Regularity and uniqueness in the Calculus of Variations

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To my mother, to Asaf and, of course,
to the rest of my family,
with a special loving place in my memory
to my grandmother, Rubén, José and Soledad.

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Abstract

This thesis is about regularity and uniqueness of minimizers of integral functionals of the form

$$\mathfrak{F}(u) := \int_\Omega F(\nabla u(x))\,\mathrm{d}x,$$

where $F \in C^2(\mathbb{R}^{N\times n})$ is a strongly quasiconvex integrand with p-growth, $\Omega \subseteq \mathbb{R}^n$ is an open bounded domain and $u \in \mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$ for some boundary datum $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

The first contribution of this work is a full regularity result, up to the boundary, for global minimizers of $\mathfrak{F}$ provided that the boundary condition g satisfies that $\|\nabla g\|_{\mathrm{L}^p} < \varepsilon$ for some $\varepsilon > 0$ depending only on n, N, the parameters given by the strong quasiconvexity and p-growth conditions and, most importantly, on an arbitrary but fixed constant $M > 0$ for which we require that $\|\nabla g\|_{0,\alpha} < M$. Furthermore, when the domain Ω is star-shaped, we extend the regularity result to the case of $\mathrm{W}^{1,p}$-local minimizers.

On the other hand, for the case of global minimizers we exploit the compactness provided by the aforementioned regularity result to establish the main contribution of this thesis: we prove that, under *essentially* the same smallness assumptions over the boundary condition g that we mentioned above, the minimizer of $\mathfrak{F}$ in $\mathrm{W}^{1,p}_g$ is unique. This result appears in contrast to the non-uniqueness examples previously given by Spadaro [Spa09], for which the boundary conditions are required to be suitably large.

Another contribution of this work is a new proof that we provide for the sufficiency result for strong local minima, established previously by Grabovsky and Mengesha. As in [GM09], we consider a set of admissible functions in which part of the boundary is allowed to be free and we assume strong quasiconvexity at the free boundary. In addition to the new proof, in the case of full Dirichlet boundary conditions and when $p = 2$, we remove one of the coercivity conditions imposed in [GM09]. The result states that, under the standard strong quasiconvexity and p-growth assumptions on the integrand, C^1 extremals at which the second variation is strongly positive are L^p-local minimizers.

In connection with this, we also provide a further full (interior) regularity result for Lipschitz solutions of the weak Euler-Lagrange equation at which the second variation is strongly positive, provided their derivative is of vanishing mean oscillation.

Contents

Introduction

One of the forces by which Mathematics is driven is our interest to describe and understand what we observe from nature. This work focuses on the fundamental problems of regularity and uniqueness of minimizers of quasiconvex integral functionals. We work in the framework of functionals of the form

$$\mathfrak{F}(u) := \int_{\Omega} F(\nabla u(x))\,\mathrm{d}x,$$

defined for suitable Sobolev mappings $u \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$, where Ω is a bounded open subset of $\mathbb{R}^n$ and, in general, $n, N > 1$ and $1 < p < \infty$. The results that we present here, however, concern mainly the case $p \geq 2$. Classical examples of such integrals include the Dirichlet integral, the minimal surface integral and stored-energy functions for hyper-elastic materials. In the particular context of elasticity, the importance of minimizers relies on the second law of thermodynamics, according to which every body shall deform in search of equilibria, at which the potential energy is minimal.

The main results of this thesis establish that, under suitable assumptions on the integrand F, the minimizers of $\mathfrak{F}$ that satisfy certain small Dirichlet boundary conditions are smooth and, furthermore, they are unique. This provides positive answers, in this particular situation, to long-standing questions in the Calculus of Variations [Bal02]. In addition, in this work we also discuss the fundamental problem of finding sufficient conditions to guarantee that weak solutions of the Euler-Lagrange equation are strong local minimizers of the corresponding integral functional. In this spirit, we have developed a new proof for an outstanding sufficiency result already present in the literature [GM09]. With this, we aim at obtaining stronger versions of this result and we further discuss some of its possible extensions.

As we shall see along this work, the subjects of existence, regularity and uniqueness in the Calculus of Variations are intrinsically related to each other. In particular, the assumptions on the integrand F required to ensure existence of minimizers play also an essential role to establish regularity and, in the particular context of this thesis, having regularity of solutions to variational problems is a core ingredient while obtaining uniqueness. Furthermore, as we will discuss in depth in Chapter 2, the regularity of solutions to the weak Euler-Lagrange equation is also a key ingredient in finding a suitable set of sufficient conditions for strong local minimizers.

On the other hand, despite the non-linear character of the problems that we address here, our results rely strongly on the classical regularity theory for linear systems of partial differential equations. In addition, regarding uniqueness, a key ingredient for us has been the fundamental theory of Fredholm operators. We find in this a reminder of the way in which Mathematics most often evolves: by building upon the solid structure that has been constructed by other mathematicians for centuries and that, luckily for us, has been rigorously consolidated over the last hundred years. The references to the literature that we often quote in this thesis are an essential part of it, since most developments in Mathematics can only be achieved after understanding the background and by assuming our role in the construction of the *spiral of knowledge.*

In the spirit of taking a glimpse at the historical context that lies behind this thesis, we recall that references to what is now called *Calculus of Variations* can already be found in Virgil's legendary Aeneid poem from 29-19 B.C. He portrays princess Dido's mathematical skills, describing how she acquired a big portion of land, where the city of Carthage was founded, by using her intuitive knowledge that, among all the planar regions sharing a fixed perimeter, the circle is the one enclosing the largest area. The mathematical problem behind this is related to the isoperimetric problems studied by the school of Pappus in ancient Greece.

Modern Calculus of Variations goes back to the 17^{th} century, with questions formulated by Fermat, in the area of geometrical optics; Newton, concerning bodies moving in fluids, and Galileo, who posed the seminal problem of the *brachistochrone*: to find the shape of the path, joining two points in space, along which a particle will slide down the fastest under the effects of gravity and without friction. This problem was solved by Johann Bernoulli in 1696 and, soon after, also by his brother Jakob, L'Hospital, Leibniz and Newton [Bal98, Dac04].

A fundamental step in the development of the Calculus of Variations was taken in the 18th century when Euler and Lagrange derived a systematic method of approaching variational problems by establishing the Euler-Lagrange equations. As a result of his work with Lagrange, it was Euler who gave the name of Calculus of Variations to this field of Mathematics. In Section 1.5 of this work we discuss the way in which the Euler-Lagrange equations provide a necessary condition for a function to be a minimizer.

On the other hand, Riemann reformulated the problem of finding solutions to the Laplace equation $\Delta u = 0$, constrained to Dirichlet boundary conditions, by approaching it from the point of view of minimizing the *Dirichlet integral*, which turned out to be equivalent to solving the partial differential equation. While trying to solve variational problems like this, for many years the direct method was implemented by assuming that, if a variational functional is bounded below, then it should attain its minimum value. However, in 1895 Weierstrass provided an example showing that this was not necessarily the case: letting $\mathcal{A} := \big\{u \in C^1[-1,1] : u(-1) = -1 \text{ and } u(1) = 1\big\}$, it is not difficult to see that the functional $\mathfrak{F}\colon \mathcal{A} \to \mathbb{R}$ given by

$$\mathfrak{F}(u) := \int_{-1}^{1} |xu'(x)| \, \mathrm{d}x$$

is such that $\inf_{\mathcal{A}} \mathfrak{F} = 0$, but there cannot be a function $u \in \mathcal{A}$ such that $\mathfrak{F}(u) = 0$.

In 1900 Hilbert successfully proved the existence of minimizers for the Dirichlet integral [Hil04]. Soon after, and exploiting the newly developed Lebesgue integral, Tonelli established the concept of semicontinuity, which is the underlying idea behind the application of the direct method and not the continuity of the functional, as was originally thought.

Also in 1900 Hilbert presented, at the International Conference of Mathematicians in Paris, a list of twenty-three problems that he considered significant for the progress of Mathematics in the 20th century. The 19th, 20th and 23rd of these problems were devoted to the Calculus of Variations and it is the 19th of them that can be seen as the forerunner of the regularity theory. It asks whether the solutions to regular problems in the Calculus of Variations are always necessarily analytic [Hil02].

By independent means, de Giorgi and Nash solved the problem affirmatively for the scalar case, i.e., when the admissible functions are real valued (see [DG57, Nas58]). More precisely, they proved the Hölder continuity of solutions to elliptic differential equations in divergence form. The application to the minima of functionals is straightforward, since they all satisfy the (weak) Euler-Lagrange equation. The ellipticity condition assumed by de Giorgi translates into the convexity of the integrand involved in the functional. On the other hand, the de Giorgi-Nash theorem was later extended in various contexts. In particular, Ladyzhenskaya and Ural'ceva used their result to settle Hilbert's 19^{th} problem in the scalar case [LU68].

Soon after, it was de Giorgi himself who proved that his regularity theorem does not extend to the vectorial case, by constructing a linear elliptic system with bounded measurable coefficients and whose solutions are discontinuous [DG68]. Giusti and Miranda [GM68] extended this example to non-linear systems with regular coefficients and to minimizers of functionals. Similar examples were found by Maz'ya about at the same time [Maz68].

Simultaneously, and by adapting the ideas of de Giorgi [DG61], Federer [Fed69] and Almgren [Alm68] in the study of minimal surfaces, Morrey was able to show that weak solutions of non-linear elliptic systems are regular outside a subset of Lebesgue measure zero of their domain. With this, he established the fundamental concept of *partial regularity*, which appeared as a reinterpretation of his paper *Partial regularity results for non-linear elliptic systems* [Mor68]. This notion still lies at the heart of the regularity theory.

In 1975, Nečas gave an example of a regular variational problem in high dimensions that admits a non-differentiable Lipschitz map as a solution [Neč77]. This made it clear that we cannot expect to have full regularity in the general multidimensional case. In fact, Šverák and Yan established that minimizers can be non-Lipschitz and even unbounded [ŠY02]. In 1986, Evans proved [Eva86] that quasiconvexity, a concept introduced by Morrey in 1952 in connection with his work on sequential weak lower semicontinuity and generalizing the notion of convexity [Mor52], is also (in a suitable strict form) closely related to the partial regularity of minimizers.

Many improvements of Evans' result were developed, for example, by Acerbi-Fusco [AF87, AF89a], Evans-Gariepy [EG87], Fusco-Hutchinson [FH85] and Giaquinta-Modica [GM86]. In 2003, Kristensen and Taheri extended the partial regularity results, available for absolute minimizers, to the case of certain local minimizers of integral functionals [KT03].

In contrast to all these regularity results, Müller and Šverák constructed remarkable examples of Lipschitz solutions to the weak Euler-Lagrange equation with strongly quasiconvex integrands, such that they are nowhere C^1 [MŠ03]. Then, Kristensen and Taheri modified the Müller-Šverák example to establish that it is possible to construct a strongly quasiconvex integrand with strictly positive second variation, such that the corresponding weak Euler-Lagrange equation admits Lipschitz solutions that are nowhere C^1 [KT03]. Furthermore, Székelyhidi extended further this new example by Kristensen-Taheri to the case of polyconvex integrands [Szé04]. These results imply that the partial regularity that both global and local minimizers satisfy is inherent to their nature as minimizers and cannot be deduced solely from the fact that they satisfy the weak Euler-Lagrange equation.

In this brief review of the history of the Calculus of Variations, it stands out that there has been a vibrant and renewed interest in the area since the second half of the 20$^{\text{th}}$ century. This was partly motivated by problems arising in non-linear elasticity theory and the subject remains very fertile with many questions still waiting for an answer. In addition, the motivation of solving variational problems related to the deformations applied to an elastic body in space is one of the reasons why the vectorial case, in which $n, N > 1$, becomes particularly relevant in this scenario [Bal98, Bal02].

The framework in which we address the subjects of regularity and uniqueness of minimizers in this work is under the natural assumptions of strong quasiconvexity and polynomial growth of the smooth integrand. These conditions are natural in the sense that they guarantee the existence of minimizers by the direct method. In Chapter 1 we set the general assumptions on the integrands that are considered throughout the rest of the thesis and we motivate each one of them, including the way in which they enable us to apply the direct method and how the Euler-Lagrange equation is rigorously derived under these hypotheses. In addition, we state other notions of convexity relevant in the Calculus of Variations and we further discuss them for the particular case of quadratic functions. These are of special interest on their own and, for the purposes of this work, essential while using Taylor Approximation Theorem for the non-linear problems that are our subject of study. Finally, we devote the last section of Chapter 1 to discuss the notion of smooth domains in space. The need to work with sufficiently regular domains comes not only from the frequent use of the Sobolev Embedding Theorem that we make, but in particular for the results concerning regularity up to the boundary in the subsequent chapters.

The second chapter of this work is related to the problem of finding sufficient conditions for a function to be a strong local minimizer. The examples of Müller-Šverák, Kristensen-Taheri and Székelyhidi show that there can be solutions to the weak Euler-Lagrange equation that are not regular on any open subset of their domain, while Evans' and Kristensen-Taheri's regularity results show that minimizers should be partially regular. It is then a natural goal to understand when is it that the solutions of the Euler-Lagrange equation furnish minimizers of the given functional. A full answer to this question for the scalar case was established by Weierstrass in the 19^{th} century. By then, it was clear that it is essential to consider the possibility of the existence of multiple local minimizers that are not global minimizers. Weierstrass realized the significance of giving possibly different norms to the space of admissible functions, in order to determine whether a given function minimizes a variational problem in a local sense. This also gave rise to the two different notions of weak and strong local minimizers. He established a fundamental set of sufficient conditions for a smooth extremal to be a local minimizer in the case of one single independent variable. His work was then generalized by Hestenes, in 1948, for the case of one dependent variable [Hes48].

Many years had to pass before the question of finding sufficient conditions for local minimizers (in a strong sense), could find a solution in the vectorial case. Motivated by problems from non-linear elasticity, Ball posed this fundamental question. He conjectured that the natural generalization of Weierstrass' conditions had to be based on quasiconvexity notions, that had already been proven to be also necessary for the existence of minimizers (see [Bal98, Section 6.2] and [BM84a]).

Taheri used Hestenes' method to deal with the problem of L^r-local minima, remarking that the results hold in the vectorial case as well under the assumption that the integrand is convex [Tah01]. In 2008, Y. Grabovsky and T. Mengesha fully settled the conjecture of Ball that strong quasiconvexity and strong positivity of the second variation are sufficient for a C^1 solution of the weak Euler-Lagrange equation to be a strong local minimizer.

In Chapter 2 of this work we present a new proof of Grabovsky-Mengesha's theorem, stated here as Theorem 55. The motivation behind providing this new approach is to simplify the proof with the aim of relaxing the a priori regularity assumptions made on the potential local minimizer. The proof that we give here consists essentially of appropriately exploiting a result by K. Zhang [Zha92], according to which smooth solutions of the weak Euler-Lagrange equation minimize the functional locally in the domain. On the other hand, this new proof of the sufficiency theorem allows us to remove some coercivity conditions imposed on the integrand, at least for the quadratic case and while assuming Dirichlet boundary conditions.

In connection with the a priori regularity assumption on the extremal imposed in Grabovsky-Mengesha's sufficiency result, and recalling that both global and strong local minimizers (in the sense considered in [KT03]) are known to be only partially regular, a natural question is whether it is possible to relax the regularity assumption on the extremal so that the strong positivity of the second variation and the standard growth and strong quasiconvexity hypotheses over the smooth integrand, still ensure that the extremal is a strong local minimizer. We have already mentioned that the examples given by Müller & Šverák, Kristensen & Taheri and Székelyhidi make it clear that Lipschitz continuity of the extremal is not enough. On the other hand, it is also an important and natural question under what conditions we can guarantee full regularity (up to the boundary) as required in Grabovsky-Mengesha's theorem. Motivated by these questions, in Chapter 3 of this work we establish a full (interior) regularity result for Lipschitz extremals at which the second variation is strongly positive, provided their derivative is of class VMO, meaning that its mean oscillations converge to zero uniformly in the space.

Chapter 3 appears then in deep connection with the issue of regularity of minimizers. In a similar spirit, an important question is that of knowing when we can ensure full regularity for the minimizers of integral functionals under the standard assumptions of strong quasiconvexity and polynomial growth.

In Chapter 4 we establish that global minimizers are smooth up to the boundary if we assume Dirichlet boundary conditions that are small in their $\mathrm{W}^{1,p}$ norm, where $p \geq 2$. The precise statement corresponds to Theorem 76. This is related to the work of Zhang [Zha91], where he gives hypotheses, different from the ones considered here, under which the smooth solution to the Euler-Lagrange equations, given by the Implicit Function Theorem, coincides with the global minimizer obtained by the direct method. There, the derivative of the boundary condition is also assumed to be small but in the more restrictive sense of C^4 (see also [Val88, Theorem 7.1]). The regularity result given in Theorem 76 is one of the main contributions of this work. We emphasize that the results in this chapter are obtained without imposing any conditions on the higher order derivatives of the boundary conditions or on the second variation.[1] Furthermore, the regularity proof is obtained via a direct argument, hence

[1] See [Zha92].

providing precise estimates for the Hölder coefficients of the minimizers, which turn out to have powerful consequences regarding their uniqueness.

In addition, in Theorem 84 we also obtain a full regularity result, up to the boundary, for $W^{1,p}$-local minimizers where the boundary condition is small in the $W^{1,p}$ sense and the domain is smooth and star-shaped. The idea behind the full regularity result for local minimizers is mainly to reduce the problem to the case of global minimizers. However, in order to achieve this, we also have to obtain some energy estimates for local minimizers that are new in the literature and that we introduce here in Proposition 81. These estimates were inspired by those obtained by Taheri in [Tah03, Proposition 2.1].

In the context of elasticity theory, the hypothesis of having small boundary conditions to ensure full regularity of the state of minimum energy is consistent with what we would expect from a real object being subject to a small deformation, which is that no fractures or defects of any kind would arise.

With the same physical motivation, in Chapter 5 we approach another fundamental question regarding uniqueness of minimizers in the Calculus of Variations. As discussed by Ball [Bal02, Section 2.6], for mixed boundary value problems of elasticity (where part of the boundary is allowed to move freely) it is expected to have non-uniqueness of the equilibrium solutions. An intuitively clear example is related to the buckling of rods, plates and shells.

Knops and Stuart gave an elegant proof to establish uniqueness of smooth *stationary points*[2] for the case when the boundary conditions are linear and the domain is star-shaped [KS84]. Taheri generalized these results by removing the regularity assumption on the deformation [Tah01]. In contrast, Spadaro gave a negative answer to the question of uniqueness posed by Ball [Bal02, Problem 8] and constructed a remarkable example of a strongly polyconvex integrand that admits at least two minimizers [Spa09].

Chapter 5 of this work begins with a brief discussion of Spadaro's example and then it is devoted to establishing a positive uniqueness result for global minimizers. This comes in compliance with the full regularity result obtained in Chapter 4 and it is also motivated by the fact that under small displacements of the boundary, the deformations should not behave too differently from the case where there is no displacement at all, for which the unique minimizer is the null function. The exact statement corresponds to Theorem 103. We expect

[2] See Definition 82

this result, and the technique used to prove it, to open the door to more general uniqueness results in the Calculus of Variations.

In this context we remark that, unless otherwise stated, all the results that appear in Chapters 4 and 5 were developed by the author of this thesis in collaboration with Jan Kristensen and they are part of a more ambitious project, currently in progress. Additionally, the results and proofs that appear in Chapters 2 and 3 were developed by the author of this work, with the exception of those preliminary statements and proofs that we include to make the text more self-contained. However, a precise reference to the corresponding literature has been given in all these cases.

At the end of this thesis we include five appendices. The first one is devoted to fix some of the standard notation used in the thesis. In Appendices B and C we compile, respectively, the standard results and notation from function spaces and theory of linear operators that we use throughout the rest of the text. The last two appendices are aimed at stating technical results used within the regularity proofs. In Appendix D we show how to construct moduli of continuity with specific properties that we often make use of. In Appendix E we define the auxiliary function V and we encompass here the notation and properties necessary to handle the polynomial non-quadratic growth of the integrands. The results compiled in the appendices, as well as in Chapter 1 of this work, comprise standard tools in the Calculus of Variations and we have included them here for convenience of the reader, specifying the corresponding references to the existent literature.

This work is written at a stage of Mathematics in which the Calculus of Variations is still a very fertile field and there are numerous questions waiting for answers regarding regularity, uniqueness and even existence of minimizers for certain integral functionals trying to model, for example, interpenetration of matter [Bal02, Problem 1]. We expect that the estimates obtained and the techniques used here can be applied to other problems related to the three main questions that we address here regarding sufficient conditions for strong local minima, regularity and uniqueness in the Calculus of Variations. Some of these improvements include, of course, higher regularity of minimizers using Schauder estimates, as well as to gain some control on the level of non-uniqueness that can arise from allowing, for example, mixed boundary conditions.

Chapter 1

Some elements of the Calculus of Variations

A fundamental problem in the Calculus of Variations concerns multiple integrals of the form

$$\mathfrak{F}(u) := \int_\Omega F(\nabla u(x))\,\mathrm{d}x, \tag{1.1}$$

where $u\colon \Omega \to \mathbb{R}^N$ lies in a suitable set of Sobolev mappings and $\Omega \subseteq \mathbb{R}^n$ is an open and bounded domain. Here, $\nabla u(x)$ denotes the Jacobi matrix consisting of the weak partial derivatives of u evaluated at x and arranged in the usual way as an $N \times n$ matrix. In addition, the integrand F is a smooth function to which we will shortly impose more precise restrictions. Typical examples of such integrals include the Dirichlet integral, the minimal surface integral and stored energy functions for hyper-elastic materials, although only the former fits into the framework considered in this work. See, for example, [Bal77] and [Dac08].

The answers to the questions of existence, regularity and uniqueness of minimizers of the functional $\mathfrak{F}$ naturally depend on the choice of admissible functions and on the properties of the integrand F. In this context, we now introduce the general framework in which most of the results of this work are presented. The rest of this chapter is devoted to motivate each of the assumptions that we now introduce and to give a brief summary of the main tools of the Calculus of Variations that are consistently used later on.

Throughout this work, we consider variational problems for which the set of admissible solutions is contained in the Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ for some $1 < p < \infty$.

In Chapter 3, we will actually assume that the admissible functions are in $\mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$. However, in Chapters 2, 4 and 5, we restrict ourselves to the case $p \in [2, \infty)$. The subquadratic case, in which $p \in (1, 2)$, has been broadly studied, for example, in [AF89b], [Bec11] and [CFM98]. We remark, however, that the new results presented in Chapters 2, 4 and 5 of this work are still not known to hold for $p \in (1, 2)$.

Regarding the integrand, we assume that $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is such that:

(H0) $F \in C^2(\mathbb{R}^{N\times n})$;

(H1) F has polynomial p-growth, i.e., for some fixed $p \in (1, \infty)$ there exists a constant $c_1 > 0$ such that, for every $z \in \mathbb{R}^{N\times n}$, $|F(z)| \leq c_1(1 + |z|^p)$ and

(H2) F is strongly p-quasiconvex, meaning that there is $c_2 > 0$ such that, for every $z \in \mathbb{R}^{N\times n}$ and every $\varphi \in \mathrm{W}_0^{1,\infty}(Q, \mathbb{R}^N)$, it holds that

$$c_2 \int_Q |V(\nabla\varphi)|^2 \,\mathrm{d}x \leq \int_Q \big(F(z + \nabla\varphi) - F(z)\big) \,\mathrm{d}x,$$

where the function $V\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is given by

$$V(z) := (1 + |z|^2)^{\frac{p-2}{4}} z. \tag{1.2}$$

For the case $p \in [2, \infty)$, the function V is such that $|V|^2$ is bounded above and below by scalar multiples of $|\cdot|^2 + |\cdot|^p$. We will use very often this property throughout this text. However, it is worth mentioning that, if $p \in (1, 2)$, the notion of strong quasiconvexity is still defined as in (H2), and it is particularly important to define it in that way. In Appendix E we state precisely all the properties of V and, to maintain generality, we emphasize there what are the differences and the common features between the V function defined for $p \in (1, 2)$ and $p \geq 2$, respectively.

On the other hand, the main reason behind assuming (H0) is that we approach the non-linear problem of proving regularity for minimizers of $\mathfrak{F}$, which is non-linear, by approximating F by its second order Taylor polynomial. This way, we can make use of the good regularity properties that are already known for solutions to linear elliptic systems of equations. As we shall see in Section 1.4, (H1)-(H2) provide F with a mean-coercivity property. This sets a suitable scenario to obtain existence of minimizers and, as we establish in the subsequent chapters, regularity and also uniqueness in some particular situations. The importance of (H1) and (H2) will become evident during this chapter.

1.1 The Direct Method in the Calculus of Variations

For the purposes of this work, we shall mainly be interested in studying the minimizers of $\mathfrak{F}$ lying in the admissible set of functions $\mathcal{A} := \mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$, with $u_0 \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ a given function. This means that the minimizers will be constrained by the condition that $u = u_0$ on $\partial\Omega$ in the sense that $u - u_0 \in \mathrm{W}^{1,p}_0$.[1]

Under these conditions, the subject of existence of minimizers can be formulated in terms of looking for a function $\overline{u} \in \mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$ such that

$$\mathfrak{F}(\overline{u}) = m := \inf \left\{ \mathfrak{F}(u) : u \in \mathrm{W}^{1,p}_{u_0}\left(\Omega, \mathbb{R}^N\right) \right\}. \tag{1.3}$$

The existence of minimizers in such a space can be obtained if we have that the functional $\mathfrak{F}$ is coercive and **sequentially weakly lower semicontinuous**, meaning that, whenever $u_j \rightharpoonup \overline{u}$ in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ with $1 \leq p < \infty$, we have

$$\mathfrak{F}(\overline{u}) \leq \liminf_{j \to \infty} \mathfrak{F}(u_j). \tag{1.4}$$

When $p = \infty$, we require that $\mathfrak{F}$ is **sequentially weakly* lower semicontinuous**, i.e., such that (1.4) holds whenever $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$.

Assuming that the infimum m is a real number, the Direct Method consists in finding a *minimizing sequence* (u_j), with $u_j \in \mathcal{A}$, such that $\mathfrak{F}(u_j) \to m$. Then, if such sequence is bounded in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$, we extract a weakly convergent subsequence, for convenience not relabelled, and a function $\overline{u}$ such that $u_j \rightharpoonup \overline{u}$ in $\mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$. The fact that $\overline{u} \in \mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$ follows from Mazur's Lemma, thanks to which we can ensure that the weak closure of a convex subset of a Banach space is equal to its strong closure. The importance of $\mathfrak{F}$ being lower semicontinuous is now evident, since it enables us to conclude that

$$m \leq \mathfrak{F}(\overline{u}) \leq \liminf_{j \to \infty} \mathfrak{F}(u_j) = m$$

[1] The results in Chapter 2 will be established for a more general class of admissible functions, allowing part of the boundary to be free. The terminology will be specified in the relevant context.

or, in other words, that $\overline{u}$ is a minimizer. By convexity of the mapping $z \mapsto |z|^p$, a sufficient condition to ensure that $m > -\infty$ is to assume pointwise coercivity in the sense that, for some $a \in \mathbb{R}$ and some $b > 0$,

$$a + b|z|^p \leq F(z) \tag{1.5}$$

for every $z \in \mathbb{R}^{N\times n}$. This implies, in particular, that $\mathfrak{F}$ is bounded below on $\mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$. Furthermore, the coercivity condition (1.5) guarantees that, if (u_j) is a minimizing sequence in $\mathrm{W}^{1,p}_{u_0}$, then (∇u_j) is bounded in L^p. Whereby, since $(u_j - u_0) \subseteq \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N)$, by Poincaré inequality we have that, for some constant $c > 0$,

$$\int_\Omega |u_j - u_0|^p \,\mathrm{d}x \leq c \int_\Omega |\nabla u_j - \nabla u_0|^p \,\mathrm{d}x$$

and, therefore, (u_j) is bounded in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$. Hence, in order to apply the aforementioned direct method to obtain existence of minimizers, we are only left to ensure that $m < \infty$. This can be easily achieved, for example, by requiring that $\mathfrak{F}(v) < \infty$ for some $v \in \mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$. In particular, it is also implied by the imposed growth condition

$$F(z) \leq c|z|^p + d \tag{1.6}$$

for some constants $c, d > 0$.

The weak lower semicontinuity of $\mathfrak{F}$ clearly plays a central role in establishing existence of minimizers. It turns out that this property is deeply linked to the quasiconvexity of the integrand F and, more precisely, to assumption (H2). This fact has lead to a vast investigation in convex analysis within the context of the Calculus of Variations and we give a brief overview of it in the following section.

1.2 Weak lower semicontinuity and quasiconvexity

While working on a Lipschitz domain, it can be shown that a sufficient condition for the functional $\mathfrak{F}$ to have a minimum is the convexity of the integrand F.[2] Fortunately, this is far from being a necessary restriction in the vectorial case, i.e., when $n > 1$ or $N > 1$. The

[2]See Theorem 3.30 in [Dac08].

following result, due essentially to Morrey, aims at stating this.[3]

Theorem 1 *Let* Ω *be an open set,* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *a continuous function and let*

$$\mathfrak{F}(u) := \int_{\Omega} F(\nabla u)\,\mathrm{d}x$$

for $u\colon \Omega \to \mathbb{R}^N$*. Furthermore, assume that there is* $u_0 \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^N)$ *such that* $\mathfrak{F}(u_0) < \infty$*. If* $\mathfrak{F}$ *is sequentially weak** *lower semicontinuous in* $\mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^N)$*, then*

$$F(\xi_0) \leq \frac{1}{|D|}\int_D F(\xi_0 + \nabla\varphi)\,\mathrm{d}x$$

for every $\varphi \in \mathrm{W}_0^{1,\infty}(D,\mathbb{R}^N)$*, every* $\xi_0 \in \mathbb{R}^{N\times n}$ *and for every open and bounded set* $D \subseteq \mathbb{R}^n$*.*

We refer the reader to [Dac08, Lemma 3.18, Theorem 8.1] for a proof of this result. Given such necessary condition to ensure lower semicontinuity, the following concept introduced by Morrey becomes one of the pillars in the study of minimizers in the Calculus of Variations.

Definition 2 *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *be a locally bounded and Borel measurable function. We say that* F *is* ***quasiconvex at*** $\xi_0 \in \mathbb{R}^{N\times n}$ *if and only if*

$$F(\xi_0) \leq \frac{1}{|D|}\int_D F(\xi_0 + \nabla\varphi)\,\mathrm{d}x \tag{1.7}$$

for every $\varphi \in \mathrm{W}_0^{1,\infty}(D,\mathbb{R}^N)$ *and for every open and bounded set* $D \subseteq \mathbb{R}^n$ *with* $\mathcal{L}^n(\partial D) = 0$*. In addition, we say that* F *is* ***quasiconvex*** *if and only if it is quasiconvex at* ξ_0 *for every* $\xi_0 \in \mathbb{R}^{N\times n}$*.*

We emphasize that condition (H2) means, in particular, that we shall assume F to be quasiconvex. However, the statement in (H2) is stronger and we will discuss the reason for this in the following section.

In addition, observe that the definition of quasiconvexity states that an integrand F is quasiconvex if and only if every affine function, say $a\colon \mathbb{R}^n \to \mathbb{R}^N$, minimizes the functional $\mathfrak{F}$ on the Sobolev space $\mathrm{W}_a^{1,\infty}(D,\mathbb{R}^N)$ with $D \subseteq \mathbb{R}^n$ open, bounded and such that $\mathcal{L}^n(\partial D) = 0$. We also remark that it is equivalent for a function F to be quasiconvex in the sense

[3] See [Mor52] and [Mor66].

of Definition 2, than to say that inequality (1.7) holds for every $\xi \in \mathbb{R}^{N\times n}$ and every $\psi \in \mathrm{W}_0^{1,\infty}(B,\mathbb{R}^N)$ in the particular case in which $D = B$. See [Dac08, Proposition 5.11]. This fact will be constantly used throughout the rest of this text.

On the other hand, under suitable growth and coercivity conditions, quasiconvexity can also be shown to be a sufficient condition for weak lower semicontinuity in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ (or sequential weak* lower semicontinuity if $p = \infty$). This was established by Morrey [Mor52] and remarkable generalizations were later achieved by Meyers [Mey65], Acerbi-Fusco [AF84] and Marcellini [Mar85]. We remark that, for the case $1 < p < \infty$, the growth conditions required to prove that quasiconvexity implies weak lower semicontinuity in $\mathrm{W}^{1,p}$ are that, for some constant $c > 0$ and some $1 \le q < p$,

$$-c(1 + |z|^q) \le F(z) \le c(1 + |z|^p).$$

Furthermore, this condition is optimal in the sense that we cannot reduce it, in general, to assume that $|F(z)| \le c(1 + |z|^p)$. An example for this is due to Tartar and it consists in considering the quasiconvex (actually polyaffine)[4] function $F(z) = \det z$ for $n = N = p = 2$. See [Dac08, Example 8.6].

On the other hand, under Dirichlet boundary conditions, say $u_0 \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, quasiconvex integrands satisfying $|F(z)| \le c(1+|z|^p)$, corresponding to assumption (H1), can also be shown to be weakly lower semicontinuous in $\mathrm{W}^{1,p}_{u_0}$, so that this scenario constitutes a natural context in which we can expect to obtain existence of minimizers. More precisely, we have the following.

Theorem 3 *Let Ω be an open set, $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ a continuous function satisfying the growth condition* (H1) *for some $p \in (1,\infty)$ and take $u_0 \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Let*

$$\mathfrak{F}(u) := \int_\Omega F(\nabla u)\,\mathrm{d}x$$

for $u\colon \Omega \to \mathbb{R}^N$. Then, if

$$F(\xi_0) \le \frac{1}{|B|}\int_B F(\xi_0 + \nabla\varphi)\,\mathrm{d}x$$

for every $\varphi \in \mathrm{W}_0^{1,p}(B,\mathbb{R}^N)$ and every $\xi_0 \in \mathbb{R}^{N\times n}$, then $\mathfrak{F}$ is sequentially weakly lower

[4]See Definition 6.

semicontinuous in $\mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^N)$.

We remark here that, since we are no longer assuming pointwise coercivity, we need to impose a further condition to ensure that the infimum m is finite. This is where the stronger version of quasiconvexity, namely (H2), comes into play. We shall discuss this in Section 1.4.

In compliance with this, we now recall the following important fact about quasiconvex functions with polynomial growth. We will often use this result in the rest of the text.

Proposition 4 *Let* $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ *be a continuous function satisfying the growth condition* (H1)*, i.e., that there is a constant* $c>0$ *such that*

$$|F(z)|\leq c(1+|z|^p)$$

for some $p\in(1,\infty)$ *and every* $z\in\mathbb{R}^{N\times n}$.
Then, F is quasiconvex (in the sense of Definition 2) if and only if

$$F(\xi_0)\leq\frac{1}{|D|}\int_D F(\xi_0+\nabla\varphi)\,\mathrm{d}x$$

holds for every $\varphi\in\mathrm{W}^{1,p}_0(D,\mathbb{R}^N)$ *and for every open and bounded set* $D\subseteq\mathbb{R}^n$ *with* $\mathcal{L}^n(\partial D)=0$.

This result was first established by Ball & Murat [BM84b, Proposition 2.4]. In order to prove it, we use the fact that $\mathrm{W}^{1,\infty}_0(\Omega,\mathbb{R}^N)\subseteq\mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$ to show that quasiconvexity is a necessary condition for (4) to hold and, for the reverse implication, we invoke Vitali's Convergence Theorem, together with an approximation argument. We refer the reader to [Son14] for further results concerning growth conditions and quasiconvexity.

We now use the following section to introduce some further definitions related to convexity.

1.3 Other notions of convexity

The importance of quasiconvex functions has been made clear given the equivalence of the quasiconvexity of the integrand with the lower semicontinuity of the functional $\mathfrak{F}$. However, since this notion is not defined pointwise, it is often hard to verify whether a given function F is quasiconvex.[5] In this context, it is useful to consider a slightly weaker condition, called

[5] See also [Kri99].

rank one convexity, and a particular case of it, namely *separate convexity*. In addition, we introduce the concept of *polyconvexity*, which is stronger than quasiconvexity and is satisfied by many of the integrands coming from problems motivated by elasticity theory.

Definition 5 *Let $F\colon \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$. We say that F is* ***separately convex*** *if and only if, letting $x = (x_1, x_2, ..., x_k)$, the function*

$$x_i \mapsto F(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_k)$$

ix convex for every fixed vector $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_k) \in \mathbb{R}^{k-1}$.

Definition 6 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{\infty\}$.*

(i) *We say that F is* ***rank one convex*** *if and only if*

$$F(\lambda\xi + (1-\lambda)\eta) \leq \lambda F(\xi) + (1-\lambda)F(\eta)$$

for every $\lambda \in [0,1]$ and for every $\xi, \eta \in \mathbb{R}^{N\times n}$ such that $\operatorname{rank}\{\xi - \eta\} \leq 1$.

(ii) *Let $T(\xi)$ denote the vector whose entries are all the $s \times s$ minors of ξ for $1 \leq s \leq \min\{n, N\}$ in some fixed order. We say that F is* ***polyconvex*** *if and only if there exists a convex function g such that*

$$F(\xi) = g(T(\xi)).$$

Remark 7 *Given $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$, we can use the tensorial notation to construct the matrix $a \otimes b \in \mathbb{R}^{N\times n}$ given by*

$$(a \otimes b)_{ij} := a_i b_j$$

for every $1 \leq i \leq N$ and every $1 \leq j \leq n$. Under this convention, a function F as above is rank one convex if and only if the function $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\varphi(t) := F(\xi + ta \otimes b),$$

is convex for every $\xi \in \mathbb{R}^{N\times n}$ and for every $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$.

The following result, due to Morrey [Mor52], summarizes the relationship between the different notions of convexity that we have defined.

Theorem 8 *Let $F\colon \mathbb{R}^n \to \mathbb{R}$. Then the following statements are true.*

(i) *F convex $\Rightarrow F$ polyconvex $\Rightarrow F$ quasiconvex $\Rightarrow F$ rank one convex.*

(ii) *If $N = 1$ or $n = 1$, then all these notions are equivalent.*

(iii) *If $F \in C^2(\mathbb{R}^{N\times n})$, then rank one convexity is equivalent to the Legendre-Hadamard (or ellipticity) condition, namely, that*

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 F(\xi)}{\partial \xi^i_\alpha \partial \xi^j_\beta} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0$$

for every $\lambda \in \mathbb{R}^N$, $\mu \in \mathbb{R}^n$ and $\xi = (\xi^i_\alpha)^{1\leq i\leq N}_{1\leq \alpha\leq n} \in \mathbb{R}^{N\times n}$.

(iv) *If F is separately convex (and therefore also if it is convex, polyconvex, quasiconvex or rank one convex), then F is locally Lipschitz.*

The proof of this theorem can be found in [Dac08, Theorem 5.3].

Remark 9 *That every convex function is quasiconvex is a straightforward consequence of Jensen's inequality. On the other hand, it is possible to find polyconvex functions that are not convex. A simple example when $n = N = 2$ is $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined as $F(\xi) := \det \xi$. This function is not convex but it is polyaffine, meaning that both F and $-F$ are polyconvex. In addition, Šverák provided a fundamental example showing that, for any $n \geq 2$ and $N \geq 3$, we can find a rank one convex function that is not quasiconvex.*[6] *However, the question of extending Šverák's example to the case where $n \geq N = 2$ is still open.*

In addition, it is worth observing that rank one convex and polyconvex functions are defined so that they admit the value of infinity. This is not the case for quasiconvex functions, since a good extension of the definition of quasiconvexity for F admitting infinite values, would also need to be proven equivalent to the weak lower semicontinuity of the functional in some sense. Ball-Murat [BM84b] and Dacorogna-Fusco [DF85] have suggested definitions that can be shown to be necessary for weak lower semicontinuity. However, it seems to be a difficult problem to establish whether they are also sufficient. Nevertheless, if $F : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{\infty\}$, it can also be shown that

$$F \text{ convex } \Rightarrow F \text{ polyconvex } \Rightarrow F \text{ rank one convex.}$$

[6] See [Dac08, p. 219].

1.4 Integrands with polynomial growth and strong quasiconvexity

As we mentioned at the end of Section 1.2, an integrand $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ with p-growth, meaning that

$$|F(z)| \leq c(1 + |z|^p),$$

is quasiconvex if and only if the associated functional $\mathfrak{F}$ is weakly lower semicontinuous in the Dirichlet class $\mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$ with $u_0 \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$. This is, therefore, a natural assumption to make in this framework.

In this section we recall some further properties of functions with p-growth and we motivate the assumption (H2) of strong quasiconvexity, seen also as a coercivity condition, by observing that (H1) and (H2) together are sufficient conditions to ensure existence of minimizers.

Before proceeding with this, we state first the following elementary proposition that will be constantly used throughout the rest of the text

Proposition 10 *Let $F\colon \mathbb{R}^k \to \mathbb{R}$ be a separately convex function such that*

$$|F(z)| \leq c\,(1 + |z|^p)$$

for some fixed constant $c > 0$, some $p \in [1, \infty)$ and for every $z \in \mathbb{R}^k$. Then, there exists a constant $\tilde{c} > 0$ such that, for every $z, w \in \mathbb{R}^k$, it holds that

$$|F(z) - F(w)| \leq \tilde{c}\left(1 + |z|^{p-1} + |w|^{p-1}\right)|x - y|\,.$$

In addition, if

$$|F(z)| \leq c\,(|z|^p)\,,$$

then

$$|F(z) - F(w)| \leq \tilde{c}\left(|z|^{p-1} + |w|^{p-1}\right)|x - y|\,.$$

This result should be related to part (iv) of Theorem 8. The proof of the first part, which relies essentially in a suitable manipulation of the convexity condition of F, can be found in [Dac08, Proposition 2.32]. The proof of the second part is completely analogous. See also

[Fus80], [Mar85] and [Mor66].

A straightforward consequence of this result, that we will also use frequently in this work, is the following.

Corollary 11 *Let* $F\colon \mathbb{R}^k \to \mathbb{R}$ *be a* C^1 *and separately convex function such that*

$$|F(z)| \leq c\,(1+|z|^p)$$

for some fixed constant $c>0$*, some* $p\in[1,\infty)$ *and for every* $z\in\mathbb{R}^k$*. Then, there exists a constant* $\tilde{c}>0$ *such that, for every* $z\in\mathbb{R}^k$*, it holds that*

$$\left|F'(z)\right| \leq \tilde{c}\left(1+|z|^{p-1}\right).$$

Going back to the subject of existence of minimizers and the role of strong quasiconvexity for the case of integrands with p-growth, we observe that, from (1.5), it follows that there is a constant $\tilde{a}=\tilde{a}(a,u_0)>0$ such that, for every $u\in \mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^N)$,

$$b\int_\Omega |\nabla u|^p \,\mathrm{d}x - \tilde{a} \leq \int_\Omega F(\nabla u)\,\mathrm{d}x. \tag{1.8}$$

An inequality of this type is usually called a **Gårding inequality**. When in presence of it, we say that F is **mean-coercive** on $\mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^N)$. C.Y. Chen and Kristensen established a characterization of mean-coercivity in terms of a strict notion of quasiconvexity. See [Che13, Theorem 6.1.1] and [YZ01] for a special case. We state here a partial version of their result, which is all we need to ensure existence of minimizers under p-growth and Dirichlet boundary conditions. The necessity is the simplest one to obtain and it can be derived in the same was as (4.150) in the proof of Theorem 83.

Theorem 12 *Let* $F:\mathbb{R}^{N\times n}\to\mathbb{R}$ *be continuous and let* $p\in(1,\infty)$*. Assume that* $c_1>0$ *is a constant such that* $|F(z)|\leq c_1(1+|z|^p)$ *for every* $z\in\mathbb{R}^{N\times n}$*. Let* $u_0\in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$*. Then,* F *is mean-coercive on* $\mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^{N\times n})$ *if and only if there exists* $\varepsilon>0$ *such that* $F-\varepsilon|\cdot|^p$ *is quasiconvex at some* $\xi_0\in\mathbb{R}^{N\times n}$*. If* $p\geq 2$*, the result holds for* $F-\varepsilon(|\cdot|^2+|\cdot|^p)$ *with the obvious adjustment to the notion of mean coercivity.*

We emphasize that, from the strong convexity of the function $z\mapsto|z|^p$ it follows that, for

$p \geq 2$, $F - \varepsilon|\cdot|^p$ is quasiconvex if and only if

$$\varepsilon \int_\Omega |\nabla\varphi|^p \,\mathrm{d}x \leq \int_\Omega F\big(z + \nabla\varphi\big) - F\big(z\big) \,\mathrm{d}x$$

for every $z \in \mathbb{R}^{N\times n}$ and every $\varphi \in \mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^N)$. Indeed, it is enough to observe that, for $f(x) := |x|^p$, we have for every $x, y \in \mathbb{R}^n$ that $f(y) \geq f(x) + f'(x)[y - x] + 2^{2-p} f(y - x)$.

In order to conclude the motivation behind the assumption (H2) which, for the case $p \in [2, \infty)$, is essentially equivalent to assuming that $F - c_2\left(|\cdot|^2 + |\cdot|^p\right)$ is quasiconvex, we now state a useful technical lemma. It establishes that, if F is an integrand with p-growth as in (H1), then we can construct an integrand $\tilde{F}$ such that $|\tilde{F}(z)| \leq \tilde{c}_2 |V(z)|^2$ for every $z \in \mathbb{R}^{N\times n}$ and for which we can define a minimization problem equivalent to that of minimizing the functional $\mathfrak{F}$. Given that $\tilde{F}$ will be bounded above by the function $|V|^2$, it will prove to be very useful while obtaining Gårding and Caccioppoli inequalities to have the mean coercivity given by the strong quasiconvexity (H2), in terms of the V function, and not just assuming that $F - c|\cdot|^p$ is quasiconvex.

Before stating the lemma, we define first a more general class of auxiliary functions, of which V is a particular case. Let $\beta > 0$ be arbitrary and, for $k \in \mathbb{N}^+$, define the function $V_\beta \colon \mathbb{R}^k \to \mathbb{R}^k$ by

$$V_\beta(\xi) := (1 + |\xi|^2)^{\frac{\beta-1}{2}}\xi. \tag{1.9}$$

Since we will mostly be dealing with $\beta = \frac{p}{2}$, where $p \in [2, \infty)$, we use the abbreviation $V = V_{\frac{p}{2}}$.[7] In addition, while V_β depends, strictly speaking, on the dimension of the domain, say k, it is easy to verify that $|V_{\beta,k}(\xi)| = V_{\beta,1}(|\xi|)$, where the $V_{\beta,k}$ and $V_{\beta,1}$ are defined on $\mathbb{R}^k$ and $\mathbb{R}$, respectively. However, for simplicity of the notation, we shall not make any distinction on the dimension of the domain of V_β and write simply $|V_\beta(\xi)| = V_\beta(|\xi|)$.

Having fixed the terminology in this way, we now state the aforementioned result, which is due to Acerbi and Fusco [AF87, Lemma 2.3].

[7] The auxiliary function V can also be defined, in the same way, for $p \in (1, 2)$ and its properties are particularly useful in this case to establish, for example, regularity of minimizers or solutions to elliptic problems.

Lemma 13 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be such that it satisfies* (H0) − (H2)*. Let $m > 0$ be fixed and take $z_0 \in \mathbb{R}^{N\times n}$ such that $|z_0| \leq m$. Define, for $z \in \mathbb{R}^{N\times n}$,*

$$F_0(z) := F(z_0 + z) - F(z_0) - \langle F'(z_0), z \rangle = \int_0^1 (1-t)F''(z_0 + tz)[z,z]\,\mathrm{d}t.$$

Then, there exists $0 < k = k(c_1, c_2, m)$ such that

(a) $|F_0(z)| \leq k|V(z)|^2$ *and*

(b) $|F_0'(z)| \leq k|V_{p-1}(z)|$.

The idea of the proof of this result is to consider separately the cases $|z| < 1$ and $|z| \geq 1$. The quadratic term of the function $|V|^2$ appears while considering $|z| < 1$, since we can then control $|F''(z_0 + tz)|$ and exploit the quadratic nature of the form $z \mapsto F''(z_0 + tz)[z,z]$. The proof of (2.12) in Chapter 2 follows a similar spirit.

The strategy of expressing the estimates for the integrand F in terms of the function V goes at least back to [Cam87, S.1]. Acerbi and Fusco settled the importance of the V function in [AF89b], since it is in the subquadratic case that the comparison between an integrand and its second degree Taylor polynomial require more of this auxiliary function.

1.5 The Euler-Lagrange Equations

Associated with any variational problem is that of finding necessary and sufficient conditions that enable us to characterize the functions that minimize the functional $\mathfrak{F}$. In this context, it is of particular importance to observe that, under appropriate regularity conditions on F and on the minimizers of $\mathfrak{F}$, these satisfy the Euler-Lagrange equations in the following sense.

Theorem 14 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be of class C^1. Assume that F satisfies the growth condition* (H1) *and, in addition,*

$$|F'(z)| \leq c\left(1 + |z|^{p-1}\right).$$

Let $u_0 \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ be a minimizer of $\mathfrak{F}$ over the Dirichlet class $\mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$. Then,

$$\int_\Omega \langle F'(\nabla u), \nabla\varphi \rangle \,\mathrm{d}x = 0 \tag{1.10}$$

for every $\varphi \in \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N)$. *Conversely, if* F *is convex and* u *satisfies (1.10), then* u *is a minimizer of* $\mathfrak{F}$.

We establish here this result and we refer the reader to [Dac08, Theorem 3.37] for alternative ways of stating the assumptions.

Proof. The proof relies on computing the Gâteaux derivative of the functional $\mathfrak{F}$. For a fixed $\varphi \in \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N)$ we consider the real mapping $f : [-1, 1] \to \mathbb{R}$ given by

$$f(\varepsilon) := \mathfrak{F}(u + \varepsilon\varphi).$$

Observe that $\varepsilon = 0$ is a minimizer of f. Therefore, if the limit exists, we will have that

$$0 = f'(0) = \lim_{\varepsilon\to 0} \frac{\mathfrak{F}(u + \varepsilon\varphi) - \mathfrak{F}(u)}{\varepsilon}. \tag{1.11}$$

It only remains to observe that, thanks to the growth condition imposed on F', we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\mathfrak{F}(u + \varepsilon\varphi) - \mathfrak{F}(u)}{\varepsilon} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \big(F(\nabla u + \varepsilon\nabla\varphi) - F(\nabla u)\big)\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} F(\nabla u + t\varepsilon\nabla\varphi)\,\mathrm{d}t\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int_\Omega \int_0^1 \big\langle F'(\nabla u + t\varepsilon\nabla\varphi), \nabla\varphi\big\rangle\,\mathrm{d}t\,\mathrm{d}x \\
&\leq c \lim_{\varepsilon\to 0} \int_\Omega \int_0^1 \Big(1 + |\nabla u + t\varepsilon\nabla\varphi|^{p-1}\Big)\,|\nabla\varphi|\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

Therefore, by Dominated Convergence Theorem we know that the limit in (1.11) exists and the first part of the theorem is proved.

For the second part of the theorem it is enough to use the characterization of differentiable convex functions in terms of their first derivative. □

We recall that a function u for which (1.10) holds is said to satisfy the **weak form of the Euler-Lagrange equation.** In this case, we also say that u is an F-**extremal**.

Having established the above theorem, it is now also easy to obtain, under higher regularity assumptions on u, the classical form of the Euler-Lagrange equation. We state this in the following corollary.

Corollary 15 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 function and assume that $u \in C^2(\overline{\Omega}, \mathbb{R}^N)$ is a minimizer of the associated functional $\mathfrak{F}$. Then u satisfies*

$$\sum_{\alpha=0}^{n} \frac{\partial}{\partial x_\alpha}\left(\frac{\partial}{\partial z_\alpha^i} F(\nabla u)\right) = 0$$

on Ω for every $1 \leq i \leq N$. In other words,

$$\mathrm{div} F'(\nabla u) = 0.$$

1.6 Quadratic forms

We devote this section to the convexity properties of symmetric bilinear forms. This case has received particular attention given that the associated Euler-Lagrange equations are linear. In addition, it is also specially important for the purposes of this work, given that, for an integrand $F\colon \mathbb{R}^{N\times n}$ of class C^2, $F''(z_0)$ induces a symmetric bilinear form for every $z_0 \in \mathbb{R}^{N\times n}$.

We first fix the notation by considering, for a symmetric matrix $A \in \mathbb{R}^{(N\times n)\times(N\times n)}$, the symmetric bilinear form $\mathcal{A}\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ defined by

$$\mathcal{A}(z) = A[z, z] := \langle Az, z\rangle .$$

We then have the following result.

Theorem 16 *Let $\mathcal{A}\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a symmetric bilinear form given by $\mathcal{A}(\xi) := A[\xi, \xi]$. The following statements are then equivalent*

(i) *$\mathcal{A}$ is rank-one convex;*

(ii) *$\mathcal{A}(a \otimes b) \geq 0$ for every $a \in \mathbb{R}^N$ and every $b \in \mathbb{R}^n$;*

(iii) *$\mathcal{A}$ is quasiconvex and*

(iv) *$\mathcal{A}$ is quasiconvex at 0.*

The equivalence between (i) and (ii) and between (iii) and (iv) follow directly from the definition of $\mathcal{A}$. For a proof of the equivalence between (i) and (iii), we refer the reader to [Dac08, Theorem 5.25].

Remark 17 *Considering the quadratic growth of bilinear forms, it is clear that the Theorem also holds if we replace rank-one convexity and quasiconvexity by strong rank-one convexity and strong quasiconvexity, respectively. In particular, we will have that $\mathcal{A}$ satisfies the strong Legendre-Hadamard condition if and only if it is strongly quasiconvex, if and only if, there exists $\lambda > 0$ such that, for every $w \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$,*

$$\int_\Omega A[\nabla w, \nabla w] \, \mathrm{d}x \geq \lambda \int_\Omega |\nabla w|^2 \, \mathrm{d}x. \tag{1.12}$$

In connection with this, we now establish the corresponding Gårding's inequality for the case in which A does not necessarily have constant coefficients, so that it also depends on x. For this result, we follow the proof of Theorem 3.42 in [GM12].

Theorem 18 (Gårding's inequality) *Let $(A_t)_{t\in\mathbb{R}}$ be a parametrized family of equicontinuous quadratic forms in $\overline{\Omega}$, with $A_t \colon \overline{\Omega}\times\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n} \to \mathbb{R}$. Suppose that there is $\Lambda > 0$ such that $|A_t| \leq \Lambda$ for every $t \in \mathbb{R}$ and that (A_t) admits a uniform modulus of continuity ω on $\overline{\Omega}$. Assume further that, for every $x \in \overline{\Omega}$, each $A_t(x)$ satisfies the quasiconvexity condition (1.12) for some $\lambda > 0$ independent of $x \in \overline{\Omega}$ and $t \in \mathbb{R}$. Then, the bilinear forms on $\mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$ defined by*

$$\mathbb{A}_t(w, w) := \int_\Omega A_t[\nabla w, \nabla w] \, \mathrm{d}x \tag{1.13}$$

are weakly coercive uniformly in t, i.e., there exist $\lambda_0 > 0$ and $\lambda_1 > 0$ such that, for every $t \in \mathbb{R}$,

$$\mathbb{A}_t(w, w) \geq \lambda_0 \int_\Omega |\nabla w|^2 \, \mathrm{d}x - \lambda_1 \int_\Omega |w|^2 \, \mathrm{d}x \tag{1.14}$$

for every $w \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$. Furthermore, λ_0 and λ_1 depend exclusively on λ, Λ, ω, and Ω.

Proof. Let ω be a modulus of continuity such that, for every $t \in \mathbb{R}$ and for every $x, x_0 \in \overline{\Omega}$,

$$|A_t(x) - A_t(x_0)| \leq \omega(|x - x_0|). \tag{1.15}$$

In addition, let $r > 0$ and $x_0 \in \overline{\Omega}$ fixed, as well as $w \in \mathrm{W}_0^{1,2}(\Omega(x_0, r), \mathbb{R}^N)$.[8] Then, from the

[8] See Appendix A.

quasiconvexity of the quadratic form $A_t(x_0)$ and (1.15), we deduce that, for every $t \in \mathbb{R}$,

$$\int_{\Omega(x_0,r)} A_t[\nabla w, \nabla w]\,\mathrm{d}x = \int_{\Omega(x_0,r)} A_t(x_0)[\nabla w, \nabla w]\,\mathrm{d}x + \int_{\Omega(x_0,r)} (A_t - A_t(x_0))[\nabla w, \nabla w]\,\mathrm{d}x$$
$$\geq \lambda \int_{\Omega(x_0,r)} |\nabla w|^2\,\mathrm{d}x - \omega(r) \int_{\Omega(x_0,r)} |\nabla w|^2\,\mathrm{d}x. \tag{1.16}$$

Now choose $r > 0$ such that $\lambda_0^* := \lambda - \omega(r) > 0$ and cover $\overline{\Omega}$ with finitely many balls $B(x_k, r)$, where $1 \leq k \leq M$. Fix a partition of unity (φ_k^2) subordinated to the covering $\{B(x_k, r)\}$. This means that, for every $1 \leq k \leq M$, φ_k is a non-negative smooth function such that

$$\operatorname{support}(\varphi_k) \subseteq B(x_k, r) \qquad \text{and} \qquad \sum_{k=1}^{M} \varphi_k^2 = 1 \text{ on } \overline{\Omega}. \tag{1.17}$$

This way, if $w \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$ and we extend w by 0 outside Ω,

$$w = \sum_{k=1}^{M} (w\varphi_k^2) \qquad \text{and} \qquad \operatorname{supp}(w\varphi_k^2) \subseteq \Omega(x_k, r).$$

Then,

$$\int_\Omega \sum_{k=1}^{M} \varphi_k^2 A_t[\nabla w, \nabla w]\,\mathrm{d}x = \sum_{k=1}^{M} \int_\Omega A_t[\varphi_k \nabla w, \varphi_k \nabla w]\,\mathrm{d}x$$
$$= \sum_{k=1}^{M} \int_\Omega A_t[\nabla(\varphi_k w), \nabla(\varphi_k w)]\,\mathrm{d}x - \sum_{k=1}^{M} \int_\Omega A_t[w \otimes \nabla\varphi_k, \varphi_k \nabla w]\,\mathrm{d}x$$
$$- \sum_{k=1}^{M} \int_\Omega A_t[\varphi_k \nabla w, w \otimes \nabla\varphi_k]\,\mathrm{d}x + \sum_{k=1}^{M} \int_\Omega A_t[w \otimes \nabla\varphi_k, w \otimes \nabla\varphi_k]\,\mathrm{d}x. \tag{1.18}$$

On the other hand, (1.16) implies that

$$\begin{aligned}\sum_{k=1}^{M}\int_\Omega A_t[\nabla(\varphi_k w),\nabla(\varphi_k w)]\,\mathrm{d}x \geq &\lambda_0^*\sum_{k=1}^{M}\int_\Omega |\nabla(\varphi_k w)|^2\,\mathrm{d}x\\ =&\lambda_0^*\sum_{k+1}^{M}\int_\Omega \left(\varphi_k^2|\nabla w|^2+|w|^2|\nabla\varphi_k|^2+2\langle\varphi_k\nabla w, w\otimes\nabla\varphi_k\rangle\right)\,\mathrm{d}x\\ \geq&\lambda_0^*\sum_{k+1}^{M}\int_\Omega \left(\varphi_k^2|\nabla w|^2+|w|^2|\nabla\varphi_k|^2-2|\varphi_k||\nabla w||w||\nabla\varphi_k|\right)\,\mathrm{d}x.\end{aligned}$$

We now apply Young's inequality $2ab\leq\delta a^2+\frac{b^2}{\delta}$ to the last term above with $a=|\nabla w|$ and $b=\varphi_k|w||\nabla\varphi_k|$ to deduce that

$$\sum_{k=1}^{M}\int_\Omega A_t[\nabla(\varphi_k w),\nabla(\varphi_k w)]\,\mathrm{d}x \geq (\lambda_0^*-\delta)\int_\Omega|\nabla w|^2\,\mathrm{d}x - c_\delta\int_\Omega |w|^2\,\mathrm{d}x. \tag{1.19}$$

We finally estimate the last three terms in (1.18) as follows. We first observe that

$$\left|\int_\Omega A_t\left[w\otimes\nabla\varphi_k, w\otimes\nabla\varphi_k\right]\,\mathrm{d}x\right| \leq c\sup_\Omega|A_t|\int_\Omega|w|^2\,\mathrm{d}x$$

and, using Young's inequality as before, we also have

$$\left|\int_\Omega \left(A_t\left[w\otimes\nabla\varphi_k,\varphi_k\nabla w\right]+A_t\left[\varphi_k\nabla w, w\otimes\nabla\varphi_k\right]\right)\,\mathrm{d}x\right| \leq \delta\int_\Omega|\nabla w|^2\,\mathrm{d}x + c\,\frac{\sup|A_t|^2}{\delta}\int_\Omega|w|^2\,\mathrm{d}x. \tag{1.20}$$

We now chose $\delta<\frac{\lambda_0^*}{3}$ and, by putting $\lambda_0:=\lambda_0^*-3\delta$ and $\lambda_1:=\lambda_1(\delta,\Lambda,\Omega,\omega)$, none of which depend on t, we conclude from (1.18)-(1.20) that $\mathbb{A}_t$ is weakly coercive. □

We conclude this section with the following observation, which concerns the quasiconvexity of the bilinear forms given by $F''(z_0)$, for a fixed z_0, whenever F satisfies the assumptions (H0)-(H2). This is naturally associated to the fact that the Hessian matrix of a convex function is positive semi-definite. Given that the fundamental tool that we will make use of to establish, for example, regularity of a given class of minimizers, it is essential to count on the strong quasiconvexity of $F''(z_0)$ for each $z_0\in\mathbb{R}^{N\times n}$.

Proposition 19 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a function satisfying* (H0) *and* (H2)*. Then, for every $z_0 \in \mathbb{R}^{N\times n}$ and every $\varphi \in \mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^N)$, it holds that*

$$2c_2 \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \le \int_\Omega F''(z_0)[\nabla\varphi, \nabla\varphi] \,\mathrm{d}x.$$

In other words, $F''(z_0)[\cdot,\cdot] - 2c_2|\cdot|^2$ is quasiconvex.

Proof. By (H2) we have that, for every $\varphi \in \mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^N)$, $t = 0$ minimizes the real valued function

$$\mathcal{J}(t) := \int_\Omega \left(F(z_0 + t\nabla\varphi) - F(z_0) - c_2|V(t\nabla\varphi)|^2\right) \,\mathrm{d}x. \tag{1.21}$$

Hence,

$$0 \le \mathcal{J}''(0) = \int_\Omega \left(F''(z_0)[\nabla\varphi, \nabla\varphi] - 2c_2|\nabla\varphi|^2\right) \,\mathrm{d}x. \tag{1.22}$$

This concludes the proof. □

1.7 Regular domains in the space

Many results in the Calculus of Variations require the reference domain to have nice properties. This can be due to the need of applying, for example, the embedding theorems (see Section B.2). In addition, when proving regularity up to the boundary of solutions to variational problems, the requirement that the boundary of the domain is *regular* naturally arises. In this section we specify the exact meaning of a domain having a regular or smooth boundary and we also define other regularity notions in the context of domains in the space, which are among the most general conditions required to ensure that the embedding theorems remain valid.

To specify this more clearly, we state here the following definition.

Definition 20 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $\alpha \in [0,1]$, $k \in \mathbb{N}^+$. We say that Ω is of class $C^{k,\alpha}$ if and only if there exists a function $\vartheta\colon \mathbb{R}^n \to \mathbb{R}$ such that $\vartheta \in C^{k,\alpha}$ and the following properties are satisfied.*

$$\Omega = \{x \in \mathbb{R}^n : \vartheta(x) < 0\};$$

$$\mathbb{R}^n \backslash \overline{\Omega} = \{x \in \mathbb{R}^n : \vartheta(x) > 0\} \text{ and}$$

$$\nabla\vartheta(x) \neq 0 \text{ for all } x \in \partial\Omega.$$

*In this case, we say that ϑ is a **defining function** for Ω.*

We remark that, if Ω is at least of class C^1, the vector field $\nabla\vartheta$ is normal to $\partial\Omega$ at each point of the boundary. Observe also that, for any $x_0 \in \partial\Omega$, we make the convention that $\nabla\vartheta(x_0)$ is an *outward normal* to $\partial\Omega$ at x_0. On the other hand, if Ω is of class C^k with $k \geq 2$, we can further assume that $|\nabla\vartheta| = 1$ on $\partial\Omega$.

We further observe that, for $k \geq 1$, a standard application of the Implicit Function Theorem implies that if ϑ is a defining function for Ω, then $\partial\Omega$ is locally the graph of a function of class $C^{k,\alpha}$. This turns out to be an equivalent definition of $C^{k,\alpha}$ sets. Indeed, to prove the converse implication, we observe that if $\partial\Omega$ is the graph of a function f_{x_0} in a neighbourhood V_{x_0} for each $x_0 \in \partial\Omega$, and if the positive x_n axis points out of the domain, then the function

$$\rho_{x_0}(x) := x_n - f_{x_0}(x_1, ..., x_{n-1})$$

behaves like a defining function for Ω near the point x_0, with $\rho_{x_0} > 0$ in $V_{x_0} \cap (\mathbb{R}^n \backslash \overline{\Omega})$ and $\rho_{x_0} < 0$ in $V_{x_0} \cap \Omega$. We can then cover the compact set $\partial\Omega$ by a finite collection of open sets $(V_i)_{i=1}^m$, each with its associated defining function ρ_i. We now consider a partition of unity $(\varphi_i)_{i=1}^m$ subordinated to the covering (V_i), meaning that each φ_i is a C^∞ function such that $\text{support}(\varphi_i) \subseteq V_i$ and $\sum_{i=1}^m \varphi_i \equiv 1$ on $\partial\Omega$. It then follows that the function

$$\rho(x) := \sum_{i=1}^m \varphi_i(x)\rho_i(x),$$

which is defined in a neighbourhood of $\partial\Omega$, satisfies the properties of a defining function on that neighbourhood. It is now easy to extend ρ to the whole space $\mathbb{R}^n$, with which the equivalence between the two notions of $C^{k,\alpha}$ sets is proved. These ideas are based in [KP99, Section 1.2], to which we refer the reader for further details on defining functions that generate outward unit normal vectors for $\partial\Omega$, as well as for other equivalent definitions of $C^{k,\alpha}$ sets and their properties.

Although the main results of this text are mainly set in the context of Lipschitz or smooth domains, many of the general properties that we will use actually hold for a more general class of subsets of $\mathbb{R}^n$. With this motivation, we consider the following definition.

Definition 21 *A domain $\Omega \subseteq \mathbb{R}^n$ is said to have the* ***cone property*** *if and only if each point of Ω is the vertex of a cone contained in Ω along with its closure, the cone being represented by the inequality*

$$\sum_{j=1}^{n-1} x_i^2 < a x_n^2 \tag{1.23}$$

for $0 < x_n < b$ in some Cartesian coordinate system and where $a, b > 0$ are constants.

It is easy to verify that any domain of class $C^{0,1}$ has the cone property. In addition, since we will be working on the boundary of Ω, we will need to make use, in particular, of a boundary version of Poincaré's inequality and, more generally, to take averages of integrals over domains of the form $\Omega(x_0, R)$. Given this, we shall state another auxiliary lemma that will show that $|\Omega(x_0, R)|$ is roughly proportional to $|B(x_0, R)|$, up to some constants that will not depend on x_0 or R.

Lemma 22 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with the cone property. Then, there exists R_Ω, which can be taken to be $R_\Omega = \mathrm{diam}(\Omega)$, and a constant $c_\Omega > 0$, such that for every $0 < R < R_\Omega$ and every $x_0 \in \partial\Omega$,*

$$c_\Omega R^n \omega_n \leq |\Omega(x_0, R)| \leq R^n \omega_n,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

The notation and results stated in this preliminary chapter will be used throughout the rest of this work. We now proceed to the main body of this text.

