Brokers and Informed Traders: dealing with toxic flow and extracting trading signals

Álvaro Cartea[a,b], Leandro Sánchez-Betancourt[a,b]

[a] *Mathematical Institute, University of Oxford, Oxford, UK*
[b] *Oxford-Man Institute of Quantitative Finance, Oxford, UK*

Abstract

We derive optimal strategies for an informed trader and a broker. The broker streams bespoke quotes to the informed trader and to a noise trader, and also trades in the lit market. The flow of the noise trader is uninformative and the broker trades with the noise trader at a profit, on average. On the other hand, the informed trader has privileged information about the trend in the price of the asset, so the broker trades with the informed trader at a loss, on average. These losses are payment for toxic flow from which the broker tries to learn the trend signal. The signal is one of the key ingredients in the broker's trading strategy to internalise (i.e., how much of the flow she keeps in her books), to externalise (i.e., how much she unwinds in a lit exchange), and to speculate in the lit market. We obtain explicit solutions to the control problems of both the informed trader, and the broker. The broker's dynamic strategy is a linear combination of four processes: the broker's inventory, the informed trader's inventory, the trend signal, and the uninformed trader's rate of trading with the broker. We demonstrate the efficacy of the trading strategy of the broker in several scenarios and perform robustness analysis against misspecification of model parameters. We employ simulations to conclude that the strategy we derived outperforms current market practices.

Keywords: Informed trading, toxic flow, noise trading, intermediaries, hedging, brokers, trading signal

1. Introduction

This paper studies the strategy of a broker who makes liquidity to her clients and also takes liquidity in a lit market. The broker's clients are an informed trader and an uninformed trader (i.e., a noise trader), both of whom trade on bespoke quotes that are streamed to them by the broker. The informed trader has privileged information about the trend component of the price dynamics of the asset; thus, the broker trades with him at a loss, on average. On the other hand, the uninformed trader has no informational advantage and their trades are exogenous and non-directional; thus, the broker earns the quoted spread to the uninformed, on average.

In our model, the broker maximises expected wealth from her trading activities, while penalising inventory holdings and penalising for model ambiguity. The broker uses the flow of the informed trader to learn the trend component of the price dynamics. However, the broker acknowledges that the learned signal may be incorrect, so the broker entertains alternative models to resolve model uncertainty and to derive a strategy that is robust to model misspecification. Finally, the penalty for inventory holdings protects her strategy from inventory risk, in particular toxic inventory.

Similarly, the informed trader maximises expected wealth from his trading activities, while penalising inventory holdings and penalising for model ambiguity. In this case, model ambiguity arises because the informed trader does not know how the broker externalises her inventory in the lit market. The broker's

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trades in the lit market impact the price of the asset, so the informed trader acknowledges that his model of price dynamics is incorrectly specified. Also, the informed trader's inventory penalty controls how much inventory risk he is willing to bear throughout the trading horizon. Despite his speculative trades are informed by a private signal of the price of the asset, his inventory is exposed to the random innovations in the price of the asset, which is in addition to the risk borne from the unknown price impact exerted by the broker's trades in the lit market.

We derive optimal strategies for the broker and the informed trader for finite-time trading horizons. The broker derives the informed trader's trading strategy and uses the informed trader's flow to learn the private signal, which she cannot perfectly learn due to model uncertainty. The strategy of the broker determines the optimal level of internalisation of flow in her books, from both the informed and uninformed traders, and how much she trades in the lit market. The broker trades in the lit market for two reasons. First, to externalise (i.e., hedge) inventory risk and to unwind toxic flow acquired from the informed trader. Second, to execute speculative trades that are informed by the signal the broker learns from the toxic flow of the informed trader.

The trading strategy we derive for the broker is a linear combination of the inventory of the broker, the inventory of the informed trader, the trading volume of the informed trader (which is linear in the trend signal), and the trading volume of the uninformed trader. We show that as the broker becomes less confident about the signal she learns, the less reactive her trading strategy is to the signal. The broker's strategy is benchmarked against other strategies (e.g., the broker immediately externalises the flow of the informed, internalises the flow of the uninformed, which is gradually externalised with a time-weighted-average strategy, i.e., TWAP). We use simulations to show the superior performance of the broker's internalisation and externalisation strategies we develop, where we also study the performance of the strategy when various model parameters are misspecified. We provide examples of a broker in a foreign exchange market and show that the outperformance against benchmarks is between 15 and 68 US dollars (USD) per million USD traded.

There is extensive literature on informed and uninformed trading. Early work is that by Grossman and Stiglitz (1980); Kyle (1985, 1989), see also O'Hara (1998). More recently, Di Maggio et al. (2019) and Barbon et al. (2019) study the role of brokers in information diffusion in the stock market. In their study of fire sales, they find evidence that brokers' leakage of information leads to higher price impacts in the equity market and more costly executions for the sale originator. In our paper, the broker disseminates the informed trader's signal through her trading in the lit market, which affects the price of the asset. We find that the optimal behaviour of the broker is the sum of a number of terms, one of which is a fraction of the order flow traded by the informed traders, which can be interpreted as the information leakage observed in empirical studies.

Our work is at the interface of two lines of research. One, brokerage strategies to internalise and externalise flow. Two, strategies that use market signals to speculate in the market and to enhance the performance of general trading strategies such as execution and market making.

In the literature, there is very little work in the first line of research. Butz and Oomen (2019) develops a model for internalisation in foreign exchange markets, which is also relevant in other asset classes. Their focus is different from ours. The authors characterise the internalisation horizon of the broker and show how brokers skew their quotes to manage inventory risk – the broker does not engage in proprietary trading.[1] In Barzykin et al. (2023), the authors work with two types of controls: (i) the quotes offered to clients (à la Avellaneda–Stoikov), and (ii) the speed at which the dealer offloads inventory (à la Cartea–Jaimungal). They find an inventory range that determines the dealer's preference to internalise or externalise inventory. They show that the dealer's value function is continuous and the unique viscosity solution to a Hamilton–Jacobi equation. Also, the functional form of their strategy in terms of the state variables is not known, further regularity studies are left as open problems, and the implementation is numerical. In contrast, we formulate a framework where all controls are trading speeds. We obtain explicit formulae for the optimal controls and derive conditions on model parameters for which the solutions we find are classical solutions. Moreover, in our work, the broker classifies her clients into "uninformed" and "informed", and the broker's quotes are increasing in the size of the trade; in line with standard assumptions in the algorithmic trading literature, see Cartea et al. (2015) and Guéant (2016). Also, in our paper, the broker offsets toxic flow with other

[1]In their paper, the internalisation horizon is defined as the length of time a given trade forms part of the broker's risk position before it is offset by another trade in the opposite direction.

incoming trades and by taking liquidity in the lit market. Another key feature is that our model focuses on how the broker learns market signals from the informed trader's toxic flow to execute speculative trades in the lit market.

In the second line of research, there are several works that study optimal trading strategies with market signals. Cartea et al. (2014) develops a market making model when the trend of the underlying asset includes a mean-reverting alpha component, as in this paper. Cartea and Jaimungal (2016) shows how to incorporate a general model of signals for which they derive closed-form execution strategies; see also Casgrain and Jaimungal (2019) which shows how to trade with latent factors that cause prices to jump and diffuse. Similarly, the work of Lehalle and Neuman (2019) shows how to trade with signals and market impact. A particular case of their model is one where the signal is mean-reverting, as the alpha component in this paper. In Cartea and Wang (2020) the market maker employs the alpha signal to make decisions in her liquidity provision strategy; e.g., minimise adverse selection costs, execute speculative trades, and limit inventory risk. More recently, Neuman and Voß (2022) studies optimal execution when trades have price impact and the strategy uses a signal, of finite variation, to predict prices. Also, Bellani et al. (2021) looks at trade execution with short-term signals and Belak et al. (2018) employs non-Markovian finite variation signals. Cartea et al. (2022) includes signals by augmenting the signature of the market with market indices. Recently, Micheli and Neuman (2022) studies a Stackelberg game between a slow trader and a better informed high-frequency trader where the aggregated order flow of both traders impacts the price. They show that the high-frequency trader can adopt predatory or cooperative strategies depending on the information and the order flow.

Other market signals used in the algorithmic trading literature are those based on book volume imbalance; e.g., Cartea et al. (2018) in equity markets and Michael et al. (2022) in options markets. Finally, the work of Bank and Körber (2022) develops a framework for Merton's optimal investment problem which uses the theory of Meyer σ-fields to incorporate signals about jumps in prices in the investment strategies of the investor.

The remainder of this paper proceeds as follows. Section 2 presents the stochastic optimal control problems solved by the broker and the informed trader and derives the strategies in closed form. Section 3 benchmarks the performance of the trading strategies with other strategies. Finally, Section 4 concludes.

2. The model

Trading takes place in a lit market and with a broker. The lit market operates a limit order book (LOB) that is accessible to takers and makers of liquidity and the broker makes liquidity by streaming quotes to her clients. In our model, we interpret the activity of the informed trader as that of a group of informed traders, and similarly for the uninformed trader. The key aspect is that the broker knows the identity of each trader, so she can classify their flow as informed or uninformed, and stream bespoke quotes to them.

Let $T > 0$ be a finite-time horizon and $\mathfrak{T} = [0, T]$. We work in a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathfrak{T}}, \mathbb{P})$ satisfying the usual conditions and supporting three independent Brownian motions W^S, W^α, W^U.[2]

The midprice $(S_t)_{t\in\mathfrak{T}}$ of the asset in the lit market satisfies

$$\mathrm{d}S_t = \left(b\,\nu_t^B + \alpha_t\right)\mathrm{d}t + \sigma^s\,\mathrm{d}W_t^S\,, \quad S_0 \in \mathbb{R}^+\,, \tag{2.1}$$

$$\mathrm{d}\alpha_t = -\kappa^\alpha\,\alpha_t\,\mathrm{d}t + \sigma^\alpha\,\mathrm{d}W_t^\alpha\,, \quad \alpha_0 \in \mathbb{R}\,, \tag{2.2}$$

where b, σ^s, κ^α, σ^α are non-negative constants.[3] In our model, information and pricing pressures are impounded in the price of the asset through the drift of the midprice in (2.1), which consists of two terms. The first term is the impact that the broker's trades have on the midprice, where $(\nu_t^B)_{t\in\mathfrak{T}}$ denotes the broker's speed of trading in the lit market and b is the price impact parameter. The second term in the drift of the midprice is the alpha component with dynamics in (2.2). This component represents liquidity taking trades and limit order activity, both of which impound information in the price of the asset.

The broker charges for liquidity in a similar way to how liquidity providers are compensated in the LOB. However, because there is no anonymity when takers request quotes, the broker streams bespoke quotes

[2] The assumption of independence does not influence the optimal trading strategies we derive – we return to this point below.

[3] For models with an alpha component see Cartea et al. (2014), Cartea and Jaimungal (2016), Lehalle and Neuman (2019), Cartea and Wang (2020), Micheli et al. (2021).

to the informed and to the uninformed trader who trade with the broker at speeds $(\nu_t^I)_{t\in\mathfrak{T}}$ and $(\nu_t^U)_{t\in\mathfrak{T}}$, respectively. For each type of trader, the broker specifies the cost of liquidity as a function of their rate of trading. The quotes (i.e., execution prices if there is a trade) for the informed and uninformed traders are

$$\hat{S}_t^I = S_t + k^I\,\nu_t^I \qquad \text{and} \qquad \hat{S}_t^U = S_t + k^U\,\nu_t^U\,, \tag{2.3}$$

respectively, where the cost of liquidity parameter $k^I > 0$ is known by the informed trader and $k^U > 0$ is known by the uninformed; when the trading rate is positive (negative) the trader buys (sells) the asset.

In the remainder of this section, subsection 2.1 solves the informed trader's problem, and subsection 2.2 describes and solves the problem faced by the broker who trades with an informed trader and with an uninformed trader. Mathematically, the first part of the model (informed trader), in isolation, extends the work of Cartea and Jaimungal (2016); Neuman and Voß (2022) to account for ambiguity aversion; here we find a classical solution for which we provide a verification theorem. In the second part of the model (broker): (i) we find the Stackelberg equilibrium strategy for the two-player game between the broker and the informed trader; (ii) we characterise existence and uniqueness of the classical solution to the control problem of the broker with the existence and uniqueness of a matrix differential Riccati equation, and, under mild conditions on model parameters, we show existence and uniqueness of the matrix Riccati equation; finally, (iii) we provide insights on how the performance of the trading strategies change with mispecification of model parameters and inventory, and study the optimal discount to offer both clients to balance the tradeoff between attracting order flow and minimising loses.

2.1. Informed trader's strategy

The informed trader benefits from superior information, he knows the alpha component (2.2) of the midprice.[4] However, the informed trader does not know the broker's trading rate ν^B in the lit market. The filtration of the informed trader $(\mathcal{F}_t^I)_{t\in\mathfrak{T}}$ is given by

$$\mathcal{F}_t^I := \sigma\left[(S_u)_{u\le t}\,,(\alpha_u)_{u\le t}\right]\,, \tag{2.4}$$

which does not contain $\left(\nu_t^B\right)_{t\in\mathfrak{T}}$. The informed trader fixes $\mathbb{P}^I$ to be a probability measure where

$$\mathrm{d}S_t = \alpha_t\,\mathrm{d}t + \sigma^s\,\mathrm{d}\tilde{W}_t^S\,, \quad S_0\in\mathbb{R}^+\,, \tag{2.5}$$

$$\mathrm{d}\alpha_t = -\kappa^\alpha\,\alpha_t\,\mathrm{d}t + \sigma^\alpha\,\mathrm{d}W_t^\alpha\,, \quad \alpha_0\in\mathbb{R}\,, \tag{2.6}$$

with $(\tilde{W}_t^S)_{t\in\mathfrak{T}}$ and $(W_t^\alpha)_{t\in\mathfrak{T}}$ are $\mathbb{P}^I$-Brownian motions independent of each other.[5] Next, given $(\nu_t^B)_{t\in\mathfrak{T}}$, the informed trader uses the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^I}\bigg|_t = \exp\left\{-\frac{1}{2}\int_0^t \left(b\,\nu_u^B/\sigma^s\right)^2\mathrm{d}u + \int_0^t \left(b\,\nu_u^B/\sigma^s\right)\mathrm{d}\tilde{W}_u^S\right\} \tag{2.7}$$

to characterise the measure $\mathbb{P}$, see (2.1) and (2.2).

