

Absorptions in Combinatorics

Yangyang Cheng

Wadham College



University of Oxford

A thesis presented for the degree of

Doctor of Philosophy

Michaelmas 2024

I dedicate this thesis to my family for their support and encouragement.

Copyright © 2024 by Yangyang Cheng
All Rights Reserved

Acknowledgements

I would like to thank my supervisor, Peter Keevash, for his support and guidance throughout my research. Without him, it would have been very difficult for me to complete this thesis smoothly. I also learned a great deal from him during my research, both academically and in real life.

I would like to thank Katherine Staden; we worked together on many projects, and I learned a lot from her through our cooperation.

I would like to thank Jie Han for his support and valuable suggestions during my research.

I would like to thank Guanghui Wang, my master's supervisor. Without his support, it would have been very difficult for me to have the opportunity to study at Oxford.

I would like to thank my parents for their love and encouragement throughout my life. Thanks to Dongting Chen for her support and companionship over the years.

I would like to thank Allan Lo, who kindly shared key ideas about absorptions with me when I was still a master's student, ultimately leading me to explore the absorption technique.

Finally, most of the work presented in this thesis was generously supported by an ERC PhD studentship.

Declaration

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I declare that the work contained in this thesis has not been submitted towards any other degree or qualification at the University of Oxford or at any other university or institution. Chapter 2 is based on joint work with Jie Han, Bing Wang and Guanghui Wang. Chapter 3 to 5 are based on joint work with Katherine Staden.

Abstract

This thesis mainly focuses on the following transversal problem, which is closely related to the absorption technique in extremal combinatorics: Given a collection of simple graphs $G_1, \ldots, G_t$ defined on the same vertex set V, where each graph G_i $(1 \leq i \leq t)$ satisfies some property P, can we find a copy of any graph F of order $|V|$ with at most t edges in V, such that the edges of F are chosen from different G_i's?

We provide frameworks and techniques for such problems when each G_i $(1 \leq i \leq t)$ is dense in V. First, we generalize the traditional absorption technique to graph collections, which allows us to modify the original absorption structure to obtain a new absorption structure that is helpful for transversal problems. Second, we establish regularity tools for graph collections. One of the key lemmas we prove is a blow-up result for transversal structures. This allows us to demonstrate more complicated results when the absorption technique does not apply.

We will further show two direct applications. One is a new proof of the asymptotic solution to the transversal Pósa-Seymour conjecture. The other is a stability result for Ahroni's conjecture on transversal Hamilton cycles.

Table of Contents

1 Introduction **1**

1.1 Graph Embeddings . . . 1

1.2 Transversal Embeddings . . . 3

1.3 Absorption Technique for Transversals . . . 5

1.4 Regularity theory for Transversals . . . 7

1.5 Summary of Results . . . 9

1.6 Notation and Terminology . . . 10

2 A General Framework for Transversal Factor Problem **13**

2.1 Introduction . . . 13

2.1.1 Main results . . . 13

2.2 A general framework for transversal F-factors . . . 15

2.3 Preliminary . . . 19

2.4 Transversal Absorption Technique . . . 21

2.4.1 transversal Absorption Lemma . . . 21

2.4.2 Enumeration of Rainbow Absorbers . . . 22

2.4.2.1 Rainbow T_k-absorber . . . 23

2.4.2.2 Rainbow edge-absorber . . . 24

2.5 Rainbow Almost Cover . . . 25

2.6 The proof of Theorem 2.2.5 . . . 28

2.7 The proofs of Theorems 2.1.2 and 2.1.3 . . . 28

2.8 The proof of Theorem 2.1.4 . . . 30

2.9 The proof of Theorem 2.1.5 . . . 32

2.10 Proof of Auxiliary Results . . . 32

3 Regularity Method and Blow-Up Lemma for Transversals **38**

3.1 Introduction . . . 38

3.1.1 The regularity-blow-up method for transversals . . . 38

3.1.2 Transversals in uniformly dense graph collections . . . 40

3.1.3 A transversal blow-up lemma . . . 42

3.1.4 Applications to embedding in uniformly dense hypergraphs 43
3.1.5 The statement of our main result 47
3.2 The regularity-blow-up method for transversals 48
3.2.1 Regularity 48
3.2.2 The regularity lemma for graph collections 51
3.2.3 Templates 53
3.3 Embedding lemmas 56
3.4 Proof of the transversal blow-up lemma 68
3.4.1 Sketch of the proof 68
3.4.2 Proof of Theorem 3.3.5 70
3.4.3 Proof of Theorem 3.1.8 79
3.5 Proof of Theorems 3.1.2 and 3.1.6 79
3.6 Concluding remarks 82
3.7 Appendix 85
3.7.1 Weak hypergraph regularity lemmas 85

4 New Proof of the Asymptotic Transversal Pósa-Seymour Conjecture 89
4.1 Introduction 89
4.1.1 Transversal embedding of spanning graphs 89
4.1.2 Powers of Hamilton cycles 90
4.1.3 Probabilistic tools 91
4.1.4 Sketch of the proof 91
4.2 A regularity lemma for graph collections 93
4.3 Tools for absorbing and connecting 94
4.3.1 Named graphs 94
4.3.2 Connecting and absorbing 96
4.4 Covering via fractional tilings 104
4.5 Proof of Theorem 4.1.2 113

5 Stability of Transversal Hamilton Cycles 121
5.1 Introduction 121
5.1.1 Transversal embedding of Hamilton cycles 121

5.1.2 Stability for transversal Hamilton cycles and paths 122
5.1.3 Sketch of the proof of Theorems 5.1.3 and 5.1.4 124
5.2 Extremality and stability 126
5.2.1 Extremal and stable graphs 126
5.2.2 Stable graph collections 127
5.3 Absorbing 134
5.4 The stable case 140
5.5 The extremal case 145
5.6 Proofs of Theorems 5.1.3 and 5.1.4 153

6 Open Problems and Discussions **155**

Bibliography **157**

1 | Introduction

In this chapter, we survey the development of the absorption technique and introduce our contribution to its generalization. We also provide a summary of each chapter of the thesis and introduce the notation and terminology in the final section.

1.1 Graph Embeddings

Given any graph H, one of the main themes in graph theory is to find conditions on G that ensure G contains H as a subgraph. For example, when H is fixed, the Turán problem asks for the minimum number m such that if $|E(G)| \geq m$, then G contains H as a subgraph.

We can also ask for the minimum number m such that if $\delta(G) \geq m$, then G contains H as a subgraph, where $\delta(G)$ denotes the minimum degree of G. Here, H is no longer fixed, and the problem becomes interesting when H has the same order as G, i.e., H is spanning in G. This is a central topic in graph embedding theory.

As an example, we have the following classic Dirac's theorem:

Theorem 1.1.1. *(Dirac, 1952) If $n \geq 3$ and G is a graph of order n with $\delta(G) \geq n/2$, then G contains a Hamiltonian cycle.*

This theorem has many generalizations, for example, the Ore's theorem (Ore, 1960) and Chvátal's theorem (Chvátal, 1972). These two theorems were proved using basic graph theory techniques, which can be found in any standard textbooks. However, proving the analogous Dirac-type theorems in hypergraphs is much more difficult. First, we need some notation for cycles in hypergraphs. Given $1 \leq \ell < k$, a k-graph $\mathcal{H}$ is called an *ℓ-cycle* if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges share exactly ℓ vertices. A $(k-1)$-cycle is often called a tight cycle, while a 1-cycle is often called a loose cycle. We say that a k-graph contains a Hamiltonian ℓ-cycle if it contains an ℓ-cycle as a spanning subhypergraph. Let $\mathcal{H}$ be any k-graph. For any $1 \leq d \leq k-1$ and any vertex set S of size d, define $d(S)$ to be the number of k-edges that contain S. Define $\delta_d(\mathcal{H})$ to be the minimum $d(S)$ over all d-sets S.

For tight cycles, Rödl, Ruciński, and Szemerédi showed the following:

Theorem 1.1.2. *(Rödl, Szemerédi, & Ruciński, 2008) For every positive integer $k \geq 3$ and $\varepsilon > 0$, there exists an $n_0 > 0$ such that if $n > n_0$, then every k-graph $\mathcal{H}$ with $\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{2} + \varepsilon\right) n$*

contains a tight Hamiltonian cycle.

Their proof uses a powerful technique called the absorption technique, which is also the main focus of this thesis. We will postpone the discussion of this technique to the next section. Using a much more involved argument, they further showed:

Theorem 1.1.3. *(Rödl, Ruciński, & Szemerédi, 2011) There exists an $n_0 > 0$ such that if $n > n_0$, then every 3-graph $\mathcal{H}$ with $\delta_2(\mathcal{H}) \geq \lfloor \frac{n}{2} \rfloor$ contains a tight Hamiltonian cycle.*

For $k \geq 4$, it is still unknown if the same lower bound $\lfloor \frac{n}{2} \rfloor$ holds for Theorem 1.1.2 when n is sufficiently large. For loose cycles, (Kühn & Osthus, 2006) proved that every sufficiently large 3-graph $\mathcal{H}$ with $\delta_2(\mathcal{H}) \geq \frac{n}{4} + o(n)$ contains a loose Hamiltonian cycle. (Czygrinow & Molla, 2014) further showed that $\delta_2(\mathcal{H}) \geq \frac{n}{4}$ is sufficient. For k-graphs ($k \geq 4$), (Han & Zhao, 2015) proved that $\delta_{k-1}(\mathcal{H}) \geq \frac{n}{2k-2}$ is sufficient for loose Hamiltonian cycles in any k-graph $\mathcal{H}$ of sufficiently large order. There have been several other results for Hamiltonian ℓ-cycles in k-graphs $\mathcal{H}$ under the $\delta_d(\mathcal{H})$ condition where $1 \leq d \leq k-1$; see the survey (Zhao, 2016).

Given any fixed graph F and any positive integer n divisible by $|F|$, let G be any graph of order n. We say H is an F-factor of G if H is a spanning subgraph of G, and each component of H is isomorphic to F. The following theorem is usually called the Hajnal-Szemerédi theorem:

Theorem 1.1.4. *(Hajnal & Szemerédi, 1970) Let $r, n \in \mathbf{N}$ be two positive integers such that $r|n$. Every graph G of order n with $\delta(G) \geq \left(1 - \frac{1}{r}\right) n$ contains a K_r-factor.*

This theorem has been extended to arbitrary F-factors; see (Alon & Yuster, 1996), (Komlós, Sárközy, & Szemerédi, 2001), and (Kühn & Osthus, 2009).

Let H be any graph. For any $k \geq 2$, the k-th power of H is a graph obtained by adding an edge between any two vertices in H that have distance at most k. One of the most influential results in graph embeddings, which generalizes both Dirac's theorem and the Hajnal-Szemerédi theorem, is the following Pósa-Seymour conjecture. It was proved by Komlós, Sárkäzy and Szemerédi for sufficiently large graphs:

Theorem 1.1.5. *(Komlós, Sárközy, & Szemerédi, 1998b) For each $k \geq 1$, there exists an $n_0 \geq 1$ such that if G is a graph of order $n \geq n_0$ and $\delta(G) \geq \left(1 - \frac{1}{k+1}\right) n$, then G contains the k-th power of a Hamiltonian cycle.*

When $k = 1$, this is Dirac's theorem. When $k = 2$, this is Pósa's conjecture. In general, this theorem implies the Hajnal-Szemerédi theorem since one can find a K_{k+1}-factor in G under the same condition when n is divisible by $k+1$. The original proof of the Pósa-Seymour conjec-

ture used the Regularity Lemma and the Blow-up Lemma, so the n_0 produced in the proof is very large, at least of tower-type growth in some parameters. Later, in (Levitt, Sárközy, & Szemerédi, 2010) and (Simao, Asif, & Szemerédi, 2014), the lower bound for n_0 was substantially improved. In fact, in (Levitt et al., 2010), the authors used a technique that avoids the use of the Regularity Lemma, resulting in a much better bound for n_0. This technique is now called the absorption technique. One advantage of the absorption technique is that it typically provides a much better lower bound for the order of the graphs than proofs using the regularity method.

1.2 Transversal Embeddings

One of the earliest problems in extremal combinatorics is the transversal problem of Latin squares, originating from Euler's research in the 18th century. A Latin square is an $n \times n$ array filled with n symbols such that each symbol appears exactly once in every row and column. A transversal is a collection of n cells of the Latin square that do not share the same row, column, or symbol. One of the main conjectures about transversals in Latin squares is the Ryser conjecture (Ryser, 1967) from the 1960s, which asserts that every Latin square of odd size has a transversal. The Ryser conjecture has many generalizations, and many related problems have been proposed. For recent progress on these problems, see a comprehensive survey by Pokrovskiy (Pokrovskiy, 2022).

Let $K_{n,n}$ be an edge-colored complete bipartite graph. $K_{n,n}$ is properly colored if every two incident edges are colored distinctly. A rainbow matching M in $K_{n,n}$ is a matching such that every two edges in M are colored distinctly. By identifying the rows and columns as the two partite sets of the complete bipartite graph $K_{n,n}$, and identifying the symbols as colors on the edges of $K_{n,n}$, the Ryser conjecture is equivalent to the following form:

Conjecture 1.2.1 ((Ryser, 1967)). *Every properly edge-colored $K_{n,n}$ using exactly n colors, where n is odd, contains a perfect rainbow matching.*

If we do not require n to be odd, then we have the following Brualdi-Stein Conjecture:

Conjecture 1.2.2 ((Brualdi, Ryser, et al., 1991), (Stein, 1975)). *Every properly edge-colored $K_{n,n}$ using exactly n colors contains a rainbow matching of size $n-1$.*

(Montgomery, 2023) proved that the Brualdi-Stein Conjecture holds for sufficiently large n, which is a breakthrough in this field. The Ryser conjecture and the Brualdi-Stein conjecture

motivate the following meta-problem:

Problem 1.2.3. *Given a collection of simple graphs $G_1, \ldots, G_t$ defined on the same vertex set V, where each graph G_i $(1 \leq i \leq t)$ satisfies some property P, for any graph F of order $|V|$ with at most t edges, can we find a copy of F in V such that the edges of F are chosen from different G_i's?*

We say such a copy of H is a *rainbow* copy of H and if $t = |E(F)|$, we say it is a *transversal* copy of H. As an example, one of the most famous generalizations of the Brualdi-Stein Conjecture is the following Aharoni-Berger Conjecture:

Conjecture 1.2.4 ((Aharoni & Berger, 2009))**.** *Given a collection of matchings $M_1, \ldots, M_n$ on the same vertex set V, where $|M_i| \geq n+1$ for each $1 \leq i \leq n$, we can find a rainbow matching M of size n.*

Note that if we identify $M_1, \ldots, M_n$ as a properly colored multigraph on V, then the Aharoni-Berger conjecture is equivalent to the existence of a rainbow matching in a properly colored multigraph where the number of colors is n and each color induces a matching of size $n+1$. Thus, the Ryser conjecture is a special case of the Aharoni-Berger conjecture. The Aharoni-Berger conjecture has been verified asymptotically in (Pokrovskiy, 2018); however, the full conjecture remains open.

There have been several other types of problems involving the search for rainbow matchings in properly colored multigraphs. In these problems, it is often required that the graphs in the collection are sparse, meaning that either each graph is a single matching or its maximum degree is bounded. But what if the graphs in the collection are dense? For example, if the minimum degree of each graph is at least some linear fraction of $|V|$, and we aim to find more complex structures.

In this direction, Aharoni proposed the following conjecture:

Conjecture 1.2.5. *(Aharoni, DeVos, de la Maza, Montejano, & Šámal, 2020) Given a collection of simple graphs $\mathbf{G} = G_1, \ldots, G_n$ defined on the same vertex set V, where $|V| = n$, if $\delta(G_i) \geq \frac{n}{2}$ for each $1 \leq i \leq n$, then there exists a transversal Hamiltonian cycle.*

Aharoni's conjecture was verified asymptotically by (Cheng, Wang, & Zhao, 2021), and later fully proved by (Joos & Kim, 2020). In the same paper, Joos and Kim posed the following general question for hypergraph families:

Question 1.2.6. *Let F be an m-edge hypergraph with vertex set V, and $\mathcal{H}$ be a family of*

hypergraphs. Suppose $H_1, \ldots, H_m$ is a collection of hypergraphs on the vertex set V, with $H_i \in \mathcal{H}$ for each $i \in [m]$. What conditions on $\mathcal{H}$ guarantee a transversal copy of F in $\mathcal{H}$?

This question has received significant attention, especially when H_i $(1 \leq i \leq m)$ is k-uniform and F is a spanning structure such as a Hamiltonian cycle or an f-factor, where f is a fixed hypergraph. In this setting, we are interested in finding the minimum threshold $\delta_{d,F}$ such that if $\delta_d(H_i) \geq \delta_{d,F}|V|$ for each i, then we can find a transversal copy of F. Here, $\delta_d(H_i)$ denotes $\min_{S \subseteq V, |S|=d} |\{e \mid S \subseteq e \in E(H_i)\}|$.

In (Cheng, Han, Wang, & Wang, 2023), we asymptotically answered this question when F is a clique factor in both the graph and directed graph cases. We also determined $\delta_{d,F}$ $(1 \leq d \leq k-1)$ when F is a perfect matching in k-uniform hypergraphs. The technique we used is a variant of the absorption technique for graph collections, first introduced in (Cheng, Wang, & Zhao, 2021). We will further discuss this technique in the next section. Using a different absorption technique, (Montgomery, Müyesser, & Pehova, 2022) determined the asymptotic threshold for general f-factors and bounded-degree spanning trees in simple graphs. Later, in (Cheng, Han, Wang, Wang, & Yang, 2021), we generalized the rainbow Hamiltonian cycle problem from simple graphs to k-uniform hypergraphs by asymptotically determining $\delta_{k-1,HC}$, where HC is the tight Hamiltonian cycle. In (Cheng, Han, et al., 2021), we also proposed problems to determine δ_{d,HC_ℓ}, where $1 \leq d \leq k-1$ and HC_ℓ is a Hamiltonian ℓ-cycle. Recently, in (Gupta, Hamann, Müyesser, Parczyk, & Sgueglia, 2023), the authors made progress on our problem by asymptotically determining most of these thresholds.

1.3 Absorption Technique for Transversals

Given a graph H of order n, let G be any graph of order n under some condition P, where n is sufficiently large. Suppose that we are going to embed the graph H into G. The idea of the standard absorption technique is as follows: first, we find a decomposition of H into $H_1 \cup H_2$, where $|H_1| \ll |H_2|$, $V(H_1) \cap V(H_2) = \emptyset$. Here, $|H_1| \ll |H_2|$ means that the order of H_1 is much smaller than that of H_2.

Next, using the probabilistic method, we construct a subgraph $H_0' \subseteq G$ that is a copy of some subgraph $H_0 \subseteq H_1$ with the properties that $|H_0| \approx |H_1|$, and for every vertex set $S \subseteq G \backslash V(H_0')$ of size $|V(H_1 - H_0)|$, $H_0' \cup S$ contains a copy of H_1. Finally, we extend H_0' to a copy of $H_0 \cup H_2$

in G, leaving exactly $|V(H_1-H_0)|$ vertices unused in G. By absorbing all the remaining vertices in G, we ultimately obtain a copy of H.

The absorption technique was first introduced in (Rödl et al., 2008), where they proved Theorem 1.1.2 without using regularity tools. Later, in (Levitt et al., 2010), the authors proved Theorem 1.1.5 by adapting the absorption technique, which provided a better lower bound for n_0 than the original proof, as it did not rely on the regularity lemma. Over time, the absorption technique has been further developed and applied by various researchers, leading to the resolution of many conjectures. For example, (Keevash, 2014) used algebraic absorption to prove the existence conjecture for designs, which marked a significant breakthrough in design theory. In (Barber, Kühn, Lo, & Osthus, 2016) and (Glock, Kühn, Lo, & Osthus, 2017), the authors developed iterative absorption, which allowed them to establish numerous new results in graph decompositions, including a new proof of the existence conjecture. More recent applications of the absorption technique include the resolution of Ringel's conjecture (see (Montgomery, Pokrovskiy, & Sudakov, 2021), and independently (Keevash & Staden, 2020)), and the even case of the Ryser-Brualdi-Stein conjecture (see (Montgomery, 2023)). For further applications of the absorption technique beyond those mentioned here, see the surveys (Simonovits & Szemerédi, 2019) and (Zhao, 2016).

The absorption technique has many variants, most of them do not fit neatly into the sketch provided above. However, in general, these variants are all developed to handle the remaining vertices. Specifically, we first embed a large part H_1 of H into G such that $|H_1| \approx |H|$, and then absorb or insert the remaining vertices into H_1 to complete it into a copy of H. In this sense, we can broadly say that the absorption technique is concerned with handling the exceptional vertices once a large part of the base graph has been embedded. It is worth noting that the traditional regularity method combined with the blow-up lemma also fits into this framework. However, in the literature, the term "absorption technique" is often used to refer to techniques that are free of regularity methods. In this thesis, we do not differentiate them rigorously; we will also discuss the regularity method combined with the blow-up lemma and its generalizations to transversals.

One of the main contributions of this thesis is to develop the absorption technique for transversals. Given a collection of simple graphs $\mathbf{G} = G_1, \dots, G_m$ defined on V under some condition $\mathbf{P}$, we say that the index set $[m]$ is the color set. Suppose we want to find a transversal copy of

H where $|E(H)| = m$. As in the non-transversal case, we first find a decomposition of H into $H_1 \cup H_2$, where $|H_1| \ll |H_2|$ and $V(H_1) \cap V(H_2) = \emptyset$. Next, we find a rainbow copy H_0' of $H_0 \subseteq H_1$ such that $|H_0| \approx |H_1|$. Denote the color set of H_0 by C_0. The rainbow copy H_0' is typically constructed using probabilistic methods such that, for each color set $C' \subseteq [m] \setminus C_0$ and each vertex set $V' \subseteq V \setminus V(H_0)$, the vertex set $V(H_0') \cup V'$ spans a rainbow copy of H_1 using the color set $C_0 \cup C'$. Finally, we extend H_0' to be a rainbow copy of $H_0 \cup H_2$ with the remaining vertex set V' and remaining color set C'. In the final step, we absorb V' and C' into H_0' to obtain a rainbow copy of $H_1 \cup H_2$ using the color set $[m]$. This is exactly a transversal copy of H.

1.4 Regularity theory for Transversals

The regularity method originates from the famous Szemerédi regularity lemma and has been developed by many authors to solve problems in graph embedding theory over the past several decades. For example, the original Pósa-Seymour conjecture was solved for large graphs using regularity theory (Komlós, Sárközy, & Szemerédi, 1998a). A key lemma frequently used in this theory is the Blow-up lemma (see (Komlós, 1999)), which roughly states that we can embed any sparse graph into a super-regular graph, even when the target area for some vertices is restricted. To embed a spanning graph H into a large base graph G under some condition $\mathbf{P}$, we first apply the regularity lemma to G, thereby obtaining a partition of G. We then define a reduced graph R according to this regular partition. The reduced graph R usually almost inherits the condition $\mathbf{P}$, allowing us to find a subgraph M within R as needed. Next, we use the Blow-up lemma to "blow up" M into a subgraph H' of H such that $|H'| \approx |H|$. One of the main difficulties in using the regularity method is handling the remaining vertices in $V(H) \setminus V(H')$. This requires additional techniques, but the key idea is to connect and insert the remaining pieces by using the restricted area conditions provided by the Blow-up lemma.

In order to characterize stability as well as provide tight bounds for transversal embedding problems, the author, together with Katherine Staden, developed regularity theory for transversals in a recent paper (Cheng & Staden, 2023). The main contribution of that paper is a transversal blow-up lemma. The full statement is somewhat involved and will not be introduced here; instead, it will be presented in full in Chapter 4. However, we provide a simplified bipartite

version as follows:

We say a graph H with n vertices is *μ-separable* if we can delete a set S of size at most μn from H such that each component in $H - S$ has size at most μn. Let $\varepsilon \ll \delta$, an (ε, δ) super-regular triple C, A, B is a 3-partite 3-uniform hypergraph H defined on (C, A, B) such that (1) for every $X \subseteq A, Y \subseteq B, Z \subseteq C$, the number of 3-edges inside (Z, X, Y) is at least $\delta|X||Y||Z|$; (2) for every vertex $v \in A(B$ or $C)$, the number of 3-edges containing v is at least $\delta|B||C|(\delta|A||C|$ or $\delta|A||B|)$. Given a 3-partite 3-uniform hypergraph H on (C, A, B), define G_H to be an edge-colored multigraph whose vertex sets are (A, B), and an edge $e = xy \in G_H$ with color c if and only if $c, x, y \in E(H)$.

The bipartite case of our blow-up lemma is as follows:

Theorem 1.4.1. *Let $0 < \mu \ll \varepsilon \ll \delta, \frac{1}{\Delta} < 1$, and let H be a 3-partite 3-uniform hypergraph defined on (C, A, B), where $\delta n \leq |C| \leq \Delta n$, such that H is (ε, δ)-super-regular. Let T be a graph defined on (A, B) with $\Delta(T) \leq \Delta$, $|E(T)| \leq |C|$, and T is μ-separable. Then, we can find a rainbow T inside G_H.*

There is no evidence showing that the separable condition in Theorem 1.4.1 is necessary. Thus the following conjecture for transversal blow-ups should hold:

Conjecture 1.4.2. *Let $0 < \varepsilon \ll \delta, \frac{1}{\Delta} < 1$, and let H be a 3-partite 3-uniform hypergraph defined on (C, A, B), where $\delta n \leq |C| \leq \Delta n$, such that H is (ε, δ)-super-regular. Let T be a graph defined on (A, B) with $\Delta(T) \leq \Delta$ and $|E(T)| \leq |C|$. Then, we can find a rainbow T inside G_H.*

If true, this conjecture would directly generalize the original blow-up lemma for bipartite graphs. By using this transversal blow-up lemma, we can adapt the traditional ideas of regularity and blow-up to dense graph families. First, we identify a graph family $\mathbf{G} = G_1, \ldots, G_{\gamma n}$ as a 3-uniform hypergraph $H_{\mathbf{G}}$, where $V(H_{\mathbf{G}}) = V(\mathbf{G})$ and $E(H_{\mathbf{G}}) = (i, u, v) \mid (u, v) \in G_i$. Next, we adapt the weak regularity lemma to $H_{\mathbf{G}}$, thus obtaining a regularity partition of $V(H_{\mathbf{G}})$ such that most triples are either almost empty or super-regular according to certain parameters.

Therefore, as in the common regularity method, whenever we find an appropriate subgraph inside the reduced graph, we can use the transversal blow-up lemma to "blow it up" and obtain a large rainbow subgraph. We can then use the restricted conditions to connect and insert pieces, making it a transversal graph as required. This idea is particularly useful for handling cases when the graph to be embedded is somewhat complex and the absorption method does

not work.

1.5 Summary of Results

In Chapter 2, we present a general framework for embedding transversal factors. One of the main results is the following asymptotic transversal Hajnal-Szemerédi theorem:

Theorem 1.5.1. *For any $\varepsilon > 0$, there exists an n_0 such that for $n \geq n_0$ and $r|n$, if $\boldsymbol{G} = G_1, \dots, G_t$ is a family of graphs defined on the same vertex set V where $|V| = n$, $t = \binom{r}{2}\frac{n}{r}$, and $\delta(G_i) \geq \left(1 - \frac{1}{r} + \varepsilon\right) n$ for $1 \leq i \leq t$, then $\boldsymbol{G}$ contains a transversal K_r-factor.*

In Chapter 3, we develop regularity theory for transversals. One of the key lemmas we prove is the transversal blow-up lemma. The full statement is long and involved, so we will postpone it to Chapter 3. To state our second main result, we introduce the following definitions:

A 3-uniform hypergraph H is a 1-expansion of a simple graph G if H is obtained by adding one vertex v_{xy} to each edge $xy \in E(G)$, where $v_e \neq v_{e'}$ if $e \neq e'$. Given $d, \eta > 0$, a (single) graph with a vertex set V of size n is (d, η)-*dense* if for all $A, B \subseteq V$, we have $e(G[A, B]) \geq d|A||B| - \eta n^2$.

As an application of the transversal blow-up lemma, we prove the following theorem, which generalizes the $k = 3$ case of the result in (Lenz, Mubayi, & Mycroft, 2016):

Theorem 1.5.2. *For all $\Delta, d, \alpha > 0$, there exist $\eta, \mu > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geq n_0$. Let G be a (d, η)-dense 3-graph on n vertices with $d_G(v) \geq \alpha n^2$ for all $v \in V(G)$. Then G contains a copy of the 1-expansion of any given μ-separable graph H with $\Delta(H) \leq \Delta$ and $|V(H)| + |E(H)| \leq n$.*

In Chapter 4, we provide a new proof of the following transversal Pósa-Seymour conjecture. Our proof is based on the transversal blow-up lemma, which is independent of that in (Gupta et al., 2023).

Theorem 1.5.3 ((Gupta et al., 2023))**.** *For all integers $k \geq 1$ and $\gamma > 0$, there exists n_0 such that the following holds for all integers $n \geq n_0$. Let $\boldsymbol{G} = (G_1, \dots, G_{(k-1)n})$ be a collection of graphs on a common vertex set of size n. Suppose that $\delta(G_i) \geq \left(\frac{k-1}{k} + \gamma\right) n$ for $1 \leq i \leq (k-1)n$. Then $\boldsymbol{G}$ contains a transversal copy of C_n^{k-1}.*

In Chapter 5, we prove a stability result for Aharoni's conjecture (Theorem 1.2.5). The proof combines the transversal absorption technique with the transversal blow-up lemma. Roughly

speaking, we show that if the graph collection $G_1, \dots, G_n$ defined on the same vertex set V, where $|V| = n$ and $\delta(G_i) \geq \left(\frac{1}{2} - o(1)\right) n$ for each $1 \leq i \leq n$, does not contain a transversal Hamilton cycle, then this collection must be close to one of a set of defined collections. The formal statement is postponed to Chapter 5.

1.6 Notation and Terminology

For reals x, a, b, we write $x = a \pm b$ if we have $a - b \leq x \leq a + b$. For any two constants $\alpha, \beta \in (0, 1)$, we write $\alpha \ll \beta$ if there exists a function $\alpha_0 = \alpha_0(\beta)$ such that the subsequent arguments hold for all $0 < \alpha \leq \alpha_0$. When we write multiple constants in a hierarchy, we mean that they are chosen from right to left. For any positive integer a, let $[a] := \{1, 2, \dots, a\}$. For any two positive integers $a \leq b$, let $[a, b] = \{a \leq x \leq b : x \in \mathbb{Z}\}$ and $[a] := [1, a]$. Given a set X and a positive integer b, write $\binom{X}{b}$ to denote the set of b-element subsets of X.

Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$ and let $v(G) := |V(G)|$ and $e(G) := |E(G)|$ be their sizes. Given any vertex $v \in V(G)$, let the *neighbourhood* $N_G(v)$ of v be the set of vertices that are incident to v and let $d_G(v) := |N_G(v)|$ be its *degree*. For any vertex set $U \subseteq V(G)$, let $G[U]$ be the induced graph of G on U, i.e. the graph with vertex set U and all the edges of G with both endpoints in U. Let $G - U := G[V(G) \setminus U]$ and $G - v := G - \{v\}$.

Let $k \geq 2$ be an integer and let $G = (V, E)$ be any k-graph, so each edge consists of k vertices. We use $V(G) := V$ to denote its vertex set and $E(G) := E$ to denote its edge set. Let $v(G) := |V(G)|$ and $e(G) := |E(G)|$ be their sizes. For any vertex $v \in V(G)$, let $N_G(v)$ be the *neighbourhood* of v, that is, the set of $(k-1)$-tuples of v that are incident to v and let $d_G(v) := |N_G(v)|$ be the *degree* of v. Sometimes we use $d(v)$ for short if G is obvious from the text. We write $\delta(G) := \min_{v \in V(G)} d_G(v)$ and $\Delta(G) := \max_{v \in V(G)} d_G(v)$ for the minimum and maximum degree of G. For each vertex $v \in V(G)$ and subset $S \subseteq \binom{V(G)}{k-1}$, let $N_G(v, S) := N_G(v) \cap S$ and $d_G(v, S) := |N_G(v, S)|$. For any vertex set $U \subseteq V(G)$, let $G[U]$ be the induced hypergraph of G on U, i.e. the hypergraph with vertex set U and all the edges of G with vertices only in U. Let $G - U := G[V(G) \setminus U]$. For a subhypergraph H of G, let $G \setminus H$ be the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$. For any (not necessarily disjoint) vertex sets $U_1, \dots, U_k \subseteq V(G)$, we write $G[U_1, \dots, U_k]$ for the k-graph with vertex set

$U_1 \cup \ldots \cup U_k$ and edge set $E_G(U_1, \ldots, U_k)$, which is the set of edges of G with one vertex in U_i for each $i \in [k]$. Let $e_G(U_1, \ldots, U_k)$ be the number of k-tuples $(x_1, \ldots, x_k) \in U_1 \times \ldots \times U_k$ for which $\{x_1, \ldots, x_k\} \in E_G(U_1, \ldots, U_k)$ (note that when these sets are not disjoint, edges may be counted more than once). When $k = 2$ and G is a simple graph. For each vertex $v \in V(G)$ and subset $S \subseteq V(G)$, let $N_G(v, S) = N_G(v) \cap S$ and $d_G(v, S) = |N_G(v, S)|$. Let $E_G(X, Y)$ be the set of edges with one endpoint in X and another in Y and let $e_G(X, Y) = |E_G(X, Y)|$. We write $E_G(X) := E_G(X, X) = E(G[X])$.

We say that a statement about a graph on n vertices holds *with high probability* if it holds with probability tending to 1 as $n \to \infty$. We use script letters (e.g. $\mathscr{C}$) to denote sets of colours and bold letters (e.g. $\boldsymbol{G}$) to denote graph collections. It will be convenient to represent a graph collection in three equivalent ways:

- a collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ of graphs on the same vertex set V. We call $\mathscr{C}$ the *colour set* of $\boldsymbol{G}$;
- an edge-coloured graph $\mathcal{G}$ on vertex set V with edge set $\bigcup_{c \in \mathscr{C}} G_c$, where xy has a multiset of colours consisting of those $c \in \mathscr{C}$ for which $xy \in G_c$. We say that $\mathcal{G}$ is the *edge-coloured graph of* $\boldsymbol{G}$ (but rarely use this representation);
- a 3-graph $G^{(3)}$ on vertex set $V \cup \mathscr{C}$ with edges xyc whenever $xy \in G_c$. We say that $G^{(3)}$ is the 3-*graph of* $\boldsymbol{G}$.

We can define the collection for hypergraphs similarly.

Given two 2-graphs H and G, a *graph homomorphism (from H to G)* is a map $\phi : V(H) \to V(G)$ such that $\phi(x)\phi(y) \in E(G)$ whenever $xy \in E(H)$. An injective graph homomorphism is an *embedding (of H into G)*. Given a graph H and a graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ on vertex set V, a *transversal embedding (of H into $\boldsymbol{G}$)* is a pair $\tau : V(H) \to V$ and $\sigma : E(H) \to \mathscr{C}$ of injective maps such that $\tau(x)\tau(y) \in E(G_{\sigma(xy)})$ whenever $xy \in E(H)$.

Given a graph H and a graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ on vertex set V, a *transversal embedding (of H into $\boldsymbol{G}$)* is a pair $\tau : V(H) \to V$ and $\sigma : E(H) \to \mathscr{C}$ of injective maps such that $\tau(x)\tau(y) \in E(G_{\sigma(xy)})$ whenever $xy \in E(H)$.

Given any collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ of graphs on a common vertex set V, we call $\mathscr{C}$ the *colour set* of $\boldsymbol{G}$. For any two sets $X, Y \subseteq V$, let $E_{\boldsymbol{G}}(X, Y)$ be the multiset of edges of any colour with one endpoint in X and another in Y and $e_{\boldsymbol{G}}(X, Y) = |E_{\boldsymbol{G}}(X, Y)| = \sum_{G \in \boldsymbol{G}} e_G(X, Y)$. Given any edge-coloured graph H with $V(H) \subseteq V$ with edge-colouring

$\sigma : E(H) \to \mathscr{C}$, we write $\mathrm{col}(H) := \bigcup_{e \in E(H)} \sigma(e)$ for the set of colours used on H.

Given a set X and positive integer k, we write

- $\binom{X}{k}$ for the set of all k-subsets of X,
- X^k for the set of all k-tuples of elements of X,
- $(X)_k$ for the set of all k-tuples of *distinct* elements of X.

We use script letters e.g. $\mathscr{C}, \mathscr{A}$ to denote sets of colours and bold letters e.g. $\boldsymbol{G}, \boldsymbol{J}$ to denote graph collections.

2 | A General Framework for Transversal Factor Problem

2.1 Introduction

2.1.1 Main results

In this chapter, we study transversal clique-factors under a few different contexts. Our first result is an asymptotical version of the transversal Hajnal–Szemerédi theorem.

Theorem 2.1.1. *For every $\varepsilon > 0$ and $t \in \mathbf{N}$, there exists $n_0 \in \mathbf{N}$ such that the following holds for all integers $n \geq n_0$ and $n \in t\mathbf{N}$. Let $m = \frac{n}{t}\binom{t}{2}$ and $\boldsymbol{G} = \{G_1, \ldots, G_m\}$ be an n-vertex graph collection. If $\delta(G_i) \geq (1 - \frac{1}{t} + \varepsilon)n$ for each i, then $\boldsymbol{G}$ contains a transversal K_t-factor.*

The minimum degree conditions are asymptotically best possible, as seen by setting all G_i to be identical and then referring to the optimality of Hajnal-Szemerédi theorem.

In fact, this naive construction serves as a simple lower bound for all "transversal" problems, certifying that the transversal version is at least "as hard as" the single host graph version (although the aforementioned transversal Mantel's theorem says that the transversal version can be strictly "harder").

It is natural to seek analogues of the Hajnal–Szemerédi theorem in the digraph and oriented graph settings, where we consider factors of directed cliques, namely, tournaments. We consider digraphs with no loops and at most one edge in each direction between every pair of vertices. Let T_k be the transitive tournament on k vertices, where a *transitive tournament* is an orientation of a complete graph D with the property that if xy and yz are arcs in D with $x \neq z$, then the arc xz is also in D. The minimum semi-degree $\delta^0(G)$ of a digraph G is the minimum of its minimum out-degree $\delta^+(G)$ and its minimum in-degree $\delta^-(G)$.

(Czygrinow, DeBiasio, Kierstead, & Molla, 2015) proved that every digraph G on n vertices with $\delta^+(G) \geq (1 - 1/k)n$ contains a perfect T_k-tiling for integers n, k with $k \mid n$. Let $\mathcal{T}_k$ be the family of all tournaments on k vertices, Treglown (Treglown, 2015) proved that given an integer $k \geq 3$, there exists an $n_0 \in \mathbf{N}$ such that the following holds. Suppose $T \in \mathcal{T}_k$, and D is a digraph on $n \geq n_0$ vertices where $k \mid n$. If $\delta^0(D) \geq (1 - 1/k)n$, then there exists a T-factor. For more results, see (Czygrinow, Kierstead, & Molla, 2014; Treglown, 2015). We prove the

following extensions of Theorem 2.1.1 for digraphs.

Theorem 2.1.2. *For every integer $k \geq 3$ and real $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that the following holds for all integers $n \geq n_0$ and $n \in k\mathbf{N}$. If $\boldsymbol{D} = \{D_1, \dots, D_m\}$, $m = \frac{n}{k}\binom{k}{2}$, is a collection of n-vertex digraphs on the same vertex set such that $\delta^+(D_i) \geq (1 - \frac{1}{k} + \varepsilon)n$, then $\boldsymbol{D}$ contains a transversal T_k-factor.*

Theorem 2.1.3. *For every integer $k \geq 3$, $T \in \mathcal{T}_k$ and real $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that the following holds for all integers $n \geq n_0$ and $n \in k\mathbf{N}$. If $\boldsymbol{D} = \{D_1, \dots, D_m\}$, $m = \frac{n}{k}\binom{k}{2}$, is a collection of n-vertex digraphs on the same vertex set such that $\delta^0(D_i) \geq (1 - \frac{1}{k} + \varepsilon)n$, then $\boldsymbol{D}$ contains a transversal T-factor.*

Note that each of these theorems implies Theorem 2.1.1 if we give a complete direction to each graph stated in Theorem 2.1.1.

We next discuss the partite setting. Suppose $V_1, \dots, V_k$ are disjoint vertex sets each of order n, and G is a k-partite graph on vertex classes $V_1, \dots, V_k$ (that is, G is a graph on the vertex set $V_1 \cup \dots \cup V_k$ such that no edge of G has both end vertices in the same class). We define the partite minimum degree of G, denoted by $\delta'(G)$, to be the largest m such that every vertex has at least m neighbours in each part other than its own, i.e.

$$\delta'(G) := \min_{i \in [k]} \min_{v \in V_i} \min_{j \in [k] \setminus \{i\}} |N(v) \cap V_j|,$$

where $N(v)$ denotes the neighbourhood of v. (Fischer, 1999) conjectured that if $\delta'(G) \geq (1 - 1/k)n$, then G has a K_k-factor. An approximate version of this conjecture assuming the degree condition $\delta'(G) \geq (1 - 1/k + o(1))n$ was proved independently by (Keevash & Mycroft, 2015a), and by (Lo & Markström, 2013) (a corrected exact version was given by (Keevash & Mycroft, 2015b), in fact, they obtain a more general result for r-partite graphs with $r \geq k$). We extend this approximate version to the transversal setting.

Theorem 2.1.4. *For every $\varepsilon > 0$ and integer k, there exists $n_0 \in \mathbf{N}$ such that the following holds for all integers $n \geq n_0$. If $\boldsymbol{G} = \{G_1, \dots, G_{n\binom{k}{2}}\}$ is a collection of k-partite graphs with a common partition $V_1, \dots, V_k$ each of size n such that $\delta'(G_i) \geq (1 - \frac{1}{k} + \varepsilon)n$, then $\boldsymbol{G}$ contains a transversal K_k-factor.*

A *matching* in H is a collection of vertex-disjoint edges of H. A *perfect matching* in H is a matching that covers all vertices of H. Given $d \in [k-1]$, the minimum d-degree of a k-graph H,

denoted by $\delta_d(H)$, is defined as the minimum of $d_H(S)$ over all d-sets S of $V(H)$ where $d_H(S)$ denotes the number of edges containing S. Another main result of this Chapter is to settle the transversal version of minimum d-degree-type results for perfect matchings in k-graphs for all $d \in [k-1]$, in a sense that the minimum d-degree condition which *forces a perfect matching in a single k-graph is essentially sufficient to force a transversal perfect matching in a k-graph collection*. Note that (Joos & Kim, 2020) proved that $\delta(G_i) \geq n/2$ guarantees a transversal perfect matching in an n-vertex graph collection.

It is well-known that perfect matchings are closely related to its fractional counterpart. Given a k-graph H, a *fractional matching* is a function $f : E(H) \to [0,1]$, subject to the requirement that $\sum_{e:v\in e} f(e) \leq 1$, for every $v \in V(H)$. Furthermore, if equality holds for every $v \in V(H)$, then we call the fractional matching *perfect*. Denote the maximum size of a fractional matching of H by $\nu^*(H) = \max_f \Sigma_{e\in E(H)} f(e)$. Let $c_{k,d}$ be the minimum d-degree threshold for perfect fractional matchings in k-graphs, namely, for every $\varepsilon > 0$ and sufficiently large $n \in \mathbf{N}$, every n-vertex k-graph H with $\delta_d(H) \geq (c_{k,d} + \epsilon)\binom{n-d}{k-d}$ contains a perfect fractional matching. It is known that (Alon et al., 2012) every n-vertex k-graph H with $\delta_d(H) \geq (\max\{c_{k,d}, 1/2\} + o(1))\binom{n-d}{k-d}$ has a perfect matching, and this condition is asymptotically best possible. However, determining the parameter $c_{k,d}$ is a major open problem in this field and we refer to (Frankl & Kupavskii, 2018) for related results and discussions.

Theorem 2.1.5. *For every $\varepsilon > 0$ and integer $d \in [k-1]$, there exists $n_0 \in \mathbf{N}$, such that the following holds for all integers $n \geq n_0$ and $n \in k\mathbf{N}$. Every n-vertex k-graph collection $\boldsymbol{G} = \{G_1, \ldots, G_{\frac{n}{k}}\}$ with $\delta_d(G_i) \geq (\max\{c_{k,d}, \frac{1}{2}\} + \varepsilon)\binom{n-d}{k-d}$ for each i contains a transversal perfect matching.*

2.2 A general framework for transversal F-factors

The naive idea of our proof is to build a rainbow absorption structure and a rainbow almost F-factor. Moreover, the rainbow absorption structure must be able to deal with (i.e., absorb) an arbitrary leftover of vertices, as well as *a leftover of colors*.

To state our general theorem for transversal F-factors, we need some general notation that captures all of our contexts. We shall consider a *directed k-graph* (Dk-graph) H, with edge set $E(H) \subseteq \binom{V(H)}{k} \times \{+,-\}$, that is, each edge consists of k vertices and a direction taken from

$\{+,-\}$. This way a directed (2-)graph can be recognized as a graph with an *ordered* vertex set, and edges following (or against) the order of the enumeration are oriented by $+$ (or $-$).

Given a Dk-graph $H=(V,E)$, for $E'\subseteq E$, we write $H[E']$ for the subgraph of H with edge set E' and vertex set $\cup_{e\in E'}E'$. For $V'\subseteq V$, if $H'\subseteq H$ contains all edges of H with vertices in V', then H' is an *induced subgraph* of H. Denote it by $H[V']$. If there is an F-tiling in H whose vertex set is V', then we say that V' spans an F-tiling. Given another Dk-graph $H_1=(V_1,E_1)$, we set $H\cup H_1:=(V\cup V_1, E\cup E_1)$.

Given a Dk-graph F with b vertices and f edges, a Dk-graph collection $\boldsymbol{G}=\{G_1,\dots,G_{\frac{n}{b}f}\}$ on vertex V and a subset $V'\subseteq V$. Let $\boldsymbol{G}[V']=\{G_1[V'],\dots,G_{\frac{n}{b}f}[V']\}$ be the *induced* Dk-graph collection on V'. If $|V'|\in b\mathbf{N}$ and there exists a rainbow perfect F-tiling inside $\boldsymbol{G}[V']$ whose color set is $C\subseteq[nf/b]$, then we say that V' *spans* a rainbow F-tiling in $\boldsymbol{G}$ with color set C. Let A_1 and A_2 be two rainbow Dk-graphs in $\boldsymbol{G}$ with color set C_1 and C_2 respectively, we set $A_1\cup A_2$ to be a Dk-graph with vertex set $V(A_1)\cup V(A_2)$, edge set $E(A_1)\cup E(A_2)$ and color set $C_1\cup C_2$. The following general minimum degree condition captures all of our contexts.

Definition 2.2.1. *Let H be a Dk-graph and $d\in[k-1]$, a minimum d-degree $\delta_d^*(H)$ corresponding to a certain (implicit) degree rule can be defined as follows. There exists $\ell\in\mathbf{N}$ such that for any d-set $S\subseteq V(H)$, let*

$$N^*(S):=\{E_1,\dots,E_\ell\},$$

where E_i is a set of edges of H that each contains S as a subset. Let $\deg^*(S):=\min_{i\in[\ell]}|E_i|$ and $\delta_d^*(H):=\min_{S\subseteq\binom{V}{d}}\deg^*(S)$.

We list all the instances of minimum degrees used in this Chapter.

- When H is a k-graph, we can take $\ell=1$ and E_1 as all edges containing S, thus $\deg^*(S)=|E_1|$ and $\delta_d^*(H)$ represents the standard minimum d-degree of H.
- When H is a k-partite 2-graph, we can take $\ell=k-1$ where E_i consists of the edges from $S=\{v\}$ to V_i for $i\in[k]\backslash\{j:v\in V_j\}$, thus $\deg^*(S)=\min\{|E_i|:i\in[k]\backslash\{j:v\in V_j\}\}$ and $\delta_1^*(H)$ represents the minimum partite degree of H.
- When H is a directed graph and $S=\{v\}$ for a given vertex v, we can take $\ell=1$ and take E_1 as the sets of out(in)-edges, thus, $\deg^*(S)=|E_1|$ and $\delta_1^*(H)$ represents the minimum out(in)-degree of H.

- When H is a directed graph and $S = \{v\}$ for a given vertex v, we can take $\ell = 2$ and take E_1, E_2 as the sets of in-edges and out-edges respectively, thus, $\deg^*(S) = \min\{|E_1|, |E_2|\}$ and $\delta_1^*(H)$ represents the minimum semi-degree of H.

Throughout the rest of this Chapter, let F be a Dk-graph with b vertices and f edges. We first define an absorber without colors. Given a set B of b vertices, a Dk-graph $A^0 = A_1^0 \cup A_2^0$ is called an *F-absorber for B* if

- $V(A^0) = B \dot\cup L$[1],
- A_1^0 is an F-factor on L and A_2^0 is an F-factor on $B \cup L$.

Note that $|V(A^0)|$ is always a constant. Naturally, we give the definition of rainbow F-absorber as follows.

Definition 2.2.2 (Rainbow F-absorber)**.** *Let $\boldsymbol{G} = \{G_1, \dots, G_{nf/b}\}$ be a Dk-graph collection on V and F be a Dk-graph with b vertices and f edges. For every b-set B in V and every f-set C in $[nf/b]$, $A = A_1 \cup A_2$ is called a rainbow F-absorber for (B, C) if*

- $V(A) = B \dot\cup L$,
- *A_1 is a rainbow F-factor on L with color set C_1 and A_2 is a rainbow F-factor on $B \cup L$ with color set $C_1 \cup C$.*

A rainbow F-absorber for (B, C) works in the following way: A_1 is a rainbow F-tiling with color set C_1, thus, if the vertices of B and the colors C are available, then we can switch to A_2 and get a larger rainbow F-tiling. For example, Let $\boldsymbol{G} = \{G_1, \dots, G_n\}$ be a graph collection on n-vertex set V. For any $B = \{v_1, v_2, v_3\}$ and $C = \{i, j, k\}$, we construct a rainbow triangle-absorber A for (B, C) where $V(A) = B \cup \{v_4, \dots, v_{12}\}$. $\{v_4v_7v_8, v_5v_9v_{10}, v_6v_{11}v_{12}\}$ is a family of rainbow triangles with color set C_1, which serves as A_1. $\{v_1v_7v_8, v_2v_9v_{10}, v_3v_{11}v_{12}, v_4v_5v_6\}$ is a family of rainbow triangles with color set $C_1 \cup \{i, j, k\}$, which serves as A_2.

Now we introduce one of the main parameters $c_{d,F}^{\mathrm{abs},*}$. Roughly speaking, it is the minimum degree threshold such that all b-sets are contained in many rainbow F-absorbers.

Definition 2.2.3 ($c_{d,F}^{\mathrm{abs},*}$: Rainbow absorption threshold)**.** *Fix an F-absorber $A^0 = A_1^0 \cup A_2^0$ and let m be the number of vertex disjoint copies of F in A_2^0. Let $c_{d,F,A^0} \in (0, 1)$ be the infimum of reals $c > 0$ such that for every $\epsilon > 0$ there exists $\epsilon' > 0$ such that the following holds for sufficiently large $n \in \mathbf{N}$ where $d \in [k-1]$. Let $\boldsymbol{G} = \{G_1, \dots, G_{nf/b}\}$ be an n-vertex Dk-graph collection on V. If $\delta_d^*(G_i) \geq (c + \epsilon)\binom{n-d}{k-d}$ for $i \in [nf/b]$, then for every b-set B in V and every*

[1] As usual, $A \dot\cup B$ denotes the disjoint union of A and B.

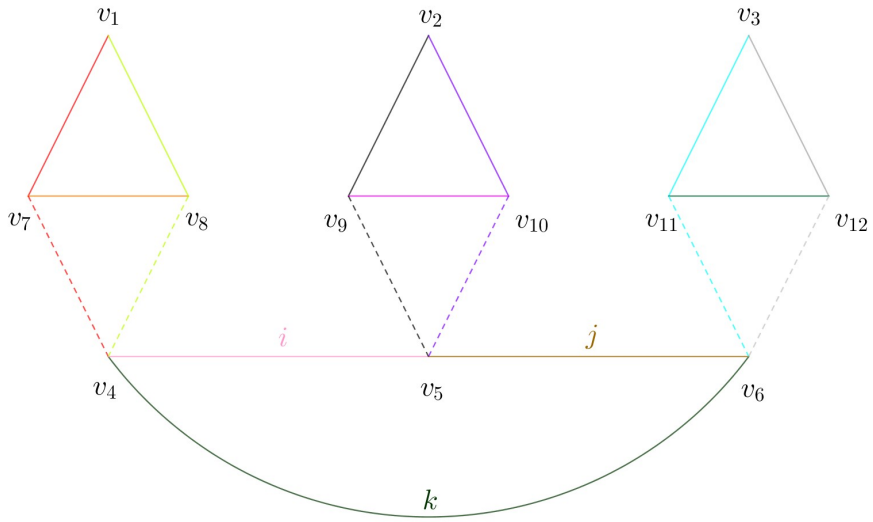


Figure 2.1: A rainbow triangle-absorber A for (B, C)

f-set C in $[nf/b]$ with the form $[(i-1)f, if]$ for some $i \in [n/b]$, there are at least $\epsilon' n^{(m-1)(b+1)}$ rainbow F-absorbers A with color set $C(A_1^0) \cup C$ whose underlying graph is isomorphic to A^0 such that $C(A_1^0) = [(i_1-1)f+1, i_1 f] \cup [(i_2-1)f+1, i_2 f] \cup \cdots \cup [(i_{m-1}-1)f+1, i_{m-1} f]$ where $i_j \in [n/b]$ for each $j \in [m-1]$ and $i_{j_1} \neq i_{j_2}$ for distinct $j_1, j_2 \in [m-1]$. Let $c_{d,F}^{\mathrm{abs},*} := \inf c_{d,F,A^0}$ where the infimum is over all F-absorbers A^0.

We next define a threshold parameter for the rainbow almost F-factor in a similar fashion. We use the following auxiliary b-graph H_F. Given a Dk-graph F with b vertices and f edges, and an n-vertex Dk-graph collection $\boldsymbol{H} = \{H_1, \ldots, H_f\}$ on V, let H_F be the (undirected) b-graph with vertex set $V(H_F) = V$ and edge set $E(H_F) = \{V(F') : F' \text{ is a rainbow copy of } F \text{ with color set } [f]\}$.

Definition 2.2.4 ($c_{d,F}^{\mathrm{cov},*}$: Rainbow almost F-factor threshold)**.** *Let $c_{d,F}^{\mathrm{cov},*} \in (0,1)$ be the infimum of reals $c > 0$ such that for every $\epsilon > 0$, the following holds for sufficiently large $n \in \mathbf{N}$. Let $\boldsymbol{H} = \{H_1, \ldots, H_f\}$ be an n-vertex Dk-graph collection. If $\delta_d^*(H_i) \geq (c+\epsilon)\binom{n-d}{k-d}$ for every $i \in [f]$, then the b-graph H_F has a perfect fractional matching.*

The property of "having a perfect fractional matching" is required in H_F, which is a single host graph. This definition (and our proofs supporting it) establishes a close relation between the rainbow F-factor problem and the classical F-factor problem with no colors.

Now we are ready to state our general result on transversal F-factors.

Theorem 2.2.5. *Let F be a Dk-graph with b vertices and f edges. For any $\epsilon > 0$ and integer $d \in [k-1]$, the following holds for sufficiently large $n \in b\mathbf{N}$. Let $\boldsymbol{G} = \{G_1, \ldots, G_{nf/b}\}$ be an*

n-vertex Dk-graph collection on V. If $\delta_d^*(G_i) \geq (\max\{c_{d,F}^{\mathrm{abs},*}, c_{d,F}^{\mathrm{cov},*}\} + \epsilon)\binom{n-d}{k-d}$ for $i \in [nf/b]$, then $\boldsymbol{G}$ contains a transversal F-factor.

Theorem 2.2.5 reduces the transversal F-factor problem to two subproblems, namely, the enumeration of rainbow F-absorbers and the study of perfect fractional matchings in H_F. In our proofs of Theorems 2.1.1–2.1.5, the first subproblem is done by greedy constructions of the K_t-absorbers with the minimum degree condition. Note that the second subproblem is trivial by definition for Theorem 2.1.5. For Theorems 2.1.1–2.1.4, we achieve it by converting the problem to the setting of complexes (downward-closed hypergraphs) and then applying a result of (Keevash & Mycroft, 2015a) on perfect fractional matchings, which is a nice application of the Farkas' Lemma for linear programming.

To conclude, we remark that the main benefit from Theorem 2.2.5 is that both of these two subproblems only concern *finitely many* colors. From this aspect, Theorem 2.2.5 irons out significant difficulties on the transversal spanning structure problem due to an unbound number of colors.

2.3 Preliminary

For a hypergraph H, the 2-degree of a pair of vertices is the number of edges containing this pair and $\Delta_2(H)$ denotes the maximum 2-degree in H. For reals a, b and c, we write $a = (1 \pm b)c$ for $(1-b)c \leq a \leq (1+b)c$. We need the following result which was attributed to (Pippenger & Spencer, 1989)(see Theorem 4.7.1 in (Alon & Spencer, 2008)), following Frankl and Rödl. A *cover* in a hypergraph H is a set of edges such that each vertex of H is in at least one edge of the set.

Lemma 2.3.1 ((Pippenger & Spencer, 1989))**.** *For every integer $k \geq 2$, $r \geq 1$ and $a > 0$, there exist $\gamma = \gamma(k, r, a) > 0$ and $d_0 = d_0(d, r, a)$ such that the following holds for every $n \in \mathbf{N}$ and $D \geq d_0$. Every k-graph $H = (V, E)$ on V of n vertices in which all vertices have positive degrees and which satisfies the following conditions:*

- *For all vertices $x \in V$ but at most γn of them, $d_H(x) = (1 \pm \gamma)D$.*
- *For all $x \in V$, $d_H(x) < rD$.*
- *$\Delta_2(H) < \gamma D$.*

contains a cover of at most $(1+a)(n/k)$ edges.

The following well-known concentration results, i.e. Chernoff bounds, can be found in Appendix A in (Alon & Spencer, 2008) and Theorem 2.8, inequality (2.9) and (2.11) in (Janson, Łuczak, & Ruciński, 2000). Denote a binomial random variable with parameters n and p by $Bi(n,p)$.

Lemma 2.3.2 (Chernoff Inequality for small deviation)**.** *If $X = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are mutually independent random variables, each X_i has Bernoulli distribution with expectation p_i and $\alpha \leq 3/2$, then*

$$\mathbf{P}[|X - \mathbf{E}[X]| \geq \alpha \mathbf{E}[X]] \leq 2e^{-\frac{\alpha^2}{3}\mathbf{E}[X]}.$$

In particular, when $X \sim Bi(n,p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbf{P}[|X - np| \geq \lambda] \leq e^{-\Omega(\lambda^2/(np))}.$$

Lemma 2.3.3 (Chernoff Inequality for large deviation)**.** *If $X = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are mutually independent random variables, each random variable X_i has Bernoulli distribution with expectation p_i and $x \geq 7\mathbf{E}[X]$, then*

$$\mathbf{P}[X \geq x] \leq e^{-x}.$$

We also need the Janson's inequality to provide an exponential upper bound for the lower tail of a sum of dependent zero-one random variables.

Lemma 2.3.4 (Theorem 8.7.2 in (Alon & Spencer, 2008))**.** *Let Γ be a finite set and $p_i \in [0,1]$ be a real for $i \in \Gamma$. Let Γ_p be a random subset of Γ such that the elements are chosen independently with $\mathbf{P}[i \in \Gamma_p] = p_i$ for $i \in \Gamma$. Let M be a family of subsets of Γ. For every $A_i \in M$, let $I_{A_i} = 1$ if $A_i \subseteq \Gamma_p$ and 0 otherwise. Let B_i be the event that $A_i \subseteq \Gamma_p$. For $A_i, A_j \in M$, we write $i \sim j$ if B_i and B_j are not pairwise independent, in other words, $A_i \cap A_j \neq \emptyset$. Define $X = \Sigma_{A_i \in M} I_{A_i}$, $\lambda = \mathbf{E}[X]$, $\Delta = \sum_{i \sim j} \mathbf{P}[B_i \wedge B_j]$, then*

$$\mathbf{P}[X \leq (1-\gamma)\lambda] < e^{-\gamma^2 \lambda/[2+(\Delta/\lambda)]}.$$

2.4 Transversal Absorption Technique

2.4.1 transversal Absorption Lemma

In this section we prove a transversal version of the absorption lemma via probabilistic method. The only difference is that we use the following auxiliary hypergraph which makes it applicable to the transversal setting.

Definition 2.4.1. *We call a hypergraph H a $(1,b)$-graph, if $V(H)$ can be partitioned into $A\cup B$ and $E(H)$ is a family of $(1+b)$-sets each of which contains exactly one vertex in A and b vertices in B.*

For a $(1,b)$-graph H with partition $A\dot\cup B$, a $(1,d)$-subset D of $V(H)$ is a $(d+1)$-tuple where $|D\cap A|=1$ and $|D\cap B|=d$. A $(1,b)$-graph H with partition classes A,B is *balanced* if $b|A|=|B|$. We say that a set $S\subseteq V(H)$ is balanced if $b|S\cap A|=|S\cap B|$.

Given an n-vertex Dk-graph collection $\boldsymbol{G}=\{G_1,\dots,G_{nf/b}\}$ on V, we first construct a sequence of hypergraphs $H'_{F_1},\dots,H'_{F_{n/b}}$, each of which is a b-graph with vertex set $V(H'_{F_i})=V$ and edge set $E(H'_{F_i})=\{e\in\binom{V}{b}: e \text{ spans a rainbow copy of } F \text{ with color set } I_i=[(i-1)f+1,if]\}$. We define an auxiliary $(1,b)$-graph $H_{\boldsymbol{G}}$ of $\boldsymbol{G}$ as follows.

Definition 2.4.2. *Let $H_{\boldsymbol{G}}$ be an auxiliary $(1,b)$-graph of $\boldsymbol{G}$ with vertex set $V'=[n/b]\cup V$ and edge set $\{\{i\}\cup e: i\in[n/b], e\in H'_{F_i}\}$.*

For any edge $e\in E(H_{\boldsymbol{G}})$, If $A\subseteq V(H_{\boldsymbol{G}})$ and $|A|$ is divisible by $b+1$, then $A\in\binom{(b+1)n}{a}$ is an *absorber* for e if $e\subseteq A$, there is a perfect matching in $H_{\boldsymbol{G}}[A]$ and there is a perfect matching in $H_{\boldsymbol{G}}[A\setminus e]$. Let $\mathcal{L}(e)$ denote the set of absorbers for e in $H_{\boldsymbol{G}}$.

Lemma 2.4.3 (transversal Absorption Lemma)**.** *Let F be a Dk-graph with b vertices and f edges and A^0 be a rainbow F-absorber. The maximum vertex-disjoint copies of F of A^0 is m. For every $\varepsilon>0$, there exist γ,γ_1 and n_0 such that the following holds for all integers $n\geq n_0$. Suppose that $\boldsymbol{G}=\{G_1,\dots,G_{\frac{n}{b}f}\}$ is an n-vertex Dk-graph collection on V and $\delta^*_d(G_i)\geq(c^{\mathrm{abs},*}_{d,F}+\varepsilon)\binom{n-d}{k-d}$ and $H_{\boldsymbol{G}}$ is the auxiliary $(1,b)$-graph of $\boldsymbol{G}$, then there exists a matching M in $H_{\boldsymbol{G}}$ with size at most $2\gamma(m-1)n$ such that for every balanced set $U\subseteq([n/b]\cup V)\setminus V(M)$ of size at most $\gamma_1 n$, $V(M)\cup U$ spans a matching in $H_{\boldsymbol{G}}$.*

Proof. Let $1/n\ll\gamma_1\ll\alpha\ll\gamma\ll\varepsilon'\ll\varepsilon$. Note that a matching of size m in $H_{\boldsymbol{G}}$ corresponds to a rainbow F-absorber in $\boldsymbol{G}$. Choose a family $\mathcal{F}$ of matchings of size $m-1$ from $H_{\boldsymbol{G}}$ by

including each matching of size $m-1$ independently at random with probability

$$p = \gamma/n^{(m-1)(b+1)-1}.$$

Note that $|\mathcal{F}|, |\mathcal{L}(e) \cap \mathcal{F}|$ are binomial random variables with expectations

$$\mathbf{E}|\mathcal{F}| \leq \gamma n \text{ and}$$

$$\mathbf{E}|\mathcal{L}(e) \cap \mathcal{F}| \geq \gamma\varepsilon' n \text{ for any } e \in E(H_{\boldsymbol{G}}).$$

The latter inequality holds since for any edge e of $H_{\boldsymbol{G}}$, $|\mathcal{L}(e)| \geq \varepsilon' n^{(m-1)(b+1)}$ by the minimum degree assumption and Definition 2.2.3. By Lemma 2.3.2, with probability $1-o(1)$, the family $\mathcal{F}$ satisfies the following properties.

i. $|\mathcal{F}| \leq 2\mathbf{E}|\mathcal{F}| \leq 2\gamma n$,

ii. $|\mathcal{L}(e) \cap \mathcal{F}| \geq \frac{1}{2}\mathbf{E}|\mathcal{L}(e) \cap \mathcal{F}| \geq \frac{1}{2}\gamma\varepsilon' n$ for any $e \in E(H_{\boldsymbol{G}})$.

Moreover, we can also bound the expected number of pairs of intersecting members of $\mathcal{F}$ by

$$n^{(m-1)(b+1)}(m-1)^2(b+1)^2 n^{(m-1)(b+1)-1}p^2 \leq \frac{1}{8}\gamma\varepsilon' n.$$

Thus, by Markov's inequality, we derive that with probability at least $1/2$, $\mathcal{F}$ contains at most $\frac{1}{4}\gamma\varepsilon' n$ pairs of intersecting members of $\mathcal{F}$. Remove one member from each of the intersecting pairs in $\mathcal{F}$. Thus, the resulting family, say $\mathcal{F}'$, consists of pairwise disjoint matchings of size $m-1$ that satisfies

i. $|\mathcal{F}'| \leq 2\gamma n$,

ii. $|\mathcal{L}(e) \cap \mathcal{F}| \geq \frac{1}{2}\gamma\varepsilon' n - \frac{1}{4}\gamma\varepsilon' n \geq \alpha n$ for any $e \in E(H_{\boldsymbol{G}})$.

Therefore, the union of members in $\mathcal{F}'$ is a matching in $H_{\boldsymbol{G}}$ of size at most $2\gamma(m-1)n$ and can (greedily) absorb a balanced set U of size at most $\gamma_1 n$ since $\gamma_1 \ll \alpha$. □

2.4.2 Enumeration of Rainbow Absorbers

In this section, we give two examples of rainbow absorbers. The first one is as follows. Recall that T_k is the transitive tournament on k vertices.

2.4.2.1 Rainbow T_k-absorber

For the proof of Theorem 2.1.2, we show that

$$c^{\mathrm{abs},+}_{1,T_k} \leq 1 - \frac{1}{k}. \tag{2.1}$$

For any k-set $S = \{u_1, u_2, \ldots, u_k\}$ in V and every $\binom{k}{2}$-set $C = [(j-1)\binom{k}{2} + 1, j\binom{k}{2}]$ where $j \in [\frac{n}{k}]$, we define a rainbow T_k-absorber A for (S, C) as follows.

- $A_1 = A - S = \{T_k^1, \ldots, T_k^k\}$ is a rainbow T_k-tiling with $C(T_k^i) = [(j_i - 1)\binom{k}{2} + 1, j_i\binom{k}{2}]$ where $j_i \in [n/k]$ for $i \in [k]$ and $A_2 = \{T_k^0, T_k^{1'}, \ldots, T_k^{k'}\}$ is a rainbow T_k-tiling with $C(T_k^0) = C$ and $C(T_k^{i'}) = [(j_i - 1)\binom{k}{2} + 1, j_i\binom{k}{2}]$ where $j_i \in [n/k]$ for $i \in [k]$.
- We can choose a rainbow T_k^0 that is isomorphic to T_k with color set C such that $V(T_k^0) = \{v_1, v_2, \ldots, v_k\}$ where $v_i \in V(T_k^i)$ and $V(T_k^{i'}) = V(T_k^i) \backslash \{v_i\} \cup \{u_i\}, i \in [k]$.
- $c(u_i x) = c(v_i x)$ for each $x \in V(A_1 \backslash V(T_k^0))$, and $c(xy)$ in T_k^i is the same as $c(xy)$ in $T_k^{i'}$ for $i \in [k]$ and $x, y \in V(A_1 \backslash V(T_k^0))$.

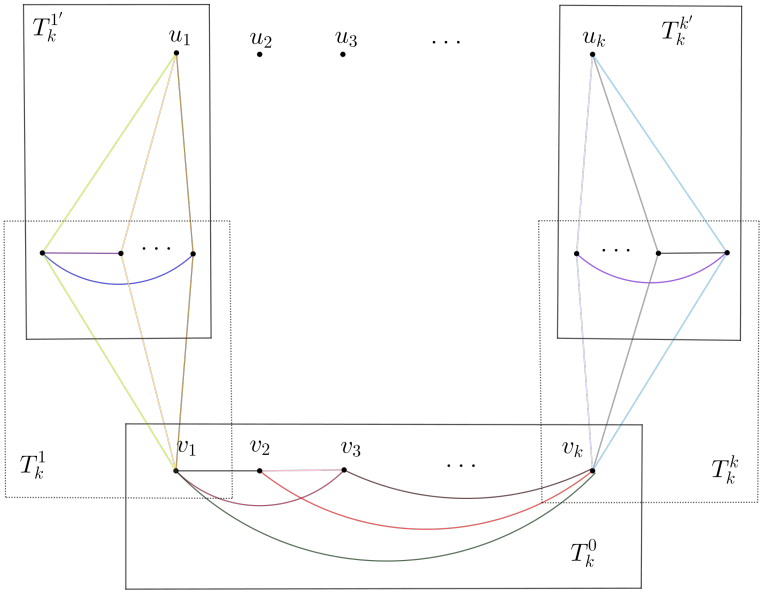


Figure 2.2: Illustration of the rainbow absorbers (with directions omitted)

Suppose $\delta(D_i^+) \geq (1 - 1/k + \varepsilon)n$ for $i \in [\frac{n}{k}\binom{k}{2}]$. For any k-set S in V and every $\binom{k}{2}$-set $C = [(j-1)\binom{k}{2} + 1, j\binom{k}{2}]$ where $j \in [\frac{n}{k}]$, we denote the family of rainbow T_k-absorbers for (S, C) by $\mathcal{A}(S, C)$.

Claim 2.4.4. *For any k-set $S = \{u_1, u_2, \ldots, u_k\}$ in V and every $\binom{k}{2}$-set $C = [(j-1)\binom{k}{2} +$*

$1, j\binom{k}{2}]$ *where* $j \in [\frac{n}{k}]$, *we have* $|\mathcal{A}(S, C)| \geq \varepsilon^{k^2+k} n^{k^2+k}$.

Proof. Fixing a k-set $S = \{u_1, u_2, \ldots, u_k\}$ in V and a $\binom{k}{2}$-set $C = [(j-1)\binom{k}{2} + 1, j\binom{k}{2}]$ for some $j \in [\frac{n}{k}]$, we construct rainbow absorbers for (S, C). We choose $[(j_i - 1)\binom{k}{2} + 1, j_i\binom{k}{2}]$ for $i \in [k]$ arbitrarily, there are $(\frac{n}{k} - 1)(\frac{n}{k} - 2) \cdots (\frac{n}{k} - k) \geq \varepsilon^k n^k$ choices. Next, we choose a rainbow T_k^0 with color set C. Due to the minimum out-degree of D_i, the number of choices for T_k^0 is at least

$$(n-k)\left((1-\frac{1}{k}+\varepsilon)n - (k+1)\right) \cdots \left(\frac{1}{k} + (k-1)\varepsilon n - (2k-1)\right) \geq \varepsilon^k n^k.$$

Now we fix one such $U = \{v_1, v_2, \ldots, v_k\}$. For each $i \in [k]$ and each pair $\{u_i, v_i\}$, suppose we succeed in choosing a set S_i such that S_i is disjoint to $W_{i-1} = \cup_{j\in[i-1]} S_j \cup S \cup U$, then $V(T_k^{i'}) = S_i \cup \{u_i\}$ spans a rainbow T_k in $\boldsymbol{D}$ with color set $[(j_i - 1)\binom{k}{2} + 1, j_i\binom{k}{2}]$ while so does $V(T_k^i) = S_i \cup \{v_i\}$.

