

Intersections of hypergraphs

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Abstract

Given two weighted k -uniform hypergraphs G, H of order n , how much (or little) can we make them overlap by placing them on the same vertex set? If we place them at random, how concentrated is the distribution of the intersection? The aim of this paper is to investigate these questions.

1 Introduction

The discrepancy of a set of points in a subset of Euclidean space measures how uniformly the points are spread through the set. For instance, the discrepancy of a set of n points in a square of area n can be defined as the maximum difference between the area of a subsquare and the number of points from the set that it contains. Discrepancy theory in the geometric setting has been studied since the work of Weyl [37] on sequences, and is of interest in areas including number theory and combinatorics, as well as having applications in computational geometry and numerical integration (see for instance the books by Beck and Chen [6], Kuipers and Niederreiter [28] and Drmota and Tichy [16]).

In the discrete context, a similar notion of discrepancy for hypergraphs was introduced by Erdős and Spencer [20], and measures the extent to which

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the edges of a hypergraph are uniformly distributed (inside the complete graph).¹

Erdős and Spencer showed that the edges of a k -uniform hypergraph can not be distributed too uniformly: for every k -uniform hypergraph on n vertices, there is a subset S in which the number of edges differs from $\frac{1}{2}\binom{|S|}{k}$ by at least $c_k n^{(k+1)/2}$. Random hypergraphs show that his bound is optimal up to a constant factor. In the case of graphs (i.e. $k = 2$), Erdős, Goldberg, Pach and Spencer [19] later extended this to graphs of any density p , where the measure of discrepancy is the maximum difference between the number of edges in a subset S and the expected $p\binom{|S|}{2}$. More general results, including an extension of the Erdős-Spencer result to arbitrary densities when $k \geq 3$, were proved in [10].

The aim of this paper is to study the discrepancy of *pairs* of hypergraphs. The discrepancy of a pair hypergraphs, introduced in [11], measures the extent to which the edges of the two hypergraphs are uniformly and independently distributed. Given k -uniform hypergraphs G and H with n vertices and densities p, q , the discrepancy of the pair G, H is the maximum size, over all bijections between their vertex sets, of the difference between their intersection and $pq\binom{n}{k}$ (the expected intersection under a random mapping). For instance, G and H have discrepancy 0 if their intersection has the same size for any placement of both hypergraphs onto the same vertex set; on the other hand, if G and H are isomorphic to the same incomplete graph then their discrepancy will be large, as any isomorphism between them will give a much larger than average intersection.

In light of the results of Erdős and Spencer [20], it is natural to expect that every pair of (unweighted) k -uniform hypergraphs of moderate density should have large discrepancy (of order at least $n^{(k+1)/2}$), and we conjectured in [11] that this should be the case. For $k = 2$, this conjecture was proved in [11], but for $k = 3$ we will show here that there is a counterexample (see section 1.2); for $k \geq 4$, the conjecture is still open.

In this paper, we will mainly be interested in the discrepancy of pairs of *weighted* hypergraphs. It turns out that, for weighted hypergraphs the picture is dramatically different from the unweighted case:

- For every $k \geq 1$, there is a set of k nontrivial weighted k -uniform hypergraphs such that every pair has discrepancy 0.

¹There are a number of other standard ways to measure discrepancy for discrete structures: see Beck and Sós [7], Chazelle [13] and Matoušek [30].

On the other hand, if we take one additional hypergraph, there must be a pair with large discrepancy:

- For every $k \geq 1$, and every set of $k + 1$ nontrivial normalised weighted hypergraphs, there is some pair that has discrepancy at least $c_k n^{(k+1)/2}$.

As we shall see in Section 1.2, both results are special cases of much more general results (Theorem 16 and Theorem 3, respectively) on the discrepancy of pairs of weighted hypergraphs. We will also be interested in the size of the intersection when two (weighted) k -uniform hypergraphs are placed at random onto the same vertex set. For sequences ($k = 1$) and graphs ($k = 2$), the distribution of this intersection has been extensively studied in the statistical literature, and central limit theorems have been proved under various conditions. Here, we work with general k , but prove only a lower bound (Theorem 4) on the concentration of the distribution.

The rest of the paper is organized as follows: after giving some background in Section 1.1, we discuss the discrepancy of pairs of weighted hypergraphs and present our results in Section 1.2. We give notation and some useful tools in Section 2, and define the W -vector in Section 3. We study the effects of a single transposition in Section 4; we prove Theorems 3 and 4 in Section 5; and Theorem 16 is proved in Section 6. We conclude in Section 7 with some comments and open problems.

We work throughout the paper with weighted hypergraphs. A *weighted k -uniform hypergraph with vertex set V* is a function $w : V^{(k)} \rightarrow \mathbb{R}$, i.e. a weighting on the k -sets in V . An *unweighted k -uniform hypergraph* is a subset of $V^{(k)}$, and can be identified with the weighted hypergraph given by the indicator function for its edges. The *density* of w is $d(w) = w(V)/\binom{n}{k}$, where $w(V) = \sum_{e \in V^{(k)}} w(e)$. We shall assume that all hypergraphs have n vertices unless otherwise stated (and, in particular, that $|V| = n$).

1.1 Discrepancy of a single hypergraph

In this section we give some background on the discrepancy of a single hypergraph.

Let G be an unweighted k -uniform hypergraph with weight function w . If $S \subset V$ is chosen uniformly at random from all sets of some fixed size, we have

$$\mathbb{E}w(S) = d(w) \binom{|S|}{k}. \quad (1)$$

It therefore makes sense to define the *discrepancy* of G by

$$\text{disc}(G) = \max_{S \subseteq V} \left| w(S) - d(w) \binom{|S|}{k} \right|. \quad (2)$$

The discrepancy measure how far $w(S)$ can deviate from (1), but does not indicate whether the number of edges is greater or less than we expect. We therefore define the *positive discrepancy* $\text{disc}^+(G)$ by

$$\text{disc}^+(G) = \max_{S \subseteq V} \left(w(S) - d(w) \binom{|S|}{k} \right).$$

and the *negative discrepancy* $\text{disc}^-(G)$ by

$$\text{disc}^-(G) = \max_{S \subseteq V} \left(d(w) \binom{|S|}{k} - w(S) \right).$$

Clearly $\text{disc}(G) = \max\{\text{disc}^+(G), \text{disc}^-(G)\}$, and it follows from (1) that both positive and negative discrepancy are nonnegative. (We note that notions of signed discrepancy have been considered in other contexts: see Erdős, Faudree, Rousseau and Schelp [18], Krivelevich [27] and Keevash and Sudakov [25].)

The discrepancy of graphs and hypergraphs was introduced by Erdős and Spencer [20], who showed that every (unweighted) k -uniform hypergraph G of order n and density $1/2$ has

$$\text{disc}(G) \geq c_k n^{(k+1)/2}. \quad (3)$$

For $k = 2$ (i.e. for graphs), Erdős, Goldberg, Pach and Spencer [19] extended (3) to arbitrary density, showing that if G has order n and density p , where $p \in (2/(n-1), 1 - 2/(n-1))$, then

$$\text{disc}(G) \geq c \sqrt{p(1-p)} n^{3/2}. \quad (4)$$

By considering random graphs in $\mathcal{G}(n, 1/2)$, it can be seen that the discrepancy of a graph on n vertices can be as small as $O(n^{3/2})$; thus (3) is optimal up to the constant. However, the one-sided discrepancies can be smaller: $K_{n/2, n/2}$ has positive discrepancy $O(n)$, while its complement $2K_{n/2}$ has negative discrepancy $O(n)$; on the other hand, both graphs have discrepancy $\Omega(n^2)$ in the other direction. Bollobás and Scott [10] showed that this

tradeoff is unavoidable: for every graph G of order n , with $p\binom{n}{2}$ edges, where $p(1-p) \geq 1/n$, we have

$$\text{disc}^+(G)\text{disc}^-(G) \geq cp(1-p)n^3. \quad (5)$$

Note that (4) follows immediately. A similar result to (5) also holds for k -uniform hypergraphs [10]: for every hypergraph H of order n and density p , where $p(1-p) \geq 1/n$,

$$\text{disc}^+(H)\text{disc}^-(H) \geq c_k p(1-p)n^{k+1}. \quad (6)$$

It follows immediately that $\text{disc}(H) \geq c'_k \sqrt{p(1-p)}n^{(k+1)/2}$, which extends (3) to general densities.

