Heights via anabelian geometry and local Bloch–Kato Selmer sets



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A thesis submitted for the degree of
Doctor of Philosophy

Michaelmas 2017–2018

To JDH

Acknowledgements

I am grateful to Jonathan Pridham for helpful discussions about comparison theorems for fundamental groups, and to Fabrizzio Andreatta, Christiaan Peters and Joseph Steenbrink for taking the time to answer my questions on various technical aspects of their papers and books. I am also grateful to many of my fellow DPhil students in the Mathematical Institute in Oxford, in particular to my officemates Claudio Llosa Isenrich and Giles Gardam, for helping me to think from a topological viewpoint. I am also grateful to Adam Morgan for introducing me to Breen's theory of cubical structures and helping me to understand its relation to the ideas discussed in this dissertation. Lastly, I am very grateful to my two examiners, Damian Rössler and Gerd Faltings, for their careful reading of the initial version of this thesis and helpful suggestions.

Among the many people who have helped shape my thoughts into the present thesis, however, two deserve special mention. Firstly, I owe a great deal to Netan Dogra, in particular for suggesting how to reduce theorem 1.3.4 to the case of curves, but also more generally for our numerous informative and lively discussions, from which I have learned so much. Finally, and most notably, I am greatly indebted to my DPhil advisor, Minhyong Kim, for his wise and gentle guidance as I trod the long path towards writing this thesis.

Abstract

We study the problem of describing local components of height functions on abelian varieties over characteristic 0 local fields as functions on spaces of torsors under various realisations of a 2-step unipotent motivic fundamental group naturally associated to the defining line bundle. To this end, we present three main theorems giving such a description in terms of the $\mathbb{Q}_\ell$- and $\mathbb{Q}_p$-pro-unipotent étale realisations when the base field is p-adic, and in terms of the $\mathbb{R}$-pro-unipotent Betti–de Rham realisation when the base field is archimedean.

In the course of proving the p-adic instance of these theorems, we develop a new technique for studying local non-abelian Bloch–Kato Selmer sets, working with certain explicit cosimplicial group models for these sets and using methods from homotopical algebra. Among other uses, these models enable us to construct a non-abelian generalisation of the Bloch–Kato exponential sequence under minimal conditions.

On the geometric side, we also prove a number of foundational results on local constancy or analyticity of various non-abelian Kummer maps for pro-unipotent étale or Betti–de Rham fundamental groups of arbitrary smooth varieties.

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Chapter 1

Introduction

1.1 Anabelian geometry and local heights

1.1.1 Historical background: the anabelian programmes of Grothendieck and Deligne

1.1.1.1 Grothendieck's étale fundamental group

In his letter to Faltings [Gro97], Grothendieck proposed a programme of studying the arithmetic and Diophantine geometry of hyperbolic curves Y over characteristic 0 fields F by means of their profinite étale fundamental groups $\pi_1^{\text{ét}}(Y_{\overline{F}})$ [Sza09, Definition 5.4.1] along with their outer Galois actions [Sza09, Remark 5.6.3(1)]. When one uses an F-rational point $y \in Y(F)$ as the basepoint for the fundamental group, the outer Galois action canonically lifts to a genuine continuous action, and the link with Diophantine geometry comes very concretely via a certain *non-abelian Kummer map*

$$Y(F) \to \mathrm{H}^1(G_F, \pi_1^{\text{ét}}(Y_{\overline{F}}; y)),$$

taking values in the non-abelian continuous[1] Galois cohomology set $\mathrm{H}^1(G_F, \pi_1^{\text{ét}}(Y_{\overline{F}}; y))$ classifying profinite torsors under $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)$ with compatible continuous Galois action; the non-abelian Kummer map sends an F-rational point $z \in Y(F)$ to the class of the profinite étale torsor of paths $\pi_1^{\text{ét}}(Y_{\overline{F}}; y, z)$ from y to z.

In the case that F is a number field, Grothendieck's *section conjecture* posits that the link between the fundamental group and the Diophantine geometry of Y provided by the non-abelian Kummer map should be very strong indeed: when Y is proper, the map should be bijective, and for general hyperbolic Y it should be injective (this much was already proved in [Gro97]), with the only extraneous elements of

[1] All Galois cohomology sets appearing in this dissertation will be continuous Galois cohomology, and we will use H^1 to denote such sets.

$\mathrm{H}^1(G_F, \pi_1^{\text{ét}}(Y_{\overline{F}}; y))$ coming from the F-rational cusps [Sti12]. It was apparently hoped that a proof of the section conjecture would enable one to find an alternative proof of the Mordell conjecture, which had been earlier proven by Faltings [Fal83].

In practice, the highly non-abelian nature of $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)$ makes studying it particularly difficult, and it is convenient to replace it with an appropriately linearised version of the fundamental group which is much closer to being abelian. Such a version is provided by the *$\mathbb{Q}_p$-pro-unipotent completion* of $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)$, namely the pro-unipotent group $U^{\text{ét}}/\mathbb{Q}_p$ admitting a universal continuous group homomorphism $\pi_1^{\text{ét}}(Y_{\overline{F}}; y) \to U^{\text{ét}}(\mathbb{Q}_p)$—universality ensures that the Galois action on $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)$ extends to a continuous algebraic action on $U^{\text{ét}}$ (see proposition 4.0.3). Despite this somewhat abstract characterisation, the $\mathbb{Q}_p$-pro-unipotent completion is relatively straightforward to understand: it is an iterated central extension of the vector groups $\mathbb{Q}_p \otimes_{\widehat{\mathbb{Z}}} \pi_1^{\text{ét}}(Y_{\overline{F}}; y)_{[n]}$, where $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)_{[n]}$ is the nth partial quotient of the descending central series filtration on $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)$ (by closed subgroups).

The group $U^{\text{ét}}/\mathbb{Q}_p$, henceforth referred to as the *$\mathbb{Q}_p$-pro-unipotent étale fundamental group of* (Y, y), admits an alternative description as the Tannaka group [DM82] classifying unipotent $\mathbb{Q}_p$-local systems[2] on $Y_{\overline{F}}$ (for any smooth connected variety Y/F), and this description ensures that we have a $\mathbb{Q}_p$-pro-unipotent version of the earlier non-abelian Kummer map

$$Y(F) \to \mathrm{H}^1(G_F, U^{\text{ét}}(\mathbb{Q}_p)),$$

again sending a point $z \in Y(F)$ to the class of the $\mathbb{Q}_p$-pro-unipotent torsor of paths from y to z provided by Tannaka's theory[3]. Although one cannot hope that this non-abelian Kummer map will be bijective in any generality, it is nonetheless very useful in the study of the arithmetic and Diophantine geometry of Y.

1.1.1.2 Deligne's motivic fundamental group

One can use the same Tannakian formalism to define other algebro-geometric theories of the pro-unipotent fundamental group, akin to the various Weil cohomology theories

[2] A $\mathbb{Z}/p^r\mathbb{Z}$-local system on Y is an étale sheaf of $\mathbb{Z}/p^r\mathbb{Z}$-modules (on $Y_{\overline{F}}$ for the étale topology) which étale-locally is constant and free of finite rank. A $\mathbb{Z}_p$-local system is a formal inverse limit of $\mathbb{Z}/p^r\mathbb{Z}$-local systems where the restriction maps are base change along the ring morphisms $\mathbb{Z}/p^{r+1}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z}$, while the category of $\mathbb{Q}_p$-local systems (or *lisse sheaves*) is the formal $\mathbb{Q}_p$-linearisation of the $\mathbb{Z}_p$-linear category of $\mathbb{Z}_p$-local systems. A $\mathbb{Q}_p$-local system is said to be *unipotent* just when it is an iterated extension of constant local systems.

[3] That is, the $\mathbb{Q}_p$-scheme of tensor-isomorphisms from the fibre functor at y to that at z, see [DM82, Theorem 3.2(a)], endowed with the evident G_F-action coming from the action on the fibre functors.

one has access to in addition to étale cohomology. For example, to every complex embedding $\sigma\colon \overline{F} \hookrightarrow \mathbb{C}$ there is associated a $\mathbb{Q}$-*pro-unipotent Betti fundamental group* $U^{\mathrm{B}}/\mathbb{Q}$ classifying unipotent $\mathbb{Q}$-local systems on the topological space $Y(\mathbb{C})$. More concretely, the Betti fundamental group U^{B} is the $\mathbb{Q}$-*Malčev completion* of the usual topological fundamental group $\pi_1^{\mathrm{top}}(Y(\mathbb{C}); y)$—this is just the $\mathbb{Q}$-pro-unipotent completion of the discrete group $\pi_1^{\mathrm{top}}(Y(\mathbb{C}); y)$, and can be concretely described as the affine group scheme

$$U^{\mathrm{B}} = \mathrm{Spec}\left(\varinjlim \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}\pi_1^{\mathrm{top}}(Y(\mathbb{C}); y)/J^n, \mathbb{Q})\right),$$

where J is the augmentation ideal of the Hopf algebra $\mathbb{Q}\pi_1^{\mathrm{top}}(Y(\mathbb{C}); y)$.

Just like the étale fundamental group, the Betti fundamental group carries more structure than simply that of a pro-unipotent group, this time in the form of a mixed Hodge structure (on its affine ring), constructed explicitly by Hain [Hai87a] using Chen's non-abelian de Rham theorem. Thus, there is also a *Betti–de Rham non-abelian Kummer map*

$$Y(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{Q}}, U^{\mathrm{B}}(\mathbb{Q}))$$

taking values in the set $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Q}}, U^{\mathrm{B}}(\mathbb{Q}))$ of torsors under U^{B} with compatible mixed Hodge structure, the map taking as usual a point z to the class of the Tannakian path-torsor from y to z, endowed with Hain's mixed Hodge structure [HZ87, Proposition 3.21].

These two theories of the fundamental group are not independent, however, and are in fact related by a canonical *comparison isomorphism*

$$U^{\text{ét}} \cong U^{\mathrm{B}}_{\mathbb{Q}_p}$$

(depending on σ). We can view the pair $(U^{\text{ét}}, U^{\mathrm{B}})$ (with the comparison isomorphism, Galois action on $U^{\text{ét}}$ and mixed Hodge structure on U^{B}) as a single object in a more structured category of pro-unipotent groups, which should bear a stronger relation to the arithmetic geometry of Y than either of $U^{\text{ét}}$ or U^{B} separately.

Even more generally, one can consider simultaneously a plethora of different types of fundamental group (étale, Betti, de Rham, log-crystalline), each endowed with various kinds of extra structures, and with various comparison isomorphisms between them. The study of such ensembles is the focus of the motivic philosophy of Deligne[4]

[4]It is somewhat anachronistic to place log-crystalline fundamental groups within Deligne's framework, since [Del89] was written when the theory of log geometry [Kat89] was still in its infancy, and in particular predates both the development of Hyodo–Kato cohomology [HK94] and log-crystalline fundamental groups [Shi00b]. Indeed, the crystalline considerations in [Del89] are limited to the good

[Del89], which posits that these ensembles should be viewed as a system of various realisations of a putative *pro-unipotent motivic fundamental group* U^{mot}. The target of the functor $(Y, y) \mapsto U^{\mathrm{mot}}$ should be a category of pro-unipotent group-objects in an as-yet-conjectural $\mathbb{Q}$-linear Tannakian category of mixed motives over F, and this functor should constitute a universal theory of pro-unipotent fundamental groups, in much the same way that the usual theory of mixed motives should constitute a universal Weil cohomology theory.

Of course, in the absence of a well-defined category of mixed motives, one cannot in general make sense of these fundamental groups purely motivically, and must content oneself with considering the interplay between the various realisations. While notable exceptions exist—Deligne and Goncharov in [DG05] construct genuinely motivic fundamental groups of punctured projective lines inside Levine's abelian category of mixed Tate motives [Lev93], and more recently Dan-Cohen and Schlank construct motivic path-spaces inside Voevodsky's derived category of motives [DCS17]—for the most part, the practical applications of Grothendieck's and Deligne's philosophies have come through consideration of the explicit realisations.

1.1.2 The method of Chabauty–Kim

Perhaps the most striking application of the anabelian ideas of Grothendieck and Deligne has been in Kim's programme of *non-abelian Chabauty* [Kim09], which aims to prove specific instances of Diophantine finiteness theorems for a hyperbolic curve $Y/\mathbb{Q}$ by consideration of the non-abelian Kummer maps associated to the maximal n-step unipotent quotients U_n of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of Y, where p is a prime of good reduction for Y. In this method, one aims to constrain the locus of $Y(\mathbb{Q}) \subseteq Y(\mathbb{Q}_p)$ by finding functions on the local non-abelian cohomology set $\mathrm{H}^1(G_{\mathbb{Q}_p}, U_n(\mathbb{Q}_p))$—or more accurately on a subset $\mathrm{H}^1_f(G_{\mathbb{Q}_p}, U_n(\mathbb{Q}_p))$ cut out by local Selmer conditions—which vanish on the image of global cohomology. When $n >> 0$, any one of various well-known conjectures (Bloch–Kato, Fontaine–Mazur, Janssen) implies the existence of such functions, thereby conditionally realising Grothendieck's dream of an anabelian proof of the Siegel–Faltings theorem on the finiteness of $Y(\mathbb{Z}_S)$. This proof can be even made unconditional in many cases: when $Y = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ [Kim05]; when Y is a once-punctured CM elliptic curve [Kim10]; when Y is a complete

reduction case, where a good theory already existed. Despite the ahistoricity of doing so, it seems reasonable to incorporate Shiho's log-crystalline fundamental group into Deligne's programme, since it adds to the constellation of specific theories of fundamental groups available at the time.

hyperbolic curve whose Jacobian has CM [CK10]; or more recently when Y is any complete hyperbolic curve which is a solvable cover of $\mathbb{P}^1_{\mathbb{Q}}$ [EH17].

At the other end of the spectrum, an explicit study of $\mathrm{H}^1_f(G_{\mathbb{Q}_p}, U_n(\mathbb{Q}_p))$ for n small leads to explicit methods for constraining the locus of $Y(\mathbb{Q}) \subseteq Y(\mathbb{Q}_p)$, and in practice often completely computing it. For instance, when $n = 1$ and Y is proper, the Chabauty–Kim method recovers Coleman's explicit Chabauty method [Col85]. The fundamental group $U_1(\mathbb{Q}_p) = V_pJ$ is the $\mathbb{Q}_p$-linear Tate module of the Jacobian J of Y, $\mathrm{H}^1_f(G_{\mathbb{Q}_p}, V_pJ) \cong \mathrm{Lie}(J_{\mathbb{Q}_p})$ is the Lie algebra of $J_{\mathbb{Q}_p}$, and the linear functions on $\mathrm{Lie}(J_{\mathbb{Q}_p})$ produced by Chabauty–Coleman agree with those produced by Chabauty–Kim.

When $n = 2$, the constraints on $Y(\mathbb{Q}) \subseteq Y(\mathbb{Q}_p)$ produced by non-abelian Chabauty involve finer arithmetic invariants. A detailed study of this case was made in [BD16, BD17]; among other results, it is shown in [BD16] that when $\mathrm{rk}(J(\mathbb{Q})) = \dim_{\mathbb{Q}}(J)$, the constraints produced by non-abelian Chabauty involve not only quadratic functions on $\mathrm{Lie}(J_{\mathbb{Q}_p})$, but also the local component of the p-adic height function in the sense of Nekovář [Nek93]. This raises the rather tantalising prospect that the constraints produced by non-abelian Chabauty for $n > 2$ should involve quantities that play the role of higher p-adic height functions.

1.1.3 Theme: Local heights as functions on local Selmer sets

Another example of the link between 2-step unipotent fundamental groups and local heights appears in a paper of Balakrishnan, Dan-Cohen, Kim and Wewers.

Theorem 1.1.1. *[BDCKW14] Let E be an elliptic curve over a p-adic[5] field K (i.e. a finite extension of $\mathbb{Q}_p$), and $U_2/\mathbb{Q}_\ell$ for $\ell \neq p$ the 2-step unipotent étale fundamental group of the punctured curve $E^\circ = E \setminus \{0\}$ based at a rational tangent vector at the origin. Then the map $\mathbb{Q}_\ell(1) \hookrightarrow U_2$ induced from the inclusion of the cusp[6] is injective and induces a bijection*

$$\mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^1(G_K, U_2(\mathbb{Q}_\ell)),$$

[5]We have interchanged the roles of p and ℓ from [BDCKW14]—for us p will always be the residue characteristic of the base field, and ℓ the residue characteristic of the field of coefficients for the fundamental groups involved.

[6]More precisely, the local ring of E at the origin is isomorphic to $K[\![t]\!]$, where t is the local parameter dual to the chosen tangent vector $\overrightarrow{t}$. This induces maps $\mathrm{Spec}(K[\![t]\!]) \hookrightarrow E$ and $\mathrm{Spec}(K((t))) \hookrightarrow E^\circ$. Since the Galois group of $\overline{K}((t))$ is canonically isomorphic to $\widehat{\mathbb{Z}}(1)$, this induces a G_K-equivariant map $\widehat{\mathbb{Z}}(1) \to \pi_1^{\text{ét}}(E^\circ_{\overline{K}}; \overrightarrow{t})$, and we are interested in the induced map on 2-step $\mathbb{Q}_\ell$-unipotent completions.

and the composite map

$$E^\circ(K) \to \mathrm{H}^1(G_K, U_2(\mathbb{Q}_\ell)) \xleftarrow{\sim} \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

takes values in $\mathbb{Q} \subseteq \mathbb{Q}_\ell$*, and is a Néron function on* $E(K)$ *with divisor* $2[0]$*.*

In this theorem we see a first concrete instance of what will be the main theme of this thesis: *that local components of height functions admit particularly natural interpretations as functions on spaces of torsors under 2-step unipotent fundamental groups*. In contrast with the papers [BD16, BD17, BDCKW14], our focus in expounding this theme will be on the local components of the classical $\mathbb{R}$-valued height rather than its p-adic counterpart, and so we will also be interested in developing analogues of theorem 1.1.1 not only when $\ell = p$, but also when the local base field K is archimedean. In keeping with Deligne's philosophy, this latter case will revolve around the study of the $\mathbb{R}$-pro-unipotent Betti fundamental group (with mixed Hodge structure) of varieties over $\mathbb{C}$, which plays the role of the $\mathbb{Q}_\ell$-pro-unipotent étale fundamental group (with local Galois action) over a p-adic field.

1.1.4 Setup and overview of results

With this theme in mind, let us now describe our general setup, where the link between local heights and 2-step unipotent fundamental groups arises most naturally, but which is also sufficiently general to allow us to study, for example, heights on curves of genus ≥ 1. We will fix an abelian variety A over a characteristic 0 local field K, and consider a line bundle L/A. On the one hand, such a line bundle induces a local height function on A, in the form of a *Néron function* on the complement of a corresponding divisor—we will see shortly that this can naturally be reinterpreted as a function on (the total space of) L with log poles along the zero section. On the other hand, there is a natural anabelian invariant associated to this setup, namely the pro-unipotent fundamental group U of the complement $L^\times = L \setminus 0$ of the zero section in L. From the fibration sequence

$$(\mathbb{G}_m, 1) \hookrightarrow (L^\times, \widetilde{0}) \twoheadrightarrow (A, 0)$$

(where $\widetilde{0} \in L^\times(K)$ is any K-rational basepoint in the fibre over $0 \in A(K)$), we see that U is a central extension of the fundamental group of A (i.e. the homology $h_1(A) = h^1(A)^*$) by the fundamental group of $\mathbb{G}_m$ (i.e. $\mathbb{Q}(1)$)[7].

[7]This assertion on the structure of U is certainly true on the level of Deligne's systems of realisations, e.g. by comparison with Betti fundamental groups. We make no assertion as to whether this is true on a more refined motivic level.

In this thesis, we will prove three theorems (1.1.6, 1.2.3 and 1.1.7) concretely relating the fundamental group U to local heights, of the following kinds:

- when K is p-adic, the local height can be realised as a function on the local non-abelian Galois cohomology set $\mathrm{H}^1(G_K, U(\mathbb{Q}_\ell))$, where U is the $\mathbb{Q}_\ell$-unipotent étale fundamental group of $L^\times$ for some $\ell \neq p$;
- when K is p-adic, the local height can be realised as a function on the local Bloch–Kato Selmer set $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$, where U is the $\mathbb{Q}_p$-unipotent étale fundamental group of $L^\times$; and
- when K is archimedean, the local height can be realised as a function on the set $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R}))$ classifying torsors under $U/\mathbb{R}$ with mixed Hodge structure, where U is the $\mathbb{R}$-unipotent Betti fundamental group of $L^\times$.

Let us remark also that in the first and last of these theorems, the space of torsors will be one-dimensional (in an appropriate sense), so that the local height is in fact the *only* information seen by the non-abelian Kummer map. The second theorem is more subtle, as the local Bloch–Kato Selmer set $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ contains plenty of analytic information, and a significant part of formulating the appropriate p-adic instance of these theorems lies in finding the right way of quotienting out the analytic part of the cohomology so that the quotient is one-dimensional and describes exactly the local height. Since developing the appropriate machinery for dealing with the p-adic case is the major part of this thesis, we will for the time being just discuss the ℓ-adic and archimedean cases, returning to discuss the p-adic case in more detail later in this introduction.

Remark 1.1.2. The problem of giving a motivic interpretation of local heights was also studied in [Sch94], where the emphasis was on giving a description of height pairings on cycles of complementary codimension in terms of motivic cohomology. Of course, a motivic description of the height pairing $\langle -, - \rangle \colon A(F) \otimes A^\vee(F) \to \mathbb{R}$ on an abelian variety A over a number field F gives rise to a description of the canonical height function induced by a line bundle L/A in terms of the usual map $\phi_L \colon A \to A^\vee$ via the formula [BG06, Proposition 9.3.6 & Corollary 9.3.7]

$$\widehat{h}_L(z) = \langle z, \phi_L(z) \rangle + \frac{1}{2} \langle z, L \otimes [-1]^* L^{-1} \rangle.$$

It is interesting that both Scholl's approach and ours require consideration of mildly non-linear invariants associated to the line bundle: in our case the fundamental group

of $L^\times$; and in Scholl's the pairing on motivic cohomology $h^1(A)$ induced by ϕ_L. Indeed, after performing suitable Tate twists, this pairing should be none other than the commutator pairing in the motivic fundamental group of $L^\times$, which is a central extension of $h_1(A)$ by $\mathbb{Q}(1)$.

Remark 1.1.3. In the particular case that the line bundle L is algebraically equivalent to the trivial bundle, the corresponding $\mathbb{G}_m$-torsor $L^\times$ canonically has the structure of a semiabelian variety with $\widetilde{0}$ as its identity, in which case the fundamental group of $L^\times$ is abelian (e.g. the étale fundamental group is the Tate module), and many of the arguments we will present in this dissertation reduce to more well-known abelian arguments. The chief advantage of our non-abelian techniques, then, is that it gives us a natural framework for studying line bundles that are not algebraically trivial, for instance ample or anti-ample bundles.

1.1.4.1 Local heights, following Néron and Bombieri–Gubler

To lead up to a precise statement of the ℓ-adic and archimedean main theorems, let us now explain the precise notion of local heights that will appear in this thesis. Since we will be working directly with line bundles rather than divisors, it will be convenient for us to follow the approach of Bombieri and Gubler in working with local metrisations on line bundles [BG06, Section 2.7] instead of Weil functions on complements of divisors [Lan83, Section 10.2]. To briefly recall what is meant by this, let us fix a characteristic 0 local field K with absolute value $|\cdot|$, and write $v = -\log|\cdot|$; we assume that v is normalised to be $\mathbb{Q}$-valued if K is non-archimedean. If L is a line bundle on a smooth connected variety Y/K, a *continuous log-metric* on L is a continuous function $\lambda_L \colon L^\times(\overline{K}) \to \mathbb{R}$ which satisfies the relation

$$\lambda_L(\alpha z) = \lambda_L(z) + v(\alpha)$$

for all $z \in L^\times(\overline{K})$ and $\alpha \in \overline{K}^\times$; here $L^\times = L \setminus 0$ denotes the complement of the zero section in L. Such a λ_L is then the negative logarithm of a metric in the sense of [BG06, Definition 2.7.1]. A continuous log-metric always exists by [BG06, Proposition 2.7.5], and if s is a non-zero section of L then $\lambda_L \circ s$ is a Weil function on Y associated to the Cartier divisor $\operatorname{div}(s)$ [Lan83, Section 10.2].

In the particular case that $Y = A$ is an abelian variety, there is a canonical choice of continuous log-metric λ_L on L up to additive constants, namely the log-metric such that for any (equivalently just one) non-zero section s, $\lambda_L \circ s$ is a Néron function on A associated to the Cartier divisor $\operatorname{div}(s)$ [Lan83, Section 11.1]. We will call such a

log-metric a *Néron log-metric* on L, and if we are provided with a point $\widetilde{0} \in L^\times(\overline{K})$, we will unambiguously refer to *the* Néron log-metric as being the unique one such that $\lambda_L(\widetilde{0}) = 0$.

Remark 1.1.4. Néron log-metrics are local components of heights in the following sense: if A is an abelian variety over a number field F and L/A is a line bundle with a distinguished point $\widetilde{0} \in L^\times(\overline{F})$ in the fibre over $0 \in A(\overline{F})$, then the function $L^\times(\overline{F}) \to \mathbb{R}$ given by an appropriately normalised sum of the Néron log-metrics at all places of $\overline{F}$ (is well-defined and) on the fibre over some $z \in A(\overline{F})$ is constant, equal to the canonical height $\widehat{h}_L(z)$ (by the corresponding result for Néron functions [Lan83, Theorem 11.1.6]).

Remark 1.1.5. In the spirit of Mazur and Tate [MT83], this description of the global height admits a natural refinement of a more Arakelov-theoretic nature when restricted to F-rational points[8]. Specifically, the Néron log-metrics at finite places are all $\mathbb{Q}$-valued, so the collection of all local metrics defines naturally a global function

$$\lambda_L \colon L^\times(F) \to \widehat{\mathrm{Div}}_{\mathbb{Q}}(\mathcal{O}_F) := \bigoplus_{v \nmid \infty} \mathbb{Q} \oplus \bigoplus_{v \mid \infty} \mathbb{R}$$

valued in a $\mathbb{Q}$-linearised analogue of the Arakelov divisor group of $\mathcal{O}_F$. By again evaluating this function on lifts of elements of $A(F)$, we thereby obtain a function

$$h_L \colon A(F) \to \widehat{\mathrm{Pic}}_{\mathbb{Q}}(\mathcal{O}_F) := \widehat{\mathrm{Div}}_{\mathbb{Q}}(\mathcal{O}_F)/\mathrm{div}(F^\times)$$

valued in the corresponding analogue of the Arakelov Picard group, through which the canonical height factors via the natural degree map $\deg \colon \widehat{\mathrm{Pic}}_{\mathbb{Q}}(\mathcal{O}_F) \twoheadrightarrow \mathbb{R}$.

It turns out (though we shall not prove this in this thesis) that this lift of the global height is a simultaneous quadratic refinement of Grothendieck's monodromy pairing on component groups [GRL67, Théorème 7.1(b)], and of Mazur and Tate's circle pairing [MT83, Section 3.5.2]. More precisely, h_L is always a quadratic function[9], and in the particular case that $L^\times$ underlies a biextension E of two abelian varieties A, B by $\mathbb{G}_m$, the map

$$h_L \colon A(F) \times B(F) \to \widehat{\mathrm{Pic}}_{\mathbb{Q}}(\mathcal{O}_F)$$

is a bilinear pairing which, on the one hand, when restricted to $A(F) \times B(F)^0$ takes values in the usual Arakelov Picard group $\widehat{\mathrm{Pic}}(\mathcal{O}_F)$ and agrees with the Mazur–Tate

[8] We assume here that $\widetilde{0} \in L^\times(F)$ is chosen F-rational.

[9] That is, satisfies the relation $h_L(x+y+z) - h_L(x+y) - h_L(y+z) - h_L(z+x) + h_L(x) + h_L(y) + h_L(z) - h_L(0) = 0$

circle pairing associated to E [MT83, Section 3.5.2][10], and on the other hand, when composed with the natural map $\widehat{\mathrm{Pic}}_{\mathbb{Q}}(\mathcal{O}_F) \to \bigoplus_{v\nmid\infty} \mathbb{Q}/\mathbb{Z}$ agrees with the composite

$$A(F) \times B(F) \to \bigoplus_{v\nmid\infty} \Phi_{A,v}(\kappa_v) \times \Phi_{B,v}(\kappa_v) \to \bigoplus_{v\nmid\infty} \mathbb{Q}/\mathbb{Z}$$

where the right-hand arrow is the sum of the local Grothendieck monodromy pairings. Here, $\Phi_{A,v}/\kappa_v$ denotes the group-scheme of components of the fibre of the Néron model of A over the residue field κ_v at v, the implicit maps $A(F) \to \Phi_{A,v}(\kappa_v)$ send an F-point of A to the component containing its reduction, $B(F)^0$ denotes the intersection of the kernels of all the maps $B(F) \to \Phi_{B,v}(\kappa_v)$ for $v \nmid \infty$, and the valuations at finite places v are normalised to have image $\mathbb{Z}$. In fact, the quadratic refinement of Grothendieck's monodromy pairing which we produce has already been considered by Breen [Bre83, Section 3], using the theory of cubical structures on torsors.

Just as for Néron functions [Lan83, Theorem 11.1.1], the Néron log-metrics are uniquely characterised by a list of properties (see lemmas 3.0.2 and 9.3.1), which is how we will prove that the functions we construct in our main theorems are indeed Néron log-metrics.

1.1.4.2 The ℓ-adic and archimedean main theorems

Now that we have made precise the form of local heights best adapted to our setup, we are now able to state our first main theorem, which is a more or less direct generalisation of theorem 1.1.1.

Theorem 1.1.6. *Let A/K be an abelian variety over a p-adic field K, L/A a line bundle, and $\widetilde{0} \in L^\times(K)$ a basepoint in the complement $L^\times = L \setminus 0$ of the zero section lying over $0 \in A(K)$. Let $U/\mathbb{Q}_\ell$ be the $\mathbb{Q}_\ell$-unipotent étale fundamental group of $L^\times$ based at $\widetilde{0}$ (where $\ell \neq p$). Then the natural map $\mathbb{Q}_\ell(1) \hookrightarrow U$ induced by the inclusion of the fibre over $0 \in A(K)$ induces a bijection*

$$\mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)),$$

and the composite map

$$L^\times(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) \xleftarrow{\sim} \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

[10]The presentation in [MT83] assumes that A and B have Tamagawa number 1 at all finite places—the setup in [MT87, Section 2] covers how to define this pairing in the general case. The assumption in the latter paper that A and B are dual and that E is the Poincaré biextension is unnecessary.

is the Néron log-metric on $(L, \widetilde{0})$ (and in particular is $\mathbb{Q}$-valued).

Here the isomorphism $\mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$ is the one arising from Kummer theory, normalised so that the composite

$$K^\times \to \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

is the ($\mathbb{Q}$-valued) valuation on K.

The archimedean analogue of theorem 1.1.6 in terms of the $\mathbb{R}$-pro-unipotent Betti fundamental group (with its mixed Hodge structure) then has a very similar statement. For the sake of simplicity, we will focus only on the case $K = \mathbb{C}$.

Theorem 1.1.7. *Let $A/\mathbb{C}$ be an abelian variety, L/A a line bundle, and $\widetilde{0} \in L^\times(\mathbb{C})$ a basepoint in the complement $L^\times = L \setminus 0$ of the zero section lying over $0 \in A(\mathbb{C})$. Let $U/\mathbb{R}$ be the $\mathbb{R}$-unipotent Betti fundamental group of $L^\times$ based at $\widetilde{0}$. Then the natural map $\mathbb{R}(1) \hookrightarrow U$ induced by the inclusion of the fibre over $0 \in A(\mathbb{C})$ induces a bijection*

$$\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R})),$$

and the composite map

$$L^\times(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R})) \xleftarrow{\sim} \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}$$

is the Néron log-metric on $(L, \widetilde{0})$.

Here $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, W(\mathbb{R}))$ denotes the set of isomorphism classes of torsors under W with compatible mixed Hodge structure. The isomorphism $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}$ here is normalised so that the composite

$$\mathbb{C}^\times \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}$$

is the valuation $v = -\log|\cdot|$ on $\mathbb{C}$.

1.2 Local Bloch–Kato Selmer sets and homotopical algebra

1.2.1 Definitions

Our p-adic analogue of theorem 1.1.6 will describe p-adic local heights as functions on spaces of torsors under $\mathbb{Q}_p$-pro-unipotent fundamental groups. However, we cannot hope that anything as naive as theorem 1.1.6 will hold verbatim when $\ell = p$, not least because the Galois cohomology $\mathrm{H}^1(G_K, \mathbb{Q}_p(1))$ is much larger than 1-dimensional.

Indeed, in the $\ell = p$ setting, the correct way to recover the valuation on $K^\times$ from Galois cohomology is to consider the quotient

$$\mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) := \mathrm{H}^1_g(G_K, \mathbb{Q}_p(1))/\mathrm{H}^1_e(G_K, \mathbb{Q}_p(1))$$

of local Bloch–Kato Selmer groups [BK90, Equation 3.7.2]. This is one-dimensional, and the composite

$$K^\times \to \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$$

with the Kummer map is, for a suitable normalisation of the final isomorphism, the valuation on K. This motivates what will be our adaptation of theorem 1.1.6 to the p-adic setting: we will define a certain "quotient" $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$ of local Bloch–Kato Selmer sets for a general (usually de Rham) representation of G_K on a finitely generated pro-unipotent group $U/\mathbb{Q}_p$, and when $U/\mathbb{Q}_p$ is the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of $L^\times$ as earlier, we will realise our local heights as functions on $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$ via the mod H^1_e non-abelian Kummer map

$$L^\times(K) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)).$$

Indeed, $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$ will prove in this case to be 1-dimensional, analogously to in theorems 1.1.6 and 1.1.7.

Let us now give the definition of the local Bloch–Kato Selmer sets by mimicking the definitions given in [BK90, Equation 3.7.2] for linear representations. Just like for non-abelian Galois cohomology, the local Bloch–Kato Selmer sets will not in general carry a natural group structure, and will be merely pointed sets. Fortunately, despite this limitation, the relative quotients of these sets also admit a natural definition along the same lines as the sets themselves.

Definition 1.2.1 (Local Bloch–Kato Selmer sets)**.** Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group (see definition-lemma 4.0.1). We define the *local Bloch–Kato Selmer (pointed) sets* $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ to be the respective kernels of the natural maps

$$\begin{aligned}
&\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}^{\varphi=1})) \\
&\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}})) \\
&\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}}))
\end{aligned}$$

on continuous cohomology. Here $\mathsf{B}_{\mathrm{cris}}$ and B_{st} denote the usual period rings, and the continuous Galois action on, for example, $U(\mathsf{B}_{\mathrm{st}}) = \mathrm{Hom}(\mathrm{Spec}(\mathsf{B}_{\mathrm{st}}), U)$ is the "semilinear" one induced from the actions of G_K on B_{st} and on U.

More generally, we let $\sim_{\mathrm{H}^1_e}$, $\sim_{\mathrm{H}^1_f}$ and $\sim_{\mathrm{H}^1_g}$ be the equivalence relations on the cohomology set $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ whose equivalence classes are the fibres of the respective maps above (so that $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ is the $\sim_{\mathrm{H}^1_*}$-equivalence class of the distinguished point). We then define the *local Bloch–Kato Selmer quotients* to be the (pointed) quotient sets:

$$\mathrm{H}^1_{f/e}(G_K, U(\mathbb{Q}_p)) := \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))/\sim_{\mathrm{H}^1_e};$$
$$\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) := \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))/\sim_{\mathrm{H}^1_e};$$
$$\mathrm{H}^1_{g/f}(G_K, U(\mathbb{Q}_p)) := \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))/\sim_{\mathrm{H}^1_f}.$$

Remark 1.2.2. The original definition of H^1_g in [BK90, Equation 3.7.2] differs from ours in that the semistable period ring B_{st} is replaced by the de Rham one B_{dR}. However, we will see in lemma 4.4.5 that, at least when U is de Rham (see definition-lemma 4.2.2)—for instance the pro-unipotent fundamental group of any smooth K-variety—there is no difference between the two definitions, so we are free to adopt the one which is best suited to our techniques. When U is replaced by a p-adic representation V, this is proved in unpublished work by Hyodo [Hyo], and later in other places, e.g. [Ber02].

1.2.1.1 The p-adic main theorem

A major part of this dissertation will be concerned with developing tools for the study of non-abelian Bloch–Kato Selmer sets and their quotients, but before we outline how we intend to develop these tools, let us use the local Bloch–Kato Selmer quotients to give a precise statement of our p-adic analogue of theorem 1.1.6.

Theorem 1.2.3. *Let A/K be an abelian variety over a p-adic field K, L/A a line bundle, and $\widetilde{0} \in L^\times(K)$ a basepoint in the complement $L^\times = L \setminus 0$ of the zero section lying over $0 \in A(K)$. Let $U/\mathbb{Q}_p$ be the $\mathbb{Q}_p$-unipotent étale fundamental group of $L^\times$ based at $\widetilde{0}$. Then the natural map $\mathbb{Q}_p(1) \hookrightarrow U$ induced by the inclusion of the fibre over $0 \in A(K)$ induces a bijection*

$$\mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)),$$

and the (well-defined) composite map

$$L^\times(K) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xleftarrow{\sim} \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$$

is the Néron log-metric on $(L, \widetilde{0})$ (and in particular is $\mathbb{Q}$-valued).

Here the isomorphism $\mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$ *is normalised so that the composite*

$$K^\times \to \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$$

is the ($\mathbb{Q}$*-valued) valuation on* K*.*

Remark 1.2.4. Informally, theorem 1.2.3 shows that when $U/\mathbb{Q}_p$ is the unipotent fundamental group of $L^\times$ as above, then $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$ is codimension 1 in $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ and the gap between H^1_e and H^1_g describes the local component of the height function. It is natural to wonder where the intermediate set H^1_f sits between these two sets.

In fact, in our setup it will already be equal to H^1_e, as we shall see in corollary 6.3.3 (see remark 6.3.4). Correspondingly, according to Bloch and Kato's ℓ-adic analogies for local Bloch–Kato Selmer groups [BK90, Equation 3.7.1], one would expect for the $\mathbb{Q}_\ell$-unipotent fundamental group of $L^\times$ to satisfy $\mathrm{H}^1_{\mathrm{nr}}(G_K, U(\mathbb{Q}_\ell)) = 1$ where

$$\mathrm{H}^1_{\mathrm{nr}}(G_K, U(\mathbb{Q}_\ell)) := \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1(I_K, U(\mathbb{Q}_\ell))\right)$$

with $I_K \trianglelefteq G_K$ the inertia subgroup—this is also true, and proven in proposition 4.1.6 (see remark 4.1.7).

1.2.2 Cosimplicial models for local Bloch–Kato Selmer sets

Ultimately, our aim is to use the local Bloch–Kato Selmer sets and quotients not just to state the p-adic analogue of the ℓ-adic theorem 1.1.6, but also to mimic closely the proof we will give in chapter 3. Thus, it will be important for us to have access to tools for the study of the various local Bloch–Kato Selmer sets $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ for a general de Rham representation on a finitely generated pro-unipotent group $U/\mathbb{Q}_p$, such as for instance the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of any smooth connected K-variety at a K-rational basepoint (see proposition 4.0.3). In particular, we will want access to methods for producing long exact sequences relating local Bloch–Kato Selmer quotients in central extensions, analogously to the long exact sequence in cohomology.

These non-abelian Bloch–Kato Selmer sets $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ have already been studied in the work of Kim [Kim09], using their moduli interpretations as classifying spaces for torsors possessing trivialisations after base change to various period rings (see section 4.4). However, our approach will differ significantly from Kim's, in that we will in general forgo these explicit moduli descriptions, instead producing certain explicit cosimplicial models for the local Bloch–Kato Selmer sets, enabling

us to bring methods from homotopical algebra to bear. The advantages of this approach are twofold: firstly, our explicit models will be cosimplicial groups, and thus will retain some of the group structure which is lost in passage to non-abelian Galois cohomology; and secondly, the models will form a kind of derived refinement of the local Bloch–Kato Selmer sets, and in particular will contain information in other degrees which is not seen directly by the local Bloch–Kato Selmer set but is nonetheless of relevance, for example in the extra terms which appear in long exact sequences.

Explicit cochain models for local Bloch–Kato Selmer groups have already been produced by Nekovář in the abelian case [Nek93, Nek00], and our treatment closely parallels his in the non-abelian context. To briefly review Nekovář's methods, let us recall that there are certain cochain complexes $\mathsf{C}^\bullet_*(V)$ for $* \in \{e, f, g\}$ associated to any de Rham representation V of G_K, whose first cohomology $\mathrm{H}^1\left(\mathsf{C}^\bullet_*(V)\right)$ is canonically identified with the corresponding local Bloch–Kato Selmer group $\mathrm{H}^1_*(G_K, V)$ (see [Nek93, Section 1.19]). The complexes $\mathsf{C}^\bullet_*(V)$ are built out of various Dieudonné functors evaluated at V; for instance, the complex $\mathsf{C}^\bullet_e(V)$ is the two-term complex

$$\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V) \xrightarrow{-\iota} \mathsf{D}_{\mathrm{dR}}(V)/F^0,$$

where ι is the natural map. The reader will note that the assertion that $\mathrm{H}^1(\mathsf{C}^\bullet_e(V)) = \mathrm{H}^1_e(G_K, V)$ is equivalent to part of the Bloch–Kato exponential sequence [BK90, Corollary 3.8.4].

In the pro-unipotent setting we will generalise this picture by associating to each de Rham representation of G_K on a finitely generated pro-unipotent group $U/\mathbb{Q}_p$ a certain *cosimplicial group* $\mathsf{D}^\bullet_*(U)(\mathbb{Q}_p)$, whose *first cohomotopy set* $\pi^1\left(\mathsf{D}^\bullet_*(U)(\mathbb{Q}_p)\right)$ is canonically identified with the corresponding local Selmer set $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ (see chapter 5 for definitions). The cosimplicial groups $\mathsf{D}^\bullet_*(U)(\mathbb{Q}_p)$ are again built out of certain non-abelian Dieudonné functors evaluated on U (see lemma 4.2.1). More precisely, we will define various cosimplicial $\mathbb{Q}_p$-algebras $\mathsf{B}^\bullet_*$ with G_K-action, built out of the standard p-adic period rings, so that the desired cosimplicial group $\mathsf{D}^\bullet_*(U)(\mathbb{Q}_p)$ is then produced by taking points of U in $\mathsf{B}^\bullet_*$ and taking G_K-invariants. In fact, we will use exactly the same setup to analyse the local Bloch–Kato quotients $\mathrm{H}^1_{g/e}$ and $\mathrm{H}^1_{f/e}$, defining cosimplicial $\mathbb{Q}_p$-algebras $\mathsf{B}^\bullet_{g/e}$ and $\mathsf{B}^\bullet_{f/e}$, and obtaining the description

$$\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p)) \cong \pi^1\left(U(\mathsf{B}^\bullet_*)^{G_K}\right)$$

also in this case.

Just as in the abelian case, these descriptions can be viewed as various instances of exponential-type exact sequences—indeed, this is exactly how we will prove them.

For instance, in the particular case $* = e$, we will prove the description via the following non-abelian analogue of the classical Bloch–Kato exponential.

Lemma 1.2.5 (Non-abelian Bloch–Kato exponential)**.** *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then there is an exact sequence*

$$1 \to U(\mathbb{Q}_p)^{G_K} \to \mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) \times \mathsf{D}_{\mathrm{dR}}^{+}(U)(K) \overset{\curvearrowright}{\to} \mathsf{D}_{\mathrm{dR}}(U)(K) \to \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \to 1,$$

where $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}$, D_{dR} and $\mathsf{D}_{\mathrm{dR}}^{+}$ denote the Dieudonné functors associated to the period rings $\mathsf{B}_{\mathrm{cris}}^{\varphi=1}$, B_{dR} and $\mathsf{B}_{\mathrm{dR}}^{+}$ respectively (see lemma 4.2.1).

The exact sequence appearing in our non-abelian Bloch–Kato exponential contains both groups and pointed sets; since such mixed exact sequences will be ubiquitous throughout this dissertation, let us fix our chosen notation.

Definition 1.2.6. An *exact sequence (of groups and pointed sets)*

$$\cdots \to X^{r-1} \to X^r \to X^{r+1} \to \cdots \to X^j \overset{\mathsf{z}}{\to} X^{j+1} \to \cdots \to X^k \overset{\curvearrowright}{\to} X^{k+1} \to \cdots$$

consists of:

- two distinguished indices $j < k$;
- an exact sequence (finite or infinite) of groups $\cdots \to X^{k-1} \to X^k$;
- an exact sequence (finite or infinite) of pointed sets $X^{k+1} \to X^{k+2} \to \cdots$; and
- a right action of X^k on X^{k+1} (not necessarily preserving the basepoint);

such that:

- X^r is abelian for $r \leq j$;
- the image of $X^j \to X^{j+1}$ is a central subgroup;
- the image of $X^{k-1} \to X^k$ is exactly the stabiliser of the basepoint of X^{k+1} under the action of X^k; and
- the orbits of the action of X^k on X^{k+1} are exactly the fibres of the map $X^{k+1} \to X^{k+2}$.

Note that in this setup, the action of X^k on the basepoint of X^{k+1} provides a map $X^k \to X^{k+1}$ making the whole sequence exact.

Remark 1.2.7. The prototypical example of such a sequence is the homotopy exact sequence associated to a fibration [BK72, Section IX.4.1], where there are exactly three non-abelian group terms ($k = j + 3$) and three pointed set terms. In fact, essentially all the exact sequences we will study in this thesis will have this same feature.

Remark 1.2.8. In our non-abelian Bloch–Kato exact sequence, the action of the group $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p)\times\mathsf{D}^{+}_{\mathrm{dR}}(U)(K)$ on the set $\mathsf{D}_{\mathrm{dR}}(U)(K)$ is given by $(x,y)\colon z\mapsto y^{-1}zx$, so that, concretely, the exponential exact sequence asserts that

$$U(\mathbb{Q}_p)^{G_K} = \mathsf{D}^{+}_{\mathrm{dR}}(U)(K)\cap\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p)$$

(inside $\mathsf{D}_{\mathrm{dR}}(U)(K)$), and that we have a canonical identification

$$\mathrm{H}^1_e(G_K,U(\mathbb{Q}_p))\cong\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p)\backslash\mathsf{D}_{\mathrm{dR}}(U)(K)/\mathsf{D}^{+}_{\mathrm{dR}}(U)(K)$$

of $\mathrm{H}^1_e(G_K,U(\mathbb{Q}_p))$ as a double-coset space.

Remark 1.2.9. An alternative description of a non-abelian Bloch–Kato exponential has been given by Sakugawa [Sak17, Theorem 1.1] following on from work of Kim [Kim09] under some auxiliary conditions including triviality of $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)$ (but without necessarily assuming U unipotent), obtaining in this case a Bloch–Kato exponential *isomorphism*

$$\mathrm{H}^1_f(G_K,U(\mathbb{Q}_p))\xrightarrow{\sim}\mathsf{D}_{\mathrm{dR}}(U)(K)/\mathsf{D}^{+}_{\mathrm{dR}}(U)(K).$$

Indeed, we will show in corollary 6.3.3 that, under this additional hypothesis, one has $\mathrm{H}^1_e(G_K,U(\mathbb{Q}_p))=\mathrm{H}^1_f(G_K,U(\mathbb{Q}_p))$, so one recovers Sakugawa's isomorphism from ours.

Remark 1.2.10. When $U=V$ is abelian, the central map

$$\mathsf{D}^{+}_{\mathrm{dR}}(V)\oplus\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V)\to\mathsf{D}_{\mathrm{dR}}(V)$$

from our exponential exact sequence (lemma 1.2.5) is the negative of the usual one. This is due to an unfortunate clash between the signs in the exact sequences in [Nek93, Section 1.19] with those of [BK90, Proposition 1.17], and between the order conventions for path-composition in the papers of Kim and of Hain. In this thesis we will follow the conventions of Kim and Deligne in composing paths in the functional order—fg is path g followed by path f—and following the sign conventions of Nekovář, pointing out discrepancies with the conventions of our sources as they arise.

1.2.2.1 A log-crystalline version

In accordance with Deligne's philosophy of the motivic fundamental group, one might wonder whether, in addition to our ℓ-adic, p-adic and Hodge-theoretic descriptions, one can give a description of the Néron log-metric on a pointed line bundle $(L, \widetilde{0})$ on an abelian variety A over a p-adic local field K in terms of the pro-unipotent log-crystalline[11] fundamental group of the special fibre of a suitable integral model of $L^\times$ [Shi00b, Definition 4.1.5]. Though this is not the main thrust of this thesis, we will in fact see that theorem 1.2.3 admits a log-crystalline reinterpretation which is approximately of this form.

To briefly explain, let us fix $(L, \widetilde{0})$, A and K as in the preceding paragraph, and assume for simplicity that A has semistable reduction. If we write $U/\mathbb{Q}_p$ for the unipotent étale fundamental group of $L^\times$ based at $\widetilde{0} \in L^\times(K)$, then our assumptions ensure that U is semistable in the sense of definition-lemma 4.2.2 (for instance by using theorem 1.3.4). The finitely generated pro-unipotent group $\mathsf{D}_{\mathrm{st}}(U)/K_0$ (lemma 4.2.1) then carries an ind-weakly admissible filtered (φ, N)-module structure on its affine ring $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U)) = \mathsf{D}_{\mathrm{st}}(\mathcal{O}(U))$ (section 4.2.2), and the local Bloch–Kato Selmer set $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ classifies torsors under $\mathsf{D}_{\mathrm{st}}(U)$ with compatible ind-weakly admissible filtered (φ, N)-module structures on their affine rings (lemma 4.4.6). We will see in lemma 4.4.7 that passing from H^1_g to $\mathrm{H}^1_{g/e}$ corresponds to forgetting the Hodge filtration on these torsors, so that $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$ has a natural log-crystalline interpretation as the moduli set $\mathrm{H}^1(\mathbf{Mod}_K^{\varphi,N}, \mathsf{D}_{\mathrm{st}}(U))$ of torsors under $\mathsf{D}_{\mathrm{st}}(U)$ with compatible ind-(φ, N)-module structures on their affine rings. This allows us to reinterpret theorem 1.2.3 in terms of the non-abelian Kummer map

$$L^\times(K) \to \mathrm{H}^1(\mathbf{Mod}_K^{\varphi,N}, \mathsf{D}_{\mathrm{st}}(U))$$

assigning to some $\widetilde{z} \in L^\times(K)$ the class of the torsor $\mathsf{D}_{\mathrm{st}}(P_{\widetilde{0},\widetilde{z}}) := \mathrm{Spec}(\mathsf{D}_{\mathrm{st}}(\mathcal{O}(P_{\widetilde{0},\widetilde{z}})))$, where $P_{\widetilde{0},\widetilde{z}}$ is the $\mathbb{Q}_p$-unipotent étale torsor of paths from $\widetilde{0}$ to $\widetilde{z}$ in $L^\times$. That this map is well-defined is proven in theorem 1.3.4.

Of course, this is only a partially log-crystalline interpretation of our main theorem; to complete the picture we would like to identify $\mathsf{D}_{\mathrm{st}}(U)$ as the pro-unipotent log-crystalline fundamental group of the special fibre of a suitable integral model[12]

[11] In [Shi00b], these fundamental groups are referred to as *crystalline* instead of *log-crystalline*—we prefer the latter terminology to emphasise that this is a theory which uses semistable models instead of smooth ones.

[12] That is, a proper log-smooth log-scheme $\mathcal{L}$ over $\mathcal{O}_K$ whose generic fibre is the total space of the projective closure $\mathbb{P}(L \oplus \mathcal{O}_A)$ of L with the log-structure coming from the union of the 0- and ∞-sections, such that the log-structure on $\mathrm{Spec}(\mathcal{O}_K)$ pulled back from $\mathcal{L}$ along the closure of $\widetilde{0} \in \overline{\mathcal{L}}(K)$ is the usual one arising from the special point.

of $L^\times$, and more generally also $\mathsf{D}_{\mathrm{st}}(P_{\widetilde{0},\widetilde{z}})$ as the log-crystalline torsor of paths between the reductions of $\widetilde{0}$ and $\widetilde{z}$. However, it seems that such an étale-log-crystalline comparison theorem for the fundamental group is still unproven or unpublished in full generality at the time of writing[13], so we will content ourselves with this partial reinterpretation.

1.3 Local constancy and analyticity of Kummer maps

It will transpire that the major part of the proof of the ℓ-adic theorem 1.1.6 lies in showing that the non-abelian Kummer map

$$L^\times(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell))$$

is locally constant for the p-adic topology on $L^\times(K)$. One might in principle be able to prove this specifically for $L^\times$ by choosing an appropriate integral model, but instead we will consider the case of a general smooth variety in place of $L^\times$, obtaining the following result, which for ease of reference also incorporates some basic properties of the fundamental group.

Theorem 1.3.1. *Let Y/K be a smooth connected variety, $y \in Y(K)$ a basepoint, and $U/\mathbb{Q}_\ell$ the pro-unipotent étale fundamental group of Y based at y, where $\ell \neq p$. Then:*

1. *U is a (potentially semistable) representation of G_K on a finitely generated $\mathbb{Q}_\ell$-pro-unipotent group, and is semistable as soon as $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Q}_\ell)$ is; and*

2. *the non-abelian Kummer map*

$$Y(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell))$$

is locally constant (for the p-adic topology).

Remark 1.3.2. When Y is a complete curve, local constancy is proven in [KT08] by relating it to the Jacobian. A similar argument using the Albanese variety in place of the Jacobian should also work for a general proper Y.

[13] [Shi00a, Theorem 3.3] contains a statement of such a comparison isomorphism in the good reduction case, and [Shi00a, Remark 3.4.1] claims an extension to the stable reduction case, but the details of both proofs are deferred to a paper which remains unpublished. The good reduction case has been worked out in [Ols11].

In proving the archimedean and p-adic analogues of theorem 1.1.6 (theorems 1.1.7 and 1.2.3), we will need corresponding results for the Hodge-theoretic and p-adic Kummer maps.

Theorem 1.3.3. *Let $Y/\mathbb{C}$ be a smooth connected variety, $y \in Y(\mathbb{C})$ a basepoint, and $U/\mathbb{R}$ the pro-unipotent Betti fundamental group of Y based at y. Then:*

1. *$U/\mathbb{R}$ carries a mixed Hodge structure with only negative weights;*
2. *$\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ is canonically a pro-C^∞ real manifold; and*
3. *the non-abelian Kummer map*

$$Y(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$$

is pro-C^∞ (and in particular continuous).

Theorem 1.3.4. *Let Y/K be a smooth connected variety, $y \in Y(K)$ a basepoint, and $U/\mathbb{Q}_p$ the pro-unipotent étale fundamental group of Y based at y. Then:*

1. *U is a de Rham (=potentially semistable) representation of G_K on a finitely generated $\mathbb{Q}_p$-pro-unipotent group, and is semistable as soon as $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Q}_p)$ is;*
2. *the image of the non-abelian Kummer map*

$$Y(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_p))$$

is contained in $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$; and

3. *the $\mathrm{H}^1_{g/e}$-valued non-abelian Kummer map*

$$Y(K) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$$

is locally constant (for the p-adic topology).

Remark 1.3.5. Despite the similar statements of the above three theorems, their proofs will be very different. Of particular note is the proof of theorem 1.3.3, which will go by relating the Hodge-theoretic Kummer map to the higher Albanese maps of Hain and Zucker [HZ87]. The proof of theorem 1.3.1 will be in section 3.1 and corollary 4.1.4; the proof of theorem 1.3.4 will be in chapter 7 and theorem 4.2.8; and the proof of theorem 1.3.3 will be in sections 9.1 and 9.2.

1.4 Overview of sections and standing assumptions

The structure of this thesis will be as follows. We will begin in chapter 2 by recalling some basic facts about pro-unipotent groups and various filtrations. These preliminaries being dispensed with, we then prove in chapter 3 the simplest instance of our main theme, namely the non-archimedean case with $\ell \neq p$. Although the proof is by-and-large straightforward, the proof strategy will reappear in the proofs of its sister theorems in either the $\ell = p$ or the archimedean cases. Thus, many of the steps in this proof will inform and motivate later lemmas in the dissertation.

The next five chapters are devoted to the proof of the main theorem in the case $\ell = p$, which is finally assembled in the very short chapter 8, following the pattern of the proof in chapter 3. The first four of these chapters, then, are devoted to developing a theory for the study of non-abelian local Bloch–Kato Selmer sets and Kummer maps to enable this proof. In chapters 4 and 5, we again review some well-trodden material, relating respectively to the basic theory of Galois representations on pro-unipotent groups, and the basic language of cosimplicial groups and their cohomotopy. These two threads are united in chapter 6, where we produce explicit cosimplicial group models for local non-abelian Bloch–Kato Selmer sets and their quotients. It is these explicit models which enable us to prove, for instance, various long exact sequences relating these sets. Chapter 7 takes somewhat a different tack, and is dedicated to the study of the pro-unipotent étale Kummer map in the case $\ell = p$. Although few of the results in this chapter are surprising, they are nonetheless necessary in that they constitute our only method for controlling the topological properties of such Kummer maps.

Chapter 9 is devoted to the proof of the final instance of our main theme, namely the archimedean case. This follows exactly the same blueprint as the other two proofs, developing a Hodge-theoretic analogue of the representation-theoretic language of the earlier parts of the thesis. The relevant Hodge theory turns out to have already been studied in the higher Albanese manifolds of Hain and Zucker [HZ87], and we make the link to their theory in proposition 9.2.4, showing that the pro-unipotent Betti–de Rham Kummer map is none other than the inverse limit of Hain and Zucker's higher Albanese maps.

The final chapter takes somewhat a different approach to the preceding chapters, and attempts to illustrate what we should expect a global version of the heretofore local results to look like. This is done in the context of ℓ-adic heights, rather than their $\mathbb{R}$-valued counterparts, and is essentially just a reworking of a particular case

of the existing theory of ℓ-adic heights due to Nekovář in a form that more closely resembles the usual Weil height machine for $\mathbb{R}$-valued heights. What is of interest in this chapter is not so much the results, which are hardly new, but in the global approach, obtaining functions directly on spaces of path-torsors for abelian varieties, rather than merely on a $\mathbb{G}_m$-torsor. The extra difficulties one encounters when trying to add up the local heights constructed in this thesis to produce a global anabelian theory of $\mathbb{R}$-valued heights—as opposed to ℓ-adic heights—seem to necessitate working with a genuinely motivic incarnation of the fundamental group, and the question of how to actually realise this plan is undoubtedly the most interesting question left unanswered by this dissertation.

Following this final chapter is appendix A, in which we address the natural algebraic structure on local Bloch–Kato Selmer sets, which is for the most part ignored in the main body of this thesis, but which plays an important role in the Chabauty–Kim method. In particular, in this section we demonstrate how to upgrade most of the p-adic theory in the thesis to be compatible with this algebraic structure.

The final appendix B contains various technical statements on minimal pointed models of hyperbolic curves which we were unable to find in the established literature.

1.4.1 Standing assumptions

Since the majority of this dissertation will be devoted to the analysis over a non-archimedean base field, let us fix some notation for the rest of the thesis.

Notation 1.4.1 (Standing assumptions)**.** Fix a prime p, a finite extension $K/\mathbb{Q}_p$, and an algebraic closure $\overline{K}/K$ determining an absolute Galois group $G_K = G_{\overline{K}|K}$. Let $\mathcal{O}_K$ denote the ring of integers, ϖ a uniformiser generating the maximal ideal $\mathfrak{m}_K = (\varpi)$, $k = \mathcal{O}_K/\varpi$ the residue field, and $v\colon K^\times \to \mathbb{Q}$ the valuation, normalised so that $v(p) = 1$. We also denote by $\mathbb{C}_K$ the completion of $\overline{K}$, by K_0 the maximal unramified subfield of K, and by $\mathsf{B}_{\mathrm{cris}}$, B_{st} and B_{dR} the period rings constructed by Fontaine [Fon94a].

A list of symbols and terms is provided at the end of this thesis for the reader's convenience.

Chapter 2

Preliminaries

Since pro-unipotent groups and their torsors will be appearing ubiquitously in this thesis (as certain completions of fundamental groups of varieties), let us begin by collecting some basic definitions and facts for the benefit of the reader. The majority of the content of this chapter is well-known and well-studied, but we will supply a few particular proofs that we were unable to find in the literature.

2.1 Pro-unipotent groups

Throughout this section, we fix a characteristic 0 *field* F.

To begin with, let us recall the definition of a pro-unipotent group.

Definition-Lemma 2.1.1 (Pro-unipotent groups)**.** We say that an affine algebraic group U/F is *unipotent* just when it satisfies any of the following equivalent properties:

- every finite dimensional representation of U is unipotent (i.e. an iterated extension of the trivial representation);
- every non-zero representation of U has a non-zero fixed vector;
- U embeds as a closed subgroup of the algebraic group of m-by-m upper triangular matrices with all diagonal elements equal to 1, for some m;
- (if $\mathrm{char}(F) = 0$) U is an iterated extension of the additive group $\mathbb{G}_a$;
- (if $\mathrm{char}(F) = 0$) U is an iterated *central* extension of the additive group $\mathbb{G}_a$.

More generally, we say that an affine group scheme U/F is *pro-unipotent* just when all of its finite dimensional quotients are unipotent; this is equivalent to requiring that every non-zero representation of U have a non-zero fixed vector.

Proof of equivalence. The first two points are equivalent since every representation of U is a union of finite dimensional representations. The equivalence of the first and third points is [Mil17, Theorem 14.5]. The fifth point trivially implies the fourth, and the fourth implies the third by [Mil17, Corollary 14.7]. Finally, the third point implies the fifth in characteristic 0 by [Mil17, Proposition 14.21 and Remark 14.24(a)]. □

Definition 2.1.2. A pro-unipotent group U/F is said to be *finitely generated* just when there is a finite subset $S \subseteq U(F)$ such that the only subgroup scheme of U containing S is U itself.

Lemma 2.1.3. *Subgroups, quotients and extensions of pro-unipotent groups are pro-unipotent. The category of pro-unipotent groups is closed under all (small) limits in the category of affine group schemes, and is equivalent to the category of pro-objects in the category of unipotent groups.*

Proof. For subgroups, quotients and extensions of unipotent groups, this is [Mil17, Corollary 14.7]. The general case is straightforward to prove from this. □

One important property of pro-unipotent groups which renders them amenable to study is that they are very closely tied to certain linear-algebraic objects, namely Lie algebras and Hopf algebras. The equivalence between these three points of view will be a recurring theme in the early part of this thesis, in which we will be defining extra structures on pro-unipotent groups equivalently as structures on their Hopf algebras or Lie algebras. The main theorem describing these connections is below (the terms involved will be defined later in this section).

Theorem 2.1.4. *There are canonical equivalences between the categories of:*

- *pro-unipotent groups U/F;*
- *pro-nilpotent Lie algebras $\mathfrak{u}$ over F; and*
- *J-adically cocomplete commutative Hopf algebras over F;*

where the equivalence between the first two categories is covariant, and the equivalence between either of these categories and the third is contravariant. More precisely, under these equivalences:

- *the pro-nilpotent Lie algebra (resp. J-adically cocomplete commutative Hopf algebra) corresponding to a pro-unipotent group U is its Lie algebra* $\mathrm{Lie}(U)$ *(resp. its affine ring $\mathcal{O}(U)$);*

- *the pro-unipotent group (resp. pro-nilpotent Lie algebra) corresponding to a J-adically cocomplete commutative Hopf algebra H is* $\mathrm{Spec}(H)$ *(resp. the space* $\mathrm{Der}_{F,\varepsilon}(H)$ *of F-derivations of H), or equivalently the affine group scheme representing the functor* $\Lambda \mapsto H^{*,\mathrm{gplike}}_{\Lambda}$ *of group-like elements of the dual Hopf algebra H^* (resp. the space* $H^{*,\mathrm{prim}}$ *of primitive elements of H^*);*

- *the pro-unipotent group (resp. J-adically cocomplete commutative Hopf algebra) corresponding to a pro-nilpotent Lie algebra $\mathfrak{u}$ is the (potentially infinite dimensional) affine space corresponding to $\mathfrak{u}$, endowed with the group structure induced by the Baker–Campbell–Hausdorff series (resp. the decompleted dual of the completed universal enveloping algebra of $\mathfrak{u}$).*

The correspondence between pro-unipotent groups and pro-nilpotent Lie algebras turns out to be particularly useful, and will appear many times in this thesis.

Definition 2.1.5 (Exponential and logarithm maps)**.** Let U/F be a pro-unipotent group. Theorem 2.1.4 asserts in particular that there is a natural isomorphism $\log\colon U \xrightarrow{\sim} \mathrm{Lie}(U)$ of F-schemes[1], which we call the *logarithm map*, and we call its inverse $\exp\colon \mathrm{Lie}(U) \xrightarrow{\sim} U$ the *exponential map*.

Let us collect here a number of useful consequences of the logarithm and exponential maps.

Corollary 2.1.6. *Let U/F be a pro-unipotent group. Then:*

- *as an F-scheme, U is isomorphic to an affine space over F (i.e. $U \simeq \mathrm{Spec}(F[\mathbf{X}])$ where $F[\mathbf{X}]$ is a polynomial algebra on a potentially infinite set of generators);*

- *$U(F)$ is Zariski-dense in U;*

- *$U(F)$, and more generally $U(\Lambda)$ for any F-algebra Λ, is uniquely divisible;*

- *any surjection (resp. injection) of pro-unipotent groups over F has a section (resp. retraction) as F-schemes; and*

- *a sequence of pro-unipotent groups is exact iff it is exact on F-points, iff it is exact on Λ-points for all F-algebras Λ, iff the sequence of Lie algebras is exact.*

[1] In this, $\mathrm{Lie}(U)$ is viewed as an F-scheme as follows. If U is unipotent, so $\mathrm{Lie}(U)$ is finite dimensional, then we view it as the associated finite dimensional affine space $\mathrm{Spec}(\mathrm{Sym}^\bullet(\mathrm{Lie}(U)^*))$, and in general if $U = \varprojlim U_i$ is an inverse limit of unipotent groups, then we view $\mathrm{Lie}(U) = \varprojlim \mathrm{Lie}(U_i)$ as the inverse limit of these finite dimensional affine spaces.

