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On affine groups admitting invariant two-point sets

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ABSTRACT

A two-point set is a subset of the plane which meets every line in exactly two points. We discuss previous work on the topological symmetries of a two-point set, and show that there exist subgroups of S^1 which do not leave any two-point set invariant. Further, we show that two-point sets may be chosen to be topological groups, in which case they are also homogeneous.

1. Introduction

Given a cardinal κ, a subset of the plane is said to be a *κ-point set* if and only if it meets every line in exactly κ many points and is said to be a *partial κ-point set* if and only if it meets every line in at most κ many points. We are particularly interested in the case that $\kappa = 2$, and we refer to such sets as *two-point sets*. The existence of two-point sets was shown by Mazurkiewicz [3]. (A French translation is available in [4].)

Chad and Suabedissen [1] have asked "What are the symmetries of a two-point set?". They showed that two-point sets may be rigid and that the isometry group of a two-point set consists of rotations only. Exploring the case of Abelian rotation groups, they showed that any subgroup of S^1 of cardinality less than $\mathfrak{c}$ may be realised as the isometry group of a two-point set, as well as showing the existence of subgroups of S^1 of size $\mathfrak{c}$ which are the isometry group of a two-point set. Our two main results are complementary to these. We will show that there exist proper subgroups of S^1 which leave no two-point set invariant. Moreover, we construct a homogeneous two-point set, where the autohomeomorphisms guaranteeing homogeneity can be taken to be affine maps which leave the two-point set invariant.

We use the variables α and β to range over ordinals and the variable κ to range over cardinals.

Throughout this paper, we will find it convenient to consider two-point sets to be subsets of the vector space $\mathbb{C}$ over $\mathbb{R}$.

2. Subgroups of S^1 and the isometry group of a two-point set

It is shown in [1] that any subgroup of S^1 of cardinality less than $\mathfrak{c}$ may be realised as the isometry group of a two-point set, and that there exist subgroups of S^1 of size $\mathfrak{c}$ which are the isometry group of a two-point set. Larman [2] proved that

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a two-point set cannot contain an arc, and so a two-point set cannot have isometry group S^1. We will now show that there exist proper subgroups of S^1 which are not the isometry group of any two-point set.

Recall that complex numbers $r_x e^{i\theta_x}$, $r_y e^{i\theta_y}$ and $r_z e^{i\theta_z}$ are collinear if and only if

$$\begin{vmatrix} r_x \cos\theta_x & r_x \sin\theta_x & 1 \\ r_y \cos\theta_y & r_y \sin\theta_y & 1 \\ r_z \cos\theta_z & r_z \sin\theta_z & 1 \end{vmatrix} = 0$$

if and only if

$$r_y r_z \sin(\theta_z - \theta_y) + r_x r_z \sin(\theta_x - \theta_z) + r_x r_y \sin(\theta_y - \theta_x) = 0.$$

Lemma 1. *Consider $\mathbb{R}$ to be a vector space over $\mathbb{Q}$. Let $x, y, z \in \mathbb{C}$ be non-collinear and such that $|x| < |y| < |z|$, and let $B \subseteq \mathbb{R}$ be such that $|B| < \mathfrak{c}$ and $\pi \notin B$ and $B \cup \{\pi\}$ is independent. Then there exist $a, b \in \mathbb{R}$ such that $B \cup \{\pi, a, b\}$ is independent and x, $ye^{i\phi_y}$, $ze^{i\phi_z}$ are collinear for some ϕ_y, ϕ_z in the span of $B \cup \{a, b\}$.*

Proof. Write $x = r_x e^{i\theta_x}$, $y = r_y e^{i\theta_y}$ and $z = r_z e^{i\theta_z}$ in polar form. Then for all $\phi_y, \phi_z \in \mathbb{R}$, we have that x, $ye^{i\phi_y}$ and $ze^{i\phi_z}$ are collinear if and only if

$$r_y r_z \sin(\theta_z - \theta_y + \phi_z - \phi_y) + r_x r_z \sin(\theta_x - \theta_z - \phi_z) + r_x r_y \sin(\theta_y - \theta_x + \phi_y)$$

is equal to zero. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(\phi_y, \phi_z)$ is equal to the above expression and let $S = f^{-1}(\{0\})$. If there exists $\langle \phi_y, \phi_z \rangle \in S$ for some ϕ_y, ϕ_z in the span of B then there is nothing left to show. We suppose then that this is not the case.

Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the respective projections of $\mathbb{R}^2$ onto its first and second coordinate axis and let V be the span of $B \cup \{\pi\}$. Then

$$\left| S \setminus \left(\pi_1^{-1}(V) \cup \pi_2^{-1}(V) \right) \right| = \mathfrak{c},$$

since $|V| = |B| + \aleph_0 < \mathfrak{c}$ and the mappings $\mathbb{R} \to \mathbb{R}$ obtained by fixing a single variable of f have countable fibres. For each $v \in V$ and each $q \in \mathbb{Q} \setminus \{0\}$, let $f_{v,q} : \mathbb{R} \to \mathbb{R}$ be defined by $f_{v,q}(\phi_y) = f(\phi_y, v + q\phi_y)$. Using the transformation $w = e^{i\phi_y}$, we can rewrite the expression defining each $f_{v,q}$ as an analytic function in w which has at most finitely many singularities, each of which is removable. These functions have at most countably many roots, for if some (transformed) $f_{v,q}$ had uncountably many roots, then by the Identity Theorem it would be identically zero, and so $\langle \phi_y, v + q\phi_y \rangle \in S$ for all $\phi_y \in B$, which is a contradiction. Hence, letting Y be the set of all ϕ_y such that ϕ_y is a zero of some $f_{v,q}$, we see that $|Y| < \mathfrak{c}$. We also define a corresponding set Z by arguing with ϕ_z instead of ϕ_y.

We may now argue as above to show that

$$\left| S \setminus \left(\pi_1^{-1}(V \cup Y) \cup \pi_2^{-1}(V \cup Z) \right) \right| = \mathfrak{c},$$

and so may set $\langle a, b \rangle \in S \setminus (\pi_1^{-1}(V \cup Y) \cup \pi_2^{-1}(V \cup Z))$. That both a and b are not in the span of $B \cup \{\pi\}$ is guaranteed by the condition

$$\langle a, b \rangle \notin \pi_1^{-1}(V) \cup \pi_2^{-1}(V),$$

and that b (respectively a) is not in the span of $B \cup \{\pi, a\}$ (respectively $B \cup \{\pi, b\}$) is guaranteed by the condition

$$\langle a, b \rangle \notin \pi_1^{-1}(Y) \cup \pi_2^{-1}(Z).$$

Hence $B \cup \{\pi, a, b\}$ is independent. □

Theorem 2. *There exist proper subgroups of S^1 which leave no two-point set invariant.*

Proof. Let $\langle \langle x_\alpha, y_\alpha, z_\alpha \rangle \colon \alpha < \mathfrak{c} \rangle$ enumerate all triples of points taken from $\mathbb{C}$ which are non-collinear and such that $|x_\alpha| < |y_\alpha| < |z_\alpha|$. Using the above lemma, we may recursively choose an increasing sequence $\langle B_\alpha \colon \alpha < \mathfrak{c} \rangle$ of independent subsets of $\mathbb{R}$ over $\mathbb{Q}$ such that:

(1) $\pi \notin B_\alpha$ and $\bigcup_{\alpha < \mathfrak{c}} B_\alpha \cup \{\pi\}$ is independent; and
(2) for each $\alpha < \mathfrak{c}$, the span of $\bigcup_{\beta \leqslant \alpha} B_\beta$ contains ϕ_y and ϕ_z such that x_α, $y_\alpha e^{i\theta_y}$ and $z_\alpha e^{i\theta_z}$ are collinear; and
(3) for each $\alpha < \mathfrak{c}$, $|B_\alpha| \leqslant 2$.

Let V be the span of $\bigcup_{\alpha < \mathfrak{c}} B_\alpha$ and note that $V \cap \pi\mathbb{Z} = \emptyset$. Let $f : V \to S^1$ be the function defined for $x \in V$ by $f(x) = e^{ix}$. Then f is easily seen to be a group homomorphism. Let $G = f(V)$. Then G is a proper subgroup of S^1, for it is clearly a subgroup of S^1, and supposing that $e^{i\pi} \in G$, we obtain the contradiction that $(1 + 2k)\pi \in V$ for some $k \in \mathbb{Z}$.