1.2 Results

We now turn to the discrepancy of pairs of hypergraphs. Given two weighted hypergraphs w, u on V , the *intersection* of w and u is naturally defined as $\langle w, u \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $V^{(k)}$. There is also a natural action of the symmetric group $S(V)$ on the space of weighted hypergraphs, given by $w_\pi(e) = w(\pi^{-1}e)$ (see Section 2 for notation).

If we permute w uniformly at random, the expected intersection with u is

$$\mathbb{E}_\pi \langle w_\pi, u \rangle = d(w)d(u) \binom{n}{k}. \quad (7)$$

This leads us to define the *positive discrepancy of the pair w, u* by

$$\text{disc}^+(w, u) = \max_\pi \langle w_\pi, u \rangle - d(w)d(u) \binom{n}{k} \quad (8)$$

and the *negative discrepancy* by

$$\text{disc}^-(w, u) = d(w)d(u) \binom{n}{k} - \min_\pi \langle w_\pi, u \rangle. \quad (9)$$

Note that both are nonnegative, by (7). The *discrepancy* $\text{disc}(w, u)$ is then defined as

$$\text{disc}(w, u) = \max\{\text{disc}^+(w, u), \text{disc}^-(w, u)\} = \max_\pi |\langle w_\pi, u \rangle - d(w)d(u) \binom{n}{k}|.$$

The discrepancy of a pair of hypergraphs was introduced in [11], and is a natural extension of the notion of discrepancy for a single hypergraph. Analogously with (4), it was shown in [11] that, for every pair of graphs G, H , of order n and densities $p, q \in (16/n, 1 - 16/n)$,

$$\text{disc}(G, H) \geq c(p, q)n^{3/2}, \quad (10)$$

where $c(p, q) = p^2(1-p)^2q^2(1-q)^2/10^{10}$.

As with the discrepancy of a single graph, the one-sided discrepancies of pairs of graphs can be quite small. For instance, consider $G = K_{n/2, n/2}$ and $H = 2K_{n/2}$: this pair has positive discrepancy $O(n)$, which is minimal up to a constant factor for dense graphs (although the negative discrepancy is $\Omega(n^2)$, which is maximal up to a constant factor). However, it was shown in [11] that there is a bound on the *product* of the two discrepancies: for every pair of graphs G, H , of order n and densities $p, q \in (16/n, 1 - 16/n)$,

$$\text{disc}^+(G, H)\text{disc}^-(G, H) \geq c(p, q)^2n^3. \quad (11)$$

Thus if either positive or negative discrepancy is small, the other must be large. The bound (11) is sharp up to the constant, as can be seen from $K_{n/2, n/2}$ and $2K_{n/2}$ or by taking $G = 2K_{n/2}$ and letting H be a random graph with fixed density. Note also that (5) is a special case of (11), as we can take $H = K_{n/2} \cup (n/2)K_1$ (which corresponds to restricting S to have size $n/2$ in (2)). The bound (10) also follows as an immediate corollary.

It seems natural to expect that bounds similar to (10) and (11) should hold for k -uniform hypergraphs: by analogy with the situation for a single hypergraph, we should expect a lower bound of form cn^{k+1} on the product of positive and negative discrepancies, which would in turn yield a bound of form $cn^{(k+1)/2}$ on $\text{disc}(G, H)$ (as in (3) and (6) above). Such a bound was conjectured in [11], but we were surprised to find the following simple counterexample for 3-uniform hypergraphs. Let V be a set of n vertices, and let $V = A \cup B$ be a partition. We let G be the 3-uniform hypergraph on V with all triples that meet both A and B , and H be a Steiner triple system. Then $\text{disc}(G, H) = 0$. (This is easily shown: in any placement of H , there must be exactly $|A||B|/2$ edges of H that meet both A and B , as each such edge contains exactly two elements from $\{ab : a \in A, b \in B\}$.) But now we can obtain an example in which both hypergraphs have density bounded away from 0 and 1 by taking H to be the union of a suitable number of edge-disjoint disjoint Steiner triple systems (see Doyen [15] or Teirlinck [35] for constructions).

For *weighted* hypergraphs, the situation is even more dramatic: we will show that there is a nontrivial set of k weighted k -uniform hypergraphs for which every pair has discrepancy 0. Note that if w is a constant function, then trivially $\text{disc}(w, u) = 0$ for every u . Indeed, if we add a constant function to w it does not affect the discrepancy (that is, $\text{disc}(w + \lambda \mathbf{1}, u) = \text{disc}(w, u)$). So, to avoid triviality, we will restrict ourselves to hypergraphs w such that $w(V) = 0$. We then have the following result.

Theorem 1. *Let $k \geq 2$. For every $n \geq 2k$ there are weighted hypergraphs w_1, \dots, w_k with vertex set $[n]$ such that $w([n]) = 0$ and $\|w_i\|_1 = \binom{n}{k}$ for every i and, for $0 \leq i < j \leq k$, we have*

$$\text{disc}(w_i, w_j) = 0.$$

Theorem 1 is a special case of a much stronger result below (Theorem 16), which gives a description of all pairs of weighted hypergraphs with discrepancy 0, and allows us to characterize collections of weighted hypergraphs satisfying Theorem 1.

If we have $k + 1$ weighted hypergraphs, however, the picture is very different: we do get a version of (11) for at least one pair, and attain the bound conjectured in [11].

Theorem 2. *For every $k \geq 1$ there are constants $c, c' > 0$ such that the following holds. Let $n \geq 2k$, and suppose that w_1, \dots, w_{k+1} are weighted k -uniform hypergraphs on $[n]$ such that $w_i([n]) = 0$ and $\|w_i\|_1 = \binom{n}{k}$ for every i . Then there are distinct i and j such that*

$$\text{disc}^+(w_i, w_j) \text{disc}^-(w_i, w_j) \geq cn^{k+1}.$$

In particular, there are $i < j$ such that,

$$\text{disc}(w_i, w_j) \geq c'n^{(k+1)/2}.$$

We will further prove (Theorem 15) that every family \mathcal{F} of weighted k -uniform hypergraphs w with $w(V) = 0$ can be partitioned into k families of hypergraphs with pairwise large discrepancy.

Theorems 2 and 15 both follow from a much stronger quantitative result (Theorem 3), which will allow us to prove a lower bound on the discrepancy of a pair of weighted hypergraphs. In order to state this result, we need to introduce the *W-vector* of a weighted hypergraph (the formal definition

will require a little work, so we defer it to Section 3). For every weighted k -uniform hypergraph w , we will define a sequence of $k + 1$ nonnegative weights W_0, \dots, W_k , giving us the W -vector $W = (W_0, \dots, W_k)$. We shall take $W_0 = |w(V)/\binom{n}{k}|$ to be the modulus of the average edge weight, and define the remaining W_i later. As we shall see in Lemma 9, it turns out that the W -vector preserves the weight of w , in that there are constants c, c' such that

$$c\|w\|_1/n^k \leq \sum_{i=0}^k W_i \leq c'\|w\|_1/n^k. \quad (12)$$

In particular, if $\|w\|_1 = \binom{n}{k}$ then some component of the W -vector is bounded away from 0. If w is nontrivial and normalised (so $w(V) = 0$ and $\|w\|_1 = \binom{n}{k}$) then one of W_1, \dots, W_k is bounded away from 0.

We can now state a quantitative version of Theorem 2.