The informed trader works on a completed filtered probability space $(\Omega,\mathcal{F},\mathbb{F}^I=(\mathcal{F}_t^I)_{t\in\mathfrak{T}},\mathbb{P}^I)$, where $\mathcal{F}_t^I$ is the filtration in (2.4). The set of admissible strategies for the informed trader is

$$\mathcal{A}^I := \left\{\nu^I = \left(\nu_t^I\right)_{t\in\mathfrak{T}}\mid \nu^I \text{ is } \mathbb{F}^I-\text{progressively measurable, and } \mathbb{E}^I\left[\int_0^T \left(\nu_s^I\right)^2\mathrm{d}s\right] < \infty\right\}. \tag{2.8}$$

Let $\nu^I\in\mathcal{A}^I$ be a given trading strategy. The informed trader's inventory satisfies

$$\mathrm{d}Q_t^I = \nu_t^I\,\mathrm{d}t\,, \quad Q_0^I = 0\,, \tag{2.9}$$

[4]The informed trader incurs a cost to obtain the alpha component. Without loss of generality, we do not include the cost of obtaining the signal in his trading strategy.

[5]The assumption of uncorrelated Brownian motions is for simplicity. In this model, the optimal strategies we derive are the same if the Brownian motions are correlated.

and his cash process $(X_t^I)_{t\in\mathfrak{T}}$ follows

$$\mathrm{d}X_t^I = -\nu_t^I \left(S_t + k^I \nu_t^I\right) \mathrm{d}t\,, \qquad X_0 = 0\,; \tag{2.10}$$

recall that $k^I > 0$ is the price that the broker charges the informed trader for liquidity.

The informed trader acknowledges that his information about the drift of the stock process (2.1) is incomplete, so he entertains alternative models for the dynamics of the stock price. Specifically, to reflect this ambiguity about the correct model, the informed trader considers candidate measures $\mathbb{Q}(y^S)$ that are equivalent to $\mathbb{P}^I$ and are characterised by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}(y^S)}{\mathrm{d}\mathbb{P}^I}\Big|_t = \exp\left\{-\frac{1}{2}\int_0^t \left(y_u^S\right)^2 \mathrm{d}u - \int_0^t y_u^S \,\mathrm{d}\tilde{W}_u^S\right\}, \tag{2.11}$$

where $\left(y_t^S\right)_{\{t\in\mathfrak{T}\}}$ is an $\mathbb{F}^I$-adapted process such that $\left(\mathrm{d}\mathbb{Q}(y^S)/\mathrm{d}\mathbb{P}^I\Big|_t\right)_{t\in\mathfrak{T}}$ is a martingale. Thus, we denote by $\boldsymbol{\mathcal{Q}}^I$ the class of alternative measures

$$\boldsymbol{\mathcal{Q}}^I = \left\{\mathbb{Q}(y^S) \mid y^S \text{ is } \mathbb{F}^I - \text{adapted and } \left(\frac{\mathrm{d}\mathbb{Q}(y^S)}{\mathrm{d}\mathbb{P}^I}\Big|_t\right)_{t\in\mathfrak{T}} \text{ is a martingale}\right\}. \tag{2.12}$$

The informed trader's performance criterion is

$$\mathfrak{H}^{\nu^I,\mathbb{Q}}(t,\alpha,q^I,S,x) = \mathbb{E}^{\mathbb{Q}(y^S)}_{t,\alpha,q^I,S,x}\left[X_T^I + Q_T^I\,S_T - a^I\left(Q_T^I\right)^2 - \phi^I\int_t^T (Q_u^I)^2\,\mathrm{d}u + \mathcal{H}^I_{t,T}\left(\mathbb{Q}(y^S)\,|\,\mathbb{P}^I\right)\right], \tag{2.13}$$

where $a^I > 0$ and $\phi^I > 0$ are the terminal and running inventory penalty parameters, and the notation $\mathbb{E}^{\mathbb{Q}(y^S)}_{t,\alpha,q^I,S,x}$ stands for conditional expectation given that at time t we have $\alpha_t = \alpha$, $Q_t^I = q^I$, $S_t = S$, and $X_t^I = x$. The value function is

$$\mathfrak{H}(t,\alpha,q^I,S,x) = \sup_{\nu^I\in\mathcal{A}^I}\;\inf_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^I}\;\mathfrak{H}^{\nu^I,\mathbb{Q}}(t,\alpha,q^I,S,x)\,. \tag{2.14}$$

Here, the last term on the right-hand side of the performance criterion is a penalty for deviating from the reference measure; see Cartea et al. (2017) for more details on ambiguity aversion. This penalty is the relative entropy from t to T, which is given by

$$\mathcal{H}^I_{t,T}\left(\mathbb{Q}\,|\,\mathbb{P}^I\right) = \frac{1}{\varphi^I}\,\log\left(\frac{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}^I\,|_T}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}^I\,|_t}\right). \tag{2.15}$$

When the trader is confident about $\mathbb{P}^I$, then the value of the ambiguity aversion parameter $\varphi^I > 0$ is small and any deviation from the reference model is costly. In the extreme $\varphi^I \to 0$, the trader is very confident about the reference measure, so he chooses $\mathbb{P}^I$ because the penalty that results from rejecting the reference measure is too high. On the other hand, if the trader is very ambiguous about the reference model, considering alternative models results in a very small penalty. In the extreme $\varphi^I \to \infty$, deviations from the reference model are costless, so the trader searches over measures that deliver the worst-case scenario. We note that the entropic penalty has the same effect as the running inventory penalty; we return to this point below.

The Hamilton–Jacobi–Bellman (HJB) equation associated to the informed trader's value function is

$$\begin{cases} \partial_t\mathfrak{H} + \mathcal{L}^\alpha\mathfrak{H} + \alpha\,\partial_S\mathfrak{H} + \frac{1}{2}\,(\sigma^s)^2\,\partial_{SS}\mathfrak{H} - \phi^I\,(q^I)^2 \\ \qquad + \sup\limits_{\nu^I}\inf\limits_{y^S}\left\{\nu^I\,\partial_{q^I}\mathfrak{H} - \nu^I\,\left(S + k^I\,\nu^I\right)\partial_x\mathfrak{H} - \sigma^s\,y^S\,\partial_S\mathfrak{H} + \frac{1}{2\,\varphi^I}\,(y^S)^2\right\} = 0\,, \\ \mathfrak{H}(T,\alpha,q^I,S,x) = x + q^I\,S - a^I\,(q^I)^2\,, \end{cases} \tag{2.16}$$

where

$$\mathcal{L}^\alpha\cdot = -\kappa^\alpha\,\alpha\,\partial_\alpha\cdot + \frac{1}{2}\,(\sigma^\alpha)^2\,\partial_{\alpha\alpha}\cdot \tag{2.17}$$

is the infinitesimal generator of the process α. The optimal controls in feedback form are

$$y^{S*} = \varphi^I \, \sigma^S \, \partial_S \mathfrak{H} \qquad \text{and} \qquad \nu^{I*} = \frac{\partial_{q^I}\mathfrak{H} - S\,\partial_x \mathfrak{H}}{2\,k^I\,\partial_x \mathfrak{H}}\,, \tag{2.18}$$

which one substitutes in (2.16) to write the partial differential equation (PDE)

$$\begin{cases} \partial_t \mathfrak{H} + \mathcal{L}^\alpha \mathfrak{H} + \alpha\,\partial_S \mathfrak{H} + \frac{1}{2}\,\left(\sigma^S\right)^2 \partial_{SS}\mathfrak{H} - \phi^I\,(q^I)^2 + \frac{\left(\partial_{q^I}\mathfrak{H} - S\,\partial_x \mathfrak{H}\right)^2}{4\,k^I\,\partial_x \mathfrak{H}} - \frac{\varphi^I\left(\sigma^S\,\partial_S \mathfrak{H}\right)^2}{2} = 0\,, \\ \mathfrak{H}(T,\alpha,q^I,S,x) = x + q^I\,S - a^I\,(q^I)^2\,. \end{cases} \tag{2.19}$$

Proposition 2.1 (Candidate to value function). *Define the $\mathcal{C}^{1,2}(\mathfrak{T}\times\mathbb{R}^4)$ function $\tilde{\mathfrak{H}}(t,\alpha,q^I,S,x)$ as*

$$\tilde{\mathfrak{H}}(t,\alpha,q^I,S,x) = x + q^I\,S + \mathfrak{h}_0(t,\alpha) + q^I\,\mathfrak{h}_1(t,\alpha) + (q^I)^2\,\mathfrak{h}_2(t) \tag{2.20}$$

with $\mathfrak{h}_2(t)$ given by

$$\mathfrak{h}_2(t) = -\sqrt{k^I\,\Phi^I}\,\frac{\zeta_1\,e^{\gamma_1\,(T-t)} + e^{-\gamma_1\,(T-t)}}{\zeta_1\,e^{\gamma_1\,(T-t)} - e^{-\gamma_1\,(T-t)}} \tag{2.21}$$

where

$$\gamma_1 = \sqrt{\frac{\Phi^I}{k^I}} \qquad \text{and} \qquad \zeta_1 = \frac{a^I + \sqrt{k^I\,\Phi^I}}{a^I - \sqrt{k^I\,\Phi^I}}\,, \tag{2.22}$$

and

$$\Phi^I = \frac{1}{2}\,\varphi^I\,(\sigma^S)^2 + \phi^I > 0\,. \tag{2.23}$$

The $\mathcal{C}^{1,2}(\mathfrak{T}\times\mathbb{R})$ function $\mathfrak{h}_1(t,\alpha)$ is given by

$$\begin{aligned} \mathfrak{h}_1(t,\alpha) &= \alpha\,\frac{1}{e^{-\gamma_1(T-t)} - \zeta_1\,e^{\gamma_1(T-t)}}\left[\frac{\zeta_1}{\kappa^\alpha+\gamma_1}\left(e^{-\kappa^\alpha(T-t)} - e^{\gamma_1(T-t)}\right) - \frac{1}{\kappa^\alpha-\gamma_1}\left(e^{-\kappa^\alpha(T-t)} - e^{-\gamma_1(T-t)}\right)\right] \\ &:= \alpha\,\mathfrak{g}(t)\,, \end{aligned} \tag{2.24}$$

and the $\mathcal{C}^{1,2}(\mathfrak{T}\times\mathbb{R})$ function $\mathfrak{h}_0(t,\alpha)$ is given by

$$\mathfrak{h}_0(t,\alpha) = \mathfrak{f}_0(t) + \alpha^2\,\mathfrak{f}_2(t) \tag{2.25}$$

where

$$\mathfrak{f}_2(t) = \int_t^T \left(\frac{\mathfrak{g}^2(u)}{4\,k^I}\right)\,e^{-2\,\kappa^\alpha(u-t)}\mathrm{d}u\,, \tag{2.26}$$

and

$$\mathfrak{f}_0(t) = \int_t^T (\sigma^\alpha)^2\,\mathfrak{f}_2(u)\mathrm{d}u\,. \tag{2.27}$$

Then, $\tilde{\mathfrak{H}}$ solves the HJB equation in (2.16).

Proof. The function $\tilde{\mathfrak{H}}(t,\alpha,q^I,S,x)$ solves the HJB equation in (2.16) if it solves the PDE in (2.19). Given that $\tilde{\mathfrak{H}}(t,\alpha,q^I,S,x) = x + q^I\,S + \mathfrak{h}(t,\alpha,q^I)$, the function $\tilde{\mathfrak{H}}$ solves the PDE in (2.19) if $\mathfrak{h}(t,\alpha,q^I)$ solves

$$\begin{cases} \partial_t \mathfrak{h} + \mathcal{L}^\alpha \mathfrak{h} + \alpha\,q^I - \phi^I\,(q^I)^2 + \frac{\left(\partial_{q^I}\mathfrak{h}\right)^2}{4\,k^I} - \frac{\varphi^I\left(\sigma^S\,q^I\right)^2}{2} = 0\,, \\ \mathfrak{h}(T,\alpha,q^I) = -a^I\,\left(q^I\right)^2\,. \end{cases} \tag{2.28}$$

Given that $\mathfrak{h}(t,\alpha,q^I) = \mathfrak{h}_0(t,\alpha) + q^I\,\mathfrak{h}_1(t,\alpha) + (q^I)^2\,\mathfrak{h}_2(t)$, the above holds if the following three ODEs are simultaneously satisfied

$$\partial_t \mathfrak{h}_0 + \mathcal{L}^\alpha \mathfrak{h}_0 + \frac{(\mathfrak{h}_1)^2}{4\,k^I} = 0\,, \tag{2.29}$$

$$\partial_t \mathfrak{h}_1 + \alpha + \mathcal{L}^\alpha \mathfrak{h}_1 + \frac{\mathfrak{h}_1\,\mathfrak{h}_2}{k^I} = 0\,, \tag{2.30}$$

$$\partial_t \mathfrak{h}_2 - \frac{1}{2}\,\varphi^I\,(\sigma^S)^2 - \phi^I + \frac{(\mathfrak{h}_2)^2}{k^I} = 0\,, \tag{2.31}$$

with $\mathfrak{h}_0(T,\alpha) = \mathfrak{h}_1(T,\alpha) = 0$ and $\mathfrak{h}_2(T) = -a^I$. It is then a short exercise to check that $\mathfrak{h}_0$, $\mathfrak{h}_1$, and $\mathfrak{h}_2$ satisfy these ODEs with the stated terminal conditions. □

Theorem 2.2. *The control problem in* (2.14) *has a classical solution. The value function in* (2.14) *is given by* $\tilde{\mathfrak{H}}(t,\alpha,q^I,S,x)$ *in* (2.20) *and the optimal trading strategy of the informed trader is the admissible Markovian control*

$$\nu_t^{I*} = \mathfrak{g}_0(t)\,\alpha_t - \mathfrak{g}_1(t)\,Q_t^{I*}\,, \tag{2.32}$$

for $\mathfrak{g}_0, \mathfrak{g}_1 : [0,T] \to \mathbb{R}$ *given by*

$$\mathfrak{g}_0(t) = \frac{\mathfrak{g}(t)}{2\,k^I}\,, \qquad \mathfrak{g}_1(t) = \gamma_1\,\frac{\zeta_1\,e^{\gamma_1\,(T-t)} + e^{-\gamma_1\,(T-t)}}{\zeta_1\,e^{\gamma_1\,(T-t)} - e^{-\gamma_1\,(T-t)}}\,. \tag{2.33}$$

The optimal change of measure is characterised by the admissible control

$$y_t^{S*} = \varphi^I\,\sigma^S\,Q_t^{I*}\,. \tag{2.34}$$

For a proof see Appendix A.