2.1.0.1 Filtrations

In order to work effectively with *a priori* infinite dimensional objects such as Hopf algebras, pro-unipotent groups or pro-nilpotent Lie algebras, it will be useful for us to put certain natural filtrations on these objects—the *J-filtration* on Hopf algebras will appear ubiquitously throughout this thesis. In the case of pro-unipotent groups or pro-nilpotent Lie algebras, these filtrations (the *descending central series* filtrations) are well-known.

Definition 2.1.7. A unipotent group U/F is called *n-step unipotent* just when it is an iterated central extension of vector groups of length at most n, i.e. U has a central series $U = U^0 \trianglerighteq U^1 \trianglerighteq \cdots \trianglerighteq U^n = 1$ of length $\leq n$. A pro-unipotent group U/F is called *n-step pro-unipotent* just when all of its finite dimensional quotients are n-step unipotent.

A pro-unipotent group U/F has a *maximal n-step pro-unipotent quotient* $U \twoheadrightarrow U_n$, and the filtration $I^\bullet U$ of U determined by the kernels of these maps is called the *descending central series* of U. (We normalise the descending central series $I^\bullet U$ such that $I^1 U = U$ and $I^{n+1}U = \ker(U \twoheadrightarrow U_n)$.)

Definition 2.1.8. A pro-nilpotent Lie algebra $\mathfrak{u}$ over F (see definition 2.1.17) is called *n-step pro-nilpotent* just when all of its finite dimensional quotients are iterated central extensions of abelian Lie algebras of length $\leq n$. The pro-nilpotent Lie algebra $\mathfrak{u}$ has a *maximal n-step pro-nilpotent quotient* $\mathfrak{u} \twoheadrightarrow \mathfrak{u}_n$, and the filtration $I^\bullet \mathfrak{u}$ of $\mathfrak{u}$ determined by the kernels of these maps is called the *descending central series* of $\mathfrak{u}$. (Again, we normalise this series such that the kernel of $\mathfrak{u} \twoheadrightarrow \mathfrak{u}_n$ is $I^{n+1}\mathfrak{u}$.)

The correponding filtration on a Hopf algebra H can be defined in several equivalent ways, see for example definition 2.1.12 and lemma 2.1.26 below. Suffice it to say that this takes the form of an increasing filtration $J_\bullet H$ by subcoalgebras called the *J-filtration* which is exhaustive when H is J-adically cocomplete (by definition).

These three filtrations are closely related via the correspondences in theorem 2.1.4, for instance the descending central series on pro-unipotent groups and pro-nilpotent Lie algebras directly correspond to one another. The correspondence between these filtrations and the J-filtration is slightly more complicated, as follows.

Lemma 2.1.9. *Let U/F be a pro-unipotent group, and for $n \geq 0$ let U_n denote the maximal n-step pro-unipotent quotient of U. Then $\mathcal{O}(U_n) \leq \mathcal{O}(U)$ is the subalgebra generated by $J_n\mathcal{O}(U)$, and the kernel of the surjection $\mathrm{Sym}^\bullet(J_n\mathcal{O}(U)) \twoheadrightarrow \mathcal{O}(U_n)$ is the ideal generated by $1-[1]$ and all elements of the form $[x][y]-[xy]$ for $x \in J_i\mathcal{O}(U)$*

and $y \in J_j\mathcal{O}(U)$ for $i + j \leq n$. (Here $[-]$ denotes the canonical inclusion $J_n\mathcal{O}(U) \hookrightarrow \mathrm{Sym}^\bullet(J_n\mathcal{O}(U))$.)

Proof. It will follow from lemma 2.1.33 and the correspondences in theorem 2.1.4 that there is a canonical isomorphism

$$\mathcal{O}(U) \xrightarrow{\sim} \mathrm{Sym}^\bullet(\mathrm{Lie}(U)^\circ),$$

where $\mathrm{Lie}(U)^\circ$ denotes the decompleted dual of the Lie algebra $\mathrm{Lie}(U)$ (see definition 2.1.22; this is the usual dual $\mathrm{Lie}(U)^*$ when U is finite dimensional). Moreover, under this isomorphism, the J-filtration on $\mathcal{O}(U)$ corresponds to the filtration on $\mathrm{Sym}^\bullet(\mathrm{Lie}(U)^\circ)$ induced by the dual of the descending central series filtration on $\mathrm{Lie}(U)$; in particular $\mathrm{Lie}(U_n)$ and $J_n\mathcal{O}(U)$ generate the same subalgebra, namely $\mathcal{O}(U_n)$.

For the second part, we firstly note that the kernel of $\mathrm{Sym}^\bullet(J_n\mathcal{O}(U)) \twoheadrightarrow \mathcal{O}(U_n)$ certainly contains the claimed ideal I. For the converse inclusion, we choose a basis (v_i) of $\mathrm{Lie}(U)^\circ$ splitting the dual of the descending central series filtration—this induces a grading on $\mathrm{Lie}(U)^\circ$, and we write $|v_i|$ for the degree of the basis element v_i. Then a basis of $J_n\mathcal{O}(U)$ is given by the finite products $\prod_j v_j$ where $\sum_j |v_j| \leq n$, and hence a basis of $\mathrm{Sym}^\bullet(J_n\mathcal{O}(U))$ is given by the elements $\prod_k \left[\prod_j v_{k,j}\right]$ where $\sum_j |v_{k,j}| \leq n$ for all k. In particular, modulo the ideal I, $\mathrm{Sym}^\bullet(J_n\mathcal{O}(U))$ is spanned by elements of the form $\prod_k [v_k]$, where each $|v_k| \leq n$. The images of these elements under the map $\mathrm{Sym}^\bullet(J_n\mathcal{O}(U)) \twoheadrightarrow \mathcal{O}(U_n) \leq \mathrm{Sym}^\bullet(\mathrm{Lie}(U)^\circ)$ are then $\prod_k v_k$, which are F-linearly independent. Hence the induced map $\mathrm{Sym}^\bullet(J_n\mathcal{O}(U))/I \twoheadrightarrow \mathcal{O}(U_n)$ is injective and hence the original map $\mathrm{Sym}^\bullet(J_n\mathcal{O}(U)) \twoheadrightarrow \mathcal{O}(U_n)$ had kernel exactly I, as desired. □

One final important property of these filtrations is that, under a certain natural assumption on the pro-unipotent group U, all of the above filtrations are levelwise finite in an appropriate sense.

Definition-Lemma 2.1.10. Let U/F be a pro-unipotent group. We say that U is *finitely generated* just when any of the following equivalent conditions hold:

- there is a finite subset $S \subseteq U(F)$ such that the only subgroup scheme of U containing S is U itself;
- for each n (or just $n = 1$), the maximal n-step pro-unipotent quotient U_n of U is finite dimensional;

- for each n (or just $n = 1$), the maximal n-step pro-nilpotent quotient of $\mathrm{Lie}(U)$ is finite dimensional; or
- for each n (or just $n = 1$), $J_n\mathcal{O}(U)$ is finite dimensional.

Proof of equivalence. The equivalence of the first three points is standard. For the fourth, either use the description of the J-filtration in lemma 2.1.26, or refer to [LM85, Proposition 1.5]. □

Corollary 2.1.11. *The class of finitely generated pro-unipotent groups is preserved under extension of the base field.*

For the remainder of this section, we will discuss in more detail the correspondences in theorem 2.1.4, in particular giving definitions of all the terms involved—the reader who is familiar with the theory of pro-unipotent groups in characteristic zero should feel free to skip the rest of this section. Our discussion draws heavily on the exposition in [Mil17].

2.1.1 Hopf algebras

We will begin by describing the correspondence between pro-unipotent groups and J-adically cocomplete commutative Hopf algebras in theorem 2.1.4. Recall that a *Hopf algebra* H over F is an F-vector space endowed with a *multiplication map* $\mu\colon H \otimes H \to H$, a *unit map* $\eta\colon F \to H$, a *comultiplication map* $\Delta\colon H \to H \otimes H$, a *counit map* $\varepsilon\colon H \to F$ and an antipode map $S\colon H \to H$ such that (μ, η) and (Δ, ε) make H into an F-algebra and an F-coalgebra in a compatible manner, and such that S obeys a certain antipode identity [Mil17, Definition 3.3]. It is well-known that there is a contravariant equivalence between the category of commutative Hopf algebras over F and the category of affine group schemes over F (e.g. [Mil17, Proposition 3.6][2]), so it just remains for us to classify which Hopf algebras are the affine rings of pro-unipotent groups. One answer to this question is given in [Mil17, Theorem 14.5], which describes those commutative Hopf algebras which are affine rings of unipotent groups as those admitting a certain "mysterious" filtration—we will exhibit a slight variant of this definition which should go some way to clarifying the role of this filtration in the theory.

[2] [Mil17, Proposition 3.6] states this with finite type assumptions on both sides, but the general case follows exactly analogously.

Definition 2.1.12 (J-filtration)**.** Let H be a Hopf algebra over F and V a (right) H-comodule with coaction $\rho\colon V \to V \otimes H$ [Mil17, Chapter 4.a]. We define an increasing *fixpoint filtration* or *J-filtration*

$$J_0V \leq J_1V \leq \dots$$

by H-sub-comodules as follows. We set

$$J_0V := \ker\left(\rho - (1_V \otimes \eta)\colon V \to V \otimes H\right)$$

to be the the largest F-subspace of V on which the coaction is trivial—this is clearly an H-sub-comodule. For $n > 0$ we define $J_nV \geq J_{n-1}V$ by $J_nV/J_{n-1}V := J_0(V/J_{n-1}V)$, or equivalently the preimage of $J_{n-1}V \otimes \mathcal{O}(U)$ under the map

$$\rho - (1_V \otimes \eta)\colon V \to V \otimes \mathcal{O}(U).$$

Example 2.1.13. If U/F is an affine group scheme, then $\mathcal{O}(U)$-comodules are the same thing as (right) representations of U [Mil17, Chapter 4.a], and for a representation V, the subrepresentation $J_0V = V^U$ is the invariant subspace of V under the U-action. In particular, U is pro-unipotent if and only if the J-filtration is exhaustive for any representation V.

Of particular interest to us is the J-filtration on the Hopf algebra H itself (viewed as a comodule via the right comultiplication action). It turns out that this filtration enjoys a number of useful properties.

Lemma 2.1.14. *Let H be a Hopf algebra over F. Then $J_\bullet H$ is an increasing filtration by sub-coalgebras with $J_0H = F$, which is invariant under Hopf algebra automorphisms. Moreover, if we endow $H \otimes H$ with the induced tensor product filtration and F with the trivial filtration, then the Hopf algebra structure morphisms for H respect the J-filtration.*

Proof. This is simplest to prove using the description of the J-filtration on H as dual to the filtration on its dual H^* induced by powers of its augmentation ideal J (see lemma 2.1.26, and section 2.1.3 for definitions). In this setting, the result is equivalent to the statement that the filtration of H^* by powers of its augmentation ideal is invariant under (pro-finite dimensional) Hopf algebra automorphisms, that each H^*/J^{n+1} is a quotient algebra of H^* with $H^*/J = F$, and that the multiplication map $\mu\colon H^* \widehat{\otimes} H^* \to H^*$ preserves the respective filtrations on either side. All of these statements are clear. □

Definition 2.1.15 (J-adically cocomplete Hopf algebra)**.** We say that a Hopf algebra H over F is *J-adically cocomplete* just when the J-filtration $J_\bullet H$ is exhaustive (i.e. $H = \bigcup_n J_n H$). We will see that this condition is equivalent to the condition referred to as *coconnectedness* in [Mil17, Definition 14.4].

Theorem 2.1.16. *An affine group scheme U/F is pro-unipotent if and only if $\mathcal{O}(U)$ is J-adically cocomplete. Thus the assignment $U \mapsto \mathcal{O}(U)$ sets up a contravariant equivalence between the category of pro-unipotent groups over F and the category of J-adically cocomplete commutative Hopf algebras.*

Proof. One direction, namely that if U is pro-unipotent then $\mathcal{O}(U)$ is J-adically cocomplete, follows immediately from example 2.1.13. Conversely, if $\mathcal{O}(U)$ is J-adically cocomplete, then for every finite dimensional quotient U_i of U, it is easy to see that the J-filtration on $\mathcal{O}(U_i)$ is the restriction of the J-filtration on $\mathcal{O}(U)$, and hence $\mathcal{O}(U_i)$ is also J-adically cocomplete. By lemma 2.1.14, the J-filtration satisfies the conditions in [Mil17, Definition 14.4], and hence by [Mil17, Theorem 14.5] U_i is unipotent, for all i. Thus U is pro-unipotent as desired. □

2.1.2 Pro-nilpotent Lie algebras

Let us now discuss the second equivalence in theorem 2.1.4, namely the equivalence between pro-unipotent groups and pro-nilpotent Lie algebras. Recall that the *Lie algebra* of an affine group scheme U over F is defined as the kernel

$$\mathrm{Lie}(U) := \ker\left(U(F[\epsilon]) \to U(F)\right),$$

where $F[\epsilon]$ is the F-algebra formed by formally adjoining to F an element ϵ with $\epsilon^2 = 0$ [Mil17, Chapter 10.b]. This is an F-vector space, and carries a canonical Lie bracket making it into a Lie algebra over F [Mil17, Definition 10.1], which can be constructed either via the adjoint representation of U [Mil17, Chapter 10.d] or via an explicit description of $\mathrm{Lie}(U)$ in terms of derivations [Mil17, Chapter 10.e][3].

When U/F is an affine algebraic group (i.e. finite type over F), $\mathrm{Lie}(U)$ is finite dimensional, while in general $\mathrm{Lie}(U)$ is an inverse limit of finite dimensional Lie algebras (if we write $U = \varprojlim U_i$ as an inverse limit of affine algebraic groups, then $\mathrm{Lie}(U) = \varprojlim \mathrm{Lie}(U_i)$ is an inverse limit of finite dimensional Lie algebras). For technical reasons, it is necessary to remember this pro-structure on the Lie algebra of affine group schemes, for which we make the following definition.

[3] [Mil17] only considers the case that U is finite type over F, but the arguments apply equally for affine group schemes.

Definition 2.1.17 (Pro-finite dimensional Lie algebras)**.** A *pro-finite dimensional Lie algebra* over F is a pro-object in the category of finite dimensional Lie algebras. A pro-finite dimensional Lie algebra is called *pro-nilpotent* just when it is an inverse limit of finite dimensional nilpotent Lie algebras.

Thus the Lie algebra of an affine group scheme U/F is canonically a pro-finite dimensional Lie algebra. If U/F is unipotent (so an iterated central extension of $\mathbb{G}_a$), then $\mathrm{Lie}(U)$ is an iterated central extension of abelian Lie algebras, and hence is nilpotent. Thus, we have seen half of the following equivalence.

Theorem 2.1.18. *The assignment $U \mapsto \mathrm{Lie}(U)$ sets up an equivalence between the category of pro-unipotent groups U/F and pro-nilpotent Lie algebras over F.*

Proof. [Mil17, Theorem 14.37] gives the equivalence between unipotent groups and nilpotent Lie algebras. For the general case, we pass to pro-objects on both sides. □

The converse part of this equivalence is particularly useful, as it gives an explicit description of the structure of pro-unipotent groups. Central to this discussion is a certain power series $M(X, Y)$ in two non-commuting variables X and Y with coefficients in $\mathbb{Q}$, called the *Baker–Campbell–Hausdorff series* and defined by

$$M(X, Y) := \log\left(\exp(X) \cdot \exp(Y)\right),$$

where $\log(Z) = \log(1 - (1 - Z))$ and $\exp(Z)$ denote the usual Taylor series [Mil17, Section 14.35]. A key property of the Baker–Campbell–Hausdorff series is that it has a particularly constrained form, namely we may write $M(X, Y) = \sum_{n \geq 1} M_n(X, Y)$, where each $M_n(X, Y)$ is a $\mathbb{Q}$-linear combination of iterated commutators of length n (i.e. expressions of the form $[Z_1, [Z_2, \ldots, [Z_{n-1}, Z_n] \ldots]]$, where $Z_i \in \{X, Y\}$ and $[\cdot, \cdot]$ denotes the commutator bracket). In particular, each $M_n(X, Y)$ makes sense when X and Y are assigned values in any Lie algebra (over any base ring containing $\mathbb{Q}$).

Construction 2.1.19. Using the Baker–Campbell–Hausdorff series, one can make any pro-nilpotent Lie algebra $\mathfrak{u}$ over F into a pro-unipotent group as follows. We write $\mathfrak{u} = \varprojlim(\mathfrak{u}_i)$ as an inverse limit of finite dimensional nilpotent Lie algebras, and hence the Baker–Campbell–Hausdorff formula defines a function $\mu_i \colon \mathfrak{u}_i \times \mathfrak{u}_i \to \mathfrak{u}_i$ for each i (since sufficiently long iterated commutators vanish in $\mathfrak{u}_i$); taking the inverse limit of these functions induces a function $\mu \colon \mathfrak{u} \times \mathfrak{u} \to \mathfrak{u}$, which defines a group law on[4] $\mathfrak{u}$.

[4]Strictly speaking, $\mathfrak{u}$ is not a vector space, but an inverse system of vector spaces—we conflate $\mathfrak{u}$ with the inverse limit of this system of vector spaces.

More generally, if Λ is an F-algebra and $\mathfrak{u}_\Lambda := \varprojlim(\Lambda \otimes \mathfrak{u}_i)$ denotes the base change of $\mathfrak{u}$ to Λ, then the Baker–Campbell–Hausdorff formula defines a group operation $\mu\colon \mathfrak{u}_\Lambda \times \mathfrak{u}_\Lambda \to \mathfrak{u}_\Lambda$ by taking an inverse limit over the quotients $\Lambda \otimes \mathfrak{u}_i$ as before. The assignment $\Lambda \mapsto \mathfrak{u}_\Lambda$ defines a functor from F-algebras to groups, and is representable by a pro-unipotent group. This is the pro-unipotent group corresponding to $\mathfrak{u}$, and describes the quasi-inverse to the functor $U \mapsto \mathrm{Lie}(U)$ in theorem 2.1.18.

2.1.3 The dual Hopf algebra

To complete our description of the relationship between pro-unipotent groups, J-adically cocomplete Hopf algebras and pro-nilpotent Lie algebras, it remains to directly relate the Lie algebras to the Hopf algebras, as in the following part of theorem 2.1.4.

Theorem 2.1.20. *The functor $H \mapsto \mathrm{Der}_{F,\varepsilon}(H)$ sending a Hopf algebra H to the Lie algebra of F-derivations of H sets up a contravariant equivalence between the category of J-adically cocomplete commutative Hopf algebras over F and the category of pro-nilpotent Lie algebras over F.*

In proving this, rather than working with (commutative) Hopf algebras H, it is more natural to work with their *dual Hopf algebras* $H^* := \mathrm{Hom}_{F\text{-linear}}(H, F)$, with the Hopf algebra operations induced by those on H. There is a small subtlety here in that the multiplication on H need not induce a comultiplication on H^*, since $(H \otimes H)^* \neq H^* \otimes H^*$ in general, and hence H^* need not be a Hopf algebra in the usual sense. Thus, rather than realising the dual H^* as an object in the category of F-vector spaces, we will realise it as an object in a certain category of pro-finite dimensional F-vector spaces, as follows.

Definition 2.1.21 (Pro-finite dimensional vector spaces)**.** The category of *pro-finite dimensional F-vector spaces* is defined to be the category of pro-objects in the category of finite dimensional F-vector spaces. Explicitly, a *pro-finite dimensional vector space* is a formal symbol $\varprojlim(V_i)$, where $(V_i)_{i\in I}$ is an inverse system of finite dimensional vector spaces (i.e. a system of vector spaces indexed by a cofiltered category I [Lan63, Section IX.1]), and the set of morphisms between such symbols is defined to be

$$\mathrm{Hom}\left(\varprojlim(V_i), \varprojlim(W_j)\right) := \varprojlim_j \varinjlim_i \mathrm{Hom}(V_i, W_j).$$

Since the forgetful functor from F-vector spaces to sets preserves filtered colimits, these hom-sets are canonically F-vector spaces, making the category of pro-finite

dimensional vector spaces into an F-linear additive category (we will see shortly that it is even abelian).

Moreover, the category of pro-finite dimensional vector spaces is a symmetric monoidal category: if $V = \varprojlim V_i$ and $W = \varprojlim W_j$ are two pro-finite rank F-modules, we define their *completed tensor product* by

$$\left(\varprojlim V_i\right) \widehat{\otimes} \left(\varprojlim W_j\right) := \varprojlim (V_i \otimes W_j),$$

and this monoidal product has a canonical choice of unit $\mathbf{1}$, namely the one-dimensional F-vector space F, viewed as a limit over the one-point category.

In order to show that the category of pro-finite dimensional F-vector spaces is well-behaved, it will be useful to relate this category to the usual category of (possibly infinite dimensional) F-vector spaces, as follows.

Definition 2.1.22 (Decompleted dual)**.** Let $V = \varprojlim V_i$ be a pro-finite dimensional F-vector space. We define its *decompleted dual* V° to be the F-vector space given by

$$V^\circ := \mathrm{Hom}(V, \mathbf{1}) = \varinjlim (V_i^*),$$

where $V_i^* = \mathrm{Hom}(V_i, F)$ is the usual dual of a finite dimensional F-vector space.

Lemma 2.1.23. *The assignment $V \mapsto V^\circ$ sets up a contravariant symmetric monoidal equivalence between the category of pro-finite dimensional F-vector spaces and the category of all F-vector spaces. In particular, the category of pro-finite dimensional F-vector spaces is abelian.*

Proof. It is easy to see that if V and W are pro-finite dimensional F-vector spaces, then $(V \widehat{\otimes} W)^\circ \cong V^\circ \otimes W^\circ$ canonically and compatibly with the symmetric monoidal structure, so that $(-)^\circ$ is a symmetric monoidal functor.

Moreover, since the category of finite dimensional F-vector spaces is self-dual, it follows that the category of pro-finite dimensional F-vector spaces is contravariantly equivalent to the category of ind-objects in the category of finite dimensional F-vector spaces. It is easy to check that the functor from this latter category to the category of all F-vector spaces sending a formal filtered colimit $\varinjlim(V_i)$ to the value of this colimit taken inside the category of all F-vector spaces is fully faithful and essentially surjective, and hence an equivalence. □

The category of pro-finite dimensional vector spaces provides the setting in which one can formally discuss the dual of a Hopf algebra, which will be a pro-finite dimensional analogue of the usual notion. More precisely, the usual definition of Hopf algebras can be expressed in terms of a vector space H carrying certain structures (maps between various tensor powers of H) and satisfying certain compatibility relations (expressible in terms of commutativity of certain diagrams)—we produce a pro-finite dimensional analogue of this definition by replacing the vector space with a pro-finite dimensional vector space, and all the tensor products appearing in the definition with completed tensor products. Similarly, one can define pro-finite dimensional analogues of a large variety of algebraic objects (e.g. algebras, ideals, coalgebras, comodules, Lie algebras) by exactly the same procedure. In fact, one can verify that the definition of pro-finite dimensional Lie algebra we gave in definition 2.1.17 is equivalent to this more categorical definition.

To fix notation, let us state here the definition of a pro-finite dimensional Hopf algebra (we will leave the exact formalisation of the compatibility relations to the reader).

Definition 2.1.24 (Pro-finite dimensional Hopf algebra)**.** A *pro-finite dimensional Hopf algebra* over F is a pro-finite dimensional F-vector space H endowed with a *multiplication map* $\mu\colon H\widehat{\otimes}H \to H$, a *unit map* $\eta\colon \mathbf{1} \to H$, a *comultiplication map* $\Delta\colon H \to H\widehat{\otimes}H$, a *counit map* $\varepsilon\colon H \to \mathbf{1}$ and an antipode map $S\colon H \to H$, which are required to satisfy the analogous conditions to those usually required of a Hopf algebra (for example, associativity of H is the requirement that $\mu \circ (1_H\widehat{\otimes}\mu) = \mu \circ (\mu \otimes 1_H)$).

Remark 2.1.25. Putting the structure of a (cocommutative) Hopf algebra on a pro-finite rank F-module H is equivalent by lemma 2.1.23 to putting a (commutative) Hopf algebra structure on H°. Thus, the category of affine group schemes over F is *covariantly* equivalent to the category of cocommutative pro-finite dimensional Hopf algebras over F.

It turns out the the J-filtration on Hopf algebras defined in definition 2.1.12 is most naturally interpreted on their duals, as follows.

Lemma 2.1.26. *Let H be a pro-finite dimensional Hopf algebra over F. Write $J \trianglelefteq H$ for the augmentation ideal of H, i.e. the kernel of the counit map $\varepsilon\colon H \to \mathbf{1}$, and for $n \in \mathbb{N}$ let J^n denote the image[5] of the n-fold multiplication map $J^{\widehat{\otimes}n} \to H$. Then*

[5]By the *image* of a morphism of pro-finite dimensional vector spaces, we simply mean the kernel of its cokernel, as in any abelian category [Lan63, Section VIII.3].

the J-filtration on the Hopf algebra H° is dual to the filtration on H by powers of J. More precisely, $J_n H^\circ = (H/J^{n+1})^\circ$ as subspaces of H°.

Proof. We proceed inductively. If we set $J_{-1}H^\circ = 0$, then the base case $n = -1$ becomes trivial. We can then proceed by induction, supposing that $(H/J^n)^\circ = J_{n-1}H^\circ$, so that $J_nH^\circ/J_{n-1}H^\circ$ is the largest F-submodule of $H^\circ/J_{n-1}H^\circ = (J^n)^\circ$ on which the H°-coaction is trivial (i.e. factors through the unit $\eta\colon \mathbf{1} \to H^\circ$). Dually, $(J_nH^\circ/J_{n-1}H^\circ)^*$ is the largest quotient of J^n on which the action of H factors through ε, namely J^n/J^{n+1}. Taking decompleted duals again, we see that $J_nH^\circ = (H/J^{n+1})^\circ$ as desired. □

Definition-Lemma 2.1.27. Let H be a pro-finite dimensional Hopf algebra over F, and write J^n for the powers of its augmentation ideal. We say that H is *J-adically complete* just when it satisfies any of the following equivalent conditions:

- $\bigcap_n J^{n+1} = 0$;
- the natural map $H \to \varprojlim(H/J^{n+1})$ is an isomorphism; or
- the Hopf algebra H° is J-adically cocomplete (see definition 2.1.15).

Proof of equivalence. By lemma 2.1.26, the first two conditions are equivalent to, respectively, $\bigcup_n J_nH^\circ = H^\circ$ and $\varinjlim_n J_nH^\circ = H^\circ$, where $J_\bullet H^\circ$ is the J-filtration of the Hopf algebra H°. These conditions are clearly equivalent to one another, and by definition also to J-adic cocompleteness of H°. □

2.1.3.1 Connection to Lie algebras

Using the language of pro-finite dimensional Hopf algebras and Lie algebras, we are now in a position to describe the correspondence between pro-nilpotent Lie algebras and J-adically cocomplete commutative Hopf algebras.

Definition 2.1.28 (Group-like and primitive elements)**.** Let $V = \varprojlim(V_i)$ be a pro-finite vector space over F. If Λ is an F-algebra, we write

$$V_\Lambda := \varprojlim(\Lambda \otimes V_i),$$

where the inverse limit is taken in the category of Λ-modules. (We will occasionally conflate V with V_F so as not to overburden the notation.) Note that if W is another pro-finite dimensional vector space, there is a canonical inclusion $V_\Lambda \otimes_\Lambda W_\Lambda \hookrightarrow (V\widehat{\otimes}W)_\Lambda$ of Λ-modules, but that this is rarely an isomorphism.

If C is a pro-finite dimensional coalgebra over F, we write

$$C_\Lambda^{\mathrm{gplike}} \subseteq C_\Lambda$$

for the subset of *group-like elements* of C_Λ, i.e. those $g \in C_\Lambda$ which satisfy $\Delta(g) = g \otimes g$ (as elements of $(C \widehat{\otimes} C)_\Lambda$). If $C = H$ is a Hopf algebra, then the Hopf algebra operation induce a group structure on $H_\Lambda^{\mathrm{gplike}}$, with the multiplication, unit and inversion maps on $H_\Lambda^{\mathrm{gplike}}$ induced by the multiplication, unit and antipode maps on H.

Finally, if the pro-finite dimensional coalgebra C is *coaugmented* (i.e. is endowed with a unit morphism $\eta\colon \mathbf{1} \to C$ of pro-finite dimensional coalgebras), then we write

$$C_\Lambda^{\mathrm{prim}} \leq C_\Lambda$$

for the Λ-submodule of *primitive elements*, i.e. those $x \in C_\Lambda$ satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$. If $C = H$ is a Hopf algebra, then the commutator bracket on H_Λ induces the structure of a Lie algebra (over Λ on $H_\Lambda^{\mathrm{prim}}$. Note that in the particular case that $\Lambda = F$, the Lie algebra H_F^{prim} canonically has the structure of a pro-finite dimensional Lie algebra H^{prim}, since it and its Lie bracket can be defined in terms of morphisms of pro-finite dimensional F-vector spaces.

Example 2.1.29. If H is a commutative Hopf algebra over F and H^* its dual Hopf algebra, then H_Λ^* is canonically identified with the Λ-module of all F-linear maps $H \to \Lambda$. The group-like elements are those elements which are ring homomorphisms, so that there is a canonical isomorphism

$$H_\Lambda^{*,\mathrm{gplike}} \cong \mathrm{Spec}(H)(\Lambda)$$

of groups. Moreover, if $\Lambda = F$, then a linear map $H \to F$ is an element of H^{prim} if and only if it is an F-derivation, so that

$$H^{\mathrm{prim}} \cong \mathrm{Der}_{F,\varepsilon}(H) \cong \mathrm{Lie}(\mathrm{Spec}(H))$$

as pro-finite dimensional Lie algebras.

Remark 2.1.30. If H is a J-adically complete Hopf algebra over F with augmentation ideal $J \trianglelefteq H$, then it follows that the usual exponential (resp. logarithm) power series converges on J_Λ (resp. $1 + J_\Lambda$) for all F-algebras Λ, and one can easily check that these define mutually inverse bijections $H_\Lambda^{\mathrm{prim}} \xrightarrow{\sim} H_\Lambda^{\mathrm{gplike}}$. Interpreting $H_\Lambda^{\mathrm{prim}}$ as the Λ-points of the corresponding pro-unipotent group $U = \mathrm{Spec}(H^\circ)$ and $H_\Lambda^{\mathrm{prim}}$ as the base change of its Lie algebra $\mathrm{Lie}(U)$, these maps describe the logarithm-exponential isomorphism between U and $\mathrm{Lie}(U)$ described in definition 2.1.5.

Remark 2.1.31. If H is a J-adically complete Hopf algebra over F, then H^{prim} is automatically pro-nilpotent. This follows since H^{prim} is contained in the augmentation ideal J (the augmentation ε factors as $-(\varepsilon\widehat{\otimes}\varepsilon)\circ(\Delta - 1_H\widehat{\otimes}\eta - \eta\widehat{\otimes}1_H)$), and hence the images of the n-fold iterated commutator maps $(H^{\mathrm{prim}})^{\widehat{\otimes}n} \to H^{\mathrm{prim}}$ are eventually zero in every finite dimensional quotient of H^{prim}.

The functor $H \mapsto H^{\mathrm{prim}}$ (dual to the functor $H^\circ \mapsto \mathrm{Der}_{F,\varepsilon}(H^\circ)$) will form one direction of the correspondence between J-adically complete cocommutative Hopf algebras and pro-nilpotent Lie algebras. The other direction of the correspondence is given by a completed universal enveloping algebra construction.

Definition 2.1.32 (Completed universal enveloping algebra)**.** Let $\mathfrak{u}$ be a pro-nilpotent Lie algebra over F. We define a J-adically complete cocommutative Hopf algebra $\widehat{\mathcal{U}}(\mathfrak{u})$ called the *completed universal enveloping algebra* of $\mathfrak{u}$, as follows. If $\mathfrak{u}$ is finite dimensional, let $\mathcal{U}(\mathfrak{u})$ denote the usual universal enveloping algebra of $\mathfrak{u}$ [Mil17, Chapter 10.j]. This is a cocommutative Hopf algebra, and in particular has an augmentation ideal $J = \ker(\varepsilon\colon \mathcal{U}(\mathfrak{u}) \to F)$, which is a Hopf ideal [Mil17, Definition 3.10]. Since $\mathcal{U}(\mathfrak{u})$ is a quotient of the tensor algebra on $\mathfrak{u}$, it follows that each quotient $\mathcal{U}(\mathfrak{u})/J^{n+1}$ is finite dimensional, and hence that $\widehat{\mathcal{U}}(\mathfrak{u}) := \varprojlim(\mathcal{U}(\mathfrak{u})/J^{n+1})$ defines a pro-finite dimensional F-vector space. It is easy to verify that the Hopf algebra operations on $\mathcal{U}(\mathfrak{u})$ induce a (pro-finite dimensional) Hopf algebra structure on $\widehat{\mathcal{U}}(\mathfrak{u})$. This is clearly cocommutative and J-adically complete.

For general $\mathfrak{u}$, we may write $\mathfrak{u} = \varprojlim \mathfrak{u}_i$ for some finite dimensional nilpotent Lie algebras $\mathfrak{u}_i$, and define $\widehat{\mathcal{U}}(\mathfrak{u}) := \varprojlim \widehat{\mathcal{U}}(\mathfrak{u}_i)$.

Note that there is a map $\mathfrak{u} \to \widehat{\mathcal{U}}(\mathfrak{u})^{\mathrm{prim}}$ of pro-finite dimensional Lie algebras induced by the maps $\mathfrak{u}_i \hookrightarrow \mathcal{U}(\mathfrak{u}_i) \to \widehat{\mathcal{U}}(\mathfrak{u}_i)$ where $\mathfrak{u}_i \hookrightarrow \mathcal{U}(\mathfrak{u}_i)$ is the usual inclusion of a Lie algebra into its universal enveloping algebra. It is easy to see that the image of this map is contained in the primitive subspace, and that it is a map of pro-finite dimensional Lie algebras.

In order to show that the two constructions $\mathfrak{u} \mapsto \widehat{\mathcal{U}}(\mathfrak{u})$ and $H \mapsto H^{\mathrm{prim}}$ are mutually inverse, we will want to show that the above map $\mathfrak{u} \to \widehat{\mathcal{U}}(\mathfrak{u})^{\mathrm{prim}}$ is in fact an isomorphism of pro-nilpotent Lie algebras. This requires a completed version of the Poincaré–Birkhoff–Witt theorem.

Lemma 2.1.33. *Let $\mathfrak{u}$ be a pro-nilpotent Lie algebra over F. Then there is a canonical isomorphism*

$$\widehat{\mathcal{U}}(\mathfrak{u}) \cong (\mathrm{Sym}^\bullet(\mathfrak{u}^\circ))^*$$

of pro-finite dimensional coaugmented coalgebras, and the filtration on $\widehat{\mathcal{U}}(\mathfrak{u})$ *induced by the powers of the augmentation ideal corresponds to the filtration on* $(\mathrm{Sym}^\bullet(\mathfrak{u}^\circ))^*$ *induced by the descending central series filtration on* $\mathfrak{u}$*.*

Proof. We assume first that $\mathfrak{u}$ is finite-dimensional and write $\mathcal{U}(\mathfrak{u})$ for the (non-completed) universal enveloping algebra. Recall from the Poincaré–Birkhoff–Witt theorem that there is a canonical isomorphism $\mathrm{Sym}_c^\bullet(\mathfrak{u}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{u})$ of coaugmented coalgebras given by sending $v_1 \dots v_k$ to the symmetrised sum $\frac{1}{k!}\sum_{\sigma\in S_k} v_{\sigma(1)} \dots v_{\sigma(k)}$. The symmetric coalgebra $\mathrm{Sym}_c^\bullet(\mathfrak{u})$ carries a canonical descending filtration $I^\bullet$ induced by the descending central series filtration on $\mathfrak{u}$—we will show this agrees with the filtration of $\mathcal{U}(\mathfrak{u})$ by powers of the augmentation ideal J.

By abuse of notation, let $I^\bullet$ also denote the filtration on $\mathcal{U}(\mathfrak{u})$ induced by that on $\mathrm{Sym}_c^\bullet(\mathfrak{u})$, so that $I^n\mathcal{U}(\mathfrak{u})$ is the span of the symmetrised sums $\frac{1}{k!}\sum_{\sigma\in S_k} v_{\sigma(1)} \dots v_{\sigma(k)}$, where each $v_i \in I^{n_i}\mathfrak{u}$ and $\sum_i n_i = n$. Thus the inclusion $I^n \leq J^n$ is obvious, since every element of each $I^{n_i}\mathrm{Lie}(U)$ can be written by definition as a linear combination of n_i-fold products of elements in $\mathfrak{u} \leq \mathcal{U}(\mathfrak{u})$.

For the converse inclusion, it suffices to prove that for such a tuple $v_1, \dots, v_k$ as above we have $v_1 \dots v_k \in I^n$, since $n \geq k$ necessarily. We proceed by induction on k, from the trivial base case $k = 0$. If $\sigma \in S_k$ denotes the transposition of some i and $i+1$, then $v_1 \dots v_k - v_{\sigma(1)} \dots v_{\sigma(k)} = v_1 \dots [v_i, v_{i+1}] \dots v_k \in I^n$ by inductive hypothesis for $k-1$, so that $v_1 \dots v_k \equiv v_{\sigma(1)} \dots v_{\sigma(k)} \bmod I^n$. Since such transpositions generate S_k, this in fact holds for any $\sigma \in S_k$, so that $v_1 \dots v_k \equiv \frac{1}{k!}\sum_{\sigma\in S_k} v_{\sigma(1)} \dots v_{\sigma(k)} \equiv 0 \bmod I^n$ as claimed.

Now if we complete $\mathrm{Sym}_c^\bullet(\mathfrak{u}) = \bigoplus_n \mathrm{Sym}^n(\mathfrak{u})$ with respect to this filtration, we obtain the pro-finite dimensional coaugmented coalgebra $\prod_n \mathrm{Sym}^n(\mathfrak{u})$, which is the dual of the symmetric algebra $\mathrm{Sym}^\bullet(\mathfrak{u}^*) = \bigoplus_n \mathrm{Sym}^n(\mathfrak{u}^*)$. Hence by completing the isomorphism $\mathrm{Sym}_c^\bullet(\mathfrak{u}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{u})$ from the Poincaré–Birkhoff–Witt theorem with respect to the filtrations on either side, we obtain an isomorphism $(\mathrm{Sym}^\bullet(\mathfrak{u}^*))^* \xrightarrow{\sim} \widehat{\mathcal{U}}(\mathfrak{u})$ of pro-finite dimensional coaugmented coalgebras.

For a general $\mathfrak{u}$ we simply note that

$$\widehat{\mathcal{U}}(\mathfrak{u}) \cong \varprojlim \widehat{\mathcal{U}}(\mathfrak{u}) \cong \varprojlim (\mathrm{Sym}^\bullet(\mathfrak{u}_i^*))^* \cong (\mathrm{Sym}^\bullet(\mathfrak{u}^\circ))^*,$$

where $\mathfrak{u}_i$ runs through the finite dimensional Lie algebra quotients of $\mathfrak{u}$. To see that this isomorphism preserves the filtrations on either side, simply note that the quotient maps $\mathfrak{u} \twoheadrightarrow \mathfrak{u}_i$ are strict for the descending central series filtration, and thus all the maps $(\mathrm{Sym}^\bullet(\mathfrak{u}^\circ))^* \twoheadrightarrow (\mathrm{Sym}^\bullet(\mathfrak{u}_i^*))^*$ and $\widehat{\mathcal{U}}(\mathfrak{u}) \twoheadrightarrow \widehat{\mathcal{U}}(\mathfrak{u}_i)$ are also strict for their respective filtrations. □

Corollary 2.1.34. *If $\mathfrak{u}$ is a pro-nilpotent Lie algebra over F, then the canonical map $\mathfrak{u} \to \widehat{\mathcal{U}}(\mathfrak{u})^{\mathrm{prim}}$ is an isomorphism of pro-nilpotent Lie algebras. If H is a J-adically complete cocommutative Hopf algebra, then there is a canonical isomorphism $H \cong \widehat{\mathcal{U}}(H^{\mathrm{prim}})$. The functors $\mathfrak{u} \mapsto \widehat{\mathcal{U}}(\mathfrak{u})$ and $H \mapsto H^{\mathrm{prim}}$ set up an equivalence between the category of pro-nilpotent Lie algebras and the category of J-adically complete cocommutative Hopf algebras over F.*

Proof. For the first part, under the isomorphism $\widehat{\mathcal{U}}(\mathfrak{u}) \cong (\mathrm{Sym}^\bullet(\mathfrak{u}^\circ))^*$ of pro-finite dimensional coaugmented coalgebras in lemma 2.1.33, the map $\mathfrak{u} \to \widehat{\mathcal{U}}(\mathfrak{u})$ corresponds to the map $\mathfrak{u} \to (\mathrm{Sym}^\bullet(\mathfrak{u}^\circ))^*$ dual to the projection map $\mathrm{Sym}^\bullet(\mathfrak{u}^\circ) \twoheadrightarrow \mathfrak{u}^\circ$. This latter map is the universal derivation for the algebra $\mathrm{Sym}^\bullet(\mathfrak{u}^\circ)$, and hence induces an isomorphism $(\mathrm{Sym}^\bullet(\mathfrak{u}^\circ))^{*,\mathrm{prim}} \cong \mathrm{Der}_{F,\varepsilon}(\mathrm{Sym}^\bullet(\mathfrak{u}^\circ)) \xleftarrow{\sim} (\mathfrak{u}^\circ)^* = \mathfrak{u}$, as desired.

For the second part, it is easy to see that the inclusion $H^{\mathrm{prim}} \hookrightarrow H$ induces a morphism $\widehat{\mathcal{U}}(H^{\mathrm{prim}}) \to H$ of J-adically complete cocommutative Hopf algebras. Moreover, by the first part, this morphism induces an isomorphism on the subspaces of primitive elements, so by remark 2.1.30 the induced map

$$\widehat{\mathcal{U}}(H^{\mathrm{prim}})^{\mathrm{gplike}}_\Lambda \to H^{\mathrm{gplike}}_\Lambda$$

is an isomorphism of groups for every F-algebra Λ. However, this says that the corresponding map $\mathrm{Spec}(\widehat{\mathcal{U}}(H^{\mathrm{prim}})^\circ) \to \mathrm{Spec}(H^\circ)$ is an isomorphism of pro-unipotent groups over F, so that $\widehat{\mathcal{U}}(H^{\mathrm{prim}}) \to H$ was an isomorphism of J-adically complete cocommutative Hopf algebras, as desired. □

2.2 Torsors under pro-unipotent groups

Throughout this section, we continue to fix a characteristic 0 field F.

As well as pro-unipotent groups, we will also be interested in studying torsors under such groups. Our chief reason for doing so is that the Tannakian formalism [DM82, Del89] provides a mechanism for associating such torsors to rational points on varieties (in many different ways). These torsors frequently come endowed with various extra structures, and the classifying map from rational points to the moduli space of such torsors (the *non-abelian Kummer maps* of the introduction) provide a powerful tool for the study of rational points.

Thus, let us begin by defining the notion of a torsor.

Definition 2.2.1. Let U/F be an affine algebraic group. A *(right) torsor under U* or *U-torsor* is a non-empty affine F-scheme P endowed with a right action $P \times U \to P$

such that the map $P \times U \to P \times P$ given by $(q, u) \mapsto (q, qu)$ is an isomorphism of F-schemes.

Remark 2.2.2. It follows from this definition that the affine ring $\mathcal{O}(P)$ of a U-torsor P carries simultaneously the structure of an F-algebra and an $\mathcal{O}(U)$-comodule, and moreover that these structures are compatible in that the $\mathcal{O}(U)$-coaction is a morphism of F-algebras (equivalently the multiplication and unit maps are morphisms of $\mathcal{O}(U)$-comodules). We refer to such a structure as the structure of a $\mathcal{O}(U)$-*comodule algebra* (the dual notion is referred to as an $\mathcal{O}(U)$-module coalgebra).

It turns out that torsors under pro-unipotent groups are particularly simple to understand, and are in fact all isomorphic. (Of course, when the torsors are endowed with various extra structures, as they will be throughout this thesis, it need not be the case that they are all isomorphic *compatibly with these structures.*)

Proposition 2.2.3. *Any torsor P under a pro-unipotent group U/F is trivial, i.e. is isomorphic to U with its right multiplication action, or equivalently $P(F) \neq \emptyset$.*

For the proof of proposition 2.2.3, let us recall the definition of the *pushout* or *base change* of torsors.

Definition 2.2.4. Let $f\colon U \to U'$ be a morphism of pro-unipotent groups over F and P/F an affine scheme with a right U-action $\rho\colon P \times U \to P$. We define the *pushout* $P \times^U U'$ *of* X *along* f to be the coequaliser of

$$\rho \times 1_{U'}\,,\ 1_P \times \mathrm{mult}\colon P \times U \times U' \rightrightarrows P \times U'$$

in the category of affine F-schemes. $P \times^U U'$ inherits a natural right U'-action, and if P is a U-torsor, so too is $P \times^U U'$.

Proof of proposition 2.2.3. When $U = \mathbb{G}_a$, the result follows from triviality of the Galois cohomology group $\mathrm{H}^1(G_F, \overline{F}) = \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(F), \mathbb{G}_a)$. When U is a central extension of some U' by $\mathbb{G}_a$, the pushout $P' := P \times^U U'$ is a U'-torsor, and the fibres of $P \to P'$ over F-points are $\mathbb{G}_a$-torsors, so that $P(F) \to P'(F)$ is surjective. The result follows for U finite-dimensional by induction, and for U general by a Zorn's lemma argument[6]. □

[6]There is a mild subtlety in complete generality, in that a filtered inverse limit of a surjective system of non-empty sets can be empty. The correct poset to consider to make the Zorn argument work consists of pairs (U', x') with U' a quotient of U and $x' \in (P \times^U U')(F)$.

Just as in the study of the affine rings of pro-unipotent groups, the J-filtration (definition 2.1.12) on the affine rings of torsors will also play an important role in our theory. This satisfies a number of useful basic properties.

Proposition 2.2.5. *If U/F is a pro-unipotent group over F and P a U-torsor, then $J_\bullet\mathcal{O}(P)$ (see definition 2.1.12) is an exhaustive filtration by sub-$\mathcal{O}(U)$-comodules, which is invariant under $\mathcal{O}(U)$-comodule automorphisms. Moreover, if we endow $\mathcal{O}(P)\otimes\mathcal{O}(P)$ and $\mathcal{O}(P)\otimes\mathcal{O}(U)$ with the induced tensor product filtrations and F with the trivial filtration, then the $\mathcal{O}(U)$-comodule algebra[7] structure morphisms for $\mathcal{O}(P)$ respect the J-filtration.*

If U is finitely generated, then each $J_n\mathcal{O}(P)$ is finite-dimensional.

Proof. Since P is trivial as a U-torsor, we have that $\mathcal{O}(P)\xrightarrow{\sim}\mathcal{O}(U)$ as $\mathcal{O}(U)$-comodule algebras and hence the desired results for $\mathcal{O}(P)$ follow from those for $\mathcal{O}(U)$ (example 2.1.13, lemma 2.1.14 and definition-lemma 2.1.10). □

One particularly important property of the J-filtration on $\mathcal{O}(P)$ is that the non-canonicity of the isomorphism $\mathcal{O}(P)\xrightarrow{\sim}\mathcal{O}(U)$ determined by some $q\in P(F)$ disappears once we pass to the corresponding J-graded algebras.

Proposition 2.2.6. *Let U/F be a pro-unipotent group and P a torsor under U. Then the graded isomorphism $\mathrm{gr}^J_\bullet\mathcal{O}(P)\xrightarrow{\sim}\mathrm{gr}^J_\bullet\mathcal{O}(U)$ induced from any trivialisation $U\xrightarrow{\sim}P$ of U-torsors is independent of the choice of trivialisation. Indeed, the unit map $\eta\colon F\to\mathcal{O}(P)$ induces an isomorphism $F\cong\mathrm{gr}^J_0(\mathcal{O}(P))$ and the canonical graded trivialisation is the composite*

$$\mathrm{gr}^J_\bullet(\mathcal{O}(P))\xrightarrow{\mathrm{gr}^J_\bullet(\rho)}\mathrm{gr}^J_\bullet(\mathcal{O}(P))\otimes\mathrm{gr}^J_\bullet(\mathcal{O}(U))\xrightarrow{\varepsilon\otimes 1}F\otimes\mathrm{gr}^J_\bullet(\mathcal{O}(U))\cong\mathrm{gr}^J_\bullet(\mathcal{O}(U)),$$

where $\rho\colon\mathcal{O}(P)\to\mathcal{O}(P)\otimes\mathcal{O}(U)$ is the coaction and $\varepsilon\colon\mathrm{gr}^J_\bullet(\mathcal{O}(P))\to F$ is projection onto $\mathrm{gr}^J_0(\mathcal{O}(P))\cong F$.

Remark 2.2.7. The significance of this discussion is as follows. The finitely generated pro-unipotent groups U and their torsors P will typically carry various kinds of extra structures on $\mathcal{O}(U)$ and $\mathcal{O}(P)$; more specifically, they will carry the structure of ind-objects in various neutralised F-linear Tannakian categories, compatibly with their Hopf algebra and $\mathcal{O}(U)$-comodule algebra structures. The canonical descriptions we

[7]Recall that an $\mathcal{O}(U)$-comodule algebra is a vector space endowed simulteously with an algebra structure and an $\mathcal{O}(U)$-comodule structure such that the $\mathcal{O}(U)$-comodule structure morphisms are morphisms of algebras (or equivalently the algebra structure morphisms are morphisms of $\mathcal{O}(U)$-coalgebras).

have given of $J_\bullet\mathcal{O}(U)$ and $J_\bullet\mathcal{O}(P)$ ensure that they are filtrations by objects in such a Tannakian category, and that the canonical isomorphism $\mathrm{gr}^J_\bullet(\mathcal{O}(P)) \xrightarrow{\sim} \mathrm{gr}^J_\bullet(\mathcal{O}(U))$ is an isomorphism of objects in the Tannakian category.

2.3 Malčev completion

The majority of the pro-unipotent groups in this thesis can be thought of as a certain kind of completion of a discrete or profinite group, known as a *Malčev completion*. We will give the definition below in terms of a universal property, but the reader is encouraged to think of this completion operation as a non-abelian version of tensoring an abelian group with a field to obtain a vector space, the point being that such an operation substantially simplifies the study of the objects involved.

Definition-Lemma 2.3.1 (Malčev completion)**.** Let Π be a discrete group and F a field of characteristic zero. Then the functor from pro-unipotent groups over F to sets defined on objects U by

$$U \mapsto \mathrm{Hom}(\Pi, U(F))$$

is representable by a pro-unipotent group Π_F and homomorphism $\Pi \to \Pi_F(F)$. This group is called the *Malčev completion* of Π, and is functorial in Π. For an F-algebra Λ, we may write $\Pi(\Lambda)$ for $\Pi_F(\Lambda)$ where the meaning is clear.

More generally, if Π is a topological group and F a topological field of characteristic zero, the functor from pro-unipotent groups over F to sets defined on objects U by

$$U \mapsto \mathrm{Hom}_{\mathrm{cts}}(\Pi, U(F))$$

is representable by a pro-unipotent group Π_F and continuous homomorphism $\Pi \to \Pi_F(F)$. This group is called the *continuous Malčev completion* of Π, and is functorial in Π.

Remark 2.3.2. Implicit in the statement of definition-lemma 2.3.1 is a certain natural topology on the F-points $U(F)$ of a pro-unipotent group U/F when F is a topological field of characteristic 0. Explicitly, if U is unipotent (so isomorphic to some affine space $\mathbb{A}^m_F$ as a variety), then the topology on $U(F) \simeq F^m$ is the natural one, and in general if $U = \varprojlim U_n$ is an inverse limit of unipotent groups, then the topology on $U(F) = \varprojlim U_n(F)$ is the inverse limit topology from each quotient. Similarly, if Λ is a topological F-algebra, the group $U(\Lambda)$ of Λ-points also carries a canonical topology, which is defined analogously.

These conventions for topologies on points of pro-unipotent groups will apply throughout this dissertation.

Proof of definition-lemma 2.3.1. It suffices to prove the case where Π and F carry topologies. The category of pro-unipotent groups over F is locally small, well-powered, and has a small cogenerating set, namely the set[8] of unipotent groups. It also has all small limits, which are preserved by the functor to the category of topological groups defined on objects by

$$U \mapsto U(F).$$

Hence by the special adjoint functor theorem [Lan78, Corollary to Theorem V.8.2], the functor $U \mapsto U(F)$ has a left adjoint $\Pi \mapsto \Pi_F$. It follows from adjunction that Π_F represents the desired functor. □

Remark 2.3.3. It follows by definition that if Π is a finitely generated discrete group, then its Malčev completion Π_F is also finitely generated for every characteristic zero field F (since the image of a generating set for Π under the map $\Pi \to \Pi_F(F)$ necessarily generates Π_F by the universal property of Π_F).

Let us now record a few basic properties of Malčev completions which are essentially immediate from the definition.

Proposition 2.3.4. *Let Π be a discrete group, and let Π_n denote its maximal torsion-free n-step nilpotent quotient. Then for any characteristic 0 field F, the canonical morphism*

$$\Pi_F \to \varprojlim \Pi_{n,F}$$

is an isomorphism of pro-unipotent groups.

Proof. If U/F is a unipotent group, then $U(F)$ is a torsion-free nilpotent group, so any morphism $\Pi \to U(F)$ factors through Π_n for some n. Hence for all such U we have

$$\mathrm{Hom}(\Pi, U(F)) \cong \varinjlim \mathrm{Hom}(\Pi_n, U(F)) \cong \varinjlim \mathrm{Hom}(\Pi_{n,F}, U) \cong \mathrm{Hom}(\varprojlim \Pi_{n,F}, U),$$

where the final bijection comes from the fact that $\mathcal{O}(U)$ is finitely generated. Taking an inverse limit, the same holds for U pro-unipotent. Hence the canonical map $\mathrm{Hom}(\varprojlim \Pi_{n,F}, U) \to \mathrm{Hom}(\Pi_F, U)$ is bijective for all U, so that $\Pi_F \to \varprojlim \Pi_{n,F}$ is an isomorphism, as desired. □

[8]Strictly, one should choose representatives of each isomorphism class of unipotent groups to ensure that this is a set.

Proposition 2.3.5. *Let Π be a finitely generated discrete group, F a characteristic zero field, and Π_F the F-Malčev completion of Π. If F'/F is a field extension, then the F'-Malčev completion $\Pi_{F'} \cong (\Pi_F)_{F'}$ is the base change of Π_F to F'.*

Proof. If U is a unipotent group over F', then $\mathrm{Lie}(U)$ is a finite dimensional nilpotent Lie algebra over F'. The restriction of scalars $\mathrm{Res}_F^{F'}\mathrm{Lie}(U)$ is then a nilpotent Lie algebra over F, and hence is a direct limit $\mathrm{Res}_F^{F'}\mathrm{Lie}(U) = \varinjlim L_i$, where the $L_i \leq \mathrm{Res}_F^{F'}\mathrm{Lie}(U)$ are finite dimensional (nilpotent) Lie subalgebras (over F). If U_i/F denotes the unipotent group with Lie algebra L_i, then $U(F') \cong \varinjlim U_i(F)$ (since both sides are the group defined on $\mathrm{Lie}(U)$ by the Baker–Campbell–Hausdorff formula), so that

$$\begin{aligned}
\mathrm{Hom}(\Pi, U(F')) &\cong \varinjlim \mathrm{Hom}(\Pi, U_i(F)) \\
&\cong \varinjlim \mathrm{Hom}(\Pi_F, U_i) \\
&\cong \varinjlim \mathrm{Hom}(\mathrm{Lie}(\Pi_F), \mathrm{Lie}(U_i)) \\
&\cong \mathrm{Hom}(\mathrm{Lie}(\Pi_F), \mathrm{Res}_F^{F'}\mathrm{Lie}(U)) \\
&\cong \mathrm{Hom}(\mathrm{Lie}(\Pi_F)_{F'}, \mathrm{Lie}(U)) \\
&\cong \mathrm{Hom}((\Pi_F)_{F'}, U),
\end{aligned}$$

where the first bijection comes from the fact that Π is finitely generated, and the fourth isomorphism from the fact that $\mathrm{Lie}(\Pi_F)$ is finitely generated (as a Lie algebra object in pro-finite dimensional F-vector spaces). Taking an inverse limit, the same holds for U pro-unipotent, so that $(\Pi_F)_{F'}$ represents the same functor as Π_F, and hence $\Pi_{F'} \cong (\Pi_F)_{F'}$ as desired. □

Proposition 2.3.6. *Let Π be a finitely generated discrete group and $\widehat{\Pi}$ its profinite completion. Then for any prime ℓ, the continuous $\mathbb{Q}_\ell$-Malčev completion $\widehat{\Pi}_{\mathbb{Q}_\ell}$ agrees with the discrete $\mathbb{Q}_\ell$-Malčev completion $\Pi_{\mathbb{Q}_\ell}$.*

Proof. If $U/\mathbb{Q}_\ell$ is a unipotent group, then $U(\mathbb{Q}_\ell) \cong \mathrm{Lie}(U)$ is a finite dimensional topological $\mathbb{Q}_\ell$-vector space which is a nilpotent Lie algebra. As a consequence of the Baker–Campbell–Hausdorff formula, $U(\mathbb{Q}_\ell) = \varinjlim U(\mathbb{Q}_\ell)_i$ is a direct limit of pro-ℓ groups corresponding to $\mathbb{Z}_\ell$-sublattices of $\mathrm{Lie}(U)$ which are stable under the group operations. Thus

$$\mathrm{Hom}(\Pi, U(\mathbb{Q}_\ell)) \cong \varinjlim \mathrm{Hom}(\Pi, U(\mathbb{Q}_\ell)_i) \cong \varinjlim \mathrm{Hom}_{\mathrm{cts}}(\widehat{\Pi}, U(\mathbb{Q}_\ell)_i) \cong \mathrm{Hom}_{\mathrm{cts}}(\widehat{\Pi}, U(\mathbb{Q}_\ell)),$$

where the first bijection comes from the fact that Π is finitely generated and the final bijection comes from the fact that $\widehat{\Pi}$ is a compact group. This isomorphism is natural in U, and hence $\Pi_{\mathbb{Q}_\ell} \cong \widehat{\Pi}_{\mathbb{Q}_\ell}$ as desired. □

2.3.1 Two constructions of the Malčev completion

As well as the definition of the Malčev completion via a universal property, there are several other equivalent definitions which have the merit of being considerably more explicit. We shall give these definitions now, and then note a few more consequences we shall need in the body of this thesis.

Lemma 2.3.7. *Let Π be a finitely generated discrete group and F a characteristic zero field. Let $F\Pi$ denote the group algebra of Π and $J \trianglelefteq F\Pi$ the augmentation ideal. Then the J-adic completion $F\Pi^\wedge = \varprojlim F\Pi/J^{n+1}$ is a J-adically complete cocommutative Hopf algebra, whose corresponding group scheme $\mathrm{Spec}(F\Pi^{\wedge\circ})$ is canonically identified with the Malčev completion Π_F.*

Proof. By proposition 2.3.5, it suffices to prove this when $F = \mathbb{Q}$. It is well-known (or easy to check) that there is a canonical isomorphism $\mathbb{Q} \otimes \Pi \xrightarrow{\sim} J/J^2$, so that in particular J/J^2 is finite dimensional. Thus every J^n/J^{n+1} is finite dimensional (as the n-fold multiplication map $(J/J^2)^{\otimes n} \twoheadrightarrow (J^n/J^{n+1})$ is surjective), and hence $\mathbb{Q}\Pi^\wedge$ is a pro-finite dimensional vector space. It is easy to check that the cocommutative Hopf algebra structure on $\mathbb{Q}\Pi$ induces a J-adically complete cocommutative Hopf algebra structure on $\mathbb{Q}\Pi^\wedge$. Thus $\mathrm{Spec}(\mathbb{Q}\Pi^{\wedge\circ})$ is a pro-unipotent group. It thus suffices to prove that this group represents the desired functor. This follows from the following chain of natural bijections:

$$\begin{aligned}\mathrm{Hom}(\Pi, U(\mathbb{Q})) &\cong \mathrm{Hom}(\Pi, \mathcal{O}(U)^{*,\mathrm{gplike}}) \\ &\cong \mathrm{Hom}(\mathbb{Q}\Pi^\wedge, \mathcal{O}(U)^*) \\ &\cong \mathrm{Hom}(\mathrm{Spec}(\mathbb{Q}\Pi^{\wedge,\mathrm{gplike}}), U),\end{aligned}$$

where implicit in the second bijection is the easily verified assertion that any homomorphism from a finitely generated discrete group to the grouplike elements of a J-adically complete cocommutative Hopf algebra over $\mathbb{Q}$ extends uniquely to a morphism of pro-finite dimensional Hopf algebras out of $\mathbb{Q}\Pi^\wedge$. □

Remark 2.3.8. If Π is a discrete (resp. topological) group and P is a Π-torsor, then for any characteristic 0 field (resp. topological field of characteristic 0) F one can also define the (continuous) F-Malčev completion of P, namely the Π_F-torsor P_F

admitting a map (resp. continuous map) $P \to P_F(F)$ compatible in an appropriate sense with the actions of Π and $\Pi_F(F)$. When Π is finitely generated and discrete, this torsor can be canonically described as follows. The free vector space FP on set P is a cocommutative $F\Pi$-module coalgebra[9], and carries a canonical filtration $J^\bullet$ corresponding to the filtration on $F\Pi$ by powers of the augmentation ideal under any trivialisation $P \xrightarrow{\sim} \Pi$ of Π-torsors. The completion $FP^\wedge = \varprojlim FP/J^{n+1}$ is then a cocommutative $F\Pi^\wedge$-module coalgebra object in pro-finite dimensional vector spaces, and hence

$$P_F := \mathrm{Spec}(FP^{\wedge\circ})$$

is an F-scheme with a Π_F-action. This P_F is the F-Malčev completion of P, as one can see for example by comparing to lemma 2.3.7 via a trivialisation $P \xrightarrow{\sim} \Pi$.

Lemma 2.3.9. *Let Π be a finitely generated torsion-free nilpotent group. Choose a Malčev basis [CMZ17, Definition 4.5] $u_1, \dots, u_r$ of Π, so that every element of Π is expressible in the form $u_1^{a_1} u_2^{a_2} \dots u_r^{a_r}$ for* unique *integers $a_1, \dots, a_r$—we write this element as $\mathbf{u}^\mathbf{a}$ where $\mathbf{a} = (a_1, \dots, a_r)$.*

Then there are polynomials $f_1(\mathbf{x}, \mathbf{y}), \dots, f_r(\mathbf{x}, \mathbf{y})$ in $2r$ variables with rational coefficients, integer-valued on $\mathbb{Z}^{2r}$, such that $\mathbf{u}^\mathbf{a} \cdot \mathbf{u}^\mathbf{b} = \mathbf{u}^{\mathbf{f}(\mathbf{a},\mathbf{b})}$ for all tuples $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^r$. If $\Pi_\mathbb{Q}$ denotes the $\mathbb{Q}$-Malčev completion of Π, then $\Pi_\mathbb{Q}(\mathbb{Q})$ is identified with the group on formal symbols $\mathbf{v}^\mathbf{a}$ with $\mathbf{a} \in \mathbb{Q}^r$, with multiplication defined by $\mathbf{v}^\mathbf{a} \cdot \mathbf{v}^\mathbf{b} = \mathbf{v}^{\mathbf{f}(\mathbf{a},\mathbf{b})}$, such that the image of $\mathbf{u}^\mathbf{a} \in \Pi$ under the canonical map $\Pi \to \Pi_\mathbb{Q}(\mathbb{Q})$ is $\mathbf{v}^\mathbf{a}$.

Proof. The classical Malčev correspondence [Khu98, Theorem 10.11] provides in particular that there is an equivalence between the category of finite dimensional nilpotent Lie algebras over $\mathbb{Q}$ and uniquely divisible nilpotent groups which are finitely generated[10], where the uniquely divisible group associated to a such Lie algebra $\mathfrak{u}$ is the group provided by the Baker–Campbell–Hausdorff series on $\mathfrak{u}$. It follows from our parallel discussion in 2.1.4 that the functor

$$U \mapsto U(\mathbb{Q})$$

sets up an equivalence between the category of $\mathbb{Q}$-unipotent groups and the category of uniquely divisible nilpotent groups which are finitely generated.

Now the operation described above defines a uniquely divisible nilpotent group which is finitely generated [CMZ17, Theorem 4.13 & proof], and hence is the $\mathbb{Q}$-points

[9] That is, an F-coalgebra with compatible $F\Pi$-module structure, cf. remark 2.2.2.

[10] That is, there is a finite subset S of G such that the only *uniquely divisible* subgroup containing S is G itself

of a unipotent group $U/\mathbb{Q}$. A straightforward generalisation of [CMZ17, Theorem 4.14] shows that any homomorphism from Π to a uniquely divisible nilpotent group which is finitely generated must factor uniquely through $U(\mathbb{Q})$, and hence $U = \Pi_{\mathbb{Q}}$ is the $\mathbb{Q}$-Malčev completion of Π. □

Corollary 2.3.10. *Let Π be a finitely generated nilpotent group and $\Pi_{\mathbb{Q}}$ its $\mathbb{Q}$-Malčev completion. Then the kernel of the map $\Pi \to \Pi_{\mathbb{Q}}(\mathbb{Q})$ is the torsion subgroup*[11] *Π_{tors}, and any element of $\Pi_{\mathbb{Q}}(\mathbb{Q})$ has a power which lies in the image of $\Pi \to \Pi_{\mathbb{Q}}(\mathbb{Q})$. The dimension of $\Pi_{\mathbb{Q}}$ is finite, equal to the Hirsch length of Π [CMZ17, Definition 4.4].*

Corollary 2.3.11. *If*

$$\Pi'' \to \Pi \to \Pi'$$

is an exact sequence of finitely generated nilpotent groups, then the induced sequence

$$\Pi''_F \to \Pi_F \to \Pi'_F$$

on F-Malčev completions is also exact, for any characteristic 0 field F.