Theorem 3. *For every $k \geq 1$ there are $c, c' > 0$ such that the following holds. For every $n \geq k$ and every pair of weighted hypergraphs $w, u : [n]^{(k)} \rightarrow \mathbb{R}$, we have*

$$\text{disc}^+(w, u)\text{disc}^-(w, u) \geq cn^{2k+1} \sum_{i=1}^k n^{-i} W_i^2 U_i^2,$$

where (W_0, \dots, W_k) and (U_0, \dots, U_k) are the W -vectors of w and u respectively. In particular,

$$\text{disc}(w, u) \geq c'n^{k+1/2} \sum_{i=1}^k n^{-i/2} W_i U_i.$$

Theorem 3 bounds the discrepancy of a pair w, u of weighted k -uniform hypergraphs in terms of the dot product of their W -vectors. As noted above, a hypergraph with $w([n]) = 0$ and $\|w\|_1 = \binom{n}{k}$ must have one of W_1, \dots, W_k bounded away from 0. Theorem 2 therefore follows immediately from Theorem 3.

In addition to bounding the discrepancy, we will also prove a result on the expectation of the intersection $|\langle w_\pi, u \rangle|$ of two weighted k -uniform hypergraphs, when π is chosen uniformly at random.

For $k = 1$ (i.e. sequences), the distribution of $\langle w_\pi, u \rangle$ has been extensively studied. Wald and Wolfowitz [36] proved a central limit theorem for $\langle w_\pi, u \rangle$ (under suitable conditions), and subsequent generalizations were given by Noether [33], Hoeffding [22], Dwass [17] and many other authors. For $k = 2$

(i.e. graphs), random intersections $\langle w_\pi, u \rangle$ arise naturally in a number of statistical applications (for instance, Barbour and Chen [4] mention applications in geography and epidemiology: see Moran [31], Geary [21], Knox [26], Mantel [29] and Hubert [23]). The distribution of $\langle w_\pi, u \rangle$ has been considered by many authors starting with Daniels [14], and including Barton and David [5], Abe [1], Barbour and Eagleson [2, 3] and Barbour and Chen [4], and there are sophisticated central limit theorems.

In this paper, we consider general k , but do not determine the limiting distribution of $|\langle w_\pi, u \rangle|$. However, we give a weak bound on the concentration of the distribution of $|\langle w_\pi, u \rangle|$, by bounding the expected value of $|\langle w_\pi, u \rangle|$.

Theorem 4. *For every $k \geq 1$ there is $c > 0$ such that, for every $n > 2k$ and every pair of weighted hypergraphs $w, u : [n]^{(k)} \rightarrow \mathbb{R}$,*

$$\mathbb{E}_\pi |\langle w_\pi, u \rangle| \geq cn^k \sum_{i=0}^k n^{-i/2} W_i U_i, \quad (13)$$

where (W_0, \dots, W_k) and (U_0, \dots, U_k) are the W -vectors of w and u respectively.

2 Notation and tools

We use standard notation: $V^{(k)}$ denotes the collection of k -sets in V ; we shall often refer to these as *edges*. We write $[n] = \{1, \dots, n\}$. For any function f , we write $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

A *weighted k -uniform hypergraph with vertex set V* is simply a function $w : V^{(k)} \rightarrow \mathbb{R}$. For $S \subset V$, we define $w(S) = \sum_{e \in S^{(k)}} w(e)$. Given weighted k -uniform hypergraphs w, u on vertex set V , we define a standard norm and inner product: $\|w\|_1 = \sum_{e \in V^{(k)}} |w(e)|$ and $\langle w, u \rangle = \sum_{e \in V^{(k)}} w(e)u(e)$. The *density* of w is $d(w) = w(V)/\binom{|V|}{k}$. We also define the constant function $\mathbf{1}$ by $\mathbf{1}(e) = 1$ for every edge e . We will feel free to move without comment between an unweighted hypergraph H , and the corresponding weight function $w = w_H$ defined by $w(e) = \mathbf{1}_{(e \in E(H))}$.

We write $\text{Sym}(V)$ for the group of all permutations of V . There is a natural action of $\text{Sym}(V)$ on weighted hypergraphs. Given a function $f : V^{(k)} \rightarrow \mathbb{R}$ and a permutation π of V , we define the function f_π by $f_\pi(e) = f(\pi^{-1}(e))$. Thus for permutations π, ρ , we have $f_{\pi\rho} = (f_\rho)_\pi$, as $f_{\pi\rho}(e) = f((\pi\rho)^{-1}(e)) = f(\rho^{-1}\pi^{-1}(e)) = f_\rho(\pi^{-1}(e)) = (f_\rho)_\pi(e)$.

We say that weighted k -uniform hypergraphs w on vertex set V and u on vertex set U are *isomorphic* if there is a bijection $f : V \rightarrow U$ such that $u(f(e)) = w(e)$ for every edge $e \in V^{(k)}$. Clearly w and w_π are isomorphic for any $\pi \in \text{Sym}(V)$.

For weighted hypergraphs w, u , the *positive discrepancy* $\text{disc}^+(w, u)$ and *negative discrepancy* $\text{disc}^-(w, u)$ are defined as in (8) and (9); we then set $\text{disc}(w, u) = \max(\text{disc}^+(w, u), \text{disc}^-(w, u))$.

Throughout the paper we will take expectations over randomly chosen vertices or edges. Unless otherwise specified, this will always be with respect to the uniform distribution. We will also adopt the convention that \mathbb{E}' and \sum' denote expectation and sum over *distinct* choices of argument: for instance if we are choosing random vertices from V , then $\mathbb{E}_{x,y}$ denotes the expectation over the $|V|^2$ possible choices of an ordered pair (x, y) , while $\mathbb{E}'_{x,y}$ denotes the expectation over the $|V|(|V| - 1)$ possible ordered pairs (x, y) such that $x \neq y$, with respect to the uniform distribution in both cases. Finally, if we take expectations with respect to a permutation π , then unless stated otherwise this will always be taken to be chosen uniformly at random from the symmetric group $\text{Sym}(V)$ on V .

It will be useful to note a few elementary facts.

Lemma 5. *Let $k \geq 1$ be fixed. There is a constant $c_k > 0$ such that every polynomial $f(x) = \sum_{i=0}^k a_i x^i$ with $\max_i |a_i| = 1$ satisfies*

$$\int_0^1 |f(x)| dx \geq c_k. \quad (14)$$

In particular, this implies

$$\sup_{x \in [0,1]} |f(x)| \geq c_k. \quad (15)$$

Proof. The proof is straightforward. For $\mathbf{a} = (a_0, \dots, a_k) \in [-1, 1]^{k+1} \setminus (-1, 1)^{k+1}$, let $F(\mathbf{a}) = \int_0^1 |\sum_{i=0}^k a_i x^i| dx$. Then F is continuous and strictly positive, and so we are done by compactness. \square

The following simple bound is proved in [10].

Lemma 6. *Let $\mathbf{a} = (a_i)_{i=1}^n$ be a sequence of real numbers and $I \subset \{1, \dots, n\}$ a subset chosen uniformly at random. Then*

$$\mathbb{E} \left| \sum_{i \in I} a_i \right| \geq \frac{\|\mathbf{a}\|_1}{\sqrt{8n}}.$$

It will also be useful to note the following elementary fact.

Proposition 7. *If X is a random variable with $\mathbb{E}X = 0$, and $c \in \mathbb{R}$, then $\mathbb{E}|X + c| \geq \max\{\mathbb{E}|X|/2, |c|\}$.*

Proof. We may assume $c > 0$. We have $\mathbb{E}X^+ = \mathbb{E}X^- = \mathbb{E}|X|/2$. But $\mathbb{E}|X + c| \geq \mathbb{E}[(X + c)\mathbf{1}_{(X \geq 0)}] \geq \mathbb{E}[X\mathbf{1}_{(X \geq 0)}] = \mathbb{E}X^+ = \mathbb{E}|X|/2$. Also, $\mathbb{E}|X + c| \geq \mathbb{E}(X + c) = \mathbb{E}X + c = c$. \square

3 The W -vector

Given a weighted hypergraph $w : V^{(k)} \rightarrow \mathbb{R}$, where V is a set of size $n \geq 2k \geq 0$, we define in this section a corresponding W -vector (W_0, \dots, W_k) , where each W_i is a nonnegative real. It will be helpful to give two different descriptions of the W -vector.

