Limit order books, diffusion approximations and reflected SPDEs: from microscopic to macroscopic models

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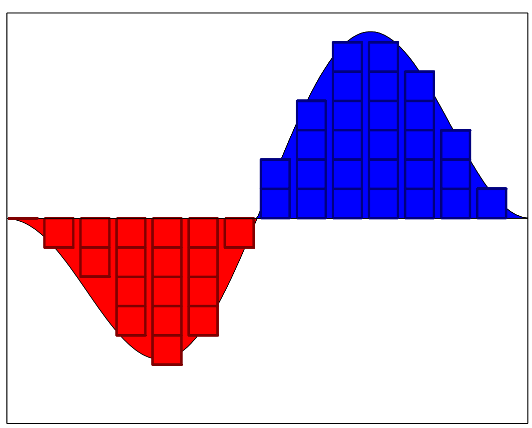
Abstract

Motivated by a zero-intelligence approach, the aim of this thesis is to unify the microscopic (discrete price and volume), mesoscopic (discrete price and continuous volume) and macroscopic (continuous price and volume) frameworks of limit order books, with a view to providing a novel yet analytically tractable description of their behaviour in a high to ultra high-frequency setting. Starting with the canonical microscopic framework, the first part of the thesis examines the limiting behaviour of the order book process when order arrival and cancellation rates are sent to infinity and when volumes are considered to be of infinitesimal size. Mathematically speaking, this amounts to establishing the weak convergence of a discrete-space process to a mesoscopic diffusion limit. This step is initially carried out in a reduced-form context, in other words, by simply looking at the best bid and ask queues, before the procedure is extended to the whole book. This subsequently leads us to the second part of the thesis, which is devoted to the transition between mesoscopic and macroscopic models of limit order books, where the general idea is to send the tick size to zero, or equivalently, to consider infinitely many price levels. The macroscopic limit is then described in terms of reflected SPDEs which typically arise in stochastic interface models. Numerical applications are finally presented, notably via the simulation of the mesocopic and macroscopic limits, which can be used as market simulators for short-term price prediction or optimal execution strategies.

Key words: *limit order book, zero-intelligence model, heavy traffic limit, diffusion approximation, local time, elastic Brownian motion, relative/absolute price grid, stochastic interface model, reflected SPDE, invariant measure, global/local order flow imbalance, mesoscopic/macroscopic system.*



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Contents

Notation for function spaces

Let E and F be arbitrary topological spaces. We define the following spaces:

$\mathbb{B}(E, F)$ - bounded functions from E to F.
$\mathbb{C}(E, F)$ - continuous functions from E to F.
$\mathbb{C}_b(E, F)$ - continuous bounded functions from E to F.
$\mathbb{C}^k(E, F)$ - k times continuously differentiable functions from E to F ($k \geq 1$).
$\mathbb{C}_b^k(E, F)$ - k times continuously differentiable bounded functions from E to F with bounded derivatives up to order k ($k \geq 1$).
$\mathbb{C}_{b,\infty}^k(E, F)$ - k times continuously differentiable bounded and vanishing at infinity functions from E to F with bounded and vanishing at infinity derivatives up to order k ($k \geq 1$).
$\mathbb{C}_\infty(E, F)$ - continuous functions from E to F vanishing at infinity.
$\mathbb{C}^{k,\sigma}(E, F)$ - Hölder functions from E to F ($k \geq 0$ is the derivative order and $0 < \sigma < 1$ is the exponent).
$\mathbb{D}(E, F)$ - Skorokhod space of càdlàg functions from E to F.
$\mathbb{L}^2(E, F)$ - Square integrable functions from E to F.

When $F = \mathbb{R}$, we omit the second argument in all the above cases.

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1 Introduction

1.1 A brief overview of limit order books

The rising prominence of order-driven markets in recent years has generated a significant interest in the modeling of limit order books. In such markets, three specific types of orders can be submitted. On the one hand, limit orders are orders to buy or sell a designated number of shares at a specified price or better. On the other hand, market orders are orders to immediately buy or sell a certain number of shares at the best available price. Finally, cancellation orders enable a market participant to cancel an existing limit order. Whilst market orders are instantly matched against the best available limit orders of the opposite quote, the collection of unexecuted and uncancelled limit orders is recorded in the limit order book, according to price and time priority. However, as pointed out by Gould et al. in [28], limit orders and market orders are intrinsically the same, given that the only difference is based on whether or not their submission results in an immediate matching. In order-driven markets, one can establish a conceptual distinction between market participants who submit limit orders, also known as liquidity makers, and those who submit market orders, the liquidity takers. In this respect, Glosten [27] perceives limit order books as a means of transferring liquidity from so-called patient market participants to less patient ones.

In limit order book terminology, the bid refers to the price of the best limit buy order, whereas the ask designates the price of the best sell order. Two other quantities of interest are the mid, which is simply the average of the bid and ask, and the spread, which corresponds to the difference between the ask and the bid. Finally, the smallest amount of asset that can be traded, referred to as the lot size, and the smallest price increment in the market, called the tick size, constitute what we call the resolution parameters of a limit order book.

Mathematically speaking, the limit order book can effectively be seen as a high-dimensional queueing system, where each queue is made up of limit orders at a specified price. This leads us to the notion of *absolute price grid*, where every single possible price level is taken into account. However, rather than assigning a queue to each price level, it is sometimes sufficient to first consider the best bid and ask queues (which correspond to different price levels over time) and subsequently track a fixed number of queues in relation to their distance to the best opposite quote. This is known as the *relative price grid* setting, and turns out to be especially pertinent when one is simply interested in the behaviour of the book in the vicinity of the best quotes (although in theory, one can examine the book at an arbitrarily large distance from the best quotes). This naturally brings us to the question of price evolution in order-driven markets, which happens to be dependent upon the current state of the limit order book as well as the incoming order flow.

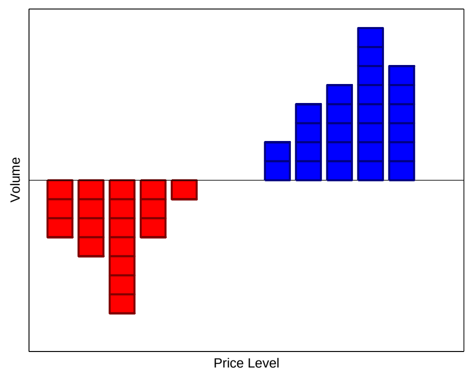


Figure 1.1: Initial state of a simplified limit order book (bid side in red and ask side in blue).

Given the initial state of the book illustrated above, suppose that an ask limit order of size 1 arrives inside the spread. This will cause the ask to decrease by one tick, as shown below:

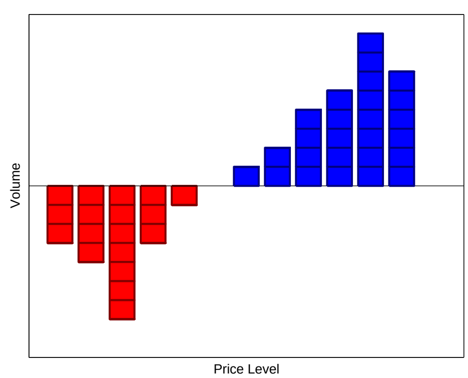


Figure 1.2: Ask decrease caused by an ask limit order inside the spread.

Now, suppose that an ask market order of size 2 is submitted. This will entail a bid decrease, as illustrated below:

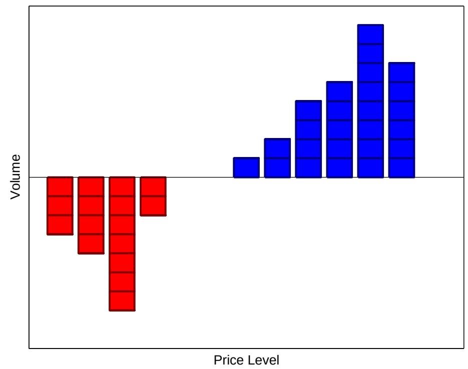


Figure 1.3: Bid decrease caused by an ask market order.

Finally, assume that a bid market order of size 2 is submitted. The ask will then naturally increase:

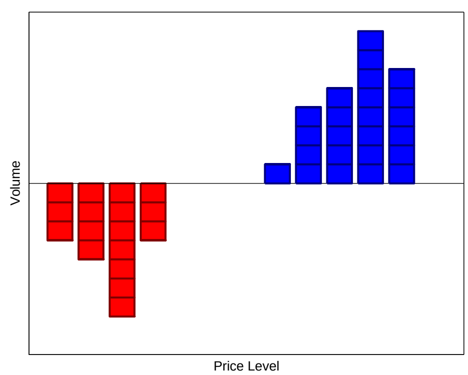


Figure 1.4: Ask increase caused by a bid market order.

Note that we have deliberately chosen three examples where there is no *exact* depletion of either one of the best queues. Indeed, it must be stated that within the literature of limit order book modeling, there is no clear consensus as to what *precisely* happens when one of the two best queues is depleted. For instance, according to Cont and de Larrard [13], one of the two following events must happen first:

(i) The best bid queue is depleted causing an *immediate* bid decrease.

(ii) The best ask queue is depleted causing an *immediate* ask increase.

In this setting, price movements (i.e. bid or ask movements) are therefore a result of a "race to the bottom" between the best bid and ask queues. However, as pointed out by Avellaneda et al. [5], the predictions of mathematical order book models based on this viewpoint are not consistent with empirical observations. More precisely, it is shown that the probability of an upward price move conditionally on the best ask queue being much smaller than the best bid queue does not increase to 1 as the best ask queue goes to zero. Two potential reasons behind these observations are then put forward. First, modern day markets are typically fragmented, with liquidity being posted on several exchanges. According to the Regulation National Market System (also known as Reg NMS) promulgated by the Securities and Exchange Commission, all market orders in the US are required to be routed to the venue with the best possible price. Consequently, if one of the two best queues is depleted, the price will not necessarily change immediately, as an order at the price level of the depleted queue may still exist at an alternative venue. Second, the existence of so-called iceberg orders (which are divided into a visible portion reported to all market participants, and a hidden portion, which is not, until the visible part of the order has effectively been fulfilled) means that the best quotes can be immediately replenished once they have been depleted. We can also add a third reason, somewhat related to the previous two, which is based on the idea that in an ultra high-frequency framework, a new limit order can still be posted at a depleted best level before the price can be updated across all the venues, thus maintaining the price at the same level.

1.2 Modeling approaches and selected literature

The considerable interest in limit order book modeling which has developed in recent years can essentially be attributed to two distinct reasons: on the one hand, the emergence of order-driven markets where the limit order book is effectively a central object of interest, and on the other hand, the increased availability of high-quality financial time series enabling one to conduct statistical analysis of various limit order book mechanisms. As a matter of fact, limit order book models have traditionally been developed by two independent schools of thought. The first one, initiated by economists, has been based on a perfect-rationality approach, while the second one, led by econophysicists and mathematicians, has been associated with a zero-intelligence framework.

Within the realm of perfect-rationality, where order flow is considered as static, the central issue has generally been related to agents conducting strategic trading decisions which maximise their individual utility. Notable models in the perfect-rationality literature include those due to Mendelson [44], who analysed the statistical behaviour of the market from a clearing house perspective, Kyle [39], where the question of insider trading with sequential auctions is addressed, Roşu [50], who introduces the notion of optimal choice between market orders and limit orders, and Almgren and Chriss [2], where the idea of optimal execution of large orders is developed.

As far as the zero-intelligence approach is concerned, the order flow is treated as dynamic and the focus is shifted to the random nature of order arrivals. One of the first models dealing with this was developed by Kruk [38], where he established a functional limit theorem for the order flow in a continuous double-auction setting. More recently, there has been a siginificant interest in modeling the book as a multiclass queueing system (Abergel and Jedidi [1], Blanchet and Chen [8], Cont et al. [15], Muni Toke [46], Huang et al. [31]), where theoretical results from the fields of queueing systems (especially Whitt [54]) and Markov chains have proven to be most indispensable. In order to deal with a market where orders are submitted at high frequency, Cont and de Larrard [14] considered a heavy traffic approximation of the order book process from a queueing theory perspective. One interesting feature in this paper is the state reduction of the order book (originally introduced in Cont and de Larrard [13]), where the emphasis is exclusively laid on the best bid and ask queues: once the bid (or ask) queue has been depleted, it takes a new value drawn from a stationary distribution representing the depth of the order book after a price change. Another paper of interest in the high-frequency setting is the work of Lakner et al. [40], where the main novelty here is to view the order book as a measure-valued process. Over the years, there have also been numerous attempts to establish PDE/SPDE limits of order books. One of the first papers to explore this direction is the one by Bovier and Černý [10], who view the book as a two-species interacting particle system and go on to prove a hydrodynamic limit for the associated empirical process. This particle system approach is also used by Dai et al. in [17] (where their limit is actually an ODE with a constant price) and by Cont in [16], whose free boundary limit is in the same spirit as the pioneering work of Lasry and Lions [41] in mean field games. In [6] and [30], Horst et al. establish functional limit theorems for two-sided order books and obtain PDE or SPDE limits depending on the initial scaling procedure. Finally, Sowers and Zheng [51] (using the results of Kim et al. [36] on stochastic Stefan problems) and Müller [45] (based on the results of Keller-Ressel [35]) model the order book as a stochastic free boundary problem.

1.3 Summary of contributions and outline

Motivated by a zero-intelligence approach, the aim of this thesis is to bridge the gap between **microscopic** (discrete volume and price), **mesoscopic** (continuous volume and discrete price) and **macroscopic** (continuous volume and price) models of limit order books. The financial context of our study is the following: we consider an order-driven market where orders and cancellations are submitted at very high frequency. Starting with a discrete-space model describing the microscopic evolution of the order book, we prove that by sending order arrival and cancellation rates to infinity and by rescaling order volumes, the behaviour of the book can be described in terms of a more tractable continuous-space jump-diffusion process, i.e. the mesoscopic limiting process. Next, by sending the tick size to zero, or, equivalently, by considering an infinite number of price levels, we derive a macroscopic SPDE limit of the previously obtained mesoscopic process.

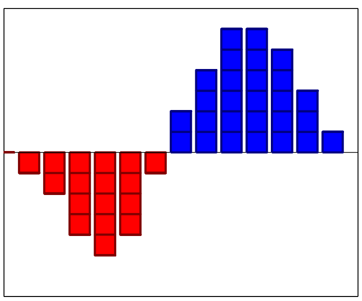
The first part of this thesis, comprised of Chapters 2 to 4, is devoted to the link between microscopic and mesoscopic limit order book models. The main results here are related to the weak convergence of a sequence of suitably rescaled discrete-space order book processes to a continuous-space jump-diffusion. Chapter 2 introduces the theoretical tools required in the derivation of these weak convergence results, which happen to be based on the theory of diffusion approximations of Markov processes via semigroup and infinitesimal generator techniques. This chapter also provides a brief overview of classically and elastically reflected diffusions, which form the basis of the limiting mesoscopic processes. Chapter 3 deals with the so-called reduced-form setting, in which we first provide a diffusion approximation of a one-sided book with one best level, before extending the result to a two-sided book with two price levels on each side. Throughout this chapter, we follow the idea developed by Cont and de Larrard [13] which consists in assuming that following a price change, the order book process takes a new value according to a stationary distribution representing the depth of the order book. Chapter 4 then covers the multidimensional order book case with a *relative price grid* and an *absolute price grid*. These two approaches are interesting in their own right, but the absolute price grid model offers an ideal starting point to the SPDE limit of the second part of the thesis. Furthermore, we show that in the relative price grid case, the price process is *exogenously* deduced as the difference of the counting processes associated with the regeneration times of the best bid and ask queues, whilst the price process is a fully fledged *endogenous* component of the absolute price grid setting. In Chapters 3 and 4, without loss of generality, we place ourselves in a framework where all order and cancellation sizes are assumed to be equal to 1. One fundamental feature of our work is based on the observation made by Avellaneda et al. [5] and presented in Section 1.1, according to which a queue depletion does not necessarily entail an immediate price change. In probabilistic terms, this corresponds to an elastic barrier, which is in actual fact a combination of a reflecting and absorbing barrier. The elastically reflected diffusion approximations obtained in this first part therefore differ from any previous existing results in the zero-intelligence order book literature, and provide an entirely novel description of the best queues at high frequency.

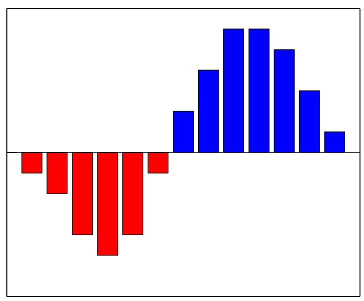
The second part of the thesis examines the transition between mesoscopic and macroscopic models of the order book. Given the multidimensional absolute price grid diffusion approximation established in the second part of Chapter 4, where an SDE mesoscopically describes the evolution of volumes at each price level, the aim here is to ask oneself what would happen if an infinite number of price levels were to be considered. The main theoretical ingredients here, presented in Chapter

5, are reflected SPDEs and stochastic interface models. Our motivation for using results from statistical mechanics stems from the **positivity of volumes constraint**, which is effectively handled by reflected SPDEs. In other applications of (unreflected) SPDEs to limit order books such as Sowers and Zheng [51] and Müller [45], the positivity of volumes is not directly given, and additional assumptions have to be made in order to ensure such a crucial property. Moreover, these papers directly postulate the form of an SPDE describing the evolution of the book, whilst we are intereseted in the actual **link** between the initial mesoscopic system of SDEs (given by the results of Chapter 4) and the macroscopic SPDE limit. In Chapter 5, we start by giving a brief presentation of reflected SPDEs, before examining their relevance in the field of stochastic interface models, where the positivity of the height variables is of utmost importance. We specifically look into the Funaki-Olla interface model presented in [22], and using techniques developed by Ambrosio et al. [3] and Zambotti [59] based on monotone gradient systems, provide an extension of a result related to the weak convergence of the suitably rescaled interface to a reflected stochastic heat equation. This result is then used in Chapter 6, which covers the application to limit order books. The idea here is to initially provide an SPDE limit in a static setting, i.e. with a constant price based on an initial profile of the book. In order to fully connect the mesoscopic and macroscopic frameworks, the last part of the thesis is ultimately devoted to the derivation of an SPDE limit in a dynamic price setting. The crucial aspect is to divide the time interval of interest into smaller periods on which the price remains constant, before piecing everything back together into a single process which fully characterises the dynamics of the limit order book in an infinite-dimensional setting. Finally, Chapter 7 introduces some numerical applications, notably the simulation of the mesoscopic and macroscopic limits using order arrival and cancellation rates estimated from tick-by-tick data. The results presented here underline the analytical tractability of the models derived throughout the thesis, which can be used as market simulators for short-term price prediction or optimal execution strategies.

Part I

Limit order books and diffusion approximations: from microscopic to mesoscopic models





2 Some results from the theory of diffusion approximations and reflected diffusions

The aim of this chapter is to introduce the theoretical tools which shall be required in the first part of the thesis. The first section covers some general results on diffusion approximations of Markov processes. We refer to Ethier and Kurtz [21] as well as Kallenberg [33] for a more detailed discussion of the topic. The second section is devoted to an overview of reflected diffusions, and more specifically reflected, elastic and regenerative elastic Brownian motions. The main references for this part are Itô and McKean [32], Karlin and Taylor [34], and Borodin and Salminen [9].

2.1 Diffusion approximations

Throughout this section, L shall denote a real Banach space with corresponding norm $||.||$.

2.1.1 Operator semigroups

We start by giving some basic definitions concerning operator semigroups:

Definition 2.1.1 *A family $T = (T(t))_{t\geq 0}$ of bounded linear operators on L is called a semigroup if:*

(i) $T(0) = Id$,

(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$.

Definition 2.1.2 *A semigroup $T = (T(t))_{t\geq 0}$ on L is said to be strongly continuous if $\lim_{t\to 0} T(t)f = f$ for all $f \in L$.*

Definition 2.1.3 *A semigroup $T = (T(t))_{t\geq 0}$ on L is said to be a contraction semigroup if $||T(t)|| \leq 1$ for all $t \geq 0$.*

Definition 2.1.4 *The infinitesimal generator of a semigroup* $T = (T(t))_{t\geq 0}$ *on* L *is the linear operator* A *defined by:*

$$Af = \lim_{t\to 0} \frac{T(t)f - f}{t}$$

for all $f \in \mathrm{Dom}(A)$*, the subspace of* L *for which this limit exists.*

Definition 2.1.5 *An operator* A *on* L *is said to be positive if* $f \geq 0$ *a.e implies* $Af \geq 0$ *a.e. A semigroup* $T = (T(t))_{t\geq 0}$ *is said to be positive if* $T(t)$ *is a positive operator for all* $t \geq 0$*.*

Definition 2.1.6 *Let* E *be a locally compact and separable metric space. A positive contraction semigroup* $T = (T(t))_{t\geq 0}$ *on* $\mathbb{C}_\infty(E)$ *is called a Feller semigroup if it satisfies:*

(i) $T(t)\mathbb{C}_\infty(E) \subset \mathbb{C}_\infty(E)$ *for all* $t \geq 0$*,*

(ii) $\lim_{t\to 0} T(t)f(x) = f(x)$ *for all* $f \in \mathbb{C}_\infty(E)$ *and* $x \in E$*.*

Remark 2.1.1 *The two previous properties imply that a Feller semigroup is necessarily strongly continuous.*

Definition 2.1.7 *A linear operator* A *on* L *is said to be closed if its graph* $\mathcal{G}(A) = \{(f, g) \in L^2 : f \in \mathrm{Dom}(A), Af = g\}$ *is a closed set.*

Remark 2.1.2 *The infinitesimal generator of a Feller semigroup is neccessarily closed.*

Definition 2.1.8 *A linear operator* A *on* L *is said to be closable if it has a closed linear extension. If* A *is closable, then the closure* $\bar{A}$ *of* A *is the minimal closed linear extension of* A*.*

Definition 2.1.9 *Let* A *be a closed linear operator on* L*. A linear subspace* $D \subset \mathrm{Dom}(A)$ *is called a core of* A *if the closure of the restriction of* A *to* D *is* A*, in other words, if the following relation holds:*

$$\overline{\{(f, Af) : f \in D\}} = \{(f, Af) : f \in \mathrm{Dom}(A)\}.$$

Definition 2.1.10 *Let* E *be a metric space and* $T = (T(t))_{t\geq 0}$ *be a semigroup on a closed subspace* $L \subset \mathbb{B}(E)$*. A Markov process* $X = (X(t))_{t\geq 0}$ *with values in* E *is said to correspond to* T *if:*

$$\mathbb{E}(f(X(t+s))|\mathcal{F}_t^X) = T_s f(X(t))$$

for all $s, t \geq 0$ *and* $f \in L$*, where* $(\mathcal{F}_t^X)_{t\geq 0}$ *denotes the augmented filtration of the process* X*.*

2.1.2 Diffusion approximation theorems

In this section we state two important results, due to Ethier and Kurtz [21], which specifically establish that convergence of generators entails convergence of the corresponding semigroups, in turn implying convergence of the relevant Markov processes. We refer to the original textbook [21] for the proofs. Let us specify that the form of convergence we are interested in is weak convergence (or convergence in distribution), that is $X_n \Rightarrow X$ when $\lim_{n\to+\infty} \mathbb{E}(f(X_n)) = \mathbb{E}(f(X))$ for all $f \in \mathbb{C}_b(\mathbb{D}([0,+\infty[,E))$.

Theorem 2.1.1 *For all $n \in \mathbb{N}^*$, let L_n be a Banach space with norm $||.||$, and let $\pi_n : L \to L_n$ be a bounded linear transformation satisfying $\sup_n ||\pi_n|| < \infty$. For all $n \in \mathbb{N}^*$, let $T_n = (T_n(t))_{t\geq 0}$ and $T = (T(t))_{t\geq 0}$ be strongly continuous contraction semigroups on L_n and L respectively, with infinitesimal generators A_n and A. Finally, let D be a core for A. Then the two following statements are equivalent:*

(i) For all $f \in L$, $T_n(t)\pi_n f \to T(t)f$ for all $t \geq 0$.

(ii) For all $f \in D$, there exists $f_n \in \mathrm{Dom}(A_n)$ for all $n \in \mathbb{N}^*$ such that $f_n \to f$ and $A_n f_n \to Af$.

Remark 2.1.3 *The notation $f_n \to f$, with $f_n \in L_n$ for all $n \in \mathbb{N}^*$ and $f \in L$, means that $\lim_{n\to+\infty} ||f_n - \pi_n f|| = 0$.*

Theorem 2.1.2 *For all $n \in \mathbb{N}^*$, let E_n be metric spaces, and let E be a locally compact and separable metric space. For all $n \in \mathbb{N}^*$, let $\eta_n : E_n \to E$ be a measurable function. Suppose that $T_n = (T_n(t))_{t\geq 0}$ is a semigroup on $\mathbb{B}(E_n)$, and let $Y_n = (Y_n(t))_{t\geq 0}$ be a Markov process with values in E_n corresponding to T_n such that $X_n = \eta_n \circ Y_n$ has sample paths in $\mathbb{D}([0,+\infty[,E)$. Let $\pi_n : \mathbb{B}(E) \to \mathbb{B}(E_n)$ be a bounded linear transformation satisfying $\sup_n ||\pi_n|| < \infty$. Finally, let $T = (T(t))_{t\geq 0}$ be a Feller semigroup on $\mathbb{C}_\infty(E)$ such that for all $f \in \mathbb{C}_\infty(E)$, $T_n(t)\pi_n f \to T(t)f$ for all $t \geq 0$. If $(X_n(0))_{n\geq 1}$ has limiting distribution $\nu \in \mathcal{P}(E)$, the set of Borel probability measures on E, then there exists a Markov process $X = (X(t))_{t\geq 0}$ corresponding to T with initial distribution ν and sample paths in $\mathbb{D}([0,+\infty[,E)$ such that $X_n \Rightarrow X$.*

The final result of interest in this section, due to Galakhov and Skubachevskii [26], specifically concerns sufficient conditions which ensure that a Waldenfels integro-differential operator with a Wentzell type boundary condition is the infinitesimal generator of a Feller semigroup.

To this end, let $Q \subset \mathbb{R}^n$ be a domain with boundary ∂Q, let $f \in \mathbb{C}^2(\bar{Q})$, and consider the following second order partial differential operator (which is a special case within the class of Waldenfels operators) $\mathcal{L}$ defined by:

$$\mathcal{L}f(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i} + c(x)(f(d(x)) - f(x)), \;\; x \in Q,$$

where $a_{ij} = a_{ji}, a_{ij}, b_i, c \in \mathbb{C}^{0,\sigma}(\bar{Q})$ for $0 < \sigma < 1$, where $c(x) \geq 0$ and where d is a continuous transformation mapping $\bar{Q}$ into itself. We also assume that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0$ for all $x \in \bar{Q}$ and $0 \neq \xi \in \mathbb{R}^n$. In addition, consider the nonlocal Wentzell boundary condition (see also Taira [52] for more details on this) $\mathcal{B}f(x) = 0$ for $x \in \partial Q$, where the operator $\mathcal{B}$ is defined as follows:

$$\mathcal{B}f(x) = \alpha(x)f(x) + \int_{\bar{Q}} (f(x) - f(y))\mu(dy) - \beta(x)\frac{\partial f}{\partial x_n} + \sum_{i=1}^{n-1} \gamma_i(x)\frac{\partial f}{\partial x_i}$$

for $\alpha, \beta, \gamma_i \in \mathbb{C}^{1,\sigma}(\partial Q)$, $\alpha(x) \geq 0, \beta(x) > 0$ for $x \in \partial Q$, and where $\mu(.)$ is a nonnegative Borel measure on $\bar{Q}$ satisfying $\lim_{\epsilon \to 0} \int_{|y|<\epsilon} |y|\mu(dy) = 0$ and $\int_{|y|\geq\epsilon} \mu(dy) < C(\epsilon)$ for $\epsilon > 0$ ($C(\epsilon)$ being a positive constant depending on ϵ). We can now state the following theorem, and refer to Galakhov and Skubachevskii [26] for its proof:

Theorem 2.1.3 *The operator $\bar{\mathcal{L}} : \mathrm{Dom}(\bar{\mathcal{L}}) \subset \mathbb{C}(\bar{Q}) \to \mathbb{C}(\bar{Q})$ is the infinitesimal generator of a Feller semigroup, which is uniquely determined by $\mathcal{L}$.*

2.2 Reflected diffusions

In this section, we give a brief presentation of two processes of interest in our subsequent diffusion approximations of discrete-space limit order book models, namely reflected and elastic Brownian motions. We also show how elastic Brownian motion can be regenerated once it has been killed, giving rise to the so-called regenerative elastic Brownian motion.

2.2.1 Reflected Brownian motion

Reflected Brownian motion represents the archetypal example of the class of reflected diffusions, as it can be found in a wide variety of interesting applications. As pointed out by Andres [4], this process can be constructed in several ways, the most natural being the following: given a standard Brownian motion $B = (B_t)_{t\geq 0}$, the reflected Brownian motion $X = (X_t)_{t\geq 0}$ can simply be obtained by taking the absolute value of B, i.e. $X = |B|$. Making use of Tanaka's formula, this gives us:

$$X_t = |B_t| = \int_0^t \mathrm{sgn}\,(B_s)dB_s + L_t,$$

where $L = (L_t)_{t\geq 0}$ is the local time at 0 of B, which is continuous, nondecreasing and with $\mathrm{supp}\,(dL) \subseteq \{t \geq 0 : X_t = 0\}$. Alternatively, we can apply Skorokhod's reflection principle to see that given a standard Brownian motion $B = (B_t)_{t\geq 0}$, there exists a unique pair (X, L) satisfying:

$$X_t = B_t + L_t,$$

where $X_t \geq 0$ for all $t \geq 0$ and where L is continuous and nondecreasing verifying $L_0 = 0$ and $\mathrm{supp}\,(dL) \subseteq \{t \geq 0 : X_t = 0\}$. L is expressed by $L_t = \sup_{0\leq s\leq t}(-B_s)^+$

in this second construction, which is an example of a so-called Skorokhod SDE, and shall form the basis of the dynamics of the Funaki-Olla interface model introduced in the second part of the thesis.

2.2.2 Elastic Brownian motion

Given a standard Brownian motion $B = (B_t)_{t\geq 0}$, let $L = (L_t)_{t\geq 0}$ be its associated local time at 0 process defined in the following way, for all $t \geq 0$:

$$L_t = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{B_s\in[-\epsilon,\epsilon]\}} ds = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{|B_s|\leq\epsilon\}} ds,$$

where the limit is taken in probability. Let ξ be a random variable, independent of B and L, following an exponential distribution with parameter γ, and define $T = \inf\{t \geq 0 : L_t > \xi\}$. Elastic Brownian motion, sometimes referred to as *partially reflected Brownian motion*, is the process $B^e = (B^e_t)_{t\geq 0}$ defined by:

$$B^e_t = \begin{cases} |B_t| & \text{if } t < T, \\ \dagger & \text{if } t \geq T, \end{cases}$$

where $\dagger$ is a cemetery point corresponding to the value of the killed process. Now and henceforth, it shall be assumed that every real-valued function f on $\mathbb{R}_+$ is extended to $\mathbb{R}_+ \cup \{\dagger\}$ using the convention $f(\dagger) = 0$. By construction, elastic Brownian motion can be seen as reflected Brownian motion killed when its local time process has reached level ξ. Given the exponential distribution of ξ, it is immediate to notice that for any bounded measurable function f, the following relation holds:

$$\mathbb{E}_x(f(B^e_t)) = \mathbb{E}_x(f(|B_t|)\mathbb{1}_{\{t<T\}}) = \mathbb{E}_x(f(|B_t|)e^{-\gamma L_t}).$$

For more details on elastic Brownian motion, we refer to Grebenkov [29] and Kostrykin et al. [37]. Now, introducing the resolvent operator $\mathcal{R}^e_\alpha$ of elastic Brownian motion defined by:

$$\mathcal{R}^e_\alpha f(x) = \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s} f(B^e_s) ds\right),$$

for any $f \in \mathbb{C}_b(\mathbb{R}_+)$, its infinitesimal generator $\mathcal{L}^e$ satisfies:

$$\mathcal{R}^e_\alpha = (\alpha - \mathcal{L}^e)^{-1},$$

for $\alpha > 0$ with $\mathrm{Dom}(\mathcal{L}^e) = \mathcal{R}^e_\alpha(\mathbb{C}_b(\mathbb{R}_+))$. Using these relations, we shall now show that $\mathcal{L}^e u = \frac{1}{2}\Delta u$, acting on the following domain:

$$\mathrm{Dom}(\mathcal{L}^e) = \left\{u \in \mathbb{C}^2_b(\mathbb{R}_+) : u'(0) = \gamma u(0)\right\}.$$

Letting $u \equiv \mathcal{R}^e_\alpha f$ for $f \in \mathbb{C}_b(\mathbb{R}_+)$ and $\alpha > 0$, it suffices to show that (Itô and McKean [32] p. 47) u is a solution of:

$$(\alpha - \mathcal{L}^e)u = f \text{ and } u \in \mathrm{Dom}(\mathcal{L}^e).$$

If we consider the first hitting time of the origin of a reflected Brownian motion (starting away from 0) defined by $\tau_0 = \inf\{t \geq 0 : |B_t| = 0\}$, we can write:

$$\begin{aligned}
u(x) &= \mathbb{E}_x\left(\int_0^T e^{-\alpha s} f(B_s^e)ds\right) \\
&= \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s}e^{-\gamma L_s} f(|B_s|)ds\right) \\
&= \mathbb{E}_x\left(\int_0^{\tau_0} e^{-\alpha s} f(|B_s|)ds\right) + \mathbb{E}_x\left(\int_{\tau_0}^{+\infty} e^{-\alpha s}e^{-\gamma L_s} f(|B_s|)ds\right) \\
&= \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s} f(|B_s|)ds\right) \\
&\quad -\mathbb{E}_x\left(e^{-\alpha\tau_0}\mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s} f(|B_{s+\tau_0}|)ds|\mathcal{F}_{\tau_0}^B\right)\right) \\
&\quad +\mathbb{E}_x\left(e^{-\alpha\tau_0}\mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s}e^{-\gamma L_{s+\tau_0}} f(|B_{s+\tau_0}|)ds|\mathcal{F}_{\tau_0}^B\right)\right) \\
&= \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s} f(|B_s|)ds\right) \\
&\quad +\mathbb{E}_x\left(e^{-\alpha\tau_0}\right)\left(u(0) - \mathbb{E}_0\left(\int_0^{+\infty} e^{-\alpha s} f(|B_s|)ds\right)\right) \\
&= \int_0^{+\infty} \frac{e^{-\sqrt{2\alpha}|x-y|} + e^{-\sqrt{2\alpha}|x+y|}}{\sqrt{2\alpha}} f(y)dy \\
&\quad +e^{-\sqrt{2\alpha}x}\left(u(0) - 2\int_0^{+\infty} \frac{e^{-\sqrt{2\alpha}y}}{\sqrt{2\alpha}} f(y)dy\right).
\end{aligned}$$

Note that we have used the strong Markov property between equalities 4 and 5. A straightforward computation then gives us $\alpha u(x) - \frac{1}{2}u''(x) = f(x)$, as required.

We now need to check that $u'(0) = \gamma u(0)$. Given that the process $(S_t - B_t)_{t\geq 0}$, where $S_t = \sup_{s\leq t} B_s$, is a reflected Brownian motion with local time S_t, and using the joint law of Brownian motion and its supremum process (Itô and McKean [32] p. 28), we can write:

$$\begin{aligned}
u(0) &= \mathbb{E}_0\left(\int_0^{+\infty} e^{-\alpha s}e^{-\gamma S_s} f(S_s - B_s)ds\right) \\
&= \int_0^{+\infty} e^{-\alpha s}\int_{\mathbb{R}^2} e^{-\gamma y} f(y-x)\mathbb{P}_0(B_s \in dx, S_s \in dy)ds
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{+\infty} e^{-\gamma y}\int_{-\infty}^{y} f(y-x)e^{-\sqrt{2\alpha}(2y-x)}dxdy \\
&= \frac{2}{\gamma+\sqrt{2\alpha}}\int_0^{+\infty} e^{-\sqrt{2\alpha}y}f(y)dy.
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
u'(0) &= -\sqrt{2\alpha}u(0) + 2\int_0^{+\infty} e^{-\sqrt{2\alpha}y}f(y)dy \\
&= 2\int_0^{+\infty} e^{-\sqrt{2\alpha}y}f(y)dy\left(\frac{-\sqrt{2\alpha}}{\gamma+\sqrt{2\alpha}}+1\right) \\
&= \gamma u(0).
\end{aligned}
$$

2.2.3 Regenerative elastic Brownian motion

As its name suggests, *regenerative elastic Brownian motion* is simply obtained by *regenerating* an elastic Brownian motion once it has been killed, with the starting point of the reborn process determined according to a certain distribution. More precisely, define a sequence $R = (R_i)_{i\geq 1}$ of i.i.d random variables with probability density function p, and another sequence $\xi = (\xi_i)_{i\geq 1}$ of i.i.d exponential random variables with parameter γ. Assume furthermore that the sequence ξ is independent of our regenerative elastic Brownian motion $B^{e,r}$. Introducing the sequence $T = (T_i)_{i\geq 1}$ of regeneration times satisfying:

$$
\begin{cases}
T_1 = \inf\{t \geq 0 : L_t > \xi_1\}, \\
T_{i+1} = \inf\{t \geq T_i : L_t - L_{T_i} > \xi_{i+1}\},
\end{cases}
$$

where $L = (L_t)_{t\geq 0}$ is the local time at 0 of the process, our regenerative elastic Brownian motion $B^{e,r}$ behaves as follows: on the interval $[0, T_1[$, $B^{e,r}$ is simply an elastic Brownian motion starting from $x \geq 0$. At time T_1, it jumps to a new position $B^{e,r}_{T_1} = R_1$ according to the density function p, and then behaves like an elastic Brownian motion starting from R_1 until time T_2, where it regenerates, and so on.

Using the result of the previous section, we will now show that the infinitesimal generator of regenerative elastic Brownian motion is the differential operator $\mathcal{L}^{e,r}u = \frac{1}{2}\Delta u$ acting on the following domain:

$$
\mathrm{Dom}(\mathcal{L}^{e,r}) = \left\{u \in \mathbb{C}^2_b(\mathbb{R}_+) : u'(0) = \gamma\left(u(0) - \int_{\mathbb{R}} u(y)p(y)dy\right)\right\}.
$$

Let $\mathcal{R}^{e,r}_\alpha$ be the resolvent operator of regenerative elastic Brownian motion defined by:

$$
\mathcal{R}^{e,r}_\alpha f(x) = \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s}f(B^{e,r}_s)ds\right),
$$

for any $f \in \mathbb{C}_b(\mathbb{R}_+)$. Defining $u \equiv \mathcal{R}_\alpha^{e,r} f$ for $f \in \mathbb{C}_b(\mathbb{R}_+)$ and $\alpha > 0$, it suffices to show that u is a solution of:

$$(\alpha - \mathcal{L}^{e,r})u = f \text{ and } u \in \text{Dom}(\mathcal{L}^{e,r}).$$

The crucial point here is to observe that up to the first regeneration time T_1, $B^{e,r}$ simply behaves like an elastic Brownian motion.

If we set $v(x) = \mathbb{E}_x\left(\int_0^{T_1} e^{-\alpha s} f(B_s^e)ds\right)$ for any $f \in \mathbb{C}_b(\mathbb{R}_+)$, we can write:

$$\begin{aligned}
u(x) &= v(x) + \mathbb{E}_x\left(\int_{T_1}^{+\infty} e^{-\alpha s} f(B_s^{e,r})ds\right) \\
&= v(x) + \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha(s+T_1)} f(B_{s+T_1}^{e,r})ds\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha T_1}\mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s} f(B_{s+T_1}^{e,r})ds|\mathcal{F}_{T_1}^{B^{e,r}}\right)\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha T_1}\int_0^{+\infty} e^{-\alpha s}\mathbb{E}_x\left(f(B_{s+T_1}^{e,r})|\mathcal{F}_{T_1}^{B^{e,r}}\right)ds\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha T_1}\int_0^{+\infty} e^{-\alpha s}\left(\int_{\mathbb{R}} \mathbb{E}_y\left(f(B_s^{e,r})\right)p(y)dy\right)ds\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha T_1}\right)\int_{\mathbb{R}} u(y)p(y)dy \\
&= v(x) + \frac{\gamma}{\sqrt{2\alpha}+\gamma}e^{-\sqrt{2\alpha}x}\int_{\mathbb{R}} u(y)p(y)dy.
\end{aligned}$$

As v satisfies $\alpha v(x) - \frac{1}{2}v''(x) = f(x)$, we have:

$$\begin{aligned}
\alpha u(x) - \frac{1}{2}u''(x) &= f(x) + \alpha\frac{\gamma}{\sqrt{2\alpha}+\gamma}e^{-\sqrt{2\alpha}x}\int_{\mathbb{R}} u(y)p(y)dy \\
&\quad -\frac{1}{2}\frac{2\gamma\alpha}{\sqrt{2\alpha}+\gamma}e^{-\sqrt{2\alpha}x}\int_{\mathbb{R}} u(y)p(y)dy \\
&= f(x).
\end{aligned}$$

As for the verification of the boundary condition, we already know that v satisfies $v'(0) = \gamma v(0)$, hence:

$$u'(0) = \gamma v(0) - \frac{\sqrt{2\alpha}\gamma}{\sqrt{2\alpha}+\gamma}\int_{\mathbb{R}} u(y)p(y)dy$$

$$
\begin{aligned}
&= \gamma\left(u(0) - \frac{\gamma}{\sqrt{2\alpha}+\gamma}\int_{\mathbb{R}} u(y)p(y)dy\right) - \frac{\sqrt{2\alpha}\gamma}{\sqrt{2\alpha}+\gamma}\int_{\mathbb{R}} u(y)p(y)dy \\
&= \gamma\left(u(0) - \int_{\mathbb{R}} u(y)p(y)dy\right).
\end{aligned}
$$

2.2.4 Regenerative elastic Brownian motion with random jumps to 0

We finally wish to consider a regenerative elastic Brownian motion where random jumps to the origin occurring at exponential times are incorporated. To this end, let $\tilde{\xi} = (\tilde{\xi})_{i\geq 1}$ be a sequence of i.i.d exponential random variables with parameter λ_0. We then define a sequence $\tilde{T} = (\tilde{T}_i)_{i\geq 1}$ of stopping times by $\tilde{T}_i = \sum_{j=1}^{i} \tilde{\xi}_j \sim \Gamma(i, \lambda_0)$.

The behaviour of our regenerative elastic Brownian motion with random jumps $X = (X_t)_{t\geq 0}$ is as follows. On the interval $[0, \tilde{T}_1[$, X is simply a regenerative elastic Brownian motion starting from $x \geq 0$. At time $\tilde{T}_1$, the process jumps to the origin, and regenerates as a new regenerative elastic Brownian motion starting from 0, until the next jump time $\tilde{T}_2$, where the same behaviour is repeated.

Building on the result of the previous section, we are now in a position to prove that the infinitesimal generator $\mathcal{L}$ of X satisfies:

$$
\mathcal{L}u(x) = \frac{1}{2}\Delta u(x) + \lambda_0(u(0) - u(x)), \;\; x \geq 0,
$$

acting on the following domain:

$$
\mathrm{Dom}(\mathcal{L}) = \left\{ u \in \mathbb{C}_b^2(\mathbb{R}_+) : u'(0) = \gamma\left(u(0) - \int_{\mathbb{R}} u(y)p(y)dy\right)\right\}.
$$

Let $\mathcal{R}_\alpha$ be the resolvent operator of X satisfying:

$$
\mathcal{R}_\alpha f(x) = \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s} f(X_s)ds\right),
$$

for any $f \in \mathbb{C}_b(\mathbb{R}_+)$. Defining $u \equiv \mathcal{R}_\alpha f$ for $f \in \mathbb{C}_b(\mathbb{R}_+)$ and $\alpha > 0$, we will show that u is a solution of:

$$
(\alpha - \mathcal{L})u = f \;\text{ and }\; u \in \mathrm{Dom}(\mathcal{L}).
$$

In this case, the key point is to see that until time $\tilde{T}_1$, X behaves like a regenerative elastic Brownian motion. We now set:

$$
v(x) = \mathbb{E}_x\left(\int_0^{\tilde{T}_1} e^{-\alpha s} f(B_s^{e,r})ds\right),
$$

for any $f \in \mathbb{C}_b(\mathbb{R}_+)$. We then have:

$$\begin{aligned}
u(x) &= v(x) + \mathbb{E}_x\left(\int_{\tilde{T}_1}^{+\infty} e^{-\alpha s} f(X_s) ds\right) \\
&= v(x) + \mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha(s+\tilde{T}_1)} f(X_{s+\tilde{T}_1}) ds\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha\tilde{T}_1}\mathbb{E}_x\left(\int_0^{+\infty} e^{-\alpha s} f(X_{s+\tilde{T}_1}) ds | \mathcal{F}^X_{\tilde{T}_1}\right)\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha\tilde{T}_1}\int_0^{+\infty} e^{-\alpha s}\mathbb{E}_x\left(f(X_{s+\tilde{T}_1}) | \mathcal{F}^X_{\tilde{T}_1}\right) ds\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha\tilde{T}_1}\int_0^{+\infty} e^{-\alpha s}\mathbb{E}_0\left(f(X_s)\right) ds\right) \\
&= v(x) + \mathbb{E}_x\left(e^{-\alpha\tilde{T}_1}\right) u(0) \\
&= v(x) + \frac{\lambda_0}{\lambda_0+\alpha} u(0).
\end{aligned}$$

Observing that $v(x) = \mathbb{E}_x\left(\int_0^{+\infty} e^{-(\alpha+\lambda_0)s} f(B^{e,r}_s) ds\right)$, v therefore satisfies:

$$(\alpha+\lambda_0)v(x) - \frac{1}{2}v''(x) = f(x).$$

This enables us to have:

$$\begin{aligned}
\alpha u(x) - \frac{1}{2}u''(x) - \lambda_0(u(0)-u(x)) &= \alpha v(x) + \frac{\alpha\lambda_0}{\lambda_0+\alpha}u(0) - \frac{1}{2}v''(x) \\
&\quad -\lambda_0 u(0) + \lambda_0 v(x) + \frac{\lambda_0^2}{\lambda_0+\alpha}u(0) \\
&= f(x).
\end{aligned}$$

As for the boundary condition, we know that:

$$v'(0) = \gamma\left(v(0) - \int_{\mathbb{R}} v(y)p(y)dy\right).$$

We ultimately obtain:

$$\begin{aligned}
u'(0) &= v'(0) \\
&= \gamma\left(u(0) - \frac{\lambda_0}{\lambda_0+\alpha}u(0) - \int_{\mathbb{R}} u(y)p(y)dy + \frac{\lambda_0}{\lambda_0+\alpha}u(0)\right) \\
&= \gamma\left(u(0) - \int_{\mathbb{R}} u(y)p(y)dy\right),
\end{aligned}$$

thus giving us the result.

3 Reduced-form limit order book models

In this chapter, we present diffusion approximations of reduced-form discrete-space limit order book models. We first examine the case of the best queue in a one-sided model, before introducing a two-sided model with the two best queues on each side. Throughout this chapter (and in the following one as well), the relevant discrete state and continuous state spaces shall be endowed with the supremum norm denoted by $\|.\|$.

3.1 A one-sided reduced-form limit order book model

3.1.1 The discrete order book process

For each $n \in \mathbb{N}^*$, consider a birth-and-death process with jumps $Y_n = (Y_n(t))_{t \geq 0}$ with values in $\mathbb{N}$, representing the discrete order book process at either one of the best levels. We assume that all order and cancellation sizes are equal to 1 without loss of generality. For $i \geq 1$, the transition probabilities are given by:

$$\begin{cases} \mathbb{P}\left(Y_n(t+h) = i+1 | Y_n(t) = i\right) = \lambda_n h + o(h), \\ \mathbb{P}\left(Y_n(t+h) = i-1 | Y_n(t) = i\right) = (\mu_n + \theta_n) h + o(h), \\ \mathbb{P}\left(Y_n(t+h) = 1 | Y_n(t) = i\right) = \lambda_n^0 h + o(h), \end{cases}$$

where λ_n, μ_n and θ_n respectively correspond to the arrival rates of limit orders, market orders and cancellations, and where λ_n^0 denotes the arrival rate of limit orders inside the spread. As for the the boundary transition probabilities, they are given as follows:

$$\begin{cases} \mathbb{P}\left(Y_n(t+h) = 1 | Y_n(t) = 0\right) = \lambda_n h + o(h), \\ \mathbb{P}\left(Y_n(t+h) = r | Y_n(t) = 0\right) = \mu_n p_n(r) h + o(h), \end{cases}$$

where p_n is the probability density function representing the depth or the order book after a price change (see Cont and de Larrard [13]). Note that these boundary transitions correspond to an *elastic* boundary, as we don't want a queue depletion to immediately cause a price change (we refer to Avellaneda et al. [5] for a thorough empirical study of this stylised fact).

