

GEOMETRIC INVARIANT THEORY FOR GRADED UNIPOTENT GROUPS AND APPLICATIONS

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Dedicated to the memory of John Roe (6 Oct 1959-9 Mar 2018), one of the founding editors of the Journal of Topology

ABSTRACT. Let U be a graded unipotent group over the complex numbers, in the sense that it has an extension \hat{U} by the multiplicative group such that the action of the multiplicative group by conjugation on the Lie algebra of U has all its weights strictly positive. Given any action of U on a projective variety X extending to an action of \hat{U} which is linear with respect to an ample line bundle on X , then provided that one is willing to replace the line bundle with a tensor power and to twist the linearisation of the action of \hat{U} by a suitable (rational) character, and provided an additional condition is satisfied which is the analogue of the condition in classical GIT that there should be no strictly semistable points for the action, we show that the \hat{U} -invariants form a finitely generated graded algebra; moreover the natural morphism from the semistable subset of X to the enveloping quotient is surjective and expresses the enveloping quotient as a geometric quotient of the semistable subset. Applying this result with X replaced by its product with the projective line gives us a projective variety which is a geometric quotient by \hat{U} of an invariant open subset of the product of X with the affine line and contains as an open subset a geometric quotient of a U -invariant open subset of X by the action of U . Furthermore these open subsets of X and its product with the affine line can be described using criteria similar to the Hilbert-Mumford criteria in classical GIT.

Mumford’s geometric invariant theory (GIT) allows us to construct and study quotients of algebraic varieties by linear actions of reductive groups [36, 38, 39, 40, 41]. When a complex reductive group G acts linearly (with respect to an ample line bundle L) on a complex projective variety X , the associated GIT quotient $X//G$ is the projective variety $\text{Proj}(\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^G)$ associated to the ring of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^G$, which is a finitely generated graded complex algebra. Geometrically the variety $X//G$ can be described as the image of a surjective morphism from an open subset X^{ss} of X , consisting of the semistable points for the action, or as X^{ss} modulo the equivalence relation \sim such that if $x, y \in X^{ss}$ then $x \sim y$ if and only if the closures of the G -orbits of x and y meet in X^{ss} . The stable points for the action form a subset X^s of X^{ss} which has a geometric quotient X^s/G which is an open subset of $X//G$. Moreover the subsets X^s and X^{ss} can be described using the Hilbert–Mumford criteria for (semi)stability. The GIT quotient $X//G$ and its open subset X^s/G can also be described in terms of symplectic geometry and a moment map [28, 37].

In suitable situations GIT can be generalised to allow us to construct GIT-like quotients for actions of non-reductive groups [9, 10, 11, 12, 18, 19, 20, 21, 23, 24, 30, 48]. However there is an immediate difficulty in extending GIT to linear actions of non-reductive groups, since now the

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ring of invariants is not necessarily finitely generated as a graded algebra, and when it is not finitely generated there is no associated projective variety.

Every affine algebraic group H has a unipotent radical $U \trianglelefteq H$ such that $R = H/U$ is reductive (over \mathbb{C} we have a semi-direct product decomposition $H \cong R \ltimes U$), and understanding GIT-theoretic questions about the action – such as whether invariants are finitely generated – often follows from understanding the action of the unipotent group U . In some cases the U -invariants happen to be finitely generated. For example, if U is the unipotent radical of a parabolic subgroup P of a complex reductive group G and an action of U on a complex projective variety X , which is linear with respect to an ample line bundle L , extends to a linear action of G , then the ring of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^U$ is finitely generated [25, 31]. In this case the ‘enveloping quotient’ $X \mathbin{\mathbb{Z}} U$ (in the sense of [18] but using the notation of [3]) is the projective variety $\text{Proj}(\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^U)$ associated to the ring of invariants, and it contains as an open subset a geometric quotient X^s/U where X^s is a U -invariant open subset of X . However there is still no analogue for $X \mathbin{\mathbb{Z}} U$ of the geometric description of $X//G$ when G is reductive as X^{ss} modulo an equivalence relation, since the natural morphism from X^{ss} to $X \mathbin{\mathbb{Z}} U$ is not in general surjective, although there are alternative geometric descriptions [31].

In this paper we consider a more general situation. Instead of taking U to be the unipotent radical of a parabolic subgroup of a complex reductive group G which acts linearly on X , we assume that U is a unipotent group over \mathbb{C} with an extension $\hat{U} = U \ltimes \mathbb{C}^*$ by \mathbb{C}^* such that the action of \mathbb{C}^* by conjugation on the Lie algebra of U has all its weights strictly positive; we call such a U a graded unipotent group. (The unipotent radical of a parabolic subgroup of a complex reductive group G always has such an extension contained in the parabolic subgroup). We are interested in linear actions of U on projective varieties X which extend to linear actions of \hat{U} . Given any action of U on a projective variety X extending to an action of \hat{U} which is linear with respect to an ample line bundle on X , then *provided* that we are willing to replace the line bundle with a tensor power and to twist the linearisation of the action of \hat{U} by a suitable (rational) character of \hat{U} , and provided an additional condition is satisfied which is the analogue of the condition in classical GIT that there should be no strictly semistable points for the action (that is, ‘semistability coincides with stability’), we find that the \hat{U} -invariants form a finitely generated algebra; moreover the natural morphism $\phi : X^{ss, \hat{U}} \rightarrow X \mathbin{\mathbb{Z}} \hat{U}$ is surjective and indeed expresses $X \mathbin{\mathbb{Z}} \hat{U}$ as a geometric quotient of $X^{ss, \hat{U}}$, so that ϕ satisfies $\phi(x) = \phi(y)$ if and only if the \hat{U} -orbits of x and y coincide in $X^{ss, \hat{U}}$. Applying this result with X replaced by $X \times \mathbb{P}^1$ gives us a projective variety $(X \times \mathbb{P}^1) \mathbin{\mathbb{Z}} \hat{U}$ which is a geometric quotient by \hat{U} of a \hat{U} -invariant open subset of $X \times \mathbb{C}$ and contains as an open subset a geometric quotient of a U -invariant open subset $X^{\hat{s}, U}$ of X by U . Furthermore the subsets $X^{s, \hat{U}} = X^{ss, \hat{U}}$ and $X^{\hat{s}, U}$ of X can be described using Hilbert–Mumford-like criteria.

This situation arises even for the Nagata counterexamples to Hilbert’s 14th problem, which provide examples of linear actions of unipotent groups U on projective space such that the corresponding U -invariants are not finitely generated. In these cases the linear action extends to a linear action of an extension $\hat{U} = U \ltimes \mathbb{C}^*$ by \mathbb{C}^* such that the action of \mathbb{C}^* by conjugation on the Lie algebra of U has all its weights strictly positive. Thus when the condition that semistability coincides with stability is satisfied, we obtain open subsets $X^{s, \hat{U}} = X^{ss, \hat{U}}$ and $X^{\hat{s}, U}$ of X , which are determined by analogues of the Hilbert–Mumford criteria, with geometric

quotients $X^{s,\hat{U}}/\hat{U}$ and $X^{\hat{s},U}/U$, such that $X^{s,\hat{U}}/\hat{U}$ is projective and $X^{\hat{s},U}/U$ is quasi-projective with a projective completion in which the complement of $X^{\hat{s},U}/U$ is $X^{s,\hat{U}}/\hat{U}$.

A related situation is studied in [7], where it is assumed that the linear action of the graded unipotent group U extends to a linear action of a general linear group $\mathrm{GL}(n)$. Here U and \hat{U} are embedded in $\mathrm{GL}(n)$ as subgroups ‘generated along the first row’ in the sense that there are integers $1 = \omega_1 < \omega_2 \leq \omega_3 \leq \dots \leq \omega_n$ and polynomials $p_{i,j}(\alpha_1, \dots, \alpha_n)$ in $\alpha_1, \dots, \alpha_n$ with complex coefficients for $1 < i < j \leq n$ such that

$$(1) \quad \hat{U} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \\ 0 & \alpha_1^{\omega_2} & p_{2,3}(\alpha) & \dots & p_{2,n}(\alpha) \\ 0 & 0 & \alpha_1^{\omega_3} & \dots & p_{3,n}(\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \alpha_1^{\omega_n} \end{pmatrix} : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^* \times \mathbb{C}^{n-1} \right\}$$

and U is the unipotent radical of \hat{U} , defined by $\alpha_1 = 1$. The main results of [7] also involve the subgroup \tilde{U} of $\mathrm{SL}(n)$ which is the intersection of $\mathrm{SL}(n)$ with the product $\hat{U}Z(\mathrm{GL}(n))$ of \hat{U} with the central one-parameter subgroup $Z(\mathrm{GL}(n)) \cong \mathbb{C}^*$ of $\mathrm{GL}(n)$. Like \hat{U} , the subgroup \tilde{U} of $\mathrm{GL}(n)$ is a semi-direct product $\tilde{U} = U \rtimes \mathbb{C}^*$ where \mathbb{C}^* acts on the Lie algebra of U with all weights strictly positive. When $\mathrm{GL}(n)$ acts linearly on a projective variety X with respect to an ample line bundle L on X , and the linearisation of the action of \tilde{U} on X is twisted by a suitable rational character χ (which is ‘well adapted’ to the action in the sense of [7]), then it is shown in [7] Theorem 1.1 that the corresponding algebra of \tilde{U} -invariants is finitely generated, and the projective variety $X \mathbin{\mathcal{Z}} \tilde{U}$ associated to this algebra of invariants is a categorical quotient of an open subset $X^{ss,\tilde{U}}$ of X by \tilde{U} and contains as an open subset a geometric quotient of an open subset $X^{s,\tilde{U}}$ of X . Applying a similar argument after replacing X with $X \times \mathbb{P}^1$ provides a projective variety $(X \times \mathbb{P}^1) \mathbin{\mathcal{Z}} \tilde{U}$ which is a categorical quotient by \tilde{U} of a \tilde{U} -invariant open subset of $X \times \mathbb{C}$ and contains as an open subset a geometric quotient of a U -invariant open subset $X^{s,U}$ of X by U .

The results of this paper are more general than those of [7] in that the linear action of the unipotent group U is only required to extend to a linear action of \hat{U} rather than to a general linear group in which U and \hat{U} are embedded in a very special way. On the other hand in [7] the additional condition that ‘semistability coincides with stability’ is not required. The removal of this additional condition is addressed in [4] (see also [5, 6]), using a partial desingularisation construction analogous to that of [29].

Let $\chi : \hat{U} \rightarrow \mathbb{C}^*$ be a character of \hat{U} with kernel containing U ; we will identify such characters χ with integers so that the integer 1 corresponds to the character which fits into the exact sequence $U \rightarrow \hat{U} \rightarrow \mathbb{C}^*$. Suppose that $\omega_{\min} = \omega_0 < \omega_1 < \dots < \omega_h = \omega_{\max}$ are the weights with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*))}(-1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X,L)^*)^{\mathbb{C}^*})$ for the action of \mathbb{C}^* on $\mathbb{P}((H^0(X,L)^*))$; when L is very ample X embeds in $\mathbb{P}((H^0(X,L)^*))$ and the line bundle L extends to the dual $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*))}(1)$ of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*))}(-1)$. We will assume that there exist at least two distinct such weights since otherwise the action of U on X is trivial. Let c be a positive integer such that

$$\omega_{\min} = \omega_0 < \frac{\chi}{c} < \omega_1;$$

we will call rational characters χ/c with this property *adapted* to the linear action of \hat{U} on L , and we will call the linearisation adapted if $\omega_0 < 0 < \omega_1$. The linearisation of the action of \hat{U} on X with respect to the ample line bundle $L^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $L_\chi^{\otimes c}$ denote this twisted linearisation. Let $X_{\min+}^{s, \mathbb{C}^*}$ denote the stable subset of X for the linear action of \mathbb{C}^* with respect to the linearisation $L_\chi^{\otimes c}$; then

$$X_{\min+}^{s, \mathbb{C}^*} = X_{\min}^0 \setminus Z_{\min}$$

where Z_{\min} is the union of the connected components of the fixed point set $X^{\mathbb{C}^*}$ for the action of \mathbb{C}^* on X given by

$$Z_{\min} = \{x \in X^{\mathbb{C}^*} \mid \mathbb{C}^* \text{ acts on } L^*|_x \text{ with weight } \omega_{\min}\}$$

and

$$X_{\min}^0 = \{x \in X \mid \lim_{t \rightarrow 0} tx \in Z_{\min}\}.$$

We set

$$X_{\min+}^{s, \hat{U}} = X \setminus \hat{U}(X \setminus X_{\min+}^{s, \mathbb{C}^*}) = \bigcap_{u \in U} uX_{\min+}^{s, \mathbb{C}^*}$$

to be the complement of the \hat{U} -sweep (or equivalently the U -sweep) of $X \setminus X_{\min+}^{s, \mathbb{C}^*}$.

In order to state the main theorem of this paper it is necessary to strengthen the condition that the linearisation is adapted. We will say that a property holds for a linear action of \hat{U} with respect to a linearisation twisted by a *well adapted* rational character if there exists $\epsilon > 0$ such that if χ/c is any rational character of \mathbb{C}^* (lifted to \hat{U}) with

$$\omega_{\min} < \frac{\chi}{c} < \omega_{\min} + \epsilon$$

then the property holds for the induced linearisation on $L^{\otimes c}$ twisted by χ .

We also require that the action of U satisfies an additional condition to which we will refer as ‘semistability coincides with stability’. More precisely, we require that whenever U' is a subgroup of U normalised by \mathbb{C}^* and ξ belongs to the Lie algebra of U but not the Lie algebra of U' , then the weight space with weight $-\omega_{\min}$ for the action of \mathbb{C}^* on $H^0(X, L)$ is contained in the image $\delta_\xi(H^0(X, L)^{U'})$ of $H^0(X, L)^{U'}$ under the infinitesimal action $\delta_\xi : H^0(X, L) \rightarrow H^0(X, L)$ of ξ on $H^0(X, L)$.

Remark 0.1. For motivation for the terminology ‘semistability coincides with stability’ see Remark 2.2, and also [4] where a slightly weaker interpretation of this terminology is used.

Theorem 0.2. Let $\hat{U} = U \rtimes \mathbb{C}^*$ be a semidirect product of the unipotent group U by \mathbb{C}^* , where the conjugation action of \mathbb{C}^* on U is such that all the weights of the induced \mathbb{C}^* -action on the Lie algebra of U are strictly positive. Suppose that \hat{U} acts linearly on a projective variety X with respect to a very ample line bundle L , and that this linear action satisfies the condition that ‘semistability coincides with stability’ as above. Suppose also that $\chi : \hat{U} \rightarrow \hat{U}/U \rightarrow \mathbb{C}^*$ is a character of \hat{U} and c is a positive integer such that the rational character χ/c is adapted to the linear action of \hat{U} on X . Then

(i) $X_{\min+}^{s, \hat{U}}$ is a \hat{U} -invariant open subvariety of X with a geometric quotient $X_{\min+}^{s, \hat{U}}/\hat{U}$ by \hat{U} which is a projective variety.

If moreover the rational character χ/c is well adapted, then

(ii) the algebra of invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{U}}$ is finitely generated, $X_{\min+}^{s, \hat{U}}$ is the corresponding (semi)stable locus for the linear action of \hat{U} on X and the enveloping quotient

$$X \mathbin{\mathbb{Z}} \hat{U} \cong X_{\min+}^{s, \hat{U}} / \hat{U}$$

is the projective variety associated to this algebra of invariants.