Proof. By proposition 2.3.5, it suffices to prove the result for $F = \mathbb{Q}$, and by corollary 2.1.6 it suffices to prove that the sequence

$$\Pi''_{\mathbb{Q}}(\mathbb{Q}) \to \Pi_{\mathbb{Q}}(\mathbb{Q}) \to \Pi'_{\mathbb{Q}}(\mathbb{Q})$$

is exact. The composite of the two maps in this sequence is zero, so it suffices to prove that the kernel of $\Pi_{\mathbb{Q}}(\mathbb{Q}) \to \Pi'_{\mathbb{Q}}(\mathbb{Q})$ is contained in the image of $\Pi''_{\mathbb{Q}}(\mathbb{Q}) \to \Pi_{\mathbb{Q}}(\mathbb{Q})$.

To do this, consider any u in the kernel of $\Pi_{\mathbb{Q}}(\mathbb{Q}) \to \Pi'_{\mathbb{Q}}(\mathbb{Q})$. By corollary 2.3.10, we may choose a power u^n which is the image of some $v \in \Pi$; the image of v in Π' is then necessarily torsion by corollary 2.3.10 again. Increasing n if necessary, we may assume that the image of v in Π' is in fact the identity, so that v lies in the image of $\Pi'' \to \Pi$, and hence u^n is the image of some $w \in \Pi''_{\mathbb{Q}}(\mathbb{Q})$. Since both $\Pi''_{\mathbb{Q}}(\mathbb{Q})$ and $\Pi_{\mathbb{Q}}(\mathbb{Q})$ are uniquely divisible, it follows that u is the image of $w^{1/n}$, as desired. □

2.4 Pro-unipotent fundamental groups

We will conclude this preliminary section by giving the two examples of pro-unipotent groups which will form the principal objects of study of this thesis, namely the *$\mathbb{Q}$-pro-unipotent Betti fundamental group* and the *$\mathbb{Q}_\ell$-pro-unipotent étale fundamental group* of a variety over an appropriate field. These can be defined somewhat glibly as Malčev

[11] The subset of torsion elements in a nilpotent group forms a subgroup [CMZ17, Theorem 2.26].

completions of the usual (discrete) Betti or (profinite) étale fundamental group of the variety in question, but are more usually defined in terms of the Tannakian formalism [DM82]. Let us make the connection between the various definitions.

Definition 2.4.1 (Tannakian fundamental group)**.** Let F be a field and $(\mathcal{C}, \otimes, \mathbf{1})$ a (neutral) Tannakian category with fibre functor $\omega\colon \mathcal{C} \to F\text{−}\mathbf{Mod}_{\text{fin}}$ [DM82, Definition 2.19]. The *Tannakian fundamental group* or *Tannaka group* G of $\mathcal{C}$ based at ω is an affine group scheme over F which is defined to be, equivalently:

- the group scheme $\mathbf{Aut}^{\otimes}(\omega)$ of tensor-compatible automorphisms of the fibre functor ω [DM82, Definition 1.12]; or
- the affine group scheme U/F (unique up to unique isomorphism) such that ω factors (up to tensor-natural isomorphism) as $\mathcal{C} \xrightarrow{\widetilde{\omega}} \mathbf{Rep}_F(U) \to F\text{−}\mathbf{Mod}_{\text{fin}}$ with $\widetilde{\omega}$ a tensor-equivalence and the final functor the forgetful functor (where $\mathbf{Rep}_F(U)$ is the category of finite dimensional F-linear representations of U) [DM82, Theorem 2.11].

More generally, if $\omega'\colon \mathcal{C} \to F\text{−}\mathbf{Mod}$ is a second fibre functor, then there is also a scheme $\mathbf{Iso}^{\otimes}(\omega, \omega')$ of tensor-compatible natural isomorphisms from ω to ω', which is a (right) torsor under the Tannakian fundamental group $\mathbf{Aut}^{\otimes}(\omega)$ by precomposition of natural isomorphisms [DM82, Theorem 3.2]. This torsor is called the *Tannakian torsor of paths* from ω to ω'.

Remark 2.4.2. If every object in a neutral Tannakian category $\mathcal{C}$ is an iterated extension of the unit object $\mathbf{1}$, then its Tannakian fundamental group is pro-unipotent (since every representation is an iterated extension of the trivial representation).

Example 2.4.3. Let $Y/\mathbb{C}$ be a smooth connected variety and $y \in Y(\mathbb{C})$ a base point. The Tannakian category $\mathrm{Uni}_{\mathbb{Q}}$ of *unipotent* $\mathbb{Q}$*-local systems on* $Y(\mathbb{C})$ is defined to be the category of locally constant sheaves of finite rank $\mathbb{Q}$-vector spaces on the topological space $Y(\mathbb{C})$ which are iterated extensions of the constant sheaf $\mathbf{1}$, endowed with the obvious fibrewise tensor structure. The point y determines a fibre functor $\mathrm{Fib}_y\colon \mathrm{Uni}_{\mathbb{Q}} \to \mathbb{Q}\text{−}\mathbf{Mod}_{\text{fin}}$ by taking the fibre of a local system at the point y. The Tannakian fundamental group of this category is called the $\mathbb{Q}$*-pro-unipotent Betti fundamental group* of (Y, y), and is denoted U^{B} (we will frequently drop the superscript when no ambiguity can arise).

This group admits a much more concrete description, as follows. A unipotent $\mathbb{Q}$-local system V is, in particular, a covering space $V \twoheadrightarrow Y(\mathbb{C})$, and hence the fibre V_y at y carries a canonical action by the usual (discrete) fundamental group

$\Pi := \pi_1^{\text{top}}(Y(\mathbb{C}); y)$. It is easy to see that the action of Π on this fibre is $\mathbb{Q}$-linear and unipotent (i.e. an iterated extension of the trivial representation of Π), and conversely that for any unipotent $\mathbb{Q}$-linear representation of Π, the corresponding covering space canonically has the structure of a unipotent $\mathbb{Q}$-local system. Thus the category of unipotent $\mathbb{Q}$-local systems is canonically equivalent to the category of unipotent representations of Π, and thus to the category of representations of the $\mathbb{Q}$-Malčev completion $\Pi_{\mathbb{Q}}$ (by the definition of Malčev completion). Thus the $\mathbb{Q}$-pro-unipotent Betti fundamental group is canonically isomorphic to the $\mathbb{Q}$-Malčev completion of the usual topological fundamental group $\pi_1^{\text{top}}(Y(\mathbb{C}); y)$. In particular, we have an explicit description of this group as $\text{Spec}(\mathbb{Q}\Pi^{\wedge\circ})$ by lemma 2.3.7, versions of which we will see repeatedly in chapter 9.

More generally, if $z \in Y(\mathbb{C})$ is a second point, then it determines a second fibre functor $\text{Fib}_z \colon \text{Uni}_{\mathbb{Q}} \to \mathbb{Q}-\mathbf{Mod}_{\text{fin}}$, and the Tannakian torsor of paths from Fib_y to Fib_z is called the *pro-unipotent torsor of Betti paths* from y to z. An argument analogous to the above establishes that this torsor is canonically identified with the Malčev completion of the usual torsor of paths from y to z, that is, the set of homotopy classes of paths from y to z in $Y(\mathbb{C})$, viewed as a Π-torsor via path-composition.

Example 2.4.4. Let Y be a connected variety over a field F with an algebraic closure $\overline{F}$, $y \in Y(F)$ a rational point of Y and ℓ a prime. We define a *$\mathbb{Z}/\ell^r\mathbb{Z}$-local system* on Y to be a sheaf of $\mathbb{Z}/\ell^r\mathbb{Z}$-modules on the small étale site of $Y_{\overline{F}}$ which locally are constant sheaves of finite rank free $\mathbb{Z}/\ell^r\mathbb{Z}$-modules. We define a *$\mathbb{Z}_\ell$-local system* on Y to be an inverse system $\cdots \to V_2 \to V_1 \to V_0$, where each V_r is a $\mathbb{Z}/\ell^r\mathbb{Z}$-local system on Y where the maps $V_r \to V_{r-1}$ induce isomorphisms $V_r/\ell V_r \xrightarrow{\sim} V_{r-1}$. The category of *$\mathbb{Q}_\ell$-local systems* on Y is defined to have the same objects as the category of $\mathbb{Z}_\ell$-local systems, but with morphisms defined by

$$\text{Hom}_{\mathbb{Q}_\ell}(V, W) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \text{Hom}(V, W).$$

The category of $\mathbb{Q}_\ell$-local systems is abelian, and we say that a $\mathbb{Q}_\ell$-local system is *unipotent* just when it is an iterated extension of the constant local system $\mathbf{1}$. The category $\text{Uni}^{\text{ét}}_{\mathbb{Q}_\ell}$ of unipotent $\mathbb{Q}_\ell$-local systems is Tannakian, and the geometric point y provides a fibre functor $\text{Fib}_y \colon \text{Uni}^{\text{ét}}_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell-\mathbf{Mod}_{\text{fin}}$—the Tannaka group of this category is called the *$\mathbb{Q}_\ell$-pro-unipotent étale fundamental group* of[12] Y based at y, and is denoted $U^{\text{ét}}$ or $U^{\text{ét}}_{\mathbb{Q}_\ell}$ (again, we will drop the superscripts and subscripts for most of this thesis).

[12]Since we've defined this group in terms of the base change $Y_{\overline{F}}$ of Y to an algebraic closure, it is perhaps more natural to call this a *geometric* fundamental group of Y.

As in the preceding example, this can be reinterpreted as a Malčev completion of another group. Recall that the *profinite étale fundamental group* $\widehat{\Pi} = \pi_1^{\text{ét}}(Y_{\overline{F}}; y)$ [Sza09, Definition 5.4.1] is a profinite group such that the category $\text{Fét}_{Y_{\overline{F}}}$ is canonically equivalent to the category of finite sets endowed with a continuous action of $\widehat{\Pi}$, so that the fibre functor $\text{Fét}_{Y_{\overline{F}}} \to \mathbf{Set}_{\text{fin}}$ determined by y on the category of finite étale covers of $Y_{\overline{F}}$ corresponds to the forgetful functor $\widehat{\Pi}\text{–}\mathbf{Set}_{\text{fin,cts}} \to \mathbf{Set}_{\text{fin}}$ [Sza09, Theorem 5.4.2]. It follows immediately from this description that the category of $\mathbb{Z}/\ell^r\mathbb{Z}$-local systems (resp. $\mathbb{Z}_\ell$-local systems, resp. $\mathbb{Q}_\ell$-local systems) is canonically equivalent to the category of continuous representations of $\widehat{\Pi}$ on finite rank free $\mathbb{Z}/\ell^r\mathbb{Z}$-modules (resp. finite rank free $\mathbb{Z}_\ell$-modules, resp. finite dimensional $\mathbb{Q}_\ell$-vector spaces). Hence $\text{Uni}_{\mathbb{Q}_\ell}^{\text{ét}}$ is canonically isomorphic to the category of continuous representations of $\widehat{\Pi}$, and hence to the category of representations of the continuous Malčev completion $\widehat{\Pi}_{\mathbb{Q}_\ell}$ of $\widehat{\Pi}$. Thus the $\mathbb{Q}_\ell$-pro-unipotent étale fundamental group is canonically isomorphic to the continuous Malčev completion of the profinite étale fundamental group $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)$.

In particular, since $\pi_1^{\text{ét}}(Y_{\overline{F}}; y)$ carries a canonical action of the Galois group $G_F = G_{\overline{F}|F}$, so too does the $\mathbb{Q}_\ell$-pro-unipotent étale fundamental group.

As with the previous example, a choice of a second rational point z defines a second fibre functor, and the Tannakian torsor of paths between these fibre functors is called the *pro-unipotent torsor of étale paths* from y to z. We also have a *profinite* torsor of étale paths from y to z [Sza09, Remark 5.5.3], which is a torsor under the profinite étale fundamental group, and it is easy to see that its continuous Malčev completion is canonically identified with the pro-unipotent torsor of étale paths arising from the Tannakian formalism.

Remark 2.4.5. If Y is a connected variety over a field F with an algebraic closure $\overline{F}$, $y \in Y(F)$ is a rational point, and $\sigma\colon \overline{F} \hookrightarrow \mathbb{C}$ is an embedding of fields, then there is a *comparison isomorphism* relating the pro-unipotent étale fundamental group of Y and the pro-unipotent Betti fundamental group of $Y_{\mathbb{C}}$. Indeed, in this setup there is a canonical isomorphism between the profinite étale fundamental group of Y and the profinite completion of the discrete Betti fundamental group of $Y_{\mathbb{C}}$ [Sza09, Theorem 5.7.4], and by propositions 2.3.6 and 2.3.5, there is an isomorphism $U^{\text{ét}} \cong U^{\text{B}}_{\mathbb{Q}_\ell}$ between their respective (continuous) $\mathbb{Q}_\ell$-Malčev completions. Note that this isomorphism is natural in the pointed variety (Y, y), but on the other hand depends rather heavily on the choice of σ.

Chapter 3

The main theorem (ℓ-adic version)

Let us begin by giving the proof of theorem 1.1.6. Although the details of the proof will present few difficulties for us, variations of the arguments outlined here will reappear in chapter 8 to prove the p-adic analogue of theorem 1.1.6.

For the remainder of this chapter let us fix an abelian variety A/K over our fixed p-adic field K and a prime number ℓ (soon to be $\neq p$). We will denote by $(L, \widetilde{0})$ a pair of a varying line bundle L/A and a basepoint $\widetilde{0} \in L^\times(K)$ in the complement $L^\times = L \setminus 0$ of the zero section lying over $0 \in A(K)$, and by $U/\mathbb{Q}_\ell$ the $\mathbb{Q}_\ell$-pro-unipotent étale fundamental group of $L^\times$ based at $\widetilde{0}$.

If we pick an embedding $K \hookrightarrow \mathbb{C}$, then $L^\times(\mathbb{C})$ is an oriented $\mathbb{C}^\times$-bundle over the torus $A(\mathbb{C})$. The homotopy exact sequence associated to a fibration ensures that

$$1 \to \pi_1^{\mathrm{top}}(\mathbb{C}^\times; 1) \xrightarrow{\mathsf{z}} \pi_1^{\mathrm{top}}(L^\times(\mathbb{C}); \widetilde{0}) \to \pi_1^{\mathrm{top}}(A(\mathbb{C}); 0) \to 1$$

is a central extension of an abelian group by an abelian group. By the comparison in remark 2.4.5, the corresponding sequence

$$1 \to \mathbb{Q}_\ell(1) \xrightarrow{\mathsf{z}} U \to V_\ell A \to 1 \tag{3.0.1}$$

on étale fundamental groups arising from the fibration sequence $(\mathbb{G}_m, 1) \hookrightarrow (L^\times, \widetilde{0}) \twoheadrightarrow (A, 0)$ can be identified (after a suitable choice of a embedding $\overline{K} \to \mathbb{C}$) with the $\mathbb{Q}_\ell$-Malčev completion of the first sequence. Hence, by corollary 2.3.11, sequence 3.0.1 is also a central extension and, in particular, U is a 2-step unipotent group.

Now, since all the unipotent groups appearing in sequence 3.0.1 are isomorphic to affine space as varieties, it follows that $\mathbb{Q}_\ell(1) \subseteq U(\mathbb{Q}_\ell)$ has the subspace topology and $U(\mathbb{Q}_\ell) \twoheadrightarrow V_\ell A$ has a continuous splitting, so that we obtain (part of) a long exact sequence

$$\mathrm{H}^0(G_K, V_\ell A) \to \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\curvearrowright} \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1(G_K, V_\ell A) \tag{3.0.2}$$

in continuous Galois cohomology (see chapter 4). The abelian terms in this sequence can be understood using well-known results from local Galois cohomology.

Lemma 3.0.1. *Let V be a representation of G_K on a $\mathbb{Q}_\ell$-vector space. Then:*

- *there is a natural isomorphism $\mathrm{H}^2(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$;*
- *(local duality) the pairing*

$$\mathrm{H}^i(G_K, V) \otimes_{\mathbb{Q}_p} \mathrm{H}^{2-i}(G_K, V^*(1)) \to \mathrm{H}^2(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

  is perfect for all i (so $\mathrm{H}^i(G_K, V) = 0$ for $i > 2$); and

- *(Euler–Poincaré characteristic formula) we have*

$$\sum_i (-1)^i \dim_{\mathbb{Q}_p} \mathrm{H}^i(G_K, V) = \begin{cases} 0 & \text{if } \ell \neq p, \\ -[K : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V & \text{if } \ell = p. \end{cases}$$

Proof. This is stated in [Bel09, Proposition 2.2] (with an incorrect sign), but the author unfortunately knows of no proof in the published literature that explicitly deals with the $\mathbb{Q}_\ell$-linear case, rather than the case of finite coefficients. We will sketch how to recover the Euler–Poincaré characteristic formula from the finite case, leaving the easier case of local duality to the reader (this part can alternatively be found in [Rub00, Theorem 4.1]).

To do this, pick a G_K-stable $\mathbb{Z}_\ell$-sublattice $\Lambda \leq V$ of full rank, so that $\mathrm{H}^i(G_K, \Lambda)$ is a finitely generated $\mathbb{Z}_\ell$-module with $\mathrm{H}^i(G_K, V) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathrm{H}^i(G_K, \Lambda)$ for all i [NSW15, Corollary 2.7.10 & Proposition 2.7.11]. The long exact sequence on cohomology associated to the short exact sequence

$$1 \to \Lambda \xrightarrow{\ell} \Lambda \to \Lambda/\ell \to 1$$

shows that $\mathrm{H}^i(G_K, \Lambda/\ell)$ is an extension of $\mathrm{H}^{i+1}(G_K, \Lambda)[\ell]$ by $\mathrm{H}^i(G_K, \Lambda)/\ell$, and hence

$$\#\mathrm{H}^i(G_K, \Lambda/\ell) = \#\mathrm{H}^i(G_K, \Lambda)[\ell] \cdot \#\mathrm{H}^{i+1}(G_K, \Lambda)[\ell] \cdot \ell^{\mathrm{rk}(\mathrm{H}^i(G_K, \Lambda))}$$

for all i. Taking the alternating product of all these identities, we see that

$$\prod_i \#\mathrm{H}^i(G_K, \Lambda/\ell)^{(-1)^i} = \ell^{\sum_i (-1)^i \mathrm{rk}(\mathrm{H}^i(G_K, \Lambda))}.$$

By the Euler–Poincaré characteristic formula with finite coefficients [NSW15, Theorem 7.3.1], the left-hand side of this identity is equal to $p^{-[K:\mathbb{Q}_p] v_p(\#\Lambda/\ell)}$, i.e. equal to 1 if $\ell \neq p$ and equal to $p^{-[K:\mathbb{Q}_p]\mathrm{rk}(\Lambda)}$ if $\ell = p$, which gives the desired result. □

Returning to the exact sequence 3.0.2, the first term vanishes since $A(K)_{\mathrm{tors}}$ is finite. By local duality, we also know that $\mathrm{H}^2(G_K, V_\ell A) = \mathrm{H}^0(G_K, V_\ell A^*(1))^* = 1$ by the same argument for the dual abelian variety $A^\vee$, and hence when $\ell \neq p$, the final term $\mathrm{H}^1(G_K, V_\ell A)$ of sequence 3.0.2 also vanishes by the Euler–Poincaré characteristic formula. Hence, in this case, $\mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell))$ is bijective, so that we have a well-defined composite map

$$\lambda_L \colon L^\times(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) \xleftarrow{\sim} \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

where the final isomorphism arises from Kummer theory.

It remains to show that the λ_L are the Néron log-metrics on our varying line bundle L (and in particular that they are $\mathbb{Q}$-valued). In order to do this, we just need to verify that these functions satisfy a certain list of properties uniquely characterising the Néron log-metric, analogous to those characterising Néron functions [Lan83, Theorem 11.1.1]. Since the functions we are interested in are not taking values in $\mathbb{R}$, the exact conditions are a little delicate.

Lemma 3.0.2. *Let A/K be an abelian variety over our fixed p-adic field K. Then for any pair $(L, \widetilde{0})$ consisting of a line bundle L/A and a basepoint $\widetilde{0} \in L^\times(K)$ lying over $0 \in A(K)$, the Néron log-metric $\lambda_L \colon L^\times(\overline{K}) \to \mathbb{R}$ (see section 1.1.4.1) is $\mathbb{Q}$-valued, and the restricted functions*

$$\lambda_L \colon L^\times(K) \to \mathbb{Q}$$

are uniquely characterised by the following properties:

0. *λ_L only depends on $(L, \widetilde{0})$ up to isomorphism;*
1. *λ_L is locally constant (for the p-adic topology on $L^\times(K)$);*
2. *$\lambda_{L_1 \otimes L_2}(P_1 \otimes P_2) = \lambda_{L_1}(P_1) + \lambda_{L_2}(P_2)$;*
3. *$\lambda_{[2]^* L}(P) = \lambda_L([2]P)$, where by abuse of notation we also denote by $[2]$ the topmost map in the pullback square*

$$\begin{array}{ccc} [2]^* L^\times & \longrightarrow & L^\times \\ \downarrow & & \downarrow \\ A & \xrightarrow{[2]} & A; \end{array}$$

4. *when $L = A \times_K \mathbb{A}^1$ is the trivial line bundle, λ_L is the composite*

$$L^\times(K) = A(K) \times K^\times \to K^\times \xrightarrow{v} \mathbb{Q}; \text{ and}$$

5. $\lambda_L(\widetilde{0}) = 0$.

More strongly, a family $(\lambda_L)_{(L,\widetilde{0})}$ of functions valued in a $\mathbb{Q}$-algebra F satisfying the above conditions with $\mathbb{Q}$ replaced by F is automatically $\mathbb{Q}$-valued, and agrees with the restriction of the Néron log-metric to $L^\times(K)$.

Proof. The fact that the Néron log-metric is $\mathbb{Q}$-valued and locally constant follows from the corresponding result for Néron functions [Lan83, Theorems 11.5.1 & 11.5.2]. The properties of the Néron log-metric then follow from the corresponding properties of Néron functions [Lan83, Theorem 11.1.1].

Conversely, if $(\lambda'_L)_{(L,\widetilde{0})}$ is another such family valued in a $\mathbb{Q}$-algebra F, then by conditions 2 and 4 we have $\lambda'_L(\alpha z) = \lambda'_L(z) + v(\alpha)$ for all $\alpha \in K^\times$ and $z \in L^\times(K)$, so that the function $\lambda_L - \lambda'_L$ is constant on the fibres of $L^\times(K) \twoheadrightarrow A(K)$ and hence factors through a unique locally constant function $\delta_L \colon A(K) \to F$. The assignment $(L, \widetilde{0}) \mapsto \delta_L$ depends only on L, and is a group homomorphism $\mathrm{Pic}(A)(K) \to \mathrm{Map}(A(K), F)$ by condition 2.

Now if L is symmetric (so $[2]^*L \simeq L^{\otimes 4}$), then $\delta_L([2]P) = 4\delta_L(P)$ for all $P \in A(K)$ by condition 3. Thus the image of δ_L is closed under multiplication by 4. Since it is also finite, this is only possible if $\delta_L = 0$. A similar argument shows that $\delta_L = 0$ also when L is antisymmetric, and hence in general by writing $L^{\otimes 2}$ for a general L as a product of a symmetric and an antisymmetric part. This shows that $\lambda'_L = \lambda_L$ as desired. □

Let us now verify that the conditions of lemma 3.0.2 are satisfied for the functions $\lambda_L \colon L^\times(K) \to \mathbb{Q}_\ell$ constructed above—for instance, condition 5 is immediate, since λ_L is defined as a composite of basepoint-preserving maps. In fact, with one exception (condition 1), the verifications will be entirely formal.

For condition 4, we note that the inclusion $\mathbb{G}_m \hookrightarrow L^\times = \mathbb{G}_m \times_K A$ has a section given by projection, and hence the inverse to the isomorphism $\mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell))$ is the map induced by projection. Naturality of the non-abelian Kummer map then gives a commuting diagram

$$\begin{array}{ccccc} K^\times \times A(K) & \xrightarrow{\text{proj}} & K^\times & \xrightarrow{v} & \mathbb{Q}_\ell \\ \downarrow & & \downarrow & & \| \\ \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) & \xrightarrow{\sim} & \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) & \xrightarrow{\sim} & \mathbb{Q}_\ell \end{array}$$

which shows that $\lambda_L(\alpha, P) = v(\alpha)$ in this case, as desired.

Condition 3 also follows from naturality of the non-abelian Kummer map, the proof being entirely contained in the commuting diagram

$$\begin{array}{ccccccc}
[2]^*L^\times(K) & \longrightarrow & \mathrm{H}^1(G_K, U'(\mathbb{Q}_\ell)) & \xleftarrow{\sim} & \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) & \xrightarrow{\sim} & \mathbb{Q}_\ell \\
\downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]_*} & & \| & & \| \\
L^\times(K) & \longrightarrow & \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) & \xleftarrow{\sim} & \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) & \xrightarrow{\sim} & \mathbb{Q}_\ell,
\end{array}$$

where $U'/\mathbb{Q}_\ell$ is the unipotent fundamental group of $[2]^*L^\times$.

Of the formal conditions, the only mildly difficult verification is that of condition 2, for which we consider the pointed $\mathbb{G}_m \times_K \mathbb{G}_m$-torsor $L_1^\times \times_A L_2^\times$ on A. Writing $U_{1,2}$ for the fundamental group of $L_1^\times \times_A L_2^\times$ based at $(\widetilde{0}_1, \widetilde{0}_2)$, the same argument as earlier establishes that the inclusion of the fibre over 0 induces an isomorphism

$$\mathrm{H}^1(G_K, \mathbb{Q}_\ell(1) \times \mathbb{Q}_\ell(1))) \xrightarrow{\sim} \mathrm{H}^1(G_K, U_{1,2}(\mathbb{Q}_\ell)),$$

and hence we have a well-defined map

$$\lambda_{L_1,L_2} \colon (L_1^\times \times_A L_2^\times)(K) \to \mathrm{H}^1(G_K, U_{1,2}(\mathbb{Q}_\ell)) \xleftarrow{\sim} \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1) \times \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell.$$

Now the tensor-multiplication map $\otimes \colon L_1^\times \times_A L_2^\times \to (L_1 \otimes L_2)^\times$ restricts to the codiagonal (multiplication) map $\mathbb{G}_m \times_K \mathbb{G}_m \to \mathbb{G}_m$ on the fibres over $0 \in A(K)$, and hence we have a commuting diagram

$$\begin{array}{ccccccc}
(L_1^\times \times_A L_2^\times)(K) & \longrightarrow & \mathrm{H}^1(G_K, U_{1,2}(\mathbb{Q}_\ell)) & \xleftarrow{\sim} & \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1) \times \mathbb{Q}_\ell(1)) & \xrightarrow{\sim} & \mathbb{Q}_\ell^{\oplus 2} \\
\downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \otimes_*} & & \downarrow{\scriptstyle \mathrm{codiag}_*} & & \downarrow{\scriptstyle \mathrm{codiag}} \\
(L_1 \otimes L_2)^\times(K) & \longrightarrow & \mathrm{H}^1(G_K, U_{1\otimes 2}(\mathbb{Q}_\ell)) & \xleftarrow{\sim} & \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) & \xrightarrow{\sim} & \mathbb{Q}_\ell
\end{array}$$

where $U_{1\otimes 2}/\mathbb{Q}_\ell$ is the unipotent fundamental group of $(L_1 \otimes L_2)^\times$ based at $\widetilde{0}_1 \otimes \widetilde{0}_2$. Reading the composites around the outside, we then see that $\mathrm{codiag} \circ \lambda_{L_1,L_2} = \lambda_{L_1 \otimes L_2} \circ \otimes$.

Repeating the same argument with the projections $\mathrm{proj}_i \colon L_1^\times \times_A L_2^\times \to L_i^\times$ for $i = 1, 2$, one sees similarly that $\mathrm{proj}_i \circ \lambda_{L_1,L_2} = \lambda_{L_i} \circ \mathrm{proj}_i$. Since the codiagonal map is the pointwise sum of the projections, we thus obtain that

$$\lambda_{L_1 \otimes L_2} \circ \otimes = \lambda_{L_1} \circ \mathrm{proj}_1 + \lambda_{L_2} \circ \mathrm{proj}_2$$

as functions on $(L_1^\times \times_A L_2^\times)(K)$, which is exactly the desired equality.

3.1 The $\mathbb{Q}_\ell$-pro-unipotent Kummer map

To conclude the proof of theorem 1.1.6, it just remains to verify condition 1, i.e. that the λ_L are locally constant. In fact, we will prove the stronger assertion in theorem

1.3.1 that the $\mathbb{Q}_\ell$-pro-unipotent Kummer map for any smooth variety Y/K is locally constant.

Informally, this should come down to a question of analytic geometry, for example in the sense of Berkovich: the K-points of the analytification Y^{Berk} (equal to $Y(K)$ by [Ber90, Theorem 3.4.1(i)], which also defines the analytification) can be covered by Berkovich analytic n-discs, which one expects to have trivial $\mathbb{Q}_\ell$-pro-unipotent étale fundamental group, and hence any two K-rational points in the same disc should have a G_K-invariant étale path between them. However, at the present time the author only knows of a proof of the triviality of the $\mathbb{Q}_\ell$-pro-unipotent étale fundamental group of the 1-dimensional disc, so we have to proceed slightly differently, using the following weaker lemma, which informally says that any two K-rational points of the Berkovich n-disc are connected by a G_K-invariant prime-to-p étale path.

Lemma 3.1.1. *Let $\mathbb{D}^n_K$ denote the n-dimensional Berkovich closed unit disc over K, and let $z \in \mathcal{O}^n_K = \mathbb{D}^n_K(K)$ be a K-rational point. If $V \twoheadrightarrow \mathbb{D}^n_{\mathbb{C}_K}$ is a finite étale Galois covering*[1] *of degree prime to p, we denote by z^*V the fibre over z of the map $V(\mathbb{C}_K) \to \mathbb{D}^n_K(\mathbb{C}_K)$—this is a finite set.*

Then for all such coverings $V \to \mathbb{D}^n_{\mathbb{C}_K}$, there is a bijection $\gamma_V \colon 0^*V \xrightarrow{\sim} z^*V$ which is natural in V and compatible with the action of G_K (the group of isometric automorphisms of $\mathbb{C}_K/K$), in that, for any $\sigma \in G_K$, the square

$$\begin{array}{ccc} 0^*V & \xrightarrow[\sim]{\gamma_V} & z^*V \\ \downarrow\wr & & \downarrow\wr \\ 0^*\sigma^*V & \xrightarrow[\sim]{\gamma_{\sigma^*V}} & z^*\sigma^*V \end{array}$$

commutes. Here σ^*V is the base change of V, pulled back along the isomorphism $\mathbb{D}^n_{\mathbb{C}_K} \xrightarrow{\sim} \sigma^*\mathbb{D}^n_{\mathbb{C}_K}$ of Berkovich spaces over $\mathbb{C}_K$ (using that it is a base change from K), and the vertical bijections are those arising from the evident bijections $V(\mathbb{C}_K) = \sigma^*V(\mathbb{C}_K)$ (using that $z \in \mathbb{D}^n_{\mathbb{C}_K}(\mathbb{C}_K)$ is G_K-fixed).

Proof. We may assume that $n \neq 0$. When $n = 1$, [Ber93, Theorem 6.3.2] shows that every finite étale Galois covering is trivial, i.e. is a disjoint union of copies of $\mathbb{D}^1_{\mathbb{C}_K}$. Since $\mathbb{D}^1_{\mathbb{C}_K}$ is connected, this occurs in an essentially unique way, and so there is a canonical identification of the fibres at 0 and at z. This is easily seen to be functorial in the cover and G_K-equivariant (by unicity of the trivialisation), so we are done when $n = 1$.

[1] That is, a finite étale morphism $V \to \mathbb{D}^n_{\mathbb{C}_K}$ which is a principal homogenous space under a finite group.

For $n > 1$ we will proceed differently. By considering a 1-dimensional linear subspace of K^n containing z, we find an embedding $\iota\colon \mathbb{D}^1_{\mathbb{C}_K} \hookrightarrow \mathbb{D}^n_{\mathbb{C}_K}$ of Berkovich spaces respecting the G_K-action on the categories of covers, taking 0 to 0 and some $w \in \mathcal{O}_K \subseteq \mathbb{D}^1_{\mathbb{C}_K}$ to z. Thus we have a bijection $z^*V \cong w^*\iota^*V$ for each prime-to-p Galois cover V of $\mathbb{D}^n_{\mathbb{C}_K}$, natural in V and compatible with the G_K-action, and similarly for 0. Hence we may define γ_V to be the composite

$$0^*V \xrightarrow{\sim} 0^*\iota^*V \xrightarrow{\gamma^1_{\iota^*V}} w^*\iota^*V \xrightarrow{\sim} z^*V,$$

where γ^1 is a G_K-equivariant natural isomorphism between the fibres at 0 and w of prime-to-p Galois covers of $\mathbb{D}^1_{\mathbb{C}_K}$. □

Corollary 3.1.2. *Let Y/K be a smooth connected variety, $y \in Y(K)$ a basepoint, and $U/\mathbb{Q}_\ell$ the pro-unipotent fundamental group of Y based at y. Then the non-abelian Kummer map*

$$Y(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell))$$

is locally constant (for the p-adic topology).

Proof. Recall that the non-abelian Kummer map for the $\mathbb{Q}_\ell$-pro-unipotent fundamental group factors through the non-abelian Kummer map

$$Y(K) \to \mathrm{H}^1(G_K, \pi_1^{(p')}(Y_{\mathbb{C}_K}; y))$$

for the prime-to-p profinite étale fundamental group. We will in fact show that this map is locally constant.

To do this, we will first show that for any $z \in Y(K)$ sufficiently close to y, there is a bijection $\gamma_V\colon y^*V \to z^*V$ for every finite étale Galois covering $V \twoheadrightarrow Y$ of degree prime to p which is G_K-equivariant and natural in V. Indeed, since Y is smooth, there is a morphism $\mathbb{D}^n_K \to Y^{\mathrm{Berk}}$ of Berkovich spaces into the analytification Y^{Berk} of Y, taking 0 to y and such that the image of $\mathbb{D}^n_K(K)$ in $Y^{\mathrm{Berk}}(K) = Y(K)$ is an open neighbourhood of y. We will prove the existence of the claimed G_K-equivariant natural transformation γ whenever $z = f(w)$ is the image of a K-rational point w of $\mathbb{D}^n_K$.

To construct γ, note that if $V \to Y_{\mathbb{C}_K}$ is a finite étale morphism, then its analytification $V^{\mathrm{Berk}} \to Y^{\mathrm{Berk}}_{\mathbb{C}_K}$ is also finite étale by [Ber90, Propositions 3.4.6(3) & 3.4.7(4)]. Moreover, the proof of [Ber90, Proposition 3.4.6] shows that the fibres of $V \to Y_{\mathbb{C}_K}$ and $V^{\mathrm{Berk}} \to Y^{\mathrm{Berk}}_{\mathbb{C}_K}$ over y (resp. z) are canonically identified with each other, so that

if V is a prime-to-p Galois cover, so too is V^{Berk}. Thus we may define γ_V to be the composite

$$y^*V = y^*V^{\mathrm{Berk}} \cong 0^*f^*V^{\mathrm{Berk}} \underset{\sim}{\overset{\gamma^n_{f^*V}}{\to}} w^*f^*V^{\mathrm{Berk}} \cong z^*V^{\mathrm{Berk}} = z^*V,$$

where γ^n is the G_K-equivariant natural bijection from lemma 3.1.1. This bijection is evidently natural in V (by functoriality of analytification and naturality of γ^n) and G_K-equivariant, as desired.

Now to show local constancy of the non-abelian Kummer map, we simply recall that the image of a point z under this map is the class of the set of natural isomorphisms from the fibre functor y^* to z^* (as functors from the category of finite étale Galois covers of $Y_{\mathbb{C}_K}$ of degree prime to p to the category of finite sets), viewed as a $\pi_1^{\text{ét}}(Y_{\mathbb{C}_K}; y)$-torsor with compatible G_K-action in the natural way. If another point z' is sufficiently close to z, then we have shown that the functors z'^* and z^* are G_K-equivariantly isomorphic, and hence their images under the non-abelian Kummer map agree, as desired. □

Chapter 4

Representations on pro-unipotent groups

In this chapter, we will explain and explore the basic theory of *(continuous) representations of G_K (the absolute Galois group of our fixed p-adic field K, see notation 1.4.1) on finitely generated pro-unipotent groups $U/\mathbb{Q}_\ell$*, which will form a mildly non-abelian generalisation of the notion of (continuous, finite-dimensional) representations of G_K on $\mathbb{Q}_\ell$-vector spaces. For the time being, ℓ here is any prime, possibly equal to p. Just as ℓ-adic linear Galois representations arise from geometry as the étale cohomology of K-varieties with coefficients in $\mathbb{Q}_\ell$, the principal source of Galois representations on pro-unipotent groups will be the pro-unipotent fundamental groups of K-varieties at K-rational basepoints.

However, just as the theory of linear Galois representations can be developed independently of their desired geometric source, we will now investigate these representations in the abstract, only making the link back to geometry in chapter 7. In this chapter in particular, we will be interested in extending the usual definitions of various classes of linear representations to the unipotent context. In extending these definitions, we will typically give three equivalent characterisations of the property of interest: one in terms of the group scheme U; one in terms of the Lie algebra $\mathrm{Lie}(U)$; and one in terms of the Hopf algebra $\mathcal{O}(U)$. The archetypal example of this is the following.

Definition-Lemma 4.0.1. Let $U/\mathbb{Q}_\ell$ be a finitely generated pro-unipotent group endowed with an action of a topological group G by algebraic group automorphisms. Then the J-filtration (see definition 2.1.12) on $\mathcal{O}(U)$ is G-stable.

We shall say that U is a *(continuous)*[1] *representation of G on a finitely generated*

[1]We will omit the word "continuous" throughout this dissertation, following from the standard practice in discussing linear ℓ-adic representations.

pro-unipotent) group just when the following equivalent conditions hold:

1. the action of G on $U(\mathbb{Q}_\ell)$ is continuous;
2. the action of G on $\mathrm{Lie}(U)$ is continuous;
3. the action of G on each $J_n\mathcal{O}(U)$ is continuous.

In this, the topology on $U(\mathbb{Q}_\ell)$ is the one discussed induced by the topology on $\mathbb{Q}_\ell$, as discussed in remark 2.3.2.

Proof of equivalence. G-invariance of the J-filtration follows from lemma 2.1.14.

The logarithm isomorphism $U \xrightarrow{\sim} \mathrm{Lie}(U)$ (of $\mathbb{Q}_\ell$-schemes) ensures that $U(\mathbb{Q}_\ell)$ is homeomorphic to $\mathrm{Lie}(U)$ compatibly with the G-action, which gives the equivalence (1)$\Leftrightarrow$(2).

For the equivalence (2)$\Leftrightarrow$(3), we note that each condition holds of U iff it holds for all of its finite-dimensional G-equivariant quotients, so that it suffices to prove the result when U is unipotent. In this case, the third condition is equivalent to the action on each $(J_n\mathcal{O}(U))^* \cong \mathcal{U}/J^{n+1}$ being continuous, where $\mathcal{U}$ is the universal enveloping algebra of $\mathrm{Lie}(U)$ and J is its augmentation ideal. Since $\mathcal{U}$ is generated as an algebra by $\mathrm{Lie}(U)$, continuity of the action on $\mathrm{Lie}(U)$ clearly implies continuity of the action on each $J_n\mathcal{O}(U)$. Conversely, $\mathrm{Lie}(U)$ injects into $(J_n\mathcal{O}(U))^*$ for $n >> 0$, and hence continuity of the action on $J_n\mathcal{O}(U)$ implies continuity on $\mathrm{Lie}(U)$. □

Remark 4.0.2. It follows immediately from the definition that any G-stable finitely generated subgroup or quotient of a representation of a topological group G on a finitely generated pro-unipotent group is also a representation, i.e. is continuous in the above sense. It also follows that any representation on a finitely generated pro-unipotent group is an inverse limit of representations on unipotent groups, so we can often prove results about general representations by reducing to the finite-dimensional case.

Of course, pro-unipotent fundamental groups are an example of such representations. This is generally well-known, but let us nonetheless briefly recall how the proof goes.

Proposition 4.0.3. *Let Y be a smooth connected variety over a characteristic 0 field F, and $y \in Y(F)$. Then the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of Y based at y is a representation of the absolute Galois group G_F on a finitely generated pro-unipotent group in the sense of definition-lemma 4.0.1.*

Proof. There are two claims here: that the fundamental group is finitely generated; and that the Galois action is continuous. This latter claim follows from [Del89, Section 13.4] by taking an inverse limit.

For the former claim, note that by the Lefschetz principle, it suffices to prove the result for $F = \mathbb{C}$, where by the comparison isomorphism it suffices to prove that the usual topological fundamental group is finitely generated. Moreover, by the Seifert–van Kampen theorem, it suffices to prove this for quasi-affine varieties. However, by [Hir75, Semi-algebraic triangulation theorem] (see also [Hir75, Remark 1.10]), any algebraic subset of $\mathbb{C}^n = \mathbb{R}^{2n}$ is homeomorphic to the complement in a finite simplicial complex of a subcomplex, and hence has finitely generated fundamental group. □

4.0.1 Serre twisting

If $U/\mathbb{Q}_\ell$ is a representation of a topological group G on a finitely generated pro-unipotent group, then we will very often be interested in the non-abelian continuous Galois cohomology set

$$\mathrm{H}^1(G, U(\mathbb{Q}_\ell)),$$

defined to be the quotient of the set of continuous cocycles $\alpha\colon G \to U(\mathbb{Q}_\ell)$ (maps satisfying $\alpha(gh) = \alpha(g)g(\alpha(h))$) modulo the right action of $U(\mathbb{Q}_\ell)$ given by

$$(\alpha \cdot u)(g) = u^{-1}\alpha(g)g(u).$$

When U is non-abelian, this in general does not have a group structure, merely a distinguished point given by the class of the trivial cocycle. In particular, we will several times encounter the problem that the fibres of a map of pointed sets are not in general determined simply by its kernel—for instance a map with trivial kernel can nonetheless fail to be injective.

To circumvent these difficulties, we will use *Serre twists* [Ser97, Section I.5.3], whose definition we now recall. Given a continuous cocycle $\alpha\colon G \to U(\mathbb{Q}_\ell)$, the *Serre twist* ${}_\alpha U/\mathbb{Q}_\ell$ of U by α is defined to be the representation of G on the same underlying pro-unipotent group, whose action on Λ-points for some $\mathbb{Q}_\ell$-algebra is given by

$$g\colon u \mapsto \alpha(g)g(u)\alpha(g)^{-1}.$$

This action is clearly algebraic (i.e. natural in Λ) and continuous on $\mathbb{Q}_\ell$-points, since α was chosen continuous.

The main basic result regarding Serre twists is the following, which allows one in practice to prove results about all fibres of maps out of Galois cohomology by just considering kernels.

Proposition 4.0.4. *[Ser97, Proposition I.35 bis][2] Let $U/\mathbb{Q}_\ell$ be a representation of a topological group G on a finitely generated pro-unipotent group, and $\alpha\colon G \to U(\mathbb{Q}_\ell)$ a continuous cocycle. Then right-multiplication by α induces a bijection*

$$\mathrm{H}^1(G, {}_\alpha U(\mathbb{Q}_\ell)) \xrightarrow{\sim} \mathrm{H}^1(G, U(\mathbb{Q}_\ell))$$

taking the distinguished point of $\mathrm{H}^1(G, {}_\alpha U(\mathbb{Q}_\ell))$ to the class $[\alpha] \in \mathrm{H}^1(G, U(\mathbb{Q}_\ell))$.

As an example, let us now use the notion of Serre twists to prove a lemma allowing one to reduce study of the Galois cohomology of representations on finitely generated pro-unipotent groups to the finite-dimensional case.

Lemma 4.0.5. *Let $U/\mathbb{Q}_\ell$ be a representation of a topological group G on a finitely generated pro-unipotent group, and write $U = \varprojlim U_n$ as an inverse limit of finite-dimensional G-equivariant quotients. Then the natural map*

$$\mathrm{H}^1(G, U(\mathbb{Q}_\ell)) \to \varprojlim \mathrm{H}^1(G, U_n(\mathbb{Q}_\ell))$$

is bijective.

Proof. We may assume the indexing set for the inverse system is $\mathbb{N}$.

Let us begin by showing surjectivity. To do this, choose an element $([\beta_n])_{n\in\mathbb{N}} \in \varprojlim \mathrm{H}^1(G, U_n(\mathbb{Q}_\ell))$, and write S_n for the set of continuous cocycles $\alpha_n\colon G \to U_n(\mathbb{Q}_\ell)$ whose class is $[\beta_n]$. If $m \geq n$ then $S_m \twoheadrightarrow S_n$ is surjective, for, if $\alpha_n \in S_n$ is any element, we may pick some $\alpha'_m \in S_m$, so that the image of α'_m in S_n differs from α_n by the right action of some $u_n \in U_n(\mathbb{Q}_\ell)$. Lifting u_n to some $u_m \in U_m(\mathbb{Q}_\ell)$ and replacing α'_m with $\alpha_m = \alpha'_m \cdot u_m$, we obtain a lift of α_n to S_m. In particular, $\varprojlim S_n$ is non-empty. But any element $\alpha \in \varprojlim S_n$ can be thought of as a continuous cocycle $\alpha\colon G \to U(\mathbb{Q}_\ell)$, and the class $[\alpha] \in \mathrm{H}^1(G, U(\mathbb{Q}_\ell))$ maps by construction to $[\beta_n]$ in each $\mathrm{H}^1(G, U_n(\mathbb{Q}_\ell))$, proving surjectivity.

Let us show next that $\mathrm{H}^1(G, U(\mathbb{Q}_\ell)) \to \varprojlim \mathrm{H}^1(G, U_n(\mathbb{Q}_\ell))$ has trivial kernel. Let $\alpha\colon G \to U(\mathbb{Q}_\ell)$ be a cocycle representing an element of the kernel, and for each n let $T_n \subseteq U_n(\mathbb{Q}_\ell)$ denote the set of elements whose coboundary is the image of α in $U_n(\mathbb{Q}_\ell)$. Each T_n is non-empty by assumption, and is a left torsor under $U_n(\mathbb{Q}_\ell)^G$ for the multiplication action. Since $U_n(\mathbb{Q}_\ell)^G$ is the $\mathbb{Q}_\ell$-points of a unipotent subgroup of U_n, it follows that the images of $U_m(\mathbb{Q}_\ell)^G \to U_n(\mathbb{Q}_\ell)^G$ stabilise for $m >> n$, i.e. the system $(U_n(\mathbb{Q}_\ell)^G)_{n\in\mathbb{N}}$ satisfies the Mittag–Leffler condition. Hence the system $(T_n)_{n\in\mathbb{N}}$

[2]Strictly, Serre only considers the case when the action of G on $U(\mathbb{Q}_\ell)$ is continuous for the *discrete* topology and the cohomology considered is non-continuous cohomology. Nonetheless, the same argument works.

of non-empty sets also satisfies the Mittag–Leffler condition, and in particular $\varprojlim T_n$ is non-empty. But by construction the coboundary of any element of $\varprojlim T_n \subseteq U(\mathbb{Q}_\ell)$ is α, so that α is a trivial cocycle as desired.

It remains to show that $\mathrm{H}^1(G, U(\mathbb{Q}_\ell)) \to \varprojlim \mathrm{H}^1(G, U_n(\mathbb{Q}_\ell))$ is actually injective. If we pick any cocycle $\alpha\colon G \to U(\mathbb{Q}_\ell)$, and write α_n for its images in each $U_n(\mathbb{Q}_\ell)$, then Serre twisting provides us a commuting square

$$\begin{array}{ccc} \mathrm{H}^1(G, {}_\alpha U(\mathbb{Q}_\ell)) & \longrightarrow & \varprojlim \mathrm{H}^1(G, {}_{\alpha_n} U_n(\mathbb{Q}_\ell)) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{H}^1(G, U(\mathbb{Q}_\ell)) & \longrightarrow & \varprojlim \mathrm{H}^1(G, U_n(\mathbb{Q}_\ell)). \end{array}$$

We've shown that the top map has trivial kernel, so that $[\alpha]$ is the unique point in its fibre under the map $\mathrm{H}^1(G, U(\mathbb{Q}_\ell)) \to \varprojlim \mathrm{H}^1(G, U_n(\mathbb{Q}_\ell))$. Since α was arbitrary, this shows that the map is injective, as desired. □

4.1 ℓ-adic Galois representations

Before we embark on our analysis of representations of G_K on finitely generated pro-unipotent groups over $\mathbb{Q}_p$, let us first examine the much simpler case of groups over $\mathbb{Q}_\ell$, where now ℓ is some prime different from p. All the results proven here will have their p-adic analogues in the following sections, and should serve to illustrate the methods we will employ in developing the basic theory.

The following definition describes how to generalise the basic notions from the theory of $\mathbb{Q}_\ell$-linear representations to the finitely generated pro-unipotent setting, as usual phrased in three equivalent ways.

Definition-Lemma 4.1.1. Let $U/\mathbb{Q}_\ell$ be a representation of G_K on a finitely generated pro-unipotent group, where $\ell \neq p$. We shall say that U is *semistable* (resp. *unramified*) just when the following equivalent conditions hold:

1. U has a (separated) G_K-stable filtration $U = U_1 \trianglerighteq U_2 \trianglerighteq \dots$ by finite-index subgroups with each U_n/U_{n+1} abelian and the action of inertia on U_n/U_{n+1} being unipotent (resp. the action of inertia on U is trivial);

2. $\mathrm{Lie}(U)$ is pro-semistable (resp. unramified), i.e. all of its finite-dimensional G_K-equivariant quotients are semistable (resp. unramified) representations;

3. each $J_n\mathcal{O}(U)$ is semistable (resp. unramified).

Proof of equivalence. The equivalence of the different definitions of unramifiedness is clear, so we only deal with the case of semistability.

For the equivalence (1)⇔(2), consider any (separated) G_K-stable filtration of U by finite-index subgroups with U_n/U_{n+1} abelian, such as for instance the descending central series filtration. The action on $\mathrm{Lie}(U)$ is pro-semistable iff the action on each $\mathrm{Lie}(U)/\mathrm{Lie}(U_n)$ is semistable, which occurs iff the action on each $\mathrm{Lie}(U_n)/\mathrm{Lie}(U_{n+1}) \cong U_n/U_{n+1}$ is semistable, as desired.

For the equivalence (2)⇔(3), we note that the third point is equivalent to ind-semistability of $\mathcal{O}(U)$, and hence each condition holds of U iff it holds for all of its finite-dimensional G_K-equivariant quotients, so that it suffices to prove the result when U is unipotent. Again, semistability of $J_n\mathcal{O}(U)$ is equivalent to semistability of $\mathcal{U}/J^{n+1}$, where $\mathcal{U}$ is the universal enveloping algebra of $\mathrm{Lie}(U)$. Since $\mathrm{Lie}(U)$ injects into $\mathcal{U}/J^{n+1}$ for $n >> 0$ and $\mathcal{U}$ is a quotient of the tensor algebra $T^{\otimes}\mathrm{Lie}(U)$, the equivalence (2)⇔(3) follows. □

Remark 4.1.2. It follows directly from the definitions that G_K-stable finitely generated pro-unipotent subgroups and quotients of semistable (resp. unramified) representations are again semistable (resp. unramified), and that a representation of G_K on a finitely generated pro-unipotent group is semistable (resp. unramified) iff all of its finite-dimensional quotients are. Moreover, the class of semistable representations is closed under central extensions, though the same is untrue for unramified representations.

Using this definition, we can prove the somewhat surprising result that semistability depends only on the abelianisation of a finitely generated pro-unipotent group, deducing as a corollary that Grothendieck's ℓ-adic monodromy theorem holds in this setting. Note that one can deduce this immediately for unipotent groups by considering the Lie algebra, but the infinite-dimensional version genuinely does require reduction to the abelianisation.

Lemma 4.1.3. *Let $U/\mathbb{Q}_\ell$ be a representation of G_K on a finitely generated pro-unipotent group, where $\ell \neq p$. Then U is semistable iff U^{ab} is.*

Proof. The "only if" direction is trivial, so let us concentrate on the converse implication. Let $U = U_1 \trianglerighteq U_2 \trianglerighteq \dots$ denote the descending central series, so that U_n are finite-index subgroups with U_n/U_{n+1} abelian. U_1/U_2 is then the abelianisation of U, and we have surjective iterated commutator maps

$$\left(U^{\mathrm{ab}}\right)^{\otimes n} \twoheadrightarrow U_n/U_{n+1}.$$

Since by assumption U^{ab} is an abelian semistable representation, so too is each U_n/U_{n+1}. Thus by definition-lemma 4.1.1 U is semistable. □

Corollary 4.1.4 (ℓ-adic monodromy theorem for pro-unipotent representations)**.** *Let $U/\mathbb{Q}_\ell$ be a representation of G_K on a finitely generated pro-unipotent group, where $\ell \neq p$. Then U is potentially semistable (i.e. becomes semistable after restriction to an open subgroup).*

Proof. This follows immediately from Grothendieck's ℓ-adic monodromy theorem, applied to the finite-dimensional U^{ab}. □

Let us conclude this section by making good on our promise in remark 1.2.4 and showing that the set of relatively unramified torsors under the $\mathbb{Q}_\ell$-unipotent fundamental group of $L^\times = L \setminus 0$ is trivial, for L a line bundle on an abelian variety A/K. In the proof, we will need the following easily-proved non-abelian analogue of the inflation-restriction sequence.

Proposition 4.1.5 (Non-abelian inflation-restriction)**.** *Let G be a profinite group acting continuously on a topological group U, and $I \trianglelefteq G$ a closed normal subgroup. Then we have an inflation-restriction exact sequence*

$$1 \to \mathrm{H}^1(G/I, U^I) \to \mathrm{H}^1(G, U) \to \mathrm{H}^1(I, U).$$

Proposition 4.1.6. *Let $U/\mathbb{Q}_\ell$ be a representation of G_K on a unipotent group, where $\ell \neq p$. Then*

$$\mathrm{H}^1_{\mathrm{nr}}(G_K, U(\mathbb{Q}_\ell)) := \mathrm{H}^1(G_K/I_K, U(\mathbb{Q}_\ell)^{I_K}) = \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1(I_K, U(\mathbb{Q}_\ell))\right)$$

is trivial iff $U(\mathbb{Q}_\ell)^{G_K} = 1$.

Proof. Since G_K/I_K is pro-cyclic, generated by Frobenius φ, and $U(\mathbb{Q}_\ell)^{I_K}$ is a direct limit of profinite groups, we know that $\mathrm{H}^1(G_K/I_K, U(\mathbb{Q}_\ell)^{I_K})$ is the orbit-space of the twisted conjugation action $u\colon w \mapsto u^{-1}w\varphi(u)$. Hence $\mathrm{H}^1_{\mathrm{nr}}(G_K, U(\mathbb{Q}_\ell)) = 1$ iff this self-action of $U(\mathbb{Q}_\ell)^{I_K}$ is transitive, and $U(\mathbb{Q}_\ell)^{G_K} = 1$ iff the stabiliser of the basepoint is trivial.

In proving the equivalence of these conditions, we are free to replace U by U^{I_K}, so we may and will assume that inertia acts trivially on U.

We now proceed by induction. When U is abelian, the twisted conjugation is just the translation action along the endomorphism $\varphi - 1$ of $U(\mathbb{Q}_\ell)$. The orbit space is

then the cokernel of $\varphi - 1$ while the point-stabiliser is the kernel of $\varphi - 1$; either of these vanishes iff the other does by dimension considerations.

In general, we write U as a G_K/I_K-equivariant central extension

$$1 \to Z \xrightarrow{\mathsf{z}} U \to Q \to 1$$

where we already know the result for Z and Q. If on the one hand the self-action of $U(\mathbb{Q}_\ell)$ is transitive, then so too is the self-action of $Q(\mathbb{Q}_\ell)$, so that by induction the latter action has trivial point-stabiliser. This forces the self-action of $Z(\mathbb{Q}_\ell)$ to also be transitive, and hence also have trivial point-stabiliser. The triviality of the point-stabilisers for the self-actions of $Q(\mathbb{Q}_\ell)$ and $Z(\mathbb{Q}_\ell)$ then implies the same for $U(\mathbb{Q}_\ell)$, as desired.

If, on the other hand, the point-stabiliser for the self-action of $U(\mathbb{Q}_\ell)$ is trivial, then the same is true for the self-action of $Z(\mathbb{Q}_\ell)$, which is thus transitive. This then forces the self-action of $Q(\mathbb{Q}_\ell)$ to have trivial point-stabiliser, so again by induction it is transitive, and combining transitivity for $Q(\mathbb{Q}_\ell)$ and $Z(\mathbb{Q}_\ell)$ we obtain transitivity for $U(\mathbb{Q}_\ell)$, as desired. □

Remark 4.1.7. It seems reasonable to expect that $U(\mathbb{Q}_\ell)^{G_K} = 1$ should hold whenever U is the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of a smooth connected K-variety (and also for certain natural quotients thereof). Indeed, this is implied by the following conjecture of weight-monodromy type.

Conjecture 4.1.8 (Weight-monodromy for the fundamental group)**.** *Let Y/K be a smooth connected variety with basepoint $y \in Y(K)$ and $U/\mathbb{Q}_\ell$ for $\ell \neq p$ the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of (Y, y). Let $U = W_{-1}U \trianglerighteq W_{-2}U \trianglerighteq \dots$ be the* weight filtration[3]*: the unique filtration on U by subgroup schemes such that the induced filtration on $U^{\mathrm{ab}} \cong \mathrm{H}_1(Y; \mathbb{Q}_\ell)$ agrees with the usual weight filtration, and the induced filtration on $\mathrm{Lie}(U)$ is strictly compatible with the Lie bracket. Assume, enlarging K if necessary, that U is semistable.*

Then each $\mathrm{gr}^W_{-n}U = W_{-n}U/W_{-n-1}U$ is abelian, finite-dimensional, and pure of weight $-n$: if $M_\bullet$ denotes the monodromy filtration on $\mathrm{gr}^W_{-n}U$ [Del80, Section 1.6], then the eigenvalues of Frobenius on $\mathrm{gr}^M_r \mathrm{gr}^W_{-n}U$ are all Weil numbers of weight $r - n$.

Let us also remark that conjecture 4.1.8 holds when Y is an abelian variety [GRL67, Corollaire IX.4.4], when $Y = \mathbb{G}_m$, or indeed when $Y = L^\times$ for a line bundle

[3]The weight filtration is explicitly constructed in [Del89, Section 13] under the additional assumption that $\mathrm{Pic}^0(X) = 0$ for a smooth compactification X of Y.

L on an abelian variety (the weight filtration in this case being the exact sequence 3.0.1 with $V_\ell A$ in degree -1 and $\mathbb{Q}_\ell(1)$ in degree -2). In particular, the conditions of proposition 4.1.6 are verified in these cases.

4.2 p-adic Galois representations, admissibility, and Dieudonné functors

Having sketched some of the basic theory of Galois representations on pro-unipotent groups over $\mathbb{Q}_\ell$ for $\ell \neq p$, let us now discuss the theory of such representations when $\ell = p$. As in the abelian case, the theory we obtain is much richer than the $\ell \neq p$ case, and will constitute the technical basis of our coming analysis of local Bloch–Kato Selmer sets.

The basic constructions and notions in abelian p-adic Hodge theory as developed by Fontaine in [Fon94b] revolve around certain *period rings* B, which are topological $\mathbb{Q}_p$-algebras with a continuous G_K-action, and consideration of the B-semilinear G_K-action on $\mathsf{B} \otimes_{\mathbb{Q}_p} V$ for a p-adic Galois representation of interest. Replacing V by a finitely generated pro-unipotent group U, the correct analogue of $\mathsf{B} \otimes_{\mathbb{Q}_p} V$ is $U(\mathsf{B}) = \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{Spec}(\mathsf{B}), U)$, given its natural topology induced from that on B and its natural "semilinear" (continuous[4]) G_K-action arising from those on B and U.

With this analogy in mind, it is now straightforward for us to define the *Dieudonné functor* D associated to a period ring B, which will take representations on finitely generated pro-unipotent groups over $\mathbb{Q}_p$ to pro-unipotent groups over the fixed field B^{G_K}. As usual, for the various period rings arising in practice, we will decorate the corresponding Dieudonné functor with the same symbols as the period ring without comment—for instance $\mathsf{D}^+_{\mathrm{dR}}$ will be the Dieudonné functor associated to $\mathsf{B}^+_{\mathrm{dR}}$.

Lemma 4.2.1. *Let $\mathsf{B}/\mathbb{Q}_p$ be a $(\mathbb{Q}_p, G_K)$-regular $\mathbb{Q}_p$-algebra [Fon94b, Definition 1.4.1] with G_K-fixed field F (or B a G_K-stable F-subalgebra of such an algebra), and let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group. Then the functor $\mathsf{D}(U)$ on F-algebras defined by*

$$\mathsf{D}(U)(\Lambda) = U(\mathsf{B} \otimes_F \Lambda)^{G_K}$$

is a pro-unipotent group over F with Lie algebra $\mathsf{D}(\mathrm{Lie}(U))$, and there is a natural injection

$$\mathsf{D}(U)_{\mathsf{B}} \hookrightarrow U_{\mathsf{B}}$$

[4]One can see continuity, for instance, by reducing to the case of U finite-dimensional and examining the action on $\mathrm{Lie}(U)(\mathsf{B}) \cong U(\mathsf{B})$.

between the base-changes to B. *In particular, if* U *is finite-dimensional, then so is* $\mathsf{D}(U)$ *and we have the inequality* $\dim_F \mathsf{D}(U) \leq \dim_{\mathbb{Q}_p} U$.

Proof. Since the formation of $U(\mathsf{B})^{G_K}$ is compatible with inverse limits in U, we may reduce to the case that U is finite-dimensional.

Using the isomorphism $\log\colon U \to \mathrm{Lie}(U)$, we obtain a natural isomorphism

$$\mathsf{D}(U)(\Lambda) \cong \big(\mathrm{Lie}(U) \otimes_{\mathbb{Q}_p} \mathsf{B} \otimes_F \Lambda\big)^{G_K} \cong \big(\mathrm{Lie}(U) \otimes_{\mathbb{Q}_p} \mathsf{B}\big)^{G_K} \otimes_F \Lambda.$$

But the right-hand side is the functor-of-points of the vector group associated to the finite-dimensional F-vector space $\mathsf{D}(\mathrm{Lie}(U)) = \big(\mathrm{Lie}(U) \otimes_{\mathbb{Q}_p} \mathsf{B}\big)^{G_K}$. Hence $\mathsf{D}(U)/F$ is a variety, isomorphic to affine space.

Now the group operation on U is induced from the Lie bracket on $\mathrm{Lie}(U)$ via the Baker–Campbell–Hausdorff formula. From this, it follows that the natural group operation on $\mathsf{D}(U)(\Lambda)$ is induced from the natural Lie bracket on $\mathsf{D}(\mathrm{Lie}(U)) \otimes_F \Lambda$ also by the Baker–Campbell–Hausdorff formula, so that $\mathsf{D}(U)$ is unipotent with Lie algebra $\mathsf{D}(\mathrm{Lie}(U))$.

Finally, the natural map $U(\mathsf{B} \otimes_F \Lambda)^{G_K} \to U(\Lambda)$ induced by the multiplication map $\mathsf{B} \otimes_F \Lambda \to \Lambda$ for any B-algebra A induces a morphism

$$\mathsf{D}(U)_{\mathsf{B}} \to U_{\mathsf{B}}$$

which, identifying both sides with their Lie algebras, is the usual one from the theory of admissible representations, and hence injective. In particular, $\dim_F \mathsf{D}(U) \leq \dim_{\mathbb{Q}_p} U$. □

Just as in abelian p-adic Hodge theory, the Dieudonné functor allows us to define certain *admissible classes* of representations on pro-unipotent groups. In fact, the notion we get is essentially already encompassed by the notion of admissibility for linear p-adic representations, as in the following lemma.

Definition-Lemma 4.2.2. Let $\mathsf{B}/\mathbb{Q}_p$ be a $(\mathbb{Q}_p, G_K)$-regular $\mathbb{Q}_p$-algebra [Fon94b, Section 1.4] with G_K-fixed field F, and let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group. The following are equivalent:

1. $\mathsf{D}(U)_{\mathsf{B}} \hookrightarrow U_{\mathsf{B}}$ is an isomorphism;

(1') (if U finite-dimensional) $\dim_F \mathsf{D}(U) = \dim_{\mathbb{Q}_p} U$;

2. $\mathrm{Lie}(U)$ is pro-B-admissible;

3. each $J_n\mathcal{O}(U)$ is B-admissible.

When these equivalent conditions hold, we will say that U is B-*admissible*. In the specific case when $\mathsf{B} = \mathsf{B}_{\mathrm{dR}}$ (resp. $\mathsf{B} = \mathsf{B}_{\mathrm{st}}$, resp. $\mathsf{B} = \mathsf{B}_{\mathrm{cris}}$), we will refer to U as being *de Rham* (resp. *semistable*, resp. *crystalline*).

Proof of equivalence. For the equivalences (1)⇔(2)⇔(3), we may reduce to the case that U is finite-dimensional, since the construction of $\mathsf{D}(U)$ and the map $\mathsf{D}(U)_{\mathsf{B}} \hookrightarrow U_{\mathsf{B}}$ commutes with inverse limits in U.

In this case, $\mathsf{D}(\mathrm{Lie}(U))$ is the Lie algebra of $\mathsf{D}(U)$ by lemma 4.2.1, so that we have $\dim_F(\mathsf{D}(U)) = \dim_F \mathsf{D}(\mathrm{Lie}(U))$, which proves the equivalence (1')⇔(2).

