-t)^*)_{s\leqslant t\leqslant 0}$, and hence (3) implies (2). ∎

4. PROOF OF (1) ⇔ (4) ⇔ (5)

The proof of equivalences (1) ⇔ (4) ⇔ (5) is rather straightforward. Nevertheless, the result will be needed subsequently and it gives an abstract argument that one can apply semigroup results on stability to the stability of evolution families.

Notice, however, that the spectrum $\sigma(\mathbf{G}_p)$ always equals a left half-plane and that the Spectral Mapping Theorem holds true for the evolution semigroups $(\mathbf{T}_p(t))_{t\geqslant 0}$, [8, Theorem 3.22; 10, Theorem 3.1; 29, Corollary 2.4]. In particular, the spectral bound $s(\mathbf{G}_p)$, the exponential growth bound $\omega(\mathbf{T}_p)$ and the exponential growth bound $\omega(\mathbf{U})$ coincide. Hence, if the evolution family $\mathbf{U}$ is only bounded but not exponentially stable, then necessarily $\sigma(\mathbf{G}_p) = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant 0\}$. Consequently, the global ABLV-Theorem can, in general, not be applied to bounded evolution semigroups.

Proof of (1) ⇔ (4) ⇔ (5). Assume that the evolution family $\mathbf{U}$ is strongly stable, and let $1 \leqslant p \leqslant \infty$. Let $f \in C_c((0,\infty); X) \subset E_p$ (where $C_c((0,\infty); X)$ is the space of continuous functions having compact support in $(0,\infty)$) and choose $R > 0$ such that $\operatorname{supp} f \subset [0, R]$. The strong continuity of $\mathbf{U}$ and the continuity of f imply that the function $U(R,\cdot)f(\cdot)$ is continuous from $[0,R]$ into X. Hence, that function has compact range in X. By assumption, $\lim_{t\to\infty}\|U(t,R)x_0\|=0$ for all $x_0\in X$. Since $\mathbf{U}$ is uniformly bounded, the convergence is uniform on compact subsets of X, so that

$$\begin{aligned}\lim_{t\to\infty}\sup_{s\in[0,R]}\|U(t+s,R)U(R,s)f(s)\| &= \lim_{t\to\infty}\sup_{s\in[0,R]}\|U(t+s,s)f(s)\|\\ &= \lim_{t\to\infty}\sup_{s\in[0,R]}\|(\mathbf{T}_p(t)f)(t+s)\|\\ &= 0,\end{aligned}$$

which in turn together with the fact that $\operatorname{supp}\mathbf{T}_p(t)f\subset[t,t+R]$ implies

$$\lim_{t\to\infty}\|(\mathbf{T}_p(t)f\|_p=0.$$

The boundedness of the semigroup $(\mathbf{T}_p(t))_{t\geqslant 0}$ and the density of $C_c((0,\infty);X)$ in E_p for all $1\leqslant p\leqslant\infty$ implies that $(\mathbf{T}_p(t))_{t\geqslant 0}$ is stable for all $1\leqslant p\leqslant\infty$. Thus (1) implies (5).

Implication (5) $\Rightarrow$ (4) is trivial.

So assume that $(\mathbf{T}_p(t))_{t\geqslant 0}$ is stable on E_p for some $1\leqslant p\leqslant\infty$. Let $s\geqslant 1$ and let $x\in X$. Define a function $f\in C(\mathbb{R}_+;X)$ by

$$f(r)=\begin{cases} e^{-(r-s)}U(r,s)x, & r\geqslant s,\\ (1-s+r)x, & s-1\leqslant r<s,\\ 0, & r<s-1.\end{cases}$$

The boundedness of $\mathbf{U}$ implies $f\in E_p$. Let $t\geqslant 0$. Then one calculates for all $r\geqslant 0$

$$(\mathbf{T}_p(t)f)(r)=\begin{cases} e^{-(r-t-s)}U(r,s)x, & r\geqslant t+s,\\ (1-s-t+r)U(r,r-t)x, & t+s-1\leqslant r<t+s,\\ 0, & r<t+s-1.\end{cases}$$

If $p=\infty$ then the stability of $\mathbf{T}_\infty$ on E_∞ implies already that

$$\begin{aligned}\lim_{t\to\infty}\|U(t,s)x\| &= \lim_{t\to\infty}\|(\mathbf{T}_\infty(t)f)(t+s)\|\\ &\leqslant \lim_{t\to\infty}\|\mathbf{T}_\infty(t)f\|_\infty=0.\end{aligned}$$

If $p < \infty$ then the stability of $\mathbf{T}_p$ on E_p implies that

$$\begin{aligned}\limsup_{t\to\infty} \int_t^{t+1} \|U(r,s)x\|^p \, dr &\leqslant e^p \limsup_{t\to\infty} \int_t^{t+1} e^{-p(r-t)} \|U(r,s)x\|^p \, dr \\ &= e^p \limsup_{t\to\infty} \int_t^{t+1} \|(\mathbf{T}_p(t-s)f)(r)\|^p \, dr \\ &\leqslant e^p \limsup_{t\to\infty} \|(\mathbf{T}_p(t)f\|_p^p = 0.\end{aligned}$$

This inequality implies $\liminf_{t\to\infty} \|U(t,s)x\| = 0$ which in turn implies, by the boundedness of $\mathbf{U}$ and the fact that $\mathbf{U}$ is an evolution family, $\limsup_{t\to\infty} \|U(t,s)x\| = 0$ (see also inequality (7)).