Chapter 2

Sufficient conditions for an extremal to be a minimizer

It is well known that, if the second variation is strongly positive, Lipschitz extremals are weak local minimizers, in the sense that they minimize the functional over small variations in $W^{1,\infty}$.[1] However, if the variations are small in the L^∞ sense, corresponding to the notion of strong local minimizers, the situation becomes more complicated. In the 19th century, Weierstrass established a fundamental set of sufficient conditions for a smooth extremal to be a strong local minimizer in the case of one single independent variable. His work was then generalized by Hestenes, in 1948, for the case of one dependent variable [Hes48]. Later, Taheri extended Hestenes' method to treat the case of L^r-local minimizers [Tah01].

Motivated by problems from non-linear elasticity, in the second half of the 20th century there was an increased interest in solving the problem for the general multidimensional case. Ball conjectured that the natural generalization of Weierstrass' conditions had to be based on quasiconvexity notions, that had already been proven to be also necessary for the existence of minimizers [Bal98, Section 6.2], [BM84a].

On the other hand, based on the example by Müller and Šverák [MŠ03] of Lipschitz extremals for quasiconvex integrands that are nowhere C^1, Kristensen and Taheri established that the same can be achieved for extremals at which, in addition, the second variation is strongly positive [KT03]. Furthermore, they also proved that strong local minimizers are partially regular, i.e., regular outside some closed subset of their domain of measure zero.

[1] See Definition 51 regarding the different notions of local minimality.

This made clear that Lipschitz continuity of the extremal is not enough for it to be a strong local minimizer: higher a priori regularity should be assumed in order to obtain a suitable set of sufficient conditions for strong local minima.

It was only until 2008 that Y. Grabovsky and T. Mengesha fully settled the conjecture of Ball [GM09]. They proved that strong quasiconvexity and strong positivity of the second variation are sufficient for a C^1 solution of the weak Euler-Lagrange equation to be a strong local minimizer.

The purpose of this chapter is to provide a new proof for Grabovsky-Mengesha Theorem in the case of homogeneous integrands (with no x or u dependence). The motivation behind this is to simplify the proof with the aim of relaxing the a priori regularity assumptions made on the potential local minimizer. On the other hand, this new proof of the sufficiency theorem allows us to remove some coercivity conditions imposed on the integrand, at least for the quadratic case and while assuming Dirichlet boundary conditions.

In the first section of this chapter we give a brief review of the Young Measure theory, which is the essential tool that we use to deal with the non-linearity of the second variation in the problem. We then expand some of the definitions already established, in order to fix the notation that allows us to consider variations with mixed boundary values. In Section 2.3 we discuss the notion of quasiconvexity at the free boundary, which is one of the sufficient (and also necessary) conditions for strong local minimizers in this context. The last section of this chapter is dedicated to the proof of the sufficiency result.

2.1 Young Measures

The importance of weak convergence for the Calculus of Variations is evident from the use of the direct method. However, while trying to establish existence of minimizers for variational problems by using weak convergent sequences, we cannot use some of the nice properties that strong convergence would enable us to use. In most cases, it is not possible to improve weak convergence to strong convergence but, fortunately, it is also not necessary to do so. In this spirit, we recall here the elementary Vitali's Convergence Theorem as a way of motivating the remaining content of this section. First, we state here the concept of equiintegrability, that we will refer to in subsequent chapters.

Definition 23 *Let $\mathcal{F} \subseteq \mathrm{L}^1(\Omega, \mathbb{R}^d)$ be a family of integrable maps. We say that $\mathcal{F}$ is equiintegrable if and only if*

$$\sup_{f\in\mathcal{F}} \int_{\{x\in\Omega\,:\,|f(x)|>t\}} |f(x)|\,\mathrm{d}x \to 0 \text{ as } t \to \infty.$$

In addition, given $1 \leq p < \infty$ we say that $\mathcal{F}$ is ***p-equiintegrable*** *if and only if the family $\{|f|^p : f \in \mathcal{F}\}$ is equiintegrable.*

It is worth recalling that a necessary condition for a function $f\colon \Omega \to \mathbb{R}^d$ to be in L^1 is that the family $\mathcal{F} := \{f\}$ is equiintegrable. Similarly, for a larger family of functions to be equiintegrable means that all the functions in the family "decay" uniformly as we consider the set where the functions become large. The following result provides a very useful characterization of the equiintegrability of a family of functions.

Theorem 24 (de la Vallé Poussin) *Let $\Omega \subseteq \mathbb{R}^n$ and let $\mathcal{F} \subseteq \mathrm{L}^p(\Omega, \mathbb{R}^N)$ be a bounded set. Then, $\mathcal{F}$ is p-equiintegrable if and only if there exists an increasing function $\Phi\colon [0,\infty) \to [0,\infty]$ such that*

$$\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$$

and

$$\sup_{u\in\mathcal{F}} \int_\Omega \Phi(|u|^p)\,\mathrm{d}x < \infty.$$

For a proof of this result we refer the reader to [FL07, Theorem 2.29].

We are now ready to state the following.

Theorem 25 (Vitali's Convergence Theorem) *Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$ such that $\mathcal{L}^n(\Omega) < \infty$. Then, $f_j \to f$ strongly in $\mathrm{L}^p(\Omega, \mathbb{R}^d)$ if and only if*

(I) *$(|f_j|^p)$ is equiintegrable and*

(II) *$f_j \to f$ in measure.*

With this result in mind we establish the following terminology, useful while referring to the properties of a weakly convergent sequence that can prevent it from converging strongly. Let $f_j \rightharpoonup f$ in L^p such that $f_j \not\to f$ in L^p.

(i) If $f_j \not\to f$ in measure, we say that (f_j) **oscillates**.

(ii) If $(|f_j|^p)$ is not equiintegrable, we say that (f_j) **concentrates** in L^p.

One of the tools that enable us to interact more easily with weakly convergent sequences are the Young Measures, introduced by L.C. Young in the 1930's [You37]. In this section, we make a brief review of this notion. Further discussion and the proofs to all the results that we mention here can be found in [FL07], unless we state otherwise.

Before defining the concept of Young Measure, we need to introduce the following definitions and notation.

Notation 26 *Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of all the Borel sets contained in Ω. We denote*

$$\mathcal{M}(\Omega, \mathbb{R}^{N\times n}) := \{\mu \colon \mathcal{B}(\Omega) \to \mathbb{R}^{N\times n} : \mu \text{ is a bounded Borel measure}\}.$$

We endow this space with the total variation norm.
To simplify the notation, for real valued measures we will write $\mathcal{M}(\Omega) := \mathcal{M}(\Omega, \mathbb{R})$.

Remark 27 *Recall that, by Riesz' Representation Theorem, if we let $C_0^0(\Omega, \mathbb{R}^{N\times n})^*$ denote the dual space of the space of continuous functions $f \colon \Omega \to \mathbb{R}^{N\times n}$ that vanish on $\partial\Omega$ equipped with the L^∞ norm, then*

$$\mathcal{M}(\Omega, \mathbb{R}^{N\times n}) \cong C_0^0(\Omega, \mathbb{R}^{N\times n})^*,$$

where $\cong$ denotes an isometric isomorphism. Furthermore, since $C_0^0(\Omega, \mathbb{R}^{N\times n})$ is a separable Banach space, by Banach-Alaoglu's Theorem we have that the closed unit ball of $\mathcal{M}(\Omega, \mathbb{R}^{N\times n})$ is weakly* sequentially compact.

Within the context of Young Measures, it will be particularly useful to focus on the following families of measures.

Notation 28 *We denote*

$$\mathcal{M}^+(\Omega) := \{\mu \in \mathcal{M}(\Omega) : \mu(B) \geq 0 \text{ for every Borel set } B \subseteq \Omega\}.$$

*In addition, we use the following notation for the space of **probability measures** on Ω.*

$$\mathcal{M}_1^+(\Omega) := \{\mu \in \mathcal{M}^+(\Omega) : \mu(\Omega) = 1\}.$$

The following definitions aim at establishing the concept of Young Measure, which is given in terms of product measures.

Definition 29 *Let $\nu\colon \Omega \to \mathcal{M}(\mathbb{R}^d)$. We say that ν is **weakly*-measurable** if and only if the mapping*

$$x \in \Omega \mapsto \langle \phi, \nu(x) \rangle = \int_{\mathbb{R}^d} \Phi(y)\, \mathrm{d}\nu(x)\, \mathrm{d}y$$

is Borel measurable for all $\Phi \in C_0^0(\mathbb{R}^d)$.

Definition 30 *Let $\mu \in \mathcal{M}^+(\Omega)$ and $\nu\colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^d)$ be weakly*-measurable. Then the generalized product measure $\mu \otimes \nu(x)$ given by*

$$\langle \Phi, \mu \otimes \nu(x) \rangle := \int_{\Omega \times \mathbb{R}^d} \Phi\, \mathrm{d}(\mu \otimes \nu(x)) = \int_{\Omega} \int_{\mathbb{R}^d} \Phi(x,z)\, \mathrm{d}\nu(x)(z)\, \mathrm{d}\mu(x)$$

*for $\Phi \in C_0^0(\Omega \times \mathbb{R}^d)$, is called a μ**-Young Measure on** Ω **with target** $\mathbb{R}^d$.*

Remark 31 *Henceforth, we write a weakly* measurable map $\nu\colon \Omega \to \mathcal{M}_1^+(\mathbb{R}^d)$ as a parametrized family of probability measures, for which we use the notation $(\nu_x)_{x\in\Omega}$, where $\nu_x := \nu(x)$. On the other hand, we will often suppress μ from the notation and we will just call the family $(\nu_x)_{x\in\Omega}$ a Young Measure.*

The following definitions and results connect the notion of Young Measure with the convergence of sequences of functions.

Definition 32 *Given a Borel map $f\colon \Omega \to \mathbb{R}^d$, we can consider the Young Measure*

$$\varepsilon_f := \mathcal{L}^n \otimes \delta_{f(x)},$$

*where $\delta_{f(x)}$ stands for the Dirac measure concentrated at $f(x) \in \mathbb{R}^d$. In this case, we call ε_f an **elementary Young Measure**.*

Recall that, given an $\mathcal{L}^n$-measurable function $f\colon \Omega \to \mathbb{R}^d$, we can modify it in an $\mathcal{L}^n$-negligible set to obtain a Borel-measurable mapping. Therefore, we can also talk about the elementary Young measure ε_f when f is merely $\mathcal{L}^n$-measurable.

The following definition is useful to establish necessary and sufficient conditions on a sequence of functions, so that their associated elementary measures admit a convergent subsequence.

Definition 33 *Let (f_j) be a sequence of measurable functions with $f_j\colon \Omega \to \mathbb{R}^d$. We say that (f_j) is* ***measure-tight*** *if and only if*

$$\lim_{t\to\infty} \sup_{j\in\mathbb{N}} \mathcal{L}^n(\{x \in \Omega : |f_j(x)| > t\}) = 0.$$

Lemma 34 *Let (f_j) be a measure-tight sequence of measurable functions. Then, there exist a subsequence (f_{j_k}) and a weak*-measurable family $(\mu_x)_{x\in\Omega} \subseteq \mathcal{M}_1^+(\mathbb{R}^d)$, such that*

$$\varepsilon_{f_{j_k}} \overset{*}{\rightharpoonup} \mathcal{L}^n \otimes \mu_x \text{ in } C_0^0(\Omega \times \mathbb{R}^d)^*.$$

The converse result is also true.

Lemma 35 *Let (f_j) be a sequence of measurable functions such that $\varepsilon_{f_{j_k}} \overset{*}{\rightharpoonup} \mathcal{L}^n \otimes \mu_x$ in $C_0^0(\Omega \times \mathbb{R}^d)^*$ with $(\mu_x)_{x\in\Omega} \subseteq \mathcal{M}_1^+(\mathbb{R}^d)$. Then, (f_j) is measure-tight.*

Definition 36 *Let (f_j) be a sequence of measurable maps and let $(\mu_x)_{x\in\Omega} \subseteq \mathcal{M}_1^+(\mathbb{R}^d)$ be a measurable family. We say that f_j* ***generates the Young Measure*** *$(\mu_x)_{x\in\Omega}$ if and only if $\varepsilon_{f_j} \overset{*}{\rightharpoonup} \mathcal{L}^n \otimes \mu_x$ in $C_0^0(\Omega \times \mathbb{R}^d)^*$. We denote this by $f_j \overset{Y}{\longrightarrow} (\mu_x)_{x\in\Omega}$.*

The following result shows that $(\mu_x)_{x\in\Omega}$ gives information about the oscillatory behaviour of (f_j) as $j \to \infty$. Therefore, we call it a **Young Measure for oscillation.**

Lemma 37 *Let $f, f_j\colon \Omega \to \mathbb{R}^d$ be measurable maps. Then, $f_j \to f$ in measure if and only if $f_j \overset{Y}{\longrightarrow} (\delta_{f(x)})_{x\in\Omega}$.*

In the same context, we also have the following useful result.

Lemma 38 *Let $f_j\colon \Omega \to \mathbb{R}^d$ and $g_j\colon \Omega \to \mathbb{R}^d$ be measurable maps such that $f_j - g_j \to 0$ in measure. If $f_j \overset{Y}{\longrightarrow} (\mu_x)_{x\in\Omega}$, then also $g_j \overset{Y}{\longrightarrow} (\mu_x)_{x\in\Omega}$.*

We will now establish a result that describes how Young Measures are helpful to investigate the way in which certain non-linearities interact with a weakly convergent sequence. With this purpose in mind, we introduce first the following notation.

Notation 39 *We use the following notation for the next two special subspaces of the space of continuous functions on* $\mathbb{R}^d$.

$$C_c^0(\mathbb{R}^d) := \{\psi\colon \mathbb{R}^d \to \mathbb{R} : \ \psi \textit{ is continuous and } \operatorname{supp}(\psi) \textit{ is compact}\};$$
$$C_b^0(\mathbb{R}^d) := \{\psi\colon \mathbb{R}^d \to \mathbb{R} : \ \psi \textit{ is continuous and bounded}\}.$$

Lemma 40 *Let* (f_j) *be a measure-tight sequence of measurable functions such that* $f_j \xrightarrow{Y} (\mu_x)_{x\in\Omega}$*. Then, for each* $\psi \in C_b^0(\mathbb{R}^d)$,

$$\psi \circ f_j \overset{*}{\rightharpoonup} \langle \psi, \mu_x \rangle$$

in $\mathrm{L}^\infty(\Omega)$.

Proof. Let $\psi \in C_b^0(\mathbb{R}^d)$ and, for each $k \in \mathbb{N}$, take $\rho_k \in C_c^0(\mathbb{R}^d)$ so that $\rho_k(z) = 1$ for $|z| \leq k$ and $0 \leq \rho_k \leq 1$. Then, $\rho_k\psi \in C_0^0(\mathbb{R}^d)$ and so, for $\varphi \in C_0^0(\Omega)$,

$$\int_\Omega \varphi(\rho_k\psi) \circ f_j \,\mathrm{d}x \underset{j\to\infty}{\longrightarrow} \int_\Omega \varphi \,\langle \rho_k\psi, \mu_x\rangle \,\mathrm{d}x.$$

Now, by Dominated Convergence Theorem, we have that

$$\int_\Omega \varphi \,\langle \rho_k\psi, \mu_x\rangle \,\mathrm{d}x \underset{k\to\infty}{\longrightarrow} \int_\Omega \varphi \,\langle \psi, \mu_x\rangle \,\mathrm{d}x$$

and

$$\left| \int_\Omega \varphi(\rho_k\psi) \circ f_j \,\mathrm{d}x - \int_\Omega \varphi\psi \circ f_j \,\mathrm{d}x \right| \leq \|\varphi\|_{\mathrm{L}^\infty} \|\psi\|_{\mathrm{L}^\infty} \mathcal{L}^n(\{x \in \Omega : |f_j| > k\}).$$

Hence, by measure-tightness of (f_j), it follows that

$$\int_\Omega \varphi\psi \circ f_j \,\mathrm{d}x \overset{j\to\infty}{\longrightarrow} \int_\Omega \varphi \,\langle \psi, \mu_x\rangle \,\mathrm{d}x.$$

Finally, since $\psi \circ f_j$ is bounded in $\mathrm{L}^\infty(\Omega)$ and $C_0^0(\Omega)$ is dense in $\mathrm{L}^1(\Omega)$, this implies the conclusion. □

We now state one of the main results of this section. It compiles some of the properties already stated for Young Measures and gives conditions under which we have continuous (or lower semicontinuous) behaviour of a given integral functional. We first recall the following definitions, which name the precise type of integrands that ensure that the variational problems under consideration are well defined.

Definition 41 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $F\colon \Omega \times \mathbb{R}^N \to [-\infty, \infty]$ be a Borel measurable function. Then, F is said to be a* ***normal integrand*** *if and only if $\xi \mapsto F(x, \xi)$ is lower semicontinuous for almost every $x \in \Omega$.*

In this definition, we require F to be Borel measurable to ensure that if $u\colon \Omega \to \mathbb{R}^N$ is a measurable function, then $g\colon \Omega \to [-\infty, \infty]$ defined as $g(x) := F(x, u(x))$ is measurable.

The most important example of normal integrands is the following.

Definition 42 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $F\colon \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$. Then, F is said to be a* ***Carathéodory function*** *if and only if*

(i) *$\xi \mapsto F(x, \xi)$ is continuous for almost every $x \in \Omega$ and*

(ii) *$x \mapsto F(x, \xi)$ is measurable for every $\xi \in \mathbb{R}^N$.*

We now state the following fundamental result.

Theorem 43 (Fundamental Theorem for Young Measures of Oscillation) *Let (f_j) be a measure-tight sequence of measurable functions $f_j\colon \Omega \to \mathbb{R}^d$. Then, there exist a subsequence (f_{j_k}) and a Young Measure $(\mu_x)_{x\in\Omega}$, so that $f_{j_k} \xrightarrow{Y} (\mu_x)_{x\in\Omega}$. Furthermore, the following statements are true:*

(I) *If $F\colon \Omega \times \mathbb{R}^d \to [-\infty, \infty]$ is a normal $\mathcal{L}^n$-integrand and $(F(\cdot, f_{j_k})^-)$ is equiintegrable, then*

$$\liminf_{k\to\infty} \int_\Omega F(x, f_{j_k})\,\mathrm{d}x \geq \int_\Omega \langle F(x, \cdot), \mu_x\rangle\,\mathrm{d}x.$$

(II) *Let $F\colon \Omega \times \mathbb{R}^d \to [0, \infty]$ be a Carathéodory $\mathcal{L}^n$-integrand. Then,*

$$\lim_{k\to\infty} \int_\Omega F(x, f_{j_k})\,\mathrm{d}x = \int_\Omega \langle F(x, \cdot), \mu_x\rangle\,\mathrm{d}x < \infty$$

if and only if $(F(\cdot, f_{j_k}))$ is equiintegrable.

We now state the following result based on Kristensen's Decomposition Theorem [Kri94], at which he established that, given a uniformly bounded sequence in $\mathrm{W}^{1,p}$, it has a subsequence that we can decompose into two further sequences such that one carries the *oscillations* and the other one the *concentrations*. The decomposition that we present here follows Kristensen's proof and provides a technical extension of his result by showing that if a bounded sequence (u_j) in $\mathrm{W}^{1,2}$ is modified by multiplying each term by a (possibly different) scalar belonging to the unit interval $(0,1)$ and if the resulting subsequence, say (ζ_j), is bounded in $\mathrm{W}^{1,p}$ for some $p \geq 2$, then the respective decompositions in the spaces $\mathrm{W}^{1,2}$ and $\mathrm{W}^{1,p}$ can be obtained in such a way that they follow the same linear relations that (u_j) and (ζ_j) satisfy.

We remark that Kristensen's Decomposition Theorem is valid also for the case $1 < p < 2$, that we skip here although the proof remains the same than the one we will perform for the sequence (u_j) in $\mathrm{W}^{1,2}$. In addition, we refer the reader to [GM09, Theorem 8.1], in which Grabovsky and Mengesha modify the decomposition result from [FMP98] and that motivated the statement that we present below.

Finally, we emphasize that we require the following version of the Decomposition Theorem in order to be able to control a sequence in $\mathrm{W}^{1,p}$ while normalizing it by both its $\mathrm{W}^{1,2}$ and its $\mathrm{W}^{1,p}$-norms in the proof of Theorem 55, which is the main subject of this chapter.

Theorem 44 (Decomposition Lemma) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and let $2 \leq p < \infty$. Let (u_j) be a sequence such that $u_j \rightharpoonup u$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$, assume that (r_j) is a sequence in $(0,1)$ and that $\zeta_j := r_j u_j$ is bounded in $\mathrm{W}^{1,p}$. Then, there exist a subsequence (u_{j_k}) and sequences $(g_k) \subseteq C_c^\infty(\Omega, \mathbb{R}^N)$, $(b_k) \subseteq \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ such that*

(a) *$g_k \rightharpoonup 0$ and $b_k \rightharpoonup 0$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$;*

(b) *$(|\nabla g_k|^2)$ is equiintegrable;*

(c) *$\nabla b_k \to 0$ in measure and*

(d) *$u_{j_k} = u + g_k + b_k$.*

In addition, (g_k) and b_k can be taken so that, for a subsequence (r_{k_j}), if $s_k := r_{k_j} g_k$ and $t_k := r_{j_k} b_k$, then

(a') *$s_k \rightharpoonup 0$ and $t_k \rightharpoonup 0$ in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$;*

(b') *$(|\nabla s_k|^p)$ is equiintegrable and*

(c') *$\nabla t_k \to 0$ in measure.*

In order to establish this result following Kristensen's proof, we state first the Helmholtz Decomposition Theorem and an auxiliary lemma.

Theorem 45 (Helmholtz Decomposition Theorem) *Let* $1 < p < \infty$ *and denote*

$$\overset{\circ}{W}{}^{1,p} := \left\{\varphi \in \mathrm{W}^{1,p}_{loc}(\mathbb{R}^n) : \nabla\varphi \in \mathrm{L}^p(\mathbb{R}^n, \mathbb{R}^n)\right\}$$

the homogeneous Sobolev space. Let

$$E^p := \left\{\nabla\varphi : \varphi \in \overset{\circ}{W}{}^{1,p}\right\}$$

and

$$B^p := \{\sigma \in \mathrm{L}^p(\mathbb{R}^n, \mathbb{R}^n) : \mathrm{div}\sigma = 0 \textit{ in the distributional sense}\}\,.$$

Then, E^p *and* B^p *are closed subspaces of* $\mathrm{L}^p = \mathrm{L}^p(\mathbb{R}^n, \mathbb{R}^n)$ *such that* $E^p \cap B^p = \{0\}$*. Furthermore, there exist bounded projections* $\mathbb{E} : E^p \to \mathrm{L}^p$, $\mathbb{B} : \mathrm{L}^p \to B^p$, *so that* $\mathrm{Id}_{\mathrm{L}^p} = \mathbb{E}+\mathbb{B}$*. In other words, for any* $v \in \mathrm{L}^p$*, there exist* $\varphi \in \overset{\circ}{W}{}^{1,p}$ *and* $\sigma \in \mathrm{L}^p$ *with* $\mathrm{div}\sigma = 0$*, with the property that*

$$v = \nabla\varphi + \sigma \tag{2.1}$$

and where $\|\nabla\varphi\|_{\mathrm{L}^p} \le C_p\|v\|_{\mathrm{L}^p}$, $\|\sigma\|_{\mathrm{L}^p} \le C_p\|v\|_{\mathrm{L}^p}$ *for some constant* $C_p > 0$.

In addition, if $v \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{R}^n) \cap \mathrm{L}^p(\mathbb{R}^n, \mathbb{R}^n)$*, then the decomposition of* v *in the space* $\mathrm{L}^2(\mathbb{R}^n, \mathbb{R}^n)$ *coincides with the decomposition of* v *in the space* $\mathrm{L}^p(\mathbb{R}^n, \mathbb{R}^n)$.

The following result follows easily from the definition of p-equiintegrability.

Lemma 46 *Let* $\Omega \subseteq \mathbb{R}^n$ *such that* $\mathcal{L}^n(\Omega) < \infty$, $1 \le p < q \le \infty$ *and let* $(f_j) \subseteq \mathrm{L}^p(\mathbb{R}^n, \mathbb{R}^m)$ *be bounded in* L^p*. Then,* (f_j) *is* $p-$*equiintegrable if and only if, for each* $\varepsilon > 0$*, there exist a sequence* $(g_j) \subseteq \mathrm{L}^q(\mathbb{R}^n, \mathbb{R}^m)$ *and a constant* $c_\varepsilon > 0$ *such that, for all* $j \in \mathbb{N}$, $\|f_j - g_j\|_{\mathrm{L}^p} < \varepsilon$ *and* $\|g_j\|_{\mathrm{L}^q} \le c_\varepsilon$.

We now proceed with the proof of the Decomposition Lemma.

Proof of Theorem 44. By considering suitable subsequences, that we do not relabel, we assume without loss of generality that $\nabla u_j \overset{Y}{\to} (\nu_x)$ and $\nabla\zeta_j \overset{Y}{\to} (\mu_x)$.

In addition, we observe that part (c') of the Theorem follows directly from part (c) and the fact that $r_j \in (0, 1)$.

Furthermore, we observe (as in [GM09, Theorem 8.1]) that (b') implies that the sequence (s_k) is bounded in $\mathrm{W}^{1,p}$ and, since $\zeta_k = r_{j_k} u + s_k + t_k$, then we also have that (t_k) is bounded in $\mathrm{W}^{1,p}$, following our initial assumption on ζ_k. This, together with (a), means that there exist subsequences of s_k and t_k, that we do not relabel, and such that they satisfy condition (a').

Given this, we are only left with establishing parts (a)-(d) and (b') of the theorem.

Step 1. Observe that, by working with the sequence $u_j - u$ instead of u_j, we can assume that $u = 0$.

Step 2. *Reduction of the problem to* $(u_j) \subseteq \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$. We begin by taking a sequence of smooth subdomains $\Omega_k \Subset \Omega_{k+1} \Subset \Omega$ such that $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. In addition, we consider cut-off functions $\rho_k \colon \Omega \to [0,1]$ with $\rho_k \in C_c^1(\Omega)$, $\mathbb{1}_{\Omega_k} \leq \rho_k \leq \mathbb{1}_\Omega$ and $|\nabla \rho_k| \leq \frac{1}{d_k}$, where $d_k := \operatorname{dist}(\Omega_k, \Omega)$.

Now, observe that for any $j, k \in \mathbb{N}$, $u_j = \rho_k u_j + (1-\rho_k)u_j$ and

$$\int_\Omega |\nabla((1-\rho_k)u_j)| \,\mathrm{d}x \leq \int_{\Omega-\Omega_k} \left(|\nabla u_j| + \frac{|u_j|}{d_k} \right) \mathrm{d}x. \tag{2.2}$$

Since $u_j \rightharpoonup 0$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$, in particular we know that $u_j \to 0$ in L^2. Hence, we can find a sequence $k_j \to \infty$ such that

$$\int_{\Omega-\Omega_{k_j}} \frac{|u_j|}{d_{k_j}} \,\mathrm{d}x \to 0$$

when $j \to \infty$. Furthermore, since (∇u_j) is bounded in L^2, by adjusting the sequence k_j if necessary, from (2.2) we can also ensure that $(1-\rho_{k_j})u_j$ is bounded in $\mathrm{W}^{1,2}$ and that $(1-\rho_{k_j})r_j u_j$ is bounded in $\mathrm{W}^{1,p}$.

In addition, we also infer that $(1-\rho_{k_j})u_j \to 0$ in L^2. The last two facts imply together that $(1-\rho_{k_j})u_j \rightharpoonup 0$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$.

What is more, using again that $|\nabla((1-\rho_{k_j})u_j)| \leq \left(|(1-\rho_{k_j})\nabla u_j| + \frac{|u_j|}{d_{k_j}} \right) \mathbb{1}_{\Omega-\Omega_{k_j}}$, we infer that this sequence converges to 0 in measure. Given this, we focus on decomposing $(\rho_{k_j} u_j)$, as we can then incorporate $(1-\rho_{k_j})u_j$ into the sequence (b_k).

Step 3. *Truncation.* We now define, for each $k \in \mathbb{N}$, the truncation $T_k \colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ at level k by

$$T_k(z) := \begin{cases} z & \text{if} \quad |z| \leq k \\ k\frac{z}{|z|} & \text{if} \quad |z| > k. \end{cases}$$

It is clear that T_k is continuous and $|T_k(z)| \leq k$ for all $z \in \mathbb{R}^{N\times n}$.

Observe that, by the Fundamental Theorem for Young Measures and the Monotone Convergence Theorem,

$$\begin{aligned} \lim_{k\to\infty} \lim_{j\to\infty} \int_\Omega |T_k(\nabla u_j)|^2 \, \mathrm{d}x &= \lim_{k\to\infty} \int_\Omega \int_{\mathbb{R}^{N\times n}} |T_k(z)|^2 \, \mathrm{d}\nu_x(z) \, \mathrm{d}x \\ &= \int_\Omega \int_{\mathbb{R}^{N\times n}} |\cdot|^2 \, \mathrm{d}\nu_x \, \mathrm{d}x. \end{aligned}$$

We also have that

$$\lim_{k\to\infty} \lim_{j\to\infty} \int_\Omega |T_k(\nabla u_j) - \nabla u_j| \, \mathrm{d}x \leq \lim_{k\to\infty} \sup_{j\in\mathbb{N}} \int_{|\nabla u_j|>k} 2|\nabla u_j| \, \mathrm{d}x = 0.$$

Given these, we can now take a subsequence $j_k \to \infty$ such that

$$\lim_{k\to\infty} \int_\Omega |T_k(\nabla u_{j_k})|^2 \, \mathrm{d}x = \int_\Omega \langle |\cdot|^2, \nu_x \rangle \, \mathrm{d}x \tag{2.3}$$

and

$$\lim_{k\to\infty} \int_\Omega |T_k(\nabla u_{j_k}) - \nabla u_{j_k}| \, \mathrm{d}x = 0. \tag{2.4}$$

Let $v_k := T_k \circ \nabla u_{j_k}$. It follows from equation (2.3) and the Fundamental Theorem for Young Measures, that (v_k) is 2-equiintegrable and, from (2.4) together with Lemma 38, we deduce that (v_k) also generates the Young Measure (ν_x). Furthermore, observe that using (2.3), (2.4), Vitali's Convergence Theorem and de la Vallée Poussin criterion for equiintegrability, we can also conclude that $(T_k \circ \nabla u_{j_k} - \nabla u_{j_k}) \to 0$ strongly in $\mathrm{L}^q(\Omega, \mathbb{R}^{N\times n})$ for $q < 2$.

Step 4. *Helmholtz decomposition.* Since $u_{j_k} \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$, we can extend u_{j_k} to $\mathbb{R}^n$ by 0 while still having $u_{j_k} \in \mathring{W}^{1,2}(\mathbb{R}^n, \mathbb{R}^N)$. We also extend $v_k \colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$ so that it is 0 outside of Ω.

Now, we apply row-wise Helmholtz decomposition in $\mathrm{L}^2(\mathbb{R}^n, \mathbb{R}^n)$ and we obtain functions $\tilde{g}_k \in \mathring{W}^{1,2}$, $\tilde{\sigma}_k \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{R}^{N\times n})$ such that:

(i) $\mathbb{E}(v_k) = \nabla \tilde{g}_k$, $\mathbb{B}(v_k) = \tilde{\sigma}_k$ defined row-wise and, hence, $v_k = \nabla \tilde{g}_k + \tilde{\sigma}_k$;

(ii) $\operatorname{div} \tilde{\sigma}_k = 0$ and

(iii) $\int_\Omega \tilde{g}_k = 0$, which we can achieve by subtracting a constant from $\tilde{g}_k$ if necessary.

We now claim that $\tilde{\sigma}_k \to 0$ in measure when $k \to \infty$. Indeed, since $v_k - \nabla u_{k_j} \to 0$ in $\mathrm{L}^q(\Omega, \mathbb{R}^{N\times n})$ for every $q \in (1,2)$, we get that

$$\|\tilde{\sigma}_k\|_{\mathrm{L}^q} = \|\mathbb{B}(v_k)\|_{\mathrm{L}^q} = \|\mathbb{B}(v_k - \nabla u_{j_k})\|_{\mathrm{L}^q} \leq c_q \|v_k - \nabla u_{j_k}\|_{\mathrm{L}^q} \to 0, \tag{2.5}$$

where $c_q > 0$ is a constant depending only on the continuity of $\mathbb{B}$.

We now proceed to prove that $\nabla \tilde{g}_k$ is 2-equiintegrable on Ω. Observe first that, since v_k is 2-equiintegrable, Lemma 46 implies that, for a given $\varepsilon > 0$, we can find a sequence $(w_k) \subseteq \mathrm{L}^3(\Omega, \mathbb{R}^{N\times n})$ such that, for all $k \in \mathbb{N}$, $\|v_k - w_k\|_{\mathrm{L}^2} < \varepsilon$ and $\|w_k\|_{\mathrm{L}^3} \leq c_\varepsilon$. On the other hand, by making $v_k = w_k = 0$ off Ω, we can also apply Helmholtz decomposition to w_k to conclude that

$$\|\nabla \tilde{g}_k - \mathbb{E}(w_k)\|_{\mathrm{L}^2} \leq c\|v_k - w_k\|_{\mathrm{L}^2} < c\varepsilon$$

and $\|\mathbb{E}(w_k)\|_{\mathrm{L}^3} \leq c_\varepsilon$ for every $k \in \mathbb{N}$. Since $\varepsilon > 0$ was arbitrary, by using Lemma 46 once again, we conclude the proof of our claim.

Observe that, since $\nabla \tilde{g}_k - \nabla u_{j_k} = v_k - \nabla u_{j_k} - \tilde{\sigma}_k = (v_k - \nabla u_{j_k}) - \tilde{\sigma}_k$, we have that $\nabla \tilde{g}_k - \nabla u_{j_k} \to 0$ in measure. In addition, (2.3) implies that $\nabla \tilde{g}_k$ is bounded in $\mathrm{L}^2(\Omega, \mathbb{R}^{N\times n})$. Hence, we can assume that $\nabla \tilde{g}_k - \nabla u_{j_k} \rightharpoonup 0$ in $\mathrm{L}^2(\Omega, \mathbb{R}^{N\times n})$ and, therefore, the same holds for $(\nabla \tilde{g}_k)$.

On the other hand, because $\int_\Omega \tilde{g}_k = 0$, by Poincaré's inequality and Rellich-Kondrachov Embedding Theorem we can conclude that there is $g \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ such that $\tilde{g}_k \rightharpoonup g$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ and, by the observations above, we further have $\tilde{g}_k \rightharpoonup 0$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$.

Now, we consider again the sequence of domains (Ω_l). Observe that, since $\tilde{g}_k \to 0$ in $\mathrm{L}^2(\Omega, \mathbb{R}^N)$, we can find a subsequence Ω_{l_k} such that, if $g_k := \rho_{l_k} \tilde{g}_k$, then

$$\nabla g_k = \rho_{l_k} \nabla \tilde{g}_k + \tilde{g}_k \otimes \nabla \rho_{l_k}$$

is 2-equiintegrable. In addition, since $\nabla\rho_{l_k} = 0$ in Ω_{l_k}, it follows that, if $b_k := u_{j_k} - g_k$, then

$$\begin{aligned}\nabla b_k = \nabla u_{j_k} - \nabla g_k &= (\nabla u_{j_k} - v_k) + (v_k - \rho_{l_k} v_k) + \rho_{l_k}(v_k - \nabla \tilde{g}_k) - \tilde{g}_k \otimes \nabla\rho_{l_k} \\ &= (\nabla u_{j_k} - v_k) + (1 - \rho_{l_k})v_k + \rho_{l_k}\tilde{\sigma}_k - \tilde{g}_k \otimes \nabla\rho_{l_k}\end{aligned}$$

converges to 0 in measure. Then, g_k and b_k are the desired functions and, since $g_k \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$, we can further assume that $(g_k) \in C_c^\infty(\Omega, \mathbb{R}^N)$, with which we complete the proof of parts (a)-(d) of the Theorem.

We now proceed with the proof of part (b'). Arguing exactly as we did in (2.3) from Step 3 and performing, once again, a slight abuse on the notation by not relabelling the corresponding subsequence, we can assume that

$$\lim_{k\to\infty} \int_\Omega |T_k(\nabla\zeta_{j_k})|^p \, \mathrm{d}x = \int_\Omega \langle |\cdot|^p, \mu_x\rangle \, \mathrm{d}x. \tag{2.6}$$

In addition, we observe that, for every $x \in \Omega$ and with $v_k = T_k \circ \nabla u_{j_k}$ as in Step 3, it holds that

$$|r_{j_k} v_k(x)| = |r_{j_k} T_k \circ \nabla u_{j_k}(x)| = \left| T_{r_{j_k}k}(r_{j_k} \nabla u_{j_k}(x)) \right| \le |T_k(r_{j_k}\nabla u_{j_k})| = |T_k(\nabla\zeta_{j_k})|\,. \tag{2.7}$$

We are using here the elementary identity $rT_k(\xi) = T_{rk}(r\xi)$ and the facts that $r_{j_k}k \le k$ and $k \mapsto |T_k(\xi)|$ is non-decreasing for every $\xi \in \mathbb{R}^{N\times n}$.

It follows from (2.6) and (2.7) that the sequence $(r_{j_k} v_k)$ is p-equiintegrable and, in particular, it is also bounded in L^p.

Furthermore, if $\mathbb{E}(v_k) = \nabla\tilde{g}_k$, then clearly $\mathbb{E}(r_{j_k} v_k) = r_{j_k}\nabla\tilde{g}_k$ by decomposing both v_k and $r_{j_k} v_k$ in $\mathrm{L}^2(\Omega, \mathbb{R}^N)$. In addition, since $r_{j_k} v_k \in \mathrm{L}^2(\Omega, \mathbb{R}^N) \cap \mathrm{L}^p(\Omega, \mathbb{R}^N)$, Helmholtz Decomposition Theorem enables us to ensure that, for some constant $c_p > 0$,

$$\|r_{j_k}\nabla\tilde{g}_k\|_{\mathrm{L}^p} = \|\mathbb{E}(r_{j_k} v_k)\|_{\mathrm{L}^p} \le c_p \|r_{j_k} v_k\|_{\mathrm{L}^p}.$$

This inequality, together with the fact that $(r_{j_k} v_k)$ is bounded in $\mathrm{L}^p(\Omega, \mathbb{R}^N)$, enables us to conclude that, for a subsequence that we do not relabel, $r_{j_k}\nabla\tilde{g}_k$ converges weakly in

$W^{1,p}(\Omega, \mathbb{R}^N)$ and, arguing exactly as we did with $\tilde{g}_k$, we can further deduce that

$$r_{j_k}\tilde{g}_k \rightharpoonup 0 \qquad \text{in} \qquad W^{1,p}(\Omega, \mathbb{R}^N). \tag{2.8}$$

To conclude the proof of (d') it is enough to observe that

$$\nabla s_k = r_{j_k}\nabla g_{j_k} = r_{j_k}\rho_{j_k}\nabla\tilde{g}_{j_k} + r_{j_k}\tilde{g}_{j_k} \otimes \nabla\rho_{j_k}$$

and use the fact that $(T_k(\nabla\zeta_{j_k}))$ is p-equiintegrable, together with (2.7) and (2.8), to proceed as we did to show that ∇g_k is 2-equiintegrable and construct that way a subsequence of (s_k) such that ∇s_k is p-equiintegrable. □

An important class of Young measures consists of those generated by gradients of Sobolev functions. We now state this definition precisely, following Kinderlehrer-Pedregal. See [KP91, KP94].

Definition 47 *Let $1 \leq p \leq \infty$ and let $(\nu_x)_{x\in\Omega}$ be a Young measure. We say that $(\nu_x)_{x\in\Omega}$ is a **gradient** p**-Young measure** if and only if there exists a sequence $(u_j) \subseteq W^{1,p}(\Omega, \mathbb{R}^N)$ such that $u_j \rightharpoonup u$ in $W^{1,p}$ if $1 \leq p < \infty$ or $u_j \overset{*}{\rightharpoonup} u$ if $p = \infty$ and $\nabla u_j \overset{Y}{\to} (\nu_x)_{x\in\Omega}$.*

One of the key features of the characterization of gradient Young measures by Kinderlehrer and Pedregal is that gradient Young measures are precisely those satisfying Jensen's inequality for all quasiconvex functions under suitable growth conditions. For a precise statement we refer the reader to [KP94, Theorems 1.1 and 1.2]. The fact that Jensen's inequality holds for quasiconvex functions and gradient p-Young measures had already been observed by Ball and Zhang [BZ90]. Their result is as follows.

Theorem 48 *Let $1 \leq p < \infty$ and let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function such that $|F(z)| \leq c_1(1 + |z|^p)$ for every $z \in \mathbb{R}^{N\times n}$. If $(\nu_x)_{x\in\Omega}$ is a gradient p-Young measure, then for almost every $x \in \Omega$ it holds that*

$$F(\overline{\nu}_x) \leq \int_{\mathbb{R}^{N\times n}} F\,\mathrm{d}\nu_x,$$

where $\overline{\nu}_x$ *is the centre of mass of the probability measure* ν_x *given by*

$$\overline{\nu}_x := \int_{\mathbb{R}^{N\times n}} z \, \mathrm{d}\nu_x(z).$$

2.2 Weak versus strong local minimizers

In order to work with mixed boundary conditions, we mention first the following concepts and conventions regarding the meaning of allowing part of the values that the minimizers take at the boundary, to be free.

We first remark that we require Ω to be a C^1 open and bounded subset of $\mathbb{R}^n$. We then assume that $\Gamma_D \subseteq \partial\Omega$ is a subset of the boundary of Ω such that the $(n-1)$-Hausdorff measure of the relative interior of Γ_D is positive (so, in particular, $\Gamma_D \neq \emptyset$). On the other hand, in order to allow free boundary values, we need to redefine the admissible functions as follows.

Definition 49 *Given* $p \in (1,\infty)$,[2] *we define the* ***set of admissible functions*** *as*

$$\mathcal{A} := \overline{\left\{u \in C^1(\overline{\Omega}, \mathbb{R}^N) : u(x) = u_0(x) \, \forall x \in \Gamma_D\right\}}^{\mathrm{W}^{1,p}},$$

where the closure is taken in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ *and* u_0 *is of class* C^1 *on some open set in* $\mathbb{R}^n$ *containing* $\overline{\Gamma_D}$.

It is clear that if $u \in \mathcal{A} \cap C^1(\overline{\Omega}, \mathbb{R}^N)$, then $u(x) = u_0(x)$ for all $x \in \overline{\Gamma_D}$. Hence, it is assumed, without loss of generality, that Γ_D is the interior of $\overline{\Gamma_D}$, relative to $\partial\Omega$. By defining $\Gamma_N := \partial\Omega - \overline{\Gamma_D}$, it holds that Γ_N is a relatively open subset of $\partial\Omega$ and $\partial\Omega = \Gamma_D \cup \overline{\Gamma_N}$. Indeed, if $x \in \partial\Omega - \overline{\Gamma_N}$, then x has an open neighbourhood in $\partial\Omega$ that does not intersect Γ_N. Therefore, this neighbourhood must belong to the interior of $\overline{\Gamma_D}$, which is Γ_D.

Definition 50 *We define the* ***space of variations*** *as the set*

$$\mathrm{Var}(\mathcal{A}) := \overline{\left\{\varphi \in C^1(\overline{\Omega}, \mathbb{R}^N) : \varphi(x) = 0 \, \forall x \in \Gamma_D\right\}}^{\mathrm{W}^{1,p}}.$$

We call $\mathrm{Var}(\mathcal{A})$ the space of variations because, for any $y_1, y_2 \in \mathcal{A}$, we have $y_1 - y_2 \in \mathrm{Var}(\mathcal{A})$. More generally, given an open set $\omega \subseteq \mathbb{R}^n$ such that $\omega \cap \Omega \neq \emptyset$, we consider the following

[2] The exponent p will be related, in this case, to the growth condition imposed on the integrand F.

space of variations defined in ω, which is naturally embedded in $\mathrm{Var}(\mathcal{A})$.

$$\mathrm{Var}(\omega,\mathbb{R}^N) := \overline{\{\varphi \in C^1(\overline{\omega},\mathbb{R}^N) : \varphi(x) = 0\, \forall x \in (\Gamma_D \cap \overline{\omega}) \cup (\partial\omega \cap \Omega)\}}^{\mathrm{W}^{1,p}}.$$

Observe that, given $\varphi \in \mathrm{Var}(\omega,\mathbb{R}^N)$, by extending φ to Ω so that it takes the value of 0 in $\Omega\backslash\omega$, we can assume that $\varphi \in \mathrm{Var}(\mathcal{A})$. Furthermore, we remark that, with this notation, $\mathrm{Var}(\mathcal{A}) = \mathrm{Var}(\Omega,\mathbb{R}^N)$.

We now recall the notions of weak and strong local minimizer, the latter being one of the core concepts of this chapter.

Definition 51 *Let $u \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. We say that u is a **weak local F-minimizer** if and only if there is a $\delta > 0$ such that*

$$\int_\Omega F(\nabla u)\,\mathrm{d}x \leq \int_\Omega F(\nabla u + \nabla\varphi)\,\mathrm{d}x$$

for every φ in $\mathrm{Var}(\mathcal{A})$ with $\|\nabla\varphi\|_{\mathrm{L}^\infty} < \delta$.

*On the other hand, we say that u is a **strong local F-minimizer** if and only if there is a $\delta > 0$ such that*

$$\int_\Omega F(\nabla u)\,\mathrm{d}x \leq \int_\Omega F(\nabla u + \nabla\varphi)\,\mathrm{d}x$$

for every φ in $\mathrm{Var}(\Omega,\mathbb{R}^N)$ with $\|\varphi\|_{\mathrm{L}^\infty} < \delta$.

For $1 < p < \infty$ the notion of strong local minimizer can be generalized by considering the L^p or the $\mathrm{W}^{1,p}$ norm of the variations. In those cases we say, respectively, that u is an L^p-local minimizer or a $\mathrm{W}^{1,p}$-local minimizer.

We also emphasize that the notion of extremal needs to be adjusted if we are considering mixed boundary values. Hence, we need to consider the following.

Definition 52 *Let $u \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. We say that u is an **F-extremal** if and only if*

$$\int_\Omega \left\langle F'(\nabla u), \nabla\varphi \right\rangle\,\mathrm{d}x = 0$$

for every $\varphi \in \mathrm{Var}(\Omega,\mathbb{R}^N)$.

2.3 Quasiconvexity at the free boundary

We now recall that, if we allow free boundary values for a minimizer $\overline{u}$, a necessary condition on the integrand that was presented by Ball and Marsden in [BM84a], is that of *quasiconvexity at the boundary*, so called because it is related to Morrey's notion of quasiconvexity.

For this reason, in addition to conditions (H0)-(H2), it is necessary to consider that the integrand F is strongly **quasiconvex on the free boundary**, meaning that, for a constant $c_2 > 0$ (which can be chosen to be the same from (H2)) it holds that

$$\text{(H2')} \qquad c_2 \int_{B^-_{\mathrm{n}(x_0)}} |V(\nabla\varphi)|^2 \,\mathrm{d}x \leq \int_{B^-_{\mathrm{n}(x_0)}} (F(\nabla u(x_0) + \nabla\varphi) - F(\nabla u(x_0)))\,\mathrm{d}x$$

for all $\varphi \in \mathrm{V}_{\mathrm{n}(x_0)}$, where

$$\mathrm{V}_{\mathrm{n}(x_0)} := \left\{\varphi \in C^\infty\left(\overline{B^-_{\mathrm{n}(x_0)}}, \mathbb{R}^N\right) : \varphi(x) = 0 \text{ on } (\partial B(x_0, 1)) \cap \overline{B^-_{\mathrm{n}(x_0)}}\right\} \tag{2.9}$$

and $\mathrm{n}(x_0)$ is the outer unit normal to $\partial\Omega$ at $x_0 \in \Gamma_N$. Here, $B^-_{\mathrm{n}(x_0)}$ is the half of the ball $B(x_0, 1)$ that lies in the half plane $\{z \in \mathbb{R}^n : \langle z - x_0, \mathrm{n}\rangle < 0\}$.

The spirit in which this new condition is shown to be necessary for strong local minima is the same in which the quasiconvexity at the interior is also proven to hold when in presence of strong local minimizers. We observe, however, that the quasiconvexity at the boundary differs from the one in the interior in the sense that it is not anymore a convexity notion. It is enough to recall, for example, that for a convex integrand F the quasiconvexity in the interior can be seen as a straightforward consequence of Jensen's inequality for probability measures. However, given $x_0 \in \Gamma_N$ and $\varphi \in \mathrm{V}_{\mathrm{n}(x_0)}$, we can follow the same ideas if we consider the probability measure defined on the space of matrices $\mathbb{R}^{N\times n}$ by

$$\langle \Phi, \nu_{\varphi,x_0}\rangle := \fint_{B^-_{\mathrm{n}(x_0)}} \Phi(\nabla u(x_0) + \nabla\varphi(x))\,\mathrm{d}x.$$

We then observe that the centre of mass of this probability measure is given by

$$\overline{\nu}_{\varphi,x_0} = \int_{R^{N\times n}} z\,\mathrm{d}\nu_{\varphi,x_0}(z) = \nabla u(x_0) + \fint_{B^-_{\mathrm{n}(x_0)}} \nabla\varphi(x)\,\mathrm{d}x.$$

Here, $\fint_{B^-_{\mathrm{n}(x_0)}} \nabla\varphi(x)\,\mathrm{d}x \neq 0$ in general, as $\varphi \in \mathrm{V}_{\mathrm{n}(x_0)}$ and it is not necessarily 0 at the boundary. On the other hand, by Jensen's inequality we will have, for a convex integrand F, that

$$F(\overline{\nu}_{\varphi,x_0}) \leq \int_{\mathbb{R}^{N\times n}} F(z)\,\mathrm{d}\nu_{\varphi,x_0} = \fint_{B^-_{\mathrm{n}(x_0)}} F(\nabla u(x_0) + \nabla\varphi(x))\,\mathrm{d}x.$$

This means, in particular, that the notion of quasiconvexity at the free boundary doesn't follow from convexity in the same way that quasiconvexity at the interior does.

Following the spirit in which the calculations above are made, we can consider the following specific example of a convex function that is not quasiconvex at the free boundary. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$F(u,v) := v$$

and let $\Omega := B^-_{\mathrm{n}(0,1)}$ be the half of the unit ball centred at zero that lies below the x-axis. We consider the mixed boundary conditions according to which the admissible test functions are precisely those in the set $\mathrm{V}_{\mathrm{n}(0)}$ defined in (2.9). It is then clear that the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\varphi(x,y) = x^2 + y^2 - 1$$

is such that $\varphi \in \mathrm{V}_{\mathrm{n}(0)}$. However, the quasiconvexity at the boundary condition is not satisfied for the convex function F at the point $(0,0)$, which lies on the free boundary, since this would imply that, for the particular φ that we defined above,

$$0 = F(0,0) \leq \fint_{B^-_{\mathrm{n}(0,1)}} F(\nabla\varphi(x,y))\,\mathrm{d}x\,\mathrm{d}y = \fint_{B^-_{\mathrm{n}(0,1)}} 2y\,\mathrm{d}x\,\mathrm{d}y,$$

which is a contradiction by definition of $B^-_{\mathrm{n}(0,1)}$.

Going back to the discussion regarding our interpretation of the notion of quasiconvexity at the free boundary, we observe that, by differentiating $t \mapsto \int_{B^-_{\mathrm{n}(x_0)}} F(\nabla u(x_0) + t\nabla\varphi(x))\,\mathrm{d}x$

we obtain, from the *quasiconvexity at the boundary* condition, that it implies

$$\begin{aligned} 0 &= \int\limits_{B^-_{\mathrm{n}(x_0)}} \langle F'(\nabla u(x_0)), \nabla\varphi(x)\rangle \, \mathrm{d}x = \Big\langle F'(\nabla u(x_0)), \int\limits_{\partial B^-_{\mathrm{n}(x_0)}} \varphi \otimes \mathrm{n}(x) \, \mathrm{d}\sigma(x) \Big\rangle \\ &= \Big\langle F'(\nabla u(x_0)), \int\limits_{\partial B^-_{\mathrm{n}(x_0)}} \varphi \, \mathrm{d}\sigma(x) \otimes \mathrm{n}(x_0) \Big\rangle, \end{aligned} \tag{2.10}$$

where the second identity above follows from the Divergence Theorem.

This enables us to give an interpretation of the quasiconvexity at the boundary as a *non-linear variational Neumann condition.* Indeed, since (2.10) holds for every $\varphi \in \mathrm{V}_{\mathrm{n}(x_0)}$, it in turn implies the Neumann boundary condition $\langle F'(\nabla u(x_0)), \mathrm{n}(x_0)\rangle = 0$ in $\mathbb{R}^N$. However, as pointed out by Ball and Marsden with an example in [BM84a], the quasiconvexity at the boundary is still a stronger notion.

We remark here that, as it turns out, the quasiconvexity on the free boundary is one of the sufficient conditions that are needed to ensure that a C^1 extremal furnishes an actual strong local minimizer.

2.4 Spatially-local minimizers: Zhang's Lemma

In this section we will establish a generalization of a theorem by K. Zhang in [Zha92]. His result states that smooth extremals are all spatially-local minimizers in a strict and convenient sense under Dirichlet boundary conditions. The generalization that we present here allows part of the boundary to take free values. Although the proof remains essentially the same as his, we state the result in this more general way aiming at using it for the new proof of Grabovsky-Mengesha's result.

Furthermore, we remark that the main idea behind Zhang's Lemma is that, if an extremal is smooth, in small subsets of its domain it is close enough to an affine function (in an uniform way). Therefore, we can exploit the strong quasiconvexity assumption on the integrand, according to which affine functions minimize the integrand under the corresponding affine boundary conditions, to obtain minimality in a local sense in space.

Theorem 53 *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfy* (H0) – (H2) *and* (H2′) *for some* $1 < p < \infty$. *If*

$u \in C^1(\overline{\Omega}, \mathbb{R}^N)$ *is an* $F-$*extremal in the sense that*

$$\int_\Omega \langle F'(\nabla u), \nabla\varphi\rangle \,\mathrm{d}x = 0$$

for every $\varphi \in \mathrm{Var}(\mathcal{A})$*, then there exists* $R > 0$ *such that*

(1) *for every* $x_0 \in \overline{\Omega}$

$$\frac{c_2}{2}\int_{\Omega(x_0,R)} |V(\nabla\varphi)|^2 \,\mathrm{d}x \leq \int_{\Omega(x_0,R)} \left(F(\nabla u + \nabla\varphi) - F(\nabla u)\right) \mathrm{d}x \tag{2.11}$$

whenever $\varphi \in \mathrm{W}_0^{1,2}(\Omega(x_0, R), \mathbb{R}^N)$ *and*

(2) *for every* $x_0 \in \Gamma_N$*, inequality (2.11) holds whenever* $\varphi \in \mathrm{Var}(\Omega(x_0, R), \mathbb{R}^N)$.

Proof. The main idea behind the proof will be an appropriate use of the quasiconvexity conditions that we are assuming under (H2) and (H2'). With this aim, we begin by defining the function $G\colon \overline{\Omega} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ as

$$\begin{aligned} G(x, z) &:= F(\nabla u(x) + z) - F(\nabla u(x)) - \langle F'(\nabla u(x)), z\rangle \\ &= \int_0^1 (1-t)F''(\nabla u(x) + tz)[z, z]\,\mathrm{d}t. \end{aligned}$$

We claim that, for a given $\varepsilon > 0$ and a fixed $J \in \mathbb{R}^{n\times n}$, there is an $R = R_\varepsilon > 0$ such that, for every $x \in \Omega(x_0, R)$ and every $z \in \mathbb{R}^{N\times n}$, if $w = z \cdot J$ and $|J - \mathrm{I}_n| < R$, then

$$E := |G(x_0, w) - G(x, z)| < \frac{\varepsilon}{4}|V(z)|^2. \tag{2.12}$$

To prove this, observe first that, since $u \in C^1(\overline{\Omega}, \mathbb{R})$, ∇u is uniformly continuous and bounded in $\overline{\Omega}$.

Motivated by the strategy originated in [AF87, Lemma II.3], we will establish (2.12) by considering the following two cases.

Case 1. If $|z| \leq 1$ and $w = z \cdot J$ with $|J - \mathrm{I}_n| < R$ as above, then, because F'' is locally uniformly continuous, we can find a modulus of continuity, say $\omega\colon [0, \infty) \to [0, 1]$, such that it is increasing, continuous, $\omega(0) = 0$ and for which there is a constant $c > 0$ with the property that

$$|F''(\nabla u(x)+tz)-F''(\nabla u(x_0)+tz)|\le c\,\omega(|x-x_0|+|z-w|)$$

for all $x,x_0\in\overline{\Omega}$, $t\in[0,1]$, $|z|\le 1$ and $w=z\cdot J$ with $|J-\mathrm{I}_n|<R$. We can further assume that c is such that

$$1+n+|F''(\nabla u(x)+tz)|\le c \tag{2.13}$$

for all $x\in\overline{\Omega}$, $t\in[0,1]$ and $|z|\le 1$.

Using this, if we fix $0<\varepsilon<1$, we can see that

$$\begin{aligned}
|G(x_0,w)-G(x,z)| &\le |G(x_0,w)-G(x_0,z)|+|G(x_0,z)-G(x,z)|\\
&\le \int_0^1 \big|F''(\nabla u(x_0)+tw)[w,w]-F''(\nabla u(x_0)+tz)[z,z]\big|\,\mathrm{d}t\\
&\quad+\int_0^1 \big|\big(F''(\nabla u(x_0)+tz)-F''(\nabla u(x)+tz)\big)[z,z]\big|\,\mathrm{d}t\\
&\le \int_0^1 \big(\big|F''(\nabla u(x_0)+tw)[w,w]-F''(\nabla u(x_0)+tz)[w,w]\big|\big)\,\mathrm{d}t\\
&\quad+\int_0^1 \big|F''(\nabla u(x_0)+tz)[w,w]-F''(\nabla u(x_0)+tz)[z,z]\big|\,\mathrm{d}t\\
&\quad+\int_0^1 \big|\big(F''(\nabla u(x_0)+tz)-F''(\nabla u(x)+tz)\big)[z,z]\big|\,\mathrm{d}t\\
&\le c\,\omega(|w-z|)|w|^2+\int_0^1 |F''(\nabla u(x_0)+tz)||w-z|(|w|+|z|)\,\mathrm{d}t\\
&\quad+c\,\omega(|x-x_0|)|z|^2\\
&\le c\,\omega(|z||J-\mathrm{I}_n|)|w|^2+|F''(\nabla u(x_0)+tz)||z||J-\mathrm{I}_n|(|w|+|z|)\\
&\quad+c\,\omega(|x-x_0|)|z|^2\\
&\le c\,\omega(|J-\mathrm{I}_n|)|J|^2|z|^2+|F''(\nabla u(x_0)+tz)||J-\mathrm{I}_n|(|J||z|^2+|z|^2)\\
&\quad+c\,\omega(|x-x_0|)|z|^2\\
&\le c\,\omega(|J-\mathrm{I}_n|)|J|^2|V(z)|^2\\
&\quad+c\,|J-\mathrm{I}_n|(|J||V(z)|^2+|V(z)|^2)\\
&\quad+c\,\omega(|x-x_0|)|V(z)|^2\\
&\le \frac{\varepsilon}{4}|V(z)|^2,
\end{aligned}$$

where the last inequality is making use of (2.13) and it holds provided that $|J-\mathrm{I}_n|$, $\omega(|J-\mathrm{I}_n|)$ and $\omega(|x-x_0|)$ are small enough. Notice that, if that is the case, we can assume $|J|<c(n)$

for a constant $c(n) > 0$.

Thus, for the case $|z| \leq 1$ we have that, if $R > 0$ is such that $c\,\omega(R)|J|^2 < \frac{\varepsilon}{8}$, then for every $x \in \Omega(x_0, R)$, $E \leq \frac{\varepsilon}{4}|V(z)|^2$, provided also that $|J - \mathrm{I}_n| < R$.

Case 2. For $|z| > 1$, we will need to make use of the Lipschitz bounds for F that are derived from (H0) – (H2) and Proposition 10. Following this, and the fact that F' is also locally uniformly continuous, we have for $x \in \Omega(x_0, R)$ that

$$
\begin{aligned}
|G(x_0, w) - G(x, z)| \leq& |G(x_0, w) - G(x_0, z)| + |G(x_0, z) - G(x, z)| \\
\leq& |F(\nabla u(x_0) + w) - F(\nabla u(x_0) + z)| + |F'(\nabla u(x_0))||w - z| \\
& + |F(\nabla u(x_0) + z) - F(\nabla u(x) + z)| + |F(\nabla u(x_0)) - F(\nabla u(x))| \\
& + |F'(\nabla u(x_0)) - F'(\nabla u(x))||z| \\
\leq& \tilde{c_1}\,(1 + |\nabla u(x_0) + w|^{p-1} + |\nabla u(x_0) + z|^{p-1})|w - z| \\
& + \tilde{c_1}\,(1 + |\nabla u(x) + z|^{p-1} + |\nabla u(x_0) + z|^{p-1})|\nabla u(x) - \nabla u(x_0)| \\
& + \tilde{c_1}\,(1 + |\nabla u(x)|^{p-1} + |\nabla u(x_0)|^{p-1})|\nabla u(x) - \nabla u(x_0)| \\
& + \tilde{c_2}\,|\nabla u(x) - \nabla u(x_0)||z| \\
\leq& c\,(1 + |z|^p)|J - \mathrm{I}_n| + C(1 + |z| + |z|^p)|\nabla u(x) - \nabla u(x_0)| \\
\leq& C\,(|J - \mathrm{I}_n| + (\mathrm{osc}_{\Omega(x_0,R)}\nabla u))|V(z)|^2,
\end{aligned}
$$

where the last inequality follows from the fact that $|z| > 1$.

Therefore, if for a given $\varepsilon > 0$ we take $R > 0$ such that

$$C\,(|J - \mathrm{I}_n| + (\mathrm{osc}_{\Omega(x_0,R)}\nabla u)) < \tfrac{\varepsilon}{4},$$

our claim follows by choosing $R = R_\varepsilon > 0$ suitable to make $E \leq \frac{\varepsilon}{4}|V(z)|^2$ for any $z \in \mathbb{R}^{N\times n}$.

We now observe that, following a similar spirit as in the above proof, by considering separately the two different cases $|z| \leq 1$ and $|z| > 1$, it is not difficult to see that there is a constant C_0 independent of x and z such that, for every $x \in \Omega$ and for every $z \in \mathbb{R}^{N\times n}$,

$$|G(x, z)| \leq C_0|V(z)|^2. \tag{2.14}$$

On the other hand, observe that since the determinant is a continuous function, for any given

$C, \varepsilon > 0$ there is a $\delta = \delta_\varepsilon \in (0, \min\{\frac{\varepsilon}{2}, 1\})$ such that, if $|J - \mathrm{I}_n| < \delta$ with $J \in \mathbb{R}^{n\times n}$, we can then ensure $|J - \mathrm{I}_n| + |1 - |\det(J)|| < \frac{\varepsilon}{4C}$. This technical observation enables us to estimate, for any $z \in \mathbb{R}^{N\times n}$ and any $J \in \mathbb{R}^{n\times n}$ with $|J - \mathrm{I}_n| < \delta$ as above, that

$$\begin{aligned}
c_2 \left||V(z\cdot J)|^2 - |V(z)|^2|\det J|\right| \leq& \, c_2 \left||V(z\cdot J)|^2 - |V(z)|^2\right| + |V(z)|^2|1 - |\det J||\\
\leq& \, C\left(|V(z\cdot J)| + |V(z)|\right)||V(z\cdot J| - |V(z)||\\
&+ C|V(z)|^2|1 - |\det J||\\
\leq& \, C|V(z)||z||J - \mathrm{I}_n|\left(1 + |z\cdot J|^2 + |z|^2\right)^{\frac{p-2}{4}} \qquad (2.15)\\
&+ C|V(z)|^2|1 - |\det J||\\
\leq& \, C|V(z)||J - \mathrm{I}_n|\left(1 + |z|^2\right)^{\frac{p-2}{4}}|z|\\
&+ C|V(z)|^2|1 - |\det J||\\
=& \, C\left(|J - \mathrm{I}_n||V(z)|^2 + |V(z)|^2|1 - |\det J||\right)\\
\leq& \, \frac{\varepsilon}{4}|V(z)|^2. \qquad (2.16)
\end{aligned}$$

We remark that inequality (2.15) follows after applying Lemma 128 (i)-(v), together with the fact that we can assume $|J| \leq C$, given that $|J - \mathrm{I}_n| < \delta < 1$. From (2.12), (2.14) and (2.16) we can infer that, for any $\varepsilon > 0$, there are $0 < \delta = \delta_\varepsilon < 1$ and $R = R_\varepsilon > 0$ such that, for any $J \in \mathbb{R}^{n\times n}$, $z \in \mathbb{R}^{N\times n}$ and any $x_0, x \in \mathbb{R}^n$, if $|J - \mathrm{I}_n| < \delta$ and $x \in \Omega(x_0, R)$, then

$$\begin{aligned}
&|G(x_0, z\cdot J) - c_2|V(z\cdot J)|^2 - (G(x,z)|\det J| - c_2|V(z)|^2|\det J|)|\\
\leq&|G(x_0, z\cdot J) - G(x,z)| + |G(x,z)|1 - |\det J|| + c_2||V(z\cdot J)|^2 - |V(z)|^2|\det J||\\
\leq&\frac{\varepsilon}{2}|V(z)|^2\\
\leq&\varepsilon|V(z)|^2|\det J|, \qquad (2.17)
\end{aligned}$$

where the last inequality follows from the local uniform continuity of the determinant and, once again, from the assumption that $|J - \mathrm{I}_n| < \delta$ for $0 < \delta < \frac{\varepsilon}{2}$.

Having obtained these preliminary estimates, we will now prove the part (2) of the Theorem and postpone the derivation of (1) until the end of the proof.

Assume that $x_0 \in \Gamma_N$. Because Ω is a set of class C^1, we can find an $R_0 > 0$, which does not depend on x_0, such that for every $0 < r < R_0$ there is a homeomorphism

$\Phi_r\colon B^-_{\mathrm{n}(x_0)}(0,1) \to \left(\frac{\Omega - x_0}{r} \cap B(0,1)\right)$. What is more, because $\partial\Omega$ is smooth and compact, we can construct the homeomorphisms Φ_r so that, given a $\delta > 0$, we can find an $R_1 \in (0, R_0)$ such that for every $0 < r \le R_1$ and for every $x_0 \in \partial\Omega$,

$$\left\|\Phi_r - Id_{B^-_{\mathrm{n}(x_0)}(0,1)}\right\|_{\mathrm{L}^\infty(B^-_{\mathrm{n}(x_0)}(0,1),\mathbb{R}^n)} + \|\nabla\Phi_r - \mathrm{I}_n\|_{\mathrm{L}^\infty(B^-_{\mathrm{n}(x_0)}(0,1),\mathbb{R}^{n\times n})} < \delta, \tag{2.18}$$

so that $\nabla\Phi_r$ converges to the identity matrix uniformly on $B^-_{\mathrm{n}(x_0)}(0,1)$ and uniformly for $x_0 \in \partial\Omega$.[3]

Having established the above estimates, after fixing $\varepsilon > 0$ we obtain $\delta > 0$ and $R > 0$ such that (2.17) is satisfied and, for such $\delta > 0$, we take R_1 so that (2.18) holds. We further assume that $||\Phi_{R_1}||_{\mathrm{L}^\infty(B^-_{\mathrm{n}(x_0)}(0,1))} \le 2$. We now let $R := \frac{1}{2}\min\{R_0, R_1\}$ and observe that, for any $y \in B^-_{\mathrm{n}(x_0)}(0,1)$, we have $R\Phi_R(y) + x_0 \in \Omega(x_0, R)$ and, therefore, (2.17) holds with $x := R\Phi_R(y) + x_0$, $z := \nabla\varphi(R\Phi_R(y) + x_0)$ and $J := \nabla\Phi_R(y)$, where φ is any function in $\mathrm{W}^{1,p}(\Omega(x_0,R),\mathbb{R}^N)$ so that $\varphi = 0$ on $\partial(B(x_0,R)) \cap \Omega$.