Now, let X be a two-point set. Since X is unbounded, we may let $\alpha < \mathfrak{c}$ be such that $\{x_\alpha, y_\alpha, z_\alpha\} \subseteq X$ and let $\phi_y, \phi_z \in V$ be such that x_α, $y_\alpha e^{i\theta_y}$ and $z_\alpha e^{i\phi_z}$ are collinear. Since $|x_\alpha| < |y_\alpha| < |z_\alpha|$, we must have that x_α, $y_\alpha e^{i\theta_y}$ and $z_\alpha e^{i\phi_z}$ are distinct, and so $e^{i\phi_y}, e^{i\phi_z} \in G$ witness that X is not invariant under G. □

3. The existence of homogeneous two-point sets

We now prove the second of our two main results, showing that a two-point set may be chosen to be topologically homogeneous.

We will say that a (partial) two-point set X is a *(partial) two-point group* if and only if it is a multiplicative subgroup of $\mathbb{C}\setminus\{0\}$ containing -1. Recall that an Abelian group G is divisible if and only if for $n\in\mathbb{N}$ and all $g\in G$ there exists $h\in G$ such that $h^n=g$; loosely speaking, a divisible group is one which contains nth roots.

For each $\langle r,\theta\rangle\in\mathbb{R}\setminus\{0\}\times\mathbb{R}$, let $T_{r,\theta}:\mathbb{C}\to\mathbb{C}$ be the function defined for $x\in\mathbb{C}$ by $T_{r,\theta}(x)=rxe^{i\theta}$, and for each $q\in\mathbb{Q}$, we let $T^q_{r,\theta}:\mathbb{C}\to\mathbb{C}$ equal $T_{r^q,q\theta}$ if $q>0$; equal the inverse of $T_{r^{-q},-q\theta}$ if $q<0$; be the identity function on $\mathbb{C}$ if $q=0$.

Lemma 3. *Let X be a partial two-point set such that $|X|<\mathfrak{c}$, let L be a line containing the origin such that $X\cap L=\emptyset$, and let $\theta_0\in\mathbb{R}$ be such that $L=\{re^{i\theta_0}\colon r\in\mathbb{R}\}$. Then there are fewer than $\mathfrak{c}$ many $r\in\mathbb{R}\setminus\{0\}$ such that $\bigcup_{n\in\mathbb{Z}}T^n_{r,\theta_0}(X)$ is not a partial two-point set.*

Proof. Suppose that $r\in\mathbb{R}\setminus\{0\}$ is such that $\bigcup_{n\in\mathbb{Z}}T^n_{r,\theta_0}(X)$ is not a partial two-point set. Then without loss of generality, there exist $r_xe^{i\theta_x},r_ye^{i\theta_y},r_ze^{i\theta_z}\in X$ and $n\leqslant m<\omega$ such that $r_xe^{i\theta_x}$, $T^n_{r,\theta_0}(r_ye^{i\theta_y})$ and $T^m_{r,\theta_0}(r_ze^{i\theta_z})$ are distinct and collinear, and so

$$r^{n+m}r_yr_z\sin\bigl(\theta_y-\theta_z+(n-m)\theta_0\bigr)+r^mr_xr_z\sin(-\theta_x+\theta_z+m\theta_0)+r^nr_xr_y\sin(\theta_x-\theta_y-n\theta_0)$$

is equal to zero. Considering the above to be a polynomial in r, it will be sufficient to assert that it is non-trivial, for then if $\bigcup_{n\in\mathbb{Z}}T^n_{s,\theta_0}(X)$ is not a partial two-point set then s is a root of one of fewer than $\mathfrak{c}$ many non-trivial polynomials. We consider three cases.

Case 1: $0<n=m$. The polynomial becomes

$$r_yr_z\sin(\theta_y-\theta_z)r^{2n}+\bigl(r_xr_z\sin(-\theta_x+\theta_z+n\theta_0)+r_xr_y\sin(\theta_x-\theta_y-n\theta_0)\bigr)r^n.$$

Noting that $n\neq 0$, the powers $2n$ and n are distinct. Supposing that both the coefficients in this polynomial vanish, then

$$\sin(\theta_y-\theta_z)=0,$$
$$r_z\sin(-\theta_x+\theta_z+n\theta_0)=r_y\sin(-\theta_x+\theta_y+n\theta_0).$$

From the first condition, it follows that $\theta_z=\theta_y+k_1\pi$ for some $k_1\in\mathbb{Z}$, and so by substituting in the second condition, we obtain the relation

$$r_z\sin(-\theta_x+\theta_y+n\theta_0+k_1\pi)=r_y\sin(-\theta_x+\theta_y+n\theta_0).$$