Next, to prove the equivalence (1)⇔(2), we simply note from the proof of lemma 4.2.1 that, via the usual logarithm maps, the injection $\mathsf{D}(U)(\Lambda) \hookrightarrow U(\Lambda)$ for a B-algebra Λ is identified with the base change to Λ of the usual map $\mathsf{B} \otimes_F \mathsf{D}(\mathrm{Lie}(U)) \hookrightarrow \mathsf{B} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(U)$. Thus the one map is an isomorphism iff the other is.

Finally, for the equivalence (2)⇔(3), we note that condition (3) is equivalent to B-admissibility of each $\mathcal{U}/J^{n+1}$, where $\mathcal{U}$ is the universal enveloping algebra of $\mathrm{Lie}(U)$ and J is the augmentation ideal. Again, the fact that $\mathcal{U}$ is a surjective image of the tensor algebra $T^{\otimes}\mathrm{Lie}(U)$ and that $\mathrm{Lie}(U)$ is a subrepresentation of $\mathcal{U}/J^{n+1}$ for $n >> 0$ justifies the equivalence (2)⇔(3). □

Remark 4.2.3. When U is B-admissible, we've seen that $\mathcal{O}(U)$ is ind-B-admissible, so that $\mathsf{D}(\mathcal{O}(U))$ is a Hopf algebra over $F = \mathsf{B}^{G_K}$. Indeed, the Λ-points of the corresponding affine group scheme are (cf. example 2.1.29)

$$\begin{aligned}\mathrm{Hom}_F(\mathsf{D}(\mathcal{O}(U)), \Lambda)^{\mathrm{gplike}} &= \mathrm{Hom}_{\mathsf{B}}(\mathsf{B} \otimes_F \mathsf{D}(\mathcal{O}(U)), \mathsf{B} \otimes_F \Lambda)^{G_K, \mathrm{gplike}} \\ &= \mathrm{Hom}_{\mathsf{B}}(\mathsf{B} \otimes_{\mathbb{Q}_p} \mathcal{O}(U), \mathsf{B} \otimes_F \Lambda)^{\mathrm{gplike}, G_K} = U(\mathsf{B} \otimes_F \Lambda)^{G_K},\end{aligned}$$

so that the corresponding affine group scheme is canonically identified with $\mathsf{D}(U)$.

Moreover, it follows from definition 2.1.12 and the fact that everything in sight is B-admissible that in this case $J_n\mathsf{D}(\mathcal{O}(U)) = \mathsf{D}(J_n\mathcal{O}(U))$, so that $\mathsf{D}(U)$ is finitely generated (see definition-lemma 2.1.10).

Remark 4.2.4. It follows directly from the definitions and [Fon94b, Proposition 1.5.2] that G_K-stable finitely generated subgroups and quotients of B-admissible representations are again B-admissible, and that a representation of G_K on a finitely generated pro-unipotent group is B-admissible iff all of its finite-dimensional quotients are.

The equivalence between B-admissibility of U and $\mathrm{Lie}(U)$ means that we immediately deduce non-abelian versions of certain basic properties of admissible representations.

Lemma 4.2.5. *Let*

$$1 \to Z \to U \to Q \to 1$$

be an exact sequence of de Rham representations of G_K on finitely generated pro-unipotent groups over $\mathbb{Q}_p$. Then the sequence remains exact on taking D_{st} and on taking $\mathsf{D}^+_{\mathrm{dR}}$.

Proof. To prove the result for D_{st}, it suffices to show that

$$1 \to \mathsf{D}_{\mathrm{st}}(Z) \to \mathsf{D}_{\mathrm{st}}(U) \to \mathsf{D}_{\mathrm{st}}(Q) \to 1$$

is exact on K_0-points. Writing the exact sequence $1 \to Z \to U \to Q \to 1$ as an inverse limit of finite-dimensional G_K-equivariant quotient sequences, we observe that this describes $\mathsf{D}_{\mathrm{st}}(Z)(K_0)$ as an inverse limit of a system of K_0-points of unipotent groups over K_0. Such a system necessarily satisfies the Mittag–Leffler condition, so we may reduce to the case that U, Z and Q are finite-dimensional.

To deal with this case, it suffices to prove exactness for the K_0-points of the Lie algebras, so, forgetting the Lie bracket, we may assume that U, Z and Q are finite-dimensional and abelian. Since by [Ber02, Théorème 0.7] all these representations are potentially semistable, this case is then dealt with in the argument (but not the statement) of [Ber02, Lemme 6.4].

To prove the result for $\mathsf{D}^+_{\mathrm{dR}}$, we may perform the same reductions to reduce to the case that U, Q and Z are all linear de Rham representations of G_K. Choose by potential semistability an open normal subgroup $G_L \trianglelefteq G_K$ such that U, Q and Z are semistable when the action is restricted to G_L. Since $\mathsf{D}_{\mathrm{st},L}(U)$ etc. are admissible filtered (φ, N)-modules and morphisms of such are strict [Fon94b, Théorème 5.3.5], we thus have that

$$0 \to \mathsf{D}^+_{\mathrm{dR},L}(Z) \to \mathsf{D}^+_{\mathrm{dR},L}(U) \to \mathsf{D}^+_{\mathrm{dR},L}(Q) \to 0$$

is exact. Since all the terms are vector spaces over a characteristic zero field, the sequence remains exact after taking invariants under the finite group $G_{L|K}$, and hence the sequence on $\mathsf{D}^+_{\mathrm{dR}}$ is exact as desired. □

Corollary 4.2.6. *If $U/\mathbb{Q}_p$ is a de Rham representation of G_K on a finitely generated pro-unipotent group which is an extension of a two semistable representations (on finitely generated pro-unipotent groups), then U itself is semistable.*

Proof. Writing U as an extension of Q by Z, both semistable, we have from lemma 4.2.5 a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathsf{D}_{\mathrm{st}}(Z)_{\mathsf{B}_{\mathrm{st}}} & \longrightarrow & \mathsf{D}_{\mathrm{st}}(U)_{\mathsf{B}_{\mathrm{st}}} & \longrightarrow & \mathsf{D}_{\mathrm{st}}(Q)_{\mathsf{B}_{\mathrm{st}}} & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \\ 1 & \longrightarrow & Z_{\mathsf{B}_{\mathrm{st}}} & \longrightarrow & U_{\mathsf{B}_{\mathrm{st}}} & \longrightarrow & Q_{\mathsf{B}_{\mathrm{st}}} & \longrightarrow & 1 \end{array}$$

of group schemes over B_{st}, with both rows exact (on Λ-points for all B_{st}-algebras Λ) and the outer vertical maps both isomorphisms. The five-lemma then shows that $\mathsf{D}_{\mathrm{st}}(U)_{\mathsf{B}_{\mathrm{st}}} \hookrightarrow U_{\mathsf{B}_{\mathrm{st}}}$ is an isomorphism, so that U is semistable as desired. □

4.2.1 Potential semistability

Just as in the ℓ-adic case, we can prove an analogue of Berger's theorem on potential semistability of de Rham representations in the setting of representations on finitely generated pro-unipotent groups. Again, the key result which allows us to reduce to the linear case is that semistability of a de Rham representation depends only on its abelianisation.

Lemma 4.2.7. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then U is semistable iff U^{ab} is.*

Proof. The same proof as lemma 4.1.3 works, the key point being that everything in sight is de Rham, so that central extensions of semistable representations remain semistable. □

Corollary 4.2.8 (p-adic monodromy theorem for pro-unipotent representations)**.** *Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group. Then U is potentially semistable (i.e. becomes semistable after restriction to an open subgroup) iff U is de Rham.*

Proof. Potentially semistable representations are clearly de Rham. Conversely, if U is de Rham, then by lemma 4.2.7 it becomes semistable as soon as U^{ab} does. But U^{ab} is a finite-dimensional linear representation, so is potentially semistable by [Ber02, Théorème 0.7]. □

4.2.2 Extra structures on Dieudonné functors

Just as in the abelian case, the pro-unipotent groups $\mathsf{D}_{\mathrm{cris}}(U)$, $\mathsf{D}_{\mathrm{st}}(U)$ and $\mathsf{D}_{\mathrm{dR}}(U)$ carry various extra structures (Frobenius, monodromy, Hodge filtration) induced from the corresponding structures on the period rings $\mathsf{B}_{\mathrm{cris}}$, B_{st} and B_{dR}. Since we will not require these extra structures in the main body of this thesis, let us merely describe in outline how these structures manifest on $\mathsf{D}_{\mathrm{st}}(U)$, assuming for simplicity that U is semistable.

Most obviously, since $\mathcal{O}(U)$ is ind-semistable, $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U)) = \mathsf{D}_{\mathrm{st}}(\mathcal{O}(U))$ then carries the structure of an ind-weakly admissible filtered (φ, N)-module, compatible with the Hopf algebra structure morphisms. As usual, these extra structures (Frobenius, monodromy, Hodge filtration) on $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$ can be viewed equivalently as structures on $\mathrm{Lie}(\mathsf{D}_{\mathrm{st}}(U))$ or on $\mathsf{D}_{\mathrm{st}}(U)$ itself. For $\mathrm{Lie}(\mathsf{D}_{\mathrm{st}}(U)) = \mathsf{D}_{\mathrm{st}}(\mathrm{Lie}(U))$ the same argument shows that it carries a pro-weakly admissible filtered (φ, N)-module structure compatible with the Lie bracket.

However, viewing these structures on $\mathsf{D}_{\mathrm{st}}(U)$ is more complicated, and indeed the Hodge filtration has no obvious direct interpretation (except as a structure on $\mathrm{Lie}(\mathsf{D}_{\mathrm{st}}(U))$ or $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$). The intepretation of Frobenius and monodromy on $\mathsf{D}_{\mathrm{st}}(U)$ is as follows.

On the level of $\mathsf{D}_{\mathrm{st}}(U)$, the semilinear Frobenius on $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$ induces a semilinear Frobenius $\varphi\colon \mathsf{D}_{\mathrm{st}}(U) \xrightarrow{\sim} \varphi^*\mathsf{D}_{\mathrm{st}}(U)$, where by abuse of notation we also denote by φ the Frobenius on K_0. The monodromy operator on $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$, which is a derivation compatible with the comultiplication, induces then a vector field on $\mathsf{D}_{\mathrm{st}}(U)$ which, when viewed as a section $\mathsf{D}_{\mathrm{st}}(U) \to T\mathsf{D}_{\mathrm{st}}(U)$ of the tangent bundle on $\mathsf{D}_{\mathrm{st}}(U)$, is a group homomorphism when the total space $T\mathsf{D}_{\mathrm{st}}(U) = \mathrm{Lie}(\mathsf{D}_{\mathrm{st}}(U)) \rtimes \mathsf{D}_{\mathrm{st}}(U)$ of the tangent bundle is given its natural group structure. The relation $N\varphi = p\varphi N$ between the Frobenius and monodromy on $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$ implies that this φ and N on $\mathsf{D}_{\mathrm{st}}(U)$ satisfy the relation $\varphi^*N \circ \varphi = p \cdot \mathrm{d}\varphi \circ N$ as morphisms $\mathsf{D}_{\mathrm{st}}(U) \to \varphi^*T\mathsf{D}_{\mathrm{st}}(U)$, where the p denotes multiplication by p in the fibres of $\varphi^*T\mathsf{D}_{\mathrm{st}}(U) \twoheadrightarrow \varphi^*\mathsf{D}_{\mathrm{st}}(U)$.

4.3 Cohomology of p-adic Galois representations

Recall from the introduction that if $U/\mathbb{Q}_p$ is a representation of G_K on a finitely generated pro-unipotent group, the *local Bloch–Kato Selmer sets* are three naturally defined pointed subsets $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ of the non-abelian Galois cohomology $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ cut out by increasingly coarse Selmer conditions (see definition 1.2.1), which in the case that U is the fundamental group

of a smooth variety Y/K should be closely tied to the geometry of Y. In order to prove theorem 1.2.3, we will need to conduct a detailed explicit study of these sets and their respective quotients, which will be carried out in chapter 6 using tools from homotopical algebra.

Before we embark on this analysis, let us briefly record a few properties of the local Bloch–Kato Selmer sets and their quotients which are essentially immediate from the definitions.

Lemma 4.3.1 (Stability of local Bloch–Kato sets under base change)**.** *Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group, and let L/K be a finite extension (embedded in the fixed algebraic closure $\overline{K}/K$). Then the restriction map*

$$\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_L, U(\mathbb{Q}_p))$$

is injective, and the preimage of $\mathrm{H}^1_g(G_L, U(\mathbb{Q}_p))$ under this map is $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$. The same applies with H^1_f or H^1_e in place of H^1_g.

Proof. To prove this, it suffices to consider only the case that L/K is finite Galois. Moreover, to prove injectivity for U it suffices by naturality of the Serre twisting bijection to prove that

$$\mathrm{H}^1(G_K, {}_\alpha U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_L, {}_\alpha U(\mathbb{Q}_p))$$

has trivial kernel for all twists ${}_\alpha U$ of U by continuous cocycles $\alpha\colon G_K \to U(\mathbb{Q}_p)$.

By non-abelian inflation-restriction, the kernel of the above map is given by $\mathrm{H}^1\left(G_{L|K}, {}_\alpha U(\mathbb{Q}_p)^{G_L}\right)$. But ${}_\alpha U(\mathbb{Q}_p)$ is a uniquely divisible group, and hence so is ${}_\alpha U(\mathbb{Q}_p)^{G_L}$. As $G_{L|K}$ is a finite group, this already implies that $\mathrm{H}^1\left(G_{L|K}, {}_\alpha U(\mathbb{Q}_p)^{G_L}\right)$ is trivial as desired.

To prove the assertion about H^1_g, it suffices to show that

$$\mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}})) \to \mathrm{H}^1(G_L, U(\mathsf{B}_{\mathrm{st}}))$$

has trivial kernel. In fact, it is even injective, as one proves by replacing $U(\mathbb{Q}_p)$ by $U(\mathsf{B}_{\mathrm{st}})$ throughout the above proof. The cases of H^1_f and H^1_e are similar. □

Lemma 4.3.2. *Let $U/\mathbb{Q}_p$ be a representation of G_K on a unipotent group and $* \in \{e, f, g\}$. Then the $\sim_{\mathrm{H}^1_*}$-equivalence class of $[\alpha] \in \mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ for a continuous cocycle $\alpha \in \mathrm{Z}^1(G_K, U(\mathbb{Q}_p)))$ is the image of $\mathrm{H}^1_*(G_K, {}_\alpha U(\mathbb{Q}_p))$ under the above bijection $\mathrm{H}^1(G_K, {}_\alpha U(\mathbb{Q}_p)) \xrightarrow{\sim} \mathrm{H}^1(G_K, U(\mathbb{Q}_p))$.*

Proof. This is obvious from, for example, commutativity of the square

$$\begin{array}{ccc} \mathrm{H}^1(G_K, {}_\alpha U(\mathbb{Q}_p)) & \longrightarrow & \mathrm{H}^1(G_K, {}_\alpha U(\mathsf{B}_{\mathrm{cris}}^{\varphi=1})) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{H}^1(G_K, U(\mathbb{Q}_p)) & \longrightarrow & \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}^{\varphi=1})). \end{array}$$

□

In applying lemma 4.3.2, it will be important for us to understand when the Serre twist ${}_\alpha U$ is de Rham. For our purposes, the following criterion will suffice.

Proposition 4.3.3. *Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group, and $\alpha\colon G_K \to U(\mathbb{Q}_p)$ a continuous cocycle whose class is in $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$. Then ${}_\alpha U$ is de Rham (resp. semistable) iff U is, and the Serre twisting bijection $\mathrm{H}^1(G_K, {}_\alpha U(\mathbb{Q}_p)) \xrightarrow{\sim} \mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ induces a bijection on H^1_g.*

Proof. The fact that $[\alpha] \in \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ says that there is some $u \in U(\mathsf{B}_{\mathrm{st}})$ (hence also in $U(\mathsf{B}_{\mathrm{dR}})$) such that $\alpha(\sigma) = u^{-1}\sigma(u)$ for all $\sigma \in G_K$. Then left-conjugation by u induces a natural G_K-equivariant isomorphism ${}_\alpha U_{\mathsf{B}_{\mathrm{st}}} \xrightarrow{\sim} U_{\mathsf{B}_{\mathrm{st}}}$ and hence also an isomorphism $\mathsf{D}_{\mathrm{st}}({}_\alpha U) \xrightarrow{\sim} \mathsf{D}_{\mathrm{st}}(U)$. Since these isomorphisms are compatible under the natural embedding $\mathsf{D}_{\mathrm{st}}(U)_{\mathsf{B}_{\mathrm{st}}} \hookrightarrow U_{\mathsf{B}_{\mathrm{st}}}$ (and similarly for ${}_\alpha U$), we have that U is semistable iff ${}_\alpha U$ is (and a similar argument works for de Rhamness). The final assertion follows from the commuting square

$$\begin{array}{ccc} \mathrm{H}^1(G_K, {}_\alpha U(\mathbb{Q}_p) & \longrightarrow & \mathrm{H}^1(G_K, {}_\alpha U(\mathsf{B}_{\mathrm{st}})) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{H}^1(G_K, U(\mathbb{Q}_p) & \longrightarrow & \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}})), \end{array}$$

where the vertical maps are the Serre twisting isomorphisms, since the right-hand vertical map identifies the distinguished points by assumption. □

Remark 4.3.4. One can show relatively straightforwardly that if $U/\mathbb{Q}_p$ is a representation of G_K on a finitely generated pro-unipotent group and we write $U = \varprojlim U_n$ as an inverse limit of finite-dimensional G_K-equivariant quotients, then the natural map

$$\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \to \varprojlim \mathrm{H}^1_g(G_K, U_n(\mathbb{Q}_p))$$

is bijective (and similarly for H^1_f and H^1_e), so that one can reduce the study of local Bloch–Kato Selmer sets to the case when U is finite-dimensional. However, it will be our preference to always work at the infinite level, not least because the corresponding statements for the Bloch–Kato quotients $\mathrm{H}^1_{g/e}$ and $\mathrm{H}^1_{f/e}$ are much less straightforward to prove directly (see lemma 6.3.5).

4.4 Torsors and H^1

A different approach to the study of local Bloch–Kato Selmer sets has been undertaken in Kim's papers [Kim05, Kim09] by viewing these sets as classifying sets for certain classes of torsors under U. This viewpoint has the distinct advantage of being the natural setting in which to describe the non-abelian Kummer map

$$Y(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_p)),$$

which is the classifying map for path-torsors when U is the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of a variety Y/K at a K-rational basepoint. Although our analysis will proceed rather differently, let us now recall this description and deduce thereby a non-abelian version of Hyodo's $\mathrm{H}^1_g = \mathrm{H}^1_{\mathrm{st}}$ theorem [Hyo].

Definition-Lemma 4.4.1. Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group, and let $P/\mathbb{Q}_p$ be a (right-)torsor under (the underlying group scheme of) U, endowed with an action of G_K by algebraic automorphisms. Then the J-filtration on $\mathcal{O}(P)$ is G_K-stable. Moreover, the following are equivalent:

1. the action of G_K on $P(\mathbb{Q}_p)$ is continuous;
2. the action of G_K on each $J_n\mathcal{O}(P)$ is continuous.

When these equivalent conditions hold, we will say that P is a *torsor under* U.

Proof of equivalence. G_K-stability of the J-filtration follows from proposition 2.2.5 and lemma 2.1.14.

If we write $U = \varprojlim U_n$ as an inverse limit of finite-dimensional G_K-equivariant quotients, then $P(\mathbb{Q}_p)$ is homeomorphic to the inverse limit of the pushouts $(P \times^U U_n)(\mathbb{Q}_p)$, so that it suffices to prove this for U finite-dimensional. As in definition-lemma 4.0.1, the second condition is equivalent to the action on each $(J_n\mathcal{O}(P))^*$ being continuous. In one direction, $P(\mathbb{Q}_p)$ is the set of group-like elements of the coalgebra $\mathcal{O}(P)^* = \varprojlim (J_n\mathcal{O}(P))^*$ and its topology is the subspace topology, so that the implication (2)$\Rightarrow$(1) is clear.

In the other direction, there is a canonical G_K-equivariant homeomorphism

$$P(\mathbb{Q}_p) \times (J_n\mathcal{O}(U))^* \xrightarrow{\sim} P(\mathbb{Q}_p) \times (J_n\mathcal{O}(P))^*$$

given by $(q, u) \mapsto (q, qu)$. When condition (1) holds, the G_K-action on the left-hand side is continuous, and hence the action on $J_n\mathcal{O}(P)$ is continuous also. $\square$

The importance of this notion of torsor is twofold. Firstly, such torsors arise naturally from geometry as torsors of étale paths between two K-rational points x, y on a connected variety X/K. Secondly, the collection of U-torsors is parametrised the continuous cohomology of $U(\mathbb{Q}_p)$, which allows one to understand them in a very concrete fashion.

Lemma 4.4.2. *Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group. Then there is a canonical bijection*

$$\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \xrightarrow{\sim} \{\textit{torsors under } U\}/\simeq .$$

Proof. This is a special case of [Kim05, Proposition 1], where the filtration W_n on $\mathcal{O}(P)$ is the J-filtration. It is worthwhile quickly recalling how this construction goes.

To a cocycle $\alpha \in \mathrm{Z}^1(G_K, U(\mathbb{Q}_p))$ we associate a U-torsor ${}_\alpha P$ whose underlying scheme is U, endowed the right action of U and the α-twisted G_K action, given on Λ-points for some $\mathbb{Q}_p$-algebra Λ by

$$g\colon u \mapsto \alpha(g)g(u).$$

Just as in section 4.0.1, this action is clearly algebraic and continuous on $\mathbb{Q}_p$-points. Moreover, this torsor has a distinguished $\mathbb{Q}_p$-point, namely $1 \in U(\mathbb{Q}_p)$, and it is easy to check that this construction induces a bijection between $\mathrm{Z}^1(G_K, U(\mathbb{Q}_p))$ and the set[5] of pointed U-torsors (where the inverse map takes (P, x) to the cocycle $g \mapsto x^{-1}g(x)$).

Now the action of $U(\mathbb{Q}_p)$ on $\mathrm{Z}^1(G_K, U(\mathbb{Q}_p))$ exactly corresponds to the change-of-basepoint action on pointed U-torsors, so the correspondence descends to a bijection between $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ and the set of isomorphism classes of U-torsors P such that $P(\mathbb{Q}_p) \neq \emptyset$. But since U is a pro-unipotent group, all torsors satisfy this and we have the desired bijection. □

In view of lemma 4.4.2, the non-abelian local Bloch–Kato Selmer sets should correspond to a subclass of torsors satisfying certain admissibility conditions with respect to B_{st}, $\mathsf{B}_{\mathrm{cris}}$ and $\mathsf{B}_{\mathrm{cris}}^{\varphi=1}$ respectively. In fact, this criterion is very simple to state (and is essentially already found in [Kim09]).

Definition-Lemma 4.4.3. Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group, $P/\mathbb{Q}_p$ be a torsor under U, and let $\mathsf{B}/\mathbb{Q}_p$ be a $(\mathbb{Q}_p, G_K)$-regular $\mathbb{Q}_p$-algebra. The following are equivalent:

1. $[P]$ lies in the kernel of $\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}))$;

[5] Technically, setoid.

2. $P(\mathsf{B})^{G_K} \neq \emptyset$;

3. (if U is B-admissible) each $J_n\mathcal{O}(P)$ is B-admissible.

When these equivalent conditions hold, we will say that P is a *(relatively)* B*-admissible torsor* under U.

Proof of equivalence. For the equivalence (1)⇔(2), we may assume that the underlying $\mathbb{Q}_p$-variety of P is U, and that the G_K action is that determined by a cocycle α. It follows from chasing definitions in lemma 4.4.2 that the image of $[\alpha]$ in $\mathrm{Z}^1(G_K, U(\mathsf{B}))$ is the coboundary of some $u \in U(\mathsf{B})$ iff u is fixed for the α-twisted action.

For the implication (2)⇒(3) when U is B-admissible, a choice of a G_K-fixed point on $P(\mathsf{B})$ gives a G_K-equivariant isomorphism $P_{\mathsf{B}} \xrightarrow{\sim} U_{\mathsf{B}}$ of B-schemes, and hence a G_K-equivariant isomorphism $\mathsf{B} \otimes_{\mathbb{Q}_p} \mathcal{O}(P) \xrightarrow{\sim} \mathsf{B} \otimes_{\mathbb{Q}_p} \mathcal{O}(U)$. Since $\mathcal{O}(U)$ is ind-B-admissible, so too is $\mathcal{O}(P)$.

For the converse implication, we recall from [Fon94b, Proposition 1.5.2] that the Dieudonné functor D on B-admissible representations commutes with tensor products and duals, so that when $\mathcal{O}(P)$ is ind-B-admissible, $\mathsf{D}(\mathcal{O}(P))$ is an algebra over the fixed field $F = \mathsf{B}^{G_K}$, and a right-comodule for $\mathsf{D}(\mathcal{O}(U))$, such that $\mathsf{D}(\mathcal{O}(P)) \otimes_F \mathsf{D}(\mathcal{O}(P)) \to \mathsf{D}(\mathcal{O}(U)) \otimes_F \mathsf{D}(\mathcal{O}(P))$ is an isomorphism. Since $\mathsf{D}(\mathcal{O}(P)) \neq 0$, this implies that $\mathsf{D}(P) := \mathrm{Spec}(\mathsf{D}(\mathcal{O}(P))$ is a torsor under the unipotent group $\mathsf{D}(U) = \mathrm{Spec}(\mathsf{D}(\mathcal{O}(U)))$ over F (remark 4.2.3). The same argument as in remark 4.2.3 shows that $\mathsf{D}(P)(\Lambda) = P(\mathsf{B} \otimes_F \Lambda)^{G_K}$ for any F-algebra Λ. But then by triviality of the $\mathsf{D}(U)$-torsor $\mathsf{D}(P)$, we know that $P(\mathsf{B})^{G_K} = \mathsf{D}(P)(F) \neq \emptyset$, as desired. □

Remark 4.4.4. Definition-lemma 4.4.3 is a non-abelian version of the well-known fact that, under the correspondence $\mathrm{H}^1(G_K, V) \cong \mathrm{Ext}^1_{G_K}(\mathbb{Q}_p, V)$ for a p-adic representation V, the kernel of $\mathrm{H}^1(G_K, V) \to \mathrm{H}^1(G_K, \mathsf{B} \otimes V)$ corresponds to the class of those extensions of $\mathbb{Q}_p$ by V which remain exact upon applying the associated Dieudonné functor D.

Using this torsor-theoretic language, we are now able to justify our use of B_{st} in place of B_{dR} in the definition of H^1_g, as promised. This is a non-abelian version of Hyodo's $\mathrm{H}^1_g = \mathrm{H}^1_{\mathrm{st}}$ theorem [Hyo].

Lemma 4.4.5. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then the kernel of $\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$ is $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$.*

Proof. The G_K-equivariant inclusion $\mathsf{B}_{\mathrm{st}} \hookrightarrow \mathsf{B}_{\mathrm{dR}}$ induced from our choice of p-adic logarithm factors the map $\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$ through $\mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}}))$, and hence its kernel clearly contains $\mathrm{H}^1_g(G_K, U(\mathsf{B}_{\mathrm{st}}))$. For the converse implication, consider any U-torsor P whose class lies in the kernel of $\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$, i.e. such that $\mathcal{O}(P)$ is ind-de Rham.

By potential semistability (corollary 4.2.8) we may pick a finite L/K such that $\mathcal{O}(U)$ is ind-semistable as a representation of G_L. Then from the canonical isomorphism $\mathrm{gr}^J_\bullet \mathcal{O}(P) \cong \mathrm{gr}^J_\bullet \mathcal{O}(U)$ (see proposition 2.2.6), we see that $\mathrm{gr}^J_\bullet \mathcal{O}(P)$, and hence $\mathcal{O}(P)$, is an ind-semistable representation of G_L. This tells us that the class of $P|_{G_L}$ lies in $\mathrm{H}^1_g(G_L, U(\mathbb{Q}_p))$, and hence by lemma 4.3.1 the class of P lies in $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ as desired. □

4.4.1 Isocrystal moduli interpretations for Bloch–Kato Selmer sets

To conclude this chapter, let us briefly discuss a second moduli interpretation for the Bloch–Kato Selmer sets H^1_e, H^1_f and H^1_g, as well as the relative quotients $\mathrm{H}^1_{g/e}$, $\mathrm{H}^1_{f/e}$ and $\mathrm{H}^1_{g/f}$ which is of a more log-crystalline flavour. As in section 4.2.2, we will assume for simplicity that U is semistable, so that $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$ is an ind-weakly admissible filtered (φ, N)-module and $\mathsf{D}_{\mathrm{st}}(U)$ carries a Frobenius φ and monodromy N. In this case, definition-lemma 4.4.3 shows that $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ has an interpretation as the set of isomorphism classes of torsors P under U such that $\mathcal{O}(P)$ is ind-semistable. Thus by the equivalence of symmetric monoidal categories between semistable representations and weakly admissible filtered (φ, N)-modules [CF00, Theorem A], we obtain an equivalent description in terms of $\mathsf{D}_{\mathrm{st}}(U)$.

Lemma 4.4.6. *Let $U/\mathbb{Q}_p$ be a semistable representation of G_K on a finitely generated pro-unipotent group. Then there is a canonical bijection between $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ and the set $\mathrm{H}^1(\mathbf{MF}_K^{\varphi,N,\mathrm{w.a.}}, \mathsf{D}_{\mathrm{st}}(U)(K_0))$ of isomorphism classes of torsors Q under $\mathsf{D}_{\mathrm{st}}(U)$ endowed with a ind-weakly admissible filtered (φ, N)-module structure on $\mathcal{O}(Q)$ compatible with the $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$-comodule algebra structure maps.*

Just as for $\mathsf{D}_{\mathrm{st}}(U)$, for such a torsor Q the Frobenius and monodromy on $\mathcal{O}(Q)$ induce a Frobenius automorphism $\varphi\colon Q \xrightarrow{\sim} \varphi^* Q$ and monodromy vector field $N\colon Q \to TQ$ on Q, compatible with the corresponding structures on $\mathsf{D}_{\mathrm{st}}(U)$ under the action map $Q \times \mathsf{D}_{\mathrm{st}}(U) \to Q$ and satisfying $\varphi^* N \circ \varphi = p \cdot \mathrm{d}\varphi \circ N$.

Lemma 4.4.7. *Let $U/\mathbb{Q}_p$ be a semistable representation of G_K on a finitely generated pro-unipotent group. Then there is a canonical bijection between:*

- $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$;
- *the set* $\mathrm{H}^1(\mathbf{Mod}_K^{\varphi,N}, \mathsf{D}_{\mathrm{st}}(U)(K_0))$ *of isomorphism classes of torsors* Q *under* $\mathsf{D}_{\mathrm{st}}(U)$ *endowed with a (non-filtered)* (φ, N)*-module structure on* $\mathcal{O}(Q)$ *compatible with the* $\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$*-comodule algebra structure maps; and*
- *the set of isomorphism classes of torsors* Q *under* $\mathsf{D}_{\mathrm{st}}(U)$ *endowed with a Frobenius* $\varphi\colon Q \xrightarrow{\sim} \varphi^* Q$ *and monodromy vector field* $N\colon Q \to TQ$ *satisfying* $\varphi^* N \circ \varphi = p \cdot \mathrm{d}\varphi \circ N$ *and compatible with the corresponding structures on* $\mathsf{D}_{\mathrm{st}}(U)$ *under the action map* $Q \times \mathsf{D}_{\mathrm{st}}(U) \to Q$.

Proof. The bijection between the latter two kinds of structures on a torsor Q arises from the usual bijection between φ-semilinear variety automorphisms of Q (resp. vector fields on Q) and φ-semilinear algebra automorphisms of $\mathcal{O}(Q)$ (resp. derivations on $\mathcal{O}(Q)$)—one checks straightforwardly that the conditions imposed on the variety side exactly correspond to those on the algebraic side.

For the equivalence of the first point with the other two, we consider the map

$$\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \xrightarrow{\sim} \mathrm{H}^1(\mathbf{MF}_K^{\varphi,N,\mathrm{w.a.}}, \mathsf{D}_{\mathrm{st}}(U)(K_0)) \to \mathrm{H}^1(\mathbf{Mod}_K^{\varphi,N}, \mathsf{D}_{\mathrm{st}}(U)(K_0)),$$

where the isomorphism is the one from lemma 4.4.6, and the second map is the one which forgets the Hodge structure on the affine ring of a $\mathsf{D}_{\mathrm{st}}(U)$-torsor. It suffices to show that this map is surjective and that its fibres are exactly the $\sim_{\mathrm{H}^1_e}$ equivalence classes, so that it factors through a bijection

$$\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xrightarrow{\sim} \mathrm{H}^1(\mathbf{Mod}_K^{\varphi,N}, \mathsf{D}_{\mathrm{st}}(U)(K_0)).$$

To show surjectivity, consider any Q with a (φ, N)-module structure on $\mathcal{O}(Q)$. It follows by definition 2.1.12 that the J-filtration $J_\bullet \mathcal{O}(Q)$ is a filtration by sub-(φ, N)-modules, and that the canonical isomorphism $\mathrm{gr}^J_\bullet(\mathcal{O}(Q)) \xrightarrow{\sim} \mathrm{gr}^J_\bullet(\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U)))$ is an isomorphism of (φ, N)-modules in each degree.

If we pick any $q \in Q(K_0)$, we obtain a trivialisation $\mathsf{D}_{\mathrm{st}}(U) \xrightarrow{\sim} Q$, and may put a Hodge filtration on $K \otimes_{K_0} \mathcal{O}(Q)$ by declaring it to be the filtration corresponding to the Hodge filtration on $K \otimes_{K_0} \mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$ under the induced isomorphism $\mathcal{O}(Q) \xrightarrow{\sim} \mathcal{O}(\mathsf{D}_{\mathrm{st}}(U))$. By construction, the canonical isomorphism $\mathrm{gr}^J_\bullet(\mathcal{O}(Q)) \xrightarrow{\sim} \mathrm{gr}^J_\bullet(\mathcal{O}(\mathsf{D}_{\mathrm{st}}(U)))$ is then a filtered isomorphism for the Hodge filtration, so that the J-graded pieces of $\mathcal{O}(Q)$, with their induced Hodge filtrations, are weakly admissible. Hence, by [CF00, Proposition 3.4], $\mathcal{O}(Q)$ itself is ind-weakly admissible, so that the

class of Q lies in the image of $\mathrm{H}^1(\mathbf{MF}_K^{\varphi,N,\mathrm{w.a.}}, \mathsf{D}_{\mathrm{st}}(U)(K_0)) \to \mathrm{H}^1(\mathbf{Mod}_K^{\varphi,N}, \mathsf{D}_{\mathrm{st}}(U)(K_0))$, as desired.

Finally, to show that the fibres of $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(\mathbf{Mod}_K^{\varphi,N}, \mathsf{D}_{\mathrm{st}}(U)(K_0))$ are exactly the $\sim_{\mathrm{H}^1_e}$ equivalence classes, consider any two semistable torsors P, P' under U, and write $\mathsf{D}_{\mathrm{st}}(P) = \mathrm{Spec}(\mathsf{D}_{\mathrm{st}}(\mathcal{O}(P)))$, respectively $\mathsf{D}_{\mathrm{st}}(P')$, for the associated $\mathsf{D}_{\mathrm{st}}(U)$-torsors. P and P' lie in the same fibre iff $\mathsf{D}_{\mathrm{st}}(P)$ and $\mathsf{D}_{\mathrm{st}}(P')$ are isomorphic as $\mathsf{D}_{\mathrm{st}}(U)$-torsors compatibly with the (φ, N)-module structure on their affine rings. This occurs iff $\mathsf{D}_{\mathrm{st}}(P)_{\mathsf{B}_{\mathrm{st}}} = P_{\mathsf{B}_{\mathrm{st}}}$ and $\mathsf{D}_{\mathrm{st}}(P')_{\mathsf{B}_{\mathrm{st}}} = P'_{\mathsf{B}_{\mathrm{st}}}$ are isomorphic as $\mathsf{D}_{\mathrm{st}}(U)_{\mathsf{B}_{\mathrm{st}}} = U_{\mathsf{B}_{\mathrm{st}}}$-torsors, compatibly with the Galois action, Frobenius and monodromy on the affine rings. This, in turn, occurs iff there is a G_K-equivariant isomorphism $P_{\mathsf{B}_{\mathrm{cris}}^{\varphi=1}} \simeq P'_{\mathsf{B}_{\mathrm{cris}}^{\varphi=1}}$ of $U_{\mathsf{B}_{\mathrm{cris}}^{\varphi=1}}$-torsors. But this final condition says exactly that the images of P and P' in $\mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}^{\varphi=1}))$ are equal, i.e. that they lie in the same $\sim_{\mathrm{H}^1_e}$-equivalence class, which is what we wanted to show. □

Chapter 5

Homotopical algebraic background

Our approach to the analysis of non-abelian Bloch–Kato sets relies on a suitable non-abelian generalisation of the homological algebra of cochain complexes and their cohomology, so that we can effectively transcribe arguments from the abelian setting to the non-abelian. This generalisation is provided by the theory of cosimplicial groups and their cohomotopy, as defined for instance in [BK72], and which subsumes the more well-known theory of non-abelian group cohomology [Ser97, Section I.5]. For the convenience of the reader, we will recall the basic setup here as expounded in [Pir14].

5.1 Cosimplicial groups and cohomotopy

Definition 5.1.1 (Cosimplicial objects)**.** Let $\mathcal{C}$ be a category. A *cosimplicial object* of $\mathcal{C}$ is a covariant functor $X^\bullet\colon \Delta \to \mathcal{C}$, where Δ is the simplex category whose objects are $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and whose morphisms are the order-preserving maps of sets (we usually write X^n for $X([n])$). A morphism of cosimplicial objects is just a natural transformation of functors, so the category of cosimplicial objects of $\mathcal{C}$ is the functor category $\mathcal{C}^\Delta$.

If we replace the simplex category Δ with the strict simplex category Δ^+ consisting of the monomorphisms in Δ, then we have the notion of a *semi-cosimplicial object*. When $\mathcal{C}$ has finite products, we define the cosimplicial object $\Gamma^\bullet(X^\bullet)$ *cogenerated by* a semi-cosimplicial object $X^\bullet\colon \Delta^+ \to \mathcal{C}$ to be, at some $[n] \in \mathrm{ob}(\Delta)$

$$\Gamma^n(X^\bullet) := \prod_{[n]\twoheadrightarrow[k]} X^k$$

where the product is taken over all order-preserving surjections from the set $[n]$. The map $\Gamma^{n'}(X^\bullet) \to \Gamma^n(X^\bullet)$ associated to a map $f\colon [n'] \to [n]$ is defined to be the unique one making commute all squares

$$\begin{array}{ccc} \prod_{[n']\to[k']} X^{k'} & \longrightarrow & \prod_{[n]\to[k]} X^{k} \\ \downarrow{\scriptstyle \mathrm{proj}_{g'}} & & \downarrow{\scriptstyle \mathrm{proj}_{g}} \\ X^{k'} & \xrightarrow{X^\bullet(f')} & X^{k} \end{array}$$

for each $g\colon [n] \twoheadrightarrow [k]$, where $[n'] \overset{g'}{\twoheadrightarrow} [k'] \overset{f'}{\hookrightarrow} [k]$ is the epi-mono factorisation of gf. There is a natural morphism $\Gamma^\bullet(X^\bullet) \to X^\bullet$ of semi-cosimplicial objects, which is terminal among all such maps to $X^\bullet$ from a cosimplicial object.

Remark 5.1.2. Often, instead of considering all morphisms appearing in a cosimplicial object $X^\bullet$, we just consider the special generating set given by the *coface maps* $d^i\colon X^{n-1} \to X^n$ and *codegeneracy maps* $s^i\colon X^{n+1} \to X^n$ for $0 \leq i \leq n$, which are those induced from the unique injection $[n-1] \hookrightarrow [n]$ (resp. surjection $[n+1] \twoheadrightarrow [n]$) which misses $i \in [n]$ (resp. hits $i \in [n]$ twice). These are related by the *cosimplicial identities*:

- $d^j d^i = d^i d^{j-1}$ for $i < j$;
- $s^j s^i = s^i s^{j+1}$ for $i \leq j$; and
- $s^j d^i = \begin{cases} d^i s^{j-1} & \text{for } i < j, \\ \mathrm{id} & \text{for } i = j, j+1, \\ d^{i-1} s^j & \text{for } i > j+1; \end{cases}$

and, conversely, maps between the X^n satisfying these identities uniquely determine the structure of a cosimplicial object.

Remark 5.1.3. The functor $\Gamma^\bullet\colon \mathcal{C}^{\Delta^+} \to \mathcal{C}^\Delta$ is, when $\mathcal{C}$ is abelian, essentially one of the two functors giving the equivalence in the cosimplicial Dold–Kan correspondence [GJ09, Corollary III.2.3][1]. More precisely, if we precompose with the functor $\mathrm{CoCh}_+(\mathcal{C}) \to \mathcal{C}^{\Delta^+}$ taking an coconnected cochain complex $A^\bullet$ to the semi-cosimplicial object with the same objects and whose coface maps $A^{n-1} \to A^n$ all zero except $d^n := (-1)^n d$, then the composite $\mathrm{CoCh}_+(\mathcal{C}) \to \mathcal{C}^\Delta$ is one half of the equivalence of categories in the Dold–Kan correspondence. Because of this, $\Gamma^\bullet$ is often called the *denormalisation* functor (e.g. in [Ols11, Kri93]), and can be variously denoted by $\mathbf{D}$, D or K in different sources[2].

[1]The Dold–Kan correspondence in [GJ09] is given for simplicial objects rather than cosimplicial objects, but since the dual of an abelian category is again an abelian category, the dual statement holds just as well. This is spelled out explicitly in [GM04].

[2]There is an apparent discrepancy in the definition of the denormalisation functor in [Kri93, Section 1.4] and the dual to the definition in [GJ09, Section III.2], even though they are both supposed to be quasi-inverses to the same normalisation functor. To save confusion, we will work exclusively with the latter description.

The preceding remark shows that the category of coconnected cochain complexes is a full subcategory of the category of cosimplicial groups, and hence one can regard cosimplicial groups as a generalisation of cochain complexes more suited to non-abelian results. Accordingly, one hopes for suitable non-abelian generalisations of the cohomology of complexes, at least in small codimension, and this is provided by the *cohomotopy sets/groups* of Bousfield–Kan.

Definition 5.1.4. [Pir14, Section 2] Let $U^\bullet$ be a cosimplicial or semi-cosimplicial group. The 0th *cohomotopy group* $\pi^0(U^\bullet)$ and 1st *cohomotopy pointed set* $\pi^1(U^\bullet)$ are defined to be:

- $\pi^0(U^\bullet)$ is the equaliser of $d^0, d^1 \colon U^0 \to U^1$;
- $\pi^1(U^\bullet) := \mathrm{Z}^1(U^\bullet)/U^0$ where

$$\mathrm{Z}^1(U^\bullet) = \{u_1 \in U^1 \mid d^1(u_1) = d^2(u_1)d^0(u_1)\}$$

  is the set of *cocycles* and U^0 acts on this set by twisted conjugation $u_0 \colon u_1 \mapsto d^1(u_0)^{-1}u_1 d^0(u_0)$. The distinguished point is just the class of $1 \in \mathrm{Z}^1(U^\bullet)$.

When $U^\bullet$ is abelian, this can be extended further to give abelian cohomotopy groups $\pi^i(U^\bullet)$ for all $i \geq 0$, defined as the cohomology groups of the (unnormalised) Moore complex $C^\bullet(U^\bullet)$, whose terms are $C^n(U^\bullet) = U^n$ and whose differential is the alternating sum $\sum_k (-1)^k d^k$ of the coface maps. It is straightforward to check this agrees with the above definition of π^0 and π^1.

Note that $\pi^0(U^\bullet)$ makes sense for cosimplicial objects in any category with equalisers, so will inherit e.g. topologies or group actions when $U^\bullet$ has them.

We will see many variants on the following example throughout this thesis.

Example 5.1.5 (Double-coset spaces)**.** Let U be a group with subgroups $U', U'' \leq U$, and consider the cosimplicial group $U^\bullet$ cogenerated by the semi-cosimplicial group

$$U' \times U'' \rightrightarrows U,$$

where $d^0(u', u'') = u'$ and $d^1(u', u'') = u''$ (and subsequent terms are the trivial group). Then (for example using remark 5.1.7 shortly) we canonically identify the cohomotopy groups of $U^\bullet$ as

$$\pi^i(U^\bullet) \cong \begin{cases} U' \cap U'' & \text{if } i = 0, \\ U''\backslash U / U' & \text{if } i = 1, \\ 1 & \text{if } i > 1 \text{ and } U \text{ abelian.} \end{cases}$$

The following lemmas are essentially restatements of the Dold–Kan and Eilenberg–Zilber theorems, which will be useful in computations.

Lemma 5.1.6. *Let $A^\bullet$ be a semi-cosimplicial abelian group (or cosimplicial object in any abelian category), and $\Gamma^\bullet(A^\bullet)$ the cosimplicial abelian group cogenerated by it. Then the canonical map $\Gamma^\bullet(A^\bullet) \to A^\bullet$ of semi-cosimplicial groups induces an isomorphism on cohomotopy groups.*

Proof. Let $N^\bullet(\Gamma^\bullet(A^\bullet)) \leq \Gamma^\bullet(A^\bullet)$ be the normalised Moore subcomplex, whose objects are the common kernels

$$N^n(\Gamma^\bullet(A^\bullet)) := \bigcap_i \ker\left(s^i \colon \Gamma^n(A^\bullet) \to \Gamma^{n-1}(A^\bullet)\right)$$

of the codegeneracy maps and whose maps are the alternating sums of the coface maps. In our case, it follows by definition that $N^n(\Gamma^\bullet(A^\bullet))$ is the copy of A^n inside the direct sum $\Gamma^n(A^\bullet) = \bigoplus_{[n]\twoheadrightarrow[k]} A^k$, so that $N^\bullet(\Gamma^\bullet(A^\bullet))$ is identified with the unnormalised Moore complex $C^\bullet(A^\bullet)$ in such a way that the canonical map $C^\bullet(\Gamma^\bullet(A^\bullet)) \twoheadrightarrow C^\bullet(A^\bullet)$ is a retract of the inclusion $N^\bullet(\Gamma^\bullet(A^\bullet)) \hookrightarrow C^\bullet(\Gamma^\bullet(A^\bullet))$. By the dual of [GJ09, Theorem III.2.5] (where our $N^\bullet(-)$ corresponds to $(-)/D(-)$), this latter map induces an isomorphism on cohomology, and hence so does the map $C^\bullet(\Gamma^\bullet(A^\bullet)) \to C^\bullet(A^\bullet)$. Hence, by definition, $\Gamma^\bullet(A^\bullet) \to A^\bullet$ induces an isomorphism on cohomotopy, as desired. □

Remark 5.1.7. Lemma 5.1.6 is also true when $A^\bullet$ is replaced by a cosimplicial group $U^\bullet$, although in this case we can only say that $\Gamma^\bullet(U^\bullet) \to U^\bullet$ induces an isomorphism on π^0 and π^1. This is proved by a straightforward chasing of definitions, which we leave to the reader.

Lemma 5.1.8. *Let $A^{\bullet,\bullet}$ be a bi-semi-cosimplicial abelian group (i.e. a functor $\Delta^+ \times \Delta^+ \to \mathbf{Ab}$). Let $C^{\bullet,\bullet}(A^{\bullet,\bullet})$ be its Moore bicomplex (formed by taking the unnormalised Moore complex in the horizontal and vertical directions), and let $\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet})$ be the bi-cosimplicial abelian group formed by taking the cosimplicial abelian group cogenerated by the semi-cosimplicial abelian groups in each of the horizontal and vertical directions. Then there is a canonical isomorphism*

$$\pi^j(d^\bullet(\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet}))) \cong \mathrm{H}^j(\mathrm{Tot}^\bullet(C^{\bullet,\bullet}(A^{\bullet,\bullet})))$$

where $d^\bullet(\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet}))$ is the diagonal cosimplicial abelian group.

Proof. By the cosimplicial Eilenberg–Zilber theorem of [GM04, Appendix A.3], there is a canonical isomorphism

$$\pi^i(d^\bullet(\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet}))) \cong \mathrm{H}^i(\mathrm{Tot}^\bullet(C^{\bullet,\bullet}(\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet}))))$$

and by lemma 5.1.6 the canonical map $C^{\bullet,\bullet}(\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet})) \to C^{\bullet,\bullet}(A^{\bullet,\bullet})$ of bicomplexes induces an isomorphism on cohomology in both the horizontal and vertical directions. Then a standard spectral sequence argument (e.g. the dual of [GJ09, Section III.2.2]), shows that the induced map on the cohomology of the totalisation is an isomorphism, as desired. □

5.2 Long exact sequences of cohomotopy

For us, the main advantage of using cosimplicial groups is that they allow us to mechanistically produce long exact sequences (in the sense of definition 1.2.6) in much the same way as one produces long exact sequences in the cohomology of cochain complexes from short exact sequences of cochains. Unlike the abelian case, of course, the exact sequences we obtain will in general be finite in length, and moreover the length of the sequence we obtain will depend on how well-behaved the terms of the short exact sequence are. For our purposes, the following four cases will suffice.

Theorem 5.2.1. *1. Suppose $U^\bullet$ is a cosimplicial group acting degree-wise transitively on a cosimplicial set $Q^\bullet$ from the right, and let $Z^\bullet \hookrightarrow U^\bullet$ be the stabiliser of a chosen basepoint $* \to Q^\bullet$. Then there is a functorially assigned exact sequence*

$$1 \to \pi^0(Z^\bullet) \to \pi^0(U^\bullet) \xrightarrow{\curvearrowright} \pi^0(Q^\bullet) \to \pi^1(Z^\bullet) \to \pi^1(U^\bullet).$$

2. Suppose

$$1 \to Z^\bullet \to U^\bullet \to Q^\bullet \to 1$$

is a degree-wise exact sequence of cosimplicial groups. Then there is a functorially assigned exact sequence

$$1 \to \pi^0(Z^\bullet) \to \pi^0(U^\bullet) \to \pi^0(Q^\bullet) \xrightarrow{\curvearrowright} \pi^1(Z^\bullet) \to \pi^1(U^\bullet) \to \pi^1(Q^\bullet).$$

3. Suppose

$$1 \to Z^\bullet \xrightarrow{\mathsf{z}} U^\bullet \to Q^\bullet \to 1$$

is a degree-wise central extension of cosimplicial groups. Then there is a functorially assigned exact sequence

$$1 \to \pi^0(Z^\bullet) \xrightarrow{\mathsf{z}} \pi^0(U^\bullet) \to \pi^0(Q^\bullet) \to \pi^1(Z^\bullet) \xrightarrow{\frown} \pi^1(U^\bullet) \to \pi^1(Q^\bullet) \to \pi^2(Z^\bullet).$$

4. *Suppose*

$$1 \to Z^\bullet \to U^\bullet \to Q^\bullet \to 1$$

is a degree-wise exact sequence of cosimplicial abelian groups. Then there is a functorially assigned exact sequence of abelian groups

$$\cdots \to \pi^{r-1}(Q^\bullet) \to \pi^r(Z^\bullet) \to \pi^r(U^\bullet) \to \pi^r(Q^\bullet) \to \pi^{r+1}(Z^\bullet) \to \cdots .$$

Proof. Part (2) is [Pir14, Proposition 2.6.1] (originally [DM00, Lemma 3.2 and Proposition 3.2.1]) and [Pir14, Lemma 2.6.2], while part (3) is [Pir14, Proposition 2.8.2] (also originally in [DM00]). Neither of these sources explicitly mention the action of $\pi^1(Z^\bullet)$ on $\pi^1(U^\bullet)$, but this is just induced by the multiplication action of $\mathrm{Z}^1(Z^\bullet)$ on $\mathrm{Z}^1(U^\bullet)$. Part (4) is well-known, in that this is just the exact sequence on cohomology for the unnormalised Moore complexes.

Part (1) we do by hand, following [Ser97, Section I.5.4]. With the exception of the coboundary map $\delta\colon \pi^0(Q^\bullet) \to \pi^1(Z^\bullet)$, the maps and group action in the sequence are just the ones induced from the maps and group actions between the Z^j, U^j and Q^j. To define δ at some $q_0 \in \pi^0(Q^\bullet)$, we choose some $u_0 \in U^0$ such that $q_0 = *u_0$. The 0-cocycle condition for q_0 says exactly that $*d^0(u_0) = *d^1(u_0)$, so that $d^1(u_0)d^0(u_0)^{-1} \in Z^1$.

Since $d^1(u_0)d^0(u_0)^{-1}$ is the coboundary of u_0^{-1} in U^1 and $Z^2 \hookrightarrow U^2$ is injective, it is even a cocycle in Z^1. Moreover, since the choice of u_0 was unique up to left multiplication by Z^0, its class $[d^1(u_0)d^0(u_0)^{-1}] \in \pi^1(Z^\bullet)$ is independent of the choice of u_0, and we take this to be $\delta(q_0)$.

It remains to check exactness, which is mostly an exercise in chasing definitions.

- An element $[z_1] \in \pi^1(Z^\bullet)$ lies in the kernel of $\pi^1(Z^\bullet) \to \pi^1(U^\bullet)$ iff z_1 is of the form $d^1(u_0)^{-1}d^0(u_0)$, which occurs iff it is in the image of δ (as $\delta(*u_0^{-1})$).

- Two elements $*u_0, *u_0' \in \pi^0(Q^\bullet)$ lie in the same fibre of δ iff there is some $z_0 \in Z^0$ such that $d^1(u_0')d^0(u_0')^{-1} = d^1(z_0^{-1}u_0)d^0(z_0^{-1}u_0)^{-1}$. This rearranges to $d^1(u_0^{-1}z_0u_0') = d^0(u_0^{-1}z_0u_0')$, so that $u_0^{-1}z_0u_0' \in \pi^0(U^\bullet)$, and hence $*u_0' = (*u_0)u_0^{-1}z_0u_0'$ lies in the same $\pi^0(U^\bullet)$-orbit as $*u_0$. This reasoning is clearly reversible.

- Exactness of $1 \to \pi^0(Z^\bullet) \to \pi^0(U^\bullet) \to \pi^0(Q^\bullet)$ follows from the fact that π^0 is a right adjoint, specifically the right adjoint of the functor taking a pointed set to the corresponding constant cosimplicial pointed set.

This completes the proof of exactness. □

Remark 5.2.2. Each of the successive sequences in theorem 5.2.1 is an extension of the previous sequences. To be precise:

- the first five terms in (2) agree with the sequence assigned in (1) to the inverse left-multiplication action of $U^\bullet$ on $Q^\bullet$ given by $u\colon q \mapsto u^{-1}q$, and the action of $\pi^0(U^\bullet)$ on $\pi^0(Q^\bullet)$ is also given by inverse left-multiplication;
- the first six terms in (3) agree with the sequence assigned in (2), with the action of $\pi^0(Q^\bullet)$ on $\pi^1(Z^\bullet)$ being given by multiplication via the connecting homomorphism $\pi^0(Q^\bullet) \to \pi^1(Z^\bullet)$;
- the first seven terms in (4) agree with the sequence assigned in (3), with the action of $\pi^1(Z^\bullet)$ on $\pi^1(U^\bullet)$ being given by multiplication.

We will also need one instance where one of the exact sequences in theorem 5.2.1 can be extended with a final 1, which intuitively should be thought of as a dimensional vanishing result for the non-existent π^2. Specifically, if $U^\bullet$ is a cosimplicial group, we denote by $U^\bullet_{\leq n}$ the semi-cosimplicial group formed by truncating $U^\bullet$ above degree n. The canonical map $U^\bullet \twoheadrightarrow U^\bullet_{\leq n}$ of semi-cosimplicial groups factors uniquely through $U^\bullet \to \Gamma^\bullet(U^\bullet_{\leq n})$, and we say that $U^\bullet$ is *cogenerated in degree* $\leq n$ if this factored map is injective in each degree. In particular, if $U^\bullet$ is the cosimplicial group cogenerated by a semi-cosimplicial group supported in degrees $\leq n$, then it is cogenerated in degree $\leq n$ in the above sense.

Lemma 5.2.3. *Suppose*

$$1 \to Z^\bullet \to U^\bullet \to Q^\bullet \to 1$$

is a sequence of cosimplicial groups which is exact in each degree, and that $Z^\bullet$ is cogenerated in degree ≤ 1. Then the exact sequence in point (2) *of theorem 5.2.1 can be extended by a final map $\pi^1(Q^\bullet) \to 1$, i.e. the map $\pi^1(U^\bullet) \to \pi^1(Q^\bullet)$ is surjective.*

Proof. Pick $q_1 \in \mathrm{Z}^1(Q^\bullet)$, and lift it to an element $u_1 \in U^1$. The fact that q_1 is a cocycle tells us that $z_2 := d^2(u_1)^{-1}d^1(u_1)d^0(u_1)^{-1}$ lies in Z^2. Now by a quick computation with the cosimplicial identities, we see that $s^0(z_2) = d^1s^0(u_1)^{-1}$, $s^1(z_2) = d^0s^0(u_1)^{-1}$ and $s^0s^0(z_2) = s^0s^1(z_2) = s^0(u_1)^{-1}$. Hence, by replacing u_1 with $u_1s^0(z_2)$ (which still

maps to $q_1 \in Q^1$ since $s^0(z_2) \in Z^1$), we may assume that $s^0(z_2) = s^1(z_2) = 1$ and $s^0 s^0(z_2) = 1$. But this says exactly that z_2 is contained in the kernel of $Z^2 \to \Gamma^2(Z^\bullet_{\leq 1})$, so that $z_2 = 1$ by assumption that $Z^\bullet$ is cogenerated in degree ≤ 1. This then shows that $d^1(u_1) = d^2(u_1)d^0(u_1)$, so that u_1 is a cocycle, and hence $[q_1] \in \pi^1(Q^\bullet)$ is the image of $[u_1] \in \pi^1(U^\bullet)$ as desired. □

5.3 Twisting

Just as one uses Serre twists to get more precise information out of long exact sequences of group cohomology, one can formulate the notion of the *twist* ${}_\beta U^\bullet$ of a cosimplicial group by to a 1-cocycle $\beta \in \mathrm{Z}^1(U^\bullet)$ [Pir14, Section 2.4]. This has the same objects and codegeneracy maps as $U^\bullet$, and the only coface maps which are changed are the $d^0 \colon U^{n-1} \to U^n$, which are replaced with the twisted maps

$$\overline{d^0} \colon u_{n-1} \mapsto d^n \cdots d^2(\beta) \cdot d^0(u_{n-1}) \cdot d^n \cdots d^2(\beta)^{-1}$$

where $d^n \cdots d^2(\beta) := \beta$ when $n = 1$.

The twisting has two key properties generalising those of Serre twists, namely that it doesn't change the first cohomotopy, and that twisting by a coboundary does not change the cosimplicial group.

Proposition 5.3.1. *[Pir14, Proposition 2.4.3] Let $U^\bullet$ be a cosimplicial group and $\beta \in \mathrm{Z}^1(U^\bullet)$. There is a canonical bijection $\pi^1({}_\beta U^\bullet) \xrightarrow{\sim} \pi^1(U^\bullet)$ given by right-multiplication by β, taking the distinguished point of $\pi^1({}_\beta U^\bullet)$ to $[\beta] \in \pi^1(U^\bullet)$.*

Proposition 5.3.2. *Let $U^\bullet$ be a cosimplicial group and $\beta' = d^1(u_0)^{-1}\beta d^0(u_0)$ two cocycles differing by the action of some $u_0 \in U^0$. Then there is an isomorphism ${}_{\beta'}U^\bullet \xrightarrow{\sim} {}_\beta U^\bullet$ canonically determined by u_0 such that the induced map on π^1 is the composite*

$$\pi^1\left({}_{\beta'}U^\bullet\right) \xrightarrow{\sim} \pi^1\left(U^\bullet\right) \xleftarrow{\sim} \pi^1\left({}_\beta U^\bullet\right),$$

and in particular is independent of u_0.

Proof. The isomorphism is given on ${}_{\beta'}U^n$ by

$$v_n \mapsto d^n \cdots d^1(u_0) \cdot v_n \cdot d^n \cdots d^1(u_0)^{-1}.$$

This is clearly degree-wise an isomorphism, so to check it gives an isomorphism of cosimplicial groups one just has to check that the coface and codegeneracy maps are

preserved. This is a straightforward exercise using the cosimplicial identities and their consequence that

$$s^i d^n \cdots d^1 = d^{n-1} \cdots d^1, \text{ and}$$
$$d^i d^n \cdots d^1 = \begin{cases} d^{n+1} \cdots d^2 d^1 & \text{if } i \neq 0, \\ d^{n+1} \cdots d^2 d^0 & \text{if } i = 0. \end{cases}$$

That the induced map on π^1 is as described is a straightforward calculation using the explicit description in proposition 5.3.1. □

5.4 Group cohomology

As explained in [Pir14, Section 2.9], cohomotopy of cosimplicial groups is a generalisation of non-abelian group cohomology. Specifically, if G is a topological group acting continuously (from the left) on another topological group U (we call U a *topological G-group*), one constructs a cosimplicial group of continuous cochains $C^\bullet(G, U)$, where $C^n(G, U) = \mathrm{Map}_{\mathrm{cts}}(G^n, U)$, where the coface maps $d^i \colon C^{n-1}(G, U) \to C^n(G, U)$ and the codegeneracy maps $s^i \colon C^{n+1}(G, U) \to C^n(G, U)$ are given by

$$d^i(f) \colon (g_1, \ldots, g_n) \mapsto \begin{cases} g_1 \cdot f(g_2, \ldots, g_n) & \text{if } i = 0, \\ f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n, \\ f(g_1, \ldots, g_{n-1}) & \text{if } i = n; \end{cases}$$
$$s^i(f) \colon (g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n).$$

By definition of non-abelian group cohomology, we have equalities

$$\pi^i(C^\bullet(G, U)) = \mathrm{H}^i(G, U)$$

for $j = 0, 1$, and for $i > 1$ when U is abelian (in fact, we even have equalities on the level of cocycles). Moreover, these descriptions are compatible with twisting, in that for a continuous cocycle $\alpha \in \mathrm{Z}^1(G, U) = \mathrm{Z}^1(C^\bullet(G, U))$, the cosimplicial group $C^\bullet(G, {}_\alpha U)$ of continuous cochains for the Serre twist ${}_\alpha U$ is equal to the twisted cosimplicial group ${}_\alpha C^\bullet(G, U)$, and the canonical isomorphism $\mathrm{H}^1(G, {}_\alpha U) \xrightarrow{\sim} \mathrm{H}^1(G, U)$ is just the canonical isomorphism on π^1 described in proposition 5.3.1.

With this in mind, we see that theorem 5.2.1 is a generalisation of various well-known long exact sequences in non-abelian group cohomology, as for example in [Ser97, Section I.5]. Since the statements there only deal with the case of discrete topological G-groups, we will trouble the reader with a sufficiently general statement for our applications.

Corollary 5.4.1 (to theorem 5.2.1)**.** *Let G be a topological group.*

1. *Suppose U is a topological G-group with acting continuously, G-equivariantly and transitively on a topological G-set Q from the right, and $Z \hookrightarrow U$ is the stabiliser of a G-fixed basepoint $* \in Q$, given the subspace topology from U. Then there is a functorially assigned exact sequence*

$$1 \to Z^G \to U^G \xrightarrow{\curvearrowright} Q^G \to \mathrm{H}^1(G, Z) \to \mathrm{H}^1(G, U).$$

2. *Suppose*

$$1 \to Z \to U \to Q \to 1$$

is an exact sequence of topological G-groups, such that $Z \subseteq U$ has the subspace topology and $U \to Q$ has a continuous splitting (as topological spaces). Then there is a functorially assigned exact sequence

$$1 \to Z^G \to U^G \to Q^G \xrightarrow{\curvearrowright} \mathrm{H}^1(G, Z) \to \mathrm{H}^1(G, U) \to \mathrm{H}^1(G, Q).$$

3. *If in the previous point Z is central in U, then the exact sequence can be extended to a functorially assigned sequence*

$$1 \to Z^G \xrightarrow{\mathsf{z}} U^G \to Q^G \to \mathrm{H}^1(G, Z) \xrightarrow{\curvearrowright} \mathrm{H}^1(G, U) \to \mathrm{H}^1(G, Q) \to \mathrm{H}^2(G, Z).$$

4. *If in the previous point U is abelian, then the exact sequence can be extended to a functorially assigned exact sequence of abelian groups*

$$\cdots \to \mathrm{H}^{r-1}(G, Q) \to \mathrm{H}^r(G, Z) \to \mathrm{H}^r(G, U) \to \mathrm{H}^r(G, Q) \to \mathrm{H}^{r+1}(G, Z) \to \cdots.$$

Proof. For points (2)–(4), the given conditions ensure that the sequence

$$1 \to C^\bullet(G, Z) \to C^\bullet(G, U) \to C^\bullet(G, Q) \to 1$$

of cosimplicial groups satisfies the conditions for the corresponding point of theorem 5.2.1. For point (1), the given conditions ensure that $C^\bullet(G, Z)$ is the stabiliser of the basepoint of $C^\bullet(G, Q)$ under the action of $C^\bullet(G, U)$, and that the action of $C^\bullet(G, U)$ on $C^\bullet(G, Q)$ is transitive in degree 0. This was all our proof of (1) required, so we are done. □

Remark 5.4.2. The first point in corollary 5.4.1 is insensitive to the topology on Q, and can be rephrased along the lines of [Ser97, Section I.5.4]. Specifically, it says that if $Z \subseteq U$ is a G-stable topological subgroup of a G-group U, given the subspace topology, then

$$1 \to Z^G \to U^G \xrightarrow{\curvearrowright} (Z \backslash U)^G \to \mathrm{H}^1(G, Z) \to \mathrm{H}^1(G, U)$$

is exact.

5.5 Cosimplicial algebras and cdgas

In order to construct the cosimplicial groups that will be of interest later, it will be necessary to have some tools to allow us to define certain cosimplicial algebras (the cosimplicial group then being produced by evaluating a group scheme on these algebras). In many cases, the construction in definition 5.1.1 of a cosimplicial algebra from a semi-cosimplicial algebra will suffice, but a few of our constructions are best described in terms of (graded-)commutative differential graded algebras following the cosimplicial monoidal Dold–Kan correspondence.