Remark 0.3. Recall that Popov [39] has shown that if H is any non-reductive group then there is an affine variety Y on which H acts such that the algebra of invariants $\mathcal{O}(Y)^H$ is not finitely generated.

Applying Theorem 0.2 after replacing X with $X \times \mathbb{P}^1$ we obtain geometric information about the action of the unipotent group U on X :

Corollary 0.4. In the situation above let \hat{U} act diagonally on $X \times \mathbb{P}^1$ where the action on \mathbb{P}^1 is via

$$\hat{u} \cdot [x : y] = [\chi_1(\hat{u})x : y]$$

where $\chi_1 : \hat{U} \rightarrow \mathbb{C}^*$ is the character of \hat{U} with kernel U which fits into the extension $\{1\} \rightarrow U \rightarrow \hat{U} \rightarrow \mathbb{C}^* \rightarrow \{1\}$, and linearise this action using the tensor product of L_{χ} with $\mathcal{O}_{\mathbb{P}^1}(M)$ for suitable $M \geq 1$. Then $(X \times \mathbb{P}^1) \mathbin{\mathbb{Z}} \hat{U}$ is a projective variety which is a geometric quotient by \hat{U} of a \hat{U} -invariant open subset of $X \times \mathbb{C}$ and contains as an open subset a geometric quotient of a U -invariant open subset $X^{\hat{s}, U}$ of X by U .

Remark 0.5. We can also deduce that the algebra $A = \oplus_{m=0}^{\infty} H^0(X \times \mathbb{P}^1, L_{m\chi}^{\otimes cm} \otimes \mathcal{O}_{\mathbb{P}^1}(M))^{\hat{U}}$ of \hat{U} -invariants on $X \times \mathbb{P}^1$ is finitely generated for a well-adapted rational character χ/c of \hat{U} when c is a sufficiently divisible positive integer. This graded algebra A can be identified with the subalgebra of the algebra of U -invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^U$ on X generated by the U -invariants in $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^U$ which are weight vectors with non-positive weights for the action of $\mathbb{C}^* \leq \hat{U}$ after twisting by the well-adapted rational character χ/c . The sections σ of L which are weight vectors with weight $-\omega_{\min}$ are all U -invariant, and after twisting by χ/c these are the only weight vectors in $H^0(X, L)$ which have non-positive (in fact strictly negative) weights. If we localise the U -invariants at any such σ then we get a finitely generated algebra of invariants $\mathcal{O}(X_{\sigma})^U$, since this algebra can be identified with the localisation of A at σ . This can be proved directly, and leads to an alternative proof of Theorem 0.2 (cf. [4]).

Theorem 0.2 has another immediate corollary:

Corollary 0.6. Let $H \cong R \ltimes U$ be a complex linear algebraic group with unipotent radical U and $R \cong H/U$ reductive, and suppose that R contains a central subgroup isomorphic to \mathbb{C}^* which acts by conjugation on the Lie algebra of U with all weights strictly positive. Let \hat{U} be the subgroup of H which is the semidirect product of U and this central one-parameter subgroup \mathbb{C}^* of R . Suppose that H acts linearly on a projective variety X with respect to an ample line bundle L , and that $\chi : H \rightarrow \mathbb{C}^*$ is a character of H , that c is a sufficiently divisible positive integer such that the restriction to \hat{U} of the rational character χ/c is well adapted for the linear action of \hat{U} on X , and that the linear action of the unipotent radical U satisfies the condition that ‘semistability coincides with stability’ as above. Then the algebra of H -invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^H$ is finitely generated, and the projective variety $X \mathbin{\mathbb{Z}} H$ associated to this algebra of invariants is a categorical quotient of an open subvariety $X^{ss, H}$ of X by H ,

and the canonical H -invariant morphism $\phi : X^{ss,H} \rightarrow X \mathbin{\mathbb{H}} H$ is surjective with $\phi(x) = \phi(y)$ if and only if the closures of the H -orbits of x and y meet in $X^{ss,H}$.

Proof. This result follows by quotienting X first by \hat{U} , using Theorem 0.2, and then by the induced linear action of the reductive group $H/\hat{U} \cong R/\mathbb{C}^*$. \square

Remark 0.7. Note that we require the rational character χ/c to be *well* adapted to ensure that the algebra of H -invariants $\bigoplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^H$ is finitely generated and that we obtain an induced ample line bundle on $X \mathbin{\mathbb{H}} H$. However, in order to obtain the open subvariety $X^{ss,H}$ of X and its categorical quotient which is the projective variety $X \mathbin{\mathbb{H}} H$, it is enough for the rational character to be merely adapted, not well adapted. Moreover both $X^{ss,H}$ and the stable locus $X^{s,H}$ (which has a geometric quotient by the H -action) are determined by Hilbert–Mumford criteria, exactly as for classical GIT.

Example: Consider the weighted projective plane $\mathbb{P}(1, 1, 2)$ which is $\mathbb{C}^3 \setminus \{0\}$ modulo the action of \mathbb{C}^* with weights 1, 1, 2. The automorphism group of $\mathbb{P}(1, 1, 2)$ is

$$\mathrm{Aut}(\mathbb{P}(1, 1, 2)) \cong R \ltimes U$$

with $R \cong GL(2)$ reductive and $U \cong (\mathbb{C}^+)^3$ unipotent; here $(\lambda, \mu, \nu) \in (\mathbb{C}^+)^3 \cong U$ acts on the weighted projective plane $\mathbb{P}(1, 1, 2)$ as $[x, y, z] \mapsto [x, y, z + \lambda x^2 + \mu xy + \nu y^2]$. The central one-parameter subgroup \mathbb{C}^* of $R \cong GL(2)$ acts on $\mathrm{Lie}(U)$ with all positive weights, and the associated extension $\hat{U} = U \ltimes \mathbb{C}^*$ can be identified with a subgroup of $\mathrm{Aut}(\mathbb{P}(1, 1, 2))$. Thus Corollary 0.6 applies to every linear action of $\mathrm{Aut}(\mathbb{P}(1, 1, 2))$ on a projective variety X with respect to an ample line bundle L after twisting by a well adapted rational character.

The weighted projective plane $\mathbb{P}(1, 1, 2)$ is a simple example of a toric variety; in fact as we shall see in §4 below, the automorphism group of any complete simplicial toric variety satisfies the conditions of Corollary 0.6.

Our first motivation for considering linear actions of groups of the form \hat{U} in this article and in [7] came from the study of jet differentials. The groups \mathbb{G}_k of k -jets of holomorphic reparametrizations of $(\mathbb{C}, 0)$ (and more generally the groups $\mathbb{G}_{k,p}$ of k -jets of holomorphic reparametrizations of $(\mathbb{C}^p, 0)$ for $p \geq 1$) play an important role in the strategy of Demailly, Siu and others [1, 8, 14, 15, 16, 22, 32, 34, 43, 44, 45] towards the Green-Griffiths conjecture on entire holomorphic curves in hypersurfaces of large degree in projective spaces. Here \mathbb{G}_k is a non-reductive complex linear algebraic group which is a semi-direct product $\mathbb{G}_k = \mathbb{U}_k \ltimes \mathbb{C}^*$ of its unipotent radical \mathbb{U}_k by \mathbb{C}^* acting with weights 1, 2, 3, \dots , k on the Lie algebra of \mathbb{U}_k , while if $p > 1$ then $\mathbb{G}_{k,p} = \mathbb{U}_{k,p} \ltimes GL(p; \mathbb{C})$ where all the weights of the central one-parameter subgroup \mathbb{C}^* of $GL(p; \mathbb{C})$ on the Lie algebra of the unipotent radical $\mathbb{U}_{k,p}$ of $\mathbb{G}_{k,p}$ are strictly positive. So the results above apply to linear actions of the reparametrization group \mathbb{G}_k and its generalizations $\mathbb{G}_{k,p}$ for $p \geq 1$). In particular the reparametrization group \mathbb{G}_k acts fibrewise in a natural way on the Semple jet bundle $J_k(T^*X) \rightarrow X$ over a complex manifold X of dimension n with fibre

$$J_{k,x} \cong \bigoplus_{j=1}^k \mathrm{Sym}^j(\mathbb{C}^n)$$

at x consisting of the k -jets of holomorphic curves at x . There is an induced action of \mathbb{G}_k on the polynomial ring $\mathcal{O}(J_{k,x})$, which can be identified with the algebra $\bigoplus_{m=0}^{\infty} H^0(\mathbb{P}(J_{k,x}), \mathcal{O}_{\mathbb{P}(J_{k,x})}(1)^{\otimes m})$ of sections of powers of the hyperplane line bundle on the associated projective space $\mathbb{P}(J_{k,x})$,

and the bundle $E_k \rightarrow X$ of Demailly-Semple invariant jet differentials of order k has fibre at x given by $(E_k)_x = \mathcal{O}(J_{k,x})^{\mathbb{U}_k}$.

The layout of this paper is as follows. §1 reviews the results of [18] and [3] on non-reductive GIT, and §2 considers the case when $\dim(U) = 1$ and proves Theorem 0.2 in this case. §3 uses these results to prove Theorem 0.2 and Corollaries 0.4 and 0.6. In §4 we observe that Corollary 0.6 applies to the automorphism groups of all complete simplicial toric varieties, while §5 discusses applications to Demailly-Semple jet differentials and their generalisations to maps $\mathbb{C}^p \rightarrow X$.

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1. CLASSICAL AND NON-REDUCTIVE GEOMETRIC INVARIANT THEORY

Let X be a complex quasi-projective variety and let G be a complex reductive group acting on X . To apply (classical) geometric invariant theory (GIT) we require a linearisation of the action; that is, a line bundle L on X and a lift \mathcal{L} of the action of G to L .

Remark 1.1. Usually L is assumed to be ample, and it makes no difference for classical GIT if we replace L with $L^{\otimes k}$ for any integer $k > 0$, so then we lose little generality in supposing that for some projective embedding $X \subseteq \mathbb{P}^n$ the action of G on X extends to an action on \mathbb{P}^n given by a representation

$$\rho : G \rightarrow GL(n+1),$$

and taking for L the hyperplane line bundle on \mathbb{P}^n .

Definition 1.2. Let X be a quasi-projective complex variety with an action of a complex reductive group G and linearisation \mathcal{L} with respect to a line bundle L on X . Then $y \in X$ is *semistable* for this linear action if there exists some $m > 0$ and $f \in H^0(X, L^{\otimes m})^G$ not vanishing at y such that the open subset

$$X_f := \{x \in X \mid f(x) \neq 0\}$$

is affine, and y is *stable* if also the action of G on X_f is closed with all stabilisers finite.

Remark 1.3. This definition comes from [36], although in [36] the terminology ‘properly stable’ is used instead of stable. When X is projective and L is ample and $f \in H^0(X, L^{\otimes m})^G$ for $m > 0$, then X_f is affine if and only if f is nonzero. The reason for introducing the requirement that X_f must be affine in Definition 1.2 above is to ensure that X^{ss} has a quasi-projective categorical quotient $X^{ss} \rightarrow X//G$, which restricts to a geometric quotient $X^s \rightarrow X^s/G$ (see [36] Theorem 1.10).

From now on in this section we will assume that X is projective and L is ample. We have an induced action of G on the homogeneous coordinate ring

$$\hat{\mathcal{O}}_L(X) = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})$$

of X . The subring $\hat{\mathcal{O}}_L(X)^G$ consisting of the elements of $\hat{\mathcal{O}}_L(X)$ left invariant by G is a finitely generated graded complex algebra because G is reductive, and the GIT quotient $X//G$ is the projective variety $\text{Proj}(\hat{\mathcal{O}}_L(X)^G)$. The subsets X^{ss} and X^s of X are characterised by the following properties (see [36, Chapter 2] or [38]).

Proposition 1.4. (Hilbert-Mumford criteria) (i) A point $x \in X$ is semistable (respectively stable) for the action of G on X if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the action of a fixed maximal torus of G .

(ii) A point $x \in X$ with homogeneous coordinates $[x_0 : \dots : x_n]$ in some coordinate system on \mathbb{P}^n is semistable (respectively stable) for the action of a maximal torus of G acting diagonally on \mathbb{P}^n with weights $\alpha_0, \dots, \alpha_n$ if and only if the convex hull

$$\text{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0 (respectively contains 0 in its interior).

Now let H be any affine algebraic group, with unipotent radical U , acting linearly on a complex projective variety X with respect to an ample line bundle L . Then the ring of invariants

$$\hat{\mathcal{O}}_L(X)^H = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H$$

is not necessarily finitely generated as a graded complex algebra, so that $\text{Proj}(\hat{\mathcal{O}}_L(X)^H)$ is not well-defined as a projective variety, although $\text{Proj}(\hat{\mathcal{O}}_L(X)^H)$ does make sense as a scheme, and the inclusion of $\hat{\mathcal{O}}_L(X)^H$ in $\hat{\mathcal{O}}_L(X)$ gives us a rational map of schemes q from X to $\text{Proj}(\hat{\mathcal{O}}_L(X)^H)$, whose image is a constructible subset of $\text{Proj}(\hat{\mathcal{O}}_L(X)^H)$ (that is, a finite union of locally closed subschemes). The action on X of the unipotent radical U of H is studied in [18] following earlier work [20, 21, 23, 24, 48].

Definition 1.5. (See [18] §4). Let $I = \bigcup_{m \geq 0} H^0(X, L^{\otimes m})^U$ and for $f \in I$ let X_f be the U -invariant affine open subset of X where f does not vanish, with $\mathcal{O}(X_f)$ its coordinate ring. A point $x \in X$ is called *naively semistable* if there exists some $f \in I$ which does not vanish at x , and the set of naively semistable points is denoted $X^{ns} = \bigcup_{f \in I} X_f$. The *finitely generated semistable set* of X is $X^{ss,fg} = \bigcup_{f \in I^{fg}} X_f$ where

$$I^{fg} = \{f \in I \mid \mathcal{O}(X_f)^U \text{ is finitely generated}\}.$$

The set of *naively stable* points of X is $X^{ns} = \bigcup_{f \in I^{ns}} X_f$ where

$$I^{ns} = \{f \in I^{fg} \mid q : X_f \longrightarrow \text{Spec}(\mathcal{O}(X_f)^U) \text{ is a geometric quotient}\},$$

and the set of *locally trivial stable* points is $X^{lts} = \bigcup_{f \in I^{lts}} X_f$ where

$$I^{lts} = \{f \in I^{fg} \mid q : X_f \longrightarrow \text{Spec}(\mathcal{O}(X_f)^U) \text{ is a locally trivial geometric quotient}\}.$$

The *enveloped quotient* of $X^{ss,fg}$ is $q : X^{ss,fg} \rightarrow q(X^{ss,fg})$, where $q : X^{ss,fg} \rightarrow \text{Proj}(\hat{\mathcal{O}}_L(X)^U)$ is the natural morphism of schemes and $q(X^{ss,fg})$ is a dense constructible subset of the *enveloping quotient*

$$X \wr U = \bigcup_{f \in I^{fg}} \text{Spec}(\mathcal{O}(X_f)^U)$$

of $X^{ss,fg}$.

Motivated by [18] 5.3.1 and 5.3.5, we call a point $x \in X$ *stable* for the linear U -action if $x \in X^{lts}$ and *semistable* if $x \in X^{ss,fg}$. We write X^s (or $X^{s,U}$) for X^{lts} , and we write X^{ss} (or $X^{ss,U}$) for $X^{ss,fg}$ (cf. [18] 5.3.7).

