

# THE DENSITY OF SHIFTED AND AFFINE EISENSTEIN POLYNOMIALS

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**ABSTRACT.** In this paper we provide a complete answer to a question by Heyman and Shparlinski concerning the natural density of polynomials which are irreducible by Eisenstein’s criterion after applying some shift. The main tool we use is a local to global principle for density computations over a free  $\mathbb{Z}$ -module of finite rank.

## 1. INTRODUCTION

Let  $\mathbb{Z}$  be the ring of rational integers. The Eisenstein irreducibility criterion [2, 7] is a very convenient tool to establish that a polynomial in  $\mathbb{Z}[x]$  is irreducible. It is a well understood fact that the density of irreducible polynomials of fixed degree  $n$  among all the polynomials of degree  $n$  is equal to one. The question which naturally arises is the following:

**Question 1.1.** What is the density of polynomials which are irreducible by the Eisenstein criterion?

More informally, how likely is it that checking whether a random polynomial is irreducible using only the Eisenstein irreducibility criterion leads to success? In [1, 3] the authors deal with Eisenstein polynomials of fixed degree with coefficients over  $\mathbb{Z}$ . They provide a complete answer to the above question in the case of monic (See [1, Theorem 1], [3, Theorem 1]) and non-monic (See [3, Theorem 2]) Eisenstein polynomials.

From now on we will specialize to the case of non-monic Eisenstein polynomials, since the proofs and methods can be easily adapted from one case to the other. In [3], the authors consider the set of polynomials of degree at most  $n$  having integer coefficients bounded in absolute value by  $B$  (the *height* of a polynomial) and give a sharp estimate for the number  $\varepsilon(B)$  of polynomials which are irreducible by the Eisenstein criterion. The *natural density* of Eisenstein polynomials is then the limit of the sequence  $\varepsilon(B)/(2B)^{n+1}$ , which fully answers Question 1.1.

As is well known, a polynomial  $f(x)$  is irreducible if and only if  $f(x + i)$  is irreducible for all  $i \in \mathbb{Z}$ . Using this simple observation, one could check irreducibility

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by trying to use the Eisenstein criterion for many  $i$ . How likely is it that this procedure works? More formally,

**Question 1.2.** What is the natural density of polynomials  $f(x)$  for which  $f(x+i)$  is irreducible by the Eisenstein criterion for some integer shift  $i$ ?

In [4], Heyman and Shparlinski address this question, giving a lower bound on this density. Nevertheless, the question regarding the exact density remained open. In this paper, we provide a complete solution to Question 1.2 using a local to global principle for densities [5, Lemma 20]. Using similar methods, we also provide a solution to the question appearing in [4, Section 7] about *affine* Eisenstein polynomials.

Our proofs are also supported by Monte Carlo experiments which we provide in Section 5.

## 2. NOTATION

In this section we fix the notation that will be used throughout the paper.

**Definition 2.1.** Let  $R$  be an integral domain and  $R[x]$  be the ring of polynomials with coefficients in  $R$ . We say that  $f(x) = \sum_{i=0}^n \alpha_i x^i \in R[x]$  of degree  $n$  is *Eisenstein with respect to a prime ideal  $p$*  or  *$p$ -Eisenstein* if

- $\alpha_n \notin p$ .
- $\alpha_i \in p$  for all  $i \in \{0, \dots, n-1\}$ .
- $\alpha_0 \notin p^2$ .

We say that  $f(x)$  is *Eisenstein* if it is Eisenstein with respect to some prime ideal  $p$ .

In this paper, we will only consider the ring of integers  $R = \mathbb{Z}$  and the rings of  $p$ -adic integers  $R = \mathbb{Z}_p$ .

**Definition 2.2.** For any subset  $A \subseteq \mathbb{Z}^d$ , we define

$$\begin{aligned}\bar{\rho}(A) &:= \limsup_{B \rightarrow \infty} \frac{|A \cap [-B, B]^d|}{(2B)^d}, \\ \underline{\rho}(A) &:= \liminf_{B \rightarrow \infty} \frac{|A \cap [-B, B]^d|}{(2B)^d}.\end{aligned}$$

If these coincide, we denote their value by  $\rho(A)$  and call it the *natural density* of  $A$ :

$$\rho(A) := \lim_{B \rightarrow \infty} \frac{|A \cap [-B, B]^d|}{(2B)^d}.$$

In what follows, we will identify the module  $R[x]_{\leq n}$  of polynomials of degree at most  $n$  with  $R^{n+1}$  by the standard basis  $\{1, x, \dots, x^n\}$ .

