

DEPARTMENT OF ECONOMICS
OxCarre (Oxford Centre for the Analysis of
Resource Rich Economies)

Manor Road Building, Manor Road, Oxford OX1 3UQ
Tel: +44(0)1865 281281 Fax: +44(0)1865 281163
reception@economics.ox.ac.uk www.economics.ox.ac.uk



OxCarre Research Paper 21

**Aggressive Oil Extraction and Precautionary Saving: Coping
with Volatility**

Revised February 2010

Frederick van der Ploeg

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Frederick van der Ploeg, University of Oxford, United Kingdom¹²

ABSTRACT

The effects of stochastic oil demand on optimal oil extraction paths and tax, spending and government debt policies are analyzed when the oil demand schedule is linear and preferences quadratic. Without prudence, optimal oil extraction is governed by the Hotelling rule and optimal budgetary policies by the tax and consumption smoothing principle. Volatile oil demand brings forward oil extraction and induces a bigger government surplus. With prudence, the government depletes oil reserves even more aggressively and engages in additional precautionary saving financed by postponing spending and bringing taxes forward, especially if it has substantial monopoly power on the oil market, gives high priority to the public spending target, is very prudent, and future oil demand has high variance. Uncertain economic prospects induce even higher precautionary saving and, if non-oil revenue shocks and oil revenue shocks are positively correlated, even more aggressive oil extraction. In contrast, prudent governments deliberately underestimate oil reserves which induce less aggressive oil depletion and less government saving, but less so if uncertainty about reserves and oil demand are positively correlated.

Keywords: Hotelling rule, tax smoothing, prudence, vigorous oil extraction, precautionary saving, taxation and under-spending, oil price volatility, uncertain economic prospects and oil reserves

JEL codes: D81, E62, H63, Q32

Revised February 2010

¹ OxCarre, Department of Economics, Manor Road, Oxford OX1 3UQ, United Kingdom. Also affiliated with the University of Amsterdam, the Tinbergen Institute, CEPR and CESifo. The support of the BP funded Oxford Centre for the Analysis of Resource Rich Economies is gratefully acknowledged.

² The detailed and constructive comments of two anonymous referees and Sören Blomquist have substantially improved an earlier version of the paper. The comments of Tony Venables are also gratefully acknowledged.

1. Introduction

Countries blessed with substantial reserves of natural resources face major challenges on how to manage their wealth efficiently. In deciding how much oil to extract today and how much in the future, resource-rich countries rely on the Hotelling rule of optimal extraction (Hotelling, 1931).³ This rule requires that one is indifferent between keeping the oil under the ground, on the one hand, and extracting, selling it and saving the oil proceeds, on the other hand. This arbitrage principle implies that the expected rate of change in marginal oil rents (revenue minus oil extraction costs) should equal the market rate of interest. The demand for oil together with the total amount of oil reserves then gives the optimal rates of oil extraction, which will be higher if demand for oil is more elastic. With a declining time path of oil proceeds, the government smoothes tax rates and public spending by saving in line with the well-known principle of tax smoothing (cf., Barro, 1979). With declining oil revenues, it is also optimal to reinvest all marginal rents from oil production in education, infrastructure, physical capital, sovereign wealth and other productive assets so that the boost to private and public consumption can be sustained as oil revenues dry up (cf., Hartwick, 1977).

The above restates standard wisdom on optimal oil extraction and management of windfall revenues (e.g., Davis et al., 2002; Barnett and Ossowski, 2003; Ossowski et al., 2008; Collier et al., 2009). A major shortcoming of these three fundamental principles of optimal oil extraction and managing the oil proceeds – the Hotelling rule, the tax smoothing principle and the Hartwick rule – is that they fail to take account of stochastic volatility of oil demand and oil prices, uncertainty about the magnitude of oil reserves, uncertainty about marginal extraction costs, and uncertainty about economic prospects. The main objective of this paper is to investigate the implications of stochastic oil demand, oil prices, oil reserves and economic prospects on the rates of optimal oil extraction, debt management and the efficient setting of tax rates and public spending when policies are conducted by a prudent government. Although the title seems at first blush an oxymoron, we show that both aggressive oil extraction and precautionary saving are optimal outcomes if prudent policy makers try to hedge against oil price volatility driven by turbulent oil demand.

To focus on the most relevant issues, we use the starkest possible model. Hence, we adopt a linear two-period model of intertemporal choice for allocating resources to private and public consumption allowing for tax collection costs/distortions, endogenous public spending, endogenous private consumption, the dynamics of oil extraction, and price-sensitive demand for oil. Future oil demand, oil reserves and shocks to future private income and government income are uncertain and normally distributed. We suppose quadratic preferences. In the absence of prudence, the principle of certainty equivalence (Theil, 1958) does not hold and the conventional Hotelling rule and principle of tax smoothing must be modified. We show that oil demand and oil price volatility brings forward oil extraction and induces a bigger government surplus. Precautionary saving/borrowing does not occur if the third derivative of the utility function is zero (Leland, 1968; Sandmo, 1970; Kimball, 1990). To introduce an element of prudence in government extraction and budgetary policies, we use a double

³ Natural resources could be oil, gas, diamonds, silver, gold, copper, bauxite, coffee, etcetera and even foreign aid. For ease of discussion, we use the shorthand 'oil' for all of them.

negative exponential transformation of future utility. This transformation function displays constant absolute prudence and is used to obtain the certainty-equivalent value of welfare to go (cf., Epstein and Zin, 1989; Weil, 1993).^{4 5} We thus investigate how the Hotelling rule for optimal oil extraction and the tax smoothing principle for optimal debt management and setting of efficient tax rates and public spending have to be modified to allow for prudence.

In more practical terms, prudence is the ability to judge between virtuous and vicious actions at a given time and place. Distinguishing when acts are courageous or reckless is an act of virtue. It should be clear that prudence does not always mean that one has to be cautious or less activist in policies. For example, we know from the literature on prudent optimal monetary policy that more volatility leads to more aggressive Taylor rules for the nominal interest rate (e.g., Sargent, 1999; Leitemo and Söderström, 2008; van der Ploeg, 2009). Indeed, we show that it is prudent to extract more oil more aggressively in the face of volatile oil demand whereas a more common-sense approach suggests that it is best to preserve oil and extract oil less quickly from the ground. What comes out depends on the particular circumstances, since when the uncertainty derives from the amount of oil that is under the ground rather than from uncertainty about future oil demand and oil prices, we show that oil is extracted less vigorously. The term prudence therefore better captures what is going on than the term cautiousness, since the latter term implies the presumption that policy becomes less active. Caution is better reserved to mean risk mitigation or the reluctance to take risks. Prudence is also different from the concept of cunning, since the latter distinguishes itself from the former in the intent with which the decisions to take action are taken. Prudence is viewed to be one of the cardinal virtues. In practical terms, prudent ministers of finance deliberately underestimate growth and thus underestimate the tax base and tax revenues for the coming year (van der Ploeg, 2010). Similarly, prudent businesses do well to estimate costs at the high end and revenues at the low end of the range of forecasts. Our result to deliberately underestimate oil reserves is related to this practical concept of prudence accounting (i.e., businesses deliberately recording inventories in their accounts at the lower of cost and net realizable value rather than at sale price).

We analyze financial buffers of sovereign wealth, not oil inventories. Although in the 1970s and early 1980s there was some hope of *global* commodity price stabilization schemes for commodities like cacao or coffee and relevant theories on optimal commodity stock-piling rules that trade off the benefits of stabilization against the costs of storage are available (Newbery and Stiglitz, 1981), such schemes have mostly been given up. Effective global governance to manage commodity inventories for stabilization purposes at the global level is tough. Other factors are that such schemes distort price signals and may be counterproductive. Hence, we suppose that there is no use of *primary commodity* buffers and focus attention at *financial* buffers. A small country that does not have access to international financial

⁴ This extends the linear-quadratic-Gaussian optimal control framework to allow for constant absolute prudence. Alternatives based on, for example, the linear-exponential-Gaussian framework to analyze precautionary saving are less easy to adopt for our purposes, since we are interested in the intratemporal tradeoff between tax cuts and public spending increases.

⁵ As is known from Kimball (1990) and others, prudence results from a positive third derivative whereas risk aversion comes from a negative second derivative of this transformation function. The double negative exponential transformation happens to display both constant absolute risk aversion and constant absolute prudence.

markets may use oil storage to cope with variable and unpredictable oil revenue streams. But oil in situ corresponds to costless storage of oil and costly physical ex situ storage of oil in containers is pointless.⁶

We consider a country that has some degree of monopoly power on the world market for its natural resources. It may be a monopolist, which only seems to be the case for very few commodities. More realistic may be that the world market for natural resources is oligopolistic. For example, the world market for resources may be characterized by Cournot or Nash-Cournot equilibrium (e.g., Salant, 1976; Pindyck, 1978; Ulph and Folie, 1980; Loury, 1986; van der Ploeg, 1987; Karp, 1992; Salo and Tahvonen, 2001). If the world market has a limited number of price-taking countries, the effective price elasticity facing each resource-exporting country is the price elasticity for the world demand for natural resources divided by the country's share of the world market for natural resources.⁷ The inverse of this effective price elasticity, i.e., the market share times the price elasticity of the world demand for oil, corresponds to the monopoly power of this oil-exporting country in the world oil market. We suppose that the country faces only one deposit of reserves and abstract from the issue of optimal sequencing of easily accessible and deeper layers of reserves (e.g., Herfindahl, 1967; Solow and Wan, 1976; Kemp and Long, 1980a, 2009; Amigues et al., 1998).⁸

Our main result is that it is optimal for prudent governments to extract oil even more aggressively at the expense of future oil production and to set up additional precautionary financial buffers to cope with volatile oil demand and prices, especially if the policy maker is very prudent and attaches a higher priority to the public spending target, the country enjoys substantial monopoly power on the global oil market, and oil demand is very turbulent. We also show that uncertainty about future public revenues or spending needs leads to even more precautionary saving and, if non-oil public revenue shocks and oil revenue shocks are positively correlated, even more aggressive oil depletion. Furthermore, we show that prudent governments deliberately underestimate their oil reserves, thereby offsetting the tendency to extract oil aggressively and reducing the government budget surplus. This effect is, however, weaker if uncertainty about reserves and oil demand are positively correlated.