We have now proved that $\lim_{t\to\infty} \|U(t,s)x\| = 0$ for all $s \geqslant 1$ and all $x \in X$. Let $s \in [0,1]$ and $x \in X$. Then we obtain $\limsup_{t\to\infty} \|U(t,s)x\| = \limsup_{t\to\infty} \|U(t,1)U(1,s)x\| = 0$ by the previous result, so that (4) implies (1). ∎

Remark 4.1. Let $(U(t,s))_{t\geqslant s\geqslant 0}$ be an evolution family on a Banach space X and define a family $(V(t,s))_{t\geqslant s} \subset \mathbf{L}(X)$ by

$$V(t,s) := \begin{cases} U(t,s) & \text{if } 0 \leqslant s \leqslant t, \\ U(t,0) & \text{if } s < 0 \leqslant t, \\ I & \text{if } s \leqslant t < 0. \end{cases}$$

Then $(V(t,s))_{t\geqslant s}$ is a (strongly continuous) evolution family. If $(U(t,s))_{t\geqslant s\geqslant 0}$ is bounded, then $(V(t,s))_{t\geqslant s}$ is bounded, and the family $(\mathbf{S}_p(t))_{t\geqslant 0}$ defined by

$$(\mathbf{S}_p(t)f)(s) := V(s, s-t)f(s-t), \quad t \geqslant 0, \ s \in \mathbb{R}$$

is a (bounded) C_0-semigroup on the space $L^p(\mathbb{R};X) =: F_p$ when $p \in [1,\infty)$ and on the space $C_0(\mathbb{R};X) = F_\infty$ when $p = \infty$.

With this construction and the previous proof of equivalences (1) ⇔ (4) ⇔ (5) the following corollary is immediate.

COROLLARY 4.2. *Let $(U(t,s))_{t\geqslant s\geqslant 0}$ be a bounded evolution family on a Banach space X, and let $(\mathbf{S}_p(t))_{t\geqslant 0}$ be defined as above. Then the following assertions are equivalent*:

(1) *The evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ is strongly stable.*
(10) *The semigroup $(\mathbf{S}_p(t))_{t\geqslant 0}$ is stable for some $1 \leqslant p \leqslant \infty$.*
(11) *The semigroup $(\mathbf{S}_p(t))_{t\geqslant 0}$ is stable for all $1 \leqslant p \leqslant \infty$.*

5. PROOF OF (1) ⇔ (6) ⇔ (7) ⇔ (8)

Before proving equivalences (1) ⇔ (6) ⇔ (7) we will study the generator $\mathbf{G}_p$ of the evolution semigroup $(\mathbf{T}_p(t))_{t\geqslant 0}$ associated with $\mathbf{U}$ on E_p. Let us first recall the following lemma [8, Proposition 3.25; 29, Lemma 1.1].

LEMMA 5.1. *Let $1\leqslant p\leqslant\infty$, and let $f, g \in E_p$. Then $g \in D(\mathbf{G}_p)$ and $\mathbf{G}_p g = f$ if and only if*

$$g(s) = -(U*f)(s) \text{ for almost all } s\geqslant 0. \tag{9}$$

As in [25, Sect. VI.4 and VI.5] we define for a Banach space X the space $L^\infty_{w^*}(\mathbb{R}_+; X^*)$ to be the space of (equivalence classes of) all bounded functions $f : \mathbb{R}_+ \to X^*$ such that $\langle x,f\rangle \in L^\infty(\mathbb{R}_+)$ for all $x \in X$. Here we say that two weak* measurable functions $f, g : \mathbb{R}_+ \to X^*$ are equivalent whenever $\langle x,f\rangle = \langle x, g\rangle$ almost everywhere and for all $x \in X$. Together with the norm

$$\|f\|_{L^\infty_{w^*}} := \sup_{\substack{x\in X\\ \|x\|\leqslant 1}} \|\langle x,f\rangle\|_\infty$$

the space $L^\infty_{w^*}(\mathbb{R}_+; X^*)$ becomes a Banach space. The following representation theorem can be found in [25, Corollary, p. 95].

THEOREM 5.2. *Let X be a Banach space. Then the two spaces $L^1(\mathbb{R}^+; X)^*$ and $L^\infty_{w^*}(\mathbb{R}_+; X^*)$ are isometrically isomorphic. For a step function $f \in L^1(\mathbb{R}_+; X)$ and a function $g \in L^\infty_{w^*}(\mathbb{R}_+; X^*)$ the duality is given by*

$$\langle f, g\rangle = \int_0^\infty \langle f(s), g(s)\rangle\, ds.$$

Proof of (1) ⇔ (6) ⇔ (7). Assume that $\mathbf{U}$ is strongly stable. Then the evolution semigroup $(\mathbf{T}_1(t))_{t\geqslant 0}$ is stable on the space $L^1(\mathbb{R}_+; X) = E_1$ by the equivalence (1) ⇔ (5). By Arendt and Batty [1, Proposition 2.1], $\operatorname{Rg}\mathbf{G}_1$ is dense in E_1, i.e. (1) implies (6).