We start by defining

$$W_0 = |d(w)| = |w(V)| / \binom{n}{k} = \binom{n}{k}^{-1} \sum_{e \in V^{(k)}} w(e) = |\mathbb{E}_e w(e)|, \quad (16)$$

where we write \mathbb{E}_e for the expectation over an edge e chosen uniformly at random over all $\binom{n}{k}$ possibilities. Clearly $W_0 = 0$ if and only if $w(V) = 0$.

For $i \geq 1$, we define W_i recursively. For each $\{x, y\} \in V^{(2)}$, the *difference weighting* w^{xy} is defined on sets $e \in (V \setminus \{x, y\})^{(k-1)}$ by

$$w^{xy}(e) = w(e \cup \{x\}) - w(e \cup \{y\}).$$

Note that $w^{xy} = -w^{yx}$. For any choice of distinct x and y , the difference weighting w^{xy} has a W -vector $(W_0^{xy}, \dots, W_{k-1}^{xy})$. We can therefore define, for $1 \leq i \leq k$,

$$W_i = \frac{1}{n(n-1)} \sum'_{x,y} W_{i-1}^{xy} = \mathbb{E}'_{x,y} W_{i-1}^{xy}, \quad (17)$$

where, as usual, we write \mathbb{E}' and \sum' for the expectation and sum over distinct indices. Note that the W -vector is well-defined, as the W -vector for a weighting of k -sets is given in terms of the W -vectors for weightings of various collections of $(k-1)$ -sets.

For example, in the trivial case $k = 0$, a weighting is just a constant w , and the W -vector is (W_0) , where $W_0 = |w|$. For $k = 1$, we have a weight

function $w : V \rightarrow \mathbb{R}$. If $|V| = n$, we have $W_0 = |\sum_{v \in V} w(v)|/n$. Now for distinct $x, y \in V$, w^{xy} is a weighting on the $(k-1)$ -sets, which in this case is just a weight (on the empty set) given by

$$w^{xy}(\emptyset) = w(\{x\} \cup \emptyset) - w(\{y\} \cup \emptyset) = w(x) - w(y),$$

so w^{xy} has W -vector (W_0^{xy}) given by $W_0^{xy} = W_0^{yx} = |w(x) - w(y)|$. We then have

$$W_1 = \frac{1}{n(n-1)} \sum'_{x,y} W_0^{xy} = \mathbb{E}'_{x,y} |w(x) - w(y)|. \quad (18)$$

Defining the W -vector by (16) and (17) will be helpful in some of the proofs below. However, we now give a second approach that allows us to write the W -vector in a form that is frequently more convenient.

We begin by choosing an arbitrary sequence $x_1, y_1, \dots, x_k, y_k$ of $2k$ distinct vertices in V . For $i = 0, \dots, k$, we define $Y_i = \{y_1, \dots, y_i\}$ and $\mathbf{s}_i = (x_1, y_1, \dots, x_i, y_i)$. We say that a set $A \in V^{(k)}$ is *compatible with \mathbf{s}_i* if $|A \cap \{x_j, y_j\}| = 1$ for $j = 1, \dots, i$. We define weighted k -uniform hypergraphs ϕ_i and ϕ_i^* on V by

$$\phi_i(A) = \phi_i(x_1, y_1, \dots, x_i, y_i; A) = \begin{cases} (-1)^{|A \cap Y_i|} & A \text{ compatible with } \mathbf{s}_i \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

and

$$\phi_i^* = \phi_i / \binom{n-2i}{k-i}. \quad (20)$$

Note that we have normalized so that $\|\phi_i^*\|_1 = \Theta(1)$.

The definitions of ϕ_i and ϕ_i^* depend on the sequence of vertices we pick for $x_1, y_1, \dots, x_i, y_i$. However, different choices give isomorphic weightings, and in practice we will always symmetrize over permutations of the vertices, as in (21) below, so our results do not depend on our particular choices.

Lemma 8. *Let $n \geq 2k \geq 1$, and suppose that w is a weighted k -uniform hypergraph on vertex set V , where $|V| = n$. Let ϕ_i and ϕ_i^* be defined as in (19) and (20). Then*

$$W_i = \mathbb{E}_\pi |\langle w_\pi, \phi_i^* \rangle|. \quad (21)$$

Proof. For $i = 0$, we have

$$\begin{aligned}
W_0 &= \binom{n}{k}^{-1} \left| \sum_{e \in V^{(k)}} w(e) \right| \\
&= \binom{n}{k}^{-1} |\langle w, \mathbf{1} \rangle| \\
&= \mathbb{E}_\pi \binom{n}{k}^{-1} |\langle w_\pi, \mathbf{1} \rangle| \\
&= \mathbb{E}_\pi |\langle w_\pi, \phi_0^* \rangle|,
\end{aligned}$$

as $\phi_0 = \mathbf{1}$, $\phi_0^* = \phi_0 / \binom{n}{k}$ and $\langle w_\pi, \mathbf{1} \rangle = \langle w, \mathbf{1} \rangle$ for every π .

We now proceed by induction on i . For $i \geq 1$, we have $W_i = \mathbb{E}'_{x,y} W_{i-1}^{xy}$. Choose a sequence $x_1, y_1, \dots, x_k, y_k$ of distinct vertices, and let $U = V \setminus \{x_1, y_1\}$. We define ϕ_i, ϕ_i^* on V as in (19) and (20), and let ψ_i, ψ_i^* be the corresponding functions for U with respect to the sequence $x_2, y_2, \dots, x_k, y_k$. Thus, for $i = 1, \dots, k$, we set $\mathbf{t}_{i-1} = (x_2, y_2, \dots, x_i, y_i)$, and define $\psi_{i-1} : U^{(k-1)} \rightarrow \mathbb{R}$ by

$$\psi_{i-1}(A) = \begin{cases} (-1)^{|A \cap \{y_2, \dots, y_i\}|} & A \text{ compatible with } \mathbf{t}_{i-1} \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

and

$$\psi_{i-1}^* = \psi_{i-1} / \binom{(n-2) - 2(i-1)}{(k-1) - (i-1)} = \psi_{i-1} / \binom{n-2i}{k-i}. \quad (23)$$

Note that for $A \in U^{(k-1)}$, it follows from (19) and (22) that

$$\psi_{i-1}(A) = \phi_i(A \cup \{x_1\}) = -\phi_i(A \cup \{y_1\});$$

so by (20) and (23) that we have

$$\psi_{i-1}^*(A) = \phi_i^*(A \cup \{x_1\}) = -\phi_i^*(A \cup \{y_1\}). \quad (24)$$

It follows by induction from (21) that, writing $(W_0^{x_1 y_1}, \dots, W_{k-1}^{x_1 y_1})$ for the W -vector of $w^{x_1 y_1}$, for $i = 1, \dots, k$,

$$W_{i-1}^{x_1 y_1} = \mathbb{E}_{\pi^* \in \text{Sym}(U)} |\langle (w^{x_1 y_1})_{\pi^*}, \psi_{i-1}^* \rangle|. \quad (25)$$

We identify $\pi^* \in \text{Sym}(U)$ with the corresponding $\pi \in \text{Sym}(V)$ that fixes x_1, y_1 and otherwise acts as π^* . Then $(w_\pi)^{x_1 y_1} = (w^{x_1 y_1})_{\pi^*}$, and so, by (24),

$$\begin{aligned}
\langle (w^{x_1 y_1})_{\pi^*}, \psi_{i-1}^* \rangle &= \sum_{e \in U^{(k-1)}} (w_\pi)^{x_1 y_1}(e) \psi_{i-1}^*(e) \\
&= \sum_{e \in U^{(k-1)}} (w_\pi(e \cup \{x_1\}) - w_\pi(e \cup \{y_1\})) \psi_{i-1}^*(e) \\
&= \sum_{e \in U^{(k-1)}} (w_\pi(e \cup \{x_1\}) \phi_i^*(e \cup \{x_1\}) + w_\pi(e \cup \{y_1\}) \phi_i^*(e \cup \{y_1\})) \\
&= \sum_{f \in V^{(k)}} w_\pi(f) \phi_i^*(f) \\
&= \langle w_\pi, \phi_i^* \rangle,
\end{aligned} \tag{26}$$

since we have $\phi_i^*(f) = \phi_i(f) = 0$ unless f is compatible with the sequence of vertices $(x_1, y_1, \dots, x_i, y_i)$.