Recall from [CC04] that if $A^\bullet$ is a dga over some base ring R, then we can identify the terms of the cosimplicial R-module $\Gamma^\bullet(A^\bullet)$ corresponding to $A^\bullet$ under the Dold–Kan correspondence as

$$\Gamma^n(A^\bullet) \cong \bigoplus_j \bigwedge^j R^n \otimes_R A^j$$

and that this inherits a natural R-algebra structure[3] from the algebra structures on $A^\bullet$ and $\bigwedge^\bullet R^n$. Moreover, when $A^\bullet$ is graded-commutative, it is easy to see that this algebra structure is commutative, so that according to [CC04] $\Gamma^\bullet(A^\bullet)$ is then a cosimplicial (commutative) R-algebra.

Remark 5.5.1. For our purposes, we will treat this construction simply as a tool for producing cosimplicial algebras, but it is worth recalling that this fits into the wider context of the *cosimplicial monoidal Dold–Kan correspondence*. Specifically, the denormalisation functor

$$\Gamma^\bullet \colon \mathrm{CoCh}_+(R) \to \mathbf{Mod}_R^\Delta$$

is a lax symmetric monoidal functor, meaning that there is a natural transformation called the *Eilenberg–Zilber* or *shuffle* map[4]

$$\nabla \colon \Gamma^\bullet(A^\bullet) \otimes \Gamma^\bullet(B^\bullet) \to \Gamma^\bullet(A^\bullet \otimes B^\bullet)$$

compatible with interchange of $A^\bullet$ and $B^\bullet$ (with respect to the usual symmetric monoidal structure on both sides). For purely abstract reasons, this means that $\Gamma^\bullet$

[3] [CC04] works only over the base ring $\mathbb{Z}$, but it is easy to see that the ring structure they define is automatically an R-algebra structure when the dg-ring is replaced by an R-dga

[4] *Eilenberg–Zilber* because the corresponding map $N^\bullet(\Gamma^\bullet(A^\bullet) \otimes \Gamma^\bullet(B^\bullet)) \to N^\bullet\Gamma^\bullet(A^\bullet \otimes B^\bullet) \cong A^\bullet \otimes B^\bullet \cong N^\bullet\Gamma^\bullet(A^\bullet) \otimes N^\bullet\Gamma^\bullet(B^\bullet)$ is the dual map to the simplicial Eilenberg–Zilber map, and *shuffle* because the simplicial Eilenberg–Zilber map has an explicit description in terms of shuffles [Lan63, Theorem VIII.8.5].

takes (c)dgas to cosimplicial (commutative) R-algebras, whose multiplication is given by the composite

$$\Gamma^\bullet(A^\bullet) \otimes \Gamma^\bullet(A^\bullet) \xrightarrow{\nabla} \Gamma^\bullet(A^\bullet \otimes A^\bullet) \xrightarrow{\Gamma^\bullet(\mathrm{mult})} \Gamma^\bullet(A^\bullet).$$

However, unlike the classical Dold–Kan correspondence, $\Gamma^\bullet$ does not induce an equivalence of categories, and (for our purposes) worse, the canonical dga structure on $N^\bullet(A^\bullet)$ for a cosimplicial R-algebra $A^\bullet$ can fail to be commutative, even when $A^\bullet$ is (since the monoidal structure on $N^\bullet$, induced from the dual Alexander–Whitney map, is not symmetric). In order to circumvent these problems properly, one needs to pass to appropriate homotopy categories on either side, and replace $N^\bullet$ with the *functor of Thom-Sullivan cochains*, as explained in [KPT08, Section 1.1].

Example 5.5.2. The particular topological $\mathbb{Q}_p$-cdga we will be interested in is $\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet$, whose underlying cochain complex is defined to be

$$\mathsf{B}_{\mathrm{st}} \xrightarrow{N} \mathsf{B}_{\mathrm{st}}$$

and whose graded algebra structure is the one induced by the obvious B_{st}-module structure on B_{st}. It has a canonical G_K-linear continuous Frobenius automorphism φ, which acts as the usual Frobenius φ on $\mathsf{B}_{\mathrm{st}}[\varepsilon]^0 = \mathsf{B}_{\mathrm{st}}$ and as $p\varphi$ on $\mathsf{B}_{\mathrm{st}}[\varepsilon]^1 = \mathsf{B}_{\mathrm{st}}$.

Following through the definition of the algebra structure in [CC04], we see that the objects of its associated cosimplicial topological $\mathbb{Q}_p$-algebra are

$$\Gamma^n(\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet) = \mathsf{B}_{\mathrm{st}}[\varepsilon_1, \dots, \varepsilon_n]$$

where ε_j are variables subject to the relation $\varepsilon_j\varepsilon_k = 0$ for all j, k. The topology on $\mathsf{B}_{\mathrm{st}}[\varepsilon_1, \dots, \varepsilon_n]$ is just the natural topology from its structure as an $(n+1)$-dimensional free B_{st}-module, its G_K-action is the natural B_{st}-semilinear one fixing $\mathbb{Q}_p[\varepsilon_1, \dots, \varepsilon_n]$, and the induced semilinear Frobenius fixes $1 \in \mathsf{B}_{\mathrm{st}}[\varepsilon_1, \dots, \varepsilon_n]$ and sends each ε_i to $p\varepsilon_i$.

Finally, we will also need to hybridise our two constructions to produce cosimplicial algebras out of semi-cosimplicial cdgas $A^{\bullet,\bullet}$.

Construction 5.5.3. Let $A^{\bullet,\bullet}$ be a semi-cosimplicial R-cdga. We produce a cosimplicial R-algebra $d^\bullet(\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet}))$ as follows. We start by replacing $A^{\bullet,\bullet}$ with the cosimplicial R-cdga cogenerated by it as in definition 5.1.1, and then replacing each of its terms by its associated cosimplicial R-algebra as above. This produces us a bicosimplicial R-algebra $\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet})$, and we let $d^\bullet(\Gamma^{\bullet,\bullet}(A^{\bullet,\bullet}))$ be the restriction to the diagonal.

It is easy to see that this simultaneously generalises both the earlier constructions, corresponding to the case when each of the individual cdgas concentrated in degree 0, and the case when the semi-cosimplicial cdga itself is concentrated in degree 0.

Chapter 6

Unipotent Bloch–Kato theory

Our aim in this chapter is to explain in detail how one can use cosimplicial groups and cosimplicial algebras to extend the analysis of local Bloch–Kato Selmer groups to the unipotent setting. All of our arguments are then essentially just rewritings of abelian arguments in such a way that they apply equally well in the unipotent setting, and the reader who is less familiar with the non-abelian theory should be able to follow the trace of these unwritten abelian arguments in the proofs below. As remarked in the introduction, the unifying theme of this chapter will be the definition of certain cosimplicial $\mathbb{Q}_p$-algebras $\mathsf{B}_*^\bullet$ for $* \in \{e, f, g, g/e, f/e\}$ such that for a de Rham representation of G_K on a finitely generated pro-unipotent group $U/\mathbb{Q}_p$, the corresponding local Bloch–Kato Selmer set or quotient $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ is computed by the first cohomotopy of the cosimplicial group $U(\mathsf{B}_*^\bullet)^{G_K}$.

From the point of view of our application, the main result in this chapter is corollary 6.3.6, which gives a long exact sequence relating the Bloch–Kato quotients $\mathrm{H}^1_{g/e}$ for central extensions of representations on finitely generated pro-unipotent groups, and plays the same role in the proof of theorem 1.2.3 as the long exact sequence in non-abelian Galois cohomology does in the proof of theorem 1.1.6 in chapter 3.

6.1 The cosimplicial algebras $\mathsf{B}_e^\bullet$, $\mathsf{B}_f^\bullet$ and $\mathsf{B}_g^\bullet$

When V is a de Rham (linear) representation of G_K, the Bloch–Kato Selmer group $\mathrm{H}^1_e(G_K, V)$ is typically studied via the Bloch–Kato exponential sequence [BK90, Corollary 3.8.4], which arises by tensoring V with the short exact sequence

$$0 \to \mathbb{Q}_p \to \mathsf{B}_{\mathrm{dR}}^+ \oplus \mathsf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathsf{B}_{\mathrm{dR}} \to 0$$

and taking the long exact sequence in Galois cohomology. Put another way, if one defines the 2-term cochain complex $\mathsf{C}_e^\bullet$ to be

$$\mathsf{C}_e^\bullet := (\mathsf{B}_{\mathrm{dR}}^+ \oplus \mathsf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathsf{B}_{\mathrm{dR}})$$

(with the map sending (x, y) to $x - y$), then one has a canonical identification

$$\mathrm{H}^1\big((\mathsf{C}_e^\bullet \otimes V)^{G_K}\big) \cong \mathrm{H}^1_e(G_K, V).$$

The advantage of this latter cohomological point of view, as noted in [Nek93, Theorem 1.23], is that one can treat both $\mathrm{H}^1_f(G_K, V)$ and $\mathrm{H}^1_g(G_K, V)$ on an equal footing by defining cochain complexes $\mathsf{C}_f^\bullet$ and $\mathsf{C}_g^\bullet$ by

$$\begin{aligned}\mathsf{C}_f^\bullet &:= (\mathsf{B}_{\mathrm{dR}}^+ \oplus \mathsf{B}_{\mathrm{cris}} \to \mathsf{B}_{\mathrm{dR}} \oplus \mathsf{B}_{\mathrm{cris}}), \text{ and}\\ \mathsf{C}_g^\bullet &:= (\mathsf{B}_{\mathrm{dR}}^+ \oplus \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{dR}} \oplus \mathsf{B}_{\mathrm{st}} \oplus \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{st}}),\end{aligned}$$

where the map in the former complex sends (x, y) to $(x - y, (\varphi - 1)y)$, and in the latter complex the first map sends (x, y) to $(x-y, (\varphi-1)y), Ny)$ and the second map sends (x, y, z) to $(p\varphi - 1)z - Ny$. With these definitions, one then has a canonical identification [Nek93, Theorem 1.23]

$$\mathrm{H}^1\big((\mathsf{C}_g^\bullet \otimes V)^{G_K}\big) \cong \mathrm{H}^1_g(G_K, V),$$

and similarly for H^1_f. Thus, in all cases, the cochain complex $(\mathsf{C}_*^\bullet \otimes V)^{G_K}$ is a homological-algebraic invariant associated to the representation V which refines the Bloch–Kato Selmer groups $\mathrm{H}^1_*(G_K, V)$.

Remark 6.1.1. All three cochain complexes $\mathsf{C}_e^\bullet$, $\mathsf{C}_f^\bullet$, $\mathsf{C}_g^\bullet$ defined above are resolutions of $\mathbb{Q}_p$: their cohomology is $\mathbb{Q}_p$ in degree 0 and vanishes in higher degrees. For $\mathsf{C}_e^\bullet$ and $\mathsf{C}_f^\bullet$, this is simply a rephrasing of the two well-known exact sequences [BK90, Proposition 1.17], except that are using different conventions for the signs of the morphisms, while for $\mathsf{C}_g^\bullet$ the surjectivity of $N\colon \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{st}}$ reduces it to the case of $\mathsf{C}_f^\bullet$.

In order to generalise this idea to the unipotent setting, we will replace the cochain complexes $\mathsf{C}_*^\bullet$ with certain cosimplicial $\mathbb{Q}_p$-algebras $\mathsf{B}_*^\bullet$ with G_K-action, such that when U is a de Rham representation of G_K on a finitely generated pro-unipotent group, the cosimplicial group $U(\mathsf{B}_*^\bullet)^{G_K}$ recovers the non-abelian Bloch–Kato Selmer set $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ as its first cohomotopy. The algebras $\mathsf{B}_*^\bullet$ themselves are quite straightforward to define, being simply built out of copies of the standard period rings.

Definition 6.1.2. We define three cosimplicial $\mathbb{Q}_p$-algebras $\mathsf{B}_e^\bullet$, $\mathsf{B}_f^\bullet$ and $\mathsf{B}_g^\bullet$ as follows:

- $\mathsf{B}_e^\bullet$ is the cosimplicial $\mathbb{Q}_p$-algebra cogenerated by the semi-cosimplicial $\mathbb{Q}_p$-algebra

$$\mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}}^{\varphi=1} \rightrightarrows \mathsf{B}_{\mathrm{dR}}$$

  (with higher terms the zero ring) where $d^0(x,y) = x$ and $d^1(x,y) = y$;

- $\mathsf{B}_f^\bullet$ is the cosimplicial $\mathbb{Q}_p$-algebra cogenerated by the semi-cosimplicial $\mathbb{Q}_p$-algebra

$$\mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}} \rightrightarrows \mathsf{B}_{\mathrm{dR}} \times \mathsf{B}_{\mathrm{cris}}$$

  where $d^0(x,y) = (x, \varphi y)$, $d^1(x,y) = (y,y)$;

- $\mathsf{B}_g^\bullet$ is the cosimplicial $\mathbb{Q}_p$-algebra constructed from the semi-cosimplicial $\mathbb{Q}_p$-cdga

$$\mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet \rightrightarrows \mathsf{B}_{\mathrm{dR}} \times \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet$$

  as in construction 5.5.3. Here $\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet$ is the $\mathbb{Q}_p$-cdga in example 5.5.2, and the maps in the semi-cosimplicial $\mathbb{Q}_p$-cdga are given by $d^0(x,y) = (x, \varphi y)$ and $d^1(x,y) = (y,y)$.

All of these cosimplicial $\mathbb{Q}_p$-algebras inherit a topology and an action of G_K from those on B_{dR}, B_{st} and $\mathsf{B}_{\mathrm{cris}}$. This action is degree-wise continuous.

Remark 6.1.3. One could just as easily use the topologically more amenable period ring $\mathsf{B}_{\max}$ in place of $\mathsf{B}_{\mathrm{cris}}$ in our theory [Col98, Section III.2], but fortunately our arguments are sufficiently insensitive to the topology that the difference is irrelevant to us.

One important property of the algebras $\mathsf{B}_e^\bullet$, $\mathsf{B}_f^\bullet$ and $\mathsf{B}_g^\bullet$ is that they are resolutions of $\mathbb{Q}_p$, in the sense that their 0th cohomotopy is canonically identified with $\mathbb{Q}_p$ (as a topological $\mathbb{Q}_p$-algebra with G_K-action), and their higher cohomotopy vanishes. Moreover, we will also see that this immediately implies that $U(\mathsf{B}_*^\bullet)$ is a resolution of $U(\mathbb{Q}_p)$ whenever $U/\mathbb{Q}_p$ is a finitely generated pro-unipotent group—this is analogous to the fact that tensoring with a linear representation is exact.

Proposition 6.1.4. *We have $\pi^0(\mathsf{B}_e^\bullet) = \pi^0(\mathsf{B}_f^\bullet) = \pi^0(\mathsf{B}_g^\bullet) = \mathbb{Q}_p$ (as a topological $\mathbb{Q}_p$-algebra with G_K-action), and $\pi^i(\mathsf{B}_e^\bullet) = \pi^i(\mathsf{B}_f^\bullet) = \pi^i(\mathsf{B}_g^\bullet) = 0$ for $i \geq 1$ (as a $\mathbb{Q}_p$-vector space/abelian group).*

Proof. For $\mathsf{B}_e^\bullet$ and $\mathsf{B}_f^\bullet$, by lemma 5.1.6 their cohomotopy groups agree with the cohomotopy groups of the generating semi-cosimplicial $\mathbb{Q}_p$-algebra, which are, respectively, the cohomology groups of the cochains $\mathsf{C}_e^\bullet$ and $\mathsf{C}_f^\bullet$ above. But these are $\mathbb{Q}_p$ in degree 0 and vanish in higher degrees by remark 6.1.1.

For $\mathsf{B}_g^\bullet$, by lemma 5.1.8 its cohomotopy groups agree with the cohomotopy groups of the totalisation of the bicomplex

$$\begin{array}{ccc} \mathsf{B}_{\mathrm{dR}}^+ \oplus \mathsf{B}_{\mathrm{st}} & \xrightarrow{\begin{pmatrix} 1 & -1 \\ 0 & \varphi-1 \end{pmatrix}} & \mathsf{B}_{\mathrm{dR}} \oplus \mathsf{B}_{\mathrm{st}} \\ \downarrow{\scriptstyle (0\ N)} & & \downarrow{\scriptstyle (0\ N)} \\ \mathsf{B}_{\mathrm{st}} & \xrightarrow{p\varphi-1} & \mathsf{B}_{\mathrm{st}}, \end{array}$$

namely $\mathsf{C}_g^\bullet$. Again, this is a resolution of $\mathbb{Q}_p$ by remark 6.1.1.

To finish, we need to check that the algebra structure, topology and G_K-action on $\pi^0(\mathsf{B}_e^\bullet)$, $\pi^0(\mathsf{B}_f^\bullet)$ and $\pi^0(\mathsf{B}_g^\bullet)$ agree with those on $\mathbb{Q}_p$. The only non-obvious one is the topology; this follows since the subspace topology on $\mathbb{Q}_p \subseteq \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}}$ is the usual one (which is automatic since $\mathbb{Q}_p$ is locally compact and $\mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}}$ is Hausdorff). □

Lemma 6.1.5. *Let $U/\mathbb{Q}_p$ be a finitely generated pro-unipotent group, and $\mathsf{B}^\bullet$ a cosimplicial topological $\mathbb{Q}_p$-algebra such that $\pi^0(\mathsf{B}^\bullet) = \mathbb{Q}_p$ (as a topological $\mathbb{Q}_p$-algebra) and $\pi^i(\mathsf{B}^\bullet) = 0$ for $i > 0$ (as a $\mathbb{Q}_p$-vector space). Then $\pi^0(U(\mathsf{B}^\bullet)) = U(\mathbb{Q}_p)$ (as a topological group), $\pi^1(U(\mathsf{B}^\bullet)) = 1$, and $\pi^i(U(\mathsf{B}^\bullet)) = 1$ for $i > 1$ when U is abelian.*

Proof. If U is abelian, then it is $\mathbb{G}_a^d$ for some d, so it suffices to consider the case $U = \mathbb{G}_a$. But then $U(\mathsf{B}^\bullet) \cong \mathsf{B}^\bullet$ as cosimplicial topological abelian groups, so the result follows by assumption.

For U non-abelian and finite-dimensional, we proceed by induction, writing U as a central extension

$$1 \to Z \xrightarrow{\mathsf{z}} U \to Q \to 1$$

where we already know the result for Z and Q. Since the sequence

$$1 \to Z(\mathsf{B}^\bullet) \xrightarrow{\mathsf{z}} U(\mathsf{B}^\bullet) \to Q(\mathsf{B}^\bullet) \to 1$$

is a central extension in each degree, we have by 5.2.1 an exact sequence

$$1 \to \pi^0(Z(\mathsf{B}^\bullet)) \xrightarrow{\mathsf{z}} \pi^0(U(\mathsf{B}^\bullet)) \to \pi^0(Q(\mathsf{B}^\bullet)) \to \pi^1(Z(\mathsf{B}^\bullet)) \xrightarrow{\curvearrowright} \pi^1(U(\mathsf{B}^\bullet)) \to \pi^1(Q(\mathsf{B}^\bullet))$$

which by our inductive assumption gives the desired description of $\pi^0(U(\mathsf{B}^\bullet))$ and $\pi^1(U(\mathsf{B}^\bullet))$. That the topology on $\pi^0(U(\mathsf{B}^\bullet))$ is the desired one follows, since after

identifying U with its Lie algebra $\mathrm{Lie}(U)$ we see that $\pi^0(U(\mathsf{B}^\bullet))$ is homeomorphic to $\pi^0\left(\mathsf{B}^\bullet \otimes \mathrm{Lie}(U)\right)$, which is homeomorphic to $\mathrm{Lie}(U) \cong U(\mathbb{Q}_p)$ by the abelian case.

For general U, we write $U = \varprojlim U_n$ as an inverse limit of finite-dimensional quotients. Since π^0 commutes with inverse limits of topological groups, we have the desired description of $\pi^0(U(\mathsf{B}^\bullet))$. To prove triviality of $\pi^1(U(\mathsf{B}^\bullet))$, it suffices to prove that the map $\pi^1(U(\mathsf{B}^\bullet)) \to \varprojlim \pi^1(U_n(\mathsf{B}^\bullet))$ has trivial kernel. Thus suppose $\alpha \in \mathrm{Z}^1(U(\mathsf{B}^\bullet)) = \varprojlim \mathrm{Z}^1(U(\mathsf{B}^\bullet))$ is an inverse limit of coboundaries. For each n, we write T_n for the set of elements of $U_n(\mathsf{B}^0)$ whose coboundary is the image of α in $\mathrm{Z}^1(U(\mathsf{B}^\bullet))$. T_n is then a left torsor under $\pi^0(U_n(\mathsf{B}^\bullet)) = U_n(\mathbb{Q}_p)$, so that the maps $T_m \to T_n$ for $m \geq n$ are all surjective and $\varprojlim T_n \neq \emptyset$. An element of the inverse limit is then an element $u \in \varprojlim U_n(\mathsf{B}^0) = U(\mathsf{B}^0)$ whose coboundary is α, so that $[\alpha] = *$ as desired. □

6.2 The non-abelian Bloch–Kato exponential

Combining proposition 6.1.4 and lemma 6.1.5 shows that, for $U/\mathbb{Q}_p$ a representation of G_K on a finitely generated pro-unipotent group, there is a G_K-equivariant exact sequence

$$1 \to U(\mathbb{Q}_p) \to U(\mathsf{B}^0_*) \xrightarrow{\curvearrowright} \mathrm{Z}^1(U(\mathsf{B}^\bullet_*)) \to 1$$

for $* \in \{e, f, g\}$. When $* = e$, the terms are given explicitly by

$$1 \to U(\mathbb{Q}_p) \to U(\mathsf{B}^+_{\mathrm{dR}}) \times U(\mathsf{B}^{\varphi=1}_{\mathrm{cris}}) \xrightarrow{\curvearrowright} U(\mathsf{B}_{\mathrm{dR}}) \to 1$$

(where the group structure on $U(\mathsf{B}_{\mathrm{dR}})$ plays no role). This suggests, following the construction in the abelian case, that a non-abelian analogue of the Bloch–Kato exponential sequence could be obtained by taking the long exact sequence on non-abelian Galois cohomology.

In order to perform this construction, we will need the following non-abelian variant of [BK90, Lemma 3.8.1].

Proposition 6.2.1. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then the map $\mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}})) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$ has trivial kernel.*

Proof. When U is finite-dimensional, we may proceed inductively. The case that U is abelian is [BK90, Lemma 3.8.1], so we need only consider the case when U is a G_K-equivariant central extension

$$1 \to Z \xrightarrow{\mathsf{z}} U \to Q \to 1$$

and we already know the lemma for Z and Q. Since all the varieties involved are affine spaces, this sequence has a splitting as varieties (not necessarily compatible with the group structure), so that $Z(\mathsf{B}^+_{\mathrm{dR}}) \subseteq U(\mathsf{B}^+_{\mathrm{dR}})$ has the subspace topology and $U(\mathsf{B}^+_{\mathrm{dR}}) \to Q(\mathsf{B}^+_{\mathrm{dR}})$ has a continuous splitting, and similarly for B_{dR}. Hence we obtain (part of) an exact sequence in cohomology:

$$\begin{array}{ccccc} \mathrm{H}^1(G_K, Z(\mathsf{B}^+_{\mathrm{dR}})) & \xrightarrow{\frown} & \mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}})) & \longrightarrow & \mathrm{H}^1(G_K, Q(\mathsf{B}^+_{\mathrm{dR}})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^1(G_K, Z(\mathsf{B}_{\mathrm{dR}})) & \xrightarrow{\frown} & \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}})) & \longrightarrow & \mathrm{H}^1(G_K, Q(\mathsf{B}_{\mathrm{dR}})). \end{array}$$

However, the assumption that U is de Rham implies that Z and Q are too, and hence that the sequence

$$1 \to \mathsf{D}_{\mathrm{dR}}(Z) \xrightarrow{\mathsf{z}} \mathsf{D}_{\mathrm{dR}}(U) \to \mathsf{D}_{\mathrm{dR}}(Q) \to 1$$

is exact (equivalently, exact on Lie algebras, or exact on K-points—see corollary 2.1.6). Thus the coboundary map $\mathsf{D}_{\mathrm{dR}}(Q)(K) = \mathrm{H}^0(G_K, Q(\mathsf{B}_{\mathrm{dR}})) \to \mathrm{H}^1(G_K, Z(\mathsf{B}_{\mathrm{dR}}))$ is zero and $\mathrm{H}^1(G_K, Z(\mathsf{B}_{\mathrm{dR}})) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$ has trivial kernel.

With this observation, it is a straightforward diagram-chase to verify as desired that $\mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}})) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$ has trivial kernel. If $u_1 \in \mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}}))$ is in the kernel, then its image in $\mathrm{H}^1(G_K, Q(\mathsf{B}^+_{\mathrm{dR}}))$ lies in the corresponding kernel, so that inductively this image is trivial and hence u_1 is the image of some $z_1 \in \mathrm{H}^1(G_K, Z(\mathsf{B}^+_{\mathrm{dR}}))$. Since the map $\mathrm{H}^1(G_K, Z(\mathsf{B}_{\mathrm{dR}})) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$ has trivial kernel, it follows that $z_1 = *$ and hence $u_1 = *$ also.

For general U, we write $U = \varprojlim U_n$ as an inverse limit of finite-dimensional G_K-equivariant quotients, which are de Rham. Since the composite

$$\mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}})) \to \varprojlim \mathrm{H}^1(G_K, U_n(\mathsf{B}^+_{\mathrm{dR}})) \to \varprojlim \mathrm{H}^1(G_K, U_n(\mathsf{B}_{\mathrm{dR}}))$$

(where the latter map has trivial kernel by the finite-dimensional case) factors through $\mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}})) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$, it would suffice for us to prove that the map $\mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}})) \to \varprojlim \mathrm{H}^1(G_K, U_n(\mathsf{B}^+_{\mathrm{dR}}))$ has trivial kernel.

To do this, suppose that $\alpha \in \mathrm{Z}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}}))$ is a continuous cocycle whose image in each $\mathrm{Z}^1(G_K, U_n(\mathsf{B}^+_{\mathrm{dR}}))$ is a coboundary, and write T_n for the set of all elements of $U_n(\mathsf{B}^+_{\mathrm{dR}})$ whose coboundary is the image of α. Then T_n is a left torsor under $U_n(\mathsf{B}^+_{\mathrm{dR}})^{G_K} = \mathsf{D}^+_{\mathrm{dR}}(U_n)(K)$. Since the $\mathsf{D}^+_{\mathrm{dR}}(U_n)$ are unipotent groups over K, it follows that the sets $(T_n)_{n\in\mathbb{N}}$ satisfy the Mittag–Leffler condition, and hence $\varprojlim T_n \neq \emptyset$. By inspection, an element of $\varprojlim T_n \subseteq \varprojlim U_n(\mathsf{B}^+_{\mathrm{dR}}) = U(\mathsf{B}^+_{\mathrm{dR}})$ is an element whose coboundary is α, so that $[\alpha] \in \mathrm{H}^1(G_K, U(\mathsf{B}^+_{\mathrm{dR}}))$ is trivial, as desired. □

Corollary 6.2.2. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group, and let $\mathrm{Z}^1(U(\mathsf{B}_g^\bullet))$ be the 1-cocycles of the cosimplicial group $U(\mathsf{B}_g^\bullet)$. Then the twisted conjugation action of $U(\mathsf{B}_g^0)$ on $\mathrm{Z}^1(U(\mathsf{B}_g^\bullet))$ is transitive with point-stabiliser $U(\mathbb{Q}_p)$, so that by corollary 5.4.1 we have an exact sequence*

$$1 \to U(\mathbb{Q}_p)^{G_K} \to U(\mathsf{B}_g^0)^{G_K} \overset{\curvearrowright}{\to} \mathrm{Z}^1(U(\mathsf{B}_g^\bullet))^{G_K} \overset{\delta}{\to} \mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_g^0)).$$

Furthermore, the image of the coboundary map is the Bloch–Kato cohomology set $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$. The same is true of $\mathsf{B}_f^\bullet$ (for H^1_f) and $\mathsf{B}_e^\bullet$ (for H^1_e).

Proof. We will just prove this for $\mathsf{B}_g^\bullet$, the other cases being similar. By proposition 6.1.4, the cosimplicial topological $\mathbb{Q}_p$-algebra $\mathsf{B}_g^\bullet$ satisfies the conditions of lemma 6.1.5, which tells us that the action of $U(\mathsf{B}_g^0)$ on $\mathrm{Z}^1(U(\mathsf{B}_g^\bullet))$ is transitive, with point-stabiliser $U(\mathbb{Q}_p)$, so that we have the claimed exact sequence.

To conclude, we want to prove that $\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_g^0))$ has kernel exactly $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$. To do this, we note that by construction $\mathsf{B}_g^0 = \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{st}}$ and that B_g^1 has a natural G_K-equivariant projection to B_{dR}, so that the map $\mathbb{Q}_p \to \mathsf{B}_g^0 = \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{st}}$ equalises the natural pair of maps $\mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{st}} \rightrightarrows \mathsf{B}_{\mathrm{dR}}$. Taking points in U and taking Galois cohomology (both of which preserve products), we see that the image of

$$\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_g^0)) = \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}^+)) \times \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}}))$$

is contained in the equaliser of

$$\mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}^+)) \times \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}})) \rightrightarrows \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}})).$$

Now on the one hand, the kernel of $\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_g^0))$ is clearly contained in $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ by definition. On the other, if $[\alpha] \in \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ then its image in $\mathrm{H}^1(G_K, U(\mathsf{B}_g^0)) = \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}^+)) \times \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}}))$ is of the form $(*, [\beta])$, where by the above discussion $[\beta]$ maps to the distinguished point $* \in \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$. But by proposition 6.2.1, this implies that $[\beta] = *$ already, so that $[\alpha]$ lies in the kernel of $\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_g^0))$ as desired. □

A convenient way to view the non-abelian Bloch–Kato exponential is as giving a canonical description of the cohomotopy of $U(\mathsf{B}_*^\bullet)^{G_K}$, as follows.

Theorem 6.2.3. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then the cohomotopy groups/sets of the cosimplicial groups $U(\mathsf{B}_e^\bullet)^{G_K}$, $U(\mathsf{B}_f^\bullet)^{G_K}$ and $U(\mathsf{B}_g^\bullet)^{G_K}$ are canonically identified as*

$$\pi^i(U(\mathsf{B}_e^\bullet)^{G_K}) = \begin{cases} U(\mathbb{Q}_p)^{G_K} & \text{if } i = 0, \\ \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) & \text{if } i = 1, \\ 1 & \text{if } i > 1 \text{ and } U \text{ abelian;} \end{cases}$$

$$\pi^i(U(\mathsf{B}_f^\bullet)^{G_K}) = \begin{cases} U(\mathbb{Q}_p)^{G_K} & \text{if } i = 0, \\ \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) & \text{if } i = 1, \\ 1 & \text{if } i > 1 \text{ and } U \text{ abelian;} \end{cases}$$

$$\pi^i(U(\mathsf{B}_g^\bullet)^{G_K}) = \begin{cases} U(\mathbb{Q}_p)^{G_K} & \text{if } i = 0, \\ \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) & \text{if } i = 1, \\ \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U(\mathbb{Q}_p)^*(1))^* & \text{if } i = 2 \text{ and } U \text{ abelian,} \\ 1 & \text{if } i > 2 \text{ and } U \text{ abelian.} \end{cases}$$

Proof. Let us do the case of $U(\mathsf{B}_g^\bullet)^{G_K}$, the other cases being similar (and simpler, since one need only work with $\mathbb{Q}_p$-algebras rather than $\mathbb{Q}_p$-cdgas). For π^0 and π^1, this is exactly the content of corollary 6.2.2 since $\mathrm{Z}^1(U(\mathsf{B}_g^\bullet))^{G_K} = \mathrm{Z}^1(U(\mathsf{B}_g^\bullet)^{G_K})$.

It remains to calculate the higher cohomotopy when $U = V$ is abelian. In this case, $U(\mathsf{B}_g^\bullet) = \mathsf{B}_g^\bullet \otimes V$ is the diagonal in the bi-cosimplicial $\mathbb{Q}_p$-representation cogenerated by the bi-semi-cosimplicial $\mathbb{Q}_p$-representation

$$\begin{array}{ccc} (\mathsf{B}^+_{\mathrm{dR}} \otimes V) \oplus (\mathsf{B}_{\mathrm{st}} \otimes V) & \rightrightarrows & (\mathsf{B}_{\mathrm{dR}} \otimes V) \oplus (\mathsf{B}_{\mathrm{st}} \otimes V) \\ {\scriptstyle (0\,0)}\downarrow\downarrow{\scriptstyle (0\,-N)} & & {\scriptstyle (0\,0)}\downarrow\downarrow{\scriptstyle (0\,-N)} \\ \mathsf{B}_{\mathrm{st}} \otimes V & \rightrightarrows & \mathsf{B}_{\mathrm{st}} \otimes V. \end{array}$$

Since the operations of passing to the associated bi-cosimplicial representation and restricting to the diagonal commute with taking G_K-invariants, the same remains true after taking G_K-invariants, and hence by lemma 5.1.8 the cohomotopy of $(\mathsf{B}_g^\bullet \otimes V)^{G_K}$ is canonically isomorphic to the cohomology of the cochain $(\mathsf{C}_g^\bullet \otimes V)^{G_K}$, where $\mathsf{C}_g^\bullet$ is the cochain at the beginning of section 6.2. Concretely, this is a cochain complex

$$\mathsf{D}^+_{\mathrm{dR}}(V) \oplus \mathsf{D}_{\mathrm{st}}(V) \to \mathsf{D}_{\mathrm{dR}}(V) \oplus \mathsf{D}_{\mathrm{st}}(V) \oplus \mathsf{D}_{\mathrm{st}}(V) \to \mathsf{D}_{\mathrm{st}}(V),$$

so the cohomology vanishes in degrees > 2, and the cohomology in degree 2 is the K_0-linear dual of the kernel of

$$(0, p\varphi^* - 1, -N^*) \colon \mathsf{D}_{\mathrm{st}}(V)^* \to \mathsf{D}_{\mathrm{dR}}(V)^* \oplus \mathsf{D}_{\mathrm{st}}(V)^* \oplus \mathsf{D}_{\mathrm{st}}(V)^*,$$

which is $(\mathsf{D}_{\mathrm{st}}(V)^*)^{N=0, p\varphi=1} = \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V^*(1))$ as desired. □

6.2.1 Compatibilities

In our study of Bloch–Kato quotients, we will need to know various compatibilities between the descriptions in theorem 6.2.3. The implicit natural isomorphisms in that theorem already tell us that the descriptions are compatible with homomorphisms of representations, but we will also want to know that the descriptions of $\pi^i(U(\mathsf{B}_e^\bullet)^{G_K})$, $\pi^i(U(\mathsf{B}_f^\bullet)^{G_K})$ and $\pi^i(U(\mathsf{B}_g^\bullet)^{G_K})$ are compatible with one another, and that they are compatible with Serre twists. Since this section consists entirely of verifying that certain obvious compatibilities are in fact true, we advise the reader to skip this section on the first reading.

To make this first compatibility precise, we observe that there are canonical G_K-equivariant cosimplicial $\mathbb{Q}_p$-algebra homomorphisms $\mathsf{B}_e^\bullet \hookrightarrow \mathsf{B}_f^\bullet \hookrightarrow \mathsf{B}_g^\bullet$, where the second embedding is induced by the obvious embedding

$$\begin{array}{ccc} \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}} & \longhookrightarrow & \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet \\ \downdownarrows & & \downdownarrows \\ \mathsf{B}_{\mathrm{dR}} \times \mathsf{B}_{\mathrm{cris}} & \longhookrightarrow & \mathsf{B}_{\mathrm{dR}} \times \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet \end{array}$$

of semi-cosimplicial $\mathbb{Q}_p$-cdgas using construction 5.5.3. There is a minor subtlety in the definition of the morphism $\mathsf{B}_e^\bullet \hookrightarrow \mathsf{B}_f^\bullet$, in that it is not induced by a morphism of the defining semi-cosimplicial $\mathbb{Q}_p$-algebras. But by the universal property of cosimplicial algebras cogenerated by semi-cosimplicial algebras, to construct this morphism it suffices to define a morphism of semi-cosimplicial $\mathbb{Q}_p$-algebras from $\mathsf{B}_e^\bullet$ to the semi-cosimplicial $\mathbb{Q}_p$-algebra defining $\mathsf{B}_f^\bullet$. Such a morphism is provided in degrees 0 and 1 by the maps

$$\begin{aligned} \mathsf{B}_e^0 = \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}}^{\varphi=1} &\longrightarrow \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}} \\ (x, y) &\longmapsto (x, y) \\ \mathsf{B}_e^1 = \mathsf{B}_{\mathrm{dR}}^+ \times \mathsf{B}_{\mathrm{cris}}^{\varphi=1} \times \mathsf{B}_{\mathrm{dR}} &\longrightarrow \mathsf{B}_{\mathrm{dR}} \times \mathsf{B}_{\mathrm{cris}} \\ (x, y, z) &\longmapsto (z, y) \end{aligned}$$

and in higher degrees by the zero morphism to the zero ring. It is then a straightforward chasing of definitions to check that this indeed defines a morphism of semi-cosimplicial $\mathbb{Q}_p$-algebras.

Equipped with these embeddings $\mathsf{B}_e^\bullet \hookrightarrow \mathsf{B}_f^\bullet \hookrightarrow \mathsf{B}_g^\bullet$, we can now state the compatibility result for theorem 6.2.3.

Theorem 6.2.4. *With respect to the descriptions in theorem 6.2.3, the maps on cohomotopy induced by the maps* $U(\mathsf{B}_e^\bullet)^{G_K} \hookrightarrow U(\mathsf{B}_f^\bullet)^{G_K} \hookrightarrow U(\mathsf{B}_g^\bullet)^{G_K}$ *coming from evaluating* U *on* $\mathsf{B}_e^\bullet \hookrightarrow \mathsf{B}_f^\bullet \hookrightarrow \mathsf{B}_g^\bullet$ *are identified as:*

- *the identity maps* $U(\mathbb{Q}_p)^{G_K} = U(\mathbb{Q}_p)^{G_K} = U(\mathbb{Q}_p)^{G_K}$ *on* π^0*;*
- *the natural inclusions* $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \hookrightarrow \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \hookrightarrow \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ *on* π^1*;*
- *the zero maps* $1 = 1 \hookrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U(\mathbb{Q}_p)^*(1))^*$ *on* π^2 *when* U *is abelian;*
- *the zero maps* $1 = 1 = 1$ *on the higher* π^i *when* U *is abelian.*

Proof. The extra compatibility relations when U is abelian are automatic. For π^0 and π^1, we just need to recall that the exact sequences assigned in theorem 5.2.1 are functorial, and hence the exact sequences in corollary 6.2.2 fit together into a commuting diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & U(\mathbb{Q}_p)^{G_K} & \longrightarrow & U(\mathsf{B}_e^0)^{G_K} & \xrightarrow{\curvearrowright} & \mathrm{Z}^1(U(\mathsf{B}_e^\bullet))^{G_K} \longrightarrow \mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \\
 & & \| & & \downarrow & & \downarrow \qquad\qquad \| \\
1 & \longrightarrow & U(\mathbb{Q}_p)^{G_K} & \longrightarrow & U(\mathsf{B}_f^0)^{G_K} & \xrightarrow{\curvearrowright} & \mathrm{Z}^1(U(\mathsf{B}_f^\bullet))^{G_K} \longrightarrow \mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \\
 & & \| & & \downarrow & & \downarrow \qquad\qquad \| \\
1 & \longrightarrow & U(\mathbb{Q}_p)^{G_K} & \longrightarrow & U(\mathsf{B}_g^0)^{G_K} & \xrightarrow{\curvearrowright} & \mathrm{Z}^1(U(\mathsf{B}_g^\bullet))^{G_K} \longrightarrow \mathrm{H}^1(G_K, U(\mathbb{Q}_p)).
\end{array}$$

The induced maps on π^1 (resp. π^0) are exactly the induced maps on the cokernel (resp. kernel) of the middle horizontal arrow, and thus are $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \hookrightarrow \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \hookrightarrow \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ (resp. the identity) as claimed. □

The second compatibility we will be interested in is compatibility with Serre twists, which is just a straightforward chasing of definitions.

Lemma 6.2.5. *Let* $U/\mathbb{Q}_p$ *be a de Rham representation of* G_K *on a finitely generated pro-unipotent group. Fix an element of* $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \cong \pi^1\left(U(\mathsf{B}_g^\bullet)^{G_K}\right)$*, and pick representatives as a Galois cocycle* $\alpha \in \mathrm{Z}^1(G_K, U(\mathbb{Q}_p))$ *and as a cocycle* $\beta \in \mathrm{Z}^1(U(\mathsf{B}_g^\bullet)^{G_K})$*. Then there is an isomorphism*

$$ {}_\beta\left(U(\mathsf{B}_g^\bullet)^{G_K}\right) \xrightarrow{\sim} \left({}_\alpha U(\mathsf{B}_g^\bullet)^{G_K}\right)$$

of cosimplicial groups such that the induced map on π^1 *makes the diagram*

$$\begin{array}{ccccc}
\pi^1\left({}_\beta\left(U(\mathsf{B}_g^\bullet)^{G_K}\right)\right) & \xrightarrow{\sim} & \pi^1\left({}_\alpha U(\mathsf{B}_g^\bullet)^{G_K}\right) & \xrightarrow{\sim} & \mathrm{H}^1_g(G_K, {}_\alpha U(\mathbb{Q}_p)) \\
\downarrow\wr & & & & \downarrow\wr \\
\pi^1\left(U(\mathsf{B}_g^\bullet)^{G_K}\right) & & \xrightarrow{\qquad\qquad\sim\qquad\qquad} & & \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))
\end{array}$$

commute, where the rightmost horizontal isomorphisms are those from theorem 6.2.3 (note that ${}_\alpha U$ is de Rham by proposition 4.3.3) and the vertical maps are the canonical identifications.

The same is true for H^1_f *(resp.* H^1_e*) when* $\mathsf{B}_g^\bullet$ *is replaced with* $\mathsf{B}_f^\bullet$ *(resp.* $\mathsf{B}_e^\bullet$*).*

Proof. We prove only the case of H^1_g, the other cases being similar. Note that it suffices to prove the result for any single choice of representative β of our fixed class, since then if β' were another choice, we would have a cosimplicial group isomorphism ${}_{\beta'}\left(U(\mathsf{B}_g^\bullet)^{G_K}\right) \xrightarrow{\sim} {}_\beta\left(U(\mathsf{B}_g^\bullet)^{G_K}\right)$ by proposition 5.3.2, so an isomorphism ${}_\beta\left(U(\mathsf{B}_g^\bullet)^{G_K}\right) \xrightarrow{\sim} \left({}_\alpha U(\mathsf{B}_g^\bullet)^{G_K}\right)$ would allow us to produce a similar isomorphism for β'; commutativity of the desired rectangle for β' follows from that for β and proposition 5.3.2.

Thus fix any choice of α. Since by assumption the class of α lies in $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) = \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_g^0))\right)$ (see the proof of corollary 6.2.2), we may choose some $u \in U(\mathsf{B}_g^0)$ whose coboundary is the Galois cocycle α. We then consider the cocycle $\beta \in \mathrm{Z}^1(U(\mathsf{B}_g^\bullet))$ which is the coboundary of u^{-1}, i.e. $\beta := d^1(u)d^0(u)^{-1}$. Since the coface and coboundary maps in $U(\mathsf{B}_g^\bullet)$ are G_K-equivariant, we thus have by straightforward calculation that

$$\beta^{-1}\sigma(\beta) = d^0(u)d^1(\alpha(\sigma))d^0(\alpha(\sigma))^{-1}d^0(u)^{-1} = 1$$

for all $\sigma \in G_K$, where the final equality comes from the fact that α takes values in $U(\mathbb{Q}_p)$, the equaliser of $d^0, d^1 \colon U(\mathsf{B}_g^0) \rightrightarrows U(\mathsf{B}_g^1)$. Hence β is G_K-fixed, so is an element of $\mathrm{Z}^1(U(\mathsf{B}_g^\bullet))^{G_K} = \mathrm{Z}^1(U(\mathsf{B}_g^\bullet)^{G_K})$. Moreover, the description of the coboundary map in theorem 5.2.1 shows that the class of β maps to that of α under our isomorphism $\pi^1\left(U(\mathsf{B}_g^\bullet)\right) \xrightarrow{\sim} \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$.

Thus we need only prove the result for this β and this α, using u to construct the desired isomorphism. Indeed, it suffices to construct from u a G_K-equivariant isomorphism

$${}_\beta\left(U(\mathsf{B}_g^\bullet)\right) \xrightarrow{\sim} {}_\alpha U(\mathsf{B}_g^\bullet)$$

of cosimplicial groups, since the G_K-invariants of the left-hand cosimplicial group is ${}_\beta\left(U(\mathsf{B}_g^\bullet)^{G_K}\right)$ as β is G_K-fixed.

To do this, since β is the coboundary of u^{-1}, proposition 5.3.2 provides an isomorphism of cosimplicial groups

$$\psi_u \colon {}_\beta\left(U(\mathsf{B}_g^\bullet)\right) \xrightarrow{\sim} U(\mathsf{B}_g^\bullet)$$

which need not be G_K-equivariant (as u^{-1} need not be G_K-fixed). However, ${}_\alpha U(\mathsf{B}_g^\bullet)$ has the same underlying cosimplicial group as $U(\mathsf{B}_g^\bullet)$, and we claim that ψ_u is G_K-equivariant when its codomain is given the α-twisted G_K-action, so that by taking G_K-fixed points we obtain the desired result.

Verifying this is a simple chasing of definitions. If $g \in G_K$ and $u_n \in {}_\beta\left(U(\mathsf{B}_g^n)\right) = U(\mathsf{B}_g^n)$, then we have

$$\begin{aligned}
\psi_u(gu_n) &= d^n \cdots d^1(u^{-1}) \cdot gu_n \cdot d^n \cdots d^1(u^{-1})^{-1} \\
&= d^n \cdots d^1(u^{-1}g(u)) \cdot g(\psi_u(u_n)) \cdot d^n \cdots d^1(u^{-1}g(u))^{-1} \\
&= d^n \cdots d^0(\alpha(g)) \cdot g(\psi_u(u_n)) \cdot d^n \cdots d^0(\alpha(g))^{-1}
\end{aligned}$$

which is the α-twisted action on ${}_\alpha U$, since the composite $d^n \cdots d^0 \colon \pi^0(U(\mathsf{B}_g^\bullet)) \to U(\mathsf{B}_g^n)$ is just the canonical inclusion $U(\mathbb{Q}_p) \hookrightarrow U(\mathsf{B}_g^n)$. This establishes the G_K-equivariance of ψ_u, and hence concludes the proof. □

6.3 Bloch–Kato quotients

We can also use similar cosimplicial techniques to analyse the Bloch–Kato quotients $\mathrm{H}^1_{g/e}$ and $\mathrm{H}^1_{f/e}$ in the non-abelian setting (the author is unfortunately currently unaware of how to make the right definitions to conduct a similar analysis of $\mathrm{H}^1_{g/f}$, but understanding the two other quotients will suffice for our purposes). Our preferred method for carrying out this analysis is to define cosimplicial algebras $\mathsf{B}^\bullet_{g/e}$ and $\mathsf{B}^\bullet_{f/e}$, so that the corresponding Bloch–Kato quotients can be described in terms of the first cohomotopy of $U(\mathsf{B}^\bullet_*)^{G_K}$, just as in theorem 6.3.2 gave the corresponding description for local Bloch–Kato Selmer sets.

Definition 6.3.1. We define two cosimplicial $\mathbb{Q}_p$-algebras $\mathsf{B}^\bullet_{f/e}$ and $\mathsf{B}^\bullet_{g/e}$ as follows:

- $\mathsf{B}^\bullet_{f/e}$ is the cosimplicial $\mathbb{Q}_p$-algebra cogenerated by the semi-cosimplicial topological $\mathbb{Q}_p$-algebra

  $$\mathsf{B}_{\mathrm{cris}} \rightrightarrows \mathsf{B}_{\mathrm{cris}}$$

  where $d^0 = \varphi$ and $d^1 = 1$;

- $\mathsf{B}^\bullet_{g/e}$ is the cosimplicial $\mathbb{Q}_p$-algebra constructed from the semi-cosimplicial $\mathbb{Q}_p$-cdga

$$\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet \rightrightarrows \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet$$

as in construction 5.5.3. Here $\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet$ is the $\mathbb{Q}_p$-cdga in example 5.5.2, and $d^0 = \varphi$ and $d^1 = 1$.

Both of these cosimplicial $\mathbb{Q}_p$-algebras inherit an action of G_K from those on B_{st} and $\mathsf{B}_{\mathrm{cris}}$. They also carry a topology, though the results we have proven up to this point mean that we will not have to concern ourselves with their topologies.

Theorem 6.3.2. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then the cohomotopy groups/sets of the cosimplicial topological groups $U(\mathsf{B}^\bullet_{f/e})^{G_K}$ and $U(\mathsf{B}^\bullet_{g/e})^{G_K}$ are canonically identified as*

$$\pi^i(U(\mathsf{B}^\bullet_{f/e})^{G_K}) = \begin{cases} \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) & \text{if } i = 0, \\ \mathrm{H}^1_{f/e}(G_K, U(\mathbb{Q}_p)) & \text{if } i = 1, \\ 1 & \text{if } i > 1 \text{ and } U \text{ abelian.} \end{cases}$$

$$\pi^i(U(\mathsf{B}^\bullet_{g/e})^{G_K}) = \begin{cases} \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) & \text{if } i = 0, \\ \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) & \text{if } i = 1, \\ \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U(\mathbb{Q}_p)^*(1))^* & \text{if } i = 2 \text{ and } U \text{ abelian,} \\ 1 & \text{if } i > 2 \text{ and } U \text{ abelian.} \end{cases}$$

Moreover, with respect to the descriptions in theorem 6.2.3, the natural maps on cohomotopy induced from $U(\mathsf{B}^\bullet_f)^{G_K} \to U(\mathsf{B}^\bullet_{f/e})^{G_K}$ and $U(\mathsf{B}^\bullet_g)^{G_K} \to U(\mathsf{B}^\bullet_{g/e})^{G_K}$ are the canonical inclusion $U(\mathbb{Q}_p)^{G_K} \hookrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p)$ in degree 0, the defining quotient map $\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \twoheadrightarrow \mathrm{H}^1_{f/e}(G_K, U(\mathbb{Q}_p))$ (and similarly for g/e) in degree 1, and the identity in higher degrees.

Before we embark on a proof of this theorem, let us collect a few straightforward consequences. Firstly, we obtain a natural criterion for the equality of H^1_e and H^1_f, thereby making good on our promise in remark 1.2.4.

Corollary 6.3.3. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group. Then $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) = \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))$ if and only if $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) = 1$.*

Proof. Note that $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p)$ and $\mathrm{H}^1_{f/e}(G_K, U(\mathbb{Q}_p))$ are, respectively, the 0th and 1st cohomotopy of the cosimplicial group $U(\mathsf{B}^\bullet_{f/e})^{G_K}$, which is cogenerated by the semi-cosimplicial group

$$\mathsf{D}_{\mathrm{cris}}(U)(K_0) \rightrightarrows \mathsf{D}_{\mathrm{cris}}(U)(K_0)$$

with coface maps $d^0 = \varphi$ and $d^1 = 1$, as defined in section 4.2.2. Hence by remark 5.1.7 we see that $\mathrm{H}^1_{f/e}(G_K, U(\mathbb{Q}_p)) = 1$ iff the twisted conjugation action $u\colon w \mapsto u^{-1}w\varphi(u)$ of $\mathsf{D}_{\mathrm{cris}}(U)(K_0)$ on itself is transitive, and that $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) = \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))$ iff it has trivial point-stabiliser.

Thus it would suffice to show that for any unipotent group D/K_0 with a semilinear Frobenius automorphism φ, the twisted conjugation self-action of D is transitive iff it has trivial point-stabiliser. This is established by the same argument as in proposition 4.1.6. □

Remark 6.3.4. Just as in remark 4.1.7, it is reasonable to expect that $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) = 1$ should hold whenever U is the $\mathbb{Q}_p$-pro-unipotent fundamental group of a smooth connected variety. Rather than explicitly formulating an analogue of conjecture 4.1.8, let us merely note that the conditions of corollary 6.3.3 hold for the $\mathbb{Q}_p$-linear Tate module of an abelian variety [BK90, Example 3.11], for $\mathbb{Q}_p(1)$ [BK90, Example 3.9], or for the fundamental group of $L^\times = L \setminus 0$ for L a line bundle on an abelian variety A/K, since by exact sequence 3.0.1 $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p)$ sits in an exact sequence

$$1 \to \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(\mathbb{Q}_p(1)) \to \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) \to \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V_pA)$$

and the outer terms vanish.

The second consequence of theorem 6.3.2 is that we can make good on our promise in remark 4.3.4 and prove that $\mathrm{H}^1_{g/e}$ preserves inverse limits.

Lemma 6.3.5. *Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group, and write $U = \varprojlim U_n$ as an inverse limit of finite-dimensional G_K-equivariant quotients. Then the natural map*

$$\mathrm{H}^1_g(G_K, \varprojlim U_n(\mathbb{Q}_p)) \to \varprojlim \mathrm{H}^1_g(G_K, U_n(\mathbb{Q}_p))$$

is bijective. The same holds with H^1_g replaced with H^1_f, H^1_e, $\mathrm{H}^1_{g/e}$ or $\mathrm{H}^1_{f/e}$.

Proof. Let us prove the lemma for $\mathrm{H}^1_{g/e}$, the other cases being similar. By theorem 6.3.2, it suffices to prove that the natural map

$$\pi^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right) \to \varprojlim \pi^1\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$$

is bijective.

To show surjectivity, we pick an element of $\varprojlim \pi^1\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ and, for each n, write S_n for the set of elements of $\mathrm{Z}^1\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ whose class is the chosen element of $\pi^1\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$. Each S_n comes with a transitive action of $U_n(\mathsf{B}^0_{g/e})^{G_K} =$

$\mathsf{D}_{\mathrm{st}}(U_n)(K_0)$. Since the system $\left(\mathsf{D}_{\mathrm{st}}(U_n)(K_0)\right)_{n\in\mathbb{N}}$ satisfies the Mittag–Leffler condition, so too does $(S_n)_{n\in\mathbb{N}}$, so that $\varprojlim S_n \neq \emptyset$. But any element of $\varprojlim S_n \subseteq \mathrm{Z}^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ represents an element of $\pi^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ mapping to the chosen element of $\varprojlim \pi^1\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$, proving surjectivity.

To show that the map has trivial kernel, pick an element $\alpha \in \mathrm{Z}^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ whose class lies in the kernel, i.e. such that the image of α in each $\mathrm{Z}^1\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ is a coboundary. Writing $T_n \subseteq U_n(\mathsf{B}^0_{g/e})^{G_K}$ for the set of elements whose coboundary is the image of α, we observe that each T_n is a left torsor under $\pi^0\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right) = \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U_n)(\mathbb{Q}_p)$, so that as before $(T_n)_{n\in\mathbb{N}}$ satisfies the Mittag–Leffler condition and $\varprojlim T_n \neq \emptyset$. Again, an element of $\varprojlim T_n \subseteq U(\mathsf{B}^0_{g/e})^{G_K}$ is an element whose coboundary is α, so we have proved triviality of the kernel.

Finally, to show injectivity, as in the proof of lemma 4.0.5, it suffices to show that the map also has trivial kernel when the cosimplicial group $U(\mathsf{B}^\bullet_{g/e})^{G_K}$ is replaced by the twist by a cocycle $\alpha \in \mathrm{Z}^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ (and each $U_n(\mathsf{B}^\bullet_{g/e})^{G_K}$ is compatibly twisted). For this, the same argument as above applies once we observe that each $U_n(\mathsf{B}^\bullet_{g/e})^{G_K}$ and hence ${}_\alpha\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ is the $\mathbb{Q}_p$-points of a cosimplicial unipotent group $\mathsf{D}^\bullet_{g/e}(U_n)/\mathbb{Q}_p$, so that the subgroups $\pi^0\left({}_\alpha\left(U_n(\mathsf{B}^\bullet_{g/e})^{G_K}\right)\right)$ are the $\mathbb{Q}_p$-points of (unipotent) algebraic subgroups of ${}_\alpha\mathsf{D}^\bullet_{g/e}(U_n)$, and hence satisfy the Mittag–Leffler condition. □

The final consequence of theorem 6.3.2 is two exact sequences relating the Bloch–Kato quotients in central extensions, which will allow us to analyse these sets in the same way we analysed non-abelian Galois cohomology in chapter 3.

Corollary 6.3.6. *Let*

$$1 \to Z \xrightarrow{\mathsf{z}} U \to Q \to 1$$

be a central extension of de Rham representations of G_K on unipotent groups over $\mathbb{Q}_p$. Then there are functorially assigned exact sequences

$$\begin{array}{l} 1 \longrightarrow Z(\mathbb{Q}_p)^{G_K} \xrightarrow{\mathsf{z}} U(\mathbb{Q}_p)^{G_K} \longrightarrow Q(\mathbb{Q}_p)^{G_K} \\ \quad \hookrightarrow \mathrm{H}^1_g(G_K, Z(\mathbb{Q}_p)) \xrightarrow{\curvearrowright} \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \longrightarrow \mathrm{H}^1_g(G_K, Q(\mathbb{Q}_p)) \\ \quad\quad \hookrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Z(\mathbb{Q}_p)^*(1))^* \end{array}$$

and

$$1 \longrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Z)(\mathbb{Q}_p) \xrightarrow{\mathsf{z}} \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) \longrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Q)(\mathbb{Q}_p) \hookrightarrow \mathrm{H}^1_{g/e}(G_K, Z(\mathbb{Q}_p)) \xrightarrow{\curvearrowright} \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \longrightarrow \mathrm{H}^1_{g/e}(G_K, Q(\mathbb{Q}_p)) \hookrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Z(\mathbb{Q}_p)^*(1))^*.$$

Moreover, when U and Q are abelian, both exact sequences continue on with the terms $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U(\mathbb{Q}_p)^*(1))^* \to \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Q(\mathbb{Q}_p)^*(1))^* \to 1$.

Proof. Take the long exact sequence in cohomotopy of the two exact sequences (which are central extensions) in proposition 6.3.7 which follows. □

Proposition 6.3.7. *Let*

$$1 \to Z \to U \to Q \to 1$$

be an exact sequence of de Rham representations of G_K on finitely generated pro-unipotent groups. Then the sequences

$$1 \to Z(\mathsf{B}^\bullet_g)^{G_K} \to U(\mathsf{B}^\bullet_g)^{G_K} \to Q(\mathsf{B}^\bullet_g)^{G_K} \to 1$$

$$1 \to Z(\mathsf{B}^\bullet_{g/e})^{G_K} \to U(\mathsf{B}^\bullet_{g/e})^{G_K} \to Q(\mathsf{B}^\bullet_{g/e})^{G_K} \to 1$$

are exact in each degree.

Proof. We will just prove this for $\mathsf{B}^\bullet_g$, the other case being similar. Observe that, by construction, each term of $U(\mathsf{B}^\bullet_g)$ is a product of terms of the form $U(\mathsf{B}_{\mathrm{dR}})$, $U(\mathsf{B}^+_{\mathrm{dR}})$ and $U(\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^n)$, where $\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^n = \mathsf{B}_{\mathrm{st}}[\varepsilon_1, \ldots, \varepsilon_n]$ is as in example 5.5.2, so that it suffices to show that the sequences

$$1 \to Z(\mathsf{B}_{\mathrm{dR}})^{G_K} \to U(\mathsf{B}_{\mathrm{dR}})^{G_K} \to Q(\mathsf{B}_{\mathrm{dR}})^{G_K} \to 1$$

$$1 \to Z(\mathsf{B}^+_{\mathrm{dR}})^{G_K} \to U(\mathsf{B}^+_{\mathrm{dR}})^{G_K} \to Q(\mathsf{B}^+_{\mathrm{dR}})^{G_K} \to 1$$

$$1 \to Z(\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^n)^{G_K} \to U(\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^n)^{G_K} \to Q(\mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^n)^{G_K} \to 1$$

are exact.

But now we can identify these sequences with the sequences

$$1 \to \mathsf{D}_{\mathrm{dR}}(Z)(K) \to \mathsf{D}_{\mathrm{dR}}(U)(K) \to \mathsf{D}_{\mathrm{dR}}(Q)(K) \to 1$$

$$1 \to \mathsf{D}^+_{\mathrm{dR}}(Z)(K) \to \mathsf{D}^+_{\mathrm{dR}}(U)(K) \to \mathsf{D}^+_{\mathrm{dR}}(Q)(K) \to 1$$

$$1 \to \mathsf{D}_{\mathrm{st}}(Z)(K_0[\boldsymbol{\varepsilon}]^n) \to \mathsf{D}_{\mathrm{st}}(U)(K_0[\boldsymbol{\varepsilon}]^n) \to \mathsf{D}_{\mathrm{st}}(Q)(K_0[\boldsymbol{\varepsilon}]^n) \to 1$$

where $K_0[\boldsymbol{\varepsilon}]^n = K_0[\varepsilon_1, \ldots, \varepsilon_n]$ subject to $\varepsilon_i \varepsilon_j = 0$ for all i, j. The exactness of these sequences is the content of lemma 4.2.5. □

The remainder of this section is devoted to a proof of theorem 6.3.2. We will restrict our attention to the calculation of the cohomotopy of $U(\mathsf{B}^\bullet_{g/e})^{G_K}$, the case of $U(\mathsf{B}^\bullet_{f/e})^{G_K}$ being similar (and simpler).

In making the argument precise, it will be convenient for this section only to permit our $\mathbb{Q}_p$-algebras (or more accurately, their homomorphisms) to be non-unital[1] and to define, for a non-unital $\mathbb{Q}_p$-algebra B and unipotent group $U/\mathbb{Q}_p$, the group $U(\mathsf{B})$ to be the kernel

$$U(\mathsf{B}) := \ker\left(U(\mathbb{Q}_p \oplus \mathsf{B}) \to U(\mathbb{Q}_p)\right),$$

where $\mathbb{Q}_p \oplus \mathsf{B}$ is given the obvious algebra structure with unit $(1, 0)$, and $\mathbb{Q}_p \oplus \mathsf{B} \to \mathbb{Q}_p$ is the natural projection. This is clearly functorial in B with respect to non-unital $\mathbb{Q}_p$-algebra homomorphisms, and extends the usual definition on unital algebras, since for a unital B there is a canonical isomorphism $\mathbb{Q}_p \oplus \mathsf{B} \cong \mathbb{Q}_p \times \mathsf{B}$.

With this definition, we will let $\mathsf{B}^\bullet_{\mathrm{dR}}$ denote the (non-unital) cosimplicial $\mathbb{Q}_p$-algebra cogenerated by the semi-cosimplicial $\mathbb{Q}_p$-algebra

$$\mathsf{B}^+_{\mathrm{dR}} \rightrightarrows \mathsf{B}_{\mathrm{dR}}$$

where $d^0 = 0$ and d^1 is the canonical inclusion. There are then G_K-equivariant sequences

$$0 \to \mathsf{B}^\bullet_{\mathrm{dR}} \to \mathsf{B}^\bullet_e \to \mathsf{B}^{\varphi=1}_{\mathrm{cris}} \to 0$$

$$0 \to \mathsf{B}^\bullet_{\mathrm{dR}} \to \mathsf{B}^\bullet_g \to \mathsf{B}^\bullet_{g/e} \to 0$$

of cosimplicial $\mathbb{Q}_p$-algebras, where $\mathsf{B}^{\varphi=1}_{\mathrm{cris}}$ denotes the constant cosimplicial $\mathbb{Q}_p$-algebra which is $\mathsf{B}^{\varphi=1}_{\mathrm{cris}}$ in each degree. We claim these sequences are levelwise exact, and split by a G_K-equivariant $\mathbb{Q}_p$-algebra homomorphism in each degree.

To see why this is, at least for the second sequence (the first being similar), we consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{B}^+_{\mathrm{dR}} & \hookrightarrow & \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet \times \mathsf{B}^+_{\mathrm{dR}} & \twoheadrightarrow & \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet & \longrightarrow & 0 \\
 & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \\
0 & \longrightarrow & \mathsf{B}_{\mathrm{dR}} & \hookrightarrow & \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet \times \mathsf{B}_{\mathrm{dR}} & \twoheadrightarrow & \mathsf{B}_{\mathrm{st}}[\boldsymbol{\varepsilon}]^\bullet & \longrightarrow & 0
\end{array}$$

[1] We follow the usual convention that "non-unital" means "not necessarily unital".

of semi-cosimplicial $\mathbb{Q}_p$-cdgas giving rise to $0 \to \mathsf{B}^\bullet_{\mathrm{dR}} \to \mathsf{B}^\bullet_g \to \mathsf{B}^\bullet_{g/e} \to 0$ through construction 5.5.3. It is clear that each row is exact and G_K-equivariantly split by a homomorphism of $\mathbb{Q}_p$-cdgas, so that the corresponding sequence of bi-cosimplicial $\mathbb{Q}_p$-algebras is G_K-equivariantly split in each bidegree by a homomorphism of $\mathbb{Q}_p$-algebras. Hence the same is true in each degree for the diagonal sequence $0 \to \mathsf{B}^\bullet_{\mathrm{dR}} \to \mathsf{B}^\bullet_g \to \mathsf{B}^\bullet_{g/e} \to 0$ as claimed.

Using this it follows that the sequences

$$1 \to U(\mathsf{B}^\bullet_{\mathrm{dR}}) \to U(\mathsf{B}^\bullet_e) \to U(\mathsf{B}^{\varphi=1}_{\mathrm{cris}}) \to 1$$

$$1 \to U(\mathsf{B}^\bullet_{\mathrm{dR}}) \to U(\mathsf{B}^\bullet_g) \to U(\mathsf{B}^\bullet_{g/e}) \to 1$$

are exact and G_K-equivariantly split in each degree, and hence that they remain exact after taking G_K-invariants. By taking the long exact sequence in cohomotopy of the G_K-invariants of the first sequence, we obtain a description of the cohomotopy groups of $U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}$ as:

- $\pi^0(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}) = 1$;
- $\pi^1(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K})$ sits in an exact sequence

$$1 \to U(\mathbb{Q}_p)^{G_K} \to \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) \overset{\curvearrowright}{\to} \pi^1(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}) \to \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \to 1; \quad (6.3.1)$$

- $\pi^i(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}) = 1$ for $i > 1$ when U is abelian.