The outline of the paper is as follows. Section 2 sets up the model. Section 3 first discusses the impact of temporary and permanent oil price shocks on the optimal budgetary policy and oil extraction plans. Since expected marginal oil rents and oil prices rise over time, oil revenues decline over time. To deal with declining oil revenue resulting from rising Hotelling scarcity rents, the government runs a surplus in order to smooth taxes and levels of public spending. If current oil demand is high, the country pumps

⁶ If oil extraction faces convex extraction costs, a case for ex situ storage might be made. In that case, oil buffers may be an alternative for financial buffers of sovereign wealth. We suppose, however, that in situ oil storage is not too costly or, alternatively, that costs of ex situ oil storage are prohibitive in which case ex situ oil storage is not an attractive option.

⁷ In our two- or three-period setting, we do not distinguish between open-loop and closed-loop solution concepts. Given the dominance of OPEC in the global oil market, the oil market may also be characterized by a Stackelberg equilibrium with the OPEC cartel as leader and a competitive fringe of other oil-producing countries as followers (Maskin and Newbery, 1980; de Groot et al., 2003).

⁸ With imperfect competition in the world oil market, simultaneous exploitation of deposits of different marginal extraction costs is possible as the market shares associated with the production of each deposit or the elasticities of demand faced by individual countries can change over time.

more oil at the expense of future oil production, especially if the shock to oil demand is more persistent, and thus current oil prices and revenues are higher today than in the future which necessitates saving by the government. Section 3 discusses the implications of stochastic shocks to the future oil price and establishes that the principle of certainty equivalence does not hold and thus that the Hotelling rule for optimal oil extraction and the tax smoothing principle need to be adjusted. Effectively, volatile oil prices make oil extraction more aggressive. Section 4 first explains how to obtain analytical results for optimal prudent policy in a linear-quadratic-Gaussian two-period setup where a constant absolute prudence utility function is used to calculate the certainty equivalent value of expected welfare to go. Section 4 then derives our main result, namely that a prudent government extracts oil reserves even more aggressively and the government needs to run an even bigger surplus to smooth private and public consumption. Sections 5 and 6 analyze what happens with uncertainty about economic prospects and the amount of oil reserves under the ground, respectively. Section 7 concludes with a summary of our results and offers some suggestions for further research.

2. The Model

To focus on the role of precautionary financial buffers, we employ a two-period framework. We set the interest rate $r \geq 0$ equal to the government's rate of time preference and denote the discount factor by $0 < \delta \equiv 1/(1+r) \leq 1$. The budget deficit at the end of the first period B is the excess of first-period public spending G_1 over tax revenues T_1 plus oil revenues R_1 and the budget surplus at the end of the second period must cover this deficit, $T_2 + R_2 - G_2 - rB = B$, so that the government's intertemporal budget constraint is given by:

$$(1) \quad B = G_1 - T_1 - R_1 = \delta(T_2 + R_2 - G_2),$$

where T_2 , R_2 and G_2 denote second-period tax revenues, oil revenues and public spending, respectively. The present value of tax plus oil revenues must thus cover the present value of public spending (plus any initial debt, which we assume to be zero). We suppose linear demand functions for oil:

$$(2) \quad N_t = \gamma_D - \gamma P_t + \varepsilon_t^D, \quad t = 1, 2,$$

where N_t denotes the demand for oil (and equals the volume of oil extraction as we abstract from storage), P_t the oil price, ε_t^D the normally distributed shock to the demand for oil in period t , $\gamma_D > 0$ indicates autonomous demand for oil, and $\gamma > 0$ is the price sensitivity of the demand for oil. We assume that the price elasticity of the demand for oil $\gamma P_t / N_t$ exceeds unity, so that marginal revenue is positive.⁹ With constant marginal cost of oil extraction $\gamma_C \geq 0$ ¹⁰, we write oil revenues (net of extraction costs) in each period as:

⁹ A ball-park estimate of the price elasticity of world demand for oil is 0.26, so that with Saudi Arabia controlling 12% of the global oil market its relevant price elasticity would be $0.26/0.12 = 2.2$ and the corresponding mark-up

$$\begin{aligned}
R_1 &\equiv (P_1 - \gamma_c)N_1 = (P_1 - \gamma_c)(\gamma_D - \gamma P_1 + \varepsilon_1^D) = \left(\gamma_P + \frac{\varepsilon_1^D - N_1}{\gamma} \right) N_1 \quad \text{and} \\
(3) \quad R_2 &\equiv (P_2 - \gamma_c)N_2 = (P_2 - \gamma_c)(\gamma_D - \gamma P_2 + \rho \varepsilon_1^D + \varepsilon_2^D) = \left(\gamma_P + \frac{\rho \varepsilon_1^D + \varepsilon_2^D - N_2}{\gamma} \right) N_2 \\
&\quad \text{with } \varepsilon_1^D = \Delta \quad \text{and } \varepsilon_2^D \sim \text{IN}(0, \sigma_D^2),
\end{aligned}$$

where $\varepsilon_1^D = \Delta$ is the realized shock to current oil demand, ε_2^D is the normally distributed shock to future oil demand, the parameter $-1 \leq \rho \leq 1$ indicates the persistence of shocks to oil demand¹¹, and $\gamma_P \equiv (\gamma_D / \gamma) - \gamma_c$.¹² Oil demand shocks translate into oil price shocks, especially if oil demand is less sensitive to the oil price (low γ). Oil reserves are exogenous and given by $\bar{N} > 0$. We abstract from oil storage. Oil extraction rates must be non-negative and satisfy the oil depletion equation:

$$(4) \quad N_1 \geq 0, \quad N_2 \geq 0, \quad N_1 + N_2 = \bar{N}.$$

Abstracting from private asset accumulation, we assume that private consumption in each period equals

$$(5) \quad C_t = 1 - T_t - \frac{1}{2} \phi T_t^2, \quad \phi > 0, \quad t = 1, 2,$$

where ϕ indicates the magnitude of the losses from tax collection and production or labor supply distortions. The undistorted level of production income has been normalized to unity. Social welfare depends on utility of private and public consumption in periods one and two:

$$(6) \quad U \equiv U_1 + \delta U_2, \quad U_t = C_t - \frac{1}{2} \psi (\bar{G} - G_t)^2, \quad \psi > 0, \quad t = 1, 2,$$

where $\bar{G} > 0$ is the target level of public spending and ψ the priority attached to this target. Preferences are thus quasi-linear in private consumption. Of course, our two-period framework is highly stylized. For example, limiting oil extraction to two periods is a bold assumption, especially when oil heritage funds use oil revenues to accumulate sovereign wealth and use the interest on this wealth to maintain the (permanent) increase in public and private consumption even after all the oil has been extracted from the ground (e.g., Davis et al., 2002; Barnett and Ossowski, 2003; Ossowski et al., 2008; Collier et al., 2009). One possibility to capture this is to have a continuation component in (6) (and also in (15)) to proxy future welfare after oil has ceased to flow at the end of period two and to suppose that all oil must be exhausted by the end of period two (e.g., due to the appearance of a cheap back-stop

on marginal extraction cost 86% (Hamilton, 2009). Less dominant oil exporters control a smaller part of the global market and thus face a higher price elasticity of the demand for their oil and enjoy less monopoly power.

¹⁰ In practice, oil extraction becomes more expensive as remaining reserves diminish as exploration companies need to go to deeper wells or pump in water or carbon dioxide to maintain pressure.

¹¹ Empirical evidence suggests that oil prices follow a near random walk and that the current spot price is a better predictor of future oil prices than the forward rate (Hamilton, 2009).

¹² A negative shock to future marginal cost of extraction is thus equivalent to a positive shock to oil demand.

technology in period three). Such a three-period framework permits a more comprehensive analysis of sovereign wealth funds and is discussed in the appendix. In the remainder of the paper we focus at the two-period analysis, since the third period does not affect our main conclusions on the implications of prudence for optimal oil extraction paths and budgetary policies. One further aspect of our model should be mentioned at the outset. Due to the linear demand functions and normally distributed demand shocks implied by (2), we may get ‘corner’ outcomes. In order to focus attention on the implications of prudence, we rule out outcomes with negative oil prices and negative extraction rates. For the same reason, we also rule out the possibility that full exhaustion of oil takes place in period 1 (or in period 2 in the three-period model of the appendix) which is less likely to occur if the initial stock of oil is large and the volatility of demand shocks is not too high. We pay this price, because it offers us simple, closed-form analytical solutions and gives clear and intuitive effects of prudence on oil extraction paths and budgetary policies.

3. Revisiting Tax Smoothing and the Hotelling Rule

To set a benchmark for our analysis of prudence in sections 4-6, we first derive the deterministic variant of the problem stated in section 3.1 and the stochastic variant in section 3.2.

3.1. Temporary and permanent deterministic shocks

In the absence of stochastic shocks, the government maximizes welfare U subject to (1)-(6) with $\varepsilon_2^D = 0$ by solving the Lagrangian problem:

$$(7) \quad \begin{aligned} & \text{Max}_{T_1, T_2, G_1, G_2, N_1, N_2} \quad \text{Min}_{\eta, \mu, \mu_1, \mu_2} \quad \left\{ 1 - T_1 - \frac{1}{2} \phi T_1^2 - \frac{1}{2} \psi (\bar{G} - G_1)^2 + \delta \left[1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \psi (\bar{G} - G_2)^2 \right] \right. \\ & \left. + \eta \left[\left(\gamma_p + \frac{\Delta - N_1}{\gamma} \right) N_1 + \delta \left(\gamma_p + \frac{\rho \Delta - N_2}{\gamma} \right) N_2 + T_1 + \delta T_2 - G_1 - \delta G_2 \right] + \mu (\bar{N} - N_1 - N_2) + \mu_1 N_1 + \mu_2 N_2 \right\}, \end{aligned}$$

where $\eta \geq 0$, $\mu \geq 0$, $\mu_1 \geq 0$ and $\mu_2 \geq 0$ denote the marginal cost of public funds (i.e., the marginal increase in social welfare from having one extra unit of government financial assets), the marginal increase in social welfare of one extra unit of oil reserves N , the co-state corresponding to N_1 , and the co-state corresponding to N_2 , respectively.