As noted above, the entropic penalisation due to model ambiguity is equivalent to the running inventory penalty in the performance criterion. This is clearly seen in (2.23) where the effect of the entropic and running inventory penalties is additive. This connection between ambiguity aversion and a penalisation of inventory was made in Cartea et al. (2017).

2.2. Broker's strategy

The broker streams buy and sell quotes to the informed and uninformed traders. The two traders do not trade in the lit market because both get better quotes from the broker or because they cannot access the lit market. The broker assumes that the informed trader trades optimally, and thus, $\nu_t^{I*} = \mathfrak{g}_0(t)\,\alpha_t - \mathfrak{g}_1(t)\,Q_t^{I*}$, which implies that knowledge of ν_t^{I*} and Q_t^{I*} determine α_t and all three stochastic processes would be known to the broker. More precisely, given ν_t^{I*} and Q_t^{I*}, then

$$\alpha_t = \frac{\mathfrak{g}_1(t)\,Q_t^{I*} + \nu_t^{I*}}{\mathfrak{g}_0(t)} = \mathfrak{g}_2(t)\,Q_t^{I*} + \mathfrak{g}_3(t)\,\nu_t^{I*} \tag{2.35}$$

for $\mathfrak{g}_3, \mathfrak{g}_2 : \mathfrak{T} \to \mathbb{R}$ defined in the equality above.

However, the broker acknowledges that the informed trader might be trading elsewhere (e.g., with other brokers and in the lit market), so the learned signal (2.35) is misspecified. Thus, the broker fixes $\mathbb{P}^B$ to be a probability measure where S_t and α_t follow (2.1), and the broker entertains alternative models for the dynamics of α_t.[6] Specifically, the broker considers candidate measures $\mathbb{Q}(y^\alpha)$ that are equivalent to $\mathbb{P}^B$ and are characterised by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}(y^\alpha)}{\mathrm{d}\mathbb{P}^B}\Big|_t = \exp\left\{-\frac{1}{2}\int_0^t (y_u^\alpha)^2\,\mathrm{d}u - \int_0^t y_u^\alpha\,\mathrm{d}W_u^\alpha\right\}, \tag{2.36}$$

where $(y_t^\alpha)_{t\in\mathfrak{T}}$ is an $\mathbb{F}^B$-adapted process such that $\left(\mathrm{d}\mathbb{Q}(y^\alpha)/\mathrm{d}\mathbb{P}^B\,\Big|_t\right)_{t\in\mathfrak{T}}$ is a martingale. Thus, we denote by $\boldsymbol{\mathcal{Q}}^B$ the class of alternative measures

$$\boldsymbol{\mathcal{Q}}^B = \left\{\mathbb{Q}(y^Q) \mid y^Q \text{ is } \mathbb{F}^B-\text{adapted and } \left(\frac{\mathrm{d}\mathbb{Q}(y^Q)}{\mathrm{d}\mathbb{P}^B}\Big|_t\right)_{t\in\mathfrak{T}} \text{ is a martingale}\right\}. \tag{2.37}$$

The broker works on a completed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^B = (\mathcal{F}_t^B)_{t\in\mathfrak{T}}, \mathbb{P}^B)$, where

$$\mathcal{F}_t^B := \sigma\left[(S_u)_{u\le t}\,, (\nu_u^I)_{u\le t}\,, (\nu_u^U)_{u\le t}\right]. \tag{2.38}$$

The set of admissible strategies for the broker is

$$\mathcal{A}^B := \left\{\nu^B = \left(\nu_t^B\right)_{t\in\mathfrak{T}} \mid \nu^B \text{ is } \mathbb{F}^B-\text{progressively measurable, and } \mathbb{E}^B\left[\int_0^T \left(\nu_s^B\right)^2\mathrm{d}s\right] < \infty\right\}. \tag{2.39}$$

[6] The broker uses (2.35) to specify the reference measure $\mathbb{P}^B$ for the dynamics of the signal.

The broker makes liquidity to the uninformed trader whose rate of trading follows the Ornstein–Uhlenbeck process

$$d\nu_t^U = -\kappa_u\, \nu_t^U\, dt + \sigma^U\, dW_t^U\,, \qquad \nu_0^U \in \mathbb{R}\,, \tag{2.40}$$

where κ_u, σ^U are constants, the Brownian motion $(W^U)_{t\in\mathfrak{T}}$ is independent of W^S, W^α, and the execution prices are in (2.3).

Let $\nu^B \in \mathcal{A}^B$ be a given trading strategy. When the broker trades in the lit market, the execution costs in the LOB are $S_t + k^B\, \nu_t^B$, where $k^B > 0$. Therefore, the broker's cash $(X_t^B)_{t\in\mathfrak{T}}$ follows

$$dX_t^B = \nu_t^U\, \left(S_t + k^U\, \nu_t^U\right)\, dt + \nu_t^I\, \left(S_t + k^I\, \nu_t^I\right)\, dt - \nu_t^B\, \left(S_t + k^B\, \nu_t^B\right)\, dt\,, \qquad X_0^B = 0\,, \tag{2.41}$$

and the inventory process $(Q_t^B)_{t\in\mathfrak{T}}$ follows

$$dQ_t^B = \left(\nu_t^B - \nu_t^U - \nu_t^I\right)\, dt\,, \qquad Q_0^B = 0\,. \tag{2.42}$$

The broker's performance criterion is

$$\begin{aligned} &\mathfrak{Q}^{\nu^B,\mathbb{Q}}(t,\alpha,q^B,\,q^I,S,x^B,\nu^U) \qquad (2.43)\\ &\quad = \mathbb{E}^{\mathbb{Q}(y^\alpha)}_{t,\alpha,q^B,q^I,S,x^B,\nu^U}\left[X_T^B + Q_T^B\, S_T - a^B\,(Q_T^B)^2 - \phi^B \int_t^T (Q_s^B)^2\, ds + \mathcal{H}^B_{t,\,T}\left(\mathbb{Q}(y^\alpha)\,|\,\mathbb{P}^B\right)\right]\,, \end{aligned}$$

and the value function is

$$\mathfrak{Q}(t,\alpha,q^B,q^I,S,x^B,\nu^U) = \sup_{\nu^B\in\mathcal{A}^B}\ \inf_{\mathbb{Q}\in\mathcal{Q}^B} \mathfrak{Q}^{\nu^B,\mathbb{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U)\,, \tag{2.44}$$

where $a^B > 0$ is a terminal liquidation parameter, and $\phi^B > 0$ is a running inventory penalty parameter. We assume that model parameters are such that $4\,\phi^B\, k^B - b^2 \geq 0$ holds. We use this last inequality to prove existence of a matrix Riccati differential equation when solving the problem of the broker.

Here, α is a state variable because the broker uses the trading rate $\nu_t^{I*} = \mathfrak{g}_0(t)\,\alpha_t - \mathfrak{g}_1(t)\, Q_t^{I*}$ of the informed trader to learn the signal α. Recall that the ambiguity is accounted for in the misspecification of the model to learn the α signal in the broker's model. The relative entropy from t to T for the broker is given by

$$\mathcal{H}^B_{t,\,T}\left(\mathbb{Q}\,|\,\mathbb{P}^B\right) = \frac{1}{\varphi^B}\,\log\left(\frac{d\mathbb{Q}/d\mathbb{P}^B\,|_T}{d\mathbb{Q}/d\mathbb{P}^B\,|_t}\right)\,. \tag{2.45}$$

When the broker is confident about the learned signal (2.35) (i.e., confident abbout the reference measure $\mathbb{P}^B$), then the value of the ambiguity aversion parameter $\varphi^B > 0$ is small, so any deviation from the reference model is costly. In the extreme $\varphi^B \to 0$, the broker is very confident about the reference measure, so she chooses $\mathbb{P}^B$ because the penalty that results from rejecting the reference measure is too high. On the other hand, if the trader is very ambiguous about the reference model (i.e., has very little confidence in the learned signal (2.35)), considering alternative models results in a very small penalty. In the extreme $\varphi^B \to \infty$, deviations from the reference model are costless, so the broker searches over measures that deliver the worst-case scenario.

Some market participants prefer to trade, or can only trade, via a broker instead of trading directly in the lit market. Here, we assume that the values of k^U and k^I are lower than the value of k^B, so the broker offers traders more competitive quotes than those available in the LOB of the lit market; this assumption does not change the results in the paper, and if we do not impose this, one can relax the restriction on the parameters that guarantee existence and uniqueness of the matrix Riccati differential equation that arises below. The broker can improve the quotes of the LOB because liquidity takers reveal their type (informed or uninformed) when trading with the broker. There is an incentive for the broker to offer attractive loss-leading quotes to the informed trader because the broker uses the flow of the informed trader to learn (albeit not perfectly) the alpha signal and the broker compensates these losses by trading in the lit market with superior information. Note that any market participant who takes liquidity from the LOB will incur the linear (in the rate of trading) cost k^B; i.e., the broker does not have a preferential rate when trading in the LOB. If the traders do not have direct access to the lit exchange, we would not require that k^U, $k^I < k^B$.

The discount the broker offers the informed trader is bounded from below, in particular, we assume that

$$k^B < \frac{k^I}{a^I} + k^I \,. \tag{2.46}$$

In the experiments below, for which $a^I = 1$, the above inequality implies that the broker does not offer more than 50% discount (in trading fees) to the informed trader. The broker can easily relax this bound when a^I is smaller. We use (2.46) to prove existence and uniqueness of the solution to a matrix Riccati differential equation with time-dependent coefficients. The parameter restrictions above are sufficient conditions for the proof of existence and uniqueness to hold. All parameters we employ in the numerical experiments satisfy the above bounds.

The broker maximises expected terminal wealth, where the final inventory is liquidated at the midprice in the lit market and pays a penalty $a^B\,(Q_T^B)^2$. When the value of the penalty parameter a^B is very large (i.e., much greater than the linear costs k of executing the order in the LOB), the strategy is curbed to ensure that by the terminal time T there is no inventory left.

The penultimate term on the right-hand side of the performance criterion (2.43) penalises running inventory, so the broker limits exposure to inventory risk. The higher is the value of the running inventory penalty parameter ϕ^B, the less exposed the broker's strategy is to inventory risk. This penalty is not a financial penalty, it will not affect the P&L of the strategy, but it will have an effect on the broker's optimal trading rate, which affects the distribution of the P&L, see Cartea et al. (2015).

The HJB equation associated with the value function (2.44) is given by

$$\begin{cases} \partial_t \mathfrak{Q} + \mathcal{L}^\alpha \mathfrak{Q} + \mathcal{L}^{\nu^U} \mathfrak{Q} + F\left(t,\alpha,q^I\right)\, \partial_{q^I}\mathfrak{Q} + \frac{1}{2}\,(\sigma^I)^2\,\partial_{q^I\,q^I}\mathfrak{Q} + \frac{1}{2}\,(\sigma^s)^2\,\partial_{SS}\mathfrak{Q} - \phi^B\,(q^B)^2 \\ \quad +\partial_{x^B}\mathfrak{Q}\,F\left(t,\alpha,q^I\right)\,\left(S + k^I\,F\left(t,\alpha,q^I\right)\right) + \partial_{x^B}\mathfrak{Q}\,\nu^U\,(S + k^U\,\nu^U) \\ \quad + \sup_{\nu^B}\inf_{y^\alpha}\Big\{ -\sigma^\alpha\,y^\alpha\,\partial_\alpha\mathfrak{Q} + \frac{1}{2\,\varphi^B}\,(y^\alpha)^2 \\ \qquad\qquad + \partial_{q^B}\mathfrak{Q}(\nu^B - F\left(t,\alpha,q^I\right) - \nu^U) + (\alpha + b\,\nu^B)\,\partial_S\mathfrak{Q} - \partial_{x^B}\mathfrak{Q}\,\nu^B(S + k^B\,\nu^B)\Big\} = 0\,, \\ \mathfrak{Q}(T,\alpha,S,q^B,q^I,x^B,\nu^U) = x + q^B\,S - a^B\,(q^B)^2\,, \end{cases} \tag{2.47}$$

where the infinitesimal operator $\mathcal{L}^\alpha$ of the alpha component is in (2.17) and $\mathcal{L}^{\nu^U}$ given by

$$\mathcal{L}^{\nu^U}\cdot = -\kappa_u\,\nu^U\,\partial_{\nu^U}\cdot + \tfrac{1}{2}\left(\sigma^U\right)^2\partial_{\nu^U\nu^U}\cdot\,, \tag{2.48}$$

is the infinitesimal generator of the trading rate ν^U of the uninformed trader.