Remark 1.6. $q(X^{ss})$ is not necessarily a subvariety of $X \wr U$ (see for example [18] §6).

Remark 1.7. If $\hat{\mathcal{O}}_L(X)^U$ is finitely generated then $X \wr U$ is the corresponding projective variety $\text{Proj}(\hat{\mathcal{O}}_L(X)^U)$ [18]. In [18] 4.2.9 and 4.2.10 it is also claimed that the enveloping quotient $X \wr U$ is always a quasi-projective variety with an ample line bundle $L_H \rightarrow X \wr U$ which pulls back to a positive tensor power of L under the natural map $q : X^{ss} \rightarrow X \wr U$. The argument given there fails in general since the morphisms $X_f \rightarrow \text{Spec}(\mathcal{O}(X_f)^U)$ for $f \in I^{ss,fg}$ are not necessarily surjective. However it is still true that the enveloping quotient $X \wr U$ has quasi-projective open subvarieties (‘inner enveloping quotients’ $X/\zeta H$) which contain the enveloped quotient $q(X^{ss})$ and have ample line bundles pulling back to positive tensor powers of L under the natural map $q : X^{ss} \rightarrow X \wr U$ (see [3] for details).

The results of [18] can be generalised to allow us to study H -invariants for linear algebraic groups H which are neither unipotent nor reductive [3, 7]. Over \mathbb{C} any linear algebraic group H is a semi-direct product $H = U \rtimes R$ where $U \subset H$ is the unipotent radical of H (its maximal unipotent normal subgroup) and $R \simeq H_r = H/U$ is a reductive subgroup of H . When H acts linearly on a projective variety X with respect to an ample line bundle L , the naively semistable and (finitely generated) semistable sets X^{nss} and $X^{ss} = X^{ss,fg}$, enveloped and enveloping quotients and inner enveloping quotients

$$q : X^{ss} \rightarrow q(X^{ss}) \subseteq X/\zeta H \subseteq X \wr H$$

are defined in [3] as for the unipotent case in Definition 1.5 and Remark 1.7. However the definition of the stable set X^s combines the unipotent and reductive cases as follows.

Definition 1.8. Let H be a linear algebraic group acting on an irreducible variety X and $L \rightarrow X$ a linearisation for the action. The *stable locus* is the open subset

$$X^s = \bigcup_{f \in I^s} X_f$$

of X^{ss} , where $I^s \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is the subset of H -invariant sections satisfying the following conditions:

- (1) the open set X_f is affine;
- (2) the action of H on X_f is closed with all stabilisers finite groups; and
- (3) the restriction of the U -enveloping quotient map

$$q_U : X_f \rightarrow \text{Spec}((S^U)_{(f)})$$

is a principal U -bundle for the action of U on X_f .

If it is necessary to indicate the group H we will write $X^{s,H}$ and $X^{ss,H}$ for X^s and X^{ss} .

Remark 1.9. This definition of stability extends the definition of stability in [18] for unipotent groups, and in the case where H is reductive, then U is trivial and the definition reduces to Mumford’s notion of properly stable points in [36]. Note that as one would hope

(i) if R is a reductive subgroup of H then it follows straight from the definition that $X^{s,R} \subseteq X^{s,H}$;

(ii) if N is a normal subgroup of H such that the canonical projection $U \rightarrow U/N_u$ splits, and if W is an H -invariant open subvariety of $X^{s,N}$ with a geometric quotient W/N which is an H/N -invariant open subvariety of $X^{s,N}/N \subseteq X/\zeta N$, where $X/\zeta N$ is an inner enveloping quotient of X by N such that a tensor power $L^{\otimes m}$ of L induces a very ample line bundle on

$X/\zeta N$ and hence an embedding of $X/\zeta N$ in the corresponding projective space with closure $\overline{X/\zeta N}$, and if $W/N \subseteq (\overline{X/\zeta N})^{s,H/N}$, then $W \subseteq X^{s,H}$.

Note also however that, in contrast with classical GIT for reductive group actions with ample linearisations, if Y is an H -invariant subvariety of X and we restrict the linearisation of the H -action on X to Y , then $Y^{s,H}$ and $Y^{ss,H}$ do not necessarily coincide with $Y \cap X^{s,H}$ and $Y \cap X^{ss,H}$.

The following result which we will need is proved in [3] Cor 3.1.20.

Proposition 1.10. *Suppose H is a linear algebraic group, X an irreducible H -variety and $L \rightarrow X$ a linearisation. If the enveloping quotient $X \twoheadrightarrow H$ is quasi-compact and complete, then for suitably divisible integers $r > 0$ the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes kr})^H$ is finitely generated and the enveloping quotient $X \twoheadrightarrow H$ is the associated projective variety; moreover the line bundle $L^{\otimes r}$ induces a very ample line bundle $L_{[H]}^{\otimes r}$ on $X \twoheadrightarrow H$ with a natural isomorphism*

$$\bigoplus_{k \geq 0} H^0(X, L^{\otimes kr})^H \cong \bigoplus_{k \geq 0} H^0(X \twoheadrightarrow H, (L_{[H]}^{\otimes r})^{\otimes k}).$$

If a linear algebraic group $H = U \rtimes R$ with unipotent radical U is a subgroup of a reductive group G then there is an induced right action of R on G/U which commutes with the left action of G . Similarly if H acts on a projective variety X then there is an induced action of $G \times R$ on $G \times_U X$ with an induced $G \times R$ -linearisation. The same is true if we replace the requirement that H is a subgroup of G with the existence of a group homomorphism $H \rightarrow G$ which is U -faithful (that is, its restriction to U is injective). We can thus consider the situation when X is a nonsingular complex projective variety acted on by a linear algebraic group $H = U \rtimes R$, while L provides a very ample linearisation of the H action defining an embedding $X \subseteq \mathbb{P}^n$, and $H \rightarrow G$ is an U -faithful homomorphism into a reductive subgroup G of $\mathrm{SL}(n+1; \mathbb{C})$ with respect to an ample line bundle L .

Remark 1.11. Suppose that L' is a $G \times R$ -linearisation over a nonsingular projective completion $\overline{G \times_U X}$ of $G \times_U X$ extending the $G \times R$ linearisation over $G \times_U X$ induced by L . Let D_1, \dots, D_r be the codimension one components of the boundary of $G \times_U X$ in $\overline{G \times_U X}$, and suppose for all sufficiently divisible N that $L'_N = L'[N \sum_{j=1}^r D_j]$ is an ample line bundle on $\overline{G \times_U X}$. Then as noted in [7] the proof of [18] Theorem 5.1.18 tells us that the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated if and only if for all sufficiently divisible N any $G \times R$ -invariant section of a positive tensor power of L'_N vanishes on every codimension one component D_j . Moreover when this happens the enveloping quotient satisfies

$$(2) \quad X \twoheadrightarrow H = \mathrm{Proj}(\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H) \cong \overline{G \times_U X} //_{L'_N} (G \times R)$$

for sufficiently divisible N .

In general even when the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H$ on X is finitely generated and (2) is true, the morphism $X \rightarrow X \twoheadrightarrow H$ is not surjective and in order to study the geometry of $X \twoheadrightarrow H$ by identifying it with $\overline{G \times_U X} //_{L'_N} (G \times R)$ we need information about the boundary $\overline{G \times_U X} \setminus G \times_U X$ of $\overline{G \times_U X}$. If, however, we are lucky enough to have a situation where for sufficiently divisible N the line bundle L'_N is ample and the boundary $\overline{G \times_U X} \setminus G \times_U X$ is unstable for the linear action of $G \times R$, then the picture is almost as well behaved as for reductive group actions on projective varieties with ample linearisations, as follows.

Definition 1.12. Let $X^{\overline{ss}} = X \cap \overline{G \times_U X}^{ss, G \times R}$ and $X^{\overline{s}} = X \cap \overline{G \times_U X}^{s, G \times R}$ where X is embedded in $G \times_U X$ in the obvious way as $x \mapsto [1, x]$.

Theorem 1.13. ([7] Thm 2.9 and [18] 5.3.1 and 5.3.5)). Let X be a complex projective variety acted on by a linear algebraic group $H = U \rtimes R$ where U is the unipotent radical of H and let L be a very ample linearisation of the H action defining an embedding $X \subseteq \mathbb{P}^n$. Let $H \rightarrow G$ be an U -faithful homomorphism into a reductive subgroup G of $\mathrm{SL}(n+1; \mathbb{C})$ with respect to an ample line bundle L . Let L' be a $G \times R$ -linearisation over a projective completion $\overline{G \times_U X}$ of $G \times_U X$ extending the $G \times R$ linearisation over $G \times_U X$ induced by L . Let D_1, \dots, D_r be the codimension 1 components of the boundary of $G \times_U X$ in $\overline{G \times_U X}$, and suppose that $L'_N = L'[N \sum_{j=1}^r D_j]$ is an ample line bundle on $\overline{G \times_U X}$ for all sufficiently divisible N . If for all sufficiently divisible N any $G \times R$ -invariant section of a positive tensor power of L'_N vanishes on the boundary of $G \times_U X$ in $\overline{G \times_U X}$, then

- (1) $X^{\overline{s}} = X^s$ and $X^{ss} = X^{\overline{ss}}$;
- (2) the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated;
- (3) the enveloping quotient $X \mathbin{\mathbb{Z}} H \cong \overline{G \times_U X} //_{L'_N} (G \times R) \cong \mathrm{Proj}(\bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^H)$ for sufficiently divisible N ;
- (4) $\overline{G \times_U X}^{ss, G \times R, L'_N} \subseteq G \times_U X$ and therefore the morphism

$$\phi : X^{\overline{ss}} \rightarrow X \mathbin{\mathbb{Z}} H$$

is surjective and $X \mathbin{\mathbb{Z}} H$ is a categorical quotient of $X^{\overline{ss}}$;

- (5) if $x, y \in X^{ss}$ then $\phi(x) = \phi(y)$ if and only if the closures of the H -orbits of x and y meet in X^{ss} ;
- (6) ϕ restricts to a geometric quotient $X^{\overline{s}} \rightarrow X^{\overline{s}}/H \subseteq X \mathbin{\mathbb{Z}} H$.

2. ACTIONS OF $\mathbb{C}^+ \rtimes \mathbb{C}^*$

We will prove Theorem 0.2 by induction on the dimension of U . In this section we will concentrate on the case when $\dim(U) = 1$ so that $U \cong \mathbb{C}^+$.

Definition 2.1. Let X be a complex projective variety equipped with a linear action (with respect to an ample line bundle L) of a semi-direct product $\hat{U} = \mathbb{C}^* \rtimes U$, where U is unipotent and the weights of the induced \mathbb{C}^* action on the Lie algebra of U are all strictly positive. When ξ is an element of the Lie algebra of U let $\delta_\xi : H^0(X, L) \rightarrow H^0(X, L)$ define the infinitesimal action of ξ on $H^0(X, L)$. We say that *semistability coincides with stability* for this linear action if whenever U' is a subgroup of U normalised by \mathbb{C}^* and ξ belongs to the Lie algebra of U but not the Lie algebra of U' , then the weight space with maximum weight $-\omega_{\min}$ for the action of \mathbb{C}^* on $H^0(X, L)$ is contained in the image $\delta_\xi(H^0(X, L)^{U'})$ of $H^0(X, L)^{U'}$ under the infinitesimal action $\delta_\xi : H^0(X, L) \rightarrow H^0(X, L)$ of ξ on $H^0(X, L)$.

Remark 2.2. When $U = \mathbb{C}^+$ has dimension one, this says that for any nonzero $\xi \in \mathrm{Lie}(U)$ the weight space with maximum weight $-\omega_{\min}$ for the action of \mathbb{C}^* on $H^0(X, L)$ is contained in the image $\delta_\xi(H^0(X, L))$ of $H^0(X, L)$ under δ_ξ . This will be the case unless δ_ξ has a Jordan block of size 1 with weight $-\omega_{\min}$; that is, unless $H^0(X, L)$ as a \hat{U} -module has a direct summand on which U acts trivially and \mathbb{C}^* acts with weight $-\omega_{\min}$. Equivalently each point in the projective space $\mathbb{P}(H^0(X, L)^*)$ represented by a weight vector of minimal weight ω_{\min} has trivial stabiliser in U .

Thus when $U = \mathbb{C}^+$ has dimension one and $X = \mathbb{P}(H^0(X, L)^*)$ is a projective space, the condition that semistability coincides with stability is equivalent to the requirement that every $x \in Z_{\min}$ has trivial stabiliser in U (cf. [4]).

Definition 2.3. Let $\chi : \hat{U} \rightarrow \mathbb{C}^*$ be a character of the semi-direct product $\hat{U} = \mathbb{C}^* \ltimes U$ acting linearly on X as above. Suppose that $\omega_{\min} = \omega_0 < \omega_1 < \dots < \omega_h = \omega_{\max}$ are the weights with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X, L)^*)}(-1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X, L)^*)^{\mathbb{C}^*})$ for the action of \mathbb{C}^* on $\mathbb{P}((H^0(X, L)^*)$. We assume that there exist at least two distinct such weights since otherwise the action of U on X is trivial. Let c be a positive integer such that

$$\omega_{\min} = \omega_0 < \frac{\chi}{c} < \omega_1;$$

such rational characters χ/c are called *adapted* to the linear action of \hat{U} on L , and the linearisation is said to be adapted if $\omega_0 < 0 < \omega_1$. Recall that the linearisation of the action of \hat{U} on X with respect to the ample line bundle $L^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $L_{\chi}^{\otimes c}$ denote this twisted linearisation. Note that the unipotent group U is contained in the kernel of χ and so the restriction of the linearisation to the action of U is unaffected by this twisting. Let $X_{\min+}^{s, \mathbb{C}^*}$ denote the stable subset of X for the linear action of \mathbb{C}^* with respect to the linearisation $L_{\chi}^{\otimes c}$, so that

$$X_{\min+}^{s, \mathbb{C}^*} = X_{\min}^0 \setminus Z_{\min}$$

where

$$Z_{\min} = \{x \in X^{\mathbb{C}^*} \mid \mathbb{C}^* \text{ acts on } L^*|_x \text{ with weight } \omega_{\min}\}$$

and

$$X_{\min}^0 = \{x \in X \mid \lim_{t \rightarrow 0} tx \in Z_{\min}\}.$$

Let

$$X_{\min+}^{s, \hat{U}} = X \setminus \hat{U}(X \setminus X_{\min+}^{s, \mathbb{C}^*}) = \bigcap_{u \in U} uX_{\min+}^{s, \mathbb{C}^*}.$$

Finally we say that a property holds for a linear action of \hat{U} with respect to a linearisation twisted by a *well adapted* rational character if there exists $\epsilon > 0$ such that if χ/c is any rational character of \mathbb{C}^* (lifted to \hat{U}) with

$$\omega_{\min} < \frac{\chi}{c} < \omega_{\min} + \epsilon$$

then the property holds for the induced linearisation on $L^{\otimes c}$ twisted by χ .

The aim of this section is to prove the following theorem, which we will use for our inductive proof of Theorem 0.2.