Optimal budgetary policies and debt management

The first-order conditions for the optimal tax rate and level of public spending following from (7) are:

$$(8) \quad 1 + \phi T_1 = 1 + \phi T_2 = \psi (\bar{G} - G_1) = \psi (\bar{G} - G_2) = \eta.$$

The marginal cost of public funds η is thus an increasing function of the tax rate; and the demand for public spending in each period is a decreasing function of the cost of public funds. We also note from (8) that it is optimal to set the current tax rate to the future tax rate and current public spending to future public spending, which extends the familiar principle of tax smoothing (Barro, 1979). It follows from the principle of optimal debt management, namely that it is optimal to intertemporally smooth the marginal costs of public funds. Upon substitution of (8) into the present-value government budget constraint (1), we solve for the optimal marginal cost of funds, tax rates and public spending shortfalls:

$$(9) \quad \eta = \left(\frac{\psi}{\psi + \phi} \right) \left[1 + \phi(\bar{G} - R_p) \right], \quad T_1 = T_2 = \frac{\psi(\bar{G} - R_p) - 1}{\psi + \phi}$$

$$\text{and } \bar{G} - G_1 = \bar{G} - G_2 = \frac{1 + \phi(\bar{G} - R_p)}{\psi + \phi},$$

where $R \equiv R_1 + \delta R_2$ defines the present value of oil revenues (i.e., oil wealth) and $R_p \equiv R / (1 + \delta)$ defines the permanent value of oil revenue or the return on oil wealth under the ground (i.e., the constant amount of revenue that yields the same present value as the actual stream of oil revenues). Having more oil reserves and thus more oil wealth R implies a lower cost of public funds, especially if tax collection costs/distortions are significant and the public spending target has high priority (high ϕ and ψ). As a result, the government can afford to cut taxes and raise spending in each period. The cut in taxes and fall in the cost of funds are relatively large compared to the boost in public spending if tax collection costs/distortions are significant and the spending target has low priority (large ϕ , small ψ). A higher target level of public spending induces bigger spending and thus a higher cost of funds and tax rate.

Equation (9) establishes the *tax smoothing principle*, which states that tax rates and public spending are smoothed over time. This implies that the current and future *non-oil primary* deficit in each period must equal permanent oil revenue or the return on oil wealth under the ground R_p :

$$(10) \quad G_1 - T_1 = G_2 - T_2 = R_p \equiv \frac{R}{1 + \delta} > 0.$$

The *full* deficit adds interest payments and subtracts oil revenues:

$$(10') \quad B = G_1 - T_1 - R_1 = R_p - R_1 = \frac{\delta}{1 + \delta} (R_2 - R_1) \text{ and } G_2 - T_2 + (1 + r)B - R_2 = (1 + r)R_1.$$

Under the optimal policies, the full deficit in period one shows a surplus if future oil revenue exceeds current oil revenue. The full deficit in period two simply equals oil revenue in period one plus interest.

Optimal oil extraction and revenue paths

At the optimum, both the current and the discounted future marginal rents from oil production are set to the marginal value of an extra unit of oil reserves (μ/η). Using $\delta = 1/(1+r)$, we obtain:

$$(11) \quad \gamma_P + \frac{\Delta - 2N_1}{\gamma} = \frac{\mu - \mu_1}{\eta}, \quad \frac{1}{1+r} \left(\gamma_P + \frac{\rho\Delta - 2N_2}{\gamma} \right) = \frac{\mu - \mu_2}{\eta}, \quad \left. \begin{array}{l} \mu_t \geq 0 \\ N_t \geq 0 \end{array} \right\} \text{c.s., } t = 1, 2.$$

We will assume that the set of parameter values is chosen in such a way that the optimum corresponds to an interior solution, so that $\mu_t \geq 0$, $t = 1, 2$ and therefore oil reserves are not completely exhausted in period two and oil extraction in period one is positive ($N_2, N_1 > 0$). We also assume that marginal oil rents are positive in each period. Hence, (11) implies the *Hotelling rule* for optimal oil extraction (Hotelling, 1931), which states that the Hotelling scarcity rent – the marginal oil rent – should rise at the rate of interest (i.e., r):¹³

$$(11') \quad \left(\gamma_P + \frac{\rho\Delta - 2N_2}{\gamma} \right) = (1+r) \left(\gamma_P + \frac{\Delta - 2N_1}{\gamma} \right).$$

Combining (4) and (11'), we obtain the optimal level of current and future oil extraction:

$$(12) \quad N_1 = \frac{1}{2+r} \bar{N} + \frac{r}{4+2r} \gamma \gamma_P + \frac{1+r-\rho}{4+2r} \Delta \quad \text{and} \quad N_2 = \frac{1+r}{2+r} \bar{N} - \frac{r}{4+2r} \gamma \gamma_P - \frac{1+r-\rho}{4+2r} \Delta.$$

The optimal oil extraction path implies the following time path for oil prices:

$$(13) \quad \begin{aligned} P_1 &= \left(\frac{4+r}{4+2r} \right) \frac{\gamma_D}{\gamma} + \left(\frac{r}{4+2r} \right) \gamma_C + \left(\frac{3+r+\rho}{4+2r} \right) \frac{\Delta}{\gamma} - \left(\frac{1}{2+r} \right) \frac{\bar{N}}{\gamma} \quad \text{and} \\ P_2 &= \left(\frac{4+3r}{4+2r} \right) \frac{\gamma_D}{\gamma} - \left(\frac{r}{4+2r} \right) \gamma_C + \left(\frac{1+r+(3+2r)\rho}{4+2r} \right) \frac{\Delta}{\gamma} - \left(\frac{1+r}{2+r} \right) \frac{\bar{N}}{\gamma}. \end{aligned}$$

The interpretation of expressions (12) and (13) is as follows. First, bigger stocks of oil reserves (higher \bar{N}) permit higher oil extraction rates. This drives down oil prices, especially if oil demand is very sensitive to the oil price (high γ). Since equation (11') says that scarcity rents must rise at the rate of interest, oil reserves have a greater impact on the future level of oil extraction and the future oil price than on current outcomes. Second, the same Hotelling logic requires that, provided that the interest rate is positive, a higher marginal cost of oil extraction (higher γ_C) leads to postponement of oil extraction and therefore today's oil price rises while the future oil price falls. Third, a permanent positive shock to oil demand (i.e., higher γ_D or $\Delta > 0$ with $\rho = 1$) brings, provided that the interest rate is positive,

¹³ If there are no capital markets but there is the possibility of *ex situ* storage of oil, the Hotelling rule needs to distinguish between the return on *in situ* oil (the rate of increase in the marginal oil rents) and the return on *ex situ* oil (benefits of price stabilization minus the marginal cost of storage). In a common-pool setting with voracious extraction, storage may also be an attractive option (Sinn, 1984).

forward oil extraction. As a result, future oil prices rise by more than current oil prices to ensure that marginal oil rents rise at the interest rate. Even without temporary oil scarcity, oil prices are thus expected to rise. Fourth, a temporary positive shock to oil demand ($\Delta > 0$ with $\rho = 0$) induces the government to pump more oil; future oil production is correspondingly lower. This pushes the current oil price above and the future oil price below its Hotelling path, especially if oil demand is relatively insensitive to oil prices (low γ).

Matters simplify if the interest rate is zero. It is then optimal to have a flat profile for the marginal oil rents, so the optimal oil extraction path (12) and time path for oil prices (13) become:

$$(12') \quad N_1 = \frac{1}{2} \bar{N} + \frac{1-\rho}{4} \Delta \quad \text{and} \quad N_2 = \frac{1}{2} \bar{N} - \frac{1-\rho}{4} \Delta.$$

$$(13') \quad P_1 = \frac{\gamma_D}{\gamma} + \left(\frac{3+\rho}{4} \right) \frac{\Delta}{\gamma} - \frac{1}{2} \frac{\bar{N}}{\gamma} \quad \text{and} \quad P_2 = \frac{\gamma_D}{\gamma} + \left(\frac{1+3\rho}{4} \right) \frac{\Delta}{\gamma} - \frac{1}{2} \frac{\bar{N}}{\gamma}.$$

With a zero interest rate, the extraction path is also flat unless a temporary oil demand shock induces pumping up more oil and causes a relatively big rise in current oil prices. Present and future oil rents are then given by:

$$(14) \quad R_1 = \left[\gamma_P + \left(\frac{3+\rho}{4} \right) \frac{\Delta}{\gamma} - \frac{\bar{N}}{2\gamma} \right] \left[\frac{1}{2} \bar{N} + \frac{1-\rho}{4} \Delta \right], \quad R_2 = \left[\gamma_P + \left(\frac{1+3\rho}{4} \right) \frac{\Delta}{\gamma} - \frac{\bar{N}}{2\gamma} \right] \left[\frac{1}{2} \bar{N} - \frac{1-\rho}{4} \Delta \right].$$

Incremental rents from oil sales today and in the future are then $\gamma_P + \bar{N} / 2\gamma$ in face of a permanent shock to oil demand ($\rho = 1$), which are higher if marginal extraction costs are low, there are more reserves to be sold, oil demand is not very price-sensitive and the interest rate is low. In contrast, a temporary boom in oil demand ($\rho = 0$) induces a temporary marginal rise in oil rents of

$$\frac{1}{4} \left(\gamma_P + \frac{\bar{N}}{\gamma} \right) \text{ followed by a marginal decline in rents of } \frac{1}{4} \left(\gamma_P - \frac{\bar{N}}{\gamma} \right).$$

The budget deficit and future outcomes

The full deficit B must equal the difference between permanent and current oil revenue, which under the optimum policies is negative (i.e., the government runs a surplus). With a zero interest rate, the full deficit can with the aid of (10') be written as:

$$(15) \quad B = \frac{1}{2} (R_2 - R_1) = -\frac{1}{2} (1-\rho) \left[\gamma_P + \left(\frac{1+\rho}{2\gamma} \right) \Delta \right] \Delta < 0.$$

Equation (15) and (10') indicate that temporary positive (negative) shocks to oil demand, i.e., $(1-\rho) \Delta$ positive (negative), require the government to save (borrow) with B negative (positive) in order to smooth private and public consumption. Permanent shocks to oil demand ($\rho = 1$) do not require such a

saving policy. However, they do lift up the whole Hotelling path of declining oil revenue and, if the interest rate were positive, they would require the government to save more. The Hotelling logic requires that falling oil revenues arising from a declining path of oil production and increasing path of Hotelling scarcity rents must be associated with a government surplus (i.e., $B < 0$).

Proposition 1: The Hotelling principle requires marginal oil rents to increase at the rate of interest, hence oil extraction and oil revenues fall and oil prices rise over time. To smooth private and public consumption, the government saves to offset declining oil revenues and on top of that saves/borrows in case current oil demand is high (low) and more oil is pumped at expense of future oil production. With permanent oil demand shocks, the declining path of oil revenues and rising path of oil prices are simply lifted upwards. The primary non-oil deficit must equal the permanent value of current and future oil revenues ($G_1 - T_1 = G_2 - T_2 = R_p$).