3.1.2 Heavy traffic diffusion approximation

We now switch our interest to the heavy traffic limit of the suitably rescaled discrete order book process. We accelerate time by a factor of n and divide the volumes by $\sqrt{n}$. More precisely, let $X_n = (X_n(t))_{t\geq 0}$ be the process on $E_n = \frac{1}{\sqrt{n}}\mathbb{N}$ defined by:

$$X_n(t) = \frac{Y_n(nt)}{\sqrt{n}}.$$

In order to obtain our scaling limit, we need to consider the three following time scales: a *fast* time scale for limit orders outside the spread and cancellations, a *relatively slower* time scale for market orders, and an *even slower* time scale for limit orders inside the spread. Financially speaking, the fast time scale for limit orders and cancellations is a simple consequence of the observation that in a high to ultra high-frequency setting, a very large number of limit orders are placed only to be cancelled within an extremely short period of time. Gai et al. [25] specifically find that a rise in the speed of trading from microseconds to nanoseconds significantly increases the order cancellation to execution ratio from 26:1 to 32:1 using NASDAQ data. The predominance of limit orders and cancellations naturally induces the necessity to consider market orders on a slower time scale. The *even slower* time scale for limit orders inside the spread can be explained in the following way: as pointed out by Cont and de Larrard [13], statistical evidence suggests the bid-ask spread is equal to one tick for more than 98% of observations on liquid stocks, thus showing that limit orders inside the spread would mostly arrive to close the spread once it has been widened. Mathematically speaking, it shall therefore be assumed that (where λ, c, μ and λ^0 are given constants and $E = \mathbb{R}_+$ is the limiting state space):

- *(A1)* $\lim_{n\to+\infty} \lambda_n = \lambda$,
- *(A2)* $\lim_{n\to+\infty} \theta_n = \lambda$,
- *(A3)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n - \theta_n) = c$,
- *(A4)* $\mu_n = \frac{\mu}{\sqrt{n}}$,
- *(A5)* $\lambda_n^0 = \frac{\lambda^0}{n}$,
- *(A6)* $c - \mu < 0$,
- *(A7)* There exists a probability density function p such that for all $f \in \mathbb{C}_b(E)$:

$$\lim_{n\to+\infty} \sum_{r\geq 0} f\left(\frac{r}{\sqrt{n}}\right) p_n(r) = \int_{\mathbb{R}} f(y)p(y)dy.$$

Assumptions $(A1)$ and $(A2)$ reflect the *fast* time scale for limit orders outside the spread and cancellations. $(A3)$ and $(A6)$ are related and ensure that in the heavy traffic regime, the service rate (cancellations and market orders) needs to be greater than the arrival rate (limit orders) so as to keep the system stable. Assumption $(A4)$ corresponds to the *relatively slower* time scale for market orders, and assumption $(A5)$ describes the *even slower* time scale for limit orders inside the spread. Finally, assumption $(A7)$ translates the fact that if R_n is a random variable with probability density function p_n, then there exists a random variable R with probability density function p such that for all $f \in \mathbb{C}_b(E)$:

$$\lim_{n\to+\infty} \mathbb{E}\left(f\left(\frac{R_n}{\sqrt{n}}\right)\right) = \mathbb{E}(f(R)).$$

Note that all quantities involved in $(A7)$ are well defined, as for all $f \in \mathbb{C}_b(E)$:

$$\sum_{r\geq 0}\left|f\left(\frac{r}{\sqrt{n}}\right)p_n(r)\right| \leq \|f\|\sum_{r\geq 0}p_n(r) < +\infty,$$

$$\int_{\mathbb{R}}\big|f(y)p(y)\big|\,dy \leq \|f\|\int_{\mathbb{R}}p(y)dy < +\infty.$$

Remark 3.1.1 *This final assumption can be illustrated via the weak convergence of a sequence of adequately rescaled geometric random variables to an exponential one. More precisely, if we consider a sequence $(R_n)_{n\geq 1}$ of geometric random variables with parameter $\alpha_n = \frac{\alpha}{\sqrt{n}}$, it is well known that it converges in distribution as $n \to \infty$ to an exponential random variable R with parameter α.*

The method we shall use to obtain the heavy traffic limit is based on a semigroup characterisation of the process X_n. Let $(T_n(t))_{t\geq 0}$ be the semigroup on $\mathbb{B}(E_n)$ defined by:

$$T_n(t)f(x) = \mathbb{E}\left(f(X_n(t))|X_n(0) = x\right),$$

for all $f \in \mathbb{B}(E_n)$ and $x \in E_n$. Furthermore, its infinitesimal generator A_n is given by, for all $f \in \mathbb{B}(E_n)$ and $x \in E_n \setminus \{0\}$:

$$\begin{aligned}
A_n f(x) &= \lim_{t\to 0}\frac{T_n(t)f(x) - f(x)}{t} \\
&= \lim_{t\to 0}\frac{1}{t}(\lambda_n nt + o(nt))\left(f\left(x + \frac{1}{\sqrt{n}}\right) - f(x)\right) \\
&\quad + \lim_{t\to 0}\frac{1}{t}(\mu_n nt + \theta_n nt + o(nt))\left(f\left(x - \frac{1}{\sqrt{n}}\right) - f(x)\right) \\
&\quad + \lim_{t\to 0}\frac{1}{t}(\lambda_n^0 nt + o(nt))\left(f\left(\frac{1}{\sqrt{n}}\right) - f(x)\right) \\
&= \lambda_n n\left(f\left(x + \frac{1}{\sqrt{n}}\right) - f(x)\right) + (\mu_n + \theta_n)n\left(f\left(x - \frac{1}{\sqrt{n}}\right) - f(x)\right) \\
&\quad + \lambda_n^0 n\left(f\left(\frac{1}{\sqrt{n}}\right) - f(x)\right).
\end{aligned}$$

When $x = 0$, A_n satisfies, for all $f \in \mathbb{B}(E_n)$:

$$\begin{aligned} A_n f(0) &= \lim_{t\to 0} \frac{T_n(t)f(0) - f(0)}{t} \\ &= \lim_{t\to 0} \frac{1}{t}(\lambda_n nt + o(nt)) \left(f\left(\frac{1}{\sqrt{n}}\right) - f(0) \right) \\ &\quad + \lim_{t\to 0} \frac{1}{t}(\mu_n nt + o(nt)) \sum_{r\in\mathbb{N}^*} p_n(r) \left(f\left(\frac{r}{\sqrt{n}}\right) - f(0) \right) \\ &= \lambda_n n \left(f\left(\frac{1}{\sqrt{n}}\right) - f(0) \right) + \mu_n n \sum_{r\in\mathbb{N}^*} p_n(r) \left(f\left(\frac{r}{\sqrt{n}}\right) - f(0) \right). \end{aligned}$$

We are now in a position to establish the weak convergence of the discrete order book process to a jump-diffusion process on E:

Theorem 3.1.1 *The E_n-valued process X_n converges weakly in $\mathbb{D}([0, +\infty[, E)$ as $n \to \infty$ to an E-valued strong Markov jump-diffusion process X with infinitesimal generator given by the closure $\bar{A}$ of the linear operator A defined by:*

$$Af(x) = \lambda f''(x) + (c - \mu) f'(x) + \lambda^0 (f(0) - f(x)), \;\; x > 0,$$

$$Af(0) = \frac{\lambda}{2} f''(0),$$

acting on the following domain:

$$\mathrm{Dom}(A) = \left\{ f \in \mathbb{C}^2_{b,\infty}(E) : f'(0) = \frac{\mu}{\lambda} \left(f(0) - \int_{\mathbb{R}} f(y)p(y)dy \right) \right\}.$$

Before proceeding to the proof, we shall require the following technical lemma:

Lemma 3.1.1 *Let $f \in \mathrm{Dom}(A)$ and define the following sequence of functions f_n on E:*

$$f_n(x) = f\left(\frac{\lambda}{\lambda_n} x\right) + \frac{x^2 e^{-x}}{\sum_{q\geq 0} \frac{q^2}{n} e^{-\frac{q}{\sqrt{n}}} p_n(q)} \left(I_f - \sum_{q\geq 0} f\left(\frac{\lambda q}{\lambda_n \sqrt{n}}\right) p_n(q) \right),$$

where $I_f = \int_E f(y)p(y)dy$. Then $\lim_{n\to+\infty} \|f_n - f\| = 0$ and f_n satisfies:

$$f_n'(0) = \frac{\mu}{\lambda_n} \left(f_n(0) - \sum_{r\geq 0} f_n\left(\frac{r}{\sqrt{n}}\right) p_n(r) \right).$$

Proof of Lemma 3.1.1: We first need to show that:

$$\lim_{n\to+\infty}\sup_{x\in E}\left|f\left(\frac{\lambda}{\lambda_n}x\right)-f(x)\right|=0.$$

Let $\epsilon>0$. As f vanishes at infinity, there exists $A>0$ such that for all $x\geq A$, $|f(x)|<\frac{\epsilon}{2}$. Using $(A1)$, there also exists $N_0\in\mathbb{N}^*$ such that for all $n\geq N_0$, $\frac{\lambda}{\lambda_n}>\frac{1}{2}$. Consequently, for all $x\geq A$ and $n\geq N_0$, we have:

$$\left|f\left(\frac{\lambda}{\lambda_n}x\right)-f(x)\right|\leq\left|f\left(\frac{\lambda}{\lambda_n}x\right)\right|+|f(x)|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Note that f is uniformly continuous on E (and in particular on $[0,2A]$), so there exists $\delta>0$ such that for all $(x,y)\in[0,2A]^2$, $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\epsilon$. Once again, according to $(A1)$, there exists $N_1\in\mathbb{N}^*$ with $N_1>N_0$ such that for all $n\geq N_1$, $\left|\frac{\lambda}{\lambda_n}-1\right|<\frac{\delta}{2A}$. For all $x\in[0,2A]$ and $n\geq N_1$, we now have:

$$\left|\frac{\lambda}{\lambda_n}x-x\right|=\left|\frac{\lambda}{\lambda_n}-1\right|.|x|<\frac{\delta}{2A}2A=\delta.$$

So for all $n\geq N_1$, we have established that $\sup_{x\in E}\left|f\left(\frac{\lambda}{\lambda_n}x\right)-f(x)\right|<\epsilon$, which enables us to deduce that $\lim_{n\to+\infty}\sup_{x\in E}\left|f\left(\frac{\lambda}{\lambda_n}x\right)-f(x)\right|=0$. As we know that $\lim_{n\to+\infty}\sum_{q\geq0}\frac{q^2}{n}e^{-\frac{q}{\sqrt{n}}}p_n(q)=\int_{\mathbb{R}}y^2e^{-y}p(y)dy$ by $(A7)$, it now suffices to show that $\lim_{n\to+\infty}\sum_{q\geq0}f\left(\frac{\lambda q}{\lambda_n\sqrt{n}}\right)p_n(q)=I_f$ in order to establish uniform convergence of f_n to f. To this end, we first prove this on an arbitrary interval $[0,b]$ for large b. As f is uniformly continuous on $[0,b]$, for all $\epsilon>0$, there exists $\delta>0$ such that for any pair $(x,y)\in[0,b]^2$, $|x-y|<\delta\Rightarrow|f(x)-(y)|<\epsilon$. According to $(A1)$, there exists $N\in\mathbb{N}^*$ such that for all $n\geq N$, $\left|\frac{\lambda}{\lambda_n}-1\right|<\frac{\delta}{b}$. We therefore have:

$$\left|\frac{\lambda q}{\lambda_n\sqrt{n}}-\frac{q}{\sqrt{n}}\right|=\left|\frac{\lambda}{\lambda_n}-1\right|\frac{q}{\sqrt{n}}<\delta,$$

and

$$\left|f\left(\frac{\lambda q}{\lambda_n\sqrt{n}}\right)-f\left(\frac{q}{\sqrt{n}}\right)\right|<\epsilon$$

for all $q\in\mathbb{N}$. We now have:

$$\left|\sum_{q\geq0}\left(f\left(\frac{\lambda q}{\lambda_n\sqrt{n}}\right)-f\left(\frac{q}{\sqrt{n}}\right)\right)p_n(q)\right|<\epsilon\sum_{q\geq0}p_n(q)=\epsilon.$$

But there also exists $B > 0$ such that for all $b \geq B$:

$$\left| \int_{[0,b]} f(y)p(y)dy - \int_E f(y)p(y)dy \right| < \epsilon,$$

which ultimately gives us uniform convergence of f_n to f on E.
Moreover, $f_n(0) = f(0)$, $f_n'(0) = \frac{\lambda}{\lambda_n} f'(0)$, and $\sum_{r \geq 0} f_n\left(\frac{r}{\sqrt{n}}\right) p_n(r) = I_f$. Consequently, we have:

$$f_n'(0) = \frac{\lambda}{\lambda_n} f'(0) = \frac{\mu}{\lambda_n}\left(f(0) - I_f\right) = \frac{\mu}{\lambda_n}\left(f_n(0) - \sum_{r \geq 0} f_n\left(\frac{r}{\sqrt{n}}\right) p_n(r)\right),$$

which concludes the proof.

□

We can now start the proof of the weak convergence result:

Proof of Theorem 3.1.1: The proof is based on the four following steps:

(1) We first prove the convergence of the generator of the discrete-space process.

(2) We then show that the limiting generator generates a Feller semigroup using Theorem 2.1.3.

(3) We deduce convergence of the semigroup of the discrete-space process using Theorem 2.1.1.

(4) We finally establish the weak convergence result using Theorem 2.1.2.

- step 1: *convergence of the sequence of generators*

Let $f \in \mathrm{Dom}(A)$, and consider the following sequence of functions f_n on E:

$$f_n(x) = f\left(\frac{\lambda}{\lambda_n} x\right) + \frac{x^2 e^{-x}}{\sum_{q \geq 0} \frac{q^2}{n} e^{-\frac{q}{\sqrt{n}}} p_n(q)}\left(I_f - \sum_{q \geq 0} f\left(\frac{\lambda q}{\lambda_n \sqrt{n}}\right) p_n(q)\right).$$

According to Lemma 3.1.1, $\lim_{n \to +\infty} \|f_n - f\| = 0$ and f_n satisfies:

$$f_n'(0) = \frac{\mu}{\lambda_n}\left(f_n(0) - \sum_{r \geq 0} f_n\left(\frac{r}{\sqrt{n}}\right) p_n(r)\right).$$

In preparation of the application of Theorem 2.1.1, we are now going to show that:

$$\lim_{n \to +\infty} \sup_{x \in E_n} |A_n f_n(x) - A f(x)| = 0.$$

For all $x \in E_n \setminus \{0\}$, we have:

$$\begin{aligned} A_n f_n(x) &= \lambda_n n \left(\frac{1}{\sqrt{n}} f_n'(x) + \frac{1}{2n} f_n''(x) + o\left(\frac{1}{n}\right) \right) \\ &\quad + \mu_n n \left(-\frac{1}{\sqrt{n}} f_n'(x) + \frac{1}{2n} f_n''(x) + o\left(\frac{1}{n}\right) \right) \\ &\quad + \theta_n n \left(-\frac{1}{\sqrt{n}} f_n'(x) + \frac{1}{2n} f_n''(x) + o\left(\frac{1}{n}\right) \right) \\ &\quad + \lambda_n^0 n \left(f_n\left(\frac{1}{\sqrt{n}}\right) - f_n(x) \right) \\ &= \frac{\lambda_n + \theta_n}{2} f_n''(x) + \left(\sqrt{n}(\lambda_n - \theta_n) - \mu\right) f_n'(x) \\ &\quad + \lambda^0 \left(f_n\left(\frac{1}{\sqrt{n}}\right) - f_n(x) \right) + \frac{\mu}{2\sqrt{n}} f_n''(x) \\ &\quad + n(\lambda_n + \theta_n) o\left(\frac{1}{n}\right) + \sqrt{n}\mu o\left(\frac{1}{n}\right). \end{aligned}$$

Consequently, we obtain:

$$\begin{aligned} |A_n f_n(x) - Af(x)| &\leq \left| \frac{\lambda_n + \theta_n}{2} f_n''(x) - \lambda f''(x) \right| \\ &\quad + \left| \sqrt{n}(\lambda_n - \theta_n) f_n'(x) - c f'(x) \right| + \mu \left| f_n'(x) - f'(x) \right| \\ &\quad + \lambda^0 \left| \left(f_n\left(\frac{1}{\sqrt{n}}\right) - f_n(x) \right) - \left(f(0) - f(x)\right) \right| \\ &\quad + \frac{\mu}{2\sqrt{n}} \left| f_n''(x) \right| + n\left(\lambda_n + \theta_n + \frac{\mu}{\sqrt{n}} \right) o\left(\frac{1}{n}\right). \end{aligned}$$

For the sake of clarity, we introduce the following quantities:

$$C_n = \frac{1}{\sum_{q \geq 0} \frac{q^2}{n} e^{-\frac{q}{\sqrt{n}}} p_n(q)} \left(I_f - \sum_{q \geq 0} f\left(\frac{\lambda q}{\lambda_n \sqrt{n}} \right) p_n(q) \right),$$

$$\epsilon_n = n \left(\lambda_n + \theta_n + \frac{\mu}{\sqrt{n}} \right) o\left(\frac{1}{n}\right), \quad u_n = \frac{\lambda}{\lambda_n},$$

as well as the function $g \in \mathbb{C}^2_{b,\infty}(E)$ defined by $g(x) = x^2 e^{-x}$. As proven in Lemma 3.1.1, we recall that $\lim_{n\to\infty} C_n = 0$. With this notation in mind, we then observe that for all $n \in \mathbb{N}^*$:

$$\left\| f_n' \right\| \leq u_n \left\| f' \right\| + |C_n| \left\| g' \right\| \quad \text{and} \quad \left\| f_n'' \right\| \leq u_n^2 \left\| f'' \right\| + |C_n| \left\| g'' \right\|.$$

We now have:

$$\begin{aligned}
\sup_{x\in E_n\setminus\{0\}} |A_n f_n(x) - Af(x)| \quad &\leq \quad \left|\frac{\lambda_n+\theta_n}{2} - \lambda\right| \left(u_n^2 \left\|f''\right\| + |C_n| \left\|g''\right\|\right) \\
&+ \quad \lambda u_n^2 \sup_{x\in E_n\setminus\{0\}} \left|f''(u_n x) - f''(x)\right| \\
&+ \quad \lambda \left|u_n^2 - 1\right| \left\|f''\right\| + \lambda C_n \left\|g''\right\| \\
&+ \quad u_n \left|\sqrt{n}(\lambda_n - \theta_n)\right| \sup_{x\in E_n\setminus\{0\}} \left|f'(u_n x) - f'(x)\right| \\
&+ \quad \left|u_n\sqrt{n}(\lambda_n-\theta_n) - c\right| \left\|f'\right\| \\
&+ \quad C_n \left|\sqrt{n}(\lambda_n-\theta_n)\right| \left\|g'\right\| \\
&+ \quad \mu u_n \sup_{x\in E_n\setminus\{0\}} \left|f'(u_n x) - f'(x)\right| \\
&+ \quad \mu |u_n - 1| \left\|f'\right\| + \mu C_n \left\|g'\right\| \\
&+ \quad \lambda^0 \left| f_n\left(\frac{1}{\sqrt{n}}\right) - f_n(0)\right| + 2\lambda^0 \|f_n - f\| \\
&+ \quad \frac{\mu}{2\sqrt{n}} \left(u_n^2 \left\|f''\right\| + |C_n| \, \|g\|\right) + \epsilon_n.
\end{aligned}$$

Using analogous arguments to those used in the proof of Lemma 3.1.1, we see that:

$$\lim_{n\to\infty} \sup_{x\in E_n\setminus\{0\}} \left|f'(u_n x) - f'(x)\right| = 0 \text{ and } \lim_{n\to\infty} \sup_{x\in E_n\setminus\{0\}} \left|f''(u_n x) - f''(x)\right| = 0.$$

According to assumptions $(A1)$, $(A2)$, $(A3)$ and the continuity of f_n at 0, we obtain:

$$\lim_{n\to+\infty} \sup_{x\in E_n\setminus\{0\}} |A_n f_n(x) - Af(x)| = 0. \tag{3.1}$$

As for the boundary case, we can write:

$$\begin{aligned}
A_n f_n(0) &= \lambda_n n \left(\frac{1}{\sqrt{n}} f_n'(0) + \frac{1}{2n} f_n''(0) + o\left(\frac{1}{n}\right)\right) \\
&\quad + \mu_n n \sum_{r\in\mathbb{N}} p_n(r) \left(f_n\left(\frac{r}{\sqrt{n}}\right) - f_n(0)\right) \\
&= \frac{\lambda_n}{2} f_n''(0) + \sqrt{n} \underbrace{\left(\lambda_n f_n'(0) + \mu \sum_{r\geq 0} f_n\left(\frac{r}{\sqrt{n}}\right) p_n(r) - \mu f_n(0)\right)}_{=0} \\
&\quad + \lambda_n n o\left(\frac{1}{n}\right) \\
&= \frac{\lambda_n}{2} f_n''(0) + \lambda_n n o\left(\frac{1}{n}\right).
\end{aligned}$$

It follows that:

$$\begin{aligned}|A_n f_n(0) - Af(0)| &\leq \frac{1}{2}\left|\lambda_n u_n^2 f''(0) + 2C_n - \lambda f''(0)\right| + \lambda_n n o\left(\frac{1}{n}\right) \\ &\leq \frac{1}{2}\left|f''(0)\right| u_n^2 \left|\lambda_n - \lambda\right| + |C_n| + \lambda_n n o\left(\frac{1}{n}\right).\end{aligned}$$

Given assumption $(A1)$:

$$\lim_{n\to+\infty} |A_n f_n(0) - Af(0)| = 0. \tag{3.2}$$

Finally, putting (3.1) and (3.2) together yields:

$$\lim_{n\to+\infty} \sup_{x\in E_n} |A_n f_n(x) - Af(x)| = 0.$$

- step 2: *$\bar{A}$ is the infinitesimal generator of a Feller semigroup*

The linear operator A is a particular case of the Waldenfels integro-differential operator introduced in Chapter 2, more specifically with constant coefficients clearly satisfying the required conditions. As for the nonlocal Wentzell boundary condition, it is immediate to see that p satisfies $\lim_{\epsilon\to 0}\int_{|y|<\epsilon} |y|p(y)dy = 0$ and $\int_{|y|\geq\epsilon} p(y)dy < C(\epsilon)$ for $\epsilon > 0$. Theorem 2.1.3 thus establishes that $\bar{A}$ is the infinitesimal generator of a Feller semigroup $T = (T(t))_{t\geq 0}$ on $\mathbb{B}(E)$.

- step 3: *convergence of the sequence of semigroups*

In order to obtain the equivalence between convergence of generators and convergence of semigroups, we need to check that the conditions of Theorem 2.1.1 are verified. To start with, $T_n = (T_n(t))_{t\geq 0}$ is a contraction semigroup as for all $f \in \mathbb{B}(E_n)$ and $x \in E_n$, we can write:

$$|T_n(t)f(x)| = |\mathbb{E}(f(X_n(t))|X_n(0) = x)| \leq \|f\|.$$

The same argument can be applied to $T = (T(t))_{t\geq 0}$, for all $f \in \mathbb{B}(E)$ and $x \in E$. Being a Feller semigroup, T is necessarily strongly continuous. As for the strong continuity of T_n, for $x > 0$, we have:

$$\begin{aligned}\sup_{x\in E_n\setminus\{0\}} |T_n(t)f(x) - f(x)| \leq\ & 2\|f\|\big(\lambda_n nt + o(nt)\big) \\ & +2\|f\|\big((\mu_n + \theta_n)nt + o(nt)\big) \\ & +2\|f\|\left(\lambda_n^0 nt + o(nt)\right).\end{aligned}$$

We therefore infer that:

$$\lim_{t\to 0} \sup_{x\in E_n\setminus\{0\}} |T_n(t)f(x) - f(x)| = 0.$$

As for the boundary, we can write:

$$|T_n(t)f(0) - f(0)| \leq 2\,\|f\|\,\big((\lambda_n + \mu_n)nt + o(nt)\big)\,,$$

from which we deduce that:

$$\lim_{t\to 0}\sup_{x\in E_n}|T_n(t)f(x) - f(x)| = 0,$$

and the strong continuity of T_n is thus established. Furthermore, define $\eta_n : E_n \to E$ with $\eta_n(x) = x$ and $\pi_n : \mathbb{B}(E) \to \mathbb{B}(E_n)$ with $\pi_n(f) = f \circ \eta_n$. The conditions of Theorem 2.1.1 are now verified, and we obtain the convergence of the sequence of semigroups.

- step 4: *weak convergence of* X_n

We can finally apply Theorem 2.1.2, taking π_n and η_n as previously defined, to conclude that there exists an E-valued Feller (thus strong Markov) process $X = (X(t))_{t\geq 0}$ with sample paths in $\mathbb{D}([0,+\infty[, E)$ corresponding to T (and therefore with generator $\bar{A}$) such that $X_n \Rightarrow X$.

□

We now see that, choosing $\gamma = \frac{\mu}{\lambda}$ in the presentation of regenerative elastic Brownian motion with random jumps to the origin carried out in Section 2.2.4 of the previous chapter, the infinitesimal generator introduced in Theorem 3.1.1 effectively corresponds to this process.

3.2 A two-sided reduced-form limit order book model

Building on the previous result, this section aims to provide a generalisation of the one-sided reduced-form diffusion approximation to a two-sided limit order book with two price levels on each side.

3.2.1 The discrete order book process

For each $n \in \mathbb{N}^*$, we define a four-dimensional process $Z_n = (X_n^2, X_n^1, Y_n^1, Y_n^2)$ with values in $\mathbb{N}^4$, representing the discrete order book process. The process X_n^2 (respectively Y_n^2) corresponds to the number of outstanding orders at the second best bid level (respectively second best ask level), whilst X_n^1 (respectively Y_n^1) is the process associated with the best bid (respectively best ask). Note that we adopt the natural convention of denoting the four components of the process in "visual" order as opposed to numerical order.

As in the previous section, it is assumed that all order and cancellation sizes are equal to 1 without loss of generality. Given a realisation (i_2, i_1, j_1, j_2) of the process

Z_n, its transitions which *do not* lead to price changes are described in the following way:

$$(i_2, i_1, j_1, j_2) \to (i_2, i_1 + 1, j_1, j_2) \text{ at rate } \lambda_n^{b,1},$$
$$(i_2, i_1, j_1, j_2) \to (i_2 + 1, i_1, j_1, j_2) \text{ at rate } \lambda_n^{b,2},$$
$$(i_2, i_1, j_1, j_2) \to (i_2, i_1, j_1 + 1, j_2) \text{ at rate } \lambda_n^{a,1},$$
$$(i_2, i_1, j_1, j_2) \to (i_2, i_1, j_1, j_2 + 1) \text{ at rate } \lambda_n^{a,2},$$
$$(i_2, i_1, j_1, j_2) \to (i_2, i_1 - 1, j_1, j_2) \text{ at rate } (\mu_n^a + \theta_n^{b,1}) \text{ for } i_1 \geq 1,$$
$$(i_2, i_1, j_1, j_2) \to (i_2, i_1, j_1 - 1, j_2) \text{ at rate } (\mu_n^b + \theta_n^{a,1}) \text{ for } j_1 \geq 1,$$
$$(i_2, i_1, j_1, j_2) \to (i_2 - 1, i_1, j_1, j_2) \text{ at rate } \theta_n^{b,2} \text{ for } i_2 \geq 1,$$
$$(i_2, i_1, j_1, j_2) \to (i_2, i_1, j_1, j_2 - 1) \text{ at rate } \theta_n^{a,2} \text{ for } j_2 \geq 1.$$

Also without loss of generality, we shall assume that the spread is constantly equal to 1 tick. As a result, we shall additionally suppose that when the best bid (respectively ask) queue is depleted, an incoming ask (respectively bid) market order causes its new value to be equal to the "previous" second best bid (respectively ask) queue. The "previous" second best bid (respectively ask) then takes a new value b_∞ (respectively a_∞). In order to maintain a constant spread, we assume the existence of a centralised market maker which *instantaneously* posts a quantity a_0 of sell limit orders (respectively b_0 buy limit orders) inside the spread. The second best ask (respectively bid) queue therefore takes the value of the "previous" best ask (respectively bid) queue. In other words, the transitions of the process Z_n which trigger price changes are given by:

$$(i_2, 0, j_1, j_2) \to (b_\infty, i_2, a_0, j_1) \text{ at rate } \mu_n^a,$$
$$(i_2, i_1, 0, j_2) \to (i_1, b_0, j_2, a_\infty) \text{ at rate } \mu_n^b.$$

3.2.2 Heavy traffic diffusion approximation

As in the one-sided model, we rescale the process by accelerating time by a factor of n and dividing the volumes by $\sqrt{n}$. Denoting $\hat{Z}_n$ the rescaled order book process on $E_n = \frac{1}{\sqrt{n}}\mathbb{N}^4$, we have:

$$\hat{Z}_n(t) = \frac{Z_n(nt)}{\sqrt{n}}.$$

We also introduce the rescaled quantities $(\hat{b}_0^n, \hat{b}_\infty^n, \hat{a}_0^n, \hat{a}_\infty^n) \in \frac{1}{\sqrt{n}}(\mathbb{N}^*)^4$ defined by:

$$\hat{b}_0^n = \frac{b_0}{\sqrt{n}}, \;\; \hat{b}_\infty^n = \frac{b_\infty}{\sqrt{n}}, \;\; \hat{a}_0^n = \frac{a_0}{\sqrt{n}}, \;\; \hat{a}_\infty^n = \frac{a_\infty}{\sqrt{n}}.$$

We stress that in order to obtain nontrivial limits, the quantities b_0, b_∞, a_0 and a_∞ implicitly depend on n. We shall assume that (where $\lambda^{b,1}$, $\lambda^{b,2}$, $\lambda^{a,1}$, $\lambda^{a,2}$, $c^{b,1}$, $c^{b,2}$, $c^{a,1}$, $c^{a,2}$, μ^b, μ^a, $\hat{b}_0$, $\hat{a}_0$, $\hat{b}_\infty$ and $\hat{a}_\infty$ are given constants, $E = \mathbb{R}_+^4$ and $z = (x_2, x_1, y_1, y_2)$):

- *(A1)* $\lim_{n\to+\infty} \lambda_n^{b,i} = \lambda^{b,i}$, for $i \in \{1,2\}$,
- *(A2)* $\lim_{n\to+\infty} \theta_n^{b,i} = \lambda^{b,i}$, for $i \in \{1,2\}$,
- *(A3)* $\lim_{n\to+\infty} \lambda_n^{a,i} = \lambda^{a,i}$, for $i \in \{1,2\}$,
- *(A4)* $\lim_{n\to+\infty} \theta_n^{a,i} = \lambda^{a,i}$, for $i \in \{1,2\}$,
- *(A5)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n^{b,i} - \theta_n^{b,i}) = c^{b,i}$, for $i \in \{1,2\}$,
- *(A6)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n^{a,i} - \theta_n^{a,i}) = c^{a,i}$, for $i \in \{1,2\}$,
- *(A7)* $\mu_n^b = \frac{\mu^b}{\sqrt{n}}$, $\mu_n^a = \frac{\mu^a}{\sqrt{n}}$,
- *(A8)* $c^{b,1} - \mu^a < 0$, $c^{a,1} - \mu^b < 0$,
- *(A9)* $\lim_{n\to+\infty} \sqrt{n}(\hat{b}_0^n - \hat{b}_0) = 0$, $\lim_{n\to+\infty} \sqrt{n}\left(\hat{a}_0^n - \hat{a}_0\right) = 0$,
- *(A10)* $\lim_{n\to+\infty} \sqrt{n}(\hat{b}_\infty^n - \hat{b}_\infty) = 0$, $\lim_{n\to+\infty} \sqrt{n}\left(\hat{a}_\infty^n - \hat{a}_\infty\right) = 0$,
- *(A11)* Defining $u_n = \frac{\lambda^{b,1}}{\lambda_n^{b,1}}$, and $\tilde{u}_n = \frac{\lambda^{a,1}}{\lambda_n^{a,1}}$, we have, for all $f \in \mathbb{C}^2_{b,\infty}(E)$:

$$\lim_{n\to+\infty} \sqrt{n} \sup_{z\in E_n} \left| f(u_n x_2, u_n x_1, \tilde{u}_n y_1, \tilde{u}_n y_2) - f(x_2, x_1, y_1, y_2) \right| = 0.$$

We now extend the semigroup approach used in the one-sided case to the two-sided order book. To this end, let $(T_n(t))_{t\geq 0}$ be the semigroup on $\mathbb{B}(E_n)$ defined by:

$$T_n(t)f(z) = \mathbb{E}\left(f(\hat{Z}_n(t)) | \hat{Z}_n(0) = z\right),$$

for all $f \in \mathbb{B}(E_n)$ and $z = (x_2, x_1, y_1, y_2) \in E_n$. Its infinitesimal generator A_n is given by, for all $f \in \mathbb{B}(E_n)$ and $z \in E_n$:

$$\begin{aligned}
A_n f(z) &= \lim_{t\to 0} \frac{T_n(t)f(z) - f(z)}{t} \\
&= \lambda_n^{b,1} n \left(f\left(x_2, x_1 + \frac{1}{\sqrt{n}}, y_1, y_2\right) - f(z) \right) \\
&\quad + \lambda_n^{b,2} n \left(f\left(x_2 + \frac{1}{\sqrt{n}}, x_1, y_1, y_2\right) - f(z) \right) \\
&\quad + \lambda_n^{a,1} n \left(f\left(x_2, x_1, y_1 + \frac{1}{\sqrt{n}}, y_2\right) - f(z) \right) \\
&\quad + \lambda_n^{a,2} n \left(f\left(x_2, x_1, y_1, y_2 + \frac{1}{\sqrt{n}}\right) - f(z) \right)
\end{aligned}$$

$$
\begin{aligned}
&+(\mu_n^a + \theta_n^{b,1})n \left(f\left(x_2, x_1 - \frac{1}{\sqrt{n}}, y_1, y_2\right) - f(z)\right) \mathbb{1}_{\{x_1 \geq \frac{1}{\sqrt{n}}\}} \\
&+(\mu_n^b + \theta_n^{a,1})n \left(f\left(x_2, x_1, y_1 - \frac{1}{\sqrt{n}}, y_2\right) - f(z)\right) \mathbb{1}_{\{y_1 \geq \frac{1}{\sqrt{n}}\}} \\
&+\theta_n^{b,2} n \left(f\left(x_2 - \frac{1}{\sqrt{n}}, x_1, y_1, y_2\right) - f(z)\right) \mathbb{1}_{\{x_2 \geq \frac{1}{\sqrt{n}}\}} \\
&+\theta_n^{a,2} n \left(f\left(x_2, x_1, y_1, y_2 - \frac{1}{\sqrt{n}}\right) - f(z)\right) \mathbb{1}_{\{y_2 \geq \frac{1}{\sqrt{n}}\}} \\
&+\mu_n^a n \left(f\left(\hat{b}_\infty^n, x_2, \hat{a}_0^n, y_1\right) - f(z)\right) \mathbb{1}_{\{x_1 = 0\}} \\
&+\mu_n^b n \left(f\left(x_1, \hat{b}_0^n, y_2, \hat{a}_\infty^n\right) - f(z)\right) \mathbb{1}_{\{y_1 = 0\}}.
\end{aligned}
$$

The heavy traffic diffusion approximation on E can then be obtained in an analogous way to the one-sided case, as shown in the following result:

Theorem 3.2.1 *The E_n-valued process $\hat{Z}_n$ converges weakly in $\mathbb{D}([0, +\infty[, E)$ as $n \to \infty$ to an E-valued strong Markov jump-diffusion process Z with infinitesimal generator given by the closure $\bar{A}$ of the linear operator A defined by, for all $z = (x_2, x_1, y_1, y_2) \in E$:*

$$
\begin{aligned}
Af(z) \;=\; & \frac{1}{2}\left(\lambda^{b,1} + \lambda^{b,1}\mathbb{1}_{\{x_1>0\}}\right)\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}\left(\lambda^{a,1} + \lambda^{a,1}\mathbb{1}_{\{y_1>0\}}\right)\frac{\partial^2 f}{\partial y_1^2} \\
&+\frac{1}{2}\left(\lambda^{b,2} + \lambda^{b,2}\mathbb{1}_{\{x_2>0\}}\right)\frac{\partial^2 f}{\partial x_2^2} + \frac{1}{2}\left(\lambda^{a,2} + \lambda^{a,2}\mathbb{1}_{\{y_2>0\}}\right)\frac{\partial^2 f}{\partial y_2^2} \\
&+(c^{b,1} - \mu^a)\mathbb{1}_{\{x_1>0\}}\frac{\partial f}{\partial x_1} + (c^{a,1} - \mu^b)\mathbb{1}_{\{y_1>0\}}\frac{\partial f}{\partial y_1} \\
&+c^{b,2}\frac{\partial f}{\partial x_2} + c^{a,2}\frac{\partial f}{\partial y_2},
\end{aligned}
$$

acting on $\mathrm{Dom}(A)$*, the space of $\mathbb{C}^2_{b,\infty}(E)$ functions satisfying:*

$$
\begin{aligned}
\left.\frac{\partial f}{\partial x_1}\right|_{x_1=0} &= \frac{\mu^a}{\lambda^{b,1}}\left(f(x_2, 0, y_1, y_2) - f(\hat{b}_\infty, x_2, \hat{a}_0, y_1)\right), \\
\left.\frac{\partial f}{\partial y_1}\right|_{y_1=0} &= \frac{\mu^b}{\lambda^{a,1}}\left(f(x_2, x_1, 0, y_2) - f(x_1, \hat{b}_0, y_2, \hat{a}_\infty)\right), \\
\left.\frac{\partial f}{\partial x_2}\right|_{x_2=0} &= \left.\frac{\partial f}{\partial y_2}\right|_{y_2=0} = 0.
\end{aligned}
$$

As in the one-sided case, we shall require a technical lemma so as to facilitate the application of Theorem 2.1.1:

Lemma 3.2.1 *Let $f \in \mathrm{Dom}(A)$ and define the following sequence of functions f_n on E:*

$$f_n(x_2, x_1, y_1, y_2) = f\left(u_n x_2, u_n x_1, \tilde{u}_n y_1, \tilde{u}_n y_2\right).$$

Then $\lim_{n\to+\infty} \|f_n - f\| = 0$ and f_n satisfies:

$$\lim_{n\to+\infty} \sqrt{n} \sup_{z\in E_n, x_1=0} \left| \frac{\partial f_n}{\partial x_1} - \frac{\mu^a}{\lambda_n^{b,1}} \left(f_n(x_2, 0, y_1, y_2) - f_n(\hat{b}_\infty^n, x_2, \hat{a}_0^n, y_1) \right) \right| = 0,$$

$$\lim_{n\to+\infty} \sqrt{n} \sup_{z\in E_n, y_1=0} \left| \frac{\partial f_n}{\partial y_1} - \frac{\mu^b}{\lambda_n^{a,1}} \left(f_n(x_2, x_1, 0, y_2) - f_n(x_1, \hat{b}_0^n, y_2, \hat{a}_\infty^n) \right) \right| = 0,$$

$$\left. \frac{\partial f_n}{\partial x_2} \right|_{x_2=0} = \left. \frac{\partial f_n}{\partial y_2} \right|_{y_2=0} = 0.$$

Proof of Lemma 3.2.1: To start with, the uniform convergence of f_n to f is based on a straightforward generalisation of the arguments presented in Lemma 3.1.1 which enabled us to obtain:

$$\lim_{n\to+\infty} \sup_{x\in E} \left| f\left(\frac{\lambda}{\lambda_n} x\right) - f(x) \right| = 0.$$

Next, we see that:

$$\begin{aligned}
\left. \frac{\partial f_n}{\partial x_1} \right|_{x_1=0} &= \frac{\mu^a}{\lambda_n^{b,1}} \left(f(u_n x_2, 0, \tilde{u}_n y_1, \tilde{u}_n y_2) - f(\hat{b}_\infty, u_n x_2, \hat{a}_0, \tilde{u}_n y_1) \right) \\
&= \frac{\mu^a}{\lambda_n^{b,1}} \left(f_n(x_2, 0, y_1, y_2) - f_n\left(\frac{1}{u_n} \hat{b}_\infty, x_2, \frac{1}{\tilde{u}_n} \hat{a}_0, y_1 \right) \right).
\end{aligned}$$

As a result, defining:

$$F_n^1(z) = \left. \frac{\partial f_n}{\partial x_1} \right|_{x_1=0} - \frac{\mu^a}{\lambda_n^{b,1}} \left(f_n(x_2, 0, y_1, y_2) - f_n(\hat{b}_\infty^n, x_2, \hat{a}_0^n, y_1) \right),$$

we have:

$$\begin{aligned}
F_n^1(z) &= \frac{\mu^a}{\lambda_n^{b,1}} \left(f_n(\hat{b}_\infty^n, x_2, \hat{a}_0^n, y_1) - f_n\left(\frac{1}{u_n} \hat{b}_\infty, x_2, \frac{1}{\tilde{u}_n} \hat{a}_0, y_1 \right) \right) \\
&= \frac{\mu^a}{\lambda_n^{b,1}} \left(f(u_n \hat{b}_\infty^n, u_n x_2, \tilde{u}_n \hat{a}_0^n, \tilde{u}_n y_1) - f(\hat{b}_\infty, u_n x_2, \hat{a}_0, \tilde{u}_n y_1) \right).
\end{aligned}$$

Using assumptions $(A9)$, $(A10)$ and $(A11)$, this implies:

$$\lim_{n\to+\infty} \sqrt{n} \sup_{z\in E_n} \left| F_n^1(z) \right| = 0.$$

Moreover, if we define:

$$F_n^2(z) = \left.\frac{\partial f_n}{\partial y_1}\right|_{y_1=0} - \frac{\mu^b}{\lambda_n^{a,1}} \left(f_n(x_2, x_1, 0, y_2) - f_n(x_1, \hat{b}_0^n, y_2, \hat{a}_\infty^n) \right),$$

we similarly show that:

$$\lim_{n\to+\infty} \sqrt{n} \sup_{z\in E_n} \left| F_n^2(z) \right| = 0.$$

Finally, it is easily seen that:

$$\left.\frac{\partial f_n}{\partial x_2}\right|_{x_2=0} = u_n \frac{\partial}{\partial x_2} f(0, u_n x_1, \tilde{u}_n y_1, \tilde{u}_n y_2) = 0,$$

$$\left.\frac{\partial f_n}{\partial y_2}\right|_{y_2=0} = \tilde{u}_n \frac{\partial}{\partial y_2} f(u_n x_2, u_n x_1, \tilde{u}_n y_1, 0) = 0,$$

which ends the proof. □

The proof of Theorem 3.2.1 has been omitted here as it is a particular case of the proof of the multidimensional diffusion approximation result presented in the next chapter.

4 Multidimensional limit order book models

This chapter aims to provide diffusion approximations of the limit order book in a multidimensional setting. We start by looking at the relative price grid framework, before examining the absolute price grid case.

4.1 A multidimensional limit order book model on a relative price grid

In this first section we derive a heavy traffic diffusion appproximation of the entire limit order book from a relative price grid point of view. One key feature of the diffusion limit in this case is the existence of nearest neighbour interactions between the different queues, leading to a more realistic representation of the limit order book's shape based on empirical considerations. Moreover, as suggested by Zovko and Farmer [63], order arrival rates now depend on the distance in ticks to the best (opposite) quote.