Theorem 2.4. *Let X be a complex projective variety equipped with a linear action (with respect to an ample line bundle L) of a semi-direct product $\hat{U} = \mathbb{C}^* \ltimes \mathbb{C}^+$, where the weight of the induced \mathbb{C}^* action on the Lie algebra of $U = \mathbb{C}^+$ is strictly positive. Suppose that the linear action of \hat{U} on X satisfies the condition that ‘semistability coincides with stability’ as above. If $\chi : \hat{U} \rightarrow \mathbb{C}^*$ is a character of \hat{U} and c is a sufficiently divisible positive integer such that the rational character χ/c is adapted for the linear action of \hat{U} with respect to L , then after twisting this linear action by χ/c we have*

- (1) $X_{\min+}^{s,\hat{U}}$ is a \hat{U} -invariant open subvariety of X with a geometric quotient $\pi : X_{\min+}^{s,\hat{U}} \rightarrow X_{\min+}^{s,\hat{U}}/\hat{U}$ by the action of \hat{U} ;
- (2) this geometric quotient $X_{\min+}^{s,\hat{U}}/\hat{U}$ is a projective variety and the tensor power $L^{\otimes c}$ of L descends to a very ample line bundle on $X_{\min+}^{s,\hat{U}}/\hat{U}$;
- (3) $X^{ss,\hat{U}} = X_{\min+}^{s,\hat{U}}$ and the geometric quotient $X_{\min+}^{s,\hat{U}}/\hat{U}$ is the enveloping quotient $X \mathbin{\mathbb{Z}}_{L_X^{\otimes c}} \hat{U}$;
- (4) the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L_{kX}^{\otimes ck})^{\hat{U}}$ is finitely generated and the enveloping quotient $X \mathbin{\mathbb{Z}}_{L_X^{\otimes c}} \hat{U} \cong \text{Proj}(\bigoplus_{k \geq 0} H^0(X, L_{kX}^{\otimes ck})^{\hat{U}})$ is the associated projective variety;
- (5) the tensor power $L^{\otimes c}$ of L induces a very ample line bundle on an inner enveloping quotient $X/\mathbb{Z} U$ for the action of U on X with a \mathbb{C}^* -equivariant embedding

$$X/\mathbb{Z} U \rightarrow \mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$$

as a quasi-projective subvariety, containing the geometric quotient $X^{s,U}/U$ as an open subvariety, with closure $\overline{X/\mathbb{Z} U}$ in $\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$;

- (6) $X_{\min+}^{s,\hat{U}}$ is a U -invariant open subvariety of $X^{s,U}$ and has a geometric quotient $X_{\min+}^{s,\hat{U}}/U$ which coincides with both the stable and semistable loci $(\overline{X/\mathbb{Z} U})^{s,\mathbb{C}^*} = (\overline{X/\mathbb{Z} U})^{ss,\mathbb{C}^*}$ for the \mathbb{C}^* action with respect to the linearisation on $\mathcal{O}_{\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)}(1)$ induced by $L_X^{\otimes c}$, so that the associated GIT quotient of $\overline{X/\mathbb{Z} U}$ by \mathbb{C}^* is given by

$$\overline{X/\mathbb{Z} U} // \mathbb{C}^* \cong (X_{\min+}^{s,\hat{U}}/U) // \mathbb{C}^* \cong X_{\min+}^{s,\hat{U}}/\hat{U} = X \mathbin{\mathbb{Z}}_{L_X^{\otimes c}} \hat{U}.$$

Remark 2.5. Note that when $\dim(U) = 1$ the hypothesis that the linearisation is adapted is sufficient; we do not need the stronger requirement that the linearisation is well adapted (cf. Remark 0.7).

In order to prove Theorem 2.4, we will first prove the theorem in the case where $X = \mathbb{P}(V)$ and $L = \mathcal{O}_{\mathbb{P}(V)}(1)$, for a finite dimensional \hat{U} -representation V . This is done by an application of Theorem 1.13 in a very simple situation with $G = SL(2)$ and $R = \mathbb{C}^*$. An explicit description of the stable locus is used to prove that the enveloping quotient map $\mathbb{P}(V)^{ss,\hat{U}} \rightarrow \mathbb{P}(V) \mathbin{\mathbb{Z}} \hat{U}$ is a geometric quotient for the \hat{U} -action on $\mathbb{P}(V)^{ss,\hat{U}}$. Theorem 2.4 then follows by embedding X into a projective space and analysing the behaviour of stability under closed immersions (cf. Remark 1.9).

2.1. The case $(X, L) = (\mathbb{P}(V), \mathcal{O}(1))$. Let V be a finite-dimensional representation of

$$\hat{U} = \hat{U}^{[\ell]} = U \rtimes \mathbb{C}^*$$

where $U = \mathbb{C}^+$ and \mathbb{C}^* acts on $\text{Lie}(U)$ with weight $\ell \geq 1$, and let $X = \mathbb{P}(V)$ with $L = \mathcal{O}(1)$ defining the canonical linearisation.

Definition 2.6. Let V_{\min} be the \mathbb{C}^* -weight space in V of minimal weight ω_0 , and let $\mathbb{P}(V)_{\min}^0$ be the open subvariety of $\mathbb{P}(V)$ consisting of points flowing to $Z_{\min} = \mathbb{P}(V_{\min})$ under the action of $t \in \mathbb{C}^*$ as $t \rightarrow 0$.

Remark 2.7. Recall that in this situation the condition that semistability coincides with stability given in Definition 2.1 is equivalent to saying that V_{\min} does not contain any fixed points for the \mathbb{C}^+ -action on V , and that $X_{\min+}^{s,\hat{U}} = \mathbb{P}(V)_{\min}^0 \setminus (U \cdot \mathbb{P}(V_{\min}))$.

We wish to prove the following proposition.

Proposition 2.8. *If V_{\min} does not contain any fixed points for the \mathbb{C}^+ -action on V , and the linearisation is twisted by an adapted rational character χ/c , then*

- (1) *there are equalities $\mathbb{P}(V)^{s, \hat{U}} = \mathbb{P}(V)^{ss, \hat{U}} = \mathbb{P}(V)_{\min}^0 \setminus (U \cdot \mathbb{P}(V_{\min}))$;*
- (2) *the enveloping quotients $\mathbb{P}(V) \mathbin{\mathbb{P}} \hat{U}$ and $\mathbb{P}(V) \mathbin{\mathbb{P}} U$ are projective varieties, and for suitably divisible integers $r > 0$ the algebras of invariants $\bigoplus_{k \geq 0} H^0(X, L^{\otimes kr})^{\hat{U}}$ and $\bigoplus_{k \geq 0} H^0(X, L^{\otimes kr})^U$ are finitely generated; and*
- (3) *the enveloping quotient map $\mathbb{P}(V)^{ss, \hat{U}} \rightarrow \mathbb{P}(V) \mathbin{\mathbb{P}} \hat{U}$ is a geometric quotient for the \hat{U} -action on $\mathbb{P}(V)^{ss, \hat{U}}$.*

In order to study the linear action of $\hat{U} = \hat{U}^{[\ell]}$ we consider the surjective homomorphism

$$\eta_\ell : \hat{U}^{[2\ell]} \rightarrow \hat{U}^{[\ell]}, \quad (u; t) \mapsto (u; t^2).$$

We can pull back the linear action of $\hat{U} = \hat{U}^{[\ell]}$ to a linear action of $\hat{U}^{[2\ell]}$ via η_ℓ . The (semi)stable loci for the linear actions of $\hat{U}^{[\ell]}$ and $\hat{U}^{[2\ell]}$ then coincide, and the same is true for the enveloping quotients. In order to prove Proposition 2.8 we may therefore work with $\hat{U}^{[2\ell]}$.

Now the $\hat{U}^{[2\ell]}$ -representation V defined by η_ℓ admits a decomposition

$$(3) \quad V \cong \bigoplus_{i=1}^q \mathbb{C}^{(a_i)} \otimes \text{Sym}^{l_i} \mathbb{C}^2,$$

of $\hat{U}^{[2\ell]}$ -modules, where

- $\mathbb{C}^{(a_i)}$ is the one dimensional representation of $\hat{U}^{[2\ell]}$ defined by the character $\hat{U}^{[2\ell]} \rightarrow \mathbb{C}^*$ of weight $a_i \in \mathbb{Z}$;
- $\text{Sym}^{l_i} \mathbb{C}^2$ is the standard irreducible representation of $G = \text{SL}(2, \mathbb{C})$ of highest weight $l_i \geq 0$, on which $\hat{U}^{[2\ell]}$ acts via the surjective homomorphism

$$\rho_\ell : \hat{U}^{[2\ell]} \rightarrow \hat{U}^{[2]}, \quad (u; t) \mapsto (u; t^\ell)$$

and the identification of $\hat{U}^{[2\ell]}$ with the Borel subgroup $B \subseteq G$ of upper triangular matrices given by

$$\hat{U}^{[2]} = \mathbb{C}^+ \rtimes \mathbb{C}^* \rightarrow B, \quad (u; t) \mapsto \begin{pmatrix} t & tu \\ 0 & t^{-1} \end{pmatrix};$$

and

- because the action of $\hat{U}^{[2\ell]}$ factors through $\eta_\ell : \hat{U}^{[2\ell]} \rightarrow \hat{U}^{[\ell]}$, we have $a_i \equiv \ell l_i \pmod{2}$ for each $i = 1, \dots, q$.

Observe that $\rho_\ell : \hat{U}^{[2\ell]} \rightarrow \hat{U}^{[2]} \cong B \subseteq G = \text{SL}(2, \mathbb{C})$ restricts to give the standard inclusion of the unipotent radical $U = \mathbb{C}^+ = (\hat{U}^{[2\ell]})_u$ of $\hat{U}^{[2\ell]}$ inside G as the subgroup of strictly upper triangular matrices, so ρ_ℓ is an $(\hat{U}^{[2\ell]})_u$ -faithful homomorphism, in the sense of Definition 1.11. The linear action of U on V extends to a linear action of G by demanding that G act on $\text{Sym}^{l_i} \mathbb{C}^2$ in the usual manner and trivially on $\mathbb{C}^{(a_i)}$ for each i .

There is therefore an isomorphism of $G \times \mathbb{C}^*$ -spaces (where $\mathbb{C}^* = \hat{U}^{[2\ell]}/U$)

$$G \times_U \mathbb{P}(V) \cong (G/U) \times \mathbb{P}(V)$$

which lifts to an isomorphism of linearisations. Here the action of $G \times \mathbb{C}^*$ on the quasi-affine variety G/U is left multiplication by G and right multiplication by \mathbb{C}^* , and the $G \times \mathbb{C}^*$ -linearisation

\mathcal{P} on $\mathbb{P}(V)$ with respect to $\mathcal{O}(1)$ is given by the G -action above and the following linear action of \mathbb{C}^* , twisted twice by χ/c :

$$(4) \quad \begin{aligned} t \cdot v &= \sum_i (t^{a_i} z_i) \otimes s_i, \quad v \in V, t \in \mathbb{C}^*, \\ v &= \sum_i z_i \otimes s_i \in \bigoplus_{i=1}^q \mathbb{C}^{(a_i)} \otimes \text{Sym}^{l_i} \mathbb{C}^2 \\ &\text{via the isomorphism (3).} \end{aligned}$$

The homomorphism ρ_ℓ embeds U into $G = \text{SL}(2; \mathbb{C})$ as a Grosshans subgroup, since there is an isomorphism $G/U \cong \mathbb{C}^2 \setminus \{0\}$ given by considering the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$ under the defining representation of G , and the inclusion $\mathbb{C}^2 \setminus \{0\} \hookrightarrow \mathbb{C}^2$ defines a nonsingular affine completion of G/U with codimension 2 complement. We may therefore construct a $G \times \mathbb{C}^*$ -equivariant nonsingular projective completion $\mathbb{P}^2 = \{[v_0 : v_1 : v_2]\}$ of G/U by adding a hyperplane at infinity defined by $v_0 = 0$ to \mathbb{C}^2 . The action of $G \times \mathbb{C}^*$ on $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ is defined by the representation given in block form by

$$(g, t) \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g \begin{pmatrix} t^{-\ell} & 0 \\ 0 & t^{-\ell} \end{pmatrix} \end{array} \right) \in \text{GL}(3; \mathbb{C}), \quad g \in G, t \in \mathbb{C}^*,$$

where $\text{GL}(3; \mathbb{C})$ acts on \mathbb{C}^3 by left multiplication. For any integer $N > 0$, this representation determines a $G \times \mathbb{C}^*$ -linearisation on $\mathcal{O}_{\mathbb{P}^2}(N) \rightarrow \mathbb{P}^2$ which restricts to the canonical linearisation on $\mathcal{O}_{G/U} \rightarrow G/U$. Let \mathcal{P}'_N denote the $G \times \mathbb{C}^*$ -linearisation over $\mathbb{P}^2 \times \mathbb{P}(V)$ given by tensoring this with \mathcal{P} . As U is a Grosshans subgroup of G , the algebras of U -invariants and \hat{U} -invariants of any positive tensor power of this linearisation are finitely generated \mathbb{C} -algebras, and the corresponding enveloping quotients

$$\mathbb{P}(V) \wr U \cong (\mathbb{P}^2 \times \mathbb{P}(V)) //_{\mathcal{P}'_N} G, \quad \mathbb{P}(V) \wr \hat{U}^{[2\ell]} \cong (\mathbb{P}^2 \times \mathbb{P}(V)) //_{\mathcal{P}'_N} (G \times \mathbb{C}^*)$$

are projective varieties. By Theorem 1.13 the stable loci $\mathbb{P}(V)^{s,U}$ and $\mathbb{P}(V)^{s,\hat{U}}$ and finitely generated semistable loci $\mathbb{P}(V)^{ss,U}$ and $\mathbb{P}(V)^{ss,\hat{U}}$ may be computed as the completely (semi)stable loci for the G and $G \times \mathbb{C}^*$ -linearisation \mathcal{P}'_N (using the Hilbert-Mumford criteria) and Proposition 2.8 is a consequence of the following lemma.

Lemma 2.9. *Under the hypotheses of Proposition 2.8, stability and semistability are equivalent for the linear action of $G \times \mathbb{C}^*$ on $\mathbb{P}^2 \times \mathbb{P}(V)$ with respect to the linearisation \mathcal{P}'_N when $N \gg 1$. Moreover if a point $p \in \mathbb{P}^2 \times \mathbb{P}(V)$ is stable for this linear action of $G \times \mathbb{C}^*$ then $p \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V)$, and if $p = ([1 : 1 : 0], [v])$ then p is stable if and only if $[v] \in \mathbb{P}(V)_{\min}^0 \setminus U\mathbb{P}(V_{\min})$.*

Proof. We shall deduce this by using the Hilbert-Mumford criteria as given in Proposition 1.4 using the maximal torus $T_1 \times T_2 \subseteq G \times \mathbb{C}^*$, where T_1 is the subgroup of diagonal matrices in G and $T_2 = \mathbb{C}^*$. The group of characters of $T_1 \times T_2$ is identified with $\mathbb{Z} \times \mathbb{Z}$ in the natural way. Introduce the following notation: for $i = 1, \dots, q$ let $e_{i,1}, e_{i,2}$ be the standard basis of \mathbb{C}^2 , so that

$$e_{i,1}^{l_i}, \dots, e_{i,1}^j e_{i,2}^{l_i-j}, \dots, e_{i,2}^{l_i} \in \text{Sym}^{l_i} \mathbb{C}^2$$

form a basis of $T_1 \times T_2$ -weight vectors in $\text{Sym}^{l_i} \mathbb{C}^2$. The fixed points in $\mathbb{P}^2 \times \mathbb{P}(V)$ for the $T_1 \times T_2$ -action, along with the corresponding rational weights for \mathcal{P}'_N , are given in Table 1.