Now, it must be the case that $\sin(-\theta_x+\theta_y+n\theta_0)=0$, for suppose not. Then if k_1 is even, we have that $\theta_y=\theta_z$ and $r_y=r_z$, which gives the contradiction $T^n_{r,\theta_0}(r_ye^{i\theta_y})=T^m_{r,\theta_0}(r_ze^{i\theta_z})$. Otherwise, k_1 is odd, and it follows from the identity $\sin(x+\pi)=-\sin(x)$ that $r_y=r_z=0$, which is also a contradiction. Letting $k_2\in\mathbb{Z}$ be such that $-\theta_x+\theta_y+n\theta_0=k_2\pi$, we see that

$$\bigl(r_xe^{i\theta_x}\bigl(r_ye^{i\theta_y}\bigr)^{-1}(-1)^{k_2}\bigr)^{1/n}=\left(\frac{r_y}{r_x}\right)^{1/n}e^{i(\theta_x-\theta_y+k_2\pi)/n}=\left(\frac{r_y}{r_x}\right)^{1/n}e^{i\theta_0}\in X\cap L,$$

which is a contradiction.

Case 2: $0=n<m$. The polynomial becomes

$$\bigl(r_yr_z\sin(\theta_y-\theta_z-m\theta_0)+r_xr_z\sin(-\theta_x+\theta_z+m\theta_0)\bigr)r^m+r_xr_y\sin(\theta_x-\theta_y),$$

as we argue as above.

Case 3: $0<n<m$. If the polynomial were trivial, then

$$\sin\bigl(\theta_y-\theta_z+(n-m)\theta_0\bigr)=\sin(-\theta_x+\theta_z+m\theta_0)=\sin(\theta_x-\theta_y-n\theta_0)=0,$$

which we have already argued is impossible. □

Lemma 4. *Let X be a divisible partial two-point group such that $|X|<\mathfrak{c}$, let L be a line containing the origin such that $X\cap L=\emptyset$, and let $\theta_0\in\mathbb{R}$ be such that $L=\{re^{i\theta_0}\colon r\in\mathbb{R}\}$. Then there are fewer than $\mathfrak{c}$ many $r\in\mathbb{R}\setminus\{0\}$ such that $\bigcup_{q\in\mathbb{Q}}T^q_{r,\theta_0}(X)$ is not a divisible partial two-point group.*

Proof. Note that $\mathbb{C}\setminus\{0\}$ is Abelian, and that for each $r\in\mathbb{R}\setminus\{0\}$, $\bigcup_{q\in\mathbb{Q}}T^q_{r,\theta_0}(X)$ is a divisible group containing -1. Suppose that there are $\mathfrak{c}$ many $r\in\mathbb{R}\setminus\{0\}$ such that $\bigcup_{q\in\mathbb{Q}}T^q_{r,\theta_0}(X)$ is not a partial two-point set. For each such r, let $q_x,q_y,q_z\in\mathbb{Q}$

and $x, y, z \in X$ be such that $T^{q_x}_{r,\theta_0}(x)$, $T^{q_y}_{r,\theta_0}(y)$ and $T^{q_z}_{r,\theta_0}(z)$ are distinct and collinear, and let q_r be the least common multiple of the (maximally reduced and positive) denominators of q_x, q_y and q_z. Then, by the Pigeonhole Principle, there exists $q \in \mathbb{Q}$ such that $q = q_r$ for $\mathfrak{c}$ many suitable r. Letting $K = \{re^{i\theta_0/q}\colon r \in \mathbb{R}\}$, we see that K is a line containing the origin such that $X \cap K = \emptyset$, and since $T_{r^{1/q},\theta/q} = T^{1/q}_{r,\theta}$, it can be argued by using the previous lemma that there are $\mathfrak{c}$ many $r \in \mathbb{R} \setminus \{0\}$ such that $\bigcup_{n\in\mathbb{Z}} T^n_{r,\theta_0/q}(X)$ is not a partial two-point set. □

Lemma 5. *Let X be a divisible partial two-point group such that $|X| < \mathfrak{c}$, and let L be a line which does not contain the origin such that $|X \cap L| < 2$. Then there are fewer than $\mathfrak{c}$ many $re^{i\theta} \in L$ such that $\bigcup_{q\in\mathbb{Q}} T^q_{r,\theta}(X)$ is not a partial two-point group.*

Proof. Let $r_0e^{i\theta_0} \in L$ be the point with minimum Euclidean distance to 0. Then $r_0 \neq 0$ and it is easily seen that

$$L = \{re^{i\theta}\colon \theta \in (\theta_0 - \pi/2, \theta_0 + \pi/2) \text{ and } r = r_0 \sec(\theta - \theta_0)\}.$$

In the following, we consider r to be an abbreviation for $r_0 \sec(\theta - \theta_0)$.