On the other hand, assume (6) and assume that $\mathbf{U}$ is not stable, or, equivalently, that there exists a bounded nontrivial complete trajectory g for $(U(-s,-t)^*)_{s\leqslant t\leqslant 0}$. Since $\mathbf{U}$ is strongly continuous, g is weak* measurable. Let $\tilde{g} \in L^\infty_{w^*}(\mathbb{R}_+; X^*)$ be defined by $\tilde{g}(t) := g(-t)$. By Theorem 5.2 the space $L^\infty_{w^*}(\mathbb{R}_+; X^*)$ is isometrically isomorphic to $(E_1)^*$ and the duality is given by

$$\langle f, \tilde{g}\rangle = \int_0^\infty \langle f(s), \tilde{g}(s)\rangle\, ds,$$

at least for all step functions $f \in E_1$. Hence, if $f \in E_1$ is a step function, then we find

$$\begin{aligned}\langle f, \mathbf{T}_1(t)^* \tilde{g} \rangle &= \int_0^\infty \langle (\mathbf{T}_1(t) f)(s), \tilde{g}(s) \rangle \, ds \\ &= \int_0^\infty \langle U(t+s,s) f(s), \tilde{g}(t+s) \rangle \, ds \\ &= \int_0^\infty \langle f(s), U(t+s,s)^* \tilde{g}(t+s) \rangle \, ds \\ &= \int_0^\infty \langle f(s), \tilde{g}(s) \rangle \, ds \\ &= \langle f, \tilde{g} \rangle.\end{aligned}$$

It follows from Hahn–Banach and the fact that the step functions are dense in E_1 that $\mathbf{T}_1(t)^* \tilde{g} = \tilde{g}$ for all $t \geqslant 0$. This implies $\tilde{g} \in D(\mathbf{G}_1^*)$ and $\mathbf{G}_1^* \tilde{g} = 0$. Thus, the point 0 belongs to the point spectrum $P\sigma(\mathbf{G}_1^*)$, a contradiction to the assumption that $\operatorname{Rg} \mathbf{G}_1$ is dense in E_1. Thus, the evolution family $\mathbf{U}$ has to be stable, i.e. (6) implies (1).

Equivalence (6) $\Leftrightarrow$ (7) is a direct consequence of Lemma 5.1. One just has to note that the operator $\mathbf{L}$ defined by

$$D(\mathbf{L}) := \{ f \in E_1 \colon -U * f \in E_1 \},$$

$$\mathbf{L} f := -U * f$$

is the (algebraic) inverse of $\mathbf{G}_1$. Hence $F = D(\mathbf{L}) = \operatorname{Rg} \mathbf{G}_1$. ∎

Remark 5.3. One could ask whether equivalence (1) $\Leftrightarrow$ (7) (or at least one implication) remains true if one replaces the Banach space E_1 by a different Banach function space E on $\mathbb{R}_+$. For example, Datko's Theorem 2.3 characterizing *exponential* stability of $\mathbf{U}$ is true for all the Banach spaces E_p.

However, equivalence (1) $\Leftrightarrow$ (7) cannot be true for E_p where $p \in (1, \infty)$. In fact, if $p \in (1, \infty)$ and X is reflexive, then E_p is reflexive [14, Theorem 1, p. 98; 23, Theorems 3.2, 3.4]. The fact that the point spectrum of $\mathbf{G}_p$ is empty and the Mean Ergodic Theorem 2.5 imply that $\mathbf{G}_p$ has dense range in E_p, even if $(U(t,s))_{t \geqslant s \geqslant 0}$ is not a stable evolution family.

An easy counterexample shows that equivalence (1) $\Leftrightarrow$ (7) is also not true for E_∞. In fact, let $X = \mathbb{C}$ and $T(t) = 1$. Then $(T * f)(t) = \int_0^t f(s) \, ds$ and the compactly supported continuous functions such that $\int_0^\infty f(s) \, ds = 0$ are dense in E_∞, but clearly $(T(t))_{t \geqslant 0}$ is not stable.

These examples show that it might be more interesting to generalize implication (7) $\Rightarrow$ (1). A positive result in this direction is given in the next proposition.

PROPOSITION 5.4. *Suppose that* $(T(t))_{t\geqslant 0}$ *is a bounded* C_0*-semigroup on a Banach space* X *and that the set*

$$\{f \in \mathrm{BUC}(\mathbb{R}_+; X) : T*f \in \mathrm{BUC}(\mathbb{R}_+; X))\}$$

is dense in $\mathrm{BUC}(\mathbb{R}_+; X)$. *Then* $(T(t))_{t\geqslant 0}$ *is stable.*