After making this substitution in (2.17), since the inequality holds for every $y \in B^-_{\mathrm{n}(x_0)}(0,1)$, we can integrate over $B^-_{\mathrm{n}(x_0)}(0,1)$ to obtain that

[3] See Theorem C.1 in [GM09] for a careful construction of the homeomorphisms Φ_r.

$$
\begin{aligned}
&\int_{B^-_{\mathrm{n}(x_0)}(0,1)} \left(F(\nabla u(x_0) + \nabla\varphi(R\Phi_R(y)+x_0)\cdot\nabla\Phi_R(y)) - F(\nabla u(x_0))\right) \mathrm{d}y \\
&- \int_{B^-_{\mathrm{n}(x_0)}(0,1)} \left\langle F'(\nabla u(x_0)), \nabla\varphi(R\Phi_R(y)+x_0)\cdot\nabla\Phi_R(y)\right\rangle \mathrm{d}y \\
&- c_2\int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V(\nabla\varphi(R\Phi_R(y)+x_0)\cdot\nabla\Phi_R(y))|^2 \,\mathrm{d}y \\
&- \int_{B^-_{\mathrm{n}(x_0)}(0,1)} F(\nabla u(R\Phi_R(y)+x_0) + \nabla\varphi(R\Phi_R(y)+x_0))|\det\nabla\Phi_R(y)|\,\mathrm{d}y \\
&+ \int_{B^-_{\mathrm{n}(x_0)}(0,1)} F(\nabla u(R\Phi_R(y)+x_0))|\det\nabla\Phi_R(y)|\,\mathrm{d}y \\
&+ \int_{B^-_{\mathrm{n}(x_0)}(0,1)} \left\langle F'(\nabla u(R\Phi_R(y)+x_0)), \nabla\varphi(R\Phi_R(y)+x_0)\right\rangle |\det\nabla\Phi_R(y)|\,\mathrm{d}y \\
&+ c_2\int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V\left(\nabla\varphi(R\Phi_R(y)+x_0)\right)|^2 |\det\nabla\Phi_R(y)|\,\mathrm{d}y \\
=&\int_{B^-_{\mathrm{n}(x_0)}(0,1)} G(x_0, \nabla\varphi(R\Phi_R(y)+x_0)\cdot\nabla\Phi_R(y))\,\mathrm{d}y \\
&- c_2\int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V(\nabla\varphi(R\Phi_R(y)+x_0)\cdot\nabla\Phi_R(y))|^2\,\mathrm{d}y \\
&- \int_{B^-_{\mathrm{n}(x_0)}(0,1)} G(R\Phi_R(y)+x_0, \nabla\varphi(R\Phi_R(y)+x_0))|\det\nabla\Phi_R(y)|\,\mathrm{d}y \\
&+ c_2\int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V\left(\nabla\varphi(R\Phi_R(y)+x_0)\right)|^2 |\det\nabla\Phi_R(y)|\,\mathrm{d}y \\
\leq&\, \varepsilon\int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V(\nabla\varphi(R\Phi_R(y)+x_0))|^2|\det\nabla\Phi_R(y)|\,\mathrm{d}y
\end{aligned}
\tag{2.19}
$$

We are interested in using the quasiconvexity at the boundary condition in order to simplify the above expression. With this aim, we define $\tilde{\varphi}\colon B^-_{\mathrm{n}(x_0)}(0,1)\to\mathbb{R}^N$ as

$$\tilde{\varphi}(y) := \frac{\varphi(R\Phi_R(y)+x_0)}{R}.$$

Observe that, since $\varphi = 0$ on $\partial(B(x_0,R))\cap\Omega$ and the homeomorphism Φ_R^{-1} "flattens" the boundary of $\frac{\Omega - x_0}{R}$ in $B(0,1)$, then

$$\tilde{\varphi} = 0 \quad \text{on} \quad \Phi_R^{-1}\left[\partial(B(0,1))\cap\frac{\Omega-x_0}{R}\right] = \partial(B(0,1))\cap B^-_{\mathrm{n}(x_0)}(0,1). \tag{2.20}$$

Hence, by approximation, $\tilde{\varphi}$ is a suitable test function for the quasiconvexity at the free boundary condition that we have. Since $\nabla\tilde{\varphi}(y) = \nabla\varphi(R\Phi_R(y) + x_0) \cdot \nabla\Phi_R(y)$, this means that

$$\begin{aligned}0 \leq &\int_{B^-_{\mathrm{n}(x_0)}(0,1)} \left(F(\nabla u(x_0) + \nabla\varphi(R\Phi_R(y) + x_0) \cdot \nabla\Phi_R(y)) - F(\nabla u(x_0))\right) \mathrm{d}y \\ &- c_2 \int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V(\nabla\varphi(R\Phi_R(y) + x_0) \cdot \nabla\Phi_R(y))|^2 \,\mathrm{d}y. \end{aligned} \tag{2.21}$$

Moreover, the weak Euler-Lagrange equation associated to the above minimality condition implies that

$$\int_{B^-_{\mathrm{n}(x_0)}(0,1)} \big\langle F'(\nabla u(x_0)), \nabla\varphi(R\Phi_R(y) + x_0) \cdot \nabla\Phi_R(y)\big\rangle \,\mathrm{d}y = 0. \tag{2.22}$$

From the expressions (2.19), (2.21) and (2.22), we deduce that

$$\begin{aligned} &- \int_{B^-_{\mathrm{n}(x_0)}(0,1)} F(\nabla u(R\Phi_R(y) + x_0) + \nabla\varphi(R\Phi_R(y) + x_0)|\det \nabla\Phi_R(y)| \,\mathrm{d}y \\ &+ \int_{B^-_{\mathrm{n}(x_0)}(0,1)} F(\nabla u(R\Phi_R(y) + x_0))|\det \nabla\Phi_R(y)| \,\mathrm{d}y \\ &+ \int_{B^-_{\mathrm{n}(x_0)}(0,1)} \big\langle F'(\nabla u(R\Phi_R(y) + x_0)), \nabla\varphi(R\Phi_R(y) + x_0)\big\rangle \,|\det \nabla\Phi_R(y)| \,\mathrm{d}y \\ &+ c_2 \int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V\left(\nabla\varphi(R\Phi_R(y) + x_0)\right)|^2 |\det \nabla\Phi_R(y)| \,\mathrm{d}y \\ \leq\, &\varepsilon \int_{B^-_{\mathrm{n}(x_0)}(0,1)} |V(\nabla\varphi(R\Phi_R(y) + x_0))|^2 |\det \nabla\Phi_R(y)| \,\mathrm{d}y. \end{aligned}$$

Applying the change of variables $x = R\Phi_R(y) + x_0$, this leads to

$$\begin{aligned} &- \int_{\Omega(x_0,R)} \big(F(\nabla u(x) + \nabla\varphi(x)) - F(\nabla u(x)) + \big\langle F'(\nabla u(x)), \nabla\varphi(x)\big\rangle\big) + c_2|V\left(\nabla\varphi(x)\right)|^2 \,\mathrm{d}x \\ \leq\, &\varepsilon \int_{\Omega(x_0,R)} |V(\nabla\varphi(x))|^2 \,\mathrm{d}x. \end{aligned} \tag{2.23}$$

Since $\varphi = 0$ on $\partial(B(x_0, R)) \cap \Omega$, in particular we have $\varphi = 0$ on $\partial(B(x_0, R)) \cap B^-_{\mathrm{n}(x_0)}(0, 1)$. Therefore, because u is an F-extremal,

$$\int_{\Omega(x_0,R)} \big\langle F'(\nabla u(x)), \nabla\varphi(x)\big\rangle \,\mathrm{d}x = 0.$$

This, together with (2.23), imply for $\varepsilon = \frac{c_2}{2}$ that

$$\int_{\Omega(x_0,R)} \left(F(\nabla u(x) + \nabla\varphi(x)) - F(\nabla u(x)) + \left\langle F'(\nabla u(x)), \nabla\varphi(x)\right\rangle\right) - c_2|V\left(\nabla\varphi(x)\right)|^2\,\mathrm{d}x$$
$$\geq -\frac{c_2}{2}\int_{\Omega(x_0,R)} |V\left(\nabla\varphi(x)\right)|^2\,\mathrm{d}x, \tag{2.24}$$

which gives us the desired inequality after adding $c_2\int_{\Omega(x_0,R)}|V\left(\nabla\varphi(x)\right)|^2\,\mathrm{d}x$ to both sides of the above expression. This concludes the proof of (2) for $x_0 \in \Gamma_N$.

On the other hand, to prove (1), if $x_0 \in \overline{\Omega}$ a simpler version of the above proof will work, since we can then use that the standard quasiconvexity holds in $\overline{\Omega}$ and take Φ_R as the identity homeomorphism in the above proof, given that there is no need, for this case, to flatten the boundary. All the other calculations follow in the exact same way. This concludes the proof of the theorem. □

Remark 54 *Let $\Omega_Q(x_0,R) := \Omega \cap Q(x_0,r)$, where $Q(x_0,r)$ is a cube with sides parallel to the coordinate axes. It is then easy to see that, if $\varphi \in \mathrm{Var}\left(\Omega_Q\left(x_0,\frac{R}{2}\right)\right)$, then by assigning φ the value of 0 in $\Omega(x_0,R)\backslash\Omega_Q(x_0,R)$, we can assume that $\varphi \in \mathrm{Var}(\Omega(x_0,R))$. Therefore, Theorem 53 remains valid if we exchange $\Omega(x_0,R)$ by $\Omega_Q(x_0,R)$ in the statement.*

2.5 Sufficiency result for strong local minima

We now establish the result related to [GM09] by Grabovsky and Mengesha. The approach that we follow here for the sufficiency theorem, consists essentially of appropriately exploiting the result of K. Zhang [Zha92] that we generalized in the previous section, according to which smooth solutions of the weak Euler-Lagrange equation minimize the functional in *small* subsets of the domain. The idea is then to partition the original domain into sufficiently small sets where we can apply Zhang's result and then add up the corresponding local estimates. This inevitably leads to obtaining an excess, which is non-linear. The purpose is then to prove that the excess converges to zero and we do that with the help of Young Measures.

We remark that, in the proof of the following result, the assumption that $u \in C^1(\overline{\Omega},\mathbb{R}^N)$ is mainly required while using the generalized version of Zhang's theorem.

Theorem 55 *Let* $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ *such that it satisfies* (H0) − (H2) *for some* $p\in[2,\infty)$. *Let* $u\in C^1\left(\overline{\Omega},\mathbb{R}^N\right)$ *be an* F*-extremal and assume that the second variation at* u *is strongly positive, meaning that there is a constant* $c_3>0$ *such that*

$$c_3\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x \le \int_\Omega F''(\nabla u)[\nabla\varphi,\nabla\varphi]\,\mathrm{d}x \tag{2.25}$$

for all $\varphi\in\mathrm{Var}(\Omega,\mathbb{R}^N)$. *In addition, if the free portion of the boundary is such that* $\Gamma_N\neq\emptyset$, *suppose that* (H2') *holds. In this case, or if* $p>2$, *further assume and that, for some constants* $c_4,c_5>0$,

(H3) $$c_4\int_\Omega |\nabla\varphi|^p\,\mathrm{d}x - c_5\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x \le \int_\Omega \left(F(\nabla u+\nabla\varphi)-F(\nabla u)\right)\,\mathrm{d}x$$

for every $\varphi\in\mathrm{Var}(\Omega,\mathbb{R}^N)$.[4] *Then,* u *is an* L^p*-local* F*-minimizer.*

Remark 56 *If the second variation is assumed to be positive in the sense of (2.25) then, by continuity, the same inequality holds for* φ *in the* $\mathrm{W}^{1,2}$*-closure of the set* $\mathrm{Var}(\Omega,\mathbb{R}^N)$.

Proof of Theorem 55. We will prove the result arguing by contradiction. Suppose that the theorem does not hold. Then, we can find a sequence $(\varphi_k)\subseteq\mathrm{Var}(\Omega,\mathbb{R}^N)$ such that $\|\varphi_k\|_{\mathrm{L}^p(\Omega,\mathbb{R}^N)}\to 0$ and

$$\int_\Omega F(\nabla u+\nabla\varphi_k)\,\mathrm{d}x < \int_\Omega F(\nabla u)\,\mathrm{d}x \tag{2.26}$$

for all $k\in\mathbb{N}$.

As in the proof of Theorem 53, we use Taylor's Approximation Theorem and define

$$\begin{aligned} G(x,z) &:= F(\nabla u(x)+z)-F(\nabla u(x))-\left\langle F'(\nabla u(x)),z\right\rangle \\ &= \int_0^1 (1-t)F''(\nabla u(x)+tz)[z,z]\,\mathrm{d}t. \end{aligned}$$

[4]We remark that, if $\Gamma_N=\emptyset$ and $p=2$, the proof that we present here removes this condition from the original result in [GM09]. On the other hand, as discussed in [GM09, S.3.2], it can be shown that (H3) also follows if we assume that F is pointwise coercive: $c_4|z|^p-c_5\le F(z)$ for all $z\in\mathbb{R}^{N\times n}$. However, (H3) is a more general assumption.

Note that, since u is an F-extremal, for every $k \in \mathbb{N}$ it holds that

$$\begin{aligned}\int_\Omega G(x, \nabla\varphi_k)\,\mathrm{d}x =& \int_\Omega \int_0^1 (1-t)F''(\nabla u + t\nabla\varphi_k)[\nabla\varphi_k, \nabla\varphi_k]\,\mathrm{d}x \\ =& \int_\Omega \big(F(\nabla u + \nabla\varphi_k) - F(\nabla u) - \langle F'(\nabla u), \nabla\varphi_k\rangle\big)\,\mathrm{d}x \\ <& 0. \end{aligned} \tag{2.27}$$

This inequality suggests the underlying idea behind this proof, which is to exploit the strong positivity of the second variation to obtain a contradiction. By a normalization argument we can construct a sequence of variations suitable for this purpose. However, such a sequence will only converge weakly to 0 in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ and, therefore, we will require the theory of Young Measures to obtain the desired convergence of terms of the form

$$\int_\Omega \int_0^1 F''(\nabla u + t\alpha_k\nabla\psi_k)[\nabla\psi_k, \nabla\psi_k]\,\mathrm{d}t\,\mathrm{d}x$$

for a suitable sequence (α_k) that we will define shortly.

Having stated this, we proceed with the first main step of the proof, which consists in showing that (φ_k) is bounded in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$. For this purpose, we treat differently the two cases $\Gamma_N = \emptyset$ and $\Gamma_N \neq \emptyset$, with the aim of providing an argument that does not make use of the coercivity assumption (H3) for the case of full Dirichlet boundary conditions. We remark, however, that if $p > 2$, condition (H3) seems to remain necessary even if $\Gamma_N = \emptyset$.

Case 1. If $\Gamma_N = \emptyset$, so that we are in the case of Dirichlet boundary conditions, we will prove the claim by obtaining a Gårding inequality from assumptions (H1) – (H2). These two, together with Proposition 10 and the fact that $\varphi_k \in \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N)$, imply that

$$c_2 \int_\Omega |\nabla\varphi_k|^p\,\mathrm{d}x \leq \int_\Omega (F(\nabla\varphi_k) - F(0))\,\mathrm{d}x \tag{2.28}$$

$$\leq \int_\Omega (F(\nabla u + \nabla\varphi_k) + F(\nabla\varphi_k) - F(\nabla u + \nabla\varphi_k) - F(0))\,\mathrm{d}x \tag{2.29}$$

$$\leq \int_\Omega (F(\nabla u + \nabla\varphi_k) + \tilde{c}(1 + |\nabla\varphi_k|^{p-1} + |\nabla u + \nabla\varphi_k|^{p-1})|\nabla u| - F(0))\,\mathrm{d}x. \tag{2.30}$$

Observe here that

$$c_2\int_\Omega \left(\frac{1}{2^{p-1}}|\nabla u+\nabla\varphi_k|^p-|\nabla u|^p\right)\,\mathrm{d}x\le c_2\int_\Omega |\nabla\varphi_k|^p\,\mathrm{d}x$$

and, on the other hand, by Young's inequality applied to $c_2c_p|\nabla u+\nabla\varphi_k|^{p-1}c_p^{-1}|\nabla u|$ with an appropriate choice of the constant c_p, we have that

$$\begin{aligned}&\tilde{c}\int_\Omega (1+|\nabla\varphi_k|^{p-1}+|\nabla u+\nabla\varphi_k|^{p-1})|\nabla u|\,\mathrm{d}x\\ \le& c\int_\Omega (1+|\nabla u+\nabla\varphi_k|^{p-1}+|\nabla u|^{p-1})|\nabla u|\,\mathrm{d}x\\ \le&\int_\Omega \left(\frac{c_2}{2^p}|\nabla u+\nabla\varphi_k|^p+c|\nabla u|^p+c|\nabla u|\right)\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\frac{c_2}{2^{p-1}}\int_\Omega |\nabla u+\nabla\varphi_k|^p\,\mathrm{d}x\le\int_\Omega\left(F(\nabla u+\nabla\varphi_k)-F(0)+c|\nabla u|+\left(c+c_2+\frac{2c^2}{c_2}\right)|\nabla u|^p\right)\,\mathrm{d}x$$

or, equivalently, there are constants $\tilde{c}_3>0$ and $\tilde{c}_4>0$ such that

$$\tilde{c}_3\int_\Omega |\nabla u+\nabla\varphi_k|^p\,\mathrm{d}x\le\int_\Omega F(\nabla u+\nabla\varphi_k)\,\mathrm{d}x+\tilde{c}_4\int_\Omega (1+|\nabla u|^p)\,\mathrm{d}x$$

for all $k\in\mathbb{N}$.

This, together with assumption (2.26) and Poincaré inequality, finally allows us to conclude that (φ_k) is bounded in $\mathrm{W}^{1,p}$.

Case 2. If $\Gamma_n\neq\emptyset$, then (φ_k) is bounded in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ by assumptions (H3) and (2.26). We remark that, the reason why we cannot proceed, as in Case 1, to obtain a Gårding inequality without this assumption, is that $\varphi_k\notin\mathrm{W}_0^{1,p}(\Omega,\mathbb{R}^N)$ and, therefore, we cannot obtain (2.30) from the quasiconvexity condition (H2).

Having established that (φ_k) is bounded in the previous two cases, we can further conclude that $\varphi_k\rightharpoonup 0$ in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$.

On the other hand, using Theorem 53 and Remark 54 we find an $R>0$ such that, if

$\Omega(x,R) := \Omega \cap Q(x,R)$,[5] then

$$\frac{c_2}{2} \int_{\Omega(x,R)} |V(\nabla\varphi)|^2 \, \mathrm{d}x \le \int_{\Omega(x,R)} (F(\nabla u + \nabla\varphi) - F(\nabla u)) \, \mathrm{d}x \tag{2.31}$$

for all $\varphi \in \mathrm{Var}(\Omega(x,R), \mathbb{R}^N)$ and all $x \in \overline{\Omega}$.

Now, for a given $r \in (0,R)$, we consider a cover for Ω consisting of non-overlapping cubes of side length $2r$, so that

$$\Omega \subseteq \bigcup_{j\in J} \overline{Q(x_j, r)}.$$

For each $j \in J$ and for $r < s < R$, consider cut-off functions $\rho_j \in C^1_c(Q(x_j, s))$ with the property that $\mathbb{1}_{Q(x_j,r)} \le \rho_j \le \mathbb{1}_{Q(x_j,s)}$ and $|\nabla\rho_j| \le \frac{2}{s-r}$.

Note that the cubes $Q(x_j, s)$ have bounded overlap since, when $s < 2r$, $Q(x_j, s)$ will intersect at most $3^n - 1$ other such cubes.

In addition, if $\varphi \in \mathrm{Var}(\Omega, \mathbb{R}^N)$, then $\rho_j\varphi \in \mathrm{Var}(\Omega(x_j, s), \mathbb{R}^N)$ and so, according to inequality (2.31),

$$\int_{\Omega(x_j,s)} \left(F(\nabla u) + \frac{c_2}{2} |V(\nabla(\rho_j\varphi))|^2 \right) \mathrm{d}x \le \int_{\Omega(x_j,s)} F(\nabla u + \nabla(\rho_j\varphi)) \, \mathrm{d}x.$$

Since u is an F-extremal, we also get that

$$\frac{c_2}{2} \int_{\Omega(x_j,s)} |V(\nabla(\rho_j\varphi))|^2 \, \mathrm{d}x \le \int_{\Omega(x_j,s)} G(x, \nabla(\rho_j\varphi)) \, \mathrm{d}x.$$

Then, since $\rho_j = 1$ on $Q(x_j, r)$, we obtain

$$\begin{aligned} &\frac{c_2}{2} \int_{\Omega(x_j,r)} |V(\nabla\varphi)|^2 \, \mathrm{d}x + \frac{c_2}{2} \int_{\Omega(x_j,s)-\Omega(x_j,r)} |V(\nabla(\rho_j\varphi))|^2 \, \mathrm{d}x \\ &\le \int_{\Omega(x_j,r)} G(x, \nabla\varphi) \, \mathrm{d}x + \int_{\Omega(x_j,s)-\Omega(x_j,r)} G(x, \nabla(\rho_j\varphi)) \, \mathrm{d}x. \end{aligned}$$

Following the proof of the estimate for G in Theorem 53, we can find a constant $c > 0$

[5] For simplicity, we change the notation and assume, only for this proof, that $\Omega(x_0, R) = \Omega_Q(x_0, R)$. See Remark 54.

such that, for every $x \in \Omega$ and $z \in \mathbb{R}^{N\times n}$,

$$|G(x,z)| \le c|V(z)|^2. \tag{2.32}$$

Using this we obtain, after adding up the previous inequalities over j, that

$$\begin{aligned}&\frac{c_2}{2}\int_\Omega |V(\nabla\varphi)|^2\,\mathrm{d}x + \frac{c_2}{2}\sum_{j\in J}\int_{\Omega(x_j,s)-\Omega(x_j,r)} |V(\nabla(\rho_j\varphi))|^2\,\mathrm{d}x\\ \le&\int_\Omega (F(\nabla u+\nabla\varphi)-F(\nabla u))\,\mathrm{d}x + c\sum_{j\in J}\int_{\Omega(x_j,s)-\Omega(x_j,r)} |V(\nabla(\rho_j\varphi))|^2\,\mathrm{d}x.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{c_2}{2}\int_\Omega |V(\nabla\varphi)|^2\,\mathrm{d}x - c\sum_{j\in J}\int_{\Omega(x_j,s)-\Omega(x_j,r)} \left(|V(\nabla\varphi)|^2 + \left|V\left(\frac{\varphi}{s-r}\right)\right|^2\right)\mathrm{d}x\\ \le&\int_\Omega (F(\nabla u+\nabla\varphi)-F(\nabla u))\,\mathrm{d}x\\ =&\int_\Omega G(x,\nabla\varphi)\,\mathrm{d}x\end{aligned} \tag{2.33}$$

for all $\varphi \in \mathrm{Var}(\Omega,\mathbb{R}^N)$, $r\in(0,R)$ and $s\in(r,\min\{2r,R\})$.

Now, let $\gamma_k := \|V(\nabla\varphi_k)\|_{\mathrm{L}^2}$. Then, $\gamma_k>0$ for all $k\in\mathbb{N}$ and (γ_k) is a bounded sequence because $p\ge 2$. We will now show that $\gamma_k\to 0$ as $k\to\infty$. Arguing by contradiction, we assume that there are $\gamma>0$ and a subsequence, that we do not relabel, such that $\gamma_k\to\gamma$. Considering a further subsequence, we may also assume that $|V(\nabla\varphi_k)|^2\mathcal{L}^n \overset{*}{\rightharpoonup} \mu$ in $C^0\left(\overline{\Omega}\right)^* \cong \mathcal{M}\left(\overline{\Omega}\right)$.

We now take $r\in(0,R)$ and the grid so that $\mu\left(\bigcup_{j\in J}(\partial(Q(x_j,r))\cap\overline{\Omega})\right) = 0$. This is possible because, *for a given* x_0, only a countable amount of cubes can be such that $\mu(\partial Q(x_0,r))>0$. To prove this, observe that for any $k\in\mathbb{N}^+$,

$$\mathcal{A}_k := \left\{r\in(0,R) : \mu\left(\partial Q(x_0,r)\right) > \frac{1}{k}\right\}$$

is a pairwise disjoint collection of subsets of $Q(x_0,R)$. Since μ is σ-additive and $\mu(Q(x_0,R))$ is a positive real number, this implies that $\mathcal{A}_k$ is finite for every $k\in\mathbb{N}^+$. Hence, the set of real numbers

$$\{r \in (0,\infty) : \mu\left(\partial Q(x_0, r)\right) > 0\} = \bigcup_{k\in\mathbb{N}} \mathcal{A}_k \tag{2.34}$$

is at most countable.

Now observe that, for $r < s < \min\{2r, R\}$, we get from inequality (2.33) applied to $\varphi = \varphi_k$, that

$$\begin{aligned}&\frac{c_2}{2}\int_\Omega |V(\nabla\varphi_k)|^2\,\mathrm{d}x - c\sum_{j\in J}\int_{\Omega(x_j,s)-\Omega(x_j,r)} \left(|V(\nabla\varphi_k)|^2 + \left|V\left(\frac{\varphi_k}{s-r}\right)\right|^2\right)\mathrm{d}x\\ \le &\int_\Omega \left(F(\nabla u + \nabla\varphi_k) - F(\nabla u)\right)\mathrm{d}x.\end{aligned}$$

Recall that, by assumption, $\varphi_k \to 0$ in $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ and, since $p \geq 2$, this implies that $V(\varphi_k) \to 0$ in $\mathrm{L}^2(\Omega,\mathbb{R}^N)$. Hence,

$$\frac{c_2}{2}\gamma^2 - c\mu\left(\overline{\Omega} \cap \bigcup_{j\in J}\left(\overline{Q(x_j,s)} - Q(x_j,r)\right)\right) \le 0$$

and, letting $s \searrow r$ in the above expression, we get

$$0 < \frac{c_2}{2}\gamma^2 = \frac{c_2}{2}\gamma^2 - c\mu\left(\overline{\Omega} \cap \bigcup_{j\in J}\partial(Q(x_j,r))\right) \le 0,$$

which is a contradiction.

Consequently, $\gamma_k = \|V(\nabla\varphi_k)\|_{\mathrm{L}^2} \to 0$.

Let $\alpha_k := \|\nabla\varphi_k\|_{\mathrm{L}^2}$ and $\beta_k := (2|\Omega|)^{\frac{1}{2}-\frac{1}{p}}\|\nabla\varphi_k\|_{\mathrm{L}^p}$. By Lemma 128 we also have that $\alpha_k \to 0$ and $\beta_k \to 0$. This way, we have reduced the problem to the case of $\mathrm{W}^{1,2}$-local minimizers. In addition we recall that, by Hölder's inequality,

$$\alpha_k = \|\nabla\varphi_k\|_{\mathrm{L}^2} \le (2|\Omega|)^{\frac{1}{2}-\frac{1}{p}}\|\nabla\varphi_k\|_{\mathrm{L}^p} = \beta_k.$$

Therefore, $r_k := \frac{\alpha_k}{\beta_k} \le 1$ for every $k \in \mathbb{N}$.

We now claim that the given sequence of variations (φ_k) is such that

$$0 \le \sup_{k\in\mathbb{N}}\frac{\beta_k^p}{\alpha_k^2} = \Lambda < \infty$$

for some real number $\Lambda > 0$. Indeed, if $p = 2$ this is trivially true and, if $p > 2$, from the coercivity condition (H3) applied to φ_k it follows, after dividing by α_k^2, that for every $k \in \mathbb{N}$,

$$\tilde{c}_4 \frac{\beta_k^p}{\alpha_k^2} - c_5 \leq \alpha_k^{-2} \int_\Omega (F(\nabla u + \nabla\varphi_k) - F(\nabla u))\, \mathrm{d}x < 0.$$

This proves that the sequence $\left(\frac{\beta_k^p}{\alpha_k^2}\right)$ is bounded and the claim follows.

We now define $\psi_k := \alpha_k^{-1}\varphi_k \in \mathrm{Var}(\Omega, \mathbb{R}^N)$. Hereby, $\int_\Omega |\nabla\psi_k|^2 = 1$ and hence we can assume, up to a subsequence, that $\psi_k \rightharpoonup \psi$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$, $|\nabla\psi_k|^2\mathcal{L}^n \overset{*}{\rightharpoonup} \tilde{\mu}$ in $C_0^0(\overline{\Omega})^*$ and that $\nabla\psi_k \xrightarrow{Y} (\nu_x)$. This, together with the the fact that $\int_\Omega |r_k\nabla\psi_k|^p = \beta_k^{-p}\int_\Omega |\nabla\varphi_k|^p = 1$ and the Decomposition Lemma, implies that, for a subsequence of (ψ_k) that we do not relabel, we can find sequences $(g_k) \subseteq \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$ and $(b_k) \subseteq \mathrm{Var}(\Omega, \mathbb{R}^N)$ such that:

- $g_k \rightharpoonup 0$ and $b_k \rightharpoonup 0$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$;
- $r_k g_k \rightharpoonup 0$ and $r_k b_k \rightharpoonup 0$ in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$;
- $(|\nabla g_k|^2)$ and $(|r_k\nabla g_k|^p)$ are both equiintegrable;
- $\nabla b_k \to 0$ in measure and
- $\psi_k = \psi + g_k + b_k$.

Let us call $f_k := \alpha_k^{-2}G(x, \alpha_k\nabla\psi_k) - \alpha_k^{-2}G(x, \alpha_k\nabla b_k)$. Then, by using (2.32) together with Proposition 10, we get that since $p \geq 2$ and $G(x, \cdot)$ is a quasiconvex function (and, therefore, also rank-one convex), there is a constant $c = c(p) > 0$ such that, for every $z, w \in \mathbb{R}^{N\times n}$ and for every $x \in \overline{\Omega}$,

$$|G(x, z) - G(x, w)| \leq c\,(|V_{p-1}(z)| + |V_{p-1}(w)|)\,|z - w|.$$

The proof of this inequality relies also on the fact that, for some constant $c > 0$,

$$c^{-1}(|z| + |z|^{p-1}) \leq |V_{p-1}(z)| \leq c(|z| + |z|^{p-1}).$$

This implies that, for any $\varepsilon > 0$, there exists a constant c_ε such that

$$\begin{aligned}|f_k| &\leq c\alpha_k^{-1}(|V_{p-1}(\alpha_k\nabla\psi_k)| + |V_{p-1}(\alpha_k\nabla b_k)|)|\nabla\psi + \nabla g_k| \\ &\leq c\left(|\nabla\psi_k| + |\nabla b_k| + \alpha_k^{p-2}(|\nabla\psi_k|^{p-1} + |\nabla b_k|^{p-1})\right)|\nabla\psi + \nabla g_k| \\ &\leq \varepsilon\left(|\nabla\psi_k|^2 + |\nabla b_k|^2 + \alpha_k^{p-2}(|\nabla\psi_k|^p + |\nabla b_k|^p)\right) + c_\varepsilon\left(|\nabla\psi + \nabla g_k|^2 + \alpha_k^{p-2}|\nabla\psi + \nabla g_k|^p\right).\end{aligned}$$

Consequently, we can observe that, for any set $A \subseteq \mathbb{R}^n$,

$$\int_A |f_k|\,\mathrm{d}x \leq \varepsilon c_1 + \tilde{c}_\varepsilon \int_A \left(|\nabla\psi + \nabla g_k|^2 + \alpha_k^{p-2}|\nabla\psi + \nabla g_k|^p\right)\mathrm{d}x. \tag{2.35}$$

Taking into account that $\left(\frac{\beta_k^p}{\alpha_k^2}\right)$ is a bounded sequence by (2.5) and that

$$\alpha_k^{p-2}|\nabla g_k|^p = \frac{\beta_k^p}{\alpha_k^2} r_k^p |\nabla g_k|^p,$$

we deduce that $(\alpha_k^{p-2}|\nabla g_k|^p)$ is equiintegrable and, hence, so is (f_k).

Now, let $\varepsilon > 0$. Since $(\nabla\psi_k)$ is measure-tight and $\nabla b_k \to 0$ in measure, we can take $m_\varepsilon > 0$ large enough so that, for every $m \geq m_\varepsilon$,

$$\int_{\{|\nabla\psi_k|\geq m\}\cup\{|\nabla b_k|\geq m\}} |f_k|\,\mathrm{d}x < \varepsilon$$

for all $k \in \mathbb{N}$.

Then, for all $m \geq m_\varepsilon$,

$$\int_{\{|\nabla\psi_k|<m\}\cap\{|\nabla b_k|<m\}} f_k\,\mathrm{d}x - \varepsilon < \int_\Omega f_k\,\mathrm{d}x.$$

We will now use the Fundamental Theorem for Young measures to take the limit inferior in both sides of the above expression and obtain that

$$\frac{1}{2}\int_\Omega \int F''(\nabla u)[z,z]\mathbb{1}_{B(0,m)}(z)\,\mathrm{d}\nu_x(z)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega f_k\,\mathrm{d}x \tag{2.36}$$

for all $m \geq m_\varepsilon$. In order to prove this claim, consider the integrand $H : \overline{\Omega} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ given by

$$H(x,z) := F''(\nabla u(x))[z,z]\mathbb{1}_{B(0,m)}(z).$$

Notice that $\mathbb{1}_{B(0,m)}(z)$ is lower semicontinuous because $B(0,m)$ is an open set. Hence, $H(x,\cdot)$ is lower semicontinuous for every $x \in \overline{\Omega}$. By part (I) of the Fundamental Theorem of Young measures (Theorem 43), this implies that, since $\nabla\psi_k \xrightarrow{Y} \nu_x$,

$$\int_{\Omega}\int F''(\nabla u)[z,z]\mathbb{1}_{B(0,m)}(z)\,\mathrm{d}\nu_x(z)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_{|\nabla\psi_k|<m} F''(\nabla u)[\nabla\psi_k,\nabla\psi_k]\,\mathrm{d}x. \tag{2.37}$$

On the other hand, the sequence of functions

$$F''(\nabla u + t\alpha_k\nabla b_k)[\nabla b_k,\nabla b_k]\mathbb{1}_{\{|\nabla b_k|<m\}\cap\{|\nabla\psi_k|<m\}}$$

is bounded in $\mathrm{L}^\infty(\Omega)$ for all $t\in[0,1]$ and, therefore, it is equiintegrable. In addition, this sequence converges to 0 in measure because $\nabla b_k \to 0$ in measure and F'' is continuous. These two facts imply, by Vitali's Convergence Theorem, that

$$F''(\nabla u + t\alpha_k\nabla b_k)[\nabla b_k,\nabla b_k]\mathbb{1}_{\{|\nabla b_k|<m\}\cap\{|\nabla\psi_k|<m\}} \to 0 \tag{2.38}$$

in $\mathrm{L}^1(\Omega)$ when $k\to\infty$ and for all $t\in[0,1]$.

It is also clear, by the Dominated Convergence Theorem, that since $\alpha_k \to 0$,

$$\left|\int_{\Omega}\int_0^1 (1-t)\left(F''(\nabla u + t\alpha_k\nabla\psi_k) - F''(\nabla u)\right)[\nabla\psi_k,\nabla\psi_k]\mathbb{1}_{\{|\nabla b_k|<m\}\cap\{|\nabla\psi_k|<m\}}\,\mathrm{d}t\,\mathrm{d}x\right| \to 0. \tag{2.39}$$

Furthermore, given that $\nabla b_k \to 0$ in measure, we have that

$$\begin{aligned}&\left|\int_{\Omega} F''(\nabla u)[\nabla\psi_k,\nabla\psi_k]\left(\mathbb{1}_{\{|\nabla\psi_k|<m\}} - \mathbb{1}_{\{|\nabla b_k|<m\}\cap\{|\nabla\psi_k|<m\}}\right)\mathrm{d}x\right|\\ \leq{}& cm^2\int_{\Omega}\mathbb{1}_{\{|\nabla\psi_k|<m\}}\left(1-\mathbb{1}_{\{|\nabla b_k|<m\}}\right)\mathrm{d}x \to 0.\end{aligned} \tag{2.40}$$

By combining (2.37)-(2.40), we obtain that (2.36) holds for all $m \geq m_\varepsilon$, as we wanted to prove.

We now claim that

$$\frac{1}{2}\int_{\Omega}\int F''(\nabla u)[z,z]\,\mathrm{d}\nu_x(z)\,\mathrm{d}x - 2\varepsilon \leq \liminf_{k\to\infty}\int_{\Omega} f_k\,\mathrm{d}x. \tag{2.41}$$

Indeed, this follows from the fact that, since ν_x is a probability measure for every $x \in \Omega$, we can find $m \geq m_\varepsilon$ large enough so that

$$\left|\int_{\Omega}\int_{\mathbb{R}^{N\times n}} F''(\nabla u(x))[z,z]\mathbb{1}_{\mathbb{R}^{N\times n}\setminus\overline{B(0,m)}}(z)\,\mathrm{d}\nu_x(z)\,\mathrm{d}x\right| < \varepsilon.$$

By letting $\varepsilon \to 0$ in (2.41), we conclude that

$$\frac{1}{2}\int_{\Omega}\int F''(\nabla u)[z,z]\,\mathrm{d}\nu_x(z)\,\mathrm{d}x \leq \liminf_{k\to\infty}\int_{\Omega} f_k\,\mathrm{d}x. \tag{2.42}$$

Next, we take $\varphi = \alpha_k b_k$ in inequality (2.33) and we recall that $c_p^{-1}(|\xi|^2 + |\xi|^p) \leq |V(\xi)|^2 \leq c_p(|\xi|^2 + |\xi|^p)$. Using that u is an F-extremal and $b_k \in \mathrm{Var}(\Omega,\mathbb{R}^N)$, after dividing by α_k^2 we get, for some constant $c_p > 0$,

$$\begin{aligned}
&\frac{c_2 c_p}{2}\int_{\Omega}\left(|\nabla b_k|^2 + \alpha_k^{p-2}|\nabla b_k|^p\right)\mathrm{d}x\\
&- c\sum_{j\in J}\int_{\Omega(x_j,s)-\Omega(x_j,r)}\left(|\nabla b_k|^2 + \alpha_k^{p-2}|\nabla b_k|^p + \frac{|b_k|^2}{(s-r)^2} + \alpha_k^{p-2}\frac{|b_k|^p}{(s-r)^p}\right)\mathrm{d}x\\
\leq& \alpha_k^{-2}\int_{\Omega}(G(x,\alpha_k\nabla b_k)\,\mathrm{d}x
\end{aligned}$$

for every s, r such that $\frac{R}{2} < r < s < R$.

Notice that

$$\alpha_k^{p-2}\left(|b_k|^p + |\nabla b_k|^p\right) = \frac{\beta_k^p}{\alpha_k^2} r_k^p\left(|b_k|^p + |\nabla b_k|^p\right).$$

Since $r_k b_k \rightharpoonup 0$ in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, we use again that $\left(\frac{\beta_k^p}{\alpha_k^2}\right)$ is bounded according to (2.5) and deduce that, for a subsequence that we do not relabel, it also holds that

$$\alpha_k^{\frac{p-2}{p}} b_k = \beta_k \alpha_k^{-\frac{2}{p}} r_k b_k \rightharpoonup 0 \qquad \text{in} \qquad \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N).$$

We can now use this, and the fact that $b_k \rightharpoonup 0$ in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$, to proceed exactly as we did to prove that $\alpha_k \to 0$ and whereby conclude that

$$\begin{aligned}
0 &\leq \frac{c_2 c_p}{2} \liminf_{k\to\infty} \int_\Omega |\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p \, \mathrm{d}x \\
&= \liminf_{k\to\infty} \left[\frac{c_2 c_p}{2} \int_\Omega \left(|\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p \right) \mathrm{d}x \right. \\
&\quad \left. -c \sum_{j\in J} \int_{\Omega(x_j,s)-\Omega(x_j,r)} \left(|\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p + \frac{|b_k|^2}{(s-r)^2} + \alpha_k^{p-2} \frac{|b_k|^p}{(s-r)^p} \right) \mathrm{d}x \right] \\
&\leq \liminf_{k\to\infty} \int_\Omega \alpha_k^{-2} G(x, \alpha_k \nabla b_k) \, \mathrm{d}x. \qquad (2.43)
\end{aligned}$$

Using this, together with (2.27) and (2.42), we get

$$\begin{aligned}
\frac{1}{2} \int_\Omega \int F''(\nabla u)[z,z] \, \mathrm{d}\nu_x(z) \, \mathrm{d}x &\leq \liminf_{k\to\infty} \int_\Omega f_k \, \mathrm{d}x + \liminf_{k\to\infty} \int_\Omega \alpha_k^{-2} G(x, \alpha_k \nabla b_k) \, \mathrm{d}x \\
&\leq \liminf_{k\to\infty} \left(\int_\Omega f_k \, \mathrm{d}x + \int_\Omega \alpha_k^{-2} G(x, \alpha_k \nabla b_k) \right) \mathrm{d}x \\
&= \liminf_{k\to\infty} \int_\Omega \alpha_k^{-2} G(x, \alpha_k \psi_k) \, \mathrm{d}x \\
&\leq 0. \qquad (2.44)
\end{aligned}$$

We now claim that

$$\frac{1}{2} \int_\Omega F''(\nabla u(x))[\overline{\nu}_x, \overline{\nu}_x] \, \mathrm{d}x + c_2 \int_\Omega \int |z - \overline{\nu}_x|^2 \, \mathrm{d}\nu_x(z) \, \mathrm{d}x \leq \frac{1}{2} \int_\Omega \int F''(\nabla u(x))[z,z] \, \mathrm{d}\nu_x(z) \, \mathrm{d}x. \qquad (2.45)$$

Indeed, since by (H2) F is strongly quasiconvex, by Proposition 19 we know that, for every $x \in \Omega$, the quadratic function

$$\eta \mapsto F''(\nabla u(x))[\eta, \eta] - 2c_2 |\eta - \overline{\nu}_x|^2$$

is quasiconvex. Hence, by Jensen's inequality from Theorem 48, we obtain (2.45) after integrating over Ω.

Therefore, since $\overline{\nu}_x = \nabla\psi(x)$ and the second variation is strongly positive,[6] we obtain from (2.44) and (2.45) that

$$\frac{c_3}{2}\int_\Omega |\nabla\psi|^2 \,\mathrm{d}x + c_2 \int_\Omega \int |z - \nabla\psi|^2 \,\mathrm{d}\nu_x(z)\,\mathrm{d}x \leq 0 \tag{2.46}$$

and thus, using Poincaré inequality in its version from Theorem 112, we can conclude that $\psi = 0$ and $\nu_x = \delta_0$. This implies that $\nabla\psi_k \rightharpoonup 0$ in $\mathrm{L}^2(\Omega, \mathbb{R}^N)$ and, by Lemma 37, we infer that $\nabla\psi_k \to 0$ in measure.

On the other hand, taking $\varphi = \alpha_k\psi_k$ in inequality (2.33), we obtain, after dividing by α_k^2, that

$$\begin{aligned}
&\frac{c_2\tilde{c_p}}{2}\int_\Omega |\nabla\psi_k|^2 \,\mathrm{d}x \\
&-c\sum_{j\in J} \int_{\Omega(x_j,s)-\Omega(x_j,r)} \left(|\nabla\psi_k|^2 + \alpha_k^{p-2}|\nabla\psi_k|^p + \frac{|\psi_k|^2}{(s-r)^2} + \alpha_k^{p-2}\frac{|\psi_k|^p}{(s-r)^p}\right)\mathrm{d}x \\
\leq &\frac{c_2\tilde{c_p}}{2}\int_\Omega \left(|\nabla\psi_k|^2 + \alpha_k^{p-2}|\nabla\psi_k|^p\right)\mathrm{d}x \\
&-c\sum_{j\in J} \int_{\Omega(x_j,s)-\Omega(x_j,r)} \left(|\nabla\psi_k|^2 + \alpha_k^{p-2}|\nabla\psi_k|^p + \frac{|\psi_k|^2}{(s-r)^2} + \alpha_k^{p-2}\frac{|\psi_k|^p}{(s-r)^p}\right)\mathrm{d}x < 0
\end{aligned}$$

for all $k \in \mathbb{N}$ and for all r, s such that $\frac{R}{2} < r < s < R$. Observe that

$$\alpha_k^{\frac{p-2}{p}}\psi_k = \beta_k\alpha_k^{-\frac{2}{p}} r_k\psi_k.$$

Hence, for a subsequence that we do not relabel, we use again (2.5) to further conclude that $\alpha_k^{\frac{p-2}{p}}\psi_k \rightharpoonup 0$ in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$. Arguing exactly as we did to prove that $\gamma_k \to 0$ and inequality (2.43), we can now take the limit when $k \to \infty$ and use the property $\int_\Omega |\nabla\psi_k|^2 \,\mathrm{d}x = 1$, to obtain that

$$0 < \frac{c_2\tilde{c_p}}{2} \leq 0,$$

which is a contradiction. This concludes the proof of the theorem. □

As an application of this theorem, in [GM09, Section 6] Grabovsky & Mengesha constructed

[6] See Remark 56.

an interesting class of strong local minimizers that are not global minimizers. Further examples of this situation can be found in [KS89, Tah05].

On the other hand, considering the partial regularity results available for both global and local minimizers that we have mentioned so far, it is a natural question whether the assumption of full regularity of the extremal is really necessary. The following chapter is motivated by this problem.

Chapter 3

Full interior regularity for a class of Lipschitz extremals

An important feature of minimizers of strongly quasiconvex integral functionals is that they can be shown to satisfy higher regularity properties. More precisely, we know from the partial regularity results of Evans [Eva86], Acerbi & Fusco [AF87, AF89b], Evans & Gariepy [EG92], Fusco & Hutchinson [FH85] and Giaquinta & Modica [GM86], to name a few, that if $u \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is a minimizer of an integral functional

$$\mathfrak{F}(u) := \int_{\Omega} F(\nabla u)\, \mathrm{d}x,$$

where $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies (H0)-(H2) for some $p \in (1,\infty)$, then there exists an open set $\Omega_0 \subseteq \Omega$ such that $|\Omega \backslash \Omega_0| = 0$ and u is of class $C^{1,\alpha}$ in Ω_0 for every $\alpha \in (0,1)$.

As we mentioned in Chapter 2, Kristensen and Taheri extended the partial regularity results, before available for global minimizers, to a certain class of local minimizers [KT03].[1] Furthermore, based on the breakthrough construction by Müller & Šverák [MŠ03], Kristensen & Taheri constructed Lipschitz extremals of quasiconvex integrands at which the second variation is positive but such that they are nowhere C^1 [KT03, Theorem 7.1]. This, together with the aforementioned regularity result, settled the fact that there are weak local minimizers that are not strong local minimizers. Furthermore, in Theorem 55 we have given a new proof for the Grabovsky-Mengesha sufficiency result, according to which C^1 extremals where

[1] The precise statement of Kristensen-Taheri regularity result is stated here as Theorem 58.

the second variation is strongly positive are strong local minimizers of quasiconvex integral functionals.

However, given that both global and strong local minimizers can only be shown to be partially regular, it is a very interesting and natural question whether we can relax the a priori regularity condition imposed on the extremal to guarantee strong local minimality.

In this chapter we establish a full interior regularity result for Lipschitz extremals at which the second variation is positive provided that, in addition, the mean oscillations of the weak derivative of the extremal converge uniformly to zero.

The idea behind being able to improve the regularity for this class of extremals is to exploit a certain local minimality property that they satisfy, which is slightly stronger than the weak local minimality that follows from Taylor's Theorem. In the first section of this chapter we specify notation and terminology to finally prove, based on a similar result by Kristensen & Taheri [KT03, Theorem 6.1], what exactly is this stronger minimality property that they satisfy. In the second section of this chapter we present the aforementioned full regularity proof. In order to enunciate the already mentioned result by Kristensen and Taheri, we must recall a more general notion of local minimizers, that we present in the following definition.

Definition 57 *Let* $u \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ *and* $q \in [1, \infty]$. *We say that* u *is a* $\mathrm{W}^{1,q}$***-local minimizer*** *if and only if there is a* $\delta > 0$ *such that*

$$\int_\Omega F(\nabla u)\, \mathrm{d}x \le \int_\Omega F(\nabla u + \nabla\varphi)\, \mathrm{d}x$$

for every φ *in* $\mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N)$ *with* $\|\nabla\varphi\|_{\mathrm{L}^q} < \delta$.

Kristensen-Taheri's result shows that, under the above conditions, any $\mathrm{W}^{1,q}$-local minimizer of class $\mathrm{W}^{1,q}_{\mathrm{loc}}$ is partially regular. More precisely, their theorem is as follows.

Theorem 58 *Suppose that* (H0) − (H2) *hold for some* $p \in [2, \infty)$. *Let* $q \in [1, \infty]$ *and suppose that* $u \in \mathrm{W}^{1,q}_{loc}(\Omega, \mathbb{R}^N) \cap \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ *is a* $\mathrm{W}^{1,q}$ *local minimizer. When* $q = \infty$, *assume in addition that*

$$\limsup_{r \to 0^+} \left(\operatorname*{ess\,sup}_{y \in B(x,r)} |\nabla u(y) - (\nabla u)_{B(x,r)}| \right) < \delta \tag{3.1}$$

holds locally uniformly in $x \in \Omega$. *Then, there exists an open set* $\Omega_0 \subseteq \Omega$ *of full* n*-dimensional* *measure such that* $u \in C^{1,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ *for every* $\alpha \in (0,1)$.

For the case $q \leq p$, the result can be established by looking at this as a problem of absolute minimizers on sets of small measure, so that partial regularity follows from Evans' result.[2] On the other hand, for $q > p$ the proof of their theorem uses strongly the fact that the minimizer u is itself of class $\mathrm{W}^{1,q}_{loc}$. However, it is still not clear whether this assumption is really necessary when $q < \infty$. If $q = \infty$ we are in the case of weak local minimizers. A consequence of Theorem 7.1 in [KT03] is that, for this case, assumption (3.1) is actually necessary, at least in a qualitative sense.

Motivated by Kristensen-Taheri's result of partial regularity for local minimizers, in this chapter we will establish that, if the second variation is strictly positive at a given Lipschitz extremal, and if the derivative of the extremal is such that its mean oscillations become arbitrarily small in a given uniform sense, then it is (fully) regular on its domain.

3.1 Lipschitz extremals that are BMO-local minimizers

In this section we will establish that Lipschitz solutions to the weak Euler-Lagrange equation at which the second variation is positive, are slightly more than merely weak local minimizers. This result is inspired by a very similar one due to Kristensen & Taheri [KT03, Theorem 6.1].

In addition, we remark that we work under the assumption of strong quasiconvexity (H2), which is not necessary to show that Lipschitz extremals with strongly positive second variation are weak local minimizers. More precisely, rank one convexity and p-growth are enough in that case.

We begin by stating the following definitions and conventions.

Notation 59 *Given* $\phi \in \mathrm{L}^1(\Omega, \mathbb{R}^N)$ *and* $B(x,r) \subseteq \Omega$, *we use the notation*

$$(\phi)_{x,r} := \fint_{B(x,r)} \phi \, \mathrm{d}x = \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} \phi(x) \, \mathrm{d}x.$$

Definition 60 *Let* $\phi \in \mathrm{L}^1(\Omega, \mathbb{R}^{N \times n})$. *We say that* ϕ *is of* ***bounded mean oscillation*** *if and only if*

$$\sup_{B(x,r) \subseteq \Omega} \fint_{B(x,r)} |\phi - (\phi)_{x,r}| \, \mathrm{d}y < \infty.$$

[2] See Theorem 83.

In this case, we define the semi-norm[3]

$$[\phi]_{\mathrm{BMO}(\Omega,\mathbb{R}^{N\times n})} := \sup_{B(x,r)\subseteq\Omega} \fint_{B(x,r)} |\phi - (\phi)_{x,r}|\,\mathrm{d}x < \infty$$

and we set

$$\mathrm{BMO}(\Omega,\mathbb{R}^{N\times n}) := \{\phi \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n}) : [\phi]_{\mathrm{BMO}(\Omega,\mathbb{R}^{N\times n})} < \infty\}.$$

The main result of this section can now be stated as follows.

Theorem 61 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a function such that it satisfies* (H0) − (H2) *for some $1 < p < \infty$. Let $\overline{u} \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^N)$ be an extremal with strongly positive second variation, i.e., for some $c_3 > 0$ and all $\varphi \in C_0^\infty(\Omega,\mathbb{R}^N)$,*

$$\int_\Omega \left\langle F'(\nabla\overline{u}), \nabla\varphi\right\rangle \mathrm{d}x = 0 \tag{3.2}$$

and

$$c_3\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x \le \int_\Omega F''(\nabla\overline{u})[\nabla\varphi,\nabla\varphi]\,\mathrm{d}x. \tag{3.3}$$

Then, there is a $\delta > 0$ such that

$$\int_\Omega F(\nabla\overline{u})\,\mathrm{d}x \le \int_\Omega F(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x$$

for every $\varphi \in C_0^\infty(\Omega,\mathbb{R}^N)$ with $[\nabla\varphi]_{\mathrm{BMO}} \le \delta$.

We remark that our statement differs from Theorem 6.1 in [KT03]theirs in that, in their result, the parameter δ that gives the local minimality, depends on a given constant $M > 0$ for which the variations φ are required to satisfy $\|\nabla\varphi\|_{\mathrm{L}^\infty} \le M$. By adapting the truncation technique from Acerbi & Fusco [AF87], we have been able to remove this additional restriction.

For the proof of Theorem 61 we will require the following definition and the subsequent lemmata, that generalize the Hardy-Littlewood-Fefferman-Stein maximal inequality to Orlicz spaces.

Definition 62 *Let $f\colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$ be an integrable map. We define the* ***Hardy-Littlewood***

[3]Observe that $[\phi]_{\mathrm{BMO}(\Omega,\mathbb{R}^N)} = 0$ if and only if ϕ is constant on each connected component of Ω.

maximal function *by*

$$f^\star(x) := \sup_{B(y,r)\ni x} \fint_{B(y,r)} |f(y)|\,\mathrm{d}y,$$

where the supremum is taken over all balls $B(y,r) \subseteq \mathbb{R}^n$ *containing* x*. Similarly, the* ***Fefferman-Stein maximal function*** *is given by*

$$f^\#(x) := \sup_{B(y,r)\ni x} \fint_{B(y,r)} |f(y) - (f)_{y,r}|\,\mathrm{d}y.$$

A useful generalization of the Hardy-Littlewood maximal inequality is the following.

Lemma 63 *Let* $\Phi\colon [0,\infty) \to [0,\infty)$ *be a continuously increasing function with* $\Phi(0) = 0$*. Assume, in addition, that* $\Phi(t) = t^p A(t)$ *for some* $p > 1$ *and some increasing function* $A\colon [0,\infty) \to [0,\infty)$*. Then, there exists a constant* $\gamma = \gamma(n,p)$ *such that*

$$\int_{\mathbb{R}^n} \Phi(|f|)\,\mathrm{d}x \le \int_{\mathbb{R}^n} \Phi(f^\star)\,\mathrm{d}x \le \gamma \int_{\mathbb{R}^n} \Phi(2|f|)\,\mathrm{d}x \tag{3.4}$$

for all $f \in \mathrm{L}^1(\mathbb{R}^n, \mathbb{R}^{N\times n})$*.*

The proof of the first inequality in this lemma follows from the fact that Φ is increasing and from Lebesgue's Differentiation Theorem, which implies that $|f(x)| \le f^\star(x)$ for almost every $x \in \mathbb{R}^n$. For a proof of the second inequality we refer the reader to [GIM95, Lemma 5.1]. We can relate both notions of maximal functions in the following way.

Lemma 64 *Let* $\Phi\colon [0,\infty) \to [0,\infty)$ *be a continuously increasing function with* $\Phi(0) = 0$*. Let* $\varepsilon > 0$ *and* $f \in \mathrm{L}^1(\mathbb{R}^n, \mathbb{R}^{N\times n})$*. Then,*

$$\int_{\mathbb{R}^n} \Phi(f^\star)\,\mathrm{d}x \le \frac{5^n}{\varepsilon} \int_{\mathbb{R}^n} \Phi\left(\frac{f^\#}{\varepsilon}\right)\,\mathrm{d}x + 2\cdot 5^{3n}\varepsilon \int_{\mathbb{R}^n} \Phi(5^n 2^{n+1} f^\star)\,\mathrm{d}x. \tag{3.5}$$

If, in addition, we have that

$$\sup_{t>0} \frac{\Phi(2t)}{\Phi(t)} < \infty,$$

we can further conclude that there is a constant $\gamma_1 = \gamma_1(n)$ *such that*

$$\int_{\mathbb{R}^n} \Phi(f^\star)\,\mathrm{d}x \le \gamma_1 \int_{\mathbb{R}^n} \Phi(f^\#)\,\mathrm{d}x \tag{3.6}$$

whenever $f \in \mathrm{L}^1(\mathbb{R}^n, \mathbb{R}^{N\times n})$ *is such that* $\int_{\mathbb{R}^n} \Phi(f^\star)\,\mathrm{d}x < \infty$.

The proof of (3.5) can be found, for example, in [KT03]. Inequality (3.6) follows easily from (3.5) under the given extra assumptions.

We can now proceed with the proof of the theorem.

Proof of Theorem 61. Let ω be a modulus of continuity for F'' on the set

$$\{\xi \in \mathbb{R}^{N\times n} : |\xi| \le 1 + \|\nabla \overline{u}\|_\infty\}.$$

As in Appendix D, we extend ω to $(1 + \|\nabla \overline{u}\|_\infty, \infty)$ so that it has the following properties:

- $\omega : [0, \infty) \to [0, \infty)$;
- ω is continuous and increasing;
- $\omega(0) = 0$ and $\omega(t) = 1$ for every $t \ge 1$;
- $\sup\limits_{t>0} \frac{\omega(2t)}{\omega(t)} < \infty$ and
- $|F''(\xi) - F''(\eta)| \le c\omega(|\xi - \eta|)$ for some constant $c = c(\|\nabla u\|_{\mathrm{L}^\infty}) > 0$ and for every $|\xi|, |\eta| \le 1 + \|\nabla \overline{u}\|_\infty$.

Now, if $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$, then by Taylor's formula, (3.2) and (3.3), we have

$$
\begin{aligned}
&\int_\Omega \left(F(\nabla\overline{u} + \nabla\varphi) - F(\nabla\overline{u})\right) \mathrm{d}x \\
=&\int_\Omega \left(F(\nabla\overline{u} + \nabla\varphi) - F(\nabla\overline{u}) - \left\langle F'(\nabla\overline{u}), \nabla\varphi\right\rangle\right) \mathrm{d}x \\
=&\int_\Omega \left(F(\nabla\overline{u} + \nabla\varphi) - F(\nabla\overline{u}) - \left\langle F'(\nabla\overline{u}), \nabla\varphi\right\rangle - \frac{1}{2}F''(\nabla\overline{u})[\nabla\varphi, \nabla\varphi]\right) \mathbb{1}_{\{\|\nabla\varphi|>1\}} \,\mathrm{d}x \\
&+ \int_\Omega \int_0^1 (1-t)\left(F''(\nabla\overline{u} + t\nabla\varphi) - F''(\nabla\overline{u})\right)[\nabla\varphi, \nabla\varphi]\mathbb{1}_{\{\|\nabla\varphi|\le 1\}} \,\mathrm{d}t\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega F''(\nabla\overline{u})[\nabla\varphi, \nabla\varphi]\,\mathrm{d}x \\
\ge&\frac{c_3}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - c\int_\Omega \left(\left(1 + |\nabla\overline{u}|^{p-1} + |\nabla\varphi|^{p-1}\right)|\nabla\varphi| + |\nabla\varphi| + |\nabla\varphi|^2\right)\mathbb{1}_{\{\|\nabla\varphi|>1\}}\,\mathrm{d}x \\
&- c\int_\Omega \omega(|\nabla\varphi|)|\nabla\varphi|^2\mathbb{1}_{\{\|\nabla\varphi|\le 1\}}\,\mathrm{d}x \\
\ge&\frac{c_3}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - c\int_\Omega \left(|\nabla\varphi|^2 + |\nabla\varphi|^p\right)\mathbb{1}_{\{\|\nabla\varphi|>1\}}\,\mathrm{d}x - c\int_\Omega \omega(|\nabla\varphi|)|\nabla\varphi|^2\mathbb{1}_{\{\|\nabla\varphi|\le 1\}}\,\mathrm{d}x,
\end{aligned} \tag{3.7}
$$

where $c = c(\|\nabla\overline{u}\|_\infty, n, p)$ and the last inequality follows from the fact that $ab^{p-1} \le a^p + b^p$ for $a, b > 0$.

We now consider two different cases.

Case 1. If $1 < p \le 2$, it follows from the above chain of inequalities that

$$
\begin{aligned}
&\int_\Omega \left(F(\nabla\overline{u} + \nabla\varphi) - F(\nabla\overline{u})\right) \mathrm{d}x \\
\ge&\frac{c_3}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - c\int_\Omega |\nabla\varphi|^2\mathbb{1}_{\{\|\nabla\varphi|>1\}}\,\mathrm{d}x - c\int_\Omega \omega(|\nabla\varphi|)|\nabla\varphi|^2\mathbb{1}_{\{\|\nabla\varphi|\le 1\}}\,\mathrm{d}x \\
\ge&\frac{c_3}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - c\int_\Omega \omega(|\nabla\varphi|)|\nabla\varphi|^2\,\mathrm{d}x.
\end{aligned} \tag{3.8}
$$

Extending φ by 0 outside of Ω, we see $\nabla\varphi$ as a map defined on $\mathbb{R}^n$.

Then, applying Lemmata 63 and 64 with $\Phi(t) = t^2\omega(t)$, we find, for a new constant $c > 0$, that

$$\begin{aligned}&\int_\Omega \left(F(\nabla\overline{u}+\nabla\varphi)-F(\nabla\overline{u})\right)\,\mathrm{d}x\\ \geq&\frac{c_3}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - c\int_\Omega \omega(c|(\nabla\varphi)^{\#}|)|(\nabla\varphi)^{\#}|^2\,\mathrm{d}x.\end{aligned}$$

It is easy to observe that

$$(\nabla\varphi)^{\#}\leq 2(\nabla\varphi)^{\star}. \tag{3.9}$$

This, and the Hardy-Littlewood-Wiener maximal inequality, imply that

$$\int_{\mathbb{R}^n}|(\nabla\varphi)^{\#}|^2\,\mathrm{d}x\leq 4\int_{\mathbb{R}^n}|(\nabla\varphi)^{\star}|^2\,\mathrm{d}x\leq c\int_{\mathbb{R}^n}|\nabla\varphi|^2\,\mathrm{d}x.$$

Putting this and (3.8) together, we conclude that

$$\int_\Omega \left(F(\nabla\overline{u}+\nabla\varphi)-F(\nabla\overline{u})\right)\,\mathrm{d}x\geq\int_\Omega\left(\frac{c_3\,c}{2}-c\,\omega(c|(\nabla\varphi)^{\#}|)\right)|(\nabla\varphi)^{\#}|^2\,\mathrm{d}x. \tag{3.10}$$

By taking $\delta>0$ small enough, the right hand side of the above expression will be non-negative when $[\nabla\varphi]_{\mathrm{BMO}}=\|(\nabla\varphi)^{\#}\|_\infty\leq\delta$. This concludes the proof for the case $1<p\leq 2$.

Case 2. If, on the other hand, $2<p<\infty$, from (3.7) we infer that

$$\begin{aligned}&\int_\Omega \left(F(\nabla\overline{u}+\nabla\varphi)-F(\nabla\overline{u})\right)\,\mathrm{d}x\\ \geq&\frac{c_3}{2}\int_\Omega|\nabla\varphi|^2\,\mathrm{d}x-c\int_\Omega\left(|\nabla\varphi|^2+|\nabla\varphi|^p\right)\,\mathrm{d}x-c\int_\Omega\omega(|\nabla\varphi|)|\nabla\varphi|^2\,\mathrm{d}x\\ \geq&\frac{c_3}{2}\int_\Omega|\nabla\varphi|^2\,\mathrm{d}x-c\int_\Omega\omega(|\nabla\varphi|)\left(|\nabla\varphi|^2+|\nabla\varphi|^p\right)\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

We extend φ by defining it like 0 outside of Ω and, for a $\delta>0$ still to be specified, we assume that $\|(\nabla\varphi)^{\#}\|_\infty<\delta<1$. Then, by Lemmata 63 and 64 applied with $\Phi(t)=t^p\omega(\tilde{c}t)$ and $\Phi(t)=t^2\omega(\tilde{c}t)$ in both directions, we use again (3.9) to obtain that, for different constants

$\tilde{c} = \tilde{c}(n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla\varphi|^p \omega(\nabla\varphi)\,\mathrm{d}x &\leq c\int_{\mathbb{R}^n} |(\nabla\varphi)^\star|^p \omega((\nabla\varphi)^\star)\,\mathrm{d}x \leq c\int_{\mathbb{R}^n} |(\nabla\varphi)^\#|^p \omega(\tilde{c}(\nabla\varphi)^\#)\,\mathrm{d}x \\ &\leq c\int_{\mathbb{R}^n} |(\nabla\varphi)^\#|^2 \omega(\tilde{c}(\nabla\varphi)^\#)\,\mathrm{d}x \leq c\int_{\mathbb{R}^n} |(\nabla\varphi)^\star|^2 \omega(\tilde{c}(\nabla\varphi)^\star)\,\mathrm{d}x \\ &\leq c\int_{\mathbb{R}^n} |\nabla\varphi|^2 \omega(\tilde{c}(\nabla\varphi))\,\mathrm{d}x. \qquad (3.12)\end{aligned}$$

It is for the third inequality above that we are using the assumption $\|(\nabla\varphi)^\#\|_\infty < \delta < 1$. The estimates obtained in (3.11) and (3.12) lead to

$$\int_\Omega (F(\nabla\overline{u} + \nabla\varphi) - F(\nabla\overline{u}))\,\mathrm{d}x \geq \frac{c_3}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - c\int_\Omega \omega(\tilde{c}|\nabla\varphi|)|\nabla\varphi|^2\,\mathrm{d}x.$$

We are now in the same situation as in (3.8) and we can conclude the proof in the same way that we did for $1 < p \leq 2$. □

3.2 Full interior regularity for Lipschitz extremals with VMO derivative

In this section we will exploit the local minimality obtained in Theorem 61 to show that Lipschitz extremals at which the second variation is strongly positive and such that the mean oscillations of their derivative become arbitrarily small in a uniform way, are smooth in their domain.

To clarify the terminology we consider a very special subspace of $\mathrm{BMO}(\Omega, \mathbb{R}^{N\times n})$ which, in a similar way as with Sobolev spaces, is the one that we obtain if we only consider functions that *vanish at infinity.* With this motivation, we establish the following definition regarding the space that consists of the closure of $C_0^0(\Omega, \mathbb{R}^{N\times n})$ in $\mathrm{BMO}(\Omega, \mathbb{R}^{N\times n})$.