4.1.1 The discrete order book process

Let $N \in \mathbb{N}^*$, and for each $n \in \mathbb{N}^*$, consider a $2N$-dimensional process $Z_n^N = (X_n^N, \ldots, X_n^1, Y_n^1, \ldots, Y_n^N)$ with values in $\mathbb{N}^{2N}$, representing the discrete order book process. For each $m \in \{1, \ldots, N\}$, the process X_n^m (respectively Y_n^m) corresponds to the number of outstanding orders at the m-th best bid level (respectively m-th best ask level). As in the previous chapter, we adopt the natural convention of denoting the $2N$ components of the process in "visual" order instead of numerical order. As far as order and cancellation sizes are concerned, they are once again assumed to be equal to 1 without loss of generality. Let $v = (i_N, \ldots, i_1, j_1, \ldots, j_N)$ be a realisation of the process Z_n^N. For $m \in \{1, \ldots, N\}$, we define $v^{b,m\pm1} = (i_N, \ldots, i_m \pm 1, \ldots, i_1, j_1, \ldots, j_N)$ and $v^{a,m\pm1} = (i_N, \ldots, i_1, j_1, \ldots, j_m \pm 1, \ldots, j_N)$. The transitions of Z_n^N which *do not* lead to price changes can be summarised in the following way:

$$v \to v^{b,m+1} \text{ at rate } \Lambda_n^{b,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m \in \{1, \ldots, N\},$$

$$v \to v^{a,m+1} \text{ at rate } \Lambda_n^{a,m}(j_{m-1}, j_m, j_{m+1}) \text{ for } m \in \{1, \ldots, N\},$$

$$v \to v^{b,m-1} \text{ at rate } \mu_n^a \mathbb{1}_{\{i_m \geq 1\}} + \Theta_n^{b,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m = 1,$$
$$v \to v^{a,m-1} \text{ at rate } \mu_n^b \mathbb{1}_{\{j_m \geq 1\}} + \Theta_n^{a,m}(j_{m-1}, j_m, j_{m+1}) \text{ for } m = 1,$$
$$v \to v^{b,m-1} \text{ at rate } \Theta_n^{b,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m \in \{2, \ldots, N\},$$
$$v \to v^{a,m-1} \text{ at rate } \Theta_n^{a,m}(j_{m-1}, j_m, j_{m+1}) \text{ for } m \in \{2, \ldots, N\}.$$

We note that the limit order arrival rates and cancellation rates now contain a nearest neighbour interaction. Heuristically speaking, we would like the queue at any given level $m \in \{1, \ldots, N\}$ to be more likely to increase if its size is less than those of the queues at levels $m-1$ and $m+1$. Conversely, we would like this same queue to be more likely to decrease when its size is greater than those of the queues at levels $m-1$ and $m+1$. This intuition can be rigorously translated into the following formal definition of the limit order arrival and cancellation rates for any given $m \in \{1, \ldots, N\}$ (with the natural convention $i_{N+1} = i_0 = j_0 = j_{N+1} = 0$):

$$\Lambda_n^{b,m}(i_{m-1}, i_m, i_{m+1}) = \lambda_n^{b,m} + \gamma_n^{b,m}\left((i_{m-1} - i_m)^+ + (i_{m+1} - i_m)^+\right),$$
$$\Lambda_n^{a,m}(j_{m-1}, j_m, j_{m+1}) = \lambda_n^{a,m} + \gamma_n^{a,m}\left((j_{m-1} - j_m)^+ + (j_{m+1} - j_m)^+\right),$$
$$\Theta_n^{b,m}(i_{m-1}, i_m, i_{m+1}) = \theta_n^{b,m} \mathbb{1}_{\{i_m \geq 1\}} + \gamma_n^{b,m}\left((i_m - i_{m-1})^+ + (i_m - i_{m+1})^+\right),$$
$$\Theta_n^{a,m}(j_{m-1}, j_m, j_{m+1}) = \theta_n^{a,m} \mathbb{1}_{\{j_m \geq 1\}} + \gamma_n^{a,m}\left((j_m - j_{m-1})^+ + (j_m - j_{m+1})^+\right).$$

As in the previous model, we assume that the spread is constantly equal to 1 tick, in such a way that when the best bid (respectively ask) queue is depleted, an incoming ask (respectively bid) market order causes its new value to be equal to the "previous" second best bid (respectively ask) queue. The "previous" second best bid (respectively ask) then takes the value of the "previous" third best bid (respectively ask) queue, and so on, whilst the "previous" N-th best bid (respectively ask) queue takes a new value equal to the constant b_∞ (respectively a_∞). In order to maintain a constant spread, we assume the existence of a centralised market maker which *instantaneously* posts a quantity a_0 of sell limit orders (respectively b_0 buy limit orders) inside the spread. Consequently, the second best ask (respectively bid) queue takes the value of the "previous" best ask (respectively bid) queue, and the N-th best ask (respectively bid) queue becomes equal to the value of the "previous" $(N-1)$-th best ask (respectively bid) queue. In other words, the transitions of the process Z_n^N which cause price changes are given by:

$$v \to (b_\infty, i_N, \ldots, i_3, i_2, a_0, j_1, \ldots, , j_{N-2}, j_{N-1}) \text{ at rate } \mu_n^a \mathbb{1}_{\{i_1 = 0\}},$$
$$v \to (i_{N-1}, i_{N-2}, \ldots, i_1, b_0, j_2, j_3, \ldots, j_N, a_\infty) \text{ at rate } \mu_n^b \mathbb{1}_{\{j_1 = 0\}}.$$

4.1.2 Heavy traffic diffusion approximation

The rescaling of the process is then conducted by accelerating time by a factor of n and dividing the volumes by $\sqrt{n}$. Denoting $\hat{Z}_n^N$ the rescaled order book process on $E_n^N = \frac{1}{\sqrt{n}}\mathbb{N}^{2N}$, we have:

$$\hat{Z}_n^N(t) = \frac{Z_n^N(nt)}{\sqrt{n}}.$$

We also introduce the rescaled quantities $(\hat{b}_0^n, \hat{b}_\infty^n, \hat{a}_0^n, \hat{a}_\infty^n) \in \frac{1}{\sqrt{n}}(\mathbb{N}^*)^4$ defined by:

$$\hat{b}_0^n = \frac{b_0}{\sqrt{n}},\ \ \hat{b}_\infty^n = \frac{b_\infty}{\sqrt{n}},\ \ \hat{a}_0^n = \frac{a_0}{\sqrt{n}},\ \ \hat{a}_\infty^n = \frac{a_\infty}{\sqrt{n}}.$$

We shall assume that (where $\lambda^{b,m}$, $\lambda^{a,m}$, $c^{b,m}$, $c^{a,m}$, μ^b, μ^a, $\gamma^{b,m}$, $\gamma^{a,m}$ for each $m \in \{1,\dots,N\}$, $\hat{b}_0$, $\hat{a}_0$, $\hat{b}_\infty$ and $\hat{a}_\infty$ are given constants, where $E^N = \mathbb{R}_+^{2N}$ and $z = (x_N,\dots,x_1,y_1,\dots,y_N)$):

- *(A1)* $\lim_{n\to+\infty} \lambda_n^{b,m} = \lambda^{b,m}$, for $m \in \{1,\dots,N\}$,
- *(A2)* $\lim_{n\to+\infty} \theta_n^{b,m} = \lambda^{b,m}$, for $m \in \{1,\dots,N\}$,
- *(A3)* $\lim_{n\to+\infty} \lambda_n^{a,m} = \lambda^{a,m}$, for $m \in \{1,\dots,N\}$,
- *(A4)* $\lim_{n\to+\infty} \theta_n^{a,m} = \lambda^{a,m}$, for $m \in \{1,\dots,N\}$,
- *(A5)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n^{b,m} - \theta_n^{b,m}) = c^{b,m}$, for $m \in \{1,\dots,N\}$,
- *(A6)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n^{a,m} - \theta_n^{a,m}) = c^{a,m}$, for $m \in \{1,\dots,N\}$,
- *(A7)* $\mu_n^b = \frac{\mu^b}{\sqrt{n}},\ \ \mu_n^a = \frac{\mu^a}{\sqrt{n}}$,
- *(A8)* $\gamma_n^{b,m} = \frac{\gamma^{b,m}}{n},\ \ \gamma_n^{a,m} = \frac{\gamma^{a,m}}{n}$, for $m \in \{1,\dots,N\}$,
- *(A9)* $c^{b,1} - \mu^a < 0,\ \ c^{a,1} - \mu^b < 0$,
- *(A10)* $\lim_{n\to+\infty} \sqrt{n}(\hat{b}_0^n - \hat{b}_0) = 0,\ \ \lim_{n\to+\infty} \sqrt{n}\left(\hat{a}_0^n - \hat{a}_0\right) = 0$,
- *(A11)* $\lim_{n\to+\infty} \sqrt{n}(\hat{b}_\infty^n - \hat{b}_\infty) = 0,\ \ \lim_{n\to+\infty} \sqrt{n}\,(\hat{a}_\infty^n - \hat{a}_\infty) = 0$,
- *(A12)* Defining $u_n = \frac{\lambda_n^{b,1}}{\lambda_n^{b,1}}$, and $\tilde{u}_n = \frac{\lambda_n^{a,1}}{\lambda_n^{a,1}}$, we have, for all $f \in \mathbb{C}^2_{b,\infty}(E^N)$:

$$\lim_{n\to+\infty} \sqrt{n} \sup_{z\in E_n^N} \left| f(u_n x_N,\dots,u_n x_1,\tilde{u}_n y_1,\dots,\tilde{u}_n y_N) - f(z)\right| = 0.$$

Moving forward, let $(T_n^N(t))_{t\geq 0}$ be the semigroup on $\mathbb{B}(E_n^N)$ defined by:

$$T_n^N(t)f(z) = \mathbb{E}\left(f(\hat{Z}_n^N(t))|\hat{Z}_n^N(0) = z\right),$$

for all $f \in \mathbb{B}(E_n^N)$ and $z = (x_N, \ldots, x_1, y_1, \ldots, y_N) \in E_n^N$. For notational convenience, we define the following vectors for all $m \in \{1, \ldots, N\}$:

$$z_m^{b,\pm} = (x_N, \ldots, x_m \pm \frac{1}{\sqrt{n}}, \ldots, x_1, y_1, \ldots, y_N),$$

$$z_m^{a,\pm} = (x_N, \ldots, x_1, y_1, \ldots, y_m \pm \frac{1}{\sqrt{n}}, \ldots, y_N),$$

as well as the vectors associated with a price change:

$$\begin{aligned} z_n^b &= (\hat{b}_\infty^n, x_N, \ldots, x_3, x_2, \hat{a}_0^n, y_1, \ldots, , y_{N-2}, y_{N-1}), \\ z_n^a &= (x_{N-1}, x_{N-2}, \ldots, x_1, \hat{b}_0^n, y_2, y_3, \ldots, y_N, \hat{a}_\infty^n), \\ z^b &= (\hat{b}_\infty, x_N, \ldots, x_3, x_2, \hat{a}_0, y_1, \ldots, , y_{N-2}, y_{N-1}), \\ z^a &= (x_{N-1}, x_{N-2}, \ldots, x_1, \hat{b}_0, y_2, y_3, \ldots, y_N, \hat{a}_\infty). \end{aligned}$$

The generator A_n^N of $(T_n^N(t))_{t\geq 0}$ is then given by, for all $f \in \mathbb{B}(E_n^N)$ and $z \in E_n^N$:

$$\begin{aligned} A_n^N f(z) &= \lim_{t\to 0} \frac{T_n^N(t)f(z) - f(z)}{t} \\ &= \sum_{m=1}^N n\Lambda_n^{b,m}(\sqrt{n}x_{m-1}, \sqrt{n}x_m, \sqrt{n}x_{m+1}) \left(f(z_m^{b,+}) - f(z)\right) \\ &+ \sum_{m=1}^N n\Lambda_n^{a,m}(\sqrt{n}y_{m-1}, \sqrt{n}y_m, \sqrt{n}y_{m+1}) \left(f(z_m^{a,+}) - f(z)\right) \\ &+ \sum_{m=1}^N n\Theta_n^{b,m}(\sqrt{n}x_{m-1}, \sqrt{n}x_m, \sqrt{n}x_{m+1}) \left(f(z_m^{b,-}) - f(z)\right) \\ &+ \sum_{m=1}^N n\Theta_n^{a,m}(\sqrt{n}y_{m-1}, \sqrt{n}y_m, \sqrt{n}y_{m+1}) \left(f(z_m^{a,-}) - f(z)\right) \\ &+ n\mu_n^a \left(f(z_1^{b,-}) - f(z)\right) \mathbb{1}_{\{x_1 \geq \frac{1}{\sqrt{n}}\}} + n\mu_n^b \left(f(z_1^{a,-}) - f(z)\right) \mathbb{1}_{\{y_1 \geq \frac{1}{\sqrt{n}}\}} \\ &+ n \left(\mu_n^a \left(f(z_n^b) - f(z)\right) \mathbb{1}_{\{x_1=0\}} + \mu_n^b \left(f(z_n^a) - f(z)\right) \mathbb{1}_{\{y_1=0\}}\right) \\ &= \sum_{m=1}^N n \left(\lambda_n^{b,m} \left(f(z_m^{b,+}) - f(z)\right) + \lambda_n^{a,m} \left(f(z_m^{a,+}) - f(z)\right)\right) \\ &+ \sum_{m=1}^N \sqrt{n}\gamma^{b,m} \left((x_{m-1} - x_m)^+ + (x_{m+1} - x_m)^+\right) \left(f(z_m^{b,+}) - f(z)\right) \\ &+ \sum_{m=1}^N \sqrt{n}\gamma^{a,m} \left((y_{m-1} - y_m)^+ + (y_{m+1} - y_m)^+\right) \left(f(z_m^{a,+}) - f(z)\right) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m=1}^{N} n\theta_n^{b,m} \left(f(z_m^{b,-}) - f(z) \right) \mathbb{1}_{\{x_m \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sum_{m=1}^{N} n\theta_n^{a,m} \left(f(z_m^{a,-}) - f(z) \right) \mathbb{1}_{\{y_m \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sum_{m=1}^{N} \sqrt{n}\gamma^{b,m} \left((x_m - x_{m-1})^+ + (x_m - x_{m+1})^+ \right) \left(f(z_m^{b,+}) - f(z) \right) \\
&+ \sum_{m=1}^{N} \sqrt{n}\gamma^{a,m} \left((y_m - y_{m-1})^+ + (y_m - y_{m+1})^+ \right) \left(f(z_m^{a,+}) - f(z) \right) \\
&+ \sqrt{n}\mu^a \left(f(z_1^{b,-}) - f(z) \right) \mathbb{1}_{\{x_1 \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sqrt{n}\mu^b \left(f(z_1^{a,-}) - f(z) \right) \mathbb{1}_{\{y_1 \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sqrt{n} \left(\mu^a \left(f(z_n^b) - f(z) \right) \mathbb{1}_{\{x_1 = 0\}} + \mu^b \left(f(z_n^a) - f(z) \right) \mathbb{1}_{\{y_1 = 0\}} \right).
\end{aligned}
$$

The multidimensional heavy traffic diffusion approximation on E^N of the entire limit order book with interacting queues is the main result of this section:

Theorem 4.1.1 *The E_n^N-valued process $\hat{Z}_n^N$ converges weakly in $\mathbb{D}([0, +\infty[, E^N)$ as $n \to \infty$ to an E^N-valued strong Markov jump-diffusion process Z^N with infinitesimal generator given by the closure $\overline{A^N}$ of the linear operator A^N defined by, for all $z = (x_N, \ldots, x_1, y_1, \ldots, y_N) \in E^N$:*

$$
\begin{aligned}
A^N f(z) \quad = \quad & \frac{1}{2} \sum_{m=1}^{N} \left(\lambda^{b,m} + \lambda^{b,m} \mathbb{1}_{\{x_m > 0\}} \right) \frac{\partial^2 f}{\partial x_m^2} \\
& + \frac{1}{2} \sum_{m=1}^{N} \left(\lambda^{a,m} + \lambda^{a,m} \mathbb{1}_{\{y_m > 0\}} \right) \frac{\partial^2 f}{\partial y_m^2} \\
& + (c^{b,1} - \mu^a) \mathbb{1}_{\{x_1 > 0\}} \frac{\partial f}{\partial x_1} + (c^{a,1} - \mu^b) \mathbb{1}_{\{y_1 > 0\}} \frac{\partial f}{\partial y_1} \\
& + \sum_{m=2}^{N} \left(c^{b,m} \frac{\partial f}{\partial x_m} + c^{a,m} \frac{\partial f}{\partial y_m} \right) \\
& + \sum_{m=1}^{N} \gamma^{b,m} (x_{m-1} + x_{m+1} - 2x_m) \frac{\partial f}{\partial x_m} \\
& + \sum_{m=1}^{N} \gamma^{a,m} (y_{m-1} + y_{m+1} - 2y_m) \frac{\partial f}{\partial y_m},
\end{aligned}
$$

acting on $\mathrm{Dom}(A^N)$*, the space of* $\mathbb{C}^2_{b,\infty}(E^N)$ *functions satisfying:*

$$\begin{cases} \left.\dfrac{\partial f}{\partial x_1}\right|_{x_1=0} = \dfrac{\mu^a}{\lambda^{b,1}}\left(f(x_N,\dots,0,y_1,\dots,y_N) - f(z^b)\right), \\ \left.\dfrac{\partial f}{\partial y_1}\right|_{y_1=0} = \dfrac{\mu^b}{\lambda^{a,1}}\left(f(x_N,\dots,x_1,0,\dots,y_N) - f(z^a)\right), \\ \left.\dfrac{\partial f}{\partial x_m}\right|_{x_m=0} = \left.\dfrac{\partial f}{\partial y_m}\right|_{y_m=0} = 0 \quad \textit{for all}\, m \in \{2,\dots,N\}. \end{cases}$$

Once again, in view of the application of Theorem 2.1.1, we shall need to make use of the following technical lemma:

Lemma 4.1.1 *Let* $f \in \mathrm{Dom}(A^N)$ *and define the following sequence of functions* f_n *on* E^N*:*

$$f_n(x_N,\dots,x_1,y_1,\dots,y_N) = f(u_n x_N,\dots,u_n x_1,\tilde{u}_n y_1,\dots,\tilde{u}_n y_N).$$

Then $\lim_{n\to+\infty}\|f_n - f\| = 0$ *and* f_n *satisfies:*

$$\begin{cases} \lim\limits_{n\to+\infty}\sqrt{n}\sup\limits_{z\in E^N_n, x_1=0}\left|\dfrac{\partial f_n}{\partial x_1} - \dfrac{\mu^a}{\lambda^{b,1}_n}\left(f_n(x_N,\dots,0,y_1,\dots,y_N) - f_n(z^b_n)\right)\right| = 0, \\ \lim\limits_{n\to+\infty}\sqrt{n}\sup\limits_{z\in E^N_n, y_1=0}\left|\dfrac{\partial f_n}{\partial y_1} - \dfrac{\mu^b}{\lambda^{a,1}_n}\left(f_n(x_N,\dots,x_1,0,\dots,y_N) - f_n(z^a_n)\right)\right| = 0, \\ \qquad\qquad\qquad \dfrac{\partial f_n}{\partial x_m} = \dfrac{\partial f_n}{\partial y_m} = 0 \quad \textit{for all}\, m \in \{2,\dots,N\}. \end{cases}$$

The proof of this lemma is omitted as it is a direct generalisation of the proof of Lemma 3.2.1. We now shift our attention to the proof of the multidimensional relative price grid weak convergence result:

Proof of Theorem 4.1.1: We follow the same four-step template as previously.

- step 1: *convergence of the sequence of generators*

Let $f \in \mathrm{Dom}(A^N)$, and consider the following sequence of functions f_n on E^N:

$$f_n(x_N,\dots,x_1,y_1,\dots,y_N) = f(u_n x_N,\dots,u_n x_1,\tilde{u}_n y_1,\dots,\tilde{u}_n y_N).$$

We are going to prove that:

$$\lim_{n\to+\infty}\sup_{z\in E^N_n}|A^N_n f_n(z) - A^N f(z)| = 0.$$

We first introduce the operators Δ^1 and Δ^2 respectively defined by:

$$(\Delta^1\phi)(m) = (\phi_{m-1} - \phi_m)^+ + (\phi_{m+1} - \phi_m)^+,$$

$$(\Delta^2\phi)(m) = (\phi_m - \phi_{m-1})^+ + (\phi_m - \phi_{m+1})^+,$$

for $m \in \{1, \ldots, N\}$ and any vector $\phi \in \mathbb{R}^N$, with the convention $\phi_0 = \phi_{N+1} = 0$.

We then have:

$$\begin{aligned}
A_n^N f_n(z) \quad = \quad & \sum_{m=1}^{N} n\lambda_n^{b,m} \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial x_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial x_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=1}^{N} n\lambda_n^{a,m} \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial y_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial y_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=1}^{N} \sqrt{n}\gamma^{b,m}(\Delta^1 x)(m) \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial x_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial x_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=1}^{N} \sqrt{n}\gamma^{a,m}(\Delta^1 y)(m) \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial y_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial y_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=1}^{N} n\theta_n^{b,m} \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial x_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial x_m^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{x_m \geq \frac{1}{\sqrt{n}}\}} \\
+ \quad & \sum_{m=1}^{N} n\theta_n^{a,m} \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial y_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial y_m^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{y_m \geq \frac{1}{\sqrt{n}}\}} \\
+ \quad & \sum_{m=1}^{N} \sqrt{n}\gamma^{b,m}(\Delta^2 x)(m) \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial x_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial x_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=1}^{N} \sqrt{n}\gamma^{a,m}(\Delta^2 y)(m) \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial y_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial y_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sqrt{n}\mu^a \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial x_1} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial x_1^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{x_1 \geq \frac{1}{\sqrt{n}}\}} \\
+ \quad & \sqrt{n}\mu^b \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial y_1} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial y_1^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{y_1 \geq \frac{1}{\sqrt{n}}\}} \\
+ \quad & \sqrt{n} \left(\mu^a \left(f_n(z_n^b) - f_n(z) \right) \mathbb{1}_{\{x_1 = 0\}} + \mu^b \left(f_n(z_n^a) - f_n(z) \right) \mathbb{1}_{\{y_1 = 0\}} \right).
\end{aligned}$$

At this point, we notice that we can naturally define the operator Δ given by:

$$\begin{aligned}
(\Delta\phi)(m) &= (\Delta^1\phi)(m) - (\Delta^2\phi)(m) \\
&= \phi_{m-1} + \phi_{m+1} - 2\phi_m,
\end{aligned}$$

for $m \in \{1, \ldots, N\}$ and any vector $\phi \in \mathbb{R}^N$, with the convention $\phi_0 = \phi_{N+1} = 0$. It therefore turns out that Δ is none other than the discrete Laplace operator.

Rearranging the terms, we can write:

$$
\begin{aligned}
A_n^N f_n(z) \quad = \quad & \frac{1}{2}\sum_{m=1}^{N}\left(\lambda_n^{b,m}+\theta_n^{b,m}\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial^2 f_n}{\partial x_m^2} \\
+ \quad & \frac{1}{2}\sum_{m=1}^{N}\left(\lambda_n^{a,m}+\theta_n^{a,m}\mathbb{1}_{\{y_m\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial^2 f_n}{\partial y_m^2} \\
+ \quad & \left(\sqrt{n}\left(\lambda_n^{b,1}-\theta_n^{b,1}\mathbb{1}_{\{x_1\geq\frac{1}{\sqrt{n}}\}}\right)-\mu^a\mathbb{1}_{\{x_1\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial x_1} \\
+ \quad & \left(\sqrt{n}\left(\lambda_n^{a,1}-\theta_n^{a,1}\mathbb{1}_{\{y_1\geq\frac{1}{\sqrt{n}}\}}\right)-\mu^b\mathbb{1}_{\{y_1\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial y_1} \\
+ \quad & \sum_{m=2}^{N}\sqrt{n}\left(\lambda_n^{b,m}-\theta_n^{b,m}\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial x_m} \\
+ \quad & \sum_{m=2}^{N}\sqrt{n}\left(\lambda_n^{a,m}-\theta_n^{a,m}\mathbb{1}_{\{y_m\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial y_m} \\
+ \quad & \sum_{m=1}^{N}\gamma^{b,m}(\Delta x)(m)\frac{\partial f_n}{\partial x_m}+\sum_{m=1}^{N}\gamma^{a,m}(\Delta y)(m)\frac{\partial f_n}{\partial y_m} \\
+ \quad & \sqrt{n}\mu^a\left(f_n(z_n^b)-f_n(z)\right)\mathbb{1}_{\{x_1=0\}} \\
+ \quad & \sqrt{n}\mu^b\left(f_n(z_n^a)-f_n(z)\right)\mathbb{1}_{\{y_1=0\}}+\epsilon_n^N(z),
\end{aligned}
$$

where we have defined:

$$
\begin{aligned}
\epsilon_n^N(z) \quad = \quad & \frac{\mu^a}{2\sqrt{n}}\frac{\partial^2 f_n}{\partial x_1^2}\mathbb{1}_{\{x_1\geq\frac{1}{\sqrt{n}}\}}+\frac{\mu^b}{2\sqrt{n}}\frac{\partial^2 f_n}{\partial y_1^2}\mathbb{1}_{\{y_1\geq\frac{1}{\sqrt{n}}\}} \\
+ \quad & \frac{1}{2\sqrt{n}}\sum_{m=1}^{N}\gamma^{b,m}\left((\Delta^1 x)(m)+(\Delta^2 x)(m)\right)\frac{\partial^2 f_n}{\partial x_m^2} \\
+ \quad & \frac{1}{2\sqrt{n}}\sum_{m=1}^{N}\gamma^{a,m}\left((\Delta^1 y)(m)+(\Delta^2 y)(m)\right)\frac{\partial^2 f_n}{\partial y_m^2} \\
+ \quad & \sum_{m=1}^{N}n\left(\lambda_n^{b,m}+\lambda_n^{a,m}+\theta_n^{b,m}\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}}+\theta_n^{a,m}\mathbb{1}_{\{y_m\geq\frac{1}{\sqrt{n}}\}}\right)o\left(\frac{1}{n}\right) \\
+ \quad & \sum_{m=1}^{N}\sqrt{n}\gamma^{b,m}\left((\Delta^1 x)(m)+(\Delta^2 x)(m)\right)o\left(\frac{1}{n}\right) \\
+ \quad & \sum_{m=1}^{N}\sqrt{n}\gamma^{a,m}\left((\Delta^1 y)(m)+(\Delta^2 y)(m)\right)o\left(\frac{1}{n}\right) \\
+ \quad & \sqrt{n}(\mu^a\mathbb{1}_{\{x_1\geq\frac{1}{\sqrt{n}}\}}+\mu^b\mathbb{1}_{\{y_1\geq\frac{1}{\sqrt{n}}\}})o\left(\frac{1}{n}\right).
\end{aligned}
$$

Before going any further, we need to provide norm estimates for the first and second partial derivatives of f_n. We have, for all $n \geq 1$ and $m \in \{1, \dots, N\}$:

$$\left\| \frac{\partial f_n}{\partial x_m} \right\| \leq u_n \left\| \frac{\partial f}{\partial x_m} \right\|, \quad \left\| \frac{\partial^2 f_n}{\partial x_m^2} \right\| \leq u_n^2 \left\| \frac{\partial^2 f}{\partial x_m^2} \right\|,$$

$$\left\| \frac{\partial f_n}{\partial y_m} \right\| \leq \tilde{u}_n \left\| \frac{\partial f}{\partial y_m} \right\|, \quad \left\| \frac{\partial^2 f_n}{\partial y_m^2} \right\| \leq \tilde{u}_n^2 \left\| \frac{\partial^2 f}{\partial y_m^2} \right\|.$$

We also need norm estimates for the differences between the first and second partial derivatives of f_n and those of f. More precisely, for all $m \in \{1, \dots, N\}$ and $n \geq 1$, we see that:

$$\left\| \frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m} \right\| \leq u_n \left\| \left.\frac{\partial f}{\partial x_m}\right|_{\tilde{z}} - \frac{\partial f}{\partial x_m} \right\| + |u_n - 1| \left\| \frac{\partial f}{\partial x_m} \right\|,$$

as well as:

$$\left\| \frac{\partial^2 f_n}{\partial x_m^2} - \frac{\partial^2 f}{\partial x_m^2} \right\| \leq u_n^2 \left\| \left.\frac{\partial^2 f}{\partial x_m^2}\right|_{\tilde{z}} - \frac{\partial^2 f}{\partial x_m^2} \right\| + \left| u_n^2 - 1 \right| \left\| \frac{\partial^2 f}{\partial x_m^2} \right\|,$$

where $\tilde{z} = (u_n x_N, \dots, u_n x_1, \tilde{u}_n y_1, \dots, \tilde{u}_n y_N)$. Once again, we have entirely analogous results for the first and second partial derivatives with respect to y_m, for all $m \in \{1, \dots, N\}$.

Adapting the steps used to show that $\lim_{n \to +\infty} \sup_{x \in E} \left| f\left(\frac{\lambda}{\lambda_n} x\right) - f(x) \right| = 0$ in Lemma 3.1.1, *mutatis mutandis*, it is straightforward to establish that, for all $m \in \{1, \dots, N\}$:

$$\lim_{n \to +\infty} \left\| \frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m} \right\| = \lim_{n \to +\infty} \left\| \frac{\partial^2 f_n}{\partial x_m^2} - \frac{\partial^2 f}{\partial x_m^2} \right\| = 0,$$

$$\lim_{n \to +\infty} \left\| \frac{\partial f_n}{\partial y_m} - \frac{\partial f}{\partial y_m} \right\| = \lim_{n \to +\infty} \left\| \frac{\partial^2 f_n}{\partial y_m^2} - \frac{\partial^2 f}{\partial y_m^2} \right\| = 0.$$

For all $z \in E_n^N$, we then have:

$$\begin{aligned} \left| A_n^N f_n(z) - A f(z) \right| \quad \leq \quad & \frac{1}{2} \sum_{m=1}^N \left| \lambda_n^{b,m} \frac{\partial^2 f_n}{\partial x_m^2} - \lambda^{b,m} \frac{\partial^2 f}{\partial x_m^2} \right| \\ + \quad & \frac{1}{2} \sum_{m=1}^N \left| \theta_n^{b,m} \mathbb{1}_{\{x_m \geq \frac{1}{\sqrt{n}}\}} \frac{\partial^2 f_n}{\partial x_m^2} - \lambda^{b,m} \mathbb{1}_{\{x_m > 0\}} \frac{\partial^2 f}{\partial x_m^2} \right| \\ + \quad & \frac{1}{2} \sum_{m=1}^N \left| \lambda_n^{a,m} \frac{\partial^2 f_n}{\partial y_m^2} - \lambda^{a,m} \frac{\partial^2 f}{\partial y_m^2} \right| \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\sum_{m=1}^{N}\left|\theta_n^{a,m}\mathbb{1}_{\{y_m\geq\frac{1}{\sqrt{n}}\}}\frac{\partial^2 f_n}{\partial y_m^2} - \lambda^{a,m}\mathbb{1}_{\{y_m>0\}}\frac{\partial^2 f}{\partial y_m^2}\right| \\
&+ \quad \sum_{m=1}^{N}\gamma^{b,m}\left|(\Delta x)(m)\right|\left|\frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m}\right| \\
&+ \quad \sum_{m=1}^{N}\gamma^{a,m}\left|(\Delta y)(m)\right|\left|\frac{\partial f_n}{\partial y_m} - \frac{\partial f}{\partial y_m}\right| \\
&+ \quad \left|\alpha_n(z)\right| + \left|\beta_n(z)\right| + \left|\gamma_n^N(z)\right| + \left|\delta_n^N(z)\right| + \left|\epsilon_n^N(z)\right|,
\end{aligned}
$$

with the quantities α_n, β_n, γ_n^N and δ_n^N being defined by:

$$
\begin{aligned}
\alpha_n(z) &= \sqrt{n}\left(\lambda_n^{b,1} - \theta_n^{b,1}\mathbb{1}_{\{x_1\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial x_1} - c^{b,1}\mathbb{1}_{\{x_1>0\}}\frac{\partial f}{\partial x_1} \\
&\quad + \mu^a\left(\mathbb{1}_{\{x_1>0\}}\frac{\partial f}{\partial x_1} - \mathbb{1}_{\{x_1\geq\frac{1}{\sqrt{n}}\}}\frac{\partial f_n}{\partial x_1}\right) \\
&\quad + \sqrt{n}\mu^a\left(f_n(z_n^b) - f_n(z)\right)\mathbb{1}_{\{x_1=0\}}, \\
\beta_n(z) &= \sqrt{n}\left(\lambda_n^{a,1} - \theta_n^{a,1}\mathbb{1}_{\{y_1\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial y_1} - c^{a,1}\mathbb{1}_{\{y_1>0\}}\frac{\partial f}{\partial y_1} \\
&\quad + \mu^b\left(\mathbb{1}_{\{y_1>0\}}\frac{\partial f}{\partial y_1} - \mathbb{1}_{\{y_1\geq\frac{1}{\sqrt{n}}\}}\frac{\partial f_n}{\partial y_1}\right) \\
&\quad + \sqrt{n}\mu^b\left(f_n(z_n^a) - f_n(z)\right)\mathbb{1}_{\{y_1=0\}}, \\
\gamma_n^N(z) &= \sum_{m=2}^{N}\left(\sqrt{n}\left(\lambda_n^{b,m} - \theta_n^{b,m}\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial x_m} - c^{b,m}\frac{\partial f}{\partial x_m}\right), \\
\delta_n^N(z) &= \sum_{m=2}^{N}\left(\sqrt{n}\left(\lambda_n^{a,m} - \theta_n^{a,m}\mathbb{1}_{\{y_m\geq\frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial y_m} - c^{a,m}\frac{\partial f}{\partial y_m}\right).
\end{aligned}
$$

Let $m \in \{1,\dots,N\}$. On the one hand, for all $n \geq 1$, we see that:

$$
\left|\lambda_n^{b,m}\frac{\partial^2 f_n}{\partial x_m^2} - \lambda^{b,m}\frac{\partial^2 f}{\partial x_m^2}\right| \leq \lambda_n^{b,m}\left|\frac{\partial^2 f_n}{\partial x_m^2} - \frac{\partial^2 f}{\partial x_m^2}\right| + \left|\lambda_n^{b,m} - \lambda^{b,m}\right|\left|\frac{\partial^2 f}{\partial x_m^2}\right|,
$$

$$
\left|\lambda_n^{a,m}\frac{\partial^2 f_n}{\partial y_m^2} - \lambda^{a,m}\frac{\partial^2 f}{\partial y_m^2}\right| \leq \lambda_n^{a,m}\left|\frac{\partial^2 f_n}{\partial y_m^2} - \frac{\partial^2 f}{\partial y_m^2}\right| + |\lambda_n^{a,m} - \lambda^{a,m}|\left|\frac{\partial^2 f}{\partial y_m^2}\right|,
$$

from which we deduce that:

$$
\lim_{n\to+\infty}\sup_{z\in E_n^N}\frac{1}{2}\sum_{m=1}^{N}\left|\lambda_n^{b,m}\frac{\partial^2 f_n}{\partial x_m^2} - \lambda^{b,m}\frac{\partial^2 f}{\partial x_m^2}\right| = 0,
$$

$$\lim_{n\to+\infty} \sup_{z\in E_n^N} \frac{1}{2}\sum_{m=1}^{N} \left|\lambda_n^{a,m}\frac{\partial^2 f_n}{\partial y_m^2} - \lambda^{a,m}\frac{\partial^2 f}{\partial y_m^2}\right| = 0.$$

On the other hand, for all $n \geq 1$, we have:

$$\begin{aligned}\left|\theta_n^{b,m}\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}}\frac{\partial^2 f_n}{\partial x_m^2} - \lambda^{b,m}\mathbb{1}_{\{x_m>0\}}\frac{\partial^2 f}{\partial x_m^2}\right| \quad &\leq \quad \theta_n^{b,m}\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}}\left|\frac{\partial^2 f_n}{\partial x_m^2} - \frac{\partial^2 f}{\partial x_m^2}\right| \\ &+ \quad \theta_n^{b,m}\left|\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}} - \mathbb{1}_{\{x_m>0\}}\right|\left|\frac{\partial^2 f}{\partial x_m^2}\right| \\ &+ \quad \left|\theta_n^{b,m} - \lambda^{b,m}\right|\mathbb{1}_{\{x_m>0\}}\left|\frac{\partial^2 f}{\partial x_m^2}\right|.\end{aligned}$$

Given that $\sup_{z\in E_n^N}\left|\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}} - \mathbb{1}_{\{x_m>0\}}\right| = 0$, we immediately conclude that:

$$\lim_{n\to+\infty} \sup_{z\in E_n^N} \frac{1}{2}\sum_{m=1}^{N} \left|\theta_n^{b,m}\mathbb{1}_{\{x_m\geq\frac{1}{\sqrt{n}}\}}\frac{\partial^2 f_n}{\partial x_m^2} - \lambda^{b,m}\mathbb{1}_{\{x_m>0\}}\frac{\partial^2 f}{\partial x_m^2}\right| = 0.$$

Naturally, we can replicate these arguments to the case of the second partial derivative with respect to y_m, and deduce that:

$$\lim_{n\to+\infty} \sup_{z\in E_n^N} \frac{1}{2}\sum_{m=1}^{N} \left|\theta_n^{a,m}\mathbb{1}_{\{y_m\geq\frac{1}{\sqrt{n}}\}}\frac{\partial^2 f_n}{\partial y_m^2} - \lambda^{a,m}\mathbb{1}_{\{y_m>0\}}\frac{\partial^2 f}{\partial y_m^2}\right| = 0.$$

As far as the two terms with the discrete Laplace operator are concerned, we simply see that:

$$\sum_{m=1}^{N} \gamma^{b,m}\left|(\Delta x)(m)\right|\left|\frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m}\right| \leq \sum_{m=1}^{N} \gamma^{b,m}\left|(\Delta x)(m)\right|\left\|\frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m}\right\|,$$

which instantly gives us:

$$\lim_{n\to+\infty} \sup_{z\in E_n^N} \sum_{m=1}^{N} \gamma^{b,m}\left|(\Delta x)(m)\right|\left|\frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m}\right| = 0.$$

Evidently, we also have:

$$\lim_{n\to+\infty} \sup_{z\in E_n^N} \sum_{m=1}^{N} \gamma^{a,m}\left|(\Delta y)(m)\right|\left|\frac{\partial f_n}{\partial y_m} - \frac{\partial f}{\partial y_m}\right| = 0.$$

Furthermore, using the decomposition $\lambda_n^{b,1} = \lambda_n^{b,1}\left(\mathbb{1}_{\{x_1=0\}} + \mathbb{1}_{\{x_1 \geq \frac{1}{\sqrt{n}}\}}\right)$, we notice that:

$$\begin{aligned}
\sup_{z \in E_n^N} |\alpha_n| \quad &\leq \quad \left|\sqrt{n}\left(\lambda_n^{b,1} - \theta_n^{b,1}\right) - c^{b,1}\right| \left\|\frac{\partial f_n}{\partial x_1}\right\| \\
&+ \quad \left|c^{b,1}\right| \left\|\frac{\partial f_n}{\partial x_1} - \frac{\partial f}{\partial x_1}\right\| + \left|c^{b,1}\right| \left\|\frac{\partial f}{\partial x_1}\right\| \underbrace{\sup_{z \in E_n^N} \left|\mathbb{1}_{\{x_1 \geq \frac{1}{\sqrt{n}}\}} - \mathbb{1}_{\{x_1 > 0\}}\right|}_{=0} \\
&+ \quad \mu^a \left(\left\|\frac{\partial f}{\partial x_1}\right\| \underbrace{\sup_{z \in E_n^N} \left|\mathbb{1}_{\{x_1 > 0\}} - \mathbb{1}_{\{x_1 \geq \frac{1}{\sqrt{n}}\}}\right|}_{=0} + \left\|\frac{\partial f_n}{\partial x_1} - \frac{\partial f}{\partial x_1}\right\| \right) \\
&+ \quad \sqrt{n} \sup_{z \in E_n^N} \left| \left(\lambda_n^{b,1} \frac{\partial f_n}{\partial x_1} + \mu^a \left(f_n(z_n^b) - f_n(z)\right)\right) \mathbb{1}_{\{x_1=0\}} \right| \\
&\leq \quad \left|\sqrt{n}\left(\lambda_n^{b,1} - \theta_n^{b,1}\right) - c^{b,1}\right| u_n \left\|\frac{\partial f}{\partial x_1}\right\| \\
&+ \quad \left(\left|c^{b,1}\right| + \mu^a\right) \left\|\frac{\partial f_n}{\partial x_1} - \frac{\partial f}{\partial x_1}\right\| \\
&+ \quad \sqrt{n} \sup_{z \in E_n^N} \left| \left(\lambda_n^{b,1} \frac{\partial f_n}{\partial x_1} + \mu^a \left(f_n(z_n^b) - f_n(z)\right)\right) \mathbb{1}_{\{x_1=0\}} \right|.
\end{aligned}$$

With similar arguments, we also have:

$$\begin{aligned}
\sup_{z \in E_n^N} |\beta_n| \quad &\leq \quad \left|\sqrt{n}\left(\lambda_n^{a,1} - \theta_n^{a,1}\right) - c^{a,1}\right| \tilde{u}_n \left\|\frac{\partial f}{\partial y_1}\right\| \\
&+ \quad \left(\left|c^{a,1}\right| + \mu^b\right) \left\|\frac{\partial f_n}{\partial y_1} - \frac{\partial f}{\partial y_1}\right\| \\
&+ \quad \sqrt{n} \sup_{z \in E_n^N} \left| \left(\lambda_n^{a,1} \frac{\partial f_n}{\partial y_1} + \mu^b \left(f_n(z_n^a) - f_n(z)\right)\right) \mathbb{1}_{\{y_1=0\}} \right|.
\end{aligned}$$

We are now in a position to apply Lemma 4.1.1, which tells us that:

$$\lim_{n \to +\infty} \sqrt{n} \sup_{z \in E_n^N, x_1=0} \left| \frac{\partial f_n}{\partial x_1} - \frac{\mu^a}{\lambda_n^{b,1}} \left(f_n(z) - f_n(z_n^b)\right) \right| = 0,$$

$$\lim_{n \to +\infty} \sqrt{n} \sup_{z \in E_n^N, y_1=0} \left| \frac{\partial f_n}{\partial y_1} - \frac{\mu^b}{\lambda_n^{a,1}} \left(f_n(z) - f_n(z_n^a)\right) \right| = 0.$$

This enables to deduce that:

$$\lim_{n \to +\infty} \sup_{z \in E_n^N} \left|\alpha_n(z)\right| = \lim_{n \to +\infty} \sup_{z \in E_n^N} \left|\beta_n(z)\right| = 0.$$

Moreover, for $m \in \{2, \ldots, N\}$, writing $1 = \mathbb{1}_{\{x_m=0\}} + \mathbb{1}_{\{x_m \geq \frac{1}{\sqrt{n}}\}}$, it follows that:

$$\begin{aligned}
\sup_{z\in E_n^N} \left|\gamma_n^N\right| &\leq \sum_{m=2}^{N} \left|\sqrt{n}\left(\lambda_n^{b,m} - \theta_n^{b,m}\right) - c^{b,m}\right| \left\|\frac{\partial f_n}{\partial x_m}\right\| \\
&+ \sum_{m=2}^{N} \left|c^{b,m}\right| \left\|\frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m}\right\| \\
&+ \sum_{m=2}^{N} \left(\sqrt{n}\lambda_n^{b,m} \sup_{z\in E_n^N} \underbrace{\left|\frac{\partial f_n}{\partial x_m}\mathbb{1}_{\{x_m=0\}}\right|}_{=0} + c^{b,m} \sup_{z\in E_n^N} \underbrace{\left|\frac{\partial f}{\partial x_m}\mathbb{1}_{\{x_m=0\}}\right|}_{=0}\right) \\
&\leq \sum_{m=2}^{N} \left|\sqrt{n}\left(\lambda_n^{b,m} - \theta_n^{b,m}\right) - c^{b,m}\right| u_n \left\|\frac{\partial f}{\partial x_m}\right\| \\
&+ \sum_{m=2}^{N} \left|c^{b,m}\right| \left\|\frac{\partial f_n}{\partial x_m} - \frac{\partial f}{\partial x_m}\right\|.
\end{aligned}$$

We analogously deduce that:

$$\begin{aligned}
\sup_{z\in E_n^N} \left|\delta_n^N\right| &\leq \sum_{m=2}^{N} \left(\left|\sqrt{n}\,(\lambda_n^{a,m} - \theta_n^{a,m}) - c^{a,m}\right| \left\|\frac{\partial f_n}{\partial y_m}\right\| + \left|c^{a,m}\right| \left\|\frac{\partial f_n}{\partial y_m} - \frac{\partial f}{\partial y_m}\right\|\right) \\
&+ \sum_{m=2}^{N} \left(\sqrt{n}\lambda_n^{a,m} \sup_{z\in E_n^N} \underbrace{\left|\frac{\partial f_n}{\partial y_m}\mathbb{1}_{\{y_m=0\}}\right|}_{=0} + c^{a,m} \sup_{z\in E_n^N} \underbrace{\left|\frac{\partial f}{\partial y_m}\mathbb{1}_{\{y_m=0\}}\right|}_{=0}\right) \\
&\leq \sum_{m=2}^{N} \left|\sqrt{n}\,(\lambda_n^{a,m} - \theta_n^{a,m}) - c^{a,m}\right| \tilde{u}_n \left\|\frac{\partial f}{\partial y_m}\right\| \\
&+ \sum_{m=2}^{N} \left|c^{a,m}\right| \left\|\frac{\partial f_n}{\partial y_m} - \frac{\partial f}{\partial y_m}\right\|.
\end{aligned}$$

Consequently, we can write:

$$\lim_{n\to+\infty} \sup_{z\in E_n^N} \left|\gamma_n^N(z)\right| = \lim_{n\to+\infty} \sup_{z\in E_n^N} \left|\delta_n^N(z)\right| = 0.$$

As it is clear that $\lim_{n\to+\infty} \sup_{z\in E_n^N} \left|\epsilon_n^N(z)\right| = 0$, we finally obtain, via assumptions $(A1)$ to $(A6)$:

$$\lim_{n\to+\infty} \sup_{z\in E_n^N} \left|A_n^N f_n(z) - A^N f(z)\right| = 0.$$

- step 2: $\overline{A^N}$ *is the infinitesimal generator of a Feller semigroup*

The linear operator A^N is a particular case of the Waldenfels integro-differential operator introduced in Chapter 2. Theorem 2.1.3 thus establishes that $\overline{A^N}$ is the infinitesimal generator of a Feller semigroup $T^N = (T^N(t))_{t\geq 0}$ on $\mathbb{B}(E^N)$.

- step 3: *convergence of the sequence of semigroups*

We now need to use the equivalence between convergence of generators and convergence of semigroups, by checking that the conditions of Theorem 2.1.1 are verified. $T_n^N = (T_n^N(t))_{t\geq 0}$ is clearly a contraction semigroup as for all $f \in \mathbb{B}(E_n^N)$ and $z \in E_n^N$, we can write:

$$|T_n^N(t)f(z)| = |\mathbb{E}(f(\hat{Z}_n^N(t))|\hat{Z}_n^N(0) = z)| \leq \|f\|.$$

This is obviously also valid for the semigroup $T^N = (T^N(t))_{t\geq 0}$, for all $f \in \mathbb{B}(E^N)$ and $z \in E^N$. As a Feller semigroup, T^N is necessarily strongly continuous. We now need to prove the strong continuity of T_n^N:

$$\begin{aligned}
\frac{1}{2\,\|f\|} \sup_{z\in E_n^N} |T_n^N(t)f(z) - f(z)| \quad \leq \quad & nt \sum_{m=1}^{N} \left(\lambda_n^{b,m} + \lambda_n^{a,m}\right) \\
& + nt \sum_{m=2}^{N} \left(\theta_n^{b,m} + \theta_n^{a,m}\right) \\
& + \sqrt{n}t \sum_{m=1}^{N} \gamma^{b,m}(\Delta^1 x)(m) \\
& + \sqrt{n}t \sum_{m=1}^{N} \gamma^{b,m}(\Delta^2 x)(m) \\
& + \sqrt{n}t \sum_{m=1}^{N} \gamma^{a,m}(\Delta^1 y)(m) \\
& + \sqrt{n}t \sum_{m=1}^{N} \gamma^{a,m}(\Delta^2 y)(m) \\
& + nt \left(\mu_n^a + \theta_n^{b,1} + \mu_n^b + \theta_n^{a,1}\right) \\
& + \left(\mu_n^a + \mu_n^b\right) nt + o(nt).
\end{aligned}$$

Therefore, we deduce that:

$$\lim_{t\to 0} \sup_{z\in E_n^N} |T_n^N(t)f(z) - f(z)| = 0,$$

which establishes the strong continuity of T_n^N. We introduce $\eta_n : E_n^N \to E^N$ with $\eta_n(z) = z$ and $\pi_n : \mathbb{B}(E^N) \to \mathbb{B}(E_n^N)$ with $\pi_n(f) = f \circ \eta_n$. Observing that the conditions of Theorem 2.1.1 are now fulfilled, we obtain the convergence of the sequence of semigroups.

- step 4: *weak convergence of* $\hat{Z}_n^N$

We are finally in a position to make use of Theorem 2.1.3, with π_n and η_n as previously defined: there exists an E^N-valued Feller (and therefore strong Markov) process $Z^N = (Z^N(t))_{t\geq 0}$ with sample paths in $\mathbb{D}([0,+\infty[, E^N)$ corresponding to T^N (and consequently with generator $\overline{A^N}$) such that $\hat{Z}_n^N \Rightarrow Z^N$.

□

Remark 4.1.1 *Given the form of the generator and its boundary conditions in Theorem 4.2.1, we see that the limiting process* $Z^N = (X^N, \ldots, X^1, Y^1, \ldots, Y^N)$ *can be formally described as a multidimensional Ornstein-Uhlenbeck process with elastic reflection/regeneration at the best levels (i.e. the processes* X^1 *and* Y^1*) and classic reflection at all the other levels* $m \in \{2, \ldots, N\}$*. Note that we use the term classic reflection so as to emphasise the difference with elastic reflection.*

Remark 4.1.2 *In this relative price grid setting, the representation of the price process can be naturally introduced by taking into consideration the regeneration times of the best queues. Indeed, the transition of the best bid queue's value to that of the second best bid queue corresponds to a price decrease, whilst the best ask queue's value jumps to that of the second best ask when there is a price increase. More specifically, let* $\xi^b = (\xi_i^b)_{i\geq 1}$ *and* $\xi^a = (\xi_i^a)_{i\geq 1}$ *be two independent sequences of i.i.d exponential random variables with respective parameters* $\frac{\mu^a}{\lambda^{b,1}}$ *and* $\frac{\mu^b}{\lambda^{a,1}}$*. We also assume that the sequence* ξ^b *is independent of* X^1 *and the sequence* ξ^a *is independent of* Y^1*. Introducing the sequence* $T^b = (T_i^b)_{i\geq 1}$ *of bid regeneration times:*

$$\begin{cases} T_1^b = \inf\left\{t \geq 0 : L_t^{X^1} > \xi_1^b\right\}, \\ T_{i+1}^b = \inf\left\{t \geq T_i^b : L_t^{X^1} - L_{T_i^b}^{X^1} > \xi_{i+1}^b\right\}, \end{cases}$$

where $L^{X^1} = (L_t^{X^1})_{t\geq 0}$ *is the local time at* 0 *of the process* X^1*, as well as the sequence* $T^a = (T_i^a)_{i\geq 1}$ *of ask regeneration times:*

$$\begin{cases} T_1^a = \inf\left\{t \geq 0 : L_t^{Y^1} > \xi_1^a\right\}, \\ T_{i+1}^a = \inf\left\{t \geq T_i^a : L_t^{Y^1} - L_{T_i^a}^{Y^1} > \xi_{i+1}^a\right\}, \end{cases}$$

where $L^{Y^1} = (L_t^{Y^1})_{t\geq 0}$ *is the local time at* 0 *of the process* Y^1*, the price process* $P = (P_t)_{t\geq 0}$ *of the limit order book can be expressed as:*

$$P_t = p + \sum_{i\geq 1} \mathbb{1}_{\{T_i^a \leq t\}} - \sum_{i\geq 1} \mathbb{1}_{\{T_i^b \leq t\}},$$

where $P_0 = p \in \mathbb{N}^*$ *is the value of the price in the initial limit order book profile.*

4.2 A multidimensional limit order book model on an absolute price grid

Throughout this section, our goal is to establish a diffusion approximation of the entire limit order book in an absolute price grid setting. In other words, we are no longer interested in the behaviour of the queues at a given distance from the best bid or ask, but we rather focus on every single level of a fixed (hence absolute) price grid. In comparison with the relative price grid model, it must be emphasised that the fundamental difference (and perhaps benefit) of the absolute price grid setup is the natural *endogeneity* of the price process. Indeed, the very nature of the relative price grid model meant that the behaviour of the price could only be *exogenously* deduced as the difference of the counting processes associated with the regeneration times of the best bid and ask queues. In the context of an absolute price grid, as we shall see shortly, the price process is a fully fledged part of the problem from the very beginning.