The minimal \mathbb{C}^* -weight for the \hat{U} -action on V is

$$\omega_{\min} = \min\{(a_i - \ell l_i)/2 \mid i = 1, \dots, q\}.$$

Let us temporarily call an index $i \in \{0, \dots, q\}$ *exceptional* if $\omega_{\min} = (a_i - \ell l_i)/2$.

Fixed point ($j = 0, \dots, l_i, i = 1, \dots, q$)	Rational weight in $\text{Hom}(T_1 \times T_2, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \times \mathbb{Q}$
$([1 : 0 : 0], [1 \otimes e_{i,1}^j e_{i,2}^{l_i-j}])$	$(2j - l_i, a_i - 2\omega_0 - 2\epsilon)$
$([0 : 1 : 0], [1 \otimes e_{i,1}^j e_{i,2}^{l_i-j}])$	$(2j - l_i, a_i - 2\omega_0 - 2\epsilon) + (N, -\ell N)$
$([0 : 0 : 1], [1 \otimes e_{i,1}^j e_{i,2}^{l_i-j}])$	$(2j - l_i, a_i - 2\omega_0 - 2\epsilon) + (-N, -\ell N)$

TABLE 1. Rational weights of the fixed points of $T_1 \times T_2 \curvearrowright \mathbb{P}^2 \times \mathbb{P}(V)$ with respect to the linearisation \mathcal{P}'_N .

Consider the rational weight $\vartheta = (2j - l_i, a_i - 2\omega_0 - 2\epsilon)$ for the fixed point $([1 : 0 : 0], [1 \otimes e_{i,1}^j e_{i,2}^{l_i-j}])$. Note that either ϑ is contained in the interior of the cone

$$C = \{(c_1, c_2) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \mid \ell c_1 + c_2 \geq 0 \text{ and } -\ell c_1 + c_2 \geq 0\},$$

or ϑ lies outside C and i, j satisfy $\omega_0 = (a_i - \ell l_i)/2$ and $j \in \{0, l_i\}$: as $0 < \epsilon < 1/2$ we see that

$$\ell(2j - l_i) + (a_i - 2\omega_0 - 2\epsilon) \begin{cases} = -2\epsilon < 0 & \text{iff } j = 0 \text{ and } \omega_0 = (a_i - \ell l_i)/2 \\ > 0 & \text{otherwise} \end{cases}$$

while

$$-\ell(2j - l_i) + (a_i - 2\omega_0 - 2\epsilon) \begin{cases} = -2\epsilon < 0 & \text{iff } j = l_i \text{ and } \omega_0 = (a_i - \ell l_i)/2 \\ > 0 & \text{otherwise.} \end{cases}$$

We also claim that $a_i - 2\omega_0 - 2\epsilon > 0$ for all $i = 1, \dots, q$. Indeed, suppose for a contradiction that $a_i - 2\omega_0 - 2\epsilon \leq 0$ for some $i = 1, \dots, q$. Because $0 < 2\epsilon < 1$ and $a_i - 2\omega_0 \in \mathbb{Z}$, this is equivalent to $a_i - 2\omega_0 \leq 0$. But $2\omega_0 \leq a_i - \ell l_i$, so $\ell l_i \leq a_i - 2\omega_0 \leq 0$. Because $\ell > 0$ we must have $l_i = 0$, and by examining the possible cases for the value of $\ell(2j - l_i) + (a_i - 2\omega_0 - 2\epsilon)$ we see that $\omega_0 = a_i/2$ and i is exceptional. This implies there is a line $\mathbb{C}^{(\omega_0)} = \mathbb{C}^{(\omega_0)} \otimes \text{Sym}^0 \mathbb{C}^2 \subseteq V_{\min}$ fixed by U , which contradicts the assumption that V_{\min} does not contain a point fixed by the U -action, and so the claim is verified.

Thus for sufficiently large $N > 0$ the weights for the rational $T_1 \times T_2$ -linearisation $\mathcal{P}'_N \rightarrow \mathbb{P}^2 \times \mathbb{P}(V)$ are arranged in the fashion of Figure 1. In particular, the weight polytope $\Delta_p \subseteq \text{Hom}(T_1 \times T_2, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a point $p = ([w_0 : w_1 : w_2], [v]) \in \mathbb{P}^2 \times \mathbb{P}(V)$ contains the origin precisely when the interior Δ_p° does and so semistability and stability for $T_1 \times T_2$ coincide.

By the Hilbert-Mumford criteria, the point p is (semi)stable for the $G \times \mathbb{C}^*$ -linearisation if and only if gp is $T_1 \times T_2$ -(semi)stable for each $g \in G$. It follows that stability and semistability are equivalent for $G \times \mathbb{C}^*$.

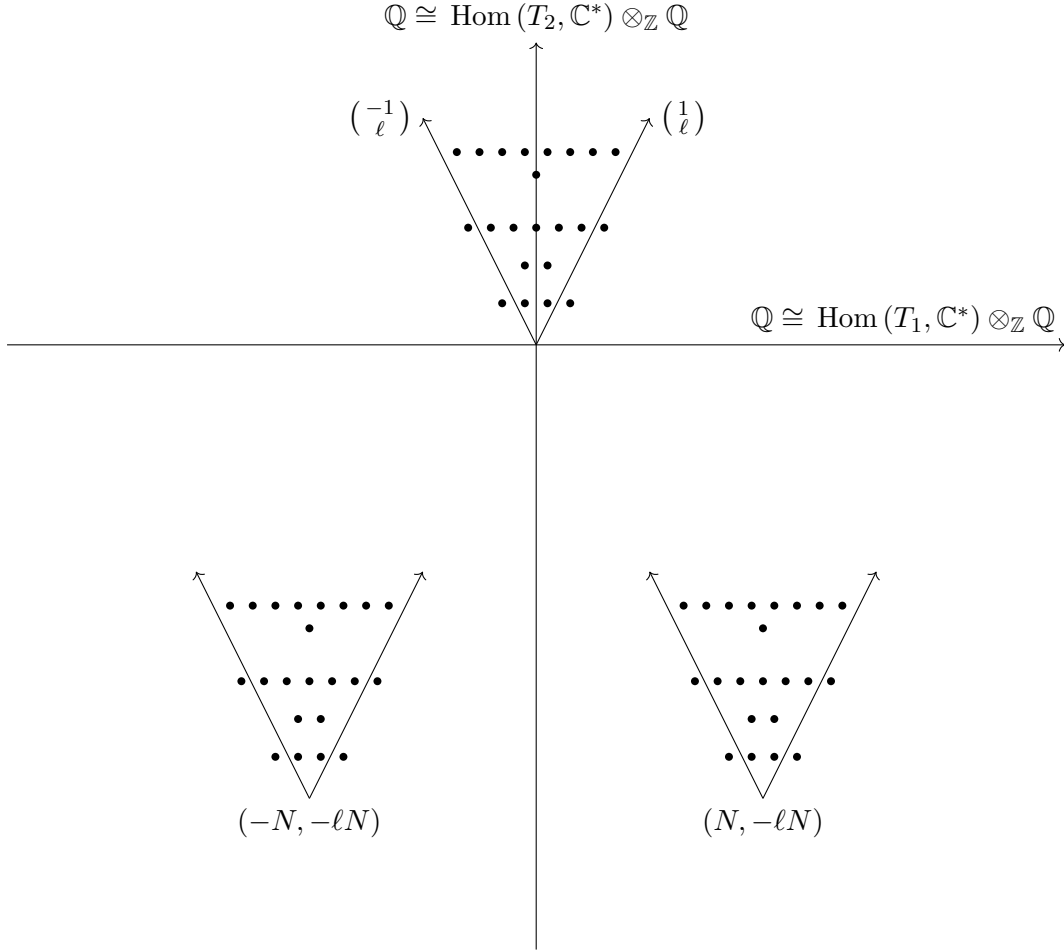
Using the isomorphism (3), write

$$v = \sum_{i=1}^q z_i \otimes s_i, \quad z_i \in \mathbb{C}^{(a_i)}, \quad 0 \neq s_i = \sum_{j=0}^{l_i} v_{i,j} e_{i,1}^j e_{i,2}^{l_i-j}, \quad v_{i,j} \in \mathbb{C}.$$

Then one finds that p is $T_1 \times T_2$ -unstable precisely when $p \notin (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V)$ (i.e. $w_0 = 0$ or $w_1 = w_2 = 0$) or else by satisfying one of the following criteria, split into three cases:

Case $w_0 w_1 w_2 \neq 0$:

Either $v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = 0)$, or $v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = l_i)$.


 FIGURE 1. Example of distribution of rational weights for $T_1 \times T_2 \curvearrowright \mathcal{P}'_N \rightarrow \mathbb{P}^2 \times \mathbb{P}(V)$.

Case $w_0 w_1 \neq 0, w_2 = 0$:

Either i exceptional $\implies v_{i,0} = 0$, or $v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = 0)$.

Case $w_0 w_2 \neq 0, w_1 = 0$:

Either i exceptional $\implies v_{i,l_i} = 0$, or $v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = l_i)$.

It follows that if p is $G \times \mathbb{C}^*$ -stable then $p \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}(V)$, and it remains to show that if $p = ([1 : 1 : 0], [v])$ then p is $G \times \mathbb{C}^*$ -stable if and only if $[v] \in \mathbb{P}(V)_{\min}^0 \setminus U\mathbb{P}(V_{\min})$, or equivalently that $gp = ([1 : g_{11} : g_{21}], [gv])$ (where $g = (g_{ij})$) is $T_1 \times T_2$ -stable for all $g \in G$ if and only if $[v] \in \mathbb{P}(V)_{\min}^0 \setminus U\mathbb{P}(V_{\min})$.

Under the isomorphism of vector spaces $V \cong \bigoplus_{i=1}^q \mathbb{C}^{(a_i)} \otimes \text{Sym}^{l_i} \mathbb{C}^2$ the weight vectors for the induced $\mathbb{C}^* \subseteq \hat{U}^{[l]}$ -action on V take the form $1 \otimes e_{1,i}^j e_{2,i}^{l_i-j}$, where $1 \leq i \leq q$ and $0 \leq j \leq l_i$, with the weight of $1 \otimes e_{1,i}^j e_{2,i}^{l_i-j}$ equal to $(a_i - \ell l_i + 2j)/2 \in \mathbb{Z}$. The weight space V_{\min} of minimal weight ω_0 is spanned by all $1 \otimes e_{2,i}^{l_i}$ with i an exceptional index, and the U -sweep $U \cdot V_{\min}$ of

V_{\min} is contained in the $\hat{U}^{[\ell]}$ -subspace

$$\bigoplus_{i \text{ exceptional}} \mathbb{C}^{(a_i)} \otimes \text{Sym}^{l_i} \mathbb{C}^2 \subseteq V.$$

Now, if $v = \sum_i z_i \otimes s_i$ with each $s_i \neq 0$, then the existence of an exceptional i with $z_i \neq 0$ and s_i not divisible by $(1, 0)$ is equivalent to $\lim_{t \rightarrow 0} t \cdot [v] \in \mathbb{P}(V_{\min})$ (where we take $t \in \mathbb{C}^* \subseteq \hat{U}^{[\ell]}$ in the limit).

The existence of a non-exceptional i such that $z_i \neq 0$ is equivalent to $v \notin \bigoplus_{i \text{ exceptional}} \mathbb{C}^{(a_i)} \otimes \text{Sym}^{l_i} \mathbb{C}^2$, which itself implies $[v] \notin U \cdot \mathbb{P}(V_{\min})$. On the other hand, because of the transitivity of the U -action on $\mathbb{C} = \mathbb{P}^1 \setminus \{[1 : 0]\}$ and the fact that V_{\min} is spanned by $1 \otimes e_{i,2}^{l_i}$ with i exceptional, we see that $[v] \in U \cdot \mathbb{P}(V_{\min})$ if and only if $v \in \bigoplus_{i \text{ exceptional}} \mathbb{C}^{(a_i)} \otimes \text{Sym}^{l_i} \mathbb{C}^2$ and there is some $(w_1, w_2) \in \mathbb{C}^2 \setminus \{0\}$ with $[w_1 : w_2] \neq [1 : 0] \in \mathbb{P}^1$ such that $s_i = (w_1, w_2)^{l_i} \in \text{Sym}^{l_i} \mathbb{C}^2$ for all exceptional i .

Therefore, applying the three cases above to $gp = ([1 : g_{11} : g_{21}], [gv])$ where $p = ([1 : 1 : 0], [v])$, we deduce that p is $G \times \mathbb{C}^*$ -stable if and only if $[v] \in \mathbb{P}(V)_{\min}^0 \setminus U\mathbb{P}(V_{\min})$, as required. \square

This completes the proof of Proposition 2.8, and of Theorem 2.4 in the special case when $(X, L) = (\mathbb{P}(V), \mathcal{O}(1))$.

2.2. Proof of Theorem 2.4 for general (X, L) . Suppose now that L is a very ample line bundle over an irreducible projective variety X equipped with a \hat{U} -linearisation, where $\hat{U} = U \rtimes \mathbb{C}^*$ for $U = \mathbb{C}^+$, let $V = H^0(X, L)^*$ and let $\gamma : X \hookrightarrow \mathbb{P}(V)$ be the canonical closed immersion. Let ω_0 be the minimal weight for the induced \mathbb{C}^* -action on V and suppose the associated weight space V_{\min} does not contain any fixed points for the U -action on V . Finally, let χ/c be an adapted rational character.

By Proposition 2.8 there is an enveloping quotient

$$q : \mathbb{P}(V)^{ss, \hat{U}} = \mathbb{P}(V)^{s, \hat{U}} \rightarrow \mathbb{P}(V) \mathbin{\mathbb{Z}} \hat{U}$$

which is a geometric quotient for the \hat{U} -action on $\mathbb{P}(V)^{ss, \hat{U}}$, and the quotient $\mathbb{P}(V) \mathbin{\mathbb{Z}} \hat{U}$ is a projective variety. Furthermore,

$$\mathbb{P}(V)^{s, \hat{U}} = \mathbb{P}(V)_{\min}^0 \setminus (U \cdot \mathbb{P}(V_{\min})),$$

from which it follows that

$$\mathbb{P}(V)^{s, \hat{U}} \cap X = \gamma^{-1}(\mathbb{P}(V)^{s, \hat{U}}) = X_{\min}^0 \setminus (U \cdot Z_{\min}).$$

Since geometric quotients behave well under closed immersions, it follows from the definitions of (semi)stability that $\mathbb{P}(V)^{s, \hat{U}} \cap X \subseteq X^{ss, \hat{U}}$, and thus that $X_{\min}^0 \setminus (U \cdot Z_{\min})$ is an open subvariety of $X^{ss, \hat{U}}$ whose image under the enveloping quotient

$$q : X^{ss, \hat{U}} \rightarrow X \mathbin{\mathbb{Z}} \hat{U}$$

is a geometric quotient for the \hat{U} -action on $X_{\min}^0 \setminus (U \cdot Z_{\min})$ that embeds naturally as a closed subvariety of $\mathbb{P}(V)^{s, \hat{U}} / \hat{U} = \mathbb{P}(V) \mathbin{\mathbb{Z}} \hat{U}$, which is projective. Hence $q(X_{\min}^0 \setminus (U \cdot Z_{\min}))$ is itself a projective variety. In particular it is complete, and since $X \mathbin{\mathbb{Z}} \hat{U}$ is separated over $\text{Spec } \mathbb{C}$ it follows that the inclusion $q(X_{\min}^0 \setminus (U \cdot Z_{\min})) \hookrightarrow X \mathbin{\mathbb{Z}} \hat{U}$ is a closed map [46, Tag 01W0]. On the other hand, because X is irreducible $q(X_{\min}^0 \setminus (U \cdot Z_{\min}))$ is a dense open subset of $X \mathbin{\mathbb{Z}} \hat{U}$ if

$X_{\min}^0 \setminus (U \cdot Z_{\min})$ is not empty, while it is easy to see that $X \not\curvearrowright \hat{U} = \emptyset$ if $X_{\min}^0 \setminus (U \cdot Z_{\min}) = \emptyset$. Hence

$$q(X_{\min}^0 \setminus (U \cdot Z_{\min})) = X \not\curvearrowright \hat{U},$$

$X \not\curvearrowright \hat{U}$ is a projective variety, and Theorem 2.4 now follows from Theorem 1.13 and Proposition 1.10, together with Proposition 2.8 which covers the case when $(X, L) = (\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$.