In the remainder of this paper, we abstract from current shocks to oil demand and focus on the implications of volatility of future oil demand in sections 3.2 and 4, uncertain economic prospects in section 5, and uncertainty about oil reserves in section 6. We thus set $\varepsilon_1^D = \Delta = 0$ from here onwards.

3.2. Stochastic shocks¹⁴

To analyze the impact of stochastic shocks, we use dynamic programming. Starting with period two, we calculate the optimal value of T_2 given the inherited debt B and the inherited stock of oil N_2 :

$$\begin{aligned}
 (16) \quad U_2^{CE} &= \text{Max}_{T_2} \text{E} \left\{ 1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \psi \left[\bar{G} - T_2 - \left(\gamma_p + \frac{\varepsilon_2^D - N_2}{\gamma} \right) N_2 + (1+r)B \right]^2 \right\} \\
 &= \text{Max}_{T_2} \left\{ 1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \psi \left[\bar{G} - T_2 - \left(\gamma_p - \frac{N_2}{\gamma} \right) N_2 + (1+r)B \right]^2 - \frac{\sigma_D^2 \psi N_2^2}{2\gamma^2} \right\}.
 \end{aligned}$$

The first-order optimality condition for the future tax rate is given by:

$$(17) \quad 1 + \phi T_2 = \psi \left[\bar{G} - T_2 - \left(\gamma_p - \frac{N_2}{\gamma} \right) N_2 + (1+r)B \right] = \eta_2.$$

¹⁴ The implications of continuous-time stochastic processes for demand and reserve uncertainty on resource extraction have been analyzed before (Pindyck, 1980). The optimal extraction of natural resources in face of stochastic fluctuations in resource prices away from a growth path has also been investigated before with the results that the ability to withhold production indefinitely and never incur the cost of production (the ‘option value’ of reserves) yields an incentive to delay the rate of production of natural resources whereas having convex (concave) costs of resource extraction speeds up (slows down) extraction (Pindyck, 1981). However, this literature does not focus on public finance and government debt issues and does not deriving stochastic Hotelling rules for prudent resource owners.

The second-period cost of funds η_2 is an increasing function of the future tax rate and the second-period demand for public goods declines with the cost of funds. Hence, the future tax rate and public spending move in opposite directions. With the aid of the envelope theorem and (17), we obtain:

$$(16') \quad U_2^{CE} = \Omega(B, N_2), \quad \Omega_B = -(1+r)\eta_2 < 0, \quad \Omega_{N_2} = (\gamma_P - 2N_2 / \gamma)\eta_2 - \sigma_D^2 \psi N_2 / \gamma^2 > 0.$$

Turning to period one and substituting (4), the government solves the following optimization problem:

$$(18) \quad \text{Max}_{T_1, B, N_1} \left\{ 1 - T_1 - \frac{1}{2} \phi T_1^2 - \frac{1}{2} \psi \left[\bar{G} - T_1 - \left(\gamma_P - \frac{N_1}{\gamma} \right) N_1 - B \right]^2 + \delta \Omega(B, \bar{N} - N_1) \right\}$$

Using (16'), the optimality conditions for the tax rate, deficit and oil extraction in period are written as:

$$(19a) \quad 1 + \phi T_1 = \psi(\bar{G} - G_1) \equiv \eta_1,$$

$$(19b) \quad \eta_1 = -\delta \Omega_B = \eta_2 \equiv \eta,$$

$$(19c) \quad (\gamma_P - 2N_1 / \gamma)\eta_1 = \delta \left[(\gamma_P - 2N_2 / \gamma)\eta_2 - \sigma_D^2 \psi N_2 / \gamma^2 \right].$$

Equation (19a) states the cost of funds in period one is an increasing function of the tax rate and public spending is a decreasing function of the cost funds. Equation (19b) states the principle of optimal debt management, which requires that the current and the future cost of funds should be the same ($\eta_1 = \eta_2$). Together with (19a), this implies that tax rates and levels of public and private spending should be smoothed over time. Equations (19c) and (19b) can be combined to give the modified Hotelling rule:

$$(19c') \quad (\gamma_P - 2N_2 / \gamma) = (1+r)(\gamma_P - 2N_1 / \gamma) + \sigma_D^2 \psi N_2 / \gamma^2 \eta_2.$$

In the absence of oil demand and oil price volatility, marginal oil rents must grow at a rate equal to the rate of interest. If these scarcity rents grow at a higher (lower) rate, it would pay to keep (extract) more oil in (from) the ground until the rate of growth in marginal oil rents and the interest rate are equalized. Equation (19c') shows that oil demand and oil price volatility implies that marginal oil rents rise by more than suggested by the standard Hotelling rule provided the government attaches some priority to its spending target. Equations (4) and (19c') together can be solved to give the optimal extraction path when oil demand and thus oil prices and revenues are volatile:

$$(20) \quad N_1 = \frac{[2 + \sigma_D^2 \psi / (\gamma \eta)] \bar{N} + r \gamma \gamma_P}{4 + 2r + \sigma_D^2 \psi / (\gamma \eta)} \quad \text{and} \quad N_2 = \frac{(2 + 2r) \bar{N} - r \gamma \gamma_P}{4 + 2r + \sigma_D^2 \psi / (\gamma \eta)}.$$

Hence, uncertainty about the future oil demand brings forward oil extraction by more than suggested by the standard Hotelling rule, especially if the government attaches a big weight to its spending target. It follows that the path of expected oil revenues becomes steeper and that the government runs a bigger surplus than dictated by the standard Hotelling logic.

Proposition 2: Higher σ_D induces higher N_1 and R_1 , lower N_2 and ER_2 , and lower P_1 and higher EP_2 , especially if ψ is relatively large. Higher σ_D lowers B , but $T_1 = ET_2$, $G_1 = EG_2$ and the non-oil primary deficit $G_1 - T_1 = EG_2 - ET_2 = R_p$ are unaffected by σ_D .

Once the shock to future oil demand is known, we can calculate the future oil price, future oil revenue, the future tax rate, and the future level of public spending:

$$\begin{aligned}
 P_2 &= EP_2 + \frac{1}{\gamma} \varepsilon_2^D, \quad R_2 = \left(EP_2 + \frac{1}{\gamma} \varepsilon_2^D \right) \left(\frac{(2+2r)\bar{N} - r\gamma\gamma_P}{4+2r+\sigma_D^2\psi/(\gamma\eta)} \right), \\
 (21) \quad T_2 &= ET_2 - \frac{1}{2} \left(\frac{\psi}{\psi+\phi} \right) \left(\frac{(2+2r)\bar{N} - r\gamma\gamma_P}{4+2r+\sigma_D^2\psi/(\gamma\eta)} \right) \frac{1}{\gamma} \varepsilon_2^D \text{ and} \\
 G_2 &= EG_2 + \frac{1}{2} \left(\frac{\phi}{\psi+\phi} \right) \left(\frac{(2+2r)\bar{N} - r\gamma\gamma_P}{4+2r+\sigma_D^2\psi/(\gamma\eta)} \right) \frac{1}{\gamma} \varepsilon_2^D.
 \end{aligned}$$

If oil prices turn out higher (lower) than expected, future oil revenues turn out higher (lower). Consequently, the tax rate is cut (increased) and public spending increased (reduced).

4. Prudent Approach to Oil Extraction and Oil Revenue Management

Although the analysis of optimal oil extraction and fiscal policies analyzed in section 3 yields policy outcomes that depend on the volatility of oil demand, it does not allow for prudent policy making.^{15 16} In practice, governments are more concerned with avoiding the welfare losses of negative future oil price shocks than benefiting from the welfare gains from positive future oil demand shocks. One way of dealing with this is to have a conservative *bird-in-hand approach*, which implies that the government only uses the interest of accumulated oil wealth to cut taxes or boost public spending (e.g., Bjerkholt, 2002; Barnett and Ossowski, 2003). Future oil revenues derived from oil reserves under the ground thus do not affect the current fiscal stance. Norway has enshrined the bird-in-hand approach and the required Stabilization Fund in law, albeit that the law allows for discretionary deviations (Harding and van der Ploeg, 2009). We prefer a less ad hoc conservative approach and therefore analyze how the Hotelling rule and the tax smoothing principle exposited in section 2 have to be modified under prudent policy making.

¹⁵ If quadratic preferences in the model of section 2 are replaced by preferences with a positive/negative third derivative, precautionary saving/borrowing would occur (e.g., Leland, 1968; Sandmo, 1970; Sibley, 1975; Zeldes, 1989; Kimball and Mankiw, 1989; Kimball, 1990).

¹⁶ Allowing for capital accumulation and stochastic dynamics of unemployment, recently a tractable derivation of the target level of precautionary assets that balances impatience, prudence, risk, intertemporal substitution and the rate of return has been derived (Carroll and Jeanne, 2009). But this does not deal with oil extraction.

In section 3 the government maximizes expected welfare $U_1 + \delta EU_2$. To allow for prudence, we assume that the government does not maximize the expected value of discounted future utility but maximizes the discounted expected value of a transformation of future utility. As Kimball (1990) has pointed out, prudence results from a positive third derivative whereas risk aversion comes from a negative second derivative of this transformation function. We will use the double negative exponential transformation function, which displays both constant absolute risk aversion and constant absolute prudence. We therefore assume that the government maximizes the following criterion:

$$(22) \quad U_1 + \delta U_2^{CE}, \quad U_2^{CE} \equiv V^{-1}(\text{EV}(U_2)) = -\ln(\text{E} \exp(-\theta U_2)) / \theta, \quad \theta > 0,$$

where U_2^{CE} denotes the certainty-equivalent value of welfare to go. The double negative exponential function $V(U_2) = -\exp(-\theta U_2)$ attaches a greater weight to adverse outcomes than beneficial outcomes. The positive parameter θ equals the coefficient of absolute prudence $-V'''(U_2) / V''(U_2) = \theta > 0$.¹⁷ For this particular specification, the prudence parameter θ happens to coincide with the constant coefficient of absolute risk aversion $-V''(U_2) / V'(U_2) = \theta > 0$, but we know from the seminal work of Kimball (1990) that it is the positive derivative of the transformation function that leads to prudent behavior and thus to the formation of precautionary buffers. The advantage of this double negative exponential transformation function is that it permits closed-form solutions. Note that as $\theta \rightarrow 0$, (22) tends to $U_1 + \delta EU_2$ which is what the government maximized in section 3.2. Our extension with $\theta > 0$ corresponds to a specific form of Epstein-Zin preferences and can be readily extended to allow for larger horizons (Epstein and Zin, 1989; Weil, 1993).¹⁸ To obtain the optimal prudent oil extraction paths and budgetary policies, we proceed by backward induction. Before we can do this, we need the following Lemma.