**Definition 2.3.** Let  $E \subseteq \mathbb{Z}^{n+1}$  be the set of degree  $n$  Eisenstein polynomials over the integers. Let  $E_p$  be the set of degree  $n$  Eisenstein polynomials over  $\mathbb{Z}_p$ .

The reader should notice that we are computing the density of shifted (or affine) Eisenstein polynomials of degree *exactly*  $n$  among polynomials of degree *at most*  $n$ . Nevertheless it is easy to see that the density of shifted (and also affine, see Remark 4.3) Eisenstein polynomials of degree less or equal than  $n$  is the same.

## 3. SHIFTED EISENSTEIN POLYNOMIALS

In this section, we determine the density of polynomials  $f(x) \in \mathbb{Z}[x]$  such that  $f(x+i)$  is Eisenstein for some shift  $i \in \mathbb{Z}$ . For this, let  $\sigma$  be the linear map defined by

$$\begin{aligned}\sigma: \mathbb{Z}^{n+1} &\longrightarrow \mathbb{Z}^{n+1} \\ f(x) &\longmapsto f(x+1).\end{aligned}$$

It is easy to see that  $\sigma$  has determinant one. Similarly, we get a determinant one map over  $\mathbb{Z}_p$  for any  $p$ , which we will also denote by  $\sigma$ .

**Definition 3.1.** Let  $\overline{E} \subseteq \mathbb{Z}^{n+1}$  be the set of degree  $n$  polynomials which are Eisenstein after applying some shift  $i \in \mathbb{Z}$ :

$$\overline{E} = \{f(x) \in \mathbb{Z}^{n+1} : f(x+i) \in E \text{ for some } i \in \mathbb{Z}\}.$$

We call these polynomials *shifted Eisenstein*.

In order to compute the density of  $\overline{E}$ , we need to consider each prime  $p$  separately. We do this by working over the  $p$ -adic integers.

**Definition 3.2.** Let  $\overline{E}_p \subseteq \mathbb{Z}_p^{n+1}$  be the set of degree  $n$  polynomials of  $\mathbb{Z}_p[x]$  which are Eisenstein after applying some shift  $i \in \mathbb{Z}_p$ :

$$\overline{E}_p = \{f(x) \in \mathbb{Z}_p^{n+1} : f(x+i) \in E_p \text{ for some } i \in \mathbb{Z}_p\}.$$

We also call these polynomials shifted Eisenstein, since it will always be clear from the context to which set we are referring.

Notice that  $\overline{E}_p \cap \mathbb{Z}^{n+1}$  are exactly the polynomials of  $\mathbb{Z}[x]$  of degree  $n$  which are shifted  $p$ -Eisenstein.

**Lemma 3.3.** *If  $n \geq 2$  and  $f(x) \in \mathbb{Z}_p^{n+1}$  is shifted Eisenstein, then it is so with respect to exactly one rational integer shift  $i \in \{0, \dots, p-1\}$ . In other words,*

$$\overline{E}_p = \bigcup_{i=0}^{p-1} \sigma^{-i} E_p.$$

*Proof.* We clearly have

$$\bigcup_{i=0}^{p-1} \sigma^{-i} E_p \subseteq \overline{E}_p.$$

The other inclusion is easy but not completely trivial.

Let  $f(x) = \sum_{i=0}^n \alpha_i x^i \in E_p$  and  $k \in \mathbb{Z}_p$ . We first show that  $f(x+kp)$  is also Eisenstein: Clearly  $f(x) = f(x+kp)$  in  $\mathbb{Z}_p/p\mathbb{Z}_p$ , so the only condition which one has to check is that the coefficient of the term of degree zero of  $f(x+kp)$  is not in  $p^2\mathbb{Z}_p$ . This coefficient is in fact  $f(kp) = \alpha_0 + \alpha_1 kp + \sum_{i=2}^n \alpha_i k^i p^i$ . Modulo  $p^2\mathbb{Z}_p$  we have that

- $\alpha_i k^i p^i$  is congruent to zero for  $i \geq 2$ ,
- $\alpha_1 kp$  is congruent to zero since  $\alpha_1$  is in  $p\mathbb{Z}_p$ ,
- $\alpha_0$  is not congruent to zero since the polynomial  $f(x)$  is Eisenstein,

from which it follows that the polynomial  $f(x + kp)$  is Eisenstein in  $\mathbb{Z}_p[x]$ .