The optimal trading rate of the broker in feedback form is

$$\nu^{B*} = \frac{b\,\partial_S\mathfrak{Q} + \partial_{q^B}\mathfrak{Q} - S\,\partial_{x^B}\mathfrak{Q}}{2\,k^B\,\partial_{x^B}\mathfrak{Q}}\,, \qquad \text{and} \qquad y^{\alpha*} = \varphi^B\,\sigma^\alpha\,\partial_\alpha\mathfrak{Q}\,, \tag{2.49}$$

so the PDE associated with the value function is given by

$$\begin{aligned} \partial_t\mathfrak{Q} + \mathcal{L}^\alpha\mathfrak{Q} + F\left(t,\alpha,q^I\right)\,\partial_{q^I}\mathfrak{Q} - \phi^B\,(q^B)^2 + \mathcal{L}^{\nu^U}\mathfrak{Q} - \partial_{q^B}\mathfrak{Q}\left(F\left(t,\alpha,q^I\right) + \nu^U\right) + \alpha\,\partial_S\mathfrak{Q} \\ + \frac{1}{2}\,(\sigma^s)^2\,\partial_{SS}\mathfrak{Q} + \partial_{x^B}\mathfrak{Q}\,\nu^U\,\left(S + k^U\,\nu^U\right) + \partial_{x^B}\mathfrak{Q}\,F\left(t,\alpha,q^I\right)\,\left(S + k^I\,F\left(t,\alpha,q^I\right)\right) \\ + \frac{\left(b\,\partial_S\mathfrak{Q} + \partial_{q^B}\mathfrak{Q} - S\,\partial_{x^B}\mathfrak{Q}\right)^2}{4\,k^B\,\partial_{x^B}\mathfrak{Q}} - \frac{1}{2}\,\varphi^B\,\left(\sigma^\alpha\,\partial_\alpha\mathfrak{Q}\right)^2 = 0\,, \end{aligned} \tag{2.50}$$

with terminal condition $\mathfrak{Q}(T,\alpha,q^B,q^I,S,x^B,\nu^U) = x^B + q^B\,S - a^B\,(q^B)^2$.

Proposition 2.3. *Define the $\mathcal{C}^{1,2}(\mathfrak{T}\times\mathbb{R}^6)$ function $\tilde{\mathfrak{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U)$ as*

$$\tilde{\mathfrak{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U) = x^B + q^B\,S + \mathfrak{q}(t,\alpha,q^B,q^I,\nu^U) \tag{2.51}$$

where

$$\mathfrak{q}(t,\alpha,q^B,q^I,\nu^U) = \mathfrak{q}_0(t) + \mathfrak{q}_1(t)\,\alpha + \mathfrak{q}_2(t)\,\alpha^2 + \mathfrak{q}_3(t)\,q^B + \mathfrak{q}_4(t)\,\left(q^B\right)^2 \tag{2.52}$$

$$+ \mathfrak{q}_5(t)\,q^I + \mathfrak{q}_6(t)\,\left(q^I\right)^2 + \mathfrak{q}_7(t)\,\nu^U + \mathfrak{q}_8(t)\,\left(\nu^U\right)^2 + \mathfrak{q}_9(t)\,\alpha\,q^B \tag{2.53}$$

$$+ \mathfrak{q}_{10}(t)\,\alpha\,q^I + \mathfrak{q}_{11}(t)\,\alpha\,\nu^U + \mathfrak{q}_{12}(t)\,q^B\,q^I + \mathfrak{q}_{13}(t)\,q^B\,\nu^U + \mathfrak{q}_{14}(t)\,q^I\,\nu^U\,. \tag{2.54}$$

For $t \in \mathfrak{T}$, let $\boldsymbol{U}, \boldsymbol{Z}_t, \boldsymbol{Q}_t, \boldsymbol{S} \in \mathbb{R}^{3\times 3}$ be given by

$$\boldsymbol{U} = \begin{pmatrix} -2\,\varphi^B\,(\sigma^\alpha)^2 & 0 & 0 \\ 0 & \frac{1}{k^B} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{Z}_t = \begin{pmatrix} -\kappa^\alpha & 0 & 0 \\ -\mathfrak{g}_0(t) & \frac{b}{2\,k^B} & \mathfrak{g}_1(t) \\ \mathfrak{g}_0(t) & 0 & -\mathfrak{g}_1(t) \end{pmatrix},$$

$$\boldsymbol{Q}_t = \begin{pmatrix} k^I\,(\mathfrak{g}_0(t))^2 & \frac{1}{2} & -k^I\,\mathfrak{g}_0(t)\,\mathfrak{g}_1(t) \\ \frac{1}{2} & \frac{b^2}{4\,k^B} - \phi^B & 0 \\ -k^I\,\mathfrak{g}_0(t)\,\mathfrak{g}_1(t) & 0 & k^I\,(\mathfrak{g}_1(t))^2 \end{pmatrix}, \qquad \boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a^B & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.55}$$

Let $\boldsymbol{P}_t : \mathfrak{T} \to \mathbb{R}^{3\times 3}$ be given by

$$\boldsymbol{P}_t = \begin{pmatrix} \mathfrak{q}_2(t) & \frac{1}{2}\mathfrak{q}_9(t) & \frac{1}{2}\mathfrak{q}_{10}(t) \\ \frac{1}{2}\mathfrak{q}_9(t) & \mathfrak{q}_4(t) & \frac{1}{2}\mathfrak{q}_{12}(t) \\ \frac{1}{2}\mathfrak{q}_{10}(t) & \frac{1}{2}\mathfrak{q}_{12}(t) & \mathfrak{q}_6(t) \end{pmatrix} = \boldsymbol{T}_t\,\boldsymbol{R}_t^{-1}, \tag{2.56}$$

where $\boldsymbol{R}_t, \boldsymbol{T}_t$ solve the linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \boldsymbol{R}_t \\ \boldsymbol{T}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{Z}_t & \boldsymbol{U} \\ -\boldsymbol{Q}_t & -\boldsymbol{Z}_t^\intercal \end{pmatrix}\begin{pmatrix} \boldsymbol{R}_t \\ \boldsymbol{T}_t \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{R}_T \\ \boldsymbol{T}_T \end{pmatrix} = \begin{pmatrix} I \\ \boldsymbol{S} \end{pmatrix}. \tag{2.57}$$

For $t \in \mathfrak{T}$ let $\tilde{\boldsymbol{Y}}_t \in \mathbb{R}^{3\times 3}$ and $\tilde{\boldsymbol{Q}}_t \in \mathbb{R}^3$ be given by

$$\tilde{\boldsymbol{Y}}_t = \begin{pmatrix} -\kappa^\alpha - \kappa_u - 2\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_2(t) & -\mathfrak{g}_0(t) + \frac{\mathfrak{q}_9(t)}{2\,k^B} & \mathfrak{g}_0(t) \\ -(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_9(t) & -\kappa_u + \frac{b+2\,\mathfrak{q}_4(t)}{2\,k^B} & 0 \\ -(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{10}(t) & \mathfrak{g}_1(t) + \frac{\mathfrak{q}_{12}(t)}{2\,k^B} & -\kappa_u - \mathfrak{g}_1(t) \end{pmatrix} \qquad \tilde{\boldsymbol{Q}}_t = \begin{pmatrix} -\mathfrak{q}_9(t) \\ -2\,\mathfrak{q}_2(t) \\ -\mathfrak{q}_{12}(t) \end{pmatrix}. \tag{2.58}$$

Let $\tilde{\boldsymbol{P}}_t : \mathfrak{T} \to \mathbb{R}^3$ be the unique solution to the linear ODE

$$0 = \frac{\mathrm{d}\tilde{\boldsymbol{P}}_t}{\mathrm{d}t} + \tilde{\boldsymbol{Y}}_t\,\tilde{\boldsymbol{P}}_t + \tilde{\boldsymbol{Q}}_t, \qquad t \in \mathfrak{T} \tag{2.59}$$

with $\tilde{\boldsymbol{P}}_T = 0$, and let $(\mathfrak{q}_{11}(t), \mathfrak{q}_{13}(t), \mathfrak{q}_{14}(t))^\intercal = \tilde{\boldsymbol{P}}_t$. The equation for $\mathfrak{q}_8(t)$ is given by

$$\mathfrak{q}_8(t) = \int_t^T \left(\kappa_u - \frac{1}{2}\,\varphi^B\,(\sigma^\alpha\,\mathfrak{q}_{11}(u))^2 - \mathfrak{q}_{13}(u) + \frac{(\mathfrak{q}_{13}(u))^2}{4\,k^B}\right) e^{-2\,\kappa_u\,(u-t)}\,\mathrm{d}u, \tag{2.60}$$

and for all $t \in \mathfrak{T}$ we have $\mathfrak{q}_1(t) = \mathfrak{q}_3(t) = \mathfrak{q}_5(t) = 0$, and

$$\mathfrak{q}_0(t) = \int_t^T \left((\sigma^\alpha)^2\,\mathfrak{q}_2(u) + \frac{(\mathfrak{q}_3(u))^2}{4\,k^B} - \frac{1}{2}\,\varphi^B\,(\sigma^\alpha\,\mathfrak{q}_1(u))^2 + \left(\sigma^U\right)^2\,\mathfrak{q}_8(u)\right)\mathrm{d}u. \tag{2.61}$$

Then, there is $\bar{\varphi} > 0$ such that for any $\varphi^B \in (0, \bar{\varphi}]$, we have that $\tilde{\mathfrak{Q}}$ solves the HJB equation (2.47).

Proof. The function $\tilde{\mathfrak{Q}}$ solves the HJB equation (2.47) if it solves the PDE in (2.50). Given the definition in (2.51), the function $\tilde{\mathfrak{Q}}$ solves the PDE in (2.50) if $\mathfrak{q}$ solves

$$\partial_t\mathfrak{q} + \mathcal{L}^\alpha\mathfrak{q} + F\left(t,\alpha,q^I\right)\,\partial_{q^I}\mathfrak{q} - \phi^B\,(q^B)^2 + \mathcal{L}^{\nu^U}\mathfrak{q} - \partial_{q^B}\mathfrak{q}\left(F\left(t,\alpha,q^I\right) + \nu^U\right) \tag{2.62}$$

$$+\,\alpha\,q^B + k^U\,(\nu^U)^2 + k^I\,F^2\left(t,\alpha,q^I\right) + \frac{\left(b\,q^B + \partial_{q^B}\mathfrak{q}\right)^2}{4\,k^B} - \frac{1}{2}\,\varphi^B\,(\sigma^\alpha\,\partial_\alpha\mathfrak{q})^2 = 0, \tag{2.63}$$

with terminal condition $\mathfrak{q}(T,\alpha,q^B,q^I,\nu^U) = -a^B\,(q^B)^2$. Given the definition of $\mathfrak{q}$ in (2.52), we have that $\mathfrak{q}$ solves the above PDE if $\mathfrak{q}_i := \mathfrak{q}_i(t)$ with $i = 0, 1, \ldots, 14$ solve

$$0 = -\frac{1}{2}\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_1^2 + (\sigma^\alpha)^2\,\mathfrak{q}_2 + \frac{\mathfrak{q}_3^2}{4\,k^B} + \left(\sigma^U\right)^2\mathfrak{q}_8 + \frac{\mathrm{d}\mathfrak{q}_0}{\mathrm{d}t}\,, \tag{2.64}$$

$$0 = -\kappa^\alpha\,\mathfrak{q}_1 - 2\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_1\,\mathfrak{q}_2 - \mathfrak{g}_0(t)\,\mathfrak{q}_3 + \mathfrak{g}_0(t)\,\mathfrak{q}_5 + \frac{\mathfrak{q}_3\,\mathfrak{q}_9}{2\,k^B} + \frac{\mathrm{d}\mathfrak{q}_1}{\mathrm{d}t}\,, \tag{2.65}$$

$$0 = k^I\,\left(\mathfrak{g}_0(t)\right)^2 + \mathfrak{g}_0(t)\,\mathfrak{q}_{10} - 2\,\kappa^\alpha\,\mathfrak{q}_2 - 2\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_2^2 - \mathfrak{g}_0(t)\,\mathfrak{q}_9 + \frac{\mathfrak{q}_9^2}{4\,k^B} + \frac{\mathrm{d}\mathfrak{q}_2}{\mathrm{d}t}\,, \tag{2.66}$$

$$0 = \frac{b\,\mathfrak{q}_3}{2\,k^B} + \frac{\mathfrak{q}_3\,\mathfrak{q}_4}{k^B} - (\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_1\,\mathfrak{q}_9 + \frac{\mathrm{d}\mathfrak{q}_3}{\mathrm{d}t}\,, \tag{2.67}$$

$$0 = \frac{b^2}{4\,k^B} - \phi^B + \frac{b\,\mathfrak{q}_4 + \mathfrak{q}_4^2}{k^B} - \frac{1}{2}\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_9^2 + \frac{\mathrm{d}\mathfrak{q}_4}{\mathrm{d}t}\,, \tag{2.68}$$

$$0 = -\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_1\,\mathfrak{q}_{10} + \mathfrak{g}_1(t)\,\mathfrak{q}_3 + \frac{\mathfrak{q}_{12}\,\mathfrak{q}_3}{2\,k^B} - \mathfrak{g}_1(t)\,\mathfrak{q}_5 + \frac{\mathrm{d}\mathfrak{q}_5}{\mathrm{d}t}\,, \tag{2.69}$$

$$0 = k^I\,\left(\mathfrak{g}_1(t)\right)^2 - \frac{1}{2}\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{10}^2 + \mathfrak{g}_1(t)\,\mathfrak{q}_{12} + \frac{\mathfrak{q}_{12}^2}{4\,k^B} - 2\,\mathfrak{g}_1(t)\,\mathfrak{q}_6 + \frac{\mathrm{d}\mathfrak{q}_6}{\mathrm{d}t}\,, \tag{2.70}$$

$$0 = -\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_1\,\mathfrak{q}_{11} - \mathfrak{q}_3 + \frac{\mathfrak{q}_{13}\,\mathfrak{q}_3}{2\,k^B} - \kappa_u\,\mathfrak{q}_7 + \frac{\mathrm{d}\mathfrak{q}_7}{\mathrm{d}t}\,, \tag{2.71}$$

$$0 = k^U - \frac{1}{2}\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{11}^2 - \mathfrak{q}_{13} + \frac{\mathfrak{q}_{13}^2}{4\,k^B} - 2\,\kappa_u\,\mathfrak{q}_8 + \frac{\mathrm{d}\mathfrak{q}_8}{\mathrm{d}t}\,, \tag{2.72}$$