Here we are using the description of the cohomotopy of $U(\mathsf{B}^\bullet_e)^{G_K}$ in theorem 6.2.3 and the fact that the higher cohomotopy of $U(\mathsf{B}^{\varphi=1}_{\mathrm{cris}})^{G_K}$ is trivial (as it is a constant cosimplicial group).

Taking the long exact sequence in cohomotopy of the G_K-invariants of the second sequence, we find that we have an exact sequence (with the final 1 coming from lemma 5.2.3)

$$\begin{array}{ccccccc} 1 & \longrightarrow & U(\mathbb{Q}_p)^{G_K} & \longrightarrow & \pi^0(U(\mathsf{B}^\bullet_{g/e})^{G_K}) & \multirow{2}{*}{$\curvearrowright$} & \\ \pi^1(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}) & \longrightarrow & \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) & \longrightarrow & \pi^1(U(\mathsf{B}^\bullet_{g/e})^{G_K}) & \longrightarrow & 1 \end{array} \quad (6.3.2)$$

and that $\pi^2(U(\mathsf{B}^\bullet_{g/e})^{G_K}) = \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U(\mathbb{Q}_p)^*(1))^*$ and $\pi^i(U(\mathsf{B}^\bullet_{g/e})^{G_K}) = 1$ for $i > 2$ when U is abelian (this proves all the extra claims in the abelian case).

Now we observe that the natural map $\mathsf{B}^\bullet_e \hookrightarrow \mathsf{B}^\bullet_g$ defined in section 6.2.1 fits into a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{B}^\bullet_{\mathrm{dR}} & \longrightarrow & \mathsf{B}^\bullet_e & \longrightarrow & \mathsf{B}^{\varphi=1}_{\mathrm{cris}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{B}^\bullet_{\mathrm{dR}} & \longrightarrow & \mathsf{B}^\bullet_g & \longrightarrow & \mathsf{B}^\bullet_{g/e} & \longrightarrow & 0, \end{array}$$

from which we obtain a morphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(\mathbb{Q}_p)^{G_K} & \longrightarrow & \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) & \overset{\curvearrowright}{\longrightarrow} & \pi^1(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}) & \longrightarrow & \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \longrightarrow 1 \\ & & \| & & \downarrow & & \| & & \downarrow \\ 1 & \longrightarrow & U(\mathbb{Q}_p)^{G_K} & \longrightarrow & \pi^0(U(\mathsf{B}^\bullet_{g/e})^{G_K}) & \overset{\curvearrowright}{\longrightarrow} & \pi^1(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}) & \longrightarrow & \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \end{array}$$

whose top row is sequence 6.3.1 and whose bottom row is part of sequence 6.3.2 (using theorem 6.2.4 for the description of the vertical arrows). A quick five-lemma-style diagram chase shows that the map $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U)(\mathbb{Q}_p) \to \pi^0(U(\mathsf{B}^\bullet_{g/e})^{G_K})$ is an isomorphism, giving us the desired description of π^0.

For π^1, we note that surjectivity of the upper-right horizontal arrow in the above diagram shows that the image of $\pi^1(U(\mathsf{B}^\bullet_{\mathrm{dR}})^{G_K}) \to \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ is $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$, and hence we obtain from sequence 6.3.2 an exact sequence of pointed sets

$$1 \to \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \to \pi^1(U(\mathsf{B}^\bullet_{g/e})^{G_K}) \to 1.$$

As usual, this doesn't imply that $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \twoheadrightarrow \pi^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ induces an isomorphism $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xrightarrow{\sim} \pi^1(U(\mathsf{B}^\bullet_{g/e})^{G_K})$, and we have to consider twists of U to complete the proof.

Thus let α be a de Rham cocycle of U, i.e. one whose class lies in $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$. As per the proof of lemma 6.2.5, we may pick some $u \in U(\mathsf{B}^0_g)$ of which α is the coboundary, and write $\beta = d^1(u)d^0(u)^{-1} \in \mathrm{Z}^1\left(U(\mathsf{B}^\bullet_g)^{G_K}\right)$ for the coboundary of u^{-1} in the cosimplicial group $U(\mathsf{B}^\bullet_g)$. Writing $\overline{u} \in U(\mathsf{B}^\bullet_{g/e})$ and $\overline{\beta} \in \mathrm{Z}\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ for the images of u and β respectively, the same argument as in the proof of lemma 6.2.5 shows that $\overline{u}$ induces a G_K-equivariant isomorphism

$$\overline{\beta}\left(U(\mathsf{B}^\bullet_{g/e})\right) \xrightarrow{\sim} {}_\alpha U(\mathsf{B}^\bullet_{g/e}),$$

compatible with the corresponding isomorphism for $\mathsf{B}^\bullet_g$. Taking G_K-invariants and π^1, we obtain a commuting diagram

$$\begin{array}{ccccc}
\mathrm{H}^1_g(G_K, {}_\alpha U(\mathbb{Q}_p)) & \xrightarrow{\sim} & \pi^1\left({}_\alpha U(\mathsf{B}^\bullet_g)^{G_K}\right) & \twoheadrightarrow & \pi^1\left({}_\alpha U(\mathsf{B}^\bullet_{g/e})^{G_K}\right) \\
\downarrow \wr & & \uparrow \wr & & \uparrow \wr \\
 & & \pi^1\left({}_\beta\left(U(\mathsf{B}^\bullet_g)^{G_K}\right)\right) & \twoheadrightarrow & \pi^1\left({}_{\overline{\beta}}\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)\right) \\
 & & \downarrow \wr & & \downarrow \wr \\
\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) & \xrightarrow{\sim} & \pi^1\left(U(\mathsf{B}^\bullet_g)^{G_K}\right) & \twoheadrightarrow & \pi^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right).
\end{array}$$

The upper-right square commutes already on the level of cosimplicial groups by the above discussion, and the lower-right square commutes by naturality of the twisting isomorphism in proposition 5.3.1; that the left-hand rectangle commutes is part of lemma 6.2.5.

Since by proposition 4.3.3 ${}_\alpha U$ is also de Rham, the composite of the top row has kernel $\mathrm{H}^1_e(G_K, {}_\alpha U(\mathbb{Q}_p))$, so that by the commutativity of the above diagram and lemma 4.3.2, the fibre of $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \twoheadrightarrow \pi^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ containing $[\alpha]$ is exactly its $\sim_{\mathrm{H}^1_e}$ equivalence class. Thus the map $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) \twoheadrightarrow \pi^1\left(U(\mathsf{B}^\bullet_{g/e})^{G_K}\right)$ factors through an isomorphism $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xrightarrow{\sim} \pi^1(U(\mathsf{B}^\bullet_{g/e})^{G_K})$ as desired. □

Chapter 7

The $\mathbb{Q}_p$-pro-unipotent Kummer map

The aim of this chapter is to prove the remaining parts of theorem 1.3.4 regarding the p-adic non-abelian Kummer map for an arbitrary smooth connected variety Y/K. In outline, our strategy for the proof is first to reduce to the case that Y is a curve, then a semistable curve, and then finally to deal with this case using the techniques developed in [AIK15]. In order to get the reduction working, we will in fact prove the following two lemmas, which together with corollary 4.2.8 establish all the assertions in theorem 1.3.4.

Lemma 7.0.1. *Let Y/K be a smooth connected variety, $y, z \in Y(K)$ two points, and $U(y,z)/\mathbb{Q}_p$ the pro-unipotent torsor of étale paths from y to z in Y. Then $\mathcal{O}(U(y,z))$ is ind-de Rham.*

Lemma 7.0.2. *Let Y/K be a (connected) open subvariety of the generic fibre of a proper integral scheme $\mathcal{X}/\mathcal{O}_K$, let $y \in Y(K)$ be a K-rational basepoint, and let $U/\mathbb{Q}_p$ be the pro-unipotent étale fundamental group of Y based at y. Write $y_s \in \mathcal{X}(k)$ for the reduction of y, and suppose that y_s is a smooth point of the special fibre, and that the tube[1] $]y_s[$ of y_s in $\mathcal{X}_K$ lies entirely inside Y.*

Then the image of $Y(K) \cap]y_s[$ under the non-abelian Kummer map

$$Y(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_p))$$

lies in $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$.

[1] Recall that the *tube* $]x_s[$ of a point x_s in the special fibre of a proper integral $\mathcal{O}_K$-scheme $\mathcal{X}$ is the preimage of x_s under the reduction map $|\mathcal{X}_\eta| \to |\mathcal{X}_s|$ from the underlying topological space of the generic fibre to that of the special fibre.

Remark 7.0.3. In reducing lemmas 7.0.1 and 7.0.2 to the case of curves, we will repeatedly use the fact that the non-abelian Kummer maps are natural in the variety: if $(Y', y') \to (Y, y)$ is a morphism of smooth connected pointed K-varieties, then this induces a G_K-equivariant homomorphism $U' \to U$ on their $\mathbb{Q}_p$-pro-unipotent étale fundamental groups, and the Kummer maps for Y' and Y fit into a commuting square

$$\begin{array}{ccc} Y'(K) & \longrightarrow & \mathrm{H}^1(G_K, U'(\mathbb{Q}_p)) \\ \downarrow & & \downarrow \\ Y(K) & \longrightarrow & \mathrm{H}^1(G_K, U(\mathbb{Q}_p)). \end{array}$$

Notice that the right-hand vertical map sends $\mathrm{H}^1_g(G_K, U'(\mathbb{Q}_p))$ into $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ and similarly for H^1_f and H^1_e.

In order to show how lemmas 7.0.1 and 7.0.2 together imply theorem 1.3.4, we need a basic proposition to provide us the models in lemma 7.0.2.

Proposition 7.0.4. *Let $Y \to \mathbb{P}^n_K$ be a connected étale open of $\mathbb{P}^n_K$ and $y \in Y(K)$ a basepoint lying over the origin. Then the conditions of lemma 7.0.2 hold for $\mathcal{X}$ the normalisation of $\mathbb{P}^n_{\mathcal{O}_K}$ in Y, after possibly rescaling the coordinates on $\mathbb{P}^n_K$.*

Proof. Let $t_1, \ldots, t_n$ be the standard local coordinates on $\mathbb{P}^n_K$ at the origin. Étaleness of $Y \to \mathbb{P}^n_K$ ensures that y extends uniquely to a $K[\![t_1, \ldots, t_n]\!]$-point of Y lying over the formal neighbourhood of the origin in $\mathbb{P}^n_K$. If we write Y locally as the vanishing locus of polynomials $P_1, \ldots, P_m \in K[t_1, \ldots, t_n, T_1, \ldots, T_m]$, so that the extension of y is given by power series $T_j(t_1, \ldots, t_n) \in K[\![t_1, \ldots, t_n]\!]$ simultaneously solving the polynomials P_j, a standard argument using Newton's method establishes that the valuations of the coefficients of the monomials appearing in the T_j are bounded linearly in the multi-index. Hence, after rescaling the coordinates t_j, we may assume that y extends to a $K \otimes_{\mathcal{O}_K} \mathcal{O}_K[\![t_1, \ldots, t_n]\!]$-point of Y.

Now let $\mathcal{X}$ be the normalisation of $\mathbb{P}^n_{\mathcal{O}_K}$ in Y, which is integral and proper over $\mathcal{O}_K$. Normality of Y ensures that there is a canonical map $Y \to \mathcal{X}$ of schemes over $\mathbb{P}^1_{\mathcal{O}_K}$; moreover this map, being birational and quasi-finite, is a Zariski-open immersion by Zariski's main theorem [Liu02, Corollary 4.4.6].

If we write y_s for the reduction of y, we get an extension $\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}, 0_s} \hookrightarrow \mathcal{O}_{\mathcal{X}, y_s}$ of local rings, where $0_s \in \mathbb{P}^n_{\mathcal{O}_K}(k)$ is the origin in the special fibre. Since both $\mathbb{P}^n_{\mathcal{O}_K}$ and $\mathcal{X}$ are normal and excellent and $\mathcal{X}$ is finite over $\mathbb{P}^n_{\mathcal{O}_K}$, this is a finite extension of excellent integrally closed local domains. In particular, the corresponding extension

$\mathcal{O}_K[\![t_1,\dots,t_n]\!] = \widehat{\mathcal{O}}_{\mathbb{P}^n_{\mathcal{O}_K},0_s} \hookrightarrow \widehat{\mathcal{O}}_{\mathcal{X},y_s}$ on the completions is also a finite extension of integrally closed domains, and our assumption that y extends to a $K\otimes_{\mathcal{O}_K}\mathcal{O}_K[\![t_1,\dots,t_n]\!]$-point ensures that this completed extension has a retraction after tensoring with K. Then $\widehat{\mathcal{O}}_{\mathbb{P}^n_{\mathcal{O}_K},0_s}[p^{-1}] \hookrightarrow \widehat{\mathcal{O}}_{\mathcal{X},y_s}[p^{-1}]$ is a finite extension of integrally closed domains with a splitting, and hence is an isomorphism. Since $\widehat{\mathcal{O}}_{\mathbb{P}^n_{\mathcal{O}_K},0_s} \hookrightarrow \widehat{\mathcal{O}}_{\mathcal{X},y_s}$ is a finite extension of integrally closed domains, it too is an isomorphism, and hence $\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K},0_s} \hookrightarrow \mathcal{O}_{\mathcal{X},y_s}$ is étale. This shows that $\mathcal{X}\to\mathbb{P}^n_{\mathcal{O}_K}$ is étale at y_s and hence $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_K)$ is smooth at y_s.

For the second part, note that the tube of y_s is the image of the generic fibre of $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{X},y_s})\to\mathcal{X}$. By the preceding discussion, $\widehat{\mathcal{O}}_{\mathcal{X},y_s}\cong\mathcal{O}_K[\![t_1,\dots,t_n]\!]$, and the map on generic fibres is given by the $K\otimes_{\mathcal{O}_K}\mathcal{O}_K[\![t_1,\dots,t_n]\!]$-point of Y above. Hence the tube of y_s is contained in Y, as desired. □

Proof of theorem 1.3.4 from lemmas 7.0.1 and 7.0.2. To show U is de Rham, combine lemma 7.0.1 and definition-lemma 4.2.2. To show that the image of the non-abelian Kummer map lands in H^1_g, combine lemma 7.0.1, definition-lemma 4.4.3 (with $\mathsf{B}=\mathsf{B}_{\mathrm{dR}}$) and lemma 4.4.5.

To show local constancy of the mod H^1_e non-abelian Kummer map, note that if $y'\in Y(K)$ is another rational point and $U'/\mathbb{Q}_p$ the pro-unipotent étale fundamental group of Y based at y', then U' is the twist of U by a cocycle representing the class in $\mathrm{H}^1(G_K,U(\mathbb{Q}_p))$ of the torsor of étale paths from y to y', and hence we obtain a commuting diagram

$$\begin{array}{ccccc} Y(K) & \longrightarrow & \mathrm{H}^1(G_K,U(\mathbb{Q}_p)) & \longrightarrow & \mathrm{H}^1(G_K,U(\mathsf{B}^{\varphi=1}_{\mathrm{cris}})) \\ \| & & \downarrow\wr & & \downarrow\wr \\ Y(K) & \longrightarrow & \mathrm{H}^1(G_K,U'(\mathbb{Q}_p)) & \longrightarrow & \mathrm{H}^1(G_K,U'(\mathsf{B}^{\varphi=1}_{\mathrm{cris}})) \end{array}$$

relating the non-abelian Kummer maps based at y and y' respectively. In particular, it suffices to show that for general (Y,y) the image of some p-adic neighbourhood of y under the non-abelian Kummer map lands in H^1_e.

To prove this, by naturality of the Kummer map (remark 7.0.3) we need only prove it when Y is replaced by some Zariski-open neighbourhood of y. Since Y is smooth, we may restrict it in such a way that it becomes an étale open subset of $\mathbb{P}^n_K$ for some n, and by proposition 7.0.4 we may choose the coordinates on $\mathbb{P}^n_K$ such that the conditions of lemma 7.0.2 are satisfied. Then the $Y(K)\cap]y_s[$ provides exactly the desired p-adic neighbourhood of y. □

7.1 Reduction to the case of curves

The first step in our proof of lemmas 7.0.1 and 7.0.2 is to reduce to the case that Y is 1-dimensional. The key input that allows us to proceed is the following lemma.

Lemma 7.1.1 (Piecewise algebraicity of paths)**.** *Let $Y/\mathbb{C}$ be a smooth variety of dimension > 0. Then for any $y, z \in Y(\mathbb{C})$ and path $\gamma\colon [0,1] \to Y(\mathbb{C})$ from y to z, there is a sequence of points $y = y_0, \dots, y_m = z \in Y(\mathbb{C})$, smooth connected algebraic curves $C_1, \dots, C_m$ in Y with $C_j(\mathbb{C}) \ni y_{j-1}, y_j$, and paths γ_j from y_{j-1} to y_j in $C_j(\mathbb{C})$, such that the composite path $\gamma_m \cdot \dots \cdot \gamma_1$ is homotopic to γ.*

Proof. By compactness, it suffices to prove that for every $t \in [0,1]$ and for every t' sufficiently close to t, there is a smooth connected curve C in Y passing through $\gamma(t)$ and $\gamma(t')$ such that $\gamma|_{[t,t']}$ is homotopic (relative to the endpoints) to a path in $C(\mathbb{C})$. (When $t' < t$, we interpret $[t,t'] = [t',t]$.)

In proving this, we are free to restrict Y to Zariski-open neighbourhoods of $\gamma(t)$, so we may assume that $f\colon Y \to \mathbb{A}^n_{\mathbb{C}}$ is an étale open subset of $\mathbb{A}^n_{\mathbb{C}}$. We may then pick a analytic open neighbourhood U of $\gamma(t)$ in $Y(\mathbb{C})$ mapping diffeomorphically onto its image in $\mathbb{A}^n(\mathbb{C})$, which we assume convex. For t' sufficiently close to t, $\gamma|_{[t,t']}$ lies entirely inside U—we will prove the condition for all such t'.

To do this, pick any 1-dimensional subspace of $\mathbb{A}^n_{\mathbb{C}}$ passing through $f\gamma(t)$ and $f\gamma(t')$, and let $C' \subseteq Y$ be the preimage of this line. This is a smooth curve. Moreover, by construction $f\gamma|_{[t,t']}$ is homotopic to the straight line path in $f(U)$ connecting $f\gamma(t)$ and $f\gamma(t')$, and hence $\gamma|_{[t,t']}$ is homotopic to a path lying in $C'(\mathbb{C})$. Taking C to be the connected component of C' containing $\gamma(t)$, we are done. □

Proposition 7.1.2. *Suppose that lemma 7.0.1 is true whenever $\dim_K(Y) = 1$. Then it is true in general.*

Proof. We may assume $\dim_K(Y) > 0$. Fix an isomorphism $\overline{K} \simeq \mathbb{C}$, so that we get an identification of the dual of $\mathcal{O}(U(y,z))$ as the completion of $\mathbb{Q}_p\pi_1(Y(\mathbb{C}); y, z)$ with respect to the filtration defined by the augmentation ideal. In particular, as a pro-finite dimensional $\mathbb{Q}_p$-vector space, it is generated by the classes of paths from y to z in $Y(\mathbb{C})$. Hence it suffices to show that, for every path γ from y to z in $Y(\mathbb{C})$, there is a pro-de Rham subrepresentation of $\mathcal{O}(U(x,y))^*$ containing the image of $[\gamma]$.

To do this, take by lemma 7.1.1 a sequence of points $y = y_0, \dots, y_m = z$ in $Y(\overline{K})$, connected smooth curves $C_1, \dots, C_m \subseteq X_{\overline{K}}$ with $C_j(\overline{K}) \ni y_{j-1}, y_j$, and paths γ_j from y_{j-1} to y_j in $C_j(\mathbb{C})$, such that γ is homotopic to the composite $\gamma_m \cdot \dots \cdot \gamma_1$. We may pass to a finite extension L/K over which all the y_j and C_j are defined.

If we let $V(y_{j-1}, y_j)$ denote the pro-unipotent torsor of étale paths from y_{j-1} to y_j in C_j, then we know by the one-dimensional case that $\mathcal{O}(V(y_{j-1}, y_j))^*$ is a pro-de Rham representation of G_L, and that its image under $\mathcal{O}(V(y_{j-1}, y_j))^* \to \mathcal{O}(U(y_{j-1}, y_j))^*$ contains $[\gamma_j]$. Hence the image of the path-composition map

$$\widehat{\bigotimes_{m \geq j \geq 1}} \mathcal{O}(V(y_{j-1}, y_j))^* \to \widehat{\bigotimes_{m \geq j \geq 1}} \mathcal{O}(U(y_{j-1}, y_j))^* \to \mathcal{O}(U(y, z))^*$$

is a pro-de Rham G_L-subrepresentation and contains $[\gamma]$. In particular, the G_K-subrepresentation generated by this image is pro-de Rham containing $[\gamma]$, as desired. □

Proposition 7.1.3. *Suppose that lemma 7.0.2 is true whenever* $\dim_K(Y) = 1$. *Then it is true in general.*

Proof. We may assume $\dim_K(Y) > 0$. Fix any $z \in Y(K) \cap]y_s[$.

Now let $\varpi, t_1, \ldots, t_n$ be a system of local parameters at y_s, where ϖ is a uniformiser of $\mathcal{O}_K$. These are automatically defined on $]y_s[$, and we may normalise them so that $t_j(y) = 0$ for $1 \leq j \leq n$ and $t_j(z) = 0$ for $2 \leq j \leq n$. Let $\mathcal{X}' \subseteq \mathcal{X}$ be the closure of the locus where $t_2, \ldots, t_n$ vanish. By construction, y_s lies in the smooth locus of $\mathcal{X}' \to \mathrm{Spec}(\mathcal{O}_K)$, and so lies in a unique irreducible component $\mathcal{X}''$ of $\mathcal{X}'$, which we endow with the reduced scheme structure.

Now let Y' (resp. Y'') be the intersections of $\mathcal{X}'$ (resp. $\mathcal{X}''$) with Y, so that they are reduced open subschemes of the respective generic fibres and Y'' is 1-dimensional. The tube of y_s in $\mathcal{X}''$ is then the intersection of its tube in $\mathcal{X}$ with $\mathcal{X}''$, and hence lies in Y''.

Moreover, by construction both y and z lie in Y', and hence also in Y'' as they both specialise to y_s. In particular, both $y, z \in Y''(K)$ lie in the tube of y_s inside $\mathcal{X}''$. Thus the conditions of lemma 7.0.2 are verified for $(Y'', \mathcal{X}'', y)$, so that we deduce from the 1-dimensional case that the image of z under the Kummer map

$$Y''(K) \to \mathrm{H}^1(G_K, U''(\mathbb{Q}_p))$$

for Y'' lies in $\mathrm{H}^1_e(G_K, U''(\mathbb{Q}_p))$, and hence we are done by naturality of the Kummer map (remark 7.0.3). □

7.2 Review of the method of Andreatta–Iovita–Kim

To prove lemmas 7.0.1 and 7.0.2 when Y is a curve, we will rely heavily on the analysis in [AIK15] for open curves with nodal reduction. That paper studies path-torsors under the $\mathbb{Q}_p$-pro-unipotent étale fundamental group by relating them to fibres of a certain universal pro-unipotent local system on Y, in much the same way that path-torsors in topology can be viewed as fibres of a universal cover. In fact, rather than working with the universal system over Y, [AIK15] extends this local system to the completion X of Y, obtaining a universal pro-unipotent local system on X with logarithmic poles on the cusps $D = X \setminus Y$, this log-geometric analogue being better adapted for studying questions of semistable reduction.

Since the methods of [AIK15] constitute the technical heart of our proof of theorem 1.3.4, let us review the salient points from that paper.

7.2.1 The universal étale family

Fix for now a smooth connected variety Y/K and a basepoint $y \in Y(K)$. We write $Y = X \setminus D$ as a complement of a reduced normal crossings divisor in a smooth proper variety, and write $X^{\log}$ for X endowed with the log-structure coming from the divisor D. We denote by $\mathrm{Uni}^{\text{ét}}_{\leq n}$ the full subcategory of the category of $\mathbb{Q}_p$-sheaves on $X^{\log}_{\overline{K}}$ for the Kummer étale topology consisting of those sheaves which are unipotent of index $\leq n$, i.e. consisting of the sheaves V admitting a filtration

$$V = V^0 \geq V^1 \geq \cdots \geq V^n = 0$$

such that each V^i/V^{i+1} is a constant $\mathbb{Q}_p$-sheaf of finite rank. Pullback along the map $y\colon \mathrm{Spec}(K)^{\mathrm{triv}} \to X^{\log}$ provides us a fibre functor

$$y^*\colon \mathrm{Uni}^{\text{ét}}_{\leq n} \to \mathbf{Mod}_{\mathrm{fin}}(\mathbb{Q}_p).$$

The category $\mathrm{Uni}^{\text{ét}} = \bigcup_n \mathrm{Uni}^{\text{ét}}_{\leq n}$ is a Tannakian category, with fibre functor y^*, and the corresponding Tannaka group is by definition the $\mathbb{Q}_p$-pro-unipotent Kummer étale fundamental group of the log-scheme $X^{\log}$ based at y, which by [Ill02, Example 4.7(c)][2] is canonically identified with the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of Y based at y.

[2] [Ill02] in fact asserts this for the profinite étale fundamental groups, rather than the $\mathbb{Q}_p$-pro-unipotent étale fundamental groups (which are the $\mathbb{Q}_p$-pro-unipotent envelopes of the former). [Ill02] also doesn't contain a proof of this fact; for a proof see e.g. [Sti02, Section 3.3.1, Example 3].

In this context, the role of the universal cover for the topological fundamental group is played by the *universal étale family* on $X^{\log}_{\overline{K}}$, defined to be the pro-object $\mathcal{E}$ of $\mathrm{Uni}^{\text{ét}}$ pro-representing the fibre functor y^*. [AIK15, Proposition 3.4] shows that this pro-representing pro-object exists when Y is a curve, and in fact it exists in general[3].

Proposition 7.2.1. *With the above notation, $y^*\colon \mathrm{Uni}^{\text{ét}}_{\leq n} \to \mathbf{Mod}_{\mathrm{fin}}(\mathbb{Q}_p)$ is representable by an object $\mathcal{E}_n \in \mathrm{Uni}^{\text{ét}}_{\leq n}$ and an element $e_n \in y^*\mathcal{E}_n$. In particular, $y^*\colon \mathrm{Uni}^{\text{ét}} \to \mathbf{Mod}_{\mathrm{fin}}(\mathbb{Q}_p)$ is pro-representable by the pro-object $\mathcal{E} = \varprojlim \mathcal{E}_n$ and $e = \varprojlim e_n \in y^*\mathcal{E}$, where the surjective transition maps $\mathcal{E}_{n+1} \twoheadrightarrow \mathcal{E}_n$ send $e_{n+1} \mapsto e_n$.*

Proof. Following [AIK15], we construct the $(\mathcal{E}_n, e_n)$ recursively, starting from $(\mathcal{E}_1, e_1) = (\mathbb{Q}_p, 1)$. Note that for any $V \in \mathrm{Uni}^{\text{ét}}$, the Yoneda Ext-group $\mathrm{Ext}^1(V, \mathbf{1})$ is finite-dimensional, as can be proved by a straightforward induction from the base case $\mathrm{Ext}^1(\mathbf{1}, \mathbf{1}) = \mathrm{H}^1_{\text{ét}}(X^{\log}_{\overline{K}}, \mathbb{Q}_p) = \mathrm{H}^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_p)$. Hence we may define recursively $\mathcal{E}_{n+1}$ to be the extension of $\mathcal{E}_n$ by the constant sheaf $\mathrm{Ext}^1(\mathcal{E}_n, \mathbb{Q}_p)^* \otimes \mathbf{1}$ whose class is the extension-class corresponding to the identity in

$$\mathrm{End}\left(\mathrm{Ext}^1(\mathcal{E}_n, \mathbf{1})\right) = \mathrm{Ext}^1(\mathcal{E}_n, \mathbf{1}) \otimes \mathrm{Ext}^1(\mathcal{E}_n, \mathbf{1})^* = \mathrm{Ext}^1(\mathcal{E}_n, \mathrm{Ext}^1(\mathcal{E}_n, \mathbf{1})^* \otimes \mathbf{1}),$$

and let $e_{n+1} \in y^*\mathcal{E}_{n+1}$ be any lift of e_n. That the pairs $(\mathcal{E}_n, e_n)$ represent the fibre functor is the content of [AIK15, Lemma 3.2]. □

The universal étale family carries additional structures which are of interest to us. Firstly, there are morphisms $\mathcal{E} \to \mathbf{1}$ (sending e to $1 \in \mathbb{Q}_p$) and $\mathcal{E} \to \mathcal{E}\widehat{\otimes}\mathcal{E}$ (sending e to $e\widehat{\otimes}e$) defining on $\mathcal{E}$ the structure of a cocommutative coalgebra object in the category of pro-objects in $\mathrm{Uni}^{\text{ét}}$ (with respect to the completed tensor product $\varprojlim V_i \widehat{\otimes} \varprojlim W_j = \varprojlim (V_i \otimes W_j)$). Secondly, there is an action of G_K on $\mathcal{E}$ lifting that on $X^{\log}_{\overline{K}}$ (sending, for each $\sigma \in G_K$, e to $e \in y^*\sigma^*\mathcal{E} = y^*\mathcal{E}$). With respect to this action, the counit and comultiplication of the coalgebra $\mathcal{E}$ are G_K-equivariant.

For us, the principal utility of the universal family is that it bundles together all the pro-unipotent torsors of étale paths at varying endpoints in Y.

Proposition 7.2.2. *Let $z \in Y(K)$ be a second rational point, and let $U(y,z)/\mathbb{Q}_p$ denote the pro-unipotent torsor of étale paths from y to z in Y. Then there is a canonical G_K-equivariant isomorphism*

$$\mathcal{O}(U(y,z))^* \cong z^*\mathcal{E}$$

[3]This was presumably known to the authors of [AIK15], although the construction in that paper seems only to work in the 1-dimensional case.

of pro-finite dimensional vector spaces with cocommutative coalgebra structure.

Proof. $U(y,z)$ is canonically identified with the pro-unipotent torsor of étale paths from y to z in $X_{\overline{K}}^{\log}$. By definition, we have a canonical isomorphism

$$\mathcal{O}(U(y,z))^* \cong \mathrm{Hom}(y^*, z^*)$$

of pro-finite dimensional vector spaces. Since $\mathcal{E}$ is a pro-representing object for y^*, we have an isomorphism of functors $y^* \cong \varinjlim \mathrm{Hom}(\mathcal{E}_n, -)$, so that by the Yoneda lemma there is a canonical isomorphism

$$\mathrm{Hom}(y^*, z^*) \cong \varprojlim \mathrm{Hom}(\mathrm{Hom}(\mathcal{E}_n, -), z^*) \cong \varprojlim z^*\mathcal{E}_n = z^*\mathcal{E}$$

of pro-finite dimensional vector spaces. The composite isomorphism $\mathcal{O}(U(y,z))^* \cong z^*\mathcal{E}$ is then an isomorphism of pro-finite dimensional $\mathbb{Q}_p$-vector spaces. Moreover, when we give y^*, z^* and each $\mathcal{E}_n$ their natural G_K-action, all of the isomorphisms involved are G_K-equivariant, and hence so too is the composite isomorphism.

To show that this is an isomorphism of coalgebras requires a little more explanation. Recall that any two functors $F, F' \colon \mathrm{Uni}^{\text{ét}} \to \mathbf{Mod}_{\mathrm{fin}}(\mathbb{Q}_p)$ have an exterior tensor product $F\widehat{\boxtimes}G \colon \mathrm{Uni}^{\text{ét}} \otimes \mathrm{Uni}^{\text{ét}} \to \mathbf{Mod}_{\mathrm{fin}}(\mathbb{Q}_p)$ defined by $(F\widehat{\boxtimes}F')(V \otimes V') := F(V) \otimes F'(V')$. It is easy to see, for example from the description of natural transformation spaces as ends, that if $F, F', G, G' \colon \mathrm{Uni}^{\text{ét}} \to \mathbf{Mod}_{\mathrm{fin}}(\mathbb{Q}_p)$ are four functors, there is a canonical isomorphism

$$\mathrm{Hom}(F\widehat{\boxtimes}F', G\widehat{\boxtimes}G') \xrightarrow{\sim} \mathrm{Hom}(F,G)\widehat{\otimes}\mathrm{Hom}(F',G')$$

of pro-finite dimensional vector spaces. Since y^* is a monoidal functor, it is endowed with an isomorphism $\xi_y \colon y^* \circ \otimes \xrightarrow{\sim} y^*\widehat{\boxtimes}y^*$ where $\otimes \colon \mathrm{Uni}^{\text{ét}} \to \mathrm{Uni}^{\text{ét}} \otimes \mathrm{Uni}^{\text{ét}}$ is the diagonal embedding, and similarly for z^*. With this identification, the comultiplication on $\mathcal{O}(U(y,z))^* \cong \mathrm{Hom}(y^*, z^*)$ is the composite

$$\mathrm{Hom}(y^*, z^*) \to \mathrm{Hom}(y^*\circ\otimes, z^*\circ\otimes) \xrightarrow{\sim} \mathrm{Hom}(y^*\widehat{\boxtimes}y^*, z^*\widehat{\boxtimes}z^*) \xrightarrow{\sim} \mathrm{Hom}(y^*, z^*)\widehat{\otimes}\mathrm{Hom}(y^*, z^*).$$

Now the isomorphism $\mathrm{Hom}(y^*, z^*) \xrightarrow{\sim} z^*\mathcal{E}$ is the map sending a natural transformation $\gamma \colon y^* \to z^*$ to $\gamma_{\mathcal{E}}(e) \in z^*\mathcal{E}$, where $\gamma_{\mathcal{E}} \colon y^*\mathcal{E} \to z^*\mathcal{E}$ is the map induced by the $(\gamma_{\mathcal{E}_n})_{n\in\mathbb{N}}$. To show this preserves comultiplication, we want to show that

$$\begin{array}{ccc} \mathrm{Hom}(y^*, z^*) & \longrightarrow & \mathrm{Hom}(y^*, z^*)\widehat{\otimes}\mathrm{Hom}(y^*, z^*) \\ \downarrow\wr & & \downarrow\wr \\ z^*\mathcal{E} & \longrightarrow & z^*\mathcal{E}\widehat{\otimes}z^*\mathcal{E} \end{array}$$

commutes. Pushing some $\gamma\colon y^* \to z^*$ both ways around the diagram, we want to show that $\xi_z \gamma_{\mathcal{E}\widehat{\otimes}\mathcal{E}} \xi_y^{-1}(e\widehat{\otimes}e) = \xi_z z^*\Delta(\gamma_{\mathcal{E}}(e))$, where $\Delta\colon \mathcal{E} \to \mathcal{E}\widehat{\otimes}\mathcal{E}$ is the comultiplication. But this follows since $\xi_y^{-1}(e\widehat{\otimes}e) = y^*\Delta(e)$ by definition of Δ and $\gamma_{\mathcal{E}\widehat{\otimes}\mathcal{E}} \circ y^*\Delta = z^*\Delta \circ \gamma_{\mathcal{E}}$ by naturality of γ. This shows that the isomorphism preserves comultiplication, and a simpler check also verifies that it preserves the counit, as desired. □

Remark 7.2.3. In fact, the Galois action on $\mathcal{E}$ allows us to descend it to a pro-$\mathbb{Q}_p$-local system on $X^{\log}$ for the Kummer étale topology, so that the G_K-action on $\mathcal{E}$ is the natural one arising on the pullback of a sheaf from $X^{\log}$ to $X^{\log}_{\overline{K}}$—by abuse of notation, we will also denote this local system by $\mathcal{E}$. Indeed, the category of $\mathbb{Q}_p$-local systems on $X^{\log}_{\overline{K}}$ (resp. $X^{\log}$) is equivalent via the fibre functor at y to the category of continuous $\mathbb{Q}_p$-representations of the profinite Kummer étale fundamental group $\pi_1^{\text{két}}(X^{\log}_{\overline{K}}; y)$ (resp. $\pi_1^{\text{két}}(X^{\log}; y)$). These two groups are related by the fundamental exact sequence (of profinite groups)

$$1 \to \pi_1^{\text{két}}(X^{\log}_{\overline{K}}; y) \to \pi_1^{\text{két}}(X^{\log}; y) \to G_K \to 1,$$

which is split canonically by the basepoint y; the pullback of local systems from $X^{\log}$ to $X^{\log}_{\overline{K}}$ corresponds to restriction of representations along $\pi_1^{\text{két}}(X^{\log}_{\overline{K}}; y) \hookrightarrow \pi_1^{\text{két}}(X^{\log}; y)$, while specifying a G_K-action on a local system on $X^{\log}_{\overline{K}}$ lifting the action on $X^{\log}_{\overline{K}}$ is the same as specifying a G_K-action on the fibre at y which is compatible with the $\pi_1^{\text{két}}(X^{\log}_{\overline{K}}; y)$-action. Hence a local system on $X^{\log}_{\overline{K}}$ with a G_K-action descends to $X^{\log}$ iff the induced action on the fibre at y is continuous—this holds for $\mathcal{E}$ by combining proposition 7.2.2 with proposition 4.0.3.

For us, the key point is that, when Y is a curve with semistable, the universal family $\mathcal{E}$ is pro-semistable in the sense of [AI13]. We will not revisit in detail the proof in [AIK15, Corollary 5.1], which revolves around comparing $\mathcal{E}$ with an analogously-defined universal isocrystal $\mathcal{E}^{\text{cris}}$ on the special fibre of an appropriate model of Y, but we shall at least briefly summarise what is meant by semistability in this context.

7.2.2 Semistable sheaves and the functor $\mathbb{D}^{\text{ar}}_{\log}$

Suppose now, continuing the notation of the previous section, that (X, D) has a *semistable model*, i.e. a pair $(\mathcal{X}, \mathcal{D})$ of a proper flat $\mathcal{O}_K$-scheme $\mathcal{X}$ and a proper flat $\mathcal{O}_K$-subscheme $\mathcal{D}$ with generic fibre (X, D), such that the union of $\mathcal{D}$ and the special fibre of $\mathcal{X}$ is a reduced normal crossings divisor on $\mathcal{X}$. We define the log-scheme $\mathcal{X}^{\log}$ to consist of $\mathcal{X}$ with the log-structure coming from the union of $\mathcal{D}$ and the special

fibre. $\mathcal{X}^{\log}$ is a fine log-scheme, and is proper and log-smooth over $\mathrm{Spec}(\mathcal{O}_K)^{\log}$, where the latter scheme is endowed with the log-structure coming from its closed point.

Now we let $\mathcal{O} = \mathbb{W}[\![Z]\!]$ be the power series ring over the ring $\mathbb{W}$ of Witt vectors of the residue field of K, and let $\mathrm{Spf}(\mathcal{O})^{\log}$ denote $\mathrm{Spf}(\mathcal{O})$ endowed with the log-structure coming from the divisor $Z = 0$. There is a morphism $\mathrm{Spf}(\mathcal{O}_K)^{\log} \to \mathrm{Spf}(\mathcal{O})^{\log}$ of formal log-schemes induced from the map $\mathcal{O} \to \mathcal{O}_K$ of $\mathbb{W}$-algebras sending Z to our fixed uniformiser ϖ of $\mathcal{O}_K$, and we let $\mathcal{O}_{\mathrm{cris}}$ denote the p-adic completion of the divided power envelope of $\mathcal{O}$ with respect to the kernel of $\mathcal{O} \to \mathcal{O}_K/p$. We assume that $\mathcal{X}^{\log}/\mathrm{Spec}(\mathcal{O}_K)^{\log}$ has a deformation to a log-smooth proper fs log-scheme $\widetilde{\mathcal{X}}^{\log}/\mathrm{Spf}(\mathcal{O})^{\log}$—such a deformation always exists when Y is a curve by [Kat89, Proposition 3.14(4)].

In this context, Andreatta and Iovita in [AI13, Section 2.2] define a site $\mathfrak{X}_K$ called Faltings' site, which sits between the Kummer étale site of $\mathcal{X}^{\log}$ and the Kummer étale site of $\mathcal{X}_K^{\log} = X^{\log}$, and a certain ind-continuous sheaf $\mathbb{B}_{\log}$ of periods on $\mathfrak{X}_K$. There is also defined in [AI13, Section 2.4.7] a notion of a *semistable $\mathbb{Q}_p$-adic sheaf* on the site $\mathfrak{X}_K$, along with a symmetric monoidal functor [AI13, Section 2.4.2 & Proposition 2.30] (also called $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}$ in [AIK15])

$$\mathbb{D}^{\mathrm{ar}}_{\log}\colon \mathrm{Sh}(\mathfrak{X}_K)_{\mathrm{ss}} \to \mathrm{Isoc}\left(\mathcal{X}_0/\mathcal{O}_{\mathrm{cris}}\right).$$

This latter category is the category of *isocrystals on* $\mathcal{X}_0 = (\mathcal{O}_K/p) \otimes_{\mathcal{O}_K} \mathcal{X}$, which we will think of as sheaves of $\widehat{\mathcal{O}}^{\mathrm{DP}}_{\widetilde{\mathcal{X}}} = \mathcal{O}_{\widetilde{\mathcal{X}}}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_{\mathrm{cris}}$-modules on $\mathcal{X}$ for the Kummer étale topology[4] endowed with a flat and topologically nilpotent log connection $\nabla_{\cdot/\mathcal{O}_{\mathrm{cris}}}$ relative to $\mathcal{O}_{\mathrm{cris}}$. In fact, $\mathbb{D}^{\mathrm{ar}}_{\log}$ takes values in an even more structured category, according to [AI13, Proposition 2.28].

Proposition 7.2.4. *Let $\mathbb{L}$ be a semistable sheaf on $\mathfrak{X}_K$. Then $\mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})$ carries the following structures, functorially in $\mathbb{L}$ and compatibly with tensor products:*

- *a lift of the log connection $\nabla_{\mathbb{L}/\mathcal{O}_{\mathrm{cris}}}$ on $\mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})$ to a flat and topologically nilpotent log connection $\nabla_{\mathbb{L}/\mathbb{W}}$ relative to $\mathbb{W}$;*
- *for each small (see [AI13, Section 2.1.2]) object U of $\mathfrak{X}_K$ and lift φ_U of Frobenius to U, a Frobenius action $\varphi_{\mathbb{L},U}$ on $\mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})|_U$ lifting the action on U and flat with respect to $\nabla_{\mathbb{L}/\mathbb{W}}$.*

[4]Recall that every sheaf on $\widetilde{\mathcal{X}}$ for the Kummer étale topology, for instance $\mathcal{O}_{\widetilde{X}}$, can be viewed as a sheaf on $\mathcal{X}$ (also for the Kummer étale topology) using the unique deformations of small Kummer étale opens as in [AI13, Section 2.3.4].

Remark 7.2.5. In addition to the structures in proposition 7.2.4, $\mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})$ carries a Hodge filtration satisfying Griffiths transversality with respect to $\nabla_{\mathbb{L}/\mathbb{W}}$. Nonetheless, it will be important that we focus only on the Frobenius and monodromy in order to prove the local constancy in lemma 7.0.2 since, intuitively, the Hodge filtration on the path-torsors depends analytically on the endpoints, and so encodes much finer information than the Frobenius and monodromy—which vary in a locally constant manner—alone.

Example 7.2.6. Suppose that $Y = \mathrm{Spec}(K)$, so that we may take $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_K)$ (with $\mathcal{D} = \emptyset$) and $\widetilde{\mathcal{X}} = \mathrm{Spf}(\mathcal{O})^{\log}$. Then the topos of sheaves on $\mathfrak{X}_K$ is equivalent to the topos of sheaves on $\mathrm{Spec}(K)$ for the étale topology, so that a $\mathbb{Q}_p$-local system on $\mathfrak{X}_K$ or on $\mathrm{Spec}(K)$ is the same thing as a continuous $\mathbb{Q}_p$-representation of G_K. With this identification, $\mathbb{B}_{\log} = \mathsf{B}_{\log}$ (also called $\widehat{\mathsf{B}_{\mathrm{st}}}$ in [Bre99]) is a certain period ring, which is an algebra over $\mathcal{O}_{\mathrm{cris}}[p^{-1}]$ containing B_{st}, and endowed with a Frobenius endomorphism φ and monodromy derivation N compatible with those on B_{st}.

The semistable sheaves on Y then correspond exactly to the representations V of G_K which are $\mathsf{B}_{\log}$-admissible in the sense of [AIK15, Section 2], while D_{st} of such a representation can be recovered by

$$\mathsf{D}_{\mathrm{st}}(V) \cong K_0 \otimes_{\mathsf{B}_{\log}^{G_K},\iota_0} (\mathsf{B}_{\log} \otimes_{\mathbb{Q}_p} V)^{G_K},$$

compatibly with Frobenius and monodromy, where $\iota_0 \colon \mathsf{B}_{\log}^{G_K} \to K_0$ is the map described in [AIK15, Section 2].

Proposition 7.2.7. *Suppose that $\mathbb{L}$ is a semistable sheaf on $\mathfrak{X}_K$, and that $z \in X(K)$ is a point extending to a section $\mathrm{Spec}(\mathcal{O}_K)^{\log} \to \mathcal{X}^{\log}$ compatible with the log structures, which we deform to a section $\widetilde{z} \colon \mathrm{Spec}(\mathcal{O}) \to \widetilde{\mathcal{X}}^{\log}$. Then the fibre $z^*(\mathbb{L})$ is a semistable representation of G_K, and we have a canonical identification*

$$\mathsf{B}_{\log}^{G_K} \otimes_{\mathcal{O}_{\mathrm{cris}}} \widetilde{z}^* \left(\mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})\right) \cong \left(\mathsf{B}_{\log} \otimes_{\mathbb{Q}_p} z^*(\mathbb{L})\right)^{G_K}$$

of $\mathsf{B}_{\log}^{G_K}$-modules, compatible with the Frobenius and monodromy on both sides, and respecting tensor products in $\mathbb{L}$.

Proof. Semistability of $\mathbb{L}$ implies [AI13, Section 2.4.4] that we have an isomorphism

$$\mathbb{B}_{\log} \otimes_{v^* \widehat{\mathcal{O}}^{\mathrm{DP}}_{\widetilde{\mathcal{X}}}} v^* \mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L}) \cong \mathbb{B}_{\log} \otimes_{\mathbb{Z}_p} \mathbb{L} \tag{7.2.1}$$

of sheaves of $\mathbb{B}_{\log}$-modules on $\mathfrak{X}_K$, compatible with the $\mathsf{B}_{\log}$-linear connections on either side and the Frobenius actions for any lift of a Frobenius on a small object of

$\mathfrak{X}_K$. Here, v^* denotes pullback of sheaves from $\mathcal{X}^{\text{két}}$ to $\mathfrak{X}_K$, and we view $\mathbb{L}$ as a sheaf on $\mathfrak{X}_K$ by pushforward from $\mathcal{X}_K^{\text{két}}$.

We wish to take fibres of this isomorphism at z. Recall that if $\mathcal{U} = \mathrm{Spec}(R_{\mathcal{U}})$ is a small [AI13, Section 2.1.2] connected affine Kummer étale open of $\mathcal{X}$ through which z factors, we write $\widetilde{\mathcal{U}} = \mathrm{Spec}(\widetilde{R_{\mathcal{U}}})$ for its deformation to $\widetilde{\mathcal{X}}$, and $\overline{\mathcal{U}} = \mathrm{Spec}(\overline{R_{\mathcal{U}}})$ for the normalisation in $\mathcal{U}$ of an inverse limit of all connected finite Kummer étale covers W of $\mathcal{U}_K$. This latter object is unique up to non-canonical isomorphism, which becomes canonical once we fix a lift of z to an $\mathcal{O}_{\overline{K}}$-point of $\overline{\mathcal{U}}$, and this choice determines an action of G_K on $\overline{\mathcal{U}}$ and $\overline{R_{\mathcal{U}}}$.

If $\mathcal{F}$ is a sheaf on $\mathfrak{X}_K$, we set

$$\mathcal{F}(\overline{\mathcal{U}}) := \mathcal{F}(\overline{R_{\mathcal{U}}}) = \varinjlim \mathcal{F}(\mathcal{U}, W),$$

where the limit is taken over all W finite Kummer étale over $\mathcal{U}_K$. As explained in [AIK15, Section 4.2], our choice of a lift of z determines a ring homomorphism $\widetilde{z}^*_{\log} \colon \mathbb{B}_{\log}(\overline{\mathcal{U}}) \to \mathsf{B}_{\log}$ extending $\widetilde{z}^* \colon \widehat{\mathcal{O}}^{\mathrm{DP}}_{\widetilde{\mathcal{X}}}(\mathcal{U}) \to \mathcal{O}_{\mathrm{cris}}$. By construction, this map is $\mathsf{B}_{\log}$-linear, G_K-equivariant, preserves the connections on either sides, and respects the action of Frobenius.

Thus, evaluating both sides of equation 7.2.1 at $\overline{\mathcal{U}}$ and tensoring over $\widetilde{z}^*_{\log}$ (and using that $\widetilde{z}^* v^* \mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})(\overline{\mathcal{U}}) = \widetilde{z}^* \mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})(\mathcal{U}) =: \widetilde{z}^* \mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})$) we obtain an isomorphism

$$\mathsf{B}_{\log} \otimes_{\mathcal{O}_{\mathrm{cris}}} \widetilde{z}^* \mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L}) \cong \mathsf{B}_{\log} \otimes_{\mathbb{Z}_p} z^* \mathbb{L}$$

compatible with all structures. In particular, this implies that $z^*\mathbb{L}$ is $\mathsf{B}_{\log}$-admissible (hence semistable), and that $\left(\mathsf{B}_{\log} \otimes_{\mathbb{Q}_p} z^*\mathbb{L}\right)^{G_K} \cong \mathsf{B}_{\log}^{G_K} \otimes_{\mathcal{O}_{\mathrm{cris}}} \widetilde{z}^* \mathbb{D}^{\mathrm{ar}}_{\log}(\mathbb{L})$ compatibly with all structures. □

7.2.3 The Kummer map for curves with semistable reduction

Suppose now that Y is a smooth curve with *semistable*[5] *reduction*, i.e. is a curve with a model $(\mathcal{X}, \mathcal{D})$ as in the previous section. Suppose moreover that the basepoint $y \in Y(K)$ doesn't lie in the tube of D. In this context[6], [AIK15, Corollary 5.1] provides that the universal étale family $\mathcal{E}$ (or more accurately the pushforward to $\mathfrak{X}_K$ of the descent of $\mathcal{E}$ described in remark 7.2.3) is pro-semistable. As an immediate

[5]Note that we permit our semistable models of curves to have exceptional divisors in the special fibre, following the convention of [Liu02].

[6]Strictly speaking, [AIK15] only explicitly deals with the case where $\mathcal{X}$ has no exceptional divisor in the special fibre. However, their proofs nowhere use this fact, so their results are still true in the generality we require.

consequence, we derive very strong restrictions on the non-abelian Kummer map for such curves, which provides us with the technical heart of our proof of theorem 1.3.4.

Proposition 7.2.8. *Suppose, in the above notation, that $z \in X(K)\backslash]D[$ is another rational point not in the tube of D, and let $U(y,z)/\mathbb{Q}_p$ be the pro-unipotent torsor of étale paths from y to z in Y. Then $\mathcal{O}(U(y,z))$ is ind-semistable. If moreover z and y lie in the same residue disc, then there is an isomorphism*

$$\mathsf{D}_{\mathrm{st}}(\mathcal{O}(U(y,z))) \cong \mathsf{D}_{\mathrm{st}}(\mathcal{O}(U(y,y)))$$

of K_0-algebras, compatible with Frobenius φ and monodromy N and the algebra structure (though not necessarily the Hodge filtration).

Proof. Extend z first to an $\mathcal{O}_K$-integral point of $\mathcal{X}$—our assumption on z ensures that the pulled-back log structure from $\mathcal{X}^{\log}$ to $\mathrm{Spec}(\mathcal{O}_K)$ along this section is the usual one. Since $\widetilde{\mathcal{X}}^{\log}$ is log-smooth over $\mathrm{Spf}(\mathcal{O})^{\log}$, we may deform z to a $\mathrm{Spf}(\mathcal{O})^{\log}$-valued point $\widetilde{z}$ of $\widetilde{\mathcal{X}}^{\log}$.

The first part follows by combining propositions 7.2.7 and 7.2.2. For the second part, it suffices by example 7.2.6 and proposition 7.2.7 to show that there is an isomorphism

$$\widetilde{y}^*\mathbb{D}^{\mathrm{ar}}_{\log}(\mathcal{E}) \cong \widetilde{z}^*\mathbb{D}^{\mathrm{ar}}_{\log}(\mathcal{E})$$

compatibly with Frobenius and monodromy for some deformation of y to a $\mathrm{Spf}(\mathcal{O})^{\log}$-valued point $\widetilde{y}$ of $\widetilde{\mathcal{X}}^{\log}$. If we let $\mathrm{Spf}(\mathcal{O}_K[\![t]\!]) \hookrightarrow \mathcal{X}$ be the inclusion of the residue disc at y_s, we may deform this to an inclusion $\mathrm{Spf}(\mathcal{O}[\![t]\!]) \hookrightarrow \widetilde{\mathcal{X}}$ over $\mathrm{Spf}(\mathcal{O})$, and we may assume that both $\widetilde{y}$ and $\widetilde{z}$ factor through $\mathrm{Spf}(\mathcal{O}[\![t]\!])$. Thus we are reduced to identifying the fibres of F-isocrystals on the deformed disc $\mathrm{Spf}(\mathcal{O}[\![t]\!])$ with the induced log-structure, as per Dwork's trick. Since we want to take care over the connections, let us take the time to spell this out explicitly. □

Lemma 7.2.9. *Let $\widetilde{\mathcal{X}} = \mathrm{Spf}(\mathcal{O}[\![t]\!]) = \mathrm{Spf}(\mathbb{W}[\![Z,t]\!])$, endowed with the log-structure coming from the special fibre and the divisor $Z = 0$, be a deformation of the 1-dimensional formal open K-disc $\mathcal{X} = \mathrm{Spf}(\mathcal{O}_K[\![t]\!])^{\log}$, and let V be an F-isocrystal on $\left(\widetilde{\mathcal{X}}/\mathbb{W}\right)^{\mathrm{cris}}_{\log}$, i.e. a finite projective $\mathcal{O}_{\mathrm{cris}}[\![t]\!]^{\mathrm{DP}}[p^{-1}]$-module of finite rank endowed with a flat and topologically nilpotent connection $\nabla_{V/\mathbb{W}}$ relative to $\mathbb{W}$ and with a horizontal Frobenius φ lifting the standard Frobenius on $\widetilde{\mathcal{X}}$. Then for any $z \in \mathfrak{m}_K$ and lift of z to a section $\widetilde{z}\colon \mathrm{Spf}(\mathbb{W}[\![Z]\!])^{\log} \to \widetilde{\mathcal{X}}$, there is a canonical identification*

$$0^*V \cong \widetilde{z}^*V$$

between the fibres of V at the origin and at $\widetilde{z}$, which is compatible with the induced Frobenius and connection on $\widetilde{z}^*V$ and on 0^*V. Moreover, this identification respects tensor products.

Proof. Write the connection on V as $\nabla_{V/\mathbb{W}} = M\mathrm{d}t + N\frac{\mathrm{d}Z}{Z}$ for some topologically nilpotent continuous $\mathbb{W}$-linear endomorphisms M, N of V. Flatness of the connection implies that M and N commute, while horizontality of Frobenius implies that $M\varphi = \varphi M$ and $N\varphi = p\varphi N$.

Assume first that $z \in p\mathcal{O}_K$, which forces $\widetilde{z}$ to lie in the divided power ideal of $\mathcal{O}_{\mathrm{cris}}$. Thus the power series

$$\mathcal{T}_{\widetilde{z}} := \exp(-\widetilde{z}M) = \sum_{r=0}^{\infty} \frac{(-\widetilde{z}M)^r}{r!}$$

converges to a well-defined $\mathbb{W}$-linear automorphism of V, with inverse $\mathcal{T}_{-\widetilde{z}} = \exp(\widetilde{z}M)$. Since M is even $\mathcal{O}_{\mathrm{cris}}[p^{-1}]$-linear and satisfies $M(tv) = tM(v) + v$, we have that the transport $\mathcal{T}_{\widetilde{z}}$ is $\mathcal{O}_{\mathrm{cris}}[p^{-1}]$-linear and satisfies $\mathcal{T}_{\widetilde{z}}(tv) = (t - \widetilde{z})\mathcal{T}_{\widetilde{z}}(v)$, so that it induces an isomorphism

$$\overline{\mathcal{T}}_{\widetilde{z}} \colon 0^*V = V/tV \xrightarrow{\sim} V/(t-\widetilde{z})V = \widetilde{z}^*V.$$

Recall that the induced connection $\widetilde{z}^*\nabla_{V/\mathbb{W}}$ on $\widetilde{z}^*V$ is then the unique map making

$$\begin{array}{ccc} V & \xrightarrow{\nabla_{V/\mathbb{W}}} & V\widehat{\otimes}_{\mathbb{W}}\omega^1_{\mathcal{O}_{\mathrm{cris}}[\![t]\!]^{\mathrm{DP}}/\mathbb{W}} \\ \downarrow & & \downarrow \\ \widetilde{z}^*V & \xrightarrow{\widetilde{z}^*\nabla_{V/\mathbb{W}}} & \widetilde{z}^*V\widehat{\otimes}_{\mathbb{W}}\omega^1_{\mathcal{O}_{\mathrm{cris}}/\mathbb{W}} \end{array}$$

commute, and that the induced Frobenius is the composite

$$V/(t-\widetilde{z})V \xrightarrow{\varphi} V/(t-\varphi(\widetilde{z}))V \xrightarrow{\overline{\mathcal{T}}_{\widetilde{z}-\varphi(\widetilde{z})}} V/(t-\widetilde{z})V.$$

A straightforward computation verifies the identities

$$\begin{aligned} \varphi(\mathcal{T}_{\widetilde{z}}(v)) &= \mathcal{T}_{\varphi(\widetilde{z})}(\varphi(v)) \\ \nabla_{V/\mathbb{W}}(\mathcal{T}_{\widetilde{z}}(v)) &= \mathcal{T}_{\widetilde{z}}(M(v))\mathrm{d}(t-\widetilde{z}) + \mathcal{T}_{\widetilde{z}}(N(v))\frac{\mathrm{d}Z}{Z} \end{aligned}$$

so that the isomorphism $\overline{\mathcal{T}}_{\widetilde{z}} \colon 0^*V \xrightarrow{\sim} \widetilde{z}^*V$ preserves the induced connection and Frobenius on either side, and is clearly compatible with tensor products in V.

Finally, if $z \in \mathfrak{m}_K$ is general, we can define $\mathcal{T}_{\widetilde{z}}$ by specifying that $\mathcal{T}_{\widetilde{z}} = \varphi^{-r} \circ \mathcal{T}_{\varphi^r(\widetilde{z})} \circ \varphi^r$ for any r large enough that $\varphi^r(z) \in p\mathcal{O}_K$. One verifies from this definition that $\mathcal{T}_{\widetilde{z}}$ still satisfies the same properties as above, and hence still induces an isomorphism $0^*V \xrightarrow{\sim} \widetilde{z}^*V$, as desired. □

Corollary 7.2.10. *Suppose that the curve Y has a regular semistable model $(\mathcal{X}, \mathcal{D})$ with y not lying in the tube of $\mathcal{D}_s$. Then lemma 7.0.2 is true for $(Y, y, \mathcal{X})$. If z is another point not lying in the tube of $\mathcal{D}_s$, then lemma 7.0.1 is true for (Y, y, z).*

Proof. The second part is immediate from proposition 7.2.8 and definition-lemma 4.4.3. For the first, the isomorphism

$$\mathsf{D}_{\mathrm{st}}(\mathcal{O}(U(y,z))) \cong \mathsf{D}_{\mathrm{st}}(\mathcal{O}(U(y,y)))$$

shows that there is a morphism $\mathsf{D}_{\mathrm{st}}(\mathcal{O}(U(y,z))) \to K_0$ of K_0-algebras which is also a morphism of ind-(φ, N)-modules. Hence there is also a G_K-equivariant morphism

$$\mathsf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \mathcal{O}(U(y,z)) \xrightarrow{\sim} \mathsf{B}_{\mathrm{st}} \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(\mathcal{O}(U(y,z))) \to \mathsf{B}_{\mathrm{st}}$$

of B_{st}-algebras, compatible with φ and N, and hence by taking points fixed by φ and killed by N, a G_K-equivariant morphism

$$\mathsf{B}_{\mathrm{cris}}^{\varphi=1} \otimes_{\mathbb{Q}_p} \mathcal{O}(U(y,z)) \to \mathsf{B}_{\mathrm{cris}}^{\varphi=1}$$

of $\mathsf{B}_{\mathrm{cris}}^{\varphi=1}$-algebras. Hence $U(y,z)(\mathsf{B}_{\mathrm{cris}}^{\varphi=1})^{G_K} \neq \emptyset$, which by definition-lemma 4.4.3 again says that the image of z under the non-abelian Kummer map lands in H^1_e. □

7.3 The case of general curves

Equipped with proposition 7.2.8, we are now in a position to prove lemmas 7.0.1 and 7.0.2 in the case $\dim_K(Y) = 1$, and thereby complete the proof of theorem 1.3.4. In order to relate this general case to the semistable case studied above, we will need the following hyperbolic version of the usual semistable reduction theorem.

Lemma 7.3.1. *Suppose that X/K is a connected smooth proper curve, and $D \subseteq X$ a reduced divisor, such that $X \setminus D$ is a geometrically connected smooth hyperbolic curve. Then there exists a finite extension L/K and a regular semistable model $(\mathcal{X}', \mathcal{D}')$ of the base-change (X_L, D_L). More specifically, if $(\mathcal{X}, \mathcal{D})$ is a proper integral $\mathcal{O}_K$-model of (X, D), then we may choose such an L and such an $(\mathcal{X}', \mathcal{D}')$ such that there is a diagram*

$$(\mathcal{X}_{\mathcal{O}_L}, \mathcal{D}_{\mathcal{O}_L}) \xleftarrow{g} (\mathcal{X}'', \mathcal{D}'') \xrightarrow{f} (\mathcal{X}', \mathcal{D}')$$

of morphisms of proper integral models of (X, D) such that g is an isomorphism away from the singular points of the special fibre of $\mathcal{X}_{\mathcal{O}_L}$ and the points of $\mathcal{D}_{\mathcal{O}_L}$ on the special fibre.

Proof. We will prove this in an appendix. L will be a field extension over which (X, D) acquires regular semistable reduction (lemma B.0.5), $(\mathcal{X}'', \mathcal{D}'')$ will be a regular normal crossings desingularisation of $(\mathcal{X}_{\mathcal{O}_L}, \mathcal{D}_{\mathcal{O}_L})$ (lemma B.0.2), and $(\mathcal{X}', \mathcal{D}')$ will be the minimal regular normal crossings model of (X_L, D_L) (theorem B.0.4), which is regular semistable by lemma B.0.5. □

Proposition 7.3.2. *Lemma 7.0.1 is true when* $\dim_K(Y) = 1$.

Proof. Note that Y is geometrically connected since it is connected and $Y(K) \neq \emptyset$. Let us write $Y = X \setminus D$ for a smooth compactification X of Y with D a reduced divisor. Enlarging D if necessary, we may assume that Y is hyperbolic. By lemma 7.3.1 (applied to the divisor $D \cup \{y, z\}$) there is a regular semistable model $(\mathcal{X}, \mathcal{D})$ of (X_L, D_L) for some finite L/K such that y and z extend to $\mathcal{O}_L$-integral points of $\mathcal{X} \setminus \mathcal{D}$. Then (Y, y, z) satisfies the condition in corollary 7.2.10, so that $\mathcal{O}(U(y, z))$ is an ind-semistable representation of G_L. But this already implies that it is ind-de Rham as a representation of G_K. □

Proposition 7.3.3. *Lemma 7.0.2 is true when* $\dim_K(Y) = 1$.

Proof. By removing a divisor from Y which doesn't meet $]y_s[$, we may assume that Y is hyperbolic. Choose by lemma 7.3.1 a field extension L, a regular semistable model $(\mathcal{X}', \mathcal{D}')$ of (X_L, D_L), and a diagram

$$(\mathcal{X}_{\mathcal{O}_L}, \mathcal{D}_{\mathcal{O}_L}) \xleftarrow{g} (\mathcal{X}'', \mathcal{D}'') \xrightarrow{f} (\mathcal{X}', \mathcal{D}')$$

of proper integral models of (X, D) where g is an isomorphism away from the singular points of the special fibre of $\mathcal{X}_{\mathcal{O}_L}$ and the points of $\mathcal{D}_{\mathcal{O}_L}$. In particular, g is an isomorphism over a Zariski neighbourhood of y_s.

Thus the tube of the reduction of y inside $\mathcal{X}_{\mathcal{O}_L}$ is contained inside the corresponding tube inside $\mathcal{X}'$, and so by corollary 7.2.10, the image of $Y(L) \cap]y_s[$ under the non-abelian Kummer map

$$Y(L) \to \mathrm{H}^1(G_L, U(\mathbb{Q}_p))$$

lands inside $\mathrm{H}^1_e(G_L, U(\mathbb{Q}_p))$. Lemma 4.3.1 (and compatibility of the non-abelian Kummer map with field extensions) shows that this descends to K, which is what we wanted to prove. □

Chapter 8

The main theorem (p-adic version)

With the assembled theory of the preceding chapters, we are now able to exactly translate the proof of theorem 1.1.6 to give a proof of its p-adic analogue, theorem 1.2.3. Let us fix an abelian variety A/K, and let us denote by $(L, \widetilde{0})$ a pair of a varying line line bundle L/A and a basepoint $\widetilde{0} \in L^\times(K)$ in the complement $L^\times = L \setminus 0$ of the zero section lying over $0 \in A(K)$. We will write $U/\mathbb{Q}_p$ for the $\mathbb{Q}_p$-pro-unipotent fundamental group of $L^\times$ based at $\widetilde{0}$.