Let now  $f(x) \in \overline{E}_p$ , then  $f(x + u)$  is Eisenstein for some  $u \in \mathbb{Z}_p$ . The inclusion

$$\overline{E}_p \subseteq \bigcup_{i=0}^{p-1} \sigma^{-i} E_p$$

will follow if we show that we can select  $u$  in  $\{0, \dots, p-1\}$ . Write  $u = kp + i$  with  $i \in \{0, \dots, p-1\}$  and  $k \in \mathbb{Z}_p$ . Using what we proved above, we see that  $f(x + u - kp) = f(x + i)$  is Eisenstein, and the inclusion follows.

We now show that the union is disjoint, i.e.  $\sigma^{-i} E_p \cap \sigma^{-j} E_p = \emptyset$  for any  $i, j \in \{0, \dots, p-1\}$  and  $i \neq j$ . Without loss of generality, we can assume  $i > j$ . Then

$$\sigma^{-i} E_p \cap \sigma^{-j} E_p = \emptyset \iff E_p \cap \sigma^{i-j} E_p = \emptyset.$$

Let  $t := j - i$  and  $\sum_{k=0}^n \alpha_k x^k = f(x) \in E_p$ , then the coefficient of the degree zero term of  $f(x + t)$  is  $f(t) = \alpha_n t^n + \sum_{k=0}^{n-1} \alpha_k t^k$ . Now, the reduction of  $\alpha_k$  modulo  $p$  is zero for any  $k < n-1$  and  $\alpha_n$  and  $t$  are invertible modulo  $p$ , so  $f(x + t)$  is not Eisenstein.  $\square$

Let  $\mu_p$  be the Haar measure measure on  $\mathbb{Z}_p^{n+1}$  normalized to have total mass 1, and  $\mu_\infty$  the Lebesgue measure on  $\mathbb{R}^{n+1}$ . (For basics on the Haar measure on the  $p$ -adic numbers, we refer to [6].)

**Lemma 3.4.** *In the above notation we have for  $n \geq 2$*

$$\mu_p(\overline{E}_p) = \frac{(p-1)^2}{p^{n+1}}.$$

*Proof.* Since  $\sigma^{-1}$  has determinant one, it does not change the  $p$ -adic volumes. Therefore, by Lemma 3.3, one has  $\mu_p(\overline{E}_p) = p \cdot \mu_p(E_p)$ . It is easy to compute the measure  $\mu_p(E_p)$  by writing  $E_p = (p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p) \times (p\mathbb{Z}_p)^{n-1} \times (\mathbb{Z}_p \setminus p\mathbb{Z}_p)$ .  $\square$

In order to obtain the density  $\rho(\overline{E})$  from the local data  $\{\mu_p(\overline{E}_p)\}_p$ , we will use the following lemma [5, Lemma 20].

**Lemma 3.5.** *Suppose  $U_\infty \subseteq \mathbb{R}^d$  is such that  $\mathbb{R}^+ \cdot U_\infty = U_\infty$ ,  $\mu_\infty(\partial U_\infty) = 0$ . Let  $U_\infty^1 = U_\infty \cap [-1, 1]^d$  and  $s_\infty = \mu_\infty(U_\infty^1)$ . Let  $U_p \subseteq \mathbb{Z}_p^d$ ,  $\mu_p(\partial U_p) = 0$  and  $s_p = \mu_p(U_p)$  for each prime  $p$ . Let  $M_\mathbb{Q}$  be the set of places of  $\mathbb{Q}$ . Moreover, suppose that*

$$(3.1) \quad \lim_{M \rightarrow \infty} \overline{\rho}(\{a \in \mathbb{Z}^d : a \in U_p \text{ for some finite prime } p \text{ greater than } M\}) = 0.$$

Let  $P: \mathbb{Z}^d \rightarrow 2^{M_\mathbb{Q}}$  be defined as  $P(a) = \{v \in M_\mathbb{Q} : a \in U_v\}$ . Then we have:

- (1)  $\sum_v s_v$  converges.
- (2) For any  $T \subseteq 2^{M_\mathbb{Q}}$ ,  $\nu(T) := \rho(P^{-1}(T))$  exists and defines a measure on  $2^{M_\mathbb{Q}}$ , which is concentrated at the finite subsets of  $M_\mathbb{Q}$ .
- (3) Let  $S$  be a finite subset of  $M_\mathbb{Q}$ , then

$$\nu(\{S\}) = \prod_{v \in S} s_v \prod_{v \notin S} (1 - s_v).$$

*Proof.* For the proof, see [5, Lemma 20].  $\square$

After showing that condition (3.1) applies, we can use Lemma 3.5 to determine the density of shifted Eisenstein polynomials over the integers.

**Theorem 3.6.** *Let  $n \geq 3$ . The density of shifted Eisenstein polynomials of degree  $n$  is*

$$(3.2) \quad \rho(\overline{E}) = 1 - \prod_{p \text{ prime}} \left( 1 - \frac{(p-1)^2}{p^{n+1}} \right).$$

*Proof.* Set  $U_p = \overline{E}_p$  for all  $p$  and  $U_\infty = \emptyset$ . The conditions  $\mu_p(\partial U_p) = 0$  hold since  $U_p$  is both closed and open. Notice that in the notation of Lemma 3.5 we have that  $P^{-1}(\{\emptyset\})$  equals the complement of  $\overline{E}$ . Therefore, if condition (3.1) is verified, we get the claim:

$$\rho(\overline{E}) = 1 - \prod_{p \text{ prime}} (1 - s_p) = 1 - \prod_{p \text{ prime}} \left( 1 - \frac{(p-1)^2}{p^{n+1}} \right).$$

Let us now show that the condition indeed holds:

$$(3.3) \quad \begin{aligned} & \lim_{M \rightarrow \infty} \bar{\rho}(\{a \in \mathbb{Z}^{n+1} : a \in \overline{E}_p \text{ for some finite prime } p \text{ greater than } M\}) \\ &= \lim_{M \rightarrow \infty} \limsup_{B \rightarrow \infty} \frac{|\bigcup_{p > M} \overline{E}_p \cap [-B, B]^{n+1}|}{(2B)^{n+1}}. \end{aligned}$$

We have  $\overline{E}_p \cap [-B, B]^{n+1} = \emptyset$  for  $p > CB^2$ , where  $C$  is a constant depending only on the degree  $n$ . One can see that using the following argument: Let  $f(x)$  be a polynomial in  $[-B, B]^{n+1}$  for which  $f(x+i)$  is Eisenstein, then [4, Lemma 3.1]

$$p^{n-1} \mid \text{Disc}(f(x+i)) = \text{Disc}(f(x)) \neq 0.$$

Now, the discriminant of  $f(x)$  is a polynomial of degree  $2n-2$  in the coefficients, whence

$$p^{n-1} \leq \text{Disc}(f(x)) \leq DB^{2n-2}$$

for some constant  $D$  depending only on  $n$ . Therefore, for  $C = D^{1/(n-1)}$ , we have  $p \leq CB^2$ . Thus, we have just shown that for fixed  $B$ , the union in (3.3) is finite, and we can bound it by

$$(3.4) \quad \begin{aligned} & \lim_{M \rightarrow \infty} \limsup_{B \rightarrow \infty} \frac{|\bigcup_{CB^2 > p > M} \overline{E}_p \cap [-B, B]^{n+1}|}{(2B)^{n+1}} \\ & \leq \lim_{M \rightarrow \infty} \limsup_{B \rightarrow \infty} \sum_{CB^2 > p > M} \frac{|\overline{E}_p \cap [-B, B]^{n+1}|}{(2B)^{n+1}}. \end{aligned}$$

Given the order of the limits, we can fix the following setting:  $M > n$  and  $B > M$ . Now let us bound  $|\overline{E}_p \cap [-B, B]^{n+1}|$  in the following two cases:

*Case  $2B < p$ .* In this case, we can consider  $[-B, B]^{n+1}$  as a subset of  $\mathbb{F}_p^{n+1}$  without losing any information. The reader should notice that modulo  $p$ , the elements of  $\overline{E}_p$  have a multiple root of order  $n$  at some  $i \in \mathbb{F}_p$ . Now, the key observation is the following: The reduction modulo  $p$  of the polynomials in  $[-B, B]^{n+1} \cap \overline{E}_p$  is contained in the set

$$S_p := \{a(x-i)^n : a \in [-B, B] \setminus \{0\} \text{ and } -nai \in [-B, B]\}.$$

This represents the condition that the degree  $n$  and  $n-1$  coefficients live in  $[-B, B]$ :

$$[-B, B]^{n+1} \cap \overline{E}_p \subseteq S_p.$$

Observe now that  $|S_p| = (2B - 1)2B \leq (2B)^2$ , since  $n$  and  $a$  are invertible modulo  $p$  (recall  $p > M > n$ ). We conclude that

$$|[-B, B^{n+1} \cap \overline{E}_p]| \leq |S_p| \leq (2B)^2.$$

Notice that this bound is uniform in  $p$ .

*Case  $2B \geq p$ .* In this case, the bound is more natural. Consider the projection map

$$\pi: \mathbb{Z}^{n+1} \longrightarrow \mathbb{F}_p^{n+1}$$

and the shift map modulo  $p$

$$\begin{aligned} \sigma^{-1}: \mathbb{F}_p^{n+1} &\longrightarrow \mathbb{F}_p^{n+1} \\ f(x) &\longmapsto f(x - 1). \end{aligned}$$

Consider the sets of polynomials  $L_p := \{ax^n : a \in \mathbb{F}_p^*\}$  and

$$(3.5) \quad \overline{L}_p = \bigcup_{i=0}^{p-1} \sigma^{-i} L_p.$$

We have  $|\overline{L}_p| \leq p^2$ .

Notice that

$$(3.6) \quad \pi([-B, B^{n+1} \cap \overline{E}_p]) \subseteq \overline{L}_p.$$

At this step, we observe that the projection is at most  $\lceil 2B/p \rceil^{n+1}$  to one, therefore we can bound  $|[-B, B^{n+1} \cap \overline{E}_p]|$  using the projection map and condition (3.6):

$$|[-B, B^{n+1} \cap \overline{E}_p]| \leq |\overline{L}_p| \cdot \lceil 2B/p \rceil^{n+1} \leq p^2 \left( \frac{2B}{p} + 1 \right)^{n+1} \leq p^2 \left( \frac{4B}{p} \right)^{n+1},$$

where the last inequality follows from  $2B \geq p$ . At the end of the day, the bound we have is of the form

$$|[-B, B^{n+1} \cap \overline{E}_p]| \leq 4^{n+1} \frac{B^{n+1}}{p^{n-1}}.$$

Let us now come back to the sum in (3.4), which we can split according to the two cases above:

$$(3.7) \quad \sum_{CB^2 > p > M} \frac{|\overline{E}_p \cap [-B, B^{n+1}]|}{(2B)^{n+1}} \leq \sum_{CB^2 > p > 2B} \frac{(2B)^2}{(2B)^{n+1}} + \sum_{2B \geq p > M} \frac{2^{n+1}}{p^{n-1}}.$$

Using the limit in  $B$ , the first sum goes to zero by the prime number theorem since  $n \geq 3$ . As  $B$  goes to infinity, the other sum becomes a converging series (again  $n \geq 3$ ) starting at the index  $M$ . Letting  $M$  go to infinity, this too goes to zero. Hence we have shown that condition (3.1) holds, and the theorem follows.  $\square$

In degree 2, the above proof does not work: Indeed, it is easily seen that  $\sum_p s_p$  diverges for  $n = 2$ , so by the first claim of Lemma 3.5, the proof we gave in degree greater or equal than 3 is doomed to fail in degree 2. However, we have a much simpler application of the lemma which shows that the density of shifted Eisenstein polynomials of degree 2 is indeed one, as Theorem 3.6 suggests.