Lemma: Let $U_2 = \alpha_0 + \alpha_1 \varepsilon_2 - \frac{1}{2} \alpha_2 \varepsilon_2^2$ and $\varepsilon_2 \sim N(0, \sigma^2)$, then the certainty-equivalent value of U_2 equals $U_2^{CE} \equiv V^{-1}(\text{EV}(U_2)) = \alpha_0 - \frac{1}{2} \left(\frac{\theta \sigma^2 \alpha_1^2}{1 - \theta \sigma^2 \alpha_2} \right) + \frac{1}{2\theta} \ln(1 - \theta \sigma^2 \alpha_2)$ provided that $\theta \sigma^2 \alpha_2 < 1$. In the limiting case $\theta \rightarrow 0$, we have $U_2^{CE} \rightarrow \alpha_0 - \frac{1}{2} \alpha_2 \sigma^2$.

Proof: The proof makes use of the conjugacy of the normal density function and the exponential function and then completing the squares. We have that

¹⁷ This bears some similarity with the theory of disappointment aversion (Gul, 1991; Aizenman, 1998), which introduces caution by assigning a bigger weight to undesirable states of nature.

¹⁸ Alternatively, our framework can be extended to longer planning horizons within the context of prudent risk-sensitive control with atemporal risk aversion (e.g., Whittle, 1990; van der Ploeg, 1993; Bommier, 2006).

$$\begin{aligned}
E[\exp(-\theta U_2)] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} \varepsilon_2^2\right) \exp(-\theta U_2) d\varepsilon_2 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} \varepsilon_2^2 - \theta \left[\alpha_0 + \alpha_1 \varepsilon_2 - \frac{1}{2} \alpha_2 \varepsilon_2^2 \right]\right) d\varepsilon_2 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} (1 - \theta\sigma^2\alpha_2) (\varepsilon_2 - \varepsilon_2^*)^2 + K\right) d\varepsilon_2,
\end{aligned}$$

where equating coefficients on ε_2 and equating the constants in the last two expressions yields

$$(23) \quad \varepsilon_2 = \varepsilon_2^* \equiv \frac{-\theta\sigma^2\alpha_1}{1 - \theta\sigma^2\alpha_2} \quad \text{and} \quad K = -\theta \left(\alpha_0 - \frac{1}{2} \frac{\theta\sigma^2\alpha_1^2}{1 - \theta\sigma^2\alpha_2} \right).$$

Since $\int_{-\infty}^{\infty} \left(\frac{1 - \theta\sigma^2\alpha_2}{2\pi\sigma^2} \right)^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2} (1 - \theta\sigma^2\alpha_2) (\varepsilon_2 - \varepsilon_2^*)^2\right) d\varepsilon_2 = 1$ must be satisfied for the normal density function, we have $E[\exp(-\theta U_2)] = \exp(K) (1 - \theta\sigma^2\alpha_2)^{-\frac{1}{2}}$ and thus

$$U_2^{CE} = -\frac{1}{\theta} \ln(E[\exp(-\theta U_2)]) = \alpha_0 - \frac{1}{2} \left(\frac{\theta\sigma^2\alpha_1^2}{1 - \theta\sigma^2\alpha_2} \right) + \frac{1}{2\theta} \ln(1 - \theta\sigma^2\alpha_2).$$

Note that $\lim_{\theta \rightarrow 0} U_2^{CE} = \alpha_0 - \frac{1}{2} \alpha_2 \sigma^2$ from using $\ln(1 - \theta\sigma^2\alpha_2) \cong -\theta\sigma^2\alpha_2$ for small θ . \square

One interpretation made by Whittle (1990) and others is that the policy maker operates under the pessimistic assumption that Nature deliberately draws shocks to hurt social welfare.¹⁹ Maximizing $U_2 + [\varepsilon_2^2 / \sigma^2 + \ln(1 - \theta\sigma^2\alpha_2)] / 2\theta$ with respect to future policies (affecting α_1 and α_2) and minimizing with respect to ε_2 yields, provided that $\theta\sigma^2\alpha_2 < 1$, $\varepsilon_2 = \varepsilon_2^*$ as given by (23) and the certainty-equivalent value of U_2 . The government that chooses its prudent policies to maximize U_2^{CE} adopts a max-min strategy to hedge against adverse outcomes, especially if the coefficient of absolute prudence θ is large and the variance of future shocks σ^2 is high.

To get a rough idea of the size of the buffer needed to cope with unexpected disturbances to the oil price ε_2 , an atemporal calculation based on our Lemma with $\alpha_2 = 0$ suggests that the optimal savings rate out of a windfall revenue equals $\Xi \nu^2 / 2$, where $\Xi \equiv \theta U_2$ is the coefficient of relative risk aversion and $\nu \equiv \sigma / U_2$ the coefficient of variation of oil prices.²⁰ Hamilton's (2009) 95%-confidence interval for predicted real oil prices gives a mean oil price of \$137 per barrel and a standard deviation of \$37.5 per

¹⁹ The Lemma extends Whittle (1990) and others to allow for $\alpha_2 \neq 0$.

²⁰ Under disappointment aversion, the magnitude of the optimal financial buffer is proportional to the coefficient of variation of commodity prices. Such first-order effects can lead to much larger buffers (Aizenman, 1998).

barrel, so $\nu = 0.27$ for a period of a quarter. A plausible range for Ξ is 1-2, so it is optimal to save between 3.75% and 7.5% of the oil windfall. If the oil bonanza lasts, say Δ quarters, oil prices are much more unpredictable as the coefficient of variation increases for a Δ -period-ahead forecast to $\nu\sqrt{\Delta}$. For an oil bonanza that lasts Δ quarters, we thus save a fraction $\Xi \Delta \nu^2/4$ of the oil bonanza in a fund. For a horizon of four years, this implies that between 30% and 60% of the oil revenues must be saved.

We now apply the Lemma to extend the analysis of section 3.2 to allow for prudence.

Period 2:

Using backward induction, we first solve given the debt inherited at the end of period one B , the

inherited stock of oil $N_2 = \bar{N} - N_1$ and the budget constraint $G_2 = T_2 + \left(\gamma_P - \frac{N_2 - \varepsilon_2^D}{\gamma} \right) N_2 - (1+r)B$

for the optimal budgetary policies of period two:

$$\begin{aligned}
 U_2^{CE} &= \text{Max}_{T_2} - \ln \left(\text{Eexp} \left[-\theta \left\{ 1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \psi \left[\bar{G} - T_2 - \left(\gamma_P + \frac{\varepsilon_2^D - N_2}{\gamma} \right) N_2 + (1+r)B \right]^2 \right\} \right] \right) / \theta \\
 (24) \quad &= \text{Max}_{T_2} \left\{ 1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \left[\psi + \frac{\theta \sigma_D^2 \psi^2 N_2^2 / \gamma^2}{1 - \theta \sigma_D^2 \psi N_2^2 / \gamma^2} \right] \left[\bar{G} - T_2 - \left(\gamma_P - \frac{N_2}{\gamma} \right) N_2 + (1+r)B \right]^2 \right. \\
 &\quad \left. + \frac{1}{2\theta} \ln (1 - \theta \sigma_D^2 \psi N_2^2 / \gamma^2) \right\}
 \end{aligned}$$

where $\varepsilon_2 \equiv \varepsilon_2^D$, $\sigma \equiv \sigma_D$, $\alpha_1 \equiv [\bar{G} - T_2 - (\gamma_2 - N_2 / \gamma) N_2 + (1+r)B] \psi N_2 / \gamma > 0$ and

$\alpha_2 = \psi N_2^2 / \gamma^2 > 0$ has been used in the Lemma to calculate the expectation in (24) under the assumption that $\theta \sigma_D^2 < \gamma^2 / \psi N_2^2$. The first-order condition for the future tax rate is:

$$(25) \quad 1 + \phi T_2 = \psi^* \left[\bar{G} - T_2 - \left(\gamma_P - \frac{N_2}{\gamma} \right) N_2 + (1+r)B \right] \equiv \eta_2 \text{ with } \psi^* = \frac{\psi}{1 - \theta \sigma_D^2 \psi N_2^2 / \gamma^2} \geq \psi.$$

Expression (25) indicates that the cost of funds increases with the tax rate as in section 3. Public spending declines with the cost of funds (η_2) and increases with the priority given to public spending (ψ) as before, but now also increases when the degree of prudence of the policy maker (θ) increases especially if oil demand is very volatile (high σ_D) and lots of oil is left in the ground (high N_2). Expression (23) in the Lemma gives the prudent estimate of the shock to future oil demand while expression (25) gives the future rate, the future cost of public funds and the budgeted future level of public spending:

$$(26a) \quad \varepsilon_2^* = -\theta \sigma_D^2 \left(\frac{\psi^* N_2}{\gamma} \right) [\bar{G} + (1+r)B - T_2 - (\gamma_2 - N_2 / \gamma) N_2] \leq 0,$$

$$(26b) \quad T_2 = \frac{\psi^* \left[(\bar{G} + (1+r)B) - (\gamma_p - N_2 / \gamma) N_2 \right] - 1}{\phi + \psi^*},$$

$$(26c) \quad \eta_2 = \left(\frac{\psi^*}{\phi + \psi^*} \right) \left\{ \phi \left[(\bar{G} + (1+r)B) - (\gamma_p - N_2 / \gamma) N_2 \right] + 1 \right\} > 0,$$

$$(26d) \quad G_2 = \bar{G} - \frac{1}{\psi} \left(\frac{\psi^*}{\phi + \psi^*} \right) \left\{ \phi \left[(\bar{G} + (1+r)B) - (\gamma_p - N_2 / \gamma) N_2 \right] + 1 \right\} > 0.$$

The interpretation of (26) is as follows. A higher public debt B at the end of the first period or a higher target level of public spending \bar{G} necessitates a higher tax rate and thus implies a higher cost of funds, which in turn induces lower public spending. A higher public debt or a higher public spending target also imply a more pessimistic view of future oil demand and thus of future oil prices and revenues. Higher autonomous oil demand or a lower marginal cost of oil extraction (higher γ_p) and more remaining oil reserves at the end of period one (higher N_2) push up oil rents and thus lower the cost of funds, which permits a lower tax rate and higher public spending. Prudence then requires the government to assume a higher level of oil demand and thus higher oil revenues.