Definition 65 *Let $\phi \in \mathrm{BMO}(\Omega, \mathbb{R}^{N\times n})$. We say that ϕ is of* ***vanishing mean oscillation*** *if and only if*

$$\lim_{\rho\to 0} \sup_{r\leq\rho} \sup_{B(x,r)\subseteq\Omega} \fint_{B(x,r)} |\phi - (\phi)_{x,r}|\,\mathrm{d}y = 0$$

and we set

$$\mathrm{VMO}(\Omega, \mathbb{R}^{N\times n}) := \{\phi \in \mathrm{BMO}(\Omega, \mathbb{R}^{N\times n}) : \ \phi \textit{ is of vanishing mean oscillation}\}.$$

Remark 66 $\mathrm{VMO}(\Omega, \mathbb{R}^{N\times n})$ *is, indeed, the closure of* $C_0^0(\Omega, \mathbb{R}^{N\times n})$ *in* $\mathrm{BMO}(\Omega, \mathbb{R}^{N\times n})$*. See [Ste93] for this and further properties of the spaces of bounded and vanishing mean oscillation.*

We would like to mention at this point that Moser states in [Mos01] that, if $u \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$ is an F-extremal of a C^2 rank one convex integrand and such that $\nabla u \in \mathrm{VMO}(\Omega, \mathbb{R}^{N\times n})$, then $u \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ for some $\alpha > 0$. We observe that, by extending this result to the boundary, it would directly imply that we can substitute the assumption of $u \in C^1(\overline{\Omega}, \mathbb{R}^N)$ by requesting that u is Lipschitz and its derivative is of vanishing mean oscillation in Theorem 55. However, while trying to incorporate these two results together, we found an inconsistency in the proof of the main result in [Mos01].

Nevertheless, the conjecture that a VMO derivative should allow us to obtain a good decay for BMO-local minimizers, and then iterate this decay to obtain higher regularity, is still an important question, specially when motivated by relaxing the a priori regularity assumptions imposed on the extremal for the sufficiency result given by Theorem 55. Therefore, in this section we give a positive answer to this problem by proving full regularity in Ω for the class of extremals that is now under consideration. We remark, however, that it is still a natural and important question whether the result that we present here can be extended up to the boundary. This would readily enable us to generalize Theorem 55.

Since the underlying idea for the regularity result that we will now establish is a BMO-local minimality condition, the proof of the following theorem follows the strategy developed by Kristensen & Taheri in their proof of Theorem 58. The argument is by contradiction using a blow-up procedure and, as we shall see, it enables us to conclude a good decay estimate of the excess, necessary for regularity, using as a starting point a Caccioppoli inequality of the first kind. This can be obtained just from the local minimality condition that the extremal satisfies. Furthermore, this is the remarkable difference between the technique used in [KT03] to obtain partial regularity of minimizers, and all the partial regularity results that had previously been obtained for global minimizers. The latter ones make use of a Caccioppoli inequality of the second kind that is obtained by iterating the one of the first kind. However, as we will see in detail in Step 3 of the following proof, this cannot be achieved in the case of local minimizers (the obstacles appearing in this case of BMO-local minimizers are the same as those arising in the context of $W^{1,q}$-local minimizers considered in [KT03]). Hence, the local minimality, together with the Caccioppoli inequality of the first kind, should suffice to conclude the proof in this case.

Despite the similarities that the following proof shares with that of Theorem 58, we remark that the ideas necessary to obtain the Caccioppoli inequality in this case (see (3.24 below) are not trivial. The main difficulty concerns obtaining suitable variations with small BMO norm that enable us to use the local minimality of u. Since the comparison maps that we construct consist, roughly speaking, of the product of a cut-off function with the extremal u, a careful treatment of the behaviour of the BMO semi-norm with respect to the product of functions is necessary. The main ideas behind the good estimates that we obtain here are inspired by the estimates obtained by Stegenga in [Ste76]. The details of this will become evident in Step 3 of the following proof.

We now proceed with the main result of this chapter.

Theorem 67 *Suppose that* (H0)-(H2) *hold for some* $2 \leq p < \infty$. *Let* $\alpha \in (0,1)$ *and assume that* $\overline{u} \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ *is an* F*-extremal with strongly positive second variation and such that* $\nabla \overline{u} \in \mathrm{VMO}(\Omega, \mathbb{R}^N)$. *Then,* $\overline{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

Recall that, under the assumptions of this statement, we have shown in Theorem 61 that $\overline{u}$ is a BMO-local F-minimizer.[4] With this in mind, we will now obtain the regularity result by establishing a suitable decay estimate for the excess

$$E(x,r) := \fint_{\Omega(x,r)} |V\left(\nabla \overline{u} - (\nabla \overline{u})_{x,r}\right)|^2 \,\mathrm{d}x.$$

The theorem will be a consequence of the following proposition, that we can then iterate to obtain full regularity of $\nabla \overline{u}$ via Campanato's characterization of Hölder continuity.

Proposition 68 *Under the assumptions of Theorem 67 we have that, for every* $m > 0$, *there exists* $C = C(m) > 0$ *with the property that for each* $\tau \in \left(0, \frac{1}{2}\right)$, *there is an* $\varepsilon = \varepsilon(m, \tau) > 0$

[4] We don't require $\nabla \overline{u} \in \mathrm{VMO}$ for Theorem 61.

such that, for a fixed $R_0 > 0$ and every $0 < r < R_0$, if $|(\nabla\overline{u})_{x,r}| \leq m$ and $E(x,r) < \varepsilon$, then

$$E(x, \tau r) < C(L)\tau^2 E(x, r).$$

Proof. Suppose that the conclusion of the proposition is false. Then, we can find $L > 0$ such that, for every $C > 0$, there is a corresponding $0 < \tau < \frac{1}{2}$ and a sequence of open balls $B(x_j, r_j) \subseteq \Omega$, such that $|(\nabla\overline{u})_{x_j,r_j}| \leq L$ and $E(x_j, r_j) \to 0$ but $E(x_j, \tau r_j) > C\tau^2 E(x_j, r_j)$. We proceed in three steps to finally obtain a contradiction for suitably large values of C.

Step 1. The blow up. We change variables and zoom in on the integrand. Let

$$B := B(0, 1), \quad \xi_j := (\nabla\overline{u})_{x_j,r_j} \quad \text{and } \lambda_j := \sqrt{E(x_j, r_j)}.$$

In addition, for $y \in B$ define

$$u_j(y) := \frac{\overline{u}(x_j + r_j y) - (\overline{u})_{x_j,r_j} - (\nabla\overline{u})_{x_j,r_j} r_j y}{r_j \lambda_j}$$

and, for $z \in \mathbb{R}^{N\times n}$,

$$\begin{aligned} F_j(z) &:= \frac{F(\xi_j + \lambda_j z) - F(\xi_j) - \lambda_j \langle F'(\xi_j), z\rangle}{\lambda_j^2} \\ &= \int_0^1 (1-t) F''(\xi_j + t\lambda_j z)[z, z]\, \mathrm{d}x. \end{aligned}$$

It is clear that $(u_j)_{0,1} = 0$, $(\nabla u_j)_{0,1} = 0$,

$$\fint_B \left(|\nabla u_j|^2 + \lambda_j^{p-2} |\nabla u_j|^p\right) \mathrm{d}x = 1 \tag{3.13}$$

and

$$\fint_{B(0,\tau)} \left(|\nabla u_j - (\nabla u_j)_{0,\tau}|^2 + \lambda_j^{p-2} |\nabla u_j - (\nabla u_j)_{0,\tau}|^p\right) \mathrm{d}x > C\tau^2. \tag{3.14}$$

Since $|\xi_j| \leq L$, it follows from Lemma 13 and from the fact that

$$|V(z)|^2 \leq c\left(|z|^2 + |z|^p\right)$$

and

$$|V_{p-1}(z)| \leq c\left(|z| + |z|^{p-1}\right),$$

that there exists $k = k(L) < \infty$ such that

$$|F_j(z)| \leq k\left(|z|^2 + \lambda_j^{p-2}|z|^p\right);$$
$$|F_j'(z)| \leq k\left(|z| + \lambda_j^{p-2}|z|^{p-1}\right). \tag{3.15}$$

In addition, condition (H2) implies that, for all $z \in \mathbb{R}^{N\times n}$ and $\varphi \in \mathrm{W}_0^{1,p}(B,\mathbb{R}^N)$,

$$\int_B \left(|\nabla\varphi|^2 + \lambda_j^{p-2}|\nabla\varphi|^p\right) \mathrm{d}x \leq \int_B \left(F_j(z+\nabla\varphi) - F_j(z)\right) \mathrm{d}x. \tag{3.16}$$

Now, write

$$I_j[u] := \int_B F_j(\nabla u)\,\mathrm{d}x.$$

Then, if $\delta > 0$ is the one given by Theorem 61, it can be easily verified that $I_j[u_j] \leq I_j[u_j+\varphi]$ holds provided that $\varphi \in \mathrm{W}_0^{1,\infty}(B,\mathbb{R}^N)$ is such that $\|\nabla\varphi\|_{\mathrm{BMO}(\Omega,\mathbb{R}^N)} \leq \delta_j := \frac{\delta}{\lambda_j}$.

On the other hand, by equation (3.13) we can assume, after extracting a subsequence, that for some $u \in \mathrm{W}^{1,2}(B,\mathbb{R}^N)$ and some $\xi_\infty \in \mathbb{R}^{N\times n}$, $u_j \rightharpoonup u$ in $\mathrm{W}^{1,2}(B,\mathbb{R}^N)$ and $\xi_j \to \xi_\infty$.

Step 2. Regularity of the limit function u. The purpose of this step is to show that the limit u satisfies

$$\int_B F''(\xi_\infty)[\nabla u, \nabla\varphi]\,\mathrm{d}x = 0 \tag{3.17}$$

for all $\varphi \in \mathrm{W}_0^{1,2}(B,\mathbb{R}^N)$. Having this, we can apply the Generalized Weyl's Lemma[5] to find a constant γ_0 depending only on n, N and L, such that

$$⨍_{B(0,r)} |\nabla u - (\nabla u)_{0,r}|^2\,\mathrm{d}x \leq \gamma_0 r^2 \tag{3.18}$$

for all $r \in \left(0, \frac{1}{2}\right]$ and then, by Campanato-Meyers characterization of Hölder continuity [Cam63], we can conclude that u is locally $C^{1,1}$. The proof of (3.17) follows exactly the same ideas than step 2 of [AF87, pp. 268-269]. We emphasize that it only uses the facts that $F \in C^2$ satisfies a strong Legendre-Hadamard condition, $|F'(\xi)| \leq c_1(1+|\xi|^{p-1})$ and $\overline{u} \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$ is an F-extremal. We reproduce their proof here as follows.

Since $\overline{u}$ is an F-extremal, the corresponding change of variables to the weak Euler-Lagrange

[5] See Theorem 72 for a more general version of it.

equation implies that, for every $\varphi \in \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N)$,

$$\int_B \left\langle F_j'(\nabla u_j), \nabla\varphi \right\rangle \, \mathrm{d}x = 0.$$

This means that u_j is an extremal for the functional I_j. By expressing this equation in terms of F, we see that it is equivalent to

$$\frac{1}{\lambda_j}\int_B \left\langle \left(F'(\xi_j + \lambda_j u_j) - F'(\xi_j)\right), \nabla\varphi \right\rangle \, \mathrm{d}x = 0. \tag{3.19}$$

We now define $B_j^+ := \{x \in B : \lambda_j|\nabla u_j(x)| > 1\}$ and $B_j^- := B \backslash B_j^+$. We claim that $|B_j^+| \leq \lambda_j^2|B|$. Indeed, it follows from (3.13) that

$$\frac{|B_j^+|}{\lambda_j^2} \leq \int_{B_j^+} |\nabla u_j|^2 + \lambda_j^{p-2}|\nabla u_j|^p \, \mathrm{d}x \leq \int_B |\nabla u_j|^2 + \lambda_j^{p-2}|\nabla u_j|^p \, \mathrm{d}x = |B|. \tag{3.20}$$

Recalling that F' has $(p-1)$-growth, the fact that $|\xi_j| \leq L$ and (3.13), we obtain that

$$\begin{aligned}
&\left| \int_{B_j^+} \left\langle \left(F'(\xi_j + \lambda_j \nabla u_j) - F'(\xi_j)\right), \nabla\varphi \right\rangle \, \mathrm{d}x \right| \\
\leq & c\int_{B_j^+} \left(1 + |\xi_j + \lambda_j\nabla u_j|^{p-1} + |\xi_j|^{p-1}\right) \|\nabla\varphi\|_{\mathrm{L}^\infty} \, \mathrm{d}x \\
\leq & c\|\nabla\varphi\|_{\mathrm{L}^\infty} \int_{B_j^+} \left(1 + L^{p-1} + |\lambda_j\nabla u_j|^{p-1}\right) \, \mathrm{d}x \\
\leq & c\|\nabla\varphi\|_{\mathrm{L}^\infty} \left(\int_B \left(\lambda_j^2|\nabla u_j|^2 + |\lambda_j\nabla u_j|^p\right) \, \mathrm{d}x + L^{p-1}|B_j^+| \right) \\
\leq & c\|\nabla\varphi\|_{\mathrm{L}^\infty} \left(1 + L^{p-1}\right) \lambda_j^2|B|.
\end{aligned}$$

On the other hand, we can rewrite the term concerning B_j^- as:

$$\begin{aligned}
&\frac{1}{\lambda_j}\int_{B_j^-} \left\langle \left(F'(\xi_j+\lambda_j\nabla u_j)-F'(\xi_j)\right), \nabla\varphi\right\rangle \mathrm{d}x \\
&=\int_{B_j^-}\int_0^1 F''(\xi_j+t\lambda_j\nabla u_j)[\nabla u_j,\nabla\varphi]\,\mathrm{d}t\,\mathrm{d}x \\
&=\int_{B_j^-}\int_0^1 \left(F''(\xi_j+t\lambda_j\nabla u_j)-F''(\xi_j)\right)[\nabla u_j,\nabla\varphi]\,\mathrm{d}t\,\mathrm{d}x+\int_{B_j^-}F''(\xi_j)[\nabla u_j,\nabla\varphi]\,\mathrm{d}x. \qquad (3.21)
\end{aligned}$$

Observe now that, by (3.20), the sequence of functions $(\mathbb{1}_{B_j^-})$ converges to $\mathbb{1}_B$ in measure as $j\to\infty$. In addition, since

$$\lambda_j^2 \fint_B |\nabla u_j|^2\,\mathrm{d}x = E(x_j,r_j)\overset{j\to\infty}{\longrightarrow} 0,$$

$(\lambda_j\nabla u_j)\to 0$ in measure. Furthermore, since F'' is continuous, this implies that

$$\mathbb{1}_{B_j^-}\int_0^1 \left|F''(\xi_j+t\lambda_j\nabla u_j)-F''(\xi_j)\right|\,\mathrm{d}t\|\nabla\varphi\|_{\mathrm{L}^\infty}\to 0 \qquad (3.22)$$

in measure as $j\to\infty$. Given that this sequence is uniformly bounded, by Vitali's Convergence Theorem we can conclude, from (3.21) and (3.22), that

$$\frac{1}{\lambda_j}\int_{B_j^-} \left\langle \left(F'(\xi_j+\lambda_j\nabla u_j)-F'(\xi_j)\right), \nabla\varphi\right\rangle \mathrm{d}x \to \int_B F''(\xi_\infty)[\nabla u,\nabla\varphi]\,\mathrm{d}x.$$

This, together with (3.19), implies that u satisfies (3.17) or, in other words, that it is $F''(\xi_\infty)$-harmonic. Recalling that $F''(\xi_\infty)[\eta,\eta]-2c_2|\eta|^2$ is quasiconvex by Theorem 19, Generalized Weyl's Lemma[6] enables us to conclude that u is C^1 on B and that (3.18) holds for some constant $\gamma_0>0$ depending only on n, N and L.

We have shown that the weak limit in $\mathrm{W}^{1,2}$ of the blown-up sequence (u_j) is a smooth function with good decay properties. This is the core of the contradiction that we shall derive in the following step.

[6] See Theorem 72 for a version up to the boundary of this classical result.

Step 3. Strong convergence. We now prove that, for every $\sigma < 1$,

$$\int_{B(0,\sigma)} \left(|\nabla(u_j - u)|^2 + \lambda_j^{p-2}|\nabla(u_j - u)|^p\right) \mathrm{d}x \to 0 \quad \text{as} \quad j \to \infty. \tag{3.23}$$

For this purpose, fix $\varsigma \in (0,1)$ and $B(x_0, r) \subseteq B$. Let $a_j : \mathbb{R}^n \to \mathbb{R}^N$ be defined by $a_j(x) := (u_j)_{x_0,r} + (\nabla u_j)_{x_0,r} \cdot (x - x_0)$. Then, $\nabla a_j = (\nabla u_j)_{x_0,r}$ and $(u_j - a_j)_{x_0,r} = 0$. Our first main goal is to show that there exists a constant $\theta < 1$ independent of j, $B(x_0, r)$ and ς, such that

$$\int_{B(x_0,\varsigma r)} \left(|\nabla u_j - (\nabla u_j)_{x_0,\varsigma r}|^2 + \lambda_j^{p-2}(|\nabla u_j - (\nabla u_j)_{x_0,\varsigma r}|^p\right) \mathrm{d}x \tag{3.24}$$

$$\begin{aligned} \leq \theta & \int_{B(x_0,r)} \left(|\nabla u_j - (\nabla u_j)_{x_0,r}|^2 + \lambda_j^{p-2}|\nabla u_j - (\nabla u_j)_{x_0,r}|^p\right) \mathrm{d}x \\ + \theta & \int_{B(x_0,r)} \left(\frac{1}{(1-\varsigma)^2 r^2}|u_j - a_j|^2 + \frac{\lambda_j^{p-2}}{(1-\varsigma)^p r^p}|u_j - a_j|^p\right) \mathrm{d}x \\ + \theta & r^n(1 - \varsigma^n)\left(|(\nabla u_j)_{x_0,r}|^2 + \lambda_j^{p-2}|(\nabla u_j)_{x_0,r}|^p\right) \end{aligned} \tag{3.25}$$

holds from a certain step $j \geq j(\varsigma, r)$, where $j(\varsigma, r) < \infty$ is allowed to depend on ς and r. This is a Caccioppoli inequality of the first kind. When $\overline{u}$ is a global minimizer, the first term in the right hand side of the inequality can be eliminated by an iteration argument, as in [EG87]. We also use this method in the direct proof of Theorem 103 of this work. However, since we are in a situation of local minimizers, $j \to \infty$ when the parameter $\varsigma \to 1^-$. Hence, we cannot perform the iteration, that requires to take values of ς that are arbitrarily close to 1.

We start by constructing the perturbations that will enable us to use the minimality of u_j for the corresponding functional I_j. Let $\rho : \mathbb{R}^n \to [0,1]$ be a Lipschitz cut-off function between $B(x_0, \varsigma r)$ and $B(x_0, r)$ so that $\mathbb{1}_{B(x_0,\varsigma r)} \leq \rho \leq \mathbb{1}_{B(x_0,r)}$ and $\|\nabla\rho\|_{\mathrm{L}^\infty} \leq 1/((1-\varsigma)r)$. Now, define

$$\varphi_j := \rho(u_j - a_j) \quad \text{and} \quad \psi_j := (1-\rho)(u_j - a_j).$$

Since $\psi_j + a_j = u_j$ outside of $B(x_0, r)$, it follows from Step 1 that $I_j[u_j] \leq I_j[\psi_j - a_j]$ provided that $\|\nabla\varphi_j\|_{\mathrm{BMO}}(B, \mathbb{R}^N) \leq \delta_j$. It is while obtaining sufficient conditions for these inequalities that we will use that $\overline{u} \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$ and $\nabla\overline{u} \in \mathrm{VMO}(\Omega, \mathbb{R}^N)$.

We take δ as in Theorem 61. We now wish to prove that there is $j = j(\varsigma, r)$ such that, for $j \geq j(\varsigma, r)$, $\|\nabla\varphi_j\|_{\mathrm{BMO}(\Omega,\mathbb{R}^N)} < \delta_j = \frac{\delta}{\lambda_j}$.

Since $\varphi_j = 0$ off $B(x_0, r)$, we can restrict our attention to

$$\sup_{B(x,s)\subseteq B(x_0,r)} \fint_{B(x,s)} |\nabla\varphi_j - (\nabla\varphi_j)_{x,s}|\,\mathrm{d}y.$$

To simplify the notation, let $v_j := u_j - a_j$ and observe that, given $B(x,s) \subseteq B(x_0,r)$,

$$\fint_{B(x,s)} |\nabla\varphi_j - (\nabla\varphi_j)_{x,s}|\,\mathrm{d}y \leq \fint_{B(x,s)} |\rho\nabla v_j - (\rho\nabla v_j)_{x,s}|\,\mathrm{d}y + \fint_{B(x,s)} |\nabla\rho\otimes v_j - (\nabla\rho\otimes v_j)_{x,s}|\,\mathrm{d}y.$$

Following the ideas from Stegenga [Ste76], we note that

$$\begin{aligned}
&\left|\fint_{B(x,s)} |\rho\nabla v_j - (\rho\nabla v_j)_{x,s}|\,\mathrm{d}y - |(\nabla v_j)_{x,s}| \fint_{B(x,s)} |\rho - (\rho)_{x,s}|\,\mathrm{d}y\right| \\
\leq &\fint_{B(x,s)} |(\rho\nabla v_j - (\rho\nabla v_j)_{x,s}) - (\nabla v_j)_{x,s}(\rho - (\rho)_{x,s})|\,\mathrm{d}y \\
\leq &\fint_{B(x,s)} |\rho(\nabla v_j - (\nabla v_j)_{x,s})|\,\mathrm{d}y + (\rho)_{x,s}(\nabla v_j)_{x,s} - (\rho\nabla v_j)_{x,s}| \\
\leq &\|\rho\|_{\mathrm{L}^\infty} \fint_{B(x,s)} |\nabla v_j - (\nabla v_j)_{x,s}|\,\mathrm{d}y + |(\rho)_{x,s}(\nabla v_j)_{x,s} - (\rho\nabla v_j)_{x,s}|.
\end{aligned} \tag{3.26}$$

We can also estimate

$$\begin{aligned}
|(\rho)_{x,s}(\nabla v_j)_{x,s} - (\rho\nabla v_j)_{x,s}| &= \left|\fint_{B(x,s)} \rho(\nabla v_j)_{x,s}\,\mathrm{d}y - \fint_{B(x,s)} \rho\nabla v_j\,\mathrm{d}y\right| \\
&\leq \fint_{B(x,s)} |\rho||\nabla v_j - (\nabla v_j)_{x,s}|\,\mathrm{d}y \\
&\leq \|\rho\|_{\mathrm{L}^\infty} \fint_{B(x,s)} |\nabla v_j - (\nabla v_j)_{x,s}|\,\mathrm{d}y,
\end{aligned} \tag{3.27}$$

from which we deduce that

$$\begin{aligned}\left| \fint_{B(x,s)} |\rho\nabla v_j - (\rho\nabla v_j)_{x,s}|\,\mathrm{d}y - |(\nabla v_j)_{x,s}| \fint_{B(x,s)} |\rho - (\rho)_{x,s}|\,\mathrm{d}y \right| &\leq 2 \fint_{B(x,s)} |\nabla v_j - (\nabla v_j)_{x,s}|\,\mathrm{d}y \\ &= 2 \fint_{B(x,s)} |\nabla u_j - (\nabla u_j)_{x,s}|\,\mathrm{d}y \\ &\leq 2\frac{\|\nabla \overline{u}\|_{\mathrm{BMO}(B(x_j,r_j),\mathbb{R}^N)}}{\lambda_j},\end{aligned} \tag{3.28}$$

where the last inequality follows, after a change of variables, from the fact that

$$B(x_j + r_j x, s r_j) \subseteq B(x_j, r_j).$$

On the other hand,

$$\begin{aligned}\left| (\nabla v_j)_{x,s} \fint_{B(x,s)} |\rho - (\rho)_{x,s}|\,\mathrm{d}y \right| &= |(\nabla v_j)_{x,s}| \left| \fint_{B(x,s)} \left| \rho - \fint_{B(x,s)} \rho(z)\,\mathrm{d}z \right| \mathrm{d}y \right| \\ &\leq |(\nabla v_j)_{x,s}| \fint_{B(x,s)} \fint_{B(x,s)} |\rho(y) - \rho(z)|\,\mathrm{d}z\,\mathrm{d}y \\ &\leq |(\nabla v_j)_{x,s}|\ \|\nabla\rho\|_{\mathrm{L}^\infty} 2s.\end{aligned} \tag{3.29}$$

Now, by the Divergence Theorem, we have that

$$\begin{aligned}|(\nabla v_j)_{x,s}|\ \|\nabla\rho\|_{\mathrm{L}^\infty} 2s \leq & \frac{2}{(1-\alpha)r} \left| \fint_{\partial B(x,s)} v_j \otimes \frac{x-y}{s} \right| \mathrm{d}\mathcal{H}^{n-1}(y) \\ \leq & \frac{2n}{(1-\alpha)r} \fint_{\partial B(x,s)} |v_j|\ \mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned} \tag{3.30}$$

To find an estimate for this term, it is convenient to consider each coordinate function $(u_j - a_j)^{(k)}$ separately. Fix $1 \leq k \leq N$ and take $\overline{x}_k \in B(x_0, r)$ and $\overline{y}_k \in B[x_0, r]$ such that

$$(u_j - a_j)^{(k)}(\overline{x}_k) = ((u_j - a_j)^{(k)})_{x_0,r} = 0 \text{ and } |(u_j - a_j)^{(k)}(\overline{y}_k)| = \sup_{B(x_0,r)} |(u_j - a_j)^k|. \tag{3.31}$$

Let $[\overline{x}_k, \overline{y}_k]$ denote the segment between $\overline{x}_k$ and $\overline{y}_k$. Then, for every $j \in \mathbb{N}$ we have that

$$\begin{aligned}
\fint_{\partial B(x,s)} |v_j| \, \mathrm{d}\mathcal{H}^{n-1}(y) \leq & \|v_j\|_{\mathrm{L}^\infty(\partial B(x,s),\mathbb{R}^N)} \\
\leq & \|v_j\|_{\mathrm{L}^\infty(B(x_0,r),\mathbb{R}^N)} \\
= & \sup_{B(x_0,r)} |u_j - a_j| \\
\leq & N \max_k \left| \int_{[\overline{x}_k, \overline{y}_k]} \nabla(u_j - a_j)^{(k)} \, \mathrm{d}y \right| \\
\leq & N \max_k \int_{[\overline{x}_k, \overline{y}_k]} \left| \nabla(u_j - a_j)^{(k)} \right| \, \mathrm{d}y \\
\leq & c(N) r^n \fint_{B(x_0,r)} |\nabla(u_j - a_j)| \, \mathrm{d}y \\
\leq & c(N) r^n \frac{\|\nabla \overline{u}\|_{\mathrm{BMO}(B(x_j,r_j),\mathbb{R}^N)}}{\lambda_j}.
\end{aligned} \tag{3.32}$$

From equations (3.30) and (3.32) we deduce that

$$|(\nabla v_j)_{x,s}| \; \|\nabla \rho\|_{\mathrm{L}^\infty} 2s \leq \frac{2n \, c(N) r^{n-1}}{(1-\varsigma)} \frac{\|\nabla \overline{u}\|_{\mathrm{BMO}(B(x_j,r_j),\mathbb{R}^N)}}{\lambda_j} \tag{3.33}$$

uniformly for $B(x,s) \subseteq B(x_0,r)$.

Using that $\nabla u \in \mathrm{VMO}(\Omega,\mathbb{R}^N)$ and equations (3.26)-(3.33), we can conclude that there are a suitable small $R_0 > 0$ and $j(\alpha, r)$ such that, for $j \geq j(\varsigma, r)$, we have

$$\fint_{B(x,s)} |\rho \nabla v_j - (\rho \nabla v_j)_{x,s}| \, \mathrm{d}y \leq \frac{\delta_j}{2} \tag{3.34}$$

for every $0 < s < r \leq R_0$.

It is still left to prove that $\fint_{B(x,s)} |\nabla \rho \otimes v_j - (\nabla \rho \otimes v_j)_{x,s}| \, \mathrm{d}y < \frac{\delta_j}{2}$ for every $j \geq j(\varsigma, r)$ and uniformly in $0 < s < r$.

Following a procedure analogous to the one above, we find that

$$\begin{aligned}
&\left| \fint_{B(x,s)} |\nabla\rho \otimes v_j - (\nabla\rho\otimes v_j)_{x,s}| \,\mathrm{d}y - |(v_j)_{x,s}| \fint_{B(x,s)} |\nabla\rho - (\nabla\rho)_{x,s}| \,\mathrm{d}y \right| \\
&\leq 2\|\nabla\rho\|_{\mathrm{L}^\infty} \fint_{B(x,s)} |v_j - (v_j)_{x,s}| \,\mathrm{d}y \\
&\leq \frac{2}{(1-\varsigma)r} \fint_{B(x,s)} |v_j - (v_j)_{x,s}| \,\mathrm{d}y \\
&\leq \frac{2}{(1-\varsigma)r} \sup_{B(x,s)} |v_j - (v_j)_{x,s}|.
\end{aligned}$$

Observe that, from the calculations in (3.32), we have

$$\begin{aligned}
\frac{2}{(1-\varsigma)r} \sup_{B(x,s)} |v_j - (v_j)_{x,s}| \leq& \frac{4}{(1-\varsigma)r} \sup_{B(x,s)} |v_j| \\
\leq& \frac{4}{(1-\varsigma)r} \|v_j\|_{\mathrm{L}^\infty(B(x_0,r),\mathbb{R}^N)} \\
\leq& \frac{4r^{n-1}}{(1-\varsigma)} \frac{\|\nabla\overline{u}\|_{\mathrm{BMO}(B(x_j,r_j),\mathbb{R}^N)}}{\lambda_j} \\
<& \frac{\delta_j}{4},
\end{aligned}$$

where the last inequality holds for $j \geq j(\varsigma, r)$ for $0 < r < R_0$.

Finally, noticing that

$$\begin{aligned}
&|(v_j)_{x,s}| \fint_{B(x,s)} |\nabla\rho - (\nabla\rho)_{x,s}| \,\mathrm{d}y \leq \|v_j\|_{\mathrm{L}^\infty(B(x,s),\mathbb{R}^N)} \|\nabla\rho\|_{\mathrm{L}^\infty(B(x,s),\mathbb{R}^N)} \\
\leq& \frac{1}{(1-\varsigma)r} \|v_j\|_{\mathrm{L}^\infty(B(x,s),\mathbb{R}^N)},
\end{aligned}$$

we can follow the ideas from equations (3.32)-(3.33) to conclude that

$$|(v_j)_{x,s}| \fint_{B(x,s)} |\nabla\rho - (\nabla\rho)_{x,s}| \,\mathrm{d}y < \frac{\delta_j}{4}$$

for $j \geq j(\varsigma, r)$. This proves that, for $0 < r < R_0$ with R_0 small enough and for $j \geq j(\varsigma, r)$, $\|\nabla\varphi_j\|_{\mathrm{BMO}(B(x,r),\mathbb{R}^N)} < \delta_j$.

Therefore, for $j \geq j(\varsigma, r)$ we have that $I_j[u_j] \leq I_j[u_j + \varphi_j]$.

Using this and (3.15), we will now show that there is $\theta \in (0,1)$ such that for $\varsigma \in (0,1)$, inequality (3.24) holds. The proof follows the standard procedure to obtain Caccioppoli inequalities for minimizers of integrands satisfying (H0)-(H2). For completeness, and for the sake of the reader, we establish it here following the derivation obtained in the proof of Proposition 4.2 in [KT03].

First, we observe that by the quasiconvexity condition obtained in (3.16), it follows that

$$\begin{aligned}
&\int_{B(x_0,\varsigma r)} \left(|\nabla u_j - \nabla a_j|^2 + \lambda_j^{p-2}|\nabla u_j - \nabla a_j|^p\right) \mathrm{d}x \\
\leq &\int_{B(x_0,r)} \left(|\nabla \varphi_j|^2 + \lambda_j^{p-2}|\nabla \varphi_j|^p\right) \mathrm{d}x \\
\leq &\int_{B(x_0,r)} (F_j(\nabla a_j + \nabla \varphi_j) - F_j(\nabla a_j)) \,\mathrm{d}x.
\end{aligned}$$

We now estimate the right hand side by using the Lipschitz estimate for F_j obtained in (3.15) and the local minimality of u_j to see that, for $j \geq j(\varsigma, r)$,

$$\begin{aligned}
&\int_{B(x_0,r)} (F_j(\nabla u_j - \nabla \psi_j) - F_j(\nabla u_j) + F(\nabla u_j) - F_j(\nabla a_j)) \,\mathrm{d}x \\
\leq &\int_{B(x_0,r)} (F_j(\nabla u_j - \nabla \psi_j) - F_j(\nabla u_j) + F_j(\nabla \psi_j + \nabla a_j) - F_j(\nabla a_j)) \,\mathrm{d}x \\
\leq c &\int_{B(x_0,r)} \left(\left(|\nabla a_j| + |\nabla u_j| + \lambda_j^{p-2}(|\nabla a_j|^{p-1} + |\nabla u_j|^{p-1})\right)|\nabla \psi_j| + |\nabla \psi_j|^2 + \lambda_j^{p-2}|\nabla \psi_j|^p\right) \mathrm{d}x,
\end{aligned}$$

where the constant $c = c(k, nN, p) > 0$.

We now use that $ab^{p-1} \leq a^p + b^p$ and triangle inequality to deduce that

$$\begin{aligned}
&\int_{B(x_0,\varsigma r)} \left(|\nabla u_j - (\nabla u_j)_{x_0,\alpha r}|^2 + \lambda_j^{p-2}(|\nabla u_j - (\nabla u_j)_{x_0,\varsigma r}|^p\right) \mathrm{d}x \\
\leq c &\int_{B(x_0,r)\setminus B(x_0,\varsigma r)} \left(|\nabla u_j - (\nabla u_j)_{x_0,r}|^2 + \lambda_j^{p-2}|\nabla u_j - (\nabla u_j)_{x_0,r}|^p\right) \mathrm{d}x \\
+ c &\int_{B(x_0,r)} \left(\frac{1}{(1-\varsigma)^2 r^2}|u_j - a_j|^2 + \frac{\lambda_j^{p-2}}{(1-\varsigma)^p r^p}|u_j - a_j|^p\right) \mathrm{d}x \\
+ c&r^n(1-\varsigma^n)\left(|(\nabla u_j)_{x_0,r}|^2 + \lambda_j^{p-2}|(\nabla u_j)_{x_0,r}|^p\right).
\end{aligned}$$

Then, the Caccioppoli inequality of the first kind, namely (3.24), follows after applying

Widman's hole-filling trick and with $\theta = c/(1+c)$.

The final stage of the proof consists in establishing the strong convergence (up to a subsequence) of ∇u_j to ∇u on $B(0,\sigma)$. The idea is to see $|\nabla u_j|^2\mathcal{L}^n$ as a convergent sequence of measures on $B(0,1)$ and then prove that the limit measure is actually $|\nabla u|^2\mathcal{L}^n$. This strategy was developed by Kristensen and Taheri [KT03] to overcome the fact that one cannot obtain a Caccioppoli inequality of the second kind, as in the case of global minimizers, for local minimizers.

We now reproduce their proof. We begin by defining the affine map

$$a(x) := (u)_{x_0,r} + (\nabla u)_{x_0,r} \cdot (x - x_0).$$

Since $u_j \rightharpoonup u$ in $\mathrm{W}^{1,2}$, it is clear that $a_j \to a$. In addition, there are a subsequence, that for convenience we do not relabel, and a positive finite Borel measure μ, such that

$$\left(|\nabla u_j|^2 + \lambda_j^{p-2}|\nabla u_j|^p\right)\mathcal{L}^n \overset{*}{\rightharpoonup} \mu$$

in $C^0(\overline{B})^*$. Furthermore, since $\lambda_j \to 0$ as $j \to \infty$, Rellich-Kondrachov Embedding theorem implies that

$$\int_{B(x_0,r)} \left(\frac{|u_j - a_j|^2}{(1-\varsigma)^2 r^2} + \lambda_j^{p-2}\frac{|u_j - a_j|^p}{(1-\varsigma)^p r^p}\right) \mathrm{d}x \to \mathrm{d}x \frac{1}{(1-\varsigma)^2 r^2} \int_{B(x_0,r)} |u-a|^2 \,\mathrm{d}x.$$

On the other hand, observe that for every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that

$$(1-\varepsilon)|\nabla u_j|^p - c_\varepsilon|\nabla a_j|^p \le |\nabla u_j - \nabla a_j|^p \le (1+\varepsilon)|\nabla u_j|^p + c_\varepsilon|\nabla a_j|^p. \qquad (3.35)$$

Furthermore, by elementary properties of the convergence of Radon measures (see, for example, [EG92, Section 1.9]), for any Borel set $A \subseteq \overline{B}$ we have that

$$\begin{aligned}
\mu(\mathrm{int}A) \le& \liminf_{j\to\infty} \int_A \left(|\nabla u_j|^2 + \lambda_j^{p-2}|\nabla u_j|^p\right) \mathrm{d}x \\
\le& \limsup_{j\to\infty} \int_A \left(|\nabla u_j|^2 + \lambda_j^{p-2}|\nabla u_j|^p\right) \mathrm{d}x \\
\le& \mu(\overline{A}).
\end{aligned}$$

It then follows from (3.24) and the auxiliary inequality (3.35) that

$$
\begin{aligned}
&(1-\varepsilon)\mu(B(x_0,\varsigma r)) + \int\limits_{B(x_0,\varsigma r)} \left(|\nabla a|^2 - 2\langle \nabla a, \nabla u\rangle\right) \mathrm{d}x \\
\leq &(1-\varepsilon)\liminf_{j\to\infty} \int\limits_{B(x_0,\varsigma r)} \left(|\nabla u_j|^2 + \lambda_j^{p-2}|\nabla u_j|^p\right) \mathrm{d}x + \lim_{j\to\infty} \int\limits_{B(x_0,\varsigma r)} \left(|\nabla a_j|^2 - 2\langle \nabla a_j, \nabla u_j\rangle\right) \mathrm{d}x \\
\leq &\liminf_{j\to\infty} \int\limits_{B(x_0,\varsigma r)} \left(|\nabla u_j - \nabla a_j|^2 + \lambda_j^{p-2}|\nabla u_j - \nabla a_j|^p\right) \mathrm{d}x \\
\leq &\limsup_{j\to\infty} \int\limits_{B(x_0,\varsigma r)} \left(|\nabla u_j - \nabla a_j|^2 + \lambda_j^{p-2}|\nabla u_j - \nabla a_j|^p\right) \mathrm{d}x \\
\leq &\theta\left((1+\varepsilon)\mu(B[x_0,r]) + \int\limits_{B(x_0,r)} \left(|\nabla a|^2 - 2\langle \nabla a, \nabla u\rangle\right) \mathrm{d}x\right) \\
&+ \frac{\theta}{(1-\varsigma)^2 r^2} \int\limits_{B(x_0,r)} |u-a|^2 \,\mathrm{d}x + \theta|\nabla a|^2 r^n (1-\varsigma^n).
\end{aligned}
$$

Since this holds for any $\varepsilon > 0$, it readily implies that

$$
\begin{aligned}
&\mu(B(x_0,\varsigma r)) - \int\limits_{B(x_0,\varsigma r)} |\nabla u|^2 \,\mathrm{d}x \\
\leq &\theta\left(\mu(B[x_0,r]) - \int\limits_{B(x_0,r)} |\nabla u|^2 \,\mathrm{d}x\right) + \frac{\theta}{(1-\alpha)^2 r^2} \int\limits_{B(x_0,r)} |u-a|^2 \,\mathrm{d}x \\
&+ \theta|\nabla a|^2 r^n (1-\alpha^n) + \int\limits_{B(x_0,r)} |\nabla u - \nabla a|^2 \,\mathrm{d}x. \qquad (3.36)
\end{aligned}
$$

Now, let $\nu := \mu - |\nabla u|\mathcal{L}^n$. It is then clear that ν is a positive, finite Borel measure on $\overline{B}$. In addition, for all but a countable amount of $r \in (0, 1-|x_0|)$ we have that $\nu(\partial B(x_0,\varsigma r)) > 0$.[7] Then, from (3.36) we deduce that, for such $r \in (0, 1-|x_0|)$,

$$
\nu(B[x_0,\varsigma r]) \leq \theta\nu(B[x_0,r]) + \left(\frac{\varepsilon_1(r)}{(1-\varsigma)^2} + \varepsilon_2(r)(1-\varsigma^n) + \varepsilon_3(r)\right) r^n \qquad (3.37)
$$

for every $\varsigma \in (0,1)$, where

$$
\varepsilon_1(r) := \frac{\theta}{r^{n+2}} \int\limits_{B(x_0,r)} |u-a|^2 \,\mathrm{d}x, \qquad \varepsilon_2(r) := \theta|\nabla a|^2
$$

[7] We can see this by following the same argument that we used in (2.34).

and

$$\varepsilon_3(r) := \frac{1}{r^n} \int_{B(x_0,r)} |\nabla u - \nabla a|^2 \, \mathrm{d}x.$$

Since u is of class C^1, $\varepsilon_3(r) \to 0$ and $\varepsilon_2(r) \to \theta|\nabla u(x_0)|^2$ as $r \to 0^+$. By Poincaré inequality, this further implies that $\varepsilon_1(r) \to 0$.

We will now prove that

$$\liminf_{r\to 0^+} \frac{\nu(B[x_0,r])}{r^n} = 0. \tag{3.38}$$

Observe first that, if $\limsup_{r\to 0^+} \frac{\nu(B[x_0,r])}{r^n} = 0$, the claim trivially follows. We therefore assume that

$$\limsup_{r\to 0^+} \frac{\nu(B[x_0,r])}{r^n} > 0. \tag{3.39}$$

This implies, in particular, that $\nu(B[x_0,r]) > 0$ for all $r > 0$ and, therefore, we can rewrite (3.37) for all but a countable amount of $r \in (0, 1 - |x_0|)$, so that

$$\frac{\nu(B[x_0,\varsigma r])}{\nu(B[x_0,r])} \le \theta + \left(\frac{\varepsilon_1(r)}{(1-\varsigma)^2} + \varepsilon_2(r)(1-\varsigma^n) + \varepsilon_3(r)\right) \frac{r^n}{\nu(B[x_0,r])}.$$

We can now take the limit superior as $r \to 0^+$ and deduce that, for $\varsigma \in (0,1)$,

$$\begin{aligned}
&\limsup_{r\to 0^+} \frac{\nu(B[x_0,\varsigma r])}{\nu(B[x_0,r])} \\
&\le \theta + \limsup_{r\to 0^+} \left(\left(\frac{\varepsilon_1(r)}{(1-\varsigma)^2} + \varepsilon_2(r)(1-\varsigma^n) + \varepsilon_3(r)\right) \frac{r^n}{\nu(B[x_0,r])}\right) \\
&\le \theta + \theta|\nabla u(x_0)|^2(1-\varsigma^n) \limsup_{r\to 0^+} \frac{r^n}{\nu(B[x_0,r])}.
\end{aligned} \tag{3.40}$$

We will now show that, if (3.39) holds, then

$$\limsup_{r\to 0^+} \frac{\nu(B[x_0,\varsigma r])}{\nu(B[x_0,r])} \ge \varsigma^n. \tag{3.41}$$

We proceed by a contradiction argument and observe that, if $\limsup\limits_{r\to 0^+} \frac{\nu(B[x_0,\varsigma r])}{\nu(B[x_0,r])} < \varsigma^n$, then we can take $\gamma < 1$ and $r_\gamma > 0$ such that

$$\nu(B[x_0,\varsigma r]) < \gamma\varsigma^n \nu(B[x_0,r])$$

for every $r \leq r_\gamma$. We can then iterate this and obtain that, for every $k \in \mathbb{N}^+$,

$$\nu(B[x_0, \varsigma^k r]) < (\gamma\varsigma^n)^k \nu(B[x_0, r]).$$

We now fix a sequence (r_j) such that $r_j \to 0^+$ and

$$\lim_{j\to\infty} \frac{\nu(B[x_0, r_j])}{r_j^n} = \limsup_{r\to 0^+} \frac{\nu(B[x_0, r])}{r^n}.$$

Since $\varsigma \in (0,1)$, we have that $(0, r_\gamma] = \bigcup_{k=0}^{\infty} [\varsigma^{k+1} r_\gamma, \varsigma^k r_\gamma]$. Hence, as $j \to \infty$ we have that

$$k_j := \sup\left\{\ k : r_j \in [\varsigma^{k+1} r_\gamma, \varsigma^k r_\gamma]\right\} \to \infty.$$

On the other hand, since $r_j \leq \varsigma^{k_j} r_\gamma$ and $\varsigma^{k_j+1} r_\gamma \leq r_j$,

$$\frac{\nu(B[x_0, r_j])}{r_j^n} \leq \frac{\nu(B[x_0, \varsigma^{k_j} r_\gamma])}{(\varsigma^{k_j+1} r_\gamma)^n} < \frac{1}{\varsigma^n}\gamma^{k_j} \frac{\nu(B[x_0, r_\gamma])}{r_\gamma^n} \to 0$$

as $j \to \infty$, which contradicts (3.39).

Thus, from (3.40) and (3.41) we conclude that

$$\varsigma^n \leq \theta + \theta|\nabla u(x_0)|^2(1-\varsigma^n) \limsup_{r\to 0^+} \frac{r^n}{\nu(B[x_0, r])}$$

for every $\varsigma \in (0,1)$. We now take the limit as $\varsigma \to 1^-$ and deduce that, since $0 < \theta < 1$, it must hold that

$$\limsup_{r\to 0^+} \frac{r^n}{\nu(B[x_0, r])} = \infty$$

and, therefore, we have that (3.38) is valid for every $x_0 \in B$, as we wanted to prove.

To conclude this step, we now fix $\sigma \in (0,1)$ and use Vitali's covering theorem (see [EG92, p.35]), as well as (3.38), to obtain for a given $\varepsilon > 0$ a countable family of disjoint open balls $\{B_i\}_{i\in I}$, such that $B_i \subseteq B$,

$$\nu\left(B[0,\sigma] \backslash \bigcup_{i\in I} B_i\right) = 0$$

and

$$\nu(B_i) < \varepsilon \mathcal{L}^n(B_i)$$

for every $i \in I$. Then, $\nu(B[0,\sigma]) \leq \varepsilon \mathcal{L}^n(B)$ and, therefore, $\nu \llcorner B = 0$. In other words,

$\mu \llcorner B = |\nabla u|^2 \mathcal{L}^n \llcorner B$, which means that

$$\int_{B(0,\sigma)} \left(|\nabla u_j|^2 + \lambda_j^{p-2} |\nabla u_j|^p \right) \mathrm{d}x \to \int_{B(0,\sigma)} |\nabla u|^2 \, \mathrm{d}x$$

for each $\sigma \in (0,1)$. This implies, in turn, that $\nabla u_j \to \nabla u$ strongly in $\mathrm{L}^2(B(0,\sigma), \mathbb{R}^{N\times n})$ and, when $p > 2$, $\lambda_j^{p-2} |\nabla u_j|^p \to 0$.

We can now pass to the limit in (3.14) to obtain that

$$\fint_{B(0,\tau)} |\nabla u - (\nabla u)_{0,\tau}|^2 \, \mathrm{d}x \geq C\tau^2,$$

which is a contradiction to (3.18) if we take $C > \gamma_0$ and $r = \tau$. This concludes the proof of the proposition. □

To conclude the proof of Theorem 67 we now require to follow a classical iteration argument performed to the decay estimate that we have obtained from Proposition 68. Using the facts that u is Lipschitz and that $\nabla u \in \mathrm{VMO}(\Omega, \mathbb{R}^N)$, it is easy to see that the iteration necessary to conclude is a particular case to the one presented in Step 5 of the proof of Theorem 76. We therefore omit it here and we refer the reader to the corresponding proof in Chapter 4 for further details. In addition, we emphasize that the assumption that $\nabla u \in \mathrm{VMO}$ is not required only for the iteration process, but it has also been essential while obtaining the Caccioppoli inequality (3.24) and, particularly, to derive the estimates of the type of (3.34).

Furthermore, we remark that, contrary to the classical partial regularity results for global and local minimizers that we have mentioned at the beginning of this chapter, the uniform decay of the mean oscillations of ∇u given by the assumption $\nabla u \in \mathrm{VMO}$ enables us to obtain full regularity in the domain Ω. A natural and interesting extension of this result is, of course, to extend the obtained regularity up to the boundary of Ω and, for that, a careful derivation of a Caccioppoli inequality, as in (3.24) should be performed.

Chapter 4

Full regularity up to the boundary of global and local minimizers

The subject of regularity in the Calculus of Variations was highlighted by Hilbert in the 19^{th} of his twenty three seminal problems from 1900. The works of de Giorgi [DG57], Ladyzenskaya & Ural'ceva [LU68] and Morrey [Mor66] can be considered as some of the forerunners in the area. For a long time, most of the proofs concerning regularity of minimizers of functionals of the form

$$\mathfrak{F}(u) := \int_{\Omega} F(\nabla u)\,\mathrm{d}x$$

were approached by dealing directly with the weak Euler-Lagrange equation associated to the functional. Giaquinta & Giusti [GG78, GG82] and Giaquinta & Modica [GM79a, GM79b] provided a new strategy to establish regularity relying essentially on an approximation argument by quasilinear systems and on the L^p-estimates available for solutions to those systems. Jost and Meier then extended some of Giaquinta & Modica's results to obtain regularity up to the boundary [JM83]. A partial regularity result for quasiconvex integrands was first obtained by Evans [Eva86] and the subsequent improvements that we have already mentioned in the Introduction of this work. On the other hand, in recent years Grotowski [Gro02], Duzaar, Grotowski & Kronz [DGK05], Kronz [Kro05], Kristensen & Mingione [KM10] and Beck [Bec11] have considerably enriched what is known about regularity up to the boundary for non-linear elliptic systems and minimizers of variational integrals.

In this chapter we focus on functionals of the type of $\mathfrak{F}$ with $u \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ for some

$2 \leq p < \infty$ and where the integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies (H0)-(H2). One of the main contributions of this work is the full regularity result up to the boundary that we present in Section 4.2.

The core assumption that we make in order to obtain full (and not only partial) regularity is that the Dirichlet boundary condition is given by a function whose $\mathrm{W}^{1,p}$ norm is sufficiently small. One of the motivations comes from elasticity theory, in the sense that we should expect that, if the deformation to which an elastic object is subject to is not very big, no singularities should arise from it.

We begin this chapter with some preliminary material from regularity theory for quasilinear systems. We then proceed to the main regularity proof for global minimizers and we conclude with the analogous result for local minimizers, for which the topology of the domain plays an important role.

4.1 Preliminaries from classical regularity theory

We now state the following version of Gehring's Lemma, which is a powerful result that enables us to improve a given reverse Hölder inequality and, in some cases, to obtain higher integrability too. This result was first established by Gehring in [Geh73, Lemma 3].

Theorem 69 (Gehring's Lemma) *Let $1 < p < \infty$, $0 < \sigma < 1$ and $K \geq 0$. Let $\mathbb{B} \subseteq \mathbb{R}^n$ be an arbitrary open ball and assume that $f, g \in \mathrm{L}^p(\mathbb{B})^+$ satisfy*

$$\fint_{\sigma B} f^p \,\mathrm{d}x \leq K \left(\fint_B f \,\mathrm{d}x \right)^p + \fint_B g^p \,\mathrm{d}x \tag{4.1}$$

for every $B \subseteq \mathbb{B}$. Then, there exists $q_0 = q(n,p,K) > p$ such that, for every $q \in [p, q_0]$ and every $r \in (0,1)$,

$$\left(\fint_{r\mathbb{B}} f^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq \frac{c(n,\sigma)}{r^{\frac{n}{q}}(1-r)^{\frac{n}{p}}} \left(\left(\fint_{\mathbb{B}} f^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \left(\fint_{\mathbb{B}} g^q \,\mathrm{d}x \right)^{\frac{1}{q}} \right). \tag{4.2}$$

In particular, if $g \in \mathrm{L}^q(\mathbb{B})$, then $f \in \mathrm{L}^q_{loc}(\mathbb{B})$ too.

A classical reference for a proof of this theorem is [Gia83, Theorems 1.1-1.2]. A more general version of it can be found, for example, in [Kri10, Lecture 2].

In what follows we will consider a domain Ω of class C^1. In addition, for what remains of this chapter we shall use the following convention.[1]

Notation 70 *If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain of class $C^{1,\alpha}$ for an $\alpha \in [0,1]$ and if ϑ is a defining function for Ω, we denote, for every $x \in \mathbb{R}^n$,*

$$\nu_x := \nabla\vartheta(x).$$

Recall that ν_x is an outward normal vector for $x \in \partial\Omega$. For ease of notation, we also write

$$\nu := \nabla\vartheta.$$

Finally, we use the following notation to denote the unitary vector of a given $\nu_x \in \mathbb{R}^n$.

$$\hat{\nu}_x := \frac{\nu_x}{|\nu_x|}$$

For functions such that $v = 0$ on $\partial\Omega$, the boundary version of Poincaré inequality (see Theorem 111) can be strengthened in the sense that the integral of v over $\Omega(x_0, R)$ can actually be estimated by the integral of $\nabla v \cdot \hat{\nu}_{x_0}$, which corresponds to the normal derivative of v in the direction ν_{x_0}.

Proposition 71 *Let $p \geq 2$. Assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded C^1 domain in $\mathbb{R}^n$. Then, there exist constants $R_\Omega > 0$ and $c > 0$, depending only on Ω, such that for every $x_0 \in \partial\Omega$, $0 < R < R_\Omega$ and every $v \in \mathrm{W}^{1,p}(\Omega(x_0, 2R), \mathbb{R}^N)$, if $v \equiv 0$ on $B(x_0, 2R) \cap \partial\Omega$, it holds that*

$$\fint_{\Omega(x_0,R)} \left|V\left(\frac{v}{R}\right)\right|^2 \mathrm{d}x \leq c \fint_{\Omega(x_0,2R)} |V(\nabla v \cdot \hat{\nu}_{x_0})|^2 \,\mathrm{d}x.$$

We prove the proposition inspired by [Bec07, Lemma 3.4]. However, we emphasize that the origins of this result can be traced back to [Cam65, Lemma 5.IV].

Proof. Without loss of generality we assume that $x_0 = 0$ and that $\nu_{x_0} = |\nu_{x_0}|\mathrm{e}_\mathrm{n}$, so that the unit normal vector to $x_0 \in \partial\Omega$ is the canonical vector e_n.

We let $\tilde{\Omega}(0, R) := \big(B^{n-1}(0, R) \times (-R, R)\big) \cap \Omega$ be the intersection of Ω with the cilinder whose

[1] See Section 1.7 for further details on the terminology of smooth domains.

centre is $x_0 = 0$ and its radius is R. We will show that

$$\fint_{\tilde{\Omega}(0,R)} \left|V\left(\frac{v}{R}\right)\right|^2 \mathrm{d}x \le c \fint_{\tilde{\Omega}(0,R)} |V(\nabla v \cdot \mathrm{e_n})|^2 \,\mathrm{d}x = c \fint_{\tilde{\Omega}(0,R)} |V(\nabla_{\mathrm{n}} v)|^2 \,\mathrm{d}x,$$

from which the desired inequality easily follows.

For $x \in \mathbb{R}^n$ we use the notation $x = (x', x_n)$ and observe that, by the Implicit Function Theorem, there exists $R_\Omega > 0$ such that, for every $0 < R < R_\Omega$, $\big(B^{n-1}(0,R) \times (-R,R)\big) \cap \partial\Omega$ is the graph of a function $h\colon B^{n-1}(0,R) \to (-R,R)$. By the compactness of $\partial\Omega$, R_Ω can be taken to be independent of x_0. Then, we can describe the set $\tilde{\Omega}(0,R)$ by

$$\tilde{\Omega}(0,R) = \{x \in \mathbb{R}^n : |x'| \le R \;\text{ and }\; -R \le x_n \le h(x')\}.$$

We now observe that, for $v \in \mathrm{W}^{1,p}(\tilde{\Omega}(0,R), \mathbb{R}^N)$, if $v \equiv 0$ on $\partial\Omega$, we can write

$$v(x) = v(x', x_n) - v(x', h(x')) = \int_{h(x')}^{x_n} \nabla_{\mathrm{n}} v(x', t)\,\mathrm{d}t.$$

Using the auxiliary convex function W, we reach the following estimates.

$$\begin{aligned}
\int_{\tilde{\Omega}(0,R)} \left|V\left(\frac{v}{R}\right)\right|^2 \mathrm{d}x &\le c \int_{\tilde{\Omega}(0,R)} \left|W\left(\frac{v}{R}\right)\right|^2 \mathrm{d}x \\
&\le c \int_{\tilde{\Omega}(0,R)} \left|W\left(\frac{1}{R}\int_{h(x')}^{x_n} \nabla_{\mathrm{n}} v(x',t)\,\mathrm{d}t\right)\right|^2 \mathrm{d}x \\
&= c \int_{\tilde{\Omega}(0,R)} \left|W\left(\frac{h(x') - x_n}{R} \fint_{x_n}^{h(x')} \nabla_{\mathrm{n}} v(x',t)\,\mathrm{d}t\right)\right|^2 \mathrm{d}x \\
&\le c \int_{\tilde{\Omega}(0,R)} \fint_{x_n}^{h(x')} \left|W\left(\frac{h(x') - x_n}{R} \nabla_{\mathrm{n}} v(x',t)\right)\right|^2 \mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

We notice that $\left|\frac{h(x')-x_n}{R}\right| \leq 2$. Then, by Lemma 128 (iv) and since $p \geq 2$, we derive from above that

$$\begin{aligned}
\int_{\tilde{\Omega}(0,R)} \left|V\left(\frac{v}{R}\right)\right|^2 \mathrm{d}x \leq & c \int_{\tilde{\Omega}(0,R)} \left(\frac{h(x')-x_n}{R}\right)^2 \fint_{x_n}^{h(x')} \left|W\left(\nabla_{\mathrm{n}} v(x',t)\right)\right|^2 \mathrm{d}t\, \mathrm{d}x \\
= & c \int_{B^{n-1}(0,R)} \int_{-R}^{h(x')} \frac{|h(x')-x_n|}{R^2} \int_{x_n}^{h(x')} \left|W\left(\nabla_{\mathrm{n}} v(x',t)\right)\right|^2 \mathrm{d}t\, \mathrm{d}x_n\, \mathrm{d}x' \\
\leq & c \int_{B^{n-1}(0,R)} \int_{-R}^{h(x')} \frac{|h(x')-x_n|}{R^2} \int_{-\sqrt{R^2-|x'|^2}}^{h(x')} \left|V\left(\nabla_{\mathrm{n}} v(x',t)\right)\right|^2 \mathrm{d}t\, \mathrm{d}x_n\, \mathrm{d}x'.
\end{aligned}$$

Observe that

$$\int_{-R}^{h(x')} \frac{|h(x')-x_n|}{R^2} \mathrm{d}x_n = \frac{h(x')^2}{2R^2} + \frac{1}{2} + \frac{h(x')}{R} \leq 3.$$

Therefore, the above chain of inequalities implies that

$$\begin{aligned}
\int_{\tilde{\Omega}(0,R)} \left|V\left(\frac{v}{R}\right)\right|^2 \mathrm{d}x \leq & c \int_{B^{n-1}(0,R)} \int_{-R}^{h(x')} \left|V\left(\nabla_{\mathrm{n}} v(x',t)\right)\right|^2 \mathrm{d}t\, \mathrm{d}x' \\
= & c \int_{\tilde{\Omega}(0,R)} \left|V\left(\nabla_{\mathrm{n}} v(x',t)\right)\right|^2 \mathrm{d}x,
\end{aligned}$$

as we wanted to prove. □

We now state a generalized version of Weyl's Lemma for $\mathcal{A}$-harmonic functions.

Theorem 72 *Let $\mathcal{A}\colon \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a symmetric bilinear form such that*

(i) $\mathcal{A}[\xi,\eta] \leq L|\xi||\eta|$ *for every* $\xi,\eta \in \mathbb{R}^{N\times n}$ *and*

(ii) $\xi \mapsto \mathcal{A}[\xi,\xi] - \frac{c_2}{2}|\xi|^2$ *is quasiconvex.*

Let $x_0 \in \partial\Omega$ and $h \in \mathrm{W}^{1,p}(\Omega(x_0,R),\mathbb{R}^N)$ an $\mathcal{A}$-harmonic function such that $h = 0$ on $B(x_0,R)\cap\partial\Omega$. Then, there is a constant $c = c(n,L,c_2) > 0$ such that, for every $r \in (0,R)$,

$$\fint_{\Omega(x_0,r)} |V(\nabla h - (\nabla h)_{x_0,r})|^2 \,\mathrm{d}x \leq c\left(\frac{r}{R}\right)^2 \fint_{\Omega(x_0,R)} |V(\nabla h - (\nabla h)_{x_0,R})|^2 \,\mathrm{d}x.$$

On the other hand, if $B(x_0,R) \Subset \Omega$, and $h \in \mathrm{W}^{1,p}(\Omega(x_0,R),\mathbb{R}^N)$ is $\mathcal{A}$-harmonic, then for *every* $r \in (0, R)$ *we have that*

$$\fint_{\Omega(x_0,r)} |V(\nabla h - (\nabla h)_{x_0,r})|^2 \, \mathrm{d}x \leq c \left(\frac{r}{R}\right)^2 \fint_{\Omega(x_0,R)} |V(\nabla h - (\nabla h)_{x_0,R})|^2 \, \mathrm{d}x.$$

The proof of this classical result relies on the difference quotient method. See [Gro02] and the references therein.

We also state the following technical but useful iteration theorem.

Theorem 73 (Iteration Theorem) *Let* $1 < p < \infty$*,* $R > 0$ *and* $f\colon \left[\frac{R}{2}, R\right] \to [0, \infty)$ *be a bounded function. Assume that, for some constants* $\theta \in (0,1)$ *and* $A, B \geq 0$*, we have that*

$$f(r) \leq \theta f(s) + \frac{A}{(s-r)^p} + B$$

for every $\frac{R}{2} \leq r < s \leq R$*. Then, there exists a constant* $0 < c = c(\theta)$ *such that*

$$f\left(\frac{R}{2}\right) \leq c\left(\frac{A}{R^p} + B\right).$$

See [Giu03, Lemma 6.1] for a proof of this result.

4.2 The case of global minimizers

In this section we present the main result of this chapter. We establish full regularity up to the boundary for minimizers of integral functionals such that the integrand satisfies the *natural* assumptions (H0)-(H2). The proof of the main theorem is direct and, while it incorporates many of the techniques developed in regularity theory over the last fifty years, the underlying idea behind it, based on Gehring's Lemma, is inspired by the regularity proofs of Giaquinta & Giusti [GG78] and Giaquinta & Modica. [GM79b]. See [Giu03, Section 9.6] for more details about the history of this particular approach to the problem of regularity.

We first establish the following elementary lemma, which is essentially a consequence of the strong quasiconvexity assumption.

Lemma 74 *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *be a continuous function satisfying* (H1) *and* (H2)*. Let* $\omega \subseteq \mathbb{R}^n$ *be a bounded domain and* $g \in \mathrm{W}^{1,p}(\omega, \mathbb{R}^N)$*. Assume that* $\overline{u} \in \mathrm{W}^{1,p}_g(\omega, \mathbb{R}^n)$ *is a minimizer of* $(\mathfrak{F}, \omega)$*. Then, there is a constant* $c > 0$*, depending only on* c_1, c_2 *and* p*, such* *that*

$$\int_\omega |V(\nabla u)|^2 \,\mathrm{d}x \le c \int_\omega |V(\nabla g)|^2 \,\mathrm{d}x.$$

Proof. Let $\tilde{F} \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be defined by

$$\tilde{F}(z) := F(z) - F(0) - \left\langle F'(0), z \right\rangle.$$

It is easy to verify that $\overline{u}$ is also a minimizer for the integrand $\tilde{F}$. Then, since $\overline{u} - g \in \mathrm{W}_0^{1,p}(\omega, \mathbb{R}^N)$, by (H1), (H2) and Lemma 13 applied with $z_0 := 0$, we have that

$$\begin{aligned} c_2 \int_\omega |V(\nabla\overline{u} - \nabla g)|^2 \,\mathrm{d}x &\le \int_\omega \left(F(\nabla\overline{u} - \nabla g) - F(0) - \left\langle F'(0), \nabla\overline{u} - \nabla g \right\rangle \right) \mathrm{d}x \\ &= \int_\omega \left(\tilde{F}(\nabla\overline{u} - \nabla g) - \tilde{F}(\nabla\overline{u}) \right) \mathrm{d}x + \int_\omega \tilde{F}(\nabla\overline{u}) \,\mathrm{d}x \\ &= - \int_\omega \int_0^1 \left\langle \tilde{F}'(\nabla\overline{u} - t\nabla g), \nabla g \right\rangle \mathrm{d}x + \int_\omega \tilde{F}(\nabla\overline{u}) \,\mathrm{d}x \\ &\le c \int_\omega \left(|V_{p-1}(\nabla\overline{u})| + |V_{p-1}(\nabla g)| \right) |\nabla g| \,\mathrm{d}x + \int_\omega \tilde{F}(\nabla g) \,\mathrm{d}x. \end{aligned}$$

The last inequality above follows from Lemma 13 and the minimality of $\overline{u}$. We now use Young's inequality from Lemma 128 (viii) and the fact that $\tilde{F} \le c\,|V|^2$ for some $0 < c = c(p)$, to deduce from above that, for any $\varepsilon' > 0$ there is a constant $c_{\varepsilon'} > 0$ such that

$$c_2 \int_\omega |V(\nabla\overline{u} - \nabla g)|^2 \,\mathrm{d}x \le \varepsilon' \int_\omega |V(\nabla\overline{u})|^2 \,\mathrm{d}x + c_{\varepsilon'} \int_\omega |V(\nabla g)|^2 \,\mathrm{d}x + c \int_\omega |V(\nabla g)|^2 \,\mathrm{d}x.$$

It then follows from triangle inequality and the subadditivity of V that, for some constants $c = c(p)$, $\tilde{c} = \tilde{c}(p)$,

$$\tilde{c} \int_\omega |V(\nabla\overline{u})|^2 \,\mathrm{d}x \le \varepsilon' \int_\omega |V(\nabla\overline{u})|^2 \,\mathrm{d}x + c_{\varepsilon'} \int_\omega |V(\nabla g)|^2 \,\mathrm{d}x + c \int_\omega |V(\nabla g)|^2 \,\mathrm{d}x. \tag{4.3}$$

By taking $\varepsilon' := \frac{\tilde{c}}{2}$ and subtracting the appropriate term from both sides above, we conclude that, for some $c > 0$,

$$\int_\omega |V(\nabla\overline{u})|^2 \,\mathrm{d}x \le c \int_\omega |V(\nabla g)|^2 \,\mathrm{d}x,$$

as required. □

The next standard lemma concerns the quasiminimality that the average of a function, over a given domain, satisfies with respect to the average distance of the function itself and any other arbitrary vector. A similar property is satisfied if we consider not the average of the function, but the average of it taken in a particular given direction of the plane.

Lemma 75 *Let $f\colon \Omega \to \mathbb{R}^m$ be such that $f \in \mathrm{L}^p(\Omega, \mathbb{R}^{N\times n})$. Then, there is a constant $c > 0$, depending only on p, such that for any $\omega \subseteq \Omega$ and every $\xi \in \mathbb{R}^{N\times n}$,*

$$\int_\omega |V(f-(f)_\omega)|^2 \,\mathrm{d}x \le c\int_\omega |V(f-\xi)|^2 \,\mathrm{d}x.$$

In addition, for every $\eta \in \mathbb{R}^N$ and every $\nu_0 \in \mathbb{R}^n$ such that $|\nu_0| = 1$,

$$\int_\omega |V(f-(f\cdot\nu_0)_\omega \otimes \nu_0)|^2 \,\mathrm{d}x \le c\int_\omega |V(f-\eta\otimes\nu_0)|^2 \,\mathrm{d}x.$$

Proof. The proof of these classical inequalities relies mainly on triangle inequality and the subadditivity properties of the function V. We adapt the ideas from [Sch08] to prove the second part of the Lemma, given that the first part is analogous and is found more often in the literature. See also (16)-(18) in [Kro05].