For the first vertex in S_1, the number of choices is at least $(1 - \frac{2}{k} + 2\varepsilon)n - 2k$, for the last vertex in S_1, the number of choices is in $k\varepsilon n - (3k-1)$. Since $\frac{1}{n} \ll \varepsilon$, the number of choices for S_1 is at least

$$\left((1 - \frac{2}{k} + 2\varepsilon)n - 2k\right) \cdots (k\varepsilon n - (3k-1)) \geq \varepsilon^{k-1} n^{k-1}.$$

Similarly, the minimum out-degree implies that for $i \in [2, k]$ there are at least $\varepsilon^{k-1} n^{k-1}$ choices for S_i and in total we obtain $\varepsilon^{k^2+k} n^{k^2+k}$ rainbow T_k-absorbers for S. □

2.4.2.2 Rainbow edge-absorber

For the proof of Theorem 2.1.5, we show that

$$c_{d,F}^{\mathrm{abs,d}} \leq \frac{1}{2}. \tag{2.2}$$

Given an n-vertex k-graph collection $\boldsymbol{G}$ on V with $\delta_d(G_i) \geq (\frac{1}{2} + \varepsilon)\binom{n-d}{k-d}$ for $i \in [n/k]$, we first construct a $(1, k)$-graph $H_{\boldsymbol{G}}$ with vertex set $[n/k] \cup V$ and edge set $\{\{i\} \cup e : e \in H_i, i \in [n/k]\}$. Next, we construct a specific rainbow edge-absorber. For any k-set $T = \{v_1, \ldots, v_k\}$ in V and every color $c_1 \in [n/k]$, we give a rainbow absorber $A = A_1 \cup A_2$ for (T, c_1) as follows.

- $A_1 = \{M_2, \ldots, M_k\}$ is a set of $k-1$ disjoint edges in $H_{\boldsymbol{G}}$ where $c_i \in M_i (i \in [2, k])$.
- There is a vertex $u_i (i \in [2, k])$ from each $V(M_i)$ such that $\{u_2, \ldots, u_k, v_1, c_1\} \in E(H_{\boldsymbol{G}})$

and $(V(M_i)\setminus\{u_i\})\cup\{v_i\} \in E(H_{\boldsymbol{G}})$ for $i \in [2,k]$. Let A_2 be $\{\{u_2,\dots,u_k,v_1,c_1\}, (V(M_2)\setminus\{u_2\})\cup\{v_2\},\dots,(V(M_k)\setminus\{u_k\})\cup\{v_k\}\}$.

For any k-set T in V and every color $c_1 \in [n/k]$, we denote the family of such rainbow edge-absorbers for (T,c_1) by $\mathcal{A}(T,c_1)$.

Claim 2.4.5. $|\mathcal{A}(T,c_1)| \geq \varepsilon^{2k-2}n^{k-1}\binom{n-1}{k-1}^k/2$.

Proof. Fix $c_1 \in [n/k]$ and $T=\{v_1,\dots,v_k\}\subseteq V$. Choose $(c_2,\dots,c_k)$ arbitrarily from $[n/k]$ and there are at least $(\frac{n}{k}-1)\cdots(\frac{n}{k}-(k-1)) \geq \varepsilon^{k-1}n^{k-1}$ choices. Fix such $(c_2,\dots,c_k)$. Next, we construct $M_2,\dots,M_k$ and note that there are at most $(k-1)\binom{n-1}{k-2} \leq \varepsilon\binom{n-1}{k-1}$ edges which contain c_1,v_1 and v_j for some $j\in[2,k]$. Due to the minimum degree assumption, there are at least $\frac{1}{2}\binom{n-1}{k-1}$ edges containing v_1 and c_1 but none of $v_2,\dots,v_k$. We fix such one edge $\{c_1,v_1,u_2,\dots,u_k\}$ and set $U_1=\{u_2,\dots,u_k\}$. For each $i\in[2,k]$ and each pair $\{u_i,v_i\}$, suppose we succeed in choosing a set U_i such that U_i is disjoint with $W_{i-1}=\cup_{j\in[i-1]}U_j\cup T$ and both $U_i\cup\{u_i,c_i\}$ and $U_i\cup\{v_i,c_i\}$ are edges in $\tilde{H}$, then for a fixed $i\in[2,k]$, we call such a choice U_i good.

Note that in each step $i\in[2,k]$, there are $k+(i-1)(k-1)\leq k^2$ vertices in W_{i-1}, thus the number of edges with color c_i intersecting u_i and at least one other vertex in W_{i-1} is at most $k^2\binom{n-1}{k-2}$. So the minimum degree assumption implies that for each $i\in[2,k]$, there are at least $2\varepsilon\binom{n-1}{k-1}-2k^2\binom{n-1}{k-2}\geq\varepsilon\binom{n-1}{k-1}$ choices for U_i and in total we obtain $\varepsilon^{2k-2}n^{k-1}\binom{n-1}{k-1}^k/2$ rainbow absorbers for (T,c_1). □

2.5 Rainbow Almost Cover

The goal of this section is to prove the following lemma, an important component of the proof of Theorem 2.2.5.

Lemma 2.5.1 (Rainbow Almost Cover Lemma)**.** *Let F be a D_k-graph with b vertices and f edges. For every $\varepsilon,\phi>0$ and integer $d\in[k-1]$, the following holds for sufficiently large $n\in b\mathbf{N}$. Suppose that $\boldsymbol{G}=\{G_1,\dots,G_{nf/b}\}$ is an n-vertex D_k-graph collection on V such that $\delta_d^*(G_i)\geq(c_{d,F}^{\mathrm{cov},*}+\varepsilon)\binom{n-d}{k-d}$ for $i\in[nf/b]$, then $\boldsymbol{G}$ contains a rainbow F-tiling covering all but at most ϕn vertices.*

For a k-graph H, a *fractional cover* is a function $\omega: V(H)\to[0,1]$, subject to the requirement $\sum_{v:v\in e}\omega(v)\geq 1$ for every $e\in E(H)$. Denote the minimum fractional cover size by $\tau^*(H)=$

$\min_\omega \Sigma_{v\in V(H)}\omega(v)$. The conclusion $\nu^*(H) = \tau^*(H)$ for any hypergraph follows from the LP-duality. For n-vertex k-graphs we trivially have $\nu^*(H) = \tau^*(H) \leq \frac{n}{k}$.

We construct another $(1, b)$-graph $\tilde{H}$ on $[\frac{nf}{b}] \cup V$ with edge set $E(\tilde{H}) = \{\{i\} \cup e : e \in H_i$ for all $i \in [nf/b]\}$. A $(1, k-1)$-subset S of $V(\tilde{H})$ contains one vertex in $[\frac{nf}{b}]$ and $k-1$ vertices in V. Let$\delta_{1,k-1}(\tilde{H}) := \min\{\deg_{\tilde{H}}(S)$: S is a $(1, k-1)$-subset of $V(\tilde{H})\}$ where $\deg_{\tilde{H}}(S)$ denotes the number of edges in $\tilde{H}$ containing S.

The proof of the following claim is by now a standard argument on fractional matchings and covers.

Claim 2.5.2. *If each H'_{F_i} contains a perfect fractional matching for $i \in [\frac{n}{b}]$, then the auxiliary $(1, b)$-graph $H_{\boldsymbol{G}}$ of $\boldsymbol{G}$ contains a perfect fractional matching.*

Proof. By the duality theorem, we transform the maximum fractional matching problem into the minimum fractional cover problem. Since $\tau^*(H_{\boldsymbol{G}}) = \nu^*(H_{\boldsymbol{G}}) \leq \frac{n}{b}$, it suffices to show that $\tau^*(H_{\boldsymbol{G}}) \geq \frac{n}{b}$ to obtain $\nu^*(H_{\boldsymbol{G}}) = \frac{n}{b}$. Let ω be the minimum fractional cover of $H_{\boldsymbol{G}}$ and take $i_1 \in [n/b]$ such that $\omega(i_1) := \min_{i\in[n/b]} \omega(i)$. We may assume that $\omega(i_1) = 1 - x < 1$, since otherwise $\omega([n/b]) \geq \frac{n}{b}$ and we are done. By definition we get $\omega(e) \geq 1 - \omega(i_1) = x$ for every $e \in H'_{F_{i_1}}$. We define a new weight function ω' on V by setting $\omega'(v) = \frac{\omega(v)}{x}$ for every vertex $v \in V$. Thus, ω' is a fractional cover of $H'_{F_{i_1}}$ because for each $e \in H'_{F_{i_1}}$, $\omega'(e) = \frac{\omega(e)}{x} \geq 1$. Recall that $H'_{F_{i_1}}$ has a perfect fractional matching, and thus $\omega'(V) \geq \tau^*(H'_{F_{i_1}}) \geq \frac{n}{b}$ which implies that $\omega(V) \geq \frac{xn}{b}$. Therefore,

$$\omega([\tfrac{n}{b}] \cup V) \geq (1 - x)\tfrac{n}{b} + \tfrac{xn}{b} = \tfrac{n}{b}.$$

Hence, $\tau^*(H_{\boldsymbol{G}}) = \frac{n}{b}$, i.e. $H_{\boldsymbol{G}}$ contains a perfect fractional matching. □

In this section, given an n-vertex Dk-graph collection $\boldsymbol{G}$, we shall construct an auxiliary $(1, b)$-graph $H_{\boldsymbol{G}}$ of $\boldsymbol{G}$ and a sequence of random subgraphs of $H_{\boldsymbol{G}}$. Then, we use the properties of them to get a "near regular" spanning subgraph for the sake of applying Lemma 2.3.1.

The proof is based on a two-round randomization which is already used in (Alon et al., 2012; Lu, Wang, & Yu, 2023; Lu, Yu, & Yuan, 2021). Since we work with balanced $(1, b)$-graphs, we need to make sure that each random graph is balanced. In order to achieve this we modify the randomization process by fixing an arbitrarily small and balanced set $S \subseteq V(H_{\boldsymbol{G}})$. This is done in Fact 2.5.3.

Let $H_{\boldsymbol{G}}$ be the auxiliary $(1,b)$-graph of $\boldsymbol{G}$ with partition classes A, B and $b|A| = |B|$ where A is the color set and $B = V$. Let $S \subseteq V(H_{\boldsymbol{G}})$ be a set of vertices such that $|S \cap A| = n^{0.99}/b$ and $|S \cap B| = n^{0.99}$. The desired subgraph H'' is obtained by two rounds of randomization. As a preparation to the first round, we choose every vertex randomly and uniformly with probability $p = n^{-0.9}$ to get a random subset R of $V(H_{\boldsymbol{G}})$. Take $n^{1.1}$ independent copies of R and denote them by R_{i+}, $i \in [n^{1.1}]$, i.e. each R_{i+} is chosen in the same way as R independently. Define $R_{i-} = R_{i+} \setminus S$ for $i \in [n^{1.1}]$.

Fact 2.5.3. *Let $n, H_{\boldsymbol{G}}, A, B, S$ and R_{i-}, R_{i+} be given as above. Then, with probability $1-o(1)$, there exist subgraphs $R_i, i \in [n^{1.1}]$, such that $R_{i-} \subseteq R_i \subseteq R_{i+}$ and R_i is balanced.*

The following two lemmas together construct the desired sparse regular k-graph we need.

Lemma 2.5.4. *Given an n-vertex Dk-graph collection $\boldsymbol{G} = \{G_1, \dots, G_{nf/b}\}$ on V, let $H_{\boldsymbol{G}}$ be the auxiliary $(1,b)$-graph of $\boldsymbol{G}$. For each $X \subseteq V(H_{\boldsymbol{G}})$, let $Y_X^+ := |\{i : X \subseteq R_{i+}\}|$ and $Y_X := |\{i : X \subseteq R_i\}|$. Let $\tilde{H}$ be with vertex set $[\frac{nf}{b}] \cup V$ and edge set $E(\tilde{H}) = \{\{i\} \cup e : e \in G_i$ for all $i \in [nf/b]\}$. Then with probability at least $1-o(1)$, we have*

i. $|R_i| = (1/b + 1 + o(1))n^{0.1}$ *for all* $i \in [n^{1.1}]$.

ii. $Y_{\{v\}} = (1+o(1))n^{0.2}$ *for* $v \in V(H_{\boldsymbol{G}}) \setminus S$ *and* $Y_{\{v\}} \le (1+o(1))n^{0.2}$ *for* $v \in S$.

iii. $Y_{\{u,v\}} \le 2$ *for all* $\{u,v\} \subseteq V(H_{\boldsymbol{G}})$.

iv. $Y_e \le 1$ *for all* $e \in E(H_{\boldsymbol{G}})$.

v. *Suppose that $V(R_i) = C_i \cup V_i$, we have $\delta_{1,d}(\tilde{H}[\bigcup_{j \in C_i}[(j-1)f+1, jf] \cup V_i]) \ge (c_{d,F}^{\mathrm{cov},*} + \varepsilon/4)\binom{|R_{i+} \cap B| - d}{k-d} - |R_{i+} \cap B \cap S|\binom{|R_{i+} \cap B| - d - 1}{k-d-1} \ge (c_{d,F}^{\mathrm{cov},*} + \varepsilon/8)\binom{|R_i \cap B| - d}{k-d}$.*

Lemma 2.5.5. *Let $n, H_{\boldsymbol{G}}, S, R_i, i \in [n^{1.1}]$ be given as in Lemma 2.5.4 such that each $H_{\boldsymbol{G}}[R_i]$ is a balanced $(1,b)$-graph and has a perfect fractional matching ω_i. Then there exists a spanning subgraph H'' of $H^* = \cup_i H_{\boldsymbol{G}}[R_i]$ such that*

- $d_{H''}(v) \le (1+o(1))n^{0.2}$ *for* $v \in S$,
- $d_{H''}(v) = (1+o(1))n^{0.2}$ *for all* $v \in V(H_{\boldsymbol{G}}) \setminus S$,
- $\Delta_2(H'') \le n^{0.1}$.

The proofs follow the lines as in (Alon et al., 2012; Lu et al., 2023; Lu, Yu, & Yuan, 2021) and thus we put them in the last section.

Proof of Lemma 2.5.1. By the definition of $c_{d,F}^{\mathrm{cov},*}$, Lemma 2.5.4 (5) and Claim 2.5.2, there exists a perfect fractional matching ω_i in every subgraph $H_{\boldsymbol{G}}[R_i]$, $i \in [n^{1.1}]$. By Lemma 2.5.5, there is

a spanning subgraph H'' of $H^* = \cup_i H_{\boldsymbol{G}}[R_i]$ such that $d_{H''}(v) \leq (1+o(1))n^{0.2}$ for each $v \in S$, $d_{H''}(v) = (1+o(1))n^{0.2}$ for all $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $\Delta_2(H'') \leq n^{0.1}$. Hence, by Lemma 2.3.1 (by setting $D = n^{0.2}$), H'' contains a cover of at most $\frac{n+n/b}{1+b}(1+a)$ edges which implies that H'' contains a matching of size at least $\frac{n+n/b}{1+b}(1-a(1+b-1))$, where a is a constant satisfying $0 < a < \phi/(1+b-1)$. Hence $H_{\boldsymbol{G}}$ contains a matching covering all but at most $\phi(n+n/b)$ vertices. □

2.6 The proof of Theorem 2.2.5

Proof. Suppose that $\frac{1}{n} \ll \phi \ll \gamma_1 \ll \gamma \ll \varepsilon' \ll \varepsilon$ where $\varepsilon', \gamma, \gamma_1$ are defined in Lemma 2.4.3 and ϕ, ε in Lemma 2.5.1. Let $H_{\boldsymbol{G}}$ be the auxiliary $(1,b)$-graph of $\boldsymbol{G}$. By Lemma 2.4.3, we get a matching M in $H_{\boldsymbol{G}}$ of size at most $2\gamma(m-1)n$ such that for every balanced set $U \subseteq [n/b] \cup V \setminus V(M)$ of size at most $\gamma_1 n$, $V(M) \cup U$ spans a matching in $H_{\boldsymbol{G}}$. Let $\boldsymbol{G}' = \{G'_1, \ldots, G'_{nf/b}\}$ be the induced Dk-graph collection of $\boldsymbol{G}$ on V' where $V' := V \setminus V(M)$. Denote the subsystem of $\boldsymbol{G}'$ by $\boldsymbol{G}'_I = \{G'_i \mid i \in I = [nf/b] \setminus \bigcup_{j \in V(M) \cap [n/b]}[(j-1)f+1, jf]\}$. We still have $\delta_d(G'_i) \geq (\max\{c^{\mathrm{abs},*}_{d,F}, c^{\mathrm{cov},*}_{d,F}\} + \frac{\varepsilon}{2})\binom{n-d}{k-d}$ for $i \in I$, since $2\gamma(m-1)n\binom{n-d-1}{k-d-1} \leq \frac{\varepsilon}{2}\binom{n-d}{k-d}$. Then, we construct the new auxiliary $(1,b)$-graph $H_{\boldsymbol{G}'_I}$ of $\boldsymbol{G}'_I$.

By Lemma 2.5.1, $H_{\boldsymbol{G}'_I}$ contains a matching M_1 covering all but at most $\phi|V'| \leq \phi(n+n/b)$ vertices. Suppose $W_1 = [n/b] \cup V \setminus (V(M) \cup V(M_1))$, hence $|W_1| \leq \phi(n+n/b) \leq \gamma_1 n$ and W_1 is balanced. By Lemma 2.4.3, $V(M) \cup W_1$ spans a matching M_2 in $H_{\boldsymbol{G}}$ and therefore $M_1 \cup M_2$ is a perfect matching in $H_{\boldsymbol{G}}$, which yields a transversal F-factor in $\boldsymbol{G}$. □

In the next few sections we prove our results in Section 1 (Theorems 2.1.2 – 2.1.5), and by Theorem 2.2.5, it suffices to specify the δ^*_d we use and bound the parameters $c^{\mathrm{abs},*}_{d,F}$ and $c^{\mathrm{cov},*}_{d,F}$. Note also that we will not present a proof of Theorem 2.1.1, as it follows from either of the two directed extensions.

2.7 The proofs of Theorems 2.1.2 and 2.1.3

For Theorem 2.1.2, to apply Theorem 2.2.5, we set δ^*_1 as the minimum out-degree δ^+. Given that $\delta^+(D_i) \geq (1-\frac{1}{k}+\varepsilon)n$ for $i \in [\frac{n}{k}\binom{k}{2}]$, by Theorem 2.2.5, it remains to prove that $c^{\mathrm{abs},+}_{1,T_k}, c^{\mathrm{cov},+}_{1,T_k} \leq 1-\frac{1}{k}$.

Similarly, for Theorem 2.1.3, we set δ_1^* as the minimum semi-degree δ^0. Given $T \in \mathcal{T}_k$ and $\delta^0(D_i) \geq (1-\frac{1}{k}+\varepsilon)n$ for $i \in [\frac{n}{k}\binom{k}{2}]$, by Theorem 2.2.5, it remains to prove that $c_{1,T}^{\mathrm{abs},0}, c_{1,T}^{\mathrm{cov},0} \leq 1-\frac{1}{k}$.

Since the proofs are similar, we only show that $c_{1,T_k}^{\mathrm{abs},+}, c_{1,T_k}^{\mathrm{cov},+} \leq 1-\frac{1}{k}$ for Theorem 2.1.2. Recall that T_k is the transitive tournament on k vertices. Note that $c_{1,T_k}^{\mathrm{abs},+} \leq 1-\frac{1}{k}$ is exactly (2.1).

We partition the n-vertex digraph collection $\boldsymbol{D}$ into $[n/k]$ subsystems $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_{n/k}$ where $\boldsymbol{D}_i = \{D_{(i-1)\binom{k}{2}+1}, \ldots, D_{i\binom{k}{2}}\}$. Define $H_{T_k,i}$ as the k-graph which consists of rainbow copies of T_k on $\boldsymbol{D}_i$ with color set $[(i-1)\binom{k}{2}+1, i\binom{k}{2}]$. We shall show that each $H_{T_k,i}$ has a perfect fractional matching.

Claim 2.7.1. *For $i \in [n/k]$, $H_{T_k,i}$ has a perfect fractional matching.*

A *k-complex* is a hypergraph J such that every edge of J has size at most k, $\emptyset \in J$ and is closed under inclusion, i.e. if $e \in J$ and $e' \subseteq e$ then $e' \in J$. We refer to the edges of size r in J as *r-edges* of J, and write J_r to denote the r-graph on $V(J)$ formed by these edges. We introduce the following notion of degree in a k-system J. For any edge e of J, the *degree* $d(e)$ of e is the number of ($|e|$+1)-edges e' of J which contains e as a subset. (Note that this is not the standard notion of degree used in k-graphs, in which the degree of a set is the number of edges containing it.) The *minimum r-degree* of J, denoted by $\delta_r(J)$, is the minimum of $d(e)$ taken over all r-edges $e \in J$. Trivially, $\delta_0(J) = |V(J)|$. So every r-edge of J is contained in at least $\delta_r(J)$ $(r+1)$-edges of J. The *degree sequence* of J is

$$\delta(J) = (\delta_0(J), \delta_1(J), \ldots, \delta_{k-1}(J)).$$

Lemma 2.7.2 (Lemma 3.6, (Keevash & Mycroft, 2015a))**.** *If the complex J satisfies $\delta(J) \geq (n, \frac{k-1}{k}n, \frac{k-2}{k}n, \ldots, \frac{1}{k}n)$, then J_k contains a perfect fractional matching.*

To prove Claim 2.7.1 we construct the *clique k-complex* J^i for each $\boldsymbol{D}_i$, $i \in [n/k]$, which has vertex set V and edge set $E(J_r^i)$ where each edge is a rainbow T_r with color set $[(i-1)\binom{k}{2}+1, (i-1)\binom{k}{2}+\binom{r}{2}]$ for each $r \in [k]$ and $i \in [n/k]$. Note that for each i, the top level J_k^i is exactly $H_{T_k,i}$. By the out-degree condition, we get

$$\delta(J^i) \geq (n, (1-1/k+\varepsilon)n, \ldots, (1/k+\varepsilon)n).$$

Therefore Claim 2.7.1 follows from Lemma 2.7.2 and we are done.

For completeness, we include the short proof of Lemma 2.7.2 given in (Keevash & Mycroft, 2015a). Given points $\mathbf{x}_1, \ldots, \mathbf{x}_s \in \mathbf{R}^d$, we define their *positive cone* as $PC(\mathbf{x}_1, \ldots, \mathbf{x}_s) := \{\sum_{j\in[s]} \lambda_j \mathbf{x}_j : \lambda_1, \ldots, \lambda_s \geq 0\}$. Recall that V is the n-vertex set, for any $S \subseteq V$, the characteristic vector $\chi(S)$ of S is the binary vector in $\mathbf{R}^n$ such that $\chi(S)_i = 1$ if and only if $i \in S$. Given a k-graph H, if H has a perfect fractional matching ω, then $\mathbf{1} \in PC(\chi(e) : e \in H)$, since $\sum_{e\in H} \omega(e)\chi(e) = \mathbf{1}$. The well-known Farkas' Lemma reads as follows.

Lemma 2.7.3 (Farkas' Lemma)**.** *Suppose $\boldsymbol{v} \in \mathbf{R}^n \backslash PC(Y)$ for some finite set $Y \subseteq \mathbf{R}^n$. Then there is some $\boldsymbol{a} \in \mathbf{R}^n$ such that $\boldsymbol{a} \cdot \boldsymbol{y} \geq 0$ for every $\boldsymbol{y} \in Y$ and $\boldsymbol{a} \cdot \boldsymbol{v} < 0$.*

Proof of Lemma 2.7.2. Suppose that J_k does not contain a perfect fractional matching, this means that $\mathbf{1} \notin PC(\chi(e) : e \in J_k)$. Then by Lemma 2.7.3, there is some $\mathbf{a} \in \mathbf{R}^n$ such that $\mathbf{a} \cdot \mathbf{1} < 0$ and $\mathbf{a} \cdot \chi(e) \geq 0$ for every $e \in J_k$. Let $V = \{v_1, \ldots, v_n\}$ and $\mathbf{a} = (a_1, \ldots, a_n)$ satisfying $a_1 \leq a_2 \leq \cdots \leq a_n$.

We first build a k-edge $e = \{v_{d_1}, \ldots, v_{d_k}\} \in J_k$ such that $d_j \leq \frac{j-1}{k}n + 1, j \in [k]$ as follows. Choose $d_1 = 1$ and having chosen $d_1, \ldots, d_j$, the choice of d_{j+1} is guaranteed by $\delta_j(J) \geq (1 - j/k)n$. As $e \in J_k$, we have $\mathbf{a} \cdot \chi(e) \geq 0$. Consider $\{S_i = \{v_i, v_{i+n/k}, \ldots, v_{i+(k-1)n/k}\} : i \in [n/k]\}$ which form a partition of V, thus $\sum_{i\in[n/k]} \mathbf{a} \cdot \chi(S_i) = \mathbf{a} \cdot \mathbf{1} < 0$. However, as the indices of vertices of e precede those of S_i one-by-one and $a_1 \leq a_2 \leq \cdots \leq a_n$, we have for each i, $\mathbf{a} \cdot \chi(S_i) \geq \mathbf{a} \cdot \chi(e) \geq 0$, a contradiction. □

2.8 The proof of Theorem 2.1.4

For Theorem 2.1.4, we set δ_1^* as the minimum partite-degree δ'. Let $\boldsymbol{G} = \{G_1, \ldots, G_{n\binom{k}{2}}\}$ be a collection of k-partite graphs with a common partition $V_1, \ldots, V_k$ each of size n such that $\delta'(G_i) \geq (1 - 1/k + \varepsilon)n$ for $i \in [n\binom{k}{2}]$. By Theorem 2.2.5, it remains to prove that $c^{\mathrm{abs},'}_{1,K_k}, c^{\mathrm{cov},'}_{1,K_k} \leq 1 - \frac{1}{k}$.

The conclusion that $c^{\mathrm{abs},'}_{1,K_k} \leq 1 - \frac{1}{k}$ can be similarly derived as (2.1) in Section 2.4.2.1 and thus omitted. We next show how to obtain $c^{\mathrm{cov},'}_{1,K_k} \leq 1 - \frac{1}{k}$.

Let H be a k-graph and let $\mathcal{P}$ be a partition of $V(H)$. Then we say a set $S \subseteq V(H)$ is *k-partite* if it has one vertex in any part of $\mathcal{P}$ and that H is *k-partite* if every edge of H is k-partite. Let V be a set of vertices, let $\mathcal{P}$ be a partition of V into k parts $V_1, \ldots, V_k$ and let J be a k-partite k-system on V. For each $0 \leq j \leq k-1$ we define the *partite minimum j-degree* $\delta_j^*(J)$ as the

largest m such that any j-edge e has at least m extensions to a $(j+1)$-edge in any part not used by e, i.e.

$$\delta_j^*(J) := \min_{e\in J_j} \min_{e\cap V_i=\emptyset} |\{v \in V_i : e\cup\{v\} \in J\}|.$$

The partite degree sequence is $\delta^*(J) = (\delta_0^*(J), \dots, \delta_{k-1}^*(J))$.

To obtain $c_{1,K_k}^{\mathrm{cov},\prime}$, we use the following lemma, which is a special case of (Keevash & Mycroft, 2015a, Lemma 7.2). Again we present the short proof of Lemma 2.8.1 given in (Keevash & Mycroft, 2015a).

Lemma 2.8.1. *Let V be a set partitioned into k parts $V_1, \dots, V_k$ each of size n and let J be a k-partite k-system on V such that*

$$\delta^*(J) \geq \left(n, \frac{(k-1)n}{k}, \frac{(k-2)n}{k}, \dots, \frac{n}{k}\right).$$

Then J_k contains a perfect fractional matching.

Proof. Suppose that J_k does not contain a perfect fractional matching, this means that $\mathbf{1} \notin PC(\chi(e) : e \in J_k)$. Then by Lemma 2.7.3, there is some $\mathbf{a} \in \mathbf{R}^{kn}$ such that $\mathbf{a}\cdot\mathbf{1} < 0$ and $\mathbf{a}\chi(e) \geq 0$ for every $e \in J_k$. Let $V = \{v_{1,1}, \dots, v_{1,n}, v_{2,1}, \dots, v_{2,n}, \dots, v_{k,1}, \dots, v_{k,n}\}$ and

$$\mathbf{a} = (a_{1,1}, \dots, a_{1,n}, a_{2,1}, \dots, a_{2,n}, \dots, a_{k,1}, \dots, a_{k,n})$$

satisfying that $a_{j,\frac{(j-1)n}{k}+1} \leq \dots \leq a_{j,n} \leq a_{j,1} \leq \dots \leq a_{j,\frac{(j-1)n}{k}}$ for each $j \in [k]$.

We first build a k-edge $e = \{v_{1,d_1}, v_{2,d_2} \dots, v_{k,d_k}\} \in J_k$ such that $d_j \leq \frac{j-1}{k}n + 1, j \in [k]$ as follows. Choose $d_1 = 1$ and having chosen $d_1, \dots, d_j$, the choice of d_{j+1} is guaranteed by $\delta_j(J) \geq (1-j/k)n$. As $e \in J_k$, we have $\mathbf{a}\cdot\chi(e) \geq 0$. Consider $\{S_i = \{v_{1,i}, v_{2,i}, \dots, v_{k,i}\} : i \in [n]\}$ which forms a partition of V, thus $\sum_{i\in[n]} \mathbf{a}\cdot\chi(S_i) = \mathbf{a}\cdot\mathbf{1} < 0$. However, we have for each i, $\mathbf{a}\cdot\chi(S_i) \geq \mathbf{a}\cdot\chi(e) \geq 0$, as $a_{j,\frac{(j-1)n}{k}+1} \leq \dots \leq a_{j,n} \leq a_{j,1} \leq \dots \leq a_{j,\frac{(j-1)n}{k}}$ for each $j \in [k]$, a contradiction. □

We partition the kn-vertex k-partite graph collection $\boldsymbol{G}$ on V into n subsystems $\boldsymbol{G}_1, \dots, \boldsymbol{G}_n$ where $\boldsymbol{G}_i = \{G_{(i-1)\binom{k}{2}+1}, \dots, G_{i\binom{k}{2}}\}$ for $i \in [n]$. The *clique k-complex* J^i of a k-partite graph collection $\boldsymbol{G}_i$ is with vertex set V and edge set $E(J_r^i)$ where each edge is a rainbow K_r with color set $[(i-1)\binom{k}{2}+1, (i-1)\binom{k}{2}+\binom{r}{2}]$ for each $r \in [k]$ and $i \in [n]$. In $\boldsymbol{G}$, by the degree condition, we get for each $i \in [n]$,

$$\delta^*(J^i) \geq \left(n, \left(1-\frac{1}{k}+\varepsilon\right)n, \ldots, \left(\frac{1}{k}+\varepsilon\right)n\right).$$

By Lemma 2.8.1, J_k^i contains a perfect fractional matching. Therefore $c_{1,K_k}^{\mathrm{cov},\prime} \leq 1-\frac{1}{k}$.

2.9 The proof of Theorem 2.1.5

Note that $c_{d,F}^{\mathrm{abs,d}} \leq \frac{1}{2}$ is given in (2.2). By Definition 2.2.4, we trivially have $c_{d,F}^{\mathrm{cov,d}} \leq c_{k,d}$ where F is an edge. By Theorem 2.2.5, the proof of Theorem 2.1.5 is completed.

2.10 Proof of Auxiliary Results

Below we restate and prove Fact 2.5.3, Lemma 2.5.4 and Lemma 2.5.5.

Fact 1. Let $n, H_{\boldsymbol{G}}, A, B, S$ and R_{i-}, R_{i+} be given as above. Then, with probability $1-o(1)$, there exist subgraphs $R_i, i \in [n^{1.1}]$, such that $R_{i-} \subseteq R_i \subseteq R_{i+}$ and R_i is balanced.

Proof. Recall that $|A| = n/b, |B| = n, |S \cap A| = n^{0.99}/b$ and $|S \cap B| = n^{0.99}$, thus

$$\mathbf{E}[|R_{i+} \cap A|] = n^{0.1}/b,$$

$$\mathbf{E}[|R_{i+} \cap A \cap S|] = n^{0.09}/b,$$

$$\mathbf{E}[|R_{i+} \cap B|] = n^{0.1},$$

$$\mathbf{E}[|R_{i+} \cap B \cap S|] = n^{0.09}.$$

By Lemma 2.3.2, we have

$$\mathbf{P}[||R_{i+} \cap A| - n^{0.1}/b| \geq n^{0.08}] \leq e^{-\Omega(n^{0.06})},$$

$$\mathbf{P}[||R_{i+} \cap A \cap S| - n^{0.09}/b| \geq n^{0.08}] \leq e^{-\Omega(n^{0.07})},$$

$$\mathbf{P}[||R_{i+} \cap B| - n^{0.1}| \geq n^{0.08}] \leq e^{-\Omega(n^{0.06})},$$

$$\mathbf{P}[||R_{i+} \cap B \cap S| - n^{0.09}/b| \geq n^{0.08}] \leq e^{-\Omega(n^{0.07})}.$$

Thus, with probability $1-o(1)$, for all $i \in [n^{1.1}]$,

$$|R_{i+} \cap A| \in [n^{0.1}/b - n^{0.08}, n^{0.1}/b + n^{0.08}],$$

$$|R_{i+} \cap A \cap S| = (1+o(1))n^{0.09}/b,$$

$$|R_{i+} \cap B| \in [n^{0.1} - n^{0.08}, n^{0.1} + n^{0.08}],$$

$$|R_{i+} \cap B \cap S| = (1+o(1))n^{0.09}.$$

Therefore, $|b|R_{i+} \cap A| - |R_{i+} \cap B|| \le (b+1)n^{0.08} < \min\{|R_{i+} \cap A \cap S|, |R_{i+} \cap B \cap S|\}$. Hence, with probability $1-o(1)$, R_i can be balanced for $i \in [n^{1.1}]$. □

Lemma 5.3. Given an n-vertex Dk-graph collection $\boldsymbol{G} = \{G_1, \dots, G_{nf/b}\}$ on V, let $H_{\boldsymbol{G}}$ be the auxiliary $(1,b)$-graph of $\boldsymbol{G}$. For each $X \subseteq V(H_{\boldsymbol{G}})$, let $Y_X^+ := |\{i : X \subseteq R_{i+}\}|$ and $Y_X := |\{i : X \subseteq R_i\}|$. Let $\tilde{H}$ be with vertex set $[\frac{nf}{b}] \cup V$ and edge set $E(\tilde{H}) = \{\{i\} \cup e : e \in G_i \text{ for all } i \in [nf/b]\}$. Then with probability at least $1-o(1)$, we have

i. $|R_i| = (1/b+1+o(1))n^{0.1}$ for all $i \in [n^{1.1}]$.

ii. $Y_{\{v\}} = (1+o(1))n^{0.2}$ for $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $Y_{\{v\}} \le (1+o(1))n^{0.2}$ for $v \in S$.

iii. $Y_{\{u,v\}} \le 2$ for all $\{u,v\} \subseteq V(H_{\boldsymbol{G}})$.

iv. $Y_e \le 1$ for all $e \in E(H_{\boldsymbol{G}})$.

v. Suppose that $V(R_i) = C_i \cup V_i$, we have $\delta_{1,d}(\tilde{H}[\bigcup_{j \in C_i}[(j-1)f+1, jf] \cup V_i]) \ge (c_{d,F}^{\text{cov},*} + \varepsilon/4)\binom{|R_{i+} \cap B|-d}{k-d}) - |R_{i+} \cap B \cap S|\binom{|R_{i+} \cap B|-d-1}{k-d-1} \ge (c_{d,F}^{\text{cov},*} + \varepsilon/8)\binom{|R_i \cap B|-d}{k-d}$.

Proof. Note that $\mathbf{E}[|R_{i+}|] = (1/b+1)n^{0.1}$, $\mathbf{E}[|R_{i-}|] = ((1/b+1)n - (1/b+1)n^{0.99})n^{-0.9} = (1/b+1)n^{0.1} - (1/b+1)n^{0.09}$. By Lemma 2.3.2, we have

$$\mathbf{P}[|\ |R_{i+}| - n^{0.1}(1/b+1)\ | \ge n^{0.095}] \le e^{-\Omega(n^{0.09})},$$

$$\mathbf{P}[|\ |R_{i-}| - ((1/b+1)n^{0.1} - (1/b+1)n^{0.09})\ | \ge n^{0.095}] \le e^{-\Omega(n^{0.09})}.$$

Hence, with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.09})}$, for the given sequence R_i in Fact 2.5.3, $i \in [n^{1.1}]$, satisfying $R_{i-} \subseteq R_i \subseteq R_{i+}$, we have $|R_i| = (1/b+1+o(1))n^{0.1}$.

For each $X \subseteq V(H_{\boldsymbol{G}})$, let $Y_X^+ := |\{i : X \subseteq R_{i+}\}|$ and $Y_X := |\{i : X \subseteq R_i\}|$. Note that the random variables Y_X^+ have binomial distributions $Bi(n^{1.1}, n^{-0.9|X|})$ with expectations $n^{1.1-0.9|X|}$

and $Y_X \leq Y_X^+$. In particular, for each $v \in V(H_G)$, $\mathbf{E}[Y_{\{v\}}^+] = n^{0.2}$, by Lemma 2.3.2, we have

$$\mathbf{P}[|\,|Y_{\{v\}}^+| - n^{0.2}\,| \geq n^{0.19}] \leq e^{-\Omega(n^{0.18})}.$$

Hence, with probability at least $1 - O(n)e^{-\Omega(n^{0.18})}$, we have $Y_{\{v\}} = (1+o(1))n^{0.2}$ for $v \in V(H_G) \setminus S$ and $Y_{\{v\}} \leq (1+o(1))n^{0.2}$ for $v \in S$.

Let $Z_{p,q} = |X \in \binom{V(H_G)}{p} : Y_X^+ \geq q|$. Then,

$$\mathbf{E}[Z_{p,q}] \leq \binom{\frac{n}{b}+n}{p}\binom{n^{1.1}}{q}(n^{-0.9pq}) \leq Cn^{p+1.1q-0.9pq}.$$

Hence, by Markov's inequality we have

$$\mathbf{P}[Z_{2,3} = 0] = 1 - \mathbf{P}[Z_{2,3} \geq 1] \geq 1 - \mathbf{E}[Z_{2,3}] = 1 - o(1),$$

$$\mathbf{P}[Z_{1+b,2} = 0] = 1 - \mathbf{P}[Z_{1+b,2} \geq 1] \geq 1 - \mathbf{E}[Z_{1+b,2}] = 1 - o(1),$$

i.e. with probability at least $1 - o(1)$, every pair $\{u, v\} \subseteq V(H_G)$ is contained in at most two sets R_{i+}, and every edge is contained in at most one set R_{i+}. Thus, the conclusions also hold for R_i.

Fix a $(1, d)$-subset $D \subseteq V(\tilde{H})$ and let $N_D(\tilde{H})$ be the neighborhood of D in $\tilde{H}$. Recall that R is obtained by choosing every vertex randomly and uniformly with probability $p = n^{-0.9}$, let DEG_D be the number of edges $\{f | f \subseteq R \text{ and } f \in N_D(\tilde{H})\}$. Therefore $DEG_D = \sum_{f \in N_D(\tilde{H})} X_f$, where $X_f = 1$ if f is in R and 0 otherwise. We have

$$\mathbf{E}[DEG_D] = d_{\tilde{H}}(D) \times (n^{-0.9})^{k-d} \geq (c_{d,F}^{\text{cov},*} + \varepsilon)\binom{n-d}{k-d}n^{-0.9(k-d)}$$

$$\geq (c_{d,F}^{\text{cov},*} + \varepsilon/3)\binom{|R \cap B| - d}{k-d} = \Omega(n^{0.1(k-d)}).$$

For two distinct intersecting edges $f_i, f_j \in N_D(\tilde{H})$ with $|f_i \cap f_j| = \ell$ for $\ell \in [k-d-1]$, the probability that both of them are in R is

$$\mathbf{P}[X_{f_i} = X_{f_j} = 1] = p^{2(k-d)-\ell},$$

for any fixed ℓ, we have

$$\Delta = \sum_{f_i \cap f_j \neq \emptyset} \mathbf{P}[X_{f_i} = X_{f_j} = 1] \leq \sum_{\ell=1}^{k-d-1} p^{2(k-d)-\ell} \binom{n-d}{k-d}\binom{k-d}{\ell}\binom{n-k}{k-d-\ell}$$

$$\leq \sum_{\ell=1}^{k-d-1} p^{2(k-d)-\ell} O(n^{2(k-d)-\ell}) = O(n^{0.1(2(k-d)-1)}).$$

Applying Lemma 2.3.4 with $\Gamma = B$, $\Gamma_p = R \cap B$ and $M = N_{\tilde{H}}(D)$(a family of $(k-d)$-sets) , we have

$$\mathbf{P}[DEG_D \leq (1-\varepsilon/12)\mathbf{E}[DEG_D]] \leq e^{-\Omega((\mathbf{E}[DEG_D])^2/\Delta)} = e^{-\Omega(n^{0.1})}.$$

Therefore by the union bound, with probability $1-o(1)$, for all $(1,d)$-subsets $D \subseteq V(\tilde{H})$, we have

$$DEG_D > (1-\varepsilon/12)\mathbf{E}[DEG_D] \geq (c_{d,F}^{\mathrm{cov},*} + \varepsilon/4)\binom{|R\cap B|-d}{k-d}.$$

Summarizing, with probability $1-o(1)$, for the sequence $R_i, i \in [n^{1.1}]$, satisfying $R_{i-} \subseteq R_i \subseteq R_{i+}$, all of the following hold.

i. $|R_i| = (1/b+1+o(1))n^{0.1}$ for all $i \in [n^{1.1}]$.

ii. $Y_{\{v\}} = (1+o(1))n^{0.2}$ for $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $Y_{\{v\}} \leq (1+o(1))n^{0.2}$ for $v \in S$.

iii. $Y_{\{u,v\}} \leq 2$ for all $\{u,v\} \subseteq V(H_{\boldsymbol{G}})$.

iv. $Y_e \leq 1$ for all $e \in E(H_{\boldsymbol{G}})$.

v. $DEG_D^{(i)} \geq (c_{d,F}^{\mathrm{cov},*} + \varepsilon/4)\binom{|R_{i+}\cap B|-d}{k-d}$ for all $(1,d)$-subsets of $D \subseteq V(\tilde{H})$ and $i \in [n^{1.1}]$.

Thus, by property (5) above, we conclude that suppose $V(R_i^+) = C_i^+ \cup V_i^+$ and $V(R_i) = C_i \cup V_i$, the following holds.

$$\delta_{1,d}(\tilde{H}[\bigcup_{j\in C_i^+}[(j-1)f+1, jf] \cup V_i^+]]) \geq (c_{d,F}^{\mathrm{cov},*} + \varepsilon/4)\binom{|R_{i+}\cap B|-d}{k-d}.$$

After the modification, we still have

$$\delta_{1,d}(\tilde{H}[\bigcup_{j\in C_i}[(j-1)f+1, jf]\cup V_i]]) \geq (c_{d,F}^{\mathrm{cov},*}+\varepsilon/4)\binom{|R_{i+}\cap B|-d}{k-d} - |R_{i+}\cap B\cap S|\binom{|R_{i+}\cap B|-d-1}{k-d-1}$$

$$\geq (c_{d,F}^{\mathrm{cov},*} + \varepsilon/8)\binom{|R_i \cap B| - d}{k-d}. \qquad \square$$

Lemma 5.4. Let $n, H_{\boldsymbol{G}}, S, R_i$, $i \in [n^{1.1}]$ be given as in Lemma 2.5.4 such that each $H_{\boldsymbol{G}}[R_i]$ is a balanced $(1,b)$-graph and has a perfect fractional matching ω_i. Then there exists a spanning subgraph H'' of $H^* = \cup_i H_{\boldsymbol{G}}[R_i]$ such that

- $d_{H''}(v) \leq (1+o(1))n^{0.2}$ for $v \in S$,
- $d_{H''}(v) = (1+o(1))n^{0.2}$ for all $v \in V(H_{\boldsymbol{G}}) \setminus S$,
- $\Delta_2(H'') \leq n^{0.1}$.

Proof. By the condition that each $H_{\boldsymbol{G}}[R_i]$ has a perfect fractional matching ω_i, we select a generalized binomial subgraph H'' of H^* by independently choosing each edge e with probability $\omega_{i_e}(e)$ where i_e is the index i such that $e \in H_{\boldsymbol{G}}[R_i]$. Recall that property (4) guarantees the uniqueness of i_e.

For $v \in V(H'')$, let $I_v = \{i : v \in R_i\}$, $E_v = \{e \in H^* : v \in e\}$ and $E_v^i = E_v \cap H_{\boldsymbol{G}}[R_i]$, then $E_v^i, i \in I_v$ forms a partition of E_v and $|I_v| = Y_{\{v\}}$. Hence, for $v \in V(H'')$,

$$d_{H''}(v) = \sum_{e \in E_v} 1 = \sum_{i \in I_v} \sum_{e \in E_v^i} X_e,$$

where X_e is the Bernoulli random variable with $X_e = 1$ if $e \in E(H'')$ and $X_e = 0$ otherwise. Thus its expectation is $\omega_{i_e}(e)$. Therefore

$$\mathbf{E}[d_{H''}(v)] = \sum_{i \in I_v} \sum_{e \in E_v^i} \omega_{i_e}(e) = \sum_{i \in I_v} 1 = Y_{\{v\}}.$$

Hence, $\mathbf{E}[d_{H''}(v)] = (1+o(1))n^{0.2}$ for $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $\mathbf{E}[d_{H''}(v)] \leq (1+o(1))n^{0.2}$ for $v \in S$. Now by Chernoff's inequality, for $v \in V(H_{\boldsymbol{G}}) \setminus S$,

$$\mathbf{P}[|d_{H''}(v) - n^{0.2}| \geq n^{0.15}] \leq e^{-\Omega(n^{0.1})},$$

and for $v \in S$,

$$\mathbf{P}[d_{H''}(v) - n^{0.2} \geq n^{0.15}] \leq e^{-\Omega(n^{0.1})},$$

Taking a union bound over all vertices, we conclude that with probability $1 - o(1)$, $d_{H''}(v) = (1+o(1))n^{0.2}$ for all $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $d_{H''}(v) \leq (1+o(1))n^{0.2}$ for $v \in S$.

Next, note that for distinct $u, v \in V(H_{\boldsymbol{G}})$,

$$d_{H''}(\{u,v\}) = \sum_{e \in E_u \cap E_v} 1 = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} X_e,$$

and

$$\mathbf{E}[d_{H''}(\{u,v\})] = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} \omega_i(e) \leq |I_u \cap I_v| \leq 2.$$

Thus, by Lemma 2.3.3,

$$\mathbf{P}[d_{H''}(\{u,v\}) \geq n^{0.1}] \leq e^{-n^{0.1}},$$

then by a union bound we have $\Delta_2(H'') \leq n^{0.1}$ with probability $1 - o(1)$. □

3 | Regularity Method and Blow-Up Lemma for Transversals

3.1 Introduction

3.1.1 The regularity-blow-up method for transversals

The so-called 'regularity-blow-up method' has been employed to prove many results concerning the embedding of a given spanning subgraph H in a large graph G. Such proofs typically run along the following lines. Apply Szemerédi's regularity lemma (Szemerédi, 1975) to obtain a constant-size 'reduced graph' R of G, which approximates the structure of G: vertices of R correspond to disjoint vertex clusters in G, and edges of R correspond to *regular* pairs of vertex clusters, which informally means that the edges between them are randomlike and therefore easy to embed into. There may be a small number of 'exceptional' vertices which are not part of this structure. Next, find a suitable subgraph H' of R which is simpler than H, and often consists of many small connected components (whereas H could be connected). Next, embed small pieces of H which connect the components of H'. At this stage, some vertices may need to be moved from the structure into the exceptional set to make clusters balanced and to ensure regular pairs have sufficiently large minimum degree (they are *superregular*). Then incorporate the exceptional vertices: for each such vertex v find a component of H' where v has many suitable neighbours and a few vertices inside the H'-structure which can be used along with v to embed a small part of H. Finally apply the *blow-up lemma* of (Komlós, Sárközy, & Szemerédi, 1997) to embed most of H into H' allowing for the restrictions imposed by the initial embedding. The blow-up lemma states that, for the purposes of embedding a possibly spanning bounded degree graph, a regular pair behaves the same as a complete bipartite graph. In this Chapter, we will provide the tools needed to apply the regularity-blow-up method to obtain transversal embeddings, following the same steps as above. The basic idea is that one can think of a graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ with common vertex set V as the 3-uniform hypergraph $G^{(3)}$ with vertex and edge sets

$$V(G^{(3)}) = V \cup \mathscr{C} \quad \text{and} \quad E(G^{(3)}) = \{xyc : xy \in G_c\}.$$

This natural idea has been noticed and used before, for example in (Joos & Kim, 2020), where it was noted that a result of (Aharoni, Georgakopoulos, & Sprüssel, 2009) on matchings in k-partite k-uniform hypergraphs can be used to find a transversal matching in a bipartite graph collection; and in (Cheng, Han, et al., 2021), where the weak regularity lemma was applied to $G^{(3)}$ to find an almost spanning clique factor. We take the idea further by using this perspective to provide the tools one would require to use the regularity-blow-up method for graph collections. To this end, we

- define regularity, superregularity, reduced graphs and provide a regularity lemma for graph collections;
- state some of the standard tools for regularity arguments, such as a 'slicing lemma' which states that regularity is inherited in subpairs of regular pairs, degree inheritance in the reduced graph, and various embedding lemmas with target sets, candidate sets and prescribed colours;
- prove a blow-up lemma for 'separable graphs', which we define shortly.

The first item is mainly a convenient reformulation of weak regularity for 3-uniform hypergraphs and does not require any new ideas; nonetheless the statements are useful to have. The third (which builds upon the tools developed in the second item) is our main contribution, and the proof uses the original blow-up lemma combined with colour absorption ideas from (Montgomery et al., 2022).

We envisage that our blow-up lemma will be useful in solving various transversal embedding problems, making it a viable alternative to the commonly used absorption technique. Furthermore, there are cases where the blow-up lemma is a more appropriate tool. For instance, when attempting to embed a given transversal 2-factor (a spanning 2-regular subgraph) or, more generally, a graph with low bandwidth, the absorption technique is not as effective. In such situations, the blow-up lemma can still be employed for successful embedding. For example, our blow-up lemma can be used to embed powers of Hamilton cycles, providing an alternative proof of a result in (Gupta et al., 2023). (Abbasi, 1998b), proving a conjecture of (El-Zahar, 1984), used the original blow-up lemma to obtain the best possible minimum degree bound in a large graph containing a given 2-factor; our blow-up lemma could be useful in proving a transversal version of this result. Another notable advantage of the blow-up lemma is its ability to aid in characterising extremal constructions. This is helpful for determining exact bounds in these

types of problems, by utilising the stability method commonly employed in non-transversal graph embeddings. This natural problem was recently raised in (Gupta et al., 2023). We discuss applications to transversal embedding problems further in Section 3.6.

The transversal blow-up lemma is also useful for embeddings in uniformly dense 3-uniform hypergraphs. We introduce these 'randomlike' hypergraphs, and our results in this direction, in Section 3.1.4. Indeed, the transversal blow-up lemma can be formulated as a blow-up lemma for 3-uniform hypergraphs. Consequently, there is also potential to employ this lemma in generalising results such as those of (Kühn & Osthus, 2006) who studied the problem of embedding loose Hamiltonian cycles under vertex degree conditions.

All these results apply to separable graphs with bounded maximum degree. Recall that an n-vertex graph H is *μ-separable* if there is $X \subseteq V(H)$ of size at most μn such that $H - X$ consists of components of size at most μn. Separable graphs (with suitable small μ) include F-factors for fixed F, trees, 2-regular graphs, powers of a Hamilton cycle, and graphs of small bandwidth.

3.1.2 Transversals in uniformly dense graph collections

Our first main result concerns transversal embeddings of spanning graphs inside a quasirandom graph collection. A 'quasirandom' condition is one possessed by a random graph of a similar density. Our condition is that for every linear subset of vertices and linear subset of colours, the total number of edges in these colours spanned by the subset is large. A graph collection satisfying this condition is said to be *uniformly dense*. We also require that the number of edges of each colour and the total degree of each vertex is $\Omega(n^2)$, where the graph collection has a common vertex set of size n. This condition cannot be completely removed, since if a colour is not present or if a vertex is isolated, we certainly cannot find a transversal copy of a spanning graph H. A graph collection satisfying both conditions is *super uniformly dense*, in analogy with regular and superregular. We are interested in the following question: for which (spanning) graphs H must any super uniformly dense graph collection contain a transversal copy of H?

Definition 3.1.1 (Uniformly dense graph collection)**.** *Given $d, \eta > 0$, we say that a graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ with vertex set V of size n is* (d, η)-dense *if for all $A, B \subseteq V$ and*

$\mathscr{C}' \subseteq \mathscr{C}$, we have

$$\sum_{c\in\mathscr{C}'} e(G_c[A,B]) \geq d|\mathscr{C}'||A||B| - \eta n^3,$$

where $e(G_c[A,B])$ denotes the number of pairs $(x,y) \in A\times B$ for which $\{x,y\}$ is an edge of G_c. We informally refer to a graph collection $\boldsymbol{G}$ on n vertices that is (d,η)-dense for parameters $0 < 1/n \ll \eta \ll d \leq 1$ as uniformly dense*, and if additionally there is $\alpha \gg \eta$ for which $\sum_{c\in\mathscr{C}} d_{G_c}(v) \geq \alpha|\mathscr{C}|n$ for all $v \in V$ and $e(G_c) \geq \alpha n^2$ for all $c \in \mathscr{C}$, then $\boldsymbol{G}$ is* super uniformly dense.

Note that the condition is vacuous unless A, B and $\mathscr{C}'$ are of linear size (in the number n of vertices). We prove that a transversal copy of a separable graph can be found in a super uniformly dense graph collection.

Theorem 3.1.2. *For all $\Delta, d, \delta, \alpha > 0$, there are $\eta, \mu > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geq n_0$. Let $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ be a (d,η)-dense graph collection on a common vertex set V of size n, where $|\mathscr{C}| \geq \delta n$, and suppose that for all $v \in V$ we have $\sum_{c\in\mathscr{C}} d_{G_c}(v) \geq \alpha|\mathscr{C}|n$, and for all $c \in \mathscr{C}$ we have $e(G_c) \geq \alpha n^2$. Then $\boldsymbol{G}$ contains a transversal copy of any given μ-separable graph H on n vertices with $\Delta(H) \leq \Delta$ and $e(H) = |\mathscr{C}|$.*

This result is a consequence of a more general transversal blow-up lemma, which we state in the next section. First, we discuss possible extensions to other colour patterns. One can ask whether a graph collection contains a copy of H with a given colour pattern, extending the monochromatic case (a copy in a single graph G_c) and rainbow case (a copy with one edge from each G_c). For example, the following was shown in (Montgomery et al., 2022). Not only does the same minimum degree $(\frac{2}{3} + o(1))3n$ which guarantees a triangle factor in a large graph with a vertex set of size $3n$ in fact guarantee a transversal copy when $|\mathscr{C}| = 3n$, it also guarantees a triangle factor where the c-th triangle lies inside G_c (has colour c), when $|\mathscr{C}| = n$. Given $d, \eta > 0$, a (single) graph with a vertex set V of size n is (d,η)-*dense* if for all $A, B \subseteq V$, we have $e(G[A,B]) \geq d|A||B| - \eta n^2$. A collection of (d,η)-dense graphs on the same vertex set is a (d,η)-dense graph collection, but the converse does not hold. It is easy to see that, for any $d > 0$, as long as η and $1/n$ are sufficiently small, such a G contains $\Omega(n^3)$ triangles: the condition applied to $A = B = V$ implies that there are $\Omega(n)$ vertices v with $d_G(v) \geq dn/2$; each one has $\Omega(n^2)$ edges in its neighbourhood.

However, there are uniformly dense graph collections with d fairly large which do not contain any monochromatic triangles, as the following example shows. (The example is essentially equivalent to one in the setting of uniformly dense hypergraphs – introduced in Section 3.1.4 – of (Reiher, Rödl, & Schacht, 2018b), which itself has its roots in work of (Erdős & Hajnal, 1972).) Form an auxiliary oriented 2-graph J by letting $V, \mathscr{C}$ be disjoint, where, say, $|V| = |\mathscr{C}| = n$ is large, and first adding edges between every pair in V, and every pair in $(V, \mathscr{C})$. Independently for each edge, choose an orientation uniformly at random. Add the pair xy to G_c with $x, y \in V$ and $c \in \mathscr{C}$ precisely when xyc is a cyclic triangle. Then, with probability tending to 1 as $n \to \infty$, $\boldsymbol{G}$ is $(\frac{1}{4}, o(1))$-dense. However, for every triple x, y, z of vertices in V and every colour $c \in \mathscr{C}$, there are at most two edges in $G_c[\{x, y, z\}]$.

This raises the question of which colour patterns of which graphs H one can expect to find in a (super) uniformly dense graph collection. If d is sufficiently large, then any bounded degree H with any colour pattern can be found; for those H which are not present when d is an arbitrary positive constant, how large must d be to guarantee such a copy of H? We will explore a generalisation of this question in Section 3.1.4.

3.1.3 A transversal blow-up lemma

In this section, we state a simplified version of our blow-up lemma for bipartite graphs. We defer the complete statement to Theorem 3.1.8 at the end of this section.

Our blow-up lemma has the advantages that its proof, sketched at the beginning of Section 3.4, is conceptually straightforward, following from the original blow-up lemma and a colour absorption tool introduced in (Montgomery et al., 2022); and the lemma should be powerful enough for all of the transversal embedding applications we have in mind. This is since the usual regularity-blow-up method is almost always successfully applied to 'non-expanding' separable graphs, since one embeds them by using the blow-up lemma on a series of small pieces. The disadvantage is that there does not seem to be a reason why the separability condition should be necessary.

Theorem 3.1.3 (Simplified transversal blow-up lemma)**.** *Let $0 < 1/n \ll \varepsilon, \mu \ll d, \delta, 1/\Delta \leq 1$. Let $\mathscr{C}$ be a set of at least δn colours and let $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ be a collection of bipartite graphs with the same vertex partition V_1, V_2, where $n \leq |V_1| \leq |V_2| \leq n/\delta$, such that*

- *for all $V_i' \subseteq V_i$ for $i = 1, 2$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|V_i'| \geq \varepsilon |V_i|$ for $i = 1, 2$ and $|\mathscr{C}'| \geq \varepsilon |\mathscr{C}|$,*

we have that

$$\sum_{c\in\mathscr{C}'} e_{G_c}(V_1', V_2') \geq d|\mathscr{C}'||V_1'||V_2'|;$$

- *for $i = 1, 2$ and every $v \in V_i$ we have $\sum_{c\in\mathscr{C}} d_{G_c}(v) \geq d|\mathscr{C}|n$ and for every $c \in \mathscr{C}$, we have $e(G_c) \geq dn^2$.*

Let H be a μ-separable bipartite graph with parts of size $|V_1|, |V_2|$, and $|\mathscr{C}|$ edges and maximum degree Δ. Then G contains a transversal copy of H.

The theorem requires that the number of vertices and edges of H are comparable, so it does not apply to very sparse H. If one adds edges to H to obtain a suitable denser graph H', and duplicates colours until there are $e(H')$ colours, the copy of H' produced by the theorem will not necessarily contain a transversal copy of H.

The general version of the theorem, Theorem 3.1.8, applies to a graph collection whose graphs have common parts $V_1, \ldots, V_r$, and a graph R with $V(R) = [r]$ such that the bipartite condition in the theorem holds between all $ij \in E(R)$, and each ij has a dedicated set of colours $\mathscr{C}_{ij}$.

3.1.4 Applications to embedding in uniformly dense hypergraphs

In this section we revisit the hypergraph perspective and discuss the consequences of our work to embeddings in 3-uniform hypergraphs. 'Weak regularity' is the straightforward generalisation of Szemerédi regularity from (2-uniform hyper)graphs to k-uniform hypergraphs (hereafter referred to as *k-graphs*), and the 'weak regularity lemma' was proved by (Chung, 2007) in much the same way as the original (Szemerédi, 1975). However, this lemma is not powerful enough to prove many of the analogues of graph results which one would like. In particular, there is no general *counting lemma* which guarantees that the number of copies of a given small graph F in a regularity partition is similar to what one would expect if pairs were replaced by uniform random hyperedges, with the same density. Thus the 'strong regularity lemma' was developed, which uses, as the name suggests, a more complicated and much stronger notion of regularity, and does have an associated counting lemma; this lemma is much more applicable than its weaker counterpart.

It was shown by (Conlon, Hàn, Person, & Schacht, 2012) and independently (Kohayakawa, Nagle, Rödl, & Schacht, 2010) that weak regularity is in fact strong enough to give a counting lemma for *linear* hypergraphs, which have the property that $|e \cap f| \leq 1$ for all distinct hyper-

edges e, f. As a transversal copy of a (simple) graph inside a graph collection $\boldsymbol{G}$ is a linear subhypergraph of the associated 3-graph $G^{(3)}$, weak regularity is an effective tool for transversal embedding problems. And conversely, the tools developed in this Chapter are useful for embedding 3-graphs.

We generalise the definition we gave in Section 3.1.2 for graphs.

Definition 3.1.4 (Uniformly dense k-graph)**.** *Let $d, \eta > 0$ and let $k \geq 2$ be an integer. A k-graph is* (d, η)-dense *if for all subsets $U_1, \ldots, U_k \subseteq V$, we have*

$$e(U_1, \ldots, U_k) \geq d|U_1| \ldots |U_k| - \eta n^k,$$

where $e(U_1, \ldots, U_k)$ is the number of k-tuples $(x_1, \ldots, x_k) \in U_1 \times \ldots \times U_k$ for which $\{x_1, \ldots, x_k\}$ is a hyperedge. We informally refer to a k-graph G on n vertices that is (d, η)-dense for parameters $0 < 1/n \ll \eta \ll d \leq 1$ as uniformly dense*. If there is $\alpha \gg \eta$ for which additionally $d_G(v) \geq \alpha n^{k-1}$ for all $v \in V(G)$, then we say that G is* super uniformly dense*.*

This is a type of quasirandomness, since a random k-graph of density at least d is $(d, o(1))$-dense with high probability. Several papers studying uniform density use the term 'quasirandom' instead.

Following a suggestion of (Erdős & Sós, 1982), a systematic treatment of extremal problems in uniformly dense hypergraphs has been started by Rödl, Reiher and Schacht, in part due to the great difficulty of such problems in general hypergraphs. In (Reiher, Rödl, & Schacht, 2018a), they fully answered the zero Turán density question for uniformly dense 3-graphs, i.e. for which F is the following true? For all $d > 0$, there exists $\eta > 0$ such that every sufficiently large (d, η)-dense 3-graph contains F as a subhypergraph. They showed that these F are precisely those with the following property. There is an ordering $v_1, \ldots, v_r$ of $V(F)$ and a colouring of the set of pairs of vertices contained in edges by red, blue and green, so that whenever $v_i v_j v_k \in E(F)$ with $i < j < k$, we have that $v_i v_j$ is red, $v_i v_k$ is blue and $v_j v_k$ is green. This set of F includes linear F and 3-partite F ((Erdős, 1964a) proved that a k-graph F has Turán density 0 if and only if F is k-partite), but is much richer than this: for example, the hypergraph obtained by removing one edge from the tight cycle on 5 vertices is such a hypergraph.

The simple argument given in Section 3.1.2 shows that for $k = 2$ and $\Delta > 0$, a sufficiently large uniformly dense 2-graph contains a copy (in fact many copies) of K_Δ. In contrast, for

$k \geq 3$ there are very simple k-graphs F which require a fairly large density d to appear in any uniformly dense k-graph. Indeed, there is a $(\frac{1}{4}, o(1))$-dense 3-graph in which $K_4^{(3)-}$, the (unique) 3-graph with 4 vertices and 3 edges, does not appear. (The example is closely related to the one given in Section 3.1.2.)

We are interested in which *spanning* hypergraphs appear in any super uniformly dense hypergraph. Note that we cannot remove 'super' since uniform density does not preclude the existence of isolated vertices.

Problem 3.1.5. *Let $\mathcal{H} = (H_\ell : \ell \in \mathbb{N})$ be a collection of k-graphs where $|V(H_\ell)| =: n_\ell \to \infty$. For which $\mathcal{H}$ is the following true? For all $d, \alpha > 0$, there exist $\eta > 0$ and $\ell_0 \in \mathbb{N}$ such that for all integers $\ell > \ell_0$, every (d, η)-dense k-graph G on n_ℓ vertices with $d_G(v) \geq \alpha n_\ell^{k-1}$ for all $v \in V(G)$ contains H_ℓ as a subhypergraph.*

For $k = 2$, the blow-up lemma of (Komlós et al., 1997) implies that every uniformly dense graph contains as a subgraph any given spanning graph H with bounded degree and sublinear bandwidth (as observed by Glock and Joos, see Theorem 9.3 in (Staden & Treglown, 2020)).

For general uniformities k, (Lenz & Mubayi, 2016) proved that uniformly dense k-graphs contain an F-factor for any given fixed size linear F. In fact they found non-linear F for which uniform density is sufficient, and 'almost-linear' F for which it is not. (Ding, Han, Sun, Wang, & Zhou, 2022) completed this work for $k = 3$ by characterising those F for which uniform density guarantees an F-factor, and for general k they also obtained a characterisation for k-partite F. (Lenz et al., 2016) showed that uniform density guarantees a loose cycle. To the best of our knowledge, there are no further results on embedding spanning F in uniformly dense hypergraphs of arbitrarily small positive density.

Our next main result generalises the result of (Lenz et al., 2016) for $k = 3$ by providing a new family of spanning H which can be found in super uniformly dense 3-graphs. An *expanded graph* is obtained from a 2-graph by replacing every edge $e = xy$ by some 3-edges $xyc_1, \dots, xyc_{t_e}$, where all the new vertices $c_1, \dots, c_{t_e}$ are distinct, and the number t_e of new edges/vertices for the edge e depends on e. If every t_e is equal to t then we call this a *t-expansion*. For example, a loose 3-uniform cycle is an expanded cycle where each edge is replaced by one expanded edge, that is, a 1-expansion of a cycle. The 1-expansion of a (simple) graph H is linear and has $|V(H)| + |E(H)|$ vertices.

Theorem 3.1.6. *For all $\Delta, d, \alpha > 0$ there are $\eta, \mu > 0$ and $n_0 \in \mathbb{N}$ such that the following*

holds for all integers $n \geq n_0$. Let G be a (d, η)-dense 3-graph on n vertices with $d_G(v) \geq \alpha n^2$ for all $v \in V(G)$. Then G contains a copy of the 1-expansion of any given μ-separable graph H with $\Delta(H) \leq \Delta$ and $|V(H)| + |E(H)| \leq n$.

Since a large 2-regular graph (a union of vertex-disjoint cycles) is $o(1)$-separable, the theorem implies that any super uniformly dense 3-graph G contains as a subhypergraph any 3-graph consisting of vertex-disjoint loose cycles.

Next, we reformulate Theorem 3.1.3 as a blow-up lemma for 3-graphs, which may be of independent interest.

Theorem 3.1.7 (Simplified weak hypergraph blow-up lemma)**.** *For all $\Delta, d, \delta > 0$ there are $\varepsilon, \mu > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geq n_0$. Let G be a 3-partite 3-graph with parts V_1, V_2, V_3 where every $|V_i| =: n_i$ and $n \leq n_1 \leq n_2 \leq n/\delta$ and $n_3 \geq \delta n$ such that G is a* weakly (ε, d)-half-superregular triple*, that is,*

(i) for all $i \in [3]$ and $V_i' \subseteq V_i$ with $|V_i'| \geq \varepsilon |V_i|$, we have $e_G(V_1', V_2', V_3') \geq d|V_1'||V_2'||V_3'|$,

(ii) $d_G(v) \geq dn^2$ for all $v \in V(G)$.

Let H be a μ-separable bipartite 2-graph with parts of size n_1, n_2, with n_3 edges and maximum degree Δ. Then G contains a copy of the 1-expansion of H, where the images of the new vertices lie in V_3.

Weak superregularity and uniform density of 3-graphs and graph collections are closely connected. Observe that if G is $(d + \sqrt{\eta}, \eta)$-dense, where $\eta \gg \varepsilon, \delta$, and V_1, V_2, V_3 is any partition with sizes as in the theorem, then the 3-partite subhypergraph G' of G induced on these parts satisfies (i). If $d_G(v) \geq \alpha n^2$ for all $v \in V$, and in addition the vertex partition is uniformly random, then with probability tending to 1 as $n \to \infty$, G' is weakly $(\varepsilon, \min\{d, \alpha/2\})$-half-superregular, say. Furthermore, if one takes any partition $V(G) = V \cup \mathscr{C}$ into parts which are not too small, then the graph collection $\boldsymbol{G}$ consisting of graphs G_c for $c \in \mathscr{C}$ with $V(G_c) = V$ and $E(G_c) = \{xy : x, y \in V \text{ and } xyc \in E(G)\}$ is (d, η)-dense. Again, if $d_G(v) \geq \alpha n^2$, a random such partition gives rise to a super uniformly dense graph collection.

As an example, any 3-graph G satisfying the hypotheses of the theorem with $n_1 = n_2$ and $n_3 = n_1 + n_2$ contains a copy of a loose Hamilton cycle. The full statement of a 3-graph version of our transversal blow-up lemma (Theorem 3.1.8) is stated in Theorem 3.6.1.

A much more general hypergraph blow-up lemma was established by (Keevash, 2011) which applies to k-graphs satisfying a strong regularity property, which is designed to be used with the

strong regularity lemma; this blow-up lemma applies to embed any bounded degree k-graph H. Nevertheless, it would be interesting to determine which graphs can be embedded into a weakly superregular triple. We discuss this further in Section 3.6.

3.1.5 The statement of our main result

The full statement of our transversal blow-up lemma is as follows. Its proof will follow from Theorem 3.3.5 which is a very similar statement written in the notation defined in Section 4.2.

Theorem 3.1.8 (Transversal blow-up lemma)**.** *For all $\nu, d, \delta, \Delta, r > 0$ where $r \geq 2$ is an integer, there exist $\varepsilon, \mu, \alpha > 0$ and $m_0 \in \mathbb{N}$ such that the following holds for all integers $m \geq m_0$.*

Suppose that $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ is a graph collection with the following properties.

- *There is a graph R with vertex set $[r]$ and a partition $\mathscr{C} = \bigcup_{e \in E(R)} \mathscr{C}_e$ where $|\mathscr{C}_e| \geq \delta m$ for all $e \in E(R)$;*
- *for all $ij \in E(R)$ and $c \in \mathscr{C}_{ij}$, G_c is bipartite with parts V_i, V_j, and V is a vertex set of size n with partition $V = V_1 \cup \ldots \cup V_r$, where $m \leq |V_i| \leq m/\delta$ for each $i \in [r]$;*
- *for all $ij \in E(R)$,*
  - *for all $V_h' \subseteq V_h$ for $h = i, j$ and $\mathscr{C}_{ij}' \subseteq \mathscr{C}_{ij}$ with $|V_h'| \geq \varepsilon |V_h|$ for $h = i, j$ and $|\mathscr{C}_{ij}'| \geq \varepsilon |\mathscr{C}_{ij}|$, we have that*

$$\sum_{c \in \mathscr{C}_{ij}'} e_{G_c}(V_i', V_j') \geq d|\mathscr{C}_{ij}'||V_i'||V_j'|;$$

  - *for $h = i, j$ and every $v \in V_h$ we have $\sum_{c \in \mathscr{C}_{ij}} d_{G_c}(v) \geq d|\mathscr{C}_{ij}|m$ and for every $c \in \mathscr{C}_{ij}$, we have $e(G_c) \geq dm^2$.*

Suppose that H is a graph with the following properties.

- *$\Delta(H) \leq \Delta$;*
- *H is μ-separable;*
- *H has vertex partition $A_1 \cup \ldots \cup A_r$ such that $|A_i| = |V_i|$ for all $i \in [r]$ and for every $xy \in E(H)$ there is $ij \in E(R)$ such that $x \in A_i$ and $y \in A_j$;*
- *$e(H[A_i, A_j]) = |\mathscr{C}_{ij}|$ for all $ij \in E(R)$;*
- *for each $i \in [r]$, there is a set $U_i \subseteq A_i$ with $|U_i| \leq \alpha m$ and for each $x \in U_i$, a set $T_x \subseteq V_i$ with $|T_x| \geq \nu m$.*

Then there is a transversal embedding of H inside $\boldsymbol{G}$ such that for every $i \in [r]$, every $x \in U_i$ is embedded inside T_x.