$$0 = 1 + \mathfrak{g}_0(t)\mathfrak{q}_{12} - 2\,\mathfrak{g}_0(t)\,\mathfrak{q}_4 - \kappa^\alpha\,\mathfrak{q}_9 + \frac{b\,\mathfrak{q}_9}{2\,k^B} - 2\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_2\,\mathfrak{q}_9 + \frac{\mathfrak{q}_4\,\mathfrak{q}_9}{k^B} + \frac{\mathrm{d}\mathfrak{q}_9}{\mathrm{d}t}\,, \tag{2.73}$$

$$0 = -2\,k^I\,\mathfrak{g}_0(t)\,\mathfrak{g}_1(t) - \kappa^\alpha\,\mathfrak{q}_{10} - \mathfrak{g}_1(t)\,\mathfrak{q}_{10} - \mathfrak{g}_0(t)\,\mathfrak{q}_{12} \tag{2.74}$$

$$\qquad - 2\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{10}\,\mathfrak{q}_2 + 2\,\mathfrak{g}_0(t)\,\mathfrak{q}_6 + \mathfrak{g}_1(t)\,\mathfrak{q}_9 + \frac{\mathfrak{q}_{12}\,\mathfrak{q}_9}{2\,k^B} + \frac{\mathrm{d}\mathfrak{q}_{10}}{\mathrm{d}t}\,, \tag{2.75}$$

$$0 = -\kappa^\alpha\,\mathfrak{q}_{11} - \kappa_u\,\mathfrak{q}_{11} - \mathfrak{g}_0(t)\,\mathfrak{q}_{13} + \mathfrak{g}_0(t)\,\mathfrak{q}_{14} - 2\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{11}\,\mathfrak{q}_2 - \mathfrak{q}_9 + \frac{\mathfrak{q}_{13}\,\mathfrak{q}_9}{2\,k^B} + \frac{\mathrm{d}\mathfrak{q}_{11}}{\mathrm{d}t}\,, \tag{2.76}$$

$$0 = \frac{b\,\mathfrak{q}_{12}}{2\,k^B} - \mathfrak{g}_1(t)\,\mathfrak{q}_{12} + 2\,\mathfrak{g}_1(t)\,\mathfrak{q}_4 + \frac{\mathfrak{q}_{12}\,\mathfrak{q}_4}{k^B} - (\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{10}\,\mathfrak{q}_9 + \frac{\mathrm{d}\mathfrak{q}_{12}}{\mathrm{d}t}\,, \tag{2.77}$$

$$0 = -\kappa_u\,\mathfrak{q}_{13} + \frac{b\,\mathfrak{q}_{13}}{2\,k^B} - 2\,\mathfrak{q}_4 + \frac{\mathfrak{q}_{13}\,\mathfrak{q}_4}{k^B} - (\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{11}\,\mathfrak{q}_9 + \frac{\mathrm{d}\mathfrak{q}_{13}}{\mathrm{d}t}\,, \tag{2.78}$$

$$0 = -\,(\sigma^\alpha)^2\,\varphi^B\,\mathfrak{q}_{10}\,\mathfrak{q}_{11} - \mathfrak{q}_{12} + \mathfrak{g}_1(t)\,\mathfrak{q}_{13} + \frac{\mathfrak{q}_{12}\,\mathfrak{q}_{13}}{2\,k^B} - \kappa_u\,\mathfrak{q}_{14} - \mathfrak{g}_1(t)\,\mathfrak{q}_{14} + \frac{\mathrm{d}\mathfrak{q}_{14}}{\mathrm{d}t}\,, \tag{2.79}$$

with $\mathfrak{q}_i(T) = 0$ for $i \neq 4$ and $\mathfrak{q}_4(T) = -a^B$. The equations for $\mathfrak{q}_2, \mathfrak{q}_4, \mathfrak{q}_6, \mathfrak{q}_9, \mathfrak{q}_{10}, \mathfrak{q}_{12}$ solve their respective ODEs if $\boldsymbol{P}_t$ given by (2.56) solves the following matrix differential Riccati equation (MDRE)

$$0 = \frac{\mathrm{d}\boldsymbol{P}_t}{\mathrm{d}t} + \boldsymbol{Z}_t^\intercal\,\boldsymbol{P}_t + \boldsymbol{P}_t\,\boldsymbol{Z}_t + \boldsymbol{P}_t\,\boldsymbol{U}\,\boldsymbol{P}_t + \boldsymbol{Q}_t\,, \qquad t \in \mathfrak{T} \tag{2.80}$$

with terminal condition $\boldsymbol{P}_T = \boldsymbol{S}$ and $\boldsymbol{U}, \boldsymbol{Z}_t, \boldsymbol{Q}_t, \boldsymbol{S} \in \mathbb{R}^{3\times 3}$ given in (2.55). We note that $\boldsymbol{Z}_t, \boldsymbol{Q}_t$ are continuous in $\mathfrak{T}$, thus, integrable and measurable functions, furthermore, $\boldsymbol{U} = \boldsymbol{U}^\intercal$ and $\boldsymbol{Q}_t = \boldsymbol{Q}_t^\intercal$ for all $t \in \mathfrak{T}$. The MDRE above is a Hermitian (or symmetric) Riccati differential equation.

Existence of a solution to (2.80) follows from Theorem 2.3 of Freiling et al. (2000) for

$$\boldsymbol{C} = \begin{pmatrix} \kappa^\alpha/\left((\sigma^\alpha)^2\varphi^B\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{2.81}$$

More precisely, in our problem we have that — using the notation of Freiling et al. (2000) — $B_{11} = \boldsymbol{Z}$, $B_{12} = \boldsymbol{U}$, $B_{21} = -\boldsymbol{Q}$, and $B_{22} = -\boldsymbol{Z}^\intercal$. Then, it follows that for the above choice of $\boldsymbol{C}$ and $\boldsymbol{D}$,

$$\boldsymbol{C} + \boldsymbol{D}\,\boldsymbol{S} + \boldsymbol{S}^\intercal\,\boldsymbol{D}^\intercal = \begin{pmatrix} \kappa^\alpha/\left((\sigma^\alpha)^2\varphi^B\right) & 0 & 0 \\ 0 & 2\,a^B & 0 \\ 0 & 0 & 1 \end{pmatrix} > 0\,, \tag{2.82}$$

that is, the resulting matrix is positive definite. Next, the matrix

$$\boldsymbol{L}_t = \begin{pmatrix} \boldsymbol{C}\,\boldsymbol{Z}_t - \boldsymbol{D}\,\boldsymbol{Q}_t & \boldsymbol{C}\,\boldsymbol{U} + \boldsymbol{Z}_t^\intercal \boldsymbol{D} - \boldsymbol{D}\,\boldsymbol{Z}_t^\intercal \\ 0 & \boldsymbol{U}^\intercal \boldsymbol{D} \end{pmatrix}, \tag{2.83}$$

satisfies that $\boldsymbol{L}_t + \boldsymbol{L}_t^\intercal$ is negative semi-definite if the following inequalities are satisfied for all values of $t \in \mathfrak{T}$: (i) $4\,k^B\,\phi^B - b^2 \geq 0$, (ii) $\mathfrak{g}_0^2(t) - 4\,c\,\mathfrak{g}_1(t)\,\kappa^\alpha - 4\,c\,\mathfrak{g}_1^2(t)\,\kappa^\alpha\,k^I \leq 0$, (iii) $\mathfrak{g}_0^2(t) - 4\,c\,\mathfrak{g}_1(t)\,\kappa^\alpha - 4\,c\,\mathfrak{g}_1^2(t)\,\kappa^\alpha\,k^I + 2\,c^2\,\mathfrak{g}_1(t)\,(\sigma^\alpha)^2\,\varphi^B + 2\,c^2\,\mathfrak{g}_1^2(t)\,(\sigma^\alpha)^2\,\varphi^B\,k^I \leq 0$, and (iv) $\mathfrak{g}_0^2(t) + 2\,c\,\mathfrak{g}_1(t)(-2\,\kappa^\alpha + c\,(\sigma^\alpha)^2\,\varphi^B) - 2\,c\,\mathfrak{g}_1^2(t)(k^B - k^I)(-2\,\kappa^\alpha + c\,(\sigma^\alpha)^2\,\varphi^B) \leq 0$, where $c = \kappa^\alpha / \left((\sigma^\alpha)^2 \varphi^B\right)$. Inequalities (i), (ii), (iii), and (iv) are obtained computing the determinants of the leading principal minors of $\boldsymbol{L}_t + \boldsymbol{L}_t^\intercal$ and employing Sylvester's criterion; we omit two additional inequalities that are trivially satisfied. Inequality (i) was assumed when we introduce the model. Inequality (ii) follows from (iii), and (iii) follows from (iv) which can be written as

$$-\frac{2\,\mathfrak{g}_1(t)}{\varphi^B}\left(\frac{\kappa^\alpha}{\sigma^\alpha}\right)^2 \left(1 - (k^B - k^I)\,\mathfrak{g}_1(t)\right) + \mathfrak{g}_0^2(t) \leq 0\,, \tag{2.84}$$

for all $t \in \mathfrak{T}$. We have that $1 - (k^B - k^I)\,\mathfrak{g}_1(t) > 0$ because in (2.46) we assumed $1 - (k^B - k^I)\,a^I/k^I > 0$, and thus, given that $\mathfrak{g}_1$ is strictly greater than zero and $\mathfrak{g}_0$ is bounded above, one sees that given κ^α and σ^α, there exists a positive value $\bar{\varphi}$ such that for all $\varphi^B \leq \bar{\varphi}$ (2.84) holds. The parameters we choose in numerical examples below satisfy the above bound, which implies that $\boldsymbol{L}_t + \boldsymbol{L}_t^\intercal \leq 0$. Given $\boldsymbol{L}_t + \boldsymbol{L}_t^\intercal \leq 0$, we use Theorem 2.3 in Freiling et al. (2000) to show that there is a solution to (2.80). Similar to part II of the proof of Theorem 3.5 in Casgrain and Jaimungal (2020), given that the solution exists and is continuous in $[0, T]$, it is bounded, and we conclude that the unique solution takes the form in (2.56), see Theorem 3.1.1 in Abou-Kandil et al. (2012). Thus, $\mathfrak{q}_i$ for $i = 2, 4, 6, 9, 10, 12$ solve their respective ODEs.

Next, trivially, $\mathfrak{q}_1 = \mathfrak{q}_3 = \mathfrak{q}_5 = 0$ solve their respective ODEs, and $\mathfrak{q}_{11}, \mathfrak{q}_{13}, \mathfrak{q}_{14}$ solve their ODEs because $\tilde{\boldsymbol{P}}$ solves (2.59). A short calculation shows that indeed (2.60) solves the ODE for $\mathfrak{q}_8$ and it follows that (2.61) solves the ODE for $\mathfrak{q}_0$. □

Theorem 2.4. *The control problem in (2.44) has a classical solution. The value function in (2.44) is given by $\tilde{\mathfrak{Q}}$ in (2.51) and the optimal trading strategy of the broker is the admissible Markovian optimal control*

$$\nu_t^{B*} = \mathfrak{r}_1(t)\,\alpha_t - \mathfrak{r}_2(t)\,Q_t^{B*} - \mathfrak{r}_3(t)\,Q_t^{I*} + \mathfrak{r}_4(t)\,\nu_t^U \tag{2.85}$$

$$= \tilde{\mathfrak{r}}_1(t)\,\nu_t^{I*} - \mathfrak{r}_2(t)\,Q_t^{B*} - \tilde{\mathfrak{r}}_3(t)\,Q_t^{I*} + \mathfrak{r}_4(t)\,\nu_t^U\,, \tag{2.86}$$

for $\mathcal{C}^1$ functions $\mathfrak{r}_1, \tilde{\mathfrak{r}}_1, \mathfrak{r}_2, \mathfrak{r}_3, \tilde{\mathfrak{r}}_3, \mathfrak{r}_4 : \mathfrak{T} \to \mathbb{R}$ given by

$$\mathfrak{r}_1(t) = \frac{\mathfrak{q}_9(t)}{2\,k^B}\,, \qquad \tilde{\mathfrak{r}}_1(t) = \frac{\mathfrak{q}_9(t)\,\mathfrak{g}_3(t)}{2\,k^B}\,, \qquad \mathfrak{r}_2(t) = -\frac{b + 2\,\mathfrak{q}_4(t)}{2\,k^B}\,, \tag{2.87}$$

$$\mathfrak{r}_3(t) = -\frac{\mathfrak{q}_{12}(t)}{2\,k^B}\,, \qquad \tilde{\mathfrak{r}}_3(t) = -\frac{\mathfrak{q}_9(t)\,\mathfrak{g}_2(t) + \mathfrak{q}_{12}(t)}{2\,k^B}\,, \qquad \mathfrak{r}_4(t) = \frac{\mathfrak{q}_{13}(t)}{2\,k^B}\,. \tag{2.88}$$

The optimal change of measure is characterised by the admissible control

$$y_t^{\alpha*} = \varphi^B\,\sigma^\alpha\,\left(\mathfrak{q}_1(t) + 2\,\mathfrak{q}_2(t)\,\alpha_t + \mathfrak{q}_9(t)\,Q_t^{B*} + \mathfrak{q}_{10}(t)\,Q_t^{I*} + \mathfrak{q}_{11}(t)\,\nu_t^U\right)\,. \tag{2.89}$$

For a proof see Appendix B.

We conclude this section with two important remarks.

Remark 2.5. The trading strategies we derived above can be thought as the equilibrium of a two-player game. In particular, they form a Stackelberg equilibrium in the sense that the informed trader optimises his performance criterion and that strategy is used to compute the leader's (broker's) optimal strategy. We emphasise that the optimisation of the follower (informed trader) does not use ν^B because this is not observable. Instead, the ambiguity averse informed trader entertains alternative models.

Remark 2.6. In the models above, the quotes offered by the broker are around the midprice in the lit market; this can be relaxed. Allowing for the values of k^I, k^U to be different from the value of k^B encodes the market

market practice where a broker (e.g., LMAX Broker) offers quotes that improve the best ask and the best bid (symmetrically) in the lit market (e.g., LMAX Exchange). An example would be to improve the best ask and the best bid in the lit market by one dollar per million traded. Alternatively, the model allows for the broker to employ quotes that are skewed (as in Barzykin et al. (2023)). Here, in the problem of the informed trader we would interpret S_t as the midprice of the quotes the broker offers her clients, and given that the optimal strategy of the informed trader does not depend on S_t, then, we interpret S_t as the midprice in the lit market for the problem of the broker.