Let $\nu_0 \in \mathbb{S}^{n-1}$ and $\eta \in \mathbb{R}^N$ arbitrary. Then,

$$\fint_\omega |V(f-(f\cdot\nu_0)_\omega\otimes\nu_0)|^2 \,\mathrm{d}x \le c\fint_\omega |V(f-\eta\otimes\nu_0)|^2 \,\mathrm{d}x + c\fint_\omega |V(((f\cdot\nu_0)_\omega-\eta)\otimes\nu_0)|^2 \,\mathrm{d}x.$$

Observe that the second term in the right hand side of this inequality is the mean integral of a constant vector. In addition, we know that, for some constant $c > 0$, $c^{-1}|W| \le |V| \le c|W|$, where W is the convex function defined in Appendix E. Therefore, recalling that $|a\otimes b| = |a||b|$ and applying Jensen's inequality, we have that

$$\begin{aligned} |V(((f\cdot\nu_0)_\omega-\eta)\otimes\nu_0)|^2 &\le c\,|V(((f\cdot\nu_0-\eta))_\omega)|^2 \\ &\le c\,|W(((f\cdot\nu_0-\eta))_\omega)|^2 \\ &\le c\fint_\omega |W(f\cdot\nu_0-\eta)|^2 \,\mathrm{d}x. \end{aligned}$$

On the other hand, since $|\nu_0| = 1$, we have that $\eta = (\nu_0\cdot\nu_0)\eta = (\eta\otimes\nu_0)\cdot\nu_0$. Therefore, by

Cauchy-Schwarz and the equivalence between V and W we have that

$$\begin{aligned}\fint_\omega |W(f\cdot\nu_0-\eta)|^2\,\mathrm{d}x =& \fint_\omega |W(f\cdot\nu_0-(\eta\otimes\nu_0)\cdot\nu_0)|^2\,\mathrm{d}x\\ \leq& c\fint_\omega |W(f-(\eta\otimes\nu_0))|^2\,\mathrm{d}x\\ \leq& c\fint_\omega |V(f-(\eta\otimes\nu_0))|^2\,\mathrm{d}x.\end{aligned}$$

By bringing together the three chains of inequalities above, we obtain the desired result. □

We are now ready to state the main regularity theorem.

Theorem 76 *Let $\alpha\in(0,1)$ and let $\Omega\subseteq\mathbb{R}^n$ be a bounded domain of class $C^{1,\alpha}$.*[2] *Suppose that* (H0) − (H2) *hold for some $p\in[2,\infty)$. Then, for every $m>0$ there exists $\varepsilon=\varepsilon(m)>0$ such that, whenever $g\in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^{N\times n})$ satisfies*

$$\|\nabla g\|_{0,\alpha}<m \quad \text{and} \quad \int_\Omega |V(\nabla g)|^2\,\mathrm{d}x<\varepsilon, \tag{4.4}$$

if $\overline{u}\in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ is a global minimizer of $\mathfrak{F}$ over the Dirichlet class $\mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N)$, then $\overline{u}\in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$ and $[\nabla\overline{u}]_{0,\alpha}\leq C$ for some constant $C=C(m,n,N,p,\Omega)>0$.

Proof. Let $\vartheta\in C^{1,\alpha}(\mathbb{R}^n)$ be a defining function for Ω and let $R_\Omega>0$ be as in Lemma 22. Take $m>0$ and fix $x_0\in\overline{\Omega}$ arbitrary. The final goal of this proof will be to find $\varepsilon>0$ such that, if (4.4) is satisfied, then there exist $0<C=C(m,n,N,p,c_1,c_2,R_\Omega)$ and $0<R=R(m,n,N,p,c_1,c_2,R_\Omega)$ such that, for every $r\in(0,R)$,

$$\tilde{E}(x_0,r):=\fint_{\Omega(x_0,r)} \left|V\left(\nabla\overline{u}_0-(\nabla\overline{u}_0)_{x_0,r}\right)\right|^2\mathrm{d}x\leq Cr^\alpha,$$

where the function $\overline{u}_0$ is defined by

$$\overline{u}_0:=\overline{u}-g. \tag{4.5}$$

The result will then follow from Campanato's characterization of Hölder continuity and the fact that, in particular $g\in C^{1,\frac{\alpha}{2}}(\overline{\Omega},\mathbb{R}^N)$. We will then be able to use Schauder estimates to

[2] Observe that we don't include the case $\alpha=1$. See (4.88).

conclude that, in fact, $\overline{u} \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$. We now proceed with the proof, that we split into five main steps.

Step 1. Preliminary higher integrability. We begin by establishing a higher integrability result for $\overline{u}_0$. The purpose of this first step is to show that, for some fixed $0 < R < R_\Omega$, some exponent $q > p$ and a constant $c > 0$, the last two depending on m, n, p and Ω, the following reverse Hölder inequality is satisfied.

$$\fint_{\Omega\left(x_0, \frac{R}{2}\right)} |\nabla \overline{u}_0|^q \,\mathrm{d}x \leq c \left(\fint_{\Omega(x_0,R)} |\nabla \overline{u}_0|^p \,\mathrm{d}x \right)^{\frac{q}{p}} + c \fint_{\Omega(x_0,R)} (1 + |\nabla g|^q) \,\mathrm{d}x. \tag{4.6}$$

The proof of this will rely on first establishing a Caccioppoli inequality and then using Gehring's Lemma. With this aim, we take $0 < R_* < R < R_\Omega$, $\frac{R_*}{2} < r < s < R_*$ and $B(y, R_*) \subseteq B(x_0, R)$ arbitrary. Let $\rho \colon \mathbb{R}^n \to [0,1]$ be a Lipschitz cut-off function so that $\mathbb{1}_{B(y,r)} \leq \rho \leq \mathbb{1}_{B(y,s)}$ and $|\nabla \rho| \leq \frac{1}{s-r}$. We now consider the following three cases, the reason being that we will need to use Poincaré-Sobolev's inequality to obtain higher integrability and we shall require different versions of it depending on whether $B(y, R_*)$ intersects $\partial\Omega$ or not. We remark that this proof covers both cases $x_0 \in \Omega$ and $x_0 \in \partial\Omega$. The overall strategy of the derivation follows Evans' proof of the Caccioppoli inequality in [Eva86].

Case 1. $B(y, R_*) \cap (\mathbb{R}^n \setminus \overline{\Omega}) \neq \emptyset$ and $B(y, R_*) \cap \overline{\Omega} \neq \emptyset$. Notice that, in this case, $\mathcal{H}^{n-1}(\partial\Omega \cap B(y, R_*)) > 0$. Define

$$\varphi := \rho \overline{u}_0; \qquad \psi := \overline{u} - \varphi = (1 - \rho)\overline{u}_0 + g.$$

Then, $\varphi \in W_0^{1,p}(\Omega(x_0,s),\mathbb{R}^N)$ and, by (H1) − (H2) and the minimality of $\overline{u}$,

$$\begin{aligned}
c_2 \int_{\Omega(y,s)} |\nabla\varphi|^p \,\mathrm{d}x \leq & \int_{\Omega(y,s)} (F(\nabla\varphi) - F(0)) \,\mathrm{d}x \\
= & \int_{\Omega(y,s)} F(\nabla\overline{u}) \,\mathrm{d}x + \int_{\Omega(y,s)} F(\nabla\varphi) - F(\nabla\overline{u}) \,\mathrm{d}x - \int_{\Omega(y,s)} F(0) \,\mathrm{d}x \\
\leq & \int_{\Omega(y,s)} F(\nabla\psi) \,\mathrm{d}x + c \int_{\Omega(y,s)} \left(1 + |\nabla\varphi|^{p-1} + |\nabla\overline{u}|^{p-1}\right) |\nabla\psi| \,\mathrm{d}x - \int_{\Omega(y,s)} F(0) \,\mathrm{d}x \\
\leq & \, c_1 \int_{\Omega(y,s)} (1 + |\nabla\psi|^p) \,\mathrm{d}x + \delta \int_{\Omega(y,s)} |\nabla\varphi|^p \,\mathrm{d}x + c_\delta \int_{\Omega(y,s)} |\nabla\psi|^p \,\mathrm{d}x \\
& + \delta \int_{\Omega(y,s)} |\nabla\overline{u}_0|^p \,\mathrm{d}x + c_\delta \int_{\Omega(y,s)} |\nabla\psi|^p \,\mathrm{d}x + c \int_{\Omega(y,s)} |\nabla g|^p \,\mathrm{d}x. \qquad (4.7)
\end{aligned}$$

The last inequality above follows from Young's and triangle inequalities for any $\delta > 0$ and its corresponding $c_\delta > 0$. By taking $\delta := \frac{c_2}{4}$ and subtracting $\delta \int_{\Omega(y,s)} |\nabla\varphi|^p \,\mathrm{d}x$ from both sides, we can deduce, using that $\varphi = \overline{u}_0$ on $\Omega(y,r)$, that

$$\begin{aligned}
\frac{3c_2}{4} \int_{\Omega(y,r)} |\nabla\overline{u}_0|^p \,\mathrm{d}x \leq & \, c \int_{\Omega(y,s)} (1 + |\nabla\psi|^p) \,\mathrm{d}x + c_\delta \int_{\Omega(y,s)} |\nabla\psi|^p \,\mathrm{d}x \\
& + \delta \int_{\Omega(y,s)} |\nabla\overline{u}_0|^p \,\mathrm{d}x + c_\delta \int_{\Omega(y,s)} |\nabla\psi|^p \,\mathrm{d}x + c \int_{\Omega(y,s)} |\nabla g|^p \,\mathrm{d}x. \qquad (4.8)
\end{aligned}$$

After subtracting $\delta \int_{\Omega(x_0,r)} |\nabla\overline{u}_0|^p \,\mathrm{d}x$ from both sides, and using that

$$|\nabla\psi| \leq |\nabla g| + (1-\rho)|\nabla\overline{u}_0| + \frac{|\overline{u}_0|}{s-r}$$

to estimate the right hand side of (4.8), we conclude that, for some constant $c > 0$,

$$\begin{aligned}
\int_{\Omega(y,r)} |\nabla\overline{u}_0|^p \,\mathrm{d}x \leq & \, c \int_{\Omega(y,s)-\Omega(y,r)} |\nabla\overline{u}_0|^p \,\mathrm{d}x + c \int_{\Omega(y,s)} \frac{|\overline{u}_0|^p}{(s-r)^p} \,\mathrm{d}x \\
& + c \int_{\Omega(y,s)} (1 + |\nabla g|^p) \,\mathrm{d}x.
\end{aligned}$$

We now use Widman's hole filling trick to conclude that, for some $\theta_1 \in (0,1)$,

$$\begin{aligned}\int_{\Omega(y,r)} |\nabla \overline{u}_0|^p \,\mathrm{d}x \leq \theta_1 \int_{\Omega(y,s)} |\nabla \overline{u}_0|^p \,\mathrm{d}x + \theta_1 \int_{\Omega(y,s)} \frac{|\overline{u}_0|^p}{(s-r)^p} \,\mathrm{d}x \\ \theta_1 \int_{\Omega(y,s)} \left(1 + |\nabla g|^p\right) \mathrm{d}x.\end{aligned} \tag{4.9}$$

Hence, by Theorem 73 there is a constant $c > 0$, depending only on θ_1, such that

$$\int_{\Omega\left(y,\frac{R_*}{2}\right)} |\nabla \overline{u}_0|^p \,\mathrm{d}x \leq c \int_{\Omega(y,R_*)} \frac{|\overline{u}_0|^p}{R_*^p} \,\mathrm{d}x + c \int_{\Omega(y,R_*)} \left(1 + |\nabla g|^p\right) \mathrm{d}x.$$

This is a Caccioppoli inequality of the second kind. In addition, since $R < R_\Omega$, there is a $c_\Omega > 0$ such that, for every $0 < r < R$ and for every $x \in \overline{\Omega}$,

$$c_\Omega \, r^n \, \omega_n \leq |\Omega(x,r)| \leq r^n \, \omega_n. \tag{4.10}$$

Therefore, we can divide by R_*^n in the previous Caccioppoli inequality and, after adjusting the constants, obtain

$$\begin{aligned}\fint_{\Omega\left(y,\frac{R_*}{2}\right)} |\nabla \overline{u}_0|^p \,\mathrm{d}x &\leq c \fint_{\Omega(y,R_*)} \frac{|\overline{u}_0|^p}{R_*^p} \,\mathrm{d}x + c \fint_{\Omega(y,R_*)} \left(1 + |\nabla g|^p\right) \mathrm{d}x \\ &\leq c \left(\fint_{\Omega(y,R_*)} |\nabla \overline{u}_0|^{\frac{np}{n+p}} \,\mathrm{d}x \right)^{\frac{n+p}{n}} + c \fint_{\Omega(y,R_*)} \left(1 + |\nabla g|^p\right) \mathrm{d}x,\end{aligned} \tag{4.11}$$

where the second inequality above follows from Poincaré-Sobolev's inequality as stated in Theorem 111.

Case 2. $B(y,R_*) \subseteq \overline{\Omega}$. In this case, we obtain a Caccioppoli inequality by subtracting from $\overline{u}_0$ a constant function as follows. Let $\tilde{u} := \overline{u}_0 - a$, where $a \in \mathbb{R}^N$ is chosen to be

$$a := \fint_{\Omega(y,R_*)} \overline{u}_0 \,\mathrm{d}x.$$

Let

$$\varphi := \rho\tilde{u}; \qquad \psi := \overline{u} - \varphi = (1-\rho)(\overline{u}_0 - a) + g + a. \tag{4.12}$$

Then, in the same way in which we obtained (4.7), we get that

$$\begin{aligned} c_2 \int\limits_{\Omega(y,s)} |\nabla\varphi|^p \,\mathrm{d}x \le c \int\limits_{\Omega(y,s)} (1+|\nabla\psi|^p)\,\mathrm{d}x + \delta \int\limits_{\Omega(y,s)} |\nabla\varphi|^p \,\mathrm{d}x + c_\delta \int\limits_{\Omega(y,s)} |\nabla\psi|^p \,\mathrm{d}x \\ + \delta \int\limits_{\Omega(y,s)} |\nabla\overline{u}_0|^p \,\mathrm{d}x + c_\delta \int\limits_{\Omega(y,s)} |\nabla\psi|^p \,\mathrm{d}x + c \int\limits_{\Omega(y,s)} |\nabla g|^p \,\mathrm{d}x. \end{aligned}$$

By using that $\psi = g + a$ on $B(y,r)$, we can also estimate, after taking $\delta > 0$ small enough and using Widman's hole-filling trick as in (4.9), that for $\theta_1 \in (0,1)$, which can be taken to be the same as in Case 1,

$$\int\limits_{\Omega(y,r)} |\nabla\overline{u}_0|^p \,\mathrm{d}x \le \theta_1 \int\limits_{\Omega(y,s)} |\nabla\overline{u}_0|^p \,\mathrm{d}x + \theta_1 \int\limits_{\Omega(y,s)} \frac{|\overline{u}_0 - a|^p}{(s-r)^p} \,\mathrm{d}x + \theta_1 \int\limits_{\Omega(y,s)} (1+|\nabla g|^p)\,\mathrm{d}x.$$

We now apply Theorem 73 and use again that $R_* < R_\Omega$ to take averages. We then obtain that, for some constant $c > 0$,

$$\begin{aligned} \fint\limits_{\Omega(y,\frac{R_*}{2})} |\nabla\overline{u}_0|^p \,\mathrm{d}x &\le c \fint\limits_{\Omega(y,R_*)} \frac{|(\overline{u}-g)-a|^p}{R_*^p} \,\mathrm{d}x + c \fint\limits_{\Omega(y,R_*)} (1+|\nabla g|^p)\,\mathrm{d}x \\ &\le c \left(\fint\limits_{\Omega(y,R_*)} |\nabla\overline{u}_0|^{\frac{np}{n+p}} \,\mathrm{d}x \right)^{\frac{n+p}{n}} + c \fint\limits_{\Omega(y,R_*)} (1+|\nabla g|^p)\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Here, the last inequality follows from the standard version of Poincaré-Sobolev's inequality for the interior and the convenient choice that we made for a.

Case 3. $B(y,R_*) \subseteq \mathbb{R}^n - \overline{\Omega}$. In this case, we just observe that the following reverse Hölder inequality trivially holds.

$$\fint\limits_{C(y,\frac{R_*}{2})} |\nabla\overline{u}_0|^p \mathbb{1}_\Omega \,\mathrm{d}x \le c \left(\fint\limits_{B(y,R_*)} |\nabla\overline{u}_0|^{\frac{np}{n+p}} \mathbb{1}_\Omega \,\mathrm{d}x \right)^{\frac{n+p}{n}} + c \fint\limits_{B(y,R_*)} (1+|\nabla g|^p)\mathbb{1}_\Omega \,\mathrm{d}x. \tag{4.14}$$

We are now in a position to apply Gehring's Lemma, thanks to which we know that the higher integrability obtained above can be improved.

We take $f := |\nabla \overline{u}_0|^{\frac{np}{n+p}} \cdot \mathbb{1}_\Omega$ and $h := (c(1+|\nabla g|^p))^{\frac{n}{n+p}} \cdot \mathbb{1}_\Omega$. Using again that $R_* < R_\Omega$, we know from (4.11)-(4.14) that, for every $B(y,R_*) \subseteq B(x_0,R)$,

$$\fint_{B\left(y,\frac{R_*}{2}\right)} f^{\frac{n+p}{n}}\,\mathrm{d}x \le c\left(\fint_{B(y,R_*)} f\,\mathrm{d}x\right)^{\frac{n+p}{n}} + \fint_{B(y,R_*)} h^{\frac{n+p}{n}}\,\mathrm{d}x. \tag{4.15}$$

Whereby, Gehring's Lemma implies that there are $q_0 > p$ with $q_0 = q_0(m,n,p,c_1,c_2)$ and $0 < c = c(m,n,p,c_1,c_2)$, such that for every $q \in [p,q_0]$,

$$\fint_{B\left(x_0,\frac{R}{2}\right)} |\nabla \overline{u}_0|^q \cdot \mathbb{1}_\Omega\,\mathrm{d}x \le c\left(\fint_{B(x_0,R)} |\nabla \overline{u}_0|^p \cdot \mathbb{1}_\Omega\,\mathrm{d}x\right)^{\frac{q}{p}} + c\fint_{B(x_0,R)} (1+|\nabla g|^q)\cdot \mathbb{1}_\Omega\,\mathrm{d}x,$$

which in turn implies (4.6), as we wanted to prove.

Step 2.1. Reverse Hölder inequality for balls near the boundary. We now fix $0 < R_1 < R_\Omega$. Assume that $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times n}$ are such that $|\zeta|, |\xi| \le m$. The purpose of this step is to show that there is a constant $c = c(m,n,p) > 0$ such that, for every $x_0 \in \overline{\Omega}$ and every $B(y,R_0) \subseteq B(x_0,R_1)$, it holds that

$$\begin{aligned}
&\fint_{\Omega\left(y,\frac{R_0}{2}\right)} |V(\nabla \overline{u} - \xi - \zeta \otimes \nu_{x_0})|^2\,\mathrm{d}x \\
&\le c\left(\fint_{\Omega(y,R_0)} \left|V\left(\nabla \overline{u} - \xi - \zeta\otimes \nu_{x_0}\right)\right|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}} \\
&\quad + c\fint_{\Omega(y,R_0)} \left(R_1^\alpha\left(1+|V_{p-1}(\nabla \overline{u})|\right) + |V(\nabla g - \xi)|^2\right)\mathrm{d}x \\
&\quad + c\fint_{\Omega(y,R_0)} |V_{p-1}(\nabla \overline{u})||\nabla g - \xi|\,\mathrm{d}x.
\end{aligned} \tag{4.16}$$

Since our ultimate purpose is to prove regularity up to the boundary, as well as to make use of the smallness of the boundary condition, we need to obtain higher integrability on balls intersecting $\partial\Omega$ and, hence, we must obtain a Caccioppoli inequality for this kind of balls too.

In order to execute this essential key step, we will define an auxiliary minimization problem associated to an integrand that we define below. For this purpose, let $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times n}$ be constant vectors such that $|\xi|, |\zeta| \leq m$ and take $x_0 \in \overline{\Omega}$. Assume, first, that $x_0 \in \partial\Omega$. Let

$$w_0 := \xi + \zeta \otimes \nu_{x_0} \in \mathbb{R}^{N\times n} \tag{4.17}$$

and define $F_{w_0} \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ by

$$\begin{aligned} F_{w_0}(z) :=& F(z + w_0) - F(w_0) - \langle F'(w_0), z\rangle \\ =& \int_0^1 (1-t)F''(tz + w_0)[z, z]\, \mathrm{d}t. \end{aligned}$$

We can then see that F_{w_0} has the following properties:

(H0′) $F_{w_0} \in C^2(\mathbb{R}^{N\times n})$;

(H1′) exists $k_1 = k_1(c_1, m) > 0$ such that, for every $z \in \mathbb{R}^{N\times n}$, $|F_{w_0}(z)| \leq k_1|V(z)|^2$ and

(H2′) F_{w_0} is strongly quasiconvex.

In order to obtain a Caccioppoli inequality, let $0 < R_1 < R_\Omega$ fixed.[3] Take $B(y, R_0) \subseteq B(x_0, R_1)$ and $\frac{R_0}{2} < r < s < R_0$. In addition, let $\rho \colon \mathbb{R}^n \to [0, 1]$ be a Lipschitz cut-off function so that $\mathbb{1}_{B(y,r)} \leq \rho \leq \mathbb{1}_{B(y,s)}$ and $|\nabla\rho| \leq \frac{1}{s-r}$.

Given that $\partial\Omega$ is not flat, we cannot find an affine function a such that $\rho(\overline{u} - a)$ works as a test function on subdomains of the form $B(y, R_0) \cap \Omega$. In addition, we don't want to apply a transformation that enables us to assume that $\partial\Omega$ is flat, since that would complicate the way in which we use the smallness assumption on the function g. This causes technical problems that we overcome by making use of the defining function of $\partial\Omega$. This idea was inspired by Kronz' work on boundary regularity.[4]

On the other hand, once we have a Caccioppoli inequality on subdomains of Ω of the form $\Omega(y, R_0)$, we will apply Poincaré-Sobolev inequality in order to finally obtain a reverse Hölder

[3] We require $R_1 < R_\Omega$ to ensure that we can assume $c_\Omega R^n\omega_n \leq |\Omega(y, R)| \leq R^n\omega_n$ for any $0 < R < R_1$.

[4] See [Kro05].

inequality and then higher integrability, by Gehring's Lemma. To do this, we will require a separate treatment depending on whether $B(y, R_0)$ intersects $\partial\Omega$ or not, since only different versions on Poincaré-Sobolev inequality can be used in each case. With this motivation, we now consider the following three possible situations.

Case 1. $B(y, R_0) \cap (\mathbb{R}^n - \overline{\Omega}) \neq \emptyset$ *and* $B(y, R_0) \cap \Omega \neq \emptyset$. We remark that, in this case, $\mathcal{H}^{n-1}(B(y, R_0) \cap \partial\Omega) > 0$.

We now define the function

$$\tilde{u}(x) := \overline{u} - w_0 \cdot x. \tag{4.18}$$

Then, $\tilde{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$. Define also

$$\varphi := \rho(\overline{u} - g - \vartheta \cdot \zeta); \qquad \psi := \tilde{u} - \varphi = (1 - \rho)(\overline{u} - g - \vartheta \cdot \zeta) + g + \vartheta \cdot \zeta - w_0 \cdot x. \tag{4.19}$$

Observe that $\varphi \in \mathrm{W}_0^{1,p}(\Omega(y, s), \mathbb{R}^N)$ is an admissible test function. Hence, by Lemma 13 we have, for some $0 < c = c(c_1, m, p)$,

$$\begin{aligned} c_2 \int_{\Omega(y,s)} |V(\nabla\varphi)|^2 \,\mathrm{d}x &\leq \int_{\Omega(y,s)} (F_{w_0}(\nabla\varphi) - F_{w_0}(0)) \,\mathrm{d}x \\ &= \int_{\Omega(y,s)} F_{w_0}(\nabla\tilde{u}) \,\mathrm{d}x + \int_{\Omega(y,s)} (F_{w_0}(\nabla\tilde{u} - \nabla\psi) - F_{w_0}(\nabla\tilde{u})) \,\mathrm{d}x \\ &= \int_{\Omega(y,s)} F_{w_0}(\nabla\tilde{u}) \,\mathrm{d}x - \int_{\Omega(y,s)} \int_0^1 \left\langle F'_{w_0}(\nabla\tilde{u} - t\nabla\psi), \nabla\psi \right\rangle \,\mathrm{d}t \,\mathrm{d}x \\ &\leq \int_{\Omega(y,s)} F_{w_0}(\nabla\tilde{u}) \,\mathrm{d}x + c \int_{\Omega(y,s)} (|V_{p-1}(\nabla\tilde{u})| + |V_{p-1}(\nabla\psi)|) \,|\nabla\psi| \,\mathrm{d}x. \end{aligned} \tag{4.20}$$

Notice that, since $\rho = 1$ on $B(y, r)$, on $\Omega(y, r)$ from the definition of w_0 we have

$$\nabla\varphi = \nabla\overline{u} - \nabla g - \zeta \otimes \nu; \quad \nabla\psi = \nabla g - \xi + \zeta \otimes (\nu - \nu_{x_0}).$$

This, together with (4.20) and Lemma 13, implies that

$$\begin{aligned}
\int_{\Omega(y,r)} |V(\nabla\overline{u} - \nabla g - \zeta\otimes\nu)|^2\,\mathrm{d}x \le c \int_{\Omega(y,s)} & F_{w_0}(\nabla\tilde{u})\,\mathrm{d}x \\
& + c \int_{\Omega(y,r)} |V_{p-1}(\nabla\tilde{u})|\,|\zeta\otimes(\nu-\nu_{x_0})|\,\mathrm{d}x \\
& + c \int_{\Omega(y,r)} |V_{p-1}(\nabla\tilde{u})|\,|\nabla g - \xi|\,\mathrm{d}x \\
& + c \int_{\Omega(y,r)} |V(\zeta\otimes(\nu-\nu_{x_0}))|^2\,\mathrm{d}x \\
& + c \int_{\Omega(y,r)} |V(\nabla g - \xi)|^2\,\mathrm{d}x \\
& + c \int_{\Omega(y,s)-\Omega(y,r)} \left(|V(\nabla\tilde{u})|^2 + |V(\nabla\psi)|^2\right)\,\mathrm{d}x \\
=:\, & I + II + III + IV + V + VI,
\end{aligned} \tag{4.21}$$

with the obvious labelling. We now estimate each term separately. We begin by observing that, since ν is $C^{0,\alpha}$, the fact that $|\zeta| \le m$ and $R_0 < R_1 < R_\Omega$ implies that there is a constant $c = c(m,p)$ such that

$$II \le c \int_{\Omega(y,r)} |V_{p-1}(\nabla\tilde{u})||x - x_0|^\alpha\,\mathrm{d}x \le c\,R_1^\alpha \int_{\Omega(y,r)} |V_{p-1}(\nabla\tilde{u})|\,\mathrm{d}x \tag{4.22}$$

and

$$IV \le c \int_{\Omega(y,r)} V(|x-x_0|^\alpha)^2\,\mathrm{d}x \le c \int_{\Omega(y,r)} V(R_1^\alpha)^2\,\mathrm{d}x. \tag{4.23}$$

To estimate VI, we begin by observing that, by triangle inequality,

$$|\nabla\psi| \le |\nabla\overline{u} - \nabla g - \zeta\otimes\nu| + \frac{|\overline{u} - g - \vartheta\cdot\zeta|}{s-r} + |\nu - \nu_{x_0}|\,|\zeta| + |\nabla g - \xi|.$$

Using this, and that

$$|\nabla\tilde{u}| \le |\nabla\overline{u} - \nabla g - \zeta\otimes\nu| + m|\nu - \nu_{x_0}| + |\nabla g - \xi|,$$

we obtain that

$$\begin{aligned}
VI \leq & c \int_{\Omega(y,s)-\Omega(y,r)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x + c \int_{\Omega(y,s)-\Omega(y,r)} \left| V\left(\frac{\overline{u} - g - \vartheta \cdot \zeta}{s-r}\right)\right|^2 \mathrm{d}x \\
& + c \int_{\Omega(y,s)-\Omega(y,r)} |V(\nabla g - \xi)|^2 \,\mathrm{d}x + c \int_{\Omega(y,s)-\Omega(y,r)} |V(\nu - \nu_{x_0})|^2 \,\mathrm{d}x \\
\leq & c \int_{\Omega(y,s)-\Omega(y,r)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x + c \cdot \int_{\Omega(y,s)-\Omega(y,r)} \left| V\left(\frac{\overline{u} - g - \vartheta \cdot \zeta}{s-r}\right)\right|^2 \mathrm{d}x \\
& + c \int_{\Omega(y,s)-\Omega(y,r)} \left(|V(\nabla g - \xi)|^2 + |V(R_1^\alpha)|^2\right) \mathrm{d}x,
\end{aligned} \tag{4.24}$$

We will now estimate I. It is in this step that we will use the minimality of $\overline{u}$.

Assuming that $\frac{R_0}{2} < r < s < R_0$ are as before, and using that $\psi = \tilde{u}$ on $\partial\Omega(y,s)$, we obtain

$$\begin{aligned}
I = & \int_{\Omega(y,s)} \left(F(\nabla\overline{u}) - F(w_0) - \left\langle F'(w_0), \nabla\tilde{u}\right\rangle\right) \mathrm{d}x \\
\leq & \int_{\Omega(y,s)} F(\nabla\psi + w_0) - F(w_0) - \left\langle F'(w_0), \nabla\psi\right\rangle \mathrm{d}x \\
= & \int_{\Omega(y,s)} F_{w_0}(\nabla\psi) \,\mathrm{d}x.
\end{aligned} \tag{4.25}$$

Using Lemma 13 leads to conclude that

$$I \leq c(m) \int_{\Omega(y,s)} |V(\nabla\psi)|^2 \,\mathrm{d}x. \tag{4.26}$$

Hence, following the same ideas that we used to derive (4.24), we obtain that

$$\begin{aligned}
I \leq & c \int_{\Omega(y,s)} |V(\nabla\psi)|^2 \,\mathrm{d}x \leq c \int_{\Omega(y,s)-\Omega(y,r)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x \\
& + c \int_{\Omega(y,s)-\Omega(y,r)} \left| V\left(\frac{\overline{u} - g - \vartheta \cdot \zeta}{s-r}\right)\right|^2 \mathrm{d}x + c \int_{\Omega(y,s)} \left(|V(\nabla g - \xi)| + |V(R_1^\alpha)|^2\right) \mathrm{d}x
\end{aligned} \tag{4.27}$$

for every $\frac{R_0}{2} < r < s < R_0$.

Considering that $0 < R_1 < R_\Omega$ and Lemma 128, we can simplify the above expression by noticing that, for a different $c = c(p, R_\Omega)$, $V(R_1^\alpha)^2 \le c\, R_1^\alpha$. From this observation, inequalities (4.21) -(4.24) and (4.27), we infer that

$$\begin{aligned}
&\int_{\Omega(y,r)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x \\
&\le c \int_{\Omega(y,s)-\Omega(y,r)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x + c \int_{\Omega(y,s)} \left| V\left(\frac{\overline{u} - g - \vartheta\cdot\zeta}{s-r}\right)\right|^2 \mathrm{d}x \\
&\quad + c \int_{\Omega(y,s)} R_1^\alpha \,(1 + |V_{p-1}(\nabla\overline{u})|)\,\mathrm{d}x + c \int_{\Omega(y,s)} |V(\nabla g - \xi)|^2 \,\mathrm{d}x \\
&\quad + c \int_{\Omega(y,s)} |V_{p-1}(\nabla\overline{u})||\nabla g - \xi| \,\mathrm{d}x.
\end{aligned} \tag{4.28}$$

We now apply Widman's hole-filling trick to conclude that, for some $\theta \in (0,1)$, $\theta = \theta(m,p)$,

$$\begin{aligned}
&\int_{\Omega(y,r)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x \\
&\le \theta \int_{\Omega(y,s)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x + \theta \int_{\Omega(y,s)} \left| V\left(\frac{\overline{u} - g - \vartheta\cdot\zeta}{s-r}\right)\right|^2 \mathrm{d}x \\
&\quad + \theta \int_{\Omega(y,s)} \left(R_1^\alpha \,(1 + |V_{p-1}(\nabla\overline{u})|) + |V(\nabla g - \xi)|^2\right) \mathrm{d}x \\
&\quad + \theta \int_{\Omega(y,s)} |V_{p-1}(\nabla\overline{u})||\nabla g - \xi| \,\mathrm{d}x.
\end{aligned} \tag{4.29}$$

Since this holds for every $\frac{R_0}{2} < r < s < R_0$, we can apply Theorem 73 to obtain the following Caccioppoli inequality of the second kind.

$$\begin{aligned}
&\int_{\Omega\left(y,\frac{R_0}{2}\right)} |V(\nabla\overline{u} - \nabla g - \zeta \otimes \nu)|^2 \,\mathrm{d}x \\
&\le c \int_{\Omega(y,R_0)} \left| V\left(\frac{\overline{u} - g - \vartheta\cdot\zeta}{R_0}\right)\right|^2 \mathrm{d}x + c \int_{\Omega(y,R_0)} \left(R_1^\alpha \,(1 + |V_{p-1}(\nabla\overline{u})|) + |V(\nabla g - \xi)|^2\right) \mathrm{d}x \\
&\quad + c \int_{\Omega(y,R_0)} |V_{p-1}(\nabla\overline{u})||\nabla g - \xi| \,\mathrm{d}x.
\end{aligned} \tag{4.30}$$

Before concluding this case, we observe that $\overline{u} - g - \vartheta\cdot\zeta = 0$ on $\partial\Omega$ and, hence, on a subset of $\partial(\Omega(y, R_0))$ of positive $\mathcal{H}^{n-1}$-measure. Therefore, we can apply Poincaré-Sobolev's inequality from Theorem 111 to the right hand side of (4.30) and, hence, obtain the following reverse Hölder inequality:

$$
\begin{aligned}
&\fint_{\Omega\left(y,\frac{R_0}{2}\right)} |V(\nabla\overline{u}-\nabla g-\zeta\otimes\nu)|^2\,\mathrm{d}x \\
&\leq c\left(\fint_{\Omega(y,R_0)} |V(\nabla\overline{u}-\nabla g-\zeta\otimes\nu)|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}} \\
&\quad + c\fint_{\Omega(y,R_0)} \left(R_1^\alpha\left(1+|V_{p-1}(\nabla\overline{u})|\right)+|V(\nabla g-\xi)|^2\right)\mathrm{d}x \\
&\quad + c\fint_{\Omega(y,R_0)} |V_{p-1}(\nabla\overline{u})||\nabla g-\xi|\,\mathrm{d}x
\end{aligned}
\tag{4.31}
$$

for some $c=c(m)$.

Using the Hölder continuity of ν once again, as well as the property $V(R_1^\alpha)^2<c\,R_1^\alpha$, we can make use of the subadditivity of V and the triangle inequality, to derive from above that

$$
\begin{aligned}
&\fint_{\Omega\left(y,\frac{R_0}{2}\right)} |V(\nabla\overline{u}-\xi-\zeta\otimes\nu_{x_0})|^2\,\mathrm{d}x \\
&\leq c\left(\fint_{\Omega(y,R_0)} |V(\nabla\overline{u}-\xi-\zeta\otimes\nu_{x_0})|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}} \\
&\quad + c\fint_{\Omega(y,R_0)} \left(R_1^\alpha\left(1+|V_{p-1}(\nabla\overline{u})|\right)+|V(\nabla g-\xi)|^2\right)\mathrm{d}x \\
&\quad + \fint_{\Omega(y,R_0)} |V_{p-1}(\nabla\overline{u})||\nabla g-\xi|\,\mathrm{d}x.
\end{aligned}
\tag{4.32}
$$

Case 2. $B(y,R_0)\subseteq\overline{\Omega}$. This case is simpler than the previous one, as we can obtain the desired Caccioppoli inequality by subtracting an affine function from $\overline{u}$ to construct the appropriate comparison maps and without having to handle the non-linearity of the function ϑ. Taking w_0 as in (4.17), we define

$$\tilde{u}(x):=\overline{u}(x)-w_0\cdot x-a_0, \tag{4.33}$$

where $a_0 \in \mathbb{R}^N$ is a constant vector chosen so that $\fint_{B(y,R_0)} \tilde{u} = 0$. We then let

$$\varphi := \rho\tilde{u}; \qquad \psi := \tilde{u} - \varphi = (1-\rho)\tilde{u}. \tag{4.34}$$

Observe that, since $B(y,s) \subseteq \Omega$, $\varphi \in \mathrm{W}_0^{1,p}(B(y,s),\mathbb{R}^N) \subseteq \mathrm{W}_0^{1,p}(\Omega,\mathbb{R}^N)$. Hence, using again the integrand F_{w_0} and following essentially the same ideas than in (4.20)-(4.30) from Case 1, we can obtain the following Caccioppoli inequality of the second kind.

$$\int_{B\left(y,\frac{R_0}{2}\right)} |V(\nabla\overline{u} - w_0)|^2 \,\mathrm{d}x \le c \int_{B(y,R_0)} \left| V\left(\frac{\overline{u} - w_0\cdot x - a_0}{R_0}\right)\right|^2 \mathrm{d}x. \tag{4.35}$$

Furthermore, using the convenient choice that we made for a_0, we can now apply Poincaré-Sobolev's inequality, in its standard version, to the relevant terms of the right hand side of (4.35). We then deduce the following, just as we did to obtain (4.32).

$$\begin{aligned}
\fint_{B\left(y,\frac{R_0}{2}\right)} |V(\nabla\overline{u} - \xi - \zeta\otimes\nu_{x_0})|^2 \,\mathrm{d}x \le{}& c\left(\fint_{B(y,R_0)} |V(\nabla\overline{u} - \xi - \zeta\otimes\nu_{x_0})|^{\frac{2n}{n+2}} \,\mathrm{d}x\right)^{\frac{n+2}{n}} \\
&+ c\fint_{B(y,R_0)} R_1^{\alpha}\left(1 + |V_{p-1}(\nabla\overline{u})|\right) \mathrm{d}x \\
\le{}& c\left(\fint_{B(y,R_0)} |V(\nabla\overline{u} - \xi - \zeta\otimes\nu_{x_0})|^{\frac{2n}{n+2}} \,\mathrm{d}x\right)^{\frac{n+2}{n}} \\
&+ c\fint_{B(y,R_0)} \left(R_1^{\alpha}\left(1 + |V_{p-1}(\nabla\overline{u})|\right) + |V(\nabla g - \xi)|^2\right) \mathrm{d}x \\
&+ \fint_{\Omega(y,R_0)} |V_{p-1}(\nabla\overline{u})||\nabla g - \xi| \,\mathrm{d}x.
\end{aligned} \tag{4.36}$$

Case 3. $B(y,R_0) \subseteq \mathbb{R}^n - \overline{\Omega}$. This is a trivial case, since we can extend $\overline{u}$ to a function that takes the value of 0 outside Ω and observe that, in this case, it trivially holds that,

$$\begin{aligned}
&\fint_{B\left(y,\frac{R_0}{2}\right)} |V(\nabla\overline{u}-\xi-\zeta\otimes\nu_{x_0})|^2\cdot\mathbb{1}_\Omega\,\mathrm{d}x\\
&\leq c\left(\fint_{B(y,R_0)} |V\left(\nabla\overline{u}-\xi-\zeta\otimes\nu_{x_0}\right)|^{\frac{2n}{n+2}}\cdot\mathbb{1}_\Omega\,\mathrm{d}x\right)^{\frac{n+2}{n}}\\
&\quad + c\fint_{B(y,R_0)} \left(R_1^\alpha\left(1+|V_{p-1}(\nabla\overline{u})|\right)+|V(\nabla g-\xi)|^2\cdot\mathbb{1}_\Omega\right)\mathrm{d}x\\
&\quad + \fint_{\Omega(y,R_0)} |V_{p-1}(\nabla\overline{u})||\nabla g-\xi|\,\mathrm{d}x. \qquad (4.37)
\end{aligned}$$

The case in which $x_0\in\Omega$ is now much simpler. We take, as before, $B(y,R_0)\subseteq B(x_0,R_1)$. We then observe that, if $B(y,R_0)\subseteq\overline{\Omega}$ or $B(y,R_0)\subseteq\mathbb{R}^n-\overline{\Omega}$, the same proof as in the previous Cases 2 and 3 enables us to conclude that inequality (4.37) is satisfied. On the other hand, if $B(y,R_0)\cap\Omega\neq\emptyset$ and $B(y,R_0)\cap(\mathbb{R}^n-\overline{\Omega})\neq\emptyset$, we can then find $x_1\in\partial\Omega$ such that $x_0\in B(x_1,2R_1)$ and $B(y,R_0)\subseteq B(x_1,2R_1)$. Therefore, we can repeat all the steps followed in the Case 1 above to conclude, as in (4.32), that

$$\begin{aligned}
&\fint_{\Omega\left(y,\frac{R_0}{2}\right)} |V(\nabla\overline{u}-\xi-\zeta\otimes\nu_{x_1})|^2\,\mathrm{d}x\\
&\leq c\left(\fint_{\Omega(y,R_0)} |V\left(\nabla\overline{u}-\xi-\zeta\otimes\nu_{x_1}\right)|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}}\\
&\quad + c\fint_{\Omega(y,R_0)} \left(R_1^\alpha\left(1+|V_{p-1}(\nabla\overline{u})|\right)+|V(\nabla g-\xi)|^2\right)\mathrm{d}x\\
&\quad + c\fint_{\Omega(y,R_0)} \left(1+|V_{p-1}(\nabla\overline{u})|\right)|\nabla g-\xi|\,\mathrm{d}x.
\end{aligned}$$

Since $|x_0-x_1|<2R_1$, we use as it is now standard that ν is $C^{0,\alpha}$ to conclude, for a

$0 < c = c(n, m, p)$, that we also have

$$\begin{aligned}
&\fint_{\Omega\left(y, \frac{R_0}{2}\right)} |V(\nabla \overline{u} - \xi - \zeta \otimes \nu_{x_0})|^2 \,\mathrm{d}x \\
&\leq c \left(\fint_{\Omega(y,R_0)} |V(\nabla \overline{u} - \xi - \zeta \otimes \nu_{x_0})|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}} \\
&\quad + c \fint_{\Omega(y,R_0)} \left(R_1^{\alpha} \left(1 + |V_{p-1}(\nabla \overline{u})|\right) + |V(\nabla g - \xi)|^2 \right) \mathrm{d}x \\
&\quad + c \fint_{\Omega(y,R_0)} |V_{p-1}(\nabla \overline{u})||\nabla g - \xi| \,\mathrm{d}x.
\end{aligned}$$

We have hence shown that such inequality is satisfied for any $x_0 \in \overline{\Omega}$, and for any $B(y, R_0) \subseteq B(x_0, R_1)$ with $0 < R_1 < R_\Omega$ arbitrary.

We will now obtain a similar result for points that are away from the boundary of Ω.

Step 2.2. Reverse Hölder inequality for interior balls. Fix $R_1 > 0$ as in Step 2.1 and assume now that $\eta \in R^{N\times n}$ is such that $|\eta| \leq m$. We will show that there is a constant $0 < c = c(m, n, p)$ such that, if $B(y, R_0) \subseteq B(x_0, R_1) \subseteq \Omega$, then

$$\fint_{B\left(y, \frac{R_0}{2}\right)} |V(\nabla \overline{u} - \eta)|^2 \,\mathrm{d}x \leq c \left(\fint_{B(y,R_0)} |V(\nabla \overline{u} - \eta)|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}}. \tag{4.38}$$

Given that $B(x_0, R_1) \subseteq \Omega$, the proof of this is exactly the same as the one we used to obtain (4.35) and then (4.36) in Case 2 of Step 2.1, but by taking η instead of w_0 and defining

$$\tilde{u} := \overline{u} - \eta \cdot x - b_0 \tag{4.39}$$

with $b_0 \in \mathbb{R}^N$ such that $\fint_{B(y,R_0)} \tilde{u} \,\mathrm{d}x = 0$.

Step 3.1 Higher integrability of the mean oscillations near the boundary via Gehring's Lemma. We now let $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times n}$ such that $|\xi|, |\zeta| \leq m$.

Observe that, since $\nu_x \neq 0$ for every $x \in \partial\Omega$, without loss of generality we can impose a further restriction on R_Ω, namely, that if $\operatorname{dist}(x, \partial\Omega) < R_\Omega$, then, for some constants

$\kappa_1, \kappa_2 > 0$, it holds that

$$\kappa_1 < |\nu_x| < \kappa_2. \tag{4.40}$$

Under this assumption, that depends exclusively on the domain Ω, we now fix $0 < R_1 < R_\Omega$. The main outcome of this step will be the existence of an exponent $\tilde{p} > 2$ such that, for every x_0 such that $\operatorname{dist}(x_0, \partial\Omega) < R_\Omega$,

$$\begin{aligned} &\fint_{\Omega\left(x_0, \frac{R_1}{2}\right)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0, \frac{R_1}{2}})|^{\tilde{p}} \, \mathrm{d}x \\ &\leq c \left(\fint_{\Omega(x_0,R_1)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,R_1} \otimes \hat{\nu}_{x_0})|^2 \, \mathrm{d}x \right)^{\frac{\tilde{p}}{2}} \\ &\quad + c\, R_1^{\alpha\tilde{p}} \left(\fint_{\Omega(x_0,R_1)} (1 + |\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R_1}|^p) \, \mathrm{d}x \right)^{\frac{\tilde{p}}{2}}. \end{aligned} \tag{4.41}$$

This establishes a reverse Hölder inequality, and hence higher integrability, for the mean oscillations of $\nabla\overline{u}_0$.

We proceed with the proof by letting $\tilde{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ be as in (4.18) and defining the functions

$$f := |V(\nabla\tilde{u})|^{\frac{2n}{n+2}} \cdot \mathbb{1}_\Omega; \tag{4.42}$$

$$h := \left(R_1^{\frac{\alpha n}{n+2}} \left(1 + |V_{p-1}(\nabla\overline{u})|^{\frac{n}{n+2}} \right) + |V(\nabla g - \xi)|^{\frac{2n}{n+2}} + |V_{p-1}(\nabla\overline{u})|^{\frac{n}{n+2}} |\nabla g - \xi|^{\frac{n}{n+2}} \right) \cdot \mathbb{1}_\Omega. \tag{4.43}$$

Then, (4.38) implies that, for every $B(y, R_0) \subseteq B(x_0, R_1)$,

$$\int_{B\left(y, \frac{R_0}{2}\right)} f^{\frac{n+2}{n}} \, \mathrm{d}x \leq c\, |\Omega(y, R_0)|^{-\frac{2}{n}} \left(\int_{B(y,R_0)} f \, \mathrm{d}x \right)^{\frac{n+2}{n}} + c \int_{B(y,R_0)} h^{\frac{n+2}{n}} \, \mathrm{d}x, \tag{4.44}$$

where $c = c(m, n, p)$. We recall that, for every $0 < t \leq R_\Omega$, and for every $x \in \overline{\Omega}$,

$$c_\Omega t^n \leq |\Omega(x, t)| \leq t^n \omega_n. \tag{4.45}$$

Therefore, after dividing by $R_0^n \cdot \omega_n$, (4.44) becomes, for some $c = c(c_\Omega, m, p)$,

$$\fint_{B\left(y,\frac{R_0}{2}\right)} f^{\frac{n+2}{n}} \,\mathrm{d}x \le c \left(\fint_{B(y,R_0)} f \,\mathrm{d}x \right)^{\frac{n+2}{n}} + c \fint_{B(y,R_0)} h^{\frac{n+2}{n}} \,\mathrm{d}x. \tag{4.46}$$

We can now apply Gehring's Lemma to conclude that there exists $q_1 > \frac{n+2}{n}$, with $q_1 = q_1(m,n,p)$, such that for every $\frac{n+2}{n} \le q \le q_1$ and for every $r \in (0,1)$,

$$\left(\fint_{B(x_0,rR_1)} f^q \,\mathrm{d}x \right)^{\frac{1}{q}} \le \frac{c(m,n,p)}{r^{\frac{n}{q}}(1-r)^{\frac{n^2}{n+2}}} \left(\left(\fint_{B(x_0,R_1)} f^{\frac{n+2}{n}} \,\mathrm{d}x \right)^{\frac{n}{n+2}} + \left(\fint_{B(x_0,R_1)} h^q \,\mathrm{d}x \right)^{\frac{1}{q}} \right). \tag{4.47}$$

By using again (4.45), this means that, for every $2 \le \tilde{p} \le \frac{2n}{n+2} q_1$, we can ensure

$$\begin{aligned}\fint_{\Omega(x_0,rR_1)} |V(\nabla \tilde{u})|^{\tilde{p}} \,\mathrm{d}x \le & \frac{c(m,n,p)}{r^n(1-r)^{\frac{n^2 q}{n+2}}} \left(\fint_{\Omega(x_0,R_1)} |V(\nabla \tilde{u})|^2 \,\mathrm{d}x \right)^{\frac{\tilde{p}}{2}} \\ & + \frac{c(m,n,p)}{r^n(1-r)^{\frac{n^2 q}{n+2}}} \fint_{\Omega(x_0,R_1)} \left(R_1^{\frac{\alpha\tilde{p}}{2}} \left(1 + |V_{p-1}(\nabla \overline{u})|^{\frac{\tilde{p}}{2}} \right) + |V(\nabla g - \xi)|^{\tilde{p}} \right) \mathrm{d}x \\ & + \frac{c(m,n,p)}{r^n(1-r)^{\frac{n^2 q}{n+2}}} \fint_{\Omega(x_0,R_1)} |V_{p-1}(\overline{u})|^{\frac{\tilde{p}}{2}} |\nabla g - \xi|^{\frac{\tilde{p}}{2}} \,\mathrm{d}x. \end{aligned} \tag{4.48}$$

On the other hand, using that V is increasing and its subadditivity properties, we can obtain a higher integrability result for the oscillations of $\nabla \overline{u}_0$ by using the above expression and the Hölder continuity of ∇g. We first record that

$$\begin{aligned} & \fint_{\Omega(x_0,rR_1)} |V(\nabla \overline{u}_0 - \zeta \otimes \nu_{x_0})|^{\tilde{p}} \,\mathrm{d}x \\ \le c & \fint_{\Omega(x_0,rR_1)} |V(\nabla \overline{u} - \xi - \zeta \otimes \nu_{x_0})|^{\tilde{p}} \,\mathrm{d}x \\ & + c \fint_{\Omega(x_0,rR_1)} |V(\nabla g - \xi)|^{\tilde{p}} \,\mathrm{d}x \end{aligned} \tag{4.49}$$

for some $c = c(p)$.

Following the same argument, we also have that

$$\left(\fint_{\Omega(x_0,R_1)} |V(\nabla \overline{u} - \xi - \zeta \otimes \nu_{x_0})|^2 \,\mathrm{d}x \right)^{\frac{\tilde{p}}{2}}$$
$$\leq c \left(\fint_{\Omega(x_0,R_1)} |V(\nabla \overline{u}_0 - \zeta \otimes \nu_{x_0})|^2 \,\mathrm{d}x \right)^{\frac{\tilde{p}}{2}} + c \left(\fint_{\Omega(x_0,R_1)} |V(\nabla g - \xi)|^2 \,\mathrm{d}x \right)^{\frac{\tilde{p}}{2}}. \tag{4.50}$$

We now take q_0 so that (4.6) holds for $p < q < q_0$ and q_1 as in (4.48). Then, we can fix $\tilde{p}$ so that

$$\tilde{p} \in \left(2, \min\left\{ \frac{2q_0}{p}, \frac{2n}{n+2} q_1 \right\} \right).$$

Combining the obtained in (4.48), (4.49) and (4.50) for $\xi = \nabla g(x_0)$, $\zeta = \frac{(\nabla(\overline{u}-g)\cdot \nu_{x_0})_{x_0,R_1}}{|\nu_{x_0}|^2}$ and $r = \frac{1}{2}$, we reach the following higher integrability, after applying Lemma 75 (i).

$$\fint_{\Omega\left(x_0,\frac{R_1}{2}\right)} \left| V\left(\nabla \overline{u}_0 - (\nabla \overline{u}_0)_{x_0,\frac{R_1}{2}} \right) \right|^{\tilde{p}} \mathrm{d}x$$
$$\leq c \fint_{\Omega\left(x_0,\frac{R_1}{2}\right)} |V(\nabla \overline{u}_0 - (\nabla \overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,R_1} \otimes \hat{\nu}_{x_0})|^{\tilde{p}} \,\mathrm{d}x$$
$$\leq c \left(\fint_{\Omega(x_0,R_1)} |V(\nabla \overline{u}_0 - (\nabla \overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,R_1} \otimes \hat{\nu}_{x_0})|^2 \,\mathrm{d}x \right)^{\frac{\tilde{p}}{2}}$$
$$+ c \fint_{\Omega(x_0,R_1)} \left(R_1^{\frac{\alpha\tilde{p}}{2}} \left(1 + |V_{p-1}(\nabla \overline{u})|^{\frac{\tilde{p}}{2}} \right) + |V(\nabla g - \nabla g(x_0))|^{\tilde{p}} \right) \mathrm{d}x$$
$$+ c \fint_{\Omega(x_0,R_1)} |V_{p-1}(\nabla \overline{u})|^{\frac{\tilde{p}}{2}} |\nabla g - \nabla g(x_0)|^{\frac{\tilde{p}}{2}} \,\mathrm{d}x + c \left(\fint_{\Omega(x_0,R_1)} |V(\nabla g - \nabla g(x_0))|^2 \,\mathrm{d}x \right)^{\frac{\tilde{p}}{2}}$$
$$\leq c \left(\fint_{\Omega(x_0,R_1)} |V(\nabla \overline{u}_0 - (\nabla \overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,R_1} \otimes \hat{\nu}_{x_0})|^2 \,\mathrm{d}x \right)^{\frac{\tilde{p}}{2}}$$
$$+ c \fint_{\Omega(x_0,R_1)} \left(R_1^{\frac{\alpha\tilde{p}}{2}} \left(1 + |V_{p-1}(\nabla \overline{u})|^{\frac{\tilde{p}}{2}} \right) + |V(\nabla g - \nabla g(x_0))|^{\tilde{p}} \right) \mathrm{d}x$$
$$+ c \fint_{\Omega(x_0,R_1)} |V_{p-1}(\nabla \overline{u})|^{\frac{\tilde{p}}{2}} |\nabla g - \nabla g(x_0)|^{\frac{\tilde{p}}{2}} \,\mathrm{d}x. \tag{4.51}$$

We remark that this holds provided $|\xi|, |\zeta| < c$ for some constant $0 < c = c(m, n, \Omega)$. Given that $|\xi| = |\nabla g(x_0)| \leq m$ and $\zeta = \frac{(\nabla(\overline{u}-g)\cdot\nu_{x_0})_{x_0,R_1}}{|\nu_{x_0}|^2}$, we recall that $\kappa_1 \leq |\nu_{x_0}| \leq \kappa_2$ to observe that this holds, in turn, provided we have the following situation.

Assumption 1 *There exists a constant* $0 < c = c(m, n, \Omega)$ *such that*

$$|(\nabla\overline{u}_0)_{x_0,R_1}| \leq c.$$

Then, we indeed have the following for a possibly different constant $c > 0$.

$$\frac{|(\nabla\overline{u}_0 \cdot \nu_{x_0})_{x_0,R_1}|}{|\nu_{x_0}|^2} \leq \frac{|(\nabla\overline{u}_0)_{x_0,R_1}|}{|\nu_{x_0}|} \leq c. \tag{4.52}$$

Even more, since by assumption $\|\nabla g\|_{0,\alpha} < m$ and because $R_1^{\frac{\alpha\tilde{p}}{2}} \leq R_1^{\alpha\tilde{p}}$, we can simplify the right hand side above and obtain

$$\begin{aligned}
&\fint_{\Omega\left(x_0,\frac{R_1}{2}\right)} \left|V\left(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,\frac{R_1}{2}}\right)\right|^{\tilde{p}} \mathrm{d}x \\
&\leq c\left(\fint_{\Omega(x_0,R_1)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,R_1}) \otimes \hat{\nu}_{x_0})|^2 \,\mathrm{d}x\right)^{\frac{\tilde{p}}{2}} \\
&\quad + c \fint_{\Omega(x_0,R_1)} R_1^{\alpha\tilde{p}}\left(1 + |V_{p-1}(\nabla\overline{u})|^{\frac{\tilde{p}}{2}}\right) \mathrm{d}x.
\end{aligned} \tag{4.53}$$

In addition, we remark that, since $|V_{p-1}(\xi)| = \frac{|V(\xi)|^2}{|\xi|}$, recalling the elementary property $a^q < 1 + a^p$ for every $1 < q < p$, and using that $p \geq 2$, this implies that $|V_{p-1}(\xi)| \leq c(1 + |\xi|^p)$. Therefore, $|V_{p-1}(\nabla\overline{u})|^{\frac{\tilde{p}}{2}} \leq c(1 + |\nabla\overline{u}|^{\frac{p\tilde{p}}{2}})$ and hence, by our choice of $\tilde{p}$, we can apply (4.6).

This, together with the fact that $\|\nabla g\|_{\mathrm{L}^\infty} < m$, enables us to derive

$$\begin{aligned}
&\fint_{\Omega\left(x_0,\frac{R_1}{2}\right)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,\frac{R_1}{2}}))|^{\tilde p}\,\mathrm{d}x \\
&\leq c\left(\fint_{\Omega(x_0,R_1)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0\cdot\hat\nu_{x_0})_{x_0,R_1})\otimes\hat\nu_{x_0})|^2\,\mathrm{d}x\right)^{\frac{\tilde p}{2}} + c\fint_{\Omega(x_0,R_1)} R_1^{\alpha\tilde p}\left(1+|\nabla\overline{u}_0|^{\frac{p\tilde p}{2}}\right)\mathrm{d}x \\
&\leq c\left(\fint_{\Omega(x_0,R_1)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0\cdot\hat\nu_{x_0})_{x_0,R_1})\otimes\hat\nu_{x_0})|^2\,\mathrm{d}x\right)^{\frac{\tilde p}{2}} + c\,R_1^{\alpha\tilde p}\left(\fint_{\Omega(x_0,R_1)} (1+|\nabla\overline{u}_0|^p)\,\mathrm{d}x\right)^{\frac{\tilde p}{2}}.
\end{aligned} \tag{4.54}$$

This and Assumption 1 together, further imply that (4.41) holds, as we wanted to show. Observe that the constant c above is such that $c = c(m,n,p)$ and does not depend on g in any way.

Step 3.2. Higher integrability of the mean oscillations in the interior. We now consider, separately, the special case in which $B(x_0,R_1)\subseteq\Omega$. Here, if $\eta\in\mathbb{R}^{N\times n}$ is such that $|\eta|\leq m$ and we take $\tilde u$ as in (4.39), we can repeat the way in which we have applied Gehring's Lemma in Step 3.1, but now using (4.38), from Step 2.2, instead of (4.16). We can then conclude that, for $\eta := (\nabla\overline{u})_{x_0,R_1}$, $|\eta|\leq c$ by Assumption 1 and so, for $\tilde p$ as above,

$$\begin{aligned}
\fint_{B\left(x_0,\frac{R_1}{2}\right)} \left|V\left(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,\frac{R_1}{2}}\right)\right|^{\tilde p}\mathrm{d}x \leq c\left(\fint_{B(x_0,R_1)} |V\left(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R_1}\right)|^2\,\mathrm{d}x\right)^{\frac{\tilde p}{2}} \\
+ cR_1^{\alpha\tilde p},
\end{aligned} \tag{4.55}$$

with $c = c(m,n,p)$.

Step 4. Linearization of the problem. This part of the proof consists in considering the second order Taylor polynomial of the translated integrand $F(\cdot+\nabla g(x_0))$. This way, by minimizing such polynomial and making use of the good estimates that we have for minimizers of linear elliptic problems, we will be able to transfer such good behaviour to the minimizer $\overline{u}$. With this in mind, let $z_0\in\mathbb{R}^{N\times n}$ be such that $|z_0|<\lambda$ for some constant $\lambda>0$ with

$\lambda = \lambda(m, n, \Omega)$. Denote

$$v_0 := \nabla g(x_0)$$

and let P be the second order Taylor polynomial of the translated integrand $F(\cdot + v_0)$ about the point z_0. That is,

$$P(z) := F(z_0 + v_0) + \big\langle F'(z_0 + v_0), z - z_0 \big\rangle + \frac{1}{2} F''(z_0 + v_0)[z - z_0, z - z_0] \tag{4.56}$$

for every $z \in \mathbb{R}^{N\times n}$. We note that, for every $z, z_1 \in \mathbb{R}^{N\times n}$,

$$P(z) - P(z_1) - \big\langle P'(z_1), z - z_1 \big\rangle = \frac{1}{2} F''(z_0 + v_0)[z - z_1, z - z_1]. \tag{4.57}$$

Let $\omega : [0, \infty) \to [0, \infty)$ be a modulus of continuity for $F''(\cdot + v_0)$ on the compact set

$$\{z \in \mathbb{R}^{N\times n} : |z| \leq 1 + \lambda\}.$$

We extend ω to $(1 + \lambda, \infty)$ so that it has the following properties:

- $\omega : [0, \infty) \to [0, \infty)$;
- ω is concave, continuous and non-decreasing;
- $\omega(0) = 0$, $\omega(t) = 1$ for every $t \geq 1$ and
- $|F''(z + v_0) - F''(w + v_0)| \leq c\omega(|z - w|)$ for some $c = c(m) > 0$ and for every $|z|, |w| \leq 1 + \lambda$.

We claim that there is a constant $0 < c = c(m)$ such that, for every $z \in \mathbb{R}^{N\times n}$ and every $x_0 \in \overline{\Omega}$,

$$|F(z + v_0) - P(z)| \leq \; c\,\omega(|z - z_0|)|V(z - z_0)|^2. \tag{4.58}$$

In order to prove this claim, we consider the following two cases.

Case 1. If $|z - z_0| \le 1 + \lambda$ then, from Taylor's Approximation Theorem we have that

$$\begin{aligned}|F(z+v_0) - P(z)| \le & \int_0^1 (1-t)|F''(z_0+v_0+t(z-z_0)) - F''(z_0+v_0)|\,\mathrm{d}t|z-z_0|^2 \\ \le & \,\omega(|z-z_0|)|z-z_0|^2 \\ \le & \,c(m)\,\omega(|z-z_0|)|V(z-z_0)|^2,\end{aligned}$$

where the last inequality follows from Lemma 128.

Case 2. If $|z - z_0| > 1 + \lambda > 1$, we use (H1) and Acerbi-Fusco's strategy from [AF87] to estimate

$$\begin{aligned}&|F(z+v_0) - P(z)| \\ =&\left|F(z+v_0) - F(z_0+v_0) - \left\langle F'(z_0+v_0), z-z_0\right\rangle - \frac{1}{2}F''(z_0+v_0)[z-z_0, z-z_0]\right| \\ \le&|F(z+v_0) - F(z_0+v_0)| + |F'(z_0+v_0)|\,|z-z_0| + \frac{1}{2}|F''(z_0+v_0)||z-z_0|^2 \\ \le&\, c\left(1 + |z_0|^{p-1} + |z|^{p-1}\right)|z-z_0| + c|z-z_0|^2 \\ \le&\, c\left(1 + |z-z_0|^{p-1}\right)|z-z_0| + c|z-z_0|^2 \\ \le&\, c\left(|z-z_0|^p + |z-z_0|^2\right) \\ =&\, c\,\omega(|z-z_0|)(|z-z_0|^2 + |z-z_0|^p)\end{aligned}$$

with $c = c(c_1, m)$. Notice that the last two inequalities follow from the fact that, in particular, $|z - z_0| > 1$. Since $2 \le p$, $|z-z_0|^2 + |z-z_0|^p \le c\,|V(z-z_0)|^2$ and the claim follows. In addition, we have also used that $|v_0| \le \|\nabla g\|_{\mathrm{L}^\infty} \le m$.

We further record that, by (H2) via Proposition 19, for every $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$

$$2c_2 \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x \le \int_\Omega F''(z_0+v_0)[\nabla\varphi, \nabla\varphi]\,\mathrm{d}x. \tag{4.59}$$

In particular, the quadratic form $F''(z_0+v_0)$ satisfies a strong ellipticity condition in the sense of Legendre-Hadamard.[5]

To make use of the Taylor approximation, we now let $0 < R < R_\Omega$ (we shall impose further conditions on R later) and take $h \in \mathrm{W}^{1,p}_{\overline{u}_0}(\Omega(x_0,R), \mathbb{R}^N)$ to be P-minimizing. Then, from

[5] See [Giu03].

Lemma 74 we infer that, for some constant $c > 0$ depending only on c_1, c_2 and p, and for every $z_0 \in \mathbb{R}^{N\times n}$,

$$\int_{\Omega(x_0,R)} |\nabla h - z_0|^2 \,\mathrm{d}x \leq c \int_{\Omega(x_0,R)} |\nabla \overline{u}_0 - z_0|^2 \,\mathrm{d}x. \tag{4.60}$$

We now make use of the L^p-estimates for $\mathcal{A}$-harmonic functions[6] to obtain that, for our fixed $\tilde{p} > 2$, there is a constant $K_1 > 0$, $K_1 = (c_2, m, n, p)$ such that

$$\int_{\Omega(x_0,R)} |\nabla h - z_0|^{\tilde{p}} \,\mathrm{d}x \leq K_1 \int_{\Omega(x_0,R)} |\nabla \overline{u}_0 - z_0|^{\tilde{p}} \,\mathrm{d}x. \tag{4.61}$$

The next aim is to obtain a suitable "approximation rate" between $\nabla \overline{u}_0$ and ∇h. The core idea behind this will be using Taylor's approximation via the following estimate, which makes use of the quasiconvexity of F and the P-minimality of h. In particular, we use that h satisfies

[6] See [Giu03, Section 10.4] and [Mor66].

the corresponding weak Euler-Lagrange equation.

$$
\begin{aligned}
&\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - \nabla h)|^2 \,\mathrm{d}x \\
\leq &\fint_{\Omega(x_0,R)} (F(\nabla\overline{u}_0 - \nabla h + z_0 + v_0) - F(z_0 + v_0)) \,\mathrm{d}x \\
= &\fint_{\Omega(x_0,R)} \big(F(\nabla\overline{u}_0 - \nabla h + z_0 + v_0) - P(\nabla\overline{u}_0 - \nabla h + z_0) + \big\langle F'(z_0 + v_0), \nabla\overline{u}_0 - \nabla h\big\rangle\big) \,\mathrm{d}x \\
&+ \fint_{\Omega(x_0,R)} \frac{1}{2}F''(z_0 + v_0)[\nabla\overline{u}_0 - \nabla h, \nabla\overline{u}_0 - \nabla h] \,\mathrm{d}x \\
= &\fint_{\Omega(x_0,R)} (F(\nabla\overline{u}_0 - \nabla h + z_0 + v_0) - P(\nabla\overline{u}_0 - \nabla h + z_0)) \,\mathrm{d}x \\
&+ \fint_{\Omega(x_0,R)} \frac{1}{2}F''(z_0 + v_0)[\nabla\overline{u}_0 - \nabla h, \nabla\overline{u}_0 - \nabla h] \,\mathrm{d}x \\
= &\fint_{\Omega(x_0,R)} (F(\nabla\overline{u}_0 - \nabla h + z_0 + v_0) - P(\nabla\overline{u}_0 - \nabla h + z_0) + P(\nabla\overline{u}_0) - P(\nabla h)) \,\mathrm{d}x \\
&- \fint_{\Omega(x_0,R)} \big\langle P'(\nabla h), \nabla\overline{u}_0 - \nabla h\big\rangle \,\mathrm{d}x \\
= &\fint_{\Omega(x_0,R)} (F(\nabla\overline{u}_0 - \nabla h + z_0 + v_0) - P(\nabla\overline{u}_0 - \nabla h + z_0)) \,\mathrm{d}x \\
&+ \fint_{\Omega(x_0,R)} (P(\nabla\overline{u}_0) - F(\nabla\overline{u}_0 + v_0)) \,\mathrm{d}x \\
&+ \fint_{\Omega(x_0,R)} (F(\nabla\overline{u}_0 + v_0) - F(\nabla\overline{u})) \,\mathrm{d}x + \fint_{\Omega(x_0,R)} (F(\nabla\overline{u}) - F(\nabla h + \nabla g)) \,\mathrm{d}x \\
&+ \fint_{\Omega(x_0,R)} (F(\nabla h + \nabla g) - F(\nabla h + v_0)) \,\mathrm{d}x + \fint_{\Omega(x_0,R)} (F(\nabla h + v_0) - P(\nabla h)) \,\mathrm{d}x \\
= &I + II + III + IV + V + VI,
\end{aligned}
\tag{4.62}
$$

with the obvious labelling.