4.2.1 The discrete order book process

We place ourselves in the framework of an absolute price grid $\{1, \ldots, N\}$, where $N \in \mathbb{N}^*$. In order to ensure a tractable as well as endogenous price process, we combine the convention of negative bid volumes and positive ask volumes used by Cont et al. [15] as well as Dai et al. [17] with the idea introduced by Lejay [42] according to which the required discrete state space is $\mathcal{Z}^N$, where $\mathcal{Z}$ is the disjoint union of $\mathbb{Z}_-$ and $\mathbb{Z}_+$. The purpose of this decomposition is to enable 0 to appear in $\mathcal{Z}$ as two distinct elements 0^- of $\mathbb{Z}_-$ and 0^+ of $\mathbb{Z}_+$. For each $n \in \mathbb{N}^*$, consider an N-dimensional process $Z_n = (Z_n^1, \ldots, Z_n^N)$ with values in $\mathcal{Z}^N$, representing the discrete order book process. As a result of the convention on volume signs, for each $m \in \{1, \ldots, N\}, |Z_n^m|$ corresponds to the number of outstanding orders at price level m. Note that as we are working on an absolute price grid, we adopt the convention of denoting the N components of the process in numerical order. Moreover, we still assume that order and cancellation sizes are equal to 1 without loss of generality. Let $i = (i_1, \ldots, i_N)$ be a realisation of the process Z_n. For $m \in \{1, \ldots, N\}$, we define $i^{m\pm1} = (i_1, \ldots, i_m \pm 1, \ldots, i_N)$. We also introduce the two following quantities:

$$b(i) = \sup\left\{l \in \{1, \ldots, N\} : i_l < 0^+\right\} \vee 0,$$

$$a(i) = \inf\left\{l \in \{1, \ldots, N\} : i_l > 0^-\right\} \wedge (N+1),$$

Let $m \in \{1, \ldots, N\}$. The transitions of Z_n which *do not* lead to price changes can be summarised in the following way:

$$i \to i^{m-1} \text{ at rate } \Lambda_n^{b,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m \leq b(i),$$

$$i \to i^{m+1} \text{ at rate } \Lambda_n^{a,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m \geq a(i),$$

$$i \to i^{m+1} \text{ at rate } \mu_n \mathbb{1}_{\{i_m \leq -1\}} + \Theta_n^{b,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m = b(i),$$

$$
\begin{aligned}
i \to i^{m-1} &\text{ at rate } \mu_n \mathbb{1}_{\{i_m \geq 1\}} + \Theta_n^{a,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m = a(i), \\
i \to i^{m+1} &\text{ at rate } \Theta_n^{b,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m < b(i), \\
i \to i^{m-1} &\text{ at rate } \Theta_n^{a,m}(i_{m-1}, i_m, i_{m+1}) \text{ for } m > a(i).
\end{aligned}
$$

As in the relative price grid setting, limit order arrival and cancellation rates exhibit a nearest neighbour interaction, and depend on the distance to the opposite best quote. As we once again assume a constant spread equal to one tick, this amounts to considering rates at a particular price level depending on the distance to the same best quote. Formally speaking, we now have (with the natural pinning convention $i_{N+1} = i_0 = 0$), for $m \leq b(i)$:

$$
\Lambda_n^{b,m}(i_{m-1}, i_m, i_{m+1}) = \lambda_n^{b(i)-m} + \gamma_n^{b(i)-m} \left((i_m - i_{m-1})^+ + (i_m - i_{m+1})^+ \right),
$$

$$
\Theta_n^{b,m}(i_{m-1}, i_m, i_{m+1}) = \theta_n^{b(i)-m} \mathbb{1}_{\{i_m \leq -1\}} + \gamma_n^{b(i)-m} \left((i_{m-1} - i_m)^+ + (i_{m+1} - i_m)^+ \right).
$$

And for $m \geq a(i)$, we have:

$$
\Lambda_n^{a,m}(i_{m-1}, i_m, i_{m+1}) = \lambda_n^{m-a(i)} + \gamma_n^{m-a(i)} \left((i_{m-1} - i_m)^+ + (i_{m+1} - i_m)^+ \right),
$$

$$
\Theta_n^{a,m}(i_{m-1}, i_m, i_{m+1}) = \theta_n^{m-a(i)} \mathbb{1}_{\{i_m \geq 1\}} + \gamma_n^{m-a(i)} \left((i_m - i_{m-1})^+ + (i_m - i_{m+1})^+ \right).
$$

One of the main challenges of this model is to provide a coherent description of events leading to a price change. When the best bid (respectively ask) queue is depleted, an incoming market order causes its new value to be equal to the "previous" second best bid (respectively ask) queue. As the spread is assumed to be constantly equal to one tick, the best ask (respectively bid) queue is then regenerated from 0^+ (respectively from 0^-). At this stage, it is important to point out that we choose these regeneration values for the sake of simplicity, but this setup could readily be extended to the case of a regenerative random distribution instead. The transitions of the process Z_n which cause price changes are thus given by:

$$
(i_1, \ldots, i_{b(i)}, \ldots, i_N)\Big|_{i_{b(i)}=0^-} \to (i_1, \ldots, i_{b(i)}, \ldots, i_N)\Big|_{i_{b(i)}=0^+} \text{ at rate } \mu_n,
$$

$$
(i_1, \ldots, i_{a(i)}, \ldots, i_N)\Big|_{i_{a(i)}=0^+} \to (i_1, \ldots, i_{a(i)}, \ldots, i_N)\Big|_{i_{a(i)}=0^-} \text{ at rate } \mu_n.
$$

The above evaluation notation is used so as to emphasise the fact that we are dealing with the same index between transitions. This feature is a simple consequence of the endogeneity of the price process in the absolute price grid setting.

Remark 4.2.1 *Unlike in the relative price grid model, we do not make the assumption of separate bid and ask indices as far as the transition rates are concerned. The very nature of the absolute price grid makes it more natural to simply use the distance to the best quotes, whether we are dealing with the bid or ask side.*

4.2.2 Heavy traffic diffusion approximation

As usual, we rescale the discrete order book process by accelerating time by a factor of n and dividing the volumes by $\sqrt{n}$. Denoting $\hat{Z}_n$ the rescaled process on $E_n^N = \frac{1}{\sqrt{n}}\mathcal{Z}^N$, we have:

$$\hat{Z}_n(t) = \frac{Z_n(nt)}{\sqrt{n}}.$$

Let $m \in \{1, \dots, N\}$. Extending what was introduced for the discrete state space, the limiting process shall be defined on $E^N = \mathcal{R}^N$, where $\mathcal{R}$ is the disjoint union of $\mathbb{R}_-$ and $\mathbb{R}_+$ (see Lejay [42]). It shall additionally be assumed that:

- *(A1)* $\lim_{n\to+\infty} \lambda_n^{b(i)-m} = \lambda^{b(i)-m}$, for $m \leq b(i)$,
- *(A2)* $\lim_{n\to+\infty} \theta_n^{b(i)-m} = \lambda^{b(i)-m}$, for $m \leq b(i)$,
- *(A3)* $\lim_{n\to+\infty} \lambda_n^{m-a(i)} = \lambda^{m-a(i)}$, for $m \geq a(i)$,
- *(A4)* $\lim_{n\to+\infty} \theta_n^{m-a(i)} = \lambda^{m-a(i)}$, for $m \geq a(i)$,
- *(A5)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n^{b(i)-m} - \theta_n^{b(i)-m}) = c^{b(i)-m}$, for $m \leq b(i)$,
- *(A6)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n^{m-a(i)} - \theta_n^{m-a(i)}) = c^{m-a(i)}$, for $m \geq a(i)$,
- *(A7)* $\mu_n = \frac{\mu}{\sqrt{n}}$,
- *(A8)* $\gamma_n^{b(i)-m} = \frac{1}{n}\gamma^{b(i)-m}$ for $m \leq b(i)$,
- *(A9)* $\gamma_n^{m-a(i)} = \frac{1}{n}\gamma^{m-a(i)}$ for $m \geq a(i)$,
- *(A10)* $c^0 - \mu < 0$,
- *(A11)* Let $z = (z_1, \dots, z_N)$ and $u_n = \frac{\lambda^0}{\lambda_n^0}$. Then, for all $f \in \mathbb{C}^2_{b,\infty}(E^N)$:

$$\lim_{n\to+\infty} \sqrt{n} \sup_{z\in E_n^N} \left|f(u_n z_1, \dots, u_n z_N) - f(z)\right| = 0.$$

Once again, we make use of the semigroup approach and introduce $(T_n^N(t))_{t\geq 0}$ as the semigroup on $\mathbb{B}(E_n^N)$ defined by:

$$T_n^N(t)f(z) = \mathbb{E}\left(f(\hat{Z}_n(t))|\hat{Z}_n(0) = z\right),$$

for all $f \in \mathbb{B}(E_n^N)$ and $z = (z_1, \dots, z_N) \in E_n^N$. For notational convenience, we define the following vector for all $m \in \{1, \dots, N\}$:

$$z_m^{\pm} = (z_1, \dots, z_m \pm \frac{1}{\sqrt{n}}, \dots, z_N),$$

For any $z = (z_1, \dots, z_N) \in E_n^N$, we also define the two price change vectors:

$$z^b = (z_1, \dots, z_{b(z)}, \dots, z_N)\Big|_{z_{b(z)}=0^+}, \quad z^a = (z_1, \dots, z_{a(z)}, \dots, z_N)\Big|_{z_{a(z)}=0^-}.$$

The crucial point here is to see that if we define z^b and z^a in terms of their *own* components, i.e. $z^b = (z_1^b, \dots, z_N^b)$ and $z^a = (z_1^a, \dots, z_N^a)$, then we have:

$$z^b_{b(z^b)} = z_{b(z)-1}, \quad z^a_{a(z^a)} = z_{a(z)+1},$$

which simply translates the fact that the price is updated in between the respective transitions. We also notice that the price process is invariant under rescaling:

$$\begin{aligned} b\left(\frac{i}{\sqrt{n}}\right) &= \sup\left\{l \in \{1, \dots, N\} : \frac{i_l}{\sqrt{n}} < 0^+\right\} \vee 0 \\ &= \sup\left\{l \in \{1, \dots, N\} : i_l < 0^+\right\} \vee 0 = b(i). \end{aligned}$$

This is evidently also true for the ask side, i.e. $a\left(\frac{i}{\sqrt{n}}\right) = a(i)$. The infinitesimal generator A_n^N of $(T_n^N(t))_{t\geq 0}$ is then given by, for all $f \in \mathbb{B}(E_n^N)$ and $z \in E_n^N$:

$$\begin{aligned} A_n^N f(z) &= \lim_{t\to 0} \frac{T_n^N(t)f(z) - f(z)}{t} \\ &= \sum_{m=1}^{b(z)} n\Lambda_n^{b,m}(\sqrt{n}z_{m-1}, \sqrt{n}z_m, \sqrt{n}z_{m+1}) \left(f(z_m^-) - f(z)\right) \\ &+ \sum_{m=a(z)}^{N} n\Lambda_n^{a,m}(\sqrt{n}z_{m-1}, \sqrt{n}z_m, \sqrt{n}z_{m+1}) \left(f(z_m^+) - f(z)\right) \\ &+ \sum_{m=1}^{b(z)} n\Theta_n^{b,m}(\sqrt{n}z_{m-1}, \sqrt{n}z_m, \sqrt{n}z_{m+1}) \left(f(z_m^+) - f(z)\right) \\ &+ \sum_{m=a(z)}^{N} n\Theta_n^{a,m}(\sqrt{n}z_{m-1}, \sqrt{n}z_m, \sqrt{n}z_{m+1}) \left(f(z_m^-) - f(z)\right) \\ &+ n\mu_n \left(f(z_{b(z)}^+) - f(z)\right) \mathbb{1}_{\{z_{b(z)} \leq -\frac{1}{\sqrt{n}}\}} \\ &+ n\mu_n \left(f(z_{a(z)}^-) - f(z)\right) \mathbb{1}_{\{z_{a(z)} \geq \frac{1}{\sqrt{n}}\}} \\ &+ n\mu_n \left(\left(f(z^b) - f(z)\right) \mathbb{1}_{\{z_{b(z)}=0^-\}} + \left(f(z^a) - f(z)\right) \mathbb{1}_{\{z_{a(z)}=0^+\}}\right) \\ &= \sum_{m=1}^{b(z)} n\lambda_n^{b(z)-m} \left(f(z_m^-) - f(z)\right) \\ &+ \sum_{m=1}^{b(z)} n\theta_n^{b(z)-m} \left(f(z_m^+) - f(z)\right) \mathbb{1}_{\{z_m \leq -\frac{1}{\sqrt{n}}\}} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m=1}^{b(z)} \sqrt{n}\gamma^{b(z)-m} \Big((z_m - z_{m-1})^+ + (z_m - z_{m+1})^+ \Big) \Big(f(z_m^-) - f(z) \Big) \\
&+ \sum_{m=1}^{b(z)} \sqrt{n}\gamma^{b(z)-m} \Big((z_{m-1} - z_m)^+ + (z_{m+1} - z_m)^+ \Big) \Big(f(z_m^+) - f(z) \Big) \\
&+ \sum_{m=a(z)}^{N} n\lambda_n^{m-a(z)} \Big(f(z_m^+) - f(z) \Big) \\
&+ \sum_{m=a(z)}^{N} n\theta_n^{m-a(z)} \Big(f(z_m^-) - f(z) \Big) \mathbb{1}_{\{z_m \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sum_{m=a(z)}^{N} \sqrt{n}\gamma^{m-a(z)} \Big((z_{m-1} - z_m)^+ + (z_{m+1} - z_m)^+ \Big) \Big(f(z_m^+) - f(z) \Big) \\
&+ \sum_{m=a(z)}^{N} \sqrt{n}\gamma^{m-a(z)} \Big((z_m - z_{m-1})^+ + (z_m - z_{m+1})^+ \Big) \Big(f(z_m^-) - f(z) \Big) \\
&+ \sqrt{n}\mu \Big(f(z_{b(z)}^+) - f(z) \Big) \mathbb{1}_{\{z_{b(z)} \leq -\frac{1}{\sqrt{n}}\}} \\
&+ \sqrt{n}\mu \Big(f(z_{a(z)}^-) - f(z) \Big) \mathbb{1}_{\{z_{a(z)} \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sqrt{n}\mu \bigg(\Big(f(z^b) - f(z) \Big) \mathbb{1}_{\{z_{b(z)} = 0^-\}} + \big(f(z^a) - f(z) \big) \mathbb{1}_{\{z_{a(z)} = 0^+\}} \bigg) .
\end{aligned}
$$

The heavy traffic diffusion approximation on E^N of the entire limit order book on an absolute price grid is formulated in the following result:

Theorem 4.2.1 *The E_n^N-valued process $\hat{Z}_n$ converges weakly in $\mathbb{D}([0, +\infty[, E^N)$ as $n \to \infty$ to an E^N-valued strong Markov jump-diffusion process Z with infinitesimal generator given by the closure $\overline{A^N}$ of the linear operator A^N defined by, for all $z = (z_1, \dots, z_N) \in E^N$:*

$$
\begin{aligned}
A^N f(z) \;=\; & \frac{1}{2} \sum_{m=1}^{b(z)} \Big(\lambda^{b(z)-m} + \lambda^{b(z)-m} \mathbb{1}_{\{z_m < 0^-\}} \Big) \frac{\partial^2 f}{\partial z_m^2} \\
& + \frac{1}{2} \sum_{m=a(z)}^{N} \Big(\lambda^{m-a(z)} + \lambda^{m-a(z)} \mathbb{1}_{\{z_m > 0^+\}} \Big) \frac{\partial^2 f}{\partial z_m^2} \\
& + (\mu - c^0) \mathbb{1}_{\{z_{b(z)} < 0^-\}} \frac{\partial f}{\partial z_{b(z)}} + (c^0 - \mu) \mathbb{1}_{\{z_{a(z)} > 0^+\}} \frac{\partial f}{\partial z_{a(z)}} \\
& - \sum_{m=1}^{b(z)-1} c^{b(z)-m} \frac{\partial f}{\partial z_m} + \sum_{a(z)+1}^{N} c^{m-a(z)} \frac{\partial f}{\partial z_m}
\end{aligned}
$$

$$+\sum_{m=1}^{b(z)} \gamma^{b(z)-m}(z_{m-1}+z_{m+1}-2z_m)\frac{\partial f}{\partial z_m}$$
$$+\sum_{m=a(z)}^{N} \gamma^{m-a(z)}(z_{m-1}+z_{m+1}-2z_m)\frac{\partial f}{\partial z_m},$$

acting on $\mathrm{Dom}(A^N)$*, the space of* $\mathbb{C}^2_{b,\infty}(E^N)$ *functions satisfying:*

$$\begin{cases} \left.\dfrac{\partial f}{\partial z_{b(z)}}\right|_{z_{b(z)}=0^-} = \dfrac{\mu}{\lambda^0}\left(f(z_1,\ldots,z_{b(z)},\ldots,z_N)\Big|_{z_{b(z)}=0^-} - f(z^b)\right), \\ \left.\dfrac{\partial f}{\partial z_{a(z)}}\right|_{z_{a(z)}=0^+} = \dfrac{\mu}{\lambda^0}\left(f(z_1,\ldots,z_{a(z)},\ldots,z_N)\Big|_{z_{a(z)}=0^+} - f(z^a)\right), \\ \left.\dfrac{\partial f}{\partial z_m}\right|_{z_m=0^-} = 0 \ \textit{for all} \ m<b(z), \ \left.\dfrac{\partial f}{\partial z_m}\right|_{z_m=0^+} = 0 \ \textit{for all} \ m>a(z). \end{cases}$$

Similarly to what was previously done, we require the following technical lemma:

Lemma 4.2.1 *Let* $f \in \mathrm{Dom}(A^N)$ *and define the following sequence of functions* f_n *on* E^N*:*

$$f_n(z_1,\ldots,z_N) = f(u_n z_1,\ldots,u_n z_N).$$

Then $\lim_{n\to+\infty}\|f_n - f\| = 0$ *and* f_n *satisfies:*

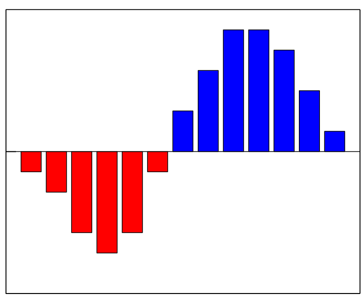
$$\begin{cases} \lim\limits_{n\to+\infty}\sqrt{n}\sup\limits_{z\in E_n^N, z_{b(z)}=0^-}\left|\dfrac{\partial f_n}{\partial z_{b(z)}} - \dfrac{\mu}{\lambda_n^0}\left(f_n(z)-f_n(z^b)\right)\right| = 0, \\ \lim\limits_{n\to+\infty}\sqrt{n}\sup\limits_{z\in E_n^N, z_{a(z)}=0^+}\left|\dfrac{\partial f_n}{\partial z_{a(z)}} - \dfrac{\mu}{\lambda_n^0}\left(f_n(z)-f_n(z^a)\right)\right| = 0, \\ \left.\dfrac{\partial f_n}{\partial z_m}\right|_{z_m=0^-} = 0 \ \textit{for all} \ m<b(z), \ \left.\dfrac{\partial f_n}{\partial z_m}\right|_{z_m=0^+} = 0 \ \textit{for all} \ m>a(z). \end{cases}$$

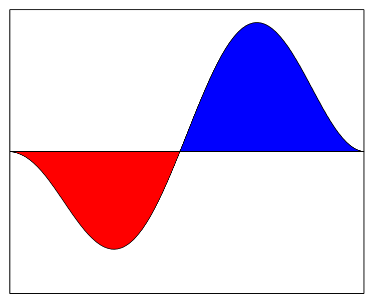
The proof of this lemma is not included as it is a direct extension of the proof of Lemma 3.2.1, whilst the proof of Theorem 4.2.1, similar in nature to that of Theorem 4.1.1, can be found in the Appendix.

Having reached this point, we stress that these multidimensional diffusion approximation results (in particular on the absolute price grid) represent the building block of the transition between mesoscopic and macroscopic models of limit order books. Simulations of the absolute price grid model are deliberately postponed to Chapter 7, as the derivation of the macroscopic limit presented in the next part of the thesis is crucial to gain a more holistic understanding of our general objective, which we recall is to provide an analytically tractable market simulating tool.

Part II

Limit order books and reflected SPDEs: from mesoscopic to macroscopic models





5 Some results from the theory of reflected SPDEs and stochastic interface models

This chapter provides an overview of the theoretical ingredients used in the transition from mesoscopic to macroscopic models of limit order books. We first give a brief presentation of reflected SPDEs in the first section, before showing their relevance in stochastic interface models in the second section. Within this context, we also establish a refinement of a weak convergence theorem proven by Funaki and Olla in [22] using results developed by Ambrosio et al. [3] and Zambotti [60] concerning monotone gradient systems. This refinement shall form the basis of our application to limit order books in the following chapter.

5.1 Overview of reflected SPDEs

Reflected SPDEs were introduced by Nualart and Pardoux [47] and Donati-Martin and Pardoux [20] in order to be able to take into consideration nonnegative solutions of SPDEs. This is evidently useful in a wide variety of applications, notably in the field of statistical mechanics and more precisely interface models, where one obvious desirable property is the positivity of the height variables. Our interest in reflected SPDEs with regard to limit order book modeling stems from the positivity of volume constraint, which is handled by such equations. Heuristically speaking, if we consider a point (x, t) where the solution $u(x, t)$ is equal to zero, the existence of a random measure η prevents it from becoming negative. We can therefore see a direct analogy with SDEs reflected at 0 with a local time term. This brief introduction to reflected SPDEs shall be carried out in the particular context of reflected solutions of the stochastic heat equation on the spatial interval $[0, 1]$ with Dirichlet boundary conditions, with an additive space-time white noise (the case of an infinite spatial interval, which falls beyond the scope of our applications, has been studied by Otobe [48]).

5.1.1 Formulation of the problem

Let W be a two-parameter Wiener process on $[0,1]\times\mathbb{R}_+$ on a complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$, where $\mathcal{F}_t$ is the σ-field generated by the random variables $W(x,r)$ for $x\in[0,1]$ and $r\in[0,t]$. W is a continuous and centred Gaussian process with covariance function given by:

$$\mathbb{E}\big(W(x,t)W(y,r)\big)=\min(x,y)\min(t,r).$$

We are interested in the following reflected stochastic heat equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}-f(x,t;u(x,t))+\sigma(x,t;u(x,t))\dot{W}(x,t)+\eta(x,t), \\ u(.,0)=u_0, \\ u(0,t)=u(1,t)=0, \end{cases} \tag{5.1}$$

where $\dot{W}$ is a space-time white noise, u_0 is a nonnegative continuous function on $[0,1]$ vanishing at 0 and 1, η is an adapted random measure on $]0,1[\times\mathbb{R}_+$ and f and σ are two measurable mappings from $[0,1]\times\mathbb{R}_+\times\mathbb{C}([0,1]\times\mathbb{R}_+)$ to $\mathbb{R}$. The precise formulation of the problem is to find a pair (u,η) such that:

(i) u is a nonnegative and continuous function of $(x,t)\in[0,1]\times\mathbb{R}_+$ and $u(x,t)$ is $\mathcal{F}_t$-measurable for all $(x,t)\in[0,1]\times\mathbb{R}_+$,

(ii) η is an adapted random measure on $]0,1[\times\mathbb{R}_+$ such that $\eta(]\epsilon,1-\epsilon[\times[0,T])<\infty$ for all $\epsilon>0$ and $T>0$,

(iii) (u,η) solves:

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}-f(x,t;u(x,t))+\sigma(x,t;u(x,t))\dot{W}(x,t)+\eta(x,t),$$

with $u(.,0)=u_0$, u_0 being a nonnegative and continuous function on $[0,1]$ vanishing at 0 and 1, and with Dirichlet boundary conditions $u(0,t)=u(1,t)=0$ for all $t\geq 0$,

(iv) η satisfies:

$$\int_{[0,1]\times\mathbb{R}_+} u(x,t)\eta(dx,dt)=0.$$

Nualart and Pardoux [47] first presented this problem in the context of a constant diffusion coefficient σ, whilst Donati-Martin and Pardoux [20] later examined the case of a general diffusion coefficient. At this point, it is important to emphasise that the third condition above is understood in the sense that for all $t\geq 0$ and any test function $\phi\in\mathbb{C}^2([0,1])$ verifying $\phi(0)=\phi(1)=0$:

$$\begin{aligned}\langle u_t,\phi\rangle &= \langle u_0,\phi\rangle+\int_0^t\langle u_r,\phi''\rangle dr-\int_0^t\langle f(x,r;u_r),\phi\rangle dr \\ &+ \int_0^t\int_0^1\phi(x)\sigma(x,r;u_r)W(dx,dr)+\int_0^t\int_0^1\phi(x)\eta(dx,dr)\ \text{a.s,}\end{aligned}$$

where $u_t = u(.,t)$ for all $t \geq 0$ and where $\langle . \rangle$ denotes the canonical scalar product in $\mathbb{L}^2([0,1])$.

Remark 5.1.1 *We notice that the fourth condition is analogous to the condition* $\int_{\mathbb{R}_+} X_t dL_t = 0$ *for reflected Brownian motion* X *and its local time* L.

We now need to specify the assumptions on the so-called external force f and the diffusion coefficient σ (as presented in Xu and Zhang [55]):

(a) For any $u, v \in \mathbb{C}([0,1] \times \mathbb{R}_+)$ and any $(x,t) \in [0,1] \times \mathbb{R}_+$ satisfying $u|_{[0,t]} = v|_{[0,t]}$, we have $f(x,t;u|_{[0,t]}(x,t)) = f(x,t;v|_{[0,t]}(x,t))$ and $\sigma(x,t;u|_{[0,t]}(x,t)) = \sigma(x,t;v|_{[0,t]}(x,t))$,

(b) For any $T, M > 0$, there exists a constant $C(T,M)$ such that for any $u, v \in \mathbb{C}([0,1] \times \mathbb{R}_+)$ and any $(x,t) \in [0,1] \times [0,T]$ verifying $\sup_{x \in [0,1], t \in [0,T]} |u(x,t)| \leq M$ and $\sup_{x \in [0,1], t \in [0,T]} |v(x,t)| \leq M$, we have:

$$\left| f(x,t;u(x,t)) - f(x,t;v(x,t)) \right| + \left| \sigma(x,t;u(x,t)) - \sigma(x,t;v(x,t)) \right| \leq C(T,M) \sup_{y \in [0,1], r \in [0,t]} \left| u(y,r) - v(y,r) \right|,$$

(c) For any $T > 0$, there exists a constant $M(T)$ such that for any $u \in \mathbb{C}([0,1] \times \mathbb{R}_+)$ and any $(x,t) \in [0,1] \times [0,T]$:

$$\left| f(x,t;u(x,t)) \right| + \left| \sigma(x,t;u(x,t)) \right| \leq M(T) \left(1 + \sup_{y \in [0,1], r \in [0,t]} \left| u(y,r) \right| \right).$$

5.1.2 Existence and uniqueness results

Nualart and Pardoux [47] first obtained the existence and uniqueness of (5.1) in the case of a constant diffusion coefficient σ. In Donati-Martin and Pardoux [20], the existence was extended to the case of a general diffusion coefficient σ, but the uniqueness was left as an open problem. In these two papers, the proof of the existence was heavily based on the use of penalised SPDEs. More recently, Xu and Zhang [55] established the uniqueness in the general diffusion coefficient framework, with the existence being dealt with via a much faster iteration method. We state their main result in the following theorem:

Theorem 5.1.1 *Under assumptions* (a),(b) *and* (c), *equation* (5.1) *has a unique solution* u *satisfying* $\mathbb{E}\left(\|u\|_t^p\right) < \infty$ *for all* $p \geq 1$, *where* $\|.\|_t$ *corresponds to the supremum norm on* $\mathbb{C}([0,1] \times [0,t])$.

Having introduced the main existence and uniqueness result for the solutions of reflected SPDEs, we now state a fundamental result established in Otobe [49] and Zambotti [56] related to the existence of an explicit invariant measure:

Theorem 5.1.2 *Suppose that the external force (or drift) f no longer depends on t, and that the diffusion coefficient σ is now a positive constant. Let $U : [0,1]\times\mathbb{R}_+ \mapsto \mathbb{R}$ be the potential associated with f, i.e. $\nabla U(x,z) = f(x,z)$, where ∇ denotes the partial derivative with respect to the second variable z. Then the reflected stochastic heat equation* (5.1) *admits an explicit invariant measure μ given by:*

$$\mu(d\psi) = Z^{-1}\exp\left(-\frac{2}{\sigma^2}\int_0^1 U(x,\psi(x))dx\right)\nu(d\psi), \quad \psi \in \mathbb{C}(]0,1[),$$

where ν is the law of the normalised Brownian excursion, otherwise known as the three-dimensional Bessel bridge on $[0,1]$, *and where Z is a finite normalisation constant.*

Remark 5.1.2 *In several applications, notably stochastic interface models, there exists a constant $\alpha > 0$ in front of the Laplace term, i.e. the system of interest now becomes:*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \alpha\frac{\partial^2 u(x,t)}{\partial x^2} - f(x,t;u(x,t)) + \sigma(x,t;u(x,t))\dot{W}(x,t) + \eta(x,t), \\ u(.,0) = u_0, \\ u(0,t) = u(1,t) = 0, \end{cases}$$

In this case, the proof of Theorem 5.1.2 can be readily extended to show that the invariant measure associated with these slightly modified dynamics is given by:

$$\tilde{\mu}(d\psi) = \widetilde{Z}^{-1}\exp\left(-\frac{2}{\sigma^2}\int_0^1 U(x,\psi(x))dx\right)\tilde{\nu}(d\psi), \quad \psi \in \mathbb{C}(]0,1[),$$

where $\tilde{\nu}$ is the law of $\alpha^{-1/2}e$, $(e_\tau)_{\tau\in[0,1]}$ being the normalised Brownian excursion, and where $\widetilde{Z}$ is a finite normalisation constant. We also stress that existence and uniquenss of the solution given by Theorem 5.1.1 can be extended to this case.

At this point, it seems interesting to give a heuristic explanation as to how one can obtain such explicit invariant measures. For the sake of simplicity, we place ourselves in the nonreflecting case, and consider a simple stochastic heat equation on $[0,1]$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + \dot{W}(x,t), \\ u(.,0) = u_0, \\ u(0,t) = u(1,t) = 0, \end{cases}$$

We suppose that $u_0 \in \mathbb{L}^2(]0,1[)$ and introduce the complete orthonormal basis $(e_k)_{k\geq 1}$ of $\mathbb{L}^2(]0,1[)$ defined by $e_k(x) = \sqrt{2}\sin(k\pi x)$ for all $k \geq 1$ and $x \in [0,1]$. Using the Fourier representations of the solution and of the space-time white noise $\dot{W}$, we can write:

$$u(x,t) = \sum_{k\geq 1}\hat{u}_t^k e_k(x),$$

$$\dot{W}(x,t) = \sum_{k\geq 1} \frac{dW_t^k}{dt} e_k(x),$$

where $(W_t^k)_{k\geq 1}$ is a family of independent standard Brownian motions, and where the derivative is taken in a distributional sense. As $e_k''(x) = -(\pi k)^2 e_k(x)$, we have:

$$\sum_{k\geq 1} d\hat{u}_t^k e_k(x) = -\frac{1}{2}\sum_{k\geq 1}(\pi k)^2 \hat{u}_t^k e_k(x)dt + \sum_{k\geq 1} dW_t^k e_k(x).$$

Let $j \geq 1$. We can therefore write:

$$\begin{aligned} d\hat{u}_t^j &= \sum_{k\geq 1} d\hat{u}_t^k \int_0^1 e_k(x)e_j(x)dx \\ &= -\frac{1}{2}\sum_{k\geq 1}(\pi k)^2 \hat{u}_t^k \int_0^1 e_k(x)e_j(x)dxdt + \sum_{k\geq 1} dW_t^k \int_0^1 e_k(x)e_j(x)dx \\ &= -\frac{(\pi j)^2}{2}\hat{u}_t^j dt + dW_t^j, \end{aligned}$$

and we immediately notice that we have now obtained a system of independent OU processes. For each $j \geq 1$, it is well known that the invariant measure associated with the j-th equation is given by $\mu_j = \mathcal{N}(0, (\pi j)^{-2})$. By independence, the (unique) invariant measure of the system is given by $\otimes_{k\geq 1}\mu_k$, which also means that the (unique) invariant measure of the solution is given by the distribution of:

$$\beta = \sum_{k\geq 1} \frac{1}{\pi k} e_k X_k,$$

where $(X_k)_{k\geq 1}$ is a sequence of i.i.d $\mathcal{N}(0,1)$ random variables. We are finally able to conclude by recognising the Karhunen-Loève representation of the Brownian bridge.

5.2 The Funaki-Olla stochastic interface model

One of the main motivations behind the study of reflected SPDEs is their relevance in the field of stochastic interface models (we refer to Funaki [24] for an exhaustive coverage of the matter). More specifically, Funaki and Olla [22] established that equilibrium fluctuations of a $\nabla\phi$ model on a hard wall converged weakly to the stationary solution of a reflected stochastic heat equation. More recently, using techniques derived from stability properties of Markov processes with log-concave invariant measures (see Ambrosio et al. [3]), Zambotti [60] studied the equilibrium fluctuations of a conservative (i.e. with conservation of the height variables) $\nabla\phi$ model on a hard wall, showing that they converged weakly to the solution of a reflected stochastic Cahn-Hilliard equation, for any sequence of initial conditions (as opposed to just the stationary case in Funaki and Olla [22]). As mentioned by Zambotti [60], his result is therefore comparatively stronger and the techniques developed in his paper can be adapted to the nonconservative case so as to correspondingly improve Funaki and Olla's result. In the following, we provide a brief overview of existing results, before writing up the proof of the improved Funaki-Olla result, which shall subsequently be the starting point (as well as link to previous diffusion approximation results) in our application to limit order books in a mesoscopic to macroscopic setting.

5.2.1 Overview of the model

Given a one-dimensional lattice $\Gamma_N = \{1, ..., N\}$, the location of the interface at any given time $t \geq 0$ is given by the height variables $\phi_t = (\phi_t(k))_{k \in \Gamma_N}$. As the height variables need to stay positive, a local time term appears in their dynamics, which are expressed in the following way:

$$\begin{aligned} d\phi_t(k) &= -\left(V'(\phi_t(k) - \phi_t(k-1)) + V'(\phi_t(k) - \phi_t(k+1))\right) dt \\ &\quad + \sqrt{2} dw_t(k) + dl_t(k), \quad k \in \Gamma_N, \end{aligned} \tag{5.2}$$

where V is a potential, $\omega = (\omega(k))_{k \in \Gamma_N}$ is a family of independent Brownian motions, and with the conditions:

(i) $\phi_t(0) = \phi_t(N+1) = 0$,

(ii) $\phi_t(k) \geq 0$,

(iii) $l_t(k)$ is continuous and nondecreasing in t,

(iv) $l_0(k) = 0$,

(v) $\int_0^\infty \phi_t(k) dl_t(k) = 0$,

for all $k \in \Gamma_N$. The fifth condition simply translates the fact that $l_t(k)$ only increases when the interface is in contact with the wall ($\phi_t(k) = 0$). In addition, the potential V is assumed to satisfy:

(a) $V \in \mathbb{C}^2(\mathbb{R})$,

(b) $V(-y) = V(y)$ for all $y \in \mathbb{R}$ (symmetry),

(c) There exist $c_- > 0$ and $c_+ > 0$ such that $c_- \leq V''(y) \leq c_+$ for all $y \in \mathbb{R}$ (strict convexity).

As we shall see in more detail later on, one fundamental ingredient of the weak convergence of the equilibrium fluctuations is the invariant measure of the initial system. As a matter of fact, the dynamics of the system of reflected SDEs (5.2) are stationary under the Gibbs measure μ_N associated with the Hamiltonian $H_N(\phi) = \sum_{k=1}^{N+1} V(\phi(k) - \phi(k-1))$ conditioned on $\phi(k)$ being nonnegative for all $k \in \Gamma_N$. In other words, μ_N is the probability measure on $\mathbb{R}_+^{\Gamma_N}$ given by:

$$\mu_N(d\phi) = Z_N^{-1} \exp\left(-\sum_{k=1}^{N+1} V(\phi(k) - \phi(k-1))\right) \prod_{k=1}^{N} \mathbb{1}_{\{\phi(k) \geq 0\}} d\phi(k),$$

with the pinning condition $\phi(0) = \phi(N+1) = 0$, and where Z_N is a finite normalisation constant. Before stating the weak convergence result, we need to introduce some notation related to the rescaling of the height variables. We follow the notation used by Zambotti [60]. Let $\Lambda_N : \mathbb{R}^N \to \mathbb{L}^2(]0,1[)$ be the rescaling map defined by:

$$\Lambda_N(\phi)(x) = \frac{1}{\sqrt{N}}\phi\left(\lfloor Nx \rfloor + 1\right), \ \ x \in]0,1[,$$

where $\lfloor . \rfloor$ denotes the floor function. We also define the following spaces:

$$H_N = \Lambda_N(\mathbb{R}^N) \subset \mathbb{L}^2(]0,1[), \ \ \Omega_N^+ = \mathbb{R}_+^N, \ \ K_N = \Lambda_N(\Omega_N^+).$$

By definition, K_N therefore corresponds to the space of nonnegative functions on $]0,1[$ which are constant on the intervals $I(k) = \left]\frac{k-1}{N}, \frac{k}{N}\right]$ for all $k \in \Gamma_N$. The rescaled interface Φ^N is now defined by:

$$\Phi_t^N = \Lambda_N(\phi_{N^2 t}), \ \ \Phi_0^N = \Lambda_N(\phi_0).$$

Using the definition of the rescaling map Λ_N, the rescaled interface can thus be expressed by:

$$\Phi_t^N(x) = \frac{1}{\sqrt{N}}\phi_{N^2 t}\left(\lfloor Nx \rfloor + 1\right), \ \ x \in]0,1[.$$

The main result of Funaki and Olla [22] is given in the following theorem:

Theorem 5.2.1 *Suppose that the rescaled interface Φ^N has initial distribution μ_N. For any $0 < s \leq T < \infty$, the law of $(\Phi_t^N)_{t\in[s,T]}$ converges weakly in $\mathbb{C}([s,T], \mathbb{L}_w^2(]0,1[)$ (where $\mathbb{L}_w^2(]0,1[)$ denotes the space $\mathbb{L}^2(]0,1[)$ endowed with the weak topology) as $N \to \infty$ to the law of the unique stationary solution of the following reflected stochastic heat equation:*

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{q}\frac{\partial^2 u(x,t)}{\partial x^2} + \sqrt{2}\dot{W}(x,t) + \eta(x,t), \tag{5.3}$$

with Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ for all $t \geq 0$, with $u \geq 0$, $d\eta \geq 0$, $\int u d\eta = 0$, and where the finite constant q is defined by:

$$q = \frac{1}{\int_{\mathbb{R}} \exp(-V(y))dy}\int_{\mathbb{R}} y^2 \exp(-V(y))dy.$$

It is vital to point out that the statement in Theorem 5.2.1 specifically means that for any $n \in \mathbb{N}$, any collection of test functions $h_1, \ldots, h_n \in \mathbb{C}^1([0,1])$, and any $0 < s \leq T < \infty$, the process $\Big(\langle \Phi_t^N, h_i\rangle, i \in \{1,\ldots,n\}\Big)_{s\leq t\leq T}$ converges weakly in $\mathbb{C}([s,T],\mathbb{R}^n)$ as $N \to \infty$ to the process $\left(\langle u_t, h_i\rangle, i \in \{1,\ldots,n\}\right)_{s\leq t\leq T}$, where $u_t \equiv u(.,t)$ by convention and where $\langle .,.\rangle$ denotes the canonical $\mathbb{L}^2(]0,1[)$ scalar product. A key component of its proof is the derivation of the so-called Boltzmann-Gibbs principle, according to which one can replace the discrete Laplace operator (i.e. the first term of the right-hand side of (5.2)) with the linear Laplacian in (5.3). This highly technical procedure described in [22] can be avoided, and the subsequent result can be improved, by adapting the techniques used by Zambotti in [60] to the nonconservative case. We present them in the following section.

5.2.2 Convergence of monotone gradient systems

This section is devoted to a weak convergence theorem introduced in Ambrosio et al. [3]. The theoretical framework of interest here deals with the convergence in law of stochastic processes associated with symmetric Dirichlet forms of gradient type and log-concave invariant measures. The idea is therefore initially based on the observation that the solutions of equations (5.2) and (5.3) are effectively in this class, which happens to be parametrised by two fundamental objects, namely the invariant measure and the scalar product of the Hilbert space which defines the gradient. We shall now see that the weak convergence of the invariant measure of (5.2) as well as a suitable convergence (to be made precise later on) of the norm induced by this scalar product imply the convergence in law of the associated processes.

Given $j \in \mathbb{N}^*$, numerous probability measures on $\mathbb{R}^j$ have a log-concave density given by:

$$\gamma(dy) = Z^{-1} \exp\left(-\frac{2}{\sigma^2} U(y)\right) dy,$$

where the potential $U : \mathbb{R}^j \to \mathbb{R} \cup \{\infty\}$ is convex and the normalisation constant $Z = \int_{\mathbb{R}^j} \exp(-U(y))dy$ is finite. In particular, all Gaussian measures are included in this class. If U has a Lipschitz-continuous gradient $\nabla U : \mathbb{R}^j \to \mathbb{R}^j$, then one can construct a Markov process X in $\mathbb{R}^j$ with γ as its invariant measure:

$$dX_t = -\nabla U(X_t)dt + \sigma dW_t, \quad X_0(y) = y,$$

where $W = (W^1, \dots, W^j)$ is a vector of j independent Wiener processes. Noticing that $-\langle \nabla U(y) - \nabla U(z), y - z\rangle \leq 0$ for any $y, z \in \mathbb{R}$ by convexity of U, the above SDE is called a monotone gradient system. One particularly important point is to understand the link between the invariant measure, the infinitesimal generator, and the Dirichlet form associated with the process X (see for example Zambotti [59] and [61]). We first recall that the infinitesimal generator $\mathcal{L}$ of X is given by:

$$\mathcal{L}f = \frac{\sigma^2}{2}\Delta f - \langle \nabla U, \nabla f\rangle.$$

Now, using integration by parts, we can write, for sufficiently smooth f and g with compact support:

$$-\int g\mathcal{L}f d\gamma = \frac{\sigma^2}{2}\int \langle \nabla f, \nabla g\rangle d\gamma = \mathcal{E}(f, g),$$

where $\mathcal{E}$ is the Dirichlet form associated with X. This interesting relationship shows that the generator, the Dirichlet form, and the law of X are equivalent in a certain sense, and subsequently forms the basis of the convergence result established by Ambrosio et al. [3].

We now present the formal framework which shall allow us to state the main result. We consider a separable Hilbert space H with scalar product $\langle .,. \rangle_H$, and a log-concave probability measure γ on H. Let K be the support of γ and A be the smallest closed affine subspace of H containing K. We then introduce the canonical decomposition $A = H^0 + h^0$, where H^0 is a closed linear subspace of H and where h^0 is the element of minimal norm in A. H^0 is endowed with the scalar product $\langle .,. \rangle_{H^0}$ induced by H. The idea is to consider an A-valued random process which is reversible with respect to γ. In this setting, the gradient of any function $f \in \mathbb{C}_b^1(A)$ is defined by:

$$\langle \nabla_{H^0} f(a), h \rangle_{H^0} = \frac{\partial}{\partial \epsilon} f(a + \epsilon h) \bigg|_{\epsilon=0},$$

for all $a \in A$ and $h \in H^0$. We denote $X_t : K^{[0,+\infty[} \to K$ the coordinate process $X_t(\omega) = \omega_t$ for all $t \geq 0$, and define:

$$\mathcal{P}_2(H) = \left\{ \mu \in \mathcal{P}(H) : \int_H \|x\|_H^2 d\mu(x) < \infty \right\},$$

where $\mathcal{P}(H)$ denotes the set of Borel probability measures on H. We now state an important result which enables us to canonically determine A-valued processes which are reversible with respect to γ:

Theorem 5.2.2

(a) The bilinear form $\mathcal{E}$ given by:

$$\mathcal{E}(u, v) = \frac{\sigma^2}{2} \int_K \langle \nabla_{H^0} u, \nabla_{H^0} v \rangle_{H^0} d\gamma, \quad u, v \in \mathbb{C}_b^1(A),$$

is closable in $\mathbb{L}^2(\gamma)$ and its closure $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a symmetric Dirichlet form.

(b) There exists a unique Markov family $(\mathbb{P}_x)_{x \in K}$ of probability measures on $K^{[0,+\infty[}$ associated with $\mathcal{E}$.

(c) The family $(\mathbb{P}_x)_{x \in K}$ is reversible with respect to γ.

(d) If $\gamma \in \mathcal{P}_2(H)$, then γ is the only invariant measure for the associated semigroup $(P_t)_{t \geq 0}$ in $\mathcal{P}_2(H)$.

As it turns out, the solutions of equations (5.2) and (5.3) (as well as the solutions of our refined equations presented in the following section) are particular cases of Markov processes characterised in this theorem. Moving on to the convergence theorem, we now consider a sequence $(\gamma_N)_{N \geq 1}$ of log-concave probability measures on H which converges weakly to γ. For each $N \geq 1$, the support of γ_N is denoted by K_N, and A_N is the smallest closed affine subspace of H containing K_N. It is assumed that $A_N \subseteq A$ for all $N \geq 1$. Once again, we can write the canonical decomposition $A_N = H_N^0 + h_N^0$, where $H_N^0 \subseteq H^0$ is a closed linear subsapce of H and h_N^0 is the element of minimal norm in A_N. Finally, let $(\mathbb{P}_x^N)_{x \in K_N}$ (respectively $(\mathbb{P}_x)_{x \in K}$) be the Markov process in $[0, +\infty[^{K_N}$ (respectively $[0, +\infty[^K$) associated with γ_N (respectively γ) in Theorem 5.2.2.

Theorem 5.2.3 *Suppose that for all $h \in H_N^0$ and $N \geq 1$ there exists a finite constant $\kappa \geq 1$ such that:*

$$\frac{1}{\kappa}\|h\|_{H^0} \leq \|h\|_{H_N^0} \leq \kappa \|h\|_{H^0}. \tag{5.4}$$

Denoting by $\Pi_N : H^0 \to H_N^0$ the orthogonal projection induced by the scalar product of H^0, we also assume that for all $h \in H^0$:

$$\lim_{N \to \infty} \|\Pi_N h\|_{H_N^0} = \|h\|_{H^0}. \tag{5.5}$$

Then if γ_N converges weakly to γ in H, for any $x_N \in K_N$ such that $x_N \to x \in K$ in H, and any $0 < s \leq T < +\infty$, we have $\mathbb{P}_{x_N}^N \to \mathbb{P}_x$ weakly in $\mathbb{C}([s, T], H_w)$.

This very powerful result amounts to saying that the weak convergence of the invariant measures and the convergence of the relevant norms entail the weak convergence of the associated processes, for any type of converging initial conditions.

5.2.3 Refinement of the Funaki-Olla convergence result

In view of our application to limit order books, and using Theorem 5.2.3, we now wish to refine Theorem 5.2.1 with the two following improvements:

(1) We would like the result to hold for any initial distribution, rather than just the stationary case.

(2) In the dynamics expressed by (5.2), we would like to consider an additional space-dependent drift term $f(k) \in \mathbb{R}^N$ as well as a general non space-dependent scaling coefficient $\sigma > 0$.

Given the same one-dimensional lattice $\Gamma_N = \{1, ..., N\}$, the dynamics of the height variables can now be expressed as:

$$\begin{aligned} d\phi_t(k) &= -\big(V'(\phi_t(k) - \phi_t(k-1)) + V'(\phi_t(k) - \phi_t(k+1))\big)\, dt \\ &\quad + N^{-3/2} f\left(\frac{k}{N}\right) dt + \sigma dw_t(k) + dl_t(k), \quad k \in \Gamma_N, \end{aligned} \tag{5.6}$$

with the same conditions (i)-(v) as previously described. We stress that the drift term $f(k)$ is mesoscopically rescaled here (as in Funaki [23] and Zambotti [57]) in order to obtain a nontrivial macroscopic drift in the limit. At this point, it is not hard to see that the dynamics of the system (5.6) are stationary under the Gibbs measure $\tilde{\mu}_N$ given by:

$$\begin{aligned}\tilde{\mu}_N(d\phi) &= \widetilde{Z}_N^{-1}\exp\left(-\frac{2}{\sigma^2}\sum_{k=1}^{N+1}V(\phi(k)-\phi(k-1))\right)\\ &\quad\times\exp\left(\frac{2}{\sigma^2}N^{-3/2}\sum_{k=1}^{N}f\left(\frac{k}{N}\right)\phi(k)\right)\prod_{k=1}^{N}\mathbb{1}_{\{\phi(k)\geq 0\}}d\phi(k)\\ &= \widetilde{Z}_N^{-1}\exp\left(\frac{2}{\sigma^2}N^{-3/2}\sum_{k=1}^{N}f\left(\frac{k}{N}\right)\phi(k)\right)\mu_N(d\phi),\end{aligned}$$

where μ_N is the invariant measure associated with the dynamics (5.2) with a general diffusion coefficient $\sigma > 0$ instead of $\sqrt{2}$, i.e.:

$$\mu_N(d\phi) = Z_N^{-1}\exp\left(-\frac{2}{\sigma^2}\sum_{k=1}^{N+1}V(\phi(k)-\phi(k-1))\right)\prod_{k=1}^{N}\mathbb{1}_{\{\phi(k)\geq 0\}}d\phi(k).$$

Once again, we work with the pinning condition $\phi(0) = \phi(N+1) = 0$, and assume that $\tilde{Z}_N$ is a finite normalisation constant. We define $H = \mathbb{L}^2(]0,1[)$ and adopt the same notation as in the two previous sections. H is endowed with the canonical scalar product $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$ and the associated norm denoted by $\|.\|$. For all $F \in \mathbb{C}_b^1(\Omega_N^+)$, we note that the Markov process $(\phi(t,\phi_0))_{t\geq 0,\phi_0\in\Omega_N^+}$ is the diffusion generated by the symmetric Dirichlet form in $\mathbb{L}^2(\Omega_N^+,\tilde{\mu}_N)$ given by the closure of:

$$\mathcal{E}_N(F,F) = \frac{\sigma^2}{2}\int_{\mathbb{R}_+^{\Gamma_N}}\|\nabla F\|^2 d\tilde{\mu}_N = \frac{\sigma^2}{2}\int_{\mathbb{R}_+^{\Gamma_N}}\sum_{k=1}^{N}\left|\frac{\partial F}{\partial\phi(k)}\right|^2 d\tilde{\mu}_N,$$

where the closability of $\mathcal{E}_N$ simply follows from the characterisation given in Theorem 5.2.2. The refinement of Theorem 5.2.1 is now given in the following result:

Theorem 5.2.4 *Suppose that the initial rescaled interface $\Phi_0^N \geq 0$ converges weakly in H as $N\to\infty$ to u_0. Then, for any $0 < s \leq T < \infty$, the law of $(\Phi_t^N)_{t\in[s,T]}$ converges weakly in $\mathbb{C}([s,T],H_w)$ (where H_w denotes the space H endowed with the weak topology) as $N\to\infty$ to the law of the unique solution of the following reflected stochastic heat equation:*

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{q}\frac{\partial^2 u(x,t)}{\partial x^2} + f(x) + \sigma\dot{W}(x,t) + \eta(x,t), \tag{5.7}$$

with Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ for all $t \geq 0$, with $u \geq 0$, $d\eta \geq 0$, $\int u d\eta = 0$, with $u(x,0) = u_0(x)$ for all $x \in [0,1]$, and where q is the finite constant given by:

$$q = \frac{1}{\int_{\mathbb{R}}\exp\left(-\frac{2}{\sigma^2}V(y)\right)dy}\int_{\mathbb{R}} y^2\exp\left(-\frac{2}{\sigma^2}V(y)\right)dy.$$

Proof of Theorem 5.2.4: The proof is based on the two-step convergence result for monotone gradient systems expressed in Theorem 5.2.3. We first need to ensure the suitable convergence of the relevant family of scalar products, by checking that conditions (5.4) and (5.5) are effectively verified. We then have to establish the weak convergence of the sequence of invariant measures associated with the dynamics (5.6) to the invariant measure of the reflected stochastic heat equation (5.7).