The main insight we obtain from (26a) is thus that a prudent government deliberately underestimates future oil demand and consequently underestimates future oil prices and oil revenues, especially if oil demand is very volatile (high $\theta\sigma_D^2$), the country has substantial monopoly power on the oil market (low γ), the priority given to public spending (ψ) is high, and the remaining stock of oil reserves (N_2) is high. With volatile oil demand, it is thus prudent to budget for a relatively high level of future public spending and thus to spend less today and tax more just in case future oil demand is less buoyant and oil revenues are less substantial than expected. In that case, the precautionary budgetary reactions – higher taxes and lower spending – are bigger.²¹ If the government has left more oil reserves for the second period, potential harm done by volatile oil demand is bigger. Hence, the government deliberately uses lower figures for future oil demand and oil revenues in the government budget.

Using the envelope theorem and (25), we obtain the certainty-equivalent value of welfare to go as a function of inherited government debt B and remaining oil reserves $N_2 = \bar{N} - N_1$:

$$(27) \quad U_2^{CE} = \Omega(B, N_2), \quad \Omega_B = -(1+r)\eta_2 < 0, \quad \Omega_{N_2} = \left(\gamma_p - \frac{2N_2}{\gamma} \right) \eta_2 - \sigma_D^2 \frac{\psi^* N_2}{\gamma^2} \left(1 + \frac{\theta\eta_2^2}{\psi^*} \right) > 0.$$

²¹ We see from equation (26d) that G_2 is an decreasing function of ψ^* and thus a decreasing function of $\theta\sigma_D^2$ and $\psi N_2^2 / \gamma^2$. Equations (26b) and (26c) indicate that T_2 and η_2 are an increasing function of ψ^* and thus an increasing function of $\theta\sigma_D^2$ and $\psi N_2^2 / \gamma^2$.

Note that if $\theta = 0$, expression (27) simplifies to (16'). The second-period cost of public funds indicates the marginal loss in future welfare from an extra unit of inherited public debt. A smaller amount of remaining oil reserves caused by aggressive oil extraction (lower N_2) has two effects on the certainty-equivalent value of expected welfare to go. The term $-(\gamma_P - 2N_2 / \gamma)\eta_2$ implies that the government has fewer funds to boost private and public consumption and therefore welfare to go will be lower. The term $\sigma_D^2 (\psi^* N_2 / \gamma^2) (1 + \theta \eta_2^2 / \psi^*)$ says that aggressive oil depletion exposes the country to less fluctuations in oil revenues which boosts social welfare for a prudent government, especially if prudence is substantial, oil demand is highly volatile, the priority given to the public spending target is high, remaining oil reserves are high and monopoly power on the oil market is substantial. For a prudent government, volatile oil demand thus weakens the incentive to leave oil under the ground.

Period 1:

Turning to period one, the government solves the optimization problem (18). Optimality conditions (19a) for the allocation of tax cuts and spending hikes and (19b) for the principle of tax and consumption smoothing are unaltered, but condition (19c') for the modified Hotelling rule becomes:

$$(19c'') \quad \left(\gamma_P - \frac{2N_2}{\gamma} \right) = (1+r) \left(\gamma_P - \frac{2N_1}{\gamma} \right) + \sigma_D^2 \frac{\psi^* N_2}{\gamma^2 \eta_2} \left(1 + \frac{\theta \eta_2^2}{\psi^*} \right).$$

The first term on the right-hand side of expression (19c'') indicates that oil extraction follows a declining path due to the rising path of Hotelling scarcity rents. The second term on the right-hand side of (19c'') reduces to the corresponding term in (19c') if there is no prudence ($\theta = 0$). Comparing (19c'') with (19c'), we see that prudence affects the modified Hotelling rule of section 3.2 in three ways. First, the term in front of the second pair of brackets on the right-hand side of (19c'') is smaller as $\psi^* < \psi$ if $\theta > 0$. Second, the term $1 + \theta \eta_2^2 / \psi^*$ is greater than unity if $\theta > 0$. Both effects result from prudence and require marginal oil rents to rise at a faster rate and oil extraction to be even more aggressive than in section 3.2, especially if oil demand is highly volatile (high σ_D^2).

By leaving less oil reserves for the future, the country becomes less vulnerable to large fluctuations in future oil demand. Oil extraction is especially aggressive if the government is more prudent (high θ), attaches a large priority to the public spending target (high ψ) and enjoys substantial monopoly power on the oil market demand (low γ). On the other hand, if public funds are scarce (high η_2), the government is less willing to depart from the Hotelling principle. Hence, oil extraction is somewhat less aggressive. As a result of pumping more oil today, the government receives relatively more oil revenue today than in the future ($R_1 > ER_2$) and thus smoothes private and public consumption by running a fiscal surplus ($B < 0$). We summarize the above in the following proposition.

Proposition 3: Higher θ raises N_1 and R_1 lowers N_2 and ER_2 , and reduces B relative to the stochastic Hotelling trajectories without prudence, (19a)-(19c) and (20), especially so if σ_D and ψ are large and γ is small.

Future oil demand uncertainty induces even more aggressive oil depletion than predicted by the Hotelling rule if a government is prudent. To ensure flat expected time profiles for the cost of funds, the tax rate, private consumption and public spending, the government needs to save even more and especially so if the government is more prudent and oil demand and prices are more volatile (higher $\theta\sigma_D^2$) and oil extraction is more upfront.

5. Uncertain Economic Prospects

Governments also face uncertainty about economic prospects. This will affect future production income and thus future private consumption and tax revenues. It would also affect future unemployment benefit bills. We capture this budgetary uncertainty by adding a second-period normally distributed shock with zero mean and variance σ_G^2 to the government budget constraint (1),

$$(1') \quad B = G_1 - T_1 - R_1 = \delta(T_2 + \varepsilon_2^G + R_2 - G_2), \quad \varepsilon_2^G \sim N(0, \sigma_G^2).$$

The stochastic term on the revenue side of the second-period government budget constraint is given by:

$$(28) \quad \varepsilon_2 \equiv \varepsilon_2^G + (N_2 / \gamma) \varepsilon_2^D \sim N(0, \sigma^2), \quad \sigma^2 \equiv \sigma_G^2 + (N_2 / \gamma)^2 \sigma_D^2 + 2(N_2 / \gamma) \text{cov}(\varepsilon_2^G, \varepsilon_2^D).$$

We can thus use the Lemma of section 4 with $\alpha_1 \equiv [\bar{G} - T_2 - (\gamma_2 - N_2 / \gamma)N_2 + (1 + r)B]\psi > 0$ and $\alpha_2 = \psi > 0$ to calculate the certainty-equivalent value of welfare to go:

$$(24') \quad U_2^{CE} = \text{Max}_{T_2} - \ln \left(E \exp \left(-\theta \left\{ 1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \psi \left[\bar{G} - T_2 - \left(\gamma_P - \frac{N_2}{\gamma} \right) N_2 + (1 + r)B - \varepsilon_2 \right]^2 \right\} \right) \right) / \theta$$

$$= \text{Max}_{T_2} \left\{ 1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \left(\frac{\psi}{1 - \theta \sigma^2 \psi} \right) \left[\bar{G} - T_2 - \left(\gamma_P - \frac{N_2}{\gamma} \right) N_2 + (1 + r)B \right]^2 + \frac{1}{2\theta} \ln(1 - \theta \sigma^2 \psi) \right\}.$$

The maximization in (24') yields (26b)-(26d) with ψ^* now defined by $\psi^* \equiv \psi / [1 - \theta \sigma^2 \psi] > \psi$ where σ^2 is given in (28). Using (23), the second-period budgeted shock to public revenue is now given by:

$$(26a') \quad \varepsilon_2^* = -\theta \sigma^2 \eta_2 = -\theta \left[\sigma_G^2 + (N_2 / \gamma)^2 \sigma_D^2 + 2(N_2 / \gamma) \text{cov}(\varepsilon_2^G, \varepsilon_2^D) \right] \eta_2 < 0.$$

A prudent government thus deliberately budgets for a negative shock to future revenues (or, equivalently, to a positive shock to the future public spending target), especially if the variance of public revenue shocks is high, the variance of oil demand shocks is high, budgetary and oil demand shocks are positively correlated and the future cost of public funds is high. Consequently, as a precaution, it sets current tax rates higher and current public spending lower to build up buffer stocks and hedge against possible adverse future economic outcomes. The certainty-equivalent value of welfare to go is given by:

$$(27') \quad \begin{aligned} U_2^{CE} &= \Omega(B, N_2), \quad \Omega_B = -(1+r)\eta_2 < 0, \\ \Omega_{N_2} &= \left(\gamma_P - \frac{2N_2}{\gamma} \right) \eta_2 - \left[\frac{N_2 \sigma_D^2}{\gamma} + \text{cov}(\varepsilon_2^G, \varepsilon_2^D) \right] \frac{\psi^*}{\gamma} \left(1 + \frac{\theta \eta_2^2}{\psi^*} \right) > 0. \end{aligned}$$

In absence of budgetary shocks, (27') simplifies to (27). Using (27'), the modified Hotelling rule becomes:

$$(19c''') \quad \left(\gamma_P - \frac{2N_2}{\gamma} \right) = (1+r) \left(\gamma_P - \frac{2N_1}{\gamma} \right) + \left[\frac{N_2 \sigma_D^2}{\gamma} + \text{cov}(\varepsilon_2^G, \varepsilon_2^D) \right] \frac{\psi^*}{\gamma \eta_2} \left(1 + \frac{\theta \eta_2^2}{\psi^*} \right)$$

Hence, uncertain economic prospects not only induce more prudent budgetary policies but also to more aggressive oil extraction provided non-oil revenue shocks are positively correlated with oil demand shocks. Both lead to bigger precautionary buffers to hedge against an uncertain future.

Proposition 4: If $\theta > 0$, higher σ_G leads to higher T_1 and lower G_1 and thus to lower B . Positive $\text{cov}(\varepsilon_2^G, \varepsilon_2^D)$ increases N_1 and R_1 , lowers N_2 and ER_2 , and reduces B relative to the prudent trajectories derived in section 4, especially if ψ is large and η small.

Uncertainty about future public revenue or future spending needs induces precautionary saving, hence a prudent government sets higher taxes and lower spending especially if budgetary shocks have high variance and if the policy maker is very prudent. Over time, tax rates are expected to fall and public spending to rise. If the government is prudent and non-oil revenue shocks are positive correlated with oil revenue shocks, oil is extracted more aggressively and the government builds bigger financial buffers to hedge more effectively against adverse future shocks to the budget. These buffers are accumulated on top of any buffers needed to cope with volatility in oil demand and the oil price.