We begin estimating (4.62) by observing that, since $\overline{u} \in (h+g) + \mathrm{W}_0^{1,p}(\Omega(x_0,R),\mathbb{R}^N)$ and $\overline{u}$ is an F-minimizer, then

$$IV < 0. \tag{4.63}$$

It is to estimate $III + V$ that we need, for this step, to consider the translated integrand

$F(\cdot + \nabla g(x_0))$ instead of simply F. The idea is that, this way, the gradient of the minimizer $\overline{u}$ can be suitably approximated by the function $\nabla\overline{u}_0 + v_0$ while $h = 0$ on $\partial\Omega$ and, hence, Theorem 72 can be applied.

With this in mind, we observe that the assumption $\|\nabla g\|_{0,\alpha} \leq m$ gives us

$$\begin{aligned} III &\leq \fint_{\Omega(x_0,R)} |F(\nabla\overline{u} - \nabla g + \nabla g(x_0)) - F(\nabla\overline{u})| \, \mathrm{d}x \\ &\leq c \fint_{\Omega(x_0,R)} \left(1 + |\nabla\overline{u}|^{p-1} + |\nabla g|^{p-1}\right) |\nabla g - \nabla g(x_0)| \, \mathrm{d}x \\ &\leq cR^{\alpha} \fint_{\Omega(x_0,R)} \left(1 + |V\left(\nabla\overline{u}_0 - z_0\right)|^2\right) \, \mathrm{d}x. \end{aligned} \tag{4.64}$$

For the last inequality we are using, once again, that $\|\nabla g\|_{0,\alpha} \leq m$, $|z|^{p-1} \leq c(1 + |z|^p)$ and $|z_0| \leq m$.

Using similar ideas, and then the L^p-estimates from (4.60) and (4.61), we derive the following.

$$\begin{aligned} V &\leq \fint_{\Omega(x_0,R)} |F(\nabla h - \nabla g) - F(\nabla h + \nabla g(x_0))| \, \mathrm{d}x \\ &\leq cR^{\alpha} \fint_{\Omega(x_0,R)} \left(1 + |V\left(\nabla h - z_0\right)|^2\right) \, \mathrm{d}x \\ &\leq cR^{\alpha} \fint_{\Omega(x_0,R)} \left(1 + |V\left(\nabla\overline{u}_0 - z_0\right)|^2\right) \, \mathrm{d}x. \end{aligned} \tag{4.65}$$

On the other hand, using (4.58) we obtain, for some $c = c(m) > 0$, that

$$I \leq c \fint_{\Omega(x_0,R)} \omega(\nabla\overline{u}_0 - \nabla h) |V(\nabla\overline{u}_0 - \nabla h)|^2 \, \mathrm{d}x. \tag{4.66}$$

Then, by taking $\tilde{p} > 2$ as in Steps 3.1-3.2 we obtain, by Hölder's inequality and concavity of

$\omega \leq 1$,

$$\begin{aligned}
I &\leq c \left(\fint_{\Omega(x_0,R)} \omega^{\frac{\tilde{p}}{\tilde{p}-2}} (\nabla \overline{u}_0 - \nabla h) \,\mathrm{d}x \right)^{\frac{\tilde{p}-2}{\tilde{p}}} \left(\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - \nabla h)|^{\tilde{p}} \,\mathrm{d}x \right)^{\frac{2}{\tilde{p}}} \\
&\leq \omega \left(\fint_{\Omega(x_0,R)} |\nabla \overline{u}_0 - \nabla h| \,\mathrm{d}x \right)^{\frac{\tilde{p}-2}{\tilde{p}}} \left(\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - \nabla h)|^{\tilde{p}} \,\mathrm{d}x. \right)^{\frac{2}{\tilde{p}}} \\
&\leq \omega \left(\left(\fint_{\Omega(x_0,R)} |\nabla \overline{u}_0 - \nabla h|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \right)^{\frac{\tilde{p}-2}{\tilde{p}}} \left(\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - \nabla h)|^{\tilde{p}} \,\mathrm{d}x \right)^{\frac{2}{\tilde{p}}}. \qquad (4.67)
\end{aligned}$$

On the other hand, we use (4.60) to estimate

$$\begin{aligned}
\fint_{\Omega(x_0,R)} |\nabla \overline{u}_0 - \nabla h|^2 \,\mathrm{d}x &\leq c \left(\fint_{\Omega(x_0,R)} |\nabla \overline{u}_0 - z_0|^2 \,\mathrm{d}x + \fint_{\Omega(x_0,R)} |\nabla h - z_0|^2 \,\mathrm{d}x \right) \\
&\leq c \fint_{\Omega(x_0,R)} |\nabla \overline{u}_0 - z_0|^2 \,\mathrm{d}x \\
&\leq c \fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - z_0)|^2 \,\mathrm{d}x \qquad (4.68)
\end{aligned}$$

and, since $\tilde{p} > 2$, by (4.61) we also have

$$\begin{aligned}
\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - \nabla h)|^{\tilde{p}} \,\mathrm{d}x &\leq c \left(\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - z_0)|^{\tilde{p}} \,\mathrm{d}x + \fint_{\Omega(x_0,R)} |V(\nabla h - z_0)|^{\tilde{p}} \,\mathrm{d}x \right) \\
&\leq c \fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - z_0)|^{\tilde{p}} \,\mathrm{d}x. \qquad (4.69)
\end{aligned}$$

It follows from (4.67)-(4.69) that, for a possibly different modulus of continuity holding the same properties as ω, we have

$$I \leq \omega \left(\left(\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - z_0)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \right)^{\frac{\tilde{p}-2}{\tilde{p}}} \left(\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - z_0)|^{\tilde{p}} \,\mathrm{d}x \right)^{\frac{2}{\tilde{p}}}. \qquad (4.70)$$

Then, in an analogous way as we did in (4.67), we apply Hölder's inequality to each of the estimates for II and VI, to derive that

$$\begin{aligned} II + VI \leq & c \fint_{\Omega(x_0,R)} \left(\omega(|\nabla\overline{u}_0 - z_0|)|V(\nabla\overline{u}_0 - z_0)|^2 + \omega(|\nabla h - z_0|)|V(\nabla h - z_0)|^2\right) \mathrm{d}x \\ \leq & c\,\omega\left(\left(\fint_{\Omega(x_0,R)} |\nabla\overline{u}_0 - z_0|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\right)^{\frac{\tilde{p}-2}{\tilde{p}}} \left(\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - z_0)|^{\tilde{p}} \,\mathrm{d}x\right)^{\frac{2}{\tilde{p}}}. \end{aligned} \tag{4.71}$$

We finally deduce from (4.62)-(4.65), (4.70) and (4.71), that

$$\begin{aligned} & \fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - \nabla h)|^2 \,\mathrm{d}x \\ & \leq c\,\omega\left(\left(\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - z_0)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\right)^{\frac{\tilde{p}-2}{\tilde{p}}} \left(\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - z_0)|^{\tilde{p}} \,\mathrm{d}x\right)^{\frac{2}{\tilde{p}}} \\ & \quad + cR^{\alpha} \fint_{\Omega(x_0,R)} \left(1 + |V(\nabla\overline{u}_0 - z_0)|^2\right) \mathrm{d}x. \end{aligned} \tag{4.72}$$

We now wish to use the reverse Hölder inequality derived in (4.41). To set ourselves in that context, we take

$$z_0 := (\nabla\overline{u}_0)_{x_0,R} \tag{4.73}$$

above. This imposes a further condition on R, since all the above calculations are only valid under the following assumption.

Assumption 2

$$|z_0| = \left|\fint_{\Omega(x_0,R)} \nabla\overline{u}_0 \,\mathrm{d}x\right| < \lambda.$$

We recall that, for x_0 near the boundary of Ω, the higher integrability estimates from Step 3.1 are given in terms of the mean oscillations with respect to the normal derivative of $\overline{u}_0$. Given that we wish to estimate the left hand side of (4.72) with the mean oscillations of $\nabla\overline{u}_0$ with respect to the whole derivative, we need to obtain those estimates handling the following three cases separately.

Case 1. Consider first $x_0 \in \partial\Omega$. We let

$$\tilde{E}(x_0, R) := \fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R})|^2 \,\mathrm{d}x$$

and we take $R_1 := 2R$ in (4.41). Then, we can combine it with (4.72) to further obtain the following estimate.

$$\begin{aligned}
&\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - \nabla h)|^2 \,\mathrm{d}x \\
&\le c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} \left(\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R})|^{\tilde{p}} \,\mathrm{d}x \right)^{\frac{2}{\tilde{p}}} \\
&\quad + cR^{\alpha} \fint_{\Omega(x_0,R)} \left(1 + |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R})|^2\right) \mathrm{d}x \\
&\le c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} \fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,2R} \otimes \hat{\nu}_{x_0})|^2 \,\mathrm{d}x \\
&\quad + c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} R^{2\alpha} \fint_{\Omega(x_0,2R)} (1 + |\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,2R}|^p) \,\mathrm{d}x \\
&\quad + cR^{\alpha} \fint_{\Omega(x_0,R)} \left(1 + |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R})|^2\right) \mathrm{d}x.
\end{aligned} \tag{4.74}$$

We use that $R = \frac{R_1}{2} < 1$ to rewrite this as follows.

$$\begin{aligned}
&\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - \nabla h)|^2 \,\mathrm{d}x \\
&\le c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} \fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,2R} \otimes \hat{\nu}_{x_0})|^2 \,\mathrm{d}x \\
&\quad + c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} R^{\alpha} \tilde{E}(x_0,4R) + cR^{\alpha} \fint_{\Omega(x_0,R)} \left(1 + |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R})|^2\right) \mathrm{d}x.
\end{aligned} \tag{4.75}$$

We now need to establish a suitable equivalence between the mean oscillations of $\nabla\overline{u}_0$ from its mean value in the direction ν_{x_0}, as we have above, and from its mean oscillations from its mean value, so that we can obtain a decay rate of such oscillations that we can finally iterate.

While it is clear that $(\nabla\overline{u}_0)_{x_0,2R}$ is a quasiminimizer for $\eta \mapsto \fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - \eta)|^2 \,\mathrm{d}x$ and that $(\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,2R}\otimes\hat{\nu}_{x_0}$ satisfies a similar property according to Lemma 75, it is not obvious how to make use of this in a way that we can go back to $\tilde{E}(x_0,2R)$ on the right hand side of the estimates. It turns out that this is the case if we consider increasing domains. We establish this by adapting some of the ideas from Remark 5.2 in [Bec11].

Since $x_0\in\partial\Omega$, we are able to use the Caccioppoli inequality that we obtained in (4.30) with $y=x_0$ and $R_0:=4R$. Hence, by Lemma 75 (ii), we have

$$\begin{aligned}
&\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0-(\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,2R}\otimes\hat{\nu}_{x_0})|^2\,\mathrm{d}x\\
&\le c\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0-\tilde{\zeta}\otimes\hat{\nu}_{x_0})|^2\,\mathrm{d}x\\
&\le c\fint_{\Omega(x_0,2R)} \left|V\left(\nabla\overline{u}_0-\frac{\tilde{\zeta}}{|\nu_{x_0}|}\otimes\nu\right)\right|^2\mathrm{d}x + c\fint_{\Omega(x_0,2R)}\left|V\left(\frac{\tilde{\zeta}}{|\nu_{x_0}|}\otimes(\nu-\nu_{x_0})\right)\right|^2\mathrm{d}x\\
&\le c\fint_{\Omega(x_0,2R)} \left|V\left(\nabla\overline{u}_0-\frac{\tilde{\zeta}}{|\nu_{x_0}|}\otimes\nu\right)\right|^2\mathrm{d}x + c\,R^\alpha\\
&\le c\fint_{\Omega(x_0,4R)} \left|V\left(\frac{\overline{u}_0-\vartheta\cdot\frac{\tilde{\zeta}}{|\nu_{x_0}|}}{4R}\right)\right|^2\mathrm{d}x + c\fint_{\Omega(x_0,4R)} R^\alpha\left(1+|V_{p-1}(\nabla\overline{u})|\right)\mathrm{d}x. \qquad (4.76)
\end{aligned}$$

where $c=c(m,p)$ and we are using that $\nu=\nabla\vartheta$ is Hölder continuous, $R<R_\Omega$ and $\frac{|\tilde{\zeta}|}{|\nu_{x_0}|}\le c$ for some constant $c>0$ depending on the usual parameters. We now let $\tilde{\zeta}:=(\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,8R}$. Applying a similar treatment to the one we used in Assumption 1, we observe that this control over $|\tilde{\zeta}|$ will hold provided we have the following.

Assumption 3 *There is a constant* $0<c=c(m,n,\Omega)$ *such that*

$$|(\nabla\overline{u}_0)_{x_0,8R}|<c.$$

In addition, we are taking $R_1=4R$ in the derivation of (4.30) for this case. Notice that we can do this because (4.30) holds after choosing any arbitrary $0<R_1<R_\Omega$ and $\frac{R}{2}$ will satisfy such a condition.

We now invoke Proposition 71, related to the boundary version of Poincaré's inequality, and apply it to the right hand side of (4.76). Then, we obtain that

$$\begin{aligned}
&\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,2R}\otimes\hat{\nu}_{x_0})|^2\,\mathrm{d}x\\
&\le c\fint_{\Omega(x_0,8R)} \left|V\left(\nabla\left(\overline{u}_0 - \vartheta\cdot\frac{\tilde{\zeta}}{|\nu_{x_0}|}\right)\cdot\hat{\nu}_{x_0}\right)\right|^2\mathrm{d}x + c\fint_{\Omega(x_0,8R)} R^\alpha\left(1+|V_{p-1}(\nabla\overline{u})|\right)\mathrm{d}x\\
&\le c\fint_{\Omega(x_0,8R)} \left|V\left(\nabla\overline{u}_0\cdot\hat{\nu}_{x_0} - \left(\frac{\tilde{\zeta}}{|\nu_{x_0}|}\otimes\nu\right)\cdot\hat{\nu}_{x_0}\right)\right|^2\mathrm{d}x + c\fint_{\Omega(x_0,8R)} R^\alpha\left(1+|\nabla\overline{u}|^p\right)\mathrm{d}x\\
&= c\fint_{\Omega(x_0,8R)} \left|V\left(\nabla\overline{u}_0\cdot\hat{\nu}_{x_0} - \frac{(\nu\cdot\hat{\nu}_{x_0})}{|\nu_{x_0}|}\tilde{\zeta}\right)\right|^2\mathrm{d}x + c\fint_{\Omega(x_0,8R)} R^\alpha\left(1+|\nabla\overline{u}|^p\right)\mathrm{d}x.
\end{aligned}$$

Exploiting the Hölder continuity of ν once again, and the facts that $|\tilde{\zeta}|\le c(m)$ and $|\nu_{x_0}|\ge\kappa_1$, this enables us to conclude the following.

$$\begin{aligned}
&\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,2R}\otimes\hat{\nu}_{x_0})|^2\,\mathrm{d}x\\
&\le c\fint_{\Omega(x_0,8R)} \left|V\left(\nabla\overline{u}_0\cdot\hat{\nu}_{x_0} - \tilde{\zeta}\right)\right|^2\mathrm{d}x + c\fint_{\Omega(x_0,8R)} R^\alpha\left(1+|\nabla\overline{u}|^p\right)\mathrm{d}x.
\end{aligned}$$

Recalling the definition of $\tilde{\zeta}$, this is easily seen to imply, by Cauchy-Schwarz inequality for the first term and triangle inequality together with Assumption 3 and $\|\nabla g\|_{\mathrm{L}^\infty}\le m$ for the second term, that

$$\begin{aligned}
&\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,2R}\otimes\hat{\nu}_{x_0})|^2\,\mathrm{d}x\\
&\le c\fint_{\Omega(x_0,8R)} \left|V\left(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,8R}\right)\right|^2\mathrm{d}x + c\fint_{\Omega(x_0,8R)} R^\alpha\left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,8R})|^2\right)\mathrm{d}x.
\end{aligned} \tag{4.77}$$

We now use (4.75) and (4.77) to conclude that

$$\begin{aligned}
\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - \nabla h)|^2\,\mathrm{d}x \le\ & c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}}\fint_{\Omega(x_0,8R)} R^\alpha\left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,8R})|^2\right)\mathrm{d}x\\
&+ cR^\alpha\fint_{\Omega(x_0,8R)}\left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,8R})|^2\right)\mathrm{d}x.
\end{aligned} \tag{4.78}$$

Case 2. $x_0 \in \Omega$ and $B(x_0, 4R) \cap (\partial\Omega) \neq \emptyset$. The key idea in this case is the same as the one we used for $x_0 \in \partial\Omega$, with the only difference being that we need to estimate the right hand side of (4.76) by approximating x_0 with a point on the boundary of Ω.

We first notice that, in the derivation of (4.30), it is not used that $x_0 \in \partial\Omega$, but only that $B(y, R_0)$ intersects both Ω and $\mathbb{R}^n \backslash \Omega$. With $y = x_0$, we can infer that (4.30) also holds in this case. We further notice that (4.54) also holds for the x_0 which is now under consideration provided $4R < R_\Omega$, because then we ensure that $\text{dist}(x_0, \partial\Omega) < R_\Omega$. These two facts imply that (4.76) is also valid, so that

$$\begin{aligned} &\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,2R} \otimes \hat{\nu}_{x_0})|^2 \,\mathrm{d}x \\ &\leq c \fint_{\Omega(x_0,4R)} \left| V\left(\frac{\overline{u}_0 - \vartheta \cdot \frac{\tilde{\zeta}}{|\nu_{x_0}|}}{4R} \right) \right|^2 \mathrm{d}x + c \fint_{\Omega(x_0,4R)} R^\alpha \left(1 + |V_{p-1}(\nabla\overline{u})|\right) \mathrm{d}x. \end{aligned} \tag{4.79}$$

We now take $x_0^* \in \partial\Omega \cap B(x_0, 4R)$. It is then clear that

$$B(x_0, 4R) \subseteq B(x_0^*, 8R) \subseteq B(x_0, 20R).$$

We also impose the following.

Assumption 4 $20R < R_\Omega$.

Then, we can estimate the right hand side of (4.79) by approximating it with the average value of the same expression over $\Omega(x_0^*, 8R)$, where we can apply Proposition 71, and then go back to an average value over $\Omega(x_0, 20R)$. In other words, by taking now $\tilde{\zeta} := (\nabla\overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,20R}$, we have that

$$\begin{aligned} &\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0 \cdot \hat{\nu}_{x_0})_{x_0,2R} \otimes \hat{\nu}_{x_0})|^2 \,\mathrm{d}x \\ &\leq c \fint_{\Omega(x_0,4R)} \left| V\left(\frac{\overline{u}_0 - \vartheta \cdot \frac{\tilde{\zeta}}{|\nu_{x_0}|}}{4R} \right) \right|^2 \mathrm{d}x + c \fint_{\Omega(x_0,4R)} R^\alpha \left(1 + |V_{p-1}(\nabla\overline{u})|\right) \mathrm{d}x \\ &\leq c \fint_{\Omega(x_0^*,8R)} \left| V\left(\frac{\overline{u}_0 - \vartheta \cdot \frac{\tilde{\zeta}}{|\nu_{x_0}|}}{8R} \right) \right|^2 \mathrm{d}x + c \fint_{\Omega(x_0,4R)} R^\alpha \left(1 + |V_{p-1}(\nabla\overline{u})|\right) \mathrm{d}x. \end{aligned} \tag{4.80}$$

Then, by the same argument that we used to obtain (4.77), but applied this time to x_0^*, we infer from above that

$$
\begin{aligned}
&\fint_{\Omega(x_0,2R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0\cdot\hat{\nu}_{x_0})_{x_0,2R}\otimes\hat{\nu}_{x_0})|^2\,\mathrm{d}x \\
&\le c \fint_{\Omega(x_0^*,16R)} \left|V\left(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0^*,16R}\right)\right|^2\,\mathrm{d}x + c\fint_{\Omega(x_0,16R)} R^\alpha\left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,16R})|^2\right)\mathrm{d}x \\
&\le c \fint_{\Omega(x_0,20R)} |V\left(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R}\right)|^2\,\mathrm{d}x \\
&\quad + c\fint_{\Omega(x_0,20R)} R^\alpha\left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R})|^2\right)\mathrm{d}x. \qquad (4.81)
\end{aligned}
$$

Hereby, in the same way as we did in Case 1, we use (4.72) to conclude that

$$
\begin{aligned}
\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - \nabla h)|^2\,\mathrm{d}x \le c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} &\fint_{\Omega(x_0,20R)} R^\alpha\left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R})|^2\right)\mathrm{d}x \\
+ cR^\alpha &\fint_{\Omega(x_0,20R)} \left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R})|^2\right)\mathrm{d}x. \qquad (4.82)
\end{aligned}
$$

Notice that, in the derivation of (4.81), we are implicitly assuming that $|\tilde{\zeta}| < c$ for some constant $c = c(m,n,\Omega) > 0$. In a similar way as we concluded the need of Assumption 1, we can ensure this by imposing the following condition, that we finally state explicitly in terms of m.

Assumption 5 $R > 0$ *is such that*

$$|(\nabla\overline{u}_0)_{x_0,20R}| < m.$$

Case 3. $x_0 \in \Omega$ and $B(x_0,4R) \subseteq \Omega$. This case is simpler, since we can use Step 3.2 to estimate $\nabla\overline{u} - \nabla h$ in terms of the mean oscillations of $\nabla\overline{u} - \nabla g$ without invoking the normal derivative. From (4.72) and the reverse Hölder inequality (4.55), it is straightforward to conclude that

$$
\begin{aligned}
\fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - \nabla h)|^2\,\mathrm{d}x \le c\,\omega(\tilde{E}(x_0,R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} &\fint_{\Omega(x_0,20R)} R^\alpha\left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R})|^2\right)\mathrm{d}x \\
+ cR^\alpha &\fint_{\Omega(x_0,20R)} \left(1+|V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R})|^2\right)\mathrm{d}x. \qquad (4.83)
\end{aligned}
$$

We remark that (4.78), (4.82) and (4.83) establish that the same estimate is valid for every $x_0 \in \overline{\Omega}$. Therefore, it is now possible to derive the decay rate that we were looking for. Indeed, the subadditivity of V and the Generalized Weyl's Lemma from Theorem 72 imply that, for some $c = c(m, n, p, R_\Omega)$,

$$\begin{aligned}
\tilde{E}(x_0, r) &= \fint_{\Omega(x_0,r)} |V(\nabla \overline{u}_0 - (\nabla \overline{u}_0)_{x_0,r})|^2 \,\mathrm{d}x \\
&\leq c \fint_{\Omega(x_0,r)} |V(\nabla \overline{u}_0 - (\nabla h)_{x_0,r})|^2 \,\mathrm{d}x \\
&\leq c \left(\frac{R}{r}\right)^n \fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - \nabla h)|^2 \,\mathrm{d}x + c \left(\frac{r}{R}\right)^2 \fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - (\nabla \overline{u}_0)_{x_0,R})|^2 \,\mathrm{d}x \\
&\leq c \left(\frac{R}{r}\right)^n \fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - \nabla h)|^2 \,\mathrm{d}x + c \left(\frac{r}{R}\right)^2 \fint_{\Omega(x_0,20R)} |V(\nabla \overline{u}_0 - (\nabla \overline{u}_0)_{x_0,20R})|^2 \,\mathrm{d}x.
\end{aligned} \tag{4.84}$$

The last inequality above follows after applying, first, the quasiminimality of $(\nabla \overline{u}_0)_{x_0,R}$ for the function $\eta \mapsto \fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - \eta)|^2 \,\mathrm{d}x$ and, then, by increasing the support and using $|\Omega(x_0, 20R)| \leq 20^n R^n \omega_n \leq c_\Omega \, |\Omega(x_0, 20R)|$ because $20R < R_\Omega$.

We conclude this step by using (4.78) and (4.84) to obtain the decay rate that we were looking for. That is,

$$\begin{aligned}
\tilde{E}(x_0, r) \leq &\, c \left(\frac{R}{r}\right)^n \left(\omega(\tilde{E}(x_0, R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} \left(R^\alpha + \tilde{E}(x_0, 20R)\right) + R^\alpha (1 + \tilde{E}(x_0, R))\right) \\
&+ c \left(\frac{r}{R}\right)^2 \tilde{E}(x_0, 20R) \\
\leq &\, c \left(\frac{R}{r}\right)^n \left(\omega(c\,\tilde{E}(x_0, 20R)^{\frac{1}{2}})^{\frac{\tilde{p}-2}{\tilde{p}}} \left(R^\alpha + \tilde{E}(x_0, 20R)\right) + R^\alpha (1 + \tilde{E}(x_0, 20R))\right) \\
&+ c \left(\frac{r}{R}\right)^2 \tilde{E}(x_0, 20R).
\end{aligned}$$

We further simplify the above to obtain, for a different modulus of continuity $\overline{\omega}$, that

$$\tilde{E}(x_0, r) \leq c \left(\frac{R}{r}\right)^n R^\alpha \tilde{E}(x_0, 20R) + c \left(\overline{\omega}(\tilde{E}(x_0, 20R)) \left(\frac{R}{r}\right)^n + \left(\frac{r}{R}\right)^2\right) \tilde{E}(x_0, 20R) \tag{4.85}$$

for every $0 < r \leq R < R_\Omega$. We remark that this is valid provided Assumptions 1-5 hold. Whereby, we now compile all these assumptions into a single one, which is the setting over which we will iterate estimate (4.85) in the next final step. In order to do this, we impose a smallness assumption on $\tilde{E}(x_0, 20R)$ that we will later specify more precisely.

Assumption 6 *There exists* $0 < \delta_1 < 1$*, depending on* m*, such that*

$$\fint_{\Omega(x_0,20R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R})| \, \mathrm{d}x < \delta_1.$$

Then, (4.85) implies

$$\tilde{E}(x_0, r) \leq c\left(\frac{R}{r}\right)^n R^\alpha \tilde{E}(x_0, 20R) + c\left(\overline{\omega}(\tilde{E}(x_0, 20R))\left(\frac{R}{r}\right)^n + \left(\frac{r}{R}\right)^2\right)\tilde{E}(x_0, 20R). \quad (4.86)$$

Furthermore, by triangle and Jensen's inequalities, together with Assumptions 5 and 6, we have that

$$\begin{aligned}
|(\nabla\overline{u}_0)_{x_0,R}| \leq & \fint_{\Omega(x_0,R)} |\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R}| \, \mathrm{d}x + |(\nabla\overline{u}_0)_{x_0,20R}| \\
\leq & c \fint_{\Omega(x_0,20R)} |\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,20R}| \, \mathrm{d}x + |(\nabla\overline{u}_0)_{x_0,20R}| \\
\leq & c + m
\end{aligned}$$

for some constant $0 < c = c(n, \Omega)$. In a completely analogous way we obtain that

$$|(\nabla\overline{u}_0)_{x_0,2R}| \leq c + m \qquad \text{and} \qquad |(\nabla\overline{u}_0)_{x_0,4R}| \leq c + m.$$

Hereby, Assumptions 1-6 will all be satisfied provided Assumptions 4-6 hold.

Step 5. Iteration. We will now iterate a refined version of the estimate that we concluded from Step 4. This will enable us to obtain the excess decay, from which regularity for $\overline{u}_0$ will follow using Campanato-Meyers characterization of Hölder continuity. Since $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$, this will in turn imply regularity for $\overline{u}$.

We first rescale and rewrite (4.86) by making $\tau = \frac{r}{R}$ so that, for $0 < \tau < \frac{1}{20}$, we have

$$\tilde{E}(x_0, \tau R) \leq \left[c\tau^{-n} R^\alpha + c\left(\tau^2 + \overline{\omega}(\tilde{E}(x_0, R))\tau^{-n}\right)\right] \tilde{E}(x_0, R). \quad (4.87)$$

Now, take τ fixed so that

$$c\tau^2 = \frac{1}{2}\tau^{2\alpha}, \tag{4.88}$$

i.e.,

$$\tau := (2c)^{-\frac{1}{2-2\alpha}}. \tag{4.89}$$

We note that it is for this step that we require $\alpha \in (0,1)$. In addition, since $c = c(m,n,N,p,c_1,c_2,\Omega)$, we can assume that c is large enough so that we have $\tau = \tau(m,n,N,p,c_1,c_2,\Omega)$ and $\tau \in \left(0, \frac{1}{20}\right)$.

We claim that there exist $0 < \varepsilon < 1$ and $R \in (0, R_\Omega)$, $R = R(\varepsilon)$ such that, if

$$\int_\Omega |\nabla g|^p \,\mathrm{d}x < \varepsilon, \tag{4.90}$$

then for every $j \in \mathbb{N}$,

$$\tilde{E}(x_0, \tau^j R) \leq \tau^{2\alpha j}\tilde{E}(x_0, R) + c_0 \left(\tau^{j-1}R\right)^\alpha \sum_{i=0}^{j-1} \tau^{i\alpha}. \tag{4.91}$$

Observe first that, since $\overline{\omega} = \overline{\omega}_m$ is increasing and $\overline{\omega}(t) \to 0$ as $t \to 0$, we can take $0 < \delta_1 < 1$ so that, for every $\delta \in (0, \delta_1]$,

$$c\,\overline{\omega}(\delta)\tau^{-n} < \frac{1}{2}\tau^{2\alpha}. \tag{4.92}$$

We now let

$$c_0 := c\tau^{-n}. \tag{4.93}$$

Notice that under assumptions (4.88) and (4.92), (4.87) will imply that

$$\tilde{E}(x_0, \tau R) \leq c_0 R^\alpha + \tau^{2\alpha}\tilde{E}(x_0, R), \tag{4.94}$$

provided $0 < R < R_\Omega$, $|(\nabla\overline{u}_0)_{x_0,R}| < m - 1$, and $\tilde{E}(x_0, R) < \delta_1 < 1$. Our goal is now to show that, if ∇g is small enough in L^p, we can find an $R > 0$ such that these properties are satisfied for every $\tau^k R$, with $k \in \mathbb{N}^+$. This will then enable us to iterate the inequality for the decay.

The main tool that we will use for this is the quasiconvexity of F via Lemma 74, thanks to which we will be able to measure the mean oscillations of $\nabla\overline{u}_0$ in terms of $\|\nabla g\|_{\mathrm{L}^p}$. Indeed, since we know that for some $c > 0$,

$$\int_\Omega |V(\nabla\overline{u}_0)|^2 \,\mathrm{d}x \le c\int_\Omega |V(\nabla\overline{u})|^2 \,\mathrm{d}x + c\int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x \le c\int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x,$$

there are constants $0 < \kappa_1 = \kappa_1(n,p)$ and $\kappa_2 = \kappa_2(n,p)$ so that, if $0 < R < R_\Omega$,

$$\left| \fint_{\Omega(x_0,R)} \nabla\overline{u}_0 \,\mathrm{d}x \right|^2 \le \frac{\kappa_1}{R^{2n}} \int_\Omega |V(\nabla\overline{u}_0)|^2 \,\mathrm{d}x \le \frac{\kappa_2}{R^{2n}} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x. \tag{4.95}$$

Even more, we can chose κ_2 so that

$$\left| \fint_{\Omega(x_0,R)} \nabla g \,\mathrm{d}x \right|^2 \le \frac{\kappa_2}{R^{2n}} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x. \tag{4.96}$$

In addition, we take κ_1 and κ_2 so that it also holds

$$\begin{aligned} \fint_{\Omega(x_0,R)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,R})|^2 \,\mathrm{d}x &\le \frac{\kappa_1}{R^n} \int_\Omega |V(\nabla\overline{u}_0)|^2 \,\mathrm{d}x \le \frac{\kappa_2}{R^n} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x \\ &\le \frac{\kappa_2}{R^{2n}} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x. \end{aligned} \tag{4.97}$$

We now choose τ as in (4.88) and δ_1 so that (4.92) is satisfied. We further assume, without loss of generality, that

$$\delta_1 < \min\left\{ \frac{2}{(m-1)^2}, \kappa_2 \right\}. \tag{4.98}$$

Define

$$\varepsilon := \frac{\delta_1 (m-1)^2}{4\kappa_2} \min\left\{ R_\Omega^{2n}, \left(\frac{\tau^\alpha (1-\tau^\alpha)\delta_1}{2c_0} \right)^{\frac{n}{\alpha}} \right\}. \tag{4.99}$$

Observe that $0 < \varepsilon = \varepsilon(m,n,N,p,c_1,c_2,\Omega)$. We can now impose the smallness condition that we require for ∇g in order to perform the iteration that leads to conclude full regularity of $\nabla\overline{u}$. We assume

$$\int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x < \varepsilon. \tag{4.100}$$

Since $\delta_1 < \frac{2}{m^2}$, this implies that

$$\kappa_2 \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x < \frac{2\kappa_2}{\delta_1(m-1)^2} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x < \frac{1}{2} \min\left\{ R_\Omega^{2n}, \left(\frac{\tau^\alpha(1-\tau^\alpha)\delta_1}{2c_0} \right)^{\frac{2n}{\alpha}} \right\}.$$

Therefore, there exists a number $R > 0$ such that

$$\begin{aligned} \kappa_2 \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x <& \frac{2\kappa_2}{\delta_1(m-1)^2} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x < \frac{1}{2} \min\left\{ R_\Omega^{2n}, \left(\frac{\tau^\alpha(1-\tau^\alpha)\delta_1}{2c_0} \right)^{\frac{2n}{\alpha}} \right\} \\ <& R^{2n} < \min\left\{ R_\Omega^{2n}, \left(\frac{\tau^\alpha(1-\tau^\alpha)\delta_1}{2c_0} \right)^{\frac{2n}{\alpha}} \right\}. \end{aligned} \tag{4.101}$$

It derives from (4.95), (4.97) and (4.101) that R satisfies the following properties

$$0 < R < R_\Omega < 1; \tag{4.102}$$

$$\left| \fint_{\Omega(x_0,R)} \nabla \overline{u}_0 \,\mathrm{d}x \right|^2 \le \frac{\kappa_2}{R^{2n}} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x < (m-1)^2; \tag{4.103}$$

$$\fint_{\Omega(x_0,R)} |V(\nabla \overline{u}_0 - (\nabla \overline{u}_0)_{x_0,R})|^2 \,\mathrm{d}x \le \frac{\kappa_2}{R^{2n}} \int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x < \frac{\delta_1}{2} \tag{4.104}$$

and

$$R < \overline{R} := \left(\frac{\tau^\alpha(1-\tau^\alpha)\delta_1}{2c_0} \right)^{\frac{1}{\alpha}}. \tag{4.105}$$

We will now prove assertion (4.91) for this choice of R. We proceed by induction on $j \in \mathbb{N}$. To prove the claim for $j = 1$ observe that, by (4.104), $\tilde{E}(x_0, R) < \delta_1$. Therefore, (4.87) and (4.92) imply that

$$\tilde{E}(x_0, \tau R) \le c_0 R^\alpha + \tau^{2\alpha} \tilde{E}(x_0, R). \tag{4.106}$$

We now assume that (4.91) holds for every $j \in \mathbb{N}$ such that $1 \le j \le k$. We will prove that the claim is true for $j := k+1$. The idea will be to iterate the process and use (4.87) with $\tau^k R$ instead of R. For this purpose, we first need to estimate $\tilde{E}(x_0, \tau^k R)$ and $|(\nabla \overline{u}_0)_{x_0,\tau^k R}|$.

Applying the strong induction hypothesis and (4.104) we have that, for every $1 \leq j \leq k$,

$$\begin{aligned}
\tilde{E}(x_0, \tau^j R) &\leq \tau^{2\alpha j} \tilde{E}(x_0, R) + c_0 \left(\tau^{j-1} R\right)^{\alpha} \sum_{i=0}^{j-1} \tau^{\alpha i} \\
&< \frac{\tau^{2\alpha j} \delta_1}{2} + c_0 (\tau^{j-1} R)^{\alpha} \frac{1}{1 - \tau^{\alpha}} \\
&\leq \delta_1 \tau^{\alpha j},
\end{aligned} \tag{4.107}$$

with the last inequality following from the fact that $\frac{c_0(\tau^{-1}R)^\alpha}{1-\tau^\alpha} < \frac{\delta_1}{2}$, which is, in turn, a consequence of (4.105). Since $0 < \tau < 1$, we conclude from above, with $j = k$, that

$$\tilde{E}(x_0, \tau^k R) < \delta_1. \tag{4.108}$$

We will now estimate $\left|(\nabla \overline{u})_{x_0, \tau^k R}\right|$. It follows from (4.103) and (4.104) that

$$\begin{aligned}
\left|(\nabla \overline{u}_0)_{x_0, \tau^k R}\right| &\leq |(\nabla \overline{u}_0)_{x_0, R}| + \sum_{j=0}^{k-1} \left|(\nabla \overline{u}_0)_{x_0, \tau^j R} - (\nabla \overline{u}_0)_{x_0, \tau^{j+1} R}\right| \\
&< m - 1 + \sum_{j=0}^{k-1} \fint_{\Omega(x_0, \tau^{j+1} R)} \left|\nabla \overline{u}_0 - (\nabla \overline{u}_0)_{x_0, \tau^j R}\right| \mathrm{d}x \\
&\leq m - 1 + \frac{c_\Omega}{\tau^n} \sum_{j=0}^{k-1} \left(\fint_{\Omega(x_0, \tau^j R)} |\nabla \overline{u}_0 - (\nabla \overline{u}_0)_{x_0, \tau^j R}|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.109}$$

Here, we have also used that $\tau^j R < R_\Omega$ and Hölder's inequality. It follows from (4.107) and (4.109) that

$$\begin{aligned}
\left|(\nabla \overline{u}_0)_{x_0, \tau^k R}\right| &\leq m - 1 + \frac{c_{\Omega,p}}{\tau^n} \sum_{j=0}^{k-1} \tilde{E}(x_0, \tau^j R)^{\frac{1}{2}} \\
&\leq m - 1 + \frac{\delta_1^{\frac{1}{2}} c_{\Omega,p}}{\tau^n} \sum_{j=0}^{k-1} \tau^{\frac{\alpha j}{2}} \\
&< m - 1 + \frac{\delta_1^{\frac{1}{2}} c_{\Omega,p}}{\tau^n (1 - \tau^{\frac{\alpha}{2}})}.
\end{aligned}$$

We impose, without loss of generality, a further condition on δ_1, namely, that

$$\delta_1 < \frac{\tau^{2n}(1 - \tau^{\frac{\alpha}{2}})^2}{2 c_{\Omega,p}^2}. \tag{4.110}$$

It is then clear that, under this assumption,

$$\left|(\nabla \overline{u}_0)_{x_0,\tau^k R}\right| < m. \tag{4.111}$$

Having established (4.108) and (4.111), we can apply estimate (4.94) with $\tau^k R$ instead of R, followed by the induction hypothesis for $j = k$, to obtain that

$$\begin{aligned}
\tilde{E}(x_0, \tau^{k+1}R) &\leq c_0 \left(\tau^k R\right)^\alpha + \tau^{2\alpha}\tilde{E}(x_0, \tau^k R) \\
&\leq c_0(\tau^k R)^\alpha + \tau^{2\alpha}\left(\tau^{2\alpha k}\tilde{E}(x_0, R) + c_0(\tau^{k-1}R)^\alpha \sum_{j=0}^{k-1}\tau^{\alpha j}\right) \\
&< \tau^{2\alpha(k+1)}\tilde{E}(x_0, R) + c_0(\tau^k R)^\alpha + c_0(\tau^k R)^\alpha \sum_{j=0}^{k-1}\tau^{\alpha j} \qquad (4.112)\\
&< \tau^{2\alpha(k+1)}\tilde{E}(x_0, R) + c_0(\tau^k R)^\alpha \sum_{j=0}^{k}\tau^{\alpha j}. \qquad (4.113)
\end{aligned}$$

This concludes the induction. We remark that it is to obtain (4.112) that it is important to keep as a factor the term $\tau^{2\alpha}$, and not just τ^α in the second term of (4.94).

We have then shown that, with the choices (4.99) and (4.110), conditions (4.102)-(4.105) are satisfied and, furthermore, claim (4.91) is valid for every $j \in \mathbb{N}$. This implies that, for every $j \in \mathbb{N}$,

$$\begin{aligned}
\tilde{E}(x_0, \tau^j R) &\leq \tau^{\alpha j}\tilde{E}(x_0, R) + c_0(\tau^{j-1}R)^\alpha \sum_{i=0}^{j-1}\tau^{\alpha j} \\
&< \tau^{\alpha j}\tilde{E}(x_0, R) + c_0\frac{(\tau^{j-1}R)^\alpha}{1-\tau^\alpha} \\
&= \tau^{\alpha j}\left(\tilde{E}(x_0, R) + c_0\frac{R^\alpha}{\tau^\alpha(1-\tau^\alpha)}\right).
\end{aligned}$$

We now take $r \in (0, R)$ arbitrary.

If $r \in (0, \tau R]$, we find $j \in \mathbb{N}$ such that $r \in [\tau^{j+1}R, \tau^j R)$ and, hence,

$$\begin{aligned}
\tilde{E}(x_0, r) &\leq \frac{c_\Omega}{\tau^n}\tilde{E}(x_0, \tau^j R) < \frac{c_\Omega}{\tau^n}\tau^{\alpha j}\left(\tilde{E}(x_0, R) + c_0\frac{R^\alpha}{\tau^\alpha(1-\tau^\alpha)}\right) \\
&\leq \left(\frac{r}{R}\right)^\alpha \frac{c_\Omega}{\tau^{n+\alpha}}\left(\tilde{E}(x_0, R) + c_0\frac{R^\alpha}{\tau^\alpha(1-\tau^\alpha)}\right).
\end{aligned}$$

On the other hand, if $r \in [\tau R, R]$, $\tilde{E}(x_0, r) \leq \frac{c_\Omega}{\tau^n} \tilde{E}(x_0, R)$. Hence, we have established that

$$\tilde{E}(x_0, r) \leq \tilde{C} \left(\frac{r}{R}\right)^{\alpha} \tag{4.114}$$

holds for every $r \in (0, R]$, with

$$\tilde{C} := \frac{c_\Omega}{\tau^{n+\alpha}} \tilde{E}(x_0, R) + c_0 c_\Omega \frac{R^\alpha}{\tau^{n+2\alpha}(1-\tau^\alpha)}.$$

Furthermore, our choice of R in (4.101) is such that R is also bounded below by a constant depending exclusively on m, n, N, p, c_1, c_2 and Ω. Therefore, we can even find a constant $C > 0$, depending only on these parameters, such that

$$\tilde{E}(x_0, r) \leq C r^\alpha.$$

In addition, since we also have that $g \in C^{1,\frac{\alpha}{2}}(\overline{\Omega}, \mathbb{R}^N)$, we obtain, for a possibly different $C > 0$ with the same properties as the one above, that

$$\begin{aligned} \fint_{\Omega(x_0,r)} |V(\nabla\overline{u} - (\nabla\overline{u})_{x_0,r})|^2 \,\mathrm{d}x &\leq c \fint_{\Omega(x_0,r)} |V(\nabla\overline{u}_0 - (\nabla\overline{u}_0)_{x_0,r})|^2 \,\mathrm{d}x \\ &\quad + c \fint_{\Omega(x_0,r)} |V(\nabla g - (\nabla g)_{x_0,r})|^2 \,\mathrm{d}x \\ &\leq C r^\alpha. \end{aligned} \tag{4.115}$$

Given that this holds for any $x_0 \in \overline{\Omega}$, we have shown that $\nabla\overline{u} \in C^{0,\frac{\alpha}{2}}(\overline{\Omega}, \mathbb{R}^{N\times n})$, by Campanato-Meyers integral characterization of Hölder continuity.

We will now use Schauder estimates, and the fact that we already know $\nabla\overline{u}$ to be continuous, to improve the Hölder continuity that we have obtained and derive that, actually, $\nabla\overline{u} \in C^{0,\alpha}$. In order to do this, we will obtain better estimates for $\fint_{\Omega(x_0,R)} |V(\nabla\overline{u} - \nabla h)| \,\mathrm{d}x$ than the ones obtained in (4.78), (4.82) and (4.83). For this purpose, we will use that $\nabla\overline{u}$ is bounded, and hence $\overline{u}$ is Lipschitz, instead of using only that $\nabla\overline{u}$ is merely higher integrable. We denote

$$M := \|\nabla\overline{u}\|_{\mathrm{L}^\infty(\Omega, \mathbb{R}^N)}. \tag{4.116}$$

As we will establish in Proposition 100, M is bounded above by some constant that only depends on $m > 0$, therefore, we can assume that $M_1 > 0$ is such that, without depending on g or $\overline{u}$ in any way,

$$|\overline{u}(x_1) - \overline{u}(x_2)| \leq M_1|x_1 - x_2|.$$

for every $x_1, x_2 \in \overline{\Omega}$.[7]

Now, let $x_0 \in \overline{\Omega}$ arbitrary for fixed. For $z_0 := (\nabla\overline{u})_{x_0,R}$, let P be the second order Taylor polynomial of the integrand F about the point z_0. Furthermore, let $h \in \mathrm{W}^{1,p}_{\overline{u}}(\Omega, \mathbb{R}^N)$ be P-minimizing. Then, just as in equation (4.62), but with $v_0 = 0$ and with $\overline{u}$ instead of $\overline{u}_0$, we have that

$$\begin{aligned}
&\fint_{\Omega(x_0,R)} |V(\nabla\overline{u} - \nabla h)|^2 \,\mathrm{d}x \\
\leq &\fint_{\Omega(x_0,R)} (F(\nabla\overline{u} - \nabla h + z_0) - P(\nabla\overline{u} - \nabla h + z_0) + P(\nabla\overline{u}) - P(\nabla h)) \,\mathrm{d}x \\
&- \fint_{\Omega(x_0,R)} \big\langle P'(\nabla h), \nabla\overline{u} - \nabla h\big\rangle \,\mathrm{d}x \\
\leq &\fint_{\Omega(x_0,R)} (F(\nabla\overline{u} - \nabla h + z_0) - P(\nabla\overline{u} - \nabla h + z_0)) \,\mathrm{d}x + \fint_{\Omega(x_0,R)} (P(\nabla\overline{u}) - F(\nabla\overline{u})) \,\mathrm{d}x \\
&+ \fint_{\Omega(x_0,R)} (F(\nabla\overline{u}) - F(\nabla h)) \,\mathrm{d}x + \fint_{\Omega(x_0,R)} (F(\nabla h) - P(\nabla h)) \,\mathrm{d}x \\
= &I + II + III + IV.
\end{aligned} \tag{4.117}$$

As in (4.62), we know that $III < 0$ because $\overline{u}$ is an F-minimizer. In addition, by global Schauder estimates for solutions to linear elliptic systems, since $\overline{u} \in C^{1,\frac{\alpha}{2}}(\Omega(x_0, R), \mathbb{R}^N)$, then $h \in C^{1,\frac{\alpha}{2}}(\Omega(x_0, R), \mathbb{R}^N)$ as well and there is a constant $c > 0$ such that

$$[\nabla h]_{0,\frac{\alpha}{2};\Omega(x_0,R)} \leq c[\nabla\overline{u}]_{0,\frac{\alpha}{2};\Omega(x_0,R)} \leq cM_1. \tag{4.118}$$

For a general and extensive treatment of the theory of Schauder estimates we refer the reader to [Sim97, Theorem 1] and [GT01, Section 6.2]. In addition, we note that the domain $\Omega(x_0, R) = \Omega \cap B(x_0, R)$ is only Lipschitz, while it should also be of class $C^{1,\alpha}$ in order to apply the theory of Schauder estimates. However, we can proceed as in [KM10] and, via a

[7] We remark that, since we are using here Proposition 100, such result will be proven using the uniform $\frac{\alpha}{2}$-Hölder continuity, and not the optimal uniform α-Hölder continuity that we will obtain as a result of this final step in the proof of Theorem 76.

regularizing procedure, obtain a set $A(x_0, R)$ of class $C^{1,\alpha}$ such that

$$\Omega(x_0, R) \subseteq A(x_0, R) \subseteq \Omega(x_0, 2R)$$

and for which

$$\partial A(x_0, R) \cap \partial\Omega(x_0, 2R) = B(x_0, R) \cap \partial\Omega.$$

We can therefore assume that $\Omega(x_0, R)$ is $C^{1,\alpha}$, as we need, and proceed with the calculations as we have done. Furthermore, (4.58) implies, as in Step 4 but recalling that we are now taking $v_0 = 0$, that

$$\begin{aligned} I &\leq c \fint_{\Omega(x_0,R)} \omega(|\nabla\overline{u} - \nabla h|)|V(\nabla\overline{u} - \nabla h)|^2 \,\mathrm{d}x \\ &\leq c \fint_{\Omega(x_0,R)} \omega(|\nabla\overline{u} - \nabla h - (\nabla\overline{u} - \nabla h)_{x_0,R}|)|V(\nabla\overline{u} - \nabla h - (\nabla\overline{u} - \nabla h)_{x_0,R})|^2 \,\mathrm{d}x. \end{aligned} \tag{4.119}$$

We observe that, for every $x \in \Omega(x_0, R)$,

$$|\nabla\overline{u}(x) - (\nabla\overline{u})_{x_0,R}| \leq \fint_{\Omega(x_0,R)} |\nabla\overline{u}(y) - \nabla\overline{u}(x)| \,\mathrm{d}y \leq cR^{\frac{\alpha}{2}}.$$

Similarly, using (4.118), we have that

$$|\nabla h(x) - (\nabla h)_{x_0,R}| \leq cM_1 R^{\frac{\alpha}{2}}.$$

Observe that cM_1 can be taken to be depending exclusively on m, n and p. We now combine these estimates with (4.119) and infer that, for some constant $c = c(m,p) > 0$,

$$I \leq c\,\omega(cM_1 R^{\frac{\alpha}{2}}) \fint_{\Omega(x_0,R)} \left(|V(\nabla\overline{u} - (\nabla\overline{u})_{x_0,R})|^2 + |V(\nabla h - (\nabla h)_{x_0,R})|^2\right) \mathrm{d}x. \tag{4.120}$$

We use again (4.58) to obtain, in a similar way than for (4.120), that

$$
\begin{aligned}
II + IV \leq & c \fint_{\Omega(x_0,R)} \left(\omega(|\nabla\overline{u} - z_0|)|V(\nabla\overline{u} - z_0)|^2 + \omega(|\nabla h - z_0|)|V(\nabla h - z_0)|^2\right) \mathrm{d}x \\
\leq & c \fint_{\Omega(x_0,R)} \left(\omega(cR^{\frac{\alpha}{2}})|V(\nabla\overline{u} - (\nabla\overline{u})_{x_0,R})|^2 + \omega(cM_1R^{\frac{\alpha}{2}})|V(\nabla h - (\nabla h)_{x_0,R})|^2\right) \mathrm{d}x.
\end{aligned}
\tag{4.121}
$$

Since $h \in \mathrm{W}^{1,p}_{\overline{u}}(\Omega, \mathbb{R}^N)$ is P-minimizing, analogous inequalities to those in (4.60) and (4.61) hold by replacing $\overline{u}_0$ by $\overline{u}$, $\tilde{p}$ by p and using L^p-estimates. Therefore, (4.117), (4.120) and (4.121) brought together imply that

$$
\fint_{\Omega(x_0,R)} |V(\nabla\overline{u} - \nabla h)|^2 \, \mathrm{d}x \leq \; c\omega(cM_1R^{\frac{\alpha}{2}}) \fint_{\Omega(x_0,R)} |V(\nabla\overline{u} - (\nabla\overline{u})_{x_0,R})|^2 \, \mathrm{d}x.
$$

Now, we proceed as in (4.84) to get that, for every $0 < r \leq R$, if

$$
E(x_0, r) := \fint_{\Omega(x_0,r)} |V(\nabla\overline{u} - (\nabla\overline{u})_{x_0,r})|^2 \, \mathrm{d}x,
$$

then

$$
\begin{aligned}
E(x_0, r) \leq & c\left(\frac{R}{r}\right)^n \fint_{\Omega(x_0,R)} |V(\nabla\overline{u} - \nabla h)|^2 \, \mathrm{d}x + c\left(\frac{r}{R}\right)^2 E(x_0, R) \\
\leq & \left(c\omega\left(cM_1R^{\frac{\alpha}{2}}\right)\left(\frac{R}{r}\right)^n + c\left(\frac{r}{R}\right)^2\right) E(x_0, R).
\end{aligned}
$$

We now write $r = \tau R$ for $0 < \tau < 1$, so that

$$
\begin{aligned}
E(x_0, \tau R) \leq & \left(c\omega\left(cM_1R^{\frac{\alpha}{2}}\right)\tau^{-n} + c\tau^2\right) E(x_0, R) \\
\leq & \tau^{2\alpha}\left(c\omega\left(cM_1R^{\frac{\alpha}{2}}\right)\tau^{-n-2\alpha} + c\tau^{2(1-\alpha)}\right) E(x_0, R).
\end{aligned}
$$

Now, let $\tau \in (0, 1)$ be fixed by

$$
c\tau^{2(1-\alpha)} = \frac{1}{2}
$$

and take $R_1 \in (0, R_\Omega)$ such that

$$c\,\omega\left(cM_1R_1^{\frac{\alpha}{2}}\right) \leq \frac{1}{2}.$$

Then we have that, for every $R \in (0, R_1]$ and every $\tau \in (0,1)$,

$$E(x_0, \tau R) \leq \tau^{2\alpha} E(x_0, R).$$

Therefore, for every $k \in \mathbb{N}$ we have that

$$E(x_0, \tau^k R) \leq \tau^{2\alpha k} E(x_0, R).$$

We now fix $0 < R < R_1$ and observe that, for any given $r \in (0, R)$, we can take $k \in \mathbb{N}$ so that

$$\tau^{k+1} R < r \leq \tau^k R$$

and hence obtain

$$E(x_0, r) \leq \tau^{-n} E(x_0, \tau^k R) < E(x_0, R)\tau^{-n-2\alpha}\left(\frac{r}{R}\right)^{2\alpha}.$$

Observe that, since the constant $cM_1 > 0$ depends exclusively on m, n and p, we can further assume that $\frac{R_1}{2} < R < R_1$ and, therefore, we have found a constant $C_1 = C_1(m,n,p) > 0$ such that

$$E(x_0, r) \leq C_1 r^{2\alpha}. \tag{4.122}$$

This implies that $\overline{u} \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$, by Campanato-Meyers characterization of Hölder continuity. $\square$

We emphasize that $[\nabla \overline{u}]_{0,\frac{\alpha}{2}}$ is bounded above by $C > 0$, as we derived in (4.115). Therefore, $[\nabla \overline{u}]_{0,\frac{\alpha}{2}}$ is bounded by a constant that, although depending on m, does not depend specifically on the boundary condition g. We also remark that, to prove this, we did not make use of Proposition 100. This fact is a key ingredient in showing that, given $m > 0$ and the corresponding $\varepsilon > 0$ provided by Theorem 76, the set of minimizers satisfying small boundary conditions in the $\mathrm{W}^{1,p}$ sense, is compact. This, in turn, is essential to show that, given a suitable boundary condition g, there is actually one and only one minimizer in the Sobolev class $W_g^{1,p}(\Omega, \mathbb{R}^N)$, which is the subject that Chapter 5 is devoted to.

4.3 The case of $W^{1,p}$-local minimizers

The regularity result in the previous section concerns global minimizers with smooth and sufficiently small boundary conditions in $W^{1,p}$. A natural question is to ask whether something similar can be obtained for local minimizers, specially in the context of the sufficiency theorem in Chapter 2, where C^1 regularity is assumed to ensure that certain extremals actually furnish strong local minimizers (more precisely, with the L^p-strong topology).

Throughout the proof of Theorem 76 we have emphasized that the minimality of $\overline{u}$ was essentially used, first, for the energy estimates from Lemma 74. This is the key idea required to ensure the smallness condition that allows us to iterate the decay of the excess. On the other hand, we used the minimality of $\overline{u}$ for the estimates leading to higher integrability in Steps 1-2.2. Finally, we exploited the fact that $\overline{u}$ is a minimizer to compare its energy with that of the $\mathbb{A}$-harmonic function h in Step 4. We remark that the rest of the proof of Theorem 76 does not use other properties of $\overline{u}$ other than the fact that it satisfies the weak Euler-Lagrange equation and it is already in $W^{1,p}(\Omega, \mathbb{R}^N)$.

4.3.1 Energy estimates

In order to obtain a full regularity result for local minimizers, we need to find a suitable alternative way of obtaining each one of the energy estimates mentioned above, even if $\overline{u}$ is only a local minimizer. The first step will be to obtain a result similar to Lemma 74. Taheri shows in [Tah03] that, if Ω is star-shaped, smooth local minimizers with affine boundary conditions are affine and hence, by strong quasiconvexity, they are unique. Motivated by his proof of this statement, we have been able to establish that an estimate similar to the one from Lemma 74 also holds for local minimizers in star-shaped domains. Before proceeding with our result, we state the following definition.

Definition 77 *Let $\omega \subseteq \mathbb{R}^n$ an open set. We say that ω is* ***star-shaped with respect to*** 0 *if and only if, for every $x \in \overline{\omega}$, the line segment*

$$\{\lambda x : 0 \leq \lambda \leq 1\}$$

is a subset of $\overline{\omega}$*. Similarly, we say that* ω *is star-shaped with respect to* $x_0 \in \omega$ *if and only if* $\omega - x_0$ *is star-shaped with respect to* 0.

In what remains of this chapter, whenever we talk about a star-shaped domain, we shall always assume, without loss of generality, that it is centred at 0. Let ω be a star-shaped domain such that $\partial\omega$ is of class C^1. Then, we can assume that there exists a strictly positive function $d\colon \mathbb{S}^{n-1} \to \mathbb{R}$ of class C^1 such that

$$\partial\omega = \left\{ y \in \mathbb{R}^n : |y| = d\left(\frac{y}{|y|}\right) \right\}. \tag{4.123}$$

It is then clear that $\omega = \{0\} \cup \{y \in \mathbb{R}^n \backslash \{0\} : |y| < d(\theta)\}$, where $\theta = \frac{y}{|y|}$. In addition, we can then see that the unit outer normal to the boundary at a point $y \in \partial\omega$ is the vector

$$\nu(y) = \alpha^{-1}(\theta)\left(\theta - (I - \theta \otimes \theta)\frac{\nabla d(\theta)}{d(\theta)}\right), \tag{4.124}$$

with

$$\alpha(\theta) = d(\theta)^{-1}\left(d(\theta)^2 + |\nabla d(\theta)|^2 - (\theta \cdot \nabla d(\theta))^2\right)^{\frac{1}{2}}. \tag{4.125}$$

We also observe that, for any $\tau > 0$, if $d_\tau := \tau d$ and $\omega_\tau := \tau\omega$, then

$$\partial\omega_\tau = \left\{ y \in \mathbb{R}^n : |y| = d_\tau\left(\frac{y}{|y|}\right) \right\}.$$

In the context of integrable functions defined on star-shaped domains, it will be very useful to consider the following representation result, which can be shown by using the co-area formula. See also [Tah03] for an alternative proof.

Proposition 78 *Let* $\omega \subseteq \mathbb{R}^n$ *be a star-shaped domain and let* $f \in \mathrm{L}^1(\omega, \mathbb{R}^N)$*. Then,*

$$\int\limits_{\omega} f(y)\,\mathrm{d}y = \int_0^1 \int\limits_{\partial\omega} \frac{d(\theta)}{\alpha(\theta)} f(\rho y)\rho^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\rho.$$

We continue with the following definition.

Definition 79 *Given a star-shaped bounded domain* ω *and a map* $v \in \mathrm{W}^{1,p}(\partial\omega, \mathbb{R}^N)$*, we define its* ***degree-one homogeneous extension*** $v^h\colon \mathbb{R}^n \to \mathbb{R}^N$ *as the mapping given by*

$$v^h(y) := \frac{|y|}{d(\theta)} v(\theta d(\theta)), \tag{4.126}$$

where $\theta := \frac{y}{|y|}$.

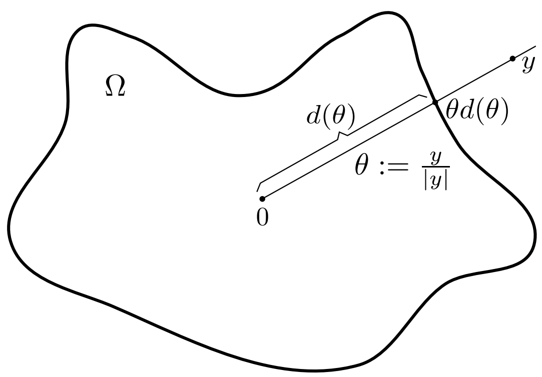

Figure 4.1: Smooth star-shaped domain.

Notice that we have defined v^h on the whole space $\mathbb{R}^n$, and not only on ω, as in [Tah03]. It can then be easily checked that

$$\nabla v^h(y) = \nabla v(\theta d(\theta)) + \left(\frac{v(\theta d(\theta))}{d(\theta)} - \nabla v(\theta d(\theta))\theta\right) \otimes \left(\theta - (I - \theta \otimes \theta)\frac{\nabla d(\theta)}{d(\theta)}\right). \tag{4.127}$$

We now establish the following lemma, that illustrates the behaviour of degree-one homogeneous extensions with respect to their integrals over dilatations of the domain.

Lemma 80 *Let* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *be a measurable function such that it satisfies* (H1). *In addition, let* $\omega \subseteq \mathbb{R}^n$ *be a star-shaped domain and* $v \in W^{1,p}(\partial\omega, \mathbb{R}^N)$. *Let* v^h *be its degree-one homogeneous extension. Then, for every* $\tau > 0$ *it holds that*

$$\int\limits_{\omega_\tau} f(\nabla v^h)\,\mathrm{d}y = \tau^n \int\limits_{\omega} f(\nabla v^h)\,\mathrm{d}y. \tag{4.128}$$

Proof. The proof will rely on Proposition 78. Indeed, we observe that if $d\colon \mathbb{S}^{n-1} \to \mathbb{R}$ is such that

$$\partial\omega = \left\{y \in \mathbb{R}^n : |y| = d\left(\frac{y}{|y|}\right)\right\}, \tag{4.129}$$

then, as before, we have that

$$\partial\omega_\tau = \left\{y \in \mathbb{R}^n : |y| = \tau d\left(\frac{y}{|y|}\right)\right\}. \tag{4.130}$$

By Proposition 78, we can write

$$\begin{aligned}\int_{\omega_\tau} f(\nabla v^h)\,\mathrm{d}y &= \int_0^1 \int_{\partial\omega_\tau} \frac{\tau d(\theta)}{\alpha(\theta)} f(\nabla v^h(\rho y))\rho^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\rho \\ &= \int_0^1 \int_{\partial\omega_\tau} \frac{\tau d(\theta)}{\alpha(\theta)} f(\nabla v^h(y))\rho^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\rho,\end{aligned}$$

where the last identity follows from the fact that $\nabla v^h(\rho y) = \nabla v^h(y)$ for every $y \in \mathbb{R}^n$. Therefore, after applying a change of variables and the corresponding invariance under dilatations of ∇v^h again, we conclude from above that

$$\begin{aligned}\int_{\omega_\tau} f(\nabla v^h)\,\mathrm{d}y =& \frac{\tau}{n}\int_{\partial\omega_\tau} \frac{d(\theta)}{\alpha(\theta)} f(\nabla v^h(y))\,\mathrm{d}\mathcal{H}^{n-1} \\ =& \frac{\tau^n}{n}\int_{\partial\omega} \frac{d(\theta)}{\alpha(\theta)} f(\nabla v^h(y))\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

By applying this formula for $\tau > 0$ arbitrary and $\tau = 1$, we conclude that

$$\int_{\omega_\tau} f(\nabla v^h)\,\mathrm{d}y = \tau^n \int_{\omega} f(\nabla v^h)\,\mathrm{d}y, \tag{4.131}$$

as we wanted to prove. □

We conclude this introduction by recalling that if $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a C^1 rank-one convex function, then for every $\lambda \in \mathbb{R}^N$ and every $\mu \in \mathbb{R}^n$ it holds that

$$F(\xi + \lambda \otimes \mu) \geq F(\xi) + \langle F'(\xi), \lambda \otimes \mu\rangle. \tag{4.132}$$

Given that quasiconvex functions are rank-one convex, the above inequality holds in particular for the class of integrands that we have now under consideration.

We are now ready to enunciate the following result.

Proposition 81 *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfy* (H0) – (H2) *for some* $p \in [2, \infty)$. *Assume that* Ω *is a star-shaped domain and let* $\overline{u} \in \mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$ *be a* $\mathrm{W}^{1,p}$*-local minimizer of* $\mathfrak{F}$. *Then,* *there is a constant $c > 0$, depending exclusively on n, such that*

$$\int_\Omega |V(\nabla\overline{u})|^2 \,\mathrm{d}y \leq c \int_\Omega |V(\nabla g^h)|^2 \,\mathrm{d}y, \tag{4.133}$$

where g^h is the one-degree homogeneous extension of $g\colon \partial\Omega \to \mathbb{R}^N$.

The idea is to exploit Ali Taheri's proof on the energy estimates that can be obtained for a stationary point in terms of its degree-one homogeneous extension.

We now recall the following definition.

Definition 82 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be of class C^1 and satisfying the p-growth condition* (H2). *Assume, in addition, that $|F'(z)| \leq c(1+|z|^{p-1})$ for some $c > 0$ and every $z \in \mathbb{R}^{N\times n}$. Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set and let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$. We say that u is a* ***stationary point*** *of the functional $\mathfrak{F}$ if and only if*

$$\frac{d}{d\varepsilon}\mathfrak{F}(u_\varepsilon,\Omega)\restriction_{\varepsilon=0} = \int_\Omega \left(F(\nabla u)\langle \mathrm{I}_{N\times n}, \nabla\varphi\rangle - \langle \nabla u^T \cdot F'(\nabla u), \nabla\varphi\rangle\right) \mathrm{d}y = 0$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n,\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subseteq \Omega$ and where $u_\varepsilon(y) := u(y+\varepsilon\varphi(y))$.

Stationary points are those for which the variation in the functional vanishes when one introduces small variations in the domain.