3.2 The regularity-blow-up method for transversals

3.2.1 Regularity

Our notion of regularity for a graph collection $\boldsymbol{G}$ is essentially weak regularity for the 3-graph $G^{(3)}$ of $\boldsymbol{G}$. We now define several related notions of weak regularity for k-graphs, which we will use for $k = 2, 3$. We usually omit *weak* and *weakly* since we will not use any stronger type of regularity.

Let G be a k-partite k-graph with classes $V_1, \dots, V_k$, which we also denote as $(V_1, \dots, V_k)_G$. We define the *density* of G to be

$$d_G(V_1, \dots, V_k) := \frac{e(G)}{|V_1| \dots |V_k|}.$$

Given $\varepsilon > 0$, we say that $(V_1, \dots, V_k)_G$ is

- *(weakly) ε-regular* if for every subhypergraph $(V_1', \dots, V_k')_G$ with $V_i' \subseteq V_i$ and $|V_i'| \geq \varepsilon |V_i|$ for all $i \in [k]$ we have

$$|d_G(V_1', \dots, V_k') - d_G(V_1, \dots, V_k)| < \varepsilon;$$

- *(weakly) (ε, d)-regular* if additionally $d_G(V_1, \dots, V_k) \geq d$;
- *(weakly) (ε, d)-superregular* if it is (weakly) (ε, d)-regular and additionally $d_G(x) \geq d(|V_1| \dots |V_k|)/|V_i|$ for all $i \in [k]$ and $x \in V_i$;
- *(weakly) (ε, d)-half-superregular* if for every subhypergraph $(V_1', \dots, V_k')_G$ with $V_i' \subseteq V_i$ and $|V_i'| \geq \varepsilon |V_i|$ for all $i \in [k]$ we have $d_G(V_1', \dots, V_k') \geq d$ and $d_G(x) \geq d(|V_1| \dots |V_k|)/|V_i|$ for all $i \in [k]$ and $x \in V_i$.

Our main results will use 'half-superregular' given its close connection to uniform density. The 2-graph blow-up lemma (Komlós et al., 1998b) uses a similar notion, but nowadays 'superregular' is generally used instead. This makes little material difference: by definition, an (ε, d)-superregular hypergraph is $(\varepsilon, d - \varepsilon)$-half-superregular, and as shown by Rödl and Ru-

ciński in the course of their alternative proof of the blow-up lemma (Rödl & Rucinski, 1999), a half-superregular 2-graph contains a spanning superregular subgraph, with weaker parameters. This proof generalises easily to k-graphs, and therefore we postpone it to the appendix.

Lemma 3.2.1. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll d, \delta, 1/k \leq 1$ where $k \geq 2$ is an integer. Suppose that G is a k-partite k-graph with parts $V_1, \dots, V_k$ where $\delta n \leq |V_i| \leq n/\delta$ for all $i \in [k]$. If G is (ε, d)-half-superregular, then G contains a spanning subhypergraph G' that is $(\varepsilon', d^2/2)$-superregular.*

Regularity for a bipartite graph collection $\boldsymbol{G}$ is defined in terms of the 3-graph $G^{(3)}$ of $\boldsymbol{G}$.

Definition 3.2.2 (Regularity, semi- and half-superregularity, superregularity)**.** *Suppose that $\boldsymbol{G}$ is a graph collection with colour set $\mathscr{C}$, where each G_c is bipartite with parts V_1, V_2. Let $G^{(3)}$ be the 3-graph of $\boldsymbol{G}$. We say that*

- *$\boldsymbol{G}$ is* (ε, d)-regular *if $G^{(3)}$ is (ε, d)-regular. That is, for all $V_i' \subseteq V_i$ with $|V_i'| \geq \varepsilon |V_i|$ for $i = 1, 2$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| \geq \varepsilon |\mathscr{C}|$, we have*

$$\left| \frac{\sum_{c \in \mathscr{C}'} e_{G_c}(V_1', V_2')}{|\mathscr{C}'||V_1'||V_2'|} - \frac{\sum_{c \in \mathscr{C}} e_{G_c}(V_1, V_2)}{|\mathscr{C}||V_1||V_2|} \right| < \varepsilon.$$

- *$\boldsymbol{G}$ is* (ε, d)-semi-superregular *if it is (ε, d)-regular and $d_{G^{(3)}}(v) = \sum_{c \in \mathscr{C}} d_{G_c}(v) \geq d|V_{3-i}||\mathscr{C}|$ for all $i \in [2]$ and $v \in V_i$.*
- *$\boldsymbol{G}$ is* (ε, d)-superregular *if $G^{(3)}$ is (ε, d)-superregular. That is, it is (ε, d)-semi-superregular and $d_{G^{(3)}}(c) = e(G_c) \geq d|V_1||V_2|$ for all $c \in \mathscr{C}$.*
- *$\boldsymbol{G}$ is* (ε, d)-half-superregular *if $G^{(3)}$ is (ε, d)-half-superregular. That is, for all $V_i' \subseteq V_i$ with $|V_i'| \geq \varepsilon |V_i|$ for $i = 1, 2$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| \geq \varepsilon |\mathscr{C}|$, we have $\sum_{c \in \mathscr{C}'} e_{G_c}(V_1', V_2') \geq d|\mathscr{C}'||V_1'||V_2'|$ and $\sum_{c \in \mathscr{C}} d_{G_c}(v) \geq d|V_{3-i}||\mathscr{C}|$ for all $i = 1, 2$ and $v \in V_i$, and $e(G_c) \geq d|V_1||V_2|$ for all $c \in \mathscr{C}$.*

Note that, if every G_c with $c \in \mathscr{C}$ is the same, then $\boldsymbol{G}$ is (ε, d)-regular if and only if G_c is (ε, d)-regular; and $\boldsymbol{G}$ is (ε, d)-superregular if and only if G_c is (ε, d)-superregular. The superregularity of $\boldsymbol{G}$ does not imply a minimum degree condition for any graph G_c in the collection, and indeed they could all have isolated vertices.

The following simple lemma shows that, in an (ε, d)-regular graph collection, most vertices – *typical* vertices – have large total degree (the sum of degrees over all colours) and typical colours have many edges.

Lemma 3.2.3 (Typical vertices and colours)**.** *Let $0 < \varepsilon \ll d \leq 1$, and let $\boldsymbol{G}$ be an (ε, d)-regular graph collection with colour set $\mathscr{C}$, where each G_c is bipartite with parts V_1, V_2. Then the following hold:*

(i) for every $i \in [2]$ and all but at most $\varepsilon|V_i|$ vertices $v \in V_i$ we have $\sum_{c\in\mathscr{C}} d_{G_c}(v) \geq (d-\varepsilon)|V_{3-i}||\mathscr{C}|$;

(ii) for all but at most $\varepsilon|\mathscr{C}|$ colours $c \in \mathscr{C}$ we have $e(G_c) \geq (d-\varepsilon)|V_1||V_2|$.

Proof. For (i), let V_1' be the set of vertices in V_1 without this property and suppose for a contradiction that $|V_1'| > \varepsilon|V_1|$. Then (ε, d)-regularity implies that $d_{G^{(3)}}(V_1', V_2, \mathscr{C}) > d - \varepsilon$, but the definition of V_1' implies that $e_{G^{(3)}}(V_1', V_2, \mathscr{C}) \leq |V_1'|(d-\varepsilon)|V_2||\mathscr{C}|$ and hence $d_{G^{(3)}}(V_1', V_2, \mathscr{C}) \leq d - \varepsilon$, a contradiction. By symmetry, the rest of (i) and (ii) are identical. □

The following lemma is a standard tool, written in our graph collection notation.

Lemma 3.2.4 (Slicing lemma)**.** *Let $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll \alpha \ll d \leq 1$, and let $\boldsymbol{G}$ be a graph collection with colour set $\mathscr{C}$, where each G_c is bipartite with parts V_1, V_2 each of size at least n, and let $V_i' \subseteq V_i$ for $i \in [2]$ and $\mathscr{C}' \subseteq \mathscr{C}$. Let $\boldsymbol{G}' = (G_c[V_1', V_2'] : c \in \mathscr{C}')$.*

(i) Suppose that $\boldsymbol{G}$ is (ε, d)-regular. Suppose $|V_i'| \geq \alpha|V_i|$ for $i \in [2]$ and $|\mathscr{C}'| \geq \alpha|\mathscr{C}|$. Then $\boldsymbol{G}'$ is $(\varepsilon/\alpha, d/2)$-regular.

(ii) Suppose that $\boldsymbol{G}$ is (ε, d)-superregular. Suppose $|V_i'| \geq (1-\alpha)|V_i|$ for $i \in [2]$ and $|\mathscr{C}'| > (1-\alpha)|\mathscr{C}|$. Then $\boldsymbol{G}'$ is $(2\varepsilon, d/2)$-superregular.

(iii) Suppose that $\boldsymbol{G}$ is (ε, d)-superregular. Given $n_1 \geq \alpha|V_1|, n_2 \geq \alpha|V_2|$ and $h \geq \alpha|\mathscr{C}|$, if $V_i' \subseteq V_i$ is a uniform random subset of size n_i for each $i = 1, 2$ and $\mathscr{C}' \subseteq \mathscr{C}$ is a uniform random subset of size h, then with high probability, $\boldsymbol{G}'$ is $(\varepsilon/\alpha, d^2/16)$-superregular.

Proof. For (i), firstly we have $d_{G^{(3)}}(V_1', V_2', \mathscr{C}') \geq d - \varepsilon \geq d/2$. Secondly, given any subset $V_i'' \subseteq V_i'$ with $|V_i''| \geq \varepsilon|V_i'|/\alpha \geq \varepsilon|V_i|$ where $i = 1, 2$ and any subset $\mathscr{C}'' \subseteq \mathscr{C}'$ with $|\mathscr{C}''| \geq \varepsilon|\mathscr{C}'|/\alpha \geq \varepsilon|\mathscr{C}|$, by regularity we have

$$
\begin{aligned}
&|d_{G^{(3)}}(V_1'', V_2'', \mathscr{C}'') - d_{G^{(3)}}(V_1', V_2', \mathscr{C}')| \\
&\leq |d_{G^{(3)}}(V_1'', V_2'', \mathscr{C}'') - d_{G^{(3)}}(V_1, V_2, \mathscr{C})| + |d_{G^{(3)}}(V_1, V_2, \mathscr{C}) - d_{G^{(3)}}(V_1', V_2', \mathscr{C}')| \leq 2\varepsilon \leq \varepsilon/\alpha.
\end{aligned}
$$

For (ii), note that $\boldsymbol{G}'$ is $(\varepsilon/(1-\alpha), d/2)$-regular and thus $(2\varepsilon, d/2)$-regular by (i). Let $G_c' := G_c[V_1', V_2']$ for each $c \in \mathscr{C}$. For each $i = 1, 2$ and $v \in V_i'$, we also have $\sum_{c\in\mathscr{C}'} d_{G_c'}(v) \geq$

$d|V_{3-i}||\mathscr{C}|-2\alpha|V_{3-i}||\mathscr{C}| \geq d|V'_{3-i}||\mathscr{C}'|/2$. Combining this with $e(G'_c) \geq d|V_1||V_2|-2\alpha|V_1||V_2| \geq d|V_1||V_2|/2$ for each $c \in \mathscr{C}'$, we get that $\boldsymbol{G}'$ is $(2\varepsilon, d/2)$-superregular.

For (iii), similarly, we always have that $\boldsymbol{G}'$ is $(\varepsilon/\alpha, d/2)$-regular by (i). Let $G'_c := G_c[V'_1, V'_2]$ for all $c \in \mathscr{C}$. We only need to show that with high probability, for each $i = 1, 2$ and $v \in V'_i$, we have $\sum_{c\in\mathscr{C}'} d_{G'_c}(v) \geq d^2|V'_{3-i}||\mathscr{C}'|/16$ and for each $c \in \mathscr{C}'$, we have $e(G'_c) \geq d^2|V_1||V_2|/16$. Suppose that $\mathscr{C}'$ has been chosen. Since $\boldsymbol{G}$ is (ε, d)-superregular, for each $c \in \mathscr{C}'$, we have $e(G_c) \geq d|V_1||V_2|$. Let $V_1^c := \{v \in V_1 : d_{G_c}(v) \geq d|V_2|/2\}$. Then we get $d|V_1||V_2| \leq |V_1^c||V_2| + (|V_1| - |V_1^c|)d|V_2|/2$ and thus $|V_1^c| \geq d|V_1|/(2-d) \geq d|V_1|/2$. A Chernoff bound implies that, with probability $1 - e^{-\Omega(n)}$, we have $|V'_1 \cap V_1^c| \geq dn_1/4$ and $|V'_2 \cap N_{G_c}(v)| \geq dn_2/4$ for each $v \in V_1^c$. Therefore, by a union bound, with high probability, we have $e(G'_c) \geq d^2 n_1 n_2/16$ for each $c \in \mathscr{C}'$. Similarly, with high probability, for each $i = 1, 2$ and $v \in V'_i$, we have $\sum_{c\in\mathscr{C}'} d_{G'_c}(v) \geq d^2 n_{3-i} h/16$. It follows that $\boldsymbol{G}'$ is $(\varepsilon/\alpha, d^2/16)$-superregular with high probability. □

3.2.2 The regularity lemma for graph collections

We use the following version of the regularity lemma for graph collections, which is obtained by applying the degree version of the weak regularity lemma (Lemma 3.7.2) to the 3-graph $G^{(3)}$ of $\boldsymbol{G}$ and cleaning up the clusters so that vertex clusters and colour clusters are separate. We postpone the derivation to the appendix.

Lemma 3.2.5 (Regularity lemma for graph collections)**.** *For all integers $L_0 \geq 1$ and every $\varepsilon, \delta > 0$, there is an $n_0 = n_0(\varepsilon, \delta, L_0)$ such that for every $d \in [0, 1)$ and every graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ on vertex set V of size $n \geq n_0$ with $\delta n \leq |\mathscr{C}| \leq n/\delta$, there exists a partition of V into $V_0, V_1, \ldots, V_L$, of $\mathscr{C}$ into $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_M$ and a spanning subgraph G'_c of G_c for each $c \in \mathscr{C}$ such that the following properties hold:*

- *(i)* $L_0 \leq L, M \leq n_0$ *and* $|V_0| + |\mathscr{C}_0| \leq \varepsilon n$*;*
- *(ii)* $|V_1| = \ldots = |V_L| = |\mathscr{C}_1| = \ldots = |\mathscr{C}_M| =: m$*;*
- *(iii)* $\sum_{c\in\mathscr{C}} d_{G'_c}(v) > \sum_{c\in\mathscr{C}} d_{G_c}(v) - (3d/\delta^2 + \varepsilon)n^2$ *for all* $v \in V$ *and* $e(G'_c) > e(G_c) - (3d/\delta^2 + \varepsilon)n^2$ *for all* $c \in \mathscr{C}$*;*
- *(iv) if, for $c \in \mathscr{C}$, the graph G'_c has an edge with both vertices in a single cluster V_i for some $i \in [L]$, then $c \in \mathscr{C}_0$;*
- *(v) for all triples $(\{h, i\}, j) \in \binom{[L]}{2} \times [M]$, we have that either $G'_c[V_h, V_i] = \emptyset$ for all $c \in \mathscr{C}_j$,*

or $\boldsymbol{G}'_{hi,j} := (G'_c[V_h, V_i] : c \in \mathscr{C}_j)$ *is* (ε, d)*-regular.*

The sets V_i are called *vertex clusters* and the sets $\mathscr{C}_j$ are called *colour clusters*, while V_0 and $\mathscr{C}_0$ are the exceptional vertex and colours sets respectively.

Definition 3.2.6 (Reduced graph collection)**.** *Given a graph collection* $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ *on* V *and parameters* $\varepsilon > 0, d \in [0, 1)$ *and* $L_0 \geq 1$*, the* reduced graph collection $\boldsymbol{R} = \boldsymbol{R}(\varepsilon, d, L_0)$, reduced 3-graph $R^{(3)} = R(\varepsilon, d, L_0)$ *and the* reduced edge-coloured graph $\mathcal{R} = \mathcal{R}(\varepsilon, d, L_0)$ *of* $\boldsymbol{G}$ *are defined as follows. Apply Lemma 4.2.1 to* $\boldsymbol{G}$ *with parameters* ε, d, L_0 *to obtain* $\boldsymbol{G}'$ *and a partition* $V_0, \dots, V_L$ *of* V *and* $\mathscr{C}_0, \dots, \mathscr{C}_M$ *of* $\mathscr{C}$ *where* V_0*,* $\mathscr{C}_0$ *are the exceptional sets and* $V_1, \dots, V_L$ *are the vertex clusters and* $\mathscr{C}_1, \dots, \mathscr{C}_M$ *are the colour clusters. Then* $\boldsymbol{R} = (R_1, \dots, R_M)$ *is a graph collection of* M *graphs each on the same vertex set* $[L]$*, where, for* $(\{h, i\}, j) \in \binom{[L]}{2} \times [M]$*, we have* $hi \in R_j$ *whenever* $\boldsymbol{G}'_{hi,j}$ *is* (ε, d)*-regular.*

Also, $R^{(3)}$ *is the* 3*-graph of* $\boldsymbol{R}$ *and* $\mathcal{R}$ *is the reduced edge-coloured graph.*

The next lemma (related to Lemma 5.5 in (Kühn, Osthus, & Townsend, 2014)) states that clusters inherit a minimum degree bound in the reduced graph from $\boldsymbol{G}$.

Lemma 3.2.7 (Degree inheritance)**.** *Suppose* $p > 0$*,* $L_0 \geq 1$ *and* $0 < 1/n \ll \varepsilon \leq d \ll \delta, \gamma, p \leq 1$*. Let* $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ *be a graph collection on a vertex set* V *of size* n *with* $\delta(G_c) \geq (p+\gamma)n$ *for all* $c \in \mathscr{C}$ *and* $\delta n \leq |\mathscr{C}| \leq n/\delta$*. Let* $\boldsymbol{R} = \boldsymbol{R}(\varepsilon, d, L_0)$ *be the reduced graph collection of* $\boldsymbol{G}$ *on* L *vertices with* M *graphs. Then*

(i) for every $i \in [L]$ *there are at least* $(1 - d^{1/4})M$ *colours* $j \in [M]$ *for which* $d_{R_j}(i) \geq (p + \gamma/2)L$*;*

(ii) for every $j \in [M]$ *there are at least* $(1 - d^{1/4})L$ *vertices* $i \in [L]$ *for which* $d_{R_j}(i) \geq (p + \gamma/2)L$*.*

Proof. To prove (i), note that for all $v \in V \setminus V_0$ we have

$$\sum_{c \in \mathscr{C}} d_{G'_c - V_0}(v) \geq \sum_{c \in \mathscr{C}} d_{G_c}(v) - (3d/\delta^2 + \varepsilon)n^2 - \varepsilon|\mathscr{C}|s \geq \sum_{c \in \mathscr{C}} d_{G_c}(v) - 4dn^2/\delta^2.$$

Let $\mathscr{D}_v$ be the collection of colours c in $\mathscr{C} \setminus \mathscr{C}_0$ for which $d_{G'_c - V_0}(v) \geq d_{G_c}(v) - \sqrt{d}n$. Then

$$\sum_{c \in \mathscr{C}} d_{G'_c - V_0}(v) \leq \sum_{c \in \mathscr{C}} d_{G_c}(v) - |\mathscr{C} \setminus \mathscr{D}_v|\sqrt{d}n$$

and therefore $|\mathscr{C} \setminus \mathscr{D}_v| \leq 4dn^2/(\delta^2\sqrt{d}n) \leq d^{1/3}n/2$ by $d \ll \delta$, so $|\mathscr{D}_v| \geq |\mathscr{C}| - d^{1/3}n/2 - \varepsilon n \geq$

$|\mathscr{C}| - d^{1/3}n$. We have $mM \leq |\mathscr{C}| \leq mM + \varepsilon n$ and $mL \leq n \leq mL + \varepsilon n$. Thus the number of clusters $\mathscr{C}_j$ containing at least one colour of $\mathscr{D}_v$ is at least

$$|\mathscr{D}_v|/m \geq M - d^{1/3}n/m \geq M - d^{1/3}L/(1-\varepsilon) \geq M - 2d^{1/3}M/\delta \geq (1-d^{1/4})M.$$

Now let $i \in [L]$ and $v \in V_i$. For each cluster $\mathscr{C}_j$ as above, choose an arbitrary colour $c_j \in \mathscr{C}_j \cap \mathscr{D}_v$. Then the number of clusters V_h containing some $u \in N_{G'_{c_j}}(v)$ is at least

$$\frac{d_{G_{c_j}}(v) - \sqrt{d}n}{m} \geq \frac{(p+\gamma-\sqrt{d})n}{m} \geq (p+\gamma/2)L.$$

But then Lemma 4.2.1(v) implies that i is adjacent to each such V_h in R_j. So for every $i \in [L]$, $d_{R_j}(i) \geq (p+\gamma/2)L$ for at least $(1-d^{1/4})M$ colours j. The proof of (ii) is similar and we omit it. □

3.2.3 Templates

We define the notion of a 'template', which is essentially a reduced graph in the transversal setting. We will use these as templates for embedding, in the same way that reduced graphs are used for embedding into a single graph.

Definition 3.2.8 (Templates)**.** *Let $0 < 1/m \leq \varepsilon \leq d, \delta \leq 1$ be parameters and let $r \in \mathbb{N}$. Suppose that*

- *R is an r-vertex graph, with vertex set $[r]$ unless otherwise specified,*
- *$\mathcal{V} = \{V_1, \dots, V_r\}$ is a set of r disjoint vertex sets with $m \leq |V_j| \leq m/\delta$ for all $j \in [r]$, whose union we denote by V,*
- *$\mathscr{C} = \bigcup_{e \in E(R)} \mathscr{C}_e$ is a colour set where $|\mathscr{C}_e| \geq \delta m$ for all $e \in E(R)$,*
- *$\boldsymbol{G}$ is a graph collection with colour set $\mathscr{C}$ where for each $c \in \mathscr{C}$, the graph G_c is the union of bipartite graphs G_c^e where G_c^e has parts V_i, V_j, over all $e = ij$ for which $c \in \mathscr{C}_e$. For each $e \in E(R)$, let $\boldsymbol{G}^e = (G_c^e : c \in \mathscr{C}_e)$.*

We say that $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ is an R-template with parameters $(m, \varepsilon, d, \delta)$ *if for every $e \in E(R)$, $\boldsymbol{G}^e$ is (ε, d)-regular. If we replace regular with semi-superregular, it is a* semi-super R-template*; if we replace regular with superregular, it is a* super R-template*; if we replace regular with half-superregular, it is a* half-super R-template*. If $\mathscr{C} = \bigcup_{e \in E(R)} \mathscr{C}_e$ is a partition, we say that the*

template is rainbow.

A transversal embedding of a graph H inside $\mathcal{F}$ *is a copy of* H *with vertices in* V *such that for every edge* e *there is a distinct* $c \in \mathscr{C}$ *such that* $e \in G_c$*. That is, there exist injections* $\tau : V(H) \to V$ *and* $\sigma : E(H) \to \mathscr{C}$ *where* $\tau(x)\tau(y) \in G_{\sigma(xy)}$ *for all* $xy \in E(H)$*.*

Note that the partition $\mathscr{C} = \bigcup_{e \in E(R)} \mathscr{C}_e$ is suppressed in the notation. Given a template $\mathcal{F}$, we explicitly use the notation in the definition unless otherwise specified. Observe that for an R-template $(\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ with parameters $(m, \varepsilon, d, \delta)$ and $v(R) = r$,

$$rm \leq |V| \leq rm/\delta.$$

Given parameters $\beta, \varepsilon', d', \delta' > 0$ with $\beta \leq 1$, $\varepsilon' \geq \varepsilon$, $d' \leq d$ and $\delta' \leq \beta\delta$, any $(m, \varepsilon, d, \delta)$ template is also a $(\beta m, \varepsilon', d', \delta')$ template. We will often take *subtemplates* of templates, meaning that the new vertex clusters, colour clusters and graphs G_c are subsets/subgraphs of the originals. Some convenient notation for this is as follows: if $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ is a template, $\mathscr{C}' \subseteq \mathscr{C}$ and $\mathcal{V}' = \{V_1', \dots, V_r'\}$ where $V_j' \subseteq V_j$ for all $j \in [r]$, we say that $\mathcal{F}' = (\mathcal{V}', \mathscr{C}', \boldsymbol{G}')$ is the *subtemplate of* $\mathcal{F}$ *induced by* $\mathcal{V}', \mathscr{C}'$ when each $\mathscr{C}_e' = \mathscr{C}_e \cap \mathscr{C}'$, and $(G')_c^e := G_c[V_i', V_j']$ is defined for each $c \in \mathscr{C}_e'$. We also say that $\mathcal{F}'$ is *obtained by deleting* $\mathscr{C} \setminus \mathscr{C}'$ *and* $V \setminus V'$.

The following straightforward lemma shows that removing a small fraction of colours and vertices from a template produces a subtemplate with slightly weaker parameters, which remains super if the original template was super.

Lemma 3.2.9 (Template slicing)**.** *Let* $0 < 1/m \ll \varepsilon \ll \varepsilon' \ll \alpha \ll d, \delta, 1/r, 1/k \leq 1$ *where* $r \geq 2$ *is an integer. Let* R *be an* r*-vertex graph and let* $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ *be an* R*-template with parameters* $(m, \varepsilon, d, \delta)$*. Let* $\mathscr{C}' \subseteq \mathscr{C}$*,* $V_i' \subseteq V_i$ *for all* $i \in [r]$ *and let* $\mathcal{F}' = (\mathcal{V}', \mathscr{C}', \boldsymbol{G}')$ *be the subtemplate of* $\mathcal{F}$ *induced by* $\mathcal{V}', \mathscr{C}'$*.*

- *(i) If every* $|\mathscr{C}_e'| \geq \alpha|\mathscr{C}_e|/k$ *and* $\alpha|V_i| \leq |V_i'| \leq k\alpha|V_i|$*, then* $\mathcal{F}'$ *is a template with parameters* $(\alpha m, \varepsilon/\alpha, d/2, \delta/k)$*.*
- *(ii) If every* $|\mathscr{C}_e \setminus \mathscr{C}_e'| \leq \alpha m$ *and* $|V_i \setminus V_i'| \leq \alpha m$*, then* $\mathcal{F}'$ *is a template with parameters* $(m/2, 2\varepsilon, d/2, \delta/2)$*. Moreover, if* $\mathcal{F}$ *is super, then* $\mathcal{F}'$ *is super.*
- *(iii) Given* $\alpha|V_i| \leq n_i \leq k\alpha|V_i|$ *for all* $i \in [r]$ *and* $h_e \geq \alpha|\mathscr{C}_e|/k$ *for all* $e \in E(R)$*, if* V_i' *is a uniform random subset of* V_i *of size* n_i *and* $\mathscr{C}_e'$ *is a uniform random subset of* $\mathscr{C}_e$ *of size* h_e *and* $\mathcal{F}$ *is super, then with high probability,* $\mathcal{F}'$ *is a super template with parameters*

$(\alpha m, \varepsilon/\alpha, d^2/16, \delta/k)$.

(iv) Suppose $\mathcal{F}$ is half-super. For all $c \in \mathscr{C}$, there is $G'_c \subseteq G_c$ such that, defining $\boldsymbol{G}' := (G'_c : c \in \mathscr{C})$, the template $(\mathcal{V}, \mathscr{C}, \boldsymbol{G}')$ is super with parameters $(m, \varepsilon', d^2/2, \delta)$.

Proof. For (i), we have $\alpha m \leq |V'_i| \leq \alpha m/(\delta/k)$ and $|\mathscr{C}'_e| \geq \alpha\delta m/k$. Also, by Lemma 3.2.4(i), for each $e \in E(R)$, $(\boldsymbol{G}')^e$ is $(\varepsilon/\alpha, d/2)$-regular. Thus $\mathcal{F}'$ is a template with parameters $(\alpha m, \varepsilon/\alpha, d/2, \delta/k)$.

For (ii), showing that $\mathcal{F}'$ is a template with the given parameters is similar to (i). Note that if $\mathcal{F}$ is super, then $(\boldsymbol{G}')^e$ is $(2\varepsilon, d/2)$-superregular for each $e \in E(R)$ by Lemma 3.2.4(ii) and thus $\mathcal{F}'$ is a super R-template with parameters $(m/2, 2\varepsilon, d/2, \delta/2)$.

For (iii), if $\mathcal{F}$ is super, then $\boldsymbol{G}^e$ is (ε, d)-superregular for each $e \in E(R)$. Thus by Lemma 3.2.4(iii), with high probability, $(\boldsymbol{G}')^e$ is $(\varepsilon/\alpha, d^2/16)$-superregular for each $e \in E(R)$. Therefore, by (i), $\mathcal{F}'$ is a super template with parameters $(\alpha m, \varepsilon/\alpha, d^2/16, \delta/k)$ with high probability.

Part (iv) follows immediately from Lemma 3.2.1. □

Edges which lie in many graphs G_c are particularly useful for embedding and thus we define the (simple, uncoloured) *thick graph* consisting of all such edges.

Definition 3.2.10 (Thick graph of a template)**.** *Given $\lambda > 0$ and an R-template $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$, let $T^\lambda_\mathcal{F}$ be the simple 2-graph with vertex set V such that $xy \in E(T^\lambda_\mathcal{F})$ whenever $xy \in G_c$ for at least $\lambda|\mathscr{C}_{ij}|$ colours $c \in \mathscr{C}_{ij}$ where $x \in V_i$, $y \in V_j$ and $ij \in E(R)$. We call $T^\lambda_\mathcal{F}$ the* λ-thick graph of $\mathcal{F}$.

The following proposition states that the thick graph of a bipartite semi-super template has a spanning half-superregular subgraph. This is very useful for embedding as one can use the usual blow-up lemma to embed a bounded degree graph into the thick graph, and then greedily assign colours.

Proposition 3.2.11. *Let $0 < 1/m \ll \varepsilon \ll \lambda \ll d, \delta, 1/r \leq 1$ where $r \geq 2$ is an integer. Let R be a graph on r vertices and let $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ be a semi-super R-template with parameters $(m, \varepsilon, d, \delta)$. Then for all $ij \in E(R)$, $T^\lambda_\mathcal{F}[V_i, V_j]$ is $(\varepsilon, d/2)$-half-superregular.*

Proof. Let $ij \in E(R)$ and let $U_i \subseteq V_i$ and $U_j \subseteq V_j$ be any two subsets such that $|U_i| \geq \varepsilon|V_i|$ and $|U_j| \geq \varepsilon|V_j|$. Let $\ell_{ij} := e(T^\lambda_\mathcal{F}[U_i, U_j])$. By regularity we have $|d_{G^{(3)}}(U_i, U_j, \mathscr{C}_{ij}) -$

$d_{G^{(3)}}(V_i, V_j, \mathscr{C}_{ij})| \leq \varepsilon$ and thus we get

$$(d-\varepsilon)|U_i||U_j||\mathscr{C}_{ij}| \leq e_{G^{(3)}}(U_i, U_j, \mathscr{C}_{ij}) \leq \ell_{ij}|\mathscr{C}_{ij}| + \lambda(|U_i||U_j| - \ell_{ij})|\mathscr{C}_{ij}|.$$

Therefore, we have $\ell_{ij}/(|U_i||U_j|) \geq (d-\varepsilon-\lambda)/(1-\lambda) \geq d/2$. Similarly, for any $x \in V_i$, let $\ell_x := d_{T^\lambda_{\mathscr{F}}}(x, V_j)$. By semi-superregularity, we have $\sum_{c\in\mathscr{C}_{ij}} d_{G_c}(x) \geq d|V_j||\mathscr{C}_{ij}|$. Then we get

$$d|V_j||\mathscr{C}_{ij}| \leq \ell_x|\mathscr{C}_{ij}| + \lambda(|V_j| - \ell_x)|\mathscr{C}_{ij}|$$

and thus $\ell_x/|V_j| \geq (d-\lambda)/(1-\lambda) \geq d/2$. The half-superregularity then follows. □

3.3 Embedding lemmas

In this section we state a series of embedding lemmas which we will combine to prove our transversal blow-up lemma, stated at the end of the section. These lemmas are also useful in their own right in applications.

Each lemma is of the following type: we are given an R-template and a bounded degree graph H whose partition matches the template. For a small number of vertices y in H we are given large *target sets* T_y where y must be embedded. The output of the lemma is a transversal embedding of H, which consists of an embedding τ of vertices and an embedding σ of colours, so $\tau(x)\tau(y) \in G_{\sigma(xy)}$ for all $xy \in E(H)$, and such that $\tau(y) \in T_y$ whenever there is a target set.

The first such lemma applies to embed a small graph H into a template. In fact we only embed some of H while finding large *candidate sets* for the rest of H. This means that we set aside some colours so that, for each unembedded vertex y and each of its embedded neighbours x, there is a distinct colour $\sigma(xy)$, and a large vertex set so that if the image of y is chosen in this set, then we can extend our embedding to a transversal embedding that uses the specified colours. That is, for every z in this set we have $\tau(x)z \in G_{\sigma(xy)}$. The proof embeds vertices one by one, at each step fixing the colours that will be used to future neighbours, and is similar to the 'partial embedding lemma', an uncoloured version, in (Böttcher, Schacht, & Taraz, 2008).

Lemma 3.3.1 (Embedding lemma with target and candidate sets)**.** *Let* $0 < 1/m \ll \varepsilon, \gamma \ll \nu' \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1$ *where* $r \geq 2$ *is an integer.*

- *Let R be a graph on vertex set $[r]$ such that $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ is an R-template with parameters $(m, \varepsilon, d, \delta)$.*
- *Let H be a graph with $\Delta(H) \leq \Delta$ for which there is a graph homomorphism $\phi : V(H) \to V(R)$ such that $|\phi^{-1}(j)| \leq \gamma m$ for all $j \in [r]$, and suppose there is a partition $V(H) = X \cup Y$ where $E(H[Y]) = \emptyset$.*
- *For every $w \in V(H)$, suppose there is a set $T_w \subseteq V_{\phi(w)}$ with $|T_w| \geq \nu m$.*

Then there are injective maps $\tau : X \to V$ and $\sigma : E(H) \to \mathscr{C}$ such that

(i) $\tau(x) \in T_x$ for all $x \in X$;

(ii) $\sigma(xy) \in \mathscr{C}_{\phi(x)\phi(y)}$ for all $xy \in E(H)$, and if $xx' \in E(H[X])$, then $\tau(x)\tau(x') \in E(G_{\sigma(xx')})$;

(iii) for all $y \in Y$ there exists $C_y \subseteq V_{\phi(y)} \setminus \tau(X)$ such that $C_y \subseteq \bigcap_{x \in N_H(y) \cap X} N_{G_{\sigma(xy)}}(\tau(x)) \cap T_y$ and $|C_y| \geq \nu' m$.

Proof. Fix an ordering of $V(H)$ where all vertices of X come before any vertex of Y, and for each $x \in X$, let $N^<(x)$ be the set of neighbours of x in H which appear before x in the ordering, and $N^>(x)$ the set of those which appear after. At step (x) (with $x \in X$), we will choose $\tau(x)$ and $\sigma(xy)$ for all $y \in N^>(x)$. Initially, define candidate sets $C_w := T_w$ for all $w \in V(H)$ and $C_{xy} := \mathscr{C}_{\phi(x)\phi(y)}$ for all $xy \in E(H)$. The following comprises step (x), which we perform for each $x \in X$ in order.

$(x,1)$ For all $y \in N^>(x)$, delete all vertices $v \in C_x$ with $\sum_{c \in C_{xy}} |N_{G_c}(v) \cap C_y| < (d - \varepsilon)|C_{xy}||C_y|$.

$(x,2)$ Choose $\tau(x) \in C_x$.

$(x,3)$ For all $u \in V(H)$, delete $\tau(x)$ from C_u.

$(x,4)$ For each $y \in N^>(x)$ in order:

$(x,y,4.1)$ delete all colours $c \in C_{xy}$ with $|N_{G_c}(\tau(x)) \cap C_y| < d|C_y|/2$,

$(x,y,4.2)$ choose $\sigma(xy) \in C_{xy}$,

$(x,y,4.3)$ for all $uv \in E(H)$, delete $\sigma(xy)$ from C_{uv},

$(x,y,4.4)$ delete all vertices $v \in C_y$ with $v \notin N_{G_{\sigma(xy)}}(\tau(x))$.

We claim that, at the end of the process, $|C_x|, |C_{xy}| \geq \nu' m$ for all $x \in V(H)$ and $xy \in E(H)$. First, let us see why the claim implies the lemma. Since candidate sets are never empty and every edge in H is incident to X, $\tau : X \to V$ and $\sigma : E(H) \to \mathscr{C}$ are defined, and by step (3)

and (4.3), they are both injections. For (i), we choose $\tau(x)$ from C_x which is always a (non-empty) subset of T_x. For (ii), we choose $\sigma(xy)$ from C_{xy} which is always a (non-empty) subset of $\mathscr{C}_{\phi(x)\phi(y)}$. If $xx' \in E(H)$ where x' appears after x in the ordering, then we choose $\tau(x')$ from $C_{x'}$ and by step (x,x',4.4) we have $\tau(x') \in N_{G_{\sigma(xx')}}(\tau(x))$. For (iii), given the claim and the fact that Y comes after X in the ordering, it suffices to show that, for all $y \in Y$ and $x \in N^<(y)$, we have $C_y \subseteq N_{G_{\sigma(xy)}}(\tau(x))$. Noting that $x \in X$, this is a consequence of step (x,y,4.4).

It remains to prove the claim. We suppose that it is true up until step (x). The vertex candidate set C_x can only shrink at steps (x,1), (3) and at step (u,x,4.4) for $u \in N^<(x)$. Proposition 3.2.3 and the fact that, currently, $|C_x| \geq \nu' m > \varepsilon |V_{\phi(x)}|$, imply that at step ($x$,1), at most $\Delta\varepsilon|C_x|$ vertices are deleted. At step (3), at most $|X|$ vertices are deleted. At step (u,x,4.4), the colour $\sigma(ux)$ was chosen from C_{ux} which has $|N_{G_{\sigma(ux)}}(\tau(u)) \cap C_x| \geq d|C_x|/2$ due to step (u,x,4.1). Thus C_x shrinks by a factor of at most $d/2$ at this step. Therefore

$$|C_x| \geq ((d/2)^\Delta - \Delta\varepsilon)|T_x| - |X| \geq ((d/2)^\Delta - \Delta\varepsilon)\nu - r\gamma)m \geq \nu(d/3)^\Delta m > \nu' m.$$

The colour candidate set C_{xy}, with x before y in the ordering, can only shrink at steps (x,y,4.1) and (4.3). At most $e(H)$ colours are lost at step (4.3). At step (x,y,4.1), we chose $\tau(x)$ from C_x which, immediately after (x,1), satisfies $\sum_{c \in C_{xy}} |N_{G_c}(\tau(x)) \cap C_y| \geq (d-\varepsilon)|C_{xy}||C_y|$. Writing $A \subseteq C_{xy}$ for the subset of colours which are not deleted at step (x,y,4.1), we have $(d-\varepsilon)|C_{xy}||C_y| \leq |A||C_y| + (d/2)(|C_{xy}| - |A|)|C_y|$ and thus $|A| \geq (d/2-\varepsilon)|C_{xy}|$. Therefore

$$|C_{xy}| \geq (d/2-\varepsilon)|\mathscr{C}_{\phi(x)\phi(y)}| - e(H) \geq (d/2-\varepsilon)\delta m - \Delta r\gamma m \geq \delta d m/3 \geq \nu' m.$$

This completes the proof of the claim, and hence of the lemma. □

The next ingredient is a version of the above lemma where we are embedding a small graph H such that a small fraction of its vertices have target sets, but additionally we now have a very small set of colours which must be used in the embedding (together with any other colours). These prescribed colours will be used on an induced matching M in H. This matching, together with its neighbours, will be embedded greedily and then Lemma 3.3.1 will apply to extend the embedding to the whole of H.

Lemma 3.3.2 (Embedding lemma with targets and prescribed colours)**.** *Let* $0 < 1/m \ll \varepsilon \ll \alpha, \gamma \ll \lambda_1, \lambda_2 \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1$ *where* $r \geq 2$ *is an integer.*

- *Let R be a graph on vertex set $[r]$ and let $(\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ be an R-template with parameters $(m, \varepsilon, d, \delta)$.*
- *Let H be a graph with $\Delta(H) \leq \Delta$ for which there is a graph homomorphism $\phi : V(H) \to V(R)$ such that $|\phi^{-1}(j)| \leq \lambda_1 m$ for all $j \in [r]$ and $e(H[\phi^{-1}(i), \phi^{-1}(j)]) \geq \lambda_2 m$ for all $ij \in E(R)$.*
- *Suppose there is a set $W \subseteq V(H)$ with $|W| \leq \alpha m$, such that for all $w \in W$ there is a set $T_w \subseteq V_{\phi(w)}$ with $|T_w| \geq \nu m$.*
- *For each $e \in E(R)$, let $\mathscr{D}_e \subseteq \mathscr{C}_e$ be a set of at most γm colours and let $\mathscr{D} = \bigcup_{e \in E(R)} \mathscr{D}_e$, and suppose that $e(G_c[V_i, V_j]) \geq d|V_i||V_j|$ for all $c \in \mathscr{D}_{ij}$ and $ij \in E(R)$, and $|\mathscr{C}_e \setminus \mathscr{D}| \geq d|\mathscr{C}_e|$ for all $e \in E(R)$.*

Then there are injective maps $\tau : V(H) \to V$ and $\sigma : E(H) \to \mathscr{C}$ such that

(i) $\tau(x) \in V_{\phi(x)}$ *for all* $x \in V(H)$*;*

(ii) $\tau(w) \in T_w$ *for all* $w \in W$*;*

(iii) for all $xx' \in E(H)$ *we have* $\tau(x)\tau(x') \in E(G_{\sigma(xx')})$*;*

(iv) $\mathscr{D} \subseteq \sigma(E(H))$.

Proof. Let $e_1, \dots, e_s$ be an arbitrary ordering of $E(R)$. Let $\mathscr{D}'_{e_i} := \mathscr{D}_{e_i} \setminus (\mathscr{D}_{e_1} \cup \dots \cup \mathscr{D}_{e_{i-1}})$ for all $i \in [s]$, so that the $\mathscr{D}'_e$ over $e \in E(R)$ are pairwise disjoint, and their union equals $\mathscr{D}$. We claim that for each $e_i = jj' \in E(R)$, we can choose a matching $M_{jj'} \subseteq H[\phi^{-1}(j), \phi^{-1}(j')]$ such that, writing $M := \bigcup_{e \in E(R)} M_e$, we have

- $V(M) \cap W = \emptyset$ and M is an induced matching in H such that for every $y \in V(H) \setminus V(M)$, there is $xx' \in E(M)$ such that $N_H(y, V(M)) \subseteq \{x, x'\}$;
- $|M_e| = |\mathscr{D}'_e|$ for each $e \in E(R)$.

We can do this greedily, as follows. Suppose we have found $M_{e_1}, \dots, M_{e_{i-1}}$ with the required properties, for some $i \geq 1$. We show how to find M_{e_i}, where we write $e_i := jj'$. Vizing's theorem states that every graph J with maximum degree Δ can be properly edge-coloured with at most $\Delta + 1$ colours. Thus J contains a matching of size $\lceil e(J)/(\Delta + 1) \rceil$. Thus $H[V_j, V_{j'}]$ contains a matching of size $\lceil \lambda_2 m/(\Delta + 1) \rceil$. Now, from this matching, delete W and any vertex at distance at most two to any vertex in a previously found matching. There are at most $r - 1$ neighbours h of j in R for which we have defined M_{jh}, and for each one we delete at most $(1 + \Delta + \Delta^2) v(M_{jh}) \leq 4\Delta^2 \gamma m$ such vertices. The total number of deleted vertices is at most $|W| + 8r\Delta^2\gamma m$, which leaves a matching M' of size at least $\lambda_2 m/(4\Delta)$. Now we

will greedily choose a suitable submatching. Suppose we have chosen a subset $E(M^q) := \{x_1y_1, \dots, x_{q-1}y_{q-1}\} \subseteq E(M')$ where $1 \leq q \leq |\mathscr{D}'_{jj'}|$ and the distance in $H \setminus M^q$ between any pair of vertices in $V(M^q)$ is at least three. To choose x_qy_q, we delete all vertices in $V(M')$ at distance in $H \setminus M^q$ at most two. The number of deleted vertices is at most $(1 + \Delta + \Delta^2)2q \leq 4\Delta^2\gamma m < \lambda_2 m/(4\Delta)$. Thus we can always choose x_qy_q from the remaining vertices of M'. We claim that $M^{q+1} := M^q \cup \{x_qy_q\}$ is a suitable extension of M^q. Indeed, x_q, y_q have no neighbours in M^q so M^{q+1} is induced, and if x_q shared a neighbour z with some other $y \in M^q$, then it would be at distance two from y, a contradiction. Thus we can obtain M_{e_i} and hence $M_{e_1}, \dots, M_{e_s}$ satisfying both required properties.

Let $X := V(M)$ and $Y := N_H(X) \setminus X$. We will embed M into the template first, defining injections $\tau : X \to V$ and $\sigma : E(M) \cup E(H[X, Y]) \to \mathscr{D}$ so that $\sigma(M_e) = \mathscr{D}'_e$ for all $e \in E(R)$. We want to choose good images for the vertices $x \in X$ so that there are many choices for each neighbour $y \in N_H(x) \setminus X$.

Now we will embed M greedily, using the fact that $e(G_c) \geq d|V_j||V_{j'}|$ for all $c \in \mathscr{D}_{jj'}$ and $\gamma \ll d$. For each $p \leq e(M)$, let $i \leq s$ be such that $e(M_{e_1}) + \dots e(M_{e_{i-1}}) \leq p < e(M_{e_1}) + \dots + e(M_{e_i})$. Let $\widetilde{M_p} := M_{e_1} \cup \dots \cup M_{e_{i-1}} \cup M'_{e_i}$ where $e(\widetilde{M_p}) = p$ and we have fixed an ordering of $E(M_{e_i})$ and M'_{e_i} is the (possibly empty) matching which consists of an initial segment. Let $X_p := V(\widetilde{M_p})$ and $Y_p := N_H(X_p) \setminus X_p$. Suppose we have found injections $\tau : X_p \to V$ and $\sigma : E(\widetilde{M_p}) \cup E(H[X_p, Y_p]) \to \mathscr{C}$ such that

- $\tau(x) \in V_{\phi(x)}$ for all $x \in X_p$;
- $\tau(x)\tau(y) \in G_{\sigma(xy)}$ for all $xy \in E(\widetilde{M_p})$;
- $\sigma(E(M_{e_h})) = \mathscr{D}'_{e_h}$ for all $h < i$ and $\sigma(E(M'_{e_i})) \subseteq \mathscr{D}'_{e_i}$;
- for all $y \in Y_p$ there exists $C_y \subseteq V_{\phi(y)}$ such that $C_y \subseteq \bigcap_{x \in N_H(y) \cap X_p} N_{G_{\sigma(xy)}}(\tau(x))$, and $|C_y| \geq d|V_{\phi(y)}|/6$.

We want to extend τ and σ by embedding an edge xx' in $M_{e_i} \setminus M'_{e_i}$ with a colour $c^* \in \mathscr{D}'_{e_i}$ and choosing colours and candidate sets for its unembedded neighbours. So fix any such c^* and let $\sigma(xx') := c^*$. Let $N(x), N(x')$ be the set of neighbours of x, x' respectively in $H \setminus M$. The first property of M implies that $N(x), N(y), V(M)$ are pairwise disjoint for all $y \in V(M) \setminus \{x, x'\}$, and similarly for x'. However, we could have $N(x) \cap N(x') \neq \emptyset$ (for example if H is a union of vertex-disjoint triangles).

First, we will define $\tau(x)$ and colours and candidate sets for every vertex in $N(x)$. Write $e_i =$

jj' and $U_h := V_h \setminus X_p$ for $h = j, j'$. We have $e(G_{c^*}[U_j, U_{j'}]) \geq d|V_j||V_{j'}| - 2\gamma m(|V_j| + |V_{j'}|) \geq d|V_j||V_{j'}|/2$. Let

$$Z(x) := \{v \in U_j : d_{G_{c^*}}(v, U_{j'}) \geq d|U_{j'}|/4\}.$$

A simple counting argument implies that $|Z(x)| \geq d|U_j|/4$. We will choose $\tau(x) \in Z(x)$ and candidate sets C_y for each $y \in N(x)$, as follows. Order $N(x)$ as $y_1, \dots, y_\ell$, where $\ell \leq \Delta$, and let $a_1, \dots, a_\ell$ be such that $\phi(y_i) = a_i$, so $a_i \in [r]$. Suppose we have found, for $i \geq 0$,

- $c_i \in \mathscr{C}_{ja_i}$ such that $c^*, c_1, \dots, c_i$ are all distinct and disjoint from $\sigma(E(\widetilde{M}_p))$ and
- a set $Z_i(x) \subseteq Z(x)$ with $|Z_i(x)| \geq (d/6)^i|Z(x)|$ and $d_{G_{c_h}}(z, V_{a_h}) \geq d|V_{a_h}|/6$ for all $h \in [i]$ and $z \in Z_i(x)$.

Now, $\boldsymbol{G}^{ja_{i+1}}[Z_i(x), V_{a_{i+1}}]$ is $(\sqrt{\varepsilon}, d/2)$-regular by Lemma 3.2.4(i) (the slicing lemma), so Lemma 3.2.3(ii) implies there are at least $(1 - \sqrt{\varepsilon})|\mathscr{C}_{ja_{i+1}}|$ colours $c \in \mathscr{C}_{ja_{i+1}}$ with $e(G_c[Z_i(x), V_{a_{i+1}}]) \geq (d/2 - \sqrt{\varepsilon})|Z_i(x)||V_{a_{i+1}}| \geq d|Z_i(x)||V_{a_{i+1}}|/3$. Let c_{i+1} be any such colour which does not lie in the set $\{c^*, c_1, \dots, c_i\} \cup \sigma(E(\widetilde{M}_p))$, which exists since the number of available colours is at least $|\mathscr{C}_{ja_{i+1}} \setminus \mathscr{D}| - 1 - \Delta \geq d|\mathscr{C}_{ja_{i+1}}|/2 \geq d\delta m/2$. By a simple counting argument, the number of vertices $z \in Z_i(x)$ with $d_{G_{c_{i+1}}}(z, V_{a_{i+1}}) \geq d|V_{a_{i+1}}|/6$ is at least $d|Z_i(x)|/6$, and we let the set of these vertices be $Z_{i+1}(x)$. Thus we can complete the iteration and find $Z_\ell(x)$. Now let $\tau(x)$ be an arbitrary vertex of $Z_\ell(x)$, and for each $i \in [\ell]$, define the candidate set $C_{y_i} := N_{G_{c_i}}(\tau(x), V_{a_i})$ and $\sigma(xy_i) := c_i$.

Next, we will define $\tau(x')$ and colours and candidate sets for every vertex in $N(x')$. For each $y \in N(x) \cap N(x')$, the candidate set for y will be a subset of C_y. For this, let $Z(x') := N_{G_{c^*}}(\tau(x), U_{j'})$, so $|Z(x')| \geq d|U_{j'}|/4$ since $\tau(x) \in Z(x)$. Write $N(x') = \{w_1, \dots, w_k\}$ and let $b_1, \dots, b_k$ be such that $\phi(w_i) = b_i$. For each $w_i \in N(x)$, we already defined C_{w_i}; for the other w_i, let $C_{w_i} := V_{b_i}$. Using the fact that $\boldsymbol{G}^{j'b_i}[Z(x'), C_{w_i}]$ is $(\sqrt{\varepsilon}, d/2)$-regular, exactly the same argument for $N(x')$ as for $N(x)$ implies that we can find

- for each $i \in [k]$, $c'_i \in \mathscr{C}_{j'b_i}$ which are distinct and disjoint from $\{c^*, c_1, \dots, c_\ell\} \cup \sigma(E(\widetilde{M}_p))$ and
- $Z_k(x') \subseteq Z(x')$ with $|Z_k(x')| \geq (d/6)^k|Z(x')|$ and $d_{G_{c'_i}}(z, C_{w_i}) \geq d|C_{w_i}|/6$ for all $i \in [k]$ and $z \in Z_k(x')$.

We let $\tau(x')$ be an arbitrary vertex of $Z_k(x')$ and for each $i \in [k]$, let $T_{w_i} := N_{G_{c'_i}}(\tau(x'), C_{w_i})$ and $\sigma(x'w_i) := c'_i$. Note that $\tau(x') \in Z(x') \subseteq N_{G_{c^*}}(\tau(x))$ so $\tau(x)\tau(x') \in G_{c^*} = G_{\sigma(xx')}$. For $y_i \in N(x) \setminus N(x')$, C_{y_i} has not changed since it was first defined, and we let $T_{y_i} := C_{y_i}$. We

have $|T_y| \geq d^2|V_{\phi(y)}|/36 \geq d^2 m/36$ for all $y \in N(x) \cup N(x')$.

Thus we can complete the iteration and embed the whole of M. This completes the required extension of τ and σ, so we have obtained $\tau : X \to V$ and $\sigma : E(M) \cup E(H[X,Y]) \to \mathscr{C}$, with $\sigma(E(M)) = \mathscr{D}$.

Finally, we extend the embedding to the whole of H by applying Lemma 3.3.1. Indeed, let $\mathcal{F}'$ be the subtemplate of $\mathcal{F}$ induced by $\mathcal{V}', \mathscr{C}'$, where $\mathcal{V}' = \{V_1', \dots, V_r'\}$ and $V_i' := V_i \setminus \tau(X)$, and $\mathscr{C}_e' := \mathscr{C}_e \setminus (\sigma(E(M) \cup E(H[X,Y]))$. Lemma 3.2.9(ii) implies that $\mathcal{F}'$ is a template with parameters $(m/2, 2\varepsilon, d/2, \delta/2)$. For each $y \in Y$, we have $|T_y \cap V'_{\phi(y)}| \geq d^2|V_{\phi(y)}|/36 - |\phi^{-1}(\phi(y))| \geq d^2 m/36 - \lambda_1 m \geq d^2 m/37$. Similarly for each $w \in W$, we have $|T_w \cap V'_{\phi(w)}| \geq (\nu - \lambda_1)m \geq \nu m/2$. For all vertices y of H for which T_y is not defined, let $T_y := V_{\phi(y)}$. Thus we can apply Lemma 3.3.1 with parameters $m/2, 2\varepsilon, 2\lambda_1, \min\{\nu, d^2/37\}, d/2, \delta/2$ playing the roles of $m, \varepsilon, \gamma, \nu, d, \delta$, target sets T_y and $Y = \emptyset$ (so no candidate sets), to find the desired embedding. Most of the required properties are immediate but we justify why (iii) holds. If $xx' \in E(M)$ then we already showed that $\tau(x)\tau(x') \in G_{\sigma(xx')}$. If $xy \in E(H) \setminus E(M)$ where $x \in X$, we have $y \in Y$, and then the choice of M guaranteed that $N_H(y, X) \subseteq \{x, x'\}$ where $xx' \in E(M)$ and, if this neighbourhood equals $\{x\}$, we chose $\tau(y) \in T_y \subseteq N_{G_{\sigma(xy)}}(\tau(x))$, while if it equals $\{x, x'\}$ we chose $\tau(y) \in T_y$ and $T_y \subseteq N_{G_{\sigma(xy)}}(\tau(x)) \cap N_{G_{\sigma(x'y)}}(\tau(x'))$. If $xy \in E(H) - X$, then this follows from Lemma 3.3.1. □

Next we state the usual blow-up lemma, which we will use in the proof of our transversal blow-up lemma. Note that the lemma is usually stated in terms of the stronger 'superregular' condition rather than 'half-superregular', but the same proof applies. Alternatively, one can apply Lemma 3.2.1.

Theorem 3.3.3 (Blow-up lemma (Komlós et al., 1998b)). *Let* $0 < 1/m \ll \varepsilon, \alpha \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1$.

- *Let* R *be a graph on vertex set* $[r]$.
- *Let* G *be a graph with vertex classes* $V_1, \dots, V_r$ *where* $m \leq |V_i| \leq m/\delta$ *for all* $i \in [r]$, *such that* $G[V_i, V_j]$ *is* (ε, d)*-half-superregular whenever* $ij \in E(R)$.
- *Let* H *be a graph with* $\Delta(H) \leq \Delta$ *for which there is a graph homomorphism* $\phi : V(H) \to V(R)$ *such that* $|\phi^{-1}(j)| \leq |V_j|$ *for all* $j \in [r]$.
- *For each* $i \in [r]$, *let* $U_i \subseteq \phi^{-1}(i)$ *with* $|U_i| \leq \alpha m$ *and suppose there is a set* $T_x \subseteq V_i$ *with* $|T_x| \geq \nu m$ *for all* $x \in U_i$.

Then there is an embedding of H inside G such that for every $i \in [r]$, every $x \in U_i$ is embedded inside T_x.

The final embedding lemma that we prove in this section applies to embed a *spanning graph* H which consists of small components and such that a small fraction of its vertices have target sets. The template $\mathcal{F}$ is required to be semi-super (every vertex has large total degree) and rainbow, since H could have many edges in every pair, and additionally $\mathcal{F}$ must contain many more colours than required for a transversal embedding.

Lemma 3.3.4 (Embedding lemma with extra colours)**.** *Let $0 < 1/m \ll \varepsilon, \mu, \alpha \ll \beta \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1/2$ where r is an integer.*

- *Let R be an r-vertex graph and let $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ be a semi-super rainbow R-template with parameters $(m, \varepsilon, d, \delta)$.*
- *Let $n := |V|$ and suppose that H is an n-vertex graph with $\Delta(H) \leq \Delta$ which is the union of vertex-disjoint components of size at most μn, and there is a graph homomorphism ϕ of H into R such that $|\phi^{-1}(j)| = |V_j|$ for all $j \in [r]$ and $e(H[\phi^{-1}(i), \phi^{-1}(j)]) \leq |\mathscr{C}_{ij}| - \beta m$ for all $ij \in E(R)$.*
- *For each $i \in [r]$, let $U_i \subseteq \phi^{-1}(i)$ with $|U_i| \leq \alpha m$ and suppose there is a set $T_x \subseteq V_i$ with $|T_x| \geq \nu m$ for each $x \in U_i$.*

Then there is a transversal embedding of H inside $\mathcal{F}$ such that for every $i \in [r]$, every $x \in U_i$ is embedded inside T_x.

Proof. Choose new parameters $\mu', \gamma, \zeta, \lambda$ satisfying $\varepsilon, \mu, \alpha \ll \gamma \ll \mu' \ll \zeta \ll \lambda \ll \beta$. Note that n and m are similar in size, since

$$rm \leq n \leq rm/\delta, \tag{3.1}$$

so in particular $\delta n/r \leq m \leq |V_j| \leq m/\delta \leq n/(r\delta)$ for all $j \in [r]$. We begin by partitioning H into parts which will later be embedded sequentially.

We claim that there is a partition $V(H) = B_0 \cup B_1 \cup \ldots \cup B_s$ such that every edge of H is inside some $H[B_i]$, and, for all $0 \leq i \leq s$ and $j \in [r]$, writing $B_{ij} := B_i \cap \phi^{-1}(j)$ and $b_{ij} := |B_{ij}|$, we

have

$$\delta^2\mu' m \leq b_{0j} \leq 3r^2\mu' m/\delta \quad \text{for all } j \in [r], \tag{3.2}$$

$$b_{ij} \leq 3r\mu' m/\delta \quad \text{for all } i \in [s] \text{ and } j \in [r], \tag{3.3}$$

$$s \leq 1/\mu'. \tag{3.4}$$

To see this, first note that our assumption is that H is the vertex-disjoint union of connected components $H_1, \dots, H_t$ each of size at most μn. Whenever we have distinct i, j such that $|V(H_i)|, |V(H_j)| \leq \mu' n$, we merge them together into a larger subgraph of H. This process terminates when each subgraph has order at most $2\mu' n$, and at most one subgraph has order at most $\mu' n$. If such a small subgraph exists, we merge it with another. Let $B'_1, \dots, B'_{s'}$ be the vertex sets of these subgraphs, so

$$\mu' n < |B'_i| \leq 3\mu' n \quad \text{for all } i \in [s'] \tag{3.5}$$

and there are no edges of H between B'_i and B'_j when $i \neq j$.

Suppose for a contradiction that there exists $j_0 \in [r]$ such that $|B'_i \cap \phi^{-1}(j_0)| < \delta^2\mu' m$ for all $i \in [s']$. Note that $\phi^{-1}(j_0)$ is covered by the B'_i and thus we obtain $s' \geq |\phi^{-1}(j_0)|/(\delta^2\mu' m) \geq 1/(\delta^2\mu')$. Then

$$n/\delta \overset{(3.1)}{>} (r-1)m/\delta \geq \sum_{j\in[r]\setminus\{j_0\}} |V_j| = \sum_{i\in[s']} |B'_i \setminus \phi^{-1}(j_0)| \overset{(3.5)}{\geq} s'(\mu' n - \delta^2\mu' m) \overset{(3.1)}{\geq} s'\mu' n/2 \geq n/(2\delta^2),$$

which is a contradiction since $\delta \leq 1/2$.

Thus, for each $j \in [r]$, there exists $i \in [s']$ such that $|B'_i \cap \phi^{-1}(j)| \geq \delta^2\mu' m$. Let $i_1, \dots, i_r$ be such that $|B'_{i_j} \cap \phi^{-1}(j)| \geq \delta^2\mu' m$ for all $j \in [r]$ and take $B_0 := \cup_{j\in[r]} B'_{i_j}$ and relabel the remaining B'_i as $B_1, \dots, B_s$. Now (3.2) and (3.3) follow by construction, (3.1) and (3.5). Finally, (3.4) follows from (3.5). This completes the proof of the claim.

In the next part of the proof, we will embed $H[B_1], \dots, H[B_s]$ in turn. For this, we first partition each V_j into sets for each stage of the embedding, and find a buffer set of colours which will be used in the final part of the embedding. Given an edge xy of G, write $c(xy) := \{c \in \mathscr{C} : xy \in G_c\}$. For each $jj' \in E(R)$ and $x \in V_j$, define $M_{jj'}(x) := \{y \in V_{j'} : |c(xy)| \geq d|\mathscr{C}_{jj'}|/2\}$. We

claim that each such set satisfies

$$|M_{jj'}(x)| \geq d|V_{j'}|/2.$$

Indeed, semi-superregularity implies that

$$d|V_{j'}||\mathscr{C}_{jj'}| \leq \sum_{c\in\mathscr{C}_{jj'}} d_{G_c}(x, V_{j'}) = \sum_{y\in V_{j'}} |c(xy)| \leq |M_{jj'}(x)||\mathscr{C}_{jj'}| + |V_{j'}|d|\mathscr{C}_{jj'}|/2,$$

as required.

For each $j \in [r]$, let $V_j^0 \cup V_j^1 \cup \ldots \cup V_j^s$ be a random partition of V_j into parts of size $b_{0j} - s\gamma m, b_{1j} + \gamma m, \ldots, b_{sj} + \gamma m$ respectively. For every $i \in [s]$, $j \in [r]$ and $x \in B_{ij}$, we will embed x into V_j^i, which has size slightly larger than necessary. For each $e \in E(R)$, let $\mathscr{C}_e^0$ be a uniform random subset of $\mathscr{C}_e$ of size $\zeta|\mathscr{C}_e|$ and let $\widetilde{\mathscr{C}_e} := \mathscr{C}_e \setminus \mathscr{C}_e^0$.

We have $b_{0j} - s\gamma m \geq (\delta^2\mu' - s\gamma)m > \delta^2\mu' m/2$ by (3.2). Thus all parts are sufficiently large for a Chernoff bound to allow us to assume that the following hold:

(C1) for all $jj' \in E(R)$, $x \in V_j$ and $y \in M_{jj'}(x)$, we have $|c(xy) \cap \mathscr{C}_{jj'}^0| \geq d|\mathscr{C}_{jj'}^0|/4$;

(C2) for all $jj' \in E(R)$ and $x \in V_j$ we have $|M_{jj'}(x) \cap V_{j'}^0| \geq d|V_{j'}^0|/4$;

(C3) for all $0 \leq i \leq s$ and $x \in V(B_i)$ where T_x is defined, we have $|T_x'| \geq \nu|V_{\phi(x)}^i|/2$, where $T_x' := T_x \cap V_{\phi(x)}^i$.

Indeed, each of these lower bounds is at most half the expectation of the quantity in question.

We iteratively construct the transversal embedding of $H[B_1 \cup \ldots \cup B_s]$. Suppose that we have obtained an embedding g_{i-1} for some $i \geq 1$, such that

(i) g_{i-1} is a transversal embedding of $H[W_{i-1}]$, where $W_{i-1} := B_1 \cup \ldots \cup B_{i-1}$;

(ii) $g_{i-1}(x) \in V_j^{i'}$ for every $x \in B_{i'j} \subseteq W_{i-1}$;

(iii) for every $jj' \in E(R)$ and every embedded pair x, y of vertices where $xy \in E(H[V_j, V_{j'}])$, the colour of xy is in $\widetilde{\mathscr{C}_{jj'}}$ and these colours are distinct over the embedding; and

(iv) if $x \in U_j \cap W_{i-1}$, then $g_{i-1}(x) \in T_x$ for every $j \in [r]$.

That is, g_{i-1} consists of injections $\tau : W_{i-1} \to V$ and $\sigma : E(H[W_{i-1}]) \to \mathscr{C}$ such that $\tau(x)\tau(y) \in G_{\sigma(xy)}$, with the stated properties.

Now we will extend g_{i-1} to a transversal embedding g_i of $W_i := W_{i-1} \cup B_i$ with the analogous properties for i. First observe that the vertices we are about to embed, that is, $B_i = W_i \setminus W_{i-1}$,

have no previously embedded neighbours since every edge of H lies inside some B_i. Therefore it suffices to find a transversal embedding of B_i when B_{ij} is embedded into V_j^i for each $j \in [r]$ which uses unused colours.

Obtain $\mathscr{C}_e^i$ from $\widetilde{\mathscr{C}_e}$ by deleting any colour we have used in the embedding g_{i-1}. Every colour cluster in the original template contains extra colours, so for every $jj' \in E(R)$, we have

$$|\mathscr{C}_{jj'}^i| \geq |\mathscr{C}_{jj'}| - \sum_{1\leq \ell \leq i-1} e(H[B_{\ell j}, B_{\ell j'}]) - \zeta|\mathscr{C}_{jj'}| \geq (1-\zeta)|\mathscr{C}_{jj'}| - e(H[B_i]) \geq \beta m/2. \quad (3.6)$$

For each $j \in [r]$ and $j' \in N_R(j)$, let

$$V_j^{i,j',\mathrm{bad}} := \{v \in V_j^i : \textstyle\sum_{c\in\mathscr{C}_{jj'}^i} d_{G_c}(v, V_{j'}^i) < 2d|V_{j'}^i||\mathscr{C}_{jj'}^i|/3\}.$$

Lemma 3.2.4(i) implies that $\boldsymbol{G}_{jj'}^i := (G_c[V_j^i, V_{j'}^i] : c \in \mathscr{C}_{jj'}^i)$ is $(\sqrt{\varepsilon}, d/2)$-regular and thus, by Lemma 3.2.3(i) and (3.3), we have that $|V_j^{i,j',\mathrm{bad}}| < \sqrt{\varepsilon}|V_j^i| < \varepsilon^{1/3}m$. For each $i \in [s]$ and $j \in [r]$, obtain a subset Y_j^i of V_j^i by removing $V_j^{i,j',\mathrm{bad}}$ for each $j' \in N_R(j)$ so that $|Y_j^i| \geq b_{ij} + \gamma m - r\varepsilon^{1/3}m \geq b_{ij} + \gamma m/2$. We further have $|Y_j^i| \leq b_{ij} + \gamma m \leq 4\mu' rm/\delta$ by (3.3). So cluster sizes differ by a factor of at most $(\gamma m/2)/(4\mu' rm/\delta) = \gamma\delta/(8\mu' r) \leq \sqrt{\gamma}$. Let $\mathcal{Y}^i := \{Y_1^i, \dots, Y_r^i\}$ and let $\mathscr{C}^i := \bigcup_{e\in E(R)} \mathscr{C}_e^i$.

We claim that the subtemplate $\mathcal{F}_i := (\mathcal{Y}^i, \mathscr{C}^i, \boldsymbol{G}^i)$ of $\mathcal{F}$ induced by $\mathcal{Y}^i$, $\mathscr{C}^i$ is a semi-super rainbow R-template with parameters $(\gamma m/2, \sqrt{\varepsilon}, d/2, \sqrt{\gamma})$. That vertex clusters have suitable sizes follows from (3.3), and that colour clusters have suitable sizes follows from (3.6). We have seen that $\boldsymbol{G}_{jj'}^i$ is $(\sqrt{\varepsilon}, d/2)$-regular. Moreover, for all $jj' \in E(R)$ and $v \in Y_j^i$, since $v \notin V_j^{i,j',\mathrm{bad}}$, we have

$$\sum_{c\in\mathscr{C}_{jj'}^i} d_{G_c}(v, Y_{j'}^i) \geq 2d|V_{j'}^i||\mathscr{C}_{jj'}^i|/3 - \varepsilon^{1/3}m|\mathscr{C}_{jj'}^i| > d|V_{j'}^i||\mathscr{C}_{jj'}^i|/2,$$

so $\boldsymbol{G}_{jj'}^i$ is $(\sqrt{\varepsilon}, d/2)$-semi-superregular. This completes the proof of the claim.

Therefore, for each $jj' \in E(R)$, we can apply Proposition 3.2.11 to see that the λ-thick graph $T_{\mathcal{F}_i}^\lambda$ is such that $T_{jj'}^i := T_{\mathcal{F}_i}^\lambda[Y_j^i, Y_{j'}^i]$ is $(\sqrt{\varepsilon}, d/4)$-half-superregular for all $jj' \in E(R)$. Apply Theorem 3.3.3 (the blow-up lemma) with target sets T_x' and parameters $\gamma m/2, \sqrt{\varepsilon}, \sqrt{\alpha}, \nu/2, d/4$ playing the roles of $m, \varepsilon, \alpha, \nu, d$ to embed $H[B^i]$ into $T_{jj'}^i$ such that for every $j \in [r]$, every

$x \in U_j \cap B_i$ is mapped to $T'_x \subseteq V^i_j$, which is possible since, by (C3), $|T'_x| \geq \nu\gamma m/2$ and $|U_j \cap B_i| \leq \alpha m$. Since for each edge of $T^i_{jj'}$, the number of unused colour graphs it lies in is

$$\lambda|\mathscr{C}^i_{jj'}| \overset{(3.6)}{>} \lambda\beta m/2 > 3\mu' r^2 m\Delta/\delta \geq \Delta \sum_{j\in[r]} b_{ij} > e(H[B_i]),$$

we can greedily assign colours to the embedding. This completes the construction of g_i which satisfies (i)–(iv).

Thus we can obtain g_s with the required properties. At the end of this process, the unembedded part of H is $H[B_0]$, and the set of vertices of V_j which are not an image of any embedded vertex of H is precisely $Z_j := V^0_j \cup \bigcup_{i\in[s]}(V^i_j \setminus g_s(B_{i,j}))$, which has size b_{0j}. We will use the colours in $\mathscr{C}^0 := \bigcup_{e\in E(R)} \mathscr{C}^0_e$ to embed each B_{0j} into Z_j.

We claim that the subtemplate $\mathcal{F}_0 := (\mathcal{Z}, \mathscr{C}^0, \boldsymbol{G}_0)$ of $\mathcal{F}$ induced by $\mathcal{Z} := (Z_j : j \in [r]), \mathscr{C}^0$ is a semi-super rainbow template with parameters $(\delta^2\mu' m, \sqrt{\varepsilon}, d^2/16, \zeta)$. Indeed, (3.2) implies that vertex clusters have sizes satisfying $\delta^2\mu' m \leq b_{0j} = |Z_j| \leq (\delta^2\mu' m)\cdot 3r^2/\delta^3 < (\delta^2\mu' m)/\zeta$. Colour clusters have size $|\mathscr{C}^0_{jj'}| = \zeta|\mathscr{C}_{jj'}| \geq \zeta\delta m > \zeta(\delta^3/3r^2)m$. Lemma 3.2.4(i) implies that $\boldsymbol{G}^0_{jj'} := (G_c[V^0_j, V^0_{j'}] : c \in \mathscr{C}^0_{jj'})$ is $(\sqrt{\varepsilon}, d/2)$-regular for every $jj' \in E(R)$. Additionally, for all $x \in Z_j \subseteq V_j$,

$$\begin{aligned}\sum_{c\in\mathscr{C}^0_{jj'}} d_{G_c}(x, Z_{j'}) = \sum_{y\in V^0_{j'}} |c(xy) \cap \mathscr{C}^0_{jj'}| &\geq \sum_{y\in M_{jj'}(x)\cap V^0_{j'}} |c(xy) \cap \mathscr{C}^0_{jj'}| \overset{(C1),(C2)}{\geq} d|V^0_{j'}|/4 \cdot d|\mathscr{C}^0_{jj'}|/4 \\ &= d^2|V^0_{j'}||\mathscr{C}^0_{jj'}|/16,\end{aligned}$$

so the template is semi-super.