**Proposition 3.7.** *The density of shifted Eisenstein polynomials of degree  $n = 2$  is one.*

*Proof.* Let again  $U_\infty = \emptyset$ . We now apply Lemma 3.5 to a truncated sequence of sets. For this, let  $M$  be a positive integer and

$$U_p = \begin{cases} \overline{E}_p & \text{if } p \leq M \\ \emptyset & \text{if } p > M. \end{cases}$$

This truncated sequence now automatically satisfies condition (3.1), and we get the density

$$\rho(\overline{E}) \geq \rho\left(\bigcup_{p \leq M} \overline{E}_p \cap \mathbb{Z}^3\right) = 1 - \prod_{p \leq M} \left(1 - \frac{(p-1)^2}{p^3}\right).$$

Letting  $M$  tend to infinity gives  $\rho(\overline{E}) = 1$ , as the product converges to zero.  $\square$

*Remark 3.8.* Even though the density of shifted Eisenstein polynomials of degree 2 is one, not all irreducible polynomials are Eisenstein for some shift (or even affine transformation): Take for example the polynomial  $f(x) = x^2 + 8x - 16$ , which is irreducible over  $\mathbb{Z}$ . Its discriminant is  $2^7$ , so it could only be shifted Eisenstein with respect to 2. But neither  $f(x)$  nor  $f(x+1) = x^2 + 10x - 7$  is 2-Eisenstein.

#### 4. AFFINE EISENSTEIN POLYNOMIALS

In [4, Section 7], the question was also raised about the density of polynomials that become Eisenstein after an arbitrary affine transformation, instead of only considering shifts. We can address this question as well, using the same methods as in Section 3.

**Definition 4.1.** For  $f(x) \in R^{n+1}$  and  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2 \times 2}$ , we define the *affine transformation of  $f$  by  $A$*  as

$$f * A := (cx + d)^n f\left(\frac{ax + b}{cx + d}\right).$$

It is easy to see that, when restricted to  $\text{GL}_2(R)$ , this is a right group action.

Like in Section 3, we consider the set of polynomials with integer coefficients that become Eisenstein after some affine transformation.

**Definition 4.2.** Let  $\tilde{E} \subseteq \mathbb{Z}^{n+1}$  be the set of degree  $n$  polynomials which become Eisenstein of degree  $n$  after some affine transformation  $A \in \mathbb{Z}^{2 \times 2}$ :

$$\tilde{E} = \{f(x) \in \mathbb{Z}^{n+1} : f * A \in E \text{ for some } A \in \mathbb{Z}^{2 \times 2}\}.$$

We call these polynomials *affine Eisenstein*.

It is easy to see that if both  $f$  and  $f * A$  have degree  $n$  and  $f * A$  is irreducible, then so is  $f$ . Hence, an affine Eisenstein polynomial is irreducible.

*Remark 4.3.* The reader should notice that also in this case, we only consider affine Eisenstein polynomials of degree *exactly*  $n$ . Nevertheless an observation is required: It could happen that a degree  $n$  polynomial becomes Eisenstein of *lower* degree after some affine transformation. Fortunately, it can be seen that a polynomial for which this happens is never irreducible. Likewise, a polynomial of degree less than  $n$  cannot become Eisenstein of degree  $n$  after an affine transformation, since any transformation that increases the degree introduces factors  $cx + d$ .

We again consider each prime separately by working over the  $p$ -adic integers.

**Definition 4.4.** Let  $\tilde{E}_p \subseteq \mathbb{Z}_p^{n+1}$  be the set of degree  $n$  polynomials of  $\mathbb{Z}_p[x]$  which become Eisenstein of degree  $n$  after some affine transformation  $A \in \mathbb{Z}_p^{2 \times 2}$ :

$$\tilde{E}_p = \{f(x) \in \mathbb{Z}_p^{n+1} : f * A \in E_p \text{ for some } A \in \mathbb{Z}_p^{2 \times 2}\}.$$

We also call these polynomials affine Eisenstein, since it will always be clear from the context to which set we are referring.

In what follows we compute the measure  $\mu_p(\tilde{E}_p)$ . For this, we need to write  $\tilde{E}_p$  as a disjoint union of transformed copies of  $E_p$  as in Lemma 3.3. The following lemma is essential for this.