6. Uncertainty about Oil Reserves and Future Oil Demand

In practice, governments cannot be sure about the exact amount of oil reserves. Although various types of natural resource reserves may be hit by events such as nationalization (e.g., Long, 1975) or catastrophic events such as the collapse of aquifers (e.g., Tsur and Zemel, 2004), we focus on geological uncertainty about the size of reserves. Earlier studies on the optimal extraction of a resource stock of unknown size (e.g., Kemp, 1976, 1977; Loury, 1978; Gilbert, 1979; Robson, 1979; Kemp and Long, 1980a, 1985, 2009; Kumar, 2005) have emphasized the importance of the precautionary motive in the optimal extraction paths. In addition, there may be a role for learning as one may wish to extract earlier layers of reserves more aggressively in order to find out more quickly the size of deeper layers of reserves.²² The

²² Kemp and Long (2009) return to the question first posed by Herfindahl (1967) and analyzed by Solow and Wan (1976), Kemp and Long (1980b) and Amigues et al. (1998) on what is the optimal order of exploitation of several deposits that differ in terms of size, extraction costs and prospect of being discovered, but extend it to allow for the situation where deeper deposits are of unknown size.

oil price should then be discounted to reflect the informational value of extraction, which implies that extraction is hastened in order to resolve stock uncertainty more quickly.²³

We do not analyze gradual learning by oil and mining companies about stocks of reserves as deeper layers of reserves are probed. Instead, we focus on a problem which may be more relevant to the government. We examine how ex-ante uncertainty about the total size of oil reserves affects optimal oil extraction paths and budgetary policies of a prudent policy maker. In the final period, the government finds out what the true level of oil reserves are. Oil reserves $N = \bar{N} + \varepsilon^N$ are stochastic and normally distributed with mean \bar{N} and variance σ_N^2 . The government budget constraint is given by:

$$(1'') \quad B = G_1 - T_1 - R_1 = \delta \left(T_2 + \varepsilon_2^G + \left(\gamma_P + \frac{\varepsilon_2^D - N + N_1}{\gamma} \right) (N - N_1) - G_2 \right), \quad N \sim N(\bar{N}, \sigma_N^2).$$

Linearizing second-period public revenue and abstracting from budgetary shocks, we obtain the compound stochastic shock in public revenue:

$$(28') \quad \begin{aligned} \varepsilon_2 &\equiv \left(\frac{\bar{N} - N_1}{\gamma} \right) \varepsilon_2^D + \left(\gamma_P - 2 \frac{\bar{N} - N_1}{\gamma} \right) \varepsilon^N \sim N(0, \sigma^2), \\ \sigma^2 &\equiv \left(\frac{\bar{N} - N_1}{\gamma} \right)^2 \sigma_D^2 + \left(\gamma_P - 2 \frac{\bar{N} - N_1}{\gamma} \right)^2 \sigma_N^2 + 2 \left(\frac{\bar{N} - N_1}{\gamma} \right) \left(\gamma_P - 2 \frac{\bar{N} - N_1}{\gamma} \right) \text{cov}(\varepsilon^N, \varepsilon_2^D). \end{aligned}$$

We allow for a potential positive covariance between remaining oil reserves and future oil demand to capture that, if unexpectedly future oil demand and the future oil price are higher than expected, less accessible reserves may become more profitable as well. Application of the Lemma of section 4 with $\alpha_1 \equiv [\bar{G} - T_2 - (\gamma_P - N_2 / \gamma)(\bar{N} - N_1) + (1+r)B]\psi > 0$ and $\alpha_2 = \psi > 0$ gives:

$$(24'') \quad U_2^{CE} = \text{Max}_{T_2} \left\{ 1 - T_2 - \frac{1}{2} \phi T_2^2 - \frac{1}{2} \left(\frac{\psi}{1 - \theta \sigma^2 \psi} \right) \left[\bar{G} - T_2 - \left(\gamma_P - \frac{\bar{N} - N_1}{\gamma} \right) (\bar{N} - N_1) + (1+r)B \right]^2 + \frac{1}{2\theta} \ln(1 - \theta \sigma^2 \psi) \right\}.$$

It follows that the future tax rate and spending level are determined as before:

$$(26b') \quad T_2 = \frac{\psi^* \left[(\bar{G} + (1+r)B) - (\gamma_P - (\bar{N} - N_1) / \gamma) (\bar{N} - N_1) \right] - 1}{\phi + \psi^*},$$

²³ This contrasts with the extraction cost surcharge that prevails in a general equilibrium economy with capital accumulation and a continuum of deposits (Solow and Wan, 1976).

$$(26d') \quad G_2 = \bar{G} - \frac{1}{\psi} \left(\frac{\psi^*}{\phi + \psi^*} \right) \left\{ \phi \left[(\bar{G} + (1+r)B) - (\gamma_P - (\bar{N} - N_1)/\gamma)(\bar{N} - N_1) \right] + 1 \right\} > 0,$$

with $\psi^* \equiv \psi / (1 - \theta \sigma^2 \psi)$ and σ^2 is defined in (28'). The second-period budgeted shock to public revenue is derived from (23) and given by:

$$(26a'') \quad \varepsilon_2^* = -\theta \left[\left(\frac{\bar{N} - N_1}{\gamma} \right)^2 \sigma_D^2 + \left(\gamma_P - 2 \frac{\bar{N} - N_1}{\gamma} \right) \sigma_N^2 + 2 \left(\frac{\bar{N} - N_1}{\gamma} \right) \left(\gamma_P - 2 \frac{\bar{N} - N_1}{\gamma} \right) \text{cov}(\varepsilon^N, \varepsilon_2^D) \right] \eta_2 < 0.$$

The government thus errs even more on the safe side by budgeting an even bigger adverse shock to public revenue, especially if remaining oil reserves are highly uncertain and positively correlated with future oil demand, if public funds are scarce (high η_2), and the policy maker is highly prudent (high θ). We can write the certainty equivalent of second-period welfare as:

$$(27'') \quad \begin{aligned} U_2^{CE} &= \Omega(B, \bar{N} - N_1), \quad \Omega_B = -(1+r)\eta_2 < 0, \\ \Omega_{\bar{N}-N_1} &= \left(\gamma_P - \frac{2(\bar{N} - N_1)}{\gamma} \right) \eta_2 \\ &- \left[(\bar{N} - N_1) \sigma_D^2 - 2 \left(\gamma_P - \frac{2(\bar{N} - N_1)}{\gamma} \right) \sigma_N^2 + \left(\gamma_P - \frac{4(\bar{N} - N_1)}{\gamma} \right) \text{cov}(\varepsilon^N, \varepsilon_2^D) \right] \frac{\psi^*}{\gamma^2} \left(1 + \frac{\theta \eta_2^2}{\psi^*} \right) > 0. \end{aligned}$$

Using (27'') to maximize $U_1 + \delta U_2^{CE}$ gives the familiar conditions for smoothing the tax rate and the cost of funds $T_1 = ET_2$ and $\eta_1 = \eta_2$ while the modified Hotelling rule becomes:

$$(19c''') \quad \begin{aligned} \left(\gamma_P - \frac{2N_2}{\gamma} \right) &= (1+r) \left(\gamma_P - \frac{2N_1}{\gamma} \right) - \\ &+ \left[(\bar{N} - N_1) \sigma_D^2 - 2 \left(\gamma_P - \frac{2(\bar{N} - N_1)}{\gamma} \right) \sigma_N^2 + \left(\gamma_P - \frac{4(\bar{N} - N_1)}{\gamma} \right) \text{cov}(\varepsilon^N, \varepsilon_2^D) \right] \frac{\psi^*}{\gamma^2 \eta_2} \left(1 + \frac{\theta \eta_2^2}{\psi^*} \right) > 0. \end{aligned}$$

If there is no uncertainty about oil reserves, the second and third term in the square brackets on the right-hand side of (19c''') are zero and the main result of section 4 is confirmed that in the face of uncertainty about future oil demand a prudent policy maker extracts oil more aggressively and must run a bigger surplus. However, if uncertainty about the quantity of oil reserves is the main concern of a prudent policy maker, the second term in the square brackets dominates so that oil is extracted in a more relaxed fashion. It is prudent to underestimate oil reserves and thus to extract less oil today and more once reserves are known, especially if oil reserves are highly uncertain (high σ_N). The underestimation of reserves is also larger if the amount of oil in the ground is large, i.e., if there has not been much oil extraction in the past (high N_1), as then oil price volatility is much more harmful. If the government cares a lot about its spending target, has substantial monopoly power on the oil market, and public funds are scarce (high ψ , low γ , high η_2), expression (19c''') indicates that the

underestimation of oil reserves will be larger also. To smooth private and public consumption, the government needs to run a smaller surplus to compensate for rising oil revenues. Finally, assuming that $\gamma_p - 4(\bar{N} - N_1) / \gamma > 0$, the third term in the square brackets on the right-hand side of expression (19c''') indicates that underestimation of oil reserves will be less pronounced if uncertainty about oil reserves and future oil demand are positively correlated.

We summarize the above in the following proposition.

Proposition 5: Higher σ_N and positive $\text{cov}(\varepsilon^N, \varepsilon_2^D)$ decrease N_1 and R_1 , increase N_2 and ER_2 , and increase B relative to the prudent trajectories derived in section 4, especially if N_1 , ψ and σ_G are large and γ and η_2 are small.

With turbulent oil demand and prices, it is prudent to extract oil more aggressively and run a bigger surplus. In contrast, with uncertainty about oil reserves, it is prudent to extract oil more slowly and to run a smaller surplus or borrow. In this case, the government accumulates fewer financial and other non-oil assets than it would have done otherwise so that it can smooth the time paths for the cost of funds as well as those of the tax rate and public spending. Note that our result that uncertainty about oil reserves slows down extraction contrasts with the result of Kemp and Long (2009) that oil extraction is speeded up. The reason is that we highlight the effects of prudence whereas Kemp and Long (2009) point to the value of learning about the magnitude of deeper layers of reserves.

7. Concluding Remarks

We constructed the starkest possible welfare-based model of optimal oil extraction and debt management and optimal setting of tax rates and public spending. In the absence of stochastic shocks, oil extraction must be governed by the Hotelling rule which requires that marginal rents from oil production must rise at the rate of interest rate. As a result of this rising path of scarcity rents, oil revenues decline so the government smooths tax rates and public spending by saving. Temporary shocks to oil demand induce pumping up more oil to benefit from the temporary higher price of oil while the government must run a bigger surplus to smooth private and public consumption. The primary non-oil deficit in each period must equal the permanent value of current and future oil revenues.