Suppose there are $\mathfrak{c}$ many $re^{i\theta} \in L$ such that $\bigcup_{q\in\mathbb{Q}} T^q_{r,\theta}(X)$ is not a partial two-point set. By the Pigeonhole Principle, we can find $x, y, z \in X$ and $q_x \leqslant q_y \leqslant q_z \in \mathbb{Q}$ such that $T^{q_x}_{r,\theta}(x)$, $T^{q_y}_{r,\theta}(y)$ and $T^{q_z}_{r,\theta}(z)$ are collinear for $\mathfrak{c}$ many $re^{i\theta} \in L$. Since X is a group, we may assume without loss of generality that $x = 1$, and by applying an affine transformation, we may assume without loss of generality that $q_x = 0$. We consider two cases.

Case 1: $0 < q_y < q_z$. Write $x = r_xe^{i\theta_x}$, $y = r_ye^{i\theta_y}$ and $z = r_ze^{i\theta_z}$ in polar form. Then for all $re^{i\theta} \in L$, $T^{q_x}_{r,\theta}(x)$, $T^{q_y}_{r,\theta}(y)$ and $T^{q_z}_{r,\theta}(z)$ are collinear if and only if

$$d(\theta) = t_x(\theta) - t_y(\theta) + t_z(\theta) = 0,$$

where d, t_x, t_y and t_z are the functions from $(\theta_0 - \pi/2, \theta_0 + \pi/2)$ to $\mathbb{R}$ defined by

$$t_x(\theta) = r^{q_y+q_z} r_y r_z \sin\bigl((q_z - q_y)\theta + \theta_z - \theta_y\bigr),$$
$$t_y(\theta) = r^{q_z} r_z \sin(q_z\theta + \theta_z),$$
$$t_z(\theta) = r^{q_y} r_y \sin(q_y\theta + \theta_y),$$
$$d(\theta) = t_x(\theta) - t_y(\theta) + t_z(\theta).$$

Since d is then an analytic function with $\mathfrak{c}$ many roots, it must be identically equal to zero.

We now show that $q_z \in \mathbb{Z}$ and $q_y \in \mathbb{Z}$. Note that as $\theta \to \theta_0 \pm \pi/2$, then $d(\theta) \to 0$ and $r \to \infty$. Hence by continuity it must be the case that

$$(q_z - q_y)(\theta_0 \pm \pi/2) + \theta_z - \theta_y \equiv 0 \pmod{\pi}, \tag{1}$$

for otherwise $|t_x(\theta)| \to \infty$ as $\theta \to \theta_0 \pm \pi/2$ and t_x dominates d, which is a contradiction. By applying L'Hôpital's Rule, we see that the limit of $t_x(\theta)$ as $\theta \to \theta_0 \pm \pi/2$ exists and is finite. Repeating the same argument, we have that

$$q_z(\theta_0 \pm \pi/2) + \theta_z \equiv 0 \pmod{\pi}, \tag{2}$$

and the limit of $t_z(\theta)$ as $\theta \to \theta_0 \pm \pi/2$ exists and is finite. In the same way, we finally see that

$$q_y(\theta_0 \pm \pi/2) + \theta_y \equiv 0 \pmod{\pi}, \tag{3}$$

and by respectively subtracting the equations represented in each of (2) and (3), our claim follows; that is, $q_z \in \mathbb{Z}$ and $q_y \in \mathbb{Z}$.