Proof. Take $x \in X$ and let $g(t) = T(t)x$ for $t\geqslant 0$. Given $\varepsilon > 0$, there exists $f \in \mathrm{BUC}(\mathbb{R}_+; X)$ such that $\|g - f\|_\infty < \varepsilon$ and $T*f$ is bounded. Then

$$\|(T*g)(t) - (T*f)(t)\| \leqslant M\varepsilon t,$$

where $M = \sup_{s\geqslant 0}\|T(s)\|$. Thus,

$$\|(T*g)(t)\| \leqslant \|(T*f)(t)\| + M\varepsilon t.$$

But for $t\geqslant 0$ we have

$$(T*g)(t) = \int_0^t T(t-s)T(s)x\,ds = tT(t)x.$$

Hence

$$\|T(t)x\| \leqslant \frac{\|(T*f)(t)\|}{t} + M\varepsilon.$$

Letting $t \to \infty$,

$$\limsup_{t\to\infty} \|T(t)x\| \leqslant M\varepsilon.$$

Sine $\varepsilon > 0$ was arbitrary, the claim follows. ∎

Proof of (7) $\Leftrightarrow$ (8) (Theorem 2.4 (i)). Let $f \in L^1(\mathbb{R}_+; X)$ be such that $T*f \in L^1(\mathbb{R}_+; X)$. Let $t\geqslant 0$. The equality

$$\begin{aligned}\int_0^t e^{-i\beta s}T*f(s)\,ds &= \int_0^t e^{-i\beta s}\int_0^s T(s-r)f(r)\,dr\,ds\\ &= \int_0^t \int_0^{t-r} e^{-i\beta s}T(s)e^{-i\beta r}f(r)\,ds\,dr\end{aligned}$$

and the fact that $\int_0^{t-r} e^{-i\beta s}T(s)e^{-i\beta r}f(r)\,ds \in D(A)$ with

$$(A - i\beta)\int_0^{t-r} e^{-i\beta s}T(s)e^{-i\beta r}f(r)\,ds = e^{-i\beta t}T(t-r)f(r) - e^{-i\beta r}f(r)$$

imply that

$$r \mapsto \int_0^{t-r} e^{-i\beta s}T(s)e^{-i\beta r}f(r)\,ds$$

is locally integrable with values in $D(A)$. Hence $\int_0^t e^{-i\beta s}T*f(s)\,ds \in D(A)$ and

$$(A - i\beta)\int_0^t e^{-i\beta s}T*f(s)\,ds = e^{-i\beta t}T*f(t) - \int_0^t e^{-i\beta s}f(s)\,ds.$$

Assume that (7) holds. From $T*f \in L^1(\mathbb{R}_+;X)$ we obtain on the one hand $\liminf_{t\to\infty}\|T*f(t)\| = 0$ and on the other hand that

$$\lim_{t\to\infty}\int_0^t e^{-i\beta s}T*f(s)\,ds = \mathbf{F}(T*f)(\beta)$$

exists in X. Since $f \in L^1(\mathbb{R}_+;X)$, the limit

$$\lim_{t\to\infty}\int_0^t e^{-i\beta s}f(s)\,ds = \mathbf{F}f(\beta)$$

exists, too, and the closedness of the operator $i\beta - A$ implies $\mathbf{F}(T*f)\times(\beta) \in D(A)$ and

$$(i\beta - A)\mathbf{F}(T*f)(\beta) = \mathbf{F}f(\beta).$$

Since (7) is equivalent to (1), $P\sigma(A) \cap i\mathbb{R} = \emptyset$ by Arendt and Batty [1, Proposition 2.1]. Thus, (7) implies (8).

In order to show the converse, let $f \in D(\mathbf{M})$ and $g \in A_+(\mathbb{R};X)$ be such that $\mathbf{M}f = g$. By definition, there exist $\tilde{f},\tilde{g} \in L^1(\mathbb{R}_+;X)$ such that $\mathbf{F}\tilde{f} = f$ and $\mathbf{F}\tilde{g} = g$. Observe that $\mathbf{F}(e^{-\alpha\cdot}T(\cdot)*\tilde{f}) = R(\alpha + i\cdot, A)f \in A_+(\mathbb{R};X)$ for all $\alpha > 0$, and that

$$\begin{aligned}\mathbf{F}(\alpha e^{-\alpha\cdot}T(\cdot)*\tilde{g}) &= \alpha R(\alpha + i\cdot, A)g\\ &= g - R(\alpha + i\cdot, A)f\\ &= \mathbf{F}(\tilde{g} - (e^{-\alpha\cdot}T(\cdot))*\tilde{f}).\end{aligned}$$

By the uniqueness of the Fourier transform we obtain that for almost all $t \in \mathbb{R}_+$ the equality

$$\alpha\int_0^t e^{-\alpha(t-s)}T(t-s)\tilde{g}(s)\,ds = \tilde{g}(t) - \int_0^t e^{-\alpha(t-s)}T(t-s)\tilde{f}(s)\,ds$$

holds. Letting α tend to 0 on both sides of this equation yields

$$\tilde{g}(t) = \int_0^t T(t-s)\tilde{f}(s)\, ds.$$

Now, if (8) holds, then, by the definition of the norm on $A_+(\mathbb{R}; X)$, the set F of all $\tilde{f} \in L^1(\mathbb{R}_+; X)$ such that $T * \tilde{f}$ belongs also to $L^1(\mathbb{R}_+; X)$ is dense in $L^1(\mathbb{R}_+; X)$. Hence, (8) implies (7). ∎

6. PROOF OF (9) ⇒ (3) ⇔ (1)

In this section we will assume that A is the generator of a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X. We will study the asymptotic behaviour of individual orbits by means of complete bounded trajectories and the set J^* defined in Eq. (4). We show in Theorem 6.3 that $\lim_{t\to\infty} \|T \times (t)x\| = 0$ when x is in a certain subspace of X defined in terms of the ranges of $i\beta - A$. One immediate corollary is the implication from (9) to (3) (Theorem 2.4(ii)), hence, by Theorem 2.2, to strong stability of the semigroup. We will show below how this result is connected to previous results in this domain.