3. Performance

Here, we showcase the performance of the optimal trading strategies we obtained in closed form. We discretise the trading window $[0, T]$, with $T = 1$, in 10,000 steps and perform 10^6 simulations. Model parameters for the price dynamics are $\alpha_0 = 0$, $S_0 = 100$, $\kappa^\alpha = 5$, $\sigma^\alpha = 1$, $\sigma^s = 1$. The price impact and penalty parameters are $k^I = k^U = 1.0 \times 10^{-3}$, $k^B = 1.2 \times 10^{-3}$, $b = 10^{-3}$, $a^I = 1$, $a^B = 1$, and $\phi^B = \phi^I = \varphi^I = \varphi^B = 10^{-2}$. Parameters for the dynamics of the uninformed trader's trading rate are $\nu_0^U = 0$, $\kappa_u = 15$, and $\sigma^U = 100$. The choice of parameter values for both traders in this example, ensure that the losses to the informed trader are of the same order of magnitude as the profits obtained from trading with the uninformed trader.[7]

Figure 1 shows sample paths for α_t, ν_t^{I*}, ν_t^U, ν_t^{B*}, Q_t^{I*}, and Q_t^{B*}. For approximately the first half of the trading horizon, the trading rates of the informed trader and the broker are similar. During this period, the broker's trading rate in the lit market is mostly driven by the alpha component, which is learnt from trading with the informed trader. As the end of the trading horizon approaches, the broker must externalise all inventory; this is clearly shown in the right-hand plot of the bottom panel. Both, the informed trader and the broker finish with a flat inventory due to the choice of the terminal inventory parameters a^I and a^B.

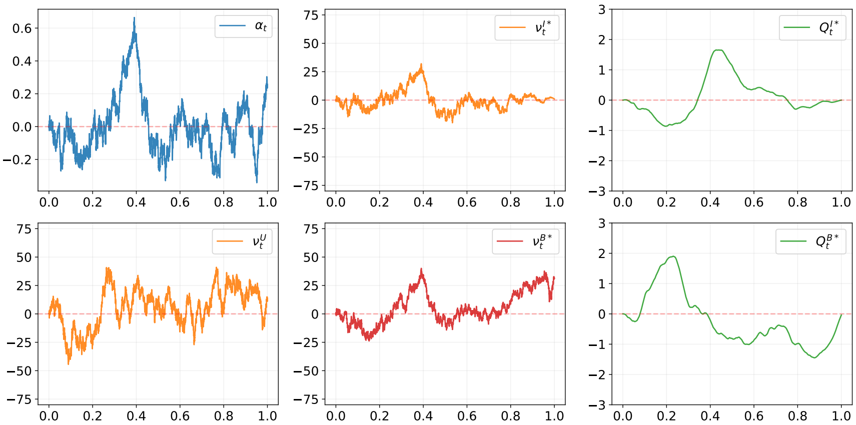

Figure 1: Sample path for α_t, ν_t^{I*}, ν_t^U, ν_t^{B*}, Q_t^{I*}, and Q_t^{B*}.

Recall that the broker's rate of trading in the lit market is

$$\nu_t^{B*} = \mathfrak{r}_1(t)\,\alpha_t - \mathfrak{r}_2(t)\,Q_t^{B*} - \mathfrak{r}_3(t)\,Q_t^{I*} + \mathfrak{r}_4(t)\,\nu_t^U\,.$$

Figure 2 plots each term of the broker's strategy displayed above. The first term is the speculative component of the broker's strategy. The second term controls the broker's inventory. The third term reacts to the informed trader's inventory. And the last term accounts for how much of the uninformed flow is immediately externalised – note that flow that is internalised will gradually be externalised via the second term of the strategy.

[7]See Cartea and Jaimungal (2016) for impact parameters in Nasdaq and Cartea et al. (2022) for impact parameters in spot FX.

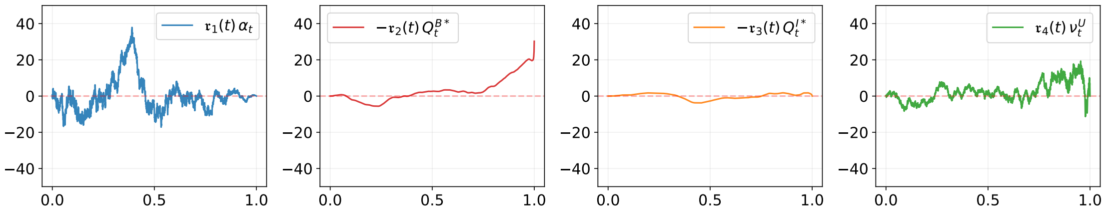

Figure 2: Sample paths for each component of the broker's trading rate ν_t^{B*}.

Next, in Figure 3 we decompose the terminal P&L of the broker. Here, given that the initial cash and inventory is zero, the P&L is equal to the terminal cash plus inventory times midprice. The first histogram in the figure shows the broker's P&L from trading with the informed trader. We see that, on average, the broker trades with the informed at a loss. These average losses are the broker's payment for order flow from the informed to learn the signal α. The second plot shows the broker's P&L from trading with the uninformed trader. On average, the broker benefits from the uninformed flow, which offsets some of the losses to the informed trader. The third plot shows the broker's P&L from trading in the lit market. On average, the broker makes a profit from this trading activity. Finally, the last plot in the figure shows the net P&L of the broker which is the aggregate of the three P&Ls from trading with informed, uninformed, and in the lit market.

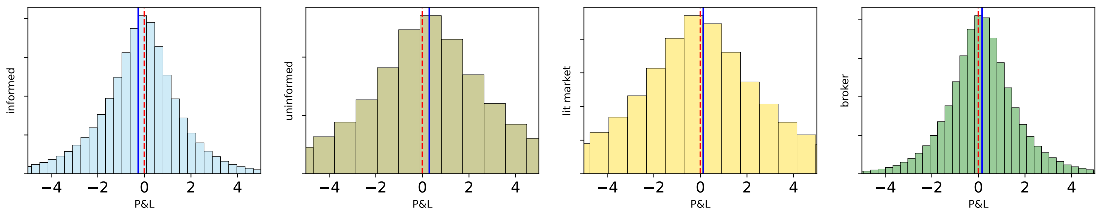

Figure 3: Broker's P&L conditional on trading with the informed (first plot), the uninformed (second plot), and in the lit market (third plot). Fourth plot shows the broker's net P&L.

Figure 4 shows the value function of the broker as a function of the ratio $k^{I,U}/k^B$; this ratio is the percentage of the liquidity costs that the broker charges to her clients in terms of the liquidity cost paid by the broker in the lit market. Recall that in the scenario discussed above $k^I = k^B$, which is the far-right point in Figure 4. To attract order flow from the informed and the uninformed traders, the broker offers them quotes that are better than those available in the lit market, that is, $k^I < k^B$ and $k^U < k^B$. Figure 4 shows the range of discount that the broker can offer both traders while remaining profitable. We leave for further work the price sensitivity of clients and how competition among brokers affects client order-flow.

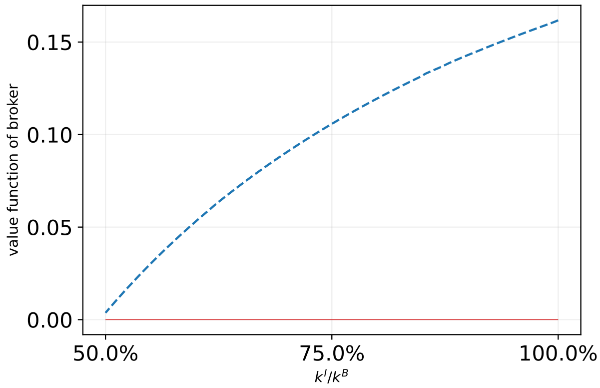

Figure 4: Value function of the broker as a function of the ratio k^I/k^B.

In Figure 4, the order flow of the informed trader reacts to the parameter k^I whereas the order flow of the uninformed trader does not. To address this, we use the value of the parameter k^U to modulate the dynamics of ν^U. Here, when the price of liquidity offered to the uninformed trader is high, the uninformed trader trades less, and when the price of liquidity is low, then the uninformed trader sends more order flow to the broker. For the next experiment, the SDE for the order flow of the uninformed trader is given by

$$d\nu_t^U = -\kappa_u \, \nu_t^U \, dt + \sigma^U \frac{k^B - k^U}{k^B - 1.0 \times 10^{-3}} \, dW_t^U \,, \qquad \nu_0^U \in \mathbb{R} \,, \tag{3.1}$$

for $k^U \in (0.5\, k^B, k^B)$. The value 1.0×10^{-3} in the denominator is the price of liquidity for the uninformed trader used in the experiments above, so when $k^U = 1.0 \times 10^{-3}$ we recover the previous case. Within this formulation, it is easy to see that as the price of liquidity increases, the volume of the uniformed orders decreases – in the limit $k^U \to k^B$ the uniformed trader does not send orders to the broker because liquidity is too expensive.

Figure 5 shows the value function of the broker as a function of the relative prices of liquidity in the lit market and those charged by the broker, i.e., $k^I, k^U/k^B$. The figure shows that the broker maximises her value function when liquidity costs charged by the broker to the informed and uniformed traders are approximately 65% of the liquidity costs in the lit market.

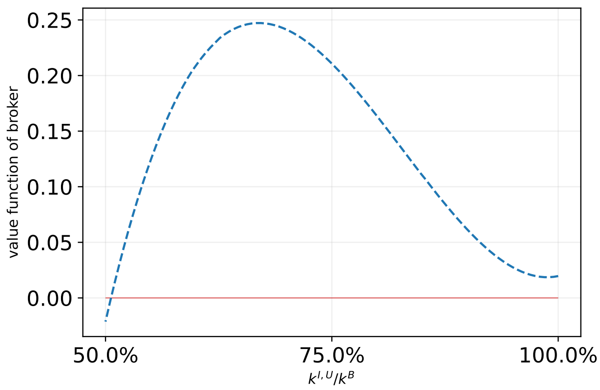

Figure 5: Value function of the broker as a function of the ratio k^I/k^B when the order flow of the uninformed trader obeys (3.1).

Lastly, Figure 6 shows the coefficients $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_4$ for three values of the broker's ambiguity aversion parameter φ^B. As the value of the ambiguity parameter φ^B increases (i.e., broker is less confident about the learned signal from the informed trader's flow) the less the internalisation-externalisation strategy will react to the learned signal. Clearly, as the broker becomes less confident about the signal she learns from the informed flow, the less she will rely on that knowledge when internalising or internalising trades. Similarly, the higher the value of φ^B, the quicker the inventory will revert to zero (see $\mathfrak{r}_2$), and the lesser the effect of the informed trader's inventory (see $\mathfrak{r}_3$), and the stronger the effect of the trading speed of the uninformed trader (see $\mathfrak{r}_4$).

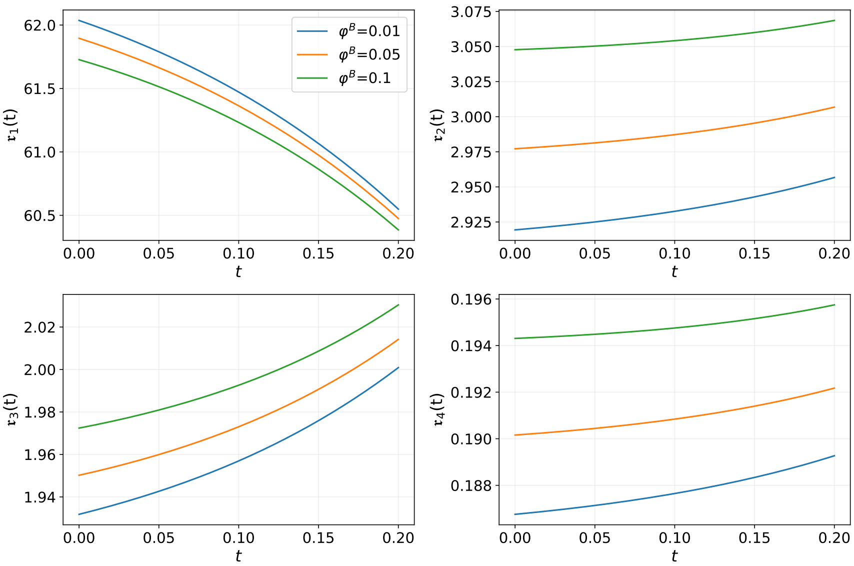

Figure 6: Functions $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_4 : \mathfrak{T} \to \mathbb{R}$ for the first fifth of the trading horizon as the as a function of thee values of the ambiguity aversion parameter φ^B.

3.1. Benchmarks

In this section, we compare the financial performance of the broker's optimal strategy with that of other strategies. We assume that $\phi^B = 10^{-10}$ and the value of the other parameters is as above. Recall that the running inventory penalty and that the ambiguity aversion penalty in the performance criterion are not financial penalties, but do affect the distribution of the terminal cash of the broker.

The strategies followed by broker in the three benchmarks are:

1. Internalise the flow of the uninformed and immediately externalise the flow of the informed, and gradually externalise inventory with TWAP:

$$\nu_t^{\mathrm{B1}} = \nu_t^I - \frac{Q_t^{B1}}{T-t} \,. \tag{3.2}$$

2. Internalise the flow of both traders, and gradually externalise inventory with TWAP:

$$\nu_t^{\mathrm{B2}} = -\frac{Q_t^{B2}}{T-t} \,. \tag{3.3}$$

3. Immediately externalise the flow of both traders:

$$\nu_t^{\mathrm{B3}} = \nu_t^I + \nu_t^U \,. \tag{3.4}$$

Table 1 reports the average outperformance of the strategy developed above over the benchmarks. The units are $\$/M$, i.e., dollars per million dollars of total volume traded. Specifically, we compute

$$\frac{1}{n}\sum_{i=1}^{n} \frac{\left(X_{T,i}^{B*} + Q_{T,i}^{B*}\, S_{T,i}^{B*}\right) - \left(X_{T,i}^{B} + Q_{T,i}^{B}\, S_{T,i}^{B}\right)}{\int_0^T S_{u,i}\, \left(|\nu_{u,i}^{I}| + |\nu_{u,i}^{U}| + |\nu_{u,i}^{B}|\right) \mathrm{d}u} \,, \tag{3.5}$$

where $i = 1, 2, \cdots, n$ with $n =$ 1,000,000 simulations, superscript $B*$ denotes the optimal strategy derived here, and superscript B is a benchmark strategy.