As before, we can apply Proposition 3.2.11 to see that the λ-thick graph $T^\lambda_{\mathcal{F}_0}$ is such that $T^0_{jj'} := T^\lambda_{\mathcal{F}^0}[Z^0_j, Z^0_{j'}]$ is $(\sqrt{\varepsilon}, d/4)$-half-superregular for each $jj' \in E(R)$. We have $\alpha m < \sqrt{\alpha}|Z_j|$, and (C3) implies that $|T'_x| \geq \nu|V^0_{\phi(x)}|/2 \geq \nu|Z_{\phi(x)}|/3$ for all $x \in V(B_0)$ with a target set. Thus we can apply Theorem 3.3.3 (the blow-up lemma) with target sets T'_x and parameters $\mu' m, \sqrt{\varepsilon}, \sqrt{\alpha}, \nu/3, d/4$ playing the roles of $m, \varepsilon, \alpha, \nu, d$ to embed $H[B_0]$ into $T^0_{jj'}$ such that for every $j \in [r]$, every $x \in U_j$ is mapped to $T'_x \subseteq V^0_j$, which is possible by (C3). The number of

colours on each edge of $T^0_{jj'}$ is at least

$$\lambda|\mathscr{C}^0_{jj'}| = \lambda\zeta|\mathscr{C}_{jj'}| \geq \lambda\zeta\delta m > 3\Delta r^3\mu' m/\delta > \Delta\sum_{j\in[r]} b_{0j} > e(H[B_0]).$$

Thus we can again greedily assign colours of $\mathscr{C}^0_{jj'}$ to obtain a transversal embedding of $H[B_0]$ using colours untouched by the previous embedding, which completes the transversal embedding of H. □

Finally we state a transversal blow-up lemma, which we prove in the next section by combining the embedding lemmas of this section. Its statement is slightly stronger than Theorem 3.1.8 since the template is super rather than half-super, but the proof of Theorem 3.1.8 is a simple matter of applying Lemma 3.2.9(iv) (essentially Lemma 3.2.1) first. The details can be found at the end of the next section.

Theorem 3.3.5 (Transversal blow-up lemma)**.** *Let* $0 < 1/m \ll \varepsilon, \mu, \alpha \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1/2$ *where* r *is an integer.*

- *Let* R *be a simple graph with vertex set* $[r]$ *and let* $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ *be a rainbow super* R*-template with parameters* $(m, \varepsilon, d, \delta)$*.*
- *Suppose that* H *is a* μ*-separable graph with* $\Delta(H) \leq \Delta$ *and there is a graph homomorphism* ϕ *of* H *into* R *such that* $|\phi^{-1}(j)| = |V_j|$ *for all* $j \in [r]$ *and* $e(H[\phi^{-1}(i), \phi^{-1}(j)]) = |\mathscr{C}_{ij}|$ *for all* $ij \in E(R)$*.*
- *For each* $i \in [r]$*, let* $U_i \subseteq \phi^{-1}(i)$ *with* $|U_i| \leq \alpha m$ *and suppose there is a set* $T_x \subseteq V_i$ *with* $|T_x| \geq \nu m$ *for each* $x \in U_i$*.*

Then there is a transversal embedding of H *inside* $\mathcal{F}$ *such that for every* $i \in [r]$*, every* $x \in U_i$ *is embedded inside* T_x*.*

3.4 Proof of the transversal blow-up lemma

3.4.1 Sketch of the proof

The proof is a combination of the usual blow-up lemma applied to the thick graph, together with the rainbow partial embedding lemmas in the previous section and the colour absorbing approach pioneered by Montgomery, Müyesser and Pehova in (Montgomery et al., 2022).

We begin by outlining the steps of the proof in (Montgomery et al., 2022), which is a common sequence of steps for embeddings using absorption, and which we shall also use. The transversal bandwidth theorem in (Chakraborti, Im, Kim, & Liu, 2023) also follows this general outline, as well as using the (usual) blow-up lemma as a key tool. Let $H = H_{\text{con}} \cup H_{\text{abs}} \cup H_{\text{app}} \cup H_{\text{col}} \cup H_{\text{vx}}$ be a suitable partition, which will be carefully chosen. These parts will be embedded in turn, each time using new colours to extend the transversal embedding.

Step 0. *Embed a small 'connecting graph'.* Embed H_{con} whose vertex set, as guaranteed by separability, is such that its removal disconnects H into very small components (Lemma 3.3.1).

Step 1. *Find a colour absorber.* Embed H_{abs} into the edge-coloured graph $\mathcal{G}$ of G, and find disjoint sets $\mathscr{A}, \mathscr{B} \subseteq \mathscr{C}$ of colours such that, given any $e(H_{\text{abs}}) - |\mathscr{A}|$ colours in $\mathscr{B}$, we can choose the colours of the edges of H_{abs} using exactly those colours and the colours in $\mathscr{A}$.

Step 2. *Use most of the colours outside $\mathscr{A} \cup \mathscr{B}$.* Embed H_{app} using most of the colours outside $\mathscr{A} \cup \mathscr{B}$ (Lemma 3.3.4).

Step 3. *Use the remaining colours outside $\mathscr{A} \cup \mathscr{B}$.* Embed H_{col} using every unused colour in $\mathscr{C} \setminus (\mathscr{A} \cup \mathscr{B})$ as well as some colours of $\mathscr{B}$ (Lemma 3.3.2).

Step 4. *Embed the remaining vertices of H using colours in $\mathscr{B}$.* Embed H_{vx} using colours of $\mathscr{B}$ (Lemma 3.3.4).

Step 5. *Use the colour absorber.* The colours for H_{abs} will consist of $\mathscr{A}$ along with the unused colours in $\mathscr{B}$.

For Step 1, we use the following lemma from (Montgomery et al., 2022) which is their key tool for colour absorption. Here we present a slightly weaker version with modified parameters for simplicity.

Lemma 3.4.1 ((Montgomery et al., 2022))**.** *Let $0 < \lambda_1 \ll \lambda_2 \ll \lambda_3 < 1$ and let ℓ, m, n be integers with $\ell = \lambda_1 m$ and $1 \leq m \leq \lambda_3^2 n/8$. Suppose that $G = (U, \mathscr{C})$ is a bipartite graph with $|U| = m$ and $|\mathscr{C}| = n$ such that $d(v, \mathscr{C}) \geq \lambda_3 n$ for each $v \in U$. Then there exist disjoint subsets $\mathscr{A}, \mathscr{B} \subseteq \mathscr{C}$ with $|\mathscr{A}| = m - \ell$ and $|\mathscr{B}| = \lambda_2 n$ such that, for every subset $\mathscr{B}^0$ of $\mathscr{B}$ with size ℓ, we can find a perfect matching between U and $\mathscr{A} \cup \mathscr{B}^0$.*

3.4.2 Proof of Theorem 3.3.5

Proof of Theorem 3.3.5. Without loss of generality, we may assume that $\varepsilon \ll \mu \ll \alpha$. We further define constants $\lambda_1, p_{\text{abs}}, p_{\text{col}}, p_{\text{vx}}, \lambda_3, \nu'$ so that altogether

$$0 < 1/m \ll \varepsilon \ll \mu \ll \alpha \ll \lambda_1 \ll p_{\text{abs}} \ll p_{\text{col}} \ll p_{\text{vx}} \ll \lambda_3 \ll \nu' \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1/2.$$

Let $p_{\text{app}} := 1-(p_{\text{vx}}+p_{\text{abs}}+p_{\text{col}})$. Let R be a graph on vertex set $[r]$, $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ a rainbow super template with parameters $(m, \varepsilon, d, \delta)$ and H a graph as in the statement. Let $n := |V|$ be the number of vertices in the template, which equals $v(H)$. Note that

$$\begin{gathered} rm \leq n \leq rm/\delta, \quad \text{and} \\ m \leq |\phi^{-1}(i)| = |V_i| \leq m/\delta \ \text{ for all } i \in [r], \quad \text{and} \quad \delta m \leq |\mathscr{C}_e| \leq \Delta m/\delta \ \text{ for all } e \in E(R), \end{gathered} \tag{3.7}$$

where the final assertion follows from the fact that every $|\mathscr{C}_{ij}| = e(H[\phi^{-1}(i), \phi^{-1}(j)]) \leq \Delta|\phi^{-1}(i)| \leq \Delta m/\delta$.

Preparation of H. First we will choose a partition $H = H_{\text{con}} \cup H_{\text{abs}} \cup H_{\text{app}} \cup H_{\text{vx}} \cup H_{\text{col}}$ such that each part $H_\circ$ has roughly a given number $p_\circ n$ of vertices, and moreover $e(H_\circ[\phi^{-1}(i), \phi^{-1}(j)]) \approx p_\circ e(H[\phi^{-1}(i), \phi^{-1}(j)])$ for all $ij \in E(R)$ and $\circ \in \{\text{abs}, \text{app}, \text{vx}, \text{col}\}$. Since H is μ-separable, there is a set X of size at most μn such that $H - X$ consists of disjoint components $H_1, \dots, H_t$, each of size at most μn. For all $\ell \in [t]$ and $ij \in E(R)$, let $H_{ij}^\ell := H_\ell[\phi^{-1}(i), \phi^{-1}(j)]$ and let $n_j^\ell := |V(H_\ell) \cap \phi^{-1}(j)|$ and $h_{ij}^\ell := e(H_{ij}^\ell)$. Let $H_{\text{con}} := H[X]$. Note that

$$|X| = |V(H_{\text{con}})| \leq \mu n \leq r\mu m/\delta \leq \sqrt{\mu} m. \tag{3.8}$$

Independently, for each $\ell \in [t]$, add $H^\ell := \bigcup_{ij \in E(R)} H_{ij}^\ell$ to $H_\circ$ with probability $p_\circ$ for $\circ \in \{\text{abs}, \text{app}, \text{vx}, \text{col}\}$. Then, for all $i \in [r]$ we have $\mathbb{E}(|V(H_\circ) \cap \phi^{-1}(i)|) = p_\circ |V_i| \pm |X| \geq (p_\circ - \sqrt{\mu})m$, and for all $ij \in E(R)$ we have

$$\begin{aligned} \mathbb{E}(e(H_\circ[\phi^{-1}(i), \phi^{-1}(j)])) = p_\circ \sum_{\ell \in [t]} h_{ij}^\ell = p_\circ e(H[\phi^{-1}(i), \phi^{-1}(j)]) \pm \Delta|X| = p_\circ |\mathscr{C}_{ij}| \pm \Delta\sqrt{\mu} m \\ \geq (p_\circ \delta - \Delta\sqrt{\mu})m. \end{aligned}$$

Now, $t \geq (1-\mu)/\mu \geq 1/(2\mu)$ and $e(H_{ij}^{\ell}) \leq \Delta n_i^{\ell} \leq \Delta\mu n$ for all $\ell \in [t]$ and $ij \in E(R)$, and each of these expectations is at least $\delta p_{\mathrm{abs}} m/2$, so a Chernoff bound implies that these values are within a multiplicative factor of $(1 \pm \frac{1}{3})$ of their expectations with probability at least

$$1 - 4\left(r + \binom{r}{2}\right)\exp\left(\frac{-(\delta p_{\mathrm{abs}} m/2)^2/9}{\frac{1}{2\mu}(\Delta\mu n)^2}\right) \geq 1 - 4r^2\exp\left(\frac{-p_{\mathrm{abs}}^2\delta^4}{18r\Delta\mu}\right) \geq \frac{1}{2}.$$

Thus we may assume that, for all $\circ \in \{\mathrm{abs}, \mathrm{app}, \mathrm{vx}, \mathrm{col}\}$,

$$n_i^{\circ} := |V(H_{\circ}) \cap \phi^{-1}(i)| = (1 \pm \tfrac{1}{2})p_{\circ}|V_i| \in [p_{\circ}m/2, 3p_{\circ}m/(2\delta)] \quad \text{for all } i \in V(R); \tag{3.9}$$

$$h_{ij}^{\circ} := e(H_{\circ}[\phi^{-1}(i), \phi^{-1}(j)]) = (1 \pm \tfrac{1}{2})p_{\circ}|\mathscr{C}_{ij}| \in [p_{\circ}\delta m/2, 3\Delta p_{\circ}m/(2\delta)] \quad \text{for all } ij \in E(R). \tag{3.10}$$

We write $H_{ij}^{\circ} := H_{\circ}[\phi^{-1}(i), \phi^{-1}(j)]$ for all $ij \in E(R)$. Every vertex $x \in V(H) \setminus V(H_{\mathrm{con}})$ lies in some $H_{\circ}$ where $\circ \in \{\mathrm{abs}, \mathrm{app}, \mathrm{vx}, \mathrm{col}\}$, and x can only have neighbours inside $H_{\circ}$ or $V(H_{\mathrm{con}})$. Note here that for each $\circ \in \{\mathrm{abs}, \mathrm{app}, \mathrm{col}, \mathrm{vx}\}$, $H_{\circ}$ is the union of vertex-disjoint components of size at most $\mu n \leq (2\mu/p_{\circ})|V(H_{\circ})| \leq \sqrt{\mu}|V(H_{\circ})|$, by (3.9).

To find a transversal embedding of H, we need to define two injective maps $\tau : V(H) \to V$ and $\sigma : E(H) \to \mathscr{C}$ where $\tau(x)\tau(y) \in G_{\sigma(xy)}$ for all $xy \in E(H)$. For this, we will define $\tau(x)$ for every vertex in H_{con}, followed by every vertex in H_{abs}, H_{app}, H_{col}, H_{vx}. When defining τ for vertices in $H_{\circ}$, we simultaneously define σ for all future incident edges except for $\circ = \mathrm{abs}$, for which colours are defined at the end. Whenever we define $\tau(y)$ for some y we always choose $\tau(y) \in V_{\phi(y)}$ and if y has a target set T_y, we ensure $\tau(y) \in T_y$. We also always choose $\sigma(xy) \in \mathscr{C}_{\phi(x)\phi(y)}$.

Step 0. We first embed H_{con} into $\mathcal{F}$ by applying Lemma 3.3.1, as follows. Recall that $X = V(H_{\mathrm{con}})$, and define $Y := N_H(X) \setminus X$. Let H'_{con} be the graph with vertex set $X \cup Y$ and edge set $E(H[X \cup Y]) \setminus E(H[Y])$. Then (3.8) implies that

$$|V(H'_{\mathrm{con}})| \leq (\Delta+1)|V(H_{\mathrm{con}})| \overset{(3.8)}{\leq} 2\Delta\sqrt{\mu}m \quad \text{and hence} \quad e(H'_{\mathrm{con}}) \leq \mu^{1/3}m. \tag{3.11}$$

For each $w \in V(H'_{\mathrm{con}})$ where T_w is not defined, let $T_w := V_{\phi(w)}$. Lemma 3.3.1 applied with $2\Delta\sqrt{\mu}$ playing the role of γ (and the other parameters the same) implies that there are injective maps $\tau : X \to V$ and $\sigma : E(H'_{\mathrm{con}}) \to \mathscr{C}$ with $\tau(x)\tau(x') \in G_{\sigma(xx')}$ for all $xx' \in E(H_{\mathrm{con}})$, and

so that $\tau(x) \in T_x$ for all $x \in X$, and so that there are candidate sets C_y for each $y \in Y$ where $C_y \subseteq V_{\phi(y)} \setminus \tau(X)$ such that $C_y \subseteq \bigcap_{x \in N_H(y) \cap X} N_{G_{\sigma(xy)}}(\tau(x)) \cap T_y$ and $|C_y| \geq \nu' m$.

We update the target sets, by defining $T_y^1 := C_y \subseteq T_y$ for $y \in Y$ where this set is defined, and $T_y^1 := T_y$ if $y \notin V(H'_{\mathrm{con}})$ and this set is defined. This updates target sets T_y^1 for all unembedded vertices y, and all target sets have size at least $\nu' m$. Note that for all $y \in Y$, if we choose $\tau(y) \in T_y^1$, then $\tau(x)\tau(y) \in G_{\sigma(xy)}$ for all $x \in N_H(y) \cap X$, so the colour $\sigma(xy)$ will be used as desired. For each $i \in [r]$, let U_i^1 be the set of vertices y in $\phi^{-1}(i)$ with a target set T_y^1. We have

$$|U_i^1| \leq |U_i| + |Y| \leq \alpha m + 2\Delta\sqrt{\mu} m \leq 2\alpha m. \tag{3.12}$$

This set will only shrink during the rest of the proof as vertices with target sets are embedded. Some target sets of vertices in U_i^1 will also shrink, but will remain large enough.

Let $n_i^{\mathrm{con}} := |V(H_{\mathrm{con}}) \cap \phi^{-1}(i)|$ for all $i \in [r]$ and $h_{ij}^{\mathrm{con}} := e(H'_{\mathrm{con}}[\phi^{-1}(i), \phi^{-1}(j)])$ for all $ij \in E(R)$. We have

$$\begin{aligned} |V_i| &= n_i^{\mathrm{con}} + n_i^{\mathrm{abs}} + n_i^{\mathrm{app}} + n_i^{\mathrm{vx}} + n_i^{\mathrm{col}} \quad \text{for all } i \in [r] \quad \text{and} \\ |\mathscr{C}_{ij}| &= e(H[\phi^{-1}(i), \phi^{-1}(j)]) = h_{ij}^{\mathrm{con}} + h_{ij}^{\mathrm{abs}} + h_{ij}^{\mathrm{app}} + h_{ij}^{\mathrm{vx}} + h_{ij}^{\mathrm{col}} \quad \text{for all } ij \in E(R). \end{aligned} \tag{3.13}$$

Let $\mathcal{F}' = (\mathcal{V}', \mathscr{C}', \boldsymbol{G}')$ be the subtemplate of $\mathcal{F}$ obtained by deleting the vertices $\tau(X)$ and the colours $\sigma(E(H'_{\mathrm{con}}))$ from $\mathcal{F}$, so, defining for all $i \in [r]$ and $e \in E(R)$

$$V_i' := V_i \setminus \tau(X), \quad \mathscr{C}_e' := \mathscr{C}_e \setminus \sigma(E(H'_{\mathrm{con}})),$$
$$\text{we have} \quad (1 - \mu^{1/4})|\mathscr{C}_e| \overset{(3.7),(3.11)}{\leq} |\mathscr{C}_e'| \overset{(3.13)}{=} h_e^{\mathrm{abs}} + h_e^{\mathrm{app}} + h_e^{\mathrm{col}} + h_e^{\mathrm{vx}}.$$

By Lemma 3.2.9(ii), $\mathcal{F}'$ is a rainbow super R-template with parameters $(m/2, 2\varepsilon, d/2, \delta/2)$.

Step 1. Next we embed (the vertices but not the colours of) H_{abs} into $\mathcal{F}'$. For each $i \in [r]$, let $V_i^{\mathrm{abs}} \subseteq V_i'$ be a uniform random subset of size n_i^{abs}. Lemma 3.2.9(iii) applied with $\mathcal{F}', p_{\mathrm{abs}}/3, 6$ playing the roles of $\mathcal{F}, \alpha, k$ implies that the subtemplate $\mathcal{F}_{\mathrm{abs}}$ of $\mathcal{F}'$ induced by $(V_i^{\mathrm{abs}} : i \in [r])$ is a rainbow super R-template with parameters $(p_{\mathrm{abs}} m/3, \sqrt{\varepsilon}, d^2/64, \delta/6)$. Furthermore, a Chernoff bound implies that we may assume $|T_y^1 \cap V_{\phi(y)}^{\mathrm{abs}}| \geq 2\nu' n_i^{\mathrm{abs}}/3 \geq \nu' |V_i^{\mathrm{abs}}|/3$ for all $y \in V(H_{\mathrm{abs}})$ with a target set T_y^1.

Let $V_i'' := V_i' \setminus V_i^{\text{abs}}$. We have

$$|V_i''| \geq |V_i| - 2\Delta\sqrt{\mu}m - n_i^{\text{abs}} \overset{(3.9)}{\geq} (1-2p_{\text{abs}})|V_i| \geq m/2. \tag{3.14}$$

Lemma 3.2.9(ii) applied with $\mathcal{F}, 2p_{\text{abs}}$ playing the roles of $\mathcal{F}, \alpha$ implies that the subtemplate $\mathcal{F}''$ induced by V_i'' is a rainbow super R-template with parameters $(m/2, 2\varepsilon, d/2, \delta/2)$, and a Chernoff bound implies that we may assume

$$|T_y^1 \cap V''_{\phi(y)}| \geq 2(\nu' m - n_i^{\text{abs}})/3 \geq \nu' m/2 \tag{3.15}$$

for all y with a target set T_y^1.

We will embed H_{abs} into the λ_3-thick graph $T^{\lambda_3}_{\mathcal{F}_{\text{abs}}}$. By Proposition 3.2.11, for every $ij \in E(R)$, $T^{\lambda_3}_{\mathcal{F}_{\text{abs}}}[V_i^{\text{abs}}, V_j^{\text{abs}}]$ is $(\sqrt{\varepsilon}, d^2/128)$-half-superregular. Apply Theorem 3.3.3 (the blow-up lemma) with target sets $T_y^1 \cap V^{\text{abs}}_{\phi(y)}$ and parameters $\sqrt{\varepsilon}, \sqrt{\alpha}, \sqrt{\nu'}, d^2/128$ playing the roles of $\varepsilon, \alpha, \nu, d$ to find an embedding τ of H_{abs} into $T^{\lambda_3}_{\mathcal{F}_{\text{abs}}}$ such that $\tau(x) \in V_{\phi(x)}$ for all $x \in V(H_{\text{abs}})$ and $\tau(y) \in T_y^1$ for every $y \in U_i^1$ with $i \in [r]$. Note that we haven't yet defined $\sigma(e)$ for any $e \in E(H_{\text{abs}})$. However, by our definition, for every such $e = xy$ we have $\tau(x)\tau(y) \in E(G_c)$ for at least $\lambda_3|\mathscr{C}_e'|$ colours $c \in \mathscr{C}_e'$. For each $e \in E(R)$, let $G_{\text{abs},e}$ be the auxiliary bipartite graph with vertex classes $Z_e := \{\tau(x)\tau(y) : xy \in E(H^e_{\text{abs}})\}$ and $\mathscr{C}_e'$, where $\{\tau(x)\tau(y), c\}$ is an edge whenever $\tau(x)\tau(y) \in E(G_c)$. Then (3.10) implies that $|Z_e| = h_e^{\text{abs}} \leq 3p_{\text{abs}}|\mathscr{C}_e|/2 \leq 2p_{\text{abs}}|\mathscr{C}_e'|$, and by construction, every $\tau(x)\tau(y)$ in Z_e has degree at least $\lambda_3|\mathscr{C}_e'|$ in $G_{\text{abs},e}$. For each $e \in E(R)$, define constants ℓ_e and p_e via

$$\ell_e := \lambda_1 h_e^{\text{abs}} \quad \text{and} \tag{3.16}$$

$$\begin{aligned} p_e|\mathscr{C}_e'| &:= (1-p_{\text{abs}})|\mathscr{C}_e'| - (h_e^{\text{abs}} - \ell_e) - h_e^{\text{app}} = h_e^{\text{vx}} + h_e^{\text{col}} - p_{\text{abs}}|\mathscr{C}_e'| + \lambda_1 h_e^{\text{abs}} \\ &\overset{(3.10)}{=} (1 \pm \tfrac{1}{4})h_e^{\text{vx}} \in [p_{\text{vx}}|\mathscr{C}_e'|/3, 3p_{\text{vx}}|\mathscr{C}_e'|]. \end{aligned} \tag{3.17}$$

Thus for each $e \in E(R)$ we can apply Lemma 3.4.1 with $G_{\text{abs},e}, Z_e, \mathscr{C}_e', \lambda_1, p_e, \lambda_3$ playing the roles of $G, U, \mathscr{C}, \lambda_1, \lambda_2, \lambda_3$ to obtain disjoint sets $\mathscr{A}_e, \mathscr{B}_e \subseteq \mathscr{C}_e'$ such that

(Q1) $|\mathscr{A}_e| = h_e^{\text{abs}} - \ell_e \overset{(3.10)}{\leq} 2p_{\text{abs}}|\mathscr{C}_e'|$;

(Q2) $|\mathscr{B}_e| = p_e|\mathscr{C}_e'|$;

(Q3) for any set $\mathscr{B}_e^0 \subseteq \mathscr{B}_e$ of size ℓ_e, there exists a colouring using colours from $\mathscr{A}_e \cup \mathscr{B}_e^0$ that

makes the embedding τ of H_e^{abs} rainbow. That is, there is a bijection $\sigma_e : E(H_e^{\mathrm{abs}}) \to \mathscr{A}_e \cup \mathscr{B}_e^0$ such that $\tau(x)\tau(y) \in G_{\sigma_e(xy)}$ for all $xy \in E(H_e^{\mathrm{abs}})$.

Preparation for Steps 2–4. Recall that the template $\mathcal{F}'' = (\mathcal{V}'', \mathscr{C}', \boldsymbol{G}'')$ has vertex clusters $\mathcal{V}'' := (V_i'' := V_i' \setminus V_i^{\mathrm{abs}} : i \in [r])$, colour clusters $(\mathscr{C}_e' : e \in E(R))$ and graphs $(\boldsymbol{G}'')^{ij} = (G_c[V_i'', V_j''] : c \in \mathscr{C}_{ij}')$, and $\mathcal{F}''$ is a rainbow super R-template with parameters $(m/2, 2\varepsilon, d/2, \delta/2)$. Let $n_i^{\mathrm{colvx}} := n_i^{\mathrm{col}} + n_i^{\mathrm{vx}}$ for each $i \in [r]$ and let $p_{\mathrm{colvx}} := p_{\mathrm{col}} + p_{\mathrm{vx}}$. During the rest of the proof, for each $\diamond \in \{\mathrm{app}, \mathrm{colvx}\}$ we will identify pairwise disjoint vertex sets $V_i^\diamond$ in the vertex clusters V_i'', $i \in [r]$, so that

$$|V_i^\diamond| = n_i^\diamond, \quad \text{and} \quad V_i'' = V_i^{\mathrm{app}} \cup V_i^{\mathrm{colvx}}, \quad \text{where} \quad \mathcal{V}^\diamond := \{V_1^\diamond, \dots, V_r^\diamond\}\ \forall \diamond \in \{\mathrm{app}, \mathrm{colvx}\}.$$

We will choose $\tau(x) \in V_{\phi(x)}^{\mathrm{app}}$ for all $x \in V(H_{\mathrm{app}})$ and $\tau(x) \in V_{\phi(x)}^{\mathrm{colvx}}$ for all $x \in V(H_{\mathrm{col}}) \cup V(H_{\mathrm{vx}})$. We will identify pairwise disjoint colour sets $\mathscr{C}^\circ$, so that

$$\mathscr{C}^\circ = \bigcup_{e \in E(R)} \mathscr{C}_e^\circ\ \forall \circ \in \{\mathrm{app}, \mathrm{col}, \mathrm{vx}\}, \quad \mathscr{C}_e' = \bigcup_{\circ \in \{\mathrm{app}, \mathrm{col}, \mathrm{vx}\}} \mathscr{C}_e^\circ$$

and for each $xy \in E(H_\circ)$, we will ensure $\sigma(xy) \in G_c$ for some $c \in \mathscr{C}_{\phi(x)\phi(y)}^\circ$. Given such sets of vertices and colours for $\circ \in \{\mathrm{app}, \mathrm{col}, \mathrm{vx}\}$, we define $\mathcal{F}_\circ := (\mathcal{V}^{\circ'}, \mathscr{C}^\circ, \boldsymbol{G}^\circ)$ to be the subtemplate of $\mathcal{F}''$ induced by $\mathcal{V}^{\circ'}, \mathscr{C}^\circ$, where $\mathrm{app}' = \mathrm{app}$ and $\mathrm{vx}' = \mathrm{col}' = \mathrm{colvx}$. Note that these templates are always rainbow R-templates with pairwise disjoint sets of colours, but $\mathcal{F}_{\mathrm{col}}$ and $\mathcal{F}_{\mathrm{vx}}$ share the same vertex clusters.

We will now define the vertex sets $V_i^{\circ'}$ for $\circ' \in \{\mathrm{app}, \mathrm{colvx}\}$, but colour sets will be defined sequentially, given the colours used in each step. For each $i \in [r]$, do the following. For each $j \in N_R(i)$, recall that $(G_c[V_i'', V_j''] : c \in \mathscr{C}_{ij}')$ is $(2\varepsilon, d/2)$-superregular. By (3.16) and (Q2), we have $|\mathscr{B}_{ij}| \geq p_{\mathrm{vx}}|\mathscr{C}_{ij}'|/2$, so by Lemma 3.2.4(i), $(G_c[V_i'', V_j''] : c \in \mathscr{B}_{ij})$ is $(4\varepsilon/p_{\mathrm{vx}}, d/4)$-regular and hence $(\sqrt{\varepsilon}/r, d/4)$-regular. By Lemma 3.2.3(i), there is a set $\overline{V}_i^j \subseteq V_i''$ with $|\overline{V}_i^j| \leq \sqrt{\varepsilon}|V_i''|/r$ such that

$$\sum_{c \in \mathscr{B}_{ij}} d_{G_c}(v) \geq d|V_j''||\mathscr{B}_{ij}|/5 \quad \text{for all } v \in V_i'' \setminus \overline{V}_i^j. \tag{3.18}$$

Let $\overline{V}_i := \bigcup_{j \in N_R(i)} \overline{V}_i^j$, so $|\overline{V}_i| \leq \sqrt{\varepsilon}|V_i''|$, and let V_i^{colvx} be a uniform random subset of

$V_i'' \setminus \overline{V}_i$ of size n_i^{colvx}, and let $V_i^{\mathrm{app}} := V_i'' \setminus V_i^{\mathrm{colvx}}$. So $|V_i^{\mathrm{app}}| = n_i^{\mathrm{app}}$. Let $T_y^2 := T_y^1 \cap V_{\phi(y)}^{\circ'(y)}$ for all $y \in U_i^1$ and $i \in [r]$ (i.e. those y for which T_y^1 has been defined), where $\circ'(y) = \mathrm{app}$ if $y \in V(H_{\mathrm{app}})$ and $\circ'(y) = \mathrm{colvx}$ if $y \in V(H_{\mathrm{col}}) \cup V(H_{\mathrm{vx}})$.

For all $c \in \mathscr{C}'_{ij}$, the superregularity of our graph collections implies that

$$\begin{aligned}\mathbb{E}(e(G_c[V_i^{\mathrm{colvx}}, V_j^{\mathrm{colvx}}])) &\geq (n_i^{\mathrm{colvx}}/|V_i''|)(n_j^{\mathrm{colvx}}/|V_j''|)(e(G_c[V_i'', V_j'']) - \sqrt{\varepsilon}|V_i''||V_j''|) \\ &\geq dn_i^{\mathrm{colvx}} n_j^{\mathrm{colvx}}/3 \overset{(3.9)}{\geq} dp_{\mathrm{vx}}^2 \delta^2 m^2/12.\end{aligned}$$

For all $i \in [r]$ and $y \in U_i^1$ we have

$$\mathbb{E}(|T_y^2|) \geq (n_i^{\circ'(y)}/|V_i''|)(|T_y^1 \cap V_i''| - \sqrt{\varepsilon}|V_i''|) \overset{(3.9),(3.15),(3.14)}{\geq} p_{\circ'(y)}\nu' m/5.$$

Further, for all $ij \in E(R)$ and $v \in V_i^{\mathrm{colvx}}$, since $v \notin \overline{V}_i^j$ we have

$$\mathbb{E}\left(\textstyle\sum_{c \in \mathscr{B}_{ij}} d_{G_c}(v, V_j^{\mathrm{colvx}})\right) \geq (n_j^{\mathrm{colvx}}/|V_j''|) \textstyle\sum_{c \in \mathscr{B}_{ij}} (d_{G_c}(v, V_j'') - \sqrt{\varepsilon}|V_j''|) \overset{(3.18)}{\geq} dn_j^{\mathrm{colvx}}|\mathscr{B}_{ij}|/11.$$

By Chernoff bounds, we may assume that each of the above quantities are close to their expectations, so

(C1) $|T_y^2| \geq p_{\circ'(y)}\nu' m/6$ for all $y \in U_i^1$ and $i \in [r]$ (i.e. all those y which have a target set);

(C2) $e(G_c[V_i^{\mathrm{colvx}}, V_j^{\mathrm{colvx}}]) \geq dn_i^{\mathrm{colvx}} n_j^{\mathrm{colvx}}/13$ for all $ij \in E(R)$ and $c \in \mathscr{C}'_{ij}$;

(C3) $\sum_{c \in \mathscr{B}_{ij}} d_{G_c}(v, V_j^{\mathrm{colvx}}) \geq dn_j^{\mathrm{colvx}}|\mathscr{B}_{ij}|/12$ for all $ij \in E(R)$ and $v \in V_i^{\mathrm{colvx}}$.

Let $m' := p_{\mathrm{vx}} m/8$. Now (3.12) and (C1) imply that, for all $i \in [r]$, the number $|U_i^1|$ of vertices $y \in \phi^{-1}(i)$ with a target set T_y^2 satisfy

$$|U_i^1| \leq 2\alpha m \leq \sqrt{\alpha} m', \quad \text{and} \quad |T_y^2| \geq p_{\mathrm{vx}}\nu' m/6 > \nu' m'. \tag{3.19}$$

For all $i \in [r]$ we have $|V_i^{\mathrm{colvx}}| = n_i^{\mathrm{colvx}} = n_i^{\mathrm{col}} + n_i^{\mathrm{vx}}$, so

$$p_{\mathrm{vx}}|V_i''|/3 \overset{(3.14)}{\leq} p_{\mathrm{vx}}|V_i|/2 \overset{(3.9)}{\leq} |V_i^{\mathrm{colvx}}| \overset{(3.9)}{\leq} 2p_{\mathrm{vx}}|V_i| \overset{(3.14)}{\leq} 3p_{\mathrm{vx}}|V_i''|. \tag{3.20}$$

We claim that the following properties about templates $\mathcal{F}_\circ$ with colour sets $\mathscr{C}_e^\circ$ for $\circ \in \{\mathrm{app}, \mathrm{col}, \mathrm{vx}\}$ hold.

(F1) Let

$$\mathscr{C}_e^{\text{app}} := \mathscr{C}'_e \setminus (\mathscr{A}_e \cup \mathscr{B}_e) \text{ for all } e \in E(R).$$

Then $\mathcal{F}_{\text{app}}$, with vertex clusters $(V_i^{\text{app}} : i \in [r])$, is a rainbow super R-template with parameters $(m/4, 4\varepsilon, d/4, \delta/4)$.

(F2) Suppose

$$\mathscr{C}_e^{\text{col}} = \mathscr{B}_e \cup \mathscr{D}_e \text{ where } \mathscr{D}_e \subseteq \mathscr{C}^{\text{app}} \text{ for all } e \in E(R).$$

Then $\mathcal{F}_{\text{col}}$, with vertex clusters $(V_i^{\text{colvx}} : i \in [r])$, is a rainbow R-template with parameters $(m', \sqrt{\varepsilon}, d/4, \delta/24)$ with the property that $e(G_c[V_i^{\text{colvx}}, V_j^{\text{colvx}}]) \geq d|V_i^{\text{colvx}}||V_j^{\text{colvx}}|/22$ for all $c \in \mathscr{D}_e$ and $ij \in E(R)$.

(F3) Suppose

$$\mathscr{C}_e^{\text{vx}} \subseteq \mathscr{B}_e \text{ and } |\mathscr{B}_e \setminus \mathscr{C}_e^{\text{vx}}| \leq 2p_{\text{col}}|\mathscr{C}'_e| \text{ for all } e \in E(R).$$

Then $\mathcal{F}_{\text{vx}}$, with colour clusters $(V_i^{\text{colvx}} : i \in [r])$, is a rainbow super R-template with parameters $(m', \sqrt{\varepsilon}, d/13, \delta/24)$.

These properties follow from Lemma 3.2.9 applied to $\mathcal{F}_\circ$ as a subtemplate of $\mathcal{F}'' = (\mathcal{V}'', \mathscr{C}', \boldsymbol{G}'')$ (which has parameters $(m/2, 2\varepsilon, d/2, \delta/2)$), as follows. First, (F1) holds by Lemma 3.2.9(ii) since $\mathcal{F}_{\text{app}}$ is a small perturbation of $\mathcal{F}''$. Indeed, (Q1) and (Q2) implies that $|\mathscr{C}'_e \setminus \mathscr{C}_e^{\text{app}}| \leq (2p_{\text{abs}}+3p_{\text{vx}})|\mathscr{C}'_e| \leq 4p_{\text{vx}}|\mathscr{C}'_e| \leq \sqrt{p_{\text{vx}}}m/2$, and $|V_i'' \setminus V_i^{\text{app}}| = |V_i^{\text{colvx}}| \leq 3p_{\text{vx}}|V_i''| \leq \sqrt{p_{\text{vx}}}m/2$ by (3.20), so we can apply the lemma with $2\varepsilon, \sqrt{p_{\text{vx}}}, d/2$ playing the roles of ε, α, d to obtain (F1).

For (F2), (Q2) and (3.16) imply that $|\mathscr{C}_e^{\text{col}}| \geq |\mathscr{B}_e| = p_e|\mathscr{C}'_e| > p_{\text{vx}}|\mathscr{C}'_e|/4$. Together with (3.20), this means that we can apply Lemma 3.2.9(i) with parameters $m/2, 2\varepsilon, p_{\text{vx}}/4, d/2, \delta/2, 12$ playing the roles of $m, \varepsilon, \alpha, d, \delta, k$ to see that $\mathcal{F}_{\text{col}}$ is a rainbow R-template with parameters $(m', \sqrt{\varepsilon}, d/4, \delta/24)$. The second property follows from (C2) since $\mathscr{D}_e \subseteq \mathscr{C}_e^{\text{app}} \subseteq \mathscr{C}'_e$.

For property (F3), we have $|\mathscr{C}_e^{\text{vx}}| \geq |\mathscr{B}_e| - 2p_{\text{col}}|\mathscr{C}'_e| \geq p_{\text{vx}}|\mathscr{C}'_e|/4$ by (3.16). Together with (3.20), this means we can apply Lemma 3.2.9(i) with the same parameters as previously to see that $\mathcal{F}_{\text{vx}}$ is a rainbow R-template with the given parameters. For superregularity, we have for all

$e = ij \in E(R)$ and $v \in V_i^{\text{colvx}}$ that

$$\sum_{c \in \mathscr{C}_{ij}^{\text{vx}}} d_{G_c}(v, V_j^{\text{colvx}}) \geq \sum_{c \in \mathscr{B}_{ij}} d_{G_c}(v, V_j^{\text{colvx}}) - 2p_{\text{col}}|\mathscr{C}'_{ij}||V''_j| \overset{(3.7),(3.18),(C3)}{\geq} dn_j^{\text{colvx}}|\mathscr{B}_{ij}|/12 - 2p_{\text{col}}\Delta m^2/\delta^2$$
$$\geq dn_j^{\text{colvx}}|\mathscr{C}_{ij}^{\text{vx}}|/13,$$

and $e(G_c[V_i^{\text{colvx}}, V_j^{\text{colvx}}]) \geq dn_i^{\text{colvx}} n_j^{\text{colvx}}/13$ for all $c \in \mathscr{C}_{ij}^{\text{vx}}$ by (C2). Thus (F1)–(F3) all hold. Now it is a matter of embedding each $H_\circ$ into its corresponding template, using suitable (unused) colours at each step. The image of $V(H_{\text{vx}})$ in each V_i will be the remaining vertices of V_i^{colvx} after embedding H_{col}, which is most of this set since H_{col} is much smaller than H_{vx}.

Step 2. We embed H_{app} into $\mathcal{F}_{\text{app}}$ using Lemma 3.3.4 (embedding lemma with extra colours) with parameters $m/4, 4\varepsilon, \sqrt{\mu}, \sqrt{\alpha}, 2p_{\text{abs}}\delta, \nu'/7, d/4, \delta/4$ playing the roles of $m, \varepsilon, \mu, \alpha, \beta, \nu, d, \delta$. For this, we recall that H_{app} is the union of components of size at most $\sqrt{\mu}|V(H_{\text{app}})|$, and there is a graph homomorphism from H_{app} into R. By (C1) and (F1), it now suffices to check that colour clusters have a suitable size to apply the lemma. Indeed, for all $e \in E(R)$ we have

$$\begin{aligned} |\mathscr{C}_e^{\text{app}}| - h_e^{\text{app}} &\overset{(F1)}{=} |\mathscr{C}'_e| - |\mathscr{A}_e| - |\mathscr{B}_e| - h_e^{\text{app}} \overset{(Q1),(Q2)}{=} |\mathscr{C}'_e| - (h_e^{\text{abs}} - \ell_e) - p_e|\mathscr{C}'_e| - h_e^{\text{app}} \\ \overset{(3.16)}{=} p_{\text{abs}}|\mathscr{C}'_e| &\in [p_{\text{abs}}\delta m/2, p_{\text{abs}}\Delta m/\delta]. \end{aligned} \tag{3.21}$$

Thus we can apply Lemma 3.3.4 to obtain $\tau(V(H^{\text{app}}))$ and $\sigma(E(H^{\text{app}}))$ where each y with a target set T_y^2 is embedded inside it.

For each $e \in E(R)$, let $\mathscr{D}_e := \mathscr{C}_e^{\text{app}} \setminus \sigma(E(H_{\text{app}}))$ and let $\mathscr{C}_e^{\text{col}} := \mathscr{B}_e \cup \mathscr{D}_e$.

Step 3. We embed H_{col} into $\mathcal{F}_{\text{col}}$ by applying Lemma 3.3.2 (embedding lemma with target sets and prescribed colours) with $m', \sqrt{\varepsilon}, \sqrt{\alpha}, \sqrt{p_{\text{abs}}}, \sqrt{p_{\text{col}}}, p_{\text{col}}, \nu', d/4, \delta/24$ playing the roles of $m, \varepsilon, \alpha, \gamma, \lambda_1, \lambda_2, \nu, d, \delta$. To see that this is possible, we have $|V(H_{\text{col}}) \cap \phi^{-1}(i)| = n_i^{\text{col}} \leq 2p_{\text{col}}m/\delta \leq \sqrt{p_{\text{col}}}m'$. We also have

$$e(H_{\text{col}}[\phi^{-1}(i), \phi^{-1}(j)]) = h_{ij}^{\text{col}} \overset{(3.10)}{\geq} p_{\text{col}}\delta m/2 \geq p_{\text{col}}m'.$$

Equation (3.19) implies that the number of vertices with target sets and the size of these target

sets are suitable. By (3.21), for every $e \in E(R)$ we have

$$|\mathscr{D}_e| = |\mathscr{C}_e^{\text{app}}| - h_e^{\text{app}} \leq p_{\text{abs}}\Delta m/\delta \leq \sqrt{p_{\text{abs}}}m'. \tag{3.22}$$

By (F2), it remains to check that $|\mathscr{C}_e^{\text{col}} \setminus \mathscr{D}| \geq d|\mathscr{C}_e^{\text{col}}|/4$. Since the original template was rainbow, $\mathscr{C}_e^{\text{col}} \setminus \mathscr{D} = \mathscr{C}_e^{\text{col}} \setminus \mathscr{D}_e = \mathscr{B}_e$. Equation (3.22) implies that $|\mathscr{B}_e| \geq |\mathscr{C}_e^{\text{col}}| - \sqrt{p_{\text{abs}}}m' \geq |\mathscr{C}_e^{\text{col}}|/2 \geq d|\mathscr{C}_e^{\text{col}}|/4$, as required. Thus we can apply Lemma 3.3.2 to obtain $\tau(V(H_{\text{col}}))$ and $\sigma(E(H_{\text{col}}))$ where each y with a target set T_y^2 (i.e. those in $U^1_{\phi(y)}$) is embedded inside it.

Let $\mathscr{C}_e^{\text{vx}} := \mathscr{B}_e \setminus \sigma(E(H_{\text{col}}))$ and $V_i^{\text{vx}} := V_i^{\text{colvx}} \setminus \tau(V(H_{\text{col}}))$, so $V_i^{\text{vx}} = n_i^{\text{vx}}$.

Step 4. Let $\mathcal{F}'_{\text{vx}}$ be the subtemplate of $\mathcal{F}_{\text{vx}}$ induced by $(V_i^{\text{vx}} : i \in [r])$. It is a small perturbation: $|V_i^{\text{colvx}} \setminus V_i^{\text{vx}}| = n_i^{\text{col}} \leq 2p_{\text{col}}m/\delta \leq p_{\text{vx}}m'$, so Lemma 3.2.9(ii) with $m', \sqrt{\varepsilon}, p_{\text{vx}}, d/13, \delta/24$ playing the roles of $m, \varepsilon, \alpha, d, \delta$ implies that $\mathcal{F}'_{\text{vx}}$ is a rainbow super template with parameters $(m'/2, 2\sqrt{\varepsilon}, d/26, \delta/48)$. Let $T_y^3 := T_y^2 \cap V_i^{\text{vx}}$ for all $i \in [r]$ and $y \in V_i^{\text{colvx}} \cap U_i^1$.

We embed H_{vx} into $\mathcal{F}'_{\text{vx}}$ by applying Lemma 3.3.4 with parameters $\sqrt{\mu}, \sqrt{\alpha}, \lambda_1^2, \nu'/3$ playing the roles of μ, α, β, ν (and template parameters as above). For this, we recall that H_{vx} is the union of vertex disjoint components of size at most $\sqrt{\mu}|V(H_{\text{vx}})|$ and there is a graph homomorphism from H_{vx} into R. By (3.19), there are suitably few vertices with target sets T_y^3, and all target sets are suitably large. By (F3), it now suffices to check that every $|\mathscr{C}_e^{\text{vx}}| - h_e^{\text{vx}}$ is large. We have

$$\begin{aligned}|\mathscr{C}_e^{\text{vx}}| - h_e^{\text{vx}} &= |\mathscr{B}_e| - h_e^{\text{col}} - h_e^{\text{vx}} \overset{(F1)}{=} |\mathscr{C}'_e| - |\mathscr{A}_e| - h_e^{\text{app}} - h_e^{\text{col}} - h_e^{\text{vx}} = h_e^{\text{abs}} - |\mathscr{A}_e| = \ell_e \\ &= \lambda_1 h_e^{\text{abs}} \geq \lambda_1 p_{\text{abs}}\delta m/2 \geq \lambda_1^2 m'/2.\end{aligned}$$

Thus we can apply Lemma 3.3.4 to obtain $\tau(V(H_{\text{vx}}))$ and $\sigma(E(H_{\text{vx}}))$ where each y with a target set T_y^2 is embedded inside it.

Let $\mathscr{C}_e^{\text{abs}} := \mathscr{A}_e \cup (\mathscr{C}_e^{\text{vx}} \setminus \sigma(E(H_{\text{vx}})))$.

Step 5. Since $\mathscr{C}'_e = \mathscr{C}_e^{\text{app}} \cup \mathscr{C}_e^{\text{col}} \cup \mathscr{C}_e^{\text{vx}} \cup \mathscr{C}_e^{\text{abs}}$ is a disjoint union, and $\mathscr{A}_e \subseteq \mathscr{C}_e^{\text{abs}} \subseteq \mathscr{A}_e \cup \mathscr{B}_e$, the set $\mathscr{C}_e^{\text{abs}} \cap \mathscr{B}_e$ must have exactly the right size:

$$|\mathscr{C}_e^{\text{abs}} \cap \mathscr{B}_e| = |\mathscr{C}'_e| - h_e^{\text{app}} - h_e^{\text{col}} - h_e^{\text{vx}} - |\mathscr{A}_e| = |\mathscr{C}'_e| - (|\mathscr{C}'_e| - h_e^{\text{abs}}) - |\mathscr{A}_e| = \ell_e.$$

Thus, by (Q3), for each $e \in E(R)$ we can find σ_e with image $\sigma_e(E(H_{\text{abs}})) = \mathscr{C}_e^{\text{abs}}$. Extending

σ by all of these σ_e completes the transversal embedding. □

3.4.3 Proof of Theorem 3.1.8

Proof of Theorem 3.1.8. We may assume without loss of generality that $1/m \ll \varepsilon, \mu, \alpha \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1/2$. Choose an additional constant ε' with $\varepsilon \ll \varepsilon' \ll \nu, d, \delta, 1/\Delta, 1/r$ such that the conclusion of Lemma 3.2.1 holds with $\varepsilon, \varepsilon', d, \delta, 3$ playing the roles of $\varepsilon, \varepsilon', d, \delta, k$. We may further assume that the conclusion of Theorem 3.3.5 holds with $\varepsilon', d^2/2$ playing the roles of ε, d and the other parameters unchanged.

Let $\boldsymbol{G}, \mathscr{C}, R, \mathcal{V} := (V_1, \dots, V_r)$ be as in the statement of Theorem 3.1.8. Let $\phi : V(H) \to V(R)$ be such that $\phi(x) = j$ whenever $x \in A_j$. So ϕ is a graph homomorphism. Our hypothesis is that $\mathcal{F} = (\mathcal{V}, \mathscr{C}, \boldsymbol{G})$ is a rainbow half-super R-template with parameters $(m, \varepsilon, d, \delta)$. Lemma 3.2.9(iv) implies that for all $c \in \mathscr{C}$ there is $G'_c \subseteq G_c$ such that, writing $\boldsymbol{G}' := (G'_c : c \in \mathscr{C})$, the template $\mathcal{F}' = (\mathcal{V}, \mathscr{C}, \boldsymbol{G}')$ is super with parameters $(m, \varepsilon', d^2/2, \delta)$.

Apply Theorem 3.3.5 to $\mathcal{F}'$ to obtain the required embedding. □

3.5 Proof of Theorems 3.1.2 and 3.1.6

In this section we prove two applications of our transversal blow-up lemma, to super uniformly dense graph collections (Theorem 3.1.2) and super uniformly dense 3-graphs (Theorem 3.1.6). The latter is an easy consequence of the former.

Proof of Theorem 3.1.2. Let $\Delta, d, \delta, \alpha > 0$ be given and let $r := \Delta + 1$. Choose additional constants such that

$$0 < 1/n_0 \ll \eta, \mu \ll \varepsilon \ll \delta_1 \ll \delta_2 \ll \ldots \ll \delta_{\binom{r}{2}+1} \ll \nu' \ll \alpha \ll \delta, d, 1/\Delta,$$

where we are assuming without loss of generality that $\alpha \ll \delta, d/1/\Delta$, and so that the conclusion of Lemma 3.2.1 holds with $3, \eta^{1/4}, \varepsilon, \alpha^2/6$ playing the roles of $k, \varepsilon, \varepsilon', d$, and Lemma 3.3.1 holds with $n_0/r, \varepsilon, \sqrt{\delta_1}, \alpha^4/72$ playing the roles of $m, \varepsilon, \gamma, d$ and Theorem 3.3.5 holds with $\delta_2^2 n_0/2, \sqrt{\varepsilon}, 2\mu, \delta_1^{1/3}, \nu', \alpha^9, \delta/2$ playing the roles of $m, \varepsilon, \mu, \alpha, \nu, d, \delta$. Let $n \geq n_0$ be an integer and let $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ be a graph collection on a vertex set V of size n and H a graph on n vertices satisfying the conditions of the theorem.

By assumption, $\delta n \le e(H) = |\mathscr{C}| \le \Delta n$. Let $m := \lfloor n/r \rfloor$. The Hajnal-Szemerédi theorem implies that there is a partition $V(H) = A_1 \cup \ldots \cup A_r$ into parts of size m and $m+1$ such that all edges go between different parts. Let $V = V_1 \cup \ldots \cup V_r$ be a random partition of V with $|V_i| = |A_i|$ for all $i \in [r]$, and let $\mathcal{V} := \{V_1, \ldots, V_r\}$. We claim that with high probability, the following hold for any $ij \in E(K_r) = \binom{[r]}{2}$:

(i) for all $V'_h \subseteq V_h$ with $|V'_h| \ge \eta^{1/4}|V_h|$ for $h = i, j$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| \ge \eta^{1/4}|\mathscr{C}|$, we have $\sum_{c\in\mathscr{C}'} e_{G_c}(V'_i, V'_j) \ge d|\mathscr{C}'||V'_i||V'_j|/2$;

(ii) for each labelling $\{g, h\} = \{i, j\}$ and $v \in V_g$, we have $\sum_{c\in\mathscr{C}} d_{G_c}(v, V_h) \ge \alpha^2|V_h||\mathscr{C}|/6$;

(iii) $e(G_c) \ge \alpha^2|V_i||V_j|/6$ for all $c \in \mathscr{C}$.

For (i), we have

$$\eta n^3 < (d/2) \cdot \eta^{3/4}\delta n^3/(2r)^2 < d\eta^{3/4}\delta n m^2/2 \le d(\eta^{1/4})^3|\mathscr{C}||V_i||V_j|/2 \le d|\mathscr{C}'||V'_i||V'_j|/2.$$

The statement then follows directly from the definition of (d, η)-dense. To prove (ii), for each vertex $v \in V$, there are at least $\alpha|\mathscr{C}|/2$ colours $c \in \mathscr{C}$ for which $d_{G_c}(v) \ge \alpha n/2$, and a Chernoff-type bound implies that, with high probability $d_{G_c}(v, V_j) \ge \alpha|V_j|/3$. Part (iii) is proved similarly by swapping colour and vertex.

Parts (i)–(iii) imply that the 3-graph $G^{(3),ij}$ of $\boldsymbol{G}^{ij} := (G_c[V_i, V_j] : c \in \mathscr{C})$ is $(\eta^{1/4}, \alpha^2/6)$-half-superregular. Lemma 3.2.1 and our choice of parameters implies that every $G^{(3),ij}$ contains a spanning subhypergraph $J^{(3),ij}$ which is $(\varepsilon, \alpha^4/72)$-superregular. Thus $\mathcal{F} := (\mathcal{V}, \mathscr{C}, \boldsymbol{J})$ is a super K_r-template with parameters $(m, \varepsilon, \alpha^4/72, \delta)$, where $\boldsymbol{J}^{ij}$ is the graph collection whose 3-graph is $J^{(3),ij}$ and $\mathscr{C}_{ij} = \mathscr{C}$ for all $ij \in E(K_r)$.

Let $d_{ij} := e(H[A_i, A_j])/n$ for all $ij \in E(K_r)$. We have $\sum_{ij} d_{ij} = e(H)/n \ge \delta$. Thus there is at least one ij such that $d_{ij} \ge \delta_{\binom{r}{2}+1}$. By the pigeonhole principle, there is $\ell \in [\binom{r}{2}]$ such that for all ij, either $d_{ij} \le \delta_\ell$, or $d_{ij} \ge \delta_{\ell+1}$. Let $P^< := \{ij \in \binom{[r]}{2} : d_{ij} \le \delta_\ell\}$. We will embed those sparse $H[A_i, A_j]$, with indices ij in $P^<$, using Lemma 3.3.1 (Embedding lemma with target and candidate sets), while the remaining (dense) pairs will be embedded using Theorem 3.3.5 (Transversal blow-up lemma) with target sets from the initial embedding of sparse pairs.

Let X be the union of non-isolated vertices in $H[A_i, A_j]$ over all $ij \in P^<$, let $Y := N_H(X)\setminus X$, and let $H^< := H[X\cup Y]\setminus H[Y]$, whose edge set is precisely those edges of H incident to X. We have $v(H^<) \le 2\binom{r}{2}\delta_\ell n \le \sqrt{\delta_\ell}m$. Apply Lemma 3.3.1 to $\mathcal{F}$ with $m, \varepsilon, \sqrt{\delta_\ell}, \nu', 1, \alpha^4/72, \delta, \Delta, r$

playing the roles of $m, \varepsilon, \gamma, \nu', \nu, d, \delta, \Delta, r$, and $T_w := V_i$ for all $w \in V(H^<) \cap A_i$ and $i \in [r]$, to find injective maps $\tau : X \to V$ and $\sigma : E(H^<) \to \mathscr{C}$ such that $\tau(x)\tau(x') \in J_{\sigma(xx')}$ for all $xx' \in E(H[X])$, for all $i \in [r]$ we have $\tau(x) \in V_i$ for all $x \in A_i$, and for all $y \in Y \cap A_i$ there exists $C_y \subseteq V_i \setminus \tau(X)$ such that $C_y \subseteq \bigcap_{x \in N_{H^<}(y) \cap X} N_{G_{\sigma(xy)}}(\tau(x))$ and $|C_y| \geq \nu' m$.

Let $\mathscr{C}' := \mathscr{C} \setminus \sigma(E(H^<))$ and let $\mathcal{V}' := \{V_1', \dots, V_r'\}$ where $V_i' := V_i \setminus \tau(X)$ for each $i \in [r]$. By Lemma 3.2.9(ii) applied with $\alpha = \Delta\sqrt{\delta_\ell}$, the template $(\mathcal{V}', \mathscr{C}', \boldsymbol{J}')$ induced by $\mathcal{V}', \mathscr{C}'$ is super with parameters $(m/2, 2\varepsilon, \alpha^4/144, \delta/2)$. Let $H^> := H - X$. Note that $H^>$ (with respect to its slightly smaller vertex set) is 2μ-separable. Now we randomly partition $\mathscr{C}'$ into $\binom{r}{2}$ (some perhaps empty) parts $\mathscr{C}' = \bigcup_{ij \in E(K_r)} \mathscr{C}''_{ij}$ such that $|\mathscr{C}''_{ij}| = e(H^>[V_i', V_j'])$ for all $ij \in \binom{[r]}{2}$. This is possible since $|\mathscr{C}'| = |\mathscr{C}| - e(H^<) = e(H) - e(H^<) = e(H^>)$. Let $\mathcal{F}'' := (\mathcal{V}', \mathscr{C}'', \boldsymbol{J}')$, where each $\mathscr{C}''_{ij}$ is as above (so $\mathscr{C}'' = \mathscr{C}'$). Each colour set is either large or empty, indeed, if $ij \in P^<$, then $\mathscr{C}''_{ij} = \emptyset$, but otherwise we have

$$|\mathscr{C}''_{ij}| \geq e(H[A_i, A_j]) - e(H^<) \geq \delta_{\ell+1} n - \Delta\sqrt{\delta_\ell} m \geq \delta_{\ell+1} n/2 \geq \delta_{\ell+1}|\mathscr{C}'|/(2\Delta) \geq \delta_{\ell+1}^2 |\mathscr{C}'|.$$

Lemma 3.2.9(iii) applied with $\delta_{\ell+1}^2, 1$ playing the roles of α, k implies that $\mathcal{F}''$ is a rainbow super template with parameters $(\delta_{\ell+1}^2 m/2, \sqrt{\varepsilon}, \alpha^9, \delta/2)$. Apply Theorem 3.3.5 (transversal blow-up lemma) with target sets C_y of size at least $\nu' m$ for the at most $\sqrt{\delta_\ell} m \leq \delta_\ell^{1/3} \delta_{\ell+1}^2 m/2$ vertices $y \in Y$, and parameters $\delta_{\ell+1}^2 m/2, \sqrt{\varepsilon}, 2\mu, \delta_1^{1/3}, \nu', \alpha^9, \delta/2$ playing the roles of $m, \varepsilon, \mu, \alpha, \nu, d, \delta$ to obtain a transversal embedding of $H^>$ inside $\mathcal{F}''$ such that every $y \in Y$ is embedded inside C_y. Together with the embedding of $H^<$, this gives a transversal embedding of H. □

Proof of Theorem 3.1.6. Let $\Delta, d, \alpha > 0$ be given. Without loss of generality, we may assume that $\alpha \ll d, 1/\Delta$. Choose constants $\eta', \mu, n_0 > 0$ such that the conclusion of Theorem 3.1.2 holds, applied with $\Delta, d, 1/5, \alpha^3$ playing the roles of $\Delta, d, \delta, \alpha$. Let $\eta := \eta'/(2\Delta)$.

Let $n \geq n_0$ be an integer and let G be a (d, η)-dense 3-graph on n vertices with $d_G(v) \geq \alpha n^2$ for all $v \in V(G)$.

Let H be a μ-separable graph with $\Delta(H) \leq \Delta$ and $v(H) + e(H) \leq n$. First, if H is small, we will enlarge it, as follows. If $e(H) > n/4 - 1$, let $H' := H$. Otherwise, obtain H' from H by successively adding an edge between isolated vertices until $e(H) > n/4 - 1$. Let H' be the obtained graph restricted to non-isolated vertices. We have $e(H') \leq n/4$, which implies that $v(H') + e(H') \leq 4e(H') \leq n$.

In both cases, we have $\Delta(H') \leq \Delta$, and H' is μ-separable; $e(H') > n/4 - 1$ and $v(H') \geq 3e(H')/\Delta > 3(n-4)/(4\Delta) > n/(2\Delta)$; and $v(H') + e(H') \leq n$.

Let $V, \mathscr{C}$ be disjoint subsets of $V(G)$ chosen uniformly at random subject to

$$|V| = v(H') > n/(2\Delta) \quad \text{and} \quad |\mathscr{C}| = e(H') > n/4 - 1.$$

Standard Chernoff-type bounds imply that, with high probability, for every $v \in V$ we have

$$e(G[\{v\}, V, \mathscr{C}]) \geq \alpha^2|V||\mathscr{C}|/6 \geq \alpha^2 n^2/(49\Delta) \geq \alpha^3 n^2 \quad \text{for all } v \in V$$
$$|\{xyc \in E(G) : x, y \in V\}| \geq \alpha^2|V|^2/12 \geq \alpha^2 n^2/(48\Delta^2) \geq \alpha^3 n^2 \quad \text{for all } c \in \mathscr{C}.$$

For each $c \in \mathscr{C}$, let G_c be the 2-graph with vertex set V and edge set $\{xy : x, y \in V \text{ and } xyc \in E(G)\}$, and let $\boldsymbol{G} := (G_c : c \in \mathscr{C})$. Thus $\sum_{c\in\mathscr{C}} d_{G_c}(v, V) = e(G[\{v\}, V, \mathscr{C}]) \geq \alpha^3 n^2$, and for every $c \in \mathscr{C}$ we have $e(G_c) \geq \alpha^3 n^2$. Since G is (d, η)-dense and $|V| \geq n/(2\Delta)$, we have that $\boldsymbol{G}$ is (d, η')-dense. Theorem 3.1.2 applied with $\Delta, d, 1/5, \alpha^3$ playing the roles of $\Delta, d, \delta, \alpha$ implies that $\boldsymbol{G}$ contains a transversal copy of H' and thus a transversal copy of H, that is, there are injective maps $\tau : V(H) \to V$ and $\sigma : E(H) \to \mathscr{C}$ so that $\tau(x)\tau(y) \in G_{\sigma(xy)}$ for every $xy \in E(H)$. Define $\rho : V(H) \cup E(H) \to V(G)$ by setting $\rho(x) := \tau(x)$ for $x \in V(H)$ and $\rho(e) := \sigma(e)$ for $e \in E(H)$. Then ρ is injective and $\rho(x)\rho(y)\rho(xy) \in E(G)$ for all $xy \in E(H)$; that is, ρ defines a copy of the 1-expansion of H in G, as required. □

3.6 Concluding remarks

We have proved a transversal blow-up lemma that embeds separable graphs into graph collections, which can be used to apply the regularity-blow-up method to transversal embedding problems. We conclude with some remarks on future directions.

Separability. In our proof of the transversal blow-up lemma, the separability condition is necessary because we need to divide the graph into linear-sized pieces and then use the blow-up lemma to embed them piece by piece. For this embedding process to work, the number of edges between different pieces should not be too large. We wonder if our transversal blow-up lemma can be generalised to embed any graph with bounded maximum degree. If such a version can be proven, it would directly generalise the original blow-up lemma for graphs (since a collection

of identical superregular pairs is a superregular collection).

Applications to hypergraph embedding. We state the full 3-graph version of our transversal blow-up lemma.

Theorem 3.6.1 (weak 3-graph blow-up lemma)**.** *Let* $0 < 1/m \ll \varepsilon, \mu, \alpha \ll \nu, d, \delta, 1/\Delta, 1/r \leq 1$.

- *Let* R *be a* 2*-graph with vertex set* $[r]$.
- *Let* G *be a* 3*-graph with parts* $V_1, \dots, V_r$ *and* V_{ij} *for* $ij \in E(R)$ *where* $m \leq |V_i| \leq m/\delta$ *for all* $i \in [r]$*, and* $|V_{ij}| \geq \delta m$ *for all* $ij \in [r]$*. Suppose that* $G[V_i, V_j, V_{ij}]$ *is weakly* (ε, d)*-(half-)superregular for all* $ij \in E(R)$.
- *Let* H *be a* μ*-separable* 2*-graph with* $\Delta(H) \leq \Delta$ *for which there is a graph homomorphism* $\phi : V(H) \to V(R)$ *with* $|\phi^{-1}(i)| = |V_i|$ *for all* $i \in [r]$ *and* $e(H[\phi^{-1}(i), \phi^{-1}(j)]) = |V_{ij}|$ *for all* $ij \in E(R)$.
- *Suppose that for each* $i \in [r]$ *there is a set* $U_i \subseteq \phi^{-1}(i)$ *with* $|U_i| \leq \alpha m$ *and* $T_x \subseteq V_{\phi(x)}$ *with* $|T_x| \geq \nu m$ *for all* $x \in U_i$.

Then G *contains a copy of the* 1*-expansion of* H*, where each vertex* $x \in V(H)$ *is mapped to* $V_{\phi(x)}$ *and the new vertex for* $xy \in E(H)$ *is mapped to* $V_{\phi(x)\phi(y)}$*; and moreover, for every* $i \in [r]$ *every* $x \in U_i$ *is mapped to* T_x.

This is a reformulation of Theorem 3.1.8 since for each $ij \in E(R)$, the 3-graph $G_{ij}^{(3)}$ of $(G_c : c \in \mathscr{C}_{ij})$ in Theorem 3.1.8 is weakly half-superregular, so we simply take $V_{ij} := \mathscr{C}_{ij}$. It would be interesting to extend Theorem 3.1.6 on embeddings in uniformly dense 3-graphs beyond 1-expansions of separable 2-graphs. It is not clear which 3-graphs one should expect to be able to embed in a weakly superregular triple (or in a uniformly dense 3-graph). Even the case of 3-partite 3-graphs seems difficult; that is, to extend Theorem 3.1.7 (simplified weak hypergraph blow-up lemma).

Problem 3.6.2. *Given* $0 < 1/n \ll \varepsilon \ll d \ll \delta \leq 1$*, which* J *are subhypergraphs of any weakly* (ε, d)*-superregular triple* $G[V_1, V_2, V_3]$*, where* $\delta n \leq |V_1|, |V_2|, |V_3| \leq n/\delta$*?*

A 3-partite 3-graph J is equivalent to an edge-coloured bipartite 2-graph $\mathcal{J}$ (where each edge can receive multiple colours) using our usual identification of one part with a set of colours. If J has $\Delta(J) \leq \Delta$, then $\Delta(\mathcal{J}) \leq \Delta$ and the colouring is Δ-bounded. (If J is linear, then the edge-colouring is proper.) Thus we would like to extend Theorem 3.1.3 (simplified weak transversal blow-up lemma). to find an embedding of $\mathcal{J}$ with a given edge-colouring up to permutation

of colours. If every colour only appears on a single edge, that is, J is an expansion of some 2-graph H where the 2-edge e is replaced by some bounded number $t_e \geq 1$ of 3-edges (which is not linear if some $t_e > 1$), it seems plausible that our techniques would work. However, the colour absorption we use from (Montgomery et al., 2022) does not seem to be able to deal with colours playing multiple roles.

Let F be a 3-partite 3-graph of fixed size. Recall that an old result of Erdős (Erdős, 1964a) implies that any large 3-graph of positive density contains a copy of F. However, the results of (Ding et al., 2022) imply that any large uniformly dense 3-graph G (whose number of vertices is a multiple of $v(F)$ and where every vertex sees a positive fraction of pairs) contains an F-factor if and only if there is a vertex $v^* \in V(F)$ such that for any two edges e, e' where e contains v^* and e' does not, e and e' share at most one vertex. Probably, the characterisation for weakly superregular triples with balanced classes is the same.

Another instructive case is the tight Hamilton cycle (whose number of vertices is divisible by 6). As a graph collection problem, this corresponds to finding a (2-graph) Hamilton cycle on $4n$ vertices with $2n$ colours, where, cyclically labelling the edges $e_1, \dots, e_{4n}$, the consecutive edges $e_{2c-1}, e_{2c}, e_{2c+1}$ are all in G_c, for each $c \in [2n]$ where indices are taken modulo $4n$. A construction in (Lenz & Mubayi, 2016) (see also (Lenz et al., 2016)) shows that a weakly superregular triple need not contain a tight Hamilton cycle, even if its density is close to $\frac{1}{8}$.

Example 3.6.3. *Let V be a vertex set of size $6n$, and let $V = V_1 \cup V_2 \cup V_3$ be a partition, where V_1, V_2, V_3 have equal size $2n$, and let $X \subseteq V$ where $|X \cap V_1|$ is odd. Independently for each distinct $i, j \in [3]$ add uniform random edges between parts V_i, V_j with probability $\frac{1}{2}$, to obtain a 3-partite 2-graph J. Now form a 3-partite 3-graph G by, for each triple $abc \in A \times B \times C$, as follows:*

- *add abc to G if $|\{a, b, c\} \cap X|$ is even and $\{a, b, c\}$ spans a triangle in J.*
- *add abc to G if $|\{a, b, c\} \cap X|$ is odd and $\{a, b, c\}$ spans an independent set in J.*

Suppose $v_1^1 v_2^1 v_3^1 \dots v_1^{2n} v_2^{2n} v_3^{2n}$ is a tight cycle H in J, where without loss of generality, $v_i^j \in V_i$ for all j. For any $1 \leq j < 2n$, Since $v_1^j v_2^j v_3^j, v_2^j v_3^j v_1^{j+1}$ are edges of H, we have that $|\{v_1^j, v_2^j, v_3^j\} \cap X|, |\{v_1^{j+1}, v_2^j, v_3^j\} \cap X|$ are both odd or both even. This implies that v_1^j, v_1^{j+1} are either both in X or neither in X. By considering the disjoint pairs $v_1^1 v_1^2, v_1^3 v_1^4, \dots, v_1^{2n-1} v_1^{2n}$, this is a contradiction to $|X \cap V_1|$ odd.

It can easily be checked using Chernoff bounds (see (Lenz & Mubayi, 2016)) that, if every

$|X \cap V_i|$ has size about n, then with high probability G is weakly $(\frac{1}{8}, o(1))$-superregular.

3.7 Appendix

3.7.1 Weak hypergraph regularity lemmas

We first introduce the following special form of the weak hypergraph regularity lemma for k-partite k-graphs. Its proof follows easily from the original lemma of Chung (Chung, 2007), which is very similar to the 2-graph regularity lemma of Szemerédi (Szemerédi, 1975).

Lemma 3.7.1. *For every $L_0, k \geq 1$ and every $\varepsilon, \delta > 0$, there is an $n_0 > 0$ such that for every k-partite k-graph G on $n \geq n_0$ vertices with parts $V_1, \dots, V_k$ where $\delta n \leq |V_i| \leq n/\delta$, there exists a refined partition $V_i = \bigcup_{j=1}^{t_i} V_{i,j}$ for each $i \in [k]$ such that the following properties hold:*

(i) $L_0 \leq t := t_1 + \ldots + t_k \leq n_0$;

(ii) $||V_{i,j}| - |V_{i',j'}|| \leq 1$ *for any* $i, i' \in [k]$, $j \in [t_i]$ *and* $j' \in [t_{i'}]$;

(iii) all but at most εt^k k*-tuples* $(V_{1,j_1}, \dots, V_{k,j_k})$ *are* ε*-regular.*

The following weak hypergraph regularity lemma (Kühn et al., 2014) is proved in the same way as the original degree form of the regularity lemma (see (Townsend, 2016)), which in turn can be derived from the standard regularity lemma via some cleaning.

Theorem 3.7.2 (Degree form of the weak hypergraph regularity lemma)**.** *For all integers $L_0 \geq 1$ and every $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon, L_0)$ such that for every $d \in [0, 1)$ and for every 3-graph $G = (V, E)$ on $n \geq n_0$ vertices there exists a partition of V into $V_0, V_1, \dots, V_L$ and a spanning subhypergraph G' of G such that the following properties hold:*

(i) $L_0 \leq L \leq n_0$ *and* $|V_0| \leq \varepsilon n$;

(ii) $|V_1| = \ldots = |V_L| =: m$;

(iii) $d_{G'}(v) > d_G(v) - (d + \varepsilon)n^2$ *for all* $v \in V$;

(iv) every edge of G' with more than one vertex in a single cluster V_i for some $i \in [L]$ has at least one vertex in V_0;

(v) for all triples $\{h, i, j\} \in \binom{[L]}{3}$, we have that $(V_h, V_i, V_j)_{G'}$ is either empty or (ε, d)-regular.