Given a bounded measurable function $F : \mathbb{R} \to X$ we define the Carleman transform $\hat{F}$ by

$$\hat{F}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} F(t)\, dt, & \operatorname{Re}\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} F(t)\, dt, & \operatorname{Re}\lambda < 0. \end{cases}$$

Clearly, the Carleman transform is an analytic function in $\mathbb{C}\setminus i\mathbb{R}$. If, as in the following, the function F is merely weakly or weak* measurable then the Carleman transform will be defined by applying functionals first. This does not change the fact that $\hat{F}$ is analytic.

Lemma 6.1. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on a Banach space X with generator A, and let $g : \mathbb{R}_- \to X^*$ be a bounded complete trajectory for $(T(t-s)^*)_{s\leqslant t\leqslant 0}$. Let $F(t) := g(t)$ when $t\leqslant 0$, and $F(t) = T(t)^* g(0)$ when $t\geqslant 0$. Denote by $\hat{F}$ the Carleman transform of F. Then, for all $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}\setminus i\mathbb{R}$, the following identity holds*:

$$\hat{F}(\mu) = R(\lambda, A^*)F(0) + (\lambda - \mu)R(\lambda, A^*)\hat{F}(\mu). \tag{10}$$

Proof. Let $\lambda \in \mathbb{C}_+$. If $\mu \in \mathbb{C}_+$, then $\hat{F}(\mu) = R(\mu, A^*)F(0)$ and (10) is just the resolvent identity. So let $\mu \in \mathbb{C}_-$ and $x \in X$. Then

$$\begin{aligned} &\langle x, (\lambda-\mu)R(\lambda, A^*)\hat{F}(\mu)\rangle \\ &\quad = -(\lambda-\mu)\int_0^\infty e^{\mu t}\langle x, R(\lambda, A^*)F(-t)\rangle\, dt \end{aligned}$$

$$
\begin{aligned}
&= -(\lambda-\mu)\int_0^\infty \int_0^\infty e^{\mu t}e^{-\lambda s}\langle x, T(s)^* F(-t)\rangle \, ds\, dt \\
&= -(\lambda-\mu)\int_0^\infty \int_0^\infty e^{\mu t}e^{-\lambda s}\langle x, F(s-t)\rangle \, ds\, dt \\
&= -(\lambda-\mu)\int_0^\infty \int_{-t}^\infty e^{\mu t}e^{-\lambda(s+t)}\langle x, F(s)\rangle \, ds\, dt \\
&= -(\lambda-\mu)\int_0^\infty e^{-(\lambda-\mu)t}\int_{-t}^0 e^{-\lambda s}\langle x, F(s)\rangle \, ds\, dt \\
&\quad -(\lambda-\mu)\int_0^\infty e^{-(\lambda-\mu)t}\langle x, R(\lambda, A^*)F(0)\rangle \, dt \\
&= -\int_0^\infty e^{\mu t}\langle x, F(-t)\rangle \, dt - \langle x, R(\lambda, A^*)F(0)\rangle \\
&= \langle x, \hat{F}(\mu)\rangle - \langle x, R(\lambda, A^*)F(0)\rangle.
\end{aligned}
$$

Since this equality holds for every $x \in X$, the claim follows. ∎

For the proof of implication (9) $\Rightarrow$ (3) $\Leftrightarrow$ (1) we have to recall the following result from complex function theory [34, Lemma 4.6; 36, Theorem E]. Note that the formulation of [36, Theorem E] is not correct and that a proof of the correct statement below (using results from [15, 33]) can be found in [34, Lemma 4.4]. However, it is not clear from his proof whether Wolf had Theorem 6.2 in mind.

THEOREM 6.2 (Edge-of-the-Wedge). *Let* $Q_b := \{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1, |\mathrm{Im}\, z| < b\}$ *be a rectangle in the complex domain, and let* $f : Q_b \setminus i\mathbb{R} \to \mathbb{C}$ *be an analytic function. Define* $g(\alpha + i\beta) := f(\alpha + i\beta) - f(-\alpha + i\beta) (\alpha + i\beta \in Q_b \cap \mathbb{C}_+)$ *and assume that*

$$\sup_{|\beta|<b} |g(\alpha + i\beta)| = O\left(\frac{1}{\alpha}\right) \quad \text{as } \alpha \searrow 0 \tag{11}$$

and

$$\lim_{\alpha \to 0} |g(\alpha + i\beta)| = 0 \quad \text{for all } |\beta| < b. \tag{12}$$

Then f has an analytic extension to Q_b.