Taheri shows that if $\overline{u} \in W^{1,p}(\Omega,\mathbb{R}^N)$ is a stationary point of $\mathfrak{F}(\cdot,\Omega)$, then for $\mathcal{L}^1$-almost every $t \in (0,1]$,

$$\mathfrak{F}(\overline{u},\Omega_t) \leq \mathfrak{F}(\overline{u}_t^h,\Omega_t), \tag{4.134}$$

where $\overline{u}_t^h$ is the homogeneous degree-one extension map corresponding to $\overline{u}\restriction_{\partial\Omega_t}$.[8] In the context of local minimizers, we wish to obtain similar energy estimates for the particular case $t = 1$. However, we cannot adapt Taheri's proof to deduce such control on the energy of the minimizer, even under smooth boundary conditions. This is so because, for (4.134) to hold, one needs $\nabla\overline{u}_t^h$ to be continuous on $\partial\Omega_t$. This will certainly be true for almost every $t \in (0,1]$. However, even though we know that $\overline{u} = g$ on $\partial\Omega$, we cannot ensure that $\nabla\overline{u}$ doesn't oscillate too much in the direction normal to $\partial\Omega$.

[8] See [Tah03].

On the other hand, Taheri observes in [Tah03] that, if $u \in \mathrm{W}^{1,p}_{\xi y}(\Omega, \mathbb{R}^N)$ is a $\mathrm{W}^{1,p}$-local minimizer of $\mathfrak{F}(\cdot, \Omega)$, then there is a $\tau > 1$ such that the map

$$v := \begin{cases} u & \text{in } \Omega \\ \xi y & \text{in } \Omega_\tau \backslash \Omega \end{cases}$$

is a $\mathrm{W}^{1,p}$-local minimizer of $\mathfrak{F}(\cdot, \Omega_\tau)$.

By definition of v, we know that this new local minimizer is of class C^1 in a neighbourhood of $\partial\Omega_\tau$. This is exactly what we need to ensure that (4.134) holds for $t = 1$ with domain Ω_τ instead of Ω. The proof of our Proposition is inspired by this idea and it will rely on considering two cases: that in which we can ensure that an extension of the local minimizer $\overline{u}$ is also a local minimizer on a slightly dilated domain Ω_τ, and the case in which this is not necessarily true. We will see in the proof below that in both situations it is possible to obtain the estimate (4.133), from which Proposition 81 follows.

Proof. For $1 < \tau < 2$ we define the map $v_\tau \in \mathrm{W}^{1,p}(\Omega_\tau, \mathbb{R}^N)$ by

$$v_\tau := \begin{cases} \overline{u} & \text{in } \Omega \\ g^h & \text{in } \overline{\Omega_\tau} \backslash \Omega \end{cases}.$$

As we have already mentioned, the proof will be split into the two following cases.

Case 1. For every $\tau \in (1, 2)$, the function v_τ is not a $\mathrm{W}^{1,p}$-local minimizer of $\mathfrak{F}(\cdot, \Omega_\tau)$. Then, there exists a monotonically decreasing sequence $\tau_j \to 1$ and a sequence $(v_j) \subseteq \mathrm{W}^{1,p}_{g^h}(\Omega_{\tau_j}, \mathbb{R}^N)$, with $v_j \to v$ in $\mathrm{W}^{1,p}(\Omega_{\tau_j}, \mathbb{R}^N)$, such that

$$\int_{\Omega_{\tau_j}} F(\nabla v_j)\,\mathrm{d}y < \int_{\Omega_{\tau_j}} F(\nabla v)\,\mathrm{d}y. \tag{4.135}$$

We now define $u_j \colon \Omega \to \mathbb{R}^N$ by

$$u_j(y) := \tau_j^{-1} v_j(\tau_j y).$$

We note that, since $v_j \in W^{1,p}_{g^h}(\Omega_{\tau_j}, \mathbb{R}^N)$, $u_j \in W^{1,p}_{g^h}(\Omega, \mathbb{R}^N) = W^{1,p}_{g}(\Omega, \mathbb{R}^N)$ for every $j \in \mathbb{N}$. We claim that $u_j \to u$ in $W^{1,p}(\Omega, \mathbb{R}^N)$. Indeed, we observe that

$$\begin{aligned}\int_\Omega |\nabla u_j(y) - \nabla u(y)|^p \,\mathrm{d}y =& \int_\Omega |\nabla v_j(\tau_j y) - \nabla u(y)|^p \,\mathrm{d}y\\ =&\tau_j^{-n} \int_{\Omega_{\tau_j}} \left|\nabla v_j(y) - \nabla u\left(\tau_j^{-1} y\right)\right|^p \mathrm{d}y\\ \le&\tau_j^{-n} \int_{\Omega_{\tau_j}} |\nabla v_j(y) - \nabla v(y)|^p \,\mathrm{d}y + \tau_j^{-n} \int_{\Omega_{\tau_j}} \left|\nabla v(y) - \nabla u\left(\tau_j^{-1} y\right)\right|^p \mathrm{d}y\\ =&\tau_j^{-n} \int_{\Omega_{\tau_j}} |\nabla v_j(y) - \nabla v(y)|^p \,\mathrm{d}y + \tau_j^{-n} \int_{\Omega_{\tau_j}} \left|\nabla v(y) - \nabla v\left(\tau_j^{-1} y\right)\right|^p \mathrm{d}y.\end{aligned}$$

When $j \to \infty$, the first term above goes to zero because $v_j \to v$ in $W^{1,p}$ and $\tau_j \to 1$. The second term goes to zero by the continuity of dilatations, which follows in turn by the density of the space of continuous functions in $W^{1,p}(\Omega_{\tau_1}, \mathbb{R}^N)$. Given that u is a $W^{1,p}$-local minimizer, this implies that, for $J \in \mathbb{N}$ large enough,

$$\begin{aligned}\tau_J^n \int_\Omega F(\nabla \overline{u}) \,\mathrm{d}y \le& \tau_J^n \int_\Omega F(\nabla u_J) \,\mathrm{d}y\\ =& \int_{\Omega_{\tau_J}} F(\nabla v_J) \,\mathrm{d}y\\ <& \int_{\Omega_{\tau_J}} F(\nabla v) \,\mathrm{d}y\\ =& \int_\Omega F(\nabla \overline{u}) \,\mathrm{d}y + \int_{\Omega_{\tau_J} \setminus \Omega} F(\nabla g^h) \,\mathrm{d}y.\end{aligned}$$

These calculations are based in Taheri's proof that $W^{1,p}$-local minimizers, with affine boundary conditions, can be extended into a dilatation of its domain, so that the extension furnishes a local minimizer of the corresponding integral functional. We now proceed in a slightly different, but still analogous, way as in [Tah03]. We conclude from the above chain of inequalities that

$$(\tau_J^n - 1) \int_\Omega F(\nabla \overline{u}) \,\mathrm{d}y \le \int_{\Omega_{\tau_J} \setminus \Omega} F(\nabla g^h) \,\mathrm{d}y = (\tau_J^n - 1) \int_\Omega F(\nabla g^h) \,\mathrm{d}y.$$

The last identity above follows from Lemma 80. We can therefore conclude that

$$\int_{\Omega} F(\nabla \overline{u})\, \mathrm{d}y \leq \int_{\Omega} F(\nabla g^h)\, \mathrm{d}y. \tag{4.136}$$

We now use that $u - g^h \in \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N)$ to deduce from (4.136), using the quasiconvexity of F, that for some $c > 0$,

$$\int_{\Omega} |V(\nabla \overline{u})|^2\, \mathrm{d}y \leq c \int_{\Omega} |V(\nabla g^h)|^2\, \mathrm{d}y, \tag{4.137}$$

just as we wanted to prove. It remains to show that such an inequality holds in the following scenario.

Case 2. There exists $\tau \in (1,2)$ such that v_τ is a $\mathrm{W}^{1,p}$-local minimizer for $\mathfrak{F}(\cdot, \Omega_\tau)$. The proof in this case will be essentially the same as in [Tah03], incorporating here the fact that, under this assumption, we know that ∇v_τ is of class C^1 in a neighbourhood of $\partial\Omega_\tau$.
For the sake of simplicity of the notation we call $v := v_\tau$. We aim at showing that

$$\int_{\Omega_\tau} F(\nabla v)\, \mathrm{d}y \leq \int_{\Omega_\tau} F(\nabla v^h)\, \mathrm{d}y. \tag{4.138}$$

We begin the proof by recalling that, since v is a local minimizer, it will satisfy the weak Euler-Lagrange equation, meaning that, for every $\varphi \in \mathrm{W}_0^{1,p}(\Omega_\tau, \mathbb{R}^N)$,

$$\frac{d}{d\varepsilon}\mathfrak{F}(v + \varepsilon\varphi, \Omega_\tau) \restriction_{\varepsilon=0} = \int_{\Omega_\tau} \langle F'(\nabla v), \nabla\varphi\rangle\, \mathrm{d}y = 0. \tag{4.139}$$

In addition, it is not difficult to verify that v satisfies the following stationarity condition, meaning that the variation of the functional under variations in the domain, also vanishes. More specifically, it holds that, if $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{supp}\varphi \subseteq \Omega_\tau$ and we consider $\psi_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\psi_\varepsilon(y) := y + \varepsilon\varphi(y)$ then, by setting $v_\varepsilon := v \circ \psi_\varepsilon$, it follows that

$$\frac{d}{d\varepsilon}\mathfrak{F}(v_\varepsilon, \Omega_\tau) \restriction_{\varepsilon=0} = \int_{\Omega_\tau} \big(F(\nabla v)\langle \mathrm{I}_{N\times n}, \nabla\varphi\rangle - \langle \nabla v^T \cdot F'(\nabla v), \nabla\varphi\rangle\big)\, \mathrm{d}y = 0. \tag{4.140}$$

We now observe that, for every $y \in \partial\Omega_\tau$, since $v \in \mathrm{W}^{1,p}_{g^h}(\Omega_\tau, \mathbb{R}^N)$,

$$\begin{aligned} v^h(y) =& \frac{|y|}{\tau d(\theta)} v(\theta \tau d(\theta)) = \frac{|y|}{\tau d(\theta)} g^h(\theta\tau d(\theta)) \\ =& \frac{|y|}{d(\theta)} g(\theta d(\theta)) = g^h(y). \end{aligned}$$

This means that $v^h = g^h$ on Ω_τ. On the other hand, using (4.127), (4.132) and Proposition 78, we can write

$$\begin{aligned} &\int_{\Omega_\tau} nF(\nabla v^h(y))\,\mathrm{d}y \\ &= \int_{\partial\Omega_\tau} \frac{d_\tau(\theta)}{\alpha(\theta)} F\left(\nabla v(y) + \left(\frac{v(y)}{d_\tau(\theta)} - \nabla v(y)\theta\right) \otimes \left(\theta - (I - \theta\otimes\theta)\frac{\nabla d_\tau(\theta)}{d(\theta)}\right)\right) \mathrm{d}\mathcal{H}^{n-1}(y) \\ &\geq \int_{\partial\Omega_\tau} \frac{d_\tau(\theta)}{\alpha(\theta)} F(\nabla v(y))\,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\partial\Omega_\tau} \frac{1}{\alpha(\theta)} \left\langle F'(\nabla v(y)), v(y) \otimes \left(\theta - (I-\theta\otimes\theta)\frac{\nabla d_\tau(\theta)}{d_\tau(\theta)}\right)\right\rangle \mathrm{d}\mathcal{H}^{n-1}(y) \\ &\quad - \int_{\partial\Omega_\tau} \frac{d_\tau(\theta)}{\alpha(\theta)} \left\langle F'(\nabla v(y)), \nabla v(y)\theta \otimes \left(\theta - (I-\theta\otimes\theta)\frac{\nabla d_\tau(\theta)}{d_\tau(\theta)}\right)\right\rangle \mathrm{d}\mathcal{H}^{n-1}(y). \end{aligned} \tag{4.141}$$

We now proceed by taking the particular choice $\varphi_\varepsilon(y) := r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right) y$ in (4.140) for $\varepsilon > 0$, where

$$r_\varepsilon(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq 1-\varepsilon \\ 1 - \frac{s-(1-\varepsilon)}{\varepsilon} & \text{if } 1-\varepsilon \leq s \leq 1 \end{cases}.$$

We then have that

$$\nabla\varphi_\varepsilon = r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right) I + \frac{1}{d_\tau(\theta)} r'_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right) y \otimes \left(\theta - (I-\theta\otimes\theta)\frac{\nabla d_\tau(\theta)}{d_\tau(\theta)}\right).$$

Substituting this in (4.140), we obtain that

$$\begin{aligned}
&\int_{\Omega_\tau} n r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right) F(\nabla v(y))\,\mathrm{d}y \\
=&-\int_{\Omega} \frac{1}{d_\tau(\theta)} r_\varepsilon'\left(\frac{|y|}{d_\tau(\theta)}\right) y\cdot\left(\theta-(I-\theta\otimes\theta)\frac{\nabla d_\tau(\theta)}{d_\tau(\theta)}\right) F(\nabla v(y))\,\mathrm{d}y \\
&+\int_{\Omega_\tau} r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right)\left\langle F'(\nabla v(y)),\nabla v(y)\right\rangle\,\mathrm{d}y \\
&+\int_{\Omega_\tau} \frac{1}{d_\tau(\theta)} r_\varepsilon'\left(\frac{|y|}{d_\tau(\theta)}\right)\left\langle F'(\nabla v(y)),\nabla v(y) y\otimes\left(\theta-(I-\theta\otimes\theta)\frac{\nabla d_\tau(\theta)}{d_\tau(\theta)}\right)\right\rangle\,\mathrm{d}y \\
=&\, I+II+III. \qquad (4.142)
\end{aligned}$$

Using again Proposition 78, we note that

$$\lim_{\varepsilon\to 0} I=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1}\rho^n\int_{\partial\Omega_\tau}\frac{d_\tau(\theta)}{\alpha(\theta)}F(\nabla v(\rho y))\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\rho. \qquad (4.143)$$

We now observe that, since $\tau>1$, $\frac{d(\theta)}{|y|}<\frac{d_\tau(\theta)}{|y|}$ for every $y\in\overline{\Omega}_\tau$. Hence, for every $y\in\partial\Omega_\tau$ and every $0<\varepsilon<1-\frac{1}{\tau}$,

$$\frac{d(\theta)}{|y|}=\frac{1}{\tau}<1-\varepsilon<1=\frac{d_\tau(\theta)}{|y|}.$$

Therefore, if $\rho\in(1-\varepsilon,1)$, it follows from above that, for $y\in\partial\Omega_\tau$,

$$d(\theta)<\rho|y|<\tau d(\theta)$$

or, equivalently, $\rho y\in\overline{\Omega}_\tau\backslash\Omega$ if $1-\varepsilon\le\rho\le 1$.

From here, the definition of $v=v_\tau$ and (4.143), we deduce that

$$\lim_{\varepsilon\to 0} I=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1}\rho^n\int_{\partial\Omega_\tau}\frac{d_\tau(\theta)}{\alpha(\theta)}F(\nabla g^h(\rho y))\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\rho.$$

We now use that ∇g^h is continuous to deduce, using Lebesgue's Differentiation Theorem and the fact that every point is a Lebesgue point for a continuous function, that the limit above exists and, even more,

$$\begin{aligned}\lim_{\varepsilon\to 0} I &= \int\limits_{\partial\Omega_\tau} \frac{d_\tau(\theta)}{\alpha(\theta)} F(\nabla g^h(y))\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \int\limits_{\partial\Omega_\tau} \frac{d_\tau(\theta)}{\alpha(\theta)} F(\nabla v(y))\,\mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned} \tag{4.144}$$

We emphasize that, for the last identity above, we are using that $v = g^h$ in a neighbourhood of $\partial\Omega_\tau$. Using an analogous argument for III, we obtain that the limit as $\varepsilon \to 0$ exists and

$$\lim_{\varepsilon\to 0} III = -\int\limits_{\partial\Omega_\tau} \frac{d_\tau(\theta)}{\alpha(\theta)} \left\langle F'(\nabla v(y)), \nabla v(y)\theta \otimes \left(\theta - (I-\theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right)\right\rangle \mathrm{d}\mathcal{H}^{n-1}(y). \tag{4.145}$$

We will now use the weak Euler Lagrange Equation to compute $\lim\limits_{\varepsilon\to 0} II$. With this purpose, we now define

$$\psi_\varepsilon := r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right) v(y). \tag{4.146}$$

We observe first that, if $y \in \partial\Omega_\tau$, $\psi_\varepsilon(y) = r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right) v(y) = r_\varepsilon(1)v(y) = 0$. This implies that $\psi_\varepsilon \in W_0^{1,p}(\Omega_\tau, \mathbb{R}^N)$. We now compute, for $y \in \Omega_\tau$,

$$\nabla\psi_\varepsilon(y) = r_\varepsilon\left(\frac{|y|}{d(\theta)}\right)\nabla v(y) + \frac{1}{d_\tau(\theta)} r_\varepsilon'\left(\frac{|y|}{d_\tau(\theta)}\right)\left(v(y)\otimes\left(\theta - (I-\theta\otimes\theta)\frac{\nabla d_\tau(\theta)}{d_\tau(\theta)}\right)\right).$$

Given that the weak Euler-Lagrange equation holds for v, as well as (4.124) and (4.125), we infer that

$$\begin{aligned}0 &= \int\limits_{\Omega_\tau} \left\langle F'(\nabla v(y)), \nabla\psi_\varepsilon(y)\right\rangle \mathrm{d}y\\ &= \int\limits_{\Omega_\tau} r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right)\left\langle F'(\nabla v(y)), \nabla v(y)\right\rangle \mathrm{d}y\\ &\quad + \int\limits_{\Omega_\tau} \frac{1}{d_\tau(\theta)} r_\varepsilon'\left(\frac{|y|}{d_\tau(\theta)}\right)\left\langle F'(\nabla v(y)), v(y)\otimes\alpha(\theta)\nu(y)\right\rangle \mathrm{d}y.\end{aligned} \tag{4.147}$$

Using this, and rewriting the second term above with the help of Proposition 78, we can follow the same argument that we used to obtain (4.144) to deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0} II &= -\lim_{\varepsilon\to 0}\int_{\Omega_\tau}\frac{1}{d_\tau(\theta)}r_\varepsilon'\left(\frac{|y|}{d_\tau(\theta)}\right)\big\langle F'(\nabla v(y)), v(y)\otimes\alpha(\theta)\nu(y)\big\rangle\,\mathrm{d}y\\ &=\int_{\partial\Omega_\tau}\big\langle F'(\nabla v(y)), v(y)\otimes\nu(y)\big\rangle\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}\tag{4.148}$$

We now observe that $r_\varepsilon\left(\frac{|y|}{d_\tau(\theta)}\right)\to\mathbb{1}_{\Omega_\tau}$ pointwise in Ω_τ so that, by Lebesgue's Dominated Convergence Theorem together with (H1)-(H2), we can pass to the limit in (4.142) and, in conjunction with (4.144), (4.145) and (4.148), we deduce that

$$\begin{aligned}&\int_{\Omega_\tau} nF(\nabla v(y))\,\mathrm{d}y\\ &=\int_{\partial\Omega_\tau}\frac{d_\tau(\theta)}{\alpha(\theta)}F(\nabla v(y))\,\mathrm{d}\mathcal{H}^{n-1}(y)+\int_{\partial\Omega_\tau}\big\langle F'(\nabla v(y)), v(y)\otimes\nu(y)\big\rangle\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad-\int_{\partial\Omega_\tau}\frac{d_\tau(\theta)}{\alpha(\theta)}\left\langle F'(\nabla v(y)),\nabla v(y)\theta\otimes\left(\theta-(I-\theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right)\right\rangle\,\mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned}$$

Comparing this to the right hand side of (4.141), we finally conclude that

$$\int_{\Omega_\tau}F(\nabla v)\,\mathrm{d}y\le\int_{\Omega_\tau}F(\nabla v^h)\,\mathrm{d}y=\int_{\Omega_\tau}F(\nabla g^h)\,\mathrm{d}y.$$

Since $v-g^h\in \mathrm{W}_0^{1,p}(\Omega_\tau,\mathbb{R}^N)$, we proceed as in Case 1 to further conclude that, for some constant $0<c=c(n)$,

$$\int_{\Omega}|V(\nabla v)|^2\,\mathrm{d}y\le\int_{\Omega_\tau}|V(\nabla v)|^2\,\mathrm{d}y\le c\int_{\Omega_\tau}|V(\nabla g^h)|^2\,\mathrm{d}y=c\tau^n\int_{\Omega}|V(\nabla g^h)|^2\,\mathrm{d}y.$$

The last identity above follows from Lemma 80. Since $\tau<2$ by assumption, this leads to the desired conclusion, with a constant $c>0$ depending exclusively on n. This concludes the proof. □

4.3.2 Full regularity of $\mathrm{W}^{1,p}$-local minimizers

In order to obtain the estimates leading to higher integrability for the case of $\mathrm{W}^{1,p}$-local minimizers, the key idea will be to reduce the problem to the case of global minimizers. Kristensen and Taheri showed in [KT03, Proposition 2.1] that $\mathrm{W}^{1,p}$-local minimizers are spatially-locally global minimizers for a class of integrands satisfying a coercivity condition. We recall that Zhang's Theorem establishes that smooth extremals are spatially-locally minimizing.[9] We can then relate Zhang's Theorem to the main result of this section, in the sense that we are dispensing with the regularity assumption by strengthening the F-extremality hypothesis to $\mathrm{W}^{1,p}$-local minimality. We now establish the following theorem based in the ideas from [Kri11, Lecture 5]. We remark that the result presented here differs from that in [KT03] in that no coercivity assumption is made on the integrand in this case.

Theorem 83 *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous and such that it satisfies* (H1) – (H2) *for some $p \in (1,\infty)$. Assume that $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ is a $\mathrm{W}^{1,p}$-local F-minimizer. Then, there exists $\eta > 0$, depending on $\overline{u}$, such that*

$$\int_\Omega F(\nabla\overline{u})\,\mathrm{d}x \leq \int_\Omega F(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x$$

for every $\varphi \in \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$ such that $|\{x \in \Omega : \varphi(x) \neq 0\}| < \eta$. In particular, $\overline{u}$ is spatially locally F-minimizing.

Proof. The first step is to obtain a Gårding inequality from conditions (H1) – (H2). We define, as usual, the auxiliary integrand

$$\tilde{F}(z) := F(z) - F(0) - \langle F'(0), z\rangle$$

and we let $\varphi \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Denote

$$A := \{x \in \Omega : \varphi(x) \neq 0\}.$$

[9] See Theorem 53.

By (H1) − (H2) and Lemma 13, we have that

$$\begin{aligned} c_2 \int\limits_A |V(\nabla\varphi)|^2 \,\mathrm{d}x &\le \int\limits_A \left(\tilde{F}(\nabla\varphi) - \tilde{F}(\nabla\overline{u} + \nabla\varphi)\right) \mathrm{d}x + \int\limits_A \tilde{F}(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x \\ &= -\int\limits_A \int_0^1 \left\langle \tilde{F}'((1-t)\nabla\overline{u} + \nabla\varphi), \nabla\overline{u} \right\rangle \mathrm{d}t\,\mathrm{d}x + \int\limits_A \tilde{F}(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x \\ &\le c \int\limits_A \left(|V_{p-1}(\nabla\overline{u})| + |V_{p-1}(\nabla\varphi)|\right) |\nabla\overline{u}|\,\mathrm{d}x + \int\limits_A \tilde{F}(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x. \end{aligned} \tag{4.149}$$

We use this, and the subadditivity of V, to observe that for some constant $\tilde{c} = \tilde{c}(p, c_2) > 0$,

$$\begin{aligned} \tilde{c} \int\limits_A |V(\nabla\varphi + \nabla\overline{u})|^2 \,\mathrm{d}x &\le c_2 \int\limits_A |V(\nabla\varphi)|^2 \,\mathrm{d}x + \int\limits_A |V(\nabla\overline{u})|^2 \,\mathrm{d}x \\ &\le c \int\limits_A \left(|V_{p-1}(\nabla\overline{u})| + |V_{p-1}(\nabla\varphi + \nabla\overline{u})|\right) |\nabla\overline{u}|\,\mathrm{d}x + \int\limits_A \tilde{F}(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x. \end{aligned} \tag{4.150}$$

Therefore, by Young's inequality from Lemma 128 (viii) we infer that, for some other constant $c > 0$, also depending on p (and on $\tilde{c}$),

$$\tilde{c} \int\limits_A |V(\nabla\varphi + \nabla\overline{u})|^2 \,\mathrm{d}x \le \frac{\tilde{c}}{2} \int\limits_A |V(\nabla\varphi + \nabla\overline{u})|^2 \,\mathrm{d}x + c \int\limits_A |V(\nabla\overline{u})|^2 \,\mathrm{d}x + \int\limits_A \tilde{F}(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x.$$

Subtracting first $\frac{\tilde{c}}{2} \int\limits_A |V(\nabla\varphi + \nabla\overline{u})|^2 \,\mathrm{d}x$ from both sides and using afterwards the subadditivity of V, this implies that, for some constant $c = c(p) > 0$,

$$\begin{aligned} \frac{\tilde{c}}{2} \int\limits_A |V(\nabla\varphi)|^2 \,\mathrm{d}x &\le c \int\limits_A |V(\nabla\varphi + \nabla\overline{u})|^2 \,\mathrm{d}x + c \int\limits_A |V(\nabla\overline{u})|^2 \,\mathrm{d}x \\ &\le c \int\limits_A |V(\nabla\overline{u})|^2 \,\mathrm{d}x + \int\limits_A F(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x. \end{aligned} \tag{4.151}$$

We now observe that, if

$$\int\limits_\Omega F(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x < \int\limits_\Omega F(\nabla\overline{u})\,\mathrm{d}x,$$

then we also have that

$$\int\limits_A F(\nabla\overline{u} + \nabla\varphi)\,\mathrm{d}x < \int\limits_A F(\nabla\overline{u})\,\mathrm{d}x.$$

Therefore, in this case, (4.151) implies that, for some constants $c_3, c > 0$, both depending on $p \geq 2$,

$$c_3 \int_A |\nabla\varphi|^p \,\mathrm{d}x \leq \frac{\tilde{c}}{2} \int_A |V(\nabla\varphi)|^2 \,\mathrm{d}x < c \int_A \left(|V(\nabla\overline{u})|^2 + F(\nabla\overline{u})\right) \,\mathrm{d}x. \tag{4.152}$$

Recall that, since $\overline{u}$ is a $\mathrm{W}^{1,p}$-local minimizer, there is a $\delta > 0$, which depends on $\overline{u}$, such that if $\|\nabla\varphi\|_{\mathrm{L}^p} < \delta$, then

$$\int_\Omega F(\nabla\overline{u}) \,\mathrm{d}x \leq \int_\Omega F(\nabla\overline{u} + \nabla\varphi) \,\mathrm{d}x. \tag{4.153}$$

We now observe that, since $\{|V(\nabla\overline{u})|^2 + F(\nabla\overline{u})\}$ is equiintegrable as a unitary subset of $L^1(\Omega, \mathbb{R}^N)$, then there exists $\eta > 0$, depending on $\delta > 0$ and $p \geq 2$, such that for every set $B \subseteq \Omega$ satisfying $|B| < \eta$, it holds that

$$\frac{c}{c_3} \left| \int_B \left(|V(\nabla\overline{u})|^2 + F(\nabla\overline{u})\right) \,\mathrm{d}x \right| < \delta.$$

Therefore, whenever A is taken such that $|A| < \eta$, we will have that $\|\nabla\varphi\|_{L^p} < \delta$ by (4.152) and, therefore,

$$\int_\Omega F(\nabla\overline{u}) \,\mathrm{d}x \leq \int_\Omega F(\nabla\overline{u} + \nabla\varphi) \,\mathrm{d}x,$$

as we wanted to prove. □

We remark that the constant η depends on δ and on the minimizer $\overline{u}$. The latter dependence arises from the p-equiintegrability of the set $\{\nabla\overline{u}\}$. This could be avoided if we had, for example, that the set of all local minimizers in $\mathrm{W}^{1,p}_g$ is equiintegrable. However, in connection with de la Vallé Poussin criterion for equiintegrability,[10] we can see that this question is closely related to the problem of higher integrability for local minimizers. This is the core difficulty in all the regularity results concerning local minimizers that have been developed so far. We recall here that the first partial regularity proof for local minimizers of quasiconvex integrands was developed in [KT03, Theorem 4.1] by obtaining the convergence of the blown-up sequence, without making use of any higher integrability results, as in Evans' partial regularity proof [Eva86]. This is also related to the full regularity proof that we presented in Theorem 67.

[10] See Theorem 24

Having established Proposition 81 and Theorem 83, it is now easy to obtain the corresponding full regularity theorem for $\mathrm{W}^{1,p}$-local minimizers if $\Omega \subseteq \mathbb{R}^n$ is a star-shaped and smooth domain. We state the result as follows.

Theorem 84 *Let $\alpha \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded star-shaped domain of class $C^{1,\alpha}$. Suppose that* (H0) – (H2) *hold for some $p \in [2,\infty)$. Then, for every $m > 0$ there exists an $\varepsilon = \varepsilon(m) > 0$ such that, whenever $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^{N\times n})$ satisfies*

$$\|\nabla g\|_{0,\alpha} < m \quad \textit{and} \quad \int_\Omega |V(\nabla g^h)|^2 \, \mathrm{d}x < \varepsilon, \tag{4.154}$$

if $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is a $\mathrm{W}^{1,p}$-local minimizer of $\mathfrak{F}$ over the Dirichlet class $\mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$, then $\overline{u} \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

Proof. Without loss of generality we assume that Ω is star-shaped with respect to $0 \in \Omega$. The whole idea behind the proof is to reduce the problem to the case of global minimizers by using Theorem 83, so that we can apply all the estimates obtained in Steps 1-4 that we established in the proof of Theorem 76. In addition, we observe that the smallness condition $\|V(\nabla g)\|^2 < \varepsilon$ was only used in Step 5 of Theorem 76 and, thanks to Proposition 81, all the calculations there will remain valid under the new assumption

$$\|\nabla g\|_{0,\alpha} < m \quad \text{and} \quad \int_\Omega |V(\nabla g^h)|^2 \, \mathrm{d}x < \varepsilon. \tag{4.155}$$

We remark that, since we assume $\|\nabla g\|_{0,\alpha} \leq m$, all the estimates and assumptions made in Step 5 can be written in terms of $\overline{u}$, instead of $\overline{u}_0$. This involves, in particular, assuming that $|(\nabla \overline{u})_{x_0,R}| \leq 2m - 1$ and $E(x_0,R) + c(m)R^{2\alpha} < \delta_1$ instead of $|(\nabla \overline{u}_0)_{x_0,R}| \leq m - 1$ and $\tilde{E}(x_0,R) < \delta_1$, in order to obtain

$$E(x_0, \tau R) \leq c_0 R^\alpha + \tau^{2\alpha} E(x_0, R)$$

instead of the estimate (4.94). Having this, we can proceed with the same iteration process performed in Step 5 of the proof of Theorem 76, by observing that (4.95)-(4.97) will now be in terms of $\overline{u}$ and g^h instead of $\overline{u}_0$ and g, respectively.

It only remains to observe that Steps 1-4 in the proof of Theorem 76 can also be performed in this case if we assume that, for η as in Theorem 83, the following conditions are satisfied:

1. $|\Omega(y, s)| < \eta$ in equation (4.7);
2. $|\Omega(y, s)| < \eta$ in equation (4.25);
3. $|\Omega(x_0, R)| < \eta$ in equation (4.63).

This is so because those were the only steps in which we used the minimality of $\overline{u}$. Since, by assumption, $s < R$ in (4.7) and (4.25), these conditions will hold provided $cR^n < \eta$ for some constant $c = c(n) > 0$. This proves that, for such $R > 0$ (which depends on $\overline{u}$) we can obtain the excess decay,

$$\tilde{E}(x_0, r) \leq \tilde{C} \left(\frac{r}{R} \right)^{\alpha}$$

that we obtained in (4.114). We remark that, in this case, $R > 0$ is not bounded below by any constant and, in addition, $\tilde{C}$ also depends on η. Therefore, $\tilde{C}$ also depends on $\overline{u}$ and, in particular, on the constant $\delta > 0$ from (4.153).

We now observe that we can repeat steps (4.116)-(4.122) from the regularity proof for global minimizers. Then, by taking $cR_1^n \in (0, \eta)$ for a suitable constant $c > 0$, we can also improve the Hölder continuity of $\overline{u}$ in this case and conclude that $\nabla \overline{u} \in C^{0,\alpha}$. We remark, however, that there will be no longer a uniform bound for $[\nabla \overline{u}]_{0,\alpha}$, because we do not have the analogous to Proposition 100 for the set of local minimizers. This concludes the proof. □

In the next chapter we will establish a uniqueness result for global minimizers satisfying small boundary conditions in the same spirit as in Theorem 76. The full regularity of such global minimizers will play a key role in proving uniqueness, given that it will provide compactness for the set of minimizers that we are interested in. We remark here that such compactness follows from the uniform continuity that we derived in Theorem 76 for the set of global minimizers. Given the lack of uniform bounds for $[\nabla \overline{u}]_{0,\alpha}$ that we have obtained from Theorem 84, no such compactness can be obtained for the corresponding class of $W^{1,p}$-local minimizers and, therefore, no uniqueness result can be obtained following the same method that we will use in Theorem 103.

Chapter 5

Uniqueness of global minimizers

With respect to uniqueness of global minimizers, Knops & Stuart [KS84], and more recently Taheri [Tah03], had established elegant results of uniqueness of minimizers for the case where the boundary displacement is linear and Ω is a star-shaped domain. Some years later, Spadaro [Spa09] gave a negative answer to the core problem of uniqueness of minimizers posed by Ball [Bal02]. He gave several examples of strictly polyconvex (and therefore quasiconvex) integrands that admit multiple minimizers, and such that they are analytic up to the boundary on domains homeomorphic to the ball and with analytic boundary conditions. However, considering the full regularity obtained in Theorem 76, it results very interesting to bring back the question of uniqueness in this particular context, thinking of the small boundary condition as a perturbation of the problem of finding a minimizer that takes the value of zero at the boundary, for which the only possible solution is the function identically zero on Ω.

In this chapter we establish a uniqueness result for global minimizers under the restriction that the gradients of their Dirichlet boundary conditions are small enough in the L^p sense. The core of the proof relies, via a compactness argument, on the result concerning full regularity up to the boundary that we obtained for this class of minimizers in the previous chapter. In addition, a key ingredient to establish uniqueness is the use of a class of Fredholm operators that arises naturally from the fact that minimizers satisfy the weak Euler-Lagrange equations and from the strong quasiconvexity condition.

In the following section we give a brief overview of Spadaro's non-uniqueness result. We then proceed with a preliminary review of the theory of Fredholm operators and, in the following section, we state and establish our contribution to this subject with a positive uniqueness result for global minimizers of quasiconvex integrands. Finally, we conclude this chapter with a brief section regarding the non-uniqueness of minimizers of non-homogeneous integrands, even under the previously favourable small boundary conditions.

5.1 Spadaro's example of non-uniqueness

Spadaro constructed in [Spa09] an example of a strictly polyconvex integrand that admits at least two analytic global minimizers under analytic boundary conditions. This provided a negative answer to the problem posed by Ball regarding the uniqueness of sufficiently smooth solutions to pure displacement boundary problems for strictly polyconvex stored-energy functions, at least for the case $u : \Omega \to \mathbb{R}^3$ where $\Omega \subseteq \mathbb{R}^2$ is a planar domain. However, in the context of elasticity theory, it is still a significant problem the case in which $\Omega \subseteq \mathbb{R}^3$ represents an elastic body in the space. See [Bal02, Problem 8] and [Bal82].

For completeness purposes we now state Spadaro's theorem for the case $u : \Omega \to \mathbb{R}^2$. The more general case that he proves in [Spa09, Theorem 1.1'] relies in a modification of the following result.

Theorem 85 *Let $\Omega := B(0,1) \subseteq \mathbb{R}^2$ and let $\mathcal{A}$ be the area functional,*

$$\mathcal{A}(u) := \int_\Omega \sqrt{1 + |\nabla u|^2 + |\det \nabla u|^2}\, \mathrm{d}x,$$

where $u \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^2)$. Then, there exist a convex C^∞ function $g\colon \mathbb{R} \to \mathbb{R}$ and a real analytic mapping $u_0\colon \partial\Omega \to \mathbb{R}^2$, such that

$$\mathfrak{A}(u) := \mathcal{A}(u) + \int_\Omega g(|\nabla u|)\, \mathrm{d}x$$

is a strictly polyconvex functional that has at least two absolute minimizers, both analytic up to the boundary, in the class $\mathrm{W}^{1,2}_{u_0}(\Omega, \mathbb{R}^2)$.

The proof of this theorem uses an elegant non-uniqueness result for the area functional $\mathcal{A}$ given by Lawson and Osserman in [LO77]. However, while the area functional comes from a polyconvex integrand, this is not strongly polyconvex and hence, to obtain a non-uniqueness result for these class of functionals, Spadaro modified Lawson-Osserman's example by including the perturbation given by the convex function $z \mapsto g(|z|)$. Once an appropriate function g has been constructed, the non-uniqueness of Theorem 85 can be derived easily from Lawson-Osserman's result. The core idea behind the non-uniqueness of minimizers for the area functional $\mathcal{A}$ is to consider, first, an analytic dumb-bell shaped curve in $\mathbb{R}^2$ denoted by γ_0. Then, the boundary condition u_0 is constructed as a double-tracing parametrization of the dilated curve $\gamma = R\gamma_0$ for some $R > 0$. Assuming uniqueness of the minimizer $u_{\min}$, Spadaro exploits the fact that its graph must be then invariant under some symmetries in $\mathbb{R}^2$ to obtain a function $u \in \mathrm{W}^{1,2}_{u_0}(\Omega, \mathbb{R}^N)$ such that, if $R > 0$ is *large enough*, then

$$\mathfrak{A}(u) < \mathfrak{A}(u_{\min}),$$

leading to the desired contradiction.

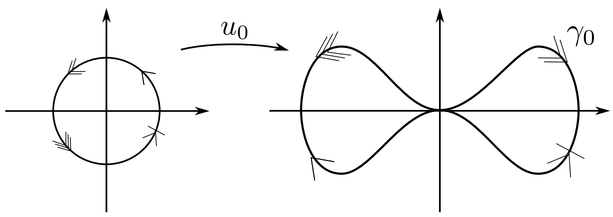

Figure 5.1: Boundary value u_0.

We remark that, in this example, the largeness of the boundary constraint is essential to establish the theorem above. Hereby, any uniqueness result should eliminate the possibility of arbitrarily large boundary conditions. This is in compliance with Theorem 103, that appears at the end of this chapter. See also Section 5.4.

In addition, given that an important motivation for the problem of uniqueness comes from elasticity theory, Spadaro also establishes that the example from Theorem 85 can be modified to construct a strictly polyconvex function $F\colon \mathbb{R}^6 \to \mathbb{R}$ and an analytic *injective* boundary condition $u_0\colon \partial\Omega \to \mathbb{R}^3$, where $\Omega = B(0,1) \subseteq \mathbb{R}^2$, such that the functional

$$\mathfrak{F}(u) := \int_\Omega F(\nabla u)\,\mathrm{d}x$$

has at least two minimizers, analytic up to the boundary, with boundary value u_0. However, we emphasize that the underlying idea to build this significant modification to his example, relies on the same argument of enlarging enough the boundary condition that we have already described.

5.2 Fredholm Operators

We devote this section to compile some relevant results concerning Fredholm operators, which will be our primary tool to establish the main result of this chapter concerning uniqueness of global minimizers. Fredholm operators are named after the Swedish mathematician Erik Ivar Fredholm. Their great importance in the study of differential equations comes from the fact that they are linear isomorphisms up to a finite dimensional space in their domain, where they are not injective, and a finite dimensional space in their image, which is all they lack to be surjective.

Throughout this section, we will assume that we are working with real vector spaces, since we do not require to consider complex spaces for what remains of this chapter.

Furthermore, we refer the reader to Appendix C for a brief compilation of the general theory of linear operators, where we recall the corresponding results that we often use in this chapter and where we specify the definitions and notation that we choose to follow here.

In order to rigorously establish the definition of Fredholm operators, we first recall the following notion.

Definition 86 *Let X and Y be Banach spaces and let $T \in \mathscr{L}(X,Y)$. We define the* ***cokernel*** *of T as the vector space*

$$\mathrm{Coker}(T) := Y/TX$$

and we call the ***codimension*** *of TX in Y the (possibly infinite) value $\dim(Y/TX)$.*

With this, we can now define the core concept of this section.

Definition 87 *Let X and Y be Banach spaces. We say that $T \in \mathscr{L}(X,Y)$ is a Fredholm operator if and only if $\mathrm{nullity(T)} := \dim(\mathrm{Ker}(T))$ is finite and TX has finite codimension in Y.*

We remark that, from this definition, it follows that every Fredholm operator has closed range. More generally, we have the following result.

Proposition 88 *Let $T \in \mathscr{L}(X,Y)$. If the range TX has finite codimension in Y, then TX is closed.*

One of the feature properties of Banach spaces that is needed to characterize Fredholm operators is the following, in which the necessity is a classical result established by by F. Riesz.

Lemma 89 *The unit ball in a Banach space X is compact if and only if X is finite dimensional.*

We will now use this result to establish the following proposition. For this, we follow the proof that appears in [Hör07, Chapter XIX].

Proposition 90 *Let $T \in \mathscr{L}(X,Y)$. Then, the following two statements are equivalent:*

(i) $\operatorname{nullity}(T) < \infty$ *and TX is closed in Y.*

(ii) *Every bounded sequence $(x_k) \subseteq X$ such that Tx_j is convergent, has a convergent subsequence.*

Proof. We first assume that (ii) holds. This implies that the unit ball of the subspace $\operatorname{Ker}(T)$ is compact. By Lemma 89, we can infer that $\operatorname{Ker}(T)$ is finite dimensional. We claim that there is a subspace $X_0 \leq X$ such that X_0 is closed and $X = \operatorname{Ker}(T) \oplus X_0$. Indeed, if $e_1, ..., e_k$ is a basis of the space $\operatorname{Ker}(T)$ then, by Hahn-Banach's Theorem, we can ensure that there are $f_1, ..., f_k \in X^*$ such that

$$f_j(e_i) = \delta_{ij} \qquad \text{for every} \qquad 1 \leq i, j \leq k.$$

We now let $p\colon X \to X \in \mathscr{L}(X)$ to be defined by

$$p(x) := \sum_{i=1}^{k} f_i(x) e_i.$$

It is then clear that $p(X) = \operatorname{Ker}(T)$. In addition,

$$p^2(x) = \sum_{j=1}^{k} f_j \left(\sum_{i=1}^{k} f_i(x) e_i \right) e_j = \sum_{j=1}^{k} \sum_{i=1}^{k} f_i(x) f_j(e_i) e_j = p(x).$$

Therefore, p is a projection of X onto $\operatorname{Ker}(T)$ and, hereby, if $X_0 := p^{-1}(\{0\})$, X_0 is a closed subspace of X with the required properties.[1] We now claim that there exists $C > 0$ such

[1] In this case, X_0 is said to be the *topological supplement* of $\operatorname{Ker}(T)$. See [RR80].

that, for every $x \in X_0$,

$$\|x\|_X \leq C\|Tx\|_Y. \tag{5.1}$$

If this was not the case, we would be able to find a sequence $(x_j) \subseteq X_0$ with $\|x_j\|_X = 1$ such that $\|Tx_j\| \leq \frac{1}{j}$. By condition (ii), and using that X_0 is closed, we would then find $x \in X_0$ and a subsequence, that we do not relabel, such that $x_j \to x$ as $j \to \infty$. Then, $\|x\|_X = 1$ and $Tx = 0$, i.e., $\|x\|_X = 1$ and $x \in \mathrm{Ker}(T) \cap X_0$, which is a contradiction. Now, since $T \restriction_{X_0}: X_0 \to TX_0$ is a linear bijection and (5.1) implies that its inverse is continuous, it follows from Banach's Bounded Inverse Theorem that $TX = TX_0$ is a closed subspace of Y. We have therefore shown that (ii) implies (i).

Conversely, if we assume (i), just as above we know that there is $X_0 \leq X$ closed and such that $X = \mathrm{Ker}(T) + X_0$. Condition (i), together with Banach's Open Mapping Theorem, implies that $T \restriction X_0$ is a top-linear isomorphism between X_0 and TX_0. Therefore, (5.1) still holds in this case for some $C > 0$. Let (x_j) be a bounded sequence in X such that (Tx_j) is convergent. We can then write $x_j = y_j + z_j$ with $y_j \in \mathrm{Ker}(T)$ and $z_j \in X_0$. By (5.1) we can ensure that (z_j) is also bounded and, therefore, (y_j) is bounded too. Furthermore, (5.1) tells us that z_j is a Cauchy sequence and, therefore, it is convergent. In addition, the bounded sequence (y_j) in the finite dimensional space $\mathrm{Ker}(T)$ has a convergent subsequence. This settles that (i) implies (ii). □

If X and Y are finite dimensional vector spaces over $\mathbb{R}$ (or $\mathbb{C}$) and $T: X \to Y$ is a linear transformation, the elementary Rank-Nullity theorem states that

$$\dim(X) - \mathrm{nullity}(T) = \dim(Y) - \dim(Y/TX).$$

This means that

$$\mathrm{nullity}(T) - \dim(Y/TX) = \dim(X) - \dim(Y)$$

is independent of T. This motivates the stability properties of the left hand side of this identity in the infinite dimensional case, that we will establish after stating the following definition.

Definition 91 *Let $T \in \mathscr{L}(X,Y)$ such that* $\mathrm{nullity}(T) < \infty$ *and TX is closed. We define the* *index of T to be*

$$\mathrm{ind}(T) := \mathrm{nullity}(T) - \dim(\mathrm{Coker}(T)).$$

The index has remarkable stability properties, some of the most important ones being summarized in the next result.

Theorem 92 *Let $T \in \mathscr{L}(X,Y)$ be such that* $\mathrm{nullity}(T) < \infty$ *and TX is closed. Then, there exists $\delta > 0$ such that, for every $S \in \mathscr{L}(X,Y)$, if $\|S\|_{\mathscr{L}(X,Y)} < \delta$, then* $\mathrm{nullity}(T+S) \leq \mathrm{nullity}(T)$*, $T+S$ has closed range and* $\mathrm{ind}(T+S) = \mathrm{ind}(T)$.

Proof. We first assume that T is bijective. It is then a top-linear isomorphism by the Open Mapping Theorem and, therefore,

$$T + S = T(I + T^{-1}S) \tag{5.2}$$

is bijective if $\|T^{-1}\|\|S\| < 1$ because, in this case, $I + T^{-1}S$ can be inverted by the Neumann series. Hence the result holds in this case.

Assume now that T is only injective. Then, (5.1) still holds and it implies that

$$\|x\|_X \leq C\|Tx\|_Y \leq C\|(T+S)x\|_Y + C\|S\|\|x\|_X. \tag{5.3}$$

Hence, $\|x\|_X \leq C\|(T+S)x\|_Y$ if $C\|S\| < \frac{1}{2}$. This means that, in this case, $T+S$ is injective and, together with Proposition 90, this also implies that $T+S$ has closed range.

In order to prove the stability of the index in this case, we consider the two following scenarios.

Case 1. If $\dim(\mathrm{Coker}(T)) < \infty$, we can then choose a finite dimensional supplementary space W of TX. Indeed, if we let $\{y_1 + TX, ..., y_k + TX\}$ be a basis for Y/TX and if $W \leq Y$ is the vector space generated by $\{y_1, ..., y_k\}$, then $Y = W \oplus TX$. We now let $Q\colon Y \to Y/W$ be the natural map defined by

$$Qy := y + W. \tag{5.4}$$

Then, QT is bijective and hence, just as before, $QT + QS$ is bijective if $\|S\|$ is small enough.

Therefore, in this case,

$$\{0\} = \operatorname{Ker}(QT + QS) = \{x \in X : (T + S)x \in W\}.$$

This, together with the fact that $T + S$ is injective, implies that

$$(T + S)X \cap W = \{0\}. \tag{5.5}$$

On the other hand, given that $QT + QS$ is onto Y/W, for every $y \in Y$ we can find $x \in X$ such that

$$y + W = (QT + QS)x = (T + S)x + W. \tag{5.6}$$

This implies, in turn, that for every $y \in Y$ there are $x \in X$ and $w \in W$ such that $y = (T + S)x + w$. We have then shown that

$$Y = (T + S)X \oplus W.$$

As a consequence of this, we have that the linear transformation $\Phi\colon W \to Y/(T+S)X$ given by

$$\Phi(w) := w + (T + S)X,$$

which is clearly well defined, is a linear homeomorphism between W and $Y/(T + S)X$. The injectivity is indeed a consequence of (5.5) and the surjectivity follows from (5.6). Given this, we can then deduce that, since T and $T + S$ are injective,

$$\operatorname{ind}(T + S) = -\dim\left(Y/(T + S)X\right) = -\dim(W) = -\dim\left(Y/TX\right) = \operatorname{ind}(T).$$

The result follows then in this case.

Case 2. If we just know that $\dim(\operatorname{Coker}(T)) > \nu$, we can choose, in a similar way as in Case 1, a subspace $W \leq Y$ of dimension $\nu + 1$ such that $W \cap TX = \{0\}$. This means that, if $Q\colon Y \to Y/W$, then $QT\colon X \to Y/W$ is injective. In addition, since TX is closed in Y by assumption, we also have that

$$(QT)X = \{Tx + W : x \in W\}$$

is closed in Y/W. Therefore, by an analogous calculation as the one in (5.3), we deduce that, if $\|S\|$ is small enough, then $QT + QS$ is injective too. Hence, in this case,

$$(T + S)X \cap W = \{0\}.$$

We now define $\Phi\colon W \to Y/(T + S)X$ by $\Phi w := w + (T + S)X$ and we claim that Φ is also injective. Indeed, if $\Phi w = 0$ in $Y/(T+S)X$, this means that $w \in (T+S)X\cap W$ and, therefore, $w = 0$. Given that Φ is a linear transformation between Banach spaces, this implies that

$$\nu + 1 = \dim(W) \leq \dim\left(Y/(T + S)X\right).$$

Whereby,

$$\operatorname{ind}(T + S) = \operatorname{nullity}(T + S) - \dim\left(Y/(T + S)X\right) \geq -(\nu + 1) > -\nu.$$

Applying respectively what we did in Cases 1 and 2, but considering $T + S$ instead of T as the injective linear operator with closed range, we see that the sets

$$\left\{S \in \mathscr{L}(X, Y) : C\|S\| < \frac{1}{2} \text{ and } \operatorname{ind}(T + S) = -\nu\right\} \quad \text{and}$$
$$\left\{S \in \mathscr{L}(X, Y) : C\|S\| < \frac{1}{2} \text{ and } \operatorname{ind}(T + S) < -\nu\right\}$$

are both open for every $\nu \geq 0$. Given that $\left\{S \in \mathscr{L}(X, Y) : C\|S\| < \frac{1}{2}\right\}$ is connected, it then follows that $\operatorname{ind}(T + S)$ must in fact be constant.

We conclude this proof by considering the general situation in which T is not necessarily injective. Assume that $N := \operatorname{Ker}(T) \neq \{0\}$. Since N is finite dimensional we can choose, exactly as in the proof of Proposition 90, a topological supplement of N, i.e., a closed subspace $X_0 \subseteq X$ such that

$$X = X_0 \oplus N. \tag{5.7}$$

It then follows, from what we have already shown, that if $\|S\|$ is small enough, then $(T+S)\restriction_{X_0}$ is injective with closed range and that

$$\dim(Y/(T+S)X_0) = \dim(Y/TX_0) = \dim(Y/TX).$$

In addition, if $N' := \operatorname{Ker}(T+S)$, then $N' \cap X_0 = \operatorname{Ker}((T+S)\restriction_{X_0}) = \{0\}$. This, together with (5.7) means that the linear mapping $\Psi \colon X \to X/X_0$ given by $\Psi x := x + X_0$ is such that $\Psi\restriction_N$ is a linear isomorphism and $\Psi\restriction_{N'}$ is an injective linear transformation. Therefore,

$$\operatorname{nullity}(T+S) = \dim(N') \leq \dim(X/X_0) = \dim(N) = \operatorname{nullity}(T). \tag{5.8}$$

Furthermore, this implies that $X = N' \oplus V \oplus X_0$ for some finite dimensional space V. In addition, if we let α and β be bases of the vector spaces V and X_0 respectively, since $(T+S)\restriction_V$ and $(T+S)\restriction_{X_0}$ are both injective, we have that $(T+S)\alpha$ is a basis of the vector space $(T+S)V$ and $(T+S)\beta$ is a basis of the vector space $(T+S)X_0$. We now extend the linearly independent set $(T+S)\alpha \cup (T+S)\beta$ to a basis, say Γ, of Y. It is then easy to see that

$$\{y + (T+S)V \oplus (T+S)X_0 : y \in \Gamma\backslash((T+S)\alpha \cup (T+S)\beta)\}$$

is a basis of $Y/((T+S)V \oplus (T+S)X_0)$,

$$\{y + (T+S)V \oplus (T+S)X_0 : y \in \Gamma\backslash((T+S)\beta)\}$$

is a basis of $Y/(T+S)X_0$ and, clearly, $(T+S)\alpha$ is a basis of $(T+S)V$, which is isomorphic to V via the bijective linear transformation $(T+S)\restriction_V \colon V \to (T+S)V$. These three facts together imply that

$$\dim(Y/(T+S)X) = \dim(Y/((T+S)V \oplus (T+S)X_0)) = \dim(Y/TX) - \dim(V).$$

Therefore,

$$\begin{aligned}\operatorname{ind}(T+S) =& \dim(N') + \dim(V) - \dim(Y/TX)\\ =& \dim(N) - \dim(Y/TX) = \operatorname{ind}(T).\end{aligned}$$

This concludes the proof. □

We now state a straightforward consequence of this result. We remark that this corollary plays a central role in the uniqueness result of global minimizers that is the main purpose of this chapter.

Corollary 93 *The set of Fredholm operators in* $\mathscr{L}(X,Y)$ *is open,* $\text{nullity}(T)$ *is upper semicontinuous and* $\text{ind}(T)$ *is constant in each component.*

Given the behaviour of the index in finite dimensional spaces, we can expect the following result, which will follow at once from the above corollary.

Theorem 94 *If* $T\colon X \to Y$ *and* $S\colon Y \to Z$ *are Fredholm operators, then so is* $ST\colon X \to Z$ *and*

$$\text{ind}(ST) = \text{ind}(T) + \text{ind}(S).$$

Proof. We first observe that T maps $\text{Ker}(ST)$ into $\text{Ker}(S)$ and that $\text{Ker}(T) \subseteq \text{Ker}(ST)$. This implies that the mapping $\tilde{T}\colon \text{Ker}(ST)/\text{Ker}(T) \to \text{Ker}(S)$ given by

$$\tilde{T}\left(x + \text{Ker}(T)\right) := Tx$$

is a well defined injective linear operator. Therefore, $\text{nullity}(ST) \leq \text{nullity}(T) + \text{nullity}(S)$. In a similar way we can prove that $\dim(\text{Coker})(ST) \leq \dim(\text{Coker})(T) + \dim(\text{Coker})(S)$. This shows that ST is a Fredholm operator. Furthermore, if I_Y is the identity operator on Y, it follows from this, in block matrix notation, that

$$\begin{pmatrix} \text{I}_Y & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \text{I}_Y \cos t & \text{I}_Y \sin t \\ -\text{I}_Y \sin t & \text{I}_Y \cos t \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & \text{I}_Y \end{pmatrix}$$

is, for every $t \in \mathbb{R}$, a Fredholm operator from $X \times Y$ into $Y \times Z$. When $t = 0$, this is the direct sum of the operators T and S, whose index is clearly $\text{ind}(T) + \text{ind}(S)$, whereas when $t = \frac{-\pi}{2}$, this is the operator $(x, y) \mapsto (-y, STx)$, whose index is $\text{ind}(ST)$. Using Corollary 93 and the continuity with respect to the parameter t for this family of operators, we conclude the proof of this result. □

We state a further consequence of Theorem 92, that will also play a central role in the main uniqueness result of this chapter.

Corollary 95 *If $T \in \mathscr{L}(X,Y)$ is a Fredholm operator and $K \in \mathscr{L}(X,Y)$ is compact, then $T+K$ is a Fredholm operator and* $\mathrm{ind}(T+K) = \mathrm{ind}(T)$.

Proof. If (x_j) is a bounded sequence in X such that $(T+K)x_j$ is convergent, then the compactness of K allows us to choose a subsequence (x_{j_k}) such that Kx_{j_k} is convergent. Thus Tx_{j_k} is convergent and, from condition (ii) in Proposition 90, we infer that (x_{j_k}) has a convergent subsequence. Therefore, by applying Proposition 90 once again, but this time to $T+K$, we obtain that $T+K$ has finite dimensional kernel and closed range. We now apply Theorem 92 to deduce from here that the index of $T+tK$ is a locally constant function of $t \in \mathbb{R}$ and, hence, it is independent of t. Therefore, $T+tK$ is always a Fredholm operator with index equal to $\mathrm{ind}(T)$. □

Despite the many further properties of Fredholm operators that can be discussed, we refer the reader to [Hör07] for a more thorough review of the subject and we move on to the following definition. This aims at establishing a particular stability result that will constitute the heart of the aforementioned uniqueness theorem. The reason for this is that it will enable us to establish that a special class of integral operators that is of interest to us, consists purely of Fredholm operators.

Definition 96 *We say that a linear operator $T\colon X \to X^*$ is positive and bounded below if there is a constant $c > 0$ such that, for every $x \in X$,*

$$\langle Tx, x\rangle \geq c\|x\|_X^2.$$

We can now establish the following result, which we can easily relate to the celebrated lemma of Lax and Milgram, given that every bounded linear operator $T\colon X \to X^*$ gives rise to a bounded bilinear form on X by defining $\Phi(x,y) := \langle Tx, y\rangle$. See [LM54].

Lemma 97 *If the bounded linear operator $T\colon X \to X^*$ is positive and bounded below, then it has a bounded inverse $T^{-1}\colon X^* \to X$.*

The proof of this result can be found in [McL00, Lemma 2.32], but we include it here for the convenience of the reader.

Proof. Since $c\|x\|_X^2 \leq |\langle Tx, x\rangle| \leq \|Tx\|_{X^*}\|x\|_X$, we have an estimate of the form

$$\|x\|_X \leq C\|Tx\|_{X^*}$$

for $x \in X$. This implies that $\mathrm{Ker}(T) = \{0\}$ and, by Corollary 116, we further deduce that $\mathrm{Im}(T)$ is closed. Likewise, $\mathrm{Ker}(T^t) = \{0\}$ so, in fact, $\mathrm{Im}(T) = X^*$ by Theorem 120. □

A straightforward consequence of this result, together with Corollary 95, is the following.

Theorem 98 *Let X be a Banach space and let $T\colon X \to X^*$ be a positive and bounded below bounded linear operator. In addition, let $K\colon X \to X^*$ be a compact operator. Then, $S := T + K$ is a Fredholm operator with zero index.*

This theorem is a classical result in the theory of Fredholm operators and it will play a central role in the proof of uniqueness of minimizers in the following section.

5.3 Uniqueness of global minimizers

In this section we will establish one of the most important results of this work concerning the uniqueness of global minimizers under the assumption of small boundary conditions in the $\mathrm{W}^{1,p}$ sense.

We begin with the following auxiliary compactness result, that we will later use to prove that the set of minimizers that are now under consideration, is compact.

Lemma 99 *Let $\alpha \in (0,1)$, $p \geq 2$ and $m, \varepsilon > 0$ be arbitrary but fixed. Then,*

$$\mathcal{B}_{m,\varepsilon} := \left\{ g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N) : \|g\|_{1,\alpha} \leq m \text{ and } \|V(\nabla g)\|_{\mathrm{L}^2} \leq \varepsilon \right\}$$

is a compact subset of $C^1(\overline{\Omega}, \mathbb{R}^N)$.

Proof. The proof of this result will essentially rely on Arzelà-Ascoli's Theorem. It is clear, from the definition of $\mathcal{B}_{m,\varepsilon}$, the continuity of the norms $\|\cdot\|_{1,\alpha}$ and $\|\cdot\|_{\mathrm{L}^2}$, and of the function V, that this is a closed subset of $C^1(\overline{\Omega}, \mathbb{R}^N)$.

We now claim that the set $\mathcal{B}_{m,\varepsilon}$ is relatively compact in $C(\overline{\Omega, \mathbb{R}^N})$. Indeed, by definition, it is clearly a bounded set in the space of continuous functions. In addition, it is also equicontinuous because, in particular, for every $g \in \mathcal{B}_{m,\varepsilon}$ we have that $\|\nabla g\|_{C(\overline{\Omega}, \mathbb{R}^N)} \leq m$. The claim then follows by Arzelà-Ascoli's Theorem.

The next step is to observe that, since $\|\nabla g\|_{0,\alpha} \leq m$ for every $g \in \mathcal{B}_{m,\varepsilon}$, by applying Arzelà-Ascoli's Theorem once again, we deduce that the set

$$\nabla \mathcal{B}_{m,\varepsilon} := \{\nabla g : g \in \mathcal{B}_{m,\varepsilon}\}$$

is also relatively compact in $C(\overline{\Omega, \mathbb{R}^N})$.

To conclude the proof of the lemma, we now let $(g_k)_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}_{m,\varepsilon}$ with respect to the C^1-norm. Since $\mathcal{B}_{m,\varepsilon}$ is relatively compact in $C(\overline{\Omega, \mathbb{R}^N})$, there exists a subsequence, say (g_{k_l}), such that it is Cauchy in the space of continuous functions. Furthermore, since (∇g_{k_l}) is a bounded sequence in $\nabla\mathcal{B}_{m,\varepsilon}$, which is also relatively compact, there exists a further subsequence of (g_k), say $(g_{k_{l_m}})$, such that $(\nabla g_{k_{l_m}})$ is Cauchy in $C(\overline{\Omega, \mathbb{R}^N})$. It is then clear that this new subsequence is Cauchy in $C^1(\overline{\Omega, \mathbb{R}^N})$ by construction. This shows that $\mathcal{B}_{m,\varepsilon}$ is relatively compact, with which we conclude the proof of the lemma. □

We can now proceed with the aforementioned result related to the compactness of the set of minimizers satisfying certain small boundary conditions. We emphasize that, in the proof of the following proposition, it is essential to use the fact that the Hölder coefficient of the gradient of the minimizer given by Theorem 76 is independent of the minimizer itself.

Proposition 100 *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *and assume that* (H0) – (H2) *are satisfied for some* $p \in [2, \infty)$. *Then, for every* $m > 0$ *there is an* $\varepsilon = \varepsilon(m) \in (0, 1)$ *such that the family of functions*

$$\mathcal{M} := \left\{ u \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N) : \begin{array}{c} u \text{ is a minimizer of } \mathfrak{F} \text{ over } \mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N) \text{ for some} \\ g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N) \text{ with } \|g\|_{1,\alpha} \le m \text{ and } \|V(\nabla g)\|_{\mathrm{L}^2} \le \varepsilon. \end{array} \right\}.$$

is a compact subset of $C^1(\overline{\Omega}, \mathbb{R}^N)$.

Proof. The proof of this result will follow the same spirit than that of Lemma 99. However, since we do not have a priori estimates for the L^∞-norms of the minimizers in $\mathcal{M}$, we must show directly that this is a bounded subset of $C^1(\overline{\Omega}, \mathbb{R}^N)$ and, more generally, relatively compact.

With this aim, let $m > 0$ be fixed and take $\varepsilon = \varepsilon(m) > 0$ as it is given by Theorem 76. Assume that $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ is such that $\|g\|_{1,\alpha} \le m$ and $\|V(\nabla g)\|_{\mathrm{L}^2} \le \varepsilon$. Let $u \in \mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$ be a minimizer of $\mathfrak{F}$. By Theorem 76 we know that $u \in C^{1,\frac{\alpha}{2}}$ and, furthermore, that for some constant $L > 0$ depending only on m, n, F'' and Ω,

$$|\nabla u(x) - \nabla u(y)| \le L|x - y|^{\frac{\alpha}{4}} \tag{5.9}$$

for every $x, y \in \overline{\Omega}$.[2]

We will use this uniform bound in the α-Hölder seminorm of the minimizers to deduce that $\mathcal{M}$ is indeed relatively compact in $C^1(\overline{\Omega}, \mathbb{R}^N)$. The initial step is to show that the set

$$\nabla\mathcal{M} := \{\nabla u : u \in \mathcal{M}\}$$

is bounded in $C(\overline{\Omega}, \mathbb{R}^N)$. Arguing by contradiction, we assume that, for every $k \in \mathbb{N}$, there are $x_k \in \overline{\Omega}$, g_k satisfying $\|g_k\|_{C^{1,\alpha}} \leq m$ and $\|V(\nabla g_k)\|_{\mathrm{L}^2} \leq \varepsilon$, as well as $u_k \in \mathcal{M}$ with $u_k \in \mathrm{W}^{1,p}_{g_k}(\Omega, \mathbb{R}^N)$, such that

$$|\nabla u_k(x_k)| > k.$$

Therefore, for any $x \in \overline{\Omega}$ we obtain, by (5.9), that

$$k \leq |\nabla u_k(x_k) - \nabla u_k(x)| + |\nabla u_k(x)| \leq L|x_k - x|^\alpha + |\nabla u_k(x)| \leq m\,\mathrm{diam}(\Omega)^\alpha + |\nabla u_k(x)|.$$

This means that the sequence $(|\nabla u_k|) \to \infty$ pointwise in $\overline{\Omega}$. Therefore, by Fatou's Lemma and Lemma 74, we infer that

$$\infty \leq \liminf_{k\to\infty} \int_\Omega |V(\nabla u_k)|^2\,\mathrm{d}x \leq c \liminf_{k\to\infty} \int_\Omega |V(\nabla g_k)|^2\,\mathrm{d}x \leq c\,\varepsilon^2,$$

which is a contradiction. We can then deduce that $\nabla\mathcal{M}$ is uniformly bounded in $C(\overline{\Omega}, \mathbb{R}^N)$ and we choose $\Lambda = \Lambda(m) > 0$ such that, for every $u \in \mathcal{M}$,

$$\|\nabla u\|_{L^\infty} \leq \Lambda. \tag{5.10}$$

Furthermore, by the uniform control over the Hölder seminorms that we have stated in (5.9), it is clear that $\nabla\mathcal{M}$ is an uniformly continuous family of functions. Whereby, by Arzelà-Ascoli's Theorem we deduce that $\nabla\mathcal{M}$ is relatively compact in $C(\overline{\Omega}, \mathbb{R}^N)$.

We now claim that $\mathcal{M}$ is also relatively compact in $C(\overline{\Omega}, \mathbb{R}^N)$. Indeed, that $\mathcal{M}$ is equicontinuous follows from (5.10) and the fact that every $u \in \mathcal{M}$ is Lipschitz with Lipschitz constant $\|\nabla u\|_\infty \leq \Lambda$.

[2] In Theorem 76 we actually proved that $u \in C^{1,\alpha}$ and that there is a uniform bound for $[\nabla \overline{u}]_{0,\alpha}$. However, in showing the latter uniform Hölder continuity we used Proposition 100. Hence, we should only make use of the uniform $\frac{\alpha}{2}$-Hölder continuity in this proof.

To show that $\mathcal{M}$ is uniformly bounded, let $x_0 \in \partial\Omega$ and take $x \in \overline{\Omega}$ arbitrary. We then have that, if $u \in \mathcal{M}$ satisfies $u \in \mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$, since $\partial\Omega$ is of class $C^{1,\alpha}$ and u is continuous in $\overline{\Omega}$, $u = g$ on $\partial\Omega$. Therefore,

$$|u(x)| \leq |u(x) - u(x_0)| + |u(x_0)| \leq \|\nabla u\|_{L^\infty}|x - x_0| + |g(x_0)| \leq \Lambda \operatorname{diam}(\Omega) + m.$$

Applying Arzelà-Ascoli's Theorem once again, we can conclude that $\mathcal{M}$ is relatively compact in $C(\overline{\Omega}, \mathbb{R}^N)$.