3. ACTIONS OF \mathbb{C}^* -EXTENSIONS OF UNIPOTENT GROUPS

Now let U be any graded unipotent group; that is, U is a unipotent group with a one-parameter group of automorphisms $\lambda : \mathbb{C}^* \rightarrow \text{Aut}(U)$ such that the weights of the induced \mathbb{C}^* action on the Lie algebra \mathfrak{u} of U are all strictly positive. Let \hat{U} be the corresponding semidirect product

$$\hat{U} = \mathbb{C}^* \ltimes U.$$

As before we assume that \hat{U} acts linearly on a projective variety X with respect to a very ample line bundle L , and that the action of U on X is not trivial. Recall from Definition 2.1 that we say that *semistability coincides with stability* for this linear action if whenever U' is a subgroup of U normalised by \mathbb{C}^* and ξ belongs to the Lie algebra of U but not the Lie algebra of U' and ξ is a weight vector for the action of \mathbb{C}^* , then the weight space with weight $-\omega_{\min}$ for the action of \mathbb{C}^* on $H^0(X, L)$ is contained in the image $\delta_\xi(H^0(X, L)^{U'})$ of $H^0(X, L)^{U'}$ under the infinitesimal action $\delta_\xi : H^0(X, L) \rightarrow H^0(X, L)$ of ξ on $H^0(X, L)$.

We are aiming to prove Theorem 0.2 by induction on $\dim(U)$; the base case when $\dim(U) = 1$ has been proved in Theorem 2.4. The following lemma will be used for the induction step.

Lemma 3.1. *Suppose that a linear action of \hat{U} on a projective variety X with respect to an ample line bundle L satisfies the condition that semistability equals stability. If U_\dagger is a subgroup of U which is normal in \hat{U} and $\hat{U}_\dagger = \mathbb{C}^* \ltimes U_\dagger$ is the subgroup of \hat{U} generated by U_\dagger and the one-parameter subgroup \mathbb{C}^* , then*

- (1) *the linear action of \hat{U} on X with respect to any positive tensor power $L^{\otimes m}$ of L satisfies the condition that semistability equals stability;*
- (2) *the restriction to \hat{U}_\dagger of the linear action of \hat{U} on X with respect to any positive tensor power $L^{\otimes m}$ of L satisfies the condition that semistability equals stability;*
- (3) *if c_\dagger is a sufficiently divisible positive integer then the induced linear action of \hat{U}/U_\dagger on the closure $\overline{X/\zeta U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})^{U_\dagger})^*)$ of an inner enveloping quotient $X/\zeta U_\dagger$ for the action of U_\dagger on X satisfies the condition that semistability equals stability with respect to the ample line bundle determined by $L^{\otimes c}$.*

Proof: (1) Suppose that U' is a subgroup of U normalised by \mathbb{C}^* and that a \mathbb{C}^* -weight vector ξ with weight a belongs to the Lie algebra of U but not the Lie algebra of U' , with corresponding infinitesimal action $\delta_\xi = \delta : H^0(X, L) \rightarrow H^0(X, L)$. By abuse of notation let δ also denote the induced infinitesimal action on $H^0(X, L^{\otimes m})$. As X is \mathbb{C}^* -invariant, the minimum weight $\omega_0^{L^{\otimes m}}$ with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the line bundle $\mathcal{O}_{\mathbb{P}((H^0(X, L^{\otimes m})^*)}(-1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X, L^{\otimes m})^*)^{\mathbb{C}^*})$ for the action of \mathbb{C}^* on $\mathbb{P}((H^0(X, L^{\otimes m})^*)$ is $m\omega_0$. Suppose that $s \in H^0(X, L^{\otimes c})^{U'}$ is a weight vector with weight $\omega_0^{L^{\otimes m}}$ for the action of \mathbb{C}^* . We want to show that there is some section $s' \in H^0(X, L^{\otimes m})^{U'}$ such that $\delta(s') = s$. Since $\omega_0^{L^{\otimes m}} = m\omega_0$ we can write s as a linear combination of monomials $s_1 \cdots s_m$ where $s_j \in H^0(X, L)$ is a weight vector

with weight ω_0 for the \mathbb{C}^* action, which implies that $\delta(s_j) = 0$ for $j = 1, \dots, m$. As the linear action of \hat{U} on X with respect to L satisfies the condition that semistability equals stability, there is $s'_1 \in H^0(X, L)^{U'}$ such that $\delta(s'_1) = s_1$. It follows that

$$\delta(s'_1 s_2 \cdots s_m) = s_1 \cdots s_m$$

where $s'_1 s_2 \cdots s_m \in H^0(X, L^{\otimes m})^{U'}$ as required.

(2) By (1) we can assume that $m = 1$ and then this follows straight from the definition of what it means for semistability to coincide with stability (Definition 2.1).

(3) Recall from Remark 1.7 the notion of an inner enveloping quotient. A subgroup of U/U_{\dagger} normalised by \mathbb{C}^* has the form U'/U_{\dagger} where U' is a subgroup of U containing U_{\dagger} and normalised by \mathbb{C}^* . A weight vector in the Lie algebra of U/U_{\dagger} which does not lie in the Lie algebra of U'/U_{\dagger} can be represented by a weight vector ξ in the Lie algebra of U not lying in the Lie algebra of U' , and the corresponding infinitesimal action on $H^0(X, L^{\otimes c_{\dagger}})^{U_{\dagger}}$ is the restriction of the infinitesimal action on $H^0(X, L^{\otimes c_{\dagger}})$ determined by ξ , so (3) follows from (1). \square

Our aim is to prove the following theorem, from which Theorem 0.2 and Corollary 0.6 will follow.

Theorem 3.2. *Let X be a complex projective variety equipped with a linear action (with respect to an ample line bundle L) of a unipotent group U with a one-parameter group of automorphisms such that the weights of the induced \mathbb{C}^* action on the Lie algebra of U are all strictly positive. Suppose that the linear action of U on X extends to a linear action of the semi-direct product $\hat{U} = \mathbb{C}^* \ltimes U$. Suppose also that the linear action of \hat{U} on X satisfies the condition that ‘semistability coincides with stability’ as above. If $\chi : \hat{U} \rightarrow \mathbb{C}^*$ is a character of \hat{U} and c is a sufficiently divisible positive integer such that the rational character χ/c is well adapted for the linear action of \hat{U} with respect to L , then after twisting this linear action by χ/c we have*

- (1) *the \hat{U} -invariant open subset $X_{\min+}^{s, \hat{U}}$ of X has a geometric quotient $\pi : X_{\min+}^{s, \hat{U}} \rightarrow X_{\min+}^{s, \hat{U}}/\hat{U}$ by the action of \hat{U} ;*
- (2) *this geometric quotient $X_{\min+}^{s, \hat{U}}/\hat{U}$ is a projective variety and the tensor power $L^{\otimes c}$ of L descends to an ample line bundle $L_{(c, \hat{U})}$ on $X_{\min+}^{s, \hat{U}}/\hat{U}$;*
- (3) *$X_{L_{\chi}^{\otimes c}}^{s, \hat{U}} = X_{L_{\chi}^{\otimes c}}^{ss, \hat{U}} = X_{\min+}^{s, \hat{U}}$;*
- (4) *the geometric quotient $X_{\min+}^{s, \hat{U}}/\hat{U}$ is the enveloping quotient $X \mathbin{\mathbb{R}}_{L_{\chi}^{\otimes c}} \hat{U}$;*
- (5) *the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}}$ is finitely generated and the enveloping quotient $X \mathbin{\mathbb{R}}_{L_{\chi}^{\otimes c}} \hat{U} \cong \text{Proj}(\bigoplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}})$ is the associated projective variety.*

Remark 3.3. Note that, for any positive integer m , Theorem 3.2 holds for a linear action of \hat{U} on X with respect to an ample line bundle L if it holds for the induced linearisation of the action of \hat{U} with respect to the line bundle $L^{\otimes m}$. To see this, we use Lemma 3.1 and observe that almost all the ingredients of the theorem are unchanged when L is replaced with $L^{\otimes m}$. The only ingredients over which we still need to take care are the concept of well adaptedness and the definition of $X_{\min+}^{s, \hat{U}}$, which both depend on the weights of the action of \mathbb{C}^* on $H^0(X, L)$ (see Definition 2.3). However since X is \mathbb{C}^* -invariant the minimum weight $\omega_{\min}^{L^{\otimes m}} = \omega_0^{L^{\otimes m}}$ with which the one-parameter subgroup $\mathbb{C}^* \leq \hat{U}$ acts on the fibres of the line bundle $\mathcal{O}_{\mathbb{P}((H^0(X, L^{\otimes m})^*)}(-1)$

over points of the connected components of the fixed point set $\mathbb{P}((H^0(X, L^{\otimes m})^*)^{\mathbb{C}^*})$ for the action of \mathbb{C}^* on $\mathbb{P}((H^0(X, L^{\otimes m})^*)$ is $m\omega_{\min}$, and by variation of GIT for the reductive group \mathbb{C}^* we know that if $\omega_{\min} = \omega_0 < \chi/c < \omega_1$ then the stable set $X_{\min+}^{s, \mathbb{C}^*}$ for the linear action of \mathbb{C}^* with respect to the linearisation $L_{\chi}^{\otimes c}$ is the same as the stable set for the linear action of \mathbb{C}^* with respect to the linearisation $(L_{\chi}^{\otimes c})^{\otimes m} = L_{m\chi}^{\otimes cm}$. So $X_{\min+}^{s, \mathbb{C}^*}$ and $X_{\min+}^{s, \hat{U}} = X \setminus \hat{U}(X \setminus X_{\min+}^{s, \mathbb{C}^*})$ are unchanged by replacing L with $L^{\otimes m}$. Finally

$$\chi/c = \omega_0 + \epsilon \quad \text{iff} \quad m\chi/c = m\omega_0 + m\epsilon = \omega_0^{L^{\otimes m}} + m\epsilon$$

where $(L_{k\chi}^{\otimes ck})^{\otimes m} = L_{mk\chi}^{\otimes ckm} = (L^{\otimes m})_{km\chi}^{\otimes ck}$ for any $k \geq 0$. Thus χ/c is well adapted for the linear action of \hat{U} with respect to L if and only if $m\chi/c$ is well adapted for the linear action of \hat{U} with respect to $L^{\otimes m}$. Note however that it is not always true that

$$\omega_0 < \chi/c < \omega_1 \quad \text{iff} \quad \omega_0^{L^{\otimes m}} < m\chi/c < \omega_1^{L^{\otimes m}}$$

since in general $\omega_1^{L^{\otimes m}} < m\omega_1$ although $\omega_0^{L^{\otimes m}} = m\omega_0$.

Proof of Theorem 3.2: We will use induction on the dimension of U to prove a slightly stronger result including

(6) the tensor power $L^{\otimes c}$ of L induces a very ample line bundle on an inner enveloping quotient $X/\!/\!U$ for the action of U on X with a \mathbb{C}^* -equivariant embedding

$$X/\!/\!U \rightarrow \mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$$

as a quasi-projective subvariety, containing the geometric quotient $X^{s, U}/U$ as an open subvariety, with closure $\overline{X/\!/\!U}$ in $\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$, and

(7) $X_{\min+}^{s, \hat{U}}$ is a U -invariant open subset of $X^{s, U}$ and has a geometric quotient $X_{\min+}^{s, \hat{U}}/U$ which is a \mathbb{C}^* -invariant open subset of $X^{s, U}/U$ and coincides with both the stable and semistable sets $(\overline{X/\!/\!U})^{s, \mathbb{C}^*} = (\overline{X/\!/\!U})^{ss, \mathbb{C}^*}$ for the \mathbb{C}^* action with respect to the linearisation on $\mathcal{O}_{\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)}(1)$ induced by $L_{\chi}^{\otimes c}$, so that the associated GIT quotient satisfies

$$\overline{X/\!/\!U} // \mathbb{C}^* \cong (X_{\min+}^{s, \hat{U}}/U) / \mathbb{C}^* \cong X_{\min+}^{s, \hat{U}} / \hat{U} = X //_{L_{\chi}^{\otimes c}} \hat{U}.$$

When $\dim(U) = 1$ so that $U = \mathbb{C}^+$, this extended version of Theorem 3.2 including (6) and (7) follows immediately from Theorem 2.4.

Now suppose that $\dim(U) > 1$ and that the extended result is true for all strictly smaller values of $\dim(U)$. We can assume without loss of generality that U is nontrivial. The centre of U is then nontrivial and isomorphic to a product of copies of \mathbb{C}^+ on which \mathbb{C}^* acts with positive weights. So U has a normal subgroup U_0 which is central in U and normal in \hat{U} and is isomorphic to \mathbb{C}^+ , such that the given one-parameter group $\mathbb{C}^* \leq \hat{U}$ of automorphisms of U preserves U_0 and acts on the Lie algebra of U_0 with positive weight. By induction on the dimension of U , we can now find a subgroup U_{\dagger} of U which is normal in \hat{U} and such that U/U_{\dagger} is one-dimensional and so isomorphic to \mathbb{C}^+ , while \hat{U}/U_{\dagger} is a semidirect product of U/U_{\dagger} by \mathbb{C}^* where \mathbb{C}^* acts on the Lie algebra of U/U_{\dagger} with strictly positive weight. Let $\hat{U}_{\dagger} = \mathbb{C}^* \ltimes U_{\dagger}$ be the subgroup of \hat{U} generated by U_{\dagger} and the one-parameter subgroup \mathbb{C}^* .