Step 1: *convergence of the scalar products*

Given that there is no conservation of the height variables, our case is in fact much more straightforward than the one presented in Zambotti [60]. We are in the so-called nondegenerate situation where $A_N = H_N^0 = H_N$ for all $N \geq 1$ and $A = H^0 = H$. All these spaces are therefore simply endowed with the canonical scalar product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ and the associated norm $\|.\|$. This is the correct Hilbert space structure which makes the equations we are studying a monotone gradient system. Condition (5.4) is therefore trivially satisfied taking $\kappa = 1$. As for condition (5.5), we first recall that $\Pi_N : H \to H_N$ is the orthogonal projection induced by the scalar product of H. We then define the linear operator $\pi_N : H \to H_N$ by:

$$\pi_N h = \sum_{i=1}^{N} N\langle h, \mathbb{1}_{I(i)}\rangle \mathbb{1}_{I(i)}.$$

Also known as the approximation operator, π_N is in fact a projection, as:

$$\begin{aligned}
\pi_N^2 h = \sum_{i=1}^{N} N\langle \pi_N h, \mathbb{1}_{I(i)}\rangle \mathbb{1}_{I(i)} &= N^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \langle h, \mathbb{1}_{I(j)}\rangle \langle \mathbb{1}_{I(j)}, \mathbb{1}_{I(i)}\rangle \mathbb{1}_{I(i)} \\
&= N^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \langle h, \mathbb{1}_{I(j)}\rangle \frac{1}{N} \mathbb{1}_{\{i=j\}} \mathbb{1}_{I(i)} \\
&= N \sum_{i=1}^{N} N\langle h, \mathbb{1}_{I(i)}\rangle \mathbb{1}_{I(i)} \\
&= \pi_N h.
\end{aligned}$$

Furthermore, it is easily seen that π_N is self-adjoint:

$$\langle \pi_N h, g\rangle = \sum_{i=1}^{N} N\langle h, \mathbb{1}_{I(i)}\rangle \langle \mathbb{1}_{I(i)}, g\rangle = \sum_{i=1}^{N} N\langle g, \mathbb{1}_{I(i)}\rangle \langle h, \mathbb{1}_{I(i)}\rangle = \langle h, \pi_N g\rangle,$$

and therefore π_N is an orthogonal projection satisfying $\|\pi_N h\| \leq \|h\|$ for all $h \in H$. Now, fix $h \in H$ and $\epsilon > 0$. By density, there exists $g \in \mathbb{C}([0,1])$ such that $\|h - g\| \leq \frac{\epsilon}{3}$. As g is uniformly continuous on $[0,1]$, there also exists $\delta_\epsilon > 0$ such that for all

$(x,y) \in [0,1]^2$, $|x-y| \leq \delta_\epsilon \Rightarrow |g(x)-g(y)| \leq \frac{\epsilon}{2}$. Let $M \in \mathbb{N}^*$ such that $1/M \leq \delta_\epsilon$ and $i \in \{1,\ldots,N\}$. For all $N \geq M$, we therefore have:

$$\left| N \int_{\frac{i-1}{N}}^{\frac{i}{N}} g(y)dy - g\left(\frac{i}{N}\right) \right| \leq \frac{\epsilon}{2}.$$

Let $x \in [0,1]$. We can then see that for all $N \geq M$:

$$\left| \pi_N g(x) - \sum_{i=1}^{N} g\left(\frac{i}{N}\right) \mathbb{1}_{I(i)}(x) \right| \leq \frac{\epsilon}{2}.$$

Reproducing this argument, we also have, for all $N \geq M$:

$$\begin{aligned}
\left|\pi_N g(x) - g(x)\right| &\leq \left| \pi_N g(x) - \sum_{i=1}^{N} g\left(\frac{i}{N}\right) \mathbb{1}_{I(i)}(x) \right| \\
&\quad + \left| \sum_{i=1}^{N} g\left(\frac{i}{N}\right) \mathbb{1}_{I(i)}(x) - g(x) \right| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

It is immediately deduced that:

$$\lim_{N\to\infty} \sup_{x\in[0,1]} \left|\pi_N g(x) - g(x)\right| = 0,$$

and consequently:

$$\lim_{N\to\infty} \|\pi_N g - g\| = 0.$$

In particular, for all $N \geq M$, we can write:

$$\|\pi_N g - g\| \leq \frac{\epsilon}{3}.$$

Finally, using the contraction property of π_N, we notice that for all $N \geq M$:

$$\begin{aligned}
\|\pi_N h - h\| &\leq \|\pi_N h - \pi_N g\| + \|\pi_N g - g\| + \|g - h\| \\
&\leq \|h - g\| + \|\pi_N g - g\| + \|g - h\| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

thus enabling us to have $\lim_{N\to\infty} \|\pi_N h - h\| = 0$. As $\Pi_N h$ is the element of minimal distance from h in H_N, we have:

$$\|\Pi_N h - h\| \leq \|\pi_N h - h\|,$$

and we subsequently deduce that $\lim_{N\to\infty} \|\Pi_N h - h\| = 0$, which is none other than condition (5.5).

Step 2: *weak convergence of the invariant measures*

We first recall that the rescaled interface is expressed by $\Phi_t^N = \Lambda_N(\phi_{N^2 t})$ for all $t \geq 0$. We next define the image (or pushforward) measures of μ_N and $\tilde{\mu}_N$ under the rescaling map Λ_N:

$$m_N = \Lambda_N^*(\mu_N), \;\; \widetilde{m}_N = \Lambda_N^*(\tilde{\mu}_N).$$

Let $(e_\tau)_{\tau \in [0,1]}$ be the normalised Brownian excursion. We denote by m be the law of $q^{1/2}e$. It is well known (for example see Zambotti [57]) that m_N converges weakly to m as $N \to \infty$ in the Skorokhod topology. We shall just give the main arguments of this proof. Let $(b_\tau)_{\tau \in [0,1]}$ be a standard Brownian bridge and consider a sequence $(X_i)_{i \in \mathbb{N}}$ of i.i.d random variables with density $\exp(-(2/\sigma^2)V(y))$ on $\mathbb{R}$. We then introduce the constant c defined by:

$$c = \mathbb{E}(X_1^2).$$

We note that in this case, $c = q$, as $\int_{\mathbb{R}} \exp(-(2/\sigma^2)V(y))dy = 1$. For all $n \in \mathbb{N}$, we define $S_n = \sum_{i=1}^n X_i$ and set $X_0 = 0$. Denoting by $\mathbb{P}_N$ the law of the vector $(S_1, \ldots, S_N)$ conditionally on the event $\{S_{N+1} = 0\}$, it is easily shown that $\mu_N = \mathbb{P}_N(.|\Omega_N^+)$. But then Donsker's Theorem and the Local Limit Theorem for the density of $S_N/\sqrt{N}$ tell us that the law of Λ_N under $\mathbb{P}_N$ converges weakly to the law of $c^{1/2}b$ as $N \to \infty$ in the Skorokhod topology. For the argument which enables us to extend this to the positive case, we refer to the full proof in [57]. In order to make use of this result, we now want to express $\widetilde{m}_N$ in terms of m_N. To this effect, let $\psi \in K_N$. As $K_N = \Lambda_N(\Omega_N^+)$ by definition, there exists $\phi \in \Omega_N^+$ such that $\psi = \Lambda_N(\phi)$. We can then write:

$$\begin{aligned}
\widetilde{m}_N(d\psi) &= \widetilde{Z}_N^{-1} \exp\left(\frac{2}{\sigma^2} N^{-3/2} \sum_{k=1}^N f\left(\frac{k}{N}\right) \Lambda_N^{-1}(\psi)(k) \right) m_N(d\psi) \\
&= \widetilde{Z}_N^{-1} \exp\left(\frac{2}{\sigma^2} N^{-3/2} \sum_{k=1}^N f\left(\frac{k}{N}\right) \phi(k) \right) m_N(d\psi) \\
&= \widetilde{Z}_N^{-1} \exp\left(\frac{2}{\sigma^2} \int_0^1 \sum_{k=1}^N f\left(\frac{k}{N}\right) N^{-1/2} \phi(k) \mathbb{1}_{I(k)}(x) dx \right) m_N(d\psi) \\
&= \widetilde{Z}_N^{-1} \exp\left(\frac{2}{\sigma^2} \int_0^1 f\left(\frac{\lfloor Nx \rfloor + 1}{N}\right) N^{-1/2} \phi\left(\lfloor Nx \rfloor + 1\right) dx \right) m_N(d\psi) \\
&= \widetilde{Z}_N^{-1} \exp\left(\frac{2}{\sigma^2} \int_0^1 f\left(\frac{\lfloor Nx \rfloor + 1}{N}\right) \Lambda_N(\phi)(x) dx \right) m_N(d\psi) \\
&= \widetilde{Z}_N^{-1} \exp\left(\frac{2}{\sigma^2} \int_0^1 f\left(\frac{\lfloor Nx \rfloor + 1}{N}\right) \psi(x) dx \right) m_N(d\psi).
\end{aligned}$$

We therefore deduce that $\widetilde{m}_N$ converges weakly to $\widetilde{m}$ as $N \to \infty$ in the Skorokhod topology, where $\widetilde{m}$ is given by:

$$\widetilde{m} = \widetilde{Z}^{-1} \exp\left(\frac{2}{\sigma^2} \int_0^1 f(x)\psi(x)dx\right) m(d\psi) = \widetilde{Z}^{-1} \exp\left(\frac{2}{\sigma^2} \langle f, \psi \rangle\right) m(d\psi),$$

and where $\widetilde{Z} = \int_K e^{(2/\sigma^2)\langle f,\psi\rangle} m(d\psi)$ is a finite normalisation constant. Thanks to Theorem 5.1.2 and Remark 5.1.2, taking $U(x, \psi(x)) = -f(x)\psi(x)$ and $\alpha = 1/q$, we then observe that $\widetilde{m}$ is effectively the invariant measure of the reflected SPDE (5.7). We then note that for all $F \in \mathbb{C}_b^1(H_N)$, the Markov process Φ_N is the diffusion generated by the symmetric Dirichlet form in $\mathbb{L}^2(K_N, \widetilde{m}_N)$ given by the closure of:

$$\widetilde{\mathcal{E}}_N(F, F) = \frac{\sigma^2}{2} \int_{K_N} \|\nabla F\|^2 d\widetilde{m}_N,$$

where the closability of $\widetilde{\mathcal{E}}_N$ once again follows from the characterisation given in Theorem 5.2.2. Finally, for all $F \in \mathbb{C}_b^1(H)$, the Markov process $(u(t, u_0))_{t \geq 0, u_0 \in K}$ is the diffusion generated by the symmetric Dirichlet form $\mathbb{L}^2(K, \widetilde{m})$ given by the closure of (for more details we refer to the weak convergence proofs given by Zambotti in [57] and [60]):

$$\widetilde{\mathcal{E}}(F, F) = \frac{\sigma^2}{2} \int_K \|\nabla F\|^2 d\widetilde{m}.$$

Being in the framework presented in Section 5.2.2, we are able to apply Theorem 5.2.3 and thus ensure the weak convergence of the associated processes.

□

6 Application to limit order books

This chapter makes use of the refinement of the Funaki-Olla weak convergence result presented in Theorem 5.2.4 within the context of the transition between mesoscopic and macroscopic models of limit order books. The main result, introduced in Section 6.2.2, generalises the weak convergence of the Funaki-Olla system to the case of a fully dynamic limit order book with price changes.

6.1 SPDE limit in a static limit order book framework

The first natural step towards bridging the gap between the mesoscopic and macroscopic settings is to consider a static order book where the price is assumed to be constant in time. The starting point is the absolute price grid model of Chapter 4.

6.1.1 Building on the absolute price grid diffusion approximation

We recall that the main result (Theorem 4.2.1) tells us that the discrete order book process $\hat{Z}_n$ converges weakly to a process Z whose generator is the closure of:

$$\begin{aligned}
A^N f(z) \quad = \quad & \frac{1}{2}\sum_{k=1}^{b(z)} \left(\lambda^{b(z)-k} + \lambda^{b(z)-k}\mathbb{1}_{\{z_k<0^-\}}\right)\frac{\partial^2 f}{\partial z_k^2} \\
& +\frac{1}{2}\sum_{k=a(z)}^{N} \left(\lambda^{k-a(z)} + \lambda^{k-a(z)}\mathbb{1}_{\{z_k>0^+\}}\right)\frac{\partial^2 f}{\partial z_k^2} \\
& +(\mu - c^0)\mathbb{1}_{\{z_{b(z)}<0^-\}}\frac{\partial f}{\partial z_{b(z)}} + (c^0 - \mu)\mathbb{1}_{\{z_{a(z)}>0^+\}}\frac{\partial f}{\partial z_{a(z)}} \\
& -\sum_{k=1}^{b(z)-1} c^{b(z)-k}\frac{\partial f}{\partial z_k} + \sum_{a(z)+1}^{N} c^{k-a(z)}\frac{\partial f}{\partial z_k} \\
& +\sum_{k=1}^{b(z)} \gamma^{b(z)-k}(z_{k-1} + z_{k+1} - 2z_k)\frac{\partial f}{\partial z_k} \\
& +\sum_{k=a(z)}^{N} \gamma^{k-a(z)}(z_{k-1} + z_{k+1} - 2z_k)\frac{\partial f}{\partial z_k},
\end{aligned}$$

for all $z = (z_1, \dots, z_N) \in E^N$, acting on $\mathrm{Dom}(A^N)$, the space of $\mathbb{C}^2_{b,\infty}(E^N)$ functions satisfying:

$$\begin{cases} \left.\dfrac{\partial f}{\partial z_{b(z)}}\right|_{z_{b(z)}=0^-} = \dfrac{\mu}{\lambda^0}\left(f(z_1, \dots, z_{b(z)}, \dots, z_N)\Big|_{z_{b(z)}=0^-} - f(z^b)\right), \\ \left.\dfrac{\partial f}{\partial z_{a(z)}}\right|_{z_{a(z)}=0^+} = \dfrac{\mu}{\lambda^0}\left(f(z_1, \dots, z_{a(z)}, \dots, z_N)\Big|_{z_{a(z)}=0^+} - f(z^a)\right), \\ \left.\dfrac{\partial f}{\partial z_k}\right|_{z_k=0^-} = 0 \text{ for all } k < b(z), \; \left.\dfrac{\partial f}{\partial z_k}\right|_{z_k=0^+} = 0 \text{ for all } k > a(z). \end{cases}$$

One fundamental observation to make here is that the diffusion coefficients in front of the partial derivatives of order 2 are space-dependent, which is incompatible with the stochastic interface model setting previously presented. In order to overcome this issue, we need to redefine the order arrival and cancellation rates described in the absolute price grid model. More precisely, these rates shall be comprised of a systemic component dictating the diffusion behaviour as well as an idiosyncratic or level-dependent component suitably rescaled so as to only appear in the drift. Furthermore, as we are in a static setting without any possible price changes, the above boundary conditions on the generator's domain become irrelevant, and we therefore need to assume that market orders are implicitly included as cancellations at the fixed best levels. This absence of elasticity at the best levels (all the queues of the absolute price grid shall simply be reflected in a classic sense) also means that we no longer require to adopt the "artificial" state-space construction as the disjoint union of $\mathbb{R}_-$ and $\mathbb{R}_+$. Given an initial profile $i = (i_1, \dots, i_N)$ of the discrete order book process Z_n, the fixed bid and ask indices are naturally defined by:

$$b(i) = \sup\left\{l \in \{1, \dots, N\} : i_l < 0\right\},$$

$$a(i) = \inf\left\{l \in \{1, \dots, N\} : i_l > 0\right\}.$$

For conventional purposes due to the pinning of the limiting process at the mid, we shall assume that the spread is constantly equal to 2 ticks, i.e. we have $a(i) = b(i)+2$, and $m(i) = (b(i) + a(i))/2 = b(i) + 1 = a(i) - 1$. In addition, the initial order book profile shall satisfy $i_k \neq 0$ for all $k \in \{1, \dots, N\} \setminus \{m(i)\}$ and $i_{m(i)} = 0$. Let $k \in \{1, \dots, N\}$. The transitions of Z_n can now be expressed by:

$$i \to i^{k-1} \text{ at rate } \Lambda_n^{b,k}(i_{k-1}, i_k, i_{k+1}) \text{ for } k \leq b(i),$$

$$i \to i^{k+1} \text{ at rate } \Lambda_n^{a,k}(i_{k-1}, i_k, i_{k+1}) \text{ for } k \geq a(i),$$

$$i \to i^{k+1} \text{ at rate } \Theta_n^{b,k}(i_{k-1}, i_k, i_{k+1}) \text{ for } k \leq b(i),$$

$$i \to i^{k-1} \text{ at rate } \Theta_n^{a,k}(i_{k-1}, i_k, i_{k+1}) \text{ for } k \geq a(i).$$

As explained above, the order arrival rates can now be decomposed as, for all $k \leq b(i)$:

$$\Lambda_n^{b,k}(i_{k-1}, i_k, i_{k+1}) = \underbrace{\lambda_n(1 + \mathbb{1}_{\{i_k=0\}})}_{\text{systemic term}} + \underbrace{\lambda_n^{b(i)-k}}_{\text{idiosyncratic term}} + \underbrace{\gamma_n(\Delta^2 i)(k)}_{\text{nearest neighbour term}},$$

where we recall that:

$$(\Delta^2 i)(k) = (i_k - i_{k-1})^+ + (i_k - i_{k+1})^+.$$

Similarly, we have for all $k \leq b(i)$:

$$\Theta_n^{b,k}(i_{k-1}, i_k, i_{k+1}) = \theta_n \mathbb{1}_{\{i_k \leq -1\}} + \theta_n^{b(i)-k} \mathbb{1}_{\{i_k \leq -1\}} + \gamma_n(\Delta^1 i)(k),$$

where:

$$(\Delta^1 i)(k) = (i_{k-1} - i_k)^+ + (i_{k+1} - i_k)^+.$$

And for all $k \geq a(i)$:

$$\Lambda_n^{a,k}(i_{k-1}, i_k, i_{k+1}) = \lambda_n(1 + \mathbb{1}_{\{i_k=0\}}) + \lambda_n^{k-a(i)} + \gamma_n(\Delta^1 i)(k),$$

$$\Theta_n^{a,k}(i_{k-1}, i_k, i_{k+1}) = \theta_n \mathbb{1}_{\{i_k \geq 1\}} + \theta_n^{k-a(i)} \mathbb{1}_{\{i_k \geq 1\}} + \gamma_n(\Delta^2 i)(k).$$

We notice that the systemic term initially appears to be state-dependent because of the presence of the indicator function $\mathbb{1}_{\{i_k=0\}}$. However, this function is absolutely necessary to account for the absence of a systemic cancellation term when $i_k = 0$, and enables us to obtain a state and space-independent diffusion coefficient in the limit. Moreover, as the best levels are now fixed, we shall write:

$$\lambda_n^{b(i)-k} = \lambda_n^{k-a(i)} = \lambda_n^k,$$

$$\theta_n^{b(i)-k} = \theta_n^{k-a(i)} = \theta_n^k.$$

The assumptions on the rates can now be summarised as follows:

- *(A1)* $\lim_{n\to+\infty} \lambda_n = \lambda$,
- *(A2)* $\lim_{n\to+\infty} \theta_n = \lambda$,
- *(A3)* $\lim_{n\to+\infty} \sqrt{n}(\lambda_n - \theta_n) = c$,
- *(A4)* $\lambda_n^k = \frac{1}{\sqrt{n}}\lambda^k$, for $k \leq b(i)$ and $k \geq a(i)$,
- *(A5)* $\theta_n^k = \frac{1}{\sqrt{n}}\theta^k$, for $k \leq b(i)$ and $k \geq a(i)$,
- *(A6)* $\gamma_n = \frac{1}{n}\gamma$,
- *(A7)* $c + \lambda^k - \theta^k < 0$ for $k \leq b(i)$ and $k \geq a(i)$.

We define the continuous state space $E^N = \mathbb{R}^N$, as well as the classically rescaled order book process $\hat{Z}_n$ on $E_n^N = \frac{1}{\sqrt{n}}\mathbb{Z}^N$:

$$\hat{Z}_n(t) = \frac{Z_n(nt)}{\sqrt{n}}.$$

Using the usual four-step template, it can be shown that, in this simplified static setting, the rescaled discrete order book process $\hat{Z}_n$ converges weakly to a strong Markov jump-diffusion process Z with infinitesimal generator given by the closure $\overline{A^N}$ of the linear operator A^N defined by, for all $z = (z_1, \ldots, z_N) \in E^N$:

$$\begin{aligned} A^N f(z) \;=\; & \lambda \sum_{k=1}^{b(z)} \frac{\partial^2 f}{\partial z_k^2} + \lambda \sum_{k=a(z)}^{N} \frac{\partial^2 f}{\partial z_k^2} - \sum_{k=1}^{b(z)} \left(c + \lambda^k - \theta^k\right) \frac{\partial f}{\partial z_k} \\ & + \sum_{k=a(z)}^{N} \left(c + \lambda^k - \theta^k\right) \frac{\partial f}{\partial z_k} + \gamma \sum_{k=1}^{b(z)} \left(z_{k-1} + z_{k+1} - 2z_k\right) \frac{\partial f}{\partial z_k} \\ & + \gamma \sum_{k=a(z)}^{N} \left(z_{k-1} + z_{k+1} - 2z_k\right) \frac{\partial f}{\partial z_k}, \end{aligned}$$

acting on $\mathrm{Dom}(A^N)$, the space of $\mathbb{C}^2_{b,\infty}(E^N)$ functions satisfying:

$$\left.\frac{\partial f}{\partial z_k}\right|_{z_m=0} = 0 \;\text{ for all }\; k \in \{1, \ldots, N\} \setminus \{m(z)\}.$$

The absence of price movements here enables us to write down the mesoscopic limit in terms of a system of reflected SDEs. Indeed, suppose the initial order book profile is now indexed by n, i.e. $i = i^n = (i_1^n, \ldots, i_N^n)$. Define $x_k^n = \frac{|i_k|}{\sqrt{n}}$ and suppose $\lim_{n\to\infty} x_k^n = x_k \geq 0$ for all $k \in \{1, \ldots, N\}$. Let $X = (X^1, \ldots, X^N)$ be an E^N-valued diffusion process with initial conditions $X_0 = (X_0^1, \ldots, X_0^N) = (x_1, \ldots, x_N)$ satisfying, for all $k \in \{1, \ldots, N\} \setminus \{m(z)\}$:

$$\begin{aligned} dX_t^k \;=\; & \gamma \left(X_t^{k-1} + X_t^{k+1} - 2X_t^k\right) dt + \left(c + \lambda^k - \theta^k\right) dt \\ & + \sqrt{2\lambda}\, dW_t^k + dL_t^{X^k}, \end{aligned}$$

where $W = (W^1, \ldots, W^N)$ is a N-dimensional Wiener process, where $L_t^{X^k}$ is the local time at zero of the process X^k, and with the pinning condition $X_t^0 = X_t^{m(z)} = X_t^{N+1} = 0$ for all $t \geq 0$. Then the form of the limiting generator A^N tells us that its associated diffusion process $Z = (Z^1, \ldots, Z^N)$ with initial conditions $Z_0 = (Z_0^1, \ldots, Z_0^N) = (-x_1, \ldots, -x_{b(z)}, 0, x_{a(z)}, \ldots, x_N)$ satisfies:

$$Z_t^k = -X_t^k \;\text{ for all }\; k \leq b(z) \;\text{ and for all }\; t \geq 0,$$

$$Z_t^k = X_t^k \;\text{ for all }\; k \geq a(z) \;\text{ and for all }\; t \geq 0,$$

$$Z_t^0 = Z_t^{m(z)} = Z_t^{N+1} = 0 \text{ for all } t \geq 0.$$

As required by the results on stochastic interface models, the space-dependence has now been transferred to the drift, and we can therefore write down an explicit invariant measure associated with these dynamics. We are now in a position to apply the refinement of the Funaki-Olla convergence result established in Theorem 5.2.4 so as to obtain an initial SPDE limit of the mesoscopic limit order book in a static setting.

6.1.2 Connection with the Funaki-Olla convergence result

To start with, in order to obtain a nontrivial macroscopic drift in the limit, we first need to rescale it in such a way that the new dynamics to be considered are written as:

$$\begin{aligned} dX_t^k &= \gamma\left(X_t^{k-1} + X_t^{k+1} - 2X_t^k\right)dt + N^{-3/2}\left(c + \lambda^{\frac{k}{N}} - \theta^{\frac{k}{N}}\right)dt \\ &\quad + \sqrt{2\lambda}dW_t^k + dL_t^{X^k}, \end{aligned}$$

with the pinning condition $X_t^0 = X_t^{m(z)} = X_t^{N+1} = 0$ for all $t \geq 0$ and with initial conditions $X_0 = (X_0^1, \ldots, X_0^N) = (x_1, \ldots, x_N)$. Next, we note that we are effectively in the framework of equation (5.6) with the quadratic (and therefore convex) potential V given by $V(y) = \frac{\gamma}{2}y^2$ for all $y \in \mathbb{R}$. It is then easily shown that these new dynamics are stationary under the Gibbs measure $\tilde{\mu}_N$ given by:

$$\begin{aligned} \tilde{\mu}_N(dX) &= \widetilde{Z}_N^{-1}\exp\left(-\frac{\gamma}{2\lambda}\sum_{k=1}^{N+1}(X^k - X^{k-1})^2\right) \\ &\quad \times \exp\left(\frac{1}{\lambda}N^{-3/2}\sum_{k=1}^{N}\left(c + \lambda^{\frac{k}{N}} - \theta^{\frac{k}{N}}\right)X^k\right)\prod_{k=1}^{N}\mathbb{1}_{\{X^k \geq 0\}}dX^k \\ &= \underbrace{\left(\widetilde{Z}_N^b\right)^{-1}\exp\left(\frac{1}{\lambda}N^{-3/2}\sum_{k=1}^{m(z)-1}\left(c + \lambda^{\frac{k}{N}} - \theta^{\frac{k}{N}}\right)X^k\right)\mu_N^b(dX)}_{\tilde{\mu}_N^b(dX)} \\ &\quad \times \underbrace{\left(\widetilde{Z}_N^a\right)^{-1}\exp\left(\frac{1}{\lambda}N^{-3/2}\sum_{k=m(z)+1}^{N}\left(c + \lambda^{\frac{k}{N}} - \theta^{\frac{k}{N}}\right)X^k\right)\mu_N^a(dX)}_{\tilde{\mu}_N^a(dX)} \\ &= \left(\tilde{\mu}_N^b \otimes \tilde{\mu}_N^a\right)(dX), \end{aligned}$$

where μ_N^b and μ_N^a are respectively expressed by:

$$\mu_N^b(dX) = \left(Z_N^b\right)^{-1}\exp\left(-\frac{\gamma}{2\lambda}\sum_{k=1}^{m(z)}(X^k - X^{k-1})^2\right)\prod_{k=1}^{m(z)-1}\mathbb{1}_{\{X^k \geq 0\}}dX^k,$$

$$\mu_N^a(dX) = (Z_N^a)^{-1} \exp\left(-\frac{\gamma}{2\lambda} \sum_{k=m(z)+1}^{N+1} (X^k - X^{k-1})^2 \right) \prod_{k=m(z)+1}^{N} \mathbb{1}_{\{X^k \geq 0\}} dX^k.$$

Naturally, we work with the pinning condition $X^0 = X^{m(z)} = X^{N+1} = 0$, and assume that Z_N^b, Z_N^a, $\widetilde{Z}_N^b$ and $\widetilde{Z}_N^a$ are finite normalisation constants. Let $\Lambda_N : \mathbb{R}^N \to \mathbb{L}^2(]0,1[)$ be the rescaling map defined by:

$$\Lambda_N(X)(x) = \frac{1}{\sqrt{N}} X^{\lfloor Nx \rfloor + 1}, \;\; x \in]0,1[,$$

and define the relevant image (or pushforward) measures under the rescaling map Λ_N:

$$\begin{aligned} \nu_N^b &= \Lambda_N^*(\mu_N^b), \;\; \nu_N^a = \Lambda_N^*(\mu_N^a), \\ \tilde{\nu}_N^b &= \Lambda_N^*(\tilde{\mu}_N^b), \;\; \nu_N^a = \Lambda_N^*(\mu_N^a), \\ \tilde{\nu}_N &= \Lambda_N^*(\tilde{\mu}_N) = \Lambda_N^*(\tilde{\mu}_N^b \otimes \tilde{\mu}_N^a). \end{aligned}$$

As in the previous chapter, the rescaling in space is conducted by considering the following intervals

$$I(k) = \left] \frac{k-1}{N}, \frac{k}{N} \right] \;\; \text{for all} \;\; k \in \Gamma_N = \{1, \ldots, N\}.$$

The rescaled mesoscopic order book (with positive bid volumes) $\widehat{X}^N$ is now defined by:

$$\widehat{X}_t^N = \Lambda_N(X_{N^2 t}), \;\; \widehat{X}_0^N = \Lambda_N(X_0).$$

Using the definition of the rescaling map Λ_N, the rescaled mesoscopic order book can thus be expressed by:

$$\widehat{X}_t^N(x) = \frac{1}{\sqrt{N}} X_{N^2 t}^{\lfloor Nx \rfloor + 1} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_{N^2 t}^k \mathbb{1}_{I(k)}(x), \;\; x \in]0,1[,$$

and we set $\widehat{X}_t^N(0) = \widehat{X}_t^N(1) = 0$ as a rescaled pinning condition. We also need to assume the convergence of the rescaled and indexed mid, i.e.:

$$\lim_{N \to \infty} \frac{m^N(z)}{N} = m \in]0,1[.$$

Let $\left(e_\tau^{l,r}\right)_{\tau \in [l,r]}$ be the normalised Brownian excursion on any finite interval $[l,r]$, and denote by $\nu^{l,r}$ be the law of $q^{1/2} e^{l,r}$, where $q = \lambda/\gamma$. Using the same techniques as in the proof of Theorem 5.2.4 as well as the results by Caravenna and Chaumont [11] which enable us to consider bridges of any length, $\tilde{\nu}_N$ converges weakly to $\nu^m = \nu^{0,m} \otimes \nu^{m,1}$ as $N \to \infty$ in the Skorokhod topology. It is then straightforward to show that ν^m is the invariant measure associated with the reflected stochastic heat equation with additional intermediate pinning at point m. A direct application of step 1 of the proof of Theorem 5.2.4 then enables us to obtain the following result:

Theorem 6.1.1 *Suppose that the initial rescaled mesoscopic order book* $\widehat{X}_0^N \geq 0$ *converges weakly in* $H = \mathbb{L}^2(]0,1[)$ *as* $N \to \infty$ *to* u_0. *Then, for any* $0 < s \leq T < \infty$, *the law of* $(\widehat{X}_t^N)_{t \in [s,T]}$ *converges weakly in* $\mathbb{C}([s,T], H_w)$ *(where* H_w *denotes the space* H *endowed with the weak topology) as* $N \to \infty$ *to the law of the unique solution of the following reflected stochastic heat equation:*

$$\frac{\partial u(x,t)}{\partial t} = \frac{\gamma}{\lambda}\frac{\partial^2 u(x,t)}{\partial x^2} + c + \lambda(x) - \theta(x) + \sqrt{2\lambda}\dot{W}(x,t) + \eta(x,t), \tag{6.1}$$

with Dirichlet boundary conditions $u(0,t) = u(m,t) = u(1,t) = 0$ *for all* $t \geq 0$, *with* $u \geq 0$, $d\eta \geq 0$, $\int u d\eta = 0$ *and with* $u(x,0) = u_0(x)$ *for all* $x \in [0,1]$.

If we wish to return to our initial convention of negative bid volumes, we can simply decompose the solution u into a negative part u^b and a positive part u^a as follows:

$$u^b(x,t) = -u(x,t) \text{ for all } x \in [0,m],$$

$$u^a(x,t) = u(x,t) \text{ for all } x \in [m,1],$$

which gives us the macroscopic order book in a static setting.

Remark 6.1.1 *We recall that the statement in Theorem 6.1.1 precisely means that for any* $n \in \mathbb{N}$, *any collection of test functions* $h_1, \ldots, h_n \in \mathbb{C}^1([0,1])$ *and any* $0 < s \leq T < \infty$, *the process* $\Big(\langle \widehat{X}_t^N, h_i\rangle, i \in \{1,\ldots,n\}\Big)_{s \leq t \leq T}$ *converges weakly in* $\mathbb{C}([s,T], \mathbb{R}^n)$ *as* $N \to \infty$ *to the process* $\big(\langle u_t, h_i\rangle, i \in \{1,\ldots,n\}\big)_{s \leq t \leq T}$, *where* $u_t \equiv u(.,t)$ *by convention and where* $\langle .,. \rangle$ *denotes the canonical* $\mathbb{L}^2(]0,1[)$ *scalar product.*

Remark 6.1.2 *As the solution of the reflected stochastic heat equation (6.1) with an intermediate condition at the mid is understood to be considered in a mild sense (as explained in Section 5.1.1 of the previous chapter), its existence and uniqueness can be established via a direct generalisation of the proof associated with the "usual" case (i.e. with boundary conditions at 0 and 1 only, see Theorem 5.1.1 of Chapter 5) presented by Xu and Zhang [55]. Indeed, the mild formulation of the problem, which we recall involves integrating against sufficiently smooth test functions, ensures that there are no second-derivative issues which could potentially arise at the mid. As far as the high-level proof strategy is concerned, the successive iteration method used in [55] can be readily adapted using Green's function associated with the second-order operator with Dirichlet boundary conditions at 0 and 1 and with an intermediate condition at the mid.*

6.2 SPDE limit in a dynamic limit order book framework

The fundamental idea behind the transition between the static and dynamic order book frameworks is to split the chosen time interval we wish to work on into smaller periods during which the price remains constant. In a high to ultra high-frequency context overwhelmingly dominated by limit orders and cancellations (see Gai et al. [25]), such a representation makes perfect sense. As mentioned by Huang et al. [31], albeit in the context of Markov chains, the problem of limit order book modeling can therefore be decomposed into two sub-problems, namely the dynamics of a static order book, and their subsequent interaction with price movements. As we shall see shortly, this procedure specifically amounts to decoupling, or *exogenising*, the *endogenous* price dynamics from the volume dynamics in the absolute price grid diffusion approximation obtained in Chapter 4.

As our aim is to provide price dynamics in an infinite-dimensional setting, it is legitimate to ask oneself what kind of signals shall be used to trigger price changes. We recall that in our previous diffusion approximation results, the notion of local time was central in determining the dynamics of the price. Indeed, the local times of the best bid and ask queues were taken into consideration and once either one of these best volume processes accumulated sufficient local time around 0, a price change would occur. Naturally, this poses a problem in an infinite-dimensional case, as there in no way of measuring the best queues. One possible solution, which shall be presented as an extension of our main result, is to fix an arbitrarily small value around both sides of the mid, and to measure the occupation time around 0 of an appropriately constructed *space-accumulated* process which can simply be interpreted as the neighbourhood of the best levels. In the first instance, however, we provide a more straightforward signal of price changes, based on the notion of *order flow imbalance*. This consists in simply comparing the difference between the volumes in the neighbourhoods of the bid and ask sides, and deciding that a price increase (respectively decrease) occurs when the imbalance is skewed in favour of the bid (respectively ask).

6.2.1 Exogenising the price in the absolute price grid model

We once again place ourselves on an absolute price grid $\{1,\dots,N\}$ and assume that the spread is constantly equal to 2 ticks for modeling convenience. We define $m^N = (m^N_M)_{M\in\mathbb{N}}$ as the sequence of mids and $b^N = (b^N_M)_{M\in\mathbb{N}}$ and $a^N = (a^N_M)_{M\in\mathbb{N}}$ as the sequence of bids and asks respectively. Given the assumption on the spread, we shall always have $b^N_M = m^N_M - 1$ and $a^N_M = m^N_M + 1$ for all $M \geq 0$. We give ourselves an initial mid $m^N_0 = m_0 N$ such that $m_0 \in]0,1[$ and $m^N_0 \in \{1,\dots,N\}$. We then introduce the sequence $\tau^N = (\tau^N_M)_{M\in\mathbb{N}}$ of price change times and set $\tau^N_0 = 0$. At this stage, it is important to stress that as we are no longer working with an endogenous price process, we do not need to adopt the convention of negative bid volumes, and all volumes are now assumed to be positive. We now define the order book process $X^{m^N_0} = (X^{1,m^N_0}\dots,X^{N,m^N_0})$ and give ourselves an initial order book profile $X^{m^N_0}_0 = (x_1,\dots,x_N)$ where $x_k > 0$ for all $k \in \{1,\dots,N\}\backslash\{m^N_0\}$ and $x_{m^N_0} = 0$.

Until the first price change time τ^N_1, we can naturally identify the best bid and ask queues from the initial profile, namely $X^{b^N_0}_t$ and $X^{a^N_0}_t$, for all $t \in [\tau^N_0, \tau^N_1[$. On this time interval, the volume dynamics are assumed to satisfy the Funaki-Olla SDE system presented in the previous section. More precisely, for all $t \in [\tau_0^N, \tau_1^N[$, and $k \in \{1, \ldots, N\}$:

$$\begin{aligned} dX_t^{k,m_0^N} &= \gamma \left(X_t^{k-1,m_0^N} + X_t^{k+1,m_0^N} - 2X_t^{k,m_0^N} \right) dt \\ &\quad + N^{-3/2} \left(c + \lambda^{\frac{k}{N}} - \theta^{\frac{k}{N}} \right) dt + \sqrt{2\lambda} dW_t^{k,0} + dL_t^{X^{k,m_0^N}}, \end{aligned}$$

where $W^0 = (W^{1,0}, \ldots, W^{N,0})$ is a N-dimensional Wiener process, where $L_t^{X^{k,m_0^N}}$ is the local time at zero of the process X^{k,m_0^N}, and with the pinning condition $X_t^{0,m_0^N} = X_t^{m_0^N,m_0^N} = X_t^{N+1,m_0^N} = 0$ and initial conditions $X_0^{m_0^N} = (x_1, \ldots, x_N)$. Having reached this point, it is vital to emphasise that we need to adequately rescale price movement increments so as to obtain a nontrivial (i.e. nonconstant) price process in the macroscopic limit. Given $\epsilon > 0$ satisfying $\epsilon << 1$, we shall therefore assume that prices move up or down by a quantity of $\epsilon^N = \epsilon N \in \{1, \ldots, N\}$ following a price change. Let Y^{N,m_0^N} be the process defined by:

$$Y_t^{N,m_0^N}(x) = \widehat{X}_t^{N,m_0^N}(x) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_{N^2 t}^{k,m_0^N} \mathbb{1}_{I(k)}(x), \quad x \in]0,1[,$$

and we naturally set $Y_t^{N,m_0^N}(0) = Y_t^{N,m_0^N}(1) = 0$ as a rescaled pinning condition. The first price change time (or first imbalance time) can now be defined by $\tau_1^N = \tau_1^{N,b} \wedge \tau_1^{N,a}$, where $\tau_1^{N,b}$ and $\tau_1^{N,a}$ are respectively given by:

$$\tau_1^{N,b} = \inf \left\{ t \geq 0 : \frac{1}{N} \sum_{i=1}^{\epsilon^N} \left(Y_t^{N,m_0^N} \left(\frac{m_0^N + i}{N} \right) - Y_t^{N,m_0^N} \left(\frac{m_0^N - i}{N} \right) \right) \geq \delta^N \right\},$$

$$\tau_1^{N,a} = \inf \left\{ t \geq 0 : \frac{1}{N} \sum_{i=1}^{\epsilon^N} \left(Y_t^{N,m_0^N} \left(\frac{m_0^N - i}{N} \right) - Y_t^{N,m_0^N} \left(\frac{m_0^N + i}{N} \right) \right) \geq \delta^N \right\}.$$

where δ^N is a positive constant satisfying $\lim_{N\to\infty} \delta^N = \delta > 0$. The new (mid) price can now be expressed as:

$$m_1^N = \left(m_0^N - \epsilon^N \right) \mathbb{1}_{\{\tau_1^N = \tau_1^{N,b}\}} + \left(m_0^N + \epsilon^N \right) \mathbb{1}_{\{\tau_1^N = \tau_1^{N,a}\}}.$$

Moving forward in time, i.e. on the interval $[\tau_1^N, \tau_2^N[$, we assume that the volume dynamics are once again given by the Funaki-Olla system, the only difference being the new pinning point m_1^N. At this stage, one uncertainty does still remain as to the specification of the initial condition following the first price change. In view of the SPDE limit, we generalise the idea suggested by Cont and de Larrard in [13] and assume that following price changes, the initial profile (with respect to the new time period) is drawn from the stationary distribution of the Funaki-Olla

SDE system. We can then define the second imbalance time and subsequently the new mid in the same manner as previously presented, and this generic behaviour can now be repeated *ad infinitum*. For the purpose of applications, however, it is necessary to fix an arbitrary $T > 0$, and to extend this informal description of the mesoscopic order book behaviour to the case of a prespecified time interval $[0, T]$.

Let $M_T^* = \max\{M \geq 0 : \sum_{j=1}^M \tau_j^N \leq T\}$. We define a sequence $R^N = (R_M^N)_{0 \leq M \leq M_T^*}$ of independent N-dimensional random vectors with distribution $\tilde{\mu}_N$ for any $M \in \{1, \ldots, M_T^*\}$ (see previous section) and deterministic initial profile $R_0^N = (x_1, \ldots, x_N)$, where for any $M \in \{1, \ldots, M_T^*\}$, R_M^N corresponds to the reinitialisation profile following the M-th price change. For any $M \in \{0, \ldots, M_T^*\}$, $Y^{N,m_M^N} = (Y_t^{N,m_M^N})_{t \in [0,T]}$ is the process defined by:

$$Y_t^{N,m_M^N}(x) = \frac{1}{\sqrt{N}} \sum_{k=1}^N X_{N^2 t}^{k,m_M^N} \mathbb{1}_{I(k)}(x), \quad x \in]0, 1[,$$

where we set the conditions $Y_t^{N,m_M^N}(0) = Y_t^{N,m_M^N}(1) = 0$, and where the process $X^{m_M^N} = (X^{1,m_M^N}, \ldots, X^{N,m_M^N}) = ((X_t^{1,m_M^N}, \ldots, X_t^{N,m_M^N}))_{t \in [0,T]}$ satisfies the Funaki-Olla dynamics with pinning at point m_M^N, i.e. for any $k \in \{1, \ldots, N\}$:

$$\begin{aligned} dX_t^{k,m_M^N} &= \gamma \left(X_t^{k-1,m_M^N} + X_t^{k+1,m_M^N} - 2X_t^{k,m_M^N} \right) dt \\ &\quad + N^{-3/2} \left(c + \lambda^{\frac{k}{N}} - \theta^{\frac{k}{N}} \right) dt + \sqrt{2\lambda} dW_t^{k,M} + dL_t^{X^{k,m_M^N}}, \end{aligned}$$

where $W^M = (W^{1,M}, \ldots, W^{N,M})$ is a N-dimensional Wiener process, where $L_t^{X^{k,m_M^N}}$ is the local time at zero of the process X^{k,m_M^N}, with the pinning condition $X_t^{0,m_M^N} = X_t^{m_M^N,m_M^N} = X_t^{N+1,m_M^N} = 0$, and initial condition $X_0^{m_M^N} = R_M^N$. Now and henceforth, these dynamics along with the initial and pinning conditions shall be referred to as the **mesoscopic system**. For any $M \in \{0, \ldots, M_T^* - 1\}$, we define the so-called $(M+1)$-th imbalance time as $\tau_{M+1}^N = \tau_{M+1}^{N,b} \wedge \tau_{M+1}^{N,a}$, where the stopping times $\tau_{M+1}^{N,b}$ and $\tau_{M+1}^{N,a}$ are respectively expressed by:

$$\tau_{M+1}^{N,b} = \inf \left\{ t \geq 0 : \frac{1}{N} \sum_{i=1}^{\epsilon^N} \left(Y_t^{N,m_M^N} \left(\frac{m_M^N + i}{N} \right) - Y_t^{N,m_M^N} \left(\frac{m_M^N - i}{N} \right) \right) \geq \delta^N \right\},$$

$$\tau_{M+1}^{N,a} = \inf \left\{ t \geq 0 : \frac{1}{N} \sum_{i=1}^{\epsilon^N} \left(Y_t^{N,m_M^N} \left(\frac{m_M^N - i}{N} \right) - Y_t^{N,m_M^N} \left(\frac{m_M^N + i}{N} \right) \right) \geq \delta^N \right\}.$$

For any $M \in \{0, \ldots, M_T^* - 1\}$, the price process can be defined by:

$$m_{M+1}^N = \left(m_M^N - \epsilon^N \right) \mathbb{1}_{\{\tau_{M+1}^N = \tau_{M+1}^{N,b}\}} + \left(m_M^N + \epsilon^N \right) \mathbb{1}_{\{\tau_{M+1}^N = \tau_{M+1}^{N,a}\}}.$$

An immediate induction enables us to rewrite this as:

$$\begin{aligned} m_{M+1}^N &= m_M^N + \epsilon^N \left(\mathbb{1}_{\{\tau_{M+1}^N = \tau_{M+1}^{N,a}\}} - \mathbb{1}_{\{\tau_{M+1}^N = \tau_{M+1}^{N,b}\}} \right) \\ &= m_0^N + \epsilon^N \sum_{j=1}^{M+1} \left(\mathbb{1}_{\{\tau_j^{N,a} < \tau_j^{N,b}\}} - \mathbb{1}_{\{\tau_j^{N,b} < \tau_j^{N,a}\}} \right) \\ &= f_{M+1}\left(\tau_1^{N,b}, \ldots, \tau_{M+1}^{N,b}, \tau_1^{N,a}, \ldots, \tau_{M+1}^{N,a}; m_0^N, \epsilon^N \right), \end{aligned}$$

where, for any $M \in \{1, \ldots, M_T^*\}$, the function $f_M : \left(\mathbb{R}_+^*\right)^{2M} \times]0,1[^2 \to]0,1[$ is defined by:

$$f_M(t_1, \ldots, t_M, t_1', \ldots, t_M'; \alpha, \beta) = \alpha + \beta \sum_{j=1}^{M} \left(\mathbb{1}_{\{t_j' < t_j\}} - \mathbb{1}_{\{t_j < t_j'\}} \right). \tag{6.2}$$

For conventional purposes, we also define the constant function $f_0 \equiv m_0$. By construction of the mesoscopic system, it is vital to observe that conditionally on m_M^N, Y^{N,m_M^N} is independent of $(Y^{N,m_0^N}, \ldots, Y^{N,m_{M-1}^N})$. Intuitively speaking, this simply means that the knowledge of the pinning point makes the system start afresh independently of its past behaviour. This shall play a central role in the proof of the main dynamic result presented in the following section.

For any $M \in \{0, \ldots, M_T^*\}$, we next define u^{m_M} to be the solution of the following reflected stochastic heat equation on $[0,T]$, with intermediate pinning at point m_M:

$$\frac{\partial u^{m_M}(x,t)}{\partial t} = \frac{\gamma}{\lambda} \frac{\partial^2 u^{m_M}(x,t)}{\partial x^2} + c + \lambda(x) - \theta(x) + \sqrt{2\lambda}\dot{W}^M(x,t) + \eta^M(x,t),$$

where $\dot{W}^M$ is a space-time white noise, η^M is the reflecting measure such that $u^{m_M} \geq 0$, $d\eta^M \geq 0$, $\int u^{m_M} d\eta^M = 0$, with Dirichlet boundary conditions $u^{m_M}(0,t) = u^{m_M}(m_M,t) = u^{m_M}(1,t) = 0$ for all $t \in [0,T]$, and with initial condition $u^{m_M}(x,0) = u_M(x)$ for all $x \in [0,1]$. It is assumed that u_M is drawn from the distribution of the relevant normalised Brownian excursion which is invariant for these dynamics for $M \in \{1, \ldots, M_T^*\}$, whilst u_0 corresponds to a deterministic initial profile. These dynamics and set of boundary and initial conditions shall be referred to as the **macroscopic system**. Analogously to the mesoscopic case, for any $M \in \{0, \ldots, M_T^* - 1\}$, we can introduce the $(M+1)$-th imbalance time as $\tau_{M+1} = \tau_{M+1}^b \wedge \tau_{M+1}^a$, where the stopping times τ_{M+1}^b and τ_{M+1}^a are respectively defined by:

$$\tau_{M+1}^b = \inf\left\{ t \geq 0 : \int_0^\epsilon \left(u^{m_M}(m_M + x, t) - u^{m_M}(m_M - x, t) \right) dx \geq \delta \right\},$$

$$\tau_{M+1}^a = \inf\left\{ t \geq 0 : \int_0^\epsilon \left(u^{m_M}(m_M - x, t) - u^{m_M}(m_M + x, t) \right) dx \geq \delta \right\}.$$

Within this macroscopic framework, for any $M \in \{0, \ldots, M_T^* - 1\}$, the price process is expressed by:

$$\begin{aligned} m_{M+1} &= (m_M - \epsilon)\, \mathbb{1}_{\{\tau_{M+1} = \tau^b_{M+1}\}} + (m_M + \epsilon)\, \mathbb{1}_{\{\tau_{M+1} = \tau^a_{M+1}\}} \\ &= m_M + \epsilon \left(\mathbb{1}_{\{\tau_{M+1} = \tau^a_{M+1}\}} - \mathbb{1}_{\{\tau_{M+1} = \tau^b_{M+1}\}} \right) \\ &= m_0 + \epsilon \sum_{j=1}^{M+1} \left(\mathbb{1}_{\{\tau^a_j < \tau^b_j\}} - \mathbb{1}_{\{\tau^b_j < \tau^a_j\}} \right) \\ &= f_{M+1}\left(\tau^b_1, \ldots, \tau^b_{M+1}, \tau^a_1, \ldots, \tau^a_{M+1}; m_0, \epsilon \right) . \end{aligned}$$

Once again, it is fundamental to emphasise that conditionally on m_M, u^{m_M} is independent of $(u^{m_0}, \ldots, u^{m_{M-1}})$, by construction of the macroscopic system, as the knowledge of the pinning point enables the system to start afresh after an imbalance time.