Since $i \geq 1$, we have

$$W_i = \mathbb{E}'_{x,y} W_{i-1}^{xy} = \mathbb{E}_\pi W_{i-1}^{\pi(x_1)\pi(y_1)},$$

since $(\pi(x_1), \pi(y_1))$ is uniformly distributed over distinct vertices x, y . Now for an edge $e \in (V \setminus \{\pi(x), \pi(y)\})^{(k-1)}$,

$$\begin{aligned}
w^{\pi(x_1)\pi(y_1)}(e) &= w(e \cup \{\pi(x_1)\}) - w(e \cup \{\pi(y_1)\}) \\
&= w_\pi(\pi^{-1}(e) \cup \{x_1\}) - w_\pi(\pi^{-1}(e) \cup \{y_1\}) \\
&= (w_\pi)^{x_1 y_1}(\pi^{-1}(e)).
\end{aligned}$$

It follows that $w^{\pi(x_1)\pi(y_1)}$ and $(w_\pi)^{x_1 y_1}$ are isomorphic and so have the same W -vector $((W_\pi^{x_1 y_1})_0, \dots, (W_\pi^{x_1 y_1})_{k-1})$. Using (25) and (26), we get

$$(W_\pi^{x_1 y_1})_{i-1} = \mathbb{E}_{\rho^* \in \text{Sym}(U)} |\langle (w_\pi)^{x_1 y_1} \rangle_{\rho^*}, \psi_{i-1}^* \rangle| = \mathbb{E}_{\rho^* \in \text{Sym}(U)} |\langle (w_\pi)_\rho, \phi_i^* \rangle|,$$

where once again we use the convention that each $\rho^* \in \text{Sym}(U)$ is identified with the $\rho \in \text{Sym}(V)$ that fixes x_1 and y_1 and otherwise acts as ρ^* . Thus

$$\begin{aligned}
W_i &= \mathbb{E}_\pi W_{i-1}^{\pi(x_1)\pi(y_1)} \\
&= \mathbb{E}_\pi (W_\pi^{x_1 y_1})_{i-1} \\
&= \mathbb{E}_{\pi, \rho^*} |\langle (w_\pi)_\rho, \phi_i^* \rangle| \\
&= \mathbb{E}_{\pi, \rho^*} |\langle w_{\rho\pi}, \phi_i^* \rangle| \\
&= \mathbb{E}_\pi |\langle w_\pi, \phi_i^* \rangle|,
\end{aligned}$$

as $\rho\pi$ is uniformly distributed over $\text{Sym}(V)$. This gives (21). \square

We remark that (21) is reminiscent of taking a Fourier transform.

In order to show that our theorems do not give trivial bounds, we need to know that the l_1 norm of a weighting is preserved up to a constant factor by its W -vector. This is the substance of the next result.

Lemma 9. *For every $k \geq 0$ there are constants $c, c' > 0$ such that the following holds. For every $n \geq 2k$, and every weighted k -uniform hypergraph w on $[n]$,*

$$cn^{-k}||w||_1 \leq \sum_{i=0}^k W_i \leq c'n^{-k}||w||_1, \quad (27)$$

where (W_0, \dots, W_k) is the W -vector of w .

We first note the following.

Proposition 10. *For every $k \geq 1$ there is $c > 0$ such the following holds. For every $n > k$, and every weighted k -uniform hypergraph w with vertex set $[n]$ such that $w([n]) = 0$,*

$$\mathbb{E}_{A,x,y}|w(A \cup \{x\}) - w(A \cup \{y\})| \geq c_k n^{-k} ||w||_1,$$

where the expectation is taken over $(k-1)$ -sets A and distinct vertices $x, y \notin A$ chosen uniformly at random.

Proof. Since we can rescale, it is enough to show that for every weighting w on $[n]^{(k)}$ with $||w||_1 = \binom{n}{k}$ and $w([n]) = 0$ we have

$$\mathbb{E}_{A,x,y}|w(A \cup \{x\}) - w(A \cup \{y\})| \geq 1/4k. \quad (28)$$

Note that $\mathbb{E}_e[w(e)^+] = \mathbb{E}_e[w(e)^-]$ and $\mathbb{E}_e|w(e)| = \mathbb{E}_e[w(e)^+ + w(e)^-] = 1$, so $\mathbb{E}_e[w(e)^+] = \mathbb{E}_e[w(e)^-] = 1/2$.

Suppose that w is nonnegative on $p\binom{n}{k}$ edges and negative on $(1-p)\binom{n}{k}$ edges. We may assume $p \leq 1/2$ or work with $-w$. Pick with replacement random edges e and f and let $e_1 \dots e_i$ be a random shortest path between them (so $e_1 = e$, $e_i = f$ and each step replaces one element of the edge). If e, f are distinct, we set $A = e_1 \cap e_2$, and let x, y be the remaining vertices of e_1, e_2 respectively; otherwise, we choose A, x, y uniformly at random. Then (A, x, y) is uniformly distributed, and

$$\begin{aligned} \mathbb{E}_{A,x,y}|w(A \cup \{x\}) - w(A \cup \{y\})| &\geq \mathbb{E}_{e,f}|w(e_2) - w(e_1)| \\ &\geq \mathbb{E}_{e,f}|w(e) - w(f)|/k. \end{aligned} \quad (29)$$

With probability $1 - p$ we have $w(f) \leq 0$. Conditioning on this event, $\mathbb{E}_{e,f}[|w(e) - w(f)| \mid w(f) \leq 0] \geq \mathbb{E}_{e,f}[w(e)^+ \mid w(f) \leq 0] = \mathbb{E}_e[w(e)^+] = 1/2$. We conclude that (without conditioning) $\mathbb{E}[|w(e) - w(f)|] \geq (1 - p) \cdot (1/2) \geq 1/4$, and (28) then follows from (29). \square

Proof of Lemma 9. Let us write $w = w_0 + w_1$ where w_0 is constant and $\sum_e w_1(e) = 0$.

Clearly $\|w\|_1 \leq \|w_0\|_1 + \|w_1\|_1$; we also have $\|\phi_i\|_1 = 2^i \binom{n-2i}{k-i}$, since there are exactly $2^i \binom{n-2i}{k-i}$ edges compatible with the sequence $x_1, y_1, \dots, x_i, y_i$. Thus $\|\phi_i^*\|_1 = 2^i$. So, by Lemma 8, we have

$$\begin{aligned} W_i &= \mathbb{E}_\pi |\langle w_\pi, \phi_i^* \rangle| \\ &= \mathbb{E}_\pi \left| \sum_e w_\pi(e) \phi_i^*(e) \right| \\ &\leq \sum_e \mathbb{E}_\pi |w_\pi(e) \phi_i^*(e)| \\ &= \sum_e |\phi_i^*(e)| \cdot \mathbb{E}_\pi |w_\pi(e)| \\ &= \sum_e |\phi_i^*(e)| \cdot \|w\|_1 / \binom{n}{k} \\ &= 2^i \|w\|_1 / \binom{n}{k}. \end{aligned}$$

Summing over i gives the upper bound in (27).