As in chapter 3, U is a G_K-equivariant central extension

$$1 \to \mathbb{Q}_p(1) \xrightarrow{\mathsf{z}} U \to V_pA \to 1$$

arising from the fibration sequence $(\mathbb{G}_m, 1) \hookrightarrow (L^\times, \widetilde{0}) \twoheadrightarrow (A, 0)$.

By theorem 1.3.4, all the groups involved are de Rham, so that by corollary 6.3.6 we have an exact sequence

$$\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(V_pA) \to \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\curvearrowright} \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1_{g/e}(G_K, V_pA).$$

Since H^1_g and H^1_e agree for the Tate module of an abelian variety, the final term vanishes, which implies by the usual dimension formulae for abelian local Bloch–Kato Selmer groups that the first term vanishes too, and thus the middle map $\mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\curvearrowright} \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$ is bijective. Since by theorem 1.3.4 the image of the non-abelian Kummer map is contained in H^1_g, we have a well-defined composite

$$\lambda_L \colon L^\times(K) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xleftarrow{\sim} \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p,$$

where the final isomorphism arises from the explicit description of the local Bloch–Kato Selmer groups of $\mathbb{Q}_p(1)$.

It remains to show that the λ_L are the Néron log-metrics on our varying line bundle L, for which it suffices to show that the conditions of lemma 3.0.2 are satisfied.

Condition 1 is provided by theorem 1.3.4, and the remaining conditions follow entirely formally; the proofs in chapter 3 hold without modification, replacing ℓ with p and H^1 with $\mathrm{H}^1_{g/e}$ throughout.

Chapter 9

The main theorem (archimedean version)

To complete the circle of ideas sketched in the introduction, let us now prove theorem 1.1.7, the archimedean instance of our results relating local heights to Kummer maps. One can prove this through direct calculation, but we will instead imitate the proofs of our other two main theorems and show that the description in theorem 1.1.7 gives a well-defined family of functions $(\lambda_L)_L$ which satisfies a certain list of conditions uniquely defining the archimedean Néron log-metric.

Central to this discussion is the mixed Hodge structure carried by the fundamental groupoids of smooth complex varieties, constructed by Hain [Hai87a] using Chen's non-abelian de Rham theorem[1]. More precisely, if $Y/\mathbb{C}$ is a smooth connected variety, Hain constructs an ind-mixed Hodge structure[2] on the decompleted dual

$$\mathbb{Z}\pi_1(Y(\mathbb{C}); y, z)^{\wedge\circ} := \varinjlim \mathrm{Hom}(\mathbb{Z}\pi_1(Y(\mathbb{C}); y, z)/J^{n+1}, \mathbb{Z})$$

of the completed path-torsor coalgebra for each pair of points $y, z \in Y(\mathbb{C})$. Here $J^\bullet$ denotes the filtration on $\mathbb{Z}\pi_1(Y(\mathbb{C}); y, z)$ induced by the filtration on $\mathbb{Z}\pi_1(Y(\mathbb{C}); y)$ (equivalently $\mathbb{Z}\pi_1(Y(\mathbb{C}); z)$) defined by powers of the augmentation ideal. This mixed

[1]Strictly speaking, [Hai87a] only defines the mixed Hodge structure when $z = y$; [HZ87, Proposition 3.21] provides the construction in general.

[2]By an *ind-mixed Hodge structure* we simply mean an ind-object in the category of mixed Hodge structures [PS08, Definition 3.1] over an appropriate ring (for us, always $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$). Since morphisms of mixed Hodge structures are strict for the weight and Hodge filtrations [PS08, Corollary 3.6], an ind-mixed Hodge structure V is equivalently a $\mathbb{Z}$-module (resp. $\mathbb{Q}$- or $\mathbb{R}$-vector space) endowed with a weight filtration on $V_\mathbb{Q}$ and a Hodge filtration on $V_\mathbb{C}$ such that V is a filtered union of finitely generated submodules V_n such the restriction of the weight and Hodge filtrations to V_n define a mixed Hodge structure in the usual sense. Morphisms of ind-mixed Hodge structures are then just $\mathbb{Z}$-module (resp. $\mathbb{Q}$- or $\mathbb{R}$-vector space) homomorphisms compatible with the weight and Hodge filtrations—these are automatically strict for both filtrations by the corresponding result for mixed Hodge structures [PS08, Corollary 3.6].

Hodge structure is compatible with the algebra structure on $\mathbb{Z}\pi_1(Y(\mathbb{C}); y, z)^{\wedge\circ}$ and with the path-composition and path-reversal maps, so that in particular when $y = z$, $\mathbb{Z}\pi_1(Y(\mathbb{C}); y)^{\wedge\circ}$ carries an ind-mixed Hodge structure compatible with its Hopf algebra structure.

In this setup, the ind-mixed Hodge structures on the path-torsors $\pi_1(Y(\mathbb{C}); y, z)$ for varying z are compatible in an appropriate sense with that on $\pi_1(Y(\mathbb{C}); y)$, so that there is a natural non-abelian Kummer map

$$Y(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \pi_1(Y(\mathbb{C}); y))$$

to the moduli set of such torsors, which we will later identify as the inverse limit of the higher Albanese maps of [HZ87]. However, it turns out that when $Y = L^{\times}$ is the complement of zero in a line bundle on an abelian variety, this non-abelian Kummer map will be bijective [HZ87, Remarks and Examples 5.45][3], so is too fine an invariant to be useful to us.

Instead, we will work with the real Malčev completion

$$U = \mathrm{Spec}(\mathbb{R}\pi_1(Y(\mathbb{C}); y)^{\wedge\circ})$$

of the Betti fundamental group and its associated non-abelian Kummer map

$$Y(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R})),$$

and it will turn out that this map will be sufficiently coarse to allow us to recover exactly the Néron log-metric on such $\mathbb{G}_m$-torsors $L^{\times}$.

9.1 Pro-unipotent groups with mixed Hodge structures

Before analysing the non-abelian Kummer map for the real unipotent fundamental group in more detail, let us briefly set out a few basic properties of unipotent groups U with mixed Hodge structure in the abstract. The main objective here is to show that, under a certain natural condition, the moduli set $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ parametrising torsors under U with compatible mixed Hodge structure canonically has the structure of a real manifold, which we will construct via an explicit description of this set.

[3] [Hai87b] only asserts this for L with non-trivial first Chern class, but in fact this is true without this restriction.

Definition 9.1.1. Let $U/\mathbb{R}$ (resp. $U/\mathbb{Q}$) be a finitely generated pro-unipotent group. By a *mixed Hodge structure* on U we shall mean an ind-mixed Hodge structure on $\mathcal{O}(U)$ compatible with the Hopf algebra operations. By a *torsor (with mixed Hodge structure)* under such a group U, we shall mean a torsor P under the underlying group scheme endowed with an ind-mixed Hodge structure on $\mathcal{O}(P)$ which is compatible with the $\mathcal{O}(U)$-comodule algebra structure. By a morphism of such groups or torsors, we shall mean a morphism of the underlying group schemes or torsors which respects the weight and Hodge filtrations on the affine rings.

Proposition 9.1.2. *Let $U/\mathbb{R}$ (resp. $U/\mathbb{Q}$) be a finitely generated pro-unipotent group with mixed Hodge structure. Then each $J_\bullet\mathcal{O}(U)$ is a mixed Hodge substructure of $\mathcal{O}(U)$. If P is a torsor under U, then each $J_\bullet\mathcal{O}(P)$ is a mixed Hodge substructure of $\mathcal{O}(P)$, and the canonical isomorphism $\mathrm{gr}^J_\bullet\mathcal{O}(P) \xrightarrow{\sim} \mathrm{gr}^J_\bullet\mathcal{O}(U)$ (see proposition 2.2.6) is an isomorphism of ind-mixed Hodge structures.*

Proof. This is obvious from definition 2.1.12 and the description of the canonical isomorphism in proposition 2.2.6 (cf. remark 2.2.7). □

9.1.1 Weight and Hodge filtrations on torsors

Central to our explicit determination of the moduli set $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ lies in describing mixed Hodge structures on torsors directly on the torsors themselves, as opposed to on their affine rings. The key definition that allows us to study mixed Hodge structures is as follows.

Definition 9.1.3. Let $U/\mathbb{R}$ be a finitely generated pro-unipotent group with mixed Hodge structure, and P a torsor under U with mixed Hodge structure. We define subsets $W_0P(\mathbb{R}) \subseteq P(\mathbb{R}) = \mathcal{O}(P)^{*,\mathrm{gplike}}$, $W_0P(\mathbb{C}) \subseteq P(\mathbb{C}) = \mathcal{O}(P)^{*,\mathrm{gplike}}_{\mathbb{C}}$ and $F^0W_0P(\mathbb{C}) \subseteq P(\mathbb{C}) = \mathcal{O}(P)^{*,\mathrm{gplike}}_{\mathbb{C}}$ by:

$$\begin{aligned} W_0P(\mathbb{R}) &= \mathcal{O}(P)^{*,\mathrm{gplike}} \cap W_0; \\ W_0P(\mathbb{C}) &= \mathcal{O}(P)^{*,\mathrm{gplike}}_{\mathbb{C}} \cap W_0; \\ F^0W_0P(\mathbb{C}) &= \mathcal{O}(P)^{*,\mathrm{gplike}}_{\mathbb{C}} \cap F^0 \cap W_0. \end{aligned}$$

We clearly have $W_0P(\mathbb{R}) \subseteq W_0P(\mathbb{C}) \supseteq F^0W_0P(\mathbb{C})$.

We will see in proposition 9.1.11 that in the particular case that $P = U$, the subsets $W_0U(\mathbb{R})$, $W_0U(\mathbb{C})$ and $F^0W_0U(\mathbb{C})$ are subgroups, and that in general the subsets $W_0P(\mathbb{R})$, $W_0P(\mathbb{C})$ and $F^0W_0P(\mathbb{C})$ are torsors under each of these respective groups.

We also make the same definitions over the base field $\mathbb{Q}$ by replacing $\mathbb{R}$ with $\mathbb{Q}$ throughout.

Since the proof of this result is relatively technical, let us for the time being describe a converse construction.

Construction 9.1.4. Let $U/\mathbb{R}$ be a finitely generated pro-unipotent group and P a torsor under U. Suppose that a mixed Hodge structure on U is given, along with subsets $W_0P(\mathbb{R}) \subseteq P(\mathbb{R})$, $W_0P(\mathbb{C}) \subseteq P(\mathbb{C})$ and $F^0W_0P(\mathbb{C}) \subseteq P(\mathbb{C})$ which are torsors under $W_0U(\mathbb{R})$, $W_0U(\mathbb{C})$ and $F^0W_0U(\mathbb{C})$ respectively, and such that $W_0P(\mathbb{R}) \subseteq W_0P(\mathbb{C}) \supseteq F^0W_0P(\mathbb{C})$. We will describe how to construct an ind-mixed Hodge structure on $\mathcal{O}(P)$ which is compatible with the $\mathcal{O}(U)$-comodule algebra operations, thereby making P into a torsor under U with mixed Hodge structure.

To construct the weight filtration on $\mathcal{O}(P)$, pick an element $q \in W_0P(\mathbb{R})$, which gives rise to an isomorphism of schemes $U \xrightarrow{\sim} P$ by the action on q. We declare the weight filtration on $\mathcal{O}(P)$ to the filtration corresponding to the weight filtration on $\mathcal{O}(U)$ under the induced isomorphism $\mathcal{O}(P) \xrightarrow{\sim} \mathcal{O}(U)$.

This filtration is independent of the choice of q, for, if q' were another choice, then it would differ from q by the action of some $u \in W_0U(\mathbb{R})$, and hence the two isomorphisms $\mathcal{O}(P) \xrightarrow{\sim} \mathcal{O}(U)$ would differ by the algebra automorphism of $\mathcal{O}(U)$ induced by right-multiplication by u on U. Since $u \in W_0U(\mathbb{R})$, this automorphism strictly preserves the weight filtration (this is easiest to see on the dual $\mathcal{O}(U)^*$, where $u^{\pm 1} \in W_0\mathcal{O}(U)^*$), and hence q and q' induce the same weight filtration on $\mathcal{O}(P)$.

Similarly, to construct the Hodge filtration on $\mathcal{O}(P)_{\mathbb{C}}$ we pick any element $r \in F^0W_0P(\mathbb{C})$ and declare the Hodge filtration on $\mathcal{O}(P)_{\mathbb{C}}$ to be the filtration corresponding to the Hodge filtration on $\mathcal{O}(U)_{\mathbb{C}}$ under the induced isomorphism $\mathcal{O}(P)_{\mathbb{C}} \xrightarrow{\sim} \mathcal{O}(U)_{\mathbb{C}}$. The same proof as above shows that this is independent of the choice of r. It also follows that the isomorphism $\mathcal{O}(P)_{\mathbb{C}} \xrightarrow{\sim} \mathcal{O}(U)_{\mathbb{C}}$ induced by r is (strictly) compatible with the weight filtration induced by our choice of q above; this is since q and r both belong to the $W_0U(\mathbb{C})$-torsor $W_0P(\mathbb{C})$, and hence induce the same weight filtration by the same argument as above.

We will see in proposition 9.1.10 that these filtrations do indeed define a mixed Hodge structure on P. One may make the same construction with $\mathbb{R}$ replaced with $\mathbb{Q}$ throughout.

Proposition 9.1.5. *The constructions in definition 9.1.3 and construction 9.1.4 are mutually inverse. That is, let $U/\mathbb{R}$ be a finitely generated pro-unipotent group and P a torsor under U, and suppose that a mixed Hodge structure on U (but not P)*

is given. Then definition 9.1.3 and construction 9.1.4 provide a canonical bijection between:

- *ind-mixed Hodge structures on $\mathcal{O}(P)$ compatible with the $\mathcal{O}(U)$-comodule algebra operations; and*
- *subsets $W_0P(\mathbb{R}) \subseteq P(\mathbb{R})$, $W_0P(\mathbb{C}) \subseteq P(\mathbb{C})$ and $F^0W_0P(\mathbb{C}) \subseteq P(\mathbb{C})$ which are torsors under $W_0U(\mathbb{R})$, $W_0U(\mathbb{C})$ and $F^0W_0U(\mathbb{C})$ respectively, such that $W_0P(\mathbb{R}) \subseteq W_0P(\mathbb{C}) \supseteq F^0W_0P(\mathbb{C})$.*

Proof. This is a straightforward verification from the definitions. In the one direction, given subsets $W_0P(\mathbb{R})$ and $F^0W_0P(\mathbb{C})$ as above, the weight and Hodge filtrations on $\mathcal{O}(P)$ were constructed from choices of elements $q \in W_0P(\mathbb{R})$ and $r \in F^0W_0P(\mathbb{C})$ in such a way that $q \in W_0\mathcal{O}(P)^*$ and $r \in F^0W_0\mathcal{O}(P)^*_{\mathbb{C}}$. Hence $W_0P(\mathbb{R})$ meets $\mathcal{O}(P)^{*,\mathrm{gplike}} \cap W_0$ non-trivially, and hence they are equal since they are both torsors under $W_0U(\mathbb{R})$. By exactly the same argument, we also have $F^0W_0P(\mathbb{C}) = \mathcal{O}(P)^{*,\mathrm{gplike}}_{\mathbb{C}} \cap F^0 \cap W_0$, so that $W_0P(\mathbb{R})$ and $F^0W_0P(\mathbb{C})$ are indeed the subsets in definition 9.1.3 associated to the ind-mixed Hodge structure constructed by construction 9.1.4.

In the other direction, suppose that we are given an ind-mixed Hodge structure on $\mathcal{O}(P)$ as above. Since the map $\mathcal{O}(P)^*\widehat{\otimes}\mathcal{O}(U)^* \xrightarrow{\sim} \mathcal{O}(P)^*\widehat{\otimes}\mathcal{O}(P)^*$ given by $(q,u) \mapsto (q,qu)$ is a bijective morphism of pro-mixed Hodge structures[4], it is strict for the weight and Hodge filtrations, and hence both the action map $\mathcal{O}(P)^*\widehat{\otimes}\mathcal{O}(U)^* \to \mathcal{O}(P)^*$ and the difference map $\mathcal{O}(P)^*\widehat{\otimes}\mathcal{O}(P)^* \to \mathcal{O}(U)^*$ are morphisms of mixed Hodge structures. In particular, if we choose any $q \in \mathcal{O}(P)^{*,\mathrm{gplike}} \cap W_0$ then the isomorphism $\mathcal{O}(P) \xrightarrow{\sim} \mathcal{O}(U)$ induced by q *strictly* preserves the weight filtration, so that the weight filtration on $\mathcal{O}(P)$ constructed by construction 9.1.4 from the subset $\mathcal{O}(P)^* \cap W_0$ agrees with the original weight filtration. Exactly the same argument applies to the Hodge filtration, which completes the proof that the constructions are mutually inverse. □

Corollary 9.1.6. *Let $U/\mathbb{R}$ be a finitely generated pro-unipotent group with mixed Hodge structure. Then there is a canonical bijection*

$$\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R})) \xrightarrow{\sim} W_0U(\mathbb{R})\backslash W_0U(\mathbb{C})/F^0W_0U(\mathbb{C}),$$

where $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ denotes the set of isomorphism classes of torsors under U.

The same holds for $\mathbb{R}$ replaced with $\mathbb{Q}$ throughout.

[4]That is, pro-objects in the category of mixed Hodge structures.

Proof. The map sends a torsor P under U to the double coset containing the element $q^{-1}r \in U(\mathbb{C})$ where $q \in W_0P(\mathbb{R})$ and $r \in F^0P(\mathbb{C})$. That this gives a bijection is left as an easy exercise to the reader. □

Remark 9.1.7. Corollary 9.1.6 is a non-abelian analogue of the classical result that if V is an $\mathbb{R}$-mixed Hodge structure then

$$\mathrm{Ext}^1(\mathbb{R}(0), V) \cong \frac{W_0V_{\mathbb{C}}}{W_0V + F^0W_0V_{\mathbb{C}}}.$$

See for instance [PS08, Theorem 3.31] (with $\mathbb{Z}$ replaced by $\mathbb{R}$).

It remains to prove that the subsets defined in definition 9.1.3 are indeed subgroups and torsors as claimed, and that the weight and Hodge filtrations in construction 9.1.4 do indeed define a mixed Hodge structure on the torsor P. The key technical input in the latter proof is a criterion which describes exactly when certain extensions of mixed Hodge structures are also mixed Hodge structures.

Lemma 9.1.8. *Let*

$$0 \to V'' \to V \to V' \to 0$$

be an exact sequence of finite dimensional $\mathbb{R}$-vector spaces, each endowed with an increasing filtration $W_\bullet$, and with a decreasing filtration $F^\bullet$ on its complexification—we assume both filtrations are separated and exhaustive. Suppose that the morphisms in the exact sequence preserve both filtrations, and that the filtrations define $\mathbb{R}$-mixed Hodge structures on V'' and V' respectively. Then the following are equivalent:

1. *the filtrations define a mixed Hodge structure on V;*
2. *the morphisms in the exact sequence are strict for the filtration $W_\bullet$, and the morphisms in the induced exact sequences*

   $$0 \to \mathrm{gr}_n^W V''_{\mathbb{C}} \to \mathrm{gr}_n^W V_{\mathbb{C}} \to \mathrm{gr}_n^W V'_{\mathbb{C}} \to 0$$

   are strict for the filtration $F^\bullet$ (this implies that the original sequence was strict for the filtration $F^\bullet$ after tensoring with $\mathbb{C}$); and
3. *the exact sequence possesses a splitting after tensoring with $\mathbb{C}$ which is strict for both filtrations simultaneously.*

The same holds for $\mathbb{R}$ replaced with $\mathbb{Q}$ throughout.

Remark 9.1.9. A version of this lemma appears as [PS08, Criterion 3.10], where it is erroneously stated that a necessary and sufficient condition for V to be a mixed Hodge structure is that the morphisms in the exact sequence be strict for the weight and Hodge filtrations. In fact, strictness for the filtrations is necessary but insufficient for V to be a mixed Hodge structure.

Proof of lemma 9.1.8. We will prove the lemma only over $\mathbb{R}$, the case of $\mathbb{Q}$ following by the same argument.

For the implication (1) $\Rightarrow$ (3), it is simplest to use the Deligne splitting [PS08, Lemma-Definition 3.4], which is a bigrading $V^{p,q}$ of $V_{\mathbb{C}}$ that simultaneously splits the weight and Hodge filtrations in that

$$F^p V_{\mathbb{C}} = \bigoplus_{i \geq p} V^{i,j}$$

and

$$W_n V_{\mathbb{C}} = \bigoplus_{i+j \leq n} V^{i,j}.$$

The Deligne splitting is functorial with respect to morphisms of mixed Hodge structures, and hence

$$0 \to V'_{\mathbb{C}} \to V_{\mathbb{C}} \to V''_{\mathbb{C}} \to 0$$

is an exact sequence of bigraded vector spaces with respect to the Deligne splittings; taking any splitting of the sequence compatible with the bigradings gives a splitting compatible with the filtrations.

For the implication (3) $\Rightarrow$ (2), the existence of the splitting ensures that the morphisms in the sequence

$$0 \to V'_{\mathbb{C}} \to V_{\mathbb{C}} \to V''_{\mathbb{C}} \to 0$$

are strict for the filtration $W_\bullet$, and hence so too are the morphisms in the original sequence before tensoring with $\mathbb{C}$. The splitting also ensures that

$$0 \to \mathrm{gr}_n^W V'_{\mathbb{C}} \to \mathrm{gr}_n^W V_{\mathbb{C}} \to \mathrm{gr}_n^W V''_{\mathbb{C}} \to 0$$

is split compatibly with the filtration $F^\bullet$, and hence the morphisms are strict for $F^\bullet$.

Finally, for the implication (2) $\Rightarrow$ (1), it suffices to prove the special case that $V = \mathrm{gr}_n^W V$, so that V' and V'' are pure Hodge structures of weight n, and we want to prove that so too is V.

We define $V^{p,q} := F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$ for $p + q = n$ (and similarly for V' and V''), so that $V^{p,q}$ is canonically isomorphic to the kernel of the map $F^p V_{\mathbb{C}} \oplus \overline{F^q V_{\mathbb{C}}} \to V_{\mathbb{C}}$ given by the sum of the inclusions. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^pV''_{\mathbb{C}} \oplus \overline{F^qV''_{\mathbb{C}}} & \longrightarrow & F^pV_{\mathbb{C}} \oplus \overline{F^qV_{\mathbb{C}}} & \longrightarrow & F^pV'_{\mathbb{C}} \oplus \overline{F^qV'_{\mathbb{C}}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V''_{\mathbb{C}} & \longrightarrow & V_{\mathbb{C}} & \longrightarrow & V'_{\mathbb{C}} & \longrightarrow & 0 \end{array}$$

we obtain a short exact sequence

$$0 \to (V'')^{p,q} \to V^{p,q} \to (V'')^{p,q} \to 0,$$

where the final 0 comes from the fact that $F^pV''_{\mathbb{C}} + \overline{F^qV''_{\mathbb{C}}} = V''_{\mathbb{C}}$ since V'' is a pure Hodge structure of weight n. Since $V'_{\mathbb{C}} = \bigoplus_{p+q=n}(V')^{p,q}$ and similarly for V'', we deduce from the above exact sequences that $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ also, so that V is a pure Hodge structure of weight n, as desired. □

Proposition 9.1.10. *The weight and Hodge filtrations constructed in construction 9.1.4 define a mixed Hodge structure on P (i.e. an ind-mixed Hodge structure on $\mathcal{O}(P)$ compatible with the $\mathcal{O}(U)$-comodule algebra structure).*

Proof. Compatibility with the $\mathcal{O}(U)$-comodule algebra structure morphisms is clear from the construction, since the isomorphisms $\mathcal{O}(P) \xrightarrow{\sim} \mathcal{O}(U)$ and $\mathcal{O}(P)_{\mathbb{C}} \xrightarrow{\sim} \mathcal{O}(U)_{\mathbb{C}}$ induced from our choices of q and r are isomorphisms of $\mathcal{O}(U)$-comodule algebras.

It remains to show that the weight and Hodge filtrations define an ind-mixed Hodge structure on $\mathcal{O}(P)$. To do this, note that by construction the induced weight and Hodge filtrations on $\mathrm{gr}^J_\bullet\mathcal{O}(P)$ agree with those on $\mathrm{gr}^J_\bullet\mathcal{O}(U)$ under the canonical isomorphism $\mathrm{gr}^J_\bullet\mathcal{O}(P) \xrightarrow{\sim} \mathrm{gr}^J_\bullet\mathcal{O}(U)$, and hence define an ind-mixed Hodge structure on $\mathrm{gr}^J_\bullet\mathcal{O}(P)$. We now show by induction on n that each $J_n\mathcal{O}(P)$ is a mixed Hodge structure (starting from the trivial base case $n = -1$), using the exact sequences

$$0 \to J_{n-1}\mathcal{O}(P) \to J_n\mathcal{O}(P) \to \mathrm{gr}^J_n\mathcal{O}(P) \to 0. \tag{9.1.1}$$

The right-hand term is a mixed Hodge structure, and our inductive hypothesis ensures that so too is the left-hand term. Now our choice of q gives rise to an isomorphism $\mathcal{O}(P)_{\mathbb{C}} \xrightarrow{\sim} \mathcal{O}(U)_{\mathbb{C}}$ strictly compatible with the weight, Hodge and J-filtrations, and hence the sequence 9.1.1 becomes isomorphic to the corresponding sequence

$$0 \to J_{n-1}\mathcal{O}(U) \to J_n\mathcal{O}(U) \to \mathrm{gr}^J_n\mathcal{O}(U) \to 0 \tag{9.1.2}$$

after tensoring with $\mathbb{C}$, in a manner strictly compatible with the weight and Hodge filtrations. Since the sequence 9.1.2 splits over $\mathbb{C}$ compatibly with both filtrations, so too does the sequence 9.1.1, and hence $J_n\mathcal{O}(P)$ is a mixed Hodge structure by lemma 9.1.8. This completes the construction of the mixed Hodge structure on P. □

Proposition 9.1.11. *Let $U/\mathbb{R}$ be a finitely generated pro-unipotent group with mixed Hodge structure and P a torsor under U (with mixed Hodge structure). Then the subsets $W_0U(\mathbb{R}) \subseteq U(\mathbb{R})$, $W_0U(\mathbb{C}) \subseteq U(\mathbb{C})$ and $F^0W_0U(\mathbb{C}) \subseteq U(\mathbb{C})$ defined in definition 9.1.3 are subgroups, and the subsets $W_0P(\mathbb{R})$, $W_0P(\mathbb{C})$ and $F^0W_0P(\mathbb{C})$ are torsors under $W_0U(\mathbb{R})$, $W_0U(\mathbb{C})$ and $F^0W_0U(\mathbb{C})$ respectively.*

The same holds when $\mathbb{R}$ is replaced with $\mathbb{Q}$ throughout.

Proof. We will just deal with the case of $F^0W_0U(\mathbb{C})$ and $F^0W_0P(\mathbb{C})$, the other cases being simpler. To see that $F^0W_0U(\mathbb{C}) \subseteq U(\mathbb{C})$ is a subgroup, we simply note that the group operations on $U(\mathbb{C}) = \mathcal{O}(U)_{\mathbb{C}}^{*,\mathrm{gplike}}$ are induced by the multiplication and antipode in the Hopf algebra $\mathcal{O}(U)_{\mathbb{C}}^{*}$, and that these operations preserve both the weight and Hodge filtrations, so that $F^0W_0U(\mathbb{C}) = \mathcal{O}(U)_{\mathbb{C}}^{*,\mathrm{gplike}} \cap F^0 \cap W_0$ is a subgroup.

To show that $F^0W_0P(\mathbb{C})$ is a torsor under $F^0W_0U(\mathbb{C})$, we note firstly that the coalgebra isomorphism $\mathcal{O}(P)^* \widehat{\otimes} \mathcal{O}(U)^* \xrightarrow{\sim} \mathcal{O}(P)^* \widehat{\otimes} \mathcal{O}(P)^*$ induced by the isomorphism $P \times U \xrightarrow{\sim} P \times P$ given by $(q, u) \mapsto (q, qu)$ is a morphism of pro-mixed Hodge structures, and hence is strict for the Hodge and weight filtrations. Thus for $q \in F^0W_0P(\mathbb{C})$ we have that $qu \in F^0W_0P(\mathbb{C})$ if and only if $u \in F^0W_0U(\mathbb{C})$, so that the action of $F^0W_0U(\mathbb{C})$ on $P(\mathbb{C})$ restricts to a transitive action on $F^0W_0P(\mathbb{C})$.

It remains to show that $F^0W_0P(\mathbb{C})$ is non-empty, i.e. that there is an algebra homomorphism $\mathcal{O}(P)_{\mathbb{C}} \to \mathbb{C}$ compatible with both the weight and Hodge filtrations. For $n \in \mathbb{N}_0$ let U_n denote the maximal n-step unipotent quotient of U and $P_n := P \times^U U_n$ the pushout. It follows from lemma 2.1.9 that $\mathcal{O}(P_n)_{\mathbb{C}} \leq \mathcal{O}(P)_{\mathbb{C}}$ is the subalgebra generated by $J_n\mathcal{O}(P)_{\mathbb{C}}$, and hence by proposition 9.1.2 is a sub-ind-mixed Hodge structure. We claim that each inclusion $\mathcal{O}(P_{n-1})_{\mathbb{C}} \hookrightarrow \mathcal{O}(P_n)_{\mathbb{C}}$ admits a retraction which is an algebra homomorphism compatible with the weight and Hodge filtrations. This completes the proof of the proposition, since then the algebra inclusion $\mathbb{C} = \mathcal{O}(P_0)_{\mathbb{C}} \hookrightarrow \mathcal{O}(P)_{\mathbb{C}} = \bigcup_n \mathcal{O}(P_n)_{\mathbb{C}}$ also admits such a retraction, as desired.

To prove the claim, we choose by lemma 9.1.8[5] a retraction of the inclusion $\mathcal{O}(P_{n-1})_{\mathbb{C}} \cap J_n\mathcal{O}(P)_{\mathbb{C}} \hookrightarrow J_n\mathcal{O}(P)_{\mathbb{C}}$ which preserves both the weight and Hodge filtrations. The chosen retraction then induces a morphism

$$\phi \colon \mathrm{Sym}^{\bullet}\left(J_n\mathcal{O}(P)_{\mathbb{C}}\right) \to \mathcal{O}(P_{n-1})_{\mathbb{C}}$$

of algebras, compatible with the weight and Hodge filtrations. Now if $x \in J_i\mathcal{O}(P)$ and $y \in J_j\mathcal{O}(P)$, we have that $\phi([x][y]) = \phi([x])\phi([y]) = \phi([xy])$—if $i, j < n$, this is

[5] In the analogous proof for $W_0P(\mathbb{R})$ one uses instead the fact that an exact sequence of $\mathbb{R}$-mixed Hodge structures is split compatibly with the weight filtration, since all the morphisms are strict.

since by construction ϕ restricts to the inclusion $\mathcal{O}(P_{n-1})_{\mathbb{C}} \cap J_n\mathcal{O}(P)_{\mathbb{C}} \hookrightarrow \mathcal{O}(P_{n-1})_{\mathbb{C}}$, while if $i = 0$ or $j = 0$, this is since $J_0\mathcal{O}(P)_{\mathbb{C}} = \mathbb{C}$ and ϕ is $\mathbb{C}$-linear. We also trivially have that $\phi([1]) = 1 = \phi(1)$.

It thus follows by lemma 2.1.9 (and the fact that $\mathcal{O}(P)_{\mathbb{C}} \simeq \mathcal{O}(U)_{\mathbb{C}}$ as J-filtered algebras) that ϕ factors uniquely through an algebra homomorphism $\overline{\phi}\colon \mathcal{O}(P_n)_{\mathbb{C}} \to \mathcal{O}(P_{n-1})_{\mathbb{C}}$. Since the quotient map $\mathrm{Sym}^\bullet(J_n\mathcal{O}(P)_{\mathbb{C}}) \twoheadrightarrow \mathcal{O}(P_n)_{\mathbb{C}}$ is the complexification of a morphism of ind-mixed Hodge structures, it is strict for the weight and Hodge filtrations, and hence $\overline{\phi}$ also preserves these filtrations. Finally, we note by construction that $\overline{\phi}$ restricts to the canonical inclusion $J_{n-1}\mathcal{O}(P)_{\mathbb{C}} \hookrightarrow \mathcal{O}(P_{n-1})_{\mathbb{C}}$, and hence is a retraction of the inclusion $\mathcal{O}(P_{n-1})_{\mathbb{C}} \hookrightarrow \mathcal{O}(P_n)_{\mathbb{C}}$, as desired. □

9.1.2 Cosimplicial groups and twists for mixed Hodge structures

Remark 9.1.12. If U is a finitely generated pro-unipotent group with mixed Hodge structure, then the description of $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ in corollary 9.1.6 can be expressed in cosimplicial language, by considering the cosimplicial group $U^\bullet_{\mathbf{MHS}}(\mathbb{R})$ cogenerated by the semi-cosimplicial group

$$F^0W_0U(\mathbb{C}) \times W_0U(\mathbb{R}) \rightrightarrows W_0U(\mathbb{C})$$

where $d^0(u, w) = u$ and $d^1(u, w) = w$. In this language, we have a canonical identification of the cohomotopy as

$$\pi^i(U^\bullet_{\mathbf{MHS}}(\mathbb{R})) = \begin{cases} F^0W_0U(\mathbb{R}) & \text{if } i = 0, \\ \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R})) & \text{if } i = 1, \\ 1 & \text{if } i \geq 2 \text{ and } U \text{ abelian.} \end{cases}$$

Thus one is tempted to call an element $\alpha \in W_0U(\mathbb{C})$ a *cocycle* for U, and define the *twist* ${}_\alpha U$ of U by α to be the finitely generated pro-unipotent group with mixed Hodge structure whose underlying group scheme is U, whose weight filtration on $\mathcal{O}({}_\alpha U) = \mathcal{O}(U)$ is that of U, and whose Hodge filtration on $\mathcal{O}({}_\alpha U)_{\mathbb{C}}$ is given by $F^p\mathcal{O}({}_\alpha U)_{\mathbb{C}} = \mathrm{Ad}^*_\alpha F^p\mathcal{O}(U)_{\mathbb{C}}$, where $\mathrm{Ad}_\alpha\colon U_{\mathbb{C}} \xrightarrow{\sim} U_{\mathbb{C}}$ is (left-)conjugation by α. To check that this defines a mixed Hodge structure on ${}_\alpha U$, we note that Ad^*_α is a Hopf algebra automorphism which acts as the identity on $\mathrm{gr}^J_\bullet(\mathcal{O}(U))$, so the induced weight and Hodge filtrations of $\mathrm{gr}^J_\bullet(\mathcal{O}({}_\alpha U))$ define a mixed Hodge structure on each graded piece. To show that the filtrations in fact define a mixed Hodge structure on each $J_n\mathcal{O}({}_\alpha U)$, one argues inductively as in construction 9.1.4, using the fact that Ad^*_α

sets up an isomorphism $\mathcal{O}({}_\alpha U)_{\mathbb{C}} \xrightarrow{\sim} \mathcal{O}(U)_{\mathbb{C}}$ compatible with the weight and Hodge filtrations simultaneously.

It follows from proposition 5.3.1 that, as expected, there is a canonical bijection $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, {}_\alpha U(\mathbb{R})) \xrightarrow{\sim} \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ (induced from the map ${}_\alpha U(\mathbb{C}) \xrightarrow{\sim} U(\mathbb{C})$ given by right-multiplication by α).

Using this cosimplicial description, we immediately deduce a result describing how the sets $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ behave in central extensions, generalising the long exact sequence on Ext-groups $\mathrm{Ext}^i(\mathbb{R}(0), V)$ for usual (abelian) mixed Hodge structures V. The non-abelian version does not, to the best of the author's knowledge, appear in the literature, though undoubtedly results of this kind are already known to experts.

Corollary 9.1.13. *Let*

$$1 \to Z \xrightarrow{\mathsf{z}} U \to Q \to 1$$

be a central extension of unipotent groups over $\mathbb{R}$ with mixed Hodge structure. Then there is a functorially assigned exact sequence

$$\begin{array}{l} 1 \longrightarrow F^0W_0Z(\mathbb{R}) \xrightarrow{\mathsf{z}} F^0W_0U(\mathbb{R}) \longrightarrow F^0W_0Q(\mathbb{R}) \\ \quad \hookrightarrow \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, Z(\mathbb{R})) \xrightarrow{\curvearrowright} \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R})) \longrightarrow \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, Q(\mathbb{R})) \longrightarrow 1 \end{array}$$

where the horizontal arrows are the induced maps.

Proof. Consider the sequence of cosimplicial groups

$$1 \to Z^\bullet_{\mathbf{MHS}} \xrightarrow{\mathsf{z}} U^\bullet_{\mathbf{MHS}} \to Q^\bullet_{\mathbf{MHS}} \to 1,$$

where $U^\bullet_{\mathbf{MHS}}$, $Q^\bullet_{\mathbf{MHS}}$ and $Z^\bullet_{\mathbf{MHS}}$ are the cosimplicial groups described in remark 9.1.12. Identifying $W_0U(\mathbb{R})$ (resp. $W_0U(\mathbb{C})$, resp. $F^0W_0U(\mathbb{C})$) with $W_0\mathrm{Lie}(U)$ (resp. $W_0\mathrm{Lie}(U)_{\mathbb{C}}$, resp. $F^0W_0\mathrm{Lie}(U)_{\mathbb{C}}$) via the logarithm map, we see that the above sequence is a degree-wise central extension. The long exact sequence on cohomotopy from theorem 5.2.1 is then the desired long exact sequence. □

9.1.3 Manifold structures on moduli of torsors

Armed with the explicit description in corollary 9.1.6, we now come to the main point of this chapter: that the set $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ for a unipotent group $U/\mathbb{R}$ with mixed Hodge structure carries a canonical topology, namely the quotient topology from its description as a double-coset space. Most crucially, under mild conditions on U, it even has canonically the structure of a C^∞ real manifold.

Definition 9.1.14. Let $U/\mathbb{R}$ (resp. $U/\mathbb{Q}$) be a pro-unipotent group with mixed Hodge structure. We say that U *has only negative weights* just when:

- $W_0\mathcal{O}(U) = \mathbb{R}$ (resp. $= \mathbb{Q}$); or equivalently
- $W_{-1}\mathrm{Lie}(U) = \mathrm{Lie}(U)$.

Remark 9.1.15. The condition that U has only negative weights is automatically satisfied for the $\mathbb{R}$-pro-unipotent Betti fundamental group of any smooth connected $\mathbb{C}$-variety (for instance by the description in [Hai87a, Section 5]). In particular, any finite-dimensional mixed-Hodge-theoretic quotient of such a U also only has negative weights.

It is this condition of negativity of weights which provides the C^∞ real manifold structure on $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R}))$, as follows.

Proposition 9.1.16. *Let $U/\mathbb{R}$ be a unipotent group with mixed Hodge structure with only negative weights. Then the C^∞ Lie group action of $F^0U(\mathbb{C}) \times U(\mathbb{R})$ on $U(\mathbb{C})$ by $x \cdot (u, w) = w^{-1}xu$ is free and proper, so that by the quotient manifold theorem [Lee03, Theorem 21.10] $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R})) \cong U(\mathbb{R})\backslash U(\mathbb{C})/F^0U(\mathbb{C})$ is canonically a C^∞ real manifold.*

Proof. In the case that U is abelian, the condition on the weights implies that $U(\mathbb{R})$ and $F^0U(\mathbb{C})$ meet trivially in $U(\mathbb{C})$. The given action is the translation by the subgroup $F^0U(\mathbb{C}) \oplus U(\mathbb{R})$, which is necessarily free and proper.

In general we may proceed inductively, writing U as a central extension

$$1 \to Z \xrightarrow{\mathsf{z}} U \to Q \to 1$$

of unipotent groups with mixed Hodge structures, where we know the result for Q and for Z. The action is free, since if some $\mathbf{u} \in F^0U(\mathbb{C}) \times U(\mathbb{R})$ had a fixed point in $U(\mathbb{C})$, its image in $F^0Q(\mathbb{C}) \times Q(\mathbb{R})$ would have a fixed point, and hence $\mathbf{u}$ would lie in $F^0Z(\mathbb{C}) \times Z(\mathbb{R}) = F^0Z(\mathbb{C}) \oplus Z(\mathbb{R})$. But $U(\mathbb{C}) \twoheadrightarrow Q(\mathbb{C})$ is a $Z(\mathbb{C})$-torsor by translation, so that the only way $\mathbf{u}$ could have a non-trivial fixed point is if it were the identity already. This proves freeness of the action.

To prove properness, it is equivalent by [Lee03, Proposition 21.5] to prove that whenever we are given sequences (x_i) in $U(\mathbb{C})$ and $(\mathbf{u}_i)$ in $F^0U(\mathbb{C}) \times U(\mathbb{R})$ such that (x_i) and $(x_i \cdot \mathbf{u}_i)$ converge, there is a convergent subsequence of $(\mathbf{u}_i)$. Given such sequences, our inductive hypothesis shows that the image of $(\mathbf{u}_i)$ in $F^0Q(\mathbb{C}) \times Q(\mathbb{R})$ has a convergent subsequence, so we may suppose without loss of generality that this

image sequence converges. Since the map $F^0U(\mathbb{C}) \times U(\mathbb{R}) \twoheadrightarrow F^0Q(\mathbb{C}) \times Q(\mathbb{R})$, being the map on $\mathbb{R}$-points of a morphism of affine spaces over $\mathbb{R}$, is continuously split, we may thus write $\mathbf{u}_i = \mathbf{u}'_i\mathbf{z}_i$, where $(\mathbf{u}'_i)$ is a convergent sequence in $F^0U(\mathbb{C}) \times U(\mathbb{R})$ and $\mathbf{z}_i \in F^0Z(\mathbb{C}) \times Z(\mathbb{R})$.

Now the sequences $(x'_i) = (x_i \cdot \mathbf{u}'_i)$ and $(x'_i \cdot \mathbf{z}_i) = (x_i \cdot \mathbf{u}_i)$ both converge. But the action of $F^0Z(\mathbb{C}) \times Z(\mathbb{R})$ on $U(\mathbb{C})$ can be identified with the translation action of the abelian Lie subgroup $F^0Z(\mathbb{C}) \times Z(\mathbb{R})$ on $U(\mathbb{C})$ (up to an automorphism of $F^0Z(\mathbb{C}) \times Z(\mathbb{R})$), and hence is proper, so that $\mathbf{z}_i$ has a convergent subsequence. Hence $(\mathbf{u}_i) = (\mathbf{u}'_i\mathbf{z}_i)$ has a convergent subsequence, as desired. □

Before we conclude this section, let us briefly analyse some of the functoriality properties of $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, -)$, for use in our later proof of theorem 1.1.7.

Lemma 9.1.17. *If $U \to W$ is a morphism of unipotent groups over $\mathbb{R}$ with mixed Hodge structure, both with only negative weights, then the induced map*

$$\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R})) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, W(\mathbb{R}))$$

is a C^∞ morphism of C^∞ manifolds. If $U \to W$ is injective (resp. surjective), then it is an immersion (resp. a submersion)[6].

Proof. Since $U(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R}))$ is a locally trivial torsor for $F^0U(\mathbb{C}) \times U(\mathbb{R})$ (for example from the proof of [Lee03, Theorem 21.10]), the first point follows from the fact that $U(\mathbb{C}) \to W(\mathbb{C})$ is C^∞ (even algebraic). Now suppose that $f\colon U \twoheadrightarrow W$ is surjective. If $\alpha \in U(\mathbb{C})$ is any point, then we have a commuting square

$$\begin{array}{ccc} \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, {}_\alpha U(\mathbb{R})) & \longrightarrow & \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, {}_{f(\alpha)}W(\mathbb{R})) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R})) & \longrightarrow & \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, W(\mathbb{R})) \end{array}$$

of C^∞ manifolds, where ${}_\alpha U$ and ${}_{f(\alpha)}W$ are the twists described in remark 9.1.12. Since the leftmost vertical map takes the distinguished point to $[\alpha]$, to prove that $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, U(\mathbb{R})) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, W(\mathbb{R}))$ is a submersion it suffices to prove (for general U, W) that it is surjective on tangent spaces at the distinguished point. But the map on tangent spaces is

$$\mathrm{Lie}(U)_{\mathbb{C}}/(F^0 + \mathrm{Lie}(U)_{\mathbb{R}}) \to \mathrm{Lie}(W)_{\mathbb{C}}/(F^0 + \mathrm{Lie}(W)_{\mathbb{R}}),$$

which is clearly surjective.

[6] In fact, the map is even an injective immersion (resp. surjective submersion)

If instead f were injective, by the same argument it would suffice to prove that the above map is injective. But we can identify the sides as the Yoneda Ext-groups $\mathrm{Ext}^1_{\mathbf{MHS}_\mathbb{R}}(\mathbb{R}, \mathrm{Lie}(U))$ (resp. $\mathrm{Ext}^1_{\mathbf{MHS}_\mathbb{R}}(\mathbb{R}, \mathrm{Lie}(W))$), so that the kernel of the map is a surjective image of $\mathrm{Hom}_{\mathbf{MHS}_\mathbb{R}}(\mathbb{R}, \mathrm{Lie}(W)/\mathrm{Lie}(U)) = F^0W_0\left(\mathrm{Lie}(W)/\mathrm{Lie}(U)\right)_\mathbb{R}$. But since $\mathrm{Lie}(W)/\mathrm{Lie}(U)$ has only negative weights, this vector space vanishes, and hence the desired map was indeed injective. □

Remark 9.1.18. When discussing the topology on $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R}))$, we have focused on unipotent groups rather than finitely generated pro-unipotent ones, even though $\mathbb{R}$-pro-unipotent fundamental groups will in general be infinite-dimensional (though still finitely generated). The principal reason for this is that one can reduce the study of $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R}))$ for U finitely generated pro-unipotent to the finite-dimensional case: if we write such a $U = \varprojlim U_n$ as an inverse limit of finite-dimensional quotients with compatible mixed Hodge structures, then the natural map

$$\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R})) \to \varprojlim \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U_n(\mathbb{R}))$$

is bijective (the same proof as lemma 6.3.5 works, using the cosimplicial description in remark 9.1.12). Hence if U has only negative weights, $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R}))$ is canonically a pro-object in the category of C^∞ manifolds (this structure is independent of the choice of U_n), whose underlying topology can be checked to agree with the quotient topology on $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R})) = U(\mathbb{R})\backslash U(\mathbb{C})/F^0U(\mathbb{C})$ induced from the pro-analytic topology on $U(\mathbb{C})$.

9.2 Higher Albanese manifolds and the $\mathbb{R}$-pro-unipotent Kummer map

In the previous section, we examined $\mathbb{Q}$- and $\mathbb{R}$-unipotent groups with mixed Hodge structure, but one can equally well study an integral version of this theory, by studying finitely generated (discrete) groups Π with a ind-mixed Hodge structure on the decompleted dual $\mathbb{Z}\Pi^{\wedge\circ} := \varinjlim \mathrm{Hom}\left(\mathbb{Z}\Pi/J^{n+1}, \mathbb{Z}\right)$ compatible with the Hopf algebra operations, or equivalently a $\mathbb{Q}$-mixed Hodge structure on the $\mathbb{Q}$-Malčev completion $\Pi_\mathbb{Q} := \mathrm{Spec}(\mathbb{Q}\Pi^{\wedge\circ})$—we call such a structure a *mixed Hodge structure* on Π. Here, as usual, J denotes the augmentation ideal of $\mathbb{Z}\Pi$.

If P is a torsor under such a group, we define a *mixed Hodge structure* on P to be an ind-mixed Hodge structure on $\mathbb{Z}P^{\wedge\circ} := \varinjlim \mathrm{Hom}\left(\mathbb{Z}P/J^{n+1}, \mathbb{Z}\right)$ compatible with the algebra operations and $\mathbb{Z}\Pi^{\wedge\circ}$-comodule structure, where J^{n+1} denotes the

filtration of $\mathbb{Z}P$ arising from the J-filtration on $\mathbb{Z}\Pi$ under any trivialisation $\Pi \xrightarrow{\sim} P$ of Π-torsors. We will refer to a P endowed with such a structure as a *torsor under* Π, where the mixed Hodge structure is taken as implied.

With these definitions, one can develop an integral version of the theory of the previous section, and again we obtain an explicit description of the classifying space for torsors.

Proposition 9.2.1. *Let Π be a finitely generated nilpotent group with mixed Hodge structure. Then there is a canonical identification of the set $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi)$ of isomorphism classes of Π-torsors with the quotient of $\Pi(\mathbb{Q}) \times W_0\Pi(\mathbb{C})$ by the action of $F^0W_0\Pi(\mathbb{C}) \times W_0\Pi(\mathbb{Q}) \times \Pi$ given by $(x, y) \cdot (u, v, w) = (\overline{w}^{-1}xv, v^{-1}yu)$, where $\overline{w}$ denotes the image of w under the map $\Pi \to \Pi(\mathbb{Q})$. Here $\Pi(\mathbb{Q})$ (resp. $\Pi(\mathbb{C})$) denotes the $\mathbb{Q}$-points (resp. $\mathbb{C}$-points) of the Malčev completion of Π.*

In particular, if Π has only negative weights, then we have a canonical bijection

$$\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi) \cong \Pi\backslash\Pi(\mathbb{C})/F^0\Pi(\mathbb{C}).$$

Proof. Let us first describe the classifying map

$$\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi) \to (\Pi(\mathbb{Q}) \times W_0\Pi(\mathbb{C})) \,/\, \left(F^0W_0\Pi(\mathbb{C}) \times W_0\Pi(\mathbb{Q}) \times \Pi\right).$$

Given a Π-torsor P with mixed Hodge structure, we may choose by proposition 9.1.5 elements $q \in P$, $x \in \Pi_{\mathbb{Q}}(\mathbb{Q})$ and $y \in W_0\Pi_{\mathbb{Q}}(\mathbb{C})$ such that

$$W_0P_{\mathbb{Q}}(\mathbb{Q}) = \overline{q}xW_0\Pi_{\mathbb{Q}}(\mathbb{Q}) \text{ and } F^0W_0P_{\mathbb{Q}}(\mathbb{C}) = \overline{q}xyF^0W_0\Pi_{\mathbb{Q}}(\mathbb{C}),$$

where $\overline{q} \in P_{\mathbb{Q}}(\mathbb{Q})$ denotes the image of q under the canonical map $P \to P_{\mathbb{Q}}(\mathbb{Q})$. We declare the image $[P]$ of P under the classifying map to be the orbit of the pair (x, y) under the specified action—this is easily seen to be independent of choices.

It remains to check that the classifying map is bijective. It is certainly surjective, since for any $(x, y) \in \Pi(\mathbb{Q}) \times W_0\Pi(\mathbb{C})$, the above construction with $P = \Pi$ (as a Π-torsor) and $q = 1$ specifies a mixed Hodge structure on $P_{\mathbb{Q}}$ (and hence on P) by proposition 9.1.5. For injectivity, suppose that we are given two Π-torsors P and P' with mixed Hodge structure, and choose elements q, x, y and q', x', y' as above. If $[P] = [P']$, then $(x', y') = (\overline{w}^{-1}xv, v^{-1}yu)$ for some $u \in F^0W_0\Pi(\mathbb{C})$, $v \in W_0\Pi(\mathbb{Q})$ and $w \in \Pi$, and we let $\phi\colon P \xrightarrow{\sim} P'$ be the unique isomorphism of Π-torsors taking q to $q'w^{-1}$. It follows that

$$\phi\left(W_0P_{\mathbb{Q}}(\mathbb{Q})\right) = \phi(\overline{q}x)W_0\Pi_{\mathbb{Q}}(\mathbb{Q}) = \overline{q}'x'v^{-1}W_0\Pi_{\mathbb{Q}}(\mathbb{Q}) = W_0P'_{\mathbb{Q}}(\mathbb{Q})$$

and similarly for $F^0W_0P_{\mathbb{Q}}(\mathbb{C})$, so that ϕ is an isomorphism of Π-torsors with mixed Hodge structures. This proves injectivity, as desired. □

Remark 9.2.2. Proposition 9.2.1 is a non-abelian generalisation of the explicit description of Ext-groups of torsion-free $\mathbb{Z}$-mixed Hodge structures, in particular that

$$\mathrm{Ext}^1(\mathbb{Z}(0), V) \cong \frac{V_{\mathbb{Q}} \oplus W_0 V_{\mathbb{C}}}{F^0 W_0 V_{\mathbb{C}} + W_0 V_{\mathbb{Q}} + V_{\mathbb{Z}}}$$

for any torsion-free $\mathbb{Z}$-mixed Hodge structure V, where $W_0 V_{\mathbb{Q}}$ is embedded in $V_{\mathbb{Q}} \oplus W_0 V_{\mathbb{C}}$ diagonally.[7]

Corollary 9.2.3. *Let Π be a finitely generated nilpotent group with mixed Hodge structure with only negative weights. Then*

$$\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi) \cong \Pi\backslash\Pi(\mathbb{C})/F^0\Pi(\mathbb{C})$$

is canonically a complex manifold, and the natural forgetful map $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, \Pi(\mathbb{R}))$ is a C^∞ map of C^∞ real manifolds.

Explicitly, the isomorphism $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi) \xrightarrow{\sim} \Pi\backslash\Pi(\mathbb{C})/F^0\Pi(\mathbb{C})$ sends the class of a torsor P to $q^{-1}r \in \Pi(\mathbb{C})$, where $q \in P$ and $r \in F^0\Pi(\mathbb{C})$ are any choices of elements.

Proof. Let $\Pi^{\mathrm{tf}} = \Pi/\Pi_{\mathrm{tors}}$ denote the image of Π under the natural map $\Pi \to \Pi(\mathbb{Q})$. It is a discrete subgroup of the Lie group $\Pi(\mathbb{R})$, so that the action of $\Pi^{\mathrm{tf}} \times F^0\Pi(\mathbb{C})$ on $\Pi(\mathbb{C})$ is free and proper by proposition 9.1.16. Hence

$$\Pi\backslash\Pi(\mathbb{C})/F^0\Pi(\mathbb{C}) = \Pi^{\mathrm{tf}}\backslash\Pi(\mathbb{C})/F^0\Pi(\mathbb{C})$$

is canonically a complex manifold. The fact that the forgetful map is C^∞ is clear, and the explicit description of the map follows from the proof of proposition 9.2.1. □

In fact, these classifying manifolds for torsors with integral mixed Hodge structures have already been studied in great detail by Hain and Zucker in [HZ87] (see also [Hai87b]). When Π_n is the n-step nilpotent quotient of the fundamental group of a smooth connected variety $Y/\mathbb{C}$ at a basepoint $y \in Y(\mathbb{C})$ (which has only negative weights), Hain and Zucker define a series of *higher Albanese manifolds*

$$\mathrm{Alb}^n(Y) := \Pi_n\backslash\Pi_n(\mathbb{C})/F^0\Pi_n(\mathbb{C})$$

and *higher Albanese maps* $Y(\mathbb{C}) \to \mathrm{Alb}^n(Y)$, generalising the usual Albanese variety and map of Y. Let us explicitly make the link between our description and theirs.

[7] This description disagrees with the description in [PS08, Theorem 3.31], which erroneously describes the Ext-groups as $W_0V_{\mathbb{C}}/(F^0W_0V_{\mathbb{C}} + V)$. This latter group in fact classifies extensions of $\mathbb{Z}(0)$ by V which admit integral splittings compatible with the weight filtration (which isn't automatic in general).

Proposition 9.2.4. *Let $Y/\mathbb{C}$ be a smooth connected variety, and let Π_n be the n-step nilpotent quotient of the fundamental group of Y based at a point $y \in Y(\mathbb{C})$, so that we have a canonical identification $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi_n) \cong \Pi_n \backslash \Pi_n(\mathbb{C})/F^0\Pi_n(\mathbb{C}) = \mathrm{Alb}^n(Y)$. Then, under this identification, the non-abelian Kummer map*

$$Y(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi_n)$$

(that is, the map assigning to each point $z \in Y(\mathbb{C})$ the torsor of paths[8] from y to z in $Y(\mathbb{C})$, pushed out along $\pi_1^{\mathrm{top}}(Y(\mathbb{C}); y) \twoheadrightarrow \Pi_n$ and endowed with the mixed Hodge structure given in [HZ87, Section 3]) is identified with the higher Albanese map $Y(\mathbb{C}) \to \mathrm{Alb}^n(Y)$.

In particular, the non-abelian Kummer map is complex analytic, and the $\mathbb{R}$-unipotent non-abelian Kummer map

$$Y(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, \Pi_n(\mathbb{R}))$$

(the composite of the above map with the map $\mathrm{H}^1(\mathbf{MHS}_{\mathbb{Z}}, \Pi_n) \twoheadrightarrow \mathrm{H}^1(\mathbf{MHS}_{\mathbb{R}}, \Pi_n(\mathbb{R}))$) is C^∞.

Proof. Let us recall first Hain and Zucker's construction of the higher Albanese map. We let $\mathbb{V}_n$ denote the nth canonical variation on Y with basepoint y. It is a variation of mixed Hodge structure, whose fibre at z is canonically identified with $\mathbb{Z}\pi_1(Y; y, z)/J^{n+1}$ with its mixed Hodge structure. The corresponding holomorphic vector bundle $V_n = \mathcal{O}_Y \otimes_{\mathbb{Z}} \mathbb{V}_n$ on $Y(\mathbb{C})$ then carries a flat holomorphic connection $\nabla_{\mathbb{C}}$ and Hodge filtration by holomorphic subbundles. Hain and Zucker prove in [HZ87, Proposition 5.24] that there is a C^∞ frame for V_n, compatible with the weight and Hodge filtrations, such that for any path γ from y to any z, the parallel transport of the frame for z^*V_n to $y^*V_n = \mathbb{C}\Pi_n/J^{n+1}$ along γ^{-1} is given by left-multiplying[9] the frame for y^*V_n by some $\tau \in \Pi_n(\mathbb{C})$. The image of z under the higher Albanese map is then declared to be the double-coset of τ.

Let now P_n denote the torsor of paths from y to z, pushed out to Π_n, so that there is a canonical identification $z^*V_n = \mathbb{C}P_n/J^{n+1}$. It is easy to see that the parallel transport $\mathbb{C}P_n/J^{n+1} \xrightarrow{\sim} \mathbb{C}\Pi_n/J^{n+1}$ along γ^{-1} is just given by left-multiplication by $[\gamma]^{-1}$. It follows from the preceding paragraph that the identification $\mathbb{C}\Pi_n/J^{n+1} \xrightarrow{\sim} \mathbb{C}P_n/J^{n+1}$ arising from the C^∞ frame for V_n is given by left-multiplication by some $r \in P_n(\mathbb{C})$—the $\mathbb{C}$-points of the Malčev completion of P_n. Then $\tau = [\gamma]^{-1}r$, with $[\gamma] \in$

[8] I.e. set of homotopy classes of paths, viewed as a torsor under the fundamental group.

[9] In [HZ87, Proposition 5.24], the action is on the right, since their conventions for path-composition are opposite to ours.

$P_n(\mathbb{Z})$ and $r \in F^0 P_n(\mathbb{C})$ (since the C^∞ frame for V_n respects the Hodge filtration), and hence by the description in corollary 9.2.3 the class of the torsor P_n is also represented by τ.

That the non-abelian Kummer map is complex analytic follows from the corresponding assertion for the higher Albanese map [HZ87, Section 5], and that the $\mathbb{R}$-unipotent non-abelian Kummer map is C^∞ follows from corollary 9.2.3. □

9.3 Proof of the main theorem

Having set up all this abstract theory and made the link to Hain and Zucker's, we are now almost ready to prove theorem 1.1.7 in an exactly analogous way to the proofs of theorems 1.1.6 and 1.2.3. The only part remaining before we can copy our earlier proof methods is a characterisation of the archimedean Néron log-metric, analogous to the non-archimedean characterisation in 3.0.2.

Lemma 9.3.1. *Let $A/\mathbb{C}$ be an abelian variety. Then the Néron log-metrics*

$$\lambda_L \colon L^\times(\mathbb{C}) \to \mathbb{R}$$

associated to pairs $(L, \widetilde{0})$ consisting of a line bundle L/A and a basepoint $\widetilde{0} \in L^\times(\mathbb{C})$ lying over $0 \in A(\mathbb{C})$,are uniquely characterised by the following properties:

0. *λ_L only depends on $(L, \widetilde{0})$ up to isomorphism;*
1. *λ_L is locally bounded (for the analytic topology on $L^\times(\mathbb{C})$);*
2. *$\lambda_{L_1 \otimes L_2}(P_1 \otimes P_2) = \lambda_{L_1}(P_1) + \lambda_{L_2}(P_2)$;*
3. *$\lambda_{[2]^* L}(P) = \lambda_L([2]P)$, where by abuse of notation we also denote by $[2]$ the topmost map in the pullback square*

$$\begin{array}{ccc} [2]^* L^\times & \longrightarrow & L^\times \\ \downarrow & & \downarrow \\ A & \xrightarrow{[2]} & A; \end{array}$$

4. *when $L = A \times_{\mathbb{C}} \mathbb{A}^1$ is the trivial line bundle, λ_L is the composite*

$$L^\times(\mathbb{C}) = A(\mathbb{C}) \times \mathbb{C}^\times \to \mathbb{C}^\times \xrightarrow{v} \mathbb{R}$$

where $v = -\log|\cdot|$; and

5. $\lambda_L(\widetilde{0}) = 0$.

In fact, the Néron log-metrics are even C^∞, not just locally bounded.

Proof. That the Néron log-metric satisfies the above properties follows from the corresponding properties of Néron functions [Lan83, Theorem 11.1.1].

That the Néron log-metric is uniquely characterised by these properties proceeds in exactly the same way as in the proof of lemma 3.0.2, with the small adjustment that the function δ_L is locally bounded with bounded image instead of being locally constant with finite image. □

With this characterisation of the Néron log-metric, theorem 1.1.7 can now be proved by a more-or-less direct translation of the proofs in chapter 3. Specifically, given a line bundle L on an abelian variety $A/\mathbb{C}$, the fibration $\mathbb{G}_m \to L^\times \to A$ gives us a central extension

$$1 \to \mathbb{R}(1) \xrightarrow{\mathsf{z}} U \to VA \to 1$$

of $\mathbb{R}$-unipotent fundamental groups, where $VA = \mathrm{H}_1(A(\mathbb{C}), \mathbb{R})$, and hence by corollary 9.1.13 an exact sequence

$$F^0W_0VA \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\frown} \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R})) \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, VA).$$

But VA, being pure of weight -1, satisfies $F^0W_0VA = 0$ and $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, VA) = VA_\mathbb{C}/F^0VA_\mathbb{C} \oplus VA = 0$, and hence the map $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R}))$ is bijective. Indeed, it is a C^∞ diffeomorphism, since it is an immersion by lemma 9.1.17 and hence a local C^∞ diffeomorphism by counting dimensions.

Now we also have a canonical C^∞ diffeomorphism $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}(1)\backslash\mathbb{C} \cong \mathbb{R}$ (the final isomorphism being minus the obvious one) and so we define a map λ_L as the composite

$$\lambda_L \colon L^\times(\mathbb{C}) \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, U(\mathbb{R})) \xleftarrow{\sim} \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}.$$

It remains to show that the λ_L are the Néron log-metrics, by verifying the properties in lemma 9.3.1.

We've already checked that the λ_L are continuous (in fact C^∞), and the verification of the remaining properties is purely formal, following exactly the pattern in the proof of theorem 1.1.6 (with $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, -)$ replacing the non-abelian Galois cohomology). The only thing we need to check is that we have normalised the isomorphism $\mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}$ correctly, for the proof of point 5. But this follows immediately from the fact that the abelian Kummer map $\mathbb{C}^\times \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{Z}, \mathbb{Z}(1)) \cong$

$\mathbb{Z}(1)\backslash\mathbb{C}$ is the logarithm map, so that the composite $\mathbb{C}^\times \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{Z}, \mathbb{Z}(1)) \to \mathrm{H}^1(\mathbf{MHS}_\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}$ is given by $z \mapsto -\mathrm{Re}\log(z) = -\log|z|$, which is the desired normalisation.

Chapter 10

An ℓ-adic Weil height machine

In this final chapter, we will illustrate how one might hope be able to make a global version of the earlier results in this thesis by setting out how one can use variants on the earlier ideas to construct global ℓ-adic[1] height functions associated to line bundles on abelian varieties, rephrasing and mildly generalising the construction of Balakrishnan and Dogra [BD16, BD17] for line bundles on Jacobians of curves with good reduction at places over ℓ. In particular, our development of the theory requires no assumptions on reduction types at places over ℓ.

For the remainder of this chapter, we fix a number field F (and algebraic closure $\overline{F}$) along with decomposition groups $G_v \leq G_F$ for each place v of F, a rational prime ℓ, and a branch of the v-adic logarithm for each place $v \mid \ell$. We also fix an abelian variety A/F, an ℓ-adic Hecke character[2] $\chi \in \mathrm{H}^1(G_F, \mathbb{Q}_\ell)$, and a splitting of the Hodge filtration on $\mathsf{D}_{\mathrm{dR},v}(V_\ell A)$ for each place $v \mid \ell$, where $V_\ell A$ is the $\mathbb{Q}_\ell$-linear Tate module of A. We take a finite set S of finite places of F containing all places of bad reduction for A and all places over ℓ; G_S denotes the maximal quotient of G_F which is unramified at places $v \notin S$.

Throughout this chapter, we will follow the natural extension of the conventions of [BK90, Equation 3.7.1], that

$$\begin{aligned}
\mathrm{H}^1_e(G_v, U(\mathbb{Q}_\ell)) &= 1 \\
\mathrm{H}^1_f(G_v, U(\mathbb{Q}_\ell)) &= \mathrm{H}^1_{\mathrm{nr}}(G_v, U(\mathbb{Q}_\ell)) \\
\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) &= \mathrm{H}^1(G_v, U(\mathbb{Q}_\ell))
\end{aligned}$$

[1] The reader may find objectionable our choice to use ℓ-adic coefficients for our heights rather than the more conventional p-adic coefficients. We can only assure such a reader that the choice is purely cosmetic, and chosen in the interests of consistency with the rest of the thesis, where p is a residue characteristic and $\mathbb{Q}_\ell$ is the coefficient field for the fundamental groups.