Remark 6.2.1 *Given these mid price dynamics associated with the mesoscopic and macroscopic systems, the assumption of a constant mesoscopic spread equal to* 2 *ticks, and the choice of initial model inputs* $m_0^N = m_0 N \in \{1, \ldots, N\}$ *and* $\epsilon^N = \epsilon N \in \{1, \ldots, N\}$*, it turns out that both the bid and ask shall coincide with the mid in the macroscopic limit. It must also be noted that the notion of continuous price associated with the macroscopic setting refers to the fact that the grid size* N *is sent to infinity rather than to any continuity property of the price process, which remains constant on nontrivial intervals between price changes.*

6.2.2 The dynamic macroscopic limit

Having fully exogenised the price in the previous section, we are now in a position to apply Theorem 6.1.1 as a building block for our macroscopic limit. Heuristically speaking, we already know that the mesoscopic order book process converges weakly to a reflected stochastic heat equation in between price changes, and the aim now is to piece everything together into a single process on $[0, T]$.

We start by specifying how Theorem 6.1.1 exactly applies to this context. By construction of the mesoscopic and macroscopic systems, for any $n \in \mathbb{N}$, any collection of test functions $h_1, \ldots, h_n \in \mathbb{C}^1([0,1])$ and any $0 < s \leq T < \infty$, we have:

$$\left(\langle Y^{N, m_0^N}, h_i \rangle, i \in \{1, \ldots, n\} \middle| \frac{m_0^N}{N} \right) \Rightarrow_{N \to \infty} \left(\langle u^{m_0}, h_i \rangle, i \in \{1, \ldots, n\} \middle| m_0 \right) ,$$

in $\mathbb{C}([s, T], \mathbb{R}^n)$, provided we have convergence of the deterministic initial profiles in $\mathbb{L}^2(]0,1[)$, i.e. if:

$$\lim_{N \to \infty} \left\| \frac{1}{\sqrt{N}} \sum_{k=1}^{N} x_k \mathbb{1}_{I(k)} - u_0 \right\| = 0 .$$

From now on, we shall indeed assume that this initial profile convergence requirement is met. In the proof of our main result below, it shall become clear that for any $M \in \{0, \ldots, M_T^*\}$, $n \in \mathbb{N}$, any collection of test functions $h_1, \ldots, h_n \in \mathbb{C}^1([0,1])$ and any $0 < s \leq T < \infty$, we also have:

$$\left(\langle Y^{N,m_M^N}, h_i \rangle, i \in \{1, \ldots, n\} \middle| \frac{m_M^N}{N} \right) \Rightarrow_{N\to\infty} \left(\langle u^{m_M}, h_i \rangle, i \in \{1, \ldots, n\} \middle| m_M \right),$$

in $\mathbb{C}([s,T], \mathbb{R}^n)$. This conditional weak convergence simply translates the fact that the mesoscopic order book process converges weakly to the macroscopic order book process conditionally on the associated pinning points, which can only take finitely many values on the corresponding price grids.

For notational convenience, we introduce the following quantities:

$$S^{N,b}\left(Y_t^{N,m_M^N}\right) = \frac{1}{N}\sum_{i=1}^{\epsilon^N}\left(Y_t^{N,m_M^N}\left(\frac{m_M^N + i}{N}\right) - Y_t^{N,m_M^N}\left(\frac{m_M^N - i}{N}\right)\right) - \delta^N,$$

$$S^{N,a}\left(Y_t^{N,m_M^N}\right) = \frac{1}{N}\sum_{i=1}^{\epsilon^N}\left(Y_t^{N,m_M^N}\left(\frac{m_M^N - i}{N}\right) - Y_t^{N,m_M^N}\left(\frac{m_M^N + i}{N}\right)\right) - \delta^N,$$

$$I^b\left(u_t^{m_M}\right) = \int_0^\epsilon \left(u^{m_M}(m_M + x, t) - u^{m_M}(m_M - x, t)\right) dx - \delta,$$

$$I^a\left(u_t^{m_M}\right) = \int_0^\epsilon \left(u^{m_M}(m_M - x, t) - u^{m_M}(m_M + x, t)\right) dx - \delta.$$

Defining $\tau(\omega) = \inf\{t > 0 : \omega_t \geq 0\}$ as a generic first-passage time for an arbitrary path $\omega \in \mathbb{D}([0,T])$, we can rewrite the mesoscopic and macroscopic imbalance times as:

$$\tau_{M+1}^{N,b} = \tau\left(S^{N,b}(Y^{N,m_M^N})\right), \quad \tau_{M+1}^{N,a} = \tau\left(S^{N,a}(Y^{N,m_M^N})\right),$$

$$\tau_{M+1}^{b} = \tau\left(I^{b}(u^{m_M})\right), \quad \tau_{M+1}^{a} = \tau\left(I^{a}(u^{m_M})\right).$$

With the convention $t_0 = 0$, for any $M \in \{0, \ldots, M_T^*\}$ and $n \in \mathbb{N}^*$, we define the functions $g_M : \mathbb{C}([0,T], \mathbb{R}^M) \times \left(\mathbb{R}_+^*\right)^M \to \mathbb{D}([0,T], \mathbb{R})$ and $g_M^n : \mathbb{C}([0,T], \mathbb{R}^{M\times n}) \times \left(\mathbb{R}_+^*\right)^M \to \mathbb{D}([0,T], \mathbb{R}^n)$ respectively given by:

$$g_M\left(\omega^0, \ldots, \omega^{M-1}, t_1, \ldots, t_M\right)(t) = \sum_{j=1}^{M} \omega^{j-1}_{t - \sum_{l=0}^{j-1} t_l} \mathbb{1}_{\{\sum_{l=0}^{j-1} t_l \leq t < \sum_{l=0}^{j} t_l\}},$$

$$g_M^n\left(\omega^{0,1}, \ldots, \omega^{0,n}, \ldots, \omega^{M-1,1}, \ldots, \omega^{M-1,n}, t_1, \ldots, t_M\right)(t) =$$
$$\left(g_M(\omega^{0,1}, \ldots, \omega^{M-1,1}, t_1, \ldots, t_M)(t), \ldots, g_M(\omega^{0,n}, \ldots, \omega^{M-1,n}, t_1, \ldots, t_M)(t)\right).$$

By construction, g_M is the function which enables us to piece together the entire order book process on the interval $[0,T]$, and g_M^n is simply the corresponding function in terms of weak convergence in the weak topology (i.e. weak convergence of the sequence of integrals against continuously differentiable functions).

For any $M \in \{0,\dots,M_T^*\}$, $(N,n) \in (\mathbb{N}^*)^2$ and any collection of test functions $h_1,\dots,h_n$ in $\mathbb{C}^1([0,1])$, we introduce the following vectorial notation:

$$\widehat{\mathcal{Y}}_M^{N,n} = \left(\langle Y^{N,m_M^N}, h_1\rangle, \dots, \langle Y^{N,m_M^N}, h_n\rangle\right), \;\; \mathcal{Y}_M^{N,n} = \left(\widehat{\mathcal{Y}}_0^{N,n}, \dots, \widehat{\mathcal{Y}}_M^{N,n}\right),$$

$$\widehat{\mathcal{U}}_M^n = \left(\langle u^{m_M}, h_1\rangle, \dots, \langle u^{m_M}, h_n\rangle\right), \;\; \mathcal{U}_M^n = \left(\widehat{\mathcal{U}}_0^n, \dots, \widehat{\mathcal{U}}_M^n\right),$$

$$\mathcal{T}_M^{N,b} = \left(\tau_1^{N,b}, \dots, \tau_M^{N,b}\right), \;\; \mathcal{T}_M^b = \left(\tau_1^b, \dots, \tau_M^b\right),$$

$$\mathcal{T}_M^{N,a} = \left(\tau_1^{N,a}, \dots, \tau_M^{N,a}\right), \;\; \mathcal{T}_M^a = \left(\tau_1^a, \dots, \tau_M^a\right),$$

$$\mathcal{T}_M^N = \left(\mathcal{T}_M^{N,b}, \mathcal{T}_M^{N,a}\right), \;\; \mathcal{T}_M = \left(\mathcal{T}_M^b, \mathcal{T}_M^a\right).$$

We can now state the main result of this chapter, which gives the weak convergence to the macroscopic limit in a fully dynamic framework:

Theorem 6.2.1 *Let $n \in \mathbb{N}^*$ and consider an arbitrary collection of test functions $h_1,\dots,h_n$ in $\mathbb{C}^1([0,1])$. We have:*

$$g_M^n\left(\mathcal{Y}_{M-1}^{N,n}, \tau_1^N, \dots, \tau_M^N\right) \xrightarrow[N\to\infty]{} g_M^n\left(\mathcal{U}_{M-1}^n, \tau_1, \dots, \tau_M\right),$$

in $\mathbb{D}([s,T],\mathbb{R}^n)$, for any $0 < s \le T < \infty$ and any $M \in \{1,\dots,M_T^*\}$.

Before moving on to the proof, we shall need to make use of the two following technical lemmas:

Lemma 6.2.1 *For any $M \in \{0,\dots,M_T^*\}$:*

$$\frac{1}{N}\sum_{i=1}^{\epsilon^N} Y_t^{N,m_M^N}\left(\frac{m_M^N \pm i}{N}\right) = \int_0^{\epsilon} Y_t^{N,m_M^N}\left(\frac{m_M^N}{N} \pm x\right) dx.$$

Proof of Lemma 6.2.1: Without loss of generality, we establish the result in the case of a minus sign. The crucial point is to remind ourselves that the model inputs m_0, ϵ and N are chosen such that $m_0^N = m_0 N \in \{1,\dots,N\}$ and $\epsilon^N = \epsilon N \in \{1,\dots,N\}$. Let us now fix an arbitrary $M \in \{0,\dots,M_T^*\}$. Setting $\mu_{M,j}^N = m_0^N - (M-2j)\epsilon^N$ for all $j \in \{0,\dots,M\}$, it is easily seen that $m_M^N(\Omega) = \{\mu_{M,j}^N, j \in \{0,\dots,M\}\}$. We shall also define $\mu_{M,j} = \frac{\mu_{M,j}^N}{N} = m_0 - (M-2j)\epsilon$ for all $j \in \{0,\dots,M\}$.

On the one hand, we have:

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{\epsilon^N} Y_t^{N,m_M^N}\left(\frac{m_M^N - i}{N}\right)\\
=\ & \sum_{j=0}^{M}\left(\frac{1}{N}\sum_{i=1}^{\epsilon^N} Y_t^{N,m_M^N}\left(\frac{\mu_{M,j}^N - i}{N}\right)\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\frac{1}{N}\sum_{i=\mu_{M,j}^N-\epsilon^N}^{\mu_{M,j}^N-1} Y_t^{N,m_M^N}\left(\frac{i}{N}\right)\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\frac{1}{N}\sum_{i=\mu_{M,j}^N-\epsilon^N}^{\mu_{M,j}^N-1} \frac{1}{\sqrt{N}}\sum_{k=1}^{N} X_{N^2t}^{k,m_M^N}\mathbb{1}_{I(k)}\left(\frac{i}{N}\right)\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\frac{1}{N\sqrt{N}}\sum_{i=\mu_{M,j}^N-\epsilon^N}^{\mu_{M,j}^N-1}\sum_{k=1}^{N} X_{N^2t}^{k,m_M^N}\mathbb{1}_{\{k=i\}}\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\frac{1}{N\sqrt{N}}\sum_{i=\mu_{M,j}^N-\epsilon^N}^{\mu_{M,j}^N-1} X_{N^2t}^{i,m_M^N}\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}.
\end{aligned}$$

On the other hand, we see that:

$$\begin{aligned}
&\int_0^{\epsilon} Y_t^{N,m_M^N}\left(\frac{m_M^N}{N} - x\right)dx\\
=\ & \sum_{j=0}^{M}\left(\int_0^{\epsilon} Y_t^{N,m_M^N}\left(\frac{\mu_{M,j}^N}{N} - x\right)dx\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\int_{\mu_{M,j}-\epsilon}^{\mu_{M,j}} Y_t^{N,m_M^N}(x)\,dx\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\int_{\mu_{M,j}-\epsilon}^{\mu_{M,j}} \frac{1}{\sqrt{N}}\sum_{k=1}^{N} X_{N^2t}^{k,m_M^N}\mathbb{1}_{I(k)}(x)dx\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\frac{1}{N\sqrt{N}}\sum_{k=\mu_{M,j}^N-\epsilon^N}^{\mu_{M,j}^N} X_{N^2t}^{k,m_M^N}\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}}\\
=\ & \sum_{j=0}^{M}\left(\frac{1}{N\sqrt{N}}\sum_{k=\mu_{M,j}^N-\epsilon^N}^{\mu_{M,j}^N-1} X_{N^2t}^{k,m_M^N}\right)\mathbb{1}_{\{m_M^N=\mu_{M,j}^N\}},
\end{aligned}$$

since the process $X^{m_M^N}$ is pinned at point m_M^N by definition, and the lemma is therefore proven.

□

Lemma 6.2.2 *Let $(h^\nu)_{\nu\geq 1}$ be a uniformly bounded sequence of continuously differentiable functions on $\mathbb{R}$, which satisfies $\lim_{\nu\to\infty} h^\nu(x) = h(x)$ for all $x \in \mathbb{R}$, and where the limiting function h is only assumed to be bounded. Then, for any Lipschitz continuous and bounded function f, we have:*

$$\lim_{\nu\to\infty} \sup_{N\geq 1} \left| \mathbb{E}\left(f\left(\langle Y_t^{N,m_M^N}, h^\nu \rangle \right)\right) - \mathbb{E}\left(f\left(\langle Y_t^{N,m_M^N}, h \rangle \right)\right) \right| = 0.$$

Proof of Lemma 6.2.2: Let $C \geq 0$ be the Lipschitz constant associated with f. We recall that $\mathbb{L}^2(]0,1[)$ is endowed with the canonical scalar product $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$ and the associated norm denoted by $\|.\|$. We can write:

$$\begin{aligned}
& \lim_{\nu\to\infty} \sup_{N\geq 1} \left| \mathbb{E}\left(f\left(\langle Y_t^{N,m_M^N}, h^\nu \rangle \right)\right) - \mathbb{E}\left(f\left(\langle Y_t^{N,m_M^N}, h \rangle \right)\right) \right| \\
\leq\ & \lim_{\nu\to\infty} \sup_{N\geq 1} \mathbb{E}\left(\left| f\left(\langle Y_t^{N,m_M^N}, h^\nu \rangle \right) - f\left(\langle Y_t^{N,m_M^N}, h \rangle \right) \right| \right) \\
\leq\ & \lim_{\nu\to\infty} \sup_{N\geq 1} C\mathbb{E}\left(\left| \langle Y_t^{N,m_M^N}, h^\nu - h \rangle \right| \right) \\
\leq\ & C \left(\lim_{\nu\to\infty} \|h^\nu - h\| \right) \left(\sup_{N\geq 1} \mathbb{E}\left(\|Y_t^{N,m_M^N}\| \right) \right),
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality between the last two lines. As the dominated convergence theorem immediately yields $\lim_{\nu\to\infty} \|h^\nu - h\| = 0$, we just have to establish that $\sup_{N\geq 1} \mathbb{E}\left(\|Y_t^{N,m_M^N}\| \right) < +\infty$ in order to conclude. Using Itô's lemma, we then have:

$$\begin{aligned}
\|Y_t^{N,m_M^N}\|^2 &= \frac{1}{N^2} \sum_{k=1}^{N} \left(X_{N^2 t}^{k,m_M^N} \right)^2 \\
&= \|Y_0^{N,m_M^N}\|^2 + \frac{2}{N^{7/2}} \int_0^t \sum_{k=1}^{N} \left(c + \lambda\tfrac{k}{N} - \theta\tfrac{k}{N} \right) X_r^{k,m_M^N} dr \\
&\quad + \frac{2\gamma}{N^2} \int_0^t \sum_{k=1}^{N} \left(X_r^{k-1,m_M^N} + X_r^{k+1,m_M^N} - 2X_r^{k,m_M^N} \right) X_r^{k,m_M^N} dr \\
&\quad + \frac{2\lambda t}{N} + \frac{2\sqrt{2\lambda}}{N^2} \int_0^t \sum_{k=1}^{N} X_r^{k,m_M^N} dW_r^{k,M},
\end{aligned}$$

where we have noted that $\int_0^t X_r^{k,m_M^N} dL_r^{X^{k,m_M^N}} = 0$ by construction of the reflected diffusion. Moving forward, we use summation by parts to see that:

$$\begin{aligned} &\sum_{k=1}^{N} \left(X_r^{k-1,m_M^N} + X_r^{k+1,m_M^N} - 2X_r^{k,m_M^N} \right) X_r^{k,m_M^N} \\ = \quad & -\left(X_r^{1,m_M^N} \right)^2 - \sum_{k=1}^{N} \left(X_r^{k+1,m_M^N} - X_r^{k,m_M^N} \right)^2 < 0 \text{ a.s.} \end{aligned}$$

Taking expectations and making use of the fact that $\left(\int_0^t \sum_{k=1}^N X_r^{k,m_M^N} dW_r^{k,M} \right)_{t\geq 0}$ is a martingale (see Funaki [23]) as well as of assumption $(A7)$, we now have:

$$\mathbb{E}\left(\|Y_t^{N,m_M^N}\|^2 \right) \quad \leq \quad \mathbb{E}\left(\|Y_0^{N,m_M^N}\|^2 \right) + \frac{2\lambda t}{N}.$$

Finally, it is shown by Funaki and Olla in [22] that $\sup_{N\geq 1} \mathbb{E}\left(\|Y_0^{N,m_M^N}\|^2 \right) < +\infty$, which concludes the proof. □

Having proven these two lemmas, we are now ready to start the proof of Theorem 6.2.1:

Proof of Theorem 6.2.1: Using an induction procedure, we are first going to prove that the following statement, denoted by $\mathcal{S}(M)$, holds for any $M \in \{1, \dots, M_T^*\}$:

$$\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N \right) \xRightarrow[N\to\infty]{} \left(\mathcal{U}_{M-1}^n, \mathcal{T}_M \right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^{M\times n} \right) \times \left(\mathbb{R}_+^* \right)^{2M}$, for any $0 < s \leq T < \infty$. Once this has been proven, the continuous mapping theorem applied to g_M^n as well as to the min and projection functions shall yield the required result. From now on, for notational clarity, we omit the N-dependency under the weak convergence arrow.

- Base case:

Using the previously introduced notation, we see that:

$$\tau_1^{N,b} = \tau\left(S^{N,b}\left(Y^{N,m_0^N} \right) \right), \quad \tau_1^{N,a} = \tau\left(S^{N,a}\left(Y^{N,m_0^N} \right) \right).$$

Moreover, using Lemma 6.2.1, we have:

$$S^{N,b}\left(Y_t^{N,m_0^N} \right) \quad = \quad \frac{1}{N} \sum_{i=1}^{\epsilon^N} \left(Y_t^{N,m_0^N}\left(m_0 + \frac{i}{N} \right) - Y_t^{N,m_0^N}\left(m_0 - \frac{i}{N} \right) \right) - \delta^N$$

$$
\begin{aligned}
&= \int_0^{\epsilon} \left(Y_t^{N,m_0^N}(m_0+x) - Y_t^{N,m_0^N}(m_0-x)\right) dx - \delta^N \\
&= \int_0^{1} \left(Y_t^{N,m_0^N}(x)\left(\mathbb{1}_{[m_0,m_0+\epsilon]}(x) - \mathbb{1}_{[m_0-\epsilon,m_0]}(x)\right)\right) dx - \delta^N \\
&= \langle Y_t^{N,m_0^N}, \mathbb{1}_{[m_0,m_0+\epsilon]}\rangle - \langle Y_t^{N,m_0^N}, \mathbb{1}_{[m_0-\epsilon,m_0]}\rangle - \delta^N \\
&= \langle Y_t^{N,m_0^N}, h_1^{m_0}\rangle - \langle Y_t^{N,m_0^N}, h_2^{m_0}\rangle - \delta^N,
\end{aligned}
$$

where we denote $h_1^{m_0} \equiv \mathbb{1}_{[m_0,m_0+\epsilon]}$ and $h_2^{m_0} \equiv \mathbb{1}_{[m_0-\epsilon,m_0]}$. We can also show that:

$$I^b\left(u_t^{m_0}\right) = \langle u_t^{m_0}, h_1^{m_0}\rangle - \langle u_t^{m_0}, h_2^{m_0}\rangle - \delta.$$

With similar arguments, we have:

$$S^{N,a}\left(Y_t^{N,m_0^N}\right) = \langle Y_t^{N,m_0^N}, h_2^{m_0}\rangle - \langle Y_t^{N,m_0^N}, h_1^{m_0}\rangle - \delta^N,$$

$$I^a\left(u_t^{m_0}\right) = \langle u_t^{m_0}, h_2^{m_0}\rangle - \langle u_t^{m_0}, h_1^{m_0}\rangle - \delta.$$

Given that the notion of weak convergence we are dealing with involves integration against continuously differentiable test functions, we need to consider the following approximation of the indicator function on any interval $[\alpha,\beta]$, defined for all $x\in\mathbb{R}$ and any steepness parameter $\nu>0$ by:

$$h^{\nu,\alpha,\beta}(x) = \frac{1}{\left(1+\exp\left(-\nu\left(x-\alpha+\frac{1}{\sqrt{\nu}}\right)\right)\right)\left(1+\exp\left(-\nu\left(\beta-x+\frac{1}{\sqrt{\nu}}\right)\right)\right)}.$$

It is immediate to check that $h^{\nu,\alpha,\beta}$ is continuously differentiable and that we have $\lim_{\nu\to\infty} h^{\nu,\alpha,\beta}(x) = \mathbb{1}_{[\alpha,\beta]}(x)$, for any $x\in\mathbb{R}$. We subsequently define $h_1^{\nu,m_0} \equiv h^{\nu,m_0,m_0+\epsilon}$ and $h_2^{\nu,m_0} \equiv h^{\nu,m_0-\epsilon,m_0}$. Now, consider f Lipschitz continuous and bounded. We also define:

$$E_{N,\nu} = \mathbb{E}\left(f\left(\mathcal{Y}_0^{N,n}, \langle Y^{N,m_0^N}, h_1^{\nu,m_0}\rangle, \langle Y^{N,m_0^N}, h_2^{\nu,m_0}\rangle\right)\right),$$

$$E_{\infty,\nu} = \lim_{N\to\infty} E_{N,\nu}, \;\; E_{N,\infty} = \lim_{\nu\to\infty} E_{N,\nu},$$

$$E = \mathbb{E}\left(f\left(\mathcal{U}_0^n, \langle u^{m_0}, h_1^{m_0}\rangle, \langle u^{m_0}, h_2^{m_0}\rangle\right)\right).$$

As h_1^{ν,m_0} and h_2^{ν,m_0} are continuously differentiable, we know that:

$$E_{\infty,\nu} = \mathbb{E}\left(f\left(\mathcal{U}_0^n, \langle u^{m_0}, h_1^{\nu,m_0}\rangle, \langle u^{m_0}, h_2^{\nu,m_0}\rangle\right)\right).$$

We also notice that for any $x\in[0,1]$,

$$\left|Y^{N,m_0^N}(x)h_1^{\nu,m_0}(x)\right| = Y^{N,m_0^N}(x)h_1^{\nu,m_0}(x) \le Y^{N,m_0^N}(x) \;\text{ a.s},$$

and that this upper bound is clearly integrable since:

$$\begin{aligned}\int_0^1 \left|Y^{N,m_0^N}(x)\right| dx &= \frac{1}{\sqrt{N}} \int_0^1 \sum_{k=1}^N X_{N^2t}^{k,m_0^N} \mathbb{1}_{I(k)}(x)dx \\ &= \frac{1}{N\sqrt{N}} \sum_{k=1}^N X_{N^2t}^{k,m_0^N} < +\infty \text{ a.s.}\end{aligned}$$

The continuity and boundedness of f along with two applications of the dominated convergence theorem therefore give us:

$$E_{N,\infty} = \mathbb{E}\left(f\left(\mathcal{Y}_0^{N,n}, \langle Y^{N,m_0^N}, h_1^{m_0}\rangle, \langle Y^{N,m_0^N}, h_2^{m_0}\rangle\right)\right).$$

Given the Lipschitz continuity of f, an immediate extension of Lemma 6.2.2 tells us that this limit is in actuality uniform in N, i.e.:

$$\lim_{\nu\to\infty} \sup_{N\geq 1} \left|E_{N,\nu} - E_{N,\infty}\right| = 0.$$

It is thus possible to apply the Moore-Osgood theorem for interchanging limits of double sequences:

$$\lim_{N\to\infty} E_{N,\infty} = \lim_{N\to\infty} \lim_{\nu\to\infty} E_{N,\nu} = \lim_{\nu\to\infty} \lim_{N\to\infty} E_{N,\nu} = \lim_{\nu\to\infty} E_{\infty,\nu}.$$

But we similarly see that for any $x \in [0,1]$,

$$\left|u^{m_0}(x)h_1^{\nu,m_0}(x)\right| = u^{m_0}(x)h_1^{\nu,m_0}(x) \leq u^{m_0}(x) \text{ a.s,}$$

and since u^{m_0} is continuous in x on $[0,1]$ and therefore bounded on $[0,1]$, we also have:

$$\int_0^1 \left|u^{m_0}(x)\right| dx < +\infty \text{ a.s.}$$

Once again, the continuity and boundedness of f as well as two applications of the dominated convergence theorem enable us to obtain:

$$\lim_{\nu\to\infty} E_{\infty,\nu} = E.$$

Putting these arguments together, it is now clear to see that:

$$\lim_{N\to\infty} E_{N,\infty} = \lim_{\nu\to\infty} E_{\infty,\nu} = E,$$

or, in other words, (according to the Portmanteau lemma), that:

$$\left(\mathcal{Y}_0^{N,n}, \langle Y^{N,m_0^N}, h_1^{m_0}\rangle, \langle Y^{N,m_0^N}, h_2^{m_0}\rangle\right) \Rightarrow \left(\mathcal{U}_0^n, \langle u^{m_0}, h_1^{m_0}\rangle, \langle u^{m_0}, h_2^{m_0}\rangle\right),$$

in $\mathbb{C}\left([s,T],\mathbb{R}^{n+2}\right)$, for any $0 < s \leq T < \infty$. Invoking the continuity of the first passage time (see Whitt [53]) and the continuous mapping theorem, we deduce that:

$$\left(\mathcal{Y}_0^{N,n}, \tau_1^{N,b}, \tau_1^{N,a}\right) \Rightarrow \left(\mathcal{U}_0^n, \tau_1^b, \tau_1^a\right),$$

in $\mathbb{C}\left([s,T],\mathbb{R}^n\right) \times \left(\mathbb{R}_+^*\right)^2$, for any $0 < s \leq T < \infty$, which establishes $\mathcal{S}(1)$.

- Inductive step:

Assume that $\mathcal{S}(M)$ holds for an arbitrary $M \in \{2,\ldots,M_T^*-1\}$. Let us now define $\mathcal{A}_M^n = \prod_{j=0}^M \prod_{p=1}^n A_j^p$, $\widehat{\mathcal{A}}_M^n = \prod_{p=1}^n A_M^p$, $\mathcal{B}_M^b = \prod_{j=1}^M B_j^b$, $\mathcal{B}_M^a = \prod_{j=1}^M B_j^a$ and $\mathcal{B}_M = \mathcal{B}_M^b \times \mathcal{B}_M^a$, where A_j^p, B_j^b and B_j^a are respective continuity sets associated with $\langle Y^{N,m_j^N}, h_p\rangle$, $\tau_j^{N,b}$ and $\tau_j^{N,a}$. We can then write:

$$\begin{aligned}
& \mathbb{P}\left(\left(\mathcal{Y}_M^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_M^n \times \mathcal{B}_M\right) \\
= & \sum_{j=0}^M \mathbb{P}\left(\left(\mathcal{Y}_M^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_M^n \times \mathcal{B}_M \middle| m_M^N = \mu_{M,j}^N\right) \times \mathbb{P}\left(m_M^N = \mu_{M,j}^N\right) \\
= & \sum_{j=0}^M \mathbb{P}\left(\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-1}^n \times \mathcal{B}_M \middle| m_M^N = \mu_{M,j}^N\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_M^{N,n} \in \widehat{\mathcal{A}}_M^n \middle| m_M^N = \mu_{M,j}^N\right) \times \mathbb{P}\left(m_M^N = \mu_{M,j}^N\right) \\
= & \sum_{j=0}^M \mathbb{P}\left(\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-1}^n \times \mathcal{B}_M, m_M^N = \mu_{M,j}^N\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_M^{N,n} \in \widehat{\mathcal{A}}_M^n \middle| m_M^N = \mu_{M,j}^N\right) \\
= & \sum_{j=0}^M \mathbb{P}\left(\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-1}^n \times \mathcal{B}_M, \frac{m_M^N}{N} = \mu_{M,j}\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_M^{N,n} \in \widehat{\mathcal{A}}_M^n \middle| \frac{m_M^N}{N} = \mu_{M,j}\right),
\end{aligned}$$

where, between the first and second equalities, we have used the independence of $\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N\right)$ and $\widehat{\mathcal{Y}}_M^{N,n}$ conditionally on m_M^N, by construction of the mesoscopic system. We then notice that:

$$\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N, \frac{m_M^N}{N}\right) = \left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N, f_M\left(\mathcal{T}_M^N; m_0, \epsilon\right)\right),$$

where we recall that f_M is defined by (6.2). Taking into consideration $\mathcal{S}(M)$, we once again invoke the continuous mapping theorem to deduce that:

$$\left(\mathcal{Y}^{N,n}_{M-1}, \mathcal{T}^N_M, \frac{m^N_M}{N}\right) \Rightarrow \left(\mathcal{U}^n_{M-1}, \mathcal{T}_M, f_M\left(\mathcal{T}_M; m_0, \epsilon\right)\right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^n\right) \times \left(\mathbb{R}^*_+\right)^{2M} \times [0,1]$, for any $0 < s \leq T < \infty$, or, in other terms:

$$\left(\mathcal{Y}^{N,n}_{M-1}, \mathcal{T}^N_M, \frac{m^N_M}{N}\right) \Rightarrow \left(\mathcal{U}^n_{M-1}, \mathcal{T}_M, m_M\right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^n\right) \times \left(\mathbb{R}^*_+\right)^{2M} \times [0,1]$, for any $0 < s \leq T < \infty$. Furthermore, as:

$$\left(\widehat{\mathcal{Y}}^{N,n}_M \middle| \frac{m^N_M}{N}\right) \Rightarrow \left(\widehat{\mathcal{U}}^n_M \middle| m_M\right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^n\right)$, for any $0 < s \leq T < \infty$, we have:

$$\begin{aligned}
& \lim_{N\to\infty} \mathbb{P}\left(\left(\mathcal{Y}^{N,n}_M, \mathcal{T}^N_M\right) \in \mathcal{A}^n_M \times \mathcal{B}_M\right) \\
= \; & \sum_{j=0}^{M} \lim_{N\to\infty} \mathbb{P}\left(\left(\mathcal{Y}^{N,n}_{M-1}, \mathcal{T}^N_M\right) \in \mathcal{A}^n_{M-1} \times \mathcal{B}_M, \frac{m^N_M}{N} = \mu_{M,j}\right) \\
& \times \lim_{N\to\infty} \mathbb{P}\left(\widehat{\mathcal{Y}}^{N,n}_M \in \widehat{\mathcal{A}}^n_M \middle| \frac{m^N_M}{N} = \mu_{M,j}\right) \\
= \; & \sum_{j=0}^{M} \mathbb{P}\left(\left(\mathcal{U}^n_{M-1}, \mathcal{T}_M\right) \in \mathcal{A}^n_{M-1} \times \mathcal{B}_M, m_M = \mu_{M,j}\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{U}}^n_M \in \widehat{\mathcal{A}}^n_M \middle| m_M = \mu_{M,j}\right) \\
= \; & \sum_{j=0}^{M} \mathbb{P}\left(\left(\mathcal{U}^n_{M-1}, \mathcal{T}_M\right) \in \mathcal{A}^n_{M-1} \times \mathcal{B}_M \middle| m_M = \mu_{M,j}\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{U}}^n_M \in \widehat{\mathcal{A}}^n_M \middle| m_M = \mu_{M,j}\right) \times \mathbb{P}\left(m_M = \mu_{M,j}\right) \\
= \; & \sum_{j=0}^{M} \mathbb{P}\left(\left(\mathcal{U}^n_M, \mathcal{T}_M\right) \in \mathcal{A}^n_M \times \mathcal{B}_M \middle| m_M = \mu_{M,j}\right) \times \mathbb{P}\left(m_M = \mu_{M,j}\right) \\
= \; & \mathbb{P}\left(\left(\mathcal{U}^n_M, \mathcal{T}_M\right) \in \mathcal{A}^n_M \times \mathcal{B}_M\right),
\end{aligned}$$

exploiting the independence of $\left(\mathcal{U}^n_M, \mathcal{T}_M\right)$ and $\widehat{\mathcal{U}}^n_M$ conditionally on m_M between the third and fourth equalities, by construction of the macroscopic system. By virtue of the Portmanteau lemma, this shows that:

$$\left(\mathcal{Y}^{N,n}_M, \mathcal{T}^N_M\right) \Rightarrow \left(\mathcal{U}^n_M, \mathcal{T}_M\right),$$

in $\mathbb{C}\left([s,T],\mathbb{R}^{(M+1)\times n}\right)\times\left(\mathbb{R}_+^*\right)^{2M}$, for any $0<s\le T<\infty$. We then observe that:

$$\mathcal{T}_{M+1}^N=\left(\mathcal{T}_M^{N,b},\tau\left(S^{N,b}\left(Y^{N,m_M^N}\right)\right),\mathcal{T}_M^{N,a},\tau\left(S^{N,a}\left(Y^{N,m_M^N}\right)\right)\right).$$

Generalising what was done in the base case and using Lemma 6.2.1, we write:

$$\begin{aligned}
&S^{N,b}\left(Y_t^{N,m_M^N}\right)\\
=\ &\frac{1}{N}\sum_{i=1}^{\epsilon^N}\left(Y_t^{N,m_M^N}\left(\frac{m_M^N}{N}+\frac{i}{N}\right)-Y_t^{N,m_M^N}\left(\frac{m_M^N}{N}-\frac{i}{N}\right)\right)-\delta^N\\
=\ &\int_0^\epsilon\left(Y_t^{N,m_M^N}\left(\frac{m_M^N}{N}+x\right)-Y_t^{N,m_M^N}\left(\frac{m_M^N}{N}-x\right)\right)dx-\delta^N\\
=\ &\sum_{j=0}^M\left(\int_0^\epsilon Y_t^{N,m_M^N}\left(\mu_{M,j}+x\right)dx\right)\mathbb{1}_{\left\{\frac{m_M^N}{N}=\mu_{M,j}\right\}}\\
&-\sum_{j=0}^M\left(\int_0^\epsilon Y_t^{N,m_M^N}\left(\mu_{M,j}-x\right)dx\right)\mathbb{1}_{\left\{\frac{m_M^N}{N}=\mu_{M,j}\right\}}-\delta^N\\
=\ &\sum_{j=0}^M\left(\int_0^1 Y_t^{N,m_M^N}(x)\,\mathbb{1}_{[\mu_{M,j},\mu_{M,j}+\epsilon]}(x)dx\right)\mathbb{1}_{\left\{\frac{m_M^N}{N}=\mu_{M,j}\right\}}\\
&-\sum_{j=0}^M\left(\int_0^1 Y_t^{N,m_M^N}(x)\,\mathbb{1}_{[\mu_{M,j}-\epsilon,\mu_{M,j}]}(x)dx\right)\mathbb{1}_{\left\{\frac{m_M^N}{N}=\mu_{M,j}\right\}}-\delta^N\\
=\ &\sum_{j=0}^M\left(\langle Y_t^{N,m_M^N},h_1^{\mu_{M,j}}\rangle-\langle Y_t^{N,m_M^N},h_2^{\mu_{M,j}}\rangle\right)\mathbb{1}_{\left\{\frac{m_M^N}{N}=\mu_{M,j}\right\}}-\delta^N,
\end{aligned}$$

where, for all $j\in\{0,\dots,M\}$, we have defined $h_1^{\mu_{M,j}}\equiv\mathbb{1}_{[\mu_{M,j},\mu_{M,j}+\epsilon]}$ and $h_2^{\mu_{M,j}}\equiv\mathbb{1}_{[\mu_{M,j}-\epsilon,\mu_{M,j}]}$. *Mutatis mutandis*, we successively establish that:

$$\begin{aligned}
S^{N,a}\left(Y_t^{N,m_M^N}\right)\ =\ &\sum_{j=0}^M\langle Y_t^{N,m_M^N},h_2^{\mu_{M,j}}\rangle\mathbb{1}_{\left\{\frac{m_M^N}{N}=\mu_{M,j}\right\}}\\
&-\sum_{j=0}^M\langle Y_t^{N,m_M^N},h_1^{\mu_{M,j}}\rangle\mathbb{1}_{\left\{\frac{m_M^N}{N}=\mu_{M,j}\right\}}-\delta^N,
\end{aligned}$$

$$I^b\left(u_t^{m_M}\right)=\sum_{j=0}^M\left(\langle u_t^{m_M},h_1^{\mu_{M,j}}\rangle-\langle u_t^{m_M},h_2^{\mu_{M,j}}\rangle\right)\mathbb{1}_{\{m_M=\mu_{M,j}\}}-\delta,$$

$$I^a\left(u_t^{m_M}\right) = \sum_{j=0}^{M} \left(\langle u_t^{m_M}, h_2^{\mu_{M,j}}\rangle - \langle u_t^{m_M}, h_1^{\mu_{M,j}}\rangle\right) \mathbb{1}_{\{m_M=\mu_{M,j}\}} - \delta.$$

Before going any further, for any $j \in \{0,\dots,M\}$, we define the continuously differentiable approximations of the above indicator functions, namely, $h_1^{\nu,\mu_{M,j}} \equiv h^{\nu,\mu_{M,j},\mu_{M,j}+\epsilon}$ and $h_2^{\nu,\mu_{M,j}} \equiv h^{\nu,\mu_{M,j}-\epsilon,\mu_{M,j}}$. With a slight abuse of notation, we also introduce the four following vectors:

$$\widetilde{\mathcal{Y}}_M^{N,\nu} = \left(\langle Y^{N,m_M^N}, h_1^{\nu,\mu_{M,j}}\rangle, \langle Y^{N,m_M^N}, h_2^{\nu,\mu_{M,j}}\rangle, j \in \{0,\dots,M\}\right),$$

$$\widetilde{\mathcal{Y}}_M^{N} = \left(\langle Y^{N,m_M^N}, h_1^{\mu_{M,j}}\rangle, \langle Y^{N,m_M^N}, h_2^{\mu_{M,j}}\rangle, j \in \{0,\dots,M\}\right),$$

$$\widetilde{\mathcal{U}}_M^{\nu} = \left(\langle u^{m_M}, h_1^{\nu,\mu_{M,j}}\rangle, \langle u^{m_M}, h_2^{\nu,\mu_{M,j}}\rangle, j \in \{0,\dots,M\}\right),$$

$$\widetilde{\mathcal{U}}_M = \left(\langle u^{m_M}, h_1^{\mu_{M,j}}\rangle, \langle u^{m_M}, h_2^{\mu_{M,j}}\rangle, j \in \{0,\dots,M\}\right).$$

Replicating the same arguments of conditional independence which enabled us to show that:

$$\left(\mathcal{Y}_M^{N,n}, \mathcal{T}_M^N\right) \Rightarrow \left(\mathcal{U}_M^n, \mathcal{T}_M\right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^{(M+1)\times n}\right) \times \left(\mathbb{R}_+^*\right)^{2M}$, for any $0 < s \le T < \infty$, it is straightforward to see that:

$$\left(\mathcal{Y}_M^{N,n}, \widetilde{\mathcal{Y}}_M^{N,\nu}, \mathcal{T}_M^N\right) \Rightarrow \left(\mathcal{U}_M^n, \widetilde{\mathcal{U}}_M^{\nu}, \mathcal{T}_M\right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^{(M+1)\times (n+2)}\right) \times \left(\mathbb{R}_+^*\right)^{2M}$, for any $0 < s \le T < \infty$. Using Lemma 6.2.2 and adapting the double limit arguments presented in the base case, we readily obtain that:

$$\left(\mathcal{Y}_M^{N,n}, \widetilde{\mathcal{Y}}_M^{N}, \mathcal{T}_M^N\right) \Rightarrow \left(\mathcal{U}_M^n, \widetilde{\mathcal{U}}_M, \mathcal{T}_M\right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^{(M+1)\times (n+2)}\right) \times \left(\mathbb{R}_+^*\right)^{2M}$, for any $0 < s \le T < \infty$. But given that $\frac{m_M^N}{N} = f_M\left(\mathcal{T}_M^N; m_0, \epsilon\right)$, we know that:

$$\left(\mathcal{Y}_M^{N,n}, \widetilde{\mathcal{Y}}_M^{N}, \mathcal{T}_M^N, \frac{m_M^N}{N}\right) \Rightarrow \left(\mathcal{U}_M^n, \widetilde{\mathcal{U}}_M, \mathcal{T}_M, m_M\right)$$

in $\mathbb{C}\left([s,T], \mathbb{R}^{(M+1)\times (n+2)}\right) \times \left(\mathbb{R}_+^*\right)^{2M} \times [0,1]$, for any $0 < s \le T < \infty$, using the continuous mapping theorem. Finally, yet another application of the continuous mapping theorem yields:

$$\left(\mathcal{Y}_M^{N,n}, \mathcal{T}_{M+1}^N\right) \Rightarrow \left(\mathcal{U}_M^n, \mathcal{T}_{M+1}\right),$$

in $\mathbb{C}\left([s,T], \mathbb{R}^{(M+1)\times n}\right) \times \left(\mathbb{R}_+^*\right)^{2(M+1)}$, for any $0 < s \le T < \infty$, and $\mathcal{S}(M+1)$ is thus established.

Now, moving on from the induction procedure, we notice that by continuity of the min and projection functions, we have:

$$\left(\mathcal{Y}^{N,n}_{M-1}, \tau^N_1, \ldots, \tau^N_M\right) \Rightarrow \left(\mathcal{U}^n_{M-1}, \tau_1, \ldots, \tau_M\right),$$

in $\mathbb{C}([s,T], \mathbb{R}^{M\times n}) \times \left(\mathbb{R}^*_+\right)^{2M}$, for any $0 < s \leq T < \infty$ and any $M \in \{1, \ldots, M^*_T\}$. But since g^n_M is measurable by construction and only has a finite number of discontinuities (which occur at the various *accumulated* imbalance times), the continuous mapping theorem gives us

$$g^n_M\left(\mathcal{Y}^{N,n}_{M-1}, \tau^N_1, \ldots, \tau^N_M\right) \xRightarrow[N\to\infty]{} g^n_M\left(\mathcal{U}^n_{M-1}, \tau_1, \ldots, \tau_M\right),$$

in $\mathbb{D}([s,T], \mathbb{R}^n)$, for any $0 < s \leq T < \infty$ and any $M \in \{1, \ldots, M^*_T\}$, and the proof is now complete.

□

Remark 6.2.2 *One potential limitation of global order flow imbalance is based on the fact that when the overall bid and ask sizes are larger, the difference between them is also likely to be larger, consequently triggering more frequent price changes. In order to address this issue, one can introduce the notion of normalised (global) order flow imbalance (described by Cartea et al. [12] and Lipton et al. [43] as a relevant price evolution signal), whereby we measure the proportional difference between the bid and ask. Using the same arguments as above, we can reproduce the proof to cover the case of this normalised order flow imbalance, which takes the following form in our current setting:*

$$\rho^{N,m^N_M}_t = \frac{\sum_{i=1}^{\epsilon^N}\left(Y^{N,m^N_M}_t\left(\frac{m^N_M - i}{N}\right) - Y^{N,m^N_M}_t\left(\frac{m^N_M + i}{N}\right)\right)}{\sum_{i=1}^{\epsilon^N}\left(Y^{N,m^N_M}_t\left(\frac{m^N_M - i}{N}\right) + Y^{N,m^N_M}_t\left(\frac{m^N_M + i}{N}\right)\right)},$$

$$\rho^{m_M}_t = \frac{\int_0^\epsilon \left(u^{m_M}(m_M - x, t) - u^{m_M}(m_M + x, t)\right) dx}{\int_0^\epsilon \left(u^{m_M}(m_M - x, t) + u^{m_M}(m_M + x, t)\right) dx}.$$

By construction, the advantage of these measures is that they are always between -1 and 1. When $\rho^{N,m^N_M}_t$ *or* $\rho^{m_M}_t$ *are close to 1, there is strong buying pressure resulting in a price increase, and when* $\rho^{N,m^N_M}_t$ *or* $\rho^{m_M}_t$ *are close to -1, we are in the presence of strong selling pressure pushing the price down.*

6.2.3 Extensions of the dynamic framework

Given the obvious drawbacks of having to resample from the invariant distribution after each price change, above all from a financial viewpoint, the first natural extension of Theorem 6.2.1 is to allow for more general reinitialisations which preserve the empirically observed (see for example Gould et al. [28]) long memory property of the book.

The case of general reinitialisations after a price change

Let $M \in \{0, \dots, M_T^*\}$. As previously, we consider the following Funaki-Olla dynamics as our starting mesoscopic system, i.e. for any $k \in \{1, \dots, N\}$:

$$\begin{aligned} dX_t^{k,m_M^N} &= \gamma\left(X_t^{k-1,m_M^N} + X_t^{k+1,m_M^N} - 2X_t^{k,m_M^N}\right)dt \\ &\quad + N^{-3/2}\left(c + \lambda^{\frac{k}{N}} - \theta^{\frac{k}{N}}\right)dt + \sqrt{2\lambda}dW_t^{k,M} + dL_t^{X^{k,m_M^N}}, \end{aligned}$$

with the pinning convention $X_t^{0,m_M^N} = X_t^{m_M^N,m_M^N} = X_t^{N+1,m_M^N} = 0$. The main difference, however, is that we now give ourselves a general reinitialisation function $r^N = (r_1^N, \dots, r_N^N)$ such that the initial condition of the system is expressed by $X_0^{k,m_M^N} = r_k^N\left(X_{\tau_M^{N-}}^{m_{M-1}^N}, \epsilon^N\right)$. Naturally, the corresponding macroscopic system to be considered is given by the following reflected stochastic heat equation:

$$\frac{\partial u^{m_M}(x,t)}{\partial t} = \frac{\gamma}{\lambda}\frac{\partial^2 u^{m_M}(x,t)}{\partial x^2} + c + \lambda(x) - \theta(x) + \sqrt{2\lambda}\dot{W}^M(x,t) + \eta^M(x,t),$$

with Dirichlet boundary conditions $u^{m_M}(0,t) = u^{m_M}(m_M,t) = u^{m_M}(1,t) = 0$ for all $t \in [0,T]$, with $u^{m_M} \geq 0$, $d\eta^M \geq 0$, $\int u^{m_M} d\eta^M = 0$ and with initial profile $u^{m_M}(x,0) = r\left(u^{m_{M-1}}(x,\tau_M^-), \epsilon\right)$. Taking into account the weak convergence results of Chapter 5, we recall that we need to ensure the appropriate notion of convergence of the initial conditions, which is more specifically weak convergence in $\mathbb{L}^2(]0,1[)$. Consequently, we need to assume that the respective mesoscopic and macroscopic reinitialisation functions r^N and r satisfy the following condition:

$$\frac{1}{N}\sqrt{\sum_{k=1}^{N}\left(r_k^N\left(X_{\tau_M^{N-}}^{m_{M-1}^N}, \epsilon^N\right)\right)^2} \Rightarrow \sqrt{\int_0^1 r\left(u^{m_{M-1}}(x,\tau_M^-), \epsilon\right)^2 dx},$$

which is none other than:

$$\|Y_0^{N,m_M^N}\| \Rightarrow \|u_0^{m_M}\|.$$

Note that this is just weak convergence of a sequence of nonnegative random variables indexed by N, i.e. weak convergence in $\mathbb{R}_+$, rather than weak convergence of a process in a path space.