**Lemma 4.5.** *Assume  $f(x) \in \mathbb{Z}_p^{n+1}$  is Eisenstein of degree  $n \geq 2$ , and let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_p^{2 \times 2}$ . Then,  $f * A$  is Eisenstein of degree  $n$  if and only if  $p \mid b$ ,  $p \nmid a$ ,  $p \nmid d$ .*

*Proof.* If we write  $f(x) = \sum_{i=0}^n \alpha_i x^i$  and  $f * A = \sum_{l=0}^n \beta_l x^l$ , then a simple calculation gives

$$(4.1) \quad \beta_l = \sum_{j=0}^l \sum_{s=l}^n \binom{n+j-s}{j} \binom{s-j}{l-j} \alpha_{s-j} d^{n-s} b^{s-l} a^{l-j} c^j.$$

Assume now that  $f * A$  is Eisenstein, so  $p \mid \beta_l$  for  $0 \leq l \leq n-1$ ,  $p^2 \nmid \beta_0$ ,  $p \nmid \beta_n$ . Consider first  $\beta_0$ . Reducing modulo  $p$  and using that  $p \mid \alpha_i$  for  $i < n$ , we see that

$$\beta_0 \equiv \alpha_n b^n \pmod{p}.$$

Since  $p \nmid \alpha_n$ , we get that  $p \mid b$ . Knowing this, we reduce  $\beta_0$  modulo  $p^2$  and get

$$\beta_0 \equiv \alpha_0 d^n + \alpha_1 d^{n-1} b \equiv \alpha_0 d^n \pmod{p^2},$$

since  $p^2 \mid \alpha_1 b$ . From this, we see that  $p^2 \nmid \beta_0$  if and only if  $p \nmid d$ .

Finally, we reduce  $\beta_n$  modulo  $p$  and get

$$\beta_n \equiv \alpha_n a^n \pmod{p},$$

from which we conclude that  $p \nmid a$ .

Vice versa, if we assume that  $p \mid b$ ,  $p \nmid a$ ,  $p \nmid d$ , the same computations as above show that  $p \nmid \beta_n$ ,  $p \mid \beta_0$ ,  $p^2 \nmid \beta_0$ , and we easily see from (4.1) that  $p \mid \beta_l$  for  $0 < l < n$ . Hence,  $f * A$  is Eisenstein.  $\square$

We denote by  $S = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_p^{2 \times 2} : p \mid b, p \nmid a, p \nmid d\}$  the set of matrices from Lemma 4.5. This is a subgroup of  $\text{GL}_2(\mathbb{Z}_p)$ . We can obtain the disjoint union decomposition of  $\tilde{E}_p$  by considering the left cosets of  $S$ , but first, we need to deal with the noninvertible matrices. It turns out that they don't matter.

**Lemma 4.6.** *Let  $n \geq 2$  and  $f(x) \in \mathbb{Z}_p^{n+1}$ . If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_p^{2 \times 2}$  is not invertible, then  $f * A$  is not Eisenstein of degree  $n$ .*

*Proof.* Assume for contradiction that  $f * A$  is Eisenstein of degree  $n$ . We write again  $f(x) = \sum_{i=0}^n \alpha_i x^i$  and  $f * A = \sum_{l=0}^n \beta_l x^l$ . We reduce modulo  $p$ :

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in \mathbb{F}_p^{2 \times 2}.$$

Since  $\det \bar{A} = 0$ , there are two cases: Either  $\bar{c} = \bar{d} = 0$ , or there is a  $\lambda \in \mathbb{F}_p$  such that  $\bar{a} = \lambda \bar{c}$  and  $\bar{b} = \lambda \bar{d}$ .



We consider the second case. Since  $f * A$  is Eisenstein, we see that  $\bar{f} * \bar{A} = \bar{\beta}_n x^n \in \mathbb{F}_p[x]$  with  $\bar{\beta}_n \neq 0$ . On the other hand,

$$\bar{f} * \bar{A} = (\bar{c}x + \bar{d})^n \bar{f} \left( \frac{\lambda \bar{c}x + \lambda \bar{d}}{\bar{c}x + \bar{d}} \right) = (\bar{c}x + \bar{d})^n \bar{f}(\lambda).$$

From this, we see that  $\bar{f}(\lambda) \neq 0$ ,  $\bar{c} \neq 0$  and  $\bar{d} = 0$ . This means that  $p \mid d$  and  $p \mid b$ , from which it follows by (4.1) that  $p^2 \mid \beta_0$ . This contradicts the assumption that  $f * A$  is Eisenstein.