In the next theorem the set J^* will be defined as in Eq. (4). Note that in the semigroup case the sets J_s^* defined in Eq. (6) do not depend on $s \geqslant 0$ and hence $J^* = J_0^*$. The following theorem is the local counterpart of implication (9) $\Rightarrow$ (3) $\Leftrightarrow$ (1).

THEOREM 6.3. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on a Banach space X with generator A, and define*

$$J := \bigcap_{\beta\in\mathbb{R}} (i\beta - A)(\overline{\mathrm{Rg}(i\beta - A)} \cap D(A)).$$

Then:

(i) *For every $x \in J$ and $x^* \in J^*$, $\langle x, x^*\rangle = 0$, i.e. $J^* \subset J^\perp$.*
(ii) *For every $x \in J$ one has $\lim_{t\to\infty} \|T(t)x\| = 0$.*

Proof. Let $x^* \in J^*$. By Proposition 3.1(iii), there exists a bounded complete trajectory $g : \mathbb{R}_- \to X^*$ for $(T(t-s)^*)_{s\leqslant t\leqslant 0}$ such that $g(0) = x^*$. Define $F(t) = g(t)$ for $t\leqslant 0$ and $F(t) = T(t)^* x^*$ for $t\geqslant 0$. Observe that then the function F is bounded and it satisfies the relation $F(t+s) = T(t)^* F(s)$ for all $s \in \mathbb{R}$ and all $t\geqslant 0$. Let $x \in J$, and let $f(t) := \langle x, F(t)\rangle$. Then $f \in L^\infty(\mathbb{R})$ and the Carleman transform is given by $\hat{f}(\lambda) = \langle x, \hat{F}(\lambda)\rangle$ $(\lambda \in \mathbb{C}\setminus i\mathbb{R})$.

Let $\beta \in \mathbb{R}$. Applying Lemma 6.1 twice, we obtain

$$\begin{aligned}
&\lim_{\alpha\to 0} |\hat{f}(\alpha + i\beta) - \hat{f}(-\alpha + i\beta)| \\
&\quad= \lim_{\alpha\to 0} |\langle x, \hat{F}(\alpha + i\beta) - \hat{F}(-\alpha + i\beta)\rangle| \\
&\quad= \lim_{\alpha\to 0} |\langle x, 2\alpha R(\alpha + i\beta, A^*)\hat{F}(-\alpha + i\beta)\rangle| \\
&\quad\leqslant \lim_{\alpha\to 0} |\langle x, 2\alpha R(\alpha + i\beta, A^*)^2 x^*\rangle| \\
&\qquad+ \lim_{\alpha\to 0} |\langle x, 4\alpha^2 R(\alpha + i\beta, A^*)^2 \hat{F}(-\alpha + i\beta)\rangle| \\
&\quad= 2\lim_{\alpha\to 0} |\langle \alpha R(\alpha + i\beta, A)^2 x, x^*\rangle| \\
&\qquad+ 4\lim_{\alpha\to 0} |\langle \alpha R(\alpha + i\beta, A)^2 x, \alpha\hat{F}(-\alpha + i\beta)\rangle|.
\end{aligned}$$

The boundedness of F implies $\limsup_{\alpha\to 0} \|\alpha\hat{F}(-\alpha + i\beta)\| < \infty$. Moreover, by assumption, there exists $y \in \overline{\mathrm{Rg}(i\beta - A)} \cap D(A)$ such that $(i\beta - A)y = x$. This and the Mean Ergodic Theorem 2.5 (applied to the semigroup) $(e^{-i\beta t}T(t))_{t\geqslant 0})$ imply

$$\begin{aligned}
\lim_{\alpha\to 0} \alpha R(\alpha + i\beta, A)^2 x &= \lim_{\alpha\to 0} \alpha R(\alpha + i\beta, A)^2 (i\beta - A)y \\
&= \lim_{\alpha\to 0} \alpha R(\alpha + i\beta, A)y \\
&\quad - \lim_{\alpha\to 0} \alpha^2 R(\alpha + i\beta, A)^2 y = 0.
\end{aligned}$$

Hence,

$$\lim_{\alpha\to 0} |\hat{f}(\alpha + i\beta) - \hat{f}(-\alpha + i\beta)| = 0 \quad \text{for all } \beta \in \mathbb{R}$$

and from Theorem 6.2 we obtain that the Carleman transform $\hat{f}$ extends to an entire function. By Prüss [31, Proposition 0.5(ii)] this implies that $f = 0$. In particular, we have $f(0) = \langle x, x^* \rangle = 0$, which proves (i).