We can now proceed as in the proof of Lemma 99 to further deduce that $\mathcal{M}$ is relatively compact in $C^1(\overline{\Omega}, \mathbb{R}^N)$.

It only remains to show that this family is closed in $C^1(\overline{\Omega}, \mathbb{R}^N)$. With this purpose, let $(u_k) \subseteq \mathcal{M}$ and assume that $u_k \overset{k\to\infty}{\longrightarrow} u$ in $C^1(\overline{\Omega}, \mathbb{R}^N)$. Since $(u_k) \subseteq \mathcal{M}$, there is a sequence $(g_k) \subseteq \mathcal{B}_{m,\varepsilon}$ such that, for every $k \in \mathbb{N}$, $u_k \in \mathrm{W}^{1,p}_{g_k}(\Omega, \mathbb{R}^N)$.

Since $\mathcal{B}_{m,\varepsilon}$ is compact, there exists $g \in \mathcal{B}_{m,\varepsilon}$ such that, for a subsequence (g_{k_l}), $g_{k_l} \overset{l\to\infty}{\longrightarrow} g$ in $C^1(\overline{\Omega}, \mathbb{R}^N)$. Therefore, $u_{k_l} - g_{k_l} \overset{l\to\infty}{\longrightarrow} u - g$ uniformly in C^1 and, in particular, in $\mathrm{W}^{1,p}$. Hence, given that $\mathrm{W}^{1,p}_0$ is closed, $u - g \in \mathrm{W}^{1,p}_0$.

Finally, the fact that u_k is a minimizer of $\mathfrak{F}$ for every $k \in \mathbb{N}$ implies that, for every $\varphi \in \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N)$,

$$\int_\Omega F(\nabla u_k)\,\mathrm{d}x \leq \int_\Omega F(\nabla u_k + \nabla\varphi)\,\mathrm{d}x. \tag{5.11}$$

By uniform convergence and the continuity of F, this clearly implies that u is a minimizer of $\mathfrak{F}$ over $\mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$. Therefore, $u \in \mathcal{M}$ and $\mathcal{M}$ is closed. This concludes the proof that $\mathcal{M}$ is compact in C^1. □

We are now ready to establish the important link between the theory of Fredholm operators and the class of minimizers that constitute the main characters of this chapter. The idea is to exploit the compactness of the set $\mathcal{M}$, to construct a compact class of Fredholm operators defined on the space of variations. We do this with the help of the linearized integrand and its quasiconvexity properties. Furthermore, we use the upper semicontinuity that the nullity of Fredholm operators satisfies, to obtain full control over the dimension of the kernels of the set of operators that we propose here.

Theorem 101 *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfy* (H0) – (H2) *for some* $p \in [2,\infty)$ *and let* $m > 0$ *arbitrary. Take* $\varepsilon = \varepsilon(m) > 0$ *as it is given by Theorem 76. In addition, assume that* $u, v \in \mathcal{M}$ *are such that* $\phi := u - v \in \mathrm{W}_0^{1,p}(\Omega,\mathbb{R}^N)$*. Then, the linear operator*

$$T_\phi^u\colon \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N) \to \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N)^*$$

given by

$$\begin{aligned}\langle T_\phi^u v, w\rangle :=& -\int_\Omega \int_0^1 (\mathrm{div} F''(\nabla u + t\nabla\phi)\cdot\nabla v)\cdot w \,\mathrm{d}t\,\mathrm{d}x \\ =& \int_\Omega \int_0^1 F''(\nabla u + t\nabla\phi)[\nabla v, \nabla w]\,\mathrm{d}t\,\mathrm{d}x \end{aligned} \tag{5.12}$$

is a Fredholm operator of zero index. Furthermore, there is a constant $K > 0$*, depending exclusively on* m*, such that*

$$\mathrm{nullity}(T_\phi^u) \le K.$$

Proof. **Step 1. Weak coercivity.** We begin the proof by recalling from (4.59) that, by (H2), for any $z_0 \in \mathbb{R}^{N\times n}$ and any $w \in \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N)$,

$$2c_2\int_\Omega |\nabla w|^2\,\mathrm{d}x \le \int_\Omega F''(z_0)[\nabla w,\nabla w]\,\mathrm{d}x. \tag{5.13}$$

We now note that this holds, in particular, for any $z_0 = \nabla u(x_0) + t\nabla\phi(x_0)$, where $x_0 \in \overline{\Omega}$, $u, u+\phi \in \mathcal{M}$, $\phi = 0$ on $\partial\Omega$ and $t \in [0,1]$.

On the other hand, having fixed $m > 0$, we know by Proposition 100 that, if $\varepsilon = \varepsilon(m)$ is the positive constant given by Theorem 76, then there are $\Lambda > 0$, $\beta \in (0,1)$ and $L > 0$ such that, for every $u \in \mathcal{M}$ and $x, x_0 \in \overline{\Omega}$,

$$|\nabla u(x)| \le \Lambda \qquad \text{and} \qquad |\nabla u(x) - \nabla u(x_0)| \le L|x - x_0|^\beta. \tag{5.14}$$

Let $\omega\colon [0,\infty) \to [0,\infty)$ be a non-decreasing modulus of continuity for F'' on the compact set

$$\{z \in \mathbb{R}^{N\times n} : |z| \le 3\Lambda\}.$$

Then, for any $x, x_0 \in \overline{\Omega}$, $u, u+\phi \in \mathcal{M}$ with $\phi = 0$ on $\partial\Omega$ and any $t \in [0,1]$, we have that

$$|F''(\nabla u(x) + t\nabla\phi(x)) - F''(\nabla u(x_0) + t\nabla\phi(x_0))| \leq \omega(3L|x - x_0|^{\beta}). \tag{5.15}$$

We now denote $A_t(x) := F''(\nabla u(x) + t\nabla\phi(x))$. By (5.13) and (5.15) we know that, for such A_t, the hypotheses of Theorem 18 are satisfied and, therefore, for some $\lambda_0, \lambda_1 > 0$ that do not depend on $t \in [0,1]$, we deduce that the associated bilinear form on $\mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$ is weakly coercive, i.e., that for every $w \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$,

$$\int_\Omega F''(\nabla u + t\nabla\phi)[\nabla w, \nabla w]\,\mathrm{d}x \geq \lambda_0 \int_\Omega |\nabla w|^2\,\mathrm{d}x - \lambda_1 \int_\Omega |w|^2\,\mathrm{d}x.$$

By $\mathcal{L}^1$-integrating with respect to $t \in (0,1)$, we further obtain that

$$\int_\Omega \int_0^1 F''(\nabla u + t\nabla\phi)[\nabla w, \nabla w]\,\mathrm{d}t\,\mathrm{d}x \geq \lambda_0 \int_\Omega |\nabla w|^2\,\mathrm{d}x - \lambda_1 \int_\Omega |w|^2\,\mathrm{d}x. \tag{5.16}$$

We now define the bounded linear operator $K\colon \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N) \to \left(\mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)\right)^*$ by

$$\langle Kv, w\rangle := \int_\Omega v \cdot w\,\mathrm{d}x.$$

It is clear, from Hölder's inequality, that K is bounded. Furthermore, it follows from (5.16) that the bounded linear operator $S^u_\phi\colon \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N) \to \left(\mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)\right)^*$ defined by

$$S^u_\phi := T^u_\phi + \lambda_1 K$$

is positive and bounded below.

We now claim that K is a compact linear operator. To verify that it is compact, let (v_k) be a bounded sequence in $\mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$. By the Sobolev Embedding Theorem, we know that there are a subsequence (v_{j_k}) and $v \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$v_{j_k} \to v \quad \text{as} \quad k \to \infty \quad \text{in} \quad \mathrm{L}^2.$$

This implies that

$$\|Kv_{j_k} - Kv\|_{(\mathrm{W}_0^{1,2})^*} = \sup_{\|w\|_{1,2}=1} |\langle K(v_{j_k} - v), w\rangle| \leq \sup_{\|w\|_{1,2}=1} \int_\Omega |v_{j_k} - v||w| \,\mathrm{d}x$$
$$\leq \left(\int_\Omega |v_{j_k} - v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The last inequality above follows from the definition of the norm in $\mathrm{W}^{1,2}$ and Hölder's inequality. Since the right hand side of this chain of inequalities converges to 0 when $k \to \infty$, we can conclude that K is in fact a compact operator. Therefore, by Theorem 98, we know that $T_\phi^u = S_\phi^u - \lambda_1 K$ is a Fredholm operator with zero index and, in particular, that $\mathrm{nullity}(T_\phi^u) \in \mathbb{N}_0$.

Step 2. Existence of the constant K**.** In order to conclude the proof, we only need to show that we can find a constant K such that, for our fixed $m > 0$, every $g \in \mathcal{B}_{m,\varepsilon(m)}$, and every pair of minimizers $u_0, v_0 \in \mathcal{M} \cap \mathrm{W}_g^{1,p}$ of $\mathfrak{F}$, if $\phi_0 := u_0 - v_0$, then

$$\mathrm{nullity}(T_{\phi_0}^{u_0}) \leq K. \tag{5.17}$$

We prove this claim by a contradiction argument. Assuming that the assertion is false, we can then find a sequence $(g_k) \subseteq \mathcal{B}_{m,\varepsilon(m)}$ and two sequences of minimizers $(u_n), (v_n) \subseteq \mathcal{M} \cap \mathrm{W}_{g_k}^{1,p}$, such that, for $\phi_n := u_n - v_n$,

$$\mathrm{nullity}(T_{\phi_n}^{u_n}) \to \infty.$$

On the other hand, we know by Proposition 100 that $\mathcal{M}$ is a compact subset of $C^1(\overline{\Omega}, \mathbb{R}^N)$. Therefore, there are subsequences (u_{n_k}) and (v_{n_k}), as well as $u_0, v_0 \in \mathcal{M}$, such that $u_{n_k} \xrightarrow{C^1} u_0$

and $v_{n_k} \xrightarrow{C^1} v_0$. Then, if $\phi_0 := u_0 - v_0$ we have that

$$\begin{aligned} &\|T^{u_{n_k}}_{\phi_{n_k}} - T^{u_0}_{\phi_0}\|_{\mathscr{L}\left(\mathrm{W}^{1,2}_0,(\mathrm{W}^{1,2}_0)^*\right)} \\ = &\sup_{\substack{\|v\|_{1,2}\le 1\\ \|w\|_{1,2}\le 1}} \left|\left\langle \left(T^{u_{n_k}}_{\phi_{n_k}} - T^{u_0}_{\phi_0}\right) v, w\right\rangle\right| \\ = &\sup_{\substack{\|v\|_{1,2}\le 1\\ \|w\|_{1,2}\le 1}} \left| \int_\Omega \int_0^1 \left(F''(\nabla u_{n_k} + t\nabla\phi_{n_k}) - F''(\nabla u_0 + t\nabla\phi_0)\right)[\nabla v, \nabla w]\, \mathrm{d}t\, \mathrm{d}x\right|. \end{aligned}$$

Since F'' is continuous, the right hand side of the above expression converges to 0 as $k\to\infty$ by Lebesgue's Dominated Convergence Theorem. We now recall that, by Corollary 93, the dimension of the kernel is upper-semicontinous on the space of Fredholm operators in $\mathscr{L}\left(\mathrm{W}^{1,2}_0, \left(\mathrm{W}^{1,2}_0\right)^*\right)$. Whereby,

$$\infty = \limsup_{k\to\infty} \operatorname{nullity}\left(T^{u_{n_k}}_{\phi_{n_k}}\right) \le \operatorname{nullity}\left(T^{u_0}_{\phi_0}\right). \tag{5.18}$$

This contradicts the fact that $T^{u_0}_{\phi_0}$ is a Fredholm operator, which holds by the first part of this proof. Then, there is a constant $K > 0$, that depends only on m, such that (5.17) holds for every pair of minimizers $u, u+\phi \in \mathcal{M}$. □

Having compactness for the set of operators that we constructed above has enabled us to find an upper bound for the dimension of their kernels, which lie in the space of admissible variations. This is the core idea behind the uniqueness result that this chapter is devoted to. Before proceeding with it, we will establish the following technical lemma, in which we find precise constants that determine the equivalence between two of the possible norms that we can assign to particular subspaces of the aforementioned (finite dimensional) kernels.

Lemma 102 *Let $m \in \mathbb{N}^+$ and let W be an m-dimensional subspace of $\mathrm{W}^{1,\infty}_0(\Omega,\mathbb{R}^N)$. Then, there are constants $\gamma_1, \gamma_2 > 0$ such that, for every $\varphi \in W$,*

$$\gamma_1 \left(\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \le \left(\int_\Omega |\nabla\varphi|^4\,\mathrm{d}x\right)^{\frac{1}{4}} \le \gamma_2\, m^{\frac{1}{2}} \left(\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

We remark that, in this lemma, we have a quantitative estimate exhibiting the role of the dimension of the space under consideration. Although we know that all norms are equivalent in a finite dimensional space, the proof of this fact does not usually provide full control over the constants that arise between the equivalent norms. However, for the uniqueness result that will follow, we require to know how precisely the behaviour between the norm inherited from $\mathrm{W}_0^{1,2}$ and the one inherited from $\mathrm{W}_0^{1,4}$ is. With this motivation, we now proceed with the proof of Lemma 102.

Proof. The first inequality of the lemma is clear and comes from Hölder's inequality by taking $\gamma_1 = |\Omega|^{-\frac{1}{4}}$.

We will now establish the existence of γ_2. We first claim that there is a basis $(\varphi_j)_{j=1}^m$ of the space W, such that for every $1 \leq j \leq m$,

$$1 = \int_\Omega |\nabla \varphi_j|^2 \, \mathrm{d}x \qquad \text{and} \qquad \int_\Omega |\nabla \varphi_j|^4 \, \mathrm{d}x < 2. \tag{5.19}$$

To prove this claim, observe that the mapping $\varphi \mapsto \int_\Omega |\nabla\varphi|^4 \, \mathrm{d}x$ is continuous in W (under any norm that we assign to this finite dimensional space). This implies that the set

$$\mathcal{C} := \left\{ \varphi \in W : \int_\Omega |\nabla\varphi|^4 \, \mathrm{d}x < 2 \right\}$$

is open in W. Notice that this set is clearly non-empty.

In addition, we observe that we can endow $W \subseteq \mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^N)$ with the norm $\varphi \mapsto \|\nabla\varphi\|_{\mathrm{L}^2}$ and with the corresponding inner product, which make of W a Hilbert space.[3]

Having established this, it is now clear that we can find an *orthonormal* basis of W, with respect to the inner product $\langle \varphi, \psi \rangle := \int_\Omega \nabla\varphi \cdot \nabla\psi \, \mathrm{d}x$, in the open set $\mathcal{C}$. We denote this basis by $(\varphi_j)_{j=1}^m$ and observe that, therefore, (5.19) is satisfied.

We will now use this basis to define the constant γ_2. Observe first that, given arbitrary $\varphi \in W$, there are scalars $a_1, ..., a_m \in \mathbb{R}$ such that $\varphi = \sum_{j=1}^m a_j \varphi_j$ and, in this case, (5.19) implies that

$$\left(\int_\Omega |\nabla\varphi|^4 \, \mathrm{d}x \right)^{\frac{1}{4}} \leq \sum_{j=1}^m |a_j| \left(\int_\Omega |\nabla\varphi_j|^4 \, \mathrm{d}x \right)^{\frac{1}{4}} \leq 2^{\frac{1}{4}} \sum_{j=1}^m |a_j| \leq 2^{\frac{1}{4}} m^{\frac{1}{2}} \left(\sum_{j=1}^m |a_j|^2 \right)^{\frac{1}{2}}. \tag{5.20}$$

[3] That this is a norm follows from Poincaré inequality.

On the other hand, it also follows from (5.19) and from the fact that (φ_j) is orthonormal, that

$$\left(\sum_{j=1}^{m}|a_j|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{m}|a_j|^2\int_{\Omega}|\nabla\varphi_j|^2\,\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_{\Omega}|\nabla\varphi|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{5.21}$$

Combining (5.20) and (5.21), we can conclude the proof of the lemma by setting $\gamma_2 := 2^{\frac{1}{4}}$.

□

We now proceed with the main result of this chapter regarding the uniqueness of minimizers. The idea of the proof will be to show that, if u and v are both minimizers of $\mathfrak{F}$ satisfying the same $\mathrm{W}^{1,2}$-small boundary condition, and if we have full control over the (finite) dimension of the space where $u-v$ is, the strong quasiconvexity of F enables us to conclude that $u - v \equiv 0$.

Theorem 103 *Lef* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *and assume that* (H0) – (H2) *are satisfied for some* $p \in [2,\infty)$. *Then, for every* $m > 0$ *there is an* $\tilde{\varepsilon} = \tilde{\varepsilon}(m) > 0$ *such that, if* $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$ *satisfies* $\|g\|_{1,\alpha} \le m$ *and* $\|V(\nabla g)\|_{\mathrm{L}^2} \le \tilde{\varepsilon}$, *then there exists a unique* $u \in W^{1,p}_g(\Omega,\mathbb{R}^N)$ *such that* u *is a minimizer of* $\mathfrak{F}$.

Proof. Let $m > 0$ and take $\varepsilon > 0$ as it is given by Theorem 76. For an $\tilde{\varepsilon} \in (0,\varepsilon)$ fixed, but still to be specified, assume that $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$ is such that $[\nabla g]_{0,\alpha} \le m$ and $\|V(\nabla g)\|_{\mathrm{L}^2} \le \tilde{\varepsilon}$. In addition, suppose that $u, v \in W^{1,p}_g(\Omega,\mathbb{R}^N)$ are global minimizers of $\mathfrak{F}$. Our aim is to show that $u \equiv v$ on Ω. We denote

$$\phi := u - v.$$

We further assume that $\omega\colon [0,\infty) \longrightarrow [0,1]$ is an increasing and concave modulus of continuity for F'' such that

$$|F''(z) - F''(w)| \le c\,\omega(|z-w|)$$

for some constant $0 < c = c(c_1,\Lambda,F'')$ and every $z, w \in B(0,\Lambda)$.[4] Here, $\Lambda > 0$ is such that, for every $u \in \mathcal{M}$, $\|\nabla u\|_\infty \le \Lambda$. Such Λ exists thanks to Proposition 100 and it only depends on m. Then, since $\phi \in W^{1,p}_0(\Omega,\mathbb{R}^N)$ and u satisfies the weak Euler-Lagrange equation, we

[4]See Appendix D for a detailed construction of a modulus of continuity with the required properties.

have that

$$\begin{aligned}
0 =& \int_\Omega \left(F(\nabla u + \nabla\phi) - F(\nabla u) - \left\langle F'(\nabla u), \nabla\phi\right\rangle\right) \mathrm{d}x \\
=& \int_\Omega \int_0^1 (1-t)F''(t\nabla\phi)[\nabla\phi, \nabla\phi] \,\mathrm{d}t \,\mathrm{d}x \\
&+ \int_\Omega \int_0^1 (1-t)\left(F''(\nabla u + t\nabla\phi) - F''(t\nabla\phi)\right)[\nabla\phi, \nabla\phi] \,\mathrm{d}t \,\mathrm{d}x \\
=& \int_\Omega \left(F(\nabla\phi) - F(0) - \left\langle F'(0), \nabla\phi\right\rangle\right) \mathrm{d}x \\
&+ \int_\Omega \int_0^1 (1-t)\left(F''(\nabla u + t\nabla\phi) - F''(t\nabla\phi)\right)[\nabla\phi, \nabla\phi] \,\mathrm{d}t \,\mathrm{d}x \\
\geq& \, c_2 \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x - \frac{c}{2}\int_\Omega \omega(\nabla u)|\nabla\phi|^2 \,\mathrm{d}x \\
\geq& \, c_2 \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x - \frac{c}{2}\left(\int_\Omega (\omega(\nabla u))^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\phi|^4 \,\mathrm{d}x\right)^{\frac{1}{2}} \qquad (5.22) \\
\geq& \, c_2 \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x - \frac{c}{2}\,\omega\left(\int_\Omega |\nabla u| \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\phi|^4 \,\mathrm{d}x\right)^{\frac{1}{2}} \qquad (5.23) \\
\geq& \, c_2 \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x - \frac{c}{2}\,\omega\left(\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\phi|^4 \,\mathrm{d}x\right)^{\frac{1}{2}}. \qquad (5.24)
\end{aligned}$$

In this chain of inequalities, (5.22) follows from the quasiconvexity of F and Hölder's inequality, (5.23) is a consequence of $\omega \leq 1$ being concave and (5.24) is derived by using that ω is increasing. In addition, we remark that all the integrals above are well defined because u and ϕ are of class $C^{1,\alpha}$ if $V(\nabla g)$ is chosen to be suitably small in L^2.

The core part of the proof will consist on showing that ϕ satisfies a reverse Hölder inequality, so that, for some $\lambda > 0$ independent of g, u and ϕ,

$$\left(\int_\Omega |\nabla\phi|^4 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq \lambda \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x. \qquad (5.25)$$

Assuming this for the moment, we see that (5.24) and (5.25) together imply that

$$0 \geq \left(c_2 - \frac{c\,\lambda}{2}\,\omega\left(\left(\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}\right)^{\frac{1}{2}}\right)\right) \int_\Omega |\nabla \phi|^2 \,\mathrm{d}x. \tag{5.26}$$

On the other hand, since ω is continuous and $\omega(0) = 0$, we know that there is a $\delta_1 = \delta_1(m) > 0$ such that, for every $0 \leq t \leq \delta_1$,

$$\omega(t) < \frac{4c_2^2}{c\lambda}. \tag{5.27}$$

In addition, we know from Lemma 74 and Lemma 128 that there is a constant $0 < c = c(n,p)$ such that

$$\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq c\left(\int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Therefore, if for such $c > 0$ we define

$$\tilde{\varepsilon} := \frac{1}{2}\min\left\{\left(\frac{\delta_1}{c}\right)^p, \varepsilon\right\}$$

and demand that $\int_\Omega |V(\nabla g)|^2 \,\mathrm{d}x \leq \tilde{\varepsilon}$, we can make use of (5.27) to conclude that

$$\left(c_2 - \frac{c\,\lambda}{2}\,\omega\left(\left(\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}\right)^{\frac{1}{2}}\right)\right) > 0. \tag{5.28}$$

This will in turn imply that $\nabla\phi = 0$ and hence, by Poincaré inequality, that $\phi = 0$, with which we will have established uniqueness of minimizers for this class of boundary conditions.

We are hence left to prove (5.25). The key idea behind the proof of this reverse Hölder inequality is to show that ϕ lies in a finite dimensional subspace of $\mathrm{W}_0^{1,2}$. More precisely, we prove that it is in the kernel of a Fredholm operator from some compact set of Fredholm operators that only depends on the data entering the definition of $\mathcal{B}_{m,\varepsilon}$.

Motivated by Theorem 101, we now aim at showing that $\phi \in \mathrm{Ker}(T_\phi^u)$, where for $u, u+\phi \in \mathcal{M} \cap \mathrm{W}_g^{1,p}$, the linear operator $T_\phi^u \colon \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N) \longrightarrow \left(\mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N)\right)^*$ is defined as in (5.12). We then take $w \in \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N)$ arbitrary and observe that, by the Fundamental Theorem of Calculus and the fact that both u and $u+\phi = v$ satisfy the weak Euler-Lagrange equation,

$$\int_{\Omega}\int_0^1 F''(\nabla u + t\nabla\phi)[\nabla\phi, \nabla w]\,\mathrm{d}t\,\mathrm{d}x = \int_{\Omega}\big\langle F'(\nabla u + \nabla\phi), \nabla w\big\rangle\,\mathrm{d}x - \int_{\Omega}\big\langle F'(\nabla u), \nabla w\big\rangle\,\mathrm{d}x = 0. \tag{5.29}$$

This shows that $T_\phi^u\phi \equiv 0$ and, therefore, $\phi \in \mathrm{Ker}(T_\phi^u)$, as we wanted to prove. Whereby, given that $\mathrm{Ker}(T_\phi^u)$ inherits the norms $w \mapsto \|\nabla w\|_{\mathrm{L}^4}$ and $w \mapsto \|\nabla w\|_{\mathrm{L}^2}$ from $\mathrm{W}_0^{1,4}$ and $\mathrm{W}_0^{1,2}$ respectively,[5] and since all the norms defined on a finite dimensional space are equivalent, we can ensure that (5.25) indeed holds for some constant $\lambda > 0$ depending only on $\mathrm{nullity}(T_\phi^u)$. Furthermore, since by Theorem 101 we also know that there exists $0 < K = K(m)$ such that

$$\mathrm{nullity}(T_\phi^u) \leq K, \tag{5.30}$$

Lemma 102 enables us to ensure that λ can even be taken to be independent of $\mathrm{nullity}(T_\phi^u)$, and depending only on K, which in turn can be completely determined by m.

Given this, the smallness restriction imposed on g in terms of $\tilde{\varepsilon}$ is well defined. Whereby, we can infer from (5.24), (5.25) and (5.28) that, for some constant $c > 0$, that depends only on m,

$$0 \geq c\int_{\Omega}|\nabla\phi|^2\,\mathrm{d}x \geq 0. \tag{5.31}$$

Therefore, using that $\phi \in \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N)$ and the corresponding Poincaré inequality, we conclude that $u - v = \phi = 0$ or, in other words, that if $\int_{\Omega}|V(\nabla g)|^2\,\mathrm{d}x \leq \tilde{\varepsilon}$, then there is a unique minimizer $u \in \mathrm{W}_g^{1,p}(\Omega, \mathbb{R}^N)$. □

With this, we have established a uniqueness result for global minimizers under certain small boundary conditions. A key ingredient to establish such uniqueness, was the fact that the set of minimizers that satisfy such suitable boundary conditions is compact in $C^1(\overline{\Omega}, \mathbb{R}^N)$. We established this in Proposition 100 by means of Arzelà-Ascoli's Theorem and, for that, it was essential to make use of the uniform bound that we had for $[\nabla\overline{u}]_{0,\alpha}$ over this class of minimizers. This bound was provided by the regularity result from Theorem 76. We have emphasized that, in contrast, we do not have such compactness for the corresponding class of $\mathrm{W}^{1,p}$-local minimizers.

[5] We recall that these are both norms on the corresponding Sobolev spaces thanks to Poincaré's inequality.

However, it is also worth remarking that, in the above proof, the assumption that u and v are minimizers was mainly used to invoke the compactness of the Fredholm operators constructed in Theorem 101 and to ensure that they satisfy the weak Euler-Lagrange equation. This suggests the possibility of applying a similar strategy to obtain further, more general, uniqueness results.

5.4 Non-uniqueness for non-homogeneous integrands

In this section we establish that, in connection with the previous theorem, it is not possible to obtain uniqueness for minimizers if the integrand depends explicitly on $x \in \Omega$, even under null boundary conditions. This can be obtained as a straightforward consequence of Theorem 85. Indeed, if we let $A\colon \mathbb{R}^4 \to \mathbb{R}$ be given by

$$A(z) := \sqrt{1 + |z|^2 + |\det z|^2},$$

then, with the notation of Theorem 85, we have that

$$\mathfrak{A}(u) = \int_\Omega A(u)\,\mathrm{d}x.$$

We now define $H\colon B(0,1) \times \mathbb{R}^4 \to \mathbb{R}$ by

$$F(x,z) := A(z + \nabla u_0(x)), \tag{5.32}$$

where u_0 is the analytic boundary condition given by Theorem 85. Then, exploiting the non-uniqueness given by this result, it is clear that the functional

$$\mathfrak{F}(u) := \int_\Omega F(x, \nabla u(x))\,\mathrm{d}x \tag{5.33}$$

admits at least two minimizers in the Sobolev class $\mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N)$, so that we cannot expect a uniqueness statement similar to Theorem 103 for these type of integrands.

Appendix A

Notation

We devote this appendix to establish some of the standard notation that is used throughout the thesis. Other conventions that are used less frequently are defined within the text in the particular context in which they are required.

For given $x \in \mathbb{R}^n$ and $r > 0$, we use the following notation:

$B(x, r)$ is the open ball with centre x and radius r.

$B[x, r]$ denotes the closure of $B(x, r)$.

$B := B(0, 1)$ denotes the unit ball in $\mathbb{R}^n$.

$Q(x, r)$ is the open cube with centre x and radius r.

$Q := Q(0, 1)$ denotes the unitary cube in $\mathbb{R}^n$.

$\Omega(x, r) := \Omega \cap B(x, r)$ for a given set $\Omega \subseteq \mathbb{R}^n$.

If $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^N$ is an integrable function, we write

$$(f)_{x,r} := \fint_{\Omega(x,r)} f(y)\,\mathrm{d}y.$$

For a given set $A \subseteq \mathbb{R}^n$, we write

$|A| = \mathcal{L}^n(A)$ to refer to the Lebesgue measure in $\mathbb{R}^n$ of the set A. Throughout the text we make use of both notations depending on the context. In addition, we write

$\mathcal{H}^k(A)$ to refer to the k-dimensional Hausdorff measure of A.

Given $N \times n$ matrices $Z, W \in \mathbb{R}^{N\times n}$ and vectors $x, y \in \mathbb{R}^n$,

$\langle Z, x\rangle$ is the vector in $\mathbb{R}^N$ that results from applying Z to the vector x;

$\langle Z, W\rangle := \mathrm{tr}(Z^t W)$;

$\langle x, y\rangle$ is the inner product in $\mathbb{R}^n$ between the two vectors x and y.

Finally, for $m \in \mathbb{N}^+$ we use the symbol I_m to denote the identity matrix in $\mathbb{R}^m$ and we write $\mathrm{e}_\mathrm{i} \in \mathbb{R}^m$ to denote the canonical vector in $\mathbb{R}^m$ whose i-th entry is 1 and the remaining $m-1$ entries are 0.

Appendix B

Spaces of functions

To fix the notation that is used in this text, we begin by establishing the following conventions.

Notation 104 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We use the following notation.*

(i) $C^0(\Omega, \mathbb{R}^N) = C(\Omega, \mathbb{R}^N)$ *is the set of continuous functions* $u\colon \Omega \to \mathbb{R}$.

(ii) $C^0(\overline{\Omega}, \mathbb{R}^N) = C(\overline{\Omega}, \mathbb{R}^N)$ *is the set of continuous functions* $u : \Omega \to \mathbb{R}^N$ *that can be continuously extended to* $\overline{\Omega}$. *If* Ω *is bounded, the norm over* $C(\overline{\Omega}, \mathbb{R}^N)$ *is given by*

$$\|u\|_{C^0} := \sup_{x\in\overline{\Omega}} |u(x)|.$$

(iii) *The support of a function* $u\colon \Omega \to \mathbb{R}^N$ *is defined as*

$$\operatorname{supp} u := \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

(iv) *We use* $C_0(\Omega, \mathbb{R}^N)$ *to denote the space of continuous functions in* Ω *such that* $\operatorname{supp}(\Omega) \subseteq \Omega$ *is a compact set.*

Regarding spaces that involve derivatives, we proceed as follows.

(v) *Let* $m \in \mathbb{N}^+$ *be a positive natural number. An element of the set*

$$\mathcal{A}_m := \left\{ a = (a_1, ..., a_n) \in \mathbb{N}^n : \sum_{j=1}^{n} a_j = m \right\} \tag{B.1}$$

is called a multi-index of order m. We also write for such elements

$$|a| = \sum_{j=1}^{n} a_j = m. \tag{B.2}$$

Then, the set of functions $u : \Omega \to \mathbb{R}^N$ *such that all its partial derivatives* $D^a u$ *are continuous for every* $a \in \mathcal{A}_k$ *and every* $0 \leq k \leq m$*, is denoted by* $C^m(\Omega, \mathbb{R}^N)$*.*

(vi) $C^m(\overline{\Omega}, \mathbb{R}^N)$ *is the set of functions in* $C^m(\Omega, \mathbb{R}^N)$ *whose derivatives up to the order* m *can be extended continuously to* $\overline{\Omega}$*. If* Ω *is bounded, it is equipped with the following norm*

$$\|u\|_{C^m} = \max_{0 \leq |a| \leq m} \sup_{x \in \overline{\Omega}} |D^a u(x)|.$$

We also consider

(vii) $C_0^m(\Omega, \mathbb{R}^N) := C^m(\Omega, \mathbb{R}^N) \cap C_0(\Omega, \mathbb{R}^N)$.

(viii) $C^\infty(\Omega, \mathbb{R}^N) := \bigcap_{m=0}^{\infty} C^m(\Omega, \mathbb{R}^N)$ *and* $C^\infty(\overline{\Omega}, \mathbb{R}^N) := \cap_{m=0}^{\infty} C^m(\overline{\Omega}, \mathbb{R}^N)$.

(ix) $C_0^\infty(\Omega, \mathbb{R}^N) := C^\infty(\Omega, \mathbb{R}^N) \cap C_0(\Omega, \mathbb{R}^N)$.

(x) *Finally, when dealing with maps* $u \colon \Omega \to \mathbb{R}$*, we omit the codomain from the notations defined above and write simply* $C(\Omega)$*,* $C^m(\Omega)$ *and* $C_0^m(\Omega)$ *for* $0 \leq m \leq \infty$*.*

B.1 Hölder spaces

In this section we compile the basic conventions regarding functions that are Hölder continuous. In addition, we state Campanato-Meyers characterization of Hölder continuity, since it is one of the pillars of the regularity theory in the Calculus of Variations.

We begin by establishing the following notation.

Definition 105 *Let* $D \subseteq \mathbb{R}^n$*,* $x_0 \in D$*,* $u \colon D \to \mathbb{R}^N$ *and* $0 < \alpha \leq 1$*. We say that* u *is* ***α-Hölder continuous with exponent α at x_0*** *if the quantity*

$$[u]_{\alpha, x_0} := \sup_{x \in D} \left\{ \frac{|u(x) - u(x_0)|}{|x - x_0|^\alpha} \right\}$$

is finite. In this case, we call $[u]_{\alpha, x_0}$ *the* α*-Hölder coefficient of* u *at* x_0 *with respect to* D*. Similarly, we say that* u *is uniformly Hölder continuous with exponent* α *in* D *if and only if* *the quantity*

$$[u]_{\alpha,D} := \sup_{x,y\in D x\neq y}\left\{\frac{|u(x)-u(y)|}{|x-y|^\alpha}\right\}$$

is finite. In addition, we say that u is locally Hölder continuous with exponent α in D if u is uniformly Hölder continuous with exponent α in compact subsets of D.

We observe that, for $\alpha = 1$, the notion of α-Hölder continuity coincides with that of Lipschitz continuity. As discussed in [GT01, Section 4.1], Hölder continuity is a quantitative measure of continuity particularly suitable for the study of partial differential equations and it can also be seen as a fractional differentiability notion. With this motivation, we define the Hölder spaces $C^{k,\alpha}(\Omega,\mathbb{R}^N)$ (or $C^{k,\alpha}(\overline{\Omega},\mathbb{R}^N)$) as the subspaces of $C^k(\Omega,\mathbb{R}^N)$ (or $C^k(\overline{\Omega},\mathbb{R}^N)$) consisting of functions whose k-th order derivatives are uniformly Hölder continuous, or locally Hölder continuous, with exponent α. We then set the following conventions.

Notation 106 *Let $k \in \mathbb{N}$ and $0 < \alpha \leq 1$. Assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set. We denote*

$$[u]_{k,\alpha,\Omega} := [\nabla^k u]_{\alpha,\Omega} = \sup_{|\beta|=k}[\nabla^\beta u]_{\alpha,\Omega}.$$

With these seminorms, we can define the related norms on the space $C^{k,\alpha}(\overline{\Omega},\mathbb{R}^N)$ by

$$\|u\|_{k,\alpha} := \|u\|_{C^k(\Omega,\mathbb{R}^N)} + \sup_{|\beta|=k}[\nabla^\beta u]_{\alpha,\Omega}.$$

We remark that $C^{k,\alpha}(\overline{\Omega},\mathbb{R}^N)$ equipped with the norm $\|\cdot\|_{k,\alpha}$ is a Banach space. By abuse of notation, we also write $C^k(\Omega,\mathbb{R}^N) = C^{k,0}(\Omega,\mathbb{R}^N)$ or, in other words, we identify the set of continuous functions with the set of Hölder continuous functions with exponent 0. Considering this, we observe that if $0 \leq \alpha \leq \beta \leq 1$, then

$$C^{k+1}(\overline{\Omega},\mathbb{R}^N) \subseteq C^{k,1}(\overline{\Omega},\mathbb{R}^N) \subseteq C^{k,\beta}(\overline{\Omega},\mathbb{R}^N) \subseteq C^{k,\alpha}(\Omega,\mathbb{R}^N) \subseteq C^k(\overline{\Omega},\mathbb{R}^N).$$

B.1.1 Campanato-Meyers integral characterization of Hölder continuity

We now state the following classical theorem that gives a necessary and sufficient condition, in terms of integral averages, for a function to be Hölder continuous. This outstanding result finds its origins in the works by S. Campanato in [Cam63] regarding Campanato spaces. Almost simultaneously, N.G. Meyers also published an equivalent characterization of Hölder continuity in [Mey64]. This way of expressing the Hölder continuity in terms of integral values has proven to be very useful in the study of regularity of solutions to partial differential equations. We now state the result in the way that we use it in this text.

Theorem 107 *Let $\Omega \subseteq \mathbb{R}^n$ be a domain with the uniform cone property. Let $p \in [1, \infty)$ and $f\colon \Omega \to \mathbb{R}^N$ be such that, for some constant $c > 0$, every $x_0 \in \overline{\Omega}$ and every $r \in (0, R)$ with $R > 0$ fixed, it holds that*

$$\int_{\Omega(x_0,r)} |f - (f)_{x_0,r}|^p \,\mathrm{d}x \leq c r^{n+p\alpha}$$

for some $\alpha \in (0, 1]$. Then, f is Hölder continuous in $\overline{\Omega}$ with exponent α.

For a proof of this result we refer the reader to [GM12, Theorem 5.5] and the original references that we have already mentioned.

B.2 Sobolev spaces

Given the nature of the variational problems considered in this work, which are in terms of the derivative of the admissible functions (or deformations), some of the main characters that come into play are the *Sobolev spaces*, introduced by S.L. Sobolev in 1938.[1]. To fix the terminology and notation, we recall that, for $1 \leq p \leq \infty$ and for an open set $\Omega \subseteq \mathbb{R}^n$, the Sobolev space $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is defined as

$$\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N) := \left\{ u\colon \Omega \to \mathbb{R}^N : \nabla u \text{ exists and } \int_{\Omega} (|u|^p + |\nabla u|^p) \,\mathrm{d}x < \infty \right\} \text{ if } 1 \leq p < \infty;$$
$$\mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N) := \left\{ u\colon \Omega \to \mathbb{R}^N : \nabla u \text{ exists and } \|u\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^N)} + \|\nabla u\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^N)} < \infty \right\},$$

where ∇u stands for the weak derivative of u. A natural way to establish a norm for $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is by defining $\|u\|_{\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)} := \|u\|_{\mathrm{L}^p(\Omega,\mathbb{R}^N)} + \|\nabla u\|_{\mathrm{L}^p(\Omega,\mathbb{R}^N)}$. With this structure, $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is a Banach space. What is more, similarly to what happens for the L^p spaces, $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is reflexive for $1 < p < \infty$ and therefore, by the Banach-Alaoglu Theorem, we can assure that if (u_j) is a bounded sequence in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$, then there exist $u \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$

[1]See [Sob38]

and a subsequence of (u_j), that we do not relabel, such that

$$u_j \rightharpoonup u.$$

Here, $u_j \rightharpoonup u$ refers to the weak convergence in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ and it is equivalent to the weak convergence, in L^p, of u_j and ∇u_j to u and ∇u respectively.

For the case $p = \infty$, we make use of the properties of the weak* convergence in L^∞ and say, for a sequence $u_j \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$, that it **converges weakly*** to a function $u \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$ if and only if

$$u_j \overset{*}{\rightharpoonup} u \quad \text{and} \quad \nabla u_j \overset{*}{\rightharpoonup} \nabla u$$

in L^∞. In this case, we write $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$. Observe that, as before, it is a consequence of the Banach-Alaoglu theorem (for L^∞ seen as the dual space of L^1) that every bounded sequence in $\mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$ has a weakly* convergent subsequence.[2]

Of particular importance are the Sobolev functions that are restricted to take certain values at the boundary of Ω in the distributional sense. In this spirit, we now consider the following definition.

Definition 108 *Let $1 \leq p < \infty$. We then set*

$$\mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N) := \overline{C_0^\infty(\Omega, \mathbb{R}^N)},$$

where the closure is taken in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$.
If $p = \infty$, we let

$$\mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^N) := \left\{ u \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N) : \exists (\varphi_k) \subseteq C_0^\infty(\Omega, \mathbb{R}^N) \textit{ s.t. } \varphi_k \overset{*}{\rightharpoonup} u \textit{ and } \nabla \varphi_k \overset{*}{\rightharpoonup} \nabla u \right\}.$$

Finally, if $u_0 \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ for some $1 \leq p \leq \infty$, we say that $u \in \mathrm{W}_{u_0}^{1,p}(\Omega, \mathbb{R}^N)$ if and only if $u - u_0 \in \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N)$.

We remark that, in the definition of $\mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^N)$, it is important to consider the weak* closure of $C_0^\infty(\Omega, \mathbb{R}^N)$ and not the strong closure because, if we only consider the strong

[2] See [FL07] for further details.

closure given by the uniform topology, we would only obtain a subspace of $C_0^1(\Omega,\mathbb{R}^N)$. In addition, we also note that, if $1 \le p < \infty$, then the weak and strong closures of $C_0^\infty(\Omega,\mathbb{R}^N)$ in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ coincide by Mazur's Lemma. We now state the following consequence of Rellich-Kondrachov Embedding Theorem, that we constantly use in this work.

Theorem 109 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary and let $1 \le p \le \infty$. Then, the embedding of $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ in $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ is compact.*

To conclude this section, we state the following standard density result.

Theorem 110 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $1 \le p < \infty$. Then, the space $C^\infty(\Omega,\mathbb{R}^N) \cap \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ is dense in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Moreover, if Ω is Lipschitz, then $C^\infty(\overline{\Omega},\mathbb{R}^N)$ is also dense in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$.*

For a proof of these two results and for further details regarding Sobolev spaces we refer the reader to [GT01, Chapter 7].

B.3 Poincaré inequalities

We conclude this appendix with a compilation of the Poincaré-type inequalities that we use along this work and the precise form in which we use them. We begin by stating a local Poincaré-Sobolev inequality that can be applied near the boundary.

Theorem 111 *Let $p \ge 2$. Assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Then, there exist constants $R_1 > 0$ and $c > 0$ such that, for every $x_0 \in \partial\Omega$, $0 < R < R_1$ and every $v \in \mathrm{W}^{1,p}(\Omega(x_0,R),\mathbb{R}^N)$, if $v \equiv 0$ on $B(x_0,R) \cap \partial\Omega$, it holds that*

$$\left(\fint_{\Omega(x_0,R)} \left| V\left(\frac{v}{R}\right)\right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \le c \left(\fint_{\Omega(x_0,R)} |V(\nabla v)|^{\frac{2n}{n+2}} \mathrm{d}x \right)^{\frac{n+2}{2n}}.$$

We refer the reader to [Kro05, Lemma 1] for a proof of this result. In connection with this, we also state the following Poincaré inequality, that can be applied to functions vanishing on a subset of the boundary of positive $\mathcal{H}^{n-1}$ measure.

Theorem 112 *Let $1 \le p < \infty$ and let $u \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Assume that $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz bounded domain and define*

$$N := \{x \in \partial\Omega : u(x) = 0\}. \tag{B.3}$$

If $\mathcal{H}^{n-1}(N) > 0$, then there is a constant $C > 0$, depending exclusively on $\mathcal{H}^{n-1}(N)$ and p, such that

$$\|u\|_{\mathrm{L}^p(\Omega,\mathbb{R}^N)} \leq C\|\nabla u\|_{\mathrm{L}^p(\Omega,\mathbb{R}^N)}.$$

The proof of this result is a consequence of [EG92, Section 5.6.3] and [Zie89, Corollary 4.5.3]. We remark that this is just a particular case of a more general class of Poincaré-type inequalities and we refer the reader to [Zie89] for a deep analysis of this subject.

We also state here the following version of the Poincaré-Sobolev inequality for interior balls.

Theorem 113 *Let $p \geq 2$. Assume that $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain. Then, there exists a constant $c > 0$ such that, for every $B(x_0, R) \subseteq \Omega$ and every $v \in \mathrm{W}^{1,p}(\Omega(x_0, R), \mathbb{R}^N)$, it holds that*

$$\left(\fint_{B(x_0,R)} \left| V\left(\frac{v - (v)_{x_0,R}}{R}\right)\right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \leq c \left(\fint_{B(x_0,R)} |V(\nabla v)|^{\frac{2n}{n+2}} \mathrm{d}x \right)^{\frac{n+2}{2n}}.$$

The proof of this classical result can be found, for example, in [Giu03, Theorem 3.17]. We further remark that, for all the results in this section, the assumption that Ω is a Lipschitz domain can be relaxed to assume that Ω has the cone property or, more generally, that it is an *extension domain* in the terminology of [Zie89, Remark 2.5.2].

Appendix C

Review on theory of Linear Operators

Definition 114 *Let X and Y be real (or complex) vector spaces and let $T\colon X \to Y$ be a linear transformation. We define the* ***kernel*** *of T as the subspace of X defined by*

$$\mathrm{Ker}(T) := \{x \in X : Tx = 0\}.$$

Likewise, we define the ***image*** *of T like the vector space*

$$\mathrm{Im}(T) = \{y \in Y : y = Tx \text{ for some } x \in X\}.$$

In addition, when X and Y are normed spaces, we say that T is ***bounded*** *if and only if there is a constant $C > 0$ such that, for every $x \in X$,*

$$\|Tx\|_Y \leq C\|x\|_X.$$

Finally, we denote

$$\mathscr{L}(X,Y) := \{T\colon X \to Y : T \text{ is a bounded linear transformation}\}.$$

We recall that, if Y is a Banach space, then $\mathscr{L}(X,Y)$ equipped with the norm

$$\|T\|_{\mathscr{L}(X,Y)} := \sup_{\|x\|=1} \frac{\|Tx\|_Y}{\|x\|_X}$$

is also a Banach space.

We further recall that if V is a subspace of X, we can naturally define the **quotient space** as the set of cosets

$$X/V := \{x + V : x \in X\}. \tag{C.1}$$

This is a vector space with the linear operations defined by

$$\lambda(x+V) := (\lambda x) + V \qquad \text{and} \qquad (x+V) + (y+V) := (x+y) + V.$$

It then follows from the definitions that every linear transformation $T\colon X \to Y$ induces an isomorphism $\Phi_T\colon X/\mathrm{Ker}(T) \to \mathrm{Im}(T)$ given by

$$\Phi_T(x + \mathrm{Ker}(T)) := Tx.$$

Regarding quotient spaces we know that, if V is closed in X, we can assign a norm to the vector space X/V by defining

$$\|x+V\|_{X/V} := \inf_{v\in V} \|x+v\|_X.$$

Furthermore, if X is a Banach space, X/V is also a Banach space with the above norm.

On the other hand, it is easy to see that, if $T \in \mathscr{L}(X,Y)$, then $\mathrm{Ker}(T) = T^{-1}\{0\}$ is closed and the induced isomorphism is bounded. In order to establish when is the inverse of the induced isomorphism Φ_T^{-1} a bounded linear transformation, we need to make use of the Open Mapping Theorem, that we state as follows.

Theorem 115 *Let X and Y be Banach spaces and suppose that $T \in \mathscr{L}(X,Y)$. If $\mathrm{Im}(T) = Y$, then T is an open mapping, i.e., the image of each open subset of X under T is an open subset of Y.*

The proof of this classical result relies in the Baire Category Theorem and can be found, for example, in [Rud91, Theorem 2.11]. The next corollary follows straight from the Open

Mapping Theorem after recalling that every closed subspace of a Banach space is again a Banach space.

Corollary 116 *Let X and Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$. Then, the following conditions are equivalent*

(i) *The subspace* $\mathrm{Im}(T)$ *is closed in* Y.

(ii) *The induced map* $\Phi_T \colon X/\mathrm{Ker}(A) \to \mathrm{Im}(T)$ *has a bounded inverse.*

(iii) *There is a constant $C > 0$ such that* $\|x + \mathrm{Ker}(A)\|_{X/\mathrm{Ker}(A)} \leq C\|Tx\|_Y$ *for every* $x \in X$.

C.1 Dual spaces

Given a normed vector space X, we define its **dual space** to be the space of all bounded linear functionals $g \colon X \to \mathbb{R}$ endowed with the natural pointwise vector space operations and we denote it by X^*. In addition, we write

$$\langle g, x \rangle := g(x).$$

We endow X^* with the standard norm for the space $\mathscr{L}(X, \mathbb{R})$ and we remark that then X^* is Banach space even if X is not complete. A fundamental result in the study of dual spaces is the Hahn-Banach Theorem, that we state here as follows.

Theorem 117 *Let V a subspace of a normed space X. Then, every functional in V^* can be extended to a functional in X^* having the same norm.*

Despite the many outstanding applications of this result, we move forward in this brief review of Operator Theory to define the following concept, that we will refer to later.

Definition 118 *Let $T \colon X \to Y$ be a linear transformation. We define the **transpose** of T as the linear map $T^t \colon Y^* \to X^*$ such that*

$$\left\langle T^t y^*, x \right\rangle := \langle y^*, Tx \rangle .$$

It can then be shown, using Hahn-Banach's Theorem, that T^t is bounded if and only if T is bounded. For a proof of this statement we refer the reader to [McL00, p.22]. While studying solutions of equations of the type $Tx = y$, it is helpful to consider the transposed equation

$$T^t y^* = x^*$$

for a given $x^* \in X^*$ and an unknown $y^* \in Y^*$. In order to describe the relationship between the two equations, we use the following terminology.

Definition 119 *Let $V \subseteq X$. The* ***annihilator*** *V^a is the closed subspace of X^* defined by*

$$V^a := \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for every } x \in V\}.$$

In a dual way, given $W \subseteq X^*$ we define the ***annihilator*** *${}^aW$ as the closed subspace of X given by*

$${}^aW := \{x \in X : \langle x^*, x\rangle = 0 \text{ for every } x^* \in W\}.$$

It is a straightforward consequence of this definition that, for any linear transformation $T\colon X \to Y$,

$$\operatorname{Ker}(T) = {}^a(\operatorname{Im}(T^t)) \qquad \text{and} \qquad \operatorname{Ker}(T^t) = (\operatorname{Im}(T))^a.$$

Having established the previous concepts, we are now ready to state the main result of this section. The main point is that if the image of T is closed, then the equation $Tx = y$ has a solution if and only if the right hand side y is annihilated by every solution y^* of the transposed equation $T^t y^* = 0$.

Theorem 120 *Let X and Y be Banach spaces. If $T \in \mathscr{L}(X, Y)$, then the following conditions are equivalent.*

(i) $\operatorname{Im}(T)$ *is closed in* Y.

(ii) $\operatorname{Im}(T^t)$ *is closed in* X^*.

(iii) $\operatorname{Im}(T) = {}^a\left(\operatorname{Ker}(T^t)\right)$.

(iv) $\operatorname{Im}(T^t)\,(\operatorname{Ker}(T))^a$.

Furthermore, if these conditions hold, there are isometric isomorphisms

$$(Y/\operatorname{Im}(T))^* \cong \operatorname{Ker}(T^t) \qquad \textit{and} \qquad X^*/\operatorname{Im}(T^t) \cong (\operatorname{Ker}(T))^*.$$

The proof of this Theorem can be found in [McL00, p.25].

C.2 Compactness

In this section we compile some outstanding notions and results regarding compactness of subsets of metric spaces. We recall that, if (X, d) is a metric space and $W \subseteq X$, an open cover of W is a family of open subsets of X whose union contains W. Then, we say that $W \subseteq X$ is **compact** if every open cover of W has a finite subcover. It is then easy to verify that every compact set is closed and bounded. In addition, if the closure of W, $\overline{W}$ is compact, we say that W is **relatively compact**.

We also recall that, given $\varepsilon > 0$ and a (possibly infinite) set of indices I, an ε-net is a subset $\{w_i\}_{i\in I} \subseteq W$ with the property that, for each $w \in W$, there is an $i \in I$ such that $d(w, w_i) < \varepsilon$. If W has a *finite* ε-net for every $\varepsilon > 0$, we say that W is **totally bounded**. It is then clear that every totally bounded set is bounded.

The relevance of these concepts relies on the following result.

Theorem 121 *Let (X, d) be a metric space and let $W \subseteq X$. The following three statements are then equivalent.*

(i) *W is relatively compact.*

(ii) *Every sequence in W has a subsequence that converges in X.*

(iii) *The subset W is totally bounded.*

For a proof of this statement we refer the reader to [Jos05, Theorem 7.40].

If we now assume that the metric space X is compact and Y is a Banach space, then the set $C(X, Y)$ of all continuous functions $f\colon X \to Y$ is a Banach space with the uniform norm

$$\|f\|_{C(X,Y)} := \|f\|_\infty = \max_{x\in X} |f(x)|_Y.$$

A fundamental concept to characterize compact subsets of $C(X, Y)$ is the following.

Definition 122 *Let $\mathcal{H} \subseteq C(X, Y)$ be a family of continuous functions. We say that $\mathcal{H}$ is **equicontinuous at** $x_0 \in X$ if, for every $\varepsilon > 0$, there is $0 < \delta = \delta(x_0, \varepsilon)$ such that, whenever $\|x - x_0\| < \delta$,*

$$\|f(x) - f(x_0)\|_Y < \varepsilon \qquad \text{for all } f \in \mathcal{H}.$$

We say that $\mathcal{H}$ is ***equicontinuous*** *if it is so at every $x_0 \in X$.*

We can now state the following result due to Cesare Arzelá and Giulio Ascoli.

Theorem 123 *Let X and Y be metric spaces such that X is compact and Y is a Banach space. A subset $\mathcal{H} \subseteq C(X,Y)$ is relatively compact in $C(X,Y)$ if and only if it is equicontinuous and bounded in $C(X,Y)$.*

For a proof of this result see [Jos05, p.56].

If we now let X and Y to be normed spaces, we can generalize the notion of compactness to linear operators in the following way.

Definition 124 *Let X and Y be normed spaces and let $K\colon X \to Y$ be a linear operator. We say that K is* ***compact*** *if it maps every bounded subset of X onto a relatively compact subset of Y.*

It follows that every compact operator is bounded and, furthermore, every linear operator whose image is a finite dimensional space is compact because, in finite dimensional normed spaces, every totally bounded set is bounded. In addition, from Theorem 121 we infer that $K\colon X \to Y$ is compact if and only if every bounded sequence $(u_j) \subseteq X$ has a subsequence (u_{j_k}) such that $(K(u_{j_k}))$ converges in Y.

On the other hand, Arzelá-Ascoli's Theorem suggests that, if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $K\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ is a compact operator, then we can expect Ku to be smoother than u. Therefore, it is not surprising that many integral operators turn out to be compact.

A class of operators of particular interest in the study of differential equations are those of the form $I + K\colon X \to X$, where I is the identity map and K is a compact operator. More precisely, they play an important role in establishing the properties of Fredholm operators.[1] Fredholm developed the first general theory of equations of the form $u + Ku = f$, with K a concrete compact operator. His method of Fredholm determinants used only techniques arising in classical analysis. A brief description of it can be found in the book on functional analysis by Riesz and Sz.-Nagy.

In order to establish the main result of this section concerning operators of the form $I+K$, we first prove the following lemma, in which we think of a vector $u \in X$ as almost orthogonal to a subspace W, even though the norm might not be induced by an inner product. Here,

[1] See Chapter 5.

and in what remains of this section, we simplify the notation by writing $\|\cdot\| = \|\cdot\|_X$, since no other norms are involved.

Lemma 125 *Let X be a normed vector space and let $W \subseteq X$ be a closed subspace. If $W \neq X$, then for each $\varepsilon > 0$ there exists $u \in X$ such that $\|u\| = 1$ and $\operatorname{dist}(u, W) \geq 1 - \varepsilon$.*

Proof. Choose $v \in X \backslash W$ arbitrary and let $d := \operatorname{dist}(v, W)$. Note that $d > 0$ because W is closed. Given $\varepsilon > 0$, select $w_\varepsilon \in W$ such that $d \leq \|v - w_\varepsilon\| \leq \frac{d}{1-\varepsilon}$. Then, set $u := \|v - w_\varepsilon\|^{-1}(v - w_\varepsilon)$. It is then clear that $\|u\| = 1$ and, for all $w \in W$, there exists $\tilde{w} \in W$ such that

$$\|v - w_\varepsilon\|(u - w) = v - w_\varepsilon - \|v - w_\varepsilon\| w = v - \tilde{w}.$$

This means that $\|v - w_\varepsilon\|\|u - w\| \geq \operatorname{dist}(v, W) = d$ and, therefore, $\|u - w\| \geq \frac{d}{\|v - w_\varepsilon\|} \geq 1 - \varepsilon$, as we wanted to prove. $\square$

We are now ready to state the main theorem concerning operators of the form $I + K$.

Theorem 126 *Let X be a normed space and assume that $K: X \to X$ is compact. Define $A: X \to X$ by $A := I + K$. We then have that the following statements are true.*

(i) *For each $m \geq 0$, the subspace $V_m := \operatorname{Ker}(A^m)$ is finite dimensional.*

(ii) *For each $m \geq 0$, the subspace $W_m := \operatorname{Im}(A^m)$ is closed.*

(iii) *There is a finite number $r \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$,*

$$\{0\} = V_0 \subsetneq V_1 \cdots \subsetneq V_r = V_{r+k}$$

and

$$X = W_0 \supsetneq W_1 \supsetneq \cdots W_r = W_{r+k}.$$

(iv) *$X = V_r \oplus W_r$.*

Proof. We first prove part (i) of the theorem. The case $m = 0$ is trivial because then $A = I$. We proceed by contradiction for the case $m = 1$. Assume then that V_1 is not finite dimensional. Using Lemma 125, we can recursively construct a sequence $(u_j) \subseteq V_1$ such that

$$\|u_j\| = 1 \qquad \text{and} \qquad \|u_j - u_k\| \geq \frac{1}{2} \text{ for } j \neq k.$$

Since K is compact, there is a subsequence (u_{j_k}) and a vector $\phi \in X$ such that $Ku_{j_k} \to \phi$. However, $u_j + Ku_j = Au_j = 0$. Hence, $u_{j_k} = -Ku_{j_k} \to \phi$ when $k \to \infty$. This is a contradiction, because (u_{j_k}) is not a Cauchy sequence. This proves that V_1 is finite dimensional. The proof of (i) follows in turn from this fact, since

$$A^m = I + L \qquad \text{with } L = \sum_{j=1}^{m} \begin{pmatrix} m \\ j \end{pmatrix} K^j \tag{C.2}$$

and L is easily seen to be compact.

We now prove (ii) and we begin by establishing that W_1 is closed. Let $f_j := Au_j \to f$ in X. Define $d_j := \mathrm{dist}(u_j, \mathrm{Ker}(A))$ and choose $v_k \in \mathrm{Ker}(A)$ such that

$$d_j \leq \|u_j - v_j\| \leq (1 + j^{-1})d_j. \tag{C.3}$$

If the sequence (d_j) is bounded, then so is (w_j) where $w_j := u_j - v_j$. Therefore, there is a subsequence w_{j_k} such that $Kw_{j_k} \to \phi$ for some $\phi \in X$. Since $Aw_j = Au_j = f_j$, it follows that $w_{j_k} = f_{j_k} - Kw_{j_k} \to f - \phi$ and thus $f = \lim_{k\to\infty} f_{j_k} = A(f - \phi) \in W_1$.

Otherwise, if (d_j) is not bounded, by possibly considering a subsequence, that we do not relabel, we can assume that $d_j \to \infty$ and $d_j > 0$ for every $j \in \mathbb{N}$. We then define $w_j := \|u_j - v_j\|^{-1}(u_j - v_j)$ so that $\|w_j\| = 1$ and $Kw_{j_k} \to \phi$ for some subsequence (w_{j_k}) and some $\phi \in X$. Since $\|Aw_j\| = \|u_j - v_j\|^{-1}\|A(u_j - v_j)\| \leq d_j^{-1}\|f_j\|$, we have that $\|f_j\| \to 0$ and $w_{j_k} = Aw_{j_k} - Kw_{j_k} \to -\phi$. In addition, $A\phi = -\lim_{k\to\infty} Aw_{j_k} = 0$, so $\phi \in \mathrm{Ker}(A)$. Consider now $\psi_j := w_j + \phi$. We then have

$$\|u_j - v_j\|\psi_j = u_j - v_j + \|u_j - v_j\|\phi = u_j - \tilde{\phi}$$

for some $\tilde{\phi} \in \mathrm{Ker}(A)$. The definition of d_j then gives that $\|u_j - v_j\|\|\psi_j\| \geq d_j$ and, hence, $\|\psi_j\| \geq \frac{d_j}{\|u_j - v_j\|} \geq \frac{1}{1+j^{-1}}$ by (C.3). This contradicts the fact that $\psi_j \to 0$ as $j \to \infty$ and, therefore, we can infer that (d_j) is bounded, so that W_1 is closed. Part (ii) then follows from (C.2), just as we did with part (i).

For a proof of statements (iii) and (iv) we refer the reader to [McL00, p.31]. □

Appendix D

Modulus of continuity

In this appendix we construct a modulus of continuity satisfying the precise properties that we required from it in the proof of Theorem 76.

The precise statement that we prove here is as follows.

Theorem 127 *Let $G\colon \mathbb{R}^m \to \mathbb{R}^k$ be a continuous function and take $m > 0$ arbitrary. Then, there is a function $\omega\colon [0,\infty) \to [0,1]$, that we call modulus of continuity of G, such that ω is increasing, concave, $\omega(t) = 1$ for every $t \geq 1$, $\lim_{t\to 0}\omega(t) = 0$ and, for a constant $k = k(m) > 0$ and for every ξ, η satisfying $|\xi|, |\eta| \leq m+1$, it holds that*

$$|G(\xi) - G(\eta)| \leq k\omega(|\xi - \eta|).$$

The existence of a modulus of continuity with the properties stated in this theorem is well known. See, for example, [GM12, Theorem 9.2]. We include a proof here for the convenience of the reader.

Proof. We let

$$\tilde{k} := 1 + \sup_{|\xi|\leq m+1} |G(\xi)|$$

and define $\tilde{\omega}\colon [0,\infty) \to [0,1]$ by

$$\tilde{\omega}(t) := \frac{1}{2\tilde{k}} \sup\left\{|G(\xi) - G(\eta)| : |\xi - \eta| \leq t \text{ and } |\xi|, |\eta| \leq m+1\right\}.$$

It is then easy to see, using that G is uniformly continuous on the compact set $\overline{B(0,m)}$, that $\lim_{t\to 0}\omega(t)=0$ and that

$$|G(\xi)-G(\eta)|\le 2\tilde{k}\omega(|\xi-\eta|)$$

for every $\xi,\eta\in\overline{B(0,m+1)}$.

We now let $\tau\colon[0,\infty)\to[0,1]$ be given by

$$\tau(t):=\max\left\{\tilde{\omega}(t),\min\{t,1\}\right\}.$$

Then, τ still has all the properties of $\tilde{\omega}$ and, in addition, it is clear that τ is well defined in the sense that $\tau(t)\le 1$ for every $t\in[0,\infty)$.

Therefore, to conclude the proof it is enough to let ω be the concave envelope of τ in $[0,\infty)$. More precisely, we define

$$\omega(t):=\inf\left\{f(t):f[0,\infty)\to[0,\infty),\ f\text{ is concave and }f(t)\ge\tau(t)\text{ for every }t\in[0,\infty)\right\}.$$

The function ω satisfies all the required properties with $k:=2\tilde{k}$. $\square$

Appendix E

The auxiliary function V and its properties

In what follows, we summarize some elementary properties of the class of functions V_β. We recall that, for $\beta > 0$ arbitrary and for $k \in \mathbb{N}^+$, we define the function $V_\beta\colon \mathbb{R}^k \to \mathbb{R}^k$ by

$$V_\beta(\xi) := \left(1 + |\xi|^2\right)^{\frac{\beta-1}{2}} \xi.$$

In addition, to simplify the notation we write $V = V_{\frac{p}{2}}$ and recall that, when $p \in [2, \infty)$, $|V|^2$ is proportional, up to a constant, to the function $|\cdot|^2 + |\cdot|^p$.

Finally, we remark that in this appendix, and in the rest of this text, we make no distinction in the notation to distinguish between different dimensions of the domain in which the function V_β is defined. An example of this is statement (ii) in the following lemma. Furthermore, we use the same syntaxes for the function V_β in statement (i), which is valid for $V_\beta\colon \mathbb{R} \to \mathbb{R}$, and in the rest of the properties that we now enunciate, which are valid for $V_\beta\colon \mathbb{R}^k \to \mathbb{R}^k$ for any $k \in \mathbb{N}^+$.

Lemma 128 *Let $\beta > 0$, $2 \leq p < \infty$ and $M > 0$. Then, there is a constant $c > 0$, depending only on β, such that for every $\xi, \eta \in \mathbb{R}^k$ and every $t \geq 0$,*

(i) *$V_\beta(t)$ is non-decreasing in $[0, \infty)$;*

(ii) *$|V_\beta(\xi)| = V_\beta(|\xi|)$;*

(iii) *$|V_\beta(\xi + \eta)| \leq c\left(|V_\beta(\xi)| + |V_\beta(\eta)|\right)$;*

(iv) $|V(t\xi)| \le \max\{t, t^{\frac{p}{2}}\}|V(\xi)|$;

(v) $c(p)|\xi-\eta| \le \frac{|V(\xi)-V(\eta)|}{(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{4}}} \le c(p,k)|\xi-\eta|$;

(vi) $\left(1+|\xi|^2+|\eta|^2\right)^{\frac{p}{2}} \le c\left(1+|V(\xi)|^2+|V(\eta)|^2\right)$;

(vii) $|V_{p-1}(\xi)|\,|\eta| \le |V(\xi)|^2+|V(\eta)|^2$;

(viii) *Young's inequality is satisfied in the sense that, for every* $\varepsilon > 0$, *there exists* $c_\varepsilon > 0$ *such that* $|V_{p-1}(\xi)||\eta| \le \varepsilon|V(\xi)|^2 + c_\varepsilon|V(\eta)|^2$;

(ix) $\max\{|\xi|, |\xi|^{\frac{p}{2}}\} \le |V(\xi)| \le 2^{\frac{p-2}{4}} \max\{|\xi|, |\xi|^{\frac{p}{2}}\}$;

$\frac{1}{2}\left(|\xi|^2+|\xi|^p\right) \le |V(\xi)|^2 \le 2^{\frac{p-2}{4}}\left(|\xi|^2+|\xi|^p\right)$;

(x) $|V(\xi-\eta)| \le c(p)|V(\xi)-V(\eta)|$;

(xi) $|V(\xi)-V(\eta)| \le c(k,p,M)|V(\xi-\eta)|$, *provided* $|\eta| \le M$.

With the exception of (ix)-(xi), all the above properties also hold if $1 < p < 2$.

The proof of this lemma requires essentially elementary computations and consider separately the cases $|\xi|, |\eta| < 1$ and $|\xi|, |\eta| \ge 1$, with all the combinations that arise from these possibilities. We refer the reader to [AF89b], [CFM98, Lemma 2.1] and [Bec11, Lemma 2.1] for further details.

Another property of the function V is that it can be estimated above and below by multiple scalars of the convex function that we define below. This is particularly useful when it comes to applying Jensen's inequality. See [Sch08, Lemma 6.2]. We let

$$W_\beta(\xi) := (1+|\xi|)^{\beta-1}\xi.$$

We observe that W_β is convex and that, for some constant $c > 0$ depending only on β,

$$c^{-1}|W_\beta| \le |V_\beta| \le c\,|W_\beta|.$$

Just as we did with the function V, we relax the notation by writing $W = W_{\frac{p}{2}}$.

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