The case  $\bar{c} = \bar{d} = 0$  is similar.  $\square$

Hence, we only need to consider the action of  $\text{GL}_2(R)$  on  $R^{n+1}$ . According to Lemma 4.5, the action of elements of  $S$  does not change whether a polynomial is Eisenstein. Therefore, to see if a polynomial  $f(x) \in \mathbb{Z}_p^{n+1}$  is affine Eisenstein, it is enough to check one representative of each left coset of  $S \subset \text{GL}_2(\mathbb{Z}_p)$ . We can list these cosets explicitly.

**Lemma 4.7.** *The subgroup  $S \subset \text{GL}_2(\mathbb{Z}_p)$  has  $p + 1$  left cosets, which are the following:*

- $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} S$  for  $i \in \{0, \dots, p-1\}$  (corresponding to shifts), and
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S$  (corresponding to the reciprocal).

*Proof.* It is easy to see that these  $p + 1$  left cosets are distinct. We need to show that every  $A = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p)$  lies in one of them.

Consider first the case  $p \mid v$ . Then,

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & v \\ s & t \end{pmatrix},$$

with  $\begin{pmatrix} u & v \\ s & t \end{pmatrix} \in S$ .

If instead  $p \nmid v$ , let  $i \equiv t/v \pmod{p}$ ,  $i \in \{0, \dots, p-1\}$ . Then,

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s - iu & t - iv \\ u & v \end{pmatrix},$$

with  $p \mid t - iv$  by choice of  $i$ , and  $p \nmid s - iu$  since the matrix has to be invertible.  $\square$

Together, Lemmata 4.5 and 4.7 say that  $f(x)$  is affine Eisenstein with respect to some  $A$  if and only if it is shifted Eisenstein with respect to some  $i \in \{0, \dots, p-1\}$ , or if its reciprocal  $x^n f(1/x)$  is Eisenstein; and these possibilities are exclusive. In other words,

$$\tilde{E}_p = \text{recip}(E_p) \sqcup \bigsqcup_{i=0}^{p-1} \sigma^{-i} E_p.$$

Since shifting and taking the reciprocal are linear maps with determinant  $\pm 1$ , they preserve the Haar measure, and we see that

$$\mu_p(\tilde{E}_p) = (p+1)\mu_p(E_p) = \frac{(p+1)(p-1)^2}{p^{n+2}}.$$

With this, we can now show the analogue of Theorem 3.6 for affine transformations.

TABLE 1. Simulations for degree  $n = 3$ .

	found	expected
irreducible	20 000	20 000
Eisenstein	1112	1112
shifted Eisenstein	3416	3353
affine Eisenstein	4360	4328

TABLE 2. Simulations for degree  $n = 4$ .

	found	expected
irreducible	20 000	20 000
Eisenstein	432	449
shifted Eisenstein	1096	1112
affine Eisenstein	1570	1547

**Theorem 4.8.** *Let  $n \geq 3$ . The density of affine Eisenstein polynomials of degree  $n$  is*

$$\rho(\tilde{E}) = 1 - \prod_{p \text{ prime}} \left( 1 - \frac{(p+1)(p-1)^2}{p^{n+2}} \right).$$

*Proof.* The proof is mostly the same as for Theorem 3.6. For the verification of condition (3.1), note that the case  $2B < p$  is unchanged from the proof of Theorem 3.6, since the reciprocal polynomial cannot be  $p$ -Eisenstein for  $p > B$ . For the case  $2B \geq p$ , we simply get an additional term in the union (3.5), and so the estimate changes to  $|\overline{L}_p| \leq p(p+1)$ . However, this doesn't affect the convergence of the sum in (3.7).  $\square$

*Remark 4.9.* Clearly, the density of affine Eisenstein polynomials of degree  $n = 2$  is one. After all, we are considering a superset of the shifted Eisenstein polynomials of Proposition 3.7.

## 5. MONTE CARLO SIMULATIONS

As in [4, Section 6], we ran some Monte Carlo simulations to verify how near our results are to the actual probability of finding a shifted (or affine) Eisenstein polynomial among all the polynomials of a given height. For degrees  $n = 3$  and 4, we tested 20 000 random polynomials of height at most 1 000 000. The results are shown in Tables 1 and 2. The first column contains the number of polynomials which were actually found by the Monte Carlo experiment, while the second column contains the expected number given by [3, Theorem 2] and Theorems 3.6 and 4.8. All the experiments seem to agree with our theoretical results.

The simulations were done using the Sage computer algebra system [8], and the code is available upon request.

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