If x belongs to J, then, by statement (i), it annihilates the set J^*. Statement (ii) follows thus from Proposition 3.1(ii). ∎

Remark 6.4. Let $Y \subset X$ be a **T**-invariant subspace and let B be the restriction of A to Y. Then, if J is defined as in Theorem 6.3,

$$J \cap Y = \bigcap_{\beta\in\mathbb{R}} (i\beta - B)(\overline{\mathrm{Rg}(i\beta - B)} \cap D(B)). \tag{13}$$

To see this it suffices to show that

$$A(\overline{\mathrm{Rg}\,A} \cap D(A)) \cap Y = B(\overline{\mathrm{Rg}\,B} \cap D(B)).$$

Of course, the inclusion is obvious. So let $x \in A(\overline{\mathrm{Rg}\,A} \cap D(A)) \cap Y$. There exists $y \in \overline{\mathrm{Rg}\,A} \cap D(A)$ such that $Ay = x$. By the Mean Ergodic Theorem 2.5, $\lim_{\lambda\to 0} \lambda R(\lambda, A)y = 0$, so

$$\begin{aligned}\lim_{\lambda\to 0} R(\lambda, B)x &= \lim_{\lambda\to 0} R(\lambda, A)x = \lim_{\lambda\to 0} AR(\lambda, A)y \\ &= \lim_{\lambda\to 0} \lambda R(\lambda, A)y - y = -y.\end{aligned}$$

Hence, $y \in Y$ and $y = -\lim_{\lambda\to 0} BR(\lambda, B)y \in \overline{\mathrm{Rg}\,B}$. This proves the second inclusion.

Equation (13) shows that in a Banach space Y we cannot produce a larger set of initial values for which Theorem 6.3(ii) holds just by extending the generator B to an operator A defined on the larger space X and generating again a bounded C_0-semigroup.

Proof of (9) ⇒ (1) (Theorem 2.4(ii)). The assumption $\overline{\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}\,(i\beta - A)} = X$ implies in particular that $\mathrm{Rg}\,(i\beta - A)$ is dense in X for all $\beta \in \mathbb{R}$. Hence

$$\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}(i\beta - A) = \bigcap_{\beta\in\mathbb{R}} (i\beta - A)(\overline{\mathrm{Rg}(i\beta - A)}) \cap D(A)).$$

By Theorem 6.3(ii), the orbit $T(\cdot)x$ is stable for all $x \in \bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}(i\beta - A)$. The density of this set implies that $(T(t))_{t\geqslant 0}$ is stable. ∎

In the Introduction we mentioned the ABLV-Theorem on stability of bounded C_0-semigroups. It is now an easy corollary of Theorem 2.4(ii).

COROLLARY 6.5 (ABLV). *Let A be the generator of a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X.* If *$\sigma(A)\cap i\mathbb{R}$ is countable, and if* $\mathrm{Rg}(i\beta - A)$ *is dense in X for all $\beta\in\mathbb{R}$ (or, equivalently, A^* has no point spectrum on $i\mathbb{R}$), then $(T(t))_{t\geqslant 0}$ is stable.*

Proof. One has to note that $\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}\,(i\beta - A)$ is dense in X by assumption and the Mittag–Leffler Theorem (cf. [18, Lemma 3.1; 19, 20]). The claim follows from Theorem 2.4(ii). ∎

Remark 6.6. We do not see any method to deduce the version of the ABLV-Theorem for individual orbits of bounded semigroups [5, Theorem 3.4] from Theorem 6.3.

From Theorem 6.3 and the Mean Ergodic Theorem 2.5 we can also immediately deduce the following property of generators of semigroups for which no nontrivial orbit tends to 0.

COROLLARY 6.7. *Assume that A is the generator of a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ such that for all $x\in X\setminus\{0\}$ one has* $\limsup_{t\to\infty}\|T(t)x\| > 0$. *Then*

$$\bigcap_{\beta\in\mathbb{R}} (i\beta - A)(\overline{\mathrm{Rg}(i\beta - A)}\cap D(A)) = \{0\}.$$

In particular, when X is reflexive, then $\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}(i\beta - A) = \{0\}$.

Remark 6.8. (a) In [4, Sect. 4] a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ satisfying the assumption of Corollary 6.7 is called *trivially asymptotically stable.* Obviously, semigroups of isometries are trivially asymptotically stable, but the class of trivially asymptotically stable C_0-semigroups is strictly larger than the class of C_0-semigroups of isometries, [4, Examples 4.1, 4.2].

(b) Recall that an operator $A\in\mathbf{L}(X)$ admits a $C^k(\mathbb{C})$-*calculus* (resp. $C^k(\mathbb{R})$-*calculus*) for some $k\in\mathbb{N}_0$ if there exists a continuous algebra homomorphism $T: C^k(\mathbb{C})\to\mathbf{L}(X)$ (resp. $T: C^k(\mathbb{R})\to\mathbf{L}(X)$) and $\Omega\subset\mathbb{C}$ (resp. $\Omega\subset\mathbb{R}$) open and bounded such that

(i) $T(1) = I$ and $T(\mathrm{id}) = A$, and
(ii) $\|T(f)\|_{\mathbf{L}(X)}\leqslant M\|f\|_{k,\Omega}$ for some $M\geqslant 0$, where

$$\|f\|_{k,\Omega} := \sum_{|\alpha|\leqslant k}\frac{1}{\alpha!}\sup_{z\in\Omega}|D^\alpha f(z)|.$$

If $A\in\mathbf{L}(X)$ admits a $C^k(\mathbb{C})$-calculus (resp. $C^k(\mathbb{R})$-calculus), then, by [11, Theorem 3.1, Remark 3.2(b)] (see also [21, Theorem 3.2]), for every $m \geqslant k+3$ (resp. $m \geqslant k+2$) one has

$$\bigcap_{\lambda \in \mathbb{C}} \mathrm{Rg}(\lambda - A)^m = \{0\}. \tag{14}$$

By Corollary 6.7, this result can be improved in several directions if, in addition, iA generates a trivially asymptotically stable bounded semigroup and if X is reflexive: first, we do not assume A to be bounded, second, the exponent $m = 1$, and third, we only take the intersection for λ on the real axis.