For the lower bound, note first that, by linearity and Proposition 10, for $k \geq 1$,

$$\begin{aligned} \mathbb{E}'_{x,y} \|w^{xy}\|_1 &= \mathbb{E}'_{x,y} \sum_{A \in (V \setminus \{x,y\})^{k-1}} |w(A \cup \{x\}) - w(A \cup \{y\})| \\ &= \binom{n-2}{k-1} \mathbb{E}'_{x,y,A} |w(A \cup \{x\}) - w(A \cup \{y\})| \\ &\geq c_k \binom{n-2}{k-1} n^{-k} \|w_1\|_1 \\ &\geq c'_k \|w_1\|_1 / n, \end{aligned} \tag{30}$$

where c_k and c'_k are constants depending only on k .

For $k = 0$, we have $W_0 = \|w\|_1$, giving (27) as required. We now argue by induction on k : we will use c, c' , etc, for constants that depend only on k . For $k \geq 1$ and distinct $x, y \in V$, we have by induction

$$W_0^{xy} + \cdots + W_{k-1}^{xy} \geq c \|w^{xy}\|_1 / n^{k-1}.$$

Since w_0 is a constant function, we also have $W_0 = \|w_0\|_1 / \binom{n}{k} \geq \|w_0\|_1 / n^k$. It follows by (30) that

$$\begin{aligned} W_0 + \cdots + W_k &= W_0 + \mathbb{E}'_{x,y}(W_0^{xy} + \cdots + W_{k-1}^{xy}) \\ &\geq c'(\|w_0\|_1 / n^k + \mathbb{E}'_{x,y}\|w^{xy}\|_1 / n^{k-1}) \\ &\geq c''(\|w_0\|_1 / n^k + \|w_1\|_1 / n^k) \\ &\geq c'''\|w\|_1 / n^k. \end{aligned}$$

□

4 Bounding in terms of transpositions

In order to prove Theorems 3 and 4, we will need bounds both on $\text{disc}(w, u)$ and on $\mathbb{E}_\pi |\langle w_\pi, u \rangle|$. These will be driven by two results bounding these quantities from below in terms of the effects of single transpositions.

Let us fix the ground set V and pick distinct vertices $x, y \in V$. Let τ be the transposition (xy) . Let w and u be two weightings of $V^{(k)}$, and choose uniformly at random two permutations π, σ . We define $\gamma(w, u)$ by

$$\gamma(w, u) = \mathbb{E}_{\pi, \sigma} |\langle w_\pi, u_\sigma \rangle - \langle w_{\tau\pi}, u_\sigma \rangle|. \quad (31)$$

Thus $\gamma(w, u)$ measures the typical effect on the inner product of exchanging x and y in one copy of V . Note that $\gamma(w, u)$ does not depend on our choice of x and y , since the expectation is taken over random permutations of the ground set for both w and u .

Our bounds will depend on the following two lemmas.

Lemma 11. *For every $k \geq 1$ there is $c > 0$ such that the following holds. For every $n \geq k$ and every pair w, u of functions from $[n]^{(k)}$ to \mathbb{R} , we have*

$$\text{disc}^+(w, u) \text{disc}^-(w, u) \geq c^2 \gamma(w, u)^2 n^2. \quad (32)$$

Lemma 12. *For every $k \geq 1$ there is $c > 0$ such that the following holds. For every $n \geq k$ and every pair w, u of functions from $[n]^{(k)}$ to \mathbb{R}*

$$\mathbb{E}_\pi |\langle w_\pi, u \rangle| \geq c\gamma(w, u)\sqrt{n}.$$

We start by setting up a framework for the proofs of Lemma 11 and Lemma 12.

Let w, u be two weightings of $[n]^{(k)}$. Let I be an index set, and suppose we have transpositions

$$\tau^i = (x_i y_i), \quad i \in I \quad (33)$$

such that the pairs $\{x_i, y_i\}_{i \in I}$ are disjoint. For $J \subset I$, we define τ^J to be the product of the transpositions $\{\tau^j : j \in J\}$ (note that the τ^j commute, and $\tau^J = (\tau^J)^{-1}$). We will want to consider the difference $|\langle w_{\tau^J}, u \rangle - \langle w, u \rangle|$ for various sets J . For $i \in I$, we define

$$\delta(i) = \langle w, u \rangle - \langle w_{\tau^i}, u \rangle.$$

For $J \subset I$, we define

$$\delta(J) = \sum_{i \in J} \delta(i)$$

and

$$\Delta(J) = \sum_{i \in J} |\delta(i)|. \quad (34)$$

If we want to specify w, u explicitly, we will write $\Delta_{w,u}$ instead of Δ , and so on. However, we drop indices when they are not necessary.

For a set $e \in V^{(k)}$, let

$$\text{tr}(e) = \{i \in I : |e \cap \{x_i, y_i\}| = 1\}.$$

Note that $i \in \text{tr}(e)$ if and only if $\tau^i(e) \neq e$, and $\text{tr}(\tau^J(e)) = \text{tr}(e)$ for any J . We decompose $\langle w, u \rangle - \langle w_{\tau^J}, u \rangle$ as follows.

Proposition 13. *Let $n \geq k \geq 1$, let w, u be weightings on $[n]^{(k)}$, and let I be an index set for transpositions τ^i as in (33). For every $J \subset I$, we have*

$$\langle w, u \rangle - \langle w_{\tau^J}, u \rangle = \frac{1}{2} \sum_{e \in [n]^{(k)}} (w(e) - w(\tau^J e))(u(e) - u(\tau^J e)).$$

In particular, if $\tau = (xy)$,

$$\langle w, u \rangle - \langle w_\tau, u \rangle = \langle w^{xy}, u^{xy} \rangle. \quad (35)$$

Proof. Note that, for any permutation π and any $w : V^{(k)} \rightarrow \mathbb{R}$, we have $\sum_{e \in [n]^{(k)}} f(e) = \sum_{e \in [n]^{(k)}} f(\pi e)$. So

$$\begin{aligned}
\langle w, u \rangle - \langle w_{\tau^J}, u \rangle &= \sum_e (w(e)u(e) - w_{\tau^J}(e)u(e)) \\
&= \frac{1}{2} \sum_e (w(e)u(e) - w_{\tau^J}(e)u(e)) \\
&\quad + \frac{1}{2} \sum_e (w(\tau^J e)u(\tau^J e) - w_{\tau^J}(\tau^J e)u(\tau^J e)) \\
&= \frac{1}{2} \sum_e (w(e)u(e) - w(\tau^J e)u(e)) \\
&\quad + \frac{1}{2} \sum_e (w(\tau^J e)u(\tau^J e) - w(e)u(\tau^J e)) \\
&= \frac{1}{2} \sum_e (w(e) - w(\tau^J e))(u(e) - u(\tau^J e)),
\end{aligned}$$

where all sums are over $[n]^{(k)}$.

To prove (35), note that $w(e) - w(\tau e) = 0$ unless $|e \cap \{x, y\}| = 1$. So, writing $W = [n] \setminus \{x, y\}$,

$$\begin{aligned}
&\frac{1}{2} \sum_{e \in [n]^{(k)}} (w(e) - w(\tau e))(u(e) - u(\tau e)) \\
&= \frac{1}{2} \sum_{f \in W^{(k-1)}, v \in \{x, y\}} (w(f \cup \{v\}) - w(f \cup \{\tau v\}))(u(f \cup \{v\}) - u(f \cup \{\tau v\})) \\
&= \sum_{f \in W^{(k-1)}} (w(f \cup \{x\}) - w(f \cup \{y\}))(u(f \cup \{x\}) - u(f \cup \{y\})) \\
&= \sum_{f \in W^{(k-1)}} w^{xy}(f)u^{xy}(f) \\
&= \langle w^{xy}, u^{xy} \rangle.
\end{aligned}$$

□

We also consider the expected effect of a randomly chosen set of transpositions.