The proof of Lemma 4.2.1 (the regularity lemma for graph collections) is a routine but tedious consequence of this theorem applied to the 3-graph $G^{(3)}$ of a graph collection $\boldsymbol{G}$.

Proof of Lemma 4.2.1. By increasing L_0 and decreasing ε, δ as necessary, we may assume with-

out loss of generality that $0 < 1/L_0 \ll \varepsilon \ll \delta \ll 1$. Let $L_0' := 2L_0/\delta$. Choose new constants ε', α, which we may assume satisfy $0 < 1/L_0' \ll \varepsilon' \ll \alpha \ll \varepsilon$. Let n_0' be obtained from Theorem 3.7.2 applied with parameters ε', L_0'. We may assume that $1/n_0' \ll 1/L_0'$ by increasing n_0'. Altogether,

$$0 < 1/n_0' \ll 1/L_0' \ll \varepsilon' \ll \alpha \ll \varepsilon \ll \delta \ll 1.$$

Let $n_0 := 2n_0'/\alpha$. Let $n \geq n_0$ be an integer and suppose that $\boldsymbol{G}$ is a graph collection on a vertex set V of size n and with colour set $\mathscr{C}$, where $\delta n \leq |\mathscr{C}| \leq n/\delta$. Let $d \in [0, 1)$ and let $G^{(3)}$ be the 3-graph of $\boldsymbol{G}$ with vertex set $U := V \cup \mathscr{C}$. Theorem 3.7.2 implies that there is a partition $U_0, U_1, \ldots, U_K$ of U and a spanning subhypergraph $G^{(3)'}$ of $G^{(3)}$ such that

(a) $L_0' \leq K \leq n_0'$ and $|U_0| \leq \varepsilon'|U|$;

(b) $|U_1| = \ldots = |U_K| =: m'$;

(c) $d_{G^{(3)'}}(u) > d_{G^{(3)}}(u) - (2d + \varepsilon')|U|^2$ for all $u \in U$;

(d) every edge of $G^{(3)'}$ with more than one vertex in a single cluster U_i has at least one vertex in U_0;

(e) for all triples $\{h, i, j\} \in \binom{[K]}{3}$, we have that $(U_h, U_i, U_j)_{G^{(3)'}}$ is either empty or $(\varepsilon', 2d)$-regular.

Partition each cluster U_i, $i \in [K]$, into $1/\alpha$ subclusters of size at most m so that all but at most two subclusters of U_i have size exactly m and the property that they lie entirely within V, or within $\mathscr{C}$. If a subcluster does not have this property, add it to U_0. The new exceptional set has size at most $|U_0| + 2\alpha m'K$, and we let V_0 be its intersection with V, and $\mathscr{C}_0$ its intersection with $\mathscr{C}$. Relabel the subclusters so that those which are subsets of V are $V_1, \ldots, V_L$ and those which are subsets of $\mathscr{C}$ are $\mathscr{C}_1, \ldots, \mathscr{C}_M$. Let G_c' be the graph with vertex set V and edge set $\{xy : xyc \in G^{(3)'}\}$ for all $c \in \mathscr{C}$. We claim that the properties of the lemma are satisfied.

For (i), we have $m'K \leq |U|$ so

$$|V_0| + |\mathscr{C}_0| \leq |U_0| + 2\alpha m'K \overset{(a)}{\leq} (\varepsilon' + 2\alpha)|U| \leq (\varepsilon' + 2\alpha)(n + n/\delta) \leq \varepsilon n.$$

Also, $L_0 \leq L_0' \leq L_0'/\alpha \leq K/\alpha \leq L + M \leq 2K/\alpha \leq 2n_0'/\alpha = n_0$, proving the required upper bound for both L and M. Furthermore, $m(L + M) \leq n + |\mathscr{C}|$ so

$$L \geq \frac{n - |V_0|}{m} \geq \frac{(n - |V_0|)(L + M)}{n + |\mathscr{C}|} \geq \frac{(n - |V_0|)L_0'}{n + n/\delta} \geq \delta L_0'/2 = L_0$$

and similarly for M. Part (ii) follows by construction.

For (iii), we have $\sum_{c\in\mathscr{C}} d_{G'_c}(v) = d_{G^{(3)\prime}}(v)$ for all $v \in V$, and $e(G'_c) = d_{G^{(3)\prime}}(c)$ for all $c \in \mathscr{C}$, and similarly for G_c. Further, $(2d+\varepsilon')(n+|\mathscr{C}|)^2 < (2d+\varepsilon')(1+1/\delta)^2 n^2 \leq (3d/\delta^2 + 2\varepsilon'/\delta^2)n^2 \leq (3d/\delta^2+\varepsilon)n^2$. Thus (c) implies the required.

For (iv), if G'_c has an edge xy with $x, y \in V_i$ for some $i \in [L]$, then $x, y \in U_{i'}$ for the cluster $U_{i'}$ containing V_i. Since $xyc \in E(G^{(3)\prime})$, (d) implies that $c \in U_0$. Thus $c \in \mathscr{C}_0$, as required.

For (v), suppose that $(\{h,i\},j) \in \binom{[L]}{2} \times M$ and $\boldsymbol{G}'_{hi,j}$ is non-empty. Let $h', i', j' \in [K]$ be such that $V_h \subseteq U_{h'}$, $V_i \subseteq U_{i'}$ and $\mathscr{C}_j \subseteq U_{j'}$. Since $\boldsymbol{G}'_{hi,j}$ is non-empty, we have that $(U_{h'}, U_{i'}, U_{j'})_{G^{(3)\prime}}$ is non-empty. So h', i', j' are distinct by (d). Part (e) implies that $(U_{h'}, U_{i'}, U_{j'})_{G^{(3)\prime}}$ is $(\varepsilon', 2d)$-regular. Therefore, for each $c \in U_{j'}$, letting J_c be the bipartite graph with partition $(U_{h'}, U_{i'})$ and edge set $\{xy : x \in U_{h'}, y \in U_{i'}, xyc \in G^{(3)\prime}\}$, we have that $\boldsymbol{J} := (J_c : c \in U_{j'})$ is $(\varepsilon', 2d)$-regular. Lemma 3.2.4(i) (the slicing lemma) implies that $\boldsymbol{G}'_{hi,j}$ is $(\varepsilon'/\alpha, d)$-regular and thus (ε, d)-regular. This completes the proof. □

We conclude the appendix with a proof of Lemma 3.2.1, which states that a half-superregular (k-graph) k-tuple contains a spanning superregular subhypergraph. The proof is the same as the one in (Rödl & Rucinski, 1999) for $k = 2$.

Proof of Lemma 3.2.1. Choose a new parameter ε'' such that $0 < \varepsilon \ll \varepsilon'' \ll \varepsilon'$. Apply Lemma 3.7.1 to G with parameters $L_0 := 1, \varepsilon'', \delta$ to obtain $n_0 > 0$. Increasing n_0, n and decreasing ε if necessary, we may assume that $1/n \ll \varepsilon \ll 1/n_0 \ll \varepsilon''$. Now let $G = (V_1, \ldots, V_k)_G$ be an (ε, d)-half-superregular k-graph with $\delta n \leq |V_i| \leq n/\delta$ for all $i \in [k]$. Lemma 3.7.1 implies that there is a refinement $V_i = \bigcup_{j=1}^{t_i} V_{i,j}$ for each $i \in [k]$ such that $t := t_1 + \ldots + t_k \leq n_0$, every pair of subparts differ in size by at most one, and all but at most $\varepsilon'' t^3$ k-tuples $(V_{1,j_1}, \ldots, V_{k,j_k})$ are ε''-regular.

Obtain a spanning subhypergraph G' of G by, for each ε''-regular triple $(X_1, \ldots, X_k)$ of subparts, removing every k-edge in $(X_1, \ldots, X_k)$ independently at random with probability $1 - d/d'$, where d' is the density of $(X_1, \ldots, X_k)$. (Note that by $|X_i| \geq |V_i|/n_0 \geq \varepsilon|V_i|$ for each $i \in [k]$, we have $d' \geq d$ due to half-superregularity.) We claim that, with high probability,

- $d_{G'}(v) \geq d^2(|V_1| \ldots |V_k|)/(2|V_i|)$ for any $i \in [k]$ and $v \in V_i$,
- any ε''-regular k-tuple (of subparts) in G is $2\varepsilon''$-regular in G',
- any ε''-regular k-tuple (of subparts) in G has density $d \pm 2\varepsilon''$ in G'.

This follows from Chernoff bounds.

Now we claim that, given that the above hold, G' is $(\varepsilon', d^2/2)$-superregular. We only need to check that G' is $(\varepsilon', d^2/2)$-regular. Let $(A_1, \ldots, A_k)$ be any k-tuple where $A_i \subseteq V_i$ and $|A_i| \geq \varepsilon'|V_i|$ for all $i \in [k]$. For each $i \in [k]$, we have partition $A_i = \bigcup_{j=1}^{t_i}(A_i \cap V_{i,j})$. We say a part $A_{i,j} := A_i \cap V_{i,j}$ is *big* if $|A_i \cap V_{i,j}| \geq \varepsilon''|V_{i,j}|$ and otherwise it is *small*. Note that in total we have at most $\varepsilon''(|V_1| + \ldots + |V_k|) \leq \varepsilon'' kn/\delta$ vertices in small parts. Also note that we have at most $\varepsilon'' t^k$ non-regular k-tuples that contain in total at most $\varepsilon''(n/\delta)^k$ edges. Thus

$$e_{G'}(A_1, \ldots, A_k) = \sum_{\substack{(A_{1,j_1}, \ldots, A_{k,j_k}) \text{ all big and} \\ (V_{1,j_1}, \ldots, V_{k,j_k}) \ \varepsilon''\text{-regular}}} e_{G'}(A_{1,j_1}, \ldots, A_{k,j_k}) \pm \left(2\varepsilon'' kn/\delta \cdot (n/\delta)^{k-1} + \varepsilon''(n/\delta)^k\right).$$

For every summand index $(j_1, \ldots, j_k)$, we have $e_{G'}(A_{1,j_1}, \ldots, A_{k,j_k}) = (d \pm 2\varepsilon'')|A_{1,j_1}| \ldots |A_{k,j_k}|$ by $2\varepsilon''$-regularity. Thus

$$e_{G'}(A_1, \ldots, A_k) = (d \pm 2\varepsilon'')|A_1| \ldots |A_k| \pm 3\varepsilon'' kn^k/\delta^k = (d \pm \varepsilon')|A_1| \ldots |A_k|.$$

This implies that G' is $(\varepsilon', d^2/2)$-superregular and thus we finish the proof. □

4 | New Proof of the Asymptotic Transversal Pósa-Seymour Conjecture

4.1 Introduction

4.1.1 Transversal embedding of spanning graphs

To state the main results in this Chapter, we define the parameters δ_H and δ_H^{col}, where

- δ_H is the minimum δ such that for all $\alpha > 0$, any sufficiently large n-vertex graph G with $\delta(G) \geq (\delta + \alpha)n$ contains a copy of H; and
- δ_H^{col} is the minimum δ such that for all $\alpha > 0$, any collection $G_1, \dots, G_{e(H)}$ of sufficiently large n-vertex graphs with $\delta(G_i) \geq (\delta + \alpha)n$ contains a transversal copy of H.

Given a graph F with $v(F)|n$, an n-vertex F-*factor* H is a spanning graph consisting of vertex-disjoint copies of F. Hajnal and Szemerédi proved that $\delta_H = (k-1)/k$ when $F = K_k$, which was generalised by (Kühn & Osthus, 2009) to arbitrary F. The author together with Han, Wang and Wang (Cheng et al., 2023) and independently, (Montgomery et al., 2022), showed that the Hajnal-Szemerédi theorem is approximately colour-blind. This is the main part of Chapter 2. The authors of (Montgomery et al., 2022) further determined δ_H^{col} for H an F-factor, for any given small F. Their result implies that the theorem of Kühn and Osthus is not approximately colour-blind: for example, when F is the disjoint union $K_{2,3} \cup C_4$, we have $\delta_H = \frac{4}{9}$ while $\delta_H^{\text{col}} = \frac{1}{2}$. They also investigated transversal spanning trees, showing that a well-known result of (Komlós, Sárkózy, & Szemerédi, 2001) is colour-blind.

(Gupta et al., 2023) showed that the Pósa-Seymour conjecture on the minimum degree needed to guarantee a given power of a Hamilton cycle is approximately colour-blind. In this Chapter, we will give a new proof of this theorem.

A far-reaching (approximate) generalisation of all the results in this section was obtained by (Chakraborti, Im, et al., 2023): a transversal 'bandwidth theorem' which states that the 'bandwidth theorem' is colour-blind. A graph H has *bandwidth* b is there is an ordering $v_1, \dots, v_m$ of its vertices so that $|i - j| \leq b$ whenever $v_i v_j \in E(H)$. So a graph with small bandwidth does not 'expand' very much; and F-factors and powers of Hamilton cycles have small bandwidth. Answering a question of Bollobás and Komlós, it was shown by (Böttcher, Schacht, & Taraz,

2009) that when H is a k-chromatic graph of bounded maximum degree and sublinear bandwidth, we have $\delta_H = (k-1)/k$. The transversal generalisation obtained in (Chakraborti, Im, et al., 2023) is as follows.

Theorem 4.1.1 (Transversal bandwidth theorem (Chakraborti, Im, et al., 2023))**.** *For all $k, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, the following holds. Let H be a k-chromatic graph on n vertices with $\Delta(H) \leq \Delta$ and with bandwidth at most βn. Let $\boldsymbol{G} = (G_1, \dots, G_{e(H)})$ be a graph collection on a common vertex set of size n, with $\delta(\boldsymbol{G}) \geq \left(\frac{k-1}{k} + \gamma\right) n$. Then $\boldsymbol{G}$ contains a transversal copy of H.*

As we have seen, this minimum degree bound is sometimes tight, but not always, as we would expect for such a general result. Thus, despite the power of this theorem, there are many interesting cases where the threshold is unknown, and one purpose of this Chapter is to demonstrate a method that provides some hope in finding it.

4.1.2 Powers of Hamilton cycles

Given $k \geq 2$, the $(k-1)$-th power H^{k-1} of a graph H is a graph with vertex set $V(H)$ in which two vertices are adjacent if and only if their distance in H is less than k. For a long time, a major open problem was the Pósa-Seymour conjecture (Erdős, 1964b; Seymour, 1974), which states that $\delta(G) > \frac{k-1}{k}n$ guarantees a copy of C_n^{k-1} in an n-vertex graph G (and this is best possible). The conjecture was eventually proved for large n by (Komlós et al., 1998b), who used their newly-developed blow-up lemma. As mentioned in Chapter 1, there are later proofs (Châu, DeBiasio, & Kierstead, 2011; Levitt et al., 2010) using the absorbing method which do not require such a large n.

An approximate transversal version of this theorem was recently proved by (Gupta et al., 2023), in a paper about finding transversal copies of graphs and hypergraphs H with a 'cyclic' structure. Their main theorem guarantees a transversal copy of such an H given that three properties hold, two of which are uncoloured *Connecting* and *Absorbing* properties. That these two properties hold is implied by existing results in the literature, in particular (Pavez-Signé, Sanhueza-Matamala, & Stein, 2021). Note that C_n^{k-1} is $2(k-1)$-regular, so $e(C_n^{k-1}) = (k-1)n$.

Theorem 4.1.2 ((Gupta et al., 2023))**.** *For all integers $k \geq 1$ and $\gamma > 0$, there exists n_0 such that the following holds for all integers $n \geq n_0$. Let $\boldsymbol{G} = (G_1, \dots, G_{(k-1)n})$ be a graph collection on a common vertex set of size n. Suppose that $\delta(\boldsymbol{G}) \geq \left(\frac{k-1}{k} + \gamma\right) n$. Then $\boldsymbol{G}$ contains a*

transversal copy of C_n^{k-1}.

We recall that Theorem 4.1.1, the transversal bandwidth theorem, was obtained by Chakraborti, Im, Kim and Liu in (Chakraborti, Im, et al., 2023), and implies this result and e.g. the result on F-factors from (Montgomery et al., 2022).

In this Chapter, we apply our regularity-blow-up method for transversals to give a new proof of Theorem 4.1.2. Our focus is on the use of the method rather than the end result – since this is not new – and the goal is to demonstrate an application of the method so that it may be used for further applications, in particular to obtain generalisations of classical embedding results where the regularity-blow-up method has been applied. The ideal situation is that in such a proof, any regularity-blow-up tools can be swapped out for our transversal versions. As discussed above, the proof of Theorem 4.1.1 in (Chakraborti, Im, et al., 2023) uses the classical regularity-blow-up method, in a rather different way, and this is another approach that could yield further results which we think are amenable to the methods we use here.

4.1.3 Probabilistic tools

A random variable Y is (μ, C)-*dominated* if there are constants $\mu_1, \dots, \mu_m$ with $\sum_{i\in[m]} \mu_i \leq \mu$ and we can write $Y = \sum_{i\in[m]} Y_i$ such that $|Y_i| \leq C$ for all $i \in [m]$ and conditional on any given values of Y_j for $j < i$, we have $\mathbb{E}|Y_i| < \mu_i$. The following inequality is a consequence of Freedman's inequality.

Lemma 4.1.3. *If* Y *is* (μ, C)*-dominated, then* $\mathbb{P}(|Y| > 2\mu) < 2e^{-\mu/(6C)}$.

4.1.4 Sketch of the proof

At a high level, the proof follows the same structure as the uncoloured proof of (Komlós et al., 1998a), which states that, given any integer $k \geq 2$, a large n-vertex graph G with $\delta(G) \geq (\frac{k-1}{k} + \gamma)n$ contains a copy of $C_n^{(k-1)}$. However, there are technical complications arising from colours to be overcome, and these require new ideas. We begin by sketching the proof in (Komlós et al., 1998a). Given a graph F, an F-*tiling* is a collection of vertex-disjoint copies of F.

Step 1. *Apply the regularity lemma and find an almost perfect* K_k*-tiling in the reduced graph.*

The reduced graph inherits the minimum degree of G, so the Hajnal-Szemerédi theorem implies

it has as large a K_k-tiling as possible. In each copy of K_k in the reduced graph, after removing a small fraction of bad vertices, one can apply the blow-up lemma to find a spanning copy of the $(k-1)$-th power P_a^{k-1} of a path. Since these copies of K_k cover most of the graph, it remains to connect up these copies and incorporate any uncovered vertices. We wait to apply the blow-up lemma until this has been achieved.

Step 2. *Connect consecutive k-cliques in the template.*

Since the reduced graph inherits the minimum degree from G, it is fairly straightforward to show that given any pair A, B of K_ks in the reduced graph, one can find a sequence of clusters where the first k form A, the last k form B, and every k consecutive clusters span a clique. That is, a $(k-1)$-path where repeated vertices are allowed. The embedding lemma now applies to find a copy of P_a^{k-1} which uses this sequence as a guide and connects A with B. Moreover, this can be done in a robust way so that there are many choices for the ends of these paths.

Step 3. *Incorporate exceptional vertices.*

After possibly removing a few more vertices to ensure that clusters have equal size, the remaining task is to incorporate the small fraction of vertices outside our structure. Since each one v has large degree, there are many copies of K_k in the tiling where v has many neighbours in all of its clusters. Thus these exceptional v can be allocated to such a cluster so that no cluster receives too many. Each such v can be slotted into a path.

Step 4. *Apply the blow-up lemma.*

Finally we apply the blow-up lemma to find a copy of P_a^{k-1} in each K_k in the tiling, including the restrictions that the ends of the path must lie in the large sets we found earlier for connecting, and various other vertices must be embedded in given large sets so that exceptional v can be added between them.

The idealised plan is to follow these same steps, but replace the regularity lemma, embedding lemma and blow-up lemma with our transversal versions Lemmas 4.2.1 and 3.3.1 and Theorem 3.3.5 from (Cheng & Staden, 2023). We now describe where this fails and how we proceed instead.

The first major obstacle is in Step 1. Now, the reduced graph only inherits the minimum degree of the graphs in $\boldsymbol{G}$ in a weaker sense: every vertex cluster has large degree in *most* colour clusters, and every colour cluster has many edges incident to *most* vertex clusters (Lemma 3.2.7). Here, 'most' means a $1-d^{1/4}$ fraction where $d \gg \varepsilon$ and ε is the parameter governing the reg-

ularity and d the minimum density in a regular pair. It would be easy to obtain a K_k-tiling in $\boldsymbol{R}$ covering $1 - O(d^{1/4})$ fraction of clusters (using e.g. (Montgomery et al., 2022)), but this is not good enough for Step 3 since it will leave a $O(d^{1/4})$ fraction of exceptional vertices, which is too many to be 'absorbed' by clusters when applying the blow-up lemma in Step 4 (it should be ζ fraction where $\zeta \ll d$). Thus we need to work harder in Step 1, and we instead find a 'fractional' K_k-tiling covering a $1 - \zeta$ fraction, which suffices (Lemma 4.4.1).

The other major obstacle is in Step 3. Now we have both exceptional vertices and exceptional colours, with $k-1$ times as many colours. Each time we incorporate an exceptional vertex into a K_k, we need to use $2k-2$ new (exceptional) colours on the edges incident to it, but when we add it we remove $k-1$ existing edges to get a path. Thus we need to remove $k-1$ suitable colours from the K_k and add those to the exceptional set. Moreover, we need $2k-1$ colours to choose from to guarantee that there are $2k-2$ colours which can be used to incorporate any given vertex. Therefore we can always allocate an exceptional vertex when there are $2k-1$ exceptional colours available. So in this way we can incorporate *all but two* exceptional vertices! This issue complicates the proof significantly. We need an additional step after Step 1, where we build an absorber for two given vertices v_1^*, v_2^*, which, for a given small family of sets $\mathscr{R}_j$ of $k-1$ colours, can incorporate v_1^*, v_2^* and $\mathscr{R}_j$ into the structure. These $\mathscr{R}_j$ are potential newly-exceptional colours created by the incorporation of the final exceptional vertex, before v_1^*, v_2^*. To ensure that there are only a small number of options for this set of colours, we first 'swap out' exceptional colours for pre-specified good ones that we are able to absorb.

4.2 A regularity lemma for graph collections

We will use the following regularity lemma for graph collections, which is just a convenient reformulation of the weak regularity lemma for hypergraphs applied to the 3-graph representation $G^{(3)}$ of $\boldsymbol{G}$ (see Lemma 3.7.2) .

Lemma 4.2.1 (Regularity lemma for graph collections (Cheng & Staden, 2023))**.** *For all integers $L_0 \geq 1$ and every $\varepsilon, \delta > 0$, there is an $n_0 = n_0(\varepsilon, \delta, L_0)$ such that for every $d \in [0,1)$ and every graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ on vertex set V of size $n \geq n_0$ with $\delta n \leq |\mathscr{C}| \leq n/\delta$, there exists a partition of V into $V_0, V_1, \ldots, V_L$, of $\mathscr{C}$ into $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_M$ and a spanning subgraph G_c' of G_c for each $c \in \mathscr{C}$ such that the following properties hold:*

(i) $L_0 \leq L, M \leq n_0$ *and* $|V_0| + |\mathscr{C}_0| \leq \varepsilon n$*;*

(ii) $|V_1| = \ldots = |V_L| = |\mathscr{C}_1| = \ldots = |\mathscr{C}_M| =: m$*;*

(iii) $\sum_{c\in\mathscr{C}} d_{G'_c}(v) > \sum_{c\in\mathscr{C}} d_{G_c}(v) - (3d/\delta^2 + \varepsilon)n^2$ *for all* $v \in V$ *and* $e(G'_c) > e(G_c) - (3d/\delta^2 + \varepsilon)n^2$ *for all* $c \in \mathscr{C}$*;*

(iv) if, for $c \in \mathscr{C}$*, the graph* G'_c *has an edge with both vertices in a single cluster* V_i *for some* $i \in [L]$*, then* $c \in \mathscr{C}_0$*;*

(v) for all triples $(\{h, i\}, j) \in \binom{[L]}{2} \times [M]$*, we have that either* $G'_c[V_h, V_i] = \emptyset$ *for all* $c \in \mathscr{C}_j$*, or* $\boldsymbol{G'}_{hi,j} := (G'_c[V_h, V_i] : c \in \mathscr{C}_j)$ *is* (ε, d)*-regular.*

The sets V_i are called *vertex clusters* and the sets $\mathscr{C}_j$ are called *colour clusters*, while V_0 and $\mathscr{C}_0$ are the exceptional vertex and colour sets respectively.

4.3 Tools for absorbing and connecting

4.3.1 Named graphs

We now fix an integer $k \geq 2$ and will seek the $(k-1)$-th power of a Hamilton cycle. We henceforth use *path* as shorthand for $(k-1)$-path and *cycle* as shorthand for $(k-1)$-cycle. The named graphs we will need are as follows (see also Figure 4.1).

- P_a is the *path* containing a vertices, which has vertices $x_1, \ldots, x_a$ and edges $x_i x_j$ whenever $ij \in \binom{[a]}{2}$ satisfy $|i - j| \leq k - 1$.
- C_a is the *cycle* containing a vertices, which has vertices $x_1, \ldots, x_a$ and edges $x_i x_j$ whenever $ij \in \binom{[a+k-1]}{2}$ satisfy $|i - j| \leq k - 1$, where $x_{a+\ell} := a_\ell$ for $\ell \in [k-1]$.
- Q_a is the *midpath* obtained from P_a by removing all edges between the first k vertices, and between the last k vertices (each of these sets of vertices form a clique in P_a).
- $A_a(\{(v_1, i_1), \ldots, (v_t, i_t)\})$ is obtained from Q_a by adding a vertex v_j which is adjacent to $x_{i_1-k+2}, \ldots, x_{i_1+k-1}$ for each $j \in [t]$. The vertices v_j are not necessarily distinct. Write $A_{4k-2}(v) := A_{4k-2}(\{v, 2k-1\})$.
- $F_a(i_1, \ldots, i_t)$ is an $\{i_1, \ldots, i_t\}$*-missing path* if it is obtained from $P_a = x_1 \ldots x_a$ by removing $x_{\ell-h+1}x_{\ell-h+k}$ for all $\ell \in \{i_1, \ldots, i_t\}$ and $h \in [k-1]$.

In using the last two definitions, if $|i_j - i_{j'}| \geq 2k$ and $k < i_j < a - k$, then all of the added/removed edges are distinct. Each of the graphs $P_a, Q_a, F_a(i_1, \ldots, i_t)$ has a natural ordering of vertices and we often write, say, $Q_a = x_1 \ldots x_a$ for this ordering.

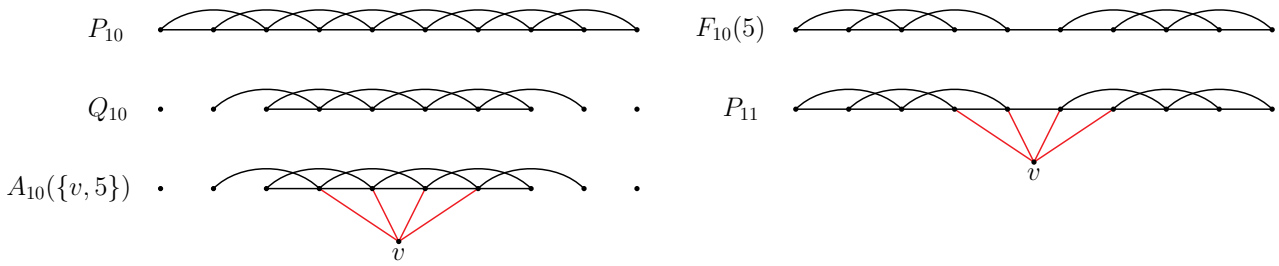


Figure 4.1: Named graphs in the case $k = 3$.

Note that

$$e(P_{a_1}) + \ldots + e(P_{a_r}) + r\binom{k}{2} = e(C_{a_1+\ldots+a_r}) \quad \text{and} \quad e(Q_a) = e(P_{a-2k}) + 2\binom{k}{2}. \tag{4.1}$$

Observation 4.3.1.

(i) If we concatenate copies of paths and midpaths as $P_{a_1}Q_{a_2}P_{a_3}$ where all vertices are distinct except the last k-tuple of P_{a_1} is the first k-tuple of Q_{a_2} and the last k-tuple of Q_{a_2} is the first k-tuple of P_{a_3}, we obtain a copy of $P_{a_1+a_2+a_3}$.

(ii) Let $F = F_{ka}(i_1, \ldots, i_t) = x_1 \ldots x_{ka}$ where every index pair satisfies $|i_j - i_{j'}| \geq 2k$ and every index satisfies $k < i_j < ka - k$. Then

$$|\{x_p x_r \in E(F) : p \equiv \ell \text{ and } r \equiv \ell' \mod k\}|$$

$$= \begin{cases} 2a - 1 & \text{if } |\ell - \ell'| \not\equiv 1 \\ 2a - 1 - t + |\{j \in [t] : i_j \equiv \ell + 1\}| & \text{if } \ell - \ell' \equiv 1. \end{cases}$$

Let $y_1, \ldots, y_t$ be new vertices then add the edges $x_{i_\ell - h+1}y_\ell, \ldots, x_{i_\ell-h+k}y_\ell$ for all $\ell \in \{i_1, \ldots, i_t\}$. The graph obtained is a path $P_{ka+t} = x_1 \ldots x_{i_1} y_1 x_{i_1+1} \ldots x_{i_t} y_t x_{i_t+1} \ldots x_{ka}$. (See the bottom right of Figure 4.1.)

(iii) An edge-coloured copy A of $A_{4k-2}(v)$ where all colours on edges not containing v are distinct, the colours incident to v are distinct and every colour used on $x_{2k-h}x_{3k-1-h}$ for $h \in [k-1]$ is also used by some xv has the property that $A - v$ is a rainbow copy of Q_{4k-2} and A contains a spanning rainbow copy of $Q_{4k-1} = x_1 \ldots x_{2k-1} v x_{2k} \ldots x_{4k-2}$ and they have the same start and end k-tuple.

4.3.2 Connecting and absorbing

We will use the following connecting lemma of Komlós, Sárközy and Szemerédi.

Lemma 4.3.2 (Connecting lemma (Komlós et al., 1998a)). *Let $0 < 1/N \ll \gamma, 1/k \leq 1/2$. Let A, B be copies of K_k in a kN-vertex graph G with $\delta(G) \geq (k-1+\gamma)N$. Let $\{x_1, \dots, x_k\}$ be a labelling of $V(A)$ and let $\{y_1, \dots, y_k\}$ be a labelling of $V(B)$. Then there exist $t \leq 5k^2$ and a graph homomorphism $\psi : V(P_{kt}) \to V(G)$ such that the initial k-tuple is mapped to $(x_1, \dots, x_k)$ and the final k-tuple is mapped to $(y_1, \dots, y_k)$.*

The next two lemmas in this section will assume the following setting, which, after some initial steps, is the setting we find ourselves in during the proof of Theorem 4.1.2.

Setting 4.3.3. *Let $0 < 1/n \ll 1/N \ll \varepsilon \ll \eta \ll d, d' \ll \gamma, \delta, 1/k \leq 1/2$. Let $L := kN$ and $M := (k-1)L$. Suppose that*

- *$\boldsymbol{G} = (G_c : c \in \mathscr{C})$ is a graph collection on a common vertex set V of size n with $\delta(\boldsymbol{G}) \geq (\frac{k-1}{k} + \gamma)n$;*
- *$\boldsymbol{R} = (R_\chi : \chi \in [M])$ is a graph collection on a common vertex set $[L]$ and*
  - *for all $i \in [L]$ we have $d_{R_\chi}(i) \geq (\frac{k-1}{k} + \gamma/2)L$ for all but at most $d'M$ colours $\chi \in [M]$, and*
  - *for all $\chi \in [M]$ we have $d_{R_\chi}(i) \geq (\frac{k-1}{k} + \gamma/2)L$ for all but at most $d'L$ vertices $i \in [L]$;*
- *$hi \in R_\chi$ whenever $(G'_c[V_h, V_i] : c \in \mathscr{C}_\chi)$ is (ε, d)-regular, where $G'_c \subseteq G_c$;*
- *there is a partition $V = V_0 \cup V_1 \cup \ldots \cup V_L$ where $\{U_h^j : j \in [N], h \in [k]\} = \{V_i : i \in [L]\}$ is a labelling such that $U_1^j, \dots, U_k^j$ span a clique in $\bigcup_{\chi \in [M]} R_\chi$ for each $j \in [L]$;*
- *there is a partition $\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1 \cup \ldots \cup \mathscr{C}_M$;*
- *$|V_0| + |\mathscr{C}_0| \leq \varepsilon n$ and the other vertex classes all have the same size m, and the other colour classes all have the same size m' with $\delta m \leq m' \leq m/\delta$.*

The first lemma will be used to build absorbers and connecting paths.

Lemma 4.3.4. *Suppose that we are in Setting 4.3.3. Let $X_1, \dots, X_\ell$ be such that $X_j \subseteq V_j$ with $|X_j| \geq \eta m$ for each $j \in [\ell]$ and let $x_1, \dots, x_h \in V$, where $h + \ell \leq k$. Let $W \subseteq V$ and $\mathscr{W} \subseteq \mathscr{C}$ with $|W|, |\mathscr{W}| \leq \varepsilon n$.*

(i) Let $b_1, \dots, b_h \in \mathscr{C}$. There exists $x \in V \setminus W$ and $X'_j \subseteq X_j$ and distinct $a_1, \dots, a_\ell \in \mathscr{C} \setminus \mathscr{W}$ with $|X'_j| \geq d|X_j|/5$ such that $X'_j \subseteq N_{G_{a_j}}(x)$ for all $j \in [\ell]$, and $xx_i \in G_{b_i}$ for all $i \in [h]$.

(ii) There are $g \in [L]$ and $X \subseteq V_g$ and distinct $b_1, \dots, b_h \in \mathscr{C} \setminus \mathscr{W}$ with $|X| \geq dm/5$ and $\chi \in [M]$ such that $gj \in E(R_\chi)$ for all $j \in [\ell]$ and $X \subseteq N_{G_{b_i}}(x_i)$ for all $i \in [h]$.

Proof. We prove (i). There are at most $\sqrt{\varepsilon}M$ colour indices $i \in [M]$ for which $|\mathscr{W} \cap \mathscr{C}_i| \geq \sqrt{\varepsilon}m$. By our assumption on $\boldsymbol{R}$, of the remaining indices, there is a $\chi \in [M]$ (in fact we can take almost any χ) such that $d_{R_\chi}(j) \geq (\frac{k-1}{k} + \gamma/2)L$ for every $j \in [\ell]$. Without loss of generality, we may suppose $\chi = 1$. Let $\mathcal{N}$ be the common neighbourhood of $[\ell]$ in R_1. Thus

$$|\mathcal{N}| \geq \ell \left(\frac{k-1}{k} + \gamma/2 \right) L - (\ell - 1)L = (1 - \ell/k + \ell\gamma/2)L.$$

Given $j \in [\ell]$ and $i \in \mathcal{N}$, since $ij \in R_1$, the collection $(G'_c[V_i, V_j] : c \in \mathscr{C}_1)$ is (ε, d)-regular. Let B^i_j be the set of vertices x in V_i with $\sum_{c \in \mathscr{C}_1} d_{G'_c}(x, X_j) \leq d|X_j|m'/4$. Lemma 3.2.4(i) implies that $(G'_c[V_i, X_j] : c \in \mathscr{C}_1)$ is $(\sqrt{\varepsilon}, d/2)$-regular. By Lemma 3.2.3(i), $|B^i_j| \leq \sqrt{\varepsilon}m$. Let $U := \bigcup_{i \in \mathcal{N}} (V_i \setminus \bigcup_{j \in [\ell]} B^i_j)$. Then

$$|U| \geq (1 - \ell\sqrt{\varepsilon})m|\mathcal{N}| \geq (1 - \ell\sqrt{\varepsilon})(1 - \ell/k + \ell\gamma/2)mL \geq (1 - \varepsilon^{1/3})(1 - \ell/k)n,$$

where we used the fact that $mL \geq (1 - \varepsilon)n$. Now let $U' := U \cap \bigcap_{i \in [h]} N_{G_{b_i}}(x_i)$. Then

$$|U'| \geq |U| + \left(h \left(\frac{k-1}{k} + \gamma \right) n - (h-1)n \right) - n \geq \left(1 - \frac{h+\ell}{k} + \gamma/2 \right) n \geq \gamma n/2.$$

Choose any $x \in U' \setminus W$ which is possible since $\gamma \gg \varepsilon$. Then $xv_i \in G_{b_i}$ for all $i \in [h]$, as required. We will choose the a_j greedily and set $X'_j := N_{G_{a_j}}(x, X_j)$. Indeed, for some $j \in [\ell]$, suppose we have chosen distinct $a_1, \dots, a_{j-1} \in \mathscr{C} \setminus \mathscr{W}$ such that $|X'_{j'}| \geq d|X_j|/5$ for all $j' \in [j-1]$. Since $x \in U$, we have $\sum_{c \in \mathscr{C}_1} d_{G'_c}(x, X_j) \geq d|X_j|m'/4$. Since $|\mathscr{W} \cap \mathscr{C}_1| < \sqrt{\varepsilon}m$, colours in $\mathscr{W} \cup \{a_1, \dots, a_{j-1}\}$ contribute at most $(\sqrt{\varepsilon}m + (j-1))m \leq 2\sqrt{\varepsilon}m^2$ to this sum. By averaging, there is some colour $a_j \in \mathscr{C}_1$ distinct from these such that $d_{G'_{a_j}}(x, X_j) \geq d|X_j|/5$. Thus we can choose $a_1, \dots, a_\ell$ as required. This completes the proof.

The proof of part (ii) is similar so we give a sketch only. Again there is a colour index in $[M]$, say 1, for which every cluster $V_1, \dots, V_\ell$ has high degree. We let $\mathcal{N}$ be their common neighbourhood in R_1, and let U be the set of vertices x in clusters indexed by $\mathcal{N}$ for which $xv_i \in G_{b_i}$ for all $i \in [h]$. Then U has size at least $\gamma n/2$, so we can find a cluster V_g whose intersection with U is at least $\gamma m/3 > dm/5$, even after deleting W. We can then choose

$b_1, \dots, b_h \in \mathscr{C}_1$ greedily as before. □

Lemma 4.3.5 (Connecting and absorbing lemma)**.** *Suppose that we are in Setting 4.3.3. Let* $\mathcal{Q} = \{(\mathscr{S}_q, v_q) : q \in Q\}$ *where each* $\mathscr{S}_q \in \binom{\mathscr{C}}{k-1}$ *and* $v_q \in V$ *and* $|Q| \leq \varepsilon^2 n/N$*, and let* $W \subseteq V$ *and* $\mathscr{W} \subseteq \mathscr{C}$ *satisfy* $|W|, |\mathscr{W}| \leq \varepsilon n/2$*. Let* $j, j' \in [N]$ *be distinct. Then the following hold.*

(i) There is

- *a vertex set* $X \subseteq V \setminus W$ *of size at most* $10k^3|Q|$*,*
- *a colour set* $\mathscr{B} \subseteq \mathscr{C} \setminus \mathscr{W}$*,*
- *vertex sets* $T_h^j \subseteq U_h^j$ *and* $S_h^{j'} \subseteq U_h^{j'}$ *with* $|T_h^j|, |S_h^{j'}| \geq \eta m$ *for all* $h \in [k]$

such that the following holds.

For any $q_1, \dots, q_r \in Q$ *where* $\mathscr{S}_{q_1}, \dots, \mathscr{S}_{q_r}$ *are pairwise disjoint and* $v_{q_1}, \dots, v_{q_r}$ *are distinct, and any* $y_h \in T_h^j$ *and* $z_h \in S_h^{j'}$ *with* $h \in [k]$*, there is a transversal midpath from* $(y_1, \dots, y_k)$ *to* $(z_1, \dots, z_k)$ *with vertex set* $\{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\} \cup X \cup \{v_{q_1}, \dots, v_{q_r}\}$ *and colour set* $\mathscr{B} \cup \mathscr{S}_{q_1} \cup \dots \cup \mathscr{S}_{q_r}$*;*

(ii) there are $X, \mathscr{B}, T_h^j, T_h^{j'}$ *as in (i) with* $|X| \leq 10k^2$ *such that for any* $y_h \in T_h^j$ *and* $z_h \in S_h^{j'}$ *with* $h \in [k]$*, there is a transversal midpath from* $(y_1, \dots, y_k)$ *to* $(z_1, \dots, z_k)$ *with vertex set* $\{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\} \cup X$ *and colour set* $\mathscr{B}$*.*

Proof. We prove (i) by first finding small, flexible vertex-disjoint absorbers for each $(\mathscr{S}_q, v_q)$, before connecting them via short paths with new colours. The individual absorbers are found in the following claim.

Claim 4.3.6. *For every* $q \in Q$*, there is*

- *a vertex set* $X_q \subseteq V \setminus (W \cup \{v_q : q \in Q\})$ *of size* $2k - 2$*;*
- *a colour set* $\mathscr{B}_q \subseteq \mathscr{C} \setminus (\mathscr{W} \cup \bigcup_{q \in Q} \mathscr{S}_q)$ *of size at most* k^2*;*
- *tuples* $\boldsymbol{i}_q^-, \boldsymbol{i}_q^+ \in [L]^k$*, each one spanning a* k*-clique in* $\bigcup_{\chi \in [M]} R_\chi$*;*
- *vertex sets* $T_{h,q}^{\pm} \subseteq V_{i_h^{\pm}}$ *with* $|T_{h,q}^{\pm}| \geq \eta m$ *for* $\boldsymbol{i}_q^{\pm} = (i_1^{\pm}, \dots, i_k^{\pm})$*;*

such that $X_q, X_{q'}$ *are disjoint and* $\mathscr{B}_q, \mathscr{B}_{q'}$ *are disjoint for any* $q \neq q'$*, and for any* $y_h^{\pm} \in T_{h,q}^{\pm}$ *with* $h \in [k]$*, there is a transversal midpath from* $(y_1^-, \dots, y_k^-)$ *to* $(y_1^+, \dots, y_k^+)$ *with vertex set* $X_q \cup \{v_q\}$ *and colour set* $\mathscr{B}_q \cup \mathscr{S}_q$*, and a transversal midpath from* $(y_1^-, \dots, y_k^-)$ *to* $(y_1^+, \dots, y_k^+)$ *with vertex set* X_q *and colour set* $\mathscr{B}_q$*.*

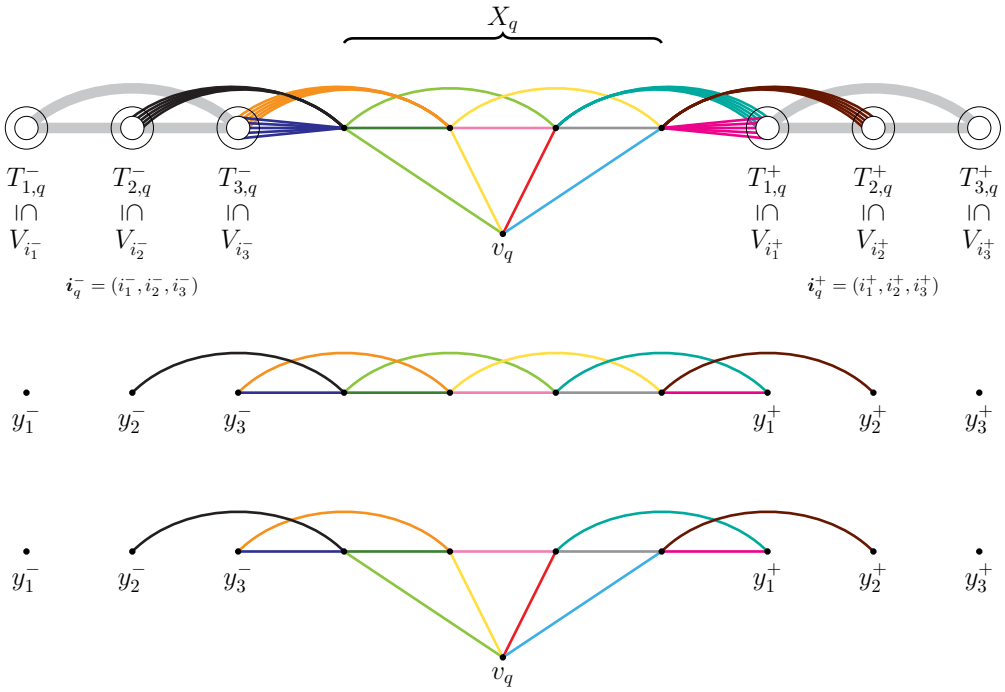


Figure 4.2: An illustration of Claim 4.3.6 for one $q \in Q$, where $\mathscr{S}_q = \{\text{red, blue}\}$. The colour set $\mathscr{A}_q$ which appears in the proof is $\{\text{green, yellow}\}$.

Proof. We will use Lemma 4.3.4 to prove the claim. We will show how to find the required objects for a single q whenever there is some set W' of at most $2\varepsilon n$ forbidden vertices and $\mathscr{W}'$ of at most $2\varepsilon n$ forbidden colours. This will suffice to find objects for all q with the required disjointness properties since in total we use $O(k^2)|Q| < \varepsilon n$ vertices and colours.

For this, fix $q \in Q$. Note that in $A = A_{4k-2}(v_q)$ (see the definition at the beginning of Section 4.3.1) if we take the natural order $A - v_q = x_1 \dots x_a$ and then reorder the vertices as

$$v_q, x_{2k-1}, x_{2k}, \dots, x_{4k-2}, x_{2k-2}, x_{2k-3}, \dots, x_1,$$

then each vertex has at most k previous neighbours (see Figure 4.1). Write $y_0, \dots, y_{4k-2}$ for this ordering. Define $r_0, \dots, r_{4k-2}$ by setting $y_i = x_{r_i}$, where $x_0 := v_q$ (so $r_0 = 0$). For each $i \in [4k-2]$, let

$$S_i^{\text{set}} := \{j \in [0, 4k-2] : j < i \text{ and } r_j \in [k] \cup [3k-1, 4k-2] \text{ and } y_j y_i \in E(A)\} \quad \text{and}$$
$$S_i^{\text{vx}} := \{j \in [0, 4k-2] : j < i \text{ and } r_j \in [k+1, 3k-2] \text{ and } y_j y_i \in E(A)\}.$$

So S_i^{set} are the indices j before i in the natural order which correspond to required vertex sets

$T^{\pm}_{h,q}$ and S^{vx}_i are the indices j before i corresponding to a required vertex $z_j \in X_q$. We have $|S^{\mathrm{set}}_i| + |S^{\mathrm{vx}}_i| \leq k$.

Initialise by setting $X_q := \emptyset$ and $\mathscr{B}_q := \emptyset$. We apply Lemma 4.3.4 to greedily find a vertex or set of vertices for each $z = y_1, \dots, y_{4k-2}$ in this order. For each $i \in [4k-2]$, if $r_i \in [k+1, 3k-2]$, we find a single vertex and add it to X_q, whereas if $r_i \in [k] \cup [3k-1, 4k-2]$ we need a set T_z of at least ηn vertices inside a single vertex cluster. Choose a set of $k-1$ colours $\mathscr{A}_q$ which is disjoint from $\mathscr{S}_q$ and $\mathscr{W}$ and distinct from each other, doubly labelled as

$$\mathscr{A}_q = \{c_{k+1,2k} = c_{0,k+1}, c_{k+2,2k+1} = c_{0,k+2}, \dots, c_{2k-1,3k-2} = c_{0,2k-1}\}$$

$$\text{and write} \quad \mathscr{S}_q = \{c_{0,2k}, \dots, c_{0,3k-2}\}.$$

Now, let $z_0 := y_0 = v_q$ and suppose we have found $T_{y_1}, \dots, T_{y_{i-1}}$ for some $i \in [4k-2]$ such that

- $T_{y_j} = \{z_j\}$ for $j \in [k+1, 3k-2]$ and the union of these sets is X_q,
- $\mathscr{B}_q = \{c_{g,j} : x_g x_j \in E(A[\{y_1, \dots, y_{i-1}\}]), 1 \leq g < j\}$ and the $c_{g,j}$ outside of $\mathscr{A}_q$ are distinct from each other and disjoint with $\mathscr{S}_q \cup \mathscr{A}_q \cup \mathscr{W}$,
- $|T_{y_j}| \geq (d/5)^{i-1} m$ for $j \in [k] \cup [3k-1, 4k-2]$ and there is i_j such that $T_{y_j} \subseteq V_{i_j}$, and, for each $j < j' < i$, we have $i_j i_{j'} \in \bigcup_\chi R_\chi$ whenever $j' \leq k$ or $j \geq 3k-1$,
- for every $z_j \in T_{y_j}$, we have that $v_q, z_1, \dots, z_{i-1}$ play the roles of $y_0, y_1, \dots, y_{i-1}$ respectively in a copy of $A[\{y_0, \dots, y_{i-1}\}]$, in which every $z_g z_j$ with $0 \leq g < j < i$ is coloured with $c_{g,j}$.

We will find T_{y_i} which extends these 'robust' partial copies of A. It suffices to find $T_{y_1}, \dots, T_{y_{4k-2}}$, since, by Observation 4.3.1(iii) (see also Figure 4.2), vertices in these sets yield suitable copies of Q_{4k-2} and Q_{4k-1}, with $\mathscr{B}_q = \{c_{g,j} : x_g x_j \in E(A - v_q)\}$ and $(d/5)^{4k-2} > \eta$.

We apply Lemma 4.3.4 to find T_{y_i}. Here, $\{T_{y_j} : j \in S^{\mathrm{set}}_i\}$ plays the role of $\{X_1, \dots, X_\ell\}$, while $\{y_j : j \in S^{\mathrm{vx}}_i\}$ plays the role of $\{x_1, \dots, x_h\}$. We therefore have $\ell + h = |S^{\mathrm{vx}}_i| + |S^{\mathrm{set}}_i| \leq k$.

If $r_i \in [k+1, 3k-2]$, we apply (i). For the colours $\{b^i_j : j \in S^{\mathrm{vx}}_i\} \subseteq \mathscr{C}$, we let $b^i_j := c_{r_j, r_i}$. Thus Lemma 4.3.4 implies there is $z_i \in V \setminus W'$ and $T'_{y_j} \subseteq T_{y_j}$ for all $j \in S^{\mathrm{set}}_i$ and distinct colours $\{a^i_j : j \in S^{\mathrm{set}}_i\} \subseteq \mathscr{C} \setminus \mathscr{W}'$ with $|T'_{y_j}| \geq d|T_{y_j}|/5$ such that $T'_{y_j} \subseteq N_{G_{a^i_j}}(z_i)$ for all $j \in S^{\mathrm{set}}_i$ and $z_i z_j \in G_{b^i_j}$ for all $j \in S^{\mathrm{vx}}_i$. Set $T_{y_i} := \{z_i\}$ and replace T_{y_j} by T'_{y_j} for all $j \in S^{\mathrm{set}}_i$ while T_{y_j} is unchanged for other $j < i$. Set $c_{r_j, r_i} := a^i_j$ for each $j \in S^{\mathrm{set}}_i$ and add these colours to $\mathscr{B}_q$.

If $r_i \in [k] \cup [3k-1, 4k-2]$, we apply (ii) to find $g_i \in [L]$ and $T_{y_i} \subseteq V_{g_i}$ and distinct colours $\{b^i_j : j \in S^{\mathrm{vx}}_i\} \in \mathscr{C} \setminus \mathscr{W}'$ with $|T_{y_i}| \geq dm/5$ and $\chi_i \in [L]$ such that $g_i j \in R_{\chi_i}$ for all $j \in S^{\mathrm{set}}_i$ and $T_{y_i} \subseteq N_{G_{b^i_j}}(x_j)$ for all $j \in S^{\mathrm{vx}}_i$. Set $c_{r_j,r_i} := b^i_j$ for each $j \in S^{\mathrm{vx}}_i$ and add these colours to $\mathscr{B}_q$.

This completes the proof of the claim. □

For (i), let $r := |Q|$ and let $q_1, \dots, q_r$ be an arbitrary ordering of Q. It suffices to find vertex-disjoint transversal copies $P^{\mathrm{con}}_{a_0}, \dots, P^{\mathrm{con}}_{a_r}$ of $P_{a_0}, \dots, P_{a_r}$ respectively, where each $a_i \leq 5k^3$, none of them use vertices in any X_q or v_q or W, none of them share colours or use colours in any $\mathscr{B}_q$ or $\mathscr{S}_q$ or $\mathscr{W}$, such that

- for each $i \in [r-1]$, $P^{\mathrm{con}}_{a_i}$ is between some $(y^+_1, \dots, y^+_k) \in \prod_{j\in[k]} T^+_{j,q_i}$ and some $(y^-_1, \dots, y^-_k) \in \prod_{h\in[k]} T^-_{h,q_{i+1}}$,
- there are $T^j_h \subseteq U^j_h$ with $|T^j_h| \geq \eta n$ for each $h \in [k]$ such that for any $y_h \in T^j_h$, $y_1 \dots y_k P^{\mathrm{con}}_{a_0}$ is a transversal path with new colours $\mathscr{D}_j$, ending at some $(y^+_1, \dots, y^+_k) \in \prod_{h\in[k]} T^+_{h,q_1}$,
- there are $S^{j'}_h \subseteq U^{j'}_h$ with $|S^{j'}_h| \geq \eta n$ for each $h \in [k]$ such that for any $z_h \in S^{j'}_h$, $P^{\mathrm{con}}_{a_r} z_1 \dots z_k$ is a transversal path with new colours $\mathscr{D}_{j'}$, starting at some $(y^-_1, \dots, y^-_k) \in \prod_{h\in[k]} T^-_{h,q_r}$.

Indeed, we let X be the union of the X_q together with the vertices of $P^{\mathrm{con}}_{a_0}, \dots, P^{\mathrm{con}}_{a_r}$, so $|X| \leq (5k^3 + 2k - 2)|Q| \leq 10k^3|Q| < \varepsilon n/2$, and $\mathscr{B}$ be the union of the $\mathscr{B}_q$ together with the colours on $P^{\mathrm{con}}_{a_0}, \dots, P^{\mathrm{con}}_{a_r}$ and $\mathscr{D}_j$ and $\mathscr{D}_{j'}$. Then, given any q_i, Claim 4.3.6 furnishes us with a midpath with vertex set $X_{q_i} \cup \{v_q\}$ whose first k vertices match with the last k of $P^{\mathrm{con}}_{a_{i-1}}$ and whose last k vertices match with the first k of $P^{\mathrm{con}}_{a_i}$, and for all other q', the midpath with vertex set $X_{q'}$ with suitable matching vertices; we interleave these midpaths with the $P^{\mathrm{con}}_{a_0}, \dots, P^{\mathrm{con}}_{a_r}$ to obtain a single midpath (see Observation 4.3.1(i)) which is as required.

To find the $P^{\mathrm{con}}_{a_i}$, recall that, for each $t \in [r]$, we have $T^{\pm}_{h,q_t} \subseteq V_{i^{\pm}_h}$ where $\boldsymbol{i}^{\pm}_{q_t} = (i^{\pm}_1, \dots, i^{\pm}_k)$ spans a clique in $\bigcup_\chi R_\chi$.

Now, fix any $t \in [0, r]$. First, choose a set of colour indices $\mathcal{C}_t$ in $[M]$ as follows. Add a colour index χ to $\mathcal{C}_t$ such that $d_{R_\chi}(i) \geq (\frac{k-1}{k} + \gamma/2)L$ for every entry i in $\boldsymbol{i}^+_{q_t}$ and $\boldsymbol{i}^-_{q_{t+1}}$, which is easy to do as at most $2kd'M$ colours are excluded. Add to $\mathcal{C}_t$ the at most $2\binom{k}{2}$ colours such that $\boldsymbol{i}^+_{q_t}$ and $\boldsymbol{i}^-_{q_{t+1}}$ span a clique using these colours as above. Let $R_{\mathrm{con}} := \bigcup_{\chi\in\mathcal{C}_t} R_\chi$. By our assumptions in Setting 4.3.3, $d_{R_{\mathrm{con}}}(i) \geq (k-1+\gamma/3)L$ for at least $L - d'L$ vertices $i \in [L]$, and by construction,

these vertices include all those in $\boldsymbol{i}^+_{q_t}$ and $\boldsymbol{i}^-_{q_{t+1}}$. Let R'_{con} be the subgraph of R_{con} induced on these vertices. So $\delta(R'_{\mathrm{con}}) \geq (k-1+\gamma/4)|V(R'_{\mathrm{con}})|$. Apply Lemma 4.3.2 (connecting lemma) in the graph R'_{con} to find a graph homomorphism $\psi_t : V(P_{a_t}) \to V(R'_{\mathrm{con}})$ such that the initial k-tuple of P_{a_t} is mapped to $\boldsymbol{i}^+_{q_t}$ and the final k-tuple to $\boldsymbol{i}^-_{q_{t+1}}$, where $a_t \leq 5k^3$. Apply Lemma 3.3.1 (Embedding lemma with target and candidate sets), with the template induced on the clusters in the image of ψ_t, with any previously used vertices and colours removed, and $Y = \emptyset$, to find the required path $P^{\mathrm{con}}_{a_t}$. Indeed, the template has parameters $(m/2, 2\sqrt{\varepsilon}, d/4, 1/2)$ since at most εm vertices and colours are removed. This completes the proof of (i).

The proof of (ii) is almost identical except we connect $(U^j_1, \dots, U^j_k)$ with $(U^{j'}_1, \dots, U^{j'}_k)$ using Lemma 4.3.2 and then use this as a guide to find a transversal path of length at most $5k^3$ with Lemma 3.3.1. There are $O_{k,N}(1)$ excluded vertices and colours to ensure that the paths are disjoint from the ones we have already found. We omit the details. □

The next ingredient, Lemma 4.3.7, is for finding good coloured neighbourhoods for vertices. Part (ii) will be used for incorporating exceptional vertices into a template, while part (i) will be used in the covering step to find a large K_k-tiling in the reduced graph.

Part (i) is essentially same as the uncoloured version which is used in (Komlós et al., 1998a) to incorporate exceptional vertices: given a single graph G with $d_G(v, V) \geq (1+\gamma)(k-1)Nm$ and $V = \bigcup_{j\in[N], i\in[k]} U^j_i$ a partition into clusters of size m, it finds many indices j such that $d_G(v, U^j_i) \geq \delta m$ for all $i \in [k]$, which follows immediately by counting. This can be applied to an exceptional vertex v to insert it 'anywhere' in a $(k-1)$-path that uses the clusters U^j_i, $i \in [k]$ cyclically. A little more effort is required for part (ii), when one wishes to insert v using $2k-2$ prescribed colours for edges to its neighbours. We would like to find some $t \in [k]$ such that we can insert v between a vertex in U^j_t and its successor in U^j_{t+1}. However, this is not possible, as it could be that for every j, the sets $U^j_1, \dots, U^j_k$ only see $2k-3$ colours. However, we can guarantee in part (ii) that among $2k-1$ colours, we can use $2k-2$ as described.

Lemma 4.3.7 (Exceptional vertices)**.** *Let $0 < 1/N \ll \gamma, 1/k$. Let $V = \{x^j_i : j \in [N], i \in [k]\}$, let $\mathscr{A}$ be a set of colours and let $(G_c : c \in \mathscr{A})$ be graphs and v a vertex such that $d_{G_c}(v, V) \geq (1+\gamma)(k-1)N$ for all $c \in \mathscr{A}$. Then the following hold:*

(i) if $|\mathscr{A}| = k$, then there exist γN values of $j \in [N]$ such that for each one there is a labelling $c_1, \dots, c_k$ of $\mathscr{A}$ such that $x^j_i v \in G_{c_i}$ for all $i \in [k]$;

(ii) if $|\mathscr{A}| = 2k-1$, then there exist γN values $j \in [N]$ such that for each one there is $t \in [k]$

and $\mathscr{B} \subseteq \mathscr{A}$ with $|\mathscr{B}| = 2k-2$ and partition $\mathscr{B} = \mathscr{A}_1 \cup \ldots \cup \mathscr{A}_k$ where $|\mathscr{A}_t| = |\mathscr{A}_{t+1}| = 1$ and all other $\mathscr{A}_i$ have size 2, and $x_i^j v \in G_c$ for all $i \in [k]$ and $c \in \mathscr{A}_i$.

Proof. Let $\mathrm{col}(xv)$ be the set of colours in $\mathscr{A}$ on the edge xv, for $x \in V$. We have

$$\sum_{j\in[N]} \sum_{i\in[k]} |\mathrm{col}(x_i^j v)| \geq \sum_{c\in\mathscr{A}} d_{G_c}(v) \geq |\mathscr{A}|(1+\gamma)(k-1)N.$$

Averaging, there are at least $\gamma N/2$ values $j \in [N]$ for which $\sum_{i\in[k]} |\mathrm{col}(x_i^j v)| \geq \lceil (1+\gamma)(k-1)|\mathscr{A}| \rceil$. We will show that, for every such j, we can find colours as specified. Fix such a j and write $U_i := \mathrm{col}(x_i^j v)$ for all $i \in [k]$, so $|U_1| + \ldots + |U_k| \geq \lceil (1+\gamma)(k-1)|\mathscr{A}| \rceil$.

For (i), when $|\mathscr{A}| = k$, we have $|U_1| + \ldots + |U_k| \geq k(k-1)+1$. We can order the sets U_i for $i \in [k]$ in increasing order of size, and then pick an arbitrary colour that has not been chosen before from each set. At step ℓ, suppose we are considering the set U_i, so we have $|U_i|$ colours to choose from, and have forbidden at most $\ell - 1$ colours in this set due to previous steps. If the algorithm fails here, then $|U_i| \leq \ell - 1$ and therefore the sum of the sizes is at most $(\ell-1)\ell + (k-\ell)k = k^2 - (k+1-\ell)\ell \leq k(k-1)$, attained when $\ell \in \{1, k\}$. This is a contradiction, and hence the algorithm can be completed.

For (ii), when $|\mathscr{A}| = 2k-1$, we have $|U_1| + \ldots + |U_k| \geq (2k-1)(k-1)+1$. We need to show that there is a $t \in [k]$ and pairwise disjoint sets $\mathscr{C}_1, \ldots, \mathscr{C}_k \subseteq \mathscr{C}$ with $|\mathscr{C}_t| = |\mathscr{C}_{t+1}| = 1$ and $|\mathscr{C}_i| = 2$ for $i \in [k] \setminus \{t, t+1\}$, and $\mathscr{C}_i \subseteq U_i$, where $\mathscr{C}_{k+1} := \mathscr{C}_1$. For this, let σ be the permutation of k such that $|U_{\sigma(1)}| \leq \ldots \leq |U_{\sigma(k)}|$, breaking ties arbitrarily. We will choose $t := \sigma(1)$. Let $2 \leq s \leq k$ be such that $\sigma(s) = t+1$, where we write $k+1$ for 1. We try to choose the $\mathscr{C}_i$ in the order $i = \sigma(1), \ldots, \sigma(k)$: choose an arbitrary colour in $U_{\sigma(1)}$ to be the single required element of $\mathscr{C}_{\sigma(1)}$, then for each $2 \leq i < s$ choose two new colours from U_i if possible, choose one new element of $\mathscr{C}_{\sigma(s)}$, then for $s < i \leq k$ choose two new colours from U_i. Suppose this algorithm fails at step i. Note that $i \geq 2$ since $|U_{\sigma(1)}| \geq 1$ by the pigeonhole principle. We have

$$|U_i| \leq \begin{cases} 2i-2 & \text{if } 1 \leq i \leq s-1 \\ 2i-3 & \text{if } s \leq i \leq k. \end{cases}$$

Suppose $i \leq s-1$, so in particular $i \leq k-1$. We have

$$(2k-1)(k-1)+1 \leq \sum_{j\in[k]} |U_{\sigma(j)}| \leq (2i-2)i+(2k-1)(k-i) \quad \text{so} \quad 2i^2-(2k+1)i+2k-2 \geq 0.$$

Therefore $i \leq \frac{k}{2}+\frac{1}{4}-\frac{1}{2}\sqrt{(k-\frac{3}{2})^2+2}$ or $i \geq \frac{k}{2}+\frac{1}{4}+\frac{1}{2}\sqrt{(k-\frac{3}{2})^2+2}$, so $i < \frac{k}{2}+\frac{1}{4}-\frac{1}{2}(k-\frac{3}{2}) = 1$ or $i > \frac{k}{2}+\frac{1}{4}+\frac{1}{2}(k-\frac{3}{2}) = k-\frac{1}{2}$, and we are done. Suppose instead $i \geq s$. Then the bound is stronger and so again $i < 1$ or $i > k-\frac{1}{2}$. So the only case to check is $i = k$. In this case we have, since $s \leq k$, that $(2k-1)(k-1)+1 < (2k-3)k$, which is a contradiction. □

4.4 Covering via fractional tilings

A *t-rainbow* copy of a graph H in a graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ is such that for each $e \in E(H)$, the image of e lies in $G_{c_e^1}, \dots, G_{c_e^t}$, and the colours $\{c_e^i : i \in [t], e \in E(H)\} \subseteq \mathscr{C}$ are distinct.

The following lemma shows that one can find an almost perfect 'fractional' t-rainbow clique tiling (defined below) in a graph collection with sufficiently high minimum degree. There is a version of this lemma in (Montgomery et al., 2022) for $t = 1$ and a perfect rainbow clique tiling (a rainbow K_k-factor) when there are slightly more colours than required, which proceeds by randomly partitioning V and then successively finding a clique factor in each part, in which every edge appears in many unused colours. This can be easily generalised to larger t. However, we need a version where

- every vertex v has high minimum degree in at least $1-\mu$ proportion of colours (not all)
- every colour c has many edges incident to at least $1-\mu$ proportion of vertices (not all)
- the required size of the tiling is $(1-\zeta)$, where $\zeta \ll \mu$.

The method of (Montgomery et al., 2022) is not sufficient for this, since when μ fraction of vertices (and thus μ fraction of colours) are left, it could be that none of these vertices have any edges in any of these colours.

It suffices for us to find a 'fractional' tiling, which we now define. Fractional ideas were also used for transversal embedding in (Cheng et al., 2023). Let $K_k(\boldsymbol{G})$ be the collection of edge-coloured copies of K_k in $\boldsymbol{G}$. More precisely, $K_k(\boldsymbol{G})$ is the set of all $K := \{x_i x_j c_{ij} : ij \in \binom{[k]}{2}\}$ where the vertices $x_1, \dots, x_k \in V$ are distinct and $x_i x_j \in G_{c_{ij}}$, while the colours $c_{ij} \in \mathscr{C}$ are

not necessarily distinct, and we write $V(K) = \{x_i : i \in [k]\}$ and define the multiset $\mathscr{C}(K) := \{c_{ij} : ij \in \binom{[k]}{2}\}$. Write $\#v(K)$ for the number of times $v \in V$ appears in K and $\#c(K)$ for the number of times $c \in \mathscr{C}$ appears in K. So $\#v(K) \in \{0,1\}$ while $\#c(K) \in \{0,\dots,\binom{k}{2}\}$. A *fractional t-rainbow K_k-tiling* in $\boldsymbol{G}$ is a function $\phi : K_k(\boldsymbol{G}) \to \mathbb{R}_{\geq 0}$ with $\phi(v) \leq t$ for all $v \in V$ and $\phi(c) \leq 1$ for all $c \in \mathscr{C}$, where

$$\phi(a) := \sum_{K \in K_k(\boldsymbol{G})} \#a(K)\phi(K) \quad \text{for all } a \in V \cup \mathscr{C}.$$

This second property implies that $\phi(K) \in [0,1]$ for all $K \in K_k(\boldsymbol{G})$. By definition,

$$\phi(\boldsymbol{G}) := \sum_{K \in K_k(\boldsymbol{G})} \phi(K) = \begin{cases} \sum_{v \in V} \phi(v)/k & \leq tn/k \\ \sum_{c \in \mathscr{C}} \phi(c)/\binom{k}{2} & \leq |\mathscr{C}|/\binom{k}{2} \end{cases}.$$

Note that for $t = 1$, the special case where each $\phi(K)$ equals 0 or 1 corresponds to a t-rainbow K_k-tiling. However, for larger t, it corresponds to a rainbow collection of K_ks where each vertex lies in at most t copies, which do not necessarily have common vertex sets.

The following lemma is our key tool and guarantees a sufficiently large fractional t-rainbow K_k-tiling as described above, in which every K_k receives a weight in $0, 1/p, 2/p, \dots, 1$ for some integer p which is not too large. The important consequence of this is that no weight is too small, which will allow us to use a fractional tiling in a reduced graph collection to give an integral tiling in the graph collection from whence it came, via Lemma 3.2.4. (If weights were as small as the regularity parameter ε, this would not be possible.) A similar requirement is present in the work of (Haxell & Rödl, 2001) on integer and fractional packings in graphs.

Lemma 4.4.1 (Fractional rainbow clique tiling with some small degrees)**.** *Let $0 < 1/n \ll \zeta \ll \mu \ll \gamma, 1/D, 1/k, 1/t \leq 1$ where $k \geq 2$ is an integer and let $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ be a graph collection on a vertex set V of size n with $|\mathscr{C}| = t\binom{k}{2}n/k$. Let $B = (V, \mathscr{C})$ be a bipartite graph with $d_B(x) \geq (1-\mu)|\mathscr{C}|$ for all $x \in V$ and $d_B(c) \geq (1-\mu)n$ for all $c \in \mathscr{C}$. Suppose that $d_{G_c}(x) \geq (\frac{k-1}{k} + \gamma)|V|$ whenever $xc \in B$. Then there exists a fractional t-rainbow K_k-tiling ϕ in $\boldsymbol{G}$ with $\phi(\boldsymbol{G}) \geq (1-\zeta)tn/k$, where there is a positive integer $p \leq \zeta^{-1/D}$ such that for all $K \in K_k(\boldsymbol{G})$, we have $p\phi(K) \in \mathbb{N}$.*

Proof. Let $1/n \ll \rho \ll \zeta$ where $\rho n/k \in \mathbb{N}$. Let $\ell := \lfloor 1/\rho \rfloor$ and $V = U_1 \cup \ldots \cup U_\ell$ be a

random partition subject to $|U_j| = \rho n$ for all $j < \ell$ (the last part may be slightly larger). Then, by a Chernoff bound, we may assume that for all $x \in V$ and $c \in \mathscr{C}$, we have $d_{G_c}(x, U_j) \geq \rho d_{G_c}(x) - n^{3/4}$ for all $j \in [\ell]$. Given $J \subseteq [\ell]$, write $U_J := \bigcup_{j\in J} U_j$. Given a function $f : A \to \mathbb{R}_{\geq 0}$ defined on a set A and $A' \subseteq A$, write $f(A') := \sum_{a\in A'} f(a)$ and $\overline{f} := 1 - f$ (that is, $\overline{f}(a) = 1 - f(a)$).

The proof will follow by iterating the next two claims. The first is a fractional version of the argument in (Montgomery et al., 2022) and allows us to fractionally tile any U_I as long as I is not too large, using colours with some 'capacity' already used up.

Claim 4.4.2. *Let $p \in \mathbb{N}$ with $\rho \ll \gamma, 1/p$, let $\mu^2 \leq \mu' \leq \sqrt{\mu}$ and let $\mathrm{Cap} : \mathscr{C} \to \{0, \frac{1}{p}, \frac{2}{p}, \dots, 1\}$. Let $I \subseteq [\ell]$ where $\mathrm{Cap}(\mathscr{C}) \geq \sqrt{\rho} n$ and $t\binom{k}{2}|I|\rho n/k \leq (1 - 2\mu'/\gamma)\mathrm{Cap}(\mathscr{C})$. Suppose that $\mathrm{Cap}(\mathscr{C} \setminus N_B(v)) \leq \mu'\mathrm{Cap}(\mathscr{C})$ for all $v \in U_I$. Then $\boldsymbol{G}$ contains a fractional t-rainbow K_k-tiling ϕ such that*

(i) $p\phi(K) \in \mathbb{N}$ *for all* $K \in K_k(\boldsymbol{G})$,

(ii) $\phi(v) = t\mathbb{1}(v \in U_I)$ *for all* $v \in V$,

(iii) $\phi(c) \leq \mathrm{Cap}(c)$ *for all* $c \in \mathscr{C}$.

Observe that (ii) implies $\phi(\boldsymbol{G}) = \sum_{v\in V} \phi(v)/k = t|U_I|/k$. The next claim is the novel part of the proof which says one can modify a given fractional tiling so that vertices are used in the same way, but for no vertex v are there too many bad colours for v (i.e. not in $N_B(v)$) with a large amount of remaining capacity.