Benchmark	outperformance (std) $\phi^B = 10^{-10}$
B1	15.1 (294.7)
B2	62.5 (567.0)
B3	68.7 (587.3)

Table 1: Average outperformance measure in USD per million USD traded.

Table 2 shows the building blocks for Table 1. All quantities reported in this paper are significantly different from zero – we reject the null hypothesis of a one-sample t-test at 99% confidence (the null hypothesis says that the mean of the normal random variable generating the data is zero). The table shows that benchmark 1 (B1) is the only benchmark that is profitable on average, which helps explain why the market often adopts such strategy.

Strategy	mean	std $\phi^B = 10^{-10}$	mean/std
Optimal	42.6	618.0	0.069
B1	26.1	633.2	0.041
B2	−26.8	736.9	−0.036
B3	−24.7	37.8	−0.652

Table 2: Average performance measures for each strategy un USD per million USD traded.

Table 3 shows the outperformance (3.5) of the optimal strategy over the three benchmarks in five scenarios. On the left, and as a reference, are the outperformances reported in Table 1. The four columns that follow report the outperformance when the mean-reversion rate of the signal is ±%10 and the volatility of the signal is ±%10 of the benchmark scenario.

Strategy	outperformance (std), $\phi^B = 10^{-10}$				
	Baseline (Table 1)	$\kappa^\alpha : +10\%$	$\kappa^\alpha : -10\%$	$\sigma^\alpha : +10\%$	$\sigma^\alpha : -10\%$
B1	15.1 (294.7)	14.6 (275.3)	15.6 (316.7)	15.6 (304.1)	14.2 (284.8)
B2	62.5 (567.0)	56.0 (529.4)	69.5 (606.3)	70.4 (589.5)	54.5 (541.8)
B3	68.7 (587.3)	71.1 (594.1)	68.5 (578.1)	68.5 (571.4)	72.7 (602.3)

Table 3: Average outperformance measure in USD per million USD traded.

Finally, Table 4 shows the outperformance (3.5) of the optimal strategy over the three benchmarks when the value of the ambiguity parameter of the broker is large, $\varphi^B = 1$, and when it is negligible, $\varphi^B = 10^{-10}$. As above, on the left, we show the outperformances reported in Table 1.

Strategy	outperformance (std)		
	Baseline (Table 1)	$\varphi^B = 10^{-10}$	$\varphi^B = 1$
B1	15.1 (294.7)	15.2 (295.1)	12.7 (290.0)
B2	62.5 (567.0)	62.4 (566.4)	61.1 (537.2)
B3	68.7 (587.3)	69.6 (587.1)	68.6 (451.7)

Table 4: Average outperformance measure in USD per million USD traded. Both cases use $\phi^B = 10^{-10}$.

We see that when the broker's level of ambiguity aversion is high, the performance of the optimal strategy is much better than current market practices.

3.2. Robustness analysis

Our model assumes that the broker knows the model and the value of the parameters of the informed trader. In particular, we assume that the broker knows the value of the terminal inventory penalty parameter a^I, and knows the value of Φ^I, which is the sum of the running inventory penalty ϕ^I and the ambiguity aversion parameter φ^I. However, it is not straightforward for the broker to learn the value of the parameters used by the informed trader. Thus, it is important that we study the changes in performance when these assumptions do not hold.

Here, we discuss the effect of parameter value misspecification on the strategy of the broker and measure how this affects the P&L of the strategy (instead of the value function). In particular, we show that misspecifying the initial inventory of the informed trader Q_0^I has an important effect on the P&L of the broker. On the other hand, the effect of misspecifying the value of other parameters, such as a^I and Φ^I is also relevant but less important for the range of parameter values we consider. For example, one expects, or the broker can learn, that the informed trader closes positions often; thus, the value of the terminal inventory parameter a^I must be high. It is straightforward to see that the value function of the broker is not too sensitive to the value of a^I once the value a^I used by the informed trader is large enough.

In Table 5, $Q_0^I = 0$ when the broker computes the inventory of the informed trader,[8] whereas in reality, the initial inventory of the informed trader is $Q_0^I \sim \mathcal{N}(0, \sigma^Q)$ for $\sigma^Q \in \{1, 2, 3, 4, 5\}$ – see Figure 1 for a typical trajectory of the inventory of the informed trader. This corresponds to the case where the broker does not know the initial inventory of the informed trader. We use measure (3.5) to show the costs of misspecifying parameter values. Here, the benchmark B uses the optimal strategy in Theorem 2.4 when the broker incorrectly uses $Q_0^I = 0$ and $B*$ uses the optimal strategy with the correct initial inventory. Our results show that the P&L of the broker considerably deteriorates as the discrepancy between the assumed and the true initial inventory of the informed trader increases.

$\sigma^Q = 1$	$\sigma^Q = 2$	$\sigma^Q = 3$	$\sigma^Q = 4$	$\sigma^Q = 5$
17.7 (683.0)	74.8 (748.3)	162.0 (855.0)	277.9 (989.2)	417.2 (1147.0)

Table 5: Average outperformance measure for each strategy un USD per million USD traded.

Next we investigate the changes in the P&L of the broker as a function of the misspecification of the value of the parameter Φ^I of the informed trader. In particular, the informed trader employs $\phi^I = \varphi^I = 10^{-2}$ and we study the cases where the broker mistakenly assumes that the informed trader uses $\tilde{\Phi}^I$ with $\tilde{\Phi}^I = m\,\Phi^I$, where $m \in \{0.1, 10, 100\}$. When $m = 0.1$, the outperformance of the strategy that does not misspecify the parameter value is not statistically different from zero, that is, the p-value of the t-test is 0.19; i.e., there is not enough evidence to reject the null hypothesis that the mean of the outperformance measure is different from zero at a confidence level of 5%. When $m \in \{10, 100\}$, the outperformance measures are statistically different from zero (the p-values of the t-tests are less than 0.01) and the outperformance (std) is 0.3 (656.3) for $m = 10$, and 2.0 (658.8) for $m = 100$. We clearly see that the effect of misspecifying the value of Φ^I is not as detrimental to the P&L as misspecifying the informed trader's initial inventory, for the range of parameters we consider.

4. Conclusions

We developed an internalisation and externalisation strategy for a broker who trades with informed and uninformed counterparties. The strategies are obtained in closed form. We showed that the broker 'pays' the informed trader for their toxic flow (i.e., trades with the informed at a loss) to learn the signal of the informed trader. The broker acknowledges that her model to learn the signal of the informed trader is misspecified, so derives a strategy that is robust to model uncertainty. The broker's optimal strategy uses this signal to optimally hedge her position in the lit market and to execute speculative trades. We conducted numerical experiments to show the superior performance of our strategy compared with a number of market practices.

[8]The inventory is given by $Q_t^I = Q_0^I + \int_0^t \nu_s^I \mathrm{d}s$. Thus, if Q_0^I is mispecified, then Q_t^I is mispecified and the trading speed of the broker uses an incorrect value for the inventory of the informed trader.

Future work will focus on extending the current framework to one where the broker faces a number of informed traders with signals that materialise over different trading horizons.

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Appendix A. Proof of verification theorem: informed trader

We start by showing that both controls, ν^{I*} and y^{S*}, are admissible. The process ν^{I*} is given by

$$\nu_t^{I*} = \mathfrak{g}_0(t)\,\alpha_t - \mathfrak{g}_1(t)\,Q_t^{I*}\,, \tag{A.1}$$

where $\mathfrak{g}_0, \mathfrak{g}_1 : [0,T] \to \mathbb{R}$ are continuous and α_t is the strong solution to the SDE in (2.2). Since Q_t^{I*} is continuous, it follows that ν_t^{I*} has càdlàg paths which given that ν_t^{I*} is adapted implies that ν_t^{I*} is progressively measurable. Given that we have strong solutions it follows that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \alpha_t^2\right] < \infty\,, \tag{A.2}$$

which is a sufficient condition to imply that ν^{I*} is square-integrable and concludes the proof that ν^{I*} is admissible. The proof of admissibility of y^{S*} relies on integrability and localisation arguments which we omit for brevity; see proof of Theorem 1 in Cheridito et al. (2007) and Theorem 1 in Cartea and Sánchez-Betancourt (2021). Next, we provide the verification argument that the function we introduce in (2.20) is indeed the value function in (2.14).

First, let $\nu^I \in \mathcal{A}^I$ and let $y_t^{S*}(\nu^I) = \varphi^I\,\sigma^S\,Q_t^{I,\nu^I}$. We let $\mathbb{Q}^{*,\nu^I}$ be the measure induced by $y_t^{S*}(\nu^I)$ and observe that under this measure

$$\mathrm{d}S_t = \left(\alpha_t - \sigma^S\,y_t^{S*}(\nu^I)\right)\mathrm{d}t + \sigma^S\,\mathrm{d}W^{\mathbb{Q}^{*,\nu^I}}\,, \tag{A.3}$$

for $W^{\mathbb{Q}^{*,\nu^I}}$ a $\mathbb{Q}^{*,\nu^I}$–Brownian motion independent of W^α. Clearly, the dynamics of α do not change under the new measure. Next, given that $\tilde{\mathfrak{H}}$ is $\mathcal{C}^{1,2}(\mathfrak{T}\times\mathbb{R}^4)$, for $(t,\alpha,q^I,S,x)\in\mathfrak{T}\times\mathbb{R}^4$ we apply Itô's lemma to obtain that

$$\begin{aligned}
&\tilde{\mathfrak{H}}(T,\alpha_T,Q_T^{I,\nu^I},S_T,X_T^I)\\
&\quad= X_T^I + Q_T^{I,\nu^I}\,S_T - a^I\left(Q_T^{I,\nu^I}\right)^2\\
&\quad= \tilde{\mathfrak{H}}(t,\alpha,q^I,S,x) + \int_t^T \sigma^S\partial_S\tilde{\mathfrak{H}}\,\mathrm{d}W^{\mathbb{Q}^{*,\nu^I}}\\
&\qquad + \int_t^T \Big\{\partial_u\tilde{\mathfrak{H}} + \mathcal{L}^\alpha\tilde{\mathfrak{H}} + \nu_u^I\partial_{q^I}\tilde{\mathfrak{H}}\\
&\qquad\quad + \left(\alpha_u - \sigma^S\,y_u^{S*}(\nu^I)\right)\partial_S\tilde{\mathfrak{H}} + \frac{1}{2}\left(\sigma^S\right)^2\partial_{SS}\tilde{\mathfrak{H}} - \nu_u^I\left(S_u - k^I\,\nu_u^I\right)\partial_x\tilde{\mathfrak{H}}\Big\}\mathrm{d}u\,.
\end{aligned}$$

Given that $\tilde{\mathfrak{H}}$ solves the PDE in (2.19) that follows from the HJB equation (2.16), we have that

$$\begin{aligned}
&\partial_u\tilde{\mathfrak{H}} + \mathcal{L}^\alpha\tilde{\mathfrak{H}} + \nu_u^I\partial_{q^I}\tilde{\mathfrak{H}} + \left(\alpha_u - \sigma^S\,y_u^{S*}(\nu^I)\right)\partial_S\tilde{\mathfrak{H}} + \frac{1}{2}\left(\sigma^S\right)^2\partial_{SS}\tilde{\mathfrak{H}} - \nu_u^I\left(S_u + k^I\,\nu_u^I\right)\partial_x\tilde{\mathfrak{H}} && \text{(A.4)}\\
&\quad\le \phi^I\left(Q_u^{I,\nu^I}\right)^2 - \frac{1}{2\,\varphi^I}\left(y_u^{S*}(\nu^I)\right)^2\,, && \text{(A.5)}
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^{*,\nu^I}}\left[X_T^I + Q_T^{I,\nu^I}\,S_T - a^I\left(Q_T^{I,\nu^I}\right)^2 - \int_t^T \phi^I\left(Q_u^{I,\nu^I}\right)^2 - \frac{1}{2\,\varphi^I}\left(y_u^{S*}(\nu^I)\right)^2\mathrm{d}u\right] && \text{(A.6)}\\
&\qquad\le \tilde{\mathfrak{H}}(t,\alpha,q^I,S,x) && \text{(A.7)}
\end{aligned}$$

and given that ν^I is an arbitrary control and $\tilde{\mathfrak{H}}(t,\alpha,q^I,S,x)$ is an upper bound, we have that

$$
\begin{aligned}
&\mathfrak{H}(t,\alpha,q^I,S,x)\\
&\quad= \sup_{\nu^I\in\mathcal{A}^I}\ \inf_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^I} \mathbb{E}^{\mathbb{Q}}\left[X_T^I + Q_T^{I,\nu^I}\,S_T - a^I\,\left(Q_T^{I,\nu^I}\right)^2 - \int_t^T \phi^I\,\left(Q_u^{I,\nu^I}\right)^2 - \frac{1}{2\,\varphi^I}\,\left(y_u^S\right)^2 \mathrm{d}u\right]\\
&\quad= \sup_{\nu^I\in\mathcal{A}^I} \mathbb{E}^{\mathbb{Q}^{*,\nu^I}}\left[X_T^I + Q_T^{I,\nu^I}\,S_T - a^I\,\left(Q_T^{I,\nu^I}\right)^2 - \int_t^T \phi^I\,\left(Q_u^{I,\nu^I}\right)^2 - \frac{1}{2\,\varphi^I}\,\left(y_u^{S*}(\nu^I)\right)^2 \mathrm{d}u\right]\\
&\quad\le \tilde{\mathfrak{H}}(t,\alpha,q^I,S,x)\,.
\end{aligned}
$$