Lemma 14. *Let $n \geq k \geq 1$, let w, u be weightings on $[n]^{(k)}$, and let I be an index set for transpositions τ^i as in (33). Let I be fixed, and let J be a random subset of I , where each $i \in I$ is taken independently with probability p . Then $\mathbb{E}(\langle w, u \rangle - \langle w_{\tau^J}, u \rangle)$ can be written as a polynomial in p of the form*

$$\delta(I)p + \sum_{i=2}^k A_i p^i, \quad (36)$$

for some real numbers A_2, \dots, A_k .

Proof. For $e \in [n]^{(k)}$, let

$$\mu_J(e) = (w(e) - w(\tau^J e))(u(e) - u(\tau^J e)).$$

By Proposition 13 we have

$$\langle w, u \rangle - \langle w_{\tau^J}, u \rangle = \frac{1}{2} \sum_e \mu_J(e). \quad (37)$$

For a given edge e , the value of $\mu_J(e)$ depends only on $J \cap \text{tr}(e)$, so

$$\mu_J(e) = \mu_{J \cap \text{tr}(e)}(e).$$

Now, for $A \subset \text{tr}(e)$, we have $\mathbb{P}[J \cap \text{tr}(e) = A] = p^{|A|}(1-p)^{|\text{tr}(e)|-|A|}$, so

$$\mathbb{E}\mu_J(e) = \sum_{A \subset \text{tr}(e)} p^{|A|}(1-p)^{|\text{tr}(e)|-|A|} \mu_A(e).$$

Hence, by (37), $\mathbb{E}(\langle w, u \rangle - \langle w_{\tau^J}, u \rangle)$ is a polynomial in p with degree at most k . Since $\mu_\emptyset(e) = 0$, the constant term is 0.

Now consider the behaviour of $\langle w, u \rangle - \langle w_{\tau^J}, u \rangle$ as $p \rightarrow 0$. For each $i \in J$ we have $\mathbb{P}(J = i) = p + O(p^2)$. As $\mathbb{P}(|J| > 1) = O(p^2)$, it follows that

$$\mathbb{E}(\langle w, u \rangle - \langle w_{\tau^J}, u \rangle) = p \sum_i \delta(i) + O(p^2),$$

and so the coefficient of p is $\sum_{i \in I} \delta(i) = \delta(I)$. □

We now prove the two lemmas stated at the beginning of the section.

Proof of Lemma 11. Adding a constant to w or u does not affect $\text{disc}^+(w, u)$, $\text{disc}^-(w, u)$ or $\gamma(w, u)$, so we may assume that $w([n]) = u([n]) = 0$. Note first that $\mathbb{E}_\pi \langle w_\pi, u \rangle = 0$ and so $\mathbb{E}_\pi [\langle w_\pi, u \rangle^+] = \mathbb{E}_\pi [\langle w_\pi, u \rangle^-] = \mathbb{E}_\pi |\langle w_\pi, u \rangle|/2$.

For fixed n , we can argue as follows. From (31) we have

$$\begin{aligned} \gamma(w, u) &= \mathbb{E}_{\pi, \sigma} |\langle w_\pi, u_\sigma \rangle - \langle w_{\tau\pi}, u_\sigma \rangle| \\ &\leq \mathbb{E}_{\pi, \sigma} (|\langle w_\pi, u_\sigma \rangle| + |\langle w_{\tau\pi}, u_\sigma \rangle|) \\ &= 2\mathbb{E}_\pi |\langle w_\pi, u \rangle|, \end{aligned}$$

and so we have

$$\begin{aligned} \min\{\text{disc}^+(w, u), \text{disc}^-(w, u)\} &\geq \min\{\mathbb{E}_\pi [\langle w_\pi, u \rangle^+], \mathbb{E}_\pi [\langle w_\pi, u \rangle^-]\} \\ &\geq \frac{1}{2} \mathbb{E}_\pi |\langle w_\pi, u \rangle| \geq \gamma(w, u)/4. \end{aligned}$$

It follows that (32) holds for any fixed constant n (and appropriate $c > 0$), and so we may assume that $n > 100k$.

Let $K \geq 2$ be a fixed constant (which we will specify later), and suppose, without loss of generality, that $\text{disc}^+(w, u) \leq \text{disc}^-(w, u)$. If $\text{disc}^+(w, u) \geq \gamma(w, u)n/10K$, we are done (with $c = 1/10K$), so we may assume that

$$\text{disc}^+(w, u) = \frac{\gamma(w, u)n}{10\alpha} \tag{38}$$

for some $\alpha \geq K$. We shall show that, for some (small) constant $c > 0$, we have

$$\text{disc}^-(w, u) \geq c\gamma(w, u)\alpha n.$$

Let $t = \lfloor n/2 \rfloor$ and let $x_1, \dots, x_t, y_1, \dots, y_t$ be a sequence of distinct vertices of V . For $I = \{1, \dots, t\}$ and each $i \in I$, let $\tau^i = (x_i y_i)$. Let π and σ be chosen independently and uniformly at random from S_n . Then (31) and linearity of expectation imply that

$$\mathbb{E} \Delta_{w_\pi, u_\sigma}(I) = t\gamma(w, u).$$

Let $I^+ = \{i : \delta_{w_\pi, u_\sigma}(i) > 0\}$ and $I^- = \{i : \delta_{w_\pi, u_\sigma}(i) \leq 0\}$, so $\delta_{w_\pi, u_\sigma}(I) = \delta_{w_\pi, u_\sigma}(I^+) + \delta_{w_\pi, u_\sigma}(I^-) = \Delta_{w_\pi, u_\sigma}(I^+) - \Delta_{w_\pi, u_\sigma}(I^-)$. Since $\mathbb{E} \delta_{w_\pi, u_\sigma}(I) = 0$ we have

$$\mathbb{E} \Delta_{w_\pi, u_\sigma}(I^+) = \mathbb{E} \Delta_{w_\pi, u_\sigma}(I^-) = \frac{t}{2} \gamma(w, u).$$

We also have $\mathbb{E}\langle w_\pi, u_\sigma \rangle = 0$, and so

$$\mathbb{E}_{\pi, \sigma}[\alpha \langle w_\pi, u_\sigma \rangle + \Delta(I^+)] = \frac{t}{2} \gamma(w, u).$$

We can therefore choose π, σ such that $\alpha \langle w_\pi, u_\sigma \rangle + \Delta_{w_\pi, u_\sigma}(I^+) \geq t\gamma(w, u)/2$. Replacing w and u by w_π, u_σ , we may therefore assume that

$$\alpha \langle w, u \rangle + \Delta_{w, u}(I^+) \geq \frac{t}{2} \gamma(w, u). \quad (39)$$

Note that this replacement does not change the value of disc^+ or of disc^- .

Now consider the effects of applying τ^J , where $J \subset I^+$ is a random subset of I^+ where each $i \in I^+$ is present independently with probability p . Lemma 14 tells us that

$$\mathbb{E}[\langle w_{\tau^J}, u \rangle - \langle w, u \rangle] = p\delta(I^+) + \sum_{i=2}^k A_i p^i \quad (40)$$

for some A_2, \dots, A_k . It follows from (39), by considering the case $p = 1/\alpha$, that we have

$$\begin{aligned} \mathbb{E}\langle w_{\tau^J}, u \rangle &= \langle w, u \rangle + \mathbb{E}[\langle w_{\tau^J}, u \rangle - \langle w, u \rangle] \\ &= \langle w, u \rangle + \Delta(I^+)/\alpha + \sum_{i=2}^k A_i/\alpha^i \\ &\geq \frac{t}{2\alpha} \gamma(w, u) + \sum_{i=2}^k A_i/\alpha^i. \end{aligned} \quad (41)$$

Now (38) implies that $\langle w_{\tau^J}, u \rangle \leq \gamma(w, u)n/10\alpha$ for any choice of J , so (41) implies that

$$\begin{aligned} \sum_{i=2}^k A_i/\alpha^i &\leq \gamma(w, u)n/10\alpha - t\gamma(w, u)/2\alpha \\ &< -\gamma(w, u)n/10\alpha. \end{aligned}$$

Since $\alpha \geq 2$, we must have $|A_i| \geq \alpha\gamma(w, u)n/20$ for some $i \geq 2$.