Claim 4.4.3. *Suppose that there is a fractional t-rainbow K_k-tiling ϕ in $\boldsymbol{G}$ with $(1 - \frac{\gamma}{6k})tn/k < \phi(\boldsymbol{G}) < (1 - \rho)tn/k$ and $p\phi(K) \in \mathbb{N}$ for all $K \in K_k(\boldsymbol{G})$. Then $\boldsymbol{G}$ contains a fractional t-rainbow K_k-tiling ψ such that*

(i) $pk(k-1)\psi(K) \in \mathbb{N}$ *for all* $K \in K_k(\boldsymbol{G})$,

(ii) $\psi(v) = \phi(v)$ *for all* $v \in V$,

(iii) $\overline{\psi}(\mathscr{C} \setminus N_B(v)) \leq (14\mu/\gamma)\overline{\psi}(\mathscr{C})$ *for all* $v \in V$.

Now we will iterate the two claims to prove the lemma. The idea is that the modified tiling from Claim 4.4.3 can be extended to new vertices using Claim 4.4.2 since no vertex is bad with respect to it.

To begin, let $\mathrm{Cap}(c) := 1$ for all $c \in \mathscr{C}$ and $I_1 = [(1 - 2\mu/\gamma)/\rho]$, where we assume that this number is an integer, which does not affect the proof. Then the hypotheses of Claim 4.4.2 hold

with $p_1 := 1$ and $\mu' := \mu$, so $\boldsymbol{G}$ contains a fractional t-rainbow K_k-tiling ϕ_1 with $\phi_1(K) \in \mathbb{N}$ for all $K \in K_k(\boldsymbol{G})$ and $\phi_i(v) = t\mathbb{1}(v \in U_{I_1})$ for all $v \in V$ and $\phi_1(\boldsymbol{G}) = t|U_{I_1}|/k = t\rho n|I_1|/k = (1-2\mu/\gamma)tn/k$. Thus we can apply Claim 4.4.3 to ϕ_1 to obtain ψ_1 which satisfies (i)–(iii).

To iterate, for each $i \in \mathbb{N}$, let

$$p_i := (k(k-1))^i, \quad \mu_i := \left(\frac{28\mu^2}{\gamma^3}\right)^i, \quad I_i := \left[\frac{1-\mu_i}{\rho}\right],$$

where again we assume $(1-\mu_i)/\rho$ is an integer, which does not affect the proof. Suppose we have a fractional t-rainbow K_k-tiling ψ_i in $\boldsymbol{G}$ with

- $p_i\psi_i(K) \in \mathbb{N}$ for all $K \in K_k(\boldsymbol{G})$,
- $\psi_i(v) = t\mathbb{1}(v \in U_{I_i})$ for all $v \in V$, and
- $\overline{\psi}_i(\mathscr{C} \setminus N_B(v)) \le (14\mu^2/\gamma^2)\overline{\psi}_i(\mathscr{C})$ for all $v \in V$,

and suppose that $\mu_{i+1} \ge \sqrt{\rho}$. Then $\overline{\psi}_i(\mathscr{C}) = |\mathscr{C}| - \binom{k}{2}\psi_i(\boldsymbol{G}) = |\mathscr{C}| - t\binom{k}{2}|I_i|\rho n/k = \mu_i|\mathscr{C}|$. Let $I := I_{i+1} \setminus I_i$. Then

$$t\binom{k}{2}|I|\rho n/k = t\binom{k}{2}(\mu_i - \mu_{i+1})n/k = (1-28\mu^2/\gamma^3)\mu_i t\binom{k}{2}n/k = (1-28\mu^2/\gamma^3)\overline{\psi}_i(\mathscr{C}).$$

Since $\mu_{i+1} \ge \sqrt{\rho}$ implies that $\mu_i t\binom{k}{2}/k \ge \sqrt{\rho}$, the conditions of Claim 4.4.2 hold with $\mathrm{Cap} := \overline{\psi}_i$ and $\mu' := 14\mu^2/\gamma^2$. Thus there is a fractional t-rainbow K_k-tiling ψ such that $p_i\psi(K) \in \mathbb{N}$ for all $K \in K_k(\boldsymbol{G})$, $\psi(v) = t\mathbb{1}(v \in U_I)$ for all $v \in V$, and $\psi(c) \le \overline{\psi}_i(c)$ for all $c \in \mathscr{C}$. Let $\phi_{i+1} := \psi_i + \psi$. Then ϕ_{i+1} is a fractional t-rainbow K_k-tiling in $\boldsymbol{G}$ since $\phi_{i+1}(v) = \psi_i(v)\mathbb{1}(v \in U_{I_i}) + \psi(v)\mathbb{1}(v \in U_I) = t\mathbb{1}(v \in U_{I_{i+1}})$, and $\phi_{i+1}(c) = \psi_i(c) + \psi(c) \le \psi_i(c) + \overline{\psi}_i(c) = 1$ for all $c \in \mathscr{C}$.

We have $\phi_{i+1}(\boldsymbol{G}) = t|U_{I_{i+1}}|/k$, which is at least $t|U_1|/k \ge t(1-2\mu/\gamma)n/k > (1-\gamma/(6k))tn/k$, and at most $t(1-\mu_{i+1})n/k < (1-\rho)tn/k$ by our assumption on μ_{i+1}. Thus we can apply Claim 4.4.3 to find ψ_{i+1} with

- $p_i k(k-1)\psi_{i+1}(K) = p_{i+1}\psi_{i+1}(K) \in \mathbb{N}$ for all $K \in K_k(\boldsymbol{G})$,
- $\psi_{i+1}(v) = \phi_{i+1}(v) = t\mathbb{1}(v \in U_{I_{i+1}})$ for all $v \in V$, and
- $\overline{\psi}_{i+1}(\mathscr{C} \setminus N_B(v)) \le (14\mu^2/\gamma^2)\overline{\psi}_{i+1}(\mathscr{C})$ for all $v \in V$.

We have shown that, as long as $\mu_{i+1} \ge \sqrt{\rho}$, we can find ψ_{i+1}. Let m be the smallest integer with $\mu_m = (28\mu^2/\gamma^3)^m \le \zeta$. So $\zeta \ge \mu^{3m}$ and $\mu_m > \sqrt{\rho}$ since $\rho \ll \zeta$. Thus we can obtain $\phi := \psi_m$.

Now, $|U_{I_m}| = (1-\mu_m)n > (1-\zeta)n$, so $\phi(v) = t$ for at least $(1-\zeta)n$ vertices v, so $\phi(\boldsymbol{G}) \geq (1-\zeta)tn/k$. Also, $p_m\phi(K) \in \mathbb{N}$ for all K. Since $\zeta \geq \mu^{3m}$, we have $p_m = (k(k-1))^m \leq \zeta^{\log(k(k-1))/(3\log\mu)} < \zeta^{-1/D}$ since $1/D > \log(k(k-1))/(3\log(1/\mu))$ which follows from $\mu \ll 1/D, 1/k$. This completes the proof of the lemma given the claims.

Proof of Claim 4.4.2. Let $\rho \ll \lambda \ll 1/p$. Write $I = \{i_1, \dots, i_{|I|}\}$. Let $j \leq |I|$ be the largest integer for which $\boldsymbol{G}[U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_j}]$ contains a fractional t-rainbow K_k-tiling ϕ_j with $\phi_j(v) = t\mathbb{1}(U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_j})$ for all $v \in V$ and $p\phi_j(K) \in \mathbb{N}$ for all $K \in K_k(\boldsymbol{G})$ and $\phi_j(c) \leq \mathrm{Cap}(c)$ for all $c \in \mathscr{C}$. We may assume $j < |I|$ or we are done. Let $\mathrm{Cap}_j(c) := \mathrm{Cap}(c) - \phi_j(c) \geq 0$ for all $c \in \mathscr{C}$. We have

$$\mathrm{Cap}_j(\mathscr{C}) \geq \mathrm{Cap}(\mathscr{C}) - \binom{k}{2}\phi_j(\boldsymbol{G}) \geq \mathrm{Cap}(\mathscr{C}) - t\binom{k}{2}|I|\rho n/k \geq (2\mu'/\gamma)\mathrm{Cap}(\mathscr{C})$$

and hence, for all $v \in V$,

$$\mathrm{Cap}_j(\mathscr{C} \setminus N_B(v)) \leq \mathrm{Cap}(\mathscr{C} \setminus N_B(v)) \leq \mu'\mathrm{Cap}(\mathscr{C}) \leq (\gamma/2)\mathrm{Cap}_j(\mathscr{C}). \tag{4.2}$$

Let T be the graph with vertex set $U_{i_{j+1}}$ where a pair xy in $U_{i_{j+1}}$ is an edge if $\sum_{c\in\mathscr{C}:xy\in G_c} \mathrm{Cap}_j(c) > \lambda\mathrm{Cap}_j(\mathscr{C})$. Then, for all $v \in U_{i_{j+1}}$, we have

$$\mathrm{Cap}_j(\mathscr{C})\left(1-\frac{\gamma}{2}\right)\left(\frac{k-1}{k}+\gamma-n^{-1/5}\right)\rho n \overset{(4.2)}{\leq} \sum_{c\in N_B(v)} \mathrm{Cap}_j(c)\left(\left(\frac{k-1}{k}+\gamma\right)\rho n - n^{3/4}\right)$$

$$\leq \sum_{c\in\mathscr{C}} \mathrm{Cap}_j(c)d_{G_c}(v, U_{i_{j+1}}) = \sum_{u\in U_{i_{j+1}}} \sum_{c\in\mathscr{C}:uv\in G_c} \mathrm{Cap}_j(c) \leq \rho n \cdot \lambda\mathrm{Cap}_j(\mathscr{C}) + d_T(v)\mathrm{Cap}_j(\mathscr{C}).$$

So $d_T(v) \geq (\frac{k-1}{k} + \gamma/3)\rho n$, where we used $\lambda, \rho \ll \gamma$. Thus the Hajnal-Szemerédi theorem implies that T has a K_k-factor (a perfect K_k-tiling). Since, using $\rho \ll \mu, \lambda$, we have $\lambda\mathrm{Cap}_j(\mathscr{C}) \geq \lambda \cdot (2\mu'/\gamma)\sqrt{\rho}n \geq t\binom{k}{2}\rho n/k$, the total capacity on each edge of T is at least the capacity required to cover the whole factor, so we can assign colours using the following greedy algorithm. For each copy of K_k in the factor in turn, with vertices $x_1, \dots, x_k$, say, for each edge x_ix_j choose colours $c^1_{ij}, \dots, c^{pt}_{ij}$ with $x_ix_j \in G_{c^h_{ij}}$ and with non-zero capacity $\mathrm{Cap}_j(c^h_{ij})$ and set $\phi(x_ix_jc^h_{ij} : ij \in \binom{[k]}{2}) = \frac{1}{p}$ for each $h \in [pt]$. Subtract $\frac{1}{p}$ from each $\mathrm{Cap}_j(c^h_{ij})$ and continue to the next clique in T. The required properties follow by construction, completing the proof. □

Proof of Claim 4.4.3. Given $\mathscr{A} \in \binom{\mathscr{C}}{k}$ and $v \in V$, we say that $K \in K_k(\boldsymbol{G})$ is

- *$\mathscr{A}$-good* if $\phi(K) \neq 0$ and $V(K) \subseteq \bigcap_{c\in\mathscr{A}} N_B(c)$;
- *$(v,\mathscr{A})$-good* if $\phi(K) \neq 0$ and there is a labelling $c_1,\dots,c_k$ of $\mathscr{A}$ and $x_1,\dots,x_k$ of $V(K)$ such that $vx_i \in G_{c_i}$ for all $i \in [k]$.

These are both properties of $V(K)$ (that is, they do not depend on $\mathscr{C}(K)$). Recall that each v has total ϕ-weight $\phi(v)$ at most t. Thus

$$\sum_{\substack{K\in K_k(\boldsymbol{G}):\\ K \text{ is } \mathscr{A}\text{-good}}} \phi(K) \geq \phi(\boldsymbol{G}) - \sum_{v\notin\bigcap_{c\in\mathscr{A}} N_B(c)} \phi(v) \geq \phi(\boldsymbol{G}) - tk\mu n \geq \left(1-\frac{\gamma}{6k}\right) tn/k - tk\mu n$$
$$\geq tn/(2k). \tag{4.3}$$

We claim that, whenever v is in an $\mathscr{A}$-good clique K, we have

$$\sum_{\substack{K\in K_k(\boldsymbol{G}):\\ K \text{ is } (v,\mathscr{A})\text{-good}}} \phi(K) \geq \gamma tn/3. \tag{4.4}$$

To see this, let $\boldsymbol{J}^v = (J_c^v : c \in \mathscr{C})$ be the auxiliary graph collection where for each $x \in V$, we replace x by $p\phi(x)$ clones and add vx' to J_c^v whenever x' is a clone of x with $xv \in G_c$. For each $K \in K_k(\boldsymbol{G})$, obtain $p\phi(K)$ disjoint clones K' of K by choosing $p\phi(K)$ distinct copies of each $y \in V(K)$ which have not been chosen before and assigning the i-th vertex to the i-th copy of K. This process is possible by the definition of $\phi(x)$: for each $x \in V$, we have $\sum_{K\in K_k(\boldsymbol{G}):x\in V(K)} p\phi(K) = p\phi(x)$ so we can always choose an unused clone x' of x. The total number of vertices in these cliques is $pk\phi(\boldsymbol{G}) < ptn$. For all $c \in N_B(v)$, we have

$$d_{J_c^v}(v) = \sum_{x\in N_{G_c}(v)} p\phi(x) \geq ptd_{G_c}(v) - p\sum_{y\in V}(t-\phi(y)) \geq pt\left(\frac{k-1}{k}+\gamma\right)n - pt\frac{\gamma n}{6k}$$
$$\geq \left(1+\frac{\gamma}{2}\right)\frac{k-1}{k}ptn.$$

Whenever v is in an $\mathscr{A}$-good clique K and $c \in \mathscr{A}$, this degree bound applies. Therefore, Lemma 4.3.7(i) implies that, there are $(\gamma/2)pk\phi(\boldsymbol{G}) \geq \gamma ptn/3$ copies of cloned cliques K' in $\boldsymbol{J}^v$ for which there is a labelling $c_1,\dots,c_k$ of $\mathscr{A}$ and $x_1',\dots,x_k'$ of $V(K')$ such that $vx_i' \in G_{c_i}$ for all $i \in [k]$. Each clique in $\boldsymbol{G}$ was cloned at most $p\phi(K) \leq p$ times. Thus (4.4) holds.

Since $\overline{\phi}(\mathscr{C}) = |\mathscr{C}| - \binom{k}{2}\phi(\boldsymbol{G})$, we have

$$\rho tn \leq \overline{\phi}(\mathscr{C}) \leq \gamma tn/6. \tag{4.5}$$

Now let $\mathscr{A}_1, \ldots, \mathscr{A}_r$ be elements in $\binom{\mathscr{C}}{k}$ such that every colour $c \in \mathscr{C}$ appears in exactly $pk(1-\phi(c))$ elements of the $\mathscr{A}_j$. So $kr = \sum_{c\in\mathscr{C}} pk(1-\phi(c))$ and hence

$$r/p = \overline{\phi}(\mathscr{C}). \tag{4.6}$$

Initially, let $\phi' \equiv \phi$. For each $i = 1, \ldots, r$, do the following to construct an edge-coloured graph L_i. (We only use ϕ' in the construction of the L_i.)

1. Among all $\mathscr{A}_i$-good K, choose one, K_i^*, randomly with probability proportional to $\phi'(K_i^*)$. Add (the coloured edges of) K_i^* to L_i and subtract $1/p$ from $\phi'(K_i^*)$.
2. For each $v \in V(K_i^*)$ in turn, do the following.

   Among all $(v, \mathscr{A}_i)$-good J, choose one, $J_{i,v}^*$, randomly with probability proportional to $\phi'(J_{i,v}^*)$. Let the corresponding labelling be $V(J_{i,v}^*) = \{x_{i,v}^1, \ldots, x_{i,v}^k\}$. Add $J_{i,v}^*$ to L_i along with the edges $vx_{i,v}^j \in G_{c_j}$, where c_j is the j-th colour of $\mathscr{A}_i$, labelled according to $(v, \mathscr{A}_i)$-goodness. Subtract $1/p$ from $\phi'(J_{i,v}^*)$.

This completes the construction of L_i.

Since every $\phi(K)$ is an integer multiple of $1/p$ and K is chosen with non-zero probability only if $\phi(K) > 0$, we have that $\phi'(K)$ is always non-negative. By (4.3) and (4.4), the sum of the $\phi'(K)$ from among the good K we are choosing is always at least $\gamma tn/3 - r/p \geq \gamma tn/6$, so this construction is possible.

We can now use the L_i to define ψ.

- For each $i \in [r]$,
  - let $\psi(K_i^*) := \phi(K_i^*) - \frac{1}{p(k-1)}$;
  - for each $v \in V(K_i^*)$ and $j \in [k]$, let $K_{i,v}^j := L_i[\{v\} \cup V(J_{i,v}^*) \setminus \{x_{i,v}^j\}]$ and $\psi(K_{i,v}^j) := \frac{1}{pk(k-1)}$;
  - for each $v \in V(K_i^*)$, let $\psi(J_{i,v}^*) := \phi(J_{i,v}^*) - \frac{1}{pk}$.
- For all other $K \in K_k(\boldsymbol{G})$, let $\psi(K) := \phi(K)$.

We will show that ψ satisfies (i)–(iii). First, note that (i) holds since every $\phi(K)$ is a multiple of

$1/p$.

For (ii), fix $x \in V$. Whenever $i \in [r]$ is such that $x \in V(K_i^*)$, we also have $x \in V(K_{i,x}^j)$ for all $j \in [k]$, so the contribution to $\psi(x)$ for each such i is

$$\psi(K_i^*) + \sum_{j\in[k]} \psi(K_{i,x}^j) = \phi(K_i^*) - \frac{1}{p(k-1)} + \sum_{j\in[k]} \frac{1}{pk(k-1)} = \phi(K_i^*).$$

Whenever $i \in [r]$ is such that $x \in V(J_{i,v}^*)$ for some $v \in V(K_i^*)$, we have $x = x_{i,v}^j$ for some $j \in [k]$, and so $x \in V(K_{i,v}^{j'})$ for all $j' \in [k] \setminus \{j\}$. Thus the contribution to $\psi(x)$ for each such i is

$$\sum_{j'\in[k]\setminus\{j\}} \psi(K_{i,v}^{j'}) + \psi(J_{i,v}^*) = (k-1)\frac{1}{pk(k-1)} + \phi(J_{i,v}^*) - \frac{1}{pk} = \phi(J_{i,v}^*).$$

For all other $i \in [r]$, the contribution to ψ for this i is the same as for ϕ. Thus $\psi(x) = \phi(x)$, proving (ii). This implies in particular that

$$\overline{\psi}(\mathscr{C}) = \overline{\phi}(\mathscr{C}). \tag{4.7}$$

It remains to show that colour weights $\psi(c)$ are all at most one (which will complete the proof that ψ is a fractional t-rainbow K_k-tiling), and to prove (iii). A very similar argument to the one for vertices shows that for each $c \in \mathscr{C}$, we have

$$\phi(c) - \psi(c) = \frac{1}{p(k-1)} \sum_{i\in[r]} \#c(K_i^*) + \frac{1}{pk} \sum_{i,v} \#c(J_{i,v}^*) - \frac{1}{pk(k-1)} \cdot (k-1)|\{i \in [r] : c \in \mathscr{A}_i\}|,$$

where for the last term, each colour c in $\mathscr{A}_i$ appears on all of the $K_{i,v}^j$ except the one where $c = c_j$. Thus, since each colour appears in $pk(1-\phi(c))$ of the $\mathscr{A}_i$, the last term equals $1-\phi(c)$. Thus

$$\overline{\psi}(c) = 1 - \psi(c) = \frac{1}{p(k-1)} \sum_{i\in[r]} \#c(K_i^*) + \frac{1}{pk} \sum_{i,v} \#c(J_{i,v}^*) \geq 0, \tag{4.8}$$

so, in particular, this finishes the proof that ψ is a fractional t-rainbow K_k-tiling.

For (iii), we need to appeal to the randomness of the construction, which will imply that, with high probability, in any large set $\mathscr{B}$ of colours, $\mathscr{B}$ does not contribute a lot more than the average to $\overline{\psi}(\mathscr{C})$. More precisely, we will show that, with high probability, $\overline{\psi}(\mathscr{B}) = O_{t,k}(1) \cdot |\mathscr{B}| \cdot \gamma^{-1}\overline{\psi}(\mathscr{C})/|\mathscr{C}|$. Here, the factor γ^{-1} arises since in Step 2 of the algorithm, we may be

choosing cliques from only a γ-proportion of all cliques. This will imply that, for each vertex v, most of the (weighted) remaining colours are good for v.

To analyse the random process, note that it is equivalent to making $p\phi(K)$ copies of each K and choosing one of these uniformly at random from among all available choices, without replacement. For uniform treatment, suppose $\mathcal{A}$ is the set of cliques from which we are choosing at a given step of the algorithm (choosing some K_i^*, or some $J_{i,v}^*$), so we choose $K \in \mathcal{A}$ randomly proportional to $\phi'(K)$, in other words, with probability $\phi'(K)/\sum_{K\in\mathcal{A}}\phi'(K)$. Let $c \in \mathscr{C}$. Then, writing $\mathbb{E}'$ and $\mathbb{P}'$ for the conditional expectation and probability, conditioned on any history before some given step, we have

$$\mathbb{E}'(\#c(K)) = \sum_{K\in\mathcal{A}} \#c(K)\frac{\phi'(K)}{\sum_{K'\in\mathcal{A}}\phi'(K')} \le \sum_{K\in\mathcal{A}} \#c(K)\frac{\phi(K)}{\sum_{K'\in\mathcal{A}}\phi(K') - r/p} \overset{(4.6)}{\le} \frac{\phi(c)}{\phi(\mathcal{A}) - \overline{\phi}(\mathscr{C})}.$$

In particular, for any colour set $\mathscr{B} \subseteq \mathscr{C}$, we have the following by linearity of expectation. Setting $\mathcal{A}$ to be the set of $\mathscr{A}_i$-good cliques, we have

$$\mathbb{E}'\left(\sum_{c\in\mathscr{B}} \#c(K_i^*)\right) \overset{(4.3),(4.5)}{\le} \frac{|\mathscr{B}|}{tn/(2k) - \gamma tn/6} \le \frac{3k|\mathscr{B}|}{tn},$$

and, given $x \in V(K_i^*)$, and setting $\mathcal{A}$ to be the set of $(x, \mathscr{A}_i)$-good cliques, we have

$$\mathbb{E}'\left(\sum_{c\in\mathscr{B}} \#c(J_{i,x}^*)\right) \overset{(4.4),(4.5)}{\le} \frac{|\mathscr{B}|}{\gamma tn/3 - \gamma tn/6} = \frac{6|\mathscr{B}|}{\gamma tn}.$$

Let $\nu := 7r/(\gamma tkpn)$ and $C := 1/(p(k-1))$ and suppose that $|\mathscr{B}_v| \ge \rho^3 n$. We have, noting that $\overline{\psi}(\mathscr{C}) = \overline{\phi}(\mathscr{C})$ due to (ii),

$$r\left(\frac{3k}{tn}\cdot\frac{1}{p(k-1)} + \frac{6}{\gamma tn}\cdot\frac{1}{pk}\right) \le \nu, \quad \text{and} \quad \frac{\nu|\mathscr{B}|}{6C} \ge \frac{r|\mathscr{B}|}{\gamma tn} \ge \frac{\rho|\mathscr{B}|}{\gamma k} \ge \frac{\rho^4 n}{\gamma k} > \sqrt{n}. \tag{4.9}$$

Given $v \in V$, write $\mathscr{B}_v := \mathscr{C} \setminus N_B(v)$. Suppose first that $|\mathscr{B}_v| < 14\mu\rho^2 tn/\gamma$. Then, by (4.5), we have $\overline{\psi}(\mathscr{B}_v) \le |\mathscr{B}_v| < 14\mu/\gamma \cdot \overline{\psi}(\mathscr{C})$, as required. Thus we may assume that $|\mathscr{B}_v| \ge 14\mu\rho^2 tn/\gamma > \rho^3 n$.

By (4.8) and (4.9), we have $\sum_{c\in\mathscr{B}_v}\mathbb{E}'(\overline{\psi}(c)) \le \nu|\mathscr{B}_v|$. We can now apply Lemma 4.1.3 with $\nu|\mathscr{B}_v|, C$ playing the roles of μ, C respectively to complete the proof. Indeed, we have $\mathbb{P}(\overline{\psi}(\mathscr{B}_v) > 2\nu|\mathscr{B}_v|) \le 2e^{-\nu|\mathscr{B}_v|/(6C)} \le e^{-\sqrt{n}}$. By a union bound, with high probability, for

every $v \in V$ with $|\mathscr{B}_v| \geq \rho^3 n$, we have

$$\overline{\psi}(\mathscr{B}_v) \leq 2\nu|\mathscr{B}_v| \leq \mu|\mathscr{C}| \cdot \frac{14r}{\gamma tkpn} = \frac{14r\mu\binom{k}{2}tn}{\gamma tk^2pn} \overset{(4.6)}{=} \frac{14\mu\binom{k}{2}\overline{\psi}(\mathscr{C})}{\gamma k^2} \leq \frac{14\mu\overline{\psi}(\mathscr{C})}{\gamma},$$

so (iii) holds (and thus there exists a ψ satisfying all properties). □

Having completed the proof of both claims, the lemma follows. □

4.5 Proof of Theorem 4.1.2

Now we can put the tools from Sections 4.3 and 4.4 together with our blow-up lemma to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Let $k \geq 2$ be an integer and let $\gamma > 0$. Choose additional constants $L_0, \varepsilon, \zeta, \eta, d$ such that $0 < 1/L_0 \ll \varepsilon \ll \zeta \ll \eta \ll d \ll \gamma, 1/k$. Apply Lemma 4.2.1 (the regularity lemma for graph collections) with parameters L_0, ε to obtain n_0. By increasing n_0 if necessary we may assume that $1/n_0 \ll 1/L_0$. Now let $n \geq n_0$ be an integer and let $\boldsymbol{G}$ be a graph collection on n vertices satisfying the conditions of the theorem. Write $\mathscr{C}$ for the colours of $\boldsymbol{G}$, so $|\mathscr{C}| = (k-1)n$.

Step 1. *Apply the regularity lemma and find a 2-rainbow K_k-factor template.*

Lemma 4.2.1 applied with parameters L_0, ε, d implies that there is a partition of V into $\widetilde{V}_0, \dots, \widetilde{V}_{\widetilde{L}}$ and of $\mathscr{C}$ into $\widetilde{\mathscr{C}}_0, \dots, \widetilde{\mathscr{C}}_{\widetilde{M}}$ and a spanning subgraph G'_c of G_c for each $c \in \mathscr{C}$ such that (i)–(v) hold, where $\widetilde{m}$ is the common size of clusters. Let $\widetilde{\boldsymbol{R}} = \boldsymbol{R}(\varepsilon, d, L_0)$ be the reduced graph collection of $\boldsymbol{G}$. By Lemma 3.2.7, for each $i \in [\widetilde{L}]$, there are at least $(1 - d^{1/4})\widetilde{M}$ colour cluster indices $\chi \in [\widetilde{M}]$ for which $d_{\widetilde{R}_\chi}(i) \geq \left(\frac{k-1}{k} + \gamma/2\right)\widetilde{L}$. We also have $\widetilde{L}\widetilde{m} + |\widetilde{V}_0| = |V| = n$ and $\widetilde{M}\widetilde{m} + |\widetilde{\mathscr{C}}_0| = |\mathscr{C}| = (k-1)n$ and $|\widetilde{V}_0| + |\widetilde{\mathscr{C}}_0| \leq \varepsilon n$, so

$$\widetilde{M} = (k-1)\widetilde{L} + \frac{(k-1)|\widetilde{V}_0| - |\widetilde{\mathscr{C}}_0|}{n - |\widetilde{V}_0|}\widetilde{L} \geq (1-2\varepsilon)(k-1)\widetilde{L}.$$

Apply Lemma 4.4.1 to $\widetilde{\boldsymbol{R}}$ with $\zeta, d^{1/4}, \gamma/2, 2, \widetilde{L}, 1$ playing the roles of $\zeta, \mu, \gamma, t, n, D$ to obtain a fractional 2-rainbow K_k-tiling ϕ of $\widetilde{\boldsymbol{R}}$ such that $\phi(\widetilde{\boldsymbol{R}}) \geq (1-\zeta)2\widetilde{L}/k$ and there is $p \in \mathbb{N}$ with $p \leq 1/\zeta$ such that, for all $K \in K_k(\widetilde{\boldsymbol{R}})$, we have $p\phi(K) \in \mathbb{N}$.

Partition each $\widetilde{V}_i$ into $2p$ clusters $\widetilde{V}_i^j$ and each $\widetilde{\mathscr{C}}_i$ into $2p$ clusters $\widetilde{\mathscr{C}}_i^j$ of equal size $m := \widetilde{m}/(2p)$, where $j \in [2p]$. Let $\boldsymbol{R} = (\widetilde{R}_i^j : i \in [\widetilde{L}], j \in [2p])$ be the graph collection obtained from $\widetilde{\boldsymbol{R}}$ as follows. Each $\widetilde{R}_i^j$ has vertex set $[\widetilde{L}] \times [2p]$. Whenever $\ell\ell' \in E(\widetilde{R}_i)$, add $(\ell, h)(\ell', h')$ to $\widetilde{R}_i^j$ for all $h, h' \in [2p]$ and $j \in [2p]$. Lemma 3.2.4(i) implies that for every $(\ell, h)(\ell', h') \in \widetilde{R}_i^j$, we have $(G'_c[\widetilde{V}_\ell^h, \widetilde{V}_{\ell'}^{h'}] : c \in \widetilde{\mathscr{C}}_i^j)$ is $(\varepsilon/(2p), d/2)$- and hence $(\sqrt{\varepsilon}, d)$-regular, using the important bounds $\varepsilon \ll \zeta \leq 1/p$. Next, for each $\widetilde{K} \in K_k(\widetilde{\boldsymbol{R}})$ and $s \in [p\phi(\widetilde{K})]$ in turn, for each $i \in V(\widetilde{K})$, choose (i, ℓ_s) with $\ell_s \in [2p]$, that has not been chosen before, and for each $ii' \in \binom{V(\widetilde{K})}{2}$, writing $\chi_{ii'} \in [\widetilde{M}]$ for the colour index such that $\chi_{ii'}$ is the colour on ii' in $\widetilde{K}$, choose two distinct $(\chi_{ii'}, \ell_{s,1}), (\chi_{ii'}, \ell_{s,2})$ with $\ell_{s,1}, \ell_{s,2} \in [2p]$ that have not been used before, and take the 2-rainbow k-clique $\widetilde{K}_s$ consisting of pairs of vertices $(i, \ell_s)(i', \ell_s)$ with $ii' \in \binom{V(\widetilde{K})}{2}$ coloured with the two colours $(\chi_{ii'}, \ell_{s,1}), (\chi_{ii'}, \ell_{s,2})$. This process is possible since ϕ is a fractional 2-rainbow K_k-tiling: for each $i \in [\widetilde{L}] = V(\widetilde{\boldsymbol{R}})$, we have $\sum_{\widetilde{K} \in K_k(\widetilde{\boldsymbol{R}}): i \in V(\widetilde{K})} p\phi(\widetilde{K}) = p\phi(i) \leq p$ so we can always choose an unused (i, ℓ_s), and for each $\chi \in [\widetilde{M}] = \mathscr{C}(\widetilde{\boldsymbol{R}})$, we have $2\sum_{\widetilde{K} \in K_k(\widetilde{\boldsymbol{R}})} \#\chi(\widetilde{K}) p\phi(\widetilde{K}) = 2p\phi(\chi) \leq 2p$ so we can always choose two distinct unused $(\chi, \ell_{s,1}), (\chi, \ell_{s,2})$. Now $J := \bigcup\{\widetilde{K}_s : \widetilde{K} \in K_k(\widetilde{\boldsymbol{R}}), s \in [p\phi(\widetilde{K})]\}$ is a 2-rainbow K_k-tiling in $\boldsymbol{R}$.

We relabel all objects, as follows. Write $V_1, \dots, V_L$ for the vertex clusters and $\mathscr{C}_1, \dots, \mathscr{C}_M$ for the colours clusters used in J, and let N be the number of copies of K_k in J. Thus from now on we write $N \times K_k$ for J. Remove any unused vertex cluster or colour cluster. Corresponding to the relabelling, the vertex set of $\boldsymbol{R}$ is $[L]$ and its colour indices are $[M]$. So

$$N = p\phi(\widetilde{\boldsymbol{R}}) \geq p(1-\zeta)2\widetilde{L}/k$$

and $L = kN$ and $M = (k-1)L$. The number of vertices covered is

$$Nkm \geq k\widetilde{m}/(2p) \cdot p(1-\zeta)2\widetilde{L}/k = (1-\zeta)\widetilde{m}\widetilde{L} \geq (1-\zeta)n - |\widetilde{V}_0| \geq (1-2\zeta)n \qquad (4.10)$$

and hence $M = N\binom{k}{2} \cdot 2m \geq (1-2\zeta)(k-1)n$. Let V_0 be the set of unused vertices and $\mathscr{C}_0$ the set of unused colours. We have $|V_0| + |\mathscr{C}_0| \leq 2\zeta n(1+k-1) = 2k\zeta n$. We can also assume, by adding further to the exceptional sets, that, without changing any of the above estimates,

$|V_0|, |\mathscr{C}_0| \geq \varepsilon n$, say. For future reference,

$$N = L/k \in \mathbb{N}, \quad M = (k-1)L, \quad \varepsilon n \leq |V_0|, |\mathscr{C}_0| \leq |V_0| + |\mathscr{C}_0| \leq 2k\zeta n \quad \text{and} \quad |\mathscr{C}_0| = (k-1)|V_0|.$$

Write $a_1^j, \dots, a_k^j$ for the vertex set of the j-th copy A_j of K_k (some $\widetilde{K}_{j'}$) in $N \times K_k$ and let U_ℓ^j be its ℓ-th vertex (some $\widetilde{V}_i^{j'}$). For each $e \in E(J)$, let $\mathscr{C}_e$ be the union of the two colour clusters $\widetilde{\mathscr{C}}_{\chi_e}^j$, $\widetilde{\mathscr{C}}_{\chi_e}^{j'}$ specified in $N \times K_k$ and let $\mathscr{C}_{jj'}^i := \mathscr{C}_{a_j^i a_{j'}^i}$. So $\{a_h^j : j \in [N], h \in [k]\} = [L]$ and $\{\mathscr{C}_1, \dots, \mathscr{C}_M\} = \bigcup_{j \in [N], \ell\ell' \in \binom{[k]}{2}} \mathscr{C}_{\ell\ell'}^j$. Note that $\mathcal{F} = (\mathcal{V}, \mathscr{C}', \boldsymbol{G}')$ is an $(N \times K_k)$-template with parameters $(m, \sqrt{\varepsilon}, d/2, 1)$ where $\mathcal{V} = (V_i : i \in [L])$, $\mathscr{C}' = \mathscr{C} \setminus \mathscr{C}_0 = \bigcup_{e \in E(N \times K_k)} \mathscr{C}_e$, and $\boldsymbol{G}' = (G'_c : c \in \mathscr{C}')$, and each $|V_i| = m$ while each $|\mathscr{C}_{jj'}^i| = 2m$.

Using the property of $\widetilde{\boldsymbol{R}}$ from Lemma 3.2.7, for all vertex clusters V_i, $i \in [L]$, there are at least $(1 - d^{1/4})\widetilde{M} \cdot 2p - 2\zeta n/m \geq (1 - 2d^{1/4})M$ colour cluster indices $\chi \in [M]$ for which

$$d_{R_\chi}(i) \geq \left(\frac{k-1}{k} + \gamma/2\right) \widetilde{L} \cdot 2p \geq \left(\frac{k-1}{k} + \gamma/3\right) L = (k - 1 + \gamma/3)N.$$

We say that any such χ is *good for* i.

We are in Setting 4.3.3 with $\gamma/2, 2d^{1/4}, \sqrt{\varepsilon}, d/2, 1/2$ playing the roles of $\gamma, d', \varepsilon, d, \delta$.

Step 2. *Construct the absorber and connect $N-1$ of the k-cliques in the template.*

Let $\mathscr{S}$ consist of $s := 40k^4N^2$ distinct colours from each $\mathscr{C}_{\ell\ell'}^j$ and let $\{\mathscr{S}_p : p \in P\}$ be the set of all $(2k-2)$-subsets of $\mathscr{S}$ where P is some index set with $|P| = \binom{sN\binom{k}{2}}{2k-2} \leq N^{7k}$. For each $p \in P$, let $\mathscr{S}_p = \mathscr{S}_{p,1} \cup \mathscr{S}_{p,2}$ be an arbitrary equipartition. Let $v_1^*, v_2^* \in V_0$ be arbitrary (recall that it is non-empty).

Apply Lemma 4.3.5 successively as follows. Apply (i) with $(j, j') = (N-1, N)$ and

$$\mathcal{Q} = \{(\mathscr{S}_{p,i}, v_i^*) : i \in [2] \text{ and } p \in P\}$$

with W consisting of all vertices in a pair of $\mathcal{Q}$ and $\mathscr{W}$ consisting of all colours in a pair of $\mathcal{Q}$. Then, for each $j = 1, 2, \dots, N-2$, apply (ii) with $(j, j') = (j, j+1)$ and W consisting of all vertices in a pair of $\mathcal{Q}$ and all vertices used in previous steps, and $\mathscr{W}$ consisting of all colours in a pair of $\mathcal{Q}$ and all colours used in previous steps. We obtain

- pairwise disjoint vertex sets $X_1, \dots, X_{N-1} \subseteq V \setminus \{v_1^*, v_2^*\}$ of size at most $20k^3|P|$,
- pairwise disjoint colour sets $\mathscr{B}_1, \dots, \mathscr{B}_{N-1} \subseteq \mathscr{C} \setminus \mathscr{S}$,

- vertex sets $T_h^j \subseteq U_h^j$ and $S_h^{j+1} \subseteq U_h^{j+1}$ with $|T_h^j|, |S_h^{j+1}| \geq \eta m$ for all $h \in [k]$ and $j \in [N-1]$

such that the following holds. For all $j \in [N-2]$,

P(j) for any $y_h \in T_h^j$ and $z_h \in S_h^{j+1}$ with $h \in [k]$, there is a transversal midpath from $(y_1, \dots, y_k)$ to $(z_1, \dots, z_k)$ with vertex set $\{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\} \cup X_j$ and colour set $\mathscr{B}_j$; and

P$(N-1)$ For any $p \in P$ and any $y_h \in T_h^{N-1}$ and $z_h \in S_h^N$ with $h \in [k]$, there is a transversal midpath from $(y_1, \dots, y_k)$ to $(z_1, \dots, z_k)$ with vertex set $\{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\} \cup X_{N-1} \cup \{v_1^*, v_2^*\}$ and colour set $\mathscr{B}_{N-1} \cup \mathscr{S}_p$.

Let $X := \bigcup_{i \in [N-1]} X_i$ and $\mathscr{B} := \bigcup_{i \in [N-1]} \mathscr{B}_i$.

After removing vertices X and colours $\mathscr{B}$ from the template $\mathcal{F}$, we have a rainbow $(N \times K_k)$-template $\mathcal{F}'$ with parameters $(m/2, 2\sqrt{\varepsilon}, d/2, 1/2)$.

Step 3. *Make classes near-equal and ensure* $|V_0| = (k-1)|\mathscr{C}_0|$.

Now we remove as few elements as possible from vertex and colour clusters so that they have the correct sizes. By Lemma 3.2.3, the following hold for any $j \in [N]$ and distinct $\ell, \ell' \in [k]$:

- the number of vertices $v \in U_\ell^j$ with $d_{G_c}(v, U_{\ell'}^j) \leq dm/5$ for at least $dm/5$ colours $c \in \mathscr{C}_{\ell\ell'}^j$ is at most $2\sqrt{\varepsilon}m$;
- the number of colours $c \in \mathscr{C}_{\ell\ell'}^j$ with $e(G_c[U_\ell^j, U_{\ell'}^j]) \leq dm/5$ is at most $2\sqrt{\varepsilon}m$.

Move all these bad vertices and colours into the exceptional sets $V_0, \mathscr{C}_0$. For each $j \in [N]$, let m' be the largest integer such that $|U_\ell^j| \geq m'$ and $|\mathscr{C}_{\ell\ell'}^j| \geq 2m'-1$ for all distinct $\ell, \ell' \in [k]$. Move arbitrary vertices from vertex clusters into V_0 so that every $|U_\ell^j| \in \{m'-1, m'\}$ and arbitrary colours from colour clusters into $\mathscr{C}_0$ so that every $|\mathscr{C}_{\ell\ell'}^j| \in \{2m'-2, 2m'-1\}$ and $|\mathscr{C}_0| = (k-1)|V_0|$. We have the disjoint unions

$$V = \left(\bigcup_{j \in [N]} \bigcup_{\ell \in [k]} U_\ell^j\right) \cup X \cup V_0 \quad \text{and} \quad \mathscr{C} = \left(\bigcup_{j \in [N]} \bigcup_{\ell\ell' \in \binom{[k]}{2}} \mathscr{C}_{\ell\ell'}^j\right) \cup \mathscr{B} \cup \mathscr{C}_0$$

and $\mathscr{S}$ is a subset of the first double union in $\mathscr{C}$. Then $m - m' \leq |X| \leq 20k^3 N < \varepsilon m$, so

$$|U_\ell^j| \geq (1-\varepsilon)m \quad \text{and} \quad |V_0| \leq n - |X| - Nk(m'-1) \overset{(4.10)}{\leq} 3k\zeta n.$$

We claim that, for each $j \in [N]$, the subtemplate $\mathcal{F}_j$ of $\mathcal{F}'$ induced by the new $\{U_\ell^j : \ell \in [k]\}$

and $\{\mathscr{C}^j_{\ell\ell'} : \ell\ell' \in \binom{[k]}{2}\}$ is a super K_k-template with parameters $(m'-1, \varepsilon^{1/3}, d/8, 1)$. Indeed, we just have to check that $\{G'_c[U^j_\ell, U^j_{\ell'}] : c \in \mathscr{C}^j_{\ell\ell'}\}$ is $(\varepsilon^{1/3}, d/8)$-superregular. Before removing vertices and colours, this collection was $(2\sqrt{\varepsilon}, d/4)$-regular. Lemma 3.2.4(i) implies that it is now $(\varepsilon^{1/3}, d/8)$-regular. Superregularity now follows since we removed any vertices and colours with small degree.

Step 4. *Incorporate exceptional vertices and all but $2k-2$ exceptional colours.*

We apply the following algorithm to greedily incorporate exceptional vertices and colours into the template. We will argue that the algorithm can run until there are $2k-2$ exceptional colours remaining.

Initialise sets $Z^j_i = \emptyset$ for $j \in [N]$ and $i \in [k]$; here Z^j_i will be the set of exceptional vertices to be incorporated into the j-th k-clique A_j between a vertex in U^j_i and a vertex in U^j_{i+1}. For each $v \in V_0 \setminus \{v^*_1, v^*_2\}$, whenever $|\mathscr{C}_0| \geq 2k-1$, choose a set $\mathscr{C}_v \subseteq \mathscr{C}_0$ of $2k-1$ exceptional colours. Let B_c be the graph with vertex set $\{v\} \cup \{(j,\ell) : j \in [N], \ell \in [k]\}$ where $v(j,\ell)$ is an edge whenever $d_{G_c}(v, U^j_\ell) \geq \eta m'$. Then $(\frac{k-1}{k}+\gamma/2)n \leq d_{G_c}(v) - |V_0| - |X| \leq d_{B_c}(v)m' + Nk\cdot\eta m'$ and so $d_{B_c}(v) \geq (\frac{k-1}{k}+\gamma/3)n/m' \geq (1+\gamma^2)(k-1)N$. Thus we can apply Lemma 4.3.7 to find, for each $v \in V_0$, $\gamma^2 N/2$ indices $j_v \in [N]$ and $t_v \in [k]$, a subset $\mathscr{C}'_v$ of $\mathscr{C}_v$ of size $2k-2$, and a partition $\mathscr{C}'_v = \mathscr{C}_{v,1} \cup \ldots \cup \mathscr{C}_{v,k}$ where $|\mathscr{C}_{v,t_v}| = |\mathscr{C}_{v,t_v+1}| = 1$ and all other $\mathscr{C}_{v,\ell}$ have size two, and $d_{G_c}(v, U^{j_v}_\ell) \geq \eta m'$ for all $\ell \in [k]$ and $c \in \mathscr{C}_{v,\ell}$.

Choose one of these j_v for which $|\bigcup_{i\in[k]} Z^{j_v}_i| \leq \sqrt{\zeta}m'$. Remove v from V_0 and add it to $Z^{j_v}_{t_v}$. Let $c_{t_v-k+2}, \ldots, c_{t_v+k-1}$ be any ordering of $\mathscr{C}'_v$ so that $i \equiv i' \mod k$ when $c_i \in \mathscr{C}_{i'}$, and for each $i \in \{-k+2, \ldots, k-1\}$, let $T^{\mathrm{ex},i}_v := N_{G_{c_i}}(v, U^{j_v}_{i'})$ be target sets for the neighbours of v, so $|T^{\mathrm{ex},i}_v| \geq \eta m'$. For each $\ell\ell' \in \{\{t_v - h + 1, t_v - h + k\} : h \in [k-1]\}$, choose a colour $c \in \mathscr{C}^{j_v}_{\ell\ell'} \setminus \mathscr{S}$, remove it, and add it to $\mathscr{C}_0$. Remove $\mathscr{C}'_v$ from $\mathscr{C}_0$.

At each step we assign one exceptional vertex by using $2k-2$ exceptional colours from a selection of $2k-1$, and we create $k-1$ new exceptional colours. Thus after each step we have $|\mathscr{C}_0| = (k-1)|V_0|$. In particular, if $V_0 \setminus \{v^*_1, v^*_2\} \neq \emptyset$, then $|\mathscr{C}_0| \geq 3k-3 \geq 2k-1$. Thus we may always apply Lemma 4.3.7, and since initially $|V_0| \leq 3k\zeta n \leq 4k^2\zeta N m'$, at most $4k^2\sqrt{\zeta}N$ of the $\bigcup_{i\in[k]} Z^j_i$ have size at least $\sqrt{\zeta}m'$ at any step, so since $\gamma \gg \zeta$, we may always choose j_v. Thus the algorithm terminates, with $V_0 = \{v^*_1, v^*_2\}$ and $\mathscr{C}^* := \mathscr{C}_0$ satisfying $|\mathscr{C}^*_0| = 2k-2$. Let

$$Z_j := \textstyle\bigcup_{i\in[k]} Z^j_i, \quad Z := \bigcup_{j\in[N]} Z_j \quad \text{and} \quad \mathscr{C}^{\mathrm{ex}} := \bigcup_{v\in Z} \mathscr{C}'_v, \quad \text{so} \quad |\mathscr{C}^{\mathrm{ex}}| = (2k-2)|Z|.$$

Each $\mathscr{C}^j_{\ell\ell'}$ has reduced in size by 0 if $|\ell - \ell'| \not\equiv 1$, and by $\sum_{i\in[k]\setminus\{\ell+1\}} |Z^j_i| = |Z_j| - |Z^j_{\ell+1}|$ if $\ell - \ell' \equiv 1$.

Step 5. *Absorb remaining $2k-2$ exceptional colours and make the final connection.*

We now incorporate the colours of $\mathscr{C}^*_0$ into the structure, by applying Lemma 4.3.5(i) to find a midpath that absorbs them. However the vertices and colours used will have to be removed from the structure, which unbalances it slightly.

Apply Lemma 4.3.5(i) as follows. Let $v_1, v_2 \in V \setminus \{v^*_1, v^*_2\}$ be distinct, let $\mathscr{C}^*_0 = \mathscr{C}^1_0 \cup \mathscr{C}^2_0$ be an equipartition and let $\mathcal{Q} = \{(\mathscr{C}^1_0, v_1), (\mathscr{C}^2_0, v_2)\}$, and let $(j, j') = (N, 1)$. By defining W and $\mathscr{W}$ in the obvious way, we obtain $X_N \subseteq V \setminus (X \cup \{v^*_1, v^*_2\} \cup Z)$ of size at most $20k^3$, $\mathscr{B}_N \subseteq \mathscr{C} \setminus (\mathscr{B} \cup \mathscr{S} \cup \mathscr{C}^{\mathrm{ex}})$, vertex sets $T^N_h \subseteq U^N_h$ and $S^1_h \subseteq U^1_h$ with $|T^N_h|, |S^1_h| \geq \eta m$ for all $h \in [k]$ such that the following holds.

P(N) for any $y_h \in T^N_h$ and $z_h \in S^1_h$ with $h \in [k]$, there is a transversal midpath from $(y_1, \dots, y_k)$ to $(z_1, \dots, z_k)$ with vertex set $\{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\} \cup X_N \cup \{v_1, v_2\}$ and colour set $\mathscr{B}_N \cup \mathscr{C}^*_0$.

By counting edges/colours in the midpaths guaranteed by **P**(1)–**P**(N), we have

$$\begin{aligned} |\mathscr{B}| + 4k - 4 &= \sum_{j\in[N-2]} e(Q_{2k+|X_j|}) + \sum_{j\in[N-1,N]} e(Q_{2k+|X_j|+2}) + \\ &\overset{(4.1)}{=} + \sum_{j\in[N-2]} \big(e(P_{|X_j|}) + 2\tbinom{k}{2}\big) + \sum_{j\in[N-1,N]} \big(e(P_{|X_j|+2}) + 2\tbinom{k}{2}\big) \\ &\overset{(4.1)}{=} e(C_{|X|+4}) + N\tbinom{k}{2} = (k-1)(|X|+4) + N\tbinom{k}{2}, \end{aligned}$$

so $|\mathscr{B}| = (k-1)|X| + N\binom{k}{2}$.

Step 6. *Repeat Steps 3 and 4, now only making colours in $\mathscr{S}$ exceptional.*

In this step, we will fix the unbalancedness created in Step 5, by equalising classes again (i.e. repeating Step 3), but this time only removing colours from $\mathscr{S}$. In this way, we will end up with $V_0 = \{v^*_1, v^*_2\}$ and each of the $2k-2$ exceptional colours will lie in $\mathscr{S}$.

Initially, $V_0 := \{v^*_1, v^*_2\}$ and $\mathscr{C}_0 := \emptyset$. Remove v_1, v_2 from their clusters. Remove X_N from the template and add it to X, and remove $\mathscr{B}_N$ from the template and add it to $\mathscr{B}$. Now remove as few arbitrary elements as possible from vertex clusters into V_0; and remove arbitrary elements

of $\mathscr{S}$ from colour clusters into $\mathscr{C}_0$ until every

$$|U_\ell^j| = m'' \quad \text{and} \quad |\mathscr{C}_{\ell\ell'}^j| = \begin{cases} 2m''-1 & \text{if } |\ell-\ell'| \not\equiv 1 \\ 2m''-1-|Z_j|+|Z_{\ell+1}^j| & \text{if } \ell-\ell' \equiv 1 \end{cases} \tag{4.11}$$

for m'' as large as possible. Recall that, at the end of Step 5, cluster sizes differed from these by at most one, where m' played the role of m''. Each midpath guaranteed by $\mathbf{P}(N)$ has $|X_N|+2k+2$ vertices, and each such vertex has degree at most $2k-2$ in the midpath. Thus $|\mathscr{B}_N| \le (20k^3+2k+2)(2k-2) < 50k^4 < |\mathscr{S}\cap\mathscr{C}_{\ell\ell'}^j|$, so there are enough colours in $\mathscr{S}$ available for this removal to work. We have $\mathscr{C}_0 \subseteq \mathscr{S}$ and $\{v_1^*, v_2^*\} \subseteq V_0$. The same arguments as in Step 3 show that $|U_\ell^j| = m'' \ge (1-2\varepsilon)m$ and $|V_0| \le kN + \binom{k}{2}N + |X_N|kN \le 30k^4N^2$.

Now, $\sum_{j\in[N]}\sum_{\ell\in[k]}(|Z_j|-|Z_{\ell+1}^j|) = (k-1)|Z|$. Since $\mathscr{C}$ is the disjoint union of the $\mathscr{C}_{\ell\ell'}^j$, $\mathscr{B}$, $\mathscr{C}^{\mathrm{ex}}$, $\mathscr{C}_0^*$ and $\mathscr{C}_0$, we have

$$\begin{aligned} |\mathscr{C}_0| &= |\mathscr{C}| - N\tbinom{k}{2}(2m''-1) + (k-1)|Z| - |\mathscr{B}| - |\mathscr{C}^{\mathrm{ex}}| - (2k-2) \\ &= (k-1)(n-|X|-Nkm''-|Z|-2) = (k-1)|V_0|. \end{aligned}$$

The final equality holds since V is the disjoint union of the U_ℓ^j, X, Z, $\{v_1,v_2\}$ and V_0.

Next, repeat Step 4, to incorporate exceptional vertices in $V_0 \setminus \{v_1^*, v_2^*\}$ by adding them to Z except that now whenever we choose a colour $c \in \mathscr{C}_{\ell\ell'}^{j_v}$ to remove into $\mathscr{C}_0$, we ensure that this colour is in $\mathscr{S}$. Each time we incorporate a vertex, we use at most one colour from each $\mathscr{C}_{\ell\ell'}^j$, so in total we require at most $|V_0| \le 30k^4N^2$ colours in $\mathscr{S}$ in each colour cluster. This is possible since $|\mathscr{S}\cap\mathscr{C}_{\ell\ell'}^{j_v}| \ge s - 50k^4 > 30k^4N^2$. The algorithm terminates, with $V_0 = \{v_1^*, v_2^*\}$ and $|\mathscr{C}_0| = 2k-2$ and, crucially, $\mathscr{C}_0 \subseteq \mathscr{S}$ so $\mathscr{C}_0 = \mathscr{S}_{p^*}$ for some $p^* \in P$. Cluster sizes still satisfy (4.11) (with respect to the enlarged Z). We have the disjoint unions

$$V = \bigcup_{j\in[N]}\Big(\bigcup_{\ell\in[k]} U_\ell^j \cup X_j \cup Z_j\Big) \cup \{v_1,v_2\} \cup \{v_1^*, v_2^*\}$$

$$\text{and} \quad \mathscr{C} = \bigcup_{j\in[N]}\Big(\bigcup_{\ell\ell'\in\binom{[k]}{2}} \mathscr{C}_{\ell\ell'}^j \cup \mathscr{B}_j\Big) \cup \mathscr{C}^{\mathrm{ex}} \cup \mathscr{C}_0^* \cup \mathscr{S}_{p^*}.$$

Moreover, for each $j \in [N]$, the subtemplate $\mathcal{F}_j'$ of $\mathcal{F}_j$ induced by the new $\{U_\ell^j : \ell \in [k]\}$ and $\{\mathscr{C}_{\ell\ell'}^j : \ell\ell' \in \binom{[k]}{2}\}$ is a super K_k-template with parameters $(m'', 2\varepsilon^{1/3}, d/16, 1)$.

Step 7. *Apply the transversal blow-up lemma to complete the transversal embedding.*

It suffices to find a transversal copy H_j of the $\{i_v : v \in Z_j\}$-missing path $F = F_{km''}(i_v : v \in Z_j) = x_1 \dots x_{km''}$ where

- $i_v \equiv t_v \mod k$ and $k < i_v < km'' - k$ for all $v \in Z_j$, and $|i_v - i_u| \geq 2k$ for all distinct $u, v \in Z_j$;
- the vertex playing the role of x_i lies in $U^j_{i'}$ where $i' \equiv i \mod k$ for every $i \in [km'']$;
- $x_i x_\ell$ uses a colour in $\mathscr{C}^j_{i'\ell'}$ where $i' \equiv i$ and $\ell' \equiv \ell \mod k$, for all distinct $i, \ell \in [km'']$.
- $x_{i_v+h} \in T^{\mathrm{ex},h}_v$ for all $h \in \{-k+2, \dots, k-1\}$ and $v \in Z_j$; and $x_h \in S^j_h$ and $x_{km''-k+h} \in T^j_h$ for all $h \in [k]$.

Indeed, if we can find such an H_j for all j, inserting each $v \in Z_j$ after i_v gives rise to a rainbow copy J_j of $P_{km''+|Z_j|}$ using colours $\bigcup_{\ell\ell' \in \binom{[k]}{2}} \mathscr{C}^j_{\ell\ell'} \cup \bigcup_{v \in Z_j} \mathscr{C}'_v$ (see Observation 4.3.1(ii)). The start- and endtuple of J_j are the same as H_j. Thus they lie in the given target sets S^j_h at the start and T^j_h at the end, and $\mathbf{P}(1)$–$\mathbf{P}(N)$ guarantee short midpaths between the endtuple of J_j and the starttuple of J_{j+1} for all $j \in [N]$, where $J_{N+1} := J_1$ and the midpath from the endtuple of J_N to the starttuple of J_1 which in addition to start- and endtuples use precisely the vertices $X_1 \cup \dots \cup X_N \cup \{v_1, v_2\} \cup \{v_1^*, v_2^*\}$, and use precisely the colours $\mathscr{B}_1 \cup \dots \cup \mathscr{B}_N \cup \mathscr{C}_0^* \cup \mathscr{S}_{p^*}$. We thus obtain a transversal cycle using all vertices V and colours $\mathscr{C}$.

To find the H_j, note that the function ϕ_j which maps x_i to $i' \in [k]$ where $i \equiv i' \mod k$ is a graph homomorphism of F into K_k when $V(K_k) = [k]$, and every $|\phi_j^{-1}(i)| = m''$. To see that $e(F[\phi_j^{-1}(\ell), \phi_j^{-1}(\ell')]) = |\mathscr{C}^j_{\ell\ell'}|$ for every $\ell\ell' \in \binom{[k]}{2}$, we have $|\mathscr{C}^j_{\ell\ell'}| = 2m'' - 1 - |Z_j| + |Z^j_{\ell+1}|$, which, by Observation 4.3.1(ii), is precisely what is required. The number of target sets is $2k + |Z_j| \leq 2\sqrt{\zeta}m''$, which, crucially, is much smaller than $\delta m''$, and each one has size at least $\eta m''/2$. Thus we can apply Theorem 3.3.5 to the template $\mathcal{F}'_j$ to find a transversal copy of F and hence the required cycle. □

5 | Stability of Transversal Hamilton Cycles

5.1 Introduction

5.1.1 Transversal embedding of Hamilton cycles

Following Joos and Kim's transversal generalisation of Dirac's theorem, there has been much progress on further questions related to the Hamiltonian case of Question 1.2.6 for graph collections on n vertices. (Cheng, Wang, & Zhao, 2021) also showed that minimum degree $(n+1)/2$ is sufficient to guarantee a rainbow cycle of every length $3, \dots, n-1$. (Bradshaw, 2020) proved analogues of the results of Joos and Kim in bipartite graph collections. (Chakraborti, Kim, Lee, & Seo, 2023) studied the problem of finding transversal Hamilton cycles in tournaments (a tournament is a complete oriented graph). (Li, Li, & Li, 2023) and (Zhang & van Dam, 2024) imposed Ore-type conditions rather than a minimum degree condition to find transversal Hamilton paths and cycles. (Bradshaw, Halasz, & Stacho, 2022) showed that minimum degree at least $n/2$ guarantees not just one transversal Hamilton cycles as per (Joos & Kim, 2020), but exponentially many.

Transversal Hamilton cycles have also been well studied in hypergraphs. Extending a result of (Rödl, Ruciński, & Szemerédi, 2006; Rödl et al., 2008), (Tang, Wang, Wang, & Yan, 2023) proved an asymptotically optimal bound on the minimum codegree guaranteeing a tight Hamilton cycle in a uniform hypergraph. The main result in (Gupta et al., 2023) implies asymptotically optimal results for other types of Hamilton cycles in uniform hypergraphs, for various degree conditions. Answering a question left open here, the notable case of vertex degree for tight 3-uniform Hamilton cycles was obtained by (Tang et al., 2023).

The problem of finding a Hamilton cycle with a given colouring in a graph collection (rather than a transversal Hamilton cycle which is *some* rainbow cycle) was studied by (Bowtell, Morris, Pehova, & Staden, 2023). It turns out that minimum degree $(\frac{1}{2} + o(1))n$ is enough to guarantee such a cycle, but at least $\lfloor n/2 \rfloor + 1$ is required. (Ferber, Han, & Mao, 2022) and (Anastos & Chakraborti, 2023) studied *robustness* of transversal Hamiltonicity. In (Anastos & Chakraborti, 2023) the threshold at which a graph collection consisting of random subgraphs of graphs satisfying Dirac's condition contains a transversal Hamilton cycle is determined. This generalises a result of (Krivelevich, Lee, & Sudakov, 2014).

5.1.2 Stability for transversal Hamilton cycles and paths

In this Chapter, we are interested in the stability phenomenon for Hamilton paths and cycles, and will show that graph collections which do not contain a Hamilton cycle, but whose graphs all have minimum degree slightly smaller than that needed to guarantee one, in fact have a special structure.

We define two extremal graphs and of order n as follows:

- the n-vertex graph is the union of two disjoint cliques of size as equal as possible;
- the n-vertex graph is a complete bipartite graph whose two parts have size as equal as possible.

Since Dirac's theorem shows that a Hamilton cycle is guaranteed in any n-vertex graph G with minimum degree $\delta(G) \geq n/2$, graphs satisfying this minimum degree condition are called *Dirac graphs*. As mentioned in the previous section, (Joos & Kim, 2020) showed that any collection of n Dirac graphs contains a transversal Hamilton cycle.

Dirac's theorem is tight: the graph does not contain a Hamilton cycle, and when n is even it has minimum degree $n/2 - 1 = \lceil n/2 \rceil - 1$. The graph does not contain a Hamilton cycle when n is odd, in which case it has minimum degree $(n-1)/2 = \lceil n/2 \rceil - 1$ and contains no Hamilton cycle. In fact, when n is odd, we can add arbitrary edges into the smaller part of without creating a Hamilton cycle. For any n, we can perturb either of these examples slightly, at the expense of slightly reducing the minimum degree, to get a non-Hamiltonian graph.

In fact, it is well-known that these are the only examples. Given $\kappa > 0$, we say that two graphs G, G' on the same vertex set of size n are *κ-close* if one can add and remove at most κn^2 edges of G to obtain G'.

Theorem 5.1.1 (Stability for Dirac's theorem, folklore)**.** *For all $\kappa > 0$, there exist $\mu > 0$ and n_0 such that the following holds for all integers $n \geq n_0$. Let G be an n-vertex graph with $\delta(G) \geq (\frac{1}{2} - \mu)n$. If G contains no Hamilton cycle, then either G is κ-close to with arbitrary edges added inside one part; or G is κ-close to .*

Given a graph collection $\boldsymbol{G}$, we write $\delta(\boldsymbol{G}) := \min_{G \in \boldsymbol{G}} \delta(G)$ for its *minimum degree*. Now suppose that a graph collection $\boldsymbol{G} = (G_1, \ldots, G_n)$ on a common vertex set of size n with $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$ has no transversal Hamilton cycle (we say $\boldsymbol{G}$ is *non-Hamiltonian*). Of course, one can take $G_i = G$ where G is non-Hamiltonian; that is, we can take $\boldsymbol{G}$ to be a collection of

identical copies of one of the graphs in Theorem 5.1.1. On the other hand, taking copies of $\overset{\bullet}{\bullet}$ and $\overset{\circ}{\circ}$ with, say, uniformly random partitions will give a graph collection which *is* Hamiltonian with high probability. However, one can combine these graphs to get further collections without a transversal Hamilton cycle.

Definition 5.1.2 ($\boldsymbol{H}_a^b$, half-split graph collection)**.** *Given integers $a, b \geq 0$, let $\boldsymbol{H}_a^b$ be the graph collection on a common vertex set of size n obtained by taking a copies of $\overset{\bullet}{\bullet}$ and b copies of $\overset{\circ}{\circ}$ where they are defined on the same equitable partition $A \cup B$.*

We say that a graph collection $\boldsymbol{J}$ on a common vertex set V of size n is half-split *if there is $A \subseteq V$ with $|A| = \lfloor n/2 \rfloor + 1$ such that $J[A] = \emptyset$ and $J[A, V \setminus A]$ is complete for all $J \in \boldsymbol{J}$.*

Note that $\delta(\boldsymbol{H}_a^b) = \lfloor n/2 \rfloor - \mathbb{1}(a > 0)$. Note that $\boldsymbol{H}_n^0$ is non-Hamiltonian since $\overset{\bullet}{\bullet}$ is, and when n is odd, $\boldsymbol{H}_0^n$ is also non-Hamiltonian since $\overset{\circ}{\circ}$ is. We claim that when b is odd and $a + b = n$, the collection $\boldsymbol{H}_a^b$ is non-Hamiltonian. Suppose not, and let C be a transversal Hamilton cycle and cyclically direct its edges. We say an edge of C is of 1-type if it comes from an $\overset{\bullet}{\bullet}$ copy and of 2-type if it comes from an $\overset{\circ}{\circ}$ copy. Note that the number of 2-type edges directed from A to B is equal to the number of 2-type edges directed from B to A. Hence, the total number of 2-type edges in C is even, which implies that b is even, a contradiction.

A half-split graph collection $\boldsymbol{J}$ on n vertices has $\delta(\boldsymbol{J}) = \lfloor n/2 \rfloor - 1$. Further, $\boldsymbol{J}$ is non-Hamiltonian since if it contained any Hamilton cycle C (with any colours), then between each pair of vertices in A on C there is at least one vertex of $V \setminus A$, so $|V \setminus A| \geq |A|$, a contradiction.

Given $\kappa > 0$, we say that two graph collections $\boldsymbol{G}, \boldsymbol{G'}$ on the same common vertex set of size n are *κ-close* if one can add and remove at most κn^3 edges of graphs in $\boldsymbol{G}$ to obtain $\boldsymbol{G'}$. Our first result shows that any non-Hamiltonian collection of n almost-Dirac graphs on n vertices is close to one of these two families.

Theorem 5.1.3. *For all $\kappa > 0$, there exist $\mu > 0$ and n_0 such that the following holds for all integers $n \geq n_0$. Let $\boldsymbol{G} = (G_1, \dots, G_n)$ be a graph collection on a common vertex set V of size n, with $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ contains no transversal Hamilton cycle, then either $\boldsymbol{G}$ is κ-close to $\boldsymbol{H}_a^b$ for some $a \in [n]$ where $b = n - a$ is odd; or $\boldsymbol{G}$ is κ-close to a half-split graph collection.*

In contrast to the Hamilton cycle case, our second result demonstrates that half-split graph collections and $\boldsymbol{H}_{n-1}^0$ are the only extremal constructions (in an approximate sense) for transversal Hamilton paths. Clearly $\boldsymbol{H}_{n-1}^0$ does not contain a transversal Hamilton path, while if we in-

crease the size of A by one when n is odd, a half-split collection of $n-1$ graphs does not contain a transversal Hamilton path either.

Theorem 5.1.4. *For all $\kappa > 0$, there exist $\mu > 0$ and n_0 such that the following holds for all integers $n \geq n_0$. Let $\boldsymbol{G} = (G_1, \dots, G_{n-1})$ be a graph collection on a common vertex set V of size n, with $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ does not contain a transversal Hamilton path, then $\boldsymbol{G}$ is either κ-close to $\boldsymbol{H}^0_{n-1}$ or $\boldsymbol{G}$ is κ-close to a half-split graph collection.*

We prove these theorems in a unified manner, combining the regularity-blow-up method from Chapter 3 and the absorption method for transversals, which uses ideas from the papers (Cheng et al., 2023; Cheng, Han, et al., 2021; Cheng, Wang, & Zhao, 2021) of the author. We give a sketch of the proofs in Section 5.1.3.

5.1.3 Sketch of the proof of Theorems 5.1.3 and 5.1.4

We will define two kinds of stability for a graph family $\boldsymbol{G} = (G_1, \dots, G_n)$ with minimum degree at least $(\frac{1}{2} - o(1))n$ (see Definition 5.2.5). We will say that

- $\boldsymbol{G}$ is *strongly stable* if $\boldsymbol{G}$ contains many graphs which are not close to containing either or , while
- $\boldsymbol{G}$ is *weakly stable* if almost all graphs in $\boldsymbol{G}$ are close to containing either or , but the vertex partitions associated with these subgraphs are not similar.

We say that $\boldsymbol{G}$ is *stable* if it is either strongly stable or weakly stable. Thus if $\boldsymbol{G}$ is not stable, most graphs in $\boldsymbol{G}$ are close to containing either  or  and their associated vertex partitions are almost the same. The first part of the proof is to show that if $\boldsymbol{G}$ is stable, then $\boldsymbol{G}$ contains a transversal Hamilton cycle. We then deal with the extremal case, where $\boldsymbol{G}$ is not stable.

Step 1. *Build the absorbing cycle when $\boldsymbol{G}$ is stable.* (Section 5.3)

We build an 'absorbing cycle' C for $\boldsymbol{G}$ with the property that C is very small and there is a set $\mathcal{A}$ consisting of *some* colour-vertex pairs (c, v) and colour-vertex-vertex triples (c, u, v) such that whenever $\mathcal{A}_0 \subseteq \mathcal{A}$ is sufficiently small compared to $|C|$, C can *absorb* all of its elements (see Definition 5.3.2). This property implies

- we can add any $(c, v) \in \mathcal{A}_0$ to the cycle C, which means there is a new cycle C' with colour set $\mathrm{col}(C') = \mathrm{col}(C) \cup \{c\}$ and vertex set $V(C') = V(C) \cup \{v\}$; and,
- for any $(c, u, v) \in \mathcal{A}_0$, whenever P is a transversal path with endpoints u, v, we can add

P into C using only the new colour c.

In fact, when $\boldsymbol{G}$ is strongly stable, $\mathcal{A}$ contains *all* pairs and triples. But in general, we need to construct an additional auxiliary set to absorb those colours and vertices which C cannot. After C is constructed, we delete C and its colour set from $\boldsymbol{G}$.

Step 2. *Use the regularity-blow-up method for transversals to cover with long paths.*

We apply the regularity lemma for graph collections (Lemma 4.2.1) to $\boldsymbol{G}$ and thus get a reduced graph collection $\boldsymbol{R}$ that inherits the minimum degree condition of $\boldsymbol{G}$. By randomly partitioning its colour set, we obtain two families $\boldsymbol{R}^1$ and $\boldsymbol{R}^2$, each using about half of the colour clusters. Since $\boldsymbol{R}^1$ and $\boldsymbol{R}^2$ inherit the original degree condition, we are able to find two almost perfect transversal matchings M_1, M_2 from $\boldsymbol{R}^1$ and $\boldsymbol{R}^2$ (see Lemma 5.2.7). The union of these matchings is 'locally' like a Hamilton cycle. We apply the transversal blow-up lemma (Theorem 3.3.5) to obtain almost spanning disjoint transversal paths inside each edge of $M_1 \cup M_2$, which cover almost all the vertices outside C.

Step 3. *Connect the paths and cover remaining vertices via the absorbing cycle.*

The last step is to use the absorbing property of C as well as the auxiliary set to connect all the paths to a transversal Hamilton cycle.

Step 4. *The extremal case.* (Section 5.5)

The remaining case is when $\boldsymbol{G}$ is not stable and thus most graphs in $\boldsymbol{G}$ are close to containing either  or  and their vertex partitions are almost the same partition $V = A \cup B$. To prove Theorem 5.1.3, we need to show that if $\boldsymbol{G}$ is not close to a half-split graph collection or close to $\boldsymbol{H}_a^b$ for some $a \in [n]$ where $b = n - a$ is odd, then $\boldsymbol{G}$ contains a transversal Hamilton cycle. For this, we first find a short path that covers atypical vertices and colours, and in some cases, balances the two sides of the partition. Next, we find partitions $A = A^0 \cup A^{10} \cup A^{11}$ and $B = B^0 \cup B^{10} \cup B^{11}$ whose sizes depend on the exact number of graphs that are close to  and to . The transversal blow-up lemma guarantees that there are spanning transversal paths in each of $\boldsymbol{G}[A^0, B^0]$, $\boldsymbol{G}[A^{10}, A^{11}]$, $\boldsymbol{G}[B^{01}, B^{11}]$ with disjoint colour sets and endpoints in any given large subsets. The structure of $\boldsymbol{G}$ allows us to find such paths which can be concatenated into a transversal Hamilton cycle, proving Theorem 5.1.3. The proof of Theorem 5.1.4 differs only at the end: we use the fact that given a collection $\boldsymbol{G}$ of $n-1$ graphs on a common vertex set of size n, the n-graph collection obtained from adding a complete graph to $\boldsymbol{G}$ has a Hamilton cycle if and only if $\boldsymbol{G}$ has a Hamilton path.