By Lemma 3.1 the linear action of \hat{U}_{\dagger} on X satisfies the condition that semistability is the same as stability. Thus by induction on the dimension of U we can assume that, for a sufficiently divisible positive integer c_{\dagger} ,

- (i) the \hat{U}_\dagger -invariant open subset $X_{\min+}^{s,\hat{U}_\dagger}$ of X has a geometric quotient $\pi : X_{\min+}^{s,\hat{U}_\dagger} \rightarrow X_{\min+}^{s,\hat{U}_\dagger}/\hat{U}_\dagger$ by the action of \hat{U}_\dagger ;
- (ii) this geometric quotient $X_{\min+}^{s,\hat{U}_\dagger}/\hat{U}_\dagger$ is a projective variety and the tensor power $L^{\otimes c_\dagger}$ of L descends to an ample line bundle $L_{(c_\dagger,\hat{U}_\dagger)}$ on $X_{\min+}^{s,\hat{U}_\dagger}/\hat{U}_\dagger$;
- (iii) the tensor power $L^{\otimes c_\dagger}$ of L induces a very ample line bundle on an inner enveloping quotient $X/\zeta U_\dagger$ for the action of U_\dagger on X with a \mathbb{C}^* -equivariant embedding

$$X/\zeta U_\dagger \rightarrow \mathbb{P}((H^0(X, L^{\otimes c_\dagger})_+^U)^*)$$

as a quasi-projective subvariety, containing the geometric quotient $X^{s,U_\dagger}/U_\dagger$ as an open subvariety, with closure $\overline{X/\zeta U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})_+^U)^*)$;

- (iv) $X_{\min+}^{s,\hat{U}_\dagger}$ is a U_\dagger -invariant open subset of X^{s,U_\dagger} and has a geometric quotient $X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger$ which is a \mathbb{C}^* -invariant open subset of $X^{s,U_\dagger}/U_\dagger$ and, if the rational character χ/c_\dagger is well adapted for the linear action of \hat{U}_\dagger with respect to L , coincides with both the stable and semistable sets $(\overline{X/\zeta U_\dagger})^{s,\mathbb{C}^*} = (\overline{X/\zeta U_\dagger})^{ss,\mathbb{C}^*}$ for the \mathbb{C}^* action with respect to the linearisation induced by $L_\chi^{\otimes c_\dagger}$ on $\mathcal{O}_{\mathbb{P}((H^0(X, L^{\otimes c_\dagger})_+^U)^*)}(1)$, so that the associated GIT quotient satisfies

$$\overline{X/\zeta U_\dagger} // \mathbb{C}^* \cong (X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger) // \mathbb{C}^* \cong X_{\min+}^{s,\hat{U}_\dagger}/\hat{U}_\dagger = X \mathcal{Q}_{L_\chi^{\otimes c_\dagger}} \hat{U}_\dagger.$$

Note that $X_{\min+}^{s,\hat{U}_\dagger}$ is a \hat{U}_\dagger -invariant open subvariety of $X_{\min+}^{s,\hat{U}_\dagger}$. We have an induced linear action of $\hat{U}/U_\dagger \cong \mathbb{C}^+ \rtimes \mathbb{C}^*$ on $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})_+^U)^*)$ which restricts to a linear action on $\overline{X/\zeta U_\dagger}$ and to the induced linear action of the open subset $X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger$ of $X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger = (\overline{X/\zeta U_\dagger})^{s,\mathbb{C}^*}$. We also have $X_{\min+}^{s,\hat{U}_\dagger} = \bigcap_{u \in U} X_{\min+}^{s,\mathbb{C}^*}$ so that

$$X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger = \bigcap_{u \in U/U_\dagger} u (X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger) = \bigcap_{u \in U/U_\dagger} u (\overline{X/\zeta U_\dagger})^{s,\mathbb{C}^*} = (\overline{X/\zeta U_\dagger})_{\min+}^{s,\hat{U}/U_\dagger}.$$

By Lemma 3.1 we can apply Theorem 2.4 to the action of \hat{U}/U_\dagger on the closure $\overline{X/\zeta U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c_\dagger})_+^U)^*)$ of the inner enveloping quotient $X/\zeta U_\dagger$ for the action of U_\dagger on X . It follows that $X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger = (\overline{X/\zeta U_\dagger})_{\min+}^{s,\hat{U}/U_\dagger}$ has a geometric quotient

$$(X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger)/(\hat{U}/U_\dagger)$$

which is then a geometric quotient for the action of \hat{U} on $X_{\min+}^{s,\hat{U}_\dagger}$. Furthermore by Theorem 2.4 this geometric quotient $(X_{\min+}^{s,\hat{U}_\dagger}/U_\dagger)/(\hat{U}/U_\dagger) = X_{\min+}^{s,\hat{U}}/\hat{U}$ is a projective variety and for a sufficiently divisible multiple c of c_\dagger the tensor power $L^{\otimes c}$ of L descends to a very ample line bundle $L_{(c,\hat{U})}$ on $X_{\min+}^{s,\hat{U}}/\hat{U}$; in addition if $\chi/c = \omega_0 + \epsilon$ where $\epsilon > 0$ is sufficiently small then $X_{\min+}^{s,\hat{U}}/\hat{U}$ is the stable set for the \hat{U}/U_\dagger -action on $\overline{X/\zeta U_\dagger}$ with respect to the linearisation induced by $L^{\otimes c}$ and twisted by the rational character χ/c , so

$$X_{\min+}^{s,\hat{U}} \subseteq X_{L_\chi^{\otimes c}}^{s,\hat{U}}$$

by Remark 1.9(ii).

Conversely if $\epsilon < \omega_1 - \omega_0$ then $X_{L^\otimes c}^{s, \hat{U}}$ is a \hat{U} -invariant subset of $X_{\min+}^{s, \mathbb{C}^*} = X_{L^\otimes c}^{s, \mathbb{C}^*}$ by Remark 1.9(i), so

$$X_{L^\otimes c}^{s, \hat{U}} \subseteq \bigcap_{u \in U} u X_{\min+}^{s, \mathbb{C}^*} = X_{\min+}^{s, \hat{U}}$$

and hence $X_{L^\otimes c}^{s, \hat{U}} = X_{\min+}^{s, \hat{U}}$.

Since the geometric quotient $X_{\min+}^{s, \hat{U}} / \hat{U} = X_{L^\otimes c}^{s, \hat{U}} / \hat{U}$ is a projective variety with a very ample line bundle $L_{(c, \hat{U})}$ induced by the tensor power $L^{\otimes c}$ of L , it follows from Proposition 1.10 that $X_{L^\otimes c}^{s, \hat{U}} = X_{L^\otimes c}^{ss, \hat{U}}$, that this geometric quotient coincides with the enveloping quotient $X \mathbin{\mathbb{Z}}_{L^\otimes c} \hat{U}$, and that if c is replaced with a sufficiently divisible multiple then the algebra of invariants $\bigoplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}}$ is finitely generated and the enveloping quotient $X \mathbin{\mathbb{Z}}_{L^\otimes c} \hat{U} \cong \text{Proj}(\bigoplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck})^{\hat{U}})$ is the associated projective variety.

We can apply Theorem 2.4 to the action of \hat{U}/U_\dagger on the closure $\overline{X/\zeta U_\dagger}$ in $\mathbb{P}((H^0(X, L^{\otimes c})^{U_\dagger})^*)$ of the inner enveloping quotient $X/\zeta U_\dagger$ for the action of U_\dagger on X . It then follows by induction that after replacing c with a sufficiently divisible multiple if necessary, we can assume that there is an inner enveloping quotient $X/\zeta U$ for the linear action of U on X with respect to the linearisation $L_\chi^{\otimes c}$ obtained by considering the induced action of the subgroup U/U_\dagger of \hat{U}/U_\dagger on $\overline{X/\zeta U_\dagger}$. We can also assume that the tensor power $L^{\otimes c}$ of L induces a very ample line bundle on $X/\zeta U$ so that there is a \mathbb{C}^* -equivariant embedding

$$X/\zeta U \rightarrow \mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$$

of $X/\zeta U$ as a quasi-projective subvariety, containing the geometric quotient $X^{s, U}/U$ as an open subvariety, with closure $\overline{X/\zeta U}$ in $\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)$, such that $X_{\min+}^{s, \hat{U}}$ is a U -invariant open subset of $X^{s, U}$ and has a geometric quotient $X_{\min+}^{s, \hat{U}}/U$ which is a \mathbb{C}^* -invariant open subset of $X^{s, U}/U$ and coincides with both the stable and semistable sets $(\overline{X/\zeta U})^{s, \mathbb{C}^*} = (\overline{X/\zeta U})^{ss, \mathbb{C}^*}$ for the \mathbb{C}^* action with respect to the linearisation on $\mathcal{O}_{\mathbb{P}((H^0(X, L^{\otimes c})^U)^*)}(1)$ induced by $L_\chi^{\otimes c}$. It then follows that the associated GIT quotient satisfies

$$\overline{X/\zeta U} // \mathbb{C}^* \cong (X_{\min+}^{s, \hat{U}}/U) // \mathbb{C}^* \cong X_{\min+}^{s, \hat{U}} / \hat{U} = X \mathbin{\mathbb{Z}}_{L^\otimes c} \hat{U}$$

and this completes the inductive proof. \square

We have now proved Theorem 0.2 and Corollary 0.5, which follow immediately from Theorem 3.2. Corollary 0.6 follows directly as well, since if a complex linear algebraic group H with unipotent radical U acts on a complex algebra A in such a way that the algebra of U -invariants A^U is finitely generated, then there is an induced action on A^U of the reductive group $R = H/U$, and the algebra of H -invariants

$$A^H = (A^U)^R$$

is finitely generated since R is reductive. In the situation of Corollary 0.6 when A is the algebra $\bigoplus_{k \geq 0} H^0(X, L_{k\chi}^{\otimes ck})$ then the associated projective variety is the enveloping quotient $X \mathbin{\mathbb{Z}} H$, and this enveloping quotient is the GIT quotient of the enveloping quotient $X \mathbin{\mathbb{Z}} \hat{U}$ by the reductive subgroup of R which is its intersection with the kernel of the character χ , with respect to the

induced linearisation. The result follows from combining Theorem 3.2 with classical GIT for the action of this reductive subgroup of R .

4. AUTOMORPHISM GROUPS OF TORIC VARIETIES

In this section we observe that if Y is a complete simplicial toric variety then its automorphism group $Aut(Y)$ satisfies the conditions of Corollary 0.6, so that every well adapted linear action of $Aut(Y)$ on a projective variety X with respect to an ample line bundle for which semistability coincides with stability has finitely generated invariants and its enveloping quotient is a geometric quotient of X^{ss} .

For this we use the description of $Aut(Y)$ given in [13]. Let Y be a complete simplicial toric variety over \mathbb{C} of dimension n , and let S be its homogeneous coordinate ring in the sense of [13]. Thus

$$S = \mathbb{C}[x_\rho : \rho \in \Delta(1)]$$

is a polynomial ring in $d = |\Delta(1)|$ variables x_ρ , one for each one-dimensional cone ρ in the fan Δ determining the toric variety Y . The homogeneous coordinate ring S is graded by setting the degree of a monomial $\prod_\rho x_\rho^{a_\rho}$ to be the class of the corresponding Weil divisor $\sum_\rho a_\rho D_\rho$ in the Chow group $A_{n-1}(Y)$, giving us the decomposition

$$S = \bigoplus_{\alpha \in A_{n-1}(Y)} S_\alpha$$

where S_α is spanned by the monomials of degree α . Then we have

$$S_\alpha = S'_\alpha \oplus S''_\alpha$$

where S'_α is spanned by the x_ρ of degree α and S''_α is spanned by the remaining monomials in S_α , each being a product of at least two variables.

Then by [13] Theorem 4.2 and Proposition 4.3, $Aut(Y)$ is an affine algebraic group fitting into an exact sequence

$$1 \rightarrow \text{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*) \rightarrow \widetilde{Aut}(Y) \rightarrow Aut(Y) \rightarrow 1$$

with $\text{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*)$ isomorphic to a product of a finite group and a torus $(\mathbb{C}^*)^{d-n}$, and the identity component $\widetilde{Aut}^0(Y)$ of $\widetilde{Aut}(Y)$ satisfies

$$\widetilde{Aut}^0(Y) \cong U \rtimes \tilde{R}$$

for

$$\tilde{R} \cong \prod_{\alpha} GL(S'_\alpha)$$

and the unipotent radical U of $\widetilde{Aut}^0(Y)$ is given by

$$U = 1 + \mathcal{N}$$

where \mathcal{N} is the ideal

$$\mathcal{N} = \bigoplus_{\alpha} \text{Hom}_{\mathbb{C}}(S'_\alpha, S''_\alpha)$$

in $\text{End}(S)$. The reductive group $\tilde{R} \cong \prod_{\alpha} GL(S'_\alpha)$ acts in the obvious way on S by identifying S with the symmetric algebra on $\bigoplus_{\alpha} S'_\alpha$, so that $r \in \tilde{R}$ acts on $\text{Hom}_{\mathbb{C}}(S'_\alpha, S''_\alpha)$ for each $\alpha \in A_{n-1}(Y)$,

and thus on \mathcal{N} , via pre-composition with the action of r on S'_α and post-composition with the induced action of r^{-1} on

$$S''_\alpha \subseteq \bigoplus_{j \geq 2} \text{Sym}^j \left(\bigoplus_\alpha S'_\alpha \right).$$

It follows that if we embed \mathbb{C}^* in $\tilde{R} = \prod_\alpha GL(S'_\alpha)$ via

$$t \mapsto (t^{-1} \text{id}_{S'_\alpha})_\alpha$$

where $\text{id}_{S'_\alpha}$ is the identity in $GL(S'_\alpha)$, then the weights of the action of \mathbb{C}^* on the Lie algebra \mathcal{N} of U are all of the form

$$t \mapsto t^{j-1}$$

for some $j \geq 2$, so that $j - 1 > 0$. Thus we obtain

Lemma 4.1. *If Y is a complete simplicial toric variety then $\text{Aut}(Y)$ is of the form*

$$\text{Aut}(Y) \cong U \rtimes R$$

where U is unipotent and R is reductive, and R contains a one-parameter subgroup $\mathbb{C}^* \leq R$ such that the action of \mathbb{C}^* on the Lie algebra of U induced by its conjugation action on U has all weights strictly positive.

As an immediate consequence of this lemma and Corollary 0.6 we have

Corollary 4.2. *Any well adapted linear action of $H = \text{Aut}(Y)$ on a projective variety X with respect to an ample line bundle L , for which semistability coincides with stability for the action of its unipotent radical U extended by the central one-parameter subgroup of $\text{Aut}(Y)/U$ described above, has finitely generated invariants when L is replaced by a tensor power $L^{\otimes c}$ for a sufficiently divisible positive integer c . Furthermore its enveloping quotient $X \mathbin{\mathbb{H}} H$ is the associated projective variety and is a categorical quotient of X^{ss} by the action of H , while the canonical morphism $\phi : X^{ss} \rightarrow X \mathbin{\mathbb{H}} H$ is surjective with $\phi(x) = \phi(y)$ if and only if the closures of the H -orbits of x and y meet in X^{ss} .*

5. JET DIFFERENTIALS AND GENERALISED DEMAILLY–SEMPLÉ JET BUNDLES

Our remaining aim is to apply our results to a family of examples involving non-reductive reparametrisation groups which arise in singularity theory and the study of jets of curves. We borrow notation from [14].

Let X be a complex n -dimensional manifold. Green and Griffiths in [22] introduced a bundle $J_k \rightarrow X$, the bundle of k -jets of germs of parametrised curves in X ; that is, the fibre over $x \in X$ is the set of equivalence classes of holomorphic maps $f : (W, 0) \rightarrow (X, x)$ where W is an open neighbourhood of 0 in \mathbb{C} , with the equivalence relation $f \sim g$ if and only if the j th derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$. If we choose local holomorphic coordinates (z_1, \dots, z_n) on an open neighbourhood $\Omega \subset X$ around x , the elements of the fibre $J_{k,x}$ are represented by the Taylor expansions

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \dots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

up to order k at $t = 0$ of \mathbb{C}^n -valued holomorphic maps

$$f = (f_1, f_2, \dots, f_n) : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, x).$$

In these coordinates we have

$$J_{k,x} \cong \left\{ (f'(0), \dots, f^{(k)}(0)/k!) \right\} \cong (\mathbb{C}^n)^k,$$

which we identify with \mathbb{C}^{nk} . Note, however, that J_k is not a vector bundle over X , since the transition functions are polynomial, but not in general linear.