[2] The role of the ℓ-adic Hecke character in the setting of $\mathbb{Q}_\ell$-valued heights is analogous to the role played implicitly by the logarithm of the norm character in the $\mathbb{R}$-valued theory—in the ℓ-adic setting, there is no canonical choice available.

when $v \nmid \ell$, for any representation of G_v on a pro-unipotent group $U/\mathbb{Q}_\ell$.

Theorem 10.0.1 (Global ℓ-adic heights)**.** *To each line bundle L/A, there is an associated function*

$$h_L \colon \mathrm{H}^1_g(G_S, V_\ell A) \to \mathbb{Q}_\ell$$

called the ℓ-adic canonical height function*. These functions satisfy the following properties:*

- *h_L depends only on the isomorphism class of L and is independent of the choice of S;*
- *h_L is a quadratic function, vanishing at $0 \in \mathrm{H}^1_g(G_S, V_\ell A)$;*
- *if L is symmetric, then h_L is a quadratic form (i.e. even);*
- *if L is antisymmetric, then h_L is linear (i.e. odd);*
- *the assignment $L \mapsto h_L$ is a homomorphism;*
- *if $f \colon A' \to A$ is a morphism of abelian varieties over F, then the composite*

$$\mathrm{H}^1_g(G_S, V_\ell A') \to \mathrm{H}^1_g(G_S, V_\ell A) \xrightarrow{h_L} \mathbb{Q}_\ell$$

is equal to the canonical height h_{f^*L} associated to the pullback of L. (For this, we assume that the splittings of the Hodge filtrations on $\mathsf{D}_{\mathrm{dR},v}(V_\ell A')$ have been chosen compatible with those on $\mathsf{D}_{\mathrm{dR},v}(V_\ell A)$, for each $v \mid \ell$.)

Remark 10.0.2. One can obtain a theory of heights more closely corresponding to the classical $\mathbb{R}$-valued theory by precomposing the ℓ-adic canonical height functions in theorem 10.0.1 with the Kummer map $A(F) \to \mathrm{H}^1_g(G_S, V_\ell A)$ to obtain a quadratic function on $A(F)$ itself. Indeed, the Kummer map sits in a short exact sequence

$$0 \to \mathbb{Q}_\ell \otimes_{\mathbb{Z}} A(F) \to \mathrm{H}^1_g(G_S, V_\ell A) \to V_\ell Ш_{A/F} \to 0,$$

where $V_\ell Ш_{A/F} = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim Ш_{A/F}[\ell^r]$ is the $\mathbb{Q}_\ell$-linear Tate module of the Tate–Shafarevich group of A/F. Of course, it is a well-known conjecture that $Ш_{A/F}$ (or even just $Ш_{A/F}[\ell^\infty]$) is finite, in which case the Kummer map provides an isomorphism $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} A(F) \xrightarrow{\sim} \mathrm{H}^1_g(G_S, V_\ell A)$ and the two senses of canonical ℓ-adic heights are equivalent. Note, however, that our construction is unconditional on finiteness of $Ш_{A/F}$.

10.1 Local ℓ-adic heights

The global ℓ-adic height we construct will be a sum of local heights for each finite place v of F, which will take the form of functions (local ℓ-adic log-metrics)

$$\lambda_{L,v}\colon \mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \to \mathbb{Q}_\ell,$$

where as usual U denotes the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of the $\mathbb{G}_m$-torsor $L^\times = L \setminus 0$ at a basepoint $\widetilde{0} \in L^\times(F)$ lying over $0 \in A(F)$. These local heights will satisfy the following properties:

- with respect to the natural action of $\mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$ on $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell))$, we have

  $$\lambda_{L,v}(P \cdot \alpha) = \lambda_{L,v}(P) + \mathrm{inv}_v(\alpha \cup \chi_v)$$

  for all $\alpha \in \mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$ and $P \in \mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell))$, where $\mathrm{inv}_v\colon \mathrm{H}^2(G_v, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mathrm{Br}(F_v)[\ell^r] \xrightarrow{\sim} \mathbb{Q}_\ell$ is the local invariant map [NSW15, Corollary 7.1.4];
- $\lambda_{L_1 \otimes L_2, v}(P_1 \otimes P_2) = \lambda_{L_1,v}(P_1) + \lambda_{L_2,v}(P_2)$; and
- if $f\colon A' \to A$ is a morphism of abelian varieties, then $\lambda_{f^*L,v}(P) = \lambda_{L,v}(f(P))$ for any line bundle L/A. (For this, we assume that the splittings of the Hodge filtrations on $\mathsf{D}_{\mathrm{dR},v}(V_\ell A')$ have been chosen compatible with those on $\mathsf{D}_{\mathrm{dR},v}(V_\ell A)$, for each $v \mid \ell$.)

10.1.1 Places $v \nmid \ell$

The construction of the local height function at places $v \nmid \ell$ is particularly straightforward. Indeed, we know from chapter 3 that the inclusion $\mathbb{G}_m \hookrightarrow L^\times$ induces an isomorphism $\mathrm{H}^1(G_v, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^1(G_v, U(\mathbb{Q}_\ell))$, so we may define the local height $\lambda_{L,v}$ as the composite

$$\mathrm{H}^1(G_v, U(\mathbb{Q}_\ell)) \xleftarrow{\sim} \mathrm{H}^1(G_v, \mathbb{Q}_\ell(1)) \xrightarrow{\cdot \cup \chi_v} \mathrm{H}^2(G_v, \mathbb{Q}_\ell(1)) \xrightarrow[\sim]{\mathrm{inv}_v} \mathbb{Q}_\ell.$$

Here, $\chi_v \in \mathrm{H}^1(G_v, \mathbb{Q}_\ell)$ denotes the restriction of the Hecke character χ, and inv_v denotes the local invariant map from local class field theory [NSW15, Corollary 7.1.4], where we identify

$$\mathrm{H}^2(G_v, \mathbb{Q}_\ell(1)) = V_\ell \mathrm{Br}(F_v) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mathrm{Br}(F_v)[\ell^r]$$

as the $\mathbb{Q}_\ell$-linear Tate module of the Brauer group of F_v.

It is clear that this local height function satisfies the above conditions; the same argument as in the proof of theorem 1.1.6 works in this case (where for the final part, we replace the morphism [2] with f).

10.1.2 Places $v \mid \ell$

At places $v \mid \ell$, the construction of the local height function is more complicated, depending not only on χ_v, but on the local splitting of the Hodge filtration we chose at the beginning of this section, which we will use to provide us with a splitting of the map $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_v, V_\ell A)$.

To construct this splitting, we first consider the exact sequence

$$0 \to \mathbb{Q}_\ell(1) \to \mathrm{Lie}(U) \to V_\ell A \to 0.$$

Since $\mathrm{Lie}(U)$ is de Rham, it becomes semistable after restriction to some open normal subgroup $G_w \trianglelefteq G_v$, corresponding to a finite Galois extension L_w/K_v. Applying the semistable Dieudonné functor over L_w, we obtain an extension

$$0 \to \mathsf{D}_{\mathrm{st},w}(\mathbb{Q}_\ell(1)) \to \mathsf{D}_{\mathrm{st},w}(\mathrm{Lie}(U)) \to \mathsf{D}_{\mathrm{st},w}(V_\ell A) \to 0$$

of (φ, N)-modules over $L_{w,0}$, endowed with a semilinear action of $G_{L_w|K_v}$ compatible with the (φ, N)-module structure. Now since the space of such extensions of (φ, N)-modules is parametrised by

$$\mathrm{Ext}^1_{(\varphi,N)}(\mathsf{D}_{\mathrm{st},w}(V_\ell A), \mathsf{D}_{\mathrm{st},w}(\mathbb{Q}_\ell(1))) = \mathrm{H}^1_{g/e}(G_w, V_\ell A^*(1)) = 0,$$

it follows that the above sequence splits as a sequence of (φ, N)-modules, and since the space of splittings is parametrised by

$$\mathrm{Hom}_{(\varphi,N)}(\mathsf{D}_{\mathrm{st},w}(V_\ell A), \mathsf{D}_{\mathrm{st},w}(\mathbb{Q}_\ell(1))) = \mathsf{D}^{\varphi=1}_{\mathrm{cris},w}(V_\ell A^*(1)) = 0,$$

it does so uniquely, and in particular, $G_{L_w|K_v}$-equivariantly. We denote the unique such section by $s^{\mathrm{st}} \colon \mathsf{D}_{\mathrm{st},w}(V_\ell A) \to \mathsf{D}_{\mathrm{st},w}(\mathrm{Lie}(U))$.

Using the comparison isomorphism $\mathsf{D}_{\mathrm{dR},w}(\mathrm{Lie}(U)) \cong L_w \otimes_{L_{w,0}} \mathsf{D}_{\mathrm{st},w}(\mathrm{Lie}(U))$ and also for $V_\ell A$, s^{st} induces a $G_{L_w|K_v}$-equivariant section of $\mathsf{D}_{\mathrm{dR},w}(\mathrm{Lie}(U)) \to \mathsf{D}_{\mathrm{dR},w}(V_\ell A)$, and hence a section of $\mathsf{D}_{\mathrm{dR},v}(\mathrm{Lie}(U)) \to \mathsf{D}_{\mathrm{dR},v}(V_\ell A)$, which we also denote by s^{st}.

We now define a morphism $s \colon \mathsf{D}_{\mathrm{dR},v}(V_\ell A)/\mathsf{D}^+_{\mathrm{dR},v}(V_\ell A) \to \mathsf{D}_{\mathrm{dR},v}(U)/\mathsf{D}^+_{\mathrm{dR},v}(U)$ as the composite

$$\mathsf{D}_{\mathrm{dR},v}(V_\ell A)/\mathsf{D}^+_{\mathrm{dR},v}(V_\ell A) \xrightarrow{s^v} \mathsf{D}_{\mathrm{dR},v}(V_\ell A) \xrightarrow{s^{\mathrm{st}}} \mathsf{D}_{\mathrm{dR},v}(\mathrm{Lie}(U)) \xrightarrow{\sim} \mathsf{D}_{\mathrm{dR},v}(U) \to \mathsf{D}_{\mathrm{dR},v}(U)/\mathsf{D}^+_{\mathrm{dR},v}(U),$$

where s^v is the chosen splitting of the Hodge filtration at v.

Proposition 10.1.1. *The morphism s constructed above is a section of the natural morphism $\mathsf{D}_{\mathrm{dR}}(U)/\mathsf{D}^+_{\mathrm{dR}}(U) \to \mathsf{D}_{\mathrm{dR}}(V_\ell A)/\mathsf{D}^+_{\mathrm{dR}}(V_\ell A)$. In particular, the map on K-points is a section of the map $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_e(G_K, V_\ell A)$ and hence of the map $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_K, V_\ell A) = \mathrm{H}^1_e(G_K, V_\ell A)$.*

Proof. This follows by a straightforward diagram-chase, using that the exponential map $\mathrm{Lie}(U) \xrightarrow{\sim} U$ fits into a commuting square

$$\begin{array}{ccc} \mathrm{Lie}(U) & \longrightarrow & \mathrm{Lie}(V_\ell A) \\ \big\downarrow{\scriptstyle \exp} & & \big\| \\ U & \longrightarrow & V_\ell A \end{array}$$

since $V_\ell A$ is abelian. □

In order to use s to produce a local height function, we will need to understand the structure of $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell))$. The key description is the following.

Proposition 10.1.2. *In the exact sequence*

$$\mathrm{H}^0(G_v, V_\ell A) \to \mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1)) \xrightarrow{\curvearrowright} \mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_v, V_\ell A) \xrightarrow{\delta_v} \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(\mathbb{Q}_\ell)^*$$

from corollary 6.3.6, both the outer maps are trivial, and the given action makes $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell))$ *into a* $\mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$*-torsor over* $\mathrm{H}^1_g(G_v, V_\ell A)$*.*

Proof. Triviality of the leftmost arrow is immediate, since $\mathrm{H}^0(G_v, V_\ell A) = 1$. To show vanishing of δ_v, we note that from the second exact sequence in corollary 6.3.6 that δ_v factors through $\mathrm{H}^1_{g/e}(G_v, V_\ell A) = 1$, so $\delta_v = 1$ also. (The two exact sequences in corollary 6.3.6 are compatible, since they are cohomotopy exact sequences induced by compatible short exact sequences of cosimplicial groups—the morphism of short exact sequences is induced from the cosimplicial algebra map $\mathsf{B}^\bullet_g \to \mathsf{B}^\bullet_{g/e}$.)

Exactness of the sequence

$$1 \to \mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1)) \xrightarrow{\curvearrowright} \mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_v, V_\ell A) \to 1$$

says that the action of $\mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$ on the fibres of $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \twoheadrightarrow \mathrm{H}^1_g(G_v, V_\ell A)$ is transitive, and free on the fibre containing the distinguished point of $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell))$. To deduce that this is actually a torsor (i.e. that the action is free), we consider the corresponding sequence for the twist ${}_\alpha U$ of U by some Galois cocycle α. The group ${}_\alpha U$ is again a de Rham representation of G_v on a unipotent group (proposition 4.3.3), and is again a central extension of $V_\ell A$ by $\mathbb{Q}_\ell(1)$. Hence the action of $\mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$ on $\mathrm{H}^1_g(G_v, {}_\alpha U(\mathbb{Q}_\ell))$ has trivial point-stabiliser for all α, so that the action of $\mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$ on $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell))$ also has trivial stabiliser at $[\alpha]$, as desired. □

Equipped with the section s and this explicit description of $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell))$, it is now straightforward to construct the local height. Specifically, the splitting s of $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \twoheadrightarrow \mathrm{H}^1_g(G_v, V_\ell A)$ gives a trivialisation of the $\mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$-torsor structure, and hence a map $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1))$. The composite

$$\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1)) \xrightarrow{\cdot\cup\chi_v} \mathrm{H}^2(G_v, \mathbb{Q}_\ell(1)) \xrightarrow[\sim]{\mathrm{inv}_v} \mathbb{Q}_\ell$$

is then the desired local height function $\lambda_{L,v}$.

It is then straightforward to check that this local height function satisfies the claimed conditions, a task which we leave to the reader.

10.2 Global Selmer sets for $\mathbb{G}_m$-torsors on abelian varieties

10.2.1 Construction of global heights

Armed with these local components of heights, let us now describe how these are to be added together to produce a global height function on $\mathrm{H}^1_g(G_S, V_\ell A)$. The key object mediating between the local heights and the global height is the *global* Bloch–Kato Selmer set $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$—defined as the intersection in $\mathrm{H}^1(G_S, U(\mathbb{Q}_\ell))$ of the preimages of the local Bloch–Kato Selmer sets $\mathrm{H}^1_g(G_v, U(\mathbb{Q}_\ell)) \subseteq \mathrm{H}^1(G_v, U(\mathbb{Q}_\ell))$ for all v—whose structure we can explicitly determine by studying cohomology of the central extension

$$1 \to \mathbb{Q}_\ell(1) \xrightarrow{\mathsf{z}} U \to V_\ell A \to 1. \tag{10.2.1}$$

We will see shortly that $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ is a $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1))$-torsor over $\mathrm{H}^1_g(G_S, V_\ell A)$, but before we give the proof of this fact, let us show how this enables us to construct the desired global $\mathbb{Q}_\ell$-valued height function on $\mathrm{H}^1_g(G_S, V_\ell A)$.

Definition-Lemma 10.2.1 (ℓ-adic canonical heights)**.** The function $\lambda_L \colon \mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell)) \to \mathbb{Q}_\ell$ given by the sum of local heights at all finite places v (is well-defined and) factors uniquely through a function

$$h_L \colon \mathrm{H}^1_g(G_S, V_\ell A) \to \mathbb{Q}_\ell,$$

which we call the *ℓ-adic canonical height* associated to L.

Proof. By definition, for $v \notin S$, the image of $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1(G_v, U(\mathbb{Q}_\ell))$ is contained in the set $\mathrm{H}^1_{\mathrm{nr}}(G_v, U(\mathbb{Q}_\ell))$ of non-ramified cohomology classes. This pointed

set is a singleton by proposition 4.1.6 (see remark 4.1.7), and hence the local heights at such places vanish on $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$—this shows that λ_L is well-defined.

Now since $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ is a $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1))$-torsor over $\mathrm{H}^1_g(G_S, V_\ell A)$ (lemma 10.2.3), to prove that λ_L factors uniquely through the desired height function h_L, it suffices to prove that λ_L is invariant under the action of $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1))$. But by construction of the local heights, λ_L transforms under the action of some $\alpha \in \mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1))$ as

$$\lambda_L(P \cdot \alpha) = \lambda_L(P) + \sum_v \mathrm{inv}_v(\alpha_v \cup \chi_v).$$

Using the cohomological calculations in [NSW15, Proposition 8.3.11], we see that $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1)) = \mathrm{H}^1(G_S, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mathrm{Br}_S(F)[\ell^r]$, where

$$\mathrm{Br}_S(F) = \ker\left(\bigoplus_{v \in S} \mathbb{Q}/\mathbb{Z} \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z}\right)$$

is the part of the Brauer group of F supported in S. It follows that we have $\sum_v \mathrm{inv}_v(\alpha_v \cup \chi_v) = 0$ for all such α, so that λ_L is invariant under the action of $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1))$ as desired. □

Let us now justify our description of the global Bloch–Kato Selmer set $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ as a torsor, beginning by showing that it is well-defined.

Proposition 10.2.2. *The action of G_F on U factors through G_S.*

Proof. It suffices to show that for each $v \notin S$ the action of the decomposition group G_v on $\mathrm{Lie}(U)$ is unramified. But since $\mathrm{Lie}(U)$ is an extension of $V_\ell A$ by $\mathbb{Q}_\ell(1)$, and such extensions are parametrised by

$$\mathrm{Ext}^1_{G_v}(V_\ell A, \mathbb{Q}_\ell(1)) = \mathrm{H}^1(G_v, V_\ell A^*(1)) = 0,$$

it follows that $\mathrm{Lie}(U) \cong \mathbb{Q}_\ell(1) \oplus V_\ell A$ as representations of G_v, so the action is unramified. □

Lemma 10.2.3. *The cohomology exact sequence*

$$\mathrm{H}^0(G_S, V_\ell A) \to \mathrm{H}^1(G_S, \mathbb{Q}_\ell(1)) \xrightarrow{\frown} \mathrm{H}^1(G_S, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1(G_S, V_\ell A) \xrightarrow{\delta} \mathrm{H}^2(G_S, \mathbb{Q}_\ell(1))$$

arising from equation (10.2.1) contains as an exact subsequence

$$1 \to \mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1)) \xrightarrow{\frown} \mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_S, V_\ell A) \to 1,$$

making $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ into a $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1))$-torsor over $\mathrm{H}^1_g(G_S, V_\ell A)$.

Proof. Freeness of the action of $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1))$ on $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ follows exactly as in the local case, since $\mathrm{H}^0(G_S, V_\ell A) = 1$ also in this case. Exactness at $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ is inherited from the cohomology exact sequence, since $\mathrm{H}^1_g(G_S, \mathbb{Q}_\ell(1)) = \mathrm{H}^1(G_S, \mathbb{Q}_\ell(1))$.

Finally, to show that $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_S, V_\ell A)$ is surjective, we first claim that the coboundary map $\delta\colon \mathrm{H}^1(G_S, V_\ell A) \to \mathrm{H}^2(G_S, \mathbb{Q}_\ell(1))$ vanishes on $\mathrm{H}^1_g(G_S, V_\ell A)$. To justify this, note that, from the description of $\mathrm{H}^2(G_S, \mathbb{Q}_\ell(1)) = V_\ell \mathrm{Br}_S(F)$ as the $\mathbb{Q}_\ell$-linear Tate module of the part of the Brauer group of F supported in S, it follows that the localisation map $\mathrm{H}^2(G_S, \mathbb{Q}_\ell(1)) \to \prod_{v\in S} \mathrm{H}^2(G_v, \mathbb{Q}_\ell(1))$ is injective, so it suffices to prove that the diagonal composite in the commuting square

$$\begin{array}{ccc} \mathrm{H}^1(G_S, V_\ell A) & \xrightarrow{\quad\delta\quad} & \mathrm{H}^2(G_S, \mathbb{Q}_\ell(1)) \\ \downarrow & & \downarrow \\ \prod_{v\in S} \mathrm{H}^1(G_v, V_\ell A) & \xrightarrow{\prod \delta_v} & \prod_{v\in S} \mathrm{H}^2(G_v, \mathbb{Q}_\ell(1)) \end{array}$$

vanishes on H^1_g. But each δ_v vanishes on $\mathrm{H}^1_g(G_v, V_\ell A)$—for $v \mid \ell$ we saw this in section 10.1.2, and for $v \nmid \ell$ this is automatic since $\mathrm{H}^1(G_v, V_\ell A) = 1$—which gives us the desired vanishing of δ on $\mathrm{H}^1(G_S, V_\ell A)$.

To now conclude surjectivity of $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_S, V_\ell A)$, we note that $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ is just the preimage of $\mathrm{H}^1_g(G_S, V_\ell A)$ under the map $\mathrm{H}^1(G_S, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1(G_S, V_\ell A)$—this is immediate from the corresponding local statements, which follow from the fact that $\mathrm{H}^1_g(G_v, \mathbb{Q}_\ell(1)) = \mathrm{H}^1(G_v, \mathbb{Q}_\ell(1))$ for each v. But now any element x of $\mathrm{H}^1_g(G_S, V_\ell A)$ lifts to $\mathrm{H}^1(G_S, U(\mathbb{Q}_\ell))$ since $\delta(x) = 1$, and any such lift lies necessarily in $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$, so $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell)) \to \mathrm{H}^1_g(G_S, V_\ell A)$ is surjective, as desired. □

Remark 10.2.4. One peculiarity of our construction of the ℓ-adic height is that it describes this height as a sum of local components only over the set S, rather than over all places—in fact, by choosing an appropriate lift of a class in $\mathrm{H}^1_g(G_S, V_\ell A)$ to $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$, it is even possible to arrange that the local components of the height vanish at all places away from ℓ. Let us explain this by describing the corresponding situation for $\mathbb{R}$-valued heights.

In the $\mathbb{R}$-valued setting, the analogous role to the set $\mathrm{H}^1_g(G_S, U(\mathbb{Q}_\ell))$ is played by the $\mathbb{Q} \otimes_{\mathbb{Z}} F^\times$-torsor $L^\times(F)_{\mathbb{Q}}$ over[3] $A(F)$ given by pushing out the $F^\times$-torsor $L^\times(F)$ along the rationalisation map $F^\times \to \mathbb{Q} \otimes_{\mathbb{Z}} F^\times$—the Néron log-metrics $\lambda_{L,v}\colon L^\times(F_v) \to \mathbb{R}$ naturally extend to functions on the corresponding rationalisation $L^\times(F_v)_{\mathbb{Q}}$, and the height of a point $z \in A(F)$ is given by choosing an arbitrary lift to $\widetilde{z} \in L^\times(F)_{\mathbb{Q}}$ and

[3]In fact, it is more natural to extend this $\mathbb{Q} \otimes_{\mathbb{Z}} F^\times$-torsor over $\mathbb{Q} \otimes_{\mathbb{Z}} A(F)$ using its cubical structure, however for illustrative purposes we will ignore this step.

taking an appropriately weighted sum of the Néron log-metrics (this is independent of the choice of lift).

In this context, the analogous statement is that the lift $\widetilde{z}$ can be chosen so that the local Néron log-metrics $\lambda_{L,v}(\widetilde{z})$ vanish at *all* finite places v. To see why this is, note that for any initial lift $\widetilde{z}_0$, the Néron log-metrics $\lambda_{L,v}(\widetilde{z}_0)$ are all $\mathbb{Q}$-valued, and almost all 0. It follows from the finiteness of the class group of F that the cokernel of the map $F^\times \to \bigoplus_{v\nmid\infty} \mathbb{Q}$ (given on the vth component by the valuation v) is torsion, and hence there is a positive integer n and a $q \in F^\times$ such that $n\lambda_{L,v}(\widetilde{z}_0) = v(q)$ for all $v \nmid \infty$. Thus if we take $\widetilde{z} = \widetilde{z}_0 \cdot q^{-1/n} \in L^\times(F)_\mathbb{Q}$, then $\lambda_{L,v}(\widetilde{z}) = 0$ for all finite v, as desired.

10.2.2 Properties of global heights

It remains to show that the global ℓ-adic height h_L constructed above satisfies the claimed conditions. Additivity in the line bundle and compatibility with morphisms of abelian varieties are immediate from the corresponding properties of local heights. Vanishing at 0 follows immediately from the fact that each local height takes $\widetilde{0}$ to 0. The fact that h_L is even (resp. odd) when L is symmetric (resp. antisymmmetric) follows from compatibility with the morphism $[-1]\colon A \to A$.

It only remains to show that h_L is quadratic. To do this, consider the morphisms $m_I\colon A^3 \to A$ given by the sum of the coordinates in a set $I \subseteq \{1,2,3\}$. For any line bundle L on A, the theorem of the cube and additivity of the map $L \mapsto h_L$ ensures that

$$\sum_{I\subseteq\{1,2,3\}} (-1)^{\#I} \cdot h_{m_I^* L} = 0$$

as functions on $\mathrm{H}^1_g(G_S, V_\ell A^3) = \mathrm{H}^1_g(G_S, V_\ell A)^3$. Thus, by functoriality, we have

$$\sum_{I\subseteq\{1,2,3\}} (-1)^{\#I} \cdot h_L\left(\sum_{i\in I} P_i\right) = 0$$

for all $(P_1, P_2, P_3) \in \mathrm{H}^1_g(G_S, V_\ell A)^3$, which says exactly that h_L is quadratic, as desired.

Appendix A

Local Selmer varieties and representability

A.1 Local Bloch–Kato Selmer presheaves

Throughout the main body of this thesis, we have avoided any discussion of the natural variety structure on local Bloch–Kato Selmer sets, which plays a crucial role in the Chabauty–Kim method [Kim05, Kim09]. In this appendix, we will address this inadequacy, by describing how our cohomotopical approach to the study of local Bloch–Kato Selmer sets produces Kim's standard variety structure on these sets in a very natural way.

Indeed, the cosimplicial group models $U(\mathsf{B}^\bullet_*)^{G_K}$ we have produced for the local Bloch–Kato Selmer sets $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ themselves have a natural algebraic structure in the sense that they are naturally the $\mathbb{Q}_p$-points of certain cosimplicial pro-unipotent groups $\mathsf{D}^\bullet_*(U)$ over $\mathbb{Q}_p$. This construction is a cosimplicial group version of the Dieudonné construction, using the cosimplicial period rings $\mathsf{B}^\bullet_*$.

Definition A.1.1. Let U be a de Rham representation of G_K on a finitely generated pro-unipotent group. For $* \in \{e, f, g, f/e, g/e\}$ we define $\mathsf{D}^\bullet_*(U)$ to be the cosimplicial pro-unipotent group over $\mathbb{Q}_p$ representing the presheaf of cosimplicial groups on $\mathbf{Aff}_{\mathbb{Q}_p}$ given by

$$\mathsf{D}^\bullet_*(U)\colon \Lambda \mapsto U(\mathsf{B}^\bullet_* \otimes_{\mathbb{Q}_p} \Lambda)^{G_K},$$

where $\mathsf{B}^\bullet_*$ is the cosimplicial period ring from definition 6.1.2 or 6.3.1 and Λ is given the trivial G_K-action. This presheaf is indeed representable by a cosimplicial pro-unipotent group: the argument of lemma 4.2.1 associates to U a series of pro-unipotent groups over the finite $\mathbb{Q}_p$-algebras $(\mathsf{B}^n_*)^{G_K}$—the Weil restrictions of these pro-unipotent groups to $\mathbb{Q}_p$ give the terms $\mathsf{D}^n_*(U)$ of the cosimplicial pro-unipotent group $\mathsf{D}^\bullet_*(U)$.

Example A.1.2. If U in definition A.1.1 is abelian, so the vector group associated to a $\mathbb{Q}_p$-vector space V, then $\mathsf{D}^\bullet_*(U)$ is the cosimplicial vector group associated to the cosimplicial $\mathbb{Q}_p$-vector space $\mathsf{D}^\bullet_*(V) := (\mathsf{B}^\bullet_* \otimes_{\mathbb{Q}_p} V)^{G_K}$.

Now on the one hand, this algebraic structure on the cosimplicial models $U(\mathsf{B}^\bullet_*)^{G_K}$ ensures that the first cohomotopy $\pi^1(U(\mathsf{B}^\bullet_*)^{G_K})$ is naturally the $\mathbb{Q}_p$-points of the first cohomotopy presheaf

$$\pi^1(\mathsf{D}^\bullet_*(U)) := \mathrm{Z}^1(\mathsf{D}^\bullet_*(U))/\mathsf{D}^0_*(U),$$

where $\mathrm{Z}^1(\mathsf{D}^\bullet_*(U)) \subseteq \mathsf{D}^1_*(U)$ is the closed subscheme cut out by the equation $d^1 = d^2 \cdot d^0$, $\mathsf{D}^0_*(U)$ acts on $\mathrm{Z}^1(\mathsf{D}^\bullet_*(U))$ by twisted conjugation as in the usual definition (5.1.4) of first cohomotopy, and the quotient is taken in the category of presheaves[1] on $\mathbf{Aff}_{\mathbb{Q}_p}$. In other words, the sections of $\pi^1(\mathsf{D}^\bullet_*(U))$ over some $\mathrm{Spec}(\Lambda)$ are just the cohomotopy of the cosimplicial group $\mathsf{D}^\bullet_*(U)(\Lambda)$.

On the other hand, the variety structure on the local Bloch–Kato Selmer sets $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p))$ (for $* \in \{e, f, g\}$) is also constructed by interpreting them as the $\mathbb{Q}_p$-points of presheaves on $\mathbf{Aff}_{\mathbb{Q}_p}$, following [Kim05].

Definition A.1.3 (Local Bloch–Kato Selmer presheaves (cf. definition 1.2.1))**.** Let $U/\mathbb{Q}_p$ be a representation of G_K on a finitely generated pro-unipotent group. If Λ is a test $\mathbb{Q}_p$-algebra, we define pointed sets $\mathrm{H}^1_e(G_K, U(\Lambda)) \subseteq \mathrm{H}^1_f(G_K, U(\Lambda)) \subseteq \mathrm{H}^1_g(G_K, U(\Lambda))$ to be the respective kernels of the natural maps

$$\mathrm{H}^1(G_K, U(\Lambda)) \to \mathrm{H}^1(G_K, U(\mathsf{B}^{\varphi=1}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \Lambda))$$

$$\mathrm{H}^1(G_K, U(\Lambda)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \Lambda))$$

$$\mathrm{H}^1(G_K, U(\Lambda)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \Lambda))$$

on continuous cohomology. In these definitions, $\Lambda = \varinjlim \Lambda_i$ is endowed with the direct limit topology coming from writing it as the limit of its finite-dimensional $\mathbb{Q}_p$-subspaces Λ_i, while $\mathsf{B}_* \otimes_{\mathbb{Q}_p} \Lambda = \varinjlim \left(\mathsf{B}_* \otimes_{\mathbb{Q}_p} \Lambda_i\right)$ is given also the direct limit topology, where each $\mathsf{B}_* \otimes_{\mathbb{Q}_p} \Lambda_i$ is given its natural topology as a finite free B_*-module.

We define the *local Bloch–Kato Selmer presheaves* $\mathrm{H}^1_e(G_K, U) \subseteq \mathrm{H}^1_f(G_K, U) \subseteq \mathrm{H}^1_g(G_K, U)$ to be the presheaves of pointed sets on $\mathbf{Aff}_{\mathbb{Q}_p}$ whose sections over some test $\mathbb{Q}_p$-algebra Λ are the pointed sets $\mathrm{H}^1_e(G_K, U(\Lambda)) \subseteq \mathrm{H}^1_f(G_K, U(\Lambda)) \subseteq \mathrm{H}^1_g(G_K, U(\Lambda))$ defined above, with the obvious transition maps.

[1] One could equally well define the first cohomotopy *sheaf* of a cosimplicial algebraic group (with respect to a suitable topology on $\mathbf{Aff}_{\mathbb{Q}_p}$) by taking the quotient $\mathrm{Z}^1(\mathsf{D}^\bullet_*(U))/\mathsf{D}^0_*(U)$ in the corresponding category of sheaves. However, since we're working with pro-unipotent groups, it in fact suffices to work with the presheaf quotient, and this choice will simplify certain proofs later in this appendix.

In fact, these two extensions of the set $\mathrm{H}^1_*(G_K, U(\mathbb{Q}_p)) \cong \pi^1\left(U(\mathsf{B}^\bullet_*)^{G_K}\right)$ to presheaves on $\mathbf{Aff}_{\mathbb{Q}_p}$ agree, in effect giving a presheaf version of theorem 6.2.3.

Theorem A.1.4 (cf. theorem 6.2.3)**.** *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then the cohomotopy presheaves of groups/sets of the cosimplicial pro-unipotent groups $\mathsf{D}^\bullet_e(U)$, $\mathsf{D}^\bullet_f(U)$ and $\mathsf{D}^\bullet_g(U)$ are canonically identified as*

$$\pi^i(\mathsf{D}^\bullet_e(U)) = \begin{cases} U^{G_K} & \text{if } i = 0, \\ \mathrm{H}^1_e(G_K, U) & \text{if } i = 1, \\ 1 & \text{if } i > 1 \text{ and } U \text{ abelian;} \end{cases}$$

$$\pi^i(\mathsf{D}^\bullet_f(U)) = \begin{cases} U^{G_K} & \text{if } i = 0, \\ \mathrm{H}^1_f(G_K, U) & \text{if } i = 1, \\ 1 & \text{if } i > 1 \text{ and } U \text{ abelian;} \end{cases}$$

$$\pi^i(\mathsf{D}^\bullet_g(U)) = \begin{cases} U^{G_K} & \text{if } i = 0, \\ \mathrm{H}^1_g(G_K, U) & \text{if } i = 1, \\ \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U^*(1))^* & \text{if } i = 2 \text{ and } U \text{ abelian,} \\ 1 & \text{if } i > 2 \text{ and } U \text{ abelian.} \end{cases}$$

Here, as always, we are conflating schemes with their functors of points, and vector spaces with (the functors of points of) their corresponding vector groups.

Proof. We proved this in chapter 6 for $\mathbb{Q}_p$-points. The point here is that the theory we set up in that section has a natural (and functorial) generalisation if we replace $\mathbb{Q}_p$ with a test $\mathbb{Q}_p$-algebra Λ throughout.

More precisely, the statements and proofs of lemma 6.1.5, proposition 6.2.1, corollary 6.2.2 and then theorem 6.2.3 hold essentially verbatim if instead of taking points of pro-unipotent groups in $\mathbb{Q}_p$ (or some $\mathbb{Q}_p$-algebra B), we take points in Λ (or $\mathsf{B}\otimes_{\mathbb{Q}_p}\Lambda$), where the action on Λ is trivial and the topology on Λ (or $\mathsf{B}\otimes_{\mathbb{Q}_p}\Lambda$) is the natural one coming from writing Λ as a direct limit of finite-dimensional $\mathbb{Q}_p$-subspaces.

Indeed, the only part of any of the proofs which isn't immediate is the base case of the induction in proposition 6.2.1 with U abelian, for which we quoted [BK90, Lemma 3.8.1]. To prove this base case, we may simply note that the map $\mathrm{H}^1(G_K, U\otimes_{\mathbb{Q}_p}\mathsf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\Lambda) \to \mathrm{H}^1(G_K, U\otimes_{\mathbb{Q}_p}\mathsf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\Lambda)$ is the directed colimit of the maps $\mathrm{H}^1(G_K, U\otimes_{\mathbb{Q}_p}\mathsf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\Lambda_i) \to \mathrm{H}^1(G_K, U\otimes_{\mathbb{Q}_p}\mathsf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\Lambda_i)$ for Λ_i ranging over the finite-dimensional $\mathbb{Q}_p$-subspaces of Λ. These maps are all injective by [BK90, Lemma 3.8.1] applied to $U\otimes_{\mathbb{Q}_p}\Lambda_i$, so by the exactness of directed colimits, the map $\mathrm{H}^1(G_K, U\otimes_{\mathbb{Q}_p}\mathsf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\Lambda) \to \mathrm{H}^1(G_K, U\otimes_{\mathbb{Q}_p}\mathsf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\Lambda)$ is injective too. This completes

the base case of the induction in proposition 6.2.1, and the inductive step proceeds exactly as before. □

Remark A.1.5. One could equally well formulate a presheaf version of theorem 6.3.2 describing the presheaf cohomotopy of $\mathsf{D}^\bullet_{f/e}(U)$ and $\mathsf{D}^\bullet_{g/e}(U)$ for $U/\mathbb{Q}_p$ a de Rham representation of G_K on a finitely generated pro-unipotent group; in particular giving a description of $\pi^1\left(\mathsf{D}^\bullet_{*/e}(U)\right)$ as a presheaf version of the set $\mathrm{H}^1_{*/e}(G_K, U(\mathbb{Q}_p))$. However, unlike theorem A.1.4, such a characterisation cannot be straightforwardly proved by making functorial the theory in chapter 6—at least when U is infinite-dimensional—so we instead prefer to *define* the presheaf

$$\mathrm{H}^1_{*/e}(G_K, U) := \pi^1\left(\mathsf{D}^\bullet_{*/e}(U)\right),$$

using theorem 6.3.2 to show that this agrees on $\mathbb{Q}_p$-points with our pre-existing definition of $\mathrm{H}^1_{*/e}(G_K, U(\mathbb{Q}_p))$.

Corollary A.1.6 (cf. corollary 6.3.6)**.** *Let*

$$1 \to Z \xrightarrow{\mathsf{z}} U \to Q \to 1$$

be a central extension of de Rham representations of G_K on unipotent groups over $\mathbb{Q}_p$. Then there are functorially assigned exact sequences

$$\begin{aligned} 1 \longrightarrow Z^{G_K} &\xrightarrow{\mathsf{z}} U^{G_K} \longrightarrow Q^{G_K} \\ &\hookrightarrow \mathrm{H}^1_g(G_K, Z) \xrightarrow{\curvearrowright} \mathrm{H}^1_g(G_K, U) \longrightarrow \mathrm{H}^1_g(G_K, Q) \\ &\hookrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Z^*(1))^* \end{aligned}$$

and

$$\begin{aligned} 1 \longrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Z) &\xrightarrow{\mathsf{z}} \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U) \longrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Q) \\ &\hookrightarrow \mathrm{H}^1_{g/e}(G_K, Z) \xrightarrow{\curvearrowright} \mathrm{H}^1_{g/e}(G_K, U) \longrightarrow \mathrm{H}^1_{g/e}(G_K, Q) \\ &\hookrightarrow \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Z^*(1))^* \end{aligned}$$

of presheaves of groups/pointed sets on $\mathbf{Aff}_{\mathbb{Q}_p}$. Moreover, when U and Q are abelian, both exact sequences continue on with the terms $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U^*(1))^* \to \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(Q^*(1))^* \to 1$. (See remark A.1.5 for the definition of the presheaf $\mathrm{H}^1_{g/e}(G_K, U)$.)

Proof. Proposition 6.3.7 provides that the sequence

$$1 \to \mathsf{D}_g^\bullet(Z) \to \mathsf{D}_g^\bullet(U) \to \mathsf{D}_g^\bullet(Q) \to 1$$

of cosimplicial pro-unipotent groups is a central extension on $\mathbb{Q}_p$-points. Hence it is a central extension of presheaves of groups, so that the first desired sequence is the cohomotopy exact sequence from theorem 5.2.1. The same argument applies for the second. □

Remark A.1.7. Using corollary A.1.6 in place of corollary 6.3.6 allows us to upgrade many of the earlier results in this thesis to presheaf versions. For instance, when $U/\mathbb{Q}_p$ is the $\mathbb{Q}_p$-unipotent étale fundamental group of a $\mathbb{G}_m$-torsor $L^\times$ on an abelian variety A/K, then the inclusion $\mathbb{G}_m \hookrightarrow L^\times$ of the fibre over 0 induces an isomorphism $\mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathrm{H}^1_{g/e}(G_K, U)$ of presheaves on $\mathbf{Aff}_{\mathbb{Q}_p}$ (since $\mathrm{H}^1_{g/e}(G_K, V_pA) = \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V_pA) = 1$ as presheaves), and hence the map $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xrightarrow{\sim} \mathbb{Q}_p$—identified in theorem 1.2.3 with the Néron log-metric on L—extends to an isomorphism of presheaves $\mathrm{H}^1_{g/e}(G_K, U) \xrightarrow{\sim} \mathbb{A}^1_{\mathbb{Q}_p}$. In fact, we will see shortly that $\mathrm{H}^1_{g/e}(G_K, U)$ and $\mathrm{H}^1_g(G_K, U)$ are varieties over $\mathbb{Q}_p$, so that the Néron log-metric is naturally an *algebraic* function on the local Bloch–Kato Selmer variety $\mathrm{H}^1_g(G_K, U)$.

A.2 Representability

In the particular case that $U = U_n$ is the maximal n-step unipotent quotient of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of a complete curve over K with good reduction, it is shown in [Kim09, p. 119] that the local Bloch–Kato Selmer presheaf $\mathrm{H}^1_f(G_K, U)$ is representable by a variety over $\mathbb{Q}_p$, specifically by the Weil restriction

$$\mathrm{Res}^K_{\mathbb{Q}_p}\left(\mathsf{D}_{\mathrm{dR}}(U)/\mathsf{D}^+_{\mathrm{dR}}(U)\right),$$

which is an affine space over $\mathbb{Q}_p$. In this section, we will examine the question of representability of the local Bloch–Kato Selmer presheaves $\mathrm{H}^1_*(G_K, U)$ for a general de Rham representation of G_K on a unipotent group $U/\mathbb{Q}_p$, using the explicit descriptions of these presheaves afforded by our cosimplicial models $\mathsf{D}^\bullet_*(U)$. When $* = f$, this is essentially the same argument as Kim's, but when $* = g$ there are extra complications that have to be circumvented.

In the proofs of representability of $\mathrm{H}^1_f(G_K, U)$ in [Kim09], a crucial role is played by the fact that, for weight reasons, $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(U) = 1$—in fact, this condition alone already suffices for representability of both $\mathrm{H}^1_e(G_K, U)$ and $\mathrm{H}^1_f(G_K, U)$, as follows.

Lemma A.2.1. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group. Then the following are equivalent:*

- $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U) = 1$;
- $\mathrm{H}^1_e(G_K, U) = \mathrm{H}^1_f(G_K, U)$; *and*
- $\pi^1\left(\mathsf{D}^\bullet_{f/e}(U)\right) = 1$.

Moreover, when these conditions hold, $\mathrm{H}^1_e(G_K, U) = \mathrm{H}^1_f(G_K, U)$ is representable by the affine space

$$\mathrm{Res}^K_{\mathbb{Q}_p}\left(\mathsf{D}_{\mathrm{dR}}(U)/\mathsf{D}^+_{\mathrm{dR}}(U)\right).$$

Proof. By following the argument of corollary 6.3.3, we see that for any test $\mathbb{Q}_p$-algebra Λ, $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(\Lambda) = 1$ iff $\pi^1\left(\mathsf{D}^\bullet_{f/e}(U)(\Lambda)\right) = 1$, giving the equivalence of the first and third points.

The second point also implies the first, since it implies in particular that we have $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) = \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))$, so that $\mathrm{H}^1_{f/e}(G_K, U(\mathbb{Q}_p)) = \pi^1\left(\mathsf{D}^\bullet_{f/e}(U)(\mathbb{Q}_p)\right) = 1$ and hence that $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) = 1$. Since $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)$ is a unipotent group, this already implies that $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U) = 1$.

To show that the first and third point together imply the second, we define $\mathsf{D}^\bullet_{\mathrm{dR}}(U)$ to be the cosimplicial unipotent group constructed from the cosimplicial $\mathbb{Q}_p$-algebra $\mathsf{B}^\bullet_{\mathrm{dR}}$ from the proof of theorem 6.3.2, following the construction of $\mathsf{D}^\bullet_*(U)$ in definition A.1.1. It follows from the proof of theorem 6.3.2 that we have a diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathsf{D}^\bullet_{\mathrm{dR}}(U) & \longrightarrow & \mathsf{D}^\bullet_e(U) & \longrightarrow & \mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathsf{D}^\bullet_{\mathrm{dR}}(U) & \longrightarrow & \mathsf{D}^\bullet_f(U) & \longrightarrow & \mathsf{D}^\bullet_{f/e}(U) & \longrightarrow & 1
\end{array}$$

whose rows are exact sequences of cosimplicial unipotent groups (for instance because they are exact on $\mathbb{Q}_p$-points). When the first and third conditions hold, both of the right-hand cosimplicial unipotent groups have trivial cohomotopy in degrees 0 and 1, so that the left-hand horizontal arrows induce isomorphisms on cohomotopy. Thus the central vertical arrow also induces an isomorphism on cohomotopy, proving the second point.

For the final part, simply note that (for instance using remark 5.1.7)

$$\pi^1\left(\mathsf{D}^\bullet_{\mathrm{dR}}(U)\right) = \mathrm{Res}^K_{\mathbb{Q}_p}\left(\mathsf{D}_{\mathrm{dR}}(U)/\mathsf{D}^+_{\mathrm{dR}}(U)\right),$$

which is isomorphic to both $\mathrm{H}^1_e(G_K, U)$ and $\mathrm{H}^1_f(G_K, U)$ in this case. □

To prove representability of $\mathrm{H}^1_g(G_K, U)$ under similar assumptions, we have to proceed differently, the main reason being that *a priori* our cosimplicial model only provides a description of $\mathrm{H}^1_g(G_K, U)$ as a quotient of the scheme $\mathrm{Z}^1(G_K, \mathsf{D}^\bullet_g(U))$ of 1-cocycles by an action of $\mathsf{D}^0_g(U)$—without knowing the structure of this scheme of 1-cocycles, it is difficult to prove any representability results for the quotient, even when the action is free.

Instead, we will follow closely the method in [Kim05, Proposition 2] and proceed inductively.

Lemma A.2.2. *Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group, and suppose that U has a central series*[2] *$W_\bullet U$ satisfying $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}\left(\mathrm{gr}^W_n(U)\right) = 1$ for all n, where $\mathrm{gr}^W_n(U) = W_nU/W_{n-1}U$ is a de Rham representation on a $\mathbb{Q}_p$-vector space. Then $\mathrm{H}^1_g(G_K, U)$ and $\mathrm{H}^1_{g/e}(G_K, U)$ are representable by affine schemes of finite type over $\mathbb{Q}_p$.*

Proof. We prove this for H^1_g, the case of $\mathrm{H}^1_{g/e}$ being similar. We denote $U_n = U/W_{-n-1}U$ and $V_n = \mathrm{gr}^W_{-n}(U)$, so that U_n is a central extension

$$1 \to V_n \xrightarrow{\mathsf{z}} U_n \to U_{n-1} \to 1.$$

We will show by induction on n that each $\mathrm{H}^1_g(G_K, U_n)$ is representable, beginning from the trivial base case $n << 0$.

To prove the inductive step, consider the exact sequence

$$U_{n-1}^{G_K} \to \mathrm{H}^1_g(G_K, V_n) \xrightarrow{\curvearrowright} \mathrm{H}^1_g(G_K, U_n) \to \mathrm{H}^1_g(G_K, U_{n-1}) \xrightarrow{\delta} \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V_n^*(1))^*$$

of presheaves on $\mathbf{Aff}_{\mathbb{Q}_p}$ given by corollary A.1.6. The action of $\mathrm{H}^1_g(G_K, V_n)$ on $\mathrm{H}^1_g(G_K, U_n)$ is free (on sections over every $\mathrm{Spec}(\Lambda)$), since the stabiliser of some $[\alpha] \in \mathrm{H}^1_g(G_K, U_n)(\Lambda)$, represented by a cocycle $\alpha \in \mathrm{Z}^1(\mathsf{D}^\bullet_g(U_n)(\Lambda))$, is

$$({}_\alpha U_{n-1}(\Lambda))^{G_K} = 1,$$

since ${}_\alpha U_{n-1}(\Lambda)$ is an iterated central extension of the abelian groups $V_m \otimes \Lambda$ for $m < n$. Thus the exact sequence

$$1 \to \mathrm{H}^1_g(G_K, V_n) \xrightarrow{\curvearrowright} \mathrm{H}^1_g(G_K, U_n) \to \ker(\delta) \to 1$$

[2] In many cases, this series may be taken to be the descending central series (with reversed indexing)—we have in mind that when U is the fundamental group of a smooth variety, it may be more natural to take the weight filtration, which agrees with the descending central series when the variety is projective.

of presheaves on $\mathbf{Aff}_{\mathbb{Q}_p}$ makes $\mathrm{H}^1_g(G_K, U_n)$ into an $\mathrm{H}^1_g(G_K, V_n)$-torsor over $\ker(\delta)$.

Now by inductive assumption, $\mathrm{H}^1_g(G_K, U_{n-1})$ is an affine $\mathbb{Q}_p$-scheme of finite type, as is the vector group $\mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V_n^*(1))^*$, so that $\ker(\delta)$ is also an affine $\mathbb{Q}_p$-scheme of finite type. In particular, the presheaf surjection $\mathrm{H}^1_g(G_K, U_n) \twoheadrightarrow \ker(\delta)$ splits, so that

$$\mathrm{H}^1_g(G_K, U_n) \simeq \ker(\delta) \times \mathrm{H}^1_g(G_K, V_n)$$

is a trivial torsor over $\ker(\delta)$, and hence is also an affine $\mathbb{Q}_p$-scheme of finite type, as desired. □

Remark A.2.3. Lemma A.2.2 is considerably weaker than lemma A.2.1 inasmuch as we make no claims as to the geometry of $\mathrm{H}^1_g(G_K, U)$—for instance, we do not claim that it is reduced, nor that it is connected. In fact, it is the author's expectation that under reasonable conditions this should again be an affine space over $\mathbb{Q}_p$.

Nonetheless, let us remark here that our representability results apply already for $U/\mathbb{Q}_p$ the unipotent étale fundamental group of a $\mathbb{G}_m$-torsor $L^\times$ on an abelian variety A/K, as studied throughout this thesis—the conditions of lemmas A.2.1 and A.2.2 were already checked in remark 6.3.4.

Appendix B

Minimal models of hyperbolic curves

In this appendix, we will set out the basic theory of minimal models of hyperbolic curves, following more or less verbatim the treatment in [Liu02] of the case of complete hyperbolic curves; nothing in this appendix is at all original, and is merely included to provide proofs of technical results we have been unable to find in the literature. As in section 7.2, we will fix a smooth connected curve Y over our fixed p-adic field K, and write it as $X \setminus D$ where X is connected, smooth and projective and D is a reduced divisor on X.

Following [Liu02, Definition 10.1.1], we will define a *model* of Y (or more properly, of (X, D)) to be a pair $(\mathcal{X}, \mathcal{D})$ consisting of a proper integral $\mathcal{O}_K$-scheme $\mathcal{X}$ and a reduced divisor $\mathcal{D}$, flat over $\mathcal{O}_K$, together with an isomorphism $(\mathcal{X}_\eta, \mathcal{D}_\eta) \xrightarrow{\sim} (X, D)$ of the generic fibre with (X, D)—these conditions already imply that $\mathcal{X}/\mathcal{O}_K$ is flat of relative dimension 1. Note that $\mathcal{D}$ is the closure in $\mathcal{X}$ of $D \subseteq X$, so the pair $(\mathcal{X}, \mathcal{D})$ is uniquely determined by $\mathcal{X}$.

A morphism of models is simply a morphism $\mathcal{X}' \to \mathcal{X}$ of $\mathcal{O}_K$-schemes compatible with the identifications of the generic fibres with X—such a morphism automatically takes $\mathcal{D}$ to $\mathcal{D}'$. Note that there is at most one such morphism (as $\mathcal{X}/\mathcal{O}_K$ separated), so that the collection of models is partially-ordered with respect to domination.

We will say that a model is *regular normal crossings* or *RNC* just when $\mathcal{X}$ is regular and $\mathcal{D} + \mathcal{X}_s$ is a normal crossings divisor on $\mathcal{X}$, where $\mathcal{X}_s$ is the (potentially non-reduced) special fibre[1]. We shall say that a model is *regular semistable* just when it is RNC and $\mathcal{X}_s$ is reduced. Equivalently, a regular $\mathcal{X}$ is semistable iff $\mathcal{X}_s$ is a reduced normal crossings divisor and the reduction map $D(\overline{K}) \to \mathcal{X}_s(\overline{k})$ is injective and has image contained in the smooth points of $\mathcal{X}_s$.

[1] We are following the convention in [Liu02] that normal crossings divisors needn't be reduced.

Remark B.0.1. It is easy to see that the blowup of an RNC model at a closed point is still RNC. In the other direction, if $\mathcal{E}$ is an exceptional divisor on an RNC model $(\mathcal{X}, \mathcal{D})$, then the contraction of $\mathcal{E}$ remains an RNC model iff $\mathcal{E}$ meets the other irreducible components of $\mathcal{X}_s$ in at most two points, which are both rational over $\mathrm{H}^0(\mathcal{E}, \mathcal{O}_\mathcal{E})$—this follows by the argument of [Liu02, Lemma 9.3.35]. Such a divisor we will call *RNC-exceptional*.

Note that by [Liu02, Theorem 9.2.2], every morphism from a regular model to an RNC model is a composite of a sequence of blowups; hence the domain is also RNC and the morphism is a composite of a sequence of contractions of RNC-exceptional divisors.

With these definitions, we will now prove analogues of the standard theorems on minimal models in the context of hyperbolic curves: that models can be desingularised to RNC models; that a minimal RNC model exists; and that after a finite extension the minimal RNC model is semistable.

Lemma B.0.2 (RNC desingularisation)**.** *Suppose that $(\mathcal{X}, \mathcal{D})$ is a model of (X, D). Then it admits an RNC desingularisation, i.e. a proper dominant morphism*

$$g\colon (\mathcal{X}', \mathcal{D}') \to (\mathcal{X}, \mathcal{D})$$

from an RNC model. Moreover, this g may be chosen to be an isomorphism away from[2] $\mathcal{X}_s^{\mathrm{sing}} \cup \mathcal{D}_s$.

Proof. [Liu02, Theorem 8.3.44] (originally [Lip78, Remark B]) shows that after repeatedly blowing up the singular locus of $\mathcal{X}$ and normalising one arrives at a regular model. The proof of [Liu02, Theorem 9.2.26] then shows that after repeatedly blowing up singular points of the special fibre and intersection points of $\mathcal{D}$ one arrives at an RNC model. From this description, we see that this g is an isomorphism away from $\mathcal{X}_s^{\mathrm{sing}} \cup \mathcal{D}_s$. □

The most complicated proof here will be that of the existence of minimal RNC models, for which we need to understand intersections with a log-canonical divisor, i.e. the sum of the divisor $\mathcal{D}$ with the usual canonical divisor.

[2] In fact, one should be able to take g an isomorphism away from the finite set of closed points where either $\mathcal{X}$ fails to be regular or $\mathcal{D} + \mathcal{X}_s$ fails to be normal crossings, which one might call an *RNC desingularisation in the strong sense*.

Proposition B.0.3. *Suppose that $(\mathcal{X}, \mathcal{D})$ is an RNC model of (X, D) as above, and let $\mathcal{K}_{\mathcal{X}/\mathcal{O}_K}$ be a canonical divisor (divisor associated to the relative canonical sheaf $\omega_{\mathcal{X}/\mathcal{O}_K}$). Then*

$$(\mathcal{K}_{\mathcal{X}/\mathcal{O}_K} + \mathcal{D}) \cdot \mathcal{X}_s = -\chi(Y),$$

where $\chi(Y)$ denotes the Euler characteristic of the K-variety Y.

If $\mathcal{E}$ is an RNC-exceptional divisor on $\mathcal{X}$, then

$$(\mathcal{K}_{\mathcal{X}/\mathcal{O}_K} + \mathcal{D}) \cdot \mathcal{E} \leq 0.$$

Proof. For the first part, [Liu02, Proposition 9.1.35] shows that $\mathcal{K}_{\mathcal{X}/\mathcal{O}_K} \cdot \mathcal{X}_s = -\chi(X)$. Since $\mathcal{D}$ meets $\mathcal{X}_s$ transversely, it follows that $\mathcal{D} \cdot \mathcal{X}_s = \deg(D)$, and hence $(\mathcal{K}_{\mathcal{X}/\mathcal{O}_K} + \mathcal{D}) \cdot \mathcal{X}_s = -\chi(X) + \deg(D) = -\chi(Y)$.

For the second part, let $f\colon \mathcal{X} \to \mathcal{X}'$ be the contraction along $\mathcal{E}$, and write x for the closed point of $\mathcal{X}'$ which is the image of $\mathcal{E}$. Write $\mathcal{D}' = f_*\mathcal{D}$, so that $\mathcal{D}$ is the strict transform of $\mathcal{D}'$, and $\mu_x(\mathcal{D}')$ for the multiplicity of x in $\mathcal{D}'$ [Liu02, Definition 9.2.19]. The assumption that $\mathcal{X}'$ is regular ensures that $\mathcal{K}_{\mathcal{X}/\mathcal{O}_K} \cdot \mathcal{E} = \mathcal{E} \cdot \mathcal{E} < 0$ [Liu02, Proposition 9.3.10(a)], while the assumption that $(\mathcal{X}', \mathcal{D}')$ is RNC ensures that $\mu_x(\mathcal{D}') \leq 1$ [Liu02, Remark 9.2.20]. Also, from [Liu02, Proposition 9.2.23] and [Liu02, Theorem 9.2.12(a)], we have $\mathcal{D} \cdot \mathcal{E} = -\mu_x(\mathcal{D}')\mathcal{E} \cdot \mathcal{E}$, so that

$$(\mathcal{K}_{\mathcal{X}/\mathcal{O}_K} + \mathcal{D}) \cdot \mathcal{E} = (1 - \mu_x(\mathcal{D}'))\mathcal{E} \cdot \mathcal{E} \leq 0$$

as desired. □

Now we can mimic the proof of [Liu02, Theorem 9.3.21] to prove the existence of minimal RNC models.

Theorem B.0.4 (Minimal RNC models of hyperbolic curves)**.** *Let (X, D) be as above, with $Y = X \setminus D$ hyperbolic. Then there is an RNC model $(\mathcal{X}, \mathcal{D})$ which is* minimal*, i.e. such that every other RNC model dominates it.*

Proof. Lemma B.0.2 shows that RNC models exist. Given any RNC model, by repeatedly contracting out RNC-exceptional divisors, we arrive at a *relatively minimal* RNC model, i.e. one which dominates no other RNC model.

We will then show that any relatively minimal RNC model $(\mathcal{X}, \mathcal{D})$ is already minimal. Suppose, for contradiction, that there were some RNC model $(\mathcal{X}', \mathcal{D}')$ not dominating $(\mathcal{X}, \mathcal{D})$. It follows then from the proof of [Liu02, Lemma 9.3.20] that there is an RNC model $(\mathcal{X}'', \mathcal{D}'')$ dominating both $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{X}', \mathcal{D}')$ along with an RNC-exceptional divisor $\mathcal{E}'$ contained in the exceptional locus of $\mathcal{X}'' \to \mathcal{X}'$ such that either:

(i) the image of $\mathcal{E}'$ in $\mathcal{X}$ is still an RNC-exceptional divisor; or

(ii) there exists another RNC-exceptional divisor $\mathcal{E}$ on $\mathcal{X}''$ and an integer $\mu > 0$ such that $(\mathcal{E}' + \mu\mathcal{E})^2 \geq 0$.

By assumption, the first of these cannot occur, so we may take $\mathcal{E}$ and μ satisfying (ii). [Liu02, Theorem 9.1.23] then shows that $\mathcal{E}' + \mu\mathcal{E} = r\mathcal{X}''_s$ for some positive rational r, and hence

$$-\chi(Y) = (\mathcal{K}_{\mathcal{X}''/\mathcal{O}_K} + \mathcal{D}'') \cdot \mathcal{X}''_s = \frac{1}{r}(\mathcal{K}_{\mathcal{X}''/\mathcal{O}_K} + \mathcal{D}'') \cdot \mathcal{E}' + \frac{\mu}{r}(\mathcal{K}_{\mathcal{X}''/\mathcal{O}_K} + \mathcal{D}'') \cdot \mathcal{E} \leq 0$$

by proposition B.0.3, which is a contradiction. □

To conclude, we now address the question of existence of semistable models, which is essentially a question of reducedness of the special fibre. Fortunately, unlike the existence of minimal RNC models, the existence of semistable models is an immediate consequence of the corresponding theorem for higher-genus curves.

Lemma B.0.5 (Potential semistable reduction)**.** *Let (X, D) be as above, with $Y = X \setminus D$ hyperbolic and geometrically connected. Then there is a finite extension L/K such that (X_L, D_L) has a regular semistable model. If a regular semistable model exists, then the minimal RNC model is semistable.*

Proof. For the first part, replacing K with L if necessary, we may assume that X itself has a regular semistable model $\mathcal{X}$—this follows from the usual semistable reduction theorem if $g(X) > 0$, and when $g(X) = 0$ by choosing L such that $X_L \simeq \mathbb{P}^1_L$. If we let $\mathcal{D}$ denote the closure of D in $\mathcal{X}$, then it follows from the proof of lemma B.0.2 that we arrive at an RNC model after a finite number of blowups at points of $\mathcal{D}_s$. However, the assumption that $\mathcal{X}$ is regular semistable ensures that $\mathcal{D}_s$ lies in the smooth locus of $\mathcal{X}_s/k$, and hence by [Liu02, Proposition 9.2.23] the multiplicity of the exceptional divisor in the pullback of the divisor $\mathcal{X}_s$ under such a blowup $\mathcal{X}' \to \mathcal{X}$ is exactly 1. In particular, the generic fibre of $\mathcal{X}'$ is also reduced, so proceeding inductively we see that this RNC desingularisation of $(\mathcal{X}, \mathcal{D})$ is also regular semistable.

For the latter part, if $(\mathcal{X}, \mathcal{D})$ is the minimal RNC model and $(\mathcal{X}', \mathcal{D}')$ any regular semistable model, we may pick a morphism $(\mathcal{X}', \mathcal{D}') \to (\mathcal{X}, \mathcal{D})$ of models. This consists of a sequence of contractions of RNC-exceptional divisors, which it is easy to see preserve reducedness of the special fibre. Hence $(\mathcal{X}, \mathcal{D})$ is regular semistable. □

List of symbols and terms

Common symbols

p	a fixed prime number, see notation 1.4.1
K	a fixed finite extension of $\mathbb{Q}_p$, see notation 1.4.1
$\overline{K}$	a fixed algebraic closure of K, see notation 1.4.1
G_K	the (absolute) Galois group of $\overline{K}/K$, see notation 1.4.1
$\mathcal{O}_K$	the ring of integers of K, see notation 1.4.1
ϖ	a fixed uniformiser of K, see notation 1.4.1
$\mathfrak{m}_K$	the maximal ideal of $\mathcal{O}_K$, see notation 1.4.1
k	the residue field of K, see notation 1.4.1
v	the valuation of K, see notation 1.4.1
$\mathbb{C}_K$	the completion of $\overline{K}$, see notation 1.4.1
$\mathsf{B}_{\mathrm{cris}}$, B_{st}, B_{dR}	the standard period rings, see notation 1.4.1

Terminology

Bloch–Kato...	
...Selmer sets, H^1_e, H^1_f, H^1_g	definition 1.2.1
...relations $\sim_{\mathrm{H}^1_e}$, $\sim_{\mathrm{H}^1_f}$, $\sim_{\mathrm{H}^1_g}$	definition 1.2.1
...quotients, $\mathrm{H}^1_{g/e}$, $\mathrm{H}^1_{f/e}$	definition 1.2.1
cohomotopy groups/sets, π^i	definition 5.1.4
comodule algebra/module coalgebra	remark 2.2.2
completed tensor product, $\widehat{\otimes}$	definition 2.1.21
cosimplicial object	definition 5.1.1
decompleted dual, $(-)^\circ$	definition 2.1.22
descending central series	definition 2.1.7 or 2.1.8
Dieudonné functor, D_*	lemma 4.2.1
exact sequence, $\xrightarrow{\mathsf{z}}$, $\overset{\curvearrowright}{\to}$	definition 1.2.6
group-like element, $(-)^{\mathrm{gplike}}$	definition 2.1.28
ind-mixed Hodge structure	footnote, page 132
J-adically...	
...cocomplete Hopf algebra	definition 2.1.15
...complete Hopf algebra	definition-lemma 2.1.27
J-filtration, fixpoint filtration	definition 2.1.12

(continuous) Malčev completion...
...of a group — definition-lemma 2.3.1
...of a torsor — remark 2.3.8
mixed Hodge structure...
...on a pro-unipotent group — definition 9.1.1
.....with only negative weights — definition 9.1.14
...on a torsor — definition 9.1.1
n-step...
...(pro-)unipotent group — definition 2.1.7
...(pro-)nilpotent Lie algebra — definition 2.1.8
primitive element, $(-)^{\mathrm{prim}}$ — definition 2.1.28
pro-finite dimensional...
...vector space — definition 2.1.21
...Lie algebra — definition 2.1.17
...Hopf algebra — definition 2.1.24
pro-nilpotent Lie algebra — definition 2.1.17
pro-unipotent...
...group — definition-lemma 2.1.1
...Betti fundamental group — example 2.4.3
...étale fundamental group — example 2.4.4
pushout — definition 2.2.4
representation on a pro-unipotent group — definition-lemma 4.0.1
...unramified representation — definition-lemma 4.1.1
...crystalline representation — definition-lemma 4.2.2
...semistable representation — definition-lemma 4.1.1 or 4.2.2
...de Rham representation — definition-lemma 4.2.2
Tannakian fundamental group, torsor of paths — definition 2.4.1
torsor...
...under a pro-unipotent group — definition 2.2.1
...of Betti paths — example 2.4.3
...of étale paths — example 2.4.4

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