5.2 Extremality and stability

5.2.1 Extremal and stable graphs

In this section, we define extremality for a single graph. A graph G is not extremal if any two half-sized sets have many edges between them.

Definition 5.2.1 (nice, extremal)**.** *Let G be a graph on a vertex set V of size n and let $\varepsilon > 0$. We say that*

- *G is* ε-nice *if for any two sets $A, B \subseteq V$ of size at least $(\frac{1}{2}-\varepsilon)n$, we have $e_G(A,B) \geq \varepsilon n^2$.*
- *G is* ε-extremal *if it is not ε^3-nice.*

Note that whenever $\varepsilon' \geq \varepsilon > 0$, an ε'-nice graph is ε-nice and hence an ε-extremal graph is ε'-extremal.

The following is a version of a well-known fact about the structure of almost Dirac graphs which forms the basis of our extremal case distinction (for example, see (Krivelevich et al., 2014)).

Lemma 5.2.2. *Suppose that $0 < 1/n \ll d \ll \varepsilon \leq 1$. Let G be a graph on a vertex set V of size n with $d_G(x) \geq (\frac{1}{2} - \varepsilon^3)n$ for all but at most dn vertices $x \in V$ which is ε-extremal. Then there is a* characteristic partition *(A, B, C) of G such that*

(i) *A, B, C partition V;*

(ii) *$|A| = |B| = (\frac{1}{2} - \varepsilon)n$;*

(iii) *one of the following holds:*

- *$d_G(a, A) \geq (\frac{1}{2} - 2\varepsilon)n$ for all $a \in A$ and $d_G(b, B) \geq (\frac{1}{2} - 2\varepsilon)n$ for all $b \in B$ and $e_G(A, B) \leq \varepsilon n^2$; here we say that G is* $(\varepsilon, \boxdot)$-extremal*;*
- *$d_G(a, B) \geq (\frac{1}{2} - 2\varepsilon)n$ for all $a \in A$ and $d_G(b, A) \geq (\frac{1}{2} - 2\varepsilon)n$ for all $b \in B$, and either $e_G(A) \leq \varepsilon n^2$ or $e_G(B) \leq \varepsilon n^2$; here we say that G is* $(\varepsilon, \boxminus)$-extremal.

Proof. By adding vertices with small degree to C, it suffices to prove that if G has $\delta(G) \geq (\frac{1}{2} - \varepsilon^3)n$, then the conclusion holds with $\varepsilon/2$ in place of ε. Let $\mu := \varepsilon^3$. Since G is ε-extremal, there are $X, Y \subseteq V$ each of size at least $(\frac{1}{2} - \mu)n$ such that $e_G(X, Y) < \mu n^2$. Let $U := X \cap Y$ and $D := V \setminus (X \cup Y)$. We divide the proof into two cases based on the size of U.

Case 1. $|U| \geq 2\sqrt{\mu}n$. Let $U_0 := \{u \in U : d_G(u, D) \leq (\frac{1}{2} - 3\sqrt{\mu})n\}$. If $|U_0| > \sqrt{\mu}n$, then since $d_G(v, X \cup Y) \geq 2\sqrt{\mu}n$ for each $v \in U_0$, we have $e_G(X, Y) \geq e_G(U, X \cup Y) \geq$

$\frac{1}{2} \cdot 2\sqrt{\mu}n \cdot \sqrt{\mu}n = \mu n^2$, a contradiction. Thus we have $|U_0| \leq \sqrt{\mu}n$ and hence $|U \setminus U_0| \geq \sqrt{\mu}n$. In particular, $U \setminus U_0 \neq \emptyset$ and thus $|D| \geq (\frac{1}{2} - 3\sqrt{\mu})n$, so $|X \cup Y| \leq (\frac{1}{2} + 3\sqrt{\mu})n$. We have

$$|U \setminus U_0| = |X| + |Y| - |X \cup Y| - |U_0| \geq 2(\tfrac{1}{2} - \mu)n - (\tfrac{1}{2} + 3\sqrt{\mu})n - \sqrt{\mu}n \geq (\tfrac{1}{2} - 5\sqrt{\mu})n.$$

Each vertex in $U \setminus U_0$ has at least $(\frac{1}{2} - 3\sqrt{\mu})n$ neighbours in $V \setminus U$. Since $d_G(v, D) \geq (\frac{1}{2} - 3\sqrt{\mu})n$ for all $v \in U \setminus U_0$, we have $e_G(U \setminus U_0, D) \geq (\frac{1}{2} - 5\sqrt{\mu})(\frac{1}{2} - 3\sqrt{\mu})n^2$ and hence $D_0 := \{x \in D : d_G(x, U \setminus U_0) \leq (\frac{1}{2} - 3\sqrt{\mu})n\}$ has size $|D_0| \leq 6\sqrt{\mu}n$. So $|D \setminus D_0| \geq (\frac{1}{2} - 9\sqrt{\mu})n$.
Now choose any $A \subseteq U \setminus U_0$ and $B \subseteq D \setminus D_0$ with $|A| = |B| = (\frac{1}{2} - \varepsilon/2)n$, and let $C := V \setminus (A \cup B)$. Then A and B are disjoint, and we have $d_G(a, B) \geq d_G(a, D \setminus D_0) - (\varepsilon/2 + 5\sqrt{\mu})n \geq (\frac{1}{2} - \varepsilon)n$ for all $a \in A$ and similarly for $d_G(b, A)$ for all $b \in B$. Finally, $e_G(A) \leq e_G(U) \leq e_G(X, Y) < \mu n^2 < \varepsilon n^2/2$. Thus G is $(\varepsilon/2, 8)$-extremal.

Case 2. $|U| < 2\sqrt{\mu}n$. This case is very similar so we only sketch the proof. It is easy to see that for all but most $2\sqrt{\mu}n$ vertices v in X we have $d_G(v, Y) \leq \sqrt{\mu}n$. Similarly, for all but most $2\sqrt{\mu}n$ vertices v in Y we have $d_G(v, X) \leq \sqrt{\mu}n$. We delete these exceptional vertices from X and from Y. Let $X_1 := X \setminus Y$ and $Y_1 := Y \setminus X$. Note that these sets are disjoint and each has size at least $(\frac{1}{2} - 5\sqrt{\mu})n$. Now it follows that for each vertex $v \in X_1$, we have $d_G(v, X_1) \geq (\frac{1}{2} - 8\sqrt{\mu})n$ and for each vertex $v \in Y_1$, we have $d_G(v, Y_1) \geq (\frac{1}{2} - 8\sqrt{\mu})n$. We choose $A \subseteq X_1$ and $B \subseteq Y_1$ of the correct sizes, and note that $e_G(A, B) \leq e_G(X, Y) \leq \mu n^2$. Thus G is $(\varepsilon/2, 9)$-extremal. □

5.2.2 Stable graph collections

We now move on to consider stability for graph collections. Lemma 5.2.2 implies that given a positive integer n and $0 < 1/n \ll \varepsilon \leq 1$ and a graph collection $\boldsymbol{G} = (G_1, \ldots, G_n)$ on a common vertex set V of size n, whenever G_i is ε-extremal, we can fix a

$$\text{characteristic partition } (A_i, B_i, C_i) \text{ of } G_i.$$

We say that a vertex $v \in V$ is *i-good* if either G_i is not ε-extremal (that is, G_i is ε^3-nice) or G_i is ε-extremal and $v \in A_i \cup B_i$. Extremality of a graph collection depends on where graphs are in relation to one another, hence we make the following definitions.

Definition 5.2.3 (crossing, similar, cross graph)**.** *Let* $0 < 1/n, \varepsilon, \delta < 1$ *where* $n \in \mathbb{N}$ *and let* $\boldsymbol{G} = (G_1, \ldots, G_n)$ *be a graph collection on a common vertex set* V *of size* n*. Given* $i, j \in [n]$ *such that* G_i *and* G_j *are both* ε*-extremal, we say that they are* δ-crossing *if* $|A_i \bigtriangleup A_j| \geq \delta n$ *and* $|A_i \bigtriangleup B_j| \geq \delta n$*. We define the* cross graph $C^{\varepsilon,\delta}_{\boldsymbol{G}}$ *to be the graph with vertex set* $[n]$ *where* i *is adjacent to* j *if and only if* G_i, G_j *are both* ε*-extremal and* δ*-crossing.*

Observation 5.2.4. *Suppose that* $0 < \varepsilon \leq \delta/8$*. If* G_i *and* G_j *are* δ*-crossing, then* $|X_i \cap Y_j| \geq \delta n/4$ *whenever* $X, Y \in \{A, B\}$*.*

Proof. Since $|A_i \bigtriangleup B_j| \geq \delta n$ and $|A_i| = |B_j|$ we get $|A_i \setminus B_j| \geq \delta n/2$. Thus $|A_i \cap A_j| = |A_i \setminus (B_j \cup C_j)| \geq \delta n/2 - 2\varepsilon n \geq \delta n/4$. The other assertions hold by symmetry. □

We can now define stability for graph collections.

Definition 5.2.5 (strongly stable, weakly stable, stable,nice)**.** *Let* $n \in \mathbb{N}$ *and* $0 < \mu, \gamma, \alpha, \varepsilon, \delta < 1$ *be parameters. Suppose that* $\boldsymbol{G} = (G_1, \ldots, G_n)$ *is a graph collection on a common vertex set* V *of size* n*. We say that*

- $\boldsymbol{G}$ *is* (γ, α)-strongly stable *if* G_i *is* α*-nice for at least* γn *colours* $i \in [n]$*;*
- $\boldsymbol{G}$ *is* (ε, δ)-weakly stable *if* $e(C^{\varepsilon,\delta}_{\boldsymbol{G}}) \geq \delta n^2$*;*
- $\boldsymbol{G}$ *is* $(\gamma, \alpha, \varepsilon, \delta)$-stable *if it is either* (γ, α)*-strongly stable or* (ε, δ)*-weakly stable (or both).*
- $\boldsymbol{G}$ *is* μ-nice *if for every* $A \subseteq V$ *of size* $\lfloor n/2 \rfloor$ *we have* $e_{\boldsymbol{G}}(A) > \mu n^3$ *and* $e_{\boldsymbol{G}}(A, V \setminus A) > \mu n^3$*.*

When the parameters are clear from the text, we simply say $\boldsymbol{G}$ *is* strongly stable*,* weakly stable*,* stable *or* nice*.*

Note that whenever $\gamma' \geq \gamma > 0$ and $\alpha' \geq \alpha > 0$, a (γ', α')-strongly stable collection is (γ, α)-strongly stable, and whenever $0 < \varepsilon' \leq \varepsilon$ and $\delta' \geq \delta > 0$, an (ε', δ')-weakly stable collection is (ε, δ)-weakly stable.

Lemma 5.2.6. *Suppose that* $0 < 1/n \ll \mu \ll \gamma, \alpha, \varepsilon, \delta < 1$*. Let* $\boldsymbol{G} = (G_1, \ldots, G_n)$ *be a graph collection on a common vertex set* V *of size* n*. If* $\boldsymbol{G}$ *is* $(\gamma, \alpha, \varepsilon, \delta)$*-stable, then* $\boldsymbol{G}$ *is* μ*-nice.*

Proof. Suppose that there is $A \subseteq V$ of size $\lfloor n/2 \rfloor$ such that $e_{\boldsymbol{G}}(A) \leq \mu n^3$. Therefore, for all but at most $\sqrt{\mu} n$ colours $i \in [n]$, we have $e_{G_i}(A) \leq \sqrt{\mu} n^2$, and hence G_i is not $\sqrt{\mu}$-nice. This implies that $\boldsymbol{G}$ is not $(1 - \sqrt{\mu}, \sqrt{\mu})$-strongly stable and hence not (γ, α)-stable. Without loss of generality, suppose that for each $i \in [(1 - \sqrt{\mu})n]$, G_i is ε-extremal with characteristic partition (A_i, B_i, C_i). Let $B := V \setminus A$. It follows that, after possibly relabelling A_i and B_i, we have

$|A \bigtriangleup A_i| \leq \delta^2 n$ and $|B \bigtriangleup B_i| \leq \delta^2 n$ for all but at most $\delta^2 n$ colours $i \in [(1-\sqrt{\mu})n]$, otherwise we have at least $\delta^4 n^3/2 > \mu n^3$ edges of $\boldsymbol{G}$ inside A by $\mu \ll \delta$, a contradiction. Delete such colours from $[(1-\sqrt{\mu})n]$ and let $\mathscr{C}$ be the remaining colour set. Thus for every $i, j \in \mathscr{C}$, the graphs G_i and G_j are not δ^2-crossing. Therefore $e(C^{\varepsilon,\delta}_{\boldsymbol{G}}) \leq \sqrt{\mu} n^2 + \delta^2 n^2 < \delta n^2$ by $\mu \ll \delta$. Thus $\boldsymbol{G}$ is not (ε, δ)-weakly stable, a contradiction.

The proof of the other case is similar and we omit it. □

The next key lemma guarantees that a nice graph collection in which all vertices have degree at least $(\frac{1}{2}-\mu)n$ in *almost all* colours contains a transversal copy of a graph H which *locally* looks like a Hamilton cycle: almost all degrees equal two. We choose H to be a union of two almost perfect matchings. It is vital that here, 'almost all' means a $1-\theta$ proportion where $\theta \ll \mu$. If we were allowed to take, say, $\theta = 3\mu$, then the very first part of the argument of Joos and Kim on the transversal minimum degree threshold for matchings (Joos & Kim, 2020) would give a very short proof even without assuming the graph collection is stable. Instead, we need to analyse the structure of maximal matchings in a collection of almost Dirac graphs.

This lemma will be applied in a reduced graph, which is why we only assume a weaker degree condition, and then the transversal blow-up lemma will be applied to obtain a transversal collection of long paths covering almost the whole of V.

Lemma 5.2.7. *Let $0 < 1/n \ll d \ll \mu, \theta \ll \gamma, \alpha, \varepsilon, \delta \ll 1$. Suppose that $\boldsymbol{G} = (G_1, \dots, G_n)$ is a graph collection defined on a common vertex set V of size n and for each vertex $v \in V$, we have $d_{G_i}(v) \geq (\frac{1}{2}-\mu)n$ for all but at most dn colours $i \in [n]$. If $\boldsymbol{G}$ is $(\gamma, \alpha, \varepsilon, \delta)$-stable, then $\boldsymbol{G}$ contains two edge-disjoint transversal matchings M_1 and M_2 in $\boldsymbol{G}$ such that $e(M_\ell) \geq (\frac{1}{2}-\theta)n$ for $\ell = 1, 2$ and $\mathrm{col}(M_1) \cap \mathrm{col}(M_2) = \emptyset$.*

The lemma is an easy consequence of the next lemma which guarantees an almost spanning transversal matching in a nice graph collection containing about $n/2$ graphs.

Lemma 5.2.8. *Let $0 < 1/n \ll d \ll \mu, \theta \ll 1$. Suppose that $\boldsymbol{G} = (G_1, \dots, G_{(\frac{1}{2}-\theta/4)n})$ is a graph collection defined on a common vertex set V of size n and for each vertex $v \in V$, we have $d_{G_i}(v) \geq (\frac{1}{2}-\mu)n$ for all but at most dn colours $i \in [n]$. If $\boldsymbol{G}$ is 100μ-nice, then $\boldsymbol{G}$ contains a transversal matching M with $e(M) \geq (\frac{1}{2}-\theta)n$.*

Proof of Lemma 5.2.7 given Lemma 5.2.8. Let $\mathscr{C}_1$ be a random set obtained by selecting each colour in $[n]$ randomly and independently with probability $\frac{1}{2}$, and let $\mathscr{C}_2 := [n] \setminus \mathscr{C}_1$.

Claim 5.2.9. *With high probability, for $\ell = 1, 2$ we have $|\mathscr{C}_\ell| = n/2 \pm n^{3/4}$ and the graph collection $\boldsymbol{G}^\ell := (G_i : i \in \mathscr{C}_\ell)$ is μ-nice.*

Proof of Claim. The first statement follows from Chernoff's inequality.

Suppose first that $\boldsymbol{G}$ is (γ, α)-strongly stable. Then $\mathscr{A} := \{i \in [n] : G_i \text{ is } \alpha\text{-nice}\}$ satisfies $|\mathscr{A}| \geq \gamma n$. By Chernoff's inequality, with high probability, we have $|\mathscr{A} \cap \mathscr{C}_\ell| \geq \gamma n/3$ and thus $\boldsymbol{G}^\ell$ is $(\gamma/2, \alpha)$-strongly stable for $\ell = 1, 2$.

Suppose instead that $\boldsymbol{G}$ is (ε, δ)-weakly stable. Let $\mathscr{A} := \{i \in [n] : G_i \text{ is } \varepsilon\text{-extremal}\}$. We have $e(C^{\varepsilon,\delta}_{\boldsymbol{G}}) \geq \delta n^2$, so in particular, $|\mathscr{A}| \geq \delta n$. Let $H := C^{\varepsilon,\delta}_{\boldsymbol{G}}$ and $\mathscr{B} := \{i \in \mathscr{A} : d_H(i, \mathscr{A}) \geq \delta n\}$. It follows that $|\mathscr{B}| \geq \delta n$. Let $\mathcal{B} := \{N_H(v, \mathscr{A}) : v \in \mathscr{B}\}$. By Chernoff's inequality, with high probability, we have $|\mathscr{B} \cap \mathscr{C}_\ell| \geq \delta n/3$ and $|N \cap \mathscr{C}_\ell| \geq \delta n/3$ for each $N \in \mathcal{B}$ and $\ell = 1, 2$. Thus with high probability, we have $e(H[\mathscr{C}_\ell]) \geq \delta^2 n^2/18$ and thus $\boldsymbol{G}^\ell$ is $(\varepsilon, \delta^2/18)$-weakly stable for $\ell = 1, 2$.

In both cases, Lemma 5.2.6 implies that $\boldsymbol{G}^\ell$ is μ-nice for $\ell = 1, 2$. □

Fix such $\mathscr{C}_1$ and $\mathscr{C}_2$. Apply Lemma 5.2.8 to $\boldsymbol{G}^1$ to obtain a transversal matching M_1 with $e(M_1) \geq (\frac{1}{2} - \theta)n$ and colours from $\mathscr{C}_1$. Remove any $xy \in E(M_1)$ from G^2_i for each $i \in \mathscr{C}_2$. Every vertex degree in every graph reduces by at most one, so $\boldsymbol{G}^2$ is still 2μ-nice. Apply Lemma 5.2.8 to $\boldsymbol{G}^2$ to obtain a transversal matching M_2 with $e(M_2) \geq (\frac{1}{2} - \theta)n$ and colours from $\mathscr{C}_2$. □

Proof of Lemma 5.2.8. Let M be a maximal transversal matching in $\boldsymbol{G}$ and suppose for a contradiction that $e(M) < (\frac{1}{2} - \theta)n$. Let $v_1, v_2 \notin V(M)$ and $c_1, c_2 \notin \mathrm{col}(M)$ be distinct such that $N_\ell := N_{G_{c_\ell}}(v_\ell)$ satisfies $|N_\ell| \geq (\frac{1}{2} - \mu)n$ for $\ell = 1, 2$. Let $U := V \setminus (V(M) \cup \{v_1, v_2\})$ and $\mathscr{C} := [(\frac{1}{2} - \theta/4)n] \setminus (\mathrm{col}(M) \cup \{c_1, c_2\})$. Given $x \in V(M)$, we write x^+ to denote the unique neighbour of x in M and given $A \subseteq V(M)$, we write $A^+ := \{x^+ : x \in A\}$. Given $A \subseteq V$, we write $\overline{A} := V \setminus A$.

Claim 5.2.10. *We have $N_\ell \subseteq V(M)$ and $(\frac{1}{2} - \mu)n \leq |N_\ell| \leq (\frac{1}{2} + 3\mu)n$ for $\ell = 1, 2$ and exactly one of the following holds:*

- *$|\overline{N_1} \bigtriangleup N_2| \leq 6\mu n$, and writing $M_\ell := M[N_\ell]$ and $X_\ell := V(M_\ell)$ for $\ell = 1, 2$, we have $e(M_\ell) \geq (\frac{1}{4} - 2\mu)n$ and $X_1 \cap X_2 = \emptyset$ and $e(G_c[X_1, X_2]) = 0$ for all $c \in \mathscr{C}$;*
- *$|N_1 \bigtriangleup N_2| \leq 6\mu n$, and there is $X \subseteq \overline{N_1 \cup N_2} \cap V(M)$ with $X^+ \subseteq N_1 \cap N_2$ such that $|\overline{N_1 \cup N_2} \setminus X|, |(N_1 \cap N_2) \setminus X^+| \leq 6\mu n$ and $|X| \geq (\frac{1}{2} - 3\mu)n$ and $e(M[X^+]) = 0$ and*

$e(G_c[X]) = 0$ *for all* $c \in \mathscr{C}$.

Proof of Claim. If there is $y \notin V(M)$ with $v_\ell y \in G_{c_\ell}$ for some $\ell = 1, 2$ then we can add $v_\ell y$ to M to get a larger matching. Thus $N_\ell \subseteq V(M)$ for $\ell = 1, 2$. Moreover, we have $N_\ell^+ \cap N_{3-\ell} = \emptyset$ since otherwise we get a 3-path $v_{3-\ell} w w^+ v_\ell$ with $\mathrm{col}(v_{3-\ell}w) = c_{3-\ell}$, $\mathrm{col}(v_\ell w^+) = c_\ell$ and $ww^+ \in E(M)$. Replacing ww^+ by $v_{3-\ell}w$ and w^+v_ℓ gives a transversal matching that is larger than M, a contradiction. Let $S_\ell := E(M[N_\ell])$ and $T_\ell := E(M[N_\ell, V - N_\ell])$ and write $s_\ell := |S_\ell|$ and $t_\ell := |T_\ell|$. Note that, since $N_\ell^+ \cap N_{3-\ell} = \emptyset$, we have

$$S_1 \cap S_2 = S_1 \cap T_2 = T_1 \cap S_2 = \emptyset. \tag{5.1}$$

Since $N_\ell \subseteq V(M)$ we have

$$2s_\ell + t_\ell = |N_\ell| \geq (\tfrac{1}{2} - \mu)n.$$

Case 1. $t_1 + t_2 \leq 2\mu n$. We will show that the first alternative holds. We have $x \in N_2$ only if $x^+ \notin N_1$ so $|N_1 \cap N_2| \leq t_1 + t_2 \leq 2\mu n$, and $|\overline{N_1 \cup N_2}| \leq n - 2(s_1 + s_2) \leq 2\mu n + t_1 + t_2 \leq 4\mu n$. So $|\overline{N_1} \bigtriangleup N_2| \leq 6\mu n$. Also $|N_2| \leq |\overline{N_1}| + 2\mu n \leq (\frac{1}{2} + 3\mu)n$ so by symmetry we have $(\frac{1}{2} - \mu)n \leq |N_\ell| \leq (\frac{1}{2} + 3\mu)n$ for $\ell = 1, 2$. Next, $e(M_\ell) = s_\ell \geq \frac{1}{2}(|N_\ell| - t_\ell) \geq (\frac{1}{4} - 2\mu)n$. We have $X_\ell = V(S_\ell)$ and since S_ℓ is a matching, $S_1 \cap S_2 = \emptyset$ implies that $X_1 \cap X_2 = \emptyset$.

Finally, suppose $E(G_c[X_1, X_2]) \neq \emptyset$ for some $c \in \mathscr{C}$. Then there are $w_\ell w_\ell^+ \in E(M_\ell)$ for $\ell = 1, 2$ with $w_1^+ w_2 \in G_c$. Thus there is a transversal path $v_1 w_1 w_1^+ w_2 w_2^+ v_2$ using c_1, c_2, c and two colours from M. Hence we can replace $w_1 w_1^+$ and $w_2 w_2^+$ by $v_1 w_1, w_1^+ w_2, v_2 w_2^+$ to obtain a larger transversal matching, a contradiction.

Case 2. $t_1 + t_2 \geq 2\mu n$. We will show that the second alternative holds. Now (5.1) implies that

$$\begin{aligned} |T_1 \cap T_2| &= t_1 + t_2 - |T_1 \cup T_2| \geq t_1 + t_2 - (e(M) - s_1 - s_2) \\ &= \frac{t_1 + t_2}{2} + \frac{2s_1 + t_1}{2} + \frac{2s_2 + t_2}{2} - e(M) \\ &\geq \mu n + (\tfrac{1}{2} - \mu)n - (\tfrac{1}{2} - \theta)n \geq \theta n. \end{aligned}$$

Since $N_\ell^+ \cap N_{3-\ell} = \emptyset$, we can choose an edge $v_3 v_3^+ \in T_1 \cap T_2$ where $v_3^+ \in N_1 \cap N_2$ and $v_3 \in \overline{N_1 \cup N_2}$. Choose any colour $c_3 \in \mathscr{C}$ such that $d_{G_{c_3}}(v_3) \geq (\frac{1}{2} - \mu)n$. Let $N_3 := N_{G_{c_3}}(v_3)$. Suppose $N_3 \cap N_\ell^+ \neq \emptyset$ for $\ell \in [2]$. Thus there exists $ww^+ \in E(M)$ such that $\mathrm{col}(v_\ell w^+) = c_\ell$,

$\mathrm{col}(v_3w) = c_3$ and hence $v_\ell w^+ w v_3 v_3^+ v_{3-\ell}$ is a transversal path with $v_\ell w^+, wv_3, v_3^+ v_{3-\ell}$ coloured by $c_\ell, c_3, c_{3-\ell}$. By replacing $v_3v_3^+, ww^+$ in M by $v_3^+v_{3-\ell}, v_\ell w^+, wv_3$, we get a larger transversal matching, a contradiction. Thus $N_3 \cap N_\ell^+ = \emptyset$. Also, $N_3 \subseteq V(M)$: if not, there is $w \notin V(M)$ such that we can replace $v_3v_3^+$ in M with $v_3w, v_3^+v_1$ using colours c_3, c_1. We have shown that $N_3, N_1^+ \cup N_2^+$ are pairwise disjoint subsets of $V(M)$. By definition, we have the partition

$$N_1^+ \cup N_2^+ = V(S_1) \cup V(S_2) \cup Y \quad \text{where} \quad Y := \{x^+ : x \in N_1, xx^+ \in T_1\} \cup \{x^+ : x \in N_2, xx^+ \in T_2\}.$$

Whenever $x \in Y$ we have $x^+ \notin Y$. So $|Y| \geq |T_1 \cup T_2|$ and hence $|V(M)| \geq |N_3| + 2s_1 + 2s_2 + |T_1 \cup T_2|$. Therefore

$$|T_1 \cap T_2| = t_1 + t_2 - |T_1 \cup T_2| \geq \sum_{\ell=1,2} (2s_\ell + t_\ell) + |N_3| - |V(M)| \geq 3(\tfrac{1}{2} - \mu)n - n \geq (\tfrac{1}{2} - 3\mu)n.$$

Let $X := \{x \in \overline{N_1 \cup N_2} : xx^+ \in T_1 \cap T_2\}$. So $X \subseteq \overline{N_1 \cup N_2}$ and $X^+ \subseteq N_1 \cap N_2$ are disjoint, so $|X| = |X^+| \geq |T_1 \cap T_2|$. Thus also $|\overline{N_1 \cup N_2} \setminus X|, |(N_1 \cap N_2) \setminus X^+| \leq n - |X| - |X^+| \leq 6\mu n$. Since $N_1 \cup N_2$ and X are disjoint, we have $|N_\ell| \leq |N_1 \cup N_2| \leq (\frac{1}{2} + 3\mu)n$ for $\ell = 1, 2$. We have $|N_1 \bigtriangleup N_2| = |N_1 \cup N_2| - |N_1 \cap N_2| \leq n - |X^+| - |X| \leq 6\mu n$. The definition of X implies that $E(M[X^+]) = \emptyset$.

Finally, suppose $e(G_c[X]) \neq 0$ for some $c \in \mathscr{C}$. Then as before we obtain a contradiction by adding such an edge xy along with v_1x^+, v_2y^+ (of colours c, c_1, c_2) and removing xx^+, y^+y. This completes the proof of the claim. □

By Claim 5.2.10, we have $N_\ell \subseteq V(M)$ and $(\frac{1}{2} - \mu)n \leq |N_\ell| \leq (\frac{1}{2} + 3\mu)n$ for $\ell = 1, 2$ and one of the following cases:

Case 1: $|\overline{N_1} \bigtriangleup N_2| \leq 6\mu n$, and writing $M_\ell := M[N_\ell]$ and $X_\ell := V(M_\ell)$ for $\ell = 1, 2$, we have $e(M_\ell) \geq (\frac{1}{4} - 2\mu)n$ and $X_1 \cap X_2 = \emptyset$ and $e(G_c[X_1, X_2]) = 0$ for all $c \in \mathscr{C}$.

It suffices to show that for all colours $c \in \mathrm{col}(M_1 \cup M_2)$, we have $e_{G_c}(X_1, X_2) \leq 13\mu n^2$. Indeed, there are at least $2(\frac{1}{4} - 2\mu)n = (\frac{1}{2} - 4\mu)n$ such colours so $e_{\boldsymbol{G}}(X_1, X_2) \leq 13\mu n^3 + 4\mu n^3 = 17\mu n^3$. Also X_1, X_2 are disjoint sets of size at least $(\frac{1}{2} - 4\mu)n$, so there is a set Y of size $n/2$ such that $e_{\boldsymbol{G}}(Y, \overline{Y}) \leq 30\mu n^3$, and therefore $\boldsymbol{G}$ is not 30μ-stable.

Now let $c \in \mathrm{col}(M_1 \cup M_2)$. Without loss of generality, there is $ww^+ \in E(M_1)$ such that $\mathrm{col}(ww^+) = c$. Choose a colour $c_3 \in \mathscr{C}$ so that $N_3 := N_{G_{c_3}}(w)$ satisfies $|N_3| \geq (\frac{1}{2} - \mu)n$. So

$N_3 \cap X_2 = \emptyset$. Now let $M' := M - ww^+ + v_1w^+$ be the transversal matching with $\mathrm{col}(M') = (\mathrm{col}(M) \setminus \{c\}) \cup \{c_1\}$ and consider new vertex pair (w, v_2) and colour pair (c_3, c_2). We have $\{c, c_2, c_3\} \cap \mathrm{col}(M') = \emptyset$. The matching M' is maximal, otherwise the maximality of M is contradicted. So Claim 5.2.10 applies to M'.

Now, $N_3 \cap X_2 = \emptyset$ so $|N_3 \cap N_2| \leq |N_2| - |X_2| \leq (\frac{1}{2}+3\mu)n - (\frac{1}{2}-4\mu)n = 7\mu n$ and in particular the second alternative cannot hold. Thus $|\overline{N_3} \bigtriangleup N_2| \leq 6\mu n$, and writing $M'_\ell := M'[N_\ell]$ and $X'_\ell := V(M'_\ell)$ for $\ell = 2, 3$, we have $e(M'_\ell) \geq (\frac{1}{4} - 2\mu)n$ and $X'_2 \cap X'_3 = \emptyset$ and $E(G_c[X'_2, X'_3]) = \emptyset$. We have $|N_1 \bigtriangleup N_3| \leq |N_1 \bigtriangleup \overline{N_2}| + |\overline{N_2} \bigtriangleup N_3| \leq 12\mu n$. Since M, M' differ by two edges we therefore have $|X_1 \bigtriangleup X'_3| + |X_2 \bigtriangleup X'_2| \leq 13\mu n$. Thus $e_{G_c}(X_1, X_2) \leq e_{G_c}(X'_3, X'_2) \leq 13\mu n^2$, as required.

Case 2: $|N_1 \bigtriangleup N_2| \leq 6\mu n$, and there is $X \subseteq \overline{N_1 \cup N_2} \cap V(M)$ with $X^+ \subseteq N_1 \cap N_2$ such that $|\overline{N_1 \cup N_2} \setminus X|, |(N_1 \cap N_2) \setminus X^+| \leq 6\mu n$ such that $|X| \geq (\frac{1}{2} - 3\mu)n$ and $e(M[X^+]) = 0$ and $e(G_c[X]) = 0$ for all $c \in \mathscr{C}$.

It suffices to show that $e_{G_c}(X) \leq 24\mu n^2$ for all $c \in \mathrm{col}(M[X, X^+])$. Indeed, there are $|X| \geq (\frac{1}{2} - 3\mu)n$ such colours, so $e_{\boldsymbol{G}}(X) \leq 24\mu n^3 + 3\mu n^3 = 27\mu n^3$. Since $(\frac{1}{2} - 3\mu)n \leq |X| \leq n - |X^+| \leq (\frac{1}{2} + 3\mu)n$, there is a set Y of size $n/2$ with $e_{\boldsymbol{G}}(Y) \leq 40\mu n^3$. Thus $\boldsymbol{G}$ is not 40μ-stable.

Let $c \in \mathrm{col}(M[X, X^+])$. Let $w \in X$ be such that $\mathrm{col}(ww^+) = c$. Choose a colour c_3 from $\mathscr{C}$ such that $N_3 := N_{G_{c_3}}(w)$ satisfies $|N_3| \geq (\frac{1}{2} - \mu)n$. So $N_3 \cap X = \emptyset$. Now let $M' := M - ww^+ + v_1w^+$ be the transversal matching with $\mathrm{col}(M') = (\mathrm{col}(M \setminus \{c\}) \cup \{c_1\}$ and consider new vertex pair (w, v_2) and colour pair (c_3, c_2). So $\{c, c_2, c_3\} \cap \mathrm{col}(M') = \emptyset$. The matching M' is maximal, otherwise the maximality of M is contradicted. So Claim 5.2.10 applies to M'. We have $N_3 \cup N_2 \subseteq \overline{X}$ so $|N_3 \cap N_2| \geq |N_3| + |N_2| - (n - |X|) \geq 2(\frac{1}{2} - \mu)n - n + |X| > (\frac{1}{2} - 5\mu)n$, so in particular the first alternative cannot hold.

So $|N_3 \bigtriangleup N_2| \leq 6\mu n$, and there is $X' \subseteq \overline{N_3 \cup N_2}$ with $|\overline{N_3 \cup N_2} \setminus X'| \leq 6\mu n$ such that $|X'| \geq (\frac{1}{2} - 3\mu)n$ and $e(M[(X')^+]) = 0$ and $e(G_c[X']) = 0$. Now,

$$\begin{aligned} |X' \bigtriangleup X| &\leq |X' \bigtriangleup \overline{N_3 \cup N_2}| + |\overline{N_3 \cup N_2} \bigtriangleup \overline{N_1 \cup N_2}| + |\overline{N_1 \cup N_2} \bigtriangleup X| \leq 12\mu n + |N_3 \bigtriangleup N_1| \\ &\leq 12\mu n + |N_3 \bigtriangleup N_2| + |N_2 \bigtriangleup N_1| \leq 24\mu n. \end{aligned}$$

Thus $e_{G_c}(X) \leq 24\mu n^2$, as required.

This completes the proof of Lemma 5.2.8. □

5.3 Absorbing

Recall that $(V)_k$ denotes the set of all k-tuples of distinct elements of V. A pair $H = (V, E)$ where V is a set and $E \subseteq (V)_k$ is called a *directed k-graph*, while if we allow E to be a multiset, it is a *directed multi-k-graph*. We use the same notation for directed k-graphs as for graphs. The following lemma is the key tool that we will use to build an absorbing structure. Similar ideas have already been used in the papers (Cheng et al., 2023; Cheng, Han, et al., 2021; Cheng, Wang, & Zhao, 2021).

Lemma 5.3.1. *Let $k, C, n \in \mathbb{N}$ suppose that $0 < 1/n \ll \gamma \ll \varepsilon \ll 1/k, 1/C \leq 1$ and let $m \in [n^C]$ and $t := \gamma n$. Let $\boldsymbol{H} = (H_1, \dots, H_t)$ be a collection of directed k-graphs and let $\boldsymbol{Z} = (Z_1, \dots, Z_m)$ be a collection of directed multi-k-graphs all defined on a common vertex set V of size n. Suppose that $e(H_i) \geq \varepsilon n^k$ for all $i \in [t]$ and for each $j \in [m]$, we have $|E(Z_j) \cap E(H_i)| \geq \varepsilon n^k$ for at least εt indices $i \in [t]$. Then there is a transversal matching M in $\boldsymbol{H}$ of size at least $(1 - \varepsilon^2/4)t$ and $|E(Z_j) \cap E(M)| \geq \varepsilon^2 t/4$ for each $j \in [m]$.*

Proof. Let $W = \{e_1, \dots, e_t\}$ be obtained by independently picking e_i from $E(H_i)$, $i \in [t]$, uniformly at random. For each $i \in [t]$ and $j \in [m]$, let $\chi_{ij} = \mathbb{1}(e_i \in Z_j)$. By definition, we have $e(H_i) \leq n^k$ for each $i \in [t]$ and thus $\mathbb{E}(\chi_{ij}) = \mathbb{P}(e_i \in Z_j) = |E(Z_j) \cap E(H_i)|/|E(H_i)| \geq \varepsilon n^k / n^k = \varepsilon$ whenever $|E(Z_j) \cap E(H_i)| \geq \varepsilon n^k$. Therefore $\mathbb{E}(|W \cap E(H_j)|) = \mathbb{E}(\sum_{i \in [t]} \chi_{ij}) = \sum_{i \in [t]} \mathbb{E}(\chi_{ij}) \geq \varepsilon^2 t$ for each $j \in [m]$. Chernoff's inequality and a union bound imply that with probability at least $1 - e^{-\sqrt{n}}$, we have $|W \cap E(H_j)| \geq \varepsilon^2 t/2$ for all $j \in [m]$.

Let Y be the number of intersecting pairs of edges in W. So $Y \leq \sum_{ii' \in \binom{[t]}{2}} Y_{ii'}$ where $Y_{ii'} = \mathbb{1}(V(e_i) \cap V(e_{i'}) \neq \emptyset)$. Let $I_{ii'}$ be the number of intersecting pairs of edges $\{a_i, a_{i'}\}$ with $a_h \in E(H_h)$ for $h = i, i'$. So $|I_{ii'}| \leq k n^{2k-1}$. Hence, for every $ii' \in \binom{[t]}{2}$, we have

$$\mathbb{E}(Y_{ii'}) = \frac{|I_{ii'}|}{e(H_i)e(H_{i'})} \leq \frac{kn^{2k-1}}{\varepsilon^2 n^{2k}} = \frac{k}{\varepsilon^2 n}$$

$$\text{and so} \quad \mathbb{E}(Y) \leq \sum_{ii' \in \binom{[t]}{2}} \mathbb{E}(Y_{ii'}) \leq \binom{t}{2} \frac{k}{\varepsilon^2 n} \leq \frac{k}{2\varepsilon^2} \gamma^2 n \leq \frac{\varepsilon^2 t}{8}.$$

Markov's inequality implies that $\mathbb{P}(Y \leq \varepsilon^2 t/4) \geq 1/2$. Thus a union bound implies that there



Figure 5.1: A c-absorbing path of (v, v) and a c-absorbing path of (v, u) for $v \neq u$.

exists W with $|W| = t$, $|W \cap E(Z_j)| \geq \varepsilon^2 t/2$ for all $j \in [m]$ and there are at most $\varepsilon^2 t/4$ pairs of intersecting edges in W. The required M is obtained by deleting one edge from each intersecting pair of W. □

Let $\boldsymbol{G} = (G_1, \ldots, G_n)$ be a graph collection on a common vertex set V of size n. For every pair x, y of distinct vertices in V, we define $L(xy) := \{i \in [n] : xy \in E(G_i)\}$ to be the set of colours appearing on xy. Let $\varepsilon > 0$ and suppose that $\delta(G) \geq (\frac{1}{2} - \varepsilon^3)n$. Recall from Lemma 5.2.2 and Section 5.2.2 that each G_i is either ε^3-nice or we have defined a characteristic partition (A_i, B_i, C_i) of G_i. Thus for every $i \in [n]$ and $v \in V$, either v is G_i-good, or not.

Definition 5.3.2 (absorbing path, absorbing cycle)**.** *Given any two not necessarily distinct vertices $u, v \in V$, a transversal path $P = v_1v_2v_3v_4$ with $u, v \notin V(P)$ is called a* c-absorbing path of (v, u) *if $c \in L(v_2v)$ and $\mathrm{col}(v_2v_3) \in L(v_3u)$ (see Figure 5.1).*

Given $\delta, \delta', \gamma, \gamma' > 0$, a transversal cycle $C = v_1v_2 \ldots v_t$ is an absorbing cycle *with parameters $(\delta, \delta', \gamma, \gamma')$ if $t \leq \gamma n$ and there exists a colour set $\mathscr{C}$ of size at least δn such that*

(i) given any colour $c \in \mathscr{C}$ and any G_c-good vertex v, for all but at most $\delta' n$ vertices $u \in V$, there are at least $\gamma' n$ disjoint c-absorbing paths of (v, u) inside C;

(ii) given any colour $c \in \mathscr{C}$, for all but at most $\delta' n$ G_c-good vertices v, there are at least $\gamma' n$ disjoint c-absorbing paths of (v, v) inside C.

It is easy to see that if C is a transversal cycle containing a c-absorbing path of (v, v) where $v \notin V(C)$ and $c \notin \mathrm{col}(C)$, then there exists a transversal cycle C' such that $V(C') = V(C) \cup \{v\}$ and $\mathrm{col}(C') = \mathrm{col}(C) \cup \{c\}$ (insert v between v_2 and v_3). Similarly, if C is a transversal cycle containing a c-absorbing path of (v, u) where u, v are the endpoints of a transversal path P, $V(P) \cap V(C) = \emptyset$ and $\mathrm{col}(P), \{c\}, \mathrm{col}(C)$ are pairwise disjoint, then there exists a transversal cycle C' such that $V(C') = V(C) \cup V(P)$ and $\mathrm{col}(C') = \mathrm{col}(C) \cup \mathrm{col}(P) \cup \{c\}$ (insert P between v_2 and v_3).

Using Lemma 5.3.1 it is quite straightforward to find an absorbing cycle when G is strongly stable, using the many nice graphs G_i guaranteed by strong stability.

Lemma 5.3.3. *Let* $0 < 1/n \ll \lambda, \mu \ll \gamma, \alpha \ll 1$ *and suppose that* $\boldsymbol{G} = (G_1, \dots, G_n)$ *is a graph collection defined on a common vertex set* V *of size* n *with* $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$. *If* $\boldsymbol{G}$ *is* (γ, α)*-stable, then there exists an absorbing cycle with parameters* $(1, 0, \lambda, \lambda^2)$.

Proof. Without loss of generality, we assume that $G_1, \dots, G_{\gamma n}$ are α-nice and $6|\lambda n$. We divide the colour set $[\lambda n/2]$ into consecutive sets $\{1, 2, 3\}, \{4, 5, 6\}, \dots, \{\lambda n/2 - 2, \lambda n/2 - 1, \lambda n/2\}$ of three. For each $i \in I := [\frac{\lambda n}{6}]$, we define

$$\mathcal{F}_i := \{(v_1, v_2, v_3, v_4) \in (V)_4 : 3(i-1) + \ell \in L(v_\ell v_{\ell+1}) \; \forall \ell \in [3]\}.$$

Our minimum degree condition implies that $e(\mathcal{F}_i) \geq n((\frac{1}{2} - \mu)n)^3 \geq n^4/9$. Now we fix a colour $c \in [n]$ and two not necessarily distinct vertices $v, v' \in V$ (which are not required to be G_c-good). We let $Z_i(c, vv')$ be the collection of $(v_1, v_2, v_3, v_4) \in \mathcal{F}_i$ for which $P = v_1v_2v_3v_4$ with the given colours is a c-absorbing path of (v, v'). Since the graph G_{3i+2} is α-nice, there are at least αn^2 ways to choose v_2v_3 with $v_2 \in N_{G_c}(v), v_3 \in N_{G_{3i+2}}(v')$ and $v_2v_3 \in E(G_{3i+2})$. When v_2, v_3 are fixed, there are at least $(\frac{1}{2} - \mu)^2 n^2$ ways to choose v_1 and v_4. Thus $|Z_i(c, vv')| \geq (\frac{1}{2} - \mu)^2 n^2 \alpha n^2 \geq \alpha n^4/5$.

We can apply Lemma 5.3.1 with $\boldsymbol{H} := (\mathcal{F}_i : i \in I)$, $\boldsymbol{Z} := (Z(c, vv') := \bigcup_{i \in I} Z_i(c, vv') : c \in [n], vv' \in V^2)$, where this is a multiset union, and parameters $|I| = t$ and $\alpha/5 = \varepsilon$, to obtain a transversal matching M in $\boldsymbol{H}$ of size at least $(1 - \alpha^2/100)\lambda n/6$ such that $|E(Z(c, vv')) \cap E(M)| \geq \alpha^2 \lambda n/600$ for all $c \in [n]$ and $vv' \in V^2$. That is, there is $I' \subseteq I$ with $|I'| \geq (1 - \alpha^2/100)\lambda n/6$ such that for each $i \in I'$ there is a transversal path $P_i = v_1^i v_2^i v_3^i v_4^i$ with $3(i-1) + \ell \in L(v_\ell^i v_{\ell+1}^i)$ for all $\ell \in [3]$, and for every $c \in [n]$ and $vv' \in V^2$, there are at least $\alpha^2 \lambda n/600$ paths P_i which are c-absorbing of (v, v').

Relabel indices so that $I' = [s]$ where $s := (1 - \alpha^2/100)\lambda n/6$. We will connect the paths $P_1, \dots, P_s$ one by one into a transversal cycle C. We first connect P_1 and P_2 into a single transversal path P_1xyP_2. For this, choose distinct unused colours $c_1, c_2 \in [\lambda n + 1, n]$ and $c_3 \in [\lambda n/2 + 1, \lambda n]$. Let $U := V \setminus V(\bigcup_{i \in [s]} V(P_i))$. Note that $d_{G_{c_1}}(v_4^1, U), d_{G_{c_2}}(v_1^2, U) \geq (\frac{1}{2} - \mu - \lambda)n \geq (\frac{1}{2} - \alpha)n$. Since $c_3 \in [\lambda n/2 + 1, \lambda n]$, the graph G_{c_3} is α-nice, so there are at least αn^2 c_3-coloured edges between $N_{G_{c_1}}(v_4^1, U)$ and $N_{G_{c_2}}(v_1^2, U)$. Thus we can choose a c_3-coloured edge xy between $N_{G_{c_1}}(v_4^1, U)$ and $N_{G_{c_2}}(v_1^2, U)$ such that $C = P_1xyP_2$ is a transversal path with colour set $\mathrm{col}(P_1) \cup \mathrm{col}(P_2) \cup \{c_1, c_2, c_3\}$.

We repeat the process above for each pair (P_i, P_{i+1}) for $i \in [s]$, where $P_{s+1} := P_1$. Each time we use two unused colours from $[\lambda n + 1, n]$ and one unused colour from $[\lambda n/2 + 1, \lambda n]$. This is possible since $2s \leq \lambda n/3$. The number of used vertices at each step is at most $6s \leq \lambda n$ so we have the same bounds on degrees into the set of used vertices. By construction, C is an absorbing cycle with parameters $(1, 0, 6s/n, \alpha^2\lambda/600)$ and hence with parameters $(1, 0, \lambda, \lambda^2)$ since $\lambda \ll \alpha$. □

To complete this section, we find an absorbing cycle when $\boldsymbol{G}$ is weakly stable. This is much more involved than the strongly stable case. Again we use Lemma 5.3.1 to construct the cycle, using the many pairs of extremal graphs which are all highly crossing, as guaranteed by weak stability.

Lemma 5.3.4. *Let $0 < 1/n \ll \lambda, \mu, \varepsilon \ll \delta < 1$. Suppose that $\boldsymbol{G} = (G_1, \dots, G_n)$ is a graph collection defined on a common vertex set V where $|V| = n$ and $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ is (ε, δ)-weakly stable, then there exists an absorbing cycle C with parameters $(\delta/3, \sqrt{\varepsilon}, \lambda, \lambda^2)$.*

Proof. For each $j \leq n/3$, define the following family.

$$\mathcal{F}_j = \{(v_1, v_2, v_3, v_4) \in (V)_4 : 3(j-1) + \ell \in L(v_\ell v_{\ell+1}) \; \forall \ell \in [3]\}.$$

We first prove the following claim, which requires only the minimum degree condition on $\boldsymbol{G}$.

Claim 5.3.5. *Let $ij \in E(C^{\varepsilon,\delta}_{\boldsymbol{G}})$ where $j + 1 \leq 3n$ and let u be a G_j-good vertex and v be a G_i-good vertex. Let $Z_j(i, vu)$ be the collection of $(v_1, v_2, v_3, v_4) \in \mathcal{F}_{(j+1)/3}$ where $P = v_1v_2v_3v_4$ is an i-absorbing path of (v, u). Then $|Z_j(i, vu)| \geq 2^{-7}\delta^2 n^4$.*

Proof of Claim. Fix any such i, j, u, v. By the definition of characteristic partition, we have $d_{G_i}(v, A_i) \geq (\frac{1}{2} - 2\varepsilon)n$ or $d_{G_i}(v, B_i) \geq (\frac{1}{2} - 2\varepsilon)n$. Without loss of generality, we assume that the former case holds since the latter case can be proved similarly. Since the graphs G_i and G_j are δ-crossing, we have $|A_i \cap A_j| \geq \delta n/4$ and $|A_i \cap B_j| \geq \delta n/4$ by Observation 5.2.4. Since $d_{G_i}(v, A_i) \geq (\frac{1}{2} - 2\varepsilon)n$, we have $d_{G_i}(v, A_j) \geq |A_i \cap A_j| - (|A_i| - d_{G_i}(v, A_i)) \geq \delta n/4 - \varepsilon n \geq \delta n/5$ by $\varepsilon \ll \delta$. Similarly, we have $d_{G_i}(v, B_j) \geq \delta n/5$. Since u is G_j-good, $u \in A_j \cup B_j$. We divide the proof into the following cases:

Case 1: $u \in A_j$ and G_j is (ε, δ)-extremal. In this case, in the graph G_j, every vertex in A_j is adjacent to all but at most εn vertices in A_j. Thus $J := G_j[N_{G_i}(v, A_j)]$ is an almost

complete graph of order at least $\delta n/5$ in the sense that each vertex in it is adjacent to all but at most εn vertices. Any choice of $x \in V(J)$, $y \in N_{G_j}(u) \cap N_J(x)$, $x' \in N_{G_{j-1}}(x)$ and $y' \in N_{G_{j+1}}(y)$ yields an i-absorbing path $x'xyy'$ of (v,u) in $\mathcal{F}_{(j+1)/3}$ (so its colours are, in order, $j-1, j, j+1$, and $i \in L(xv)$ and $j \in L(yu)$). The number of such paths is therefore at least $\delta n/5 \cdot (\delta n/5 - 2\varepsilon n) \cdot (\frac{1}{2} - \mu)^2 n^2 \geq 2^{-7}\delta^2 n^4$.

Case 2: $u \in A_j$ and G_j is (ε, δ)-extremal. In this case, in the graph G_j, every vertex in A_j is adjacent to all but at most εn vertices in B_j and every vertex in B_j is adjacent to all but at most εn vertices in A_j. Thus $J := G_j[N_{G_i}(v, A_j), N_{G_j}(u, B_j)]$ is a bipartite graph with each part of order at least $\delta n/5$ which is almost complete in the sense that each vertex is adjacent to all but at most εn vertices from other side. Any choice of $x \in V(J) \cap A_j$, $y \in N_{G_j}(u) \cap N_J(x)$, $x' \in N_{G_{j-1}}(x)$ and $y' \in N_{G_{j+1}}(y)$ yields an i-absorbing path $x'xyy'$ of (v,u) in $\mathcal{F}_{(j+1)/3}$. The same lower bound from the previous case applies to the number of such choices.

The remaining cases when $u \in B_j$ are identical, so we omit them. □

For any vertex v, recall Definition 5.2.3 and let

$$\mathscr{G}_v := \{i \in [n] : v \text{ is } G_i\text{-good}\}.$$

Since $e(C_{\boldsymbol{G}}^{\varepsilon,\delta}) \geq \delta n^2$, there is a subgraph H of $C_{\boldsymbol{G}}^{\varepsilon,\delta}$ such that $\delta(H) \geq \delta n$, so in particular, $|V(H)| > \delta n$. For each $i \in V(H)$, define the set $T_i := \{u \in V : |N_H(i) \setminus \mathscr{G}_u| \geq \delta n/2\}$. Note that since $|C_i| = 2\varepsilon n$ for each i where G_i is ε-extremal, we have $|T_i|\delta n/2 \leq \sum_{u \in V} |[n] \setminus \mathscr{G}_u| = \sum_{j \in [n]} |C_j| \leq 2\varepsilon n^2$ and thus $|T_i| \leq 4\varepsilon n/\delta \leq \sqrt{\varepsilon} n/2$ since $\varepsilon \ll \delta$. Let $\overline{T}_i := V \setminus T_i$. For each $i \in V(H)$ and $u \in \overline{T}_i$, we have $|\overline{T}_i| \geq (1 - \sqrt{\varepsilon}/2)n$ and $|\mathscr{G}_u \cap N_H(i)| \geq \delta n/2$.

Now we independently and randomly select vertices from $V(H)$ with probability $\kappa := \lambda/14$ to obtain a set $\mathscr{U}$ of colours. A Chernoff bound implies that, with high probability,

(i) $\kappa|V(H)|/2 \leq t := |\mathscr{U}| \leq 2\kappa|V(H)|$;

(ii) for every $i \in V(H)$ and $u \in \overline{T}_i$, we have $|\mathscr{G}_u \cap N_H(i, \mathscr{U})| \geq \delta\kappa n/4$.

Fix such a $\mathscr{U}$. By relabelling colours, we assume that $\mathscr{U} = \{3j - 1 : j \in [t]\}$. Let $\overline{\mathscr{U}} := V(H) \cap [3t+1, n]$. So $\mathscr{U} \cap \overline{\mathscr{U}} = \emptyset$ and $|\overline{\mathscr{U}}| \geq \delta n/2$ and $\delta(H[\overline{\mathscr{U}}]) \geq \delta n/2$ since $\lambda \ll \delta$. For each colour $j \in [t]$, let I_j be the collection of $(i, vu) \in [n] \times V^2$ with $i \in \overline{\mathscr{U}}$, $u \in \overline{T}_i$, $j \in \mathscr{G}_u \cap N_H(i, \mathscr{U})$ and v a G_i-good vertex.

Given i and u, the number of choices of j is at least $\delta\kappa n/4$ by (ii). Since $i, j \in V(H)$ and u is

G_j-good, Claim 5.3.5 implies that there are at least $2^{-7}\delta^2 n^4$ i-absorbing paths of (v,u) whose ordered vertex set is in $\mathcal{F}_{(j+1)/3}$. We can apply Lemma 5.3.1 with $\boldsymbol{H} := \{\mathcal{F}_j : j \in [t]\}$ and the directed multi-4-graph collection

$$\boldsymbol{Z} := (Z(i,vu) := \bigcup_{j \in \mathscr{G}_u \cap N_H(i,\mathscr{U})} Z_j(i,vu) : i \in \overline{\mathscr{U}}, u \in \overline{T}_i, v \text{ is } G_i\text{-good})$$

and with $2^{-7}\delta^2$ playing the role of ε. Now, $e(\mathcal{F}_j) \geq n^4/9$ for all $j \in [t]$ and $\boldsymbol{H}$ contains $\kappa\delta n/2 \leq t \leq 2\kappa n$ graphs by (i), so $t/n \ll 2^{-7}\delta^2$. For each (i,vu) for which $Z(i,vu) \in \boldsymbol{Z}$ and every fixed $j \in \mathscr{G}_u \cap N_H(i,\mathscr{U})$, we have $(j+1)/3 \in [t]$ and $|E(Z(i,vu)) \cap E(\mathcal{F}_{(j+1)/3})| \geq 2^{-7}\delta^2 n^4$. The number of such j is at least $\delta\kappa n/4 \geq \delta t/8$ by (ii). Thus there is a transversal matching M in $\boldsymbol{H}$ of size at least $(1-2^{-16}\delta^4)t$ such that $|E(Z(i,vu)) \cap E(M)| \geq 2^{-16}\delta^4 t$ for all $Z(i,vu) \in \boldsymbol{Z}$. That is, there is $I \subseteq [t]$ with $|I| \geq (1-2^{-16}\delta^4)t$ such that for each $j \in I$ there is a path $P_j = v_1^j v_2^j v_3^j v_4^j$ in $\mathcal{F}_{(j+1)/3}$ and for every $i \in \overline{\mathscr{U}}$, $u \in \overline{T}_i$ and v which is G_i-good, there are at least $2^{-16}\delta^4 t$ paths P_j which are i-absorbing of (v,u).

Relabel indices so that $I = [s]$ where $s := (1-2^{-16}\delta^4)t$. It remains to connect the $P_1, \dots, P_s$ into an absorbing cycle C. Recall that no colour in $\overline{\mathscr{U}}$ appears on any P_i. We first connect P_1 and P_2 into a single transversal path $P_1 v_4^1 xzy v_1^2 P_2$. For this, we first choose distinct colours $c_1, c_2, c_3, c_4 \in \overline{\mathscr{U}}$ where $c_1c_2 \in E(H)$. There are at least $|\overline{\mathscr{U}}| \geq \delta n/2$ choices for c_1, and given this at least $\delta(H[\overline{\mathscr{U}}]) \geq \delta n/2$ for c_2, and at least $|\overline{\mathscr{U}}| - 3 > \delta n/3$ choices for each of c_3, c_4. Next we choose unused vertices x, y such that $x \in N_{G_{c_3}}(v_4^1)$ is G_{c_1}-good and $y \in N_{G_{c_4}}(v_1^2)$ is G_{c_2}-good. There are at least $(\frac{1}{2}-\mu)n - 4e(M) - 2\varepsilon n$ choices for each of these. Finally, we choose $z \in N_{G_{c_1}}(x) \cap N_{G_{c_2}}(y)$. We claim that there are at least $\delta n/5$ choices for z.

To prove the claim, since $c_1c_2 \in E(H) \subseteq E(C_{\boldsymbol{G}}^{\varepsilon,\delta})$, both G_{c_1} and G_{c_2} are ε-extremal, so $x \in A_{c_1} \cup B_{c_1}$ and $y \in A_{c_2} \cup B_{c_2}$, and G_{c_1} and G_{c_2} are δ-crossing. By Observation 5.2.4, we have $|X_{c_1} \cap Y_{c_2}| \geq \delta n/4$ whenever $X, Y \in \{A, B\}$. By the definition of characteristic partition, there are $Z, W \in \{A, B\}$ such that $d_{G_{c_1}}(x, Z_{c_1}) \geq |Z_{c_1}| - \varepsilon n$ and $d_{G_{c_2}}(y, W_{c_2}) \geq |W_{c_2}| - \varepsilon n$. Thus $|N_{G_{c_1}}(x) \cap N_{G_{c_2}}(y)| \geq |N_{G_{c_1}}(x, Z_{c_1}) \cap N_{G_{c_2}}(y, W_{c_2})| \geq |Z_{c_1} \cap W_{c_2}| - 2\varepsilon n \geq \delta n/5$. This completes the proof of the claim.

Thus there are at least $\delta n/5$ choices for each of $c_1, c_2, c_3, c_4, x, y, z$ given any previous choices. So we can obtain the desired $P_1 xyz P_2$. We can further repeat the process for each pair (P_j, P_{j+1}) for $j \in [s]$ where $P_{s+1} := P_1$. Each time we use four unused colours in $\overline{\mathscr{U}}$ and three unused ver-

tices, so for each choice of colour at most $4s \le 4t \le 8\kappa n < \delta n/10$ colours are forbidden and $3s < \delta n/10$ vertices are forbidden. Thus we obtain a cycle C of length $7s$ which contains every P_j with $j \in [s]$ as a segment. By construction, for every colour $i \in \overline{\mathscr{U}}$ and any G_i-good vertex v, for every vertex $u \in \overline{T}_i$, there are at least $2^{-16}\delta^4 t \ge 2^{-17}\kappa\delta^5 n$ disjoint i-absorbing paths of (v,u) inside C. Moreover, since at most $2\varepsilon n$ vertices are G_i-bad, the number of vertices in $\overline{T}_i$ which are G_i-good is at least $|\overline{T}_i| - 2\varepsilon n \ge (1-\sqrt{\varepsilon}/2)n - 2\varepsilon n \ge (1-\sqrt{\varepsilon})n$. Thus C is an absorbing cycle with parameters $(\delta/3, \sqrt{\varepsilon}, 14\kappa, 2^{-17}\kappa\delta^5)$ and hence with $(\delta/3, \sqrt{\varepsilon}, \lambda, \lambda^2)$. □

5.4 The stable case

In this section we combine the results of the previous sections and use the regularity-blow-up method to find a transversal Hamilton cycle when $\boldsymbol{G}$ is stable. First, we show that the reduced graph inherits stability.

Lemma 5.4.1. *Suppose that $0 < 1/n \ll 1/L_0 \ll \varepsilon_0 \ll d \ll \mu, \alpha \ll \gamma, \varepsilon \ll \delta < 1$. Let $\boldsymbol{G} = (G_1, \dots, G_n)$ be a graph collection on a common vertex set V of size n. Let $\boldsymbol{R} = \boldsymbol{R}(\varepsilon_0, 1, L_0)$ be the reduced graph collection of $\boldsymbol{G}$. If $\boldsymbol{G}$ is $(\gamma, \alpha, \varepsilon, \delta)$-stable, then $\boldsymbol{R}$ is $(\gamma/2, \alpha^2, \varepsilon, \delta/2)$-stable.*

Proof. We may assume that $1/n \ll 1/n_0$ where $n_0 = n_0(\varepsilon, 1, L_0)$ is the constant from Lemma 4.2.1 (the regularity lemma for graph collections). Write $[L]$ for the common vertex set of $\boldsymbol{R}$ and $[M]$ for the set of colour clusters where $L_0 \le L, M \le n_0$. Thus there is a partition $V_0, \dots, V_L$ of V and $\mathscr{C}_0, \dots, \mathscr{C}_M$ of $[n]$ and a graph collection $\boldsymbol{G}'$ satisfying (i)–(v) of Lemma 4.2.1 and for $(\{h,i\}, j) \in \binom{[L]}{2} \times [M]$, we have $hi \in R_j$ whenever $\boldsymbol{G}'_{hi,j}$ is (ε_0, d)-regular, where $\boldsymbol{G}'_{hi,j} := (G'_c[V_h, V_i] : c \in \mathscr{C}_j)$. We have $Lm \le n \le Mm + \varepsilon_0 n$ and $Mm \le n \le Lm + \varepsilon_0 n$ so $|L - M| \le \varepsilon_0 n/m \le \varepsilon_0 L/(1-\varepsilon_0)$. Thus we may assume that $M = L$ at the expense of assuming the slightly worse bound $|V_0| + |\mathscr{C}_0| \le 3\varepsilon_0 n$. Given $X \subseteq V(R) = [L]$, we write $\widehat{X} := \bigcup_{j \in X} V_j$. So $|\widehat{X}| = m|X|$.

Claim 5.4.2. *Let $\alpha' \gg d$. Suppose that $i \in [L]$ and there are sets $A, B \subseteq V(R) = [L]$ of size at least $(\frac{1}{2} - \alpha')L$ such that $e_{R_i}(A,B) \le \alpha' L^2$. Then, for all but at most $2\sqrt{\alpha'}m$ colours $c \in \mathscr{C}_i$, we have $e_{G_c}(\widehat{A}, \widehat{B}) \le \sqrt{\alpha'} n^2$.*

Proof of Claim. Let t be the number of colours $c \in \mathscr{C}_i$ such that $e_{G_c}(\widehat{A}, \widehat{B}) > \sqrt{\widetilde{\alpha}} n^2$. Since

$e(G_c) - e(G'_c) \leq (3d + \varepsilon_0)n^2$ for all $c \in [n]$, we have

$$\sqrt{\alpha'}n^2 t \leq \sum_{c \in \mathscr{C}_i} e_{G_c}(\widehat{A}, \widehat{B}) \leq \sum_{c \in \mathscr{C}_i} (e_{G'_c}(\widehat{A}, \widehat{B}) + (3d + \varepsilon_0)n^2) \leq |\mathscr{C}_i| e_{R_i}(A, B)m^2 + |\mathscr{C}_i|(3d + \varepsilon_0)n^2$$
$$\leq \alpha' L^2 m^3 + 4dmn^2 \leq 2\alpha' mn^2.$$

This implies $t \leq 2\sqrt{\alpha'}m$. □

Case 1: $\boldsymbol{G}$ is (γ, α)-strongly stable. We claim that $\boldsymbol{R}$ is $(\gamma/2, \alpha^2)$-strongly stable. Suppose for a contradiction that there exists a subset $I \subseteq [L]$ with $|I| \geq (1-\gamma/2)L$ such that R_i is not α^2-nice for any $i \in I$. The claim applied with $\alpha' = \alpha^2$ implies that there are at least $(1 - 2\alpha)m$ colours $c \in \mathscr{C}_i$ such that G_i is not α-nice, since $|\widehat{A}|, |\widehat{B}| \geq (\frac{1}{2} - \alpha^2)Lm > (\frac{1}{2} - \alpha)n$. Thus the number of colours $c \in [n]$ for which G_c is not α-nice is at least $(1 - 2\alpha)m(1 - \gamma/2)L > (1 - \gamma)n$ since $\alpha \ll \gamma$. This contradicts the (γ, α)-strong stability of $\boldsymbol{G}$.

Case 2: $\boldsymbol{G}$ is (ε, δ)-weakly stable. Suppose that $\boldsymbol{R}$ is not $(\gamma/2, \alpha^2)$-strongly stable. It suffices to show that $\boldsymbol{R}$ is $(\varepsilon, \delta/2)$-weakly stable. There is $I \subseteq [L]$ with $|I| \geq (1 - \gamma/2)L$ such that for all $i \in I$, R_i is not α^2-nice. Thus R_i is $\alpha^{2/3}$-extremal, and hence ε-extremal. Lemma 3.2.7 implies that $d_{R_i}(j) \geq (\frac{1}{2} - 2\mu)L$ for all but at most $d^{1/4}L$ vertices $j \in [L]$. Lemma 5.2.2 now implies that R_i has a characteristic partition (A'_i, B'_i, C'_i). Furthermore, there are $Z, W \in \{A, B\}$ such that $e_{R_i}(Z'_i, W'_i) \leq \alpha^{2/3}L^2$ (and $Z \neq W$ if R_i is $(\alpha^{2/3}, \bullet)$-extremal, while $Z = W$ if R_i is $(\alpha^{2/3}, \circ)$-extremal). We have $|\widehat{W'_i}| = |\widehat{Z'_i}| = |Z'_i|m = (\frac{1}{2} - \alpha^{2/3})Lm \geq (\frac{1}{2} - 2\alpha^{2/3})n$. By Claim 5.4.2 applied with (W'_i, Z'_i) and $\alpha' := \alpha^{1/3}$, there is $\mathscr{B}_i \subseteq \mathscr{C}_i$ with $|\mathscr{C}_i \setminus \mathscr{B}_i| \leq 2\alpha^{1/3}m$ such that for all $c \in \mathscr{B}_i$, we have $e_{G_c}(\widehat{W'_i}, \widehat{Z'_i}) \leq \alpha^{1/3}n^2$. It follows that every such G_c is $2\alpha^{1/3}$-extremal. Thus, recalling that (A_c, B_c, C_c) is the characteristic partition of G_c, there are $Z, Y, W \in \{A, B\}$ with $\{Z, Y\} = \{A, B\}$ such that $e_{G_c}(Z_c, W_c) \leq \alpha^{1/4}n^2$ and $e_{G_c}(Y_c, W_c) \geq (\frac{1}{4} - \alpha^{1/4})n^2$. But then we must have $|\widehat{Z'_i} \bigtriangleup Z_c|, |\widehat{Y'_i} \bigtriangleup Y_c| < \varepsilon n$ or $|\widehat{Z'_i} \bigtriangleup Y_c|, |\widehat{Y'_i} \bigtriangleup Z_c| < \varepsilon n$. So either $|\widehat{A'_i} \bigtriangleup A_c|, |\widehat{B'_i} \bigtriangleup B_c| < \varepsilon n$ or $|\widehat{A'_i} \bigtriangleup B_c|, |\widehat{B'_i} \bigtriangleup A_c| < \varepsilon n$. That is, for all $c \in \mathscr{B}_i$, the characteristic partition of G_c is almost the same as the union of clusters corresponding to the characteristic partition of R_i.

Suppose now that $c, c' \in [n]$ are such that G_c and $G_{c'}$ are ε-extremal and δ-crossing and $c \in \mathscr{B}_i$ and $c' \in \mathscr{B}_j$ for some $i, j \in [L]$. We must have $i \neq j$. Then, if $|\widehat{A'_i} \bigtriangleup A_c|, |\widehat{A'_j} \bigtriangleup A_{c'}| < \varepsilon n$, we have $|A'_i \bigtriangleup A'_j|m = |\widehat{A'_i} \bigtriangleup \widehat{A'_j}| \geq |A_c \bigtriangleup A_{c'}| - |\widehat{A'_i} \bigtriangleup A_c| - |\widehat{A'_j} \bigtriangleup A_{c'}| \geq \delta n - 2\varepsilon n \geq \delta n/2 \geq$

$\delta mL/2$. The other cases are almost identical. Thus R_i and R_j are $\delta/2$-crossing. Since $\boldsymbol{G}$ is (ε, δ)-weakly stable, we have $e(C^{\varepsilon,\delta}_{\boldsymbol{G}}) \geq \delta n^2$. The number of pairs $i, j \in [L]$ with c, c' as above is therefore at least $\delta n^2/m^2 \geq \delta L^2/2$. Thus $e(C^{\varepsilon,\delta/2}_{\boldsymbol{R}}) \geq \delta L^2/2$, and hence $\boldsymbol{R}$ is $(\varepsilon, \delta/2)$-weakly stable. □

Lemma 5.4.3. *Let* $0 < 1/n \ll \mu \ll \alpha \ll \gamma, \varepsilon \ll \delta \ll 1$. *Suppose that* $\boldsymbol{G} = (G_1, \dots, G_n)$ *is a graph collection on a common vertex set* V *of size* n *and* $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$. *If* $\boldsymbol{G}$ *is* $(\gamma, \alpha, \varepsilon, \delta)$*-stable, then* $\boldsymbol{G}$ *contains a transversal Hamilton cycle.*

Proof. Choose additional parameters $n_0, L_0, \varepsilon_0, d, \theta, \beta, \lambda$ where $n_0 = n_0(\varepsilon, 1, L_0)$ is obtained from Lemma 4.2.1 and so that

$$0 < 1/n \ll 1/n_0 \ll 1/L_0 \ll \varepsilon_0 \ll d \ll \mu \ll \theta \ll \beta \ll \lambda \ll \alpha \ll \gamma, \varepsilon \ll \delta \ll 1,$$

where the previous lemmas in this section hold with suitable parameters. For any colour c and two vertices $(x, y) \in V^2$, we say the triple (c, x, y) is *absorbable* if there are at least $\lambda^2 n$ disjoint c-absorbing paths of (x, y) inside C. Similarly, we say the pair (c, x) is *absorbable* if there are at least $\lambda^2 n$ disjoint c-absorbing paths of (x, x) inside C. Let $c, c' \in [n]$ be two colours and $x, y \in V$ be two vertices. We say (c, c', x, y) is *totally absorbable* if (c, x), (c', y) and (c, x, y) are all absorbable. By Lemma 5.3.3 and Lemma 5.3.4, $\boldsymbol{G}$ has an absorbing cycle C with parameters $(\delta/3, \sqrt{\varepsilon}, \lambda, \lambda^2)$. By definition, C has length at most λn, and there exists a colour set $\mathscr{C} \subseteq [n]$ with $\mathscr{C} \cap \mathrm{col}(C) = \emptyset$ of size at least $\delta n/3$ such that

(i) given any colour $c \in \mathscr{C}$ and any G_c-good vertex v, the triple (c, v, u) is absorbable for all but at most $\sqrt{\varepsilon} n$ vertices u.

(ii) given any colour $c \in \mathscr{C}$, for all but at most $\sqrt{\varepsilon} n$ G_c-good vertices v, the pair (c, v) is absorbable.