Remark 6.2.3 *One possible specification of the reinitialisation function r is given by, for any $M \in \{1, \dots, M_T^*\}$:*

$$r\left(u^{m_{M-1}}(x,\tau_M^-), \epsilon\right) = \begin{cases} u^{m_{M-1}}(x - \epsilon\frac{1-x}{1-m_M}, \tau_M^-) \text{ for } x \in [m_M, 1],\ m_M = m_{M-1} + \epsilon, \\ u^{m_{M-1}}(x - \epsilon\frac{x}{m_M}, \tau_M^-) \text{ for } x \in [0, m_M[,\ m_M = m_{M-1} + \epsilon, \\ u^{m_{M-1}}(x + \epsilon\frac{1-x}{1-m_M}, \tau_M^-) \text{ for } x \in [m_M, 1],\ m_M = m_{M-1} - \epsilon, \\ u^{m_{M-1}}(x + \epsilon\frac{x}{m_M}, \tau_M^-) \text{ for } x \in [m_M, 1],\ m_M = m_{M-1} - \epsilon. \end{cases}$$

The construction of r is simply based on a space shift depending on the price move direction. The new pinning conditions following a price change are also satisfied using an appropriate weighting proportional to ϵ.

As a matter of fact, it turns out that Theorem 6.2.1 is readily extendable to this more general framework by noticing the following crucial point. In the inductive step of its proof, we recall that the main argument was based on the law of total probability, subsequently enabling us to use the independence of $\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N\right)$ and $\widehat{\mathcal{Y}}_M^{N,n}$ conditionally on m_M^N. In this case, by additionally conditioning on $\widehat{\mathcal{Y}}_{M-1}^{N,n}$, we notice that $\left(\mathcal{Y}_{M-2}^{N,n}, \mathcal{T}_M^N\right)$ and $\widehat{\mathcal{Y}}_M^{N,n}$ are independent by construction of the mesoscopic system. Using the law of total *conditional* probablity, we now have:

$$
\begin{aligned}
& \mathbb{P}\left(\left(\mathcal{Y}_M^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M,n} \times \mathcal{B}_M\right) \\
= \; & \mathbb{P}\left(\left(\mathcal{Y}_{M-2}^{N,n}, \widehat{\mathcal{Y}}_M^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-2}^n \times \widehat{\mathcal{A}}_M^n \times \mathcal{B}_M \middle| \widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n\right) \\
= \; & \sum_{j=0}^{M} \mathbb{P}\left(\left(\mathcal{Y}_{M-2}^{N,n}, \widehat{\mathcal{Y}}_M^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-2}^n \times \widehat{\mathcal{A}}_M^n \times \mathcal{B}_M \middle| \widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n, m_M^N = \mu_{M,j}^N\right) \\
& \times \mathbb{P}\left(m_M^N = \mu_{M,j}^N \middle| \widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n\right) \times \mathbb{P}\left(\widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n\right) \\
= \; & \sum_{j=0}^{M} \mathbb{P}\left(\left(\mathcal{Y}_{M-2}^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-2}^n \times \mathcal{B}_M \middle| \widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n, m_M^N = \mu_{M,j}^N\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_M^{N,n} \in \widehat{\mathcal{A}}_M^n \middle| \widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n, m_M^N = \mu_{M,j}^N\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n, m_M^N = \mu_{M,j}^N\right) \\
= \; & \sum_{j=0}^{M} \mathbb{P}\left(\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-1}^n \times \mathcal{B}_M, m_M^N = \mu_{M,j}^N\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_M^{N,n} \in \widehat{\mathcal{A}}_M^n \middle| \widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n, m_M^N = \mu_{M,j}^N\right) \\
= \; & \sum_{j=0}^{M} \mathbb{P}\left(\left(\mathcal{Y}_{M-1}^{N,n}, \mathcal{T}_M^N\right) \in \mathcal{A}_{M-1}^n \times \mathcal{B}_M, \frac{m_M^N}{N} = \mu_{M,j}\right) \\
& \times \mathbb{P}\left(\widehat{\mathcal{Y}}_M^{N,n} \in \widehat{\mathcal{A}}_M^n \middle| \widehat{\mathcal{Y}}_{M-1}^{N,n} \in \widehat{\mathcal{A}}_{M-1}^n, \frac{m_M^N}{N} = \mu_{M,j}\right).
\end{aligned}
$$

This argument of conditional independence can be directly plugged into the proof of Theorem 6.2.1, and naturally adapted to the case of the macroscopic system. As the rest of the proof is only slightly modified due to the additional conditioning, we do not provide any extra details here and move on to the second possible extension of this dynamic framework.

Price changes triggered via an *ad hoc* occupation measure around 0

In the microscopic and mesoscopic settings presented in Part I, the notion of local time played a fundamental role in the determination of price changes. Indeed, based on empirical evidence of limit order books in a high-frequency setting, we recall that Avellaneda et al. [5] introduced a novel stylised fact related to price elasticity once either one of the best queues was depleted. This was the specific reason behind our use of (regenerative) elastic Brownian motion throughout our diffusion approximation models. In this section, we propose an infinite-dimensional adaptation which enables us to capture this "price elasticity around 0" stylised fact.

Heuristically speaking, the idea is to measure the amount of time spent around 0 by a certain space-accumulated process around the mid. In order to maintain some sort of interaction between the bid and ask sides, we combine this idea with the order flow imbalance approach used in the previous section. This precisely amounts to saying that price changes shall only occur once the difference between the time spent around 0 by the space-accumulated process to the left of the mid (i.e. the bid) and the time spent around 0 by the space-accumuated process to the right of the mid (i.e. the ask) has exceeded a prespecified level. By doing so, we note that price changes occur less frequently than in the simple order flow imbalance case, which makes this approach potentially more relevant to an ultra high-frequency setting where the ratio of cancellations to executions is even higher than at high frequency (see Gai et al. [25]).

We start by clarifying what we exactly mean by *space-accumulated process*. Let $M \in \{0, \dots, M_T^*\}$. As previously, let Y^{N,m_M^N} and u^{m_M} be the order book processes respectively associated with the mesoscopic and macroscopic systems. We also recall that fixing an arbitrary $j \in \{0, \dots, M\}$, we define $h_1^{\mu_{M,j}} \equiv \mathbb{1}_{[\mu_{M,j},\mu_{M,j}+\epsilon]}$ and $h_2^{\mu_{M,j}} \equiv \mathbb{1}_{[\mu_{M,j}-\epsilon,\mu_{M,j}]}$. The processes $(A_r^{N,m_M^N})_{r\in[0,T]}$ and $(B_r^{N,m_M^N})_{r\in[0,T]}$ defined hereafter shall be referred to as the mesoscopic ask and bid space-accumulated processes respectively:

$$A_r^{N,m_M^N} = \sum_{j=0}^{M} \langle Y_r^{N,m_M^N}, h_1^{\mu_{M,j}} \rangle \mathbb{1}_{\{\frac{m_M^N}{N}=\mu_{M,j}\}},$$

$$B_r^{N,m_M^N} = \sum_{j=0}^{M} \langle Y_r^{N,m_M^N}, h_2^{\mu_{M,j}} \rangle \mathbb{1}_{\{\frac{m_M^N}{N}=\mu_{M,j}\}}.$$

We naturally extend this terminology to the case of the macroscopic ask and bid space-accumulated processes $(A_r^{m_M})_{r\in[0,T]}$ and $(B_r^{m_M})_{r\in[0,T]}$, respectively given by:

$$A_r^{m_M} = \sum_{j=0}^{M} \langle u_r^{m_M}, h_1^{\mu_{M,j}} \rangle \mathbb{1}_{\{m_M=\mu_{M,j}\}},$$

$$B_r^{m_M} = \sum_{j=0}^{M} \langle u_r^{m_M}, h_2^{\mu_{M,j}} \rangle \mathbb{1}_{\{m_M=\mu_{M,j}\}}.$$

As our aim here is solely to construct an *ad hoc* measure of the time spent around 0, we shall simply fix an arbitrarily small $\omega > 0$ without concerning ourselves with examining the limit as $\omega \to 0$ (as in the formal definition of local time). For any $M \in \{0, \dots, M_T^* - 1\}$, we define the so-called $(M + 1)$-th *local imbalance time* associated with the mesoscopic system as $\tau_{M+1}^N = \tau_{M+1}^{N,b} \wedge \tau_{M+1}^{N,a}$, where the stopping times $\tau_{M+1}^{N,b}$ and $\tau_{M+1}^{N,a}$ are now respectively expressed by:

$$\tau_{M+1}^{N,b} = \inf \left\{ t \geq 0 : \int_0^t \left(\mathbb{1}_{[0,\omega]} \left(A_r^{N,m_M^N} \right) - \mathbb{1}_{[0,\omega]} \left(B_r^{N,m_M^N} \right) \right) dr \geq \kappa^N \right\},$$

$$\tau_{M+1}^{N,a} = \inf \left\{ t \geq 0 : \int_0^t \left(\mathbb{1}_{[0,\omega]} \left(B_r^{N,m_M^N} \right) - \mathbb{1}_{[0,\omega]} \left(A_r^{N,m_M^N} \right) \right) dr \geq \kappa^N \right\},$$

where κ^N is a positive constant satisfying $\lim_{N\to\infty} \kappa^N = \kappa > 0$. As for the macroscopic system, the $(M+1)$-th *local imbalance time* is given by $\tau_{M+1} = \tau_{M+1}^b \wedge \tau_{M+1}^a$, with τ_{M+1}^b and τ_{M+1}^a respectively defined by:

$$\tau_{M+1}^{b} = \inf \left\{ t \geq 0 : \int_0^t \left(\mathbb{1}_{[0,\omega]} \left(A_r^{m_M} \right) - \mathbb{1}_{[0,\omega]} \left(B_r^{m_M} \right) \right) dr \geq \kappa \right\},$$

$$\tau_{M+1}^{a} = \inf \left\{ t \geq 0 : \int_0^t \left(\mathbb{1}_{[0,\omega]} \left(B_r^{m_M} \right) - \mathbb{1}_{[0,\omega]} \left(A_r^{m_M} \right) \right) dr \geq \kappa \right\}.$$

We recall that:

$$\widetilde{\mathcal{Y}}_M^N = \left(\langle Y^{N,m_M^N}, h_1^{\mu_{M,j}} \rangle, \langle Y^{N,m_M^N}, h_2^{\mu_{M,j}} \rangle, j \in \{0, \dots, M\} \right),$$

$$\widetilde{\mathcal{U}}_M = \left(\langle u^{m_M}, h_1^{\mu_{M,j}} \rangle, \langle u^{m_M}, h_2^{\mu_{M,j}} \rangle, j \in \{0, \dots, M\} \right).$$

Using the logistic approximation of the above indicator functions used in the proof of Theorem 6.2.1 as well as the Moore-Osgood theorem to interchange double limits, it can easily be shown that:

$$\widetilde{\mathcal{Y}}_M^N \Rightarrow \widetilde{\mathcal{U}}_M,$$

in $\mathbb{C}\left([s,T], \mathbb{R}^{2(M+1)}\right)$, for any $0 < s \leq T < \infty$. In order to obtain the weak convergence of the mesoscopic local imbalance times to the macroscopic local imbalance times, the crucial point is to make use of the following lemma:

Lemma 6.2.3 *Let $h \in \mathbb{C}^1(\mathbb{R};\mathbb{R}_+)$. Then for any $M \in \{0,\dots,M_T^*\}$, we have:*

$$\int_0^t \mathbb{1}_{[0,\omega]}\left(\langle Y_r^{N,m_M^N}, h\rangle\right) dr \Rightarrow \int_0^t \mathbb{1}_{[0,\omega]}\left(\langle u_r^{m_M}, h\rangle\right) dr,$$

in $\mathbb{C}\left([s,T]\right)$, for any $0 < s \le T < \infty$.

Proof of Lemma 6.2.3: We first define a sequence of continuous functions $(h^\nu)_{\nu\ge1}$ on $\mathbb{R}_+$ which approximates (in the sense of pointwise convergence) the indicator function $\mathbb{1}_{[0,\omega]}$ as $\nu \to \infty$ (as we are dealing with positive quantities, $\mathbb{1}_{[0,\omega]}$ shall refer to the restriction of itself to $\mathbb{R}_+$ by abuse of notation):

$$h^\nu(x) = \begin{cases} 1 \text{ for } x \in [0,\omega[, \\ \nu(\omega - x) + 1 \text{ for } x \in [\omega, \omega + \frac{1}{\nu}], \\ 0 \text{ for } x > \omega + \frac{1}{\nu}. \end{cases}$$

As we shall be examining weak convergence with respect to two parameters N and ν, we reintroduce the relevant subscripts under the notational arrow.

Let $M \in \{0,\dots,M_T^*\}$. Given the pointwise convergence of $(h^\nu)_{\nu\ge1}$ to $\mathbb{1}_{[0,\omega]}$ as $\nu \to \infty$, as well as the continuity of the integral functional, we have:

$$\int_0^t h^\nu\left(\langle Y_r^{N,m_M^N}, h\rangle\right) dr \Rightarrow_{\nu\to\infty} \int_0^t \mathbb{1}_{[0,\omega]}\left(\langle Y_r^{N,m_M^N}, h\rangle\right) dr,$$

in $\mathbb{C}\left([s,T]\right)$, for any $0 < s \le T < \infty$. Furthermore, still using the continuity of the integral functional along with that of h^ν, we know that:

$$\int_0^t h^\nu\left(\langle Y_r^{N,m_M^N}, h\rangle\right) dr \Rightarrow_{N\to\infty} \int_0^t h^\nu\left(\langle u_r^{m_M}, h\rangle\right) dr,$$

in $\mathbb{C}\left([s,T]\right)$, for any $0 < s \le T < \infty$. In view of these two assertions, according to Theorem 3.2 in Billingsley [7], it suffices to establish the following limit in order to prove the lemma:

$$\lim_{\nu\to\infty}\limsup_{N\to\infty}\mathbb{P}\left(\sup_{t\in[0,T]}\left|\Sigma_t^{N,\nu}\right| \ge \alpha\right) = 0,$$

for any $\alpha \ge 0$, where we have defined:

$$\Sigma_t^{N,\nu} = \int_0^t \left(h^\nu\left(\langle Y_r^{N,m_M^N}, h\rangle\right) - \mathbb{1}_{[0,\omega]}\left(\langle Y_r^{N,m_M^N}, h\rangle\right)\right) dr.$$

Let $\alpha \geq 0$. We see that:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]}\left|\Sigma_t^{N,\nu}\right| \geq \alpha\right)\\
\leq\ &\mathbb{P}\left(\sup_{t\in[0,T]}\int_0^t \left|\left(h^\nu\left(\langle Y_r^{N,m_M^N}, h\rangle\right) - \mathbb{1}_{[0,\omega]}\left(\langle Y_r^{N,m_M^N}, h\rangle\right)\right)\right| dr \geq \alpha\right)\\
\leq\ &\mathbb{P}\left(\int_0^T \left|\left(h^\nu\left(\langle Y_r^{N,m_M^N}, h\rangle\right) - \mathbb{1}_{[0,\omega]}\left(\langle Y_r^{N,m_M^N}, h\rangle\right)\right)\right| dr \geq \alpha\right)\\
\leq\ &\mathbb{P}\left(\int_0^T \mathbb{1}_{[\omega,\omega+\frac{1}{\nu}]}\left(\langle Y_r^{N,m_M^N}, h\rangle\right) dr \geq \alpha\right)\\
\leq\ &\frac{1}{\alpha}\mathbb{E}\left(\int_0^T \mathbb{1}_{[\omega,\omega+\frac{1}{\nu}]}\left(\langle Y_r^{N,m_M^N}, h\rangle\right) dr\right)\\
\leq\ &\frac{1}{\alpha}\int_0^T \mathbb{P}\left(\langle Y_r^{N,m_M^N}, h\rangle \in \left[\omega, \omega+\frac{1}{\nu}\right]\right) dr,
\end{aligned}$$

where we have used the fact that $\left|h^\nu(x) - \mathbb{1}_{[0,\omega]}(x)\right| \leq \mathbb{1}_{[\omega,\omega+\frac{1}{\nu}]}(x)$ for any $x \geq 0$ between the second and third inequalities, and Markov's inequality between the third and fourth inequalities. Using the dominated convergence theorem, we therefore know that:

$$\limsup_{N\to\infty}\mathbb{P}\left(\sup_{t\in[0,T]}\left|\Sigma_t^{N,\nu}\right| \geq \alpha\right) \leq \frac{1}{\alpha}\int_0^T \mathbb{P}\left(\langle u_r^{m_M}, h\rangle \in \left[\omega, \omega+\frac{1}{\nu}\right]\right) dr.$$

As $\left(\langle u_r^{m_M}, h\rangle \in \left[\omega, \omega+\frac{1}{\nu}\right]\right)_{\nu\geq 1}$ is a decreasing family of events, we can interchange limit and probability before using another application of the dominated convergence theorem, finally giving us:

$$\lim_{\nu\to\infty}\limsup_{N\to\infty}\mathbb{P}\left(\sup_{t\in[0,T]}\left|\Sigma_t^{N,\nu}\right| \geq \alpha\right) = 0.$$

□

A direct generalisation of Lemma 6.2.3 to the case of the corresponding product space along with the continuous mapping theorem enables us to obtain that:

$$\begin{aligned}
&\left(\int_0^t \mathbb{1}_{[0,\omega]}\left(A_r^{N,m_M^N}\right) dr, \int_0^t \mathbb{1}_{[0,\omega]}\left(B_r^{N,m_M^N}\right) dr\right)\\
\Rightarrow &\left(\int_0^t \mathbb{1}_{[0,\omega]}\left(A_r^{m_M}\right) dr, \int_0^t \mathbb{1}_{[0,\omega]}\left(B_r^{m_M}\right) dr\right),
\end{aligned}$$

in $\mathbb{C}\left([s,T],\mathbb{R}^2\right)$, for any $0 < s \leq T < \infty$. One final application of the continuous mapping theorem gives us the weak convergence of the local imbalance times. As the rest of the full proof is based on a slight adaptation of the arguments presented in the induction proof of Theorem 6.2.1, we do not give any further details here.

Remark 6.2.4 *As in the global order flow imbalance case (see Remark* 6.2.2*), it is possible to normalise the quantities involved in the measure of local order flow imbalance, by defining:*

$$\tilde{\rho}_t^{N,m_M^N} = \frac{\int_0^t \left(\mathbb{1}_{[0,\omega]}\left(B_r^{N,m_M^N}\right) - \mathbb{1}_{[0,\omega]}\left(A_r^{N,m_M^N}\right)\right) dr}{\int_0^t \left(\mathbb{1}_{[0,\omega]}\left(B_r^{N,m_M^N}\right) + \mathbb{1}_{[0,\omega]}\left(A_r^{N,m_M^N}\right)\right) dr},$$

$$\tilde{\rho}_t^{m_M} = \frac{\int_0^t \left(\mathbb{1}_{[0,\omega]}\left(B_r^{m_M}\right) - \mathbb{1}_{[0,\omega]}\left(A_r^{m_M}\right)\right) dr}{\int_0^t \left(\mathbb{1}_{[0,\omega]}\left(B_r^{m_M}\right) + \mathbb{1}_{[0,\omega]}\left(A_r^{m_M}\right)\right) dr}.$$

By contruction of these measures, one should expect even fewer price changes using normalised local order flow imbalance instead of the nonnormalised version, as we are now measuring the proportional imbalance of the volumes around 0. This intuition shall be confirmed in the numerical results presented in the following chapter.

7 Numerical applications

The general aim of this final chapter is to demonstrate the financial applicability as well as analytical tractability of the previously derived mesoscopic and macroscopic systems. We start by giving an overview of the simulation algorithms, before introducing the LOBSTER dataset as well as the subsequent parameter estimation procedure. We finally present numerical illustrations and results by plugging in the estimated parameters into the simulation algorithms.

7.1 Simulation algorithms

Throughout this section, we give a description of the simulation algorithms associated with the mesoscopic and macroscopic systems. We recall that we initially fix an arbitrary finite interval $[0,T]$ and define M_T^* as the total number of mid price changes between 0 and T. In both cases, the general simulation strategy in effect mirrors the construction of the systems exhibited in Section 6.2.1 of Chapter 6, as we need to simulate a total number of M_T^* Funaki-Olla SDE systems/reflected stochastic heat equations, where the dynamics associated with the M-th price change ($M \in \{1,\dots,M_T^*\}$) can only be simulated with the knowledge of the relevant pinning point, i.e. m_M^N and m_M respectively. We now give ourselves an arbitrary $M \in \{0,\dots,M_T^*\}$ as well as an absolute price grid $\{1,\dots,N\}$. We start by recalling that for any $k \in \{1,\dots,N\}$, the dynamics of the mesoscopic system are given by:

$$\begin{aligned} dX_t^{k,m_M^N} &= \gamma\left(X_t^{k-1,m_M^N} + X_t^{k+1,m_M^N} - 2X_t^{k,m_M^N}\right)dt \\ &\quad + \left(c + \lambda^k - \theta^k\right)dt + \sqrt{2\lambda}\,dW_t^{k,M} + dL_t^{X^{k,m_M^N}}, \end{aligned} \tag{7.1}$$

where $W^M = (W^{1,M},\dots,W^{N,M})$ is a N-dimensional Wiener process, where $L_t^{X^{k,m_M^N}}$ is the local time at zero of the process X^{k,m_M^N}, with the pinning convention $X_t^{0,m_M^N} = X_t^{m_M^N,m_M^N} = X_t^{N+1,m_M^N} = 0$, and with initial condition $X_0^{m_M^N} = R_M^N$. At this stage, we do not make any further assumptions on the initial condition so as to keep the simulation framework as general as possible. Note that as our aim here is to simulate the mesoscopic system, we do not need to rescale the drift term as in Chapter 6. The general idea is to make use of a simple Euler-Maruyama discretisation scheme, by first introducing the time grid $\{0,\dots,L\}$, where $L \in \mathbb{N}^*$, the time step $\Delta t = T/L$, and by defining $t_j = j\Delta t$ for any $j \in \{0, \ldots, L\}$. In the nonreflecting case, for any $k \in \{1, \ldots, N\}$, the Euler-Maruyama approximation of (7.1) (without the local time) is expressed by:

$$\begin{aligned} \widetilde{X}^{k,m_M^N}_{t_{j+1}} &= \widetilde{X}^{k,m_M^N}_{t_j} + \gamma \left(\widetilde{X}^{k-1,m_M^N}_{t_j} + \widetilde{X}^{k+1,m_M^N}_{t_j} - 2\widetilde{X}^{k,m_M^N}_{t_j} \right) \Delta t \\ &\quad + \left(c + \lambda^k - \theta^k \right) \Delta t + \sqrt{2\lambda} \left(W^{k,M}_{t_{j+1}} - W^{k,M}_{t_j} \right), \end{aligned} \tag{7.2}$$

with the initial condition $\widetilde{X}^{k,m_M^N}_{t_0} = X^{k,m_M^N}_0$ as well as the pinning condition $\widetilde{X}^{0,m_M^N}_{t_j} = \widetilde{X}^{m_M^N,m_M^N}_{t_j} = \widetilde{X}^{N+1,m_M^N}_{t_j} = 0$ for any $j \in \{0, \ldots, L\}$. Naturally, we need to suppose that the mid m_N^M is known prior to the start of the simulation procedure and that it belongs to the price grid $\{1, \ldots, N\}$. The reflecting term can be dealt with by using the so-called projection scheme (see for example Ding and Zhang [18]), which simply consists in taking the positive part of the right-hand side of (7.2), i.e.:

$$\begin{aligned} \bar{X}^{k,m_M^N}_{t_{j+1}} &= \widetilde{X}^{k,m_M^N}_{t_j} + \gamma \left(\widetilde{X}^{k-1,m_M^N}_{t_j} + \widetilde{X}^{k+1,m_M^N}_{t_j} - 2\widetilde{X}^{k,m_M^N}_{t_j} \right) \Delta t \\ &\quad + \left(c + \lambda^k - \theta^k \right) \Delta t + \sqrt{2\lambda} \left(W^{k,M}_{t_{j+1}} - W^{k,M}_{t_j} \right), \end{aligned} \tag{7.3}$$

with $\widetilde{X}^{k,m_M^N}_{t_{j+1}} = \max(\bar{X}^{k,m_M^N}_{t_{j+1}}, 0)$, $j \in \{0, \ldots, L-1\}$, with the initial condition $\bar{X}^{k,m_M^N}_{t_0} = \widetilde{X}^{k,m_M^N}_{t_0} = X^{k,m_M^N}_0$ and pinning condition $\widetilde{X}^{0,m_M^N}_{t_j} = \widetilde{X}^{m_M^N,m_M^N}_{t_j} = \widetilde{X}^{N+1,m_M^N}_{t_j} = \bar{X}^{0,m_M^N}_{t_j} = \bar{X}^{m_M^N,m_M^N}_{t_j} = \bar{X}^{N+1,m_M^N}_{t_j} = 0$ for any $j \in \{0, \ldots, L\}$. Note that it is also possible to adapt the scheme proposed by Diop in [19] and take the absolute value of the right-hand side of (7.2).

As far as the macroscopic system is concerned, we recall that we have, for any $M \in \{0, \ldots, M_T^*\}$:

$$\frac{\partial u^{m_M}(x,t)}{\partial t} = \frac{\gamma}{\lambda} \frac{\partial^2 u^{m_M}(x,t)}{\partial x^2} + c + \lambda(x) - \theta(x) + \sqrt{2\lambda}\dot{W}^M(x,t) + \eta^M(x,t), \tag{7.4}$$

where $\dot{W}^M$ is a space-time white noise, η^M is the reflecting measure such that $u^{m_M} \geq 0$, $d\eta^M \geq 0$, $\int u^{m_M} d\eta^M = 0$, with Dirichlet boundary conditions $u^{m_M}(0,t) = u^{m_M}(m_M, t) = u^{m_M}(1,t) = 0$ for all $t \in [0,T]$, and with initial condition $u^{m_M}(x,0) = u_M(x)$ for all $x \in [0,1]$. Once again, no additional assumptions are made on the initial condition at this point. We first explain how to simulate the above stochastic heat equation in the case without reflection. We consider the space grid $\{1, \ldots, N\}$ (which corresponds to the absolute price grid) and the time grid $\{0, \ldots, L\}$, and introduce the corresponding space and time steps given by $\Delta x = 1/N$ and $\Delta t = T/L$. We assume that the mid m_M associated with this system is known in advance and satisfies $m_M N \in \{1, \ldots, N\}$. For any $i \in \{1, \ldots, N\}$ and $j \in \{0, \ldots, L\}$, the forward or explicit Euler finite difference scheme is given by:

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{\gamma}{\lambda} \frac{u_{i-1}^j + u_{i+1}^j - 2u_i^j}{(\Delta x)^2} + c + \lambda_i - \theta_i + \sqrt{\frac{2\lambda}{\Delta x \Delta t}} Z_i^j, \tag{7.5}$$

where u_i^j is the approximation of $u(i\Delta x, j\Delta t)$, $\lambda_i = \lambda(i\Delta x)$, $\theta_i = \theta(i\Delta x)$, and where Z_i^j is an i.i.d family of unit normal random variables. The pinning condition is given by $u_0^j = u_{m_M N}^j = u_{N+1}^j = 0$ for all $j \in \{0, \dots, L\}$, and the initial condition by $u_i^0 = u_M(i\Delta x)$ for all $i \in \{1, \dots, N\}$. Rearranging the terms, the scheme can be written as:

$$u_i^{j+1} = u_i^j + \frac{\gamma}{\lambda} \frac{u_{i-1}^j + u_{i+1}^j - 2u_i^j}{(\Delta x)^2} \Delta t + (c + \lambda_i - \theta_i)\Delta t + \sqrt{\frac{2\lambda \Delta t}{\Delta x}} Z_i^j. \tag{7.6}$$

In order to consider the case with reflection, we can either use a projection scheme analogously to the mesoscopic case (7.3), or simply take the absolute value of the right-hand side, i.e.:

$$u_i^{j+1} = \left| u_i^j + \frac{\gamma}{\lambda} \frac{u_{i-1}^j + u_{i+1}^j - 2u_i^j}{(\Delta x)^2} \Delta t + (c + \lambda_i - \theta_i)\Delta t + \sqrt{\frac{2\lambda \Delta t}{\Delta x}} Z_i^j \right|, \tag{7.7}$$

with the same pinning and initial conditions as those associated with (7.5). Unlike in the mesoscopic case, the convergence of this scheme has not been studied thus far, and we stress that we only use it here from a heuristic perspective for the sole purpose of numerical illustrations.

Before each imbalance time, it is imperative to verify that the new mid effectively belongs to the price grid, i.e. $m_M^N \in \{1, \dots, N\}$ and $m_M \in \{1/N, \dots, 1\}$, and if this turns out not to be the case, the mid update is subsequently ignored. For example, in the mesoscopic case, if the mid is located at point 1 (respectively N), we ensure that a price decrease (respectively increase) is no longer possible, and continue the simulation of the system until a price increase (respectively decrease) occurs or until we reach the end of the time grid. We also need to ensure that the resampled volumes after each price change (whether we choose the invariant measure approach or a more general reinitialisation function) lie within the imbalance bounds (given by the chosen price evolution signal and the corresponding imbalance time definitions) so as to avoid a potential series of instantaneous price decreases (similar in nature to a flash crash) or increases. This is simply done via a rejection method whereby resampled volumes outside the relevant bounds are discarded until appropriate ones are obtained.

We additionally define Λ and Θ to be the vectors containing the idiosyncratic order arrival and cancellation rates, where $\Lambda = (\lambda(1), \dots, \lambda(N))$, $\Theta = (\theta(1), \dots, \theta(N))$ in the mesoscopic case and $\Lambda = (\lambda(\Delta x), \dots, \lambda(N\Delta x))$, $\Theta = (\theta(\Delta x), \dots, \theta(N\Delta x))$ in the macroscopic case. As explained in Section 6.1.1 of Chapter 6, these rates are actually expressed in terms of the distance to the mid, i.e. $\lambda(i)$ (respectively $\lambda(i\Delta x)$) corresponds to the idiosyncratic order arrival rate at a distance of i (respectively $i\Delta x$) from the mesoscopic (respectively macroscopic) mid, for any $i \in \{1, \dots, N\}$. We also recall that market orders are assumed to be included in cancellation rates. As the mesoscopic and macroscopic systems are simulated via the same general architecture, we present the algorithm using pseudo-code which covers both cases. We use global order flow imbalance as the mid evolution signal, although the algorithm presentation can readily be adapted to the case of local order flow imbalance.

Algorithm High-frequency limit order book simulation

Require: c, γ, δ, ϵ, λ, Λ, Θ, L, N, T (model and simulation parameters).

1: Define three-dimensional LOB array A;
2: Define mid array m and set $m(0) := \text{floor}(N/2)$;
3: Define imbalance times array t and set $t(0) := 0$;
4: Define accumulated imbalance times array $\bar{t}$ and set $\bar{t}(0) := 0$;
5: Define two-dimensional time-ordered LOB array $\bar{A}$;
6: Define price change number index M and initialise $M := 0$;
7: **while** $\bar{t}(M) \leq L$ **do**
8:     Define bid sum Σ_{bid} and ask sum Σ_{ask} and initialise $\Sigma_{bid} = \Sigma_{ask} := 0$;
9:     **for** $i \in \{1, \ldots, N\}$ **do**
10:         Load $A(i, 0, M)$ with desired initial LOB profile;
11:     **end for**
12:     **for** $j \in \{1, \ldots, L\}$ **do**
13:         Enforce pinning $A(0, j, M) = A(m(M), j, M) = A(N + 1, j, M) := 0$;
14:         **for** $i \in \{1, \ldots, N\} \setminus \{m(M)\}$ **do**
15:             Load $A(i, j, M)$ using approximation scheme (7.3) or (7.7);
16:         **end for**
17:         **for** $k \in \{1, \ldots, \epsilon N\}$ **do**
18:             Increment bid sum $\Sigma_{bid} += A(m(M) - k, j, M)$;
19:             Increment ask sum $\Sigma_{ask} += A(m(M) + k, j, M)$;
20:         **end for**
21:         **if** $\Sigma_{bid} - \Sigma_{ask} < -\delta$ & $m(M) - \epsilon N > 0$ **then**
22:             Set $m(M + 1) := m(M) - \epsilon N$ and $t(M + 1) := j$;
23:             **break**
24:         **else if** $\Sigma_{bid} - \Sigma_{ask} > \delta$ & $m(M) + \epsilon N < N + 1$ **then**
25:             Set $m(M + 1) := m(M) + \epsilon N$ and $t(M + 1) := j$;
26:             **break**
27:         **end if**
28:     **end for**
29:     Set $\bar{t}(M + 1) := \bar{t}(M) + t(M + 1)$;
30:     **for** $i \in \{0, \ldots, N + 1\}$ **do**
31:         **for** $j \in \{\bar{t}(M), \ldots, \bar{t}(M + 1) - 1\}$ **do**
32:             Set $\bar{A}(i, j) := A(i, j - \bar{t}(M), M)$;
33:         **end for**
34:     **end for**
35:     Increment price change number index $M += 1$;
36: **end while**

We now give a few comments about the algorithm. To start with, we recall that the initial book profile generation between lines 9 and 11 can either be done with the invariant measure of the mesoscopic/macroscopic system or with a more general specification as presented in the first part of Section 6.2.3 of Chapter 6. This is left to the discretion of the simulation user depending on the context of application.

Moreover, it is important to underline that for any $i \in \{0, \dots, N+1\}$, $j \in \{0, \dots, L\}$ and $M \in \{0, \dots, M_T^*\}$, $A(i,j,M)$ corresponds to the approximation of $X_{j\Delta t}^{i,m_M^N}$ or $u^{m_M}(i\Delta x, j\Delta t)$, and that the so-called time-ordered array $\bar{A}$ simply enables us to piece together the order book process, as required by the definition of $g_{M_T^*}$:

$$\begin{aligned}\bar{A}(i,j) &= g_{M_T^*}\left(A(i,j,0), \dots, A(i,j,M_T^*-1), t(1), \dots, t(M_T^*)\right)(j) \\ &= \sum_{M=1}^{M_T^*} A(i, j - \bar{t}(M-1), M) \mathbb{1}_{\{\bar{t}(M-1) \leq j < \bar{t}(M)\}},\end{aligned}$$

where the imbalance times and accumulated imbalance times are naturally related by $\bar{t}(M) = \sum_{l=0}^{M} t(l)$ for any $M \in \{0, \dots, M_T^*\}$.

On lines 23 and 26, the presence of the **break** commands ensure that once an imbalance has been detected, the approximation scheme associated with the current mid index M is halted and we move on to the generation of the scheme associated with the next mid index $M+1$. Simply put, this means that we do not simulate M_T^* sets of mesoscopic/macroscopic systems on the time interval $[0, T]$, but only until their respective first imbalance times occur. This is fundamental in terms of computational time complexity (which is actually $\mathcal{O}(NLM_T^*)$, by virtue of the bottom-up tabulation approach employed), above all if one wishes to make use of this simulation procedure expecting a large number of total price changes M_T^*. Finally, the general structure of the algorithm makes it flexible enough to accommodate different price evolution signals, such as local order flow imbalance introduced in the second part of Section 6.2.3 of Chapter 6, by simply making the necessary adaptations within the **if** conditions on lines 21 and 24.

7.2 Dataset description and parameter estimation

Before proceeding to the actual simulation of the mesoscopic and macroscopic systems, we propose an estimation procedure for the idiosyncratic order arrival and cancellation rates (market orders being included in the latter by assumption) represented by the previously defined vectors Λ and Θ, where we recall that $\lambda(i)$ (respectively $\lambda(i\Delta x)$) and $\theta(i)$ (respectively $\theta(i\Delta x)$) correspond to the arrival and cancellation rates at a distance of i (respectively $i\Delta x$) from the mesoscopic (respectively macroscopic) mid, for any $i \in \{1, \dots, N\}$.

The data we have at our disposal originates from the LOBSTER (Limit Order Book System, The Efficient Reconstructor) database project initiated by the Humboldt University of Berlin, which gives access to reconstructed limit order book data for all NASDAQ traded stocks between June 2007 up to the present day. For each trading day of a given ticker, LOBSTER generates two distinct files. On the one hand, a *message* file, which lists indicators for the different kinds of events which cause an update of the book (limit order arrivals and cancellations, executions or market orders, trading halts) within a prespecified price range. On the other hand, an *order book* file, which displays the evolution of the book up to a chosen number of price levels (which can go up to 200, depending on the selected ticker). Order book events are timestamped according to seconds after midnight, and the decimal precision available ranges from milliseconds to nanoseconds.

Our sample consists of data from five highly liquid stocks, namely Amazon (AMZN), Apple (AAPL), Google (GOOG), Intel (INTC) and Microsoft (MSFT), and covers the 10 best levels on each side of the book on the trading day (i.e. from 09:30 to 16:00) of June 21 2012. In the following table, we present the total number of limit orders, cancellations and market orders associated with these five stocks:

Ticker	Limit orders	Cancellations	Market orders
AMZN	131954	126375	11419
AAPL	191015	174386	34990
GOOG	71258	64980	11678
INTC	304790	286767	32483
MSFT	329566	305785	33414

Table 7.1: Total number of limit orders, cancellations and market orders on June 21 2012 for AMZN, AAPL, GOOG, INTC and MSFT.

Naturally, we observe that limit orders and cancellations overwhelmingly dominate market orders in all five cases, which is consistent with the separation of time scales used throughout our diffusion approximation results in Part I. Now and henceforth, for the purpose of our estimation procedure, we restrict ourselves to the time period between 10:00 and 15:30 so as to minimise any possible trading activity outliers due to market opening and close. As the spread is assumed to be constant in our mesoscopic and macroscopic models, we discard limit orders arriving inside the spread in the sample and estimate limit order arrival rates at a distance of $i-1$ from the *same* best quote, where $i \in \{1, \ldots, 10\}$. More precisely, for each ticker, limit order arrival rates $\hat{\lambda}(i)$ at a distance of $i-1$ from the same best quote are simply estimated as the ratio of the number of limit orders which arrived at a distance of $i-1$ from the same best quote over the total trading time of the sample (330 minutes). We proceed in a similar manner for cancellation rates $\hat{\theta}(i)$ (which we recall include market orders in our approach), as we do not adopt the traditional convention of considering cancellation rates to be proportional to the number of outstanding orders at any given level. We then take the average of the arrival and cancellation rates over the five sample tickers, and obtain the following estimates:

i	1	2	3	4	5	6	7	8	9	10
$\hat{\lambda}(i)$	1.56	1.38	0.98	0.85	0.89	0.74	0.72	0.73	0.71	0.69
$\hat{\theta}(i)$	1.67	1.31	0.75	0.72	0.57	0.51	0.53	0.41	0.33	0.24

Table 7.2: Limit order arrival and cancellation rates estimates.

As we require additional values for our simulations, we follow Zovko and Farmer [63] and Cont et al. [15] by fitting a power law function for arrivals as well as for cancellations, where the parameters of interest are deduced via least squares:

$$\min_{A,B} \sum_{i=1}^{10} \left(\hat{\lambda}(i) - \frac{A}{i^B} \right)^2,$$

$$\min_{C,D} \sum_{i=1}^{10} \left(\hat{\theta}(i) - \frac{C}{i^D} \right)^2.$$

The power law parameter estimates are given in the following table:

A	B	C	D
1.58	0.38	1.94	0.78

Table 7.3: Power law parameter estimates for arrival and cancellation rates.

Given these results, we can now provide estimates for the vectors of idiosyncratic limit order arrival and cancellation rates Λ and Θ, which are required inputs in the mesoscopic/macroscopic simulation algorithm.

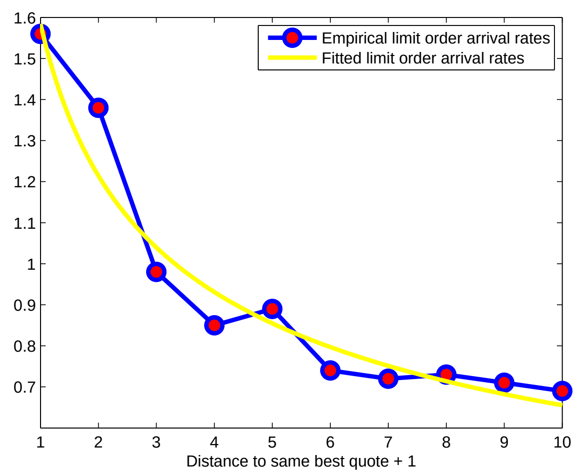


Figure 7.1: Empirical vs fitted limit order arrival rates.

We stress that these estimates correspond to the idiosyncratic or level-dependent arrival and cancellation rates presented in Section 6.1.1 of Chapter 6. We recall that the incorporation of level-dependent parameters into our model could only be done via the drift component of the mesoscopic/macroscopic system, as the weak convergence result assumed a constant diffusion coefficient. One potential limitation of this procedure is that we have to arbitrarily fix the systemic arrival

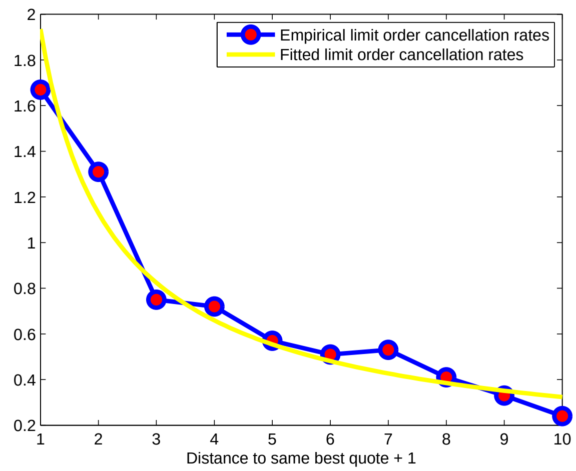


Figure 7.2: Empirical vs fitted limit order cancellation rates.

and cancellation rate λ as well as the nearest neighbour component γ which appear in the decomposition presented in Section 6.1.1 of Chapter 6, as there is no seemingly obvious way to estimate these values from the data at our disposal.

Remark 7.2.1 *Having estimated these rates, it is legitimate to ask oneself whether the Poisson assumption on arrival and cancellation rates is actually validated by the data. One fundamental feature of Poisson models is that the variance of the arrival count within each time period is equal to its mean over the corresponding period. When the sample variance is observed to be greater that the sample mean (which is in effect the arrival rate), the data is said to be overdispersed by a factor given by the so-called overdispersion parameter (OP), defined as the ratio of the sample variance over the sample mean. One can also compute the Pearson dispersion statistic (PDS), given by the ratio of the classic Pearson chi-squared statistic over the residual degrees of freedom. When the PDS is observed to be greater than 1, the model is likely to be Poisson overdispersed. In Table 7.4 below, we compute the sample mean, sample variance, OP and PDS of limit order arrivals for the 10 available levels of the MSFT ticker (and using the same day/time period as previously):*

i	1	2	3	4	5	6	7	8	9	10
Sample mean	2.89	2.44	2.18	1.80	1.11	0.91	0.75	0.83	0.79	0.65
Sample variance	12.56	9.02	7.68	7.09	4.12	2.79	2.33	2.18	1.71	1.39
OP	4.35	3.70	3.52	3.94	3.71	3.07	3.11	2.63	2.16	2.14
PDS	5.62	4.13	3.95	4.82	4.17	3.71	3.53	3.21	3.03	2.56

Table 7.4: Evidence of overdispersion for MSFT limit order arrivals.

These results suggest that the assumption of Poisson arrivals is not supported by the *data, and consequently warrant the need to consider more elaborate models where arrival rates are assumed to be stochastic.*

7.3 Illustrations and results

7.3.1 Simulation of the mesoscopic system

Using the previously obtained limit order arrival and cancellation rates estimates, we first simulate the mesoscopic system using the algorithm presented in Section 7.1. We use global order flow imbalance as the price evolution signal and fix the following model parameters:

c	γ	δ	ϵ	λ	L	N	T
-0.5	1	0.1	0.01	1	1000	100	1

Table 7.5: Model parameters for the mesoscopic system simulation.

In this case, the resampling after a price change is carried out via a discrete-space analogue of the reinitialisation function constructed in Remark 6.2.3 of Chapter 6. As for the initial profile at system initialisation, we simply make use of a deterministic sine curve. We first introduce a three-dimensional representation of the simulation, where the general evolution of the book in both time and space can be visualised:

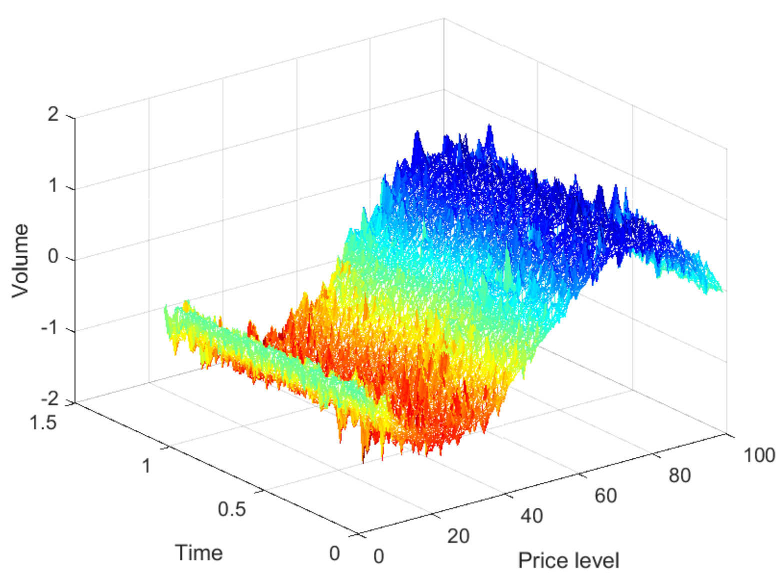


Figure 7.3: Mesoscopic system simulation.

The evolution of the price over time is plotted in the following figure:

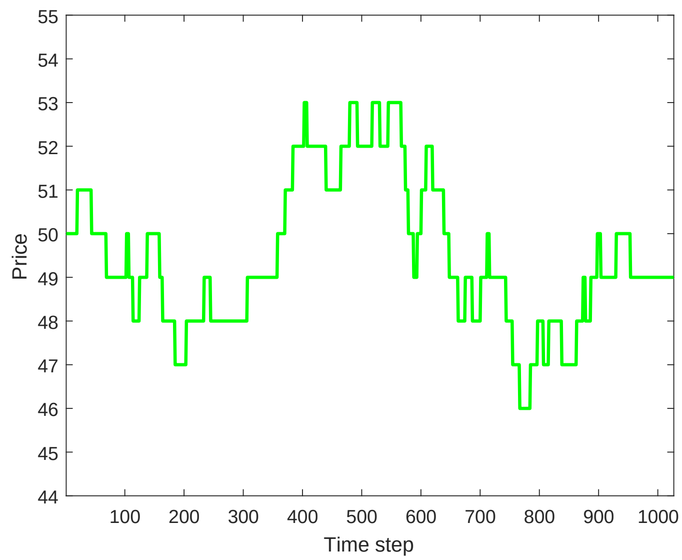


Figure 7.4: Mesoscopic system price evolution.

In order to verify the absence of autocorrelation of price returns empirically observed across limit order data, we first transform the price series into centred log-returns before plotting the corresponding sample autocorrelation function:

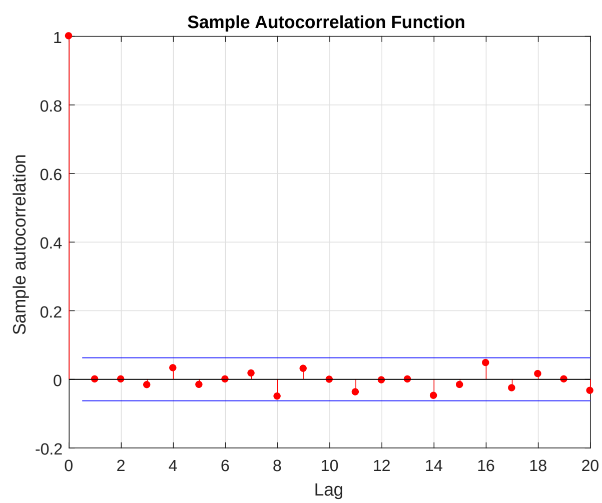


Figure 7.5: Sample autocorrelation of centred log-returns of the mesoscopic system.

As expected, the sample autocorrelation function lies within the significance bounds from the first lag onwards. We conduct a simple Ljung-Box test to further inspect the absence of autocorrelation (null hypothesis) between the first and the l-th lag, for different values of l:

l	4	8	12	16	20
P-value	0.68	0.69	0.62	0.84	0.80

Table 7.6: Ljung-Box test P-values for the series of mesoscopic centred log-returns.

The P-values show that there is little evidence of non-zero autocorrelation of centred log-returns, thus validating this basic albeit fundamental stylised fact. Finally, we present a static snapshot of the book, at time step 785 of the simulation:

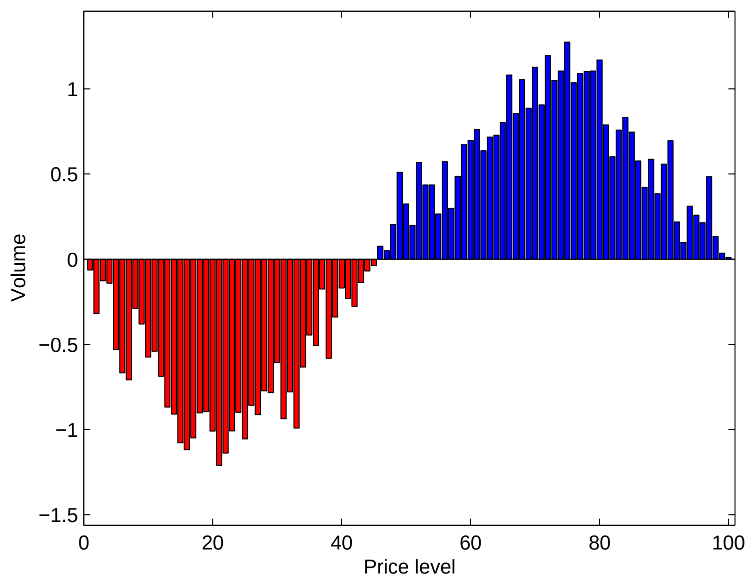


Figure 7.6: Static snapshot of the mesoscopic system at time step 785.