For the other side of the inequality, we fix $\nu^{I*}\in\mathcal{A}^I$ and observe that for any $\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^I$ given that $\tilde{\mathfrak{H}}$ solves the PDE in (2.19) and given that one can interchange inf and sup in the HJB equation (2.16), we have that

$$
\partial_u\tilde{\mathfrak{H}} + \mathcal{L}^\alpha\tilde{\mathfrak{H}} + \nu_u^{I*}\,\partial_{q^I}\tilde{\mathfrak{H}} + \left(\alpha_u - \sigma^S\,y_u^S\right)\partial_S\tilde{\mathfrak{H}} + \frac{1}{2}\left(\sigma^S\right)^2\partial_{SS}\tilde{\mathfrak{H}} - \nu_u^{I*}\left(S_u - k^I\,\nu_u^{I*}\right)\partial_x\tilde{\mathfrak{H}} \tag{A.8}
$$

$$
\ge \phi^I\left(Q_u^{I*}\right)^2 - \frac{1}{2\,\varphi^I}\left(y_u^S\right)^2\,, \tag{A.9}
$$

and by the same arguments as above, together with (A.8) we have that

$$
\mathbb{E}^{\mathbb{Q}}\left[X_T^{I*} + Q_T^{I*}\,S_T - a^I\,\left(Q_T^{I*}\right)^2 - \int_t^T \phi^I\,\left(Q_u^{I*}\right)^2 - \frac{1}{2\,\varphi^I}\,\left(y_u^S\right)^2 \mathrm{d}u\right] \ge \tilde{\mathfrak{H}}(t,\alpha,q^I,S,x)\,, \tag{A.10}
$$

and because this holds for any $\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^I$, we have that

$$
\mathfrak{H}(t,\alpha,q^I,S,x) \tag{A.11}
$$

$$
= \sup_{\nu^I\in\mathcal{A}^I}\ \inf_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^I} \mathbb{E}^{\mathbb{Q}}\left[X_T^I + Q_T^{I,\nu^I}\,S_T - a^I\,\left(Q_T^{I,\nu^I}\right)^2 - \int_t^T \phi^I\,\left(Q_u^{I,\nu^I}\right)^2 - \frac{1}{2\,\varphi^I}\,\left(y_u^S\right)^2 \mathrm{d}u\right] \tag{A.12}
$$

$$
= \inf_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^I} \mathbb{E}^{\mathbb{Q}}\left[X_T^{I*} + Q_T^{I*}\,S_T - a^I\,\left(Q_T^{I*}\right)^2 - \int_t^T \phi^I\,\left(Q_u^{I*}\right)^2 - \frac{1}{2\,\varphi^I}\,\left(y_u^S\right)^2 \mathrm{d}u\right] \tag{A.13}
$$

$$
\ge \tilde{\mathfrak{H}}(t,\alpha,q^I,S,x)\,, \tag{A.14}
$$

which concludes the proof.

Appendix B. Proof of verification theorem: broker

Similar to the proof in Appendix A, the control ν^{B*} is admissible because $\mathfrak{r}_i$ for $i\in\{1,2,3,4\}$ are continuous in $[0,T]$ and α,ν^U are progressively measurable which suffices to prove that ν^{B*} is progressively measurable. Since α,ν^U have strong solutions, it follows that

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \alpha_t^2 + \left(\nu_t^U\right)^2\right] < \infty\,, \tag{B.1}
$$

and this is a sufficient condition to show that ν^{B*} is square-integrable. As above, the proof of admissibility of $y^{\alpha*}$ relies on integrability and localisation arguments. Next, we provide the verification argument that the function we introduce in (2.51) is indeed the value function in (2.44).

Let $\nu^B\in\mathcal{A}^B$ and let $y_t^{\alpha*}(\nu^B) = \varphi^B\,\sigma^\alpha\,\left(\mathfrak{q}_1(t) + 2\,\mathfrak{q}_2(t)\,\alpha_t + \mathfrak{q}_9(t)\,Q_t^B + \mathfrak{q}_{10}(t)\,Q_t^{I*} + \mathfrak{q}_{11}(t)\,\nu_t^U\right)$. We let $\mathbb{Q}^{*,\nu^B}$ be the measure induced by $y_t^{\alpha*}(\nu^B)$ and observe that under this measure

$$
\mathrm{d}\alpha_t = \left(-\kappa^\alpha\,\alpha_t - \sigma^\alpha\,y_t^{\alpha*}(\nu^B)\right)\mathrm{d}t + \sigma^\alpha\,\mathrm{d}W^{\mathbb{Q}^{*,\nu^B}}\,, \tag{B.2}
$$

for $W^{\mathbb{Q}^{*,\nu^B}}$ a $\mathbb{Q}^{*,\nu^B}$–Brownian motion independent of W^S. Clearly, the dynamics of S do not change under the new measure. Next, given that $\tilde{\mathfrak{Q}}$ is $\mathcal{C}^{1,2}(\mathfrak{T}\times\mathbb{R}^6)$, for $(t,\alpha,q^B,q^I,S,x^B,\nu^U)\in\mathfrak{T}\times\mathbb{R}^6$ we apply Itô's lemma to obtain that

$$\begin{aligned}
&\tilde{\mathfrak{Q}}(T,\,\alpha_T,\,Q_T^{B,\nu^B},\,Q_t^{I*},S_T,X_T^B,\nu_T^U)\\
&\quad= X_T^B + Q_T^{B,\nu^B}\,S_T - a^B\,\left(Q_T^{B,\nu^B}\right)^2\\
&\quad= \tilde{\mathfrak{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U) + \int_t^T \sigma^\alpha\,\partial_\alpha\tilde{\mathfrak{Q}}\,\mathrm{d}W^{\mathbb{Q}^{*,\nu^B}}\\
&\qquad+\int_t^T \Big\{\partial_u\tilde{\mathfrak{Q}} + \mathcal{L}^S\tilde{\mathfrak{Q}} + \left(\nu_u^B-\nu_u^I-\nu_u^U\right)\partial_{q^B}\tilde{\mathfrak{Q}} + \nu_u^I\,\partial_{q^I}\tilde{\mathfrak{Q}}\\
&\qquad\qquad+\left(-\kappa^\alpha\,\alpha_u - \sigma^\alpha\,y_u^{\alpha*}(\nu^B)\right)\partial_\alpha\tilde{\mathfrak{Q}} + \frac{1}{2}\,(\sigma^\alpha)^2\,\partial_{\alpha\alpha}\tilde{\mathfrak{Q}}\\
&\qquad\qquad-\nu_u^B\,\left(S_u+k^B\,\nu_u^B\right)\partial_x\tilde{\mathfrak{Q}} + \nu_u^I\,\left(S_u+k^I\,\nu_u^I\right)\partial_x\tilde{\mathfrak{Q}} + \nu_u^U\,\left(S_u+k^U\,\nu_u^U\right)\partial_x\tilde{\mathfrak{Q}}\Big\}\mathrm{d}u\,.
\end{aligned}$$

Given that $\tilde{\mathfrak{Q}}$ solves the PDE in (2.50) that follows from the HJB equation (2.47), we have that

$$\partial_u\tilde{\mathfrak{Q}} + \mathcal{L}^S\tilde{\mathfrak{Q}} + \left(\nu_u^B-\nu_u^I-\nu_u^U\right)\partial_{q^B}\tilde{\mathfrak{Q}} + \nu_u^I\,\partial_{q^I}\tilde{\mathfrak{Q}} + \left(-\kappa^\alpha\,\alpha_u - \sigma^\alpha\,y_u^{\alpha*}(\nu^B)\right)\partial_\alpha\tilde{\mathfrak{Q}} + \frac{1}{2}\,(\sigma^\alpha)^2\,\partial_{\alpha\alpha}\tilde{\mathfrak{Q}} \tag{B.3}$$

$$-\nu_u^B\,\left(S_u+k^B\,\nu_u^B\right)\partial_x\tilde{\mathfrak{Q}} + \nu_u^I\,\left(S_u+k^I\,\nu_u^I\right)\partial_x\tilde{\mathfrak{Q}} + \nu_u^U\,\left(S_u+k^U\,\nu_u^U\right)\partial_x\tilde{\mathfrak{Q}} \tag{B.4}$$

$$\le \phi^B\left(Q_u^{B,\nu^B}\right)^2 - \frac{1}{2\,\varphi^B}\left(y_u^{\alpha*}(\nu^B)\right)^2\,, \tag{B.5}$$

which implies that

$$\mathbb{E}^{\mathbb{Q}^{*,\nu^B}}\left[X_T^B + Q_T^{B,\nu^B}\,S_T - a^B\,\left(Q_T^{B,\nu^B}\right)^2 - \int_t^T \phi^B\left(Q_u^{B,\nu^B}\right)^2 - \frac{1}{2\,\varphi^B}\left(y_u^{\alpha*}(\nu^B)\right)^2\mathrm{d}u\right] \tag{B.6}$$

$$\le \tilde{\mathfrak{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U) \tag{B.7}$$

and given that ν^B is an arbitrary control and $\tilde{\mathfrak{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U)$ is an upper bound, we have that

$$\begin{aligned}
&\mathfrak{Q}(t,\alpha,q^B,q^I,S,x^B,\nu^U)\\
&\quad=\sup_{\nu^B\in\mathcal{A}^B}\inf_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^B}\mathbb{E}^{\mathbb{Q}}\left[X_T^B + Q_T^{B,\nu^B}\,S_T - a^B\,\left(Q_T^{B,\nu^B}\right)^2 - \int_t^T \phi^B\left(Q_u^{B,\nu^B}\right)^2 - \frac{1}{2\,\varphi^B}\,(y_u^\alpha)^2\,\mathrm{d}u\right]\\
&\quad=\sup_{\nu^B\in\mathcal{A}^B}\mathbb{E}^{\mathbb{Q}^{*,\nu^B}}\left[X_T^B + Q_T^{B,\nu^B}\,S_T - a^B\,\left(Q_T^{B,\nu^B}\right)^2 - \int_t^T \phi^B\left(Q_u^{B,\nu^B}\right)^2 - \frac{1}{2\,\varphi^B}\left(y_u^{\alpha*}(\nu^B)\right)^2\mathrm{d}u\right]\\
&\quad\le\tilde{\mathfrak{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U)\,.
\end{aligned}$$

For the other side of the inequality, we fix $\nu^{B*}\in\mathcal{A}^B$ and observe that for any $\mathbb{Q}\in\boldsymbol{\mathcal{Q}}^B$ given that $\tilde{\mathfrak{Q}}$ solves the PDE in (2.50) and given that one can interchange inf and sup in the HJB equation (2.47), we have that

$$\partial_u\tilde{\mathfrak{Q}} + \mathcal{L}^S\tilde{\mathfrak{Q}} + \left(\nu_u^{B*}-\nu_u^I-\nu_u^U\right)\partial_{q^B}\tilde{\mathfrak{Q}} + \nu_u^I\,\partial_{q^I}\tilde{\mathfrak{Q}} + \left(-\kappa^\alpha\,\alpha_u - \sigma^\alpha\,y_u^{\alpha}(\nu^{B*})\right)\partial_\alpha\tilde{\mathfrak{Q}} + \frac{1}{2}\,(\sigma^\alpha)^2\,\partial_{\alpha\alpha}\tilde{\mathfrak{Q}} \tag{B.8}$$

$$-\nu_u^{B*}\,\left(S_u+k^B\,\nu_u^{B*}\right)\partial_x\tilde{\mathfrak{Q}} + \nu_u^I\,\left(S_u+k^I\,\nu_u^I\right)\partial_x\tilde{\mathfrak{Q}} + \nu_u^U\,\left(S_u+k^U\,\nu_u^U\right)\partial_x\tilde{\mathfrak{Q}} \tag{B.9}$$

$$\ge \phi^B\left(Q_u^{B*}\right)^2 - \frac{1}{2\,\varphi^B}\,(y_u^\alpha)^2\,, \tag{B.10}$$

and by the same arguments as above, together with (B.8) we have that

$$\mathbb{E}^{\mathbb{Q}}\left[X_T^{B*} + Q_T^{B*}\,S_T - a^B\,\left(Q_T^{B*}\right)^2 - \int_t^T \phi^B\left(Q_u^{B*}\right)^2 - \frac{1}{2\,\varphi^B}\,(y_u^\alpha)^2\,\mathrm{d}u\right] \ge \tilde{\mathfrak{Q}}(t,\alpha,q^B,q^I,S,x^B,\nu^U)\,, \tag{B.11}$$

and because this holds for any $\mathbb{Q} \in \boldsymbol{\mathcal{Q}}^B$, we have that

$$
\begin{aligned}
&\mathfrak{Q}(t, \alpha, q^B, q^I, S, x^B, \nu^U) && \text{(B.12)}\\
&= \sup_{\nu^B \in \mathcal{A}^B} \inf_{\mathbb{Q} \in \boldsymbol{\mathcal{Q}}^B} \mathbb{E}^{\mathbb{Q}}\left[X_T^B + Q_T^{B,\nu^B} S_T - a^B \left(Q_T^{B,\nu^B}\right)^2 - \int_t^T \phi^B \left(Q_u^{B,\nu^B}\right)^2 - \frac{1}{2\,\varphi^B} (y_u^\alpha)^2 \, \mathrm{d}u \right] && \text{(B.13)}\\
&= \inf_{\mathbb{Q} \in \boldsymbol{\mathcal{Q}}^B} \mathbb{E}^{\mathbb{Q}}\left[X_T^{B*} + Q_T^{B*} S_T - a^B \left(Q_T^{B*}\right)^2 - \int_t^T \phi^B \left(Q_u^{B*}\right)^2 - \frac{1}{2\,\varphi^B} (y_u^\alpha)^2 \, \mathrm{d}u \right] && \text{(B.14)}\\
&\geq \tilde{\mathfrak{Q}}(t, \alpha, q^B, q^I, S, x^B, \nu^U)\,, && \text{(B.15)}
\end{aligned}
$$

which concludes the proof.