Now, consider the derivative $d' = t'_x - t'_y + t'_z$, which is of course identically equal to zero, so that $d'(\theta) \to 0$ as $\theta \to \theta_0 + \pi/2$. Let

$$j_x = \frac{t'_x}{r_y r_z r^{q_y+q_z}}, \qquad j_y = \frac{t'_y}{r_z r^{q_z}}, \quad \text{and} \quad j_z = \frac{t'_z}{r_y r^{q_y}},$$

and note that $j_x(\theta)$ is equal to

$$(q_y + q_z)\frac{\sin(\theta - \theta_0)\sin((q_y - q_z)\theta + \theta_y - \theta_z)}{\cos(\theta - \theta_0)} + (q_y - q_z)\cos\bigl((q_y - q_z)\theta + \theta_y - \theta_z\bigr)$$

for all θ. Making use of L'Hôpital's Rule and (1), it can be seen that j_y and j_z are bounded and $|j_x(\theta)| \to (q_y + q_z - 1)(q_y - q_z)$ as $\theta \to \theta_0 + \pi/2$. Then $(q_y + q_z - 1)(q_y - q_z) = 0$, for otherwise $|t'_x(\theta)| \to \infty$ and $|d'(\theta)| \to \infty$ as $\theta \to \theta_0 + \pi/2$, contradicting that q_y and q_z are distinct positive integers.

Case 2: $q_y = 0$ *or* $q_y = q_z$. By applying a suitable linear transformation and possibly renaming some variables, we can argue by using the Identity Theorem that x, y and w^{q_z} are collinear for all $w \in L$. Hence the image of L under the mapping

$w \mapsto w^{q_z}$ on a suitable branch cut of the complex plane is on the line spanned by x and y, and this is only possible if $q_z = 0$, whence x, y and 1 are three distinct collinear points in X; or $q_z = 1$, whence L is the line spanned by x and y; or $q_z = -1$, whence L is the line spanned by x^{-1} and y^{-1}. In any case, by noting that $|L \cap X| = 2$, we have found a contradiction. □

Theorem 6. *There exists a divisible two-point group.*

Proof. Let $\langle L_\alpha\colon \alpha < \mathfrak{c}\rangle$ be an enumeration of all lines in the plane such that $\langle L_n\colon n < \omega\rangle$ is an enumeration of all lines in the plane which contain the origin and make an angle with the horizontal axis which is a rational multiple of π. We now construct a sequence $\langle X_\alpha\colon \alpha < \mathfrak{c}\rangle$ of subsets of the plane such that for all $\alpha < \mathfrak{c}$:

(1) $|X_\alpha| \leqslant |\alpha| + \aleph_0$; and
(2) $\bigcup_{\beta<\alpha} X_\beta$ is a divisible partial two-point group meeting every member of $\{X_\beta\colon \beta < \alpha\}$ in precisely two points.

For each $n < \omega$, let $X_n = \{e^{iq\pi}\colon q \in \mathbb{Q}\}$. Suppose that for some $\omega \leqslant \alpha < \mathfrak{c}$ we have chosen the partial sequence $\langle X_\beta\colon \beta < \alpha\rangle$. Let $n = |L_\alpha \cap \bigcup_{\beta<\alpha} X_\beta|$. We consider three cases.

Case 1: $n < 2$ *and* $0 \in L_\alpha$. Let θ_0 be the angle made by L_α with the horizontal axis and note that $n = 0$. By Lemma 4, let $r \in \mathbb{R} \setminus \{0\}$ be such that

$$X_\alpha := \bigcup_{q\in\mathbb{Q}} T^q_{r,\theta_0}\left(\bigcup_{\beta<\alpha} X_\beta\right)$$

is a divisible partial two-point group. Then $T_{r,\theta_0}(1)$ and $T_{r,\theta_0}(1)e^{i\pi}$ are distinct members of $L_\alpha \cap X_\alpha$.

Case 2: $n < 2$ *and* $0 \notin L_\alpha$. By Lemma 5, let $re^{i\theta} \in L_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta$ be such that

$$X'_\alpha := \bigcup_{q\in\mathbb{Q}} T^q_{r,\theta}\left(\bigcup_{\beta<\alpha} X_\beta\right)$$

is a divisible partial two-point group. Since

$$T_{r,\theta}(1) \in L_\alpha \cap \left(X'_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta\right),$$

we have that $|L_\alpha \cap X'_\alpha| \geqslant n + 1$. If $|L_\alpha \cap X'_\alpha| = 2$, let $X_\alpha = X'_\alpha$. Otherwise, we repeat this argument once in the obvious way to extend X'_α to a suitable X_α.

Case 3: $n = 2$. Let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

By transfinite recursion, let the X_α be defined for all $\alpha < \mathfrak{c}$. Then $\bigcup_{\alpha<\mathfrak{c}} X_\alpha$ is a divisible partial two-point group. □

Corollary 7. *There exists a homogeneous two-point set.*

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