Claim 5.4.4. *There is* $(2 - 2^{-10})\beta n \leq r \leq 2\beta n$ *and for each* $i \in [r]$*, disjoint vertex pairs* $(v_i, v_i') \in (V)_2$ *and disjoint colour pairs* $(c_i, c_i') \in (\mathscr{C})_2$ *such that the family* $\mathcal{Q} := \{(c_i, c_i', v_i, v_i') : i \in [r]\}$ *has the following properties:*

(a) (c_i, c_i', v_i, v_i') *is totally absorbable for all* $i \in [r]$*;*

(b) for every pair $(u_1, u_2) \in V^2$ *and* $c \in [n]$*, there are at least* $2^{-9}\beta n$ *values* $i \in [r]$ *such that* $c \in L(u_1 v_i)$ *and* $c_i' \in L(u_2 v_i')$.

Proof of Claim. For every $(b_1, b_2) \in (\mathscr{C})_2$, every $(u_1, u_2) \in V^2$ and $c \in \mathscr{C}$, Let $S(b_1, b_2, u_1, u_2, c)$ be the multiset of pairs $(v_1, v_2) \in (V)_2$ such that $c \in L(u_1v_1)$, $b_2 \in L(u_2v_2)$ and (b_1, b_2, v_1, v_2) is totally absorbable. We will show that $|S(b_1, b_2, u_1, u_2, c)| \geq 2^{-4}n^2$.

For this, we first count the number of choices for v_1. Note that we have $d_{G_c}(u_1) \geq (\frac{1}{2} - \mu)n$ and the number of vertices v_1 such that v_1 is not G_{b_1}-good or (b_1, v_1) is not absorbable is at most $2\varepsilon n + \sqrt{\varepsilon}n$. Thus we have at least $n/4$ choices of v_1 such that $v_1 \in N_{G_c}(u_1)$, v_1 is G_{b_1}-good and (b_1, v_1) is absorbable. Now we fix v_1. Let $N_2 := \{x \in N_{G_{b_2}}(u_2) : x \text{ is } G_{b_2}\text{-good}\}$. So $|N_2| \geq (\frac{1}{2} - \mu)n - 2\varepsilon n$. By (i), for all but at most $\sqrt{\varepsilon}n$ vertices $v_2 \in V$, the pair (v_1, v_2) has at least $\lambda^2 n$ b_1-absorbing paths inside C. Thus, we can delete at most $\sqrt{\varepsilon}n + 1$ vertices from N_2 and obtain a set N_2' such that $|N_2'| \geq n/4$, and for each $v_2 \in N_2'$, $v_2 \neq v_1$ and (b_1, b_2, v_1, v_2) is totally absorbable. Therefore, the total number of choices for (v_1, v_2) is at least $2^{-4}n^2$.

Let $\{c_i, c_i' : i \in [2\beta n]\} \subseteq \mathscr{C}$ be a collection of distinct colours. Set $\boldsymbol{H} := (\mathcal{F}_i := \{(v_1, v_2) \in (V)_2 : (c_i, c_i', v_1, v_2) \text{ is totally absorbable}\} : i \in [2\beta n])$ and define the collection $\boldsymbol{Z} := \{S(u_1, u_2, c) := \bigcup_{i \in [2\beta n]} S(c_i, c_i', u_1, u_2, c) : (u_1, u_2) \in V^2, c \in [n]\}$ of multi-2-graphs. For every $S = S(u_1, u_2, c) \in \boldsymbol{Z}$ and every index $i \in [2\beta n]$, we have $|E(S) \cap E(\mathcal{F}_i)| \geq 2^{-4}n^2$. Lemma 5.3.1 applied with $2\beta n, 2^{-4}$ playing the roles of t, ε, implies that there is a transversal matching M in $\boldsymbol{H}$ of size at least $(2 - 2^{-10})\beta n$ (and at most $2\beta n$) and $|E(S) \cap E(M)| \geq 2^{-9}\beta n$ for all $S \in \boldsymbol{Z}$. □

For the remainder of the proof, we will cover most of the unused vertices and colours by a small number of long paths. This step will use the regularity-blow-up method for transversals. Finally, we will use the absorbing cycle C and family $\mathcal{Q}$ to absorb the remaining colours and vertices and connect the long paths.

Let

$$V_{\text{abs}} := \bigcup_{i \in [r]} \{v_i, v_i'\}, \quad \mathscr{C}_{\text{abs}} := \bigcup_{i \in [r]} \{c_i, c_i'\}, \quad \mathscr{C}_{\text{rem}} := [n] \setminus (\text{col}(C) \cup \mathscr{C}_{\text{abs}}),$$
$$U := V \setminus (V(C) \cup V_{\text{abs}}) \quad \text{and} \quad \boldsymbol{J} = (J_i : i \in \mathscr{C}_{\text{rem}}) \text{ where } J_i := G_i[U].$$

So $|[n] \setminus \mathscr{C}_{\text{rem}}| = |V \setminus U| \leq \lambda n + 2\beta n \leq 2\lambda n$.

Since $\lambda \ll \gamma, \alpha, \varepsilon, \delta$, it is easy to see that since $\boldsymbol{G}$ is $(\gamma, \alpha, \varepsilon, \delta)$-stable, $\boldsymbol{J}$ is $(\gamma/2, \alpha/2, 2\varepsilon, \delta/2)$-stable, and $\delta(J_i) \geq (\frac{1}{2} - \mu - 2\lambda)n \geq (\frac{1}{2} - 3\lambda)n$ for $i \in \mathscr{C}_{\text{rem}}$. Apply Lemma 4.2.1 (the regularity lemma for graph collections) to $\boldsymbol{J}$ with parameters $(\varepsilon_0, 1, d, L_0)$. Let $\boldsymbol{R}$ be the reduced graph

of $\boldsymbol{J}$. Write $[L]$ for the common vertex set of $\boldsymbol{R}$ where $L_0 \leq L \leq n_0$, and $[M]$ for the set of colour clusters. Thus there is a partition $V_0, \dots, V_L$ of V and $\mathscr{C}_0, \dots, \mathscr{C}_M$ of $[n]$ and a graph collection $\boldsymbol{J}'$ satisfying (i)–(v) of Lemma 4.2.1. Therefore, for $(\{h,i\}, j) \in \binom{[L]}{2} \times [M]$, we have $hi \in R_j$ whenever $\boldsymbol{J}'_{hi,j}$ is (ε_0, d)-regular, where $\boldsymbol{J}'_{hi,j} := (J'_c[V_h, V_i] : c \in \mathscr{C}_j)$. We have $Lm \leq n \leq Mm + \varepsilon_0 n$ and $Mm \leq n \leq Lm + \varepsilon_0 n$ so $|L - M| \leq \varepsilon_0 n/m \leq \varepsilon_0 L/(1-\varepsilon_0)$. Thus we may assume that $M = L$ at the expense of assuming the slightly worse bound $|V_0| + |\mathscr{C}_0| \leq 3\varepsilon_0 n$. Lemma 3.2.7 implies that for each vertex $i \in [L]$, there are at least $(1 - d^{1/4})L$ colours $j \in [L]$ such that $d_{R_j}(i) \geq (\frac{1}{2} - 4\lambda)L$. By Lemma 5.4.1, $\boldsymbol{R}$ is $(\gamma/4, \alpha^2/4, 2\varepsilon, \delta/4)$-stable. Lemma 5.2.7 implies that there exist two edge-disjoint transversal matchings M_1 and M_2 in $\boldsymbol{R}$ such that $e(M_\ell) \geq (\frac{1}{2} - \theta)L$ for $\ell = 1, 2$ and $\mathrm{col}(M_1) \cap \mathrm{col}(M_2) = \emptyset$.

Now for each vertex $i \in [L]$, let $V_i = V_i^1 \cup V_i^2$ be an arbitrary equipartition. Lemma 3.2.4(i) implies that whenever $hi \in E(R_j)$ where $j \in [L]$, we have that $\boldsymbol{J}^\ell_{hi,j} = (J'_c[V_h^\ell, V_i^\ell] : c \in \mathscr{C}_j)$ is $(2\varepsilon_0, d/2)$-regular for both $\ell = 1, 2$. By Lemma 3.2.3, for each $\ell = 1, 2$ and $hi \in E(M_\ell)$, we can remove at most $2\varepsilon_0 m$ vertices in V_h^ℓ and at most $2\varepsilon_0 m$ colours $c \in \mathscr{C}_j$, where j is the colour of hi in M_ℓ, so that the remaining graph collection is $(4\varepsilon_0, d/4)$-superregular.

Apply Theorem 3.3.5 (the transversal blow-up lemma) to obtain a transversal path $P_{hi,j}$ with $2\min\{|V_h^\ell|, |V_i^\ell|\}$ vertices using colours from $\mathscr{C}_j$. (Note that we could have avoided using the blow-up lemma and instead used a tool for embedding an almost spanning structure inside a regular pair (as opposed to superregular), but it was convenient to follow the above approach.) The collection of $P_{hi,j}$ over $hi \in E(M_1) \cup E(M_2)$ (where hi has colour j in its matching) is vertex-disjoint, since the subcluster V_h^ℓ is used $d_{M_\ell}(h) \leq 1$ times, and transversal since $M_1 \cup M_2$ is transversal. The number of vertices not in any $P_{hi,j}$ is at most $4\varepsilon_0 mL + 3\varepsilon_0 n + 2(L - e(M_1) - e(M_2))m \leq 3\theta n$. Thus the number of colours not in any $P_{hi,j}$ is at most $3\theta n + e(M_1) + e(M_2) \leq 4\theta n$. We consider each vertex in U but not in any $P_{hi,j}$ to be a path (of length 0).

Now relabel so that the paths are $P_1, \dots, P_s$, so $s \leq e(M_1) + e(M_2) + 3\theta n \leq 4\theta n$, and let x_i, y_i be the startvertex and endvertex of P_i for each $i \in [s]$ (so $x_i = y_i$ if P_i has length 0). The paths $P_1, \dots, P_s$ are transversal and pairwise vertex-disjoint, and cover U. Thus the number of colours in $\mathscr{C}_{\mathrm{rem}}$ which are not used on any P_i is precisely s.

Do the following for each $i = 1, \dots, s$ in turn. Let a_i be an arbitrary unused colour, in $\mathscr{C}_{\mathrm{rem}}$. Choose an unused 4-tuple $Q_i = (c_{j_i}, c'_{j_i}, v_{j_i}, v'_{j_i}) \in \mathcal{Q}$ where $j_i \in [r]$, $a_i \in L(x_i v_{j_i})$ and

$c'_{j_i} \in L(y_i v'_{j_i})$. This is possible since Claim 5.4.4 implies there are at least $2^{-9}\beta n$ choices for Q_i, of which at most $s \leq 4\theta n$ have been used. Now, $(c_{j_i}, c'_{j_i}, v_{j_i}, v'_{j_i})$ is totally absorbable, so there are at least $\lambda^2 n$ disjoint c_{j_i}-absorbing paths of (v_{j_i}, v'_{j_i}) inside C. So we can choose one of them, $S_i := x_1^i x_2^i x_3^i x_4^i$, whose vertices have not been previously chosen since $s/n \ll \lambda$, and whose colours are, in order, b_1^i, b_2^i, b_3^i.

At the end of this process, there remains $I \subseteq [r]$ such that the $(c_j, c'_j, v_j, v'_j) \in \mathcal{Q}$ with $j \in I$ are precisely the 4-tuples which were not chosen to be some Q_i. For each one, there are at least $\lambda^2 n$ disjoint c_j-absorbing paths of (v_j, v_j) and disjoint c'_j-absorbing paths of (v'_j, v'_j) inside C. So, since $\beta \ll \lambda$, we can choose one such path T_i, T'_i for each of (v_j, c_j) and (v'_j, c'_j), which are vertex-disjoint and whose vertices have not previously been chosen.

At the end of this process, we have a collection $\{S_i : i \in [s]\}, \{T_i : i \in I\}, \{T'_i : i \in I\}$ of vertex-disjoint paths in C where, for each $i \in [s]$, S_i is a c_{j_i}-absorbing path of (v_{j_i}, v'_{j_i}); for each $i \in I$, T_i is a c_i-absorbing path of (v_i, v_i) and T'_i is a c'_i-absorbing path of (v'_i, v'_i). For each $i \in [s]$, we replace S_i by $x_1^i x_2^i v_{j_i} x_i P_i y_i v'_{j_i} x_3^i x_4^i$ with colours b_1^i, c_{j_i}, a_i, followed by the colours inherited from P_i, followed by c'_{j_i}, b_2^i, b_3^i. That is, we have replaced S_i by a path with the same endpoints, vertices $V(S_i) \cup V(P_i) \cup \{v_{j_i}, v'_{j_i}\}$ and colours $\mathrm{col}(S_i) \cup \mathrm{col}(P_i) \cup \{c_{j_i}, c'_{j_i}, a_i\}$. For each $i \in I$, we replace $T_i = y_1^i y_2^i y_3^i y_4^i$ by $y_1^i y_2^i v_i y_3^i y_4^i$ where colours are inherited except $\mathrm{col}(y_2^i v_i) = c_i$ and $\mathrm{col}(v_i y_3^i) = \mathrm{col}(y_2^i y_3^i)$. We do a similar replacement of T'_i, using new vertex v'_i and new colour c'_i. So we have replaced T_i by a path with the same endpoints, vertices $V(T_i) \cup \{v_i\}$ and colours $\mathrm{col}(T_i) \cup \{c_i\}$, and T'_i by a path with the same endpoints, vertices $V(T'_i) \cup \{v'_i\}$ and colours $\mathrm{col}(T'_i) \cup \{c'_i\}$. Thus we have obtained a cycle using vertices $V(C) \cup \bigcup_{i\in[s]} V(P_i) \cup \{v_i, v'_i : i \in [r]\} = [n]$ and colours $\mathrm{col}(C) \cup \bigcup_{i\in[s]} \mathrm{col}(P_i) \cup \{c_i, c'_i : i \in [r]\} = [n]$ where each colour is used at most once (and hence exactly once). That is, we have constructed a transversal Hamilton cycle. □

5.5 The extremal case

This section concerns the remaining case when $\boldsymbol{G}$ is not stable; so most graphs in the collection are close to containing one of the two extremal graphs  or , and moreover their characteristic partitions are similar. Throughout, we assume the following hypothesis:

(†) Suppose that

$$0 < 1/n \ll \varepsilon \ll \delta \ll \eta \ll 1.$$

Let $m = (1-3\delta)n$. Let $\boldsymbol{G} = (G_1, \dots, G_n)$ be a graph collection on a common vertex set V of size n and $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$. Suppose that for each $i \in [m]$, G_i is ε-extremal with characteristic partition (A_i, B_i, C_i) and $|A_1 \bigtriangleup A_i|, |B_1 \bigtriangleup B_i| \leq \delta n$.

We also use the following notation:

$$\mathscr{C}(H) := \{i \in [m] : G_i \text{ is } (\varepsilon, H)\text{-extremal}\} \quad \text{and} \quad \boldsymbol{G}(H) := (G_i : i \in \mathscr{C}(H)) \quad \text{for } H \in \{\boxdot, \boxminus\},$$

$$V_{\text{bad}} := C_1 \cup V^A_{\text{bad}} \cup V^B_{\text{bad}} \quad \text{where} \quad V^Z_{\text{bad}} := \{x \in Z_1 : x \notin Z_i \text{ for at least } \sqrt{\delta}n \text{ colours } i \in [m]\},$$

$$\mathscr{C}_{\text{bad}} := [m+1, n].$$

Now, $\sqrt{\delta}n|V^Z_{\text{bad}}| \leq \sum_{i\in[m]} |Z_1 \setminus Z_i| \leq m\delta n$ so

$$|V_{\text{bad}}| \leq 2\varepsilon n + 2\sqrt{\delta}m \leq 3\sqrt{\delta}n.$$

It is a consequence of Lemma 5.2.2 that $\mathscr{C}(\boxdot) \cap \mathscr{C}(\boxminus) = \emptyset$. The first result in this section is an application of the transversal blow-up lemma for embedding transversal Hamilton paths inside very dense bipartite graph collections. We note that this application is more for convenience than necessity.

Lemma 5.5.1. *Suppose that (†) holds. Let $W, Z \in \{A, B\}$. Let $W^* \subseteq W_1 \setminus V_{\text{bad}}$ and $Z^* \subseteq Z_1 \setminus V_{\text{bad}}$ where $|W^*|, |Z^*| \geq \eta n$ and $W^* \cap Z^* = \emptyset$ and $|W^*| - |Z^*| =: t \in \{0, 1\}$. Let $T := Z$ if $t = 0$ and $T := W$ if $t = 1$. Let $\mathscr{C} \subseteq [n]$ satisfy $|\mathscr{C}| = |W^*| + |Z^*| - 1$, where $\mathscr{C} \subseteq \mathscr{C}(\boxdot)$ if $W = Z$ and $\mathscr{C} \subseteq \mathscr{C}(\boxminus)$ if $W \neq Z$. Let $W^- \subseteq W^*$ and $T^+ \subseteq T^*$ with $|W^-|, |T^+| \geq \eta n$. Then there is a transversal Hamilton path in $\{G_i[W^*, Z^*] : i \in \mathscr{C}\}$ starting in W^- and ending in T^+.*

Proof. Choose a new constant ξ with $\eta \ll \xi \ll 1$. We will show that $\{G_i[W^*, Z^*] : i \in \mathscr{C}\}$ is $(\xi, 1-\xi)$-superregular.

Suppose that $W \neq Z$, so $\{W^*, Z^*\} = \{A^*, B^*\}$ and $\mathscr{C} \subseteq \mathscr{C}(\boxminus)$. For every $i \in \mathscr{C}$, we have $e_{G_i}(W_i, Z_i) \geq |W_i||Z_i| - \varepsilon n^2$ and hence $e_{G_i}(W^*, Z^*) > (1-\xi)|W^*||Z^*|$. Now let $x \in W^*$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| \geq \xi|\mathscr{C}|$ and $Z' \subseteq Z^*$ with $|Z'| \geq \xi|Z^*|$. Then $x \notin V_{\text{bad}}$, so $x \in W_i$ for all but at most $\sqrt{\delta}n$ colours in $i \in [m]$. For each such i we have $d_{G_i}(x, Z_i) \geq (\frac{1}{2} - 2\varepsilon)n = |Z_i| - \varepsilon n$ by

Lemma 5.2.2. Thus $d_{G_i}(x, Z') \geq |Z'| - \varepsilon n - |Z_1 \bigtriangleup Z_i| \geq |Z'| - 2\delta n$. Thus $\sum_{i\in\mathscr{C}'} d_{G_i}(x, Z') \geq (|\mathscr{C}'| - 2\sqrt{\delta}n)(|Z'| - 2\delta n) > (1-\xi)|\mathscr{C}'||Z'|$. An analogous statement holds for vertices in Z^*. Thus $\sum_{i\in\mathscr{C}} d_{G_i}(x, Z^*) \geq (1-\xi)|\mathscr{C}||Z^*|$ for all $x \in W^*$ and $\sum_{i\in\mathscr{C}} d_{G_i}(y, W^*) \geq (1-\xi)|\mathscr{C}||W^*|$ for all $y \in Z^*$ and $(1-\xi)|\mathscr{C}'||W'||Z'| \leq \sum_{i\in\mathscr{C}'} e_{G_i}(W', Z') \leq |\mathscr{C}'||W'||Z'|$. Thus $(G_i[W^*, Z^*] : i \in \mathscr{C})$ is $(\xi, 1-\xi)$-superregular.

Suppose instead that $W = Z$, so $\mathscr{C} \subseteq \mathscr{C}(\circledcirc)$. A very similar argument shows that, since every $G_i[Z^*]$ and hence $G_i[W^*, Z^*]$ is almost complete, the same conclusion holds here.

Let $s := |W^*| + |Z^*|$. Since $|W^*| - |Z^*| = t \in \{0, 1\}$, there is an s-vertex path P_s which is bipartite with parts A_W, A_Z of size $|W^*|, |Z^*|$ respectively, with $|\mathscr{C}| = |W^*| + |Z^*| - 1$ edges and whose first vertex lies in A_W and whose last lies in A_T. Thus we can apply Theorem 3.3.5 (the transversal blow-up lemma) with target sets W^-, T^+ for the first and last vertex of P_s respectively to obtain the required transversal Hamilton path. □

The next lemma shows that whenever there are many $(\varepsilon, \circledcirc)$-extremal graphs, we can find a short transversal path which covers bad vertices and colours.

Lemma 5.5.2. *Suppose that (†) holds, and $|\mathscr{C}(\circledcirc)| \geq \eta n$. Given any $\mathscr{F} \subseteq [n]$ with $|\mathscr{F}| \leq 1$, there is a transversal path P in $\boldsymbol{G}$ with endpoints x, y such that the following holds:*

- *(i)* $V_{\text{bad}} \subseteq V(P)$ *and* $\mathscr{C}_{\text{bad}} \setminus \mathscr{F} \subseteq \text{col}(P)$;
- *(ii)* $|V(P)| \leq 19\sqrt{\delta}n$;
- *(iii)* $x, y \notin V_{\text{bad}}$ *and there are distinct* $c, c' \in \mathscr{C}(\circledcirc) \setminus \text{col}(P)$ *such that* $x \in A_c \cap A_1$ *and* $y \in B_{c'} \cap B_1$;
- *(iv)* $\mathscr{F} \cap (\text{col}(P) \cup \{c, c'\}) = \emptyset$.

Proof. Since $|V_{\text{bad}}| \leq 3\sqrt{\delta}n$ and $|\mathscr{C}_{\text{bad}}| \leq 3\delta n$, by adding vertices to V_{bad} if necessary we may assume that $|\mathscr{C}_{\text{bad}}| < r := |V_{\text{bad}}| \leq 3\sqrt{\delta}n$. Let $A := A_1 \setminus V_{\text{bad}}$ and $B := B_1 \setminus V_{\text{bad}}$.

We will find a transversal family $\mathcal{P} = \{y_i x_i y_i' : i \in [r]\}$ of vertex-disjoint 3-vertex paths, with colour set $\{c_i, c_i' : i \in [r]\} \supseteq \mathscr{C}_{\text{bad}}$ where $V_{\text{bad}} = \{x_1, \dots, x_r\}$ and $\{y_i', y_i' : i \in [r]\} \subseteq A \cup B$. For this, let $x \in V_{\text{bad}}$ and let $c, c' \in [m]$ be distinct colours. Since $d_{G_c}(x) \geq (\frac{1}{2} - \mu)n$, there are $Z, Z' \in \{A, B\}$ such that $|N_{G_c}(x) \cap Z| \geq n/8$ and $N_{G_{c'}}(x) \cap Z'| \geq n/8$. Choose $y \in N_{G_c}(x) \cap Z$ and $y' \in N_{G_{c'}}(x) \cap Z'$. This gives the path yxy' using the given colours c, c'. We find such paths for every $x \in V_{\text{bad}}$, each time using unused vertices, which is possible since there are at least $n/8$ choices for each vertex and only $2r \leq 6\sqrt{\delta}n$ are used in total, and using

all of the $3\delta n$ colours of $\mathscr{C}_{\mathrm{bad}}$ among the $2r$ colours used in total.

Next, we connect the paths of $\mathcal{P}$ into a single short transversal path P. We start with two arbitrary paths P_1, P_2 in $\mathcal{P}$, with endpoints y_1, y_1' and y_2, y_2' respectively. Note that each of these endpoints is not in V_{bad} and is therefore G_i-good for at least $|\mathscr{C}(\beth)| - \sqrt{\delta}n - 2r \geq \eta n/2$ unused colours $i \in \mathscr{C}(\beth)$. Thus we can choose distinct unused colours $j_1, j_2 \in \mathscr{C}(\beth)$ such that y_1' is G_{j_1}-good and y_2 is G_{j_2}-good. Without loss of generality, we have the following two cases:

Case 1: $y_1' \in A_{G_{j_1}}$ and $y_2 \in A_{G_{j_2}}$. Choose a common neighbour $y \in N_{G_{j_1}}(y_1') \cap N_{G_{j_2}}(y_2) \cap B$ that avoids P_1, P_2. There are at least $n/3$ choices for y since y_1' is missing at most εn neighbours in B_{j_1} in the graph G_{j_1} and $|B_{j_1} \setminus B| \leq |V_{\mathrm{bad}}| + |B_{j_1} \bigtriangleup B_1| \leq 4\sqrt{\delta}n$, and similarly for y_2 and G_{j_2}. Let $P_{12} := P_1 y_1' y y_2 P_2$ with colour set $\mathrm{col}(P) = \mathrm{col}(P_1) \cup \{j_1, j_2\} \cup \mathrm{col}(P_2)$.

Case 2: $y_1' \in A_{G_{j_1}}$ and $y_2 \in B_{G_{j_2}}$. Choose an unused colour $j_3 \in \mathscr{C}(\beth)$, for which there are at least $\eta n/2$ choices. Choose a vertex $y \in N_{G_{j_1}}(y_1') \cap B_{G_{j_3}}$ that avoids P_1, P_2; there are at least $n/3$ choices. Choose a common neighbour $z \in N_{G_{j_3}}(y) \cap N_{G_{j_2}}(y_2) \cap A$ that is distinct from y_1' and avoids P_1, P_2; there are at least $n/3$ choices. Let $P_{12} := P_1 y_1' y z y_2 P_2$ with colour set $\mathrm{col}(P) = \mathrm{col}(P_1) \cup \{j_1, j_2, j_3\} \cup \mathrm{col}(P_2)$.

By considering P_{12}, P_3 and so on, we can find a transversal path $P_{12\ldots r}$ that includes every path in $\mathcal{P}$. For this, we need to argue that we can always choose unused vertices and colours; this is indeed true since in the first step there are at least $n/3$ choices for any vertex and at least $\eta n/2$ choices for any colour, and at most $3r < \eta n/4$ vertices and colours are used in total. We have $|V(P_{12\ldots r})| \leq 6r$.

Using very similar arguments, we can extend the path by at most one vertex and colour in $\mathscr{C}(\beth)$ so that its endpoints are in A and B. Since they do not lie in V_{bad}, there are many choices of unused colour for c, c' to satisfy (iii). The final path has length at most $6r + 1 \leq 19\sqrt{\delta}n$. To achieve (iv), we simply remove $\mathscr{F}$ from $\mathscr{C}_{\mathrm{bad}}$ and choose not to use $\mathscr{F}$ at any step, which does not affect any of the above estimates. □

The next lemma shows that some additional assumptions guarantee a short transversal path that not only covers bad vertices and colours, but also balances A_1 and B_1. These assumptions are that almost every graph is $(\varepsilon, \beth)$-extremal, and moreover both parts A_1, B_1 contain many internal edges from these graphs.

Lemma 5.5.3. *Suppose that (†) holds, and*

$$|\mathscr{C}(\overset{\bullet}{\bullet})| \le \eta n \quad \textit{and} \quad e_{\boldsymbol{G}(\overset{\circ}{\circ})}(A_1) \ge 30\eta n^3 \quad \textit{and} \quad e_{\boldsymbol{G}(\overset{\circ}{\circ})}(B_1) \ge 30\eta n^3.$$

Then $\boldsymbol{G}$ contains a transversal path P with endpoints x, y such that the following holds:

- *(i)* $V_{\mathrm{bad}} \subseteq V(P)$ *and* $\mathscr{C}_{\mathrm{bad}} \cup \mathscr{C}(\overset{\bullet}{\bullet}) \subseteq \mathrm{col}(P)$*;*
- *(ii)* $|V(P)| \le 4\eta n$*;*
- *(iii)* $|A_1 \setminus V(P)| = |B_1 \setminus V(P)|$*;*
- *(iv)* $x, y \notin V_{\mathrm{bad}}$ *and there are distinct* $c, c' \in \mathscr{C}(\overset{\circ}{\circ}) \setminus \mathrm{col}(P)$ *such that* $x \in A_c \cap A_1$ *and* $y \in B_{c'} \cap B_1$.

Proof. We have that $|\mathscr{C}(\overset{\circ}{\circ})| = m - |\mathscr{C}(\overset{\bullet}{\bullet})| \ge (1-2\eta)n$. Thus Lemma 5.5.2 applied with $\mathscr{F} := \emptyset$ implies that $\boldsymbol{G}$ contains a transversal path $\widetilde{P}$ with endpoints $\widetilde{x}, \widetilde{y}$ such that $V_{\mathrm{bad}} \subseteq V(\widetilde{P})$, $\mathscr{C}_{\mathrm{bad}} \subseteq \mathrm{col}(\widetilde{P})$, $|V(\widetilde{P})| \le 19\sqrt{\delta}n$, and $\widetilde{x}, \widetilde{y} \notin V_{\mathrm{bad}}$ and $\widetilde{x} \in A_1$ and $\widetilde{y} \in B_1 \cap B_c$ for some $c \in \mathscr{C}(\overset{\circ}{\circ}) \setminus \mathrm{col}(\widetilde{P})$.

Without loss of generality, suppose that $|A_1 \setminus V(\widetilde{P})| - |B_1 \setminus V(\widetilde{P})| := \widetilde{t} \ge 0$. Next we greedily extend $\widetilde{P}$ to a path whose colour set contains $\mathscr{C}(\overset{\bullet}{\bullet})$. Write $\{a_1, \dots, a_r\}$ for the collection of unused colours in $\mathscr{C}(\overset{\bullet}{\bullet})$ and let a_0, a_{r+1} be distinct unused colours in $\mathscr{C}(\overset{\circ}{\circ})$, for which $\widetilde{x} \in A_{a_0}$, of which there are at least $\eta n/2$ choices because $\widetilde{x} \notin V_{\mathrm{bad}}$. Choose an unused vertex $y_0 \in N_{G_{a_0}}(\widetilde{x}, B_1 \cap B_{a_1})$. There are at least $n/3$ choices since $|B_{a_0} \bigtriangleup (B_1 \cap B_{a_1})| \le 2\delta n$ by (†) and $d_{G_{a_0}}(\widetilde{x}, B_{a_0}) \ge (\frac{1}{2} - 2\varepsilon)n = |B_{a_0}| - \varepsilon n$ because $a_0 \in \mathscr{C}(\overset{\circ}{\circ})$ and $\widetilde{x} \in A_{a_0}$. Now, given $i \in [r]$ and $y_{i-1} \in B_{a_i}$, we can choose an unused vertex $y_i \in N_{G_{a_i}}(y, B_1 \cap B_{a_{i+1}})$; again there are at least $n/3$ choices since $\{a_i, a_{i+1}\} \in [m]$ and (†) implies $|B_{a_i} \bigtriangleup (B_1 \cap B_{a_{i+1}})| \le 2\delta n$, and $a_i \in \mathscr{C}(\overset{\bullet}{\bullet})$ implies $d_{G_{a_i}}(y, B_{a_i}) \ge (\frac{1}{2} - 2\varepsilon)n = |B_{a_i}| - \varepsilon n$. Thus we can obtain the transversal path $\widetilde{x} y_0 y_1 \dots y_r$ using colours $a_0, a_1, \dots, a_r$, and with $y_r \in B_1 \cap B_{a_{r+1}}$.

Let $P_1 = \widetilde{y}\widetilde{P}\widetilde{x} y_0 y_1 y_1 \dots y_r$ be the final transversal path obtained by concatenation with $\widetilde{P}$ and let $A := A_1 \setminus V(P_1)$ and $B := B_1 \setminus V(P_1)$. It satisfies (i) and has $|V(P_1)| \le |V(\widetilde{P})| + |\mathscr{C}(\overset{\bullet}{\bullet})| + 1 \le 2\eta n$, and also $||A| - |B|| \le 2\eta n$. It has endpoints $\widetilde{y} \in B_1 \cap B_c$ and $\widetilde{z} := y_r \in B_b$ where $b := a_{r+1} \in \mathscr{C}(\overset{\circ}{\circ})$. Also, $|A_1 \setminus V(P_1)| - |B_1 \setminus V(P_1)| := t \ge \widetilde{t} + 1 \ge 1$. After removing used colours, we have $\sum_{i \in \mathscr{C}(\overset{\circ}{\circ})} e_{G_i}(A) \ge 20\eta n^3$ and $\sum_{i \in \mathscr{C}(\overset{\circ}{\circ})} e_{G_i}(B) \ge 20\eta n^3$ since we consumed at most $2\eta n$ vertices from $A_1 \cup B_1$ and at most $2\eta n$ colours from $\mathscr{C}(\overset{\circ}{\circ})$ in building P_1. For each

vertex pair $uv \in \binom{V}{2}$, define

$$c(uv) := \{i \in \mathscr{C}(\circledcirc) : uv \in E(G_i)\} \quad \text{and} \quad D := \{uv : u, v \in A \text{ and } c(uv) \geq 10\eta n\}.$$

Then we have $20\eta n^3 \leq |D|n + n^2 \cdot 10\eta n$ and thus $|D| \geq 10\eta n^2$. It follows that D contains a subgraph D' with minimum degree at least $10\eta n$.

Let $z \in N_{G_b}(\widetilde{z}, V(D'))$ be an unused vertex. Such a z exists since $b \in \mathscr{C}(\circledcirc)$ and $\widetilde{z} \in B_b$ imply that $d_{G_b}(\widetilde{z}, A_b) \geq (\frac{1}{2} - 2\varepsilon)n$ and $|V(D') \cap A_b| \geq |V(D')| - |A \bigtriangleup A_b| \geq 10\eta n - 2\eta n - \delta n \geq 7\eta n$ which is larger than the $2\eta n$ vertices used so far. Greedily construct a path of unused vertices inside D' starting at z, ending at some $w \in B_{c'} \cap B_1$ where $c' \in \mathscr{C}(\circledcirc)$ is unused, and consisting of t vertices. Greedily assign unused colours from the lists guaranteed by the definition of D. The path P obtained by concatenating this transversal path with P_1 has endpoints $\widetilde{y}, w$ and has all of the required properties. □

The final lemma of this section combines the previous ones to find a transversal Hamilton cycle in three cases. The proof proceeds by using Lemma 5.5.2 or 5.5.3 to find a short transversal path covering bad vertices and colours, and then covering the remaining vertices with three long paths guaranteed by Lemma 5.5.1.

Lemma 5.5.4. *Suppose that (†) holds, along with one of the following:*

$$\begin{aligned}
&\textit{either } (i) && |\mathscr{C}(\circledcirc)| < \eta n && \textit{and} && \min\left\{e_{\boldsymbol{G}(\circledcirc)}(A_1), e_{\boldsymbol{G}(\circledcirc)}(B_1)\right\} \geq 30\eta n^3;\\
&\textit{or } (ii) && |\mathscr{C}(\circledcirc)| \geq \eta n && \textit{and} && \max\left\{e_{\boldsymbol{G}(\circledcirc)}(A_1), e_{\boldsymbol{G}(\circledcirc)}(B_1)\right\} \geq 30\eta n^3;\\
&\textit{or } (iii) && |\mathscr{C}(\circledcirc)| \geq \eta n && \textit{and} && |\mathscr{C}(\circledcirc)| \geq \eta n \quad \textit{and} \quad G_n \cong K_n.
\end{aligned}$$

Then $\boldsymbol{G}$ contains a transversal Hamilton cycle.

Proof. Suppose that (i) holds. By Lemma 5.5.3, there is a transversal path P with endpoints $x \in A_c$ and $y \in B_{c'}$ where $c, c' \in \mathscr{C}(\circledcirc) \backslash \mathrm{col}(P)$ are distinct, $\mathscr{A} := [n] \backslash (\mathrm{col}(P) \cup \{c, c'\}) \subseteq \mathscr{C}(\circledcirc)$ and $|A| = |B| \geq (\frac{1}{2} - 4\eta)n$ where $Z := Z_1 \setminus V(P)$ for $Z \in \{A, B\}$. So

$$|\mathscr{A}| = n - |\mathrm{col}(P)| - 2 = |A| + |B| + |V(P)| - |\mathrm{col}(P)| - 2 = |A| + |B| - 1.$$

Let $A^- := N_{G_{c'}}(y, A)$ and $B^+ := N_{G_c}(x, B)$. By Lemma 5.5.1 there is a transversal path P'

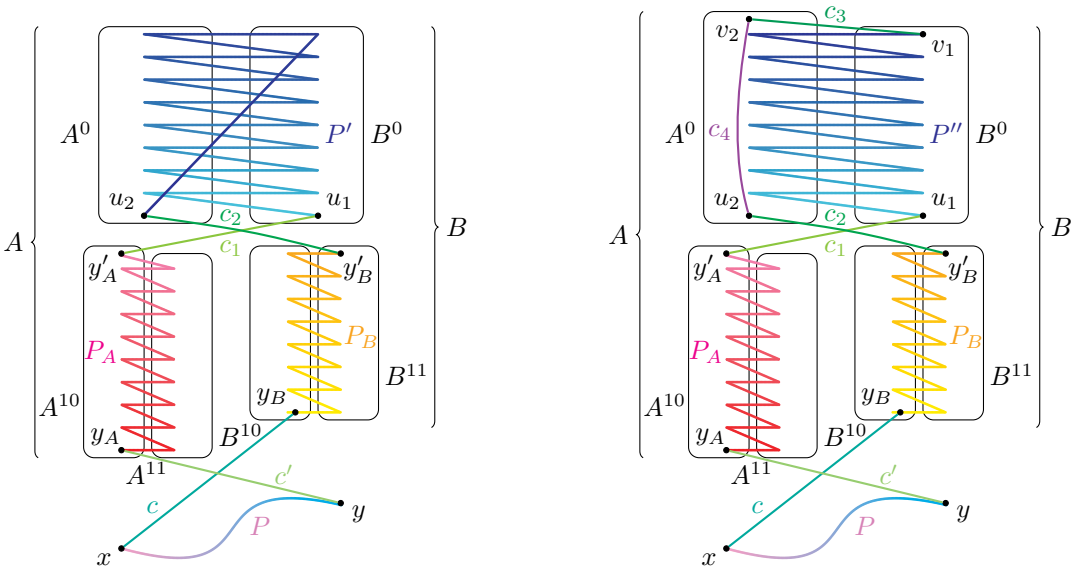


Figure 5.2: Finding a transversal Hamilton cycle in Lemma 5.5.4(ii) and (iii). Here, $|A^{10}| - |A^{11}| = 1$ and $|B^{10}| - |B^{11}| = 0$. On the left, $|A^0| - |B^0| = 0$, while on the right, $|A^0| - |B^0| = 1$.

with colour set $\mathscr{A}$ starting at some $x' \in A^-$ and ending at some $y' \in B^+$. Concatenating this with P using the connecting edges yx' of colour c' and $y'x$ of colour c, we obtain a transversal Hamilton cycle, proving Case (i).

Thus it remains to consider Cases (ii) and (iii). In Case (ii), set $\mathscr{F} := \emptyset$ and in Case (iii), set $\mathscr{F} := \{n\}$. In both cases, we have $|\mathscr{C}(\boxtimes)| \geq \eta n$. Thus we can apply Lemma 5.5.2 with $\mathscr{F}$ to obtain a transversal path P with endpoints $x, y \notin V_{\text{bad}}$ such that $V_{\text{bad}} \subseteq V(P)$ and $\mathscr{C}_{\text{bad}} \setminus \mathscr{F} \subseteq \text{col}(P)$ and $|V(P)| \leq 19\sqrt{\delta} n$, and $\mathscr{F} \cap (\text{col}(P) \cup \{c, c'\} = \emptyset$, and $x \in A_1 \cap A_c$ and $y \in B_1 \cap B_{c'}$ for some colours $c, c' \in \mathscr{C}(\boxtimes)$. Let $Z := Z_1 \setminus V(P)$ for $Z \in \{A, B\}$.

In the next step, we proceed differently in each case. In Case (ii), by symmetry, we assume that $e_{G(\boxtimes)}(A_1) \geq 30\eta n^3$. Let $D := \{uv : u, v \in A \text{ and } c(uv) \geq 10\eta n\}$ as defined in the proof of Lemma 5.5.3 where $c(uv) := \{i \in \mathscr{C}(\boxtimes) : uv \in E(G_i)\}$. There is a subgraph D' of D with minimum degree at least $10\eta n$. In Case (iii), let D' be an arbitrary clique of size $10\eta n$ inside A.

We resume a unified approach for both cases. Let $c_1, c_2 \in \mathscr{C}(\boxtimes)$ be distinct unused colours. Let $\mathscr{A}(H) := \mathscr{C}(H) \setminus (\text{col}(P) \cup \{c, c', c_1, c_2\})$ for $H \in \{\boxdot, \boxtimes\}$. We have

$$|A| + |B| - 3 = n - |V(P)| - 3 = n - |\text{col}(P)| - 4 = |\mathscr{A}(\boxdot)| + |\mathscr{A}(\boxtimes)|$$

and $||A| - |B|| \leq 5\delta n$. Choose partitions $A = A^0 \cup A^1$ and $A^1 = A^{10} \cup A^{11}$, and $B = B^0 \cup B^1$

and $B^1 = B^{10} \cup B^{11}$ such that

$$0 \leq |A^0| - |B^0| \leq 1, \quad |\mathscr{A}(\circledS)| = |A^0| + |B^0| - 1 \quad \text{and} \quad \delta(D'[V(D') \cap A^0]) \geq \eta^2 n,$$

and a partition $\mathscr{A}(\circledS) = \mathscr{C}_A \cup \mathscr{C}_B$ with

$$0 \leq |A^{10}| - |A^{11}|, |B^{10}| - |B^{11}| \leq 1, \quad |\mathscr{C}_A| = |A^{10}| + |A^{11}| - 1 \quad \text{and} \quad |\mathscr{C}_B| = |B^{10}| + |B^{11}| - 1.$$

The only non-trivial part of this is the assertion about D' which holds since $|A^0| \geq |\mathscr{C}(\circledS)|/3 \geq \eta n/3$, and we could take e.g. a random partition and appeal to a Chernoff bound. Let

$$\boldsymbol{G}^0 = (G_i[A^0, B^0] : i \in \mathscr{A}(\circledS)) \quad \text{and} \quad \boldsymbol{G}^Z = (G_i[Z^{10}, Z^{11}] : i \in \mathscr{C}_Z) \quad \text{for } Z \in \{A, B\}.$$

Lemma 5.5.1 with $W = Z$ implies that there is a transversal path P_A inside $\boldsymbol{G}^A$ with colour set $\mathscr{C}_A$ and transversal path P_B inside $\boldsymbol{G}^B$ with colour set $\mathscr{C}_B$ such that P_A starts at some $y_A \in N_{G_{c'}}(y) \cap A^{10}$ and ends at some $y'_A \in A_{c_1} \cap A^1$, and P_B starts at some $y_B \in N_{G_c}(x) \cap B^{10}$ and ends at some $y'_B \in B_{c_2} \cap B^1$. (For example, to find P_A, we take $(W^*, Z^*) = (A^{10}, A^{11})$.) To complete the proof, it suffices to find a transversal path P' in $\boldsymbol{G}^0$ with colour set $\mathscr{A}(\circledS)$, with startpoint $u_1 \in B^0$ where $c_1 \in L(u_1 y'_A)$, and endpoint $u_2 \in A^0$ where $c_2 \in L(u_2 y'_B)$. Then $xPyy_AP_Ay'_Au_1P'u_2y'_BP_By_Bx$ with colours $\mathrm{col}(P), c', \mathrm{col}(P_A), c_1, \mathrm{col}(P'), c_2, \mathrm{col}(P_B), c$ is a transversal Hamilton cycle in $\boldsymbol{G}$.

Suppose first that $|\mathscr{A}(\circledS)|$ is odd. Then $|A^0| = |B^0|$ and Lemma 5.5.1 with $(W^*, Z^*) = (B^0, A^0)$, $t = 0$, $W^- = N_{G_{c_1}}(y'_A) \cap B^0$ and $T^+ = N_{G_{c_2}}(y'_B) \cap A^0$ immediately implies the existence of the required path between $u_1 \in W^-$ and $u_2 \in T^+$ (see the left-hand side of Figure 5.2).

Suppose secondly that $|\mathscr{A}(\circledS)|$ is even. So $|A^0| = |B^0| + 1$. Choose a colour $c_3 \in \mathscr{A}(\circledS)$ and $v_2 \in A_{c_3} \cap A^0$ which is a vertex of D'. Then choose $u_2 \in N_{G_{c_2}}(y'_B) \cap A^0$ which is a neighbour of v_2 in D'. These choices are both possible since there are at least $\eta^2 n$ such neighbours in D' of which at most $2\delta n$ are forbidden due to the G_{c_2} neighbourhood condition. In Case (ii), assign an unused colour $c_4 \in \mathscr{A}(\circledS)$ to u_2v_2 using the large guaranteed colour lists. In Case (iii), assign the colour $c_4 := n$ to u_2v_2. In both cases, $u_2v_2 \in G_{c_4}$.

Now let $\boldsymbol{G}^{0\prime} = (G_i[A^0 \setminus \{u_2, v_2\}, B^0] : i \in \mathscr{A}(\circledS) \setminus \{c_3, c_4\})$. By Lemma 5.5.1 with $(W^*, Z^*) = (B^0, A^0 \setminus \{u_2, v_2\})$ and $t = 1$, there is a transversal path P'' inside $\boldsymbol{G}^{0\prime}$ with startpoint $u_1 \in$

$N_{G_{c_1}}(y'_A) \cap B^0$ and endpoint $v_1 \in N_{G_{c_3}}(v_2) \cap B^0$ using colour set $\mathscr{A}(\circledcirc) \setminus \{c_3, c_4\}$. We set $P' := u_1 P'' v_1 v_2 u_2$, using $\mathrm{col}(P''), c_3, c_4$ in that order (see the right-hand side of Figure 5.2). This completes the proof. □

5.6 Proofs of Theorems 5.1.3 and 5.1.4

Finally, we can put all the ingredients together to prove our two main theorems on stability for transversal Hamilton cycles and paths.

Proof of Theorems 5.1.3 and 5.1.4. Let $\kappa > 0$ and assume that $\kappa < 1$ or the theorems are both vacuous. Choose additional parameters $n, \mu, \alpha, \gamma, \varepsilon, \delta, \eta$ such that $n \in \mathbb{N}$ and

$$0 < 1/n \ll \mu, \alpha \ll \gamma, \varepsilon \ll \delta \ll \eta \ll \kappa < 1$$

such that the conclusions of Lemma 5.2.2 and Lemmas 5.4.3–5.5.4 hold. Let $n' \in \{n-1, n\}$ and let $\boldsymbol{G} = (G_1, \dots, G_{n'})$ be a graph collection on a common vertex set V of size n with $\delta(\boldsymbol{G}) \geq (\frac{1}{2} - \mu)n$.

If $n' = n-1$, let $G_n := K_n$ and $\boldsymbol{J} := (G_1, \dots, G_n)$. If $n' = n$, let $\boldsymbol{J} := \boldsymbol{G}$. Suppose that $\boldsymbol{J}$ does not contain a transversal Hamilton cycle. Then Lemma 5.4.3 implies that $\boldsymbol{J}$ is not $(\gamma, \alpha, \varepsilon, \delta)$-stable. Thus, without loss of generality, for every colour $i \in [(1-\gamma)n]$, the graph G_i is not α-nice and hence is ε-extremal with characteristic partition (A_i, B_i, C_i), and $e(C_{\boldsymbol{J}}^{\varepsilon,\delta}) < \delta n^2$. We may further assume that 1 is the colour of minimum degree in the cross graph $C_{\boldsymbol{J}}^{\varepsilon,\delta}$, and $[(1-\gamma)n] \setminus N_{C_{\boldsymbol{J}}^{\varepsilon,\delta}}(1) = [m]$. Let $\mathscr{C}_{\mathrm{bad}} := [m+1, n]$ be the set of excluded colours and thus $|\mathscr{C}_{\mathrm{bad}}| \leq \gamma n + 2\delta n \leq 3\delta n$. It follows from Lemma 5.2.2 that $e_{\boldsymbol{J}(\circledcirc)}(A_1, B_1) \leq \varepsilon n^3$. For every $i \in [m]$, G_i and G_1 are not δ-crossing and hence we can swap the labels of A_i and B_i if necessary to get that $|A_1 \triangle A_i|, |B_1 \triangle B_i| \leq \delta n$. That is, $\boldsymbol{J}$ satisfies (†). We use the same notation defined after (†).

Suppose that one of the following hold.

(i) $|\mathscr{C}(\overset{\bullet}{\circ})| \leq \eta n$ and either $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(A_1) \leq 30\eta n^3$ or $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(B_1) \leq 30\eta n^3$; or

(ii) $|\mathscr{C}(\overset{\bullet}{\circ})| \geq \eta n$ and $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(A_1) + e_{\boldsymbol{J}(\overset{\circ}{\circ})}(B_1) \leq 60\eta n^3$; or

(iii) $G_n = K_n$ and $|\mathscr{C}(\overset{\circ}{\circ})| < \eta n \leq |\mathscr{C}(\overset{\bullet}{\circ})|$.

We claim that in these cases, $\boldsymbol{J}$ is κ-close to, respectively,

(i) a half-split graph collection;

(ii) $\boldsymbol{H}_a^b$ where $a = |\mathscr{C}(\overset{\bullet}{\bullet})|$ and $b = |\mathscr{C}(\overset{\circ}{\circ})| \pm 1$ is odd;

(iii) $\boldsymbol{H}_n^0$.

Indeed, for (i), we can remove at most $30\eta n^3$ edges so that some $Z_1 \in \{A_1, B_1\}$ has no $\boldsymbol{J}(\overset{\circ}{\circ})$-edges, and delete all edges in graphs in $\boldsymbol{J}(\overset{\bullet}{\bullet}) \cup \mathscr{C}_{\text{bad}}$ by removing at most $\eta n^3 + 3\delta n^3$ edges. Finally, edit at most $2\delta n^3$ edges to increase $|Z_1|$ by less than εn so it has size $\lfloor n/2 \rfloor + 1$ and make $(Z_1, \overline{Z_1})$ complete in every graph in $\boldsymbol{J}(\overset{\circ}{\circ})$. The resulting graph collection is half-split and in total we have made $31\eta n^3 + 5\delta n^3 < \kappa n^3$ edits. For (ii), we can edit at most $3\delta n^3 + \varepsilon n^3 + 60\eta n^3$ edges so that for every $i \in \mathscr{C}_{\text{bad}}$ the graph G_i becomes a copy of $\overset{\bullet}{\bullet}$ whose parts contain A_1, B_1; $e_{G_i}(A_1, B_1) = 0$ for all $G_i \in \boldsymbol{J}(\overset{\bullet}{\bullet})$; and $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(A_1) + e_{\boldsymbol{J}(\overset{\circ}{\circ})}(B_1) = 0$. A further at most δn^3 edits will make $\boldsymbol{J}$ isomorphic to $\boldsymbol{H}_a^b$ where $a = |\mathscr{C}(\overset{\bullet}{\bullet})|$ and $b = |\mathscr{C}(\overset{\circ}{\circ})| \pm 1$ is odd. Thus we make at most $61\eta n^3 < \kappa \eta n^3$ edits in total. For (iii), we can make at most $2\eta n^3 + 2\delta n^3 < \kappa n^3$ edits to make $\boldsymbol{J}$ isomorphic to $\boldsymbol{H}_n^0$. This proves the claim.

Now we prove Theorem 5.1.3. Here, $\boldsymbol{J} = \boldsymbol{G}$. Lemma 5.5.4 implies that, if $|\mathscr{C}(\overset{\bullet}{\bullet})| < \eta n$, then (i) holds, while if $|\mathscr{C}(\overset{\bullet}{\bullet})| \geq \eta n$, then (ii) holds. This completes the proof of the theorem.

Finally, we prove Theorem 5.1.4. Here, $n' = n - 1$ and $\boldsymbol{J} = \boldsymbol{G} \cup \{K_n\}$, so $\boldsymbol{G}$ does not have a transversal Hamilton path if and only if $\boldsymbol{J}$ does not have a transversal Hamilton cycle. Lemma 5.5.4 implies that, if $|\mathscr{C}(\overset{\bullet}{\bullet})| < \eta n$, then (i) holds, while if $|\mathscr{C}(\overset{\bullet}{\bullet})| \geq \eta n$, then (iii) holds (and (ii) also holds in this case). This completes the proof of the theorem. □

6 | Open Problems and Discussions

It is conjectured that a tight threshold for the clique factors actually holds(see (Cheng et al., 2023)):

Conjecture 6.0.1. *(Transversal Hajnal-Szemerédi) Let* $\mathbf{G} = \{G_1, G_2, ..., G_{\frac{n}{t}\binom{t}{2}}\}$ *be a graph family defined on* V *where* $|V| = n$ *and* $t|n$*, if* $\delta(G_i) \geq (1 - \frac{1}{t})n$ *for* $i \in \frac{n}{t}\binom{t}{2}$*, then* $\mathbf{G}$ *contains a rainbow* K_r*-factor.*

A similar problem whose asymptotical threshold is determined in (Montgomery et al., 2022) is as follows:

Conjecture 6.0.2. *Let* $\mathbf{G} = \{G_1, G_2, ..., G_{\frac{n}{t}}\}$ *be a graph family defined on* V *where* $|V| = n$ *and* $t|n$*, if* $\delta(G_i) \geq (1 - \frac{1}{t})n$ *for* $i \in \frac{n}{t}$*, then* $\mathbf{G}$ *contains a* K_r*-factor* $\{F_1, ..., F_{n/t}\}$ *such that* $F_i \subseteq G_i$ *for* $1 \leq i \leq \frac{n}{t}$*.*

Even for the case when $r = 3$, there is no solution for the above two conjectures. Similarly, for Hamiltonian ℓ-cycles, we can also ask the same question. Currently, the only known tight bounds for these type of problems are Hamiltonian cycles in simple graphs and $\delta_{k-1,M}$ in k-uniform hypergraphs where M is a perfect matching(see (Lu, Wang, & Yu, 2021)). No other tight bounds are known.

In (Gupta et al., 2023), the authors asked about characterizing the stability of all such kinds of transversal embedding problems. In particular, they were asking about if the following tight threshold also holds as in the graph case (the asymptotic version has been proved in their paper, an independent proof was presented in Chapter 4):

Conjecture 6.0.3. *(Transversal Pósa-Seymour) Let* $\mathbf{G} = \{G_1, G_2, ..., G_{(k-1)n}\}$ *be a graph family defined on* V *where* $|V| = n$*, if* $\delta(G_i) \geq (1-\frac{1}{k})n$ *for* $i \in (k-1)n$*, then* $\mathbf{G}$ *contains a rainbow* $(r-1)^{th}$ *power of Hamiltonian cycle.*

As mentioned in previous Chapters, a transversal bandwidth theorem (Theorem 4.1.1) was proved in (Chakraborti, Im, et al., 2023), which gives an asymptotically tight minimum degree bound $(\chi(H) - 1)/\chi(H)$ guaranteeing the existence of a transversal copy of H for many graphs H. On the other hand, there are many graphs where this threshold is not tight. For example, when H is a 2-factor – a spanning 2-regular graph – (El-Zahar, 1984) conjectured that the minimum degree threshold should equal $(n + r)/2$ where r is the number of odd cycles. Note that if H consists only of even cycles, then $(\chi(H) - 1)/\chi(H) = \frac{1}{2}$, while otherwise it equals $\frac{2}{3}$.

A proof of El-Zahar's conjecture was announced by Abbasi for large n (Abbasi, 1998a). The following problem asks whether the transversal generalisation is true.

Problem 6.0.4 (Transversal El-Zahar)**.** *Let n be a sufficiently large integer and let H be a 2-factor on n vertices in which r cycles are odd. Let $\boldsymbol{G} = (G_1, \dots, G_n)$ be a graph collection on a common vertex set of size n. Suppose that $\delta(\boldsymbol{G}) \geq (n+r)/2$. Does $\boldsymbol{G}$ contain a transversal copy of H?*

Abbasi's proof uses the regularity-blow-up method. Later, a proof avoiding this method was found by Khan (Khan, 2011), although the broad structure of the proof is similar.

An even harder problem is a transversal version of the Bollobás-Eldridge-Catlin Conjecture (see (Bollobás & Eldridge, 1978), (Catlin, 1976)).

Problem 6.0.5 (Transversal Bollobás-Eldridge-Catlin)**.** *Let n be a sufficiently large integer and let H be an n vertex graph with $\Delta(H) \leq r$. Let $\boldsymbol{G} = (G_1, G_2, \dots, G_{e(H)})$ be a graph collection on a common vertex set of size n. Suppose that $\delta(\boldsymbol{G}) \geq \frac{rn-1}{r+1}$. Does $\boldsymbol{G}$ contain a transversal copy of H?*

If true, this would generalise Conjecture 6.0.1 and Conjecture 6.0.3. Problem 6.0.5 is likely very difficult, although Kun reported an asymptotic solution to the classical problem around 2006. However, it may be amenable in some special cases. For example, it is known when $r \leq 3$, and the proof for e.g. $r = 3$ by (Csaba, Shokoufandeh, & Szemerédi, 2003) uses the regularity lemma, and arguments closed related to the proof of the blow-up lemma.

We believe the methods developed in this thesis will be useful for the open problems listed here. Especially, it is hopeful to prove a stable version of Conjecture 6.0.1 and Conjecture 6.0.3 and thus give hope for the full solution of them.

Bibliography

Abbasi, S. (1998a). *The solution of the el-zahar problem* (Unpublished doctoral dissertation). Rutgers University.

Abbasi, S. (1998b). *Spanning subgraphs of dense graphs and a combinatorial problem on strings*. Rutgers The State University of New Jersey, School of Graduate Studies.

Aharoni, R., & Berger, E. (2009). Rainbow matchings in r-partite r-graphs. *The Electronic Journal of Combinatorics*, R119–R119.

Aharoni, R., DeVos, M., de la Maza, S., Montejano, A., & Šámal, R. (2020). A rainbow version of Mantel's theorem. *Advance in Combinatorics.*, *2, 12pp*.

Aharoni, R., Georgakopoulos, A., & Sprüssel, P. (2009). Perfect matchings in r-partite r-graphs. *European Journal of Combinatorics*, *30*(1), 39–42.

Alon, N., Frankl, P., Huang, H., Rödl, V., Ruciński, A., & Sudakov, B. (2012). Large matchings in uniform hypergraphs and the conjecture of Erdős and Samuels. *Journal of Combinatorial Theory, Series A*, *119*(6), 1200–1215.

Alon, N., & Spencer, J. (2008). The probabilistic method. *John Wiley Inc.*, *New York*.

Alon, N., & Yuster, R. (1996). H-factors in dense graphs. *Journal of Combinatorial Theory, Series B*, *66*(2), 269–282.

Anastos, M., & Chakraborti, D. (2023). Robust Hamiltonicity in families of Dirac graphs. *arXiv preprint arXiv:2309.12607*.

Barber, B., Kühn, D., Lo, A., & Osthus, D. (2016). Edge-decompositions of graphs with high minimum degree. *Advances in Mathematics*, *288*, 337–385.

Bollobás, B., & Eldridge, S. (1978). Packings of graphs and applications to computational complexity. *Journal of Combinatorial Theory, Series B*, *25*(2), 105–124.

Böttcher, J., Schacht, M., & Taraz, A. (2008). Spanning 3-colourable subgraphs of small bandwidth in dense graphs. *Journal of Combinatorial Theory, Series B*, *98*(4), 752–777.

Böttcher, J., Schacht, M., & Taraz, A. (2009). Proof of the bandwidth conjecture of Bollobás and Komlós. *Mathematische Annalen*, *343*(1), 175–205.

Bowtell, C., Morris, P., Pehova, Y., & Staden, K. (2023). Universality for transversal Hamilton cycles. *arXiv preprint arXiv:2310.04138*.

Bradshaw, P. (2020). Transversals and bipancyclicity in bipartite graph families. *arXiv preprint arXiv:2002.10014*.

Bradshaw, P., Halasz, K., & Stacho, L. (2022). From one to many rainbow Hamiltonian cycles. *Graphs and Combinatorics*, *38*(6), 188.

Brualdi, R. A., Ryser, H. J., et al. (1991). *Combinatorial matrix theory*. Springer.

Catlin, P. A. (1976). *Embedding subgraphs and coloring graphs under extremal degree conditions* (Unpublished doctoral dissertation). The Ohio State University.

Chakraborti, D., Im, S., Kim, J., & Liu, H. (2023). A bandwidth theorem for graph transversals. *arXiv preprint arXiv:2302.09637*.

Chakraborti, D., Kim, J., Lee, H., & Seo, J. (2023). Hamilton transversals in tournaments. *arXiv preprint arXiv:2307.00912*.

Châu, P., DeBiasio, L., & Kierstead, H. A. (2011). Pósa's conjecture for graphs of order at least 2×10^8. *Random Structures & Algorithms*, *39*(4), 507–525.

Cheng, Y., Han, J., Wang, B., & Wang, G. (2023). Rainbow spanning structures in graph and hypergraph systems. *Forum of Mathematics, Sigma*, *11*, e95.

Cheng, Y., Han, J., Wang, B., Wang, G., & Yang, D. (2021). Rainbow hamilton cycle in hypergraph systems. *arXiv:2111.07079*.

Cheng, Y., & Staden, K. (2023). Transversals via regularity. *arXiv preprint arXiv:2306.03595*.

Cheng, Y., Wang, G., & Zhao, Y. (2021). Rainbow pancyclicity in graph systems. *The Electronic Journal of Combinatorics*, *28*(3).

Chung, F. (2007). Regularity lemmas for hypergraphs and quasi-randomness. *Random Structures & Algorithms*, *2*, 241 - 252.

Chvátal, V. (1972). On Hamilton's ideals. *Journal of Combinatorial Theory, Series B*, *12*(2), 163–168.

Conlon, D., Hàn, H., Person, Y., & Schacht, M. (2012). Weak quasi-randomness for uniform hypergraphs. *Random Structures & Algorithms*, *40*(1), 1–38.

Csaba, B., Shokoufandeh, A., & Szemerédi, E. (2003). Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three. *Combinatorica*, *23*, 35–72.

Czygrinow, A., DeBiasio, L., Kierstead, H. A., & Molla, T. (2015). An extension of the Hajnal-Szemerédi theorem to directed graphs. *Combinatorics, Probability and Computing*, *24*(5), 754–773.

Czygrinow, A., Kierstead, H. A., & Molla, T. (2014). On directed versions of the Corrádi-Hajnal corollary. *European Journal of Combinatorics*, *42*, 1–14.

Czygrinow, A., & Molla, T. (2014). Tight codegree condition for the existence of loose hamilton cycles in 3-graphs. *SIAM Journal on Discrete Mathematics*, *28*(1), 67–76.

Ding, L., Han, J., Sun, S., Wang, G., & Zhou, W. (2022). F-factors in quasi-random hypergraphs. *Journal of the London Mathematical Society*, *106*(3), 1810–1843.

Dirac, G. A. (1952). Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, *3*(1), 69–81.

El-Zahar, M. H. (1984). On circuits in graphs. *Discrete Mathematics*, *50*, 227–230.

Erdős, P. (1964a). On extremal problems of graphs and generalized graphs. *Israel Journal of Mathematics*, *2*(3), 183–190.

Erdős, P. (1964b). Problem 9, theory of graphs and its applications. In *Proceedings of symposium on smolenice 1963* (p. 159). Publ. House Czechoslovak Academic Science.

Erdős, P., & Hajnal, A. (1972). On Ramsey like theorems, problems and results. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)* (pp. 123–140).

Erdős, P., & Sós, V. T. (1982). On Ramsey-Turán type theorems for hypergraphs. *Combinatorica*, *2*, 289–295.

Ferber, A., Han, J., & Mao, D. (2022). Dirac-type problem of rainbow matchings and Hamilton cycles in random graphs. *arXiv preprint arXiv:2211.05477*.

Fischer, E. (1999). Variants of the Hajnal-Szemerédi theorem. *Journal of Graph Theory*, *31*(4), 275–282.

Frankl, P., & Kupavskii, A. (2018). The Erdős matching conjecture and concentration inequalities. *Israel Journal of Mathematics*, *222*(1), 421-430.

Glock, S., Kühn, D., Lo, A., & Osthus, D. (2017). Hypergraph f-designs for arbitrary f. *arXiv preprint arXiv:1706.01800*.

Gupta, P., Hamann, F., Müyesser, A., Parczyk, O., & Sgueglia, A. (2023). A general approach to transversal versions of Dirac-type theorems. *Bulletin of the London Mathematical Society*, *55*(6), 2817–2839.

Hajnal, A., & Szemerédi, E. (1970). Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications* (p. 601-623).

Han, J., & Zhao, Y. (2015). Minimum codegree threshold for hamilton ℓ-cycles in k-uniform hypergraphs. *Journal of Combinatorial Theory, Series A*, *132*, 194–223.

Haxell, P. E., & Rödl, V. (2001). Integer and fractional packings in dense graphs. *Combinator-*

ica, *21*(1), 13–38.

Janson, S., Łuczak, T., & Ruciński, A. (2000). Random graphs. *John Wiley and Sons*, *New York*.

Joos, F., & Kim, J. (2020). On a rainbow version of Dirac's theorem. *Bulletin of the London Mathematical Society*, *53*(2), 498–504.

Keevash, P. (2011). A hypergraph blow-up lemma. *Random Structures & Algorithms*, *39*(3), 275–376.

Keevash, P. (2014). The existence of designs. *arXiv preprint arXiv:1401.3665*.

Keevash, P., & Mycroft, R. (2015a). A geometric theory for hypergraph matching. *Memoir of American Mathematical Society*.

Keevash, P., & Mycroft, R. (2015b). A multipartite Hajnal-Szemerédi theorem. *Journal of Combinatorial Theory, Series B*, *114*, 187-236.

Keevash, P., & Staden, K. (2020). Ringel's tree packing conjecture in quasirandom graphs. *arXiv preprint arXiv:2004.09947*.

Khan, I. (2011). *Spanning subgraphs in graphs and hypergraphs* (Unpublished doctoral dissertation). Rutgers, The State University of New Jersey.

Kohayakawa, Y., Nagle, B., Rödl, V., & Schacht, M. (2010). Weak hypergraph regularity and linear hypergraphs. *Journal of Combinatorial Theory, Series B*, *100*(2), 151–160.

Komlós, J. (1999). The blow-up lemma. *Combinatorics, Probability and Computing*, *8*(1-2), 161–176.

Komlós, J., Sárközy, G., & Szemerédi, E. (2001). Proof of the Alon-Yuster conjecture. *Discrete Mathematics*, *235*(1-3), 255–269.

Komlós, J., Sárközy, G. N., & Szemerédi, E. (1997). Blow-up lemma. *Combinatorica*, *17*, 109–123.

Komlós, J., Sárközy, G. N., & Szemerédi, E. (1998a). On the Pósa-Seymour conjecture. *Journal of Graph Theory*, *29*(3), 167–176.

Komlós, J., Sárközy, G. N., & Szemerédi, E. (1998b). Proof of the Seymour conjecture for large graphs. *Annals of Combinatorics*, *2*, 43–60.

Komlós, J., Sárkózy, G. N., & Szemerédi, E. (2001). Spanning trees in dense graphs. *Combinatorics, Probability and Computing*, *10*(5), 397–416.

Krivelevich, M., Lee, C., & Sudakov, B. (2014). Robust Hamiltonicity of Dirac graphs. *Trans-*

actions of the American Mathematical Society, *366*(6), 3095–3130.

Kühn, D., & Osthus, D. (2006). Loose hamilton cycles in 3-uniform hypergraphs of high minimum degree. *Journal of Combinatorial Theory, Series B*, *96*(6), 767–821.

Kühn, D., & Osthus, D. (2009). The minimum degree threshold for perfect graph packings. *Combinatorica*, *29*(1), 65–107.

Kühn, D., Osthus, D., & Townsend, T. (2014). Fractional and integer matchings in uniform hypergraphs. *European Journal of Combinatorics*, *38*, 83–96.

Lenz, J., & Mubayi, D. (2016). Perfect packings in quasirandom hypergraphs I. *Journal of Combinatorial Theory, Series B*, *119*, 155–177.

Lenz, J., Mubayi, D., & Mycroft, R. (2016). Hamilton cycles in quasirandom hypergraphs. *Random Structures & Algorithms*, *49*(2), 363–378.

Levitt, I., Sárközy, G. N., & Szemerédi, E. (2010). How to avoid using the regularity lemma: Pósa's conjecture revisited. *Discrete Mathematics*, *310*(3), 630–641.

Li, L., Li, P., & Li, X. (2023). Rainbow structures in a collection of graphs with degree conditions. *Journal of Graph Theory*.

Lo, A., & Markström, K. (2013). A multipartite version of the Hajnal-Szemerédi theorem for graphs and hypergraphs. *Combinatorics, Probability and Computing*, *22*(1), 97–111.

Lu, H., Wang, Y., & Yu, X. (2021). Co-degree threshold for rainbow perfect matchings in uniform hypergraphs. *arXiv:2111.00372*.

Lu, H., Wang, Y., & Yu, X. (2023). A better bound on the size of rainbow matchings. *Journal of Combinatorial Theory, Series A*, *195*, 105700.

Lu, H., Yu, X., & Yuan, X. (2021). Rainbow matchings for 3-uniform hypergraphs. *Journal of Combinatorial Theory, Series A*, *183*.

Montgomery, R. (2023). A proof of the ryser-brualdi-stein conjecture for large even n. *arXiv:2310.19779*.

Montgomery, R., Müyesser, A., & Pehova, Y. (2022). Transversal factors and spanning trees. *Advances in Combinatorics*.

Montgomery, R., Pokrovskiy, A., & Sudakov, B. (2021). A proof of Ringel's conjecture. *Geometric and Functional Analysis*, *31*(3), 663–720.

Ore, Ã. (1960). Note on hamilton circuits. *American Mathematical Monthly*, *67*(1), 55.

Pavez-Signé, M., Sanhueza-Matamala, N., & Stein, M. (2021). Towards a hypergraph version

of the Pósa-seymour conjecture. *arXiv preprint arXiv:2110.09373*.

Pippenger, N., & Spencer, J. (1989). Asymptotic behavior of the chromatic index for hypergraphs. *Journal of Combinatorial Theory, Series A*, *51*(1), 24-42.

Pokrovskiy, A. (2018). An approximate version of a conjecture of Aharoni and Berger. *Advances in Mathematics*, *333*, 1197–1241.

Pokrovskiy, A. (2022). Rainbow subgraphs and their applications. *Surveys in Combinatorics 2022*, 191–214.

Reiher, C., Rödl, V., & Schacht, M. (2018a). Hypergraphs with vanishing Turán density in uniformly dense hypergraphs. *Journal of the London Mathematical Society*, *97*(1), 77–97.

Reiher, C., Rödl, V., & Schacht, M. (2018b). On a Turán problem in weakly quasirandom 3-uniform hypergraphs. *Journal of the European Mathematical Society*, *20*(5), 1139–1159.

Rödl, V., & Rucinski, A. (1999). Perfect matchings in ε-regular graphs and the blow-up lemma. *Combinatorica*, *19*(3), 437–452.

Rödl, V., Ruciński, A., & Szemerédi, E. (2006). A Dirac-type theorem for 3-uniform hypergraphs. *Combinatorics, Probability and Computing*, *15*(1-2), 229–251.

Rödl, V., Ruciński, A., & Szemerédi, E. (2011). Dirac-type conditions for hamiltonian paths and cycles in 3-uniform hypergraphs. *Advances in Mathematics*, *227*(3), 1225–1299.

Rödl, V., Szemerédi, E., & Ruciński, A. (2008). An approximate Dirac-type theorem for k-uniform hypergraphs. *Combinatorica*, *28*(2), 229–260.

Ryser, H. J. (1967). Neuere probleme der kombinatorik. *Vorträge über Kombinatorik, Oberwolfach*, *69*(91), 35.

Seymour, P. (1974). Problem section, Combinatorics. In *Proceedings of British Combinatorial conference 1973* (pp. 201–204). Cambridge University Press.

Simao, H., Asif, J., & Szemerédi, E. (2014). Proof of the Pósa-Seymour conjecture. *Unpublished*.

Simonovits, M., & Szemerédi, E. (2019). Embedding graphs into larger graphs: results, methods, and problems. *Building Bridges II: Mathematics of László Lovász*, 445–592.

Staden, K., & Treglown, A. (2020). The bandwidth theorem for locally dense graphs. In *Forum of mathematics, sigma* (Vol. 8, p. e42).

Stein, S. K. (1975). Transversals of latin squares and their generalizations. *Pacific Journal of*

Mathematics, *59*(2), 567–576.

Szemerédi, E. (1975). *Regular partitions of graphs* (Tech. Rep.). Stanford Univ Calif Dept of Computer Science.

Tang, Y., Wang, B., Wang, G., & Yan, G. (2023). Rainbow Hamilton cycle in hypergraph system. *arXiv preprint arXiv:2302.00080*.

Townsend, T. (2016). *Extremal problems on graphs, directed graphs and hypergraphs* (Unpublished doctoral dissertation). University of Birmingham.

Treglown, A. (2015). On directed versions of the Hajnal-Szemerédi theorem. *Combinatorics, Probability and Computing*, *24*(6), 873–928.

Zhang, Y., & van Dam, E. R. (2024). Rainbow Hamiltonicity and the spectral radius. *arXiv preprint arXiv:2401.17845*.

Zhao, Y. (2016). Recent advances on Dirac-type problems for hypergraphs. *Recent trends in combinatorics*, 145–165.