Let us point out that the best possible exponent $m = 2$ in [11] is attained when A admits a $C(\mathbb{R})$-calculus. By Kantorovitz [26, Lemma 2.2], the spectrum $\sigma(A)$ is then a subset of $\mathbb{R}$. The following is an example of an operator $A \in \mathbf{L}(X)$ such that $\sigma(A) \subset \mathbb{R}$, iA generates a group of isometries (in particular a trivially asymptotically stable semigroup), but A does not admit a $C(\mathbb{R})$-calculus.

In fact, let X be the space of all functions $f \in \mathrm{BUC}(\mathbb{R})$ such that f extends to an entire function of exponential type less or equal to 1. Then X is a closed subspace of $\mathrm{BUC}(\mathbb{R})$. Moreover, the operator $A = i\frac{d}{dx}$ is bounded by Bernstein's inequality [16, p. 227]. The operator iA is the restriction to X of the generator of the C_0-group of left-shifts on $\mathrm{BUC}(\mathbb{R})$, i.e. $(e^{itA})_{t\in\mathbb{R}}$ is a group of isometries on X. However, since, by [22, p. 202], the function $t \to e^{itA^2}$ is unbounded, the operator A cannot admit a $C(\mathbb{R})$-calculus. Thus the Curtis–Neumann result is not applicable with $m = 2$, but Corollary 6.7 is.

The same example shows that Corollary 6.7 cannot be improved to $\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}(\beta - A) = \{0\}$ in nonreflexive Banach spaces. For example, the function f defined by $f(x) := (\frac{\sin x}{x})^2 (x \in \mathbb{R})$ belongs to $\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}(\beta - A)$. Indeed, when we put $g_\beta(x) = ie^{i\beta x} \int_0^x e^{-i\beta y} f(y)\, dy$ ($\beta \in \mathbb{R}$, $x \in \mathbb{R}$), then $g_\beta \in X$ and $(\beta - A)g_\beta = f$.

7. THE DISCRETE CASE

Many previous articles have shown that quite often it is possible to formulate results on the asymptotic behaviour in a similar way both for C_0-semigroups and discrete operator semigroups. The same holds for implication (9) $\Rightarrow$ (1).

Let $T \in \mathbf{L}(X)$ be a *power bounded* operator, i.e. $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. We say that the operator semigroup $(T^n)_{n\in\mathbb{N}}$ is (*strongly*) *stable* if $\lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in X$.

THEOREM 7.1. *Let $T \in \mathbf{L}(X)$ be a power bounded operator. If*

$$\overline{\bigcap_{\theta\in[0,2\pi]} \mathrm{Rg}(e^{i\theta} - T)} = X, \tag{15}$$

then the semigroup $(T^n)_{n\in\mathbb{N}}$ is stable.

We only sketch the proof which uses again the equivalence of stability and the nonexistence of bounded nontrivial complete trajectories for the dual semigroup $(T^{*n})_{n\in\mathbb{N}}$, which are now sequences $(g_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}; X^*)$ such that $g_{n+m} = T^{*n} g_m$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. As pointed out above his equivalence had actually been proved in [13, 27] for this special case.

Then the proof of Theorem 7.1 follows the lines of the proof of implication (9) $\Rightarrow$ (1) in Section 6, replacing the generator A by the operator T, and the Carleman transform (representing the resolvent of the operator A^*) by its discrete analogue

$$\hat{g}(\lambda) := \begin{cases} \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} g(n), & |\lambda| > 1, \\ -\sum_{n=-\infty}^{-1} \frac{1}{\lambda^{n+1}} g(n), & |\lambda| < 1, \end{cases}$$

where $g = (g_n)_{n\in\mathbb{Z}}$ is a bounded complete trajectory for $(T^{*n})_{n\in\mathbb{N}}$. The function $\hat{g}$ thus defined is analytic in $\mathbb{C}\setminus\{\lambda\colon\ |\lambda| = 1\}$, and $\hat{g}(\lambda) = R(\lambda, T^*)g(0)$ for $|\lambda| > 1$. Similarly to the proof of Theorem 6.3, the function $\hat{g}$ can be extended to an entire function by the assumption in Theorem 7.1 and by the corresponding analogue of the Edge-of-the-Wedge Theorem 6.2 for the unit circle [34].

Once $\hat{g}$ has been extended to an entire function, the fact that $\lim_{|\lambda|\to\infty} \|\hat{g}(\lambda)\| = 0$ and Liouville's Theorem imply that $\hat{g}$ and hence g are trivial. Hence, there does not exist a bounded nontrivial complete trajectory for $(T^{*n})_{n\in\mathbb{N}}$.

We believe that it is possible to formulate Theorem 2.2 for the discrete case by changing the notions properly, but it is not the purpose of this article to do that explicitly.

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