Let \mathbb{G}_k be the group of k -jets of biholomorphisms

$$(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0);$$

that is, the k -jets at the origin of local reparametrisations

$$t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*, \alpha_2, \dots, \alpha_k \in \mathbb{C},$$

in which the composition law is taken modulo terms t^j for $j > k$. This group acts fibrewise on J_k by substitution. A short computation shows that the action on the fibre is linear:

$$\begin{aligned} f \circ \varphi(t) &= f'(0) \cdot (\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k) + \frac{f''(0)}{2!} \cdot (\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k)^2 + \dots \\ &\quad \dots + \frac{f^{(k)}(0)}{k!} \cdot (\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k)^k \text{ (modulo } t^{k+1}) \end{aligned}$$

so the linear action of φ on the k -jet $(f'(0), \dots, f^{(k)}(0)/k!)$ is given by the following matrix multiplication:

$$(5) \quad (f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!) \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & \alpha_1\alpha_{k-1} + \dots + \alpha_{k-1}\alpha_1 \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{k-2} + \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \alpha_1^k \end{pmatrix}$$

with (i, j) th entry

$$\sum_{s_1 + \dots + s_i = j} \alpha_{s_1} \dots \alpha_{s_i}$$

for $i, j \leq k$.

There is an exact sequence of groups:

$$0 \rightarrow \mathbb{U}_k \rightarrow \mathbb{G}_k \rightarrow \mathbb{C}^* \rightarrow 0,$$

where $\mathbb{G}_k \rightarrow \mathbb{C}^*$ is the morphism $\varphi \rightarrow \varphi'(0) = \alpha_1$ in the notation used above, and

$$\mathbb{G}_k = \mathbb{U}_k \ltimes \mathbb{C}^*$$

is a semi-direct product of \mathbb{U}_k by \mathbb{C}^* . With the above identification, \mathbb{C}^* is the subgroup of diagonal matrices satisfying $\alpha_2 = \dots = \alpha_k = 0$ and \mathbb{U}_k is the unipotent radical of \mathbb{G}_k , i.e. the subgroup of matrices with $\alpha_1 = 1$. The action of $\lambda \in \mathbb{C}^*$ on k -jets is described by

$$\lambda \cdot (f'(0), f''(0), \dots, f^{(k)}(0)) = (\lambda f'(0), \lambda^2 f''(0), \dots, \lambda^k f^{(k)}(0)).$$

Let $\mathcal{E}_{k,m}^n$ denote the vector space of polynomials $Q(u_1, u_2, \dots, u_k)$, of weighted degree m , with respect to this \mathbb{C}^* action, where $u_i = f^{(i)}(0)$; that is, such that

$$Q(\lambda u_1, \lambda^2 u_2, \dots, \lambda^k u_k) = \lambda^m Q(u_1, u_2, \dots, u_k).$$

Elements of $\mathcal{E}_{k,m}^n$ have the form

$$Q(u_1, u_2, \dots, u_k) = \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=m} u_1^{\alpha_1} u_2^{\alpha_2} \dots u_k^{\alpha_k},$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are multi-indices of length n .

$\mathcal{E}_{k,m}^n$ can be identified with the fibre of the vector bundle $E_{k,m}^{GG} \rightarrow X$ introduced by Green and Griffiths in [22], whose fibres consist of polynomials on the fibres of J_k of weighted degree m with respect to the fibrewise \mathbb{C}^* action on J_k .

The action of \mathbb{G}_k naturally induces an action on the vector space

$$\mathcal{E}_k^n = \bigoplus_{m \geq 0} \mathcal{E}_{k,m}^n = \mathcal{O}(J_{k,x})$$

of polynomial functions on $J_{k,x}$. Following Demailly ([14]), we define $\tilde{\mathcal{E}}_{k,m}^n \subset \mathcal{E}_{k,m}^n$ to be the vector space of \mathbb{U}_k -invariant polynomials of weighted degree m , i.e. those which satisfy

$$Q((f \circ \varphi)', (f \circ \varphi)'', \dots, (f \circ \varphi)^{(k)}) = \varphi'(0)^m \cdot Q(f', f'', \dots, f^{(k)}).$$

Thus $\tilde{\mathcal{E}}_k^n = \bigoplus_{m \geq 0} \tilde{\mathcal{E}}_{k,m}^n = \mathcal{O}(J_{k,x})^{\mathbb{U}_k}$ consists of the polynomials functions on $J_{k,x}$ which are invariant under the induced action of \mathbb{U}_k on $\mathcal{O}(J_{k,x})$. The corresponding bundle of invariants is the Demailly-Semple bundle of algebras $E_k^n = \bigoplus_m E_{k,m}^n \subset \bigoplus_m E_{k,m}^{GG}$ with fibres $\tilde{\mathcal{E}}_k^n = \bigoplus_{m \geq 0} \tilde{\mathcal{E}}_{k,m}^n = \mathcal{O}(J_{k,x})^{\mathbb{U}_k}$.

This bundle of graded algebras $E_k^n = \bigoplus_m E_{k,m}^n$ has been an important object of study for a long time. The invariant jet differentials play a crucial role in the strategy developed by Green, Griffiths, Bloch, Ahlfors, Demailly, Siu and others to prove Kobayashi's 1970 hyperbolicity conjecture [1, 8, 14, 15, 16, 22, 32, 34, 43, 44, 45].

We can now apply Theorem 0.2, Corollary 0.4 and Remark 0.5 to linear action of \mathbb{G}_k on the projective variety associated to $J_{k,x}$. In this case we can also apply the results of [7] since \mathbb{G}_k is a subgroup of $\mathrm{GL}(k; \mathbb{C})$ which is 'generated along the first row' in the sense of [7], and the action of \mathbb{G}_k extends to $\mathrm{GL}(k; \mathbb{C})$.

We can also consider a generalised version of the Demailly-Semple jet differentials to which the results of [7] do not apply. Instead of germs of holomorphic maps $\mathbb{C} \rightarrow X$, we now consider higher dimensional holomorphic objects in X , and therefore we fix a parameter $1 \leq p \leq n$, and study germs of holomorphic maps $\mathbb{C}^p \rightarrow X$.

Again we fix the degree k of these maps, and introduce the bundle $J_{k,p} \rightarrow X$ of k -jets of germs of holomorphic maps $\mathbb{C}^p \rightarrow X$. With respect to local holomorphic coordinates near $x \in X$ the fibre over x is identified with the set of equivalence classes of holomorphic maps $f : (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}^n, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$. Equivalently the elements of the fibre $J_{k,p,x}$ are the Taylor expansions

$$f(\mathbf{u}) = x + \mathbf{u}f'(0) + \frac{\mathbf{u}^2}{2!}f''(0) + \dots + \frac{\mathbf{u}^k}{k!}f^{(k)}(0) + O(|\mathbf{u}|^{k+1})$$

around $\mathbf{u} = 0$ up to order k of \mathbb{C}^n -valued maps

$$f = (f_1, f_2, \dots, f_n) : (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}^n, x).$$

Here

$$f^{(i)}(0) \in \mathrm{Hom}(\mathrm{Sym}^i \mathbb{C}^p, \mathbb{C}^n)$$

so that in these coordinates the fibre is

$$J_{k,p,x} = \left\{ (f'(0), \dots, f^{(k)}(0)/k!) \right\} = \mathbb{C}^{n \binom{k+p}{k} - 1}$$

which is a finite dimensional vector space.

Let $\mathbb{G}_{k,p}$ be the group of k -jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$, that is, the group of biholomorphic maps

$$(6) \quad \mathbf{u} \rightarrow \varphi(\mathbf{u}) = \Phi_1 \mathbf{u} + \Phi_2 \mathbf{u}^2 + \dots + \Phi_k \mathbf{u}^k = \sum_{1 \leq i_1 + \dots + i_p \leq k} a_{i_1 \dots i_p} u_1^{i_1} \dots u_p^{i_p}$$

for which $\Phi_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$ and $\Phi_1 \in \text{Hom}(\mathbb{C}^p, \mathbb{C}^p)$ is non-degenerate. Then $\mathbb{G}_{k,p}$ admits a natural fibrewise right action on $J_{k,p}$, which consist of reparametrizing the k -jets of holomorphic p -discs. A similar computation to at (5) shows that

$$f \circ \varphi(\mathbf{u}) = (f'(0)\Phi_1)\mathbf{u} + (f'(0)\Phi_2 + \frac{f''(0)}{2!}\Phi_1^2)\mathbf{u}^2 + \dots + \dots + \sum_{i_1 + \dots + i_l = k} \left(\frac{f^{(l)}(0)}{l!} \Phi_{i_1} \dots \Phi_{i_l} \right) \mathbf{u}^l.$$

This is a linear action on the fibres $J_{k,p,x}$ with matrix given by

$$(7) \quad \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1 \Phi_2 & \dots & \\ 0 & 0 & \Phi_1^3 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & \Phi_1^k \end{pmatrix},$$

where Φ_i is a $p \times \dim(\text{Sym}^i \mathbb{C}^p)$ -matrix, the i th degree component of the map Φ and the $p \times p$ -matrix Φ_1 is invertible. Here $\Phi_{i_1} \dots \Phi_{i_l}$ is the matrix of the map $\text{Sym}^{i_1 + \dots + i_l}(\mathbb{C}^p) \rightarrow \text{Sym}^l \mathbb{C}^p$, which is induced by

$$\Phi_{i_1} \otimes \dots \otimes \Phi_{i_l} : (\mathbb{C}^p)^{\otimes i_1} \otimes \dots \otimes (\mathbb{C}^p)^{\otimes i_l} \rightarrow (\mathbb{C}^p)^{\otimes l}$$

The linear group $\mathbb{G}_{k,p}$ is generated along its first p rows, in the sense that the parameters in the first p rows are independent, and all the remaining entries are polynomials in these parameters. The only condition which the parameters must satisfy is that the determinant of the first diagonal $p \times p$ block is nonzero. Note that $\mathbb{G}_{k,p}$ is an extension of its unipotent radical $\mathbb{U}_{k,p}$ (given by $\Phi_1 = 1$ by $GL(p; \mathbb{C})$ (given by $\Phi_i = 0$ for $i > 1$), so we have an exact sequence

$$0 \rightarrow \mathbb{U}_{k,p} \rightarrow \mathbb{G}_{k,p} \rightarrow GL(p; \mathbb{C}) \rightarrow 0.$$

The central \mathbb{C}^* of $GL(p; \mathbb{C})$ corresponds to the diagonal matrices with entries t, t^2, \dots, t^k for $t \in \mathbb{C}^*$ where t^i occurs $\dim(\text{Sym}^i(\mathbb{C}^p))$ times, and these act by conjugation on the Lie algebra of $\mathbb{U}_{k,p}$ with weights $i - 1$ for $2 \leq i \leq k$. Thus by Corollary 0.6 we have

Corollary 5.1. *Any linear action of $\mathbb{G}_{k,p}$ on a projective variety X with respect to an ample line bundle L for which semistability coincides with stability for the action of $\mathbb{U}_{k,p}$ extended by the central one-parameter subgroup of $GL(p; \mathbb{C})$ has finitely generated invariants when L is replaced by a tensor power $L^{\otimes c}$ for a sufficiently divisible positive integer c and the linearisation is twisted by a well adapted rational character. Furthermore its enveloping quotient $X \mathbin{\mathbb{G}_{k,p}}$ is the associated projective variety and is a categorical quotient of X^{ss} by the action of $\mathbb{G}_{k,p}$, while the canonical morphism $\phi : X^{ss} \rightarrow X \mathbin{\mathbb{G}_{k,p}}$ is surjective with $\phi(x) = \phi(y)$ if and only if the closures of the $\mathbb{G}_{k,p}$ -orbits of x and y meet in X^{ss} .*

Definition 5.2. The generalized Demailly-Semple jet bundle $E_{k,p,m} \rightarrow X$ of invariant jet differentials of order k and weighted degree (m, \dots, m) has fibre at $x \in X$ consisting of complex-valued polynomials $Q(f'(0), f''(0)/2, \dots, f^{(k)}(0)/k!)$ on the fibre $J_{k,p,x}$ of $J_{k,p}$, which transform under any reparametrization $\phi \in \mathbb{G}_{k,p}$ of $(\mathbb{C}^p, 0)$ as

$$Q(f \circ \phi) = (J_\phi(0))^m Q(f) \circ \phi,$$

where $J_\phi(0)$ denotes the Jacobian at 0 of ϕ ; that is, $J_\phi(0) = \det \Phi_1$ when ϕ is given as at (7). Thus the generalized Demailly-Semple bundle $E_{k,p} = \oplus E_{k,p,m}$ of invariant jet differentials of order k has fibre at $x \in X$ given by the generalized Demailly-Semple algebra $\mathcal{O}(J_{k,p,x})^{\mathbb{U}_{k,p} \rtimes SL(p; \mathbb{C})}$.

We can apply Corollary 5.1 to the linear action of $\mathbb{G}_{k,p}$ on the projective space $X = \mathbb{P}(J_{k,p,x})$ with respect to the line bundle $L = \mathcal{O}_{\mathbb{P}(J_{k,p,x})}(1)$ satisfying

$$\mathcal{O}(J_{k,p,x}) = \oplus_{j \geq 0} H^0(X, L^{\otimes j}).$$

As at Remark 0.5, by considering a diagonal action on $X \times \mathbb{P}^1$, we can deduce that the algebra $\oplus_{m=0}^{\infty} H^0(X \times \mathbb{P}^1, L_{m\chi}^{\otimes cm} \otimes \mathcal{O}_{\mathbb{P}^1}(M))^{\mathbb{G}_{k,p}}$ of $\mathbb{G}_{k,p}$ -invariants on $X \times \mathbb{P}^1$ is finitely generated when $M \gg 1$ and c is a sufficiently divisible positive integer and the linear action has been twisted by a suitable rational character χ/c . This finitely generated graded algebra can be identified with the subalgebra of the generalized Demailly-Semple algebra $\mathcal{O}(J_{k,p,x})^{\mathbb{U}_{k,p} \rtimes SL(p; \mathbb{C})}$ generated by the $\mathbb{U}_{k,p} \rtimes SL(p; \mathbb{C})$ -invariants in $\oplus_{m=0}^{\infty} H^0(X, L^{\otimes cm})^{\mathbb{U}_{k,p} \rtimes SL(p; \mathbb{C})}$ which are weight vectors with non-negative weights for the action of the central one-parameter subgroup of GL_p after twisting by a suitable character χ . This twisting is such that the matrix (7) is replaced with its multiple by $(\det(\Phi_1))^{-(1/p)-\epsilon}$ for $0 < \epsilon \ll 1$, so the only weight vectors $\sigma \in H^0(X, L) = \bigoplus_{i=1}^k \text{Sym}^i(\mathbb{C}^p)$ with non-negative weights are the sections σ in $\text{Sym}^1(\mathbb{C}^p) = \mathbb{C}^p$, which have weight $p\epsilon$. It therefore follows that the localisation $\mathcal{O}(J_{k,p,x})_{\sigma}^{\mathbb{U}_{k,p} \rtimes SL(p; \mathbb{C})}$ of the generalized Demailly-Semple algebra $\mathcal{O}(J_{k,p,x})^{\mathbb{U}_{k,p} \rtimes SL(p; \mathbb{C})}$ at any such σ is finitely generated (cf. [15, 34]).

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