7.3.2 Simulation of the macroscopic system

We now move on to the simulation of the macroscopic system using the same algorithm presented in Section 7.1. Once again, we consider global order flow imbalance as the price evolution signal and fix the following model parameters:

c	γ	δ	ϵ	λ	L	N	T
-0.5	1	0.1	0.04	1	2000	25	1

Table 7.7: Model parameters for the macroscopic system simulation.

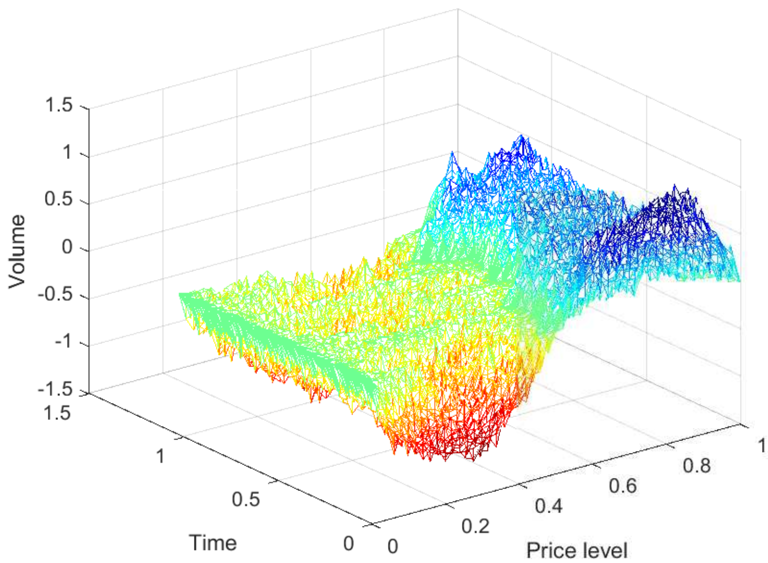


Figure 7.7: Macroscopic system simulation.

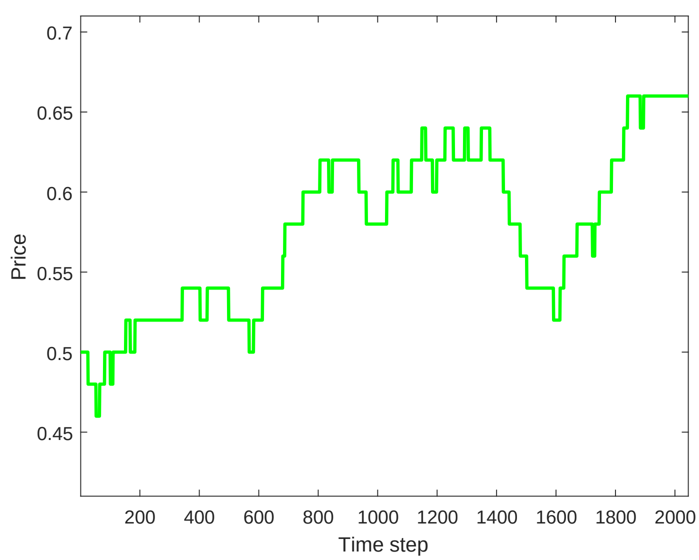


Figure 7.8: Macroscopic system price evolution.

As in the mesoscopic case, we use general reinitialisations following a price change using the space shift function presented in Remark 6.2.3 of Chapter 6.

We also compute the centred log-returns and plot the sample autocorrelation function:

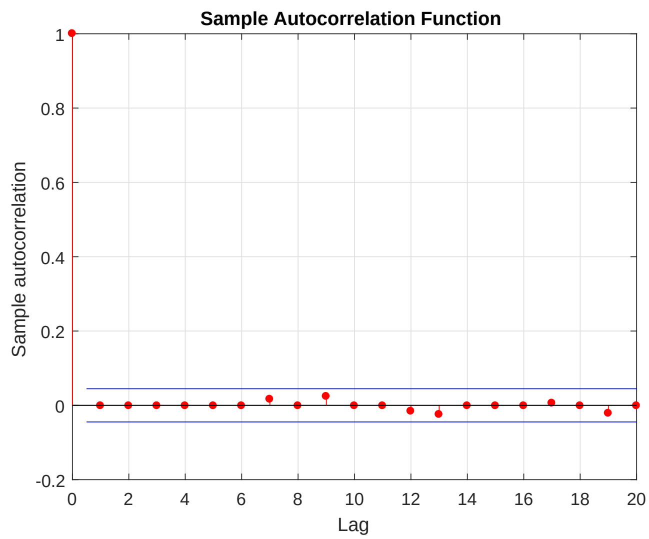


Figure 7.9: Sample autocorrelation of centred log-returns of the macroscopic system.

We carry out another Ljung-Box test to validate the absence of autocorrelation between the first and the l-th lag, and report the corresponding P-values in the following table:

l	4	8	12	16	20
P-value	0.49	0.46	0.57	0.66	0.81

Table 7.8: Ljung-Box test P-values for the series of macroscopic centred log-returns.

These results also confirm that the null hypothesis of absence of autocorrelation of centred log-returns cannot be rejected in the macroscopic case.

Moving on, the two following figures present static snapshots of the system at two different time steps:

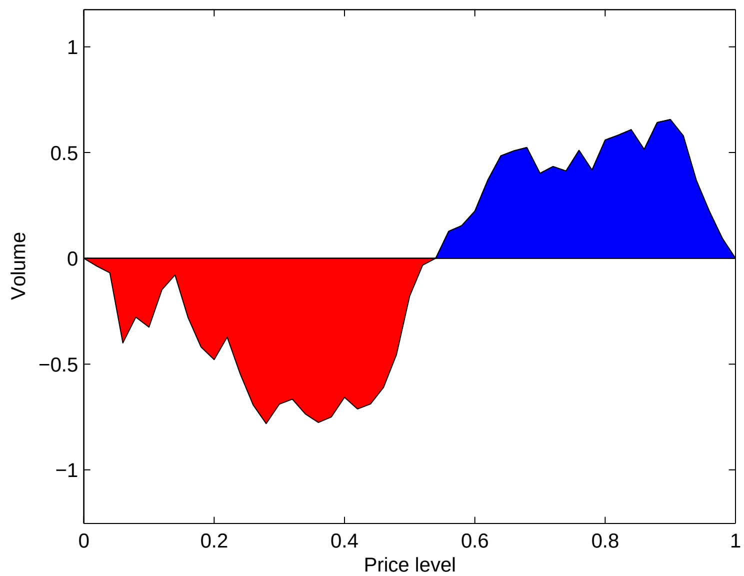


Figure 7.10: Static snapshot of the macroscopic system at time step 705.

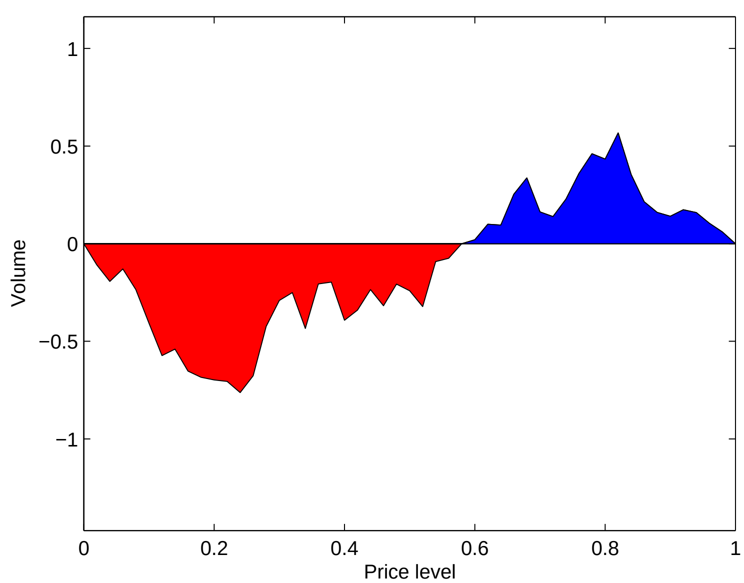


Figure 7.11: Static snapshot of the macroscopic system at time step 1745.

As these simulations have been conducted using global order flow imbalance as the price evolution signal, *ceteris paribus*, it is of great interest to compare the total number of price changes using the other signals presented in Chapter 6, namely normalised global order flow imbalance (see Remark 6.2.2) local order flow imbalance (Section 6.2.3), and normalised local order flow imbalance (Remark 6.2.4). The comparison results are presented in the following table:

Price evolution signal	Number of price changes
Global order flow imbalance	52
Normalised global order flow imbalance	37
Local order flow imbalance	16
Normalised local order flow imbalance	11

Table 7.9: Total number of price changes of the macroscopic system using different price evolution signals.

The above figures showcase two fundamental properties of these price evolution signals. On the one hand, the number of price changes in the local case is roughly equal to a third of those observed in the global case. On the other hand, normalising both global and local order flow imbalance reduces the number of price changes by approximately one quarter. In terms of applications, these observations precisely show to what extent it is necessary to tailor this simulation procedure depending on the desired time scale of analysis but also the speed of trading (i.e. high or ultra high-frequency) of the simulation user.

Finally, as we recall that the values of λ and γ are difficult to estimate from order book data, we investigate their effect on the total number of price changes in the macroscopic case, using global order flow imbalance as the price evolution signal:

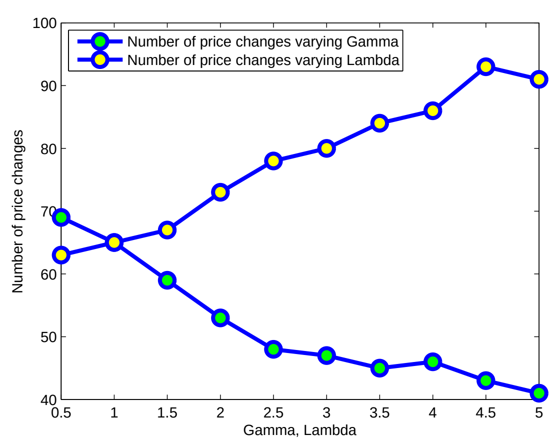


Figure 7.12: Effect of γ and λ on the number of price changes.

Given the form of the dynamics (7.4) we are simulating, keeping λ constant, it should be expected that a higher value of γ (the nearest neighbour component) has a more intense smoothing effect on the solution, thus triggering fewer price changes. Conversely, keeping γ constant, one should reasonably expect a higher value of λ (which appears in the diffusion coefficient but also in the denominator of the smoothing coefficient γ/λ) to have the opposite effect and make the solution more irregular and therefore entail more price changes. We also notice that by setting $\lambda = \gamma$, it is possible to reduce the dimensionality of the set of model parameters, as the smoothing coefficient is now equal to 1.

8 Concluding considerations

8.1 Summary of main results and contributions

We first summarise the approach which has been followed here in order to bridge the gap between microscopic, mesoscopic and macroscopic models of limit order books. In the first part of the thesis, we provided a thorough examination of the link between microscopic and mesoscopic models, proving several weak convergence results of discrete-space order book processes to more analytically tractable continuous-space jump diffusions. Using the theoretical tools introduced in Chapter 2, this step was initially carried out in Chapter 3 in a reduced-form setting, before being extended to the multidimensional case, with the relative price grid and absolute price grid diffusion approximations established in Chapter 4. Throughout the second part of the thesis, concerned with the transition between mesoscopic and macroscopic models, we first presented some fundamental results on reflected SPDEs and stochastic interface models in Chapter 5. Having proven a refined weak convergence result related to the Funaki-Olla interface model, we were then able to recontextualise the absolute price grid diffusion approximation of Chapter 4 and use it as the starting point of this refinement. This step was done in Chapter 6 and enabled us to obtain a first SPDE limit of the order book in a static setting, i.e. without any price movements over time. We then showed how the price could be exogenised, more precisely by splitting the time interval of interest into smaller periods during which the price remains constant. This subsequently gave rise to the SPDE limit in a fully dynamic framework. We finally presented some numerical applications in Chapter 7, notably the simulation of the mesoscopic and macroscopic systems using limit order arrival and cancellation rates estimated from high-frequency data.

In terms of contributions with respect to the existing zero-intelligence limit order book literature, the results obtained in this thesis provide a novel yet analytically tractable description of the behaviour of the limit order book in a high to ultra high-frequency setting. To the best of our knowledge, this is the first time reflected SPDEs have been involved in the modeling procedure of limit order books. As mentioned in the Introduction, other limit order book models making use of SPDEs are based upon completely different approaches. On the one hand, within the free boundary context proposed by Sowers and Zheng [51] and Müller [45], the price is by construction a continuous function of time. Taking into consideration the empirical observations reported by Gai et al. [25] according to which a rise in the speed of trading from microseconds to nanoseconds increases the order cancellation to execution ratio from 26:1 to 32:1, it can be argued that it is more appropriate to model the limit order book with nontrivial time intervals during which the price remains constant. On the other hand, in terms of end product and analytical tractability, it is far less computationally costly to simulate the macroscopic system presented in Chapter 6 in comparison with a free boundary problem which would require quasi-continuous resampling from a new SPDE as the price (i.e. boundary) continuously evolves over time. As shown in Chapter 7, the mesoscopic and macroscopic limits established here can be readily simulated using basic discretisation schemes. The resulting simulations can then be used as a market simulator for short-term price prediction or as an input for optimal execution problems, depending on the required field of application. Finally, other SPDE limits such as those obtained by Bayer et al. [6] involve unreflected SPDEs, which do not prevent volumes from becoming negative. By contrast, using and refining the weak convergence results derived from stochastic interface models, the constraint of positive volumes is ideally handled by reflected SPDEs without any excessive theoretical cost.

8.2 Open questions and potential refinements for future research

The first natural extension which comes to mind concerns an even further refinement of the Funaki-Olla convergence result. To start with, with a view to obtaining a more realistic description of the behaviour of the limit order book, extending Theorem 5.2.4 in Chapter 5 to the case of a space-dependent diffusion coefficient $\sigma(x)$ would enable us to avoid the order arrival and cancellation rate decompositions described at the beginning of Chapter 6. In [62], Zhang recently introduced an approximation/discretisation scheme associated with reflected SPDEs in the case of a general diffusion coefficient (depending on space but also on the solution itself). An adaptation of this result to our current order book setting could therefore prove to be most useful. Moreover, generalising Theorem 5.2.4 to the case of the semi-infinite space interval $[0, \infty[$ with the same pinning conditions at 0 and at the mid and with exponential decay around infinity could potentially lead to a more accurate description of the evolution of the limit order book over longer time intervals. Broadly speaking, the key ingredient at the heart of this extension to a semi-infinite interval is likely to be based on the weak convergence of the associated invariant measures, which remains an open question in the field of sochastic interface models at this point in time.

Another refinement of particular interest concerns the use of the notion of infinite-dimensional occupation density as a new price evolution signal. As explained in Zambotti [58], the positive random measure $\eta(dx, [0, t])$ used in reflected SPDEs is absolutely continuous with respect to the Lebesgue measure dx on $]0, 1[$, i.e. $\eta(dx, [0, t]) = \eta(x, [0, t])dx$, and the *occupation density* is characterised as a family of additive functionals of u satisfying, for all $t \geq 0$:

$$\eta(x,[0,t]) = \lim_{\epsilon \to 0} \frac{3}{4\epsilon^3} \int_0^t \mathbb{1}_{[0,\epsilon]}(u(x,s))ds,$$

where the equality holds in probability. We recall that in Section 6.2.3 of Chapter 6, we used what we referred to as an *ad hoc* occupation measure around 0 to describe local order flow imbalance, in contrast to the *official* occupation measure around 0 which is given by η. The reason for this artificial construction is that there are no established results regarding the weak convergence of the local times which appear in the mesoscopic system to η as we move into a continuous-space setting. Establishing such a result would enable one to obtain a more precise measure of local order flow imbalance, at least from a theoretical perspective. As simulations would naturally require an approximation of the previously introduced expression of η, the potential benefits of such a result from a numerical viewpoint remain to be seen.

Finally, observing that the macroscopic system obtained in Chapter 6 and simulated in Chapter 7 can be perceived as a family indexed by the fixed model parameter ϵ, it could be interesting to examine the limiting behaviour of the system as $\epsilon \to 0$. In particular, it seems legitimate to ask oneself how this limit would relate to the stochastic two-phase Stefan problem (see for example Sowers and Zheng [51]), where the evolution of the continuous-time boundary (i.e. the price in this case) depends on a gradient-type condition which shares some heuristic similarities with the price evolution signals used throughout the second part of this thesis.

Appendix

Proof of Theorem 4.2.1

- step 1: *convergence of the sequence of generators*

Let $f \in \mathrm{Dom}(A^N)$, and define the following sequence of functions f_n on E^N:

$$f_n(z_1, \dots, z_N) = f(u_n z_1, \dots, u_n z_N).$$

We need to show that:

$$\lim_{n \to +\infty} \sup_{z \in E_n^N} |A_n^N f_n(z) - A^N f(z)| = 0.$$

As in the previous case, we consider the operators Δ^1 and Δ^2 respectively defined by:

$$(\Delta^1 \phi)(m) = (\phi_{m-1} - \phi_m)^+ + (\phi_{m+1} - \phi_m)^+,$$
$$(\Delta^2 \phi)(m) = (\phi_m - \phi_{m-1})^+ + (\phi_m - \phi_{m+1})^+,$$

for $m \in \{1, \dots, N\}$ and any vector $\phi \in \mathbb{R}^N$, with the convention $\phi_0 = \phi_{N+1} = 0$.

We now write:

$$\begin{aligned}
A_n^N f_n(z) \quad = \quad & \sum_{m=1}^{b(z)} n\lambda_n^{b(z)-m} \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=1}^{b(z)} n\theta_n^{b(z)-m} \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{z_m \le -\frac{1}{\sqrt{n}}\}} \\
+ \quad & \sum_{m=1}^{b(z)} \sqrt{n}\gamma^{b(z)-m} (\Delta^2 z)(m) \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=1}^{b(z)} \sqrt{n}\gamma^{b(z)-m} (\Delta^1 z)(m) \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right) \\
+ \quad & \sum_{m=a(z)}^{N} n\lambda_n^{m-a(z)} \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m=a(z)}^{N} n\theta_n^{m-a(z)} \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{z_m \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sum_{m=a(z)}^{N} \sqrt{n}\gamma^{m-a(z)} (\Delta^1 z)(m) \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right) \\
&+ \sum_{m=a(z)}^{N} \sqrt{n}\gamma^{m-a(z)} (\Delta^2 z)(m) \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_m} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_m^2} + o\left(\frac{1}{n}\right) \right) \\
&+ \sqrt{n}\mu \left(\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_{b(z)}} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_{b(z)}^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{z_{b(z)} \leq -\frac{1}{\sqrt{n}}\}} \\
&+ \sqrt{n}\mu \left(-\frac{1}{\sqrt{n}} \frac{\partial f_n}{\partial z_{a(z)}} + \frac{1}{2n} \frac{\partial^2 f_n}{\partial z_{a(z)}^2} + o\left(\frac{1}{n}\right) \right) \mathbb{1}_{\{z_{a(z)} \geq \frac{1}{\sqrt{n}}\}} \\
&+ \sqrt{n}\mu \left(\left(f_n(z^b) - f_n(z)\right) \mathbb{1}_{\{z_{b(z)}=0^-\}} + \left(f_n(z^a) - f_n(z)\right) \mathbb{1}_{\{z_{a(z)}=0^+\}} \right).
\end{aligned}
$$

Once again, it is useful to introduce the discrete Laplace operator Δ given by:

$$
\begin{aligned}
(\Delta\phi)(m) &= (\Delta^1\phi)(m) - (\Delta^2\phi)(m) \\
&= \phi_{m-1} + \phi_{m+1} - 2\phi_m,
\end{aligned}
$$

for $m \in \{1, \ldots, N\}$ and any vector $\phi \in \mathbb{R}^N$, with the convention $\phi_0 = \phi_{N+1} = 0$. Rearranging the previous expression of the generator, we have:

$$
\begin{aligned}
A_n^N f_n(z) \quad = \quad & \frac{1}{2} \sum_{m=1}^{b(z)} \left(\lambda_n^{b(z)-m} + \theta_n^{b(z)-m} \mathbb{1}_{\{z_m \leq -\frac{1}{\sqrt{n}}\}} \right) \frac{\partial^2 f_n}{\partial z_m^2} \\
+ \; & \frac{1}{2} \sum_{m=a(z)}^{N} \left(\lambda_n^{m-a(z)} + \theta_n^{m-a(z)} \mathbb{1}_{\{z_m \geq \frac{1}{\sqrt{n}}\}} \right) \frac{\partial^2 f_n}{\partial z_m^2} \\
+ \; & \left(\sqrt{n} \left(\theta_n^0 \mathbb{1}_{\{z_{b(z)} \leq -\frac{1}{\sqrt{n}}\}} - \lambda_n^0 \right) + \mu \mathbb{1}_{\{z_{b(z)} \leq -\frac{1}{\sqrt{n}}\}} \right) \frac{\partial f_n}{\partial z_{b(z)}} \\
+ \; & \left(\sqrt{n} \left(\lambda_n^0 - \theta_n^0 \mathbb{1}_{\{z_{a(z)} \geq \frac{1}{\sqrt{n}}\}} \right) - \mu \mathbb{1}_{\{z_{a(z)} \geq \frac{1}{\sqrt{n}}\}} \right) \frac{\partial f_n}{\partial z_{a(z)}} \\
+ \; & \sum_{m=1}^{b(z)-1} \sqrt{n} \left(\theta_n^{b(z)-m} \mathbb{1}_{\{z_m \leq -\frac{1}{\sqrt{n}}\}} - \lambda_n^{b(z)-m} \right) \frac{\partial f_n}{\partial z_m} \\
+ \; & \sum_{m=a(z)+1}^{N} \sqrt{n} \left(\lambda_n^{m-a(z)} - \theta_n^{m-a(z)} \mathbb{1}_{\{z_m \geq \frac{1}{\sqrt{n}}\}} \right) \frac{\partial f_n}{\partial z_m} \\
+ \; & \sum_{m=1}^{b(z)} \gamma^{b(z)-m} (\Delta z)(m) \frac{\partial f_n}{\partial z_m} + \sum_{m=a(z)}^{N} \gamma^{m-a(z)} (\Delta z)(m) \frac{\partial f_n}{\partial z_m}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sqrt{n}\mu\left(f_n\left(z^b\right) - f_n(z)\right)\mathbb{1}_{\{z_{b(z)}=0^-\}} \\
&+ \quad \sqrt{n}\mu\left(f_n\left(z^a\right) - f_n(z)\right)\mathbb{1}_{\{z_{a(z)}=0^+\}} + \epsilon_n^N(z),
\end{aligned}
$$

where we have introduced the following residual quantity:

$$
\begin{aligned}
\epsilon_n^N(z) \quad = \quad & \frac{\mu}{2\sqrt{n}}\left(\frac{\partial^2 f_n}{\partial z_{b(z)}^2}\mathbb{1}_{\{z_{b(z)}\leq -\frac{1}{\sqrt{n}}\}} + \frac{\partial^2 f_n}{\partial z_{a(z)}^2}\mathbb{1}_{\{z_{a(z)}\geq \frac{1}{\sqrt{n}}\}}\right) \\
+ \quad & \frac{1}{2\sqrt{n}}\sum_{m=1}^{b(z)}\gamma^{b(z)-m}\left((\Delta^1 z)(m) + (\Delta^2 z)(m)\right)\frac{\partial^2 f_n}{\partial z_m^2} \\
+ \quad & \frac{1}{2\sqrt{n}}\sum_{m=a(z)}^{N}\gamma^{m-a(z)}\left((\Delta^1 z)(m) + (\Delta^2 z)(m)\right)\frac{\partial^2 f_n}{\partial z_m^2} \\
+ \quad & \sum_{m=1}^{b(z)} n\left(\lambda_n^{b(z)-m} + \theta_n^{b(z)-m}\mathbb{1}_{\{z_m\leq -\frac{1}{\sqrt{n}}\}}\right) o\left(\frac{1}{n}\right) \\
+ \quad & \sum_{m=a(z)}^{N} n\left(\lambda_n^{m-a(z)} + \theta_n^{m-a(z)}\mathbb{1}_{\{z_m\geq \frac{1}{\sqrt{n}}\}}\right) o\left(\frac{1}{n}\right) \\
+ \quad & \sum_{m=1}^{b(z)} \sqrt{n}\gamma^{b(z)-m}\left((\Delta^1 z)(m) + (\Delta^2 z)(m)\right) o\left(\frac{1}{n}\right) \\
+ \quad & \sum_{m=a(z)}^{N} \sqrt{n}\gamma^{m-a(z)}\left((\Delta^1 z)(m) + (\Delta^2 z)(m)\right) o\left(\frac{1}{n}\right) \\
+ \quad & \sqrt{n}\mu\left(\mathbb{1}_{\{z_{b(z)}\leq -\frac{1}{\sqrt{n}}\}} + \mathbb{1}_{\{z_{a(z)}\geq \frac{1}{\sqrt{n}}\}}\right) o\left(\frac{1}{n}\right).
\end{aligned}
$$

We now give norm estimates for the first and second partial derivatives of f_n. For all $n \geq 1$ and $m \in \{1, \ldots, N\}$, we have:

$$
\left\|\frac{\partial f_n}{\partial z_m}\right\| \leq u_n \left\|\frac{\partial f}{\partial z_m}\right\|, \quad \left\|\frac{\partial^2 f_n}{\partial z_m^2}\right\| \leq u_n^2 \left\|\frac{\partial^2 f}{\partial z_m^2}\right\|,
$$

We also give norm estimates for the differences between the first and second partial derivatives of f_n and those of f. For all $m \in \{1, \ldots, N\}$ and $n \geq 1$, we observe that:

$$
\left\|\frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m}\right\| \leq u_n \left\|\frac{\partial f}{\partial z_m}\bigg|_{\tilde{z}} - \frac{\partial f}{\partial z_m}\right\| + |u_n - 1| \left\|\frac{\partial f}{\partial z_m}\right\|,
$$

and:

$$
\left\|\frac{\partial^2 f_n}{\partial z_m^2} - \frac{\partial^2 f}{\partial z_m^2}\right\| \leq u_n^2 \left\|\frac{\partial^2 f}{\partial z_m^2}\bigg|_{\tilde{z}} - \frac{\partial^2 f}{\partial z_m^2}\right\| + \left|u_n^2 - 1\right| \left\|\frac{\partial^2 f}{\partial z_m^2}\right\|,
$$

where $\tilde{z} = (u_n z_1, \ldots, u_n z_N)$.
Extending the arguments used to prove that $\lim_{n\to+\infty} \sup_{x\in E} \left| f\left(\frac{\lambda}{\lambda_n} x\right) - f(x)\right| = 0$ in Lemma 3.2.1, *mutatis mutandis*, it is easily shown that, for $m \in \{1, \ldots, N\}$:

$$\lim_{n\to+\infty} \left\| \frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m} \right\| = \lim_{n\to+\infty} \left\| \frac{\partial^2 f_n}{\partial z_m^2} - \frac{\partial^2 f}{\partial z_m^2} \right\| = 0,$$

We then see that, for all $z \in E_n^N$:

$$\begin{aligned}
\left| A_n^N f_n(z) - Af(z) \right| \quad \leq \quad & \frac{1}{2} \sum_{m=1}^{b(z)} \left| \lambda_n^{b(z)-m} \frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{b(z)-m} \frac{\partial^2 f}{\partial z_m^2} \right| \\
+ \quad & \frac{1}{2} \sum_{m=1}^{b(z)} \left| \theta_n^{b(z)-m} \mathbb{1}_{\{z_m \leq -\frac{1}{\sqrt{n}}\}} \frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{b(z)-m} \mathbb{1}_{\{z_m < 0^-\}} \frac{\partial^2 f}{\partial z_m^2} \right| \\
+ \quad & \frac{1}{2} \sum_{m=a(z)}^{N} \left| \lambda_n^{m-a(z)} \frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{m-a(z)} \frac{\partial^2 f}{\partial z_m^2} \right| \\
+ \quad & \frac{1}{2} \sum_{m=a(z)}^{N} \left| \theta_n^{m-a(z)} \mathbb{1}_{\{z_m \geq \frac{1}{\sqrt{n}}\}} \frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{m-a(z)} \mathbb{1}_{\{z_m > 0^+\}} \frac{\partial^2 f}{\partial z_m^2} \right| \\
+ \quad & \sum_{m=1}^{b(z)} \gamma^{b(z)-m} \left| (\Delta z)(m) \right| \left| \frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m} \right| \\
+ \quad & \sum_{m=a(z)}^{N} \gamma^{m-a(z)} \left| (\Delta z)(m) \right| \left| \frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m} \right| \\
+ \quad & \left| \alpha_n(z) \right| + \left| \beta_n(z) \right| + \left| \gamma_n^N(z) \right| + \left| \delta_n^N(z) \right| + \left| \epsilon_n^N(z) \right|,
\end{aligned}$$

where we have introduced the quantities α_n, β_n, γ_n^N and δ_n^N defined by:

$$\begin{aligned}
\alpha_n(z) \quad = \quad & \sqrt{n} \left(\theta_n^0 \mathbb{1}_{\{z_{b(z)} \leq -\frac{1}{\sqrt{n}}\}} - \lambda_n^0 \right) \frac{\partial f_n}{\partial z_{b(z)}} + c^0 \mathbb{1}_{\{z_{b(z)} < 0^-\}} \frac{\partial f}{\partial z_{b(z)}} \\
+ \quad & \mu \left(\mathbb{1}_{\{z_{b(z)} \leq -\frac{1}{\sqrt{n}}\}} \frac{\partial f_n}{\partial z_{b(z)}} - \mathbb{1}_{\{z_{b(z)} < 0^-\}} \frac{\partial f}{\partial z_{b(z)}} \right) \\
+ \quad & \sqrt{n} \mu \left(f_n(z^b) - f_n(z) \right) \mathbb{1}_{\{z_{b(z)} = 0^-\}}, \\
\beta_n(z) \quad = \quad & \sqrt{n} \left(\lambda_n^0 - \theta_n^0 \mathbb{1}_{\{z_{a(z)} \geq \frac{1}{\sqrt{n}}\}} \right) \frac{\partial f_n}{\partial z_{a(z)}} - c^0 \mathbb{1}_{\{z_{a(z)} > 0^+\}} \frac{\partial f}{\partial z_{a(z)}} \\
+ \quad & \mu \left(\mathbb{1}_{\{z_{a(z)} > 0^+\}} \frac{\partial f}{\partial z_{a(z)}} - \mathbb{1}_{\{z_{a(z)} \geq \frac{1}{\sqrt{n}}\}} \frac{\partial f_n}{\partial z_{a(z)}} \right)
\end{aligned}$$

$$
\begin{aligned}
& + \quad \sqrt{n}\mu\left(f_n(z^a) - f_n(z)\right)\mathbb{1}_{\{z_{a(z)}=0^+\}}, \\
\gamma_n^N(z) \quad &= \quad \sum_{m=1}^{b(z)-1}\left(\sqrt{n}\left(\theta_n^{b(z)-m}\mathbb{1}_{\{z_m\leq -\frac{1}{\sqrt{n}}\}} - \lambda_n^{b(z)-m}\right)\frac{\partial f_n}{\partial z_m} + c^{b(z)-m}\frac{\partial f}{\partial z_m}\right), \\
\delta_n^N(z) \quad &= \quad \sum_{m=a(z)+1}^{N}\left(\sqrt{n}\left(\lambda_n^{m-a(z)} - \theta_n^{m-a(z)}\mathbb{1}_{\{z_m\geq \frac{1}{\sqrt{n}}\}}\right)\frac{\partial f_n}{\partial z_m} - c^{m-a(z)}\frac{\partial f}{\partial z_m}\right).
\end{aligned}
$$

For all $m \in \{1, \ldots, b(z)\}$ and $n \geq 1$, we have:

$$
\left|\lambda_n^{b(z)-m}\frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{b(z)-m}\frac{\partial^2 f}{\partial z_m^2}\right| \leq \lambda_n^{b(z)-m}\left|\frac{\partial^2 f_n}{\partial z_m^2} - \frac{\partial^2 f}{\partial z_m^2}\right| + \left|\lambda_n^{b(z)-m} - \lambda^{b(z)-m}\right|\left|\frac{\partial^2 f}{\partial z_m^2}\right|.
$$

For all $m \in \{a(z), \ldots, N\}$ and $n \geq 1$, we also have:

$$
\left|\lambda_n^{m-a(z)}\frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{m-a(z)}\frac{\partial^2 f}{\partial z_m^2}\right| \leq \lambda_n^{m-a(z)}\left|\frac{\partial^2 f_n}{\partial z_m^2} - \frac{\partial^2 f}{\partial z_m^2}\right| + \left|\lambda_n^{m-a(z)} - \lambda^{m-a(z)}\right|\left|\frac{\partial^2 f}{\partial z_m^2}\right|,
$$

which subsequently gives us:

$$
\lim_{n\to+\infty}\sup_{z\in E_n^N}\frac{1}{2}\sum_{m=1}^{b(z)}\left|\lambda_n^{b(z)-m}\frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{b(z)-m}\frac{\partial^2 f}{\partial z_m^2}\right| = 0,
$$

$$
\lim_{n\to+\infty}\sup_{z\in E_n^N}\frac{1}{2}\sum_{a(z)=1}^{N}\left|\lambda_n^{m-a(z)}\frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{m-a(z)}\frac{\partial^2 f}{\partial z_m^2}\right| = 0.
$$

Now, for all $m \in \{1, \ldots, b(z)\}$ and $n \geq 1$, we notice that:

$$
\begin{aligned}
& \left|\theta_n^{b(z)-m}\mathbb{1}_{\{z_m\leq -\frac{1}{\sqrt{n}}\}}\frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{b(z)-m}\mathbb{1}_{\{z_m<0^-\}}\frac{\partial^2 f}{\partial z_m^2}\right| \\
\leq \quad & \theta_n^{b(z)-m}\mathbb{1}_{\{z_m\leq -\frac{1}{\sqrt{n}}\}}\left|\frac{\partial^2 f_n}{\partial z_m^2} - \frac{\partial^2 f}{\partial z_m^2}\right| \\
& + \theta_n^{b(z)-m}\left|\mathbb{1}_{\{z_m\leq -\frac{1}{\sqrt{n}}\}} - \mathbb{1}_{\{z_m<0^-\}}\right|\left|\frac{\partial^2 f}{\partial z_m^2}\right| \\
& + \left|\theta_n^{b(z)-m} - \lambda^{b(z)-m}\right|\mathbb{1}_{\{z_m<0^-\}}\left|\frac{\partial^2 f}{\partial z_m^2}\right|.
\end{aligned}
$$

As $\sup_{z\in E_n^N}\left|\mathbb{1}_{\{z_m\leq\ -\frac{1}{\sqrt{n}}\}} - \mathbb{1}_{\{z_m<0^-\}}\right| = 0$, we are able to deduce that:

$$
\lim_{n\to+\infty}\sup_{z\in E_n^N}\frac{1}{2}\sum_{m=1}^{b(z)}\left|\theta_n^{b(z)-m}\mathbb{1}_{\{z_m\leq -\frac{1}{\sqrt{n}}\}}\frac{\partial^2 f_n}{\partial z_m^2} - \lambda^{b(z)-m}\mathbb{1}_{\{z_m<0^-\}}\frac{\partial^2 f}{\partial z_m^2}\right| = 0.
$$

We can obviously apply these steps to the ask side, and obtain:

$$\lim_{n\to+\infty}\sup_{z\in E_n^N}\frac{1}{2}\sum_{m=a(z)}^{N}\left|\theta_n^{m-a(z)}\mathbb{1}_{\{z_m\geq\frac{1}{\sqrt{n}}\}}\frac{\partial^2 f_n}{\partial z_m^2}-\lambda^{m-a(z)}\mathbb{1}_{\{z_m>0^+\}}\frac{\partial^2 f}{\partial z_m^2}\right|=0.$$

Moving forward, we can write:

$$\sum_{m=1}^{b(z)}\gamma^{b(z)-m}\left|(\Delta z)(m)\right|\left|\frac{\partial f_n}{\partial z_m}-\frac{\partial f}{\partial z_m}\right|\leq\sum_{m=1}^{b(z)}\gamma^{b(z)-m}\left|(\Delta z)(m)\right|\left\|\frac{\partial f_n}{\partial z_m}-\frac{\partial f}{\partial z_m}\right\|,$$

from which we establish:

$$\lim_{n\to+\infty}\sup_{z\in E_n^N}\sum_{m=1}^{b(z)}\gamma^{b(z)-m}\left|(\Delta z)(m)\right|\left|\frac{\partial f_n}{\partial z_m}-\frac{\partial f}{\partial z_m}\right|=0.$$

We can analogously see that:

$$\lim_{n\to+\infty}\sup_{z\in E_n^N}\sum_{m=a(z)}^{N}\gamma^{m-a(z)}\left|(\Delta z)(m)\right|\left|\frac{\partial f_n}{\partial z_m}-\frac{\partial f}{\partial z_m}\right|=0.$$

Making use of the fact that $\lambda_n^0=\lambda_n^0\left(\mathbb{1}_{\{z_{b(z)}=0^-\}}+\mathbb{1}_{\{z_{b(z)}\leq-\frac{1}{\sqrt{n}}\}}\right)$, we have:

$$\begin{aligned}
\sup_{z\in E_n^N}|\alpha_n| \quad &\leq \quad \left|c^0-\sqrt{n}\left(\lambda_n^0-\theta_n^0\right)\right|\left\|\frac{\partial f_n}{\partial z_{b(z)}}\right\| \\
&\quad + \quad \left|c^0\right|\left\|\frac{\partial f_n}{\partial z_{b(z)}}-\frac{\partial f}{\partial z_{b(z)}}\right\| \\
&\quad + \quad \left|c^0\right|\left\|\frac{\partial f}{\partial z_{b(z)}}\right\|\underbrace{\sup_{z\in E_n^N}\left|\mathbb{1}_{\{z_{b(z)}\leq-\frac{1}{\sqrt{n}}\}}-\mathbb{1}_{\{z_{b(z)}<0^-\}}\right|}_{=0} \\
&\quad + \quad \mu\left\|\frac{\partial f}{\partial z_{b(z)}}\right\|\underbrace{\sup_{z\in E_n^N}\left|\mathbb{1}_{\{z_{b(z)}<0^-\}}-\mathbb{1}_{\{z_{b(z)}\leq-\frac{1}{\sqrt{n}}\}}\right|}_{=0} \\
&\quad + \quad \mu\left\|\frac{\partial f_n}{\partial z_{b(z)}}-\frac{\partial f}{\partial z_{b(z)}}\right\| \\
&\quad + \quad \sqrt{n}\sup_{z\in E_n^N}\left|\left(\lambda_n^0\frac{\partial f_n}{\partial z_{b(z)}}+\mu\left(f_n(z^b)-f_n(z)\right)\right)\mathbb{1}_{\{z_{b(z)}=0^-\}}\right| \\
&\leq \quad \left|c^0-\sqrt{n}\left(\lambda_n^0-\theta_n^0\right)\right|u_n\left\|\frac{\partial f}{\partial z_{b(z)}}\right\|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left(\left|c^0\right| + \mu\right) \left\| \frac{\partial f_n}{\partial z_{b(z)}} - \frac{\partial f}{\partial z_{b(z)}} \right\| \\
&+ \quad \sqrt{n} \sup_{z \in E_n^N} \left| \left(\lambda_n^0 \frac{\partial f_n}{\partial z_{b(z)}} + \mu \left(f_n(z^b) - f_n(z) \right) \right) \mathbb{1}_{\{z_{b(z)} = 0^-\}} \right|.
\end{aligned}
$$

We also see that:

$$
\begin{aligned}
\sup_{z \in E_n^N} |\beta_n| \quad \leq \quad & \left| \sqrt{n} \left(\lambda_n^0 - \theta_n^0 \right) - c^0 \right| u_n \left\| \frac{\partial f}{\partial z_{a(z)}} \right\| \\
&+ \quad \left(\left|c^0\right| + \mu\right) \left\| \frac{\partial f_n}{\partial z_{a(z)}} - \frac{\partial f}{\partial z_{a(z)}} \right\| \\
&+ \quad \sqrt{n} \sup_{z \in E_n^N} \left| \left(\lambda_n^0 \frac{\partial f_n}{\partial z_{a(z)}} + \mu \left(f_n(z^a) - f_n(z) \right) \right) \mathbb{1}_{\{z_{a(z)} = 0^+\}} \right|.
\end{aligned}
$$

Having reached this point, Lemma 4.2.1 gives us:

$$
\lim_{n \to +\infty} \sqrt{n} \sup_{z \in E_n^N, z_{b(z)} = 0^-} \left| \frac{\partial f_n}{\partial z_{b(z)}} - \frac{\mu}{\lambda_n^0} \left(f_n(z) - f_n(z^b) \right) \right| = 0,
$$

$$
\lim_{n \to +\infty} \sqrt{n} \sup_{z \in E_n^N, z_{a(z)} = 0^+} \left| \frac{\partial f_n}{\partial z_{a(z)}} - \frac{\mu}{\lambda_n^0} \left(f_n(z) - f_n(z^a) \right) \right| = 0.
$$

We therefore have:

$$
\lim_{n \to +\infty} \sup_{z \in E_n^N} \left| \alpha_n(z) \right| = \lim_{n \to +\infty} \sup_{z \in E_n^N} \left| \beta_n(z) \right| = 0.
$$

Let $m \in \{1, \dots, b(z) - 1\}$. Given that $1 = \mathbb{1}_{\{z_m = 0^-\}} + \mathbb{1}_{\{z_m \leq -\frac{1}{\sqrt{n}}\}}$, we can write:

$$
\begin{aligned}
\sup_{z \in E_n^N} \left| \gamma_n^N \right| \quad \leq \quad & \sum_{m=1}^{b(z)-1} \left| c^{b(z)-m} - \sqrt{n} \left(\lambda_n^{b(z)-m} - \theta_n^{b(z)-m} \right) \right| \left\| \frac{\partial f_n}{\partial z_m} \right\| \\
&+ \quad \sum_{m=1}^{b(z)-1} \left| c^{b(z)-m} \right| \left\| \frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m} \right\| \\
&+ \quad \sum_{m=1}^{b(z)-1} \sqrt{n} \lambda_n^{b(z)-m} \sup_{z \in E_n^N} \underbrace{\left| \frac{\partial f_n}{\partial z_m} \mathbb{1}_{\{z_m = 0^-\}} \right|}_{=0} \\
&+ \quad \sum_{m=1}^{b(z)-1} c^{b(z)-m} \sup_{z \in E_n^N} \underbrace{\left| \frac{\partial f}{\partial z_m} \mathbb{1}_{\{z_m = 0^-\}} \right|}_{=0}
\end{aligned}
$$

$$\leq \sum_{m=1}^{b(z)-1} \left| \sqrt{n} \left(\lambda_n^{b(z)-m} - \theta_n^{b(z)-m} \right) - c^{b(z)-m} \right| u_n \left\| \frac{\partial f}{\partial z_m} \right\|$$
$$+ \sum_{m=1}^{b(z)-1} \left| c^{b(z)-m} \right| \left\| \frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m} \right\|.$$

We similarly have:

$$\begin{aligned}
\sup_{z \in E_n^N} \left| \delta_n^N \right| \leq & \sum_{m=a(z)+1}^{N} \left| \sqrt{n} \left(\lambda_n^{m-a(z)} - \theta_n^{m-a(z)} \right) - c^{m-a(z)} \right| \left\| \frac{\partial f_n}{\partial z_m} \right\| \\
& + \sum_{m=a(z)+1}^{N} \left| c^{m-a(z)} \right| \left\| \frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m} \right\| \\
& + \sum_{m=a(z)+1}^{N} \sqrt{n} \lambda_n^{m-a(z)} \sup_{z \in E_n^N} \underbrace{\left| \frac{\partial f_n}{\partial z_m} \mathbb{1}_{\{z_m = 0^+\}} \right|}_{=0} \\
& + \sum_{m=a(z)+1}^{N} c^{m-a(z)} \sup_{z \in E_n^N} \underbrace{\left| \frac{\partial f}{\partial z_m} \mathbb{1}_{\{z_m = 0^+\}} \right|}_{=0} \\
\leq & \sum_{m=a(z)+1}^{N} \left| \sqrt{n} \left(\lambda_n^{m-a(z)} - \theta_n^{m-a(z)} \right) - c^{m-a(z)} \right| u_n \left\| \frac{\partial f}{\partial z_m} \right\| \\
& + \sum_{m=a(z)+1}^{N} \left| c^{m-a(z)} \right| \left\| \frac{\partial f_n}{\partial z_m} - \frac{\partial f}{\partial z_m} \right\|.
\end{aligned}$$

As a result:

$$\lim_{n \to +\infty} \sup_{z \in E_n^N} \left| \gamma_n^N(z) \right| = \lim_{n \to +\infty} \sup_{z \in E_n^N} \left| \delta_n^N(z) \right| = 0.$$

Given that $\lim_{n \to +\infty} \sup_{z \in E_n^N} \left| \epsilon_n^N(z) \right| = 0$, and making use of assumptions $(A1)$ to $(A6)$, we are able to conclude that:

$$\lim_{n \to +\infty} \sup_{z \in E_n^N} \left| A_n^N f_n(z) - A^N f(z) \right| = 0.$$

- step 2: $\overline{A^N}$ *is the infinitesimal generator of a Feller semigroup*

Once again, the linear operator A^N is a particular case of the Waldenfels integro-differential operator introduced in Chapter 2. We can then apply Theorem 2.1.3 to see that $\overline{A^N}$ is the infinitesimal generator of a Feller semigroup $T^N = (T^N(t))_{t \geq 0}$ on $\mathbb{B}(E^N)$.

- step 3: *convergence of the sequence of semigroups*

We now exploit the equivalence between convergence of generators and convergence of semigroups, by verifying that the conditions of Theorem 2.1.1 are fulfilled. We see that $T_n^N = (T_n^N(t))_{t\geq 0}$ is a contraction semigroup as for all $f \in \mathbb{B}(E_n^N)$ and $z \in E_n^N$:

$$|T_n^N(t)f(z)| = |\mathbb{E}(f(\hat{Z}_n(t))|\hat{Z}_n(0) = z)| \leq \|f\|.$$

This is also true for the semigroup $T^N = (T^N(t))_{t\geq 0}$, for all $f \in \mathbb{B}(E^N)$ and $z \in E^N$. Being a Feller semigroup, T^N is strongly continuous. Furthermore, we have:

$$\begin{aligned}
\frac{1}{2\|f\|} \sup_{z\in E_n^N} |T_n^N(t)f(z) - f(z)| \quad &\leq \quad nt\left(\sum_{m=1}^{b(z)} \lambda_n^{b(z)-m} + \sum_{m=a(z)}^{N} \lambda_n^{m-a(z)}\right) \\
&+ \quad nt\left(\sum_{m=1}^{b(z)-1} \theta_n^{b(z)-m} + \sum_{m=a(z)+1}^{N} \theta_n^{m-a(z)}\right) \\
&+ \quad \sqrt{n}t \sum_{m=1}^{b(z)} \gamma^{b(z)-m}(\Delta^1 z)(m) \\
&+ \quad \sqrt{n}t \sum_{m=1}^{b(z)} \gamma^{b(z)-m}(\Delta^2 z)(m) \\
&+ \quad \sqrt{n}t \sum_{m=a(z)}^{N} \gamma^{m-a(z)}(\Delta^1 z)(m) \\
&+ \quad \sqrt{n}t \sum_{m=a(z)}^{N} \gamma^{m-a(z)}(\Delta^2 z)(m) \\
&+ \quad 2nt\left(2\mu_n + \theta_n^0\right) + o(nt).
\end{aligned}$$

We then see that:

$$\lim_{t\to 0} \sup_{z\in E_n^N} |T_n^N(t)f(z) - f(z)| = 0,$$

giving us the strong continuity of T_n^N. Defining $\eta_n : E_n^N \to E^N$ by $\eta_n(z) = z$ and $\pi_n : \mathbb{B}(E^N) \to \mathbb{B}(E_n^N)$ by $\pi_n(f) = f \circ \eta_n$, the conditions of Theorem 2.1.1 are now satisfied, and we can ensure the convergence of the sequence of semigroups.

- step 4: *weak convergence of* $\hat{Z}_n$

We can now apply Theorem 2.1.2, with π_n and η_n as previously defined: there exists an E^N-valued Feller (and therefore strong Markov) process $Z = (Z(t))_{t\geq 0}$ with sample paths in $\mathbb{D}([0, +\infty[, E^N)$ corresponding to T^N (and consequently with generator $\overline{A^N}$) such that $\hat{Z}_n \Rightarrow Z$.

□

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