With uncertainty about future oil demand, marginal oil rents rise faster than the rate of interest, especially if oil demand is more turbulent. As a result, the government needs to run a bigger surplus to smooth consumption. If the policy maker is prudent, it attaches greater value to avoiding negative shocks to welfare than to enjoying positive shocks to welfare. In that case, we have shown that in the face of volatile oil demand and oil prices, oil is extracted more quickly, especially if the government is relatively prudent, attaches more priority to its spending target, and has more monopoly power on the oil market. As a result of the faster rising path of oil revenues the government needs to run an even bigger surplus to ensure a flat time profile for private and public consumption.

Uncertainty about future public revenue or spending needs induces precautionary saving, hence a prudent government sets higher taxes and lower spending especially if the variance of budgetary shocks is high and the degree of prudence is large. Over time, tax rates fall and public spending rise. We have shown that, non-oil and oil revenue shocks are positively correlated, budgetary uncertainty induces a prudent policy maker to extract oil even more aggressively and as a result the government accumulates even bigger financial buffers to hedge against adverse future shocks to the budget. If oil reserves are not known exactly, we have shown that a prudent government deliberately underestimates oil reserves. As a result, oil reserves are depleted more slowly than suggested by the Hotelling rule and the government needs to save less to smooth private and public consumption. This effect is offset somewhat if uncertainty about oil reserves and uncertainty about future oil demand are positively correlated.

Summing up, the government runs a *surplus* to smooth tax rates and public spending levels to deal with: (i) rising Hotelling scarcity rents and the resulting declining path of oil revenue; (ii) temporary oil price hikes and declining paths of oil revenue resulting from temporary booms in oil demand; (iii) aggressive oil depletion and the resulting decline in oil revenue resulting from oil price volatility; (iv) the more aggressive oil depletion undertaken by a prudent policy maker facing oil price volatility; and (v) precautionary saving to hedge against uncertainty about future fiscal revenues or spending needs if non-oil public revenue shocks and oil demand shocks are positively correlated. The government runs a *deficit* to cope with: (vi) temporary falls in oil demand and a rising path of oil revenue (cf., (ii)); and (vii) the slower rate of oil extraction and postponement of oil revenues resulting from uncertainty about oil reserves and positive correlation between uncertainty about oil reserves and oil demand.

Policy makers also have to cope with uncertainty about geological conditions and discovery of new fields, future costs of oil extraction, future substitutes for oil, oil supplies held by competitors, and sensitivity of market prices with respect to the quantity of oil supplied to the market. A prudent government errs on the safe side with all of these forms of uncertainty, especially if shocks are more persistent. If the government faces habit persistence in the sense that public and private agents get hooked on high levels of spending, the government may extract oil more aggressively and accumulate more precautionary financial buffers to offset this.

Storage of oil, gas, and other commodities imply that Hotelling scarcity rents must rise at the rate of interest plus cost of storage, so that optimal depletion paths will become steeper as well. Study of optimal storage rules and management of commodity stabilization funds such as the Chilean Copper Stabilization Fund (e.g., Newbery and Stiglitz, 1981) suggests that even with large risk and low storage costs, the resulting buffer stocks are very small. This is also the case in models where agents are liquidity constrained (Deaton, 1991). However, once one allows for disappointment aversion, the optimal size of buffer stocks is much larger (Aizenman, 1998). An interesting area for further research is therefore to investigate the impact of prudence on optimal storage rules for commodities.

Policy making in developing countries requires one to take account of real-life market failures (Collier et al., 2009; van der Ploeg and Venables, 2009). For example, many developing countries suffer from capital scarcity and are slowly moving towards a steady state with higher levels of income per capita.

Building a sovereign wealth fund is then not necessarily optimal. If countries are underdeveloped and the windfall is insufficiently large and protracted, it is better to speed up the development process by using the oil to pay off foreign debt more rapidly, bring down interest rates and boost private and public investment. One must also take account of bottlenecks in productive capacity and other absorption constraints, since in many developing economies productive capacity of the construction and other non-traded sector may be insufficient to allow an efficient and productive hike in public spending on much needed infrastructure. Given that oil bonanzas typically induce appreciations of the real exchange rate (Dutch disease), these constraints are even more likely to bite in the booming construction and other non-traded sectors. Allowing for the notorious volatility of oil and other commodity prices in conjunction with absorption constraints for productive use of government spending, it may pay to save rather more of the proceeds of an oil bonanza in a fund than one would have done otherwise (e.g., Gelb and Grasmann, 2008). Much more research is required on the implications of volatility commodity prices for resource-rich developing economies with capital scarcity and absorption constraints.

Finally, our normative framework of optimal oil extraction and management of oil proceeds with volatile oil prices needs to be extended with political economy considerations. For example, in commodity price booms politicians may lose sight of value for money and invest in white elephants which are difficult to reverse when commodity prices collapse.²⁴ These problems may be especially severe if the minister of finance is weak and faces many unsustainable claims from its spending ministers. These common-pool problems can lead to rapacious natural resource depletion and excessive debt accumulation. Furthermore, prudently accumulated buffer funds can be raided for short-run electoral purposes.²⁵ Also, such funds are typically very liquid and can be raided by political rivals with different preferences about spending public goods.²⁶ More research is needed on how volatility and political economy affect oil extraction and fiscal policy in more realistic political economy settings.

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²⁴ Earlier work on white elephants as a credible form of redistribution (Robinson and Torvik, 2003) may be useful.

²⁵ Shocks to income and the tax base require a prudent government to build precautionary buffers. Such buffers offset the debt biases of the common-pool problem within the cabinet of ministers, but can be raided for opportunistic short-term electoral gains (van der Ploeg, 2010). Various other political economy distortions can also lead to raiding of buffers and thus limit accumulation of such buffers in practice as well (e.g., Tornell and Lane, 1998; Aizenman and Powell, 1998).

²⁶ With partisan preferences about different types of illiquid public investment projects, incumbent governments over-borrow and over-invest in partisan projects to prevent future investments in pet projects by political rivals and incumbents therefore put fewer resources in liquid sovereign wealth funds (Beetsma and van der Ploeg, 2008).

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Appendix: What happens after the oil has ceased to flow?

Here we briefly discuss a three-period framework to examine what happens after oil has ceased to flow at the end of period two. We replace social welfare (6) by

$$(6') \quad U \equiv U_1 + \delta U_2 + \delta^2 U_3, \quad U_t = C_t - \frac{1}{2} \psi (\bar{G} - G_t)^2, \quad \psi > 0, \quad t = 1, 2,$$

where the flow government budget constraints are now given by

$$B = G_1 - T_1 - R_1, \quad S = T_2 + R_2 - (1+r)B - G_2 \quad \text{and} \quad (1+r)S = G_3 - T_3,$$

where S denotes government saving at the end of period two. The flow budget constraint for the third period indicates that in the post-oil era taxes sovereign wealth (including interest on principal) must cover any excess of public spending over taxes, that is sovereign wealth can be used to ensure an increase in public or private consumption after oil revenues have ceased to flow. The optimal policy corresponding to this extension of section 3 must thus satisfy the Lagrangian problem:

$$\begin{aligned} \text{Max}_{T_1, ET_2, G_1, EG_2, N_1, N_2} \quad \text{Min}_{\eta, \mu, \mu_1, \mu_2} \quad & \left\{ 1 - T_1 - \frac{1}{2} \phi T_1^2 - \frac{1}{2} \psi (\bar{G} - G_1)^2 + \delta \left[1 - ET_2 - \frac{1}{2} \phi (ET_2)^2 - \frac{1}{2} \psi (\bar{G} - EG_2)^2 \right] \right. \\ & + \delta^2 \left[1 - ET_3 - \frac{1}{2} \phi (ET_3)^2 - \frac{1}{2} \psi (\bar{G} - EG_3)^2 \right] + \mu (\bar{N} - N_1 - N_2) + \mu_1 N_1 + \mu_2 N_2 \\ & \left. + \eta \left[\left(\gamma_p + \frac{\Delta - N_1}{\gamma} \right) N_1 + \delta \left(\gamma_p + \frac{\rho \Delta - N_2}{\gamma} \right) N_2 + T_1 + \delta ET_2 + \delta^2 ET_3 - G_1 - \delta EG_2 - \delta^2 EG_3 \right] \right\}. \end{aligned}$$

Given that we restrict attention to interior solutions, we find that the optimal oil extraction path (12) and the corresponding paths for optimal oil prices (13) and oil revenues (14) are unaffected but that the conditions for optimal budgetary policies are now given by:

$$(8') \quad 1 + \phi T_1 = 1 + \phi ET_2 = 1 + \phi ET_3 = \psi (\bar{G} - G_1) = \psi (\bar{G} - EG_2) = \psi (\bar{G} - EG_3) = \eta,$$

It is thus optimal to smooth taxes, private consumption and public spending across all three periods. It follows from (8') and the government budget constraint that the optimal cost of funds, tax rates and public spending shortfalls are given by:

$$(9') \quad \eta = \left(\frac{\psi}{\psi + \phi} \right) \left[1 + \phi (\bar{G} - R_p) \right], \quad T_1 = ET_2 = ET_3 = \frac{\psi (\bar{G} - R_p) - 1}{\psi + \phi}$$

$$\text{and } \bar{G} - G_1 = \bar{G} - EG_2 = \bar{G} - EG_3 = \frac{1 + \phi (\bar{G} - R_p)}{\psi + \phi},$$

where $ER \equiv R_1 + \delta ER_2$ is oil wealth as before and $R_p \equiv ER / (1 + \delta + \delta^2)$ is the permanent value of oil revenue. We thus see that an increase in oil wealth lowers the cost of public funds and thus leads to an increase in private consumption (achieved via a cut in taxes) and an increase in public consumption in all three periods, including the third period where oil revenues have dried up. Since the optimal non-oil primary deficits still equal the permanent oil revenue (cf., equation (10)), the optimal deficit at the end of period one, the optimal saving at the end of period two and the non-oil primary deficit in period three are given by:

$$(10'') \quad B = R_p - R_1, \quad S = (1+r)ER - (2+r)R_p = \delta R_p, \quad \text{and} \quad G_3 - T_3 = (1+r)S = R_p > 0.$$

Hence, the interest and principal on the sovereign wealth accumulated during the oil boom of the first two periods sustain the boost to private and public consumption in the post-oil era of period three.