It follows by (15) that for some $p \in [0, 1]$ and some $c_k > 0$ that depends only on k , we have

$$|p\delta(I^+) + \sum_{i=2}^k A_i p^i| \geq 2c_k \alpha \gamma(w, u)n. \quad (42)$$

But now, choosing p such that (42) holds, we have by (40)

$$\mathbb{E}|\langle w, u \rangle - \langle w_{\tau^J}, u \rangle| \geq 2c_k \alpha \gamma(w, u)n \quad (43)$$

and, since (38) holds, if we have chosen $K > 1/\sqrt{c_k}$, we must have, by (38) and (43),

$$\begin{aligned} \mathbb{E} \min(\langle w, u \rangle, \langle w_{\tau^J}, u \rangle) &= \mathbb{E}[\max(\langle w, u \rangle, \langle w_{\tau^J}, u \rangle) - |\langle w, u \rangle - \langle w_{\tau^J}, u \rangle|] \\ &\leq \frac{\gamma(w, u)n}{10\alpha} - 2c_k \alpha \gamma(w, u)n \\ &= \gamma(w, u)n\alpha \cdot [1/10\alpha^2 - 2c_k] \\ &\leq -c_k \alpha \gamma(w, u)n. \end{aligned}$$

In particular, there is some J such that

$$\langle w_{\tau^J}, u \rangle \leq -c_k \alpha \gamma(w, u)n$$

and so $\text{disc}^-(w, u) \geq c_k \alpha \gamma(w, u)n$, as claimed. \square

Proof of Lemma 12. We would like to argue as in the proof of Lemma 11. However, there is an important difference: in the previous proof we could replace w and u by our choice of w_π and u_σ , and then choose an advantageous set of transpositions to apply; now we must select our permutations π, σ and transpositions so that the resulting permutations are uniformly distributed.

Consider first a specific choice of π and σ , and let t, I and the transpositions τ^i be defined as in the proof of Lemma 11. Recall that the set I and the transpositions τ^i are fixed with respect to the ground set. However, $\delta(I)$ might be close to 0, which is not helpful if we want to use (36). We therefore generate random sets J_0 and J_1 of transpositions in two steps as follows:

1. Let J_0 be a random subset of I , chosen uniformly at random from all $2^{|I|}$ subsets.
2. Let $p \in [0, 1]$ be chosen uniformly at random, and let $J_1 \subset J_0$ be a random subset, where each $i \in J_0$ is taken independently with probability p .

Consider first J_0 . It follows from Lemma 6 that

$$\mathbb{E}_{J_0} |\delta(J_0)| \geq \frac{\Delta(I)}{\sqrt{8|I|}}. \quad (44)$$

Now if we condition on J_0 and p , then by Lemma 14 we have

$$\mathbb{E}_{J_1}[\langle w, u \rangle - \langle w_{\tau^{J_1}}, u \rangle \mid J_0, p] = \delta(J_0)p + \sum_{i=2}^k A_i p^i,$$

for some A_2, \dots, A_k that depend on J_0 . It then follows from Lemma 5 and the tower law for expectation that there is a constant $c_k > 0$ such that, if we condition just on J_0 , we have

$$\mathbb{E}_{p, J_1}[|\langle w, u \rangle - \langle w_{\tau^{J_1}}, u \rangle| \mid J_0] = \mathbb{E}_p[|\delta(J_0)p + \sum_{i=2}^k A_i p^i|] \geq c_k |\delta(J_0)|.$$

But now by (44) and (again) the tower law for expectation it follows that

$$\mathbb{E}_{J_0, p, J_1}[|\langle w, u \rangle - \langle w_{\tau^{J_1}}, u \rangle|] \geq \frac{c_k \Delta(I)}{\sqrt{8|I|}}.$$

This bound holds for any fixed placement of w and u . However, with a uniformly random choice of permutations π and σ , giving weightings w_π and w_σ , we have (by definition from (31) and (34))

$$\mathbb{E}_{\pi, \sigma} \Delta_{w_\pi, u_\sigma}(I) = t\gamma(w, u).$$

Thus there is a constant $c = c(k) > 0$ such that

$$\mathbb{E}_{\pi, \sigma, J_0, p, J_1}[|\langle w_\pi, u_\sigma \rangle - \langle w_{\tau^{J_1} \pi}, u_\sigma \rangle|] \geq \frac{c_k t \gamma(w, u)}{\sqrt{8t}} \geq 2c\gamma(w, u)\sqrt{n}.$$

By the triangle inequality, we have

$$\begin{aligned} 2c\gamma(w, u)\sqrt{n} &\leq \mathbb{E}_{\pi, \sigma, J_0, p, J_1}[|\langle w_\pi, u_\sigma \rangle| + |\langle w_{\tau^{J_1} \pi}, u_\sigma \rangle|] \\ &= 2\mathbb{E}_{\pi, \sigma} |\langle w_\pi, u_\sigma \rangle| \\ &= 2\mathbb{E}_\pi |\langle w_\pi, u \rangle|, \end{aligned}$$

which implies our result. \square

5 Proof of Theorems 3 and 4

We are now ready to prove our main quantitative results. We begin by proving Theorem 4; Theorem 3 will then follow easily. At the end of the section, we will deduce another result on partitioning families of hypergraphs with large pairwise discrepancy.

Proof of Theorem 4. As usual, we write $w = w_0 + w_1$ and $u = u_0 + u_1$, where u_0, w_0 are constant functions and u_1, w_1 sum to 0. Since $\langle (w_0)_\pi, u_0 \rangle = \langle w_0, u_0 \rangle = \langle w_0, u \rangle$ and $\langle (w_1)_\pi, u_0 \rangle = \langle w_0, (u_1)_\pi \rangle = 0$ for any π , it follows from Lemma 12 and Proposition 7 that, for some $c' = c'(k) > 0$,

$$\begin{aligned} \mathbb{E}_\pi |\langle w_\pi, u \rangle| &= \mathbb{E}_\pi |\langle w_0, u_0 \rangle + \langle (w_1)_\pi, u_1 \rangle| \\ &\geq \max\{|\langle w_0, u_0 \rangle|, \frac{1}{2} \mathbb{E}_\pi |\langle (w_1)_\pi, u_1 \rangle|\} \\ &\geq 2c' \max\{n^k U_0 W_0, \gamma(w_1, u_1) \sqrt{n}\} \\ &\geq c' n^k U_0 W_0 + c' \gamma(w_1, u_1) \sqrt{n}. \end{aligned}$$

It is therefore enough to prove that, for some fixed $c = c(k) > 0$,

$$\gamma(w_1, u_1) \geq c n^{k-1/2} \sum_{i=1}^k n^{-i/2} W_i U_i. \quad (45)$$

Note that $\gamma(w_1, u_1) = \gamma(w, u)$, since u_0 and w_0 are invariant under permutations.

We know from (35) that, with $\tau = (xy)$,

$$\begin{aligned} \gamma(w, u) &= \mathbb{E}_{\pi, \sigma} |\langle w_\pi, u_\sigma \rangle - \langle w_{\tau\pi}, u_\sigma \rangle| \\ &= \mathbb{E}_{\pi, \sigma} |\langle (w_\pi)^{xy}, (u_\sigma)^{xy} \rangle|. \end{aligned}$$

For $k = 1$, $(w_\pi)^{xy}$ and $(u_\sigma)^{xy}$ are nullary functions with absolute values $|w_\pi(x) - w_\pi(y)|$ and $|u_\sigma(x) - u_\sigma(y)|$. It follows from (18) that

$$\begin{aligned} \gamma(w, u) &= \mathbb{E}_{\pi, \sigma} |(w_\pi(x) - w_\pi(y))(u_\sigma(x) - u_\sigma(y))| \\ &= \mathbb{E}_\pi |w_\pi(x) - w_\pi(y)| \cdot \mathbb{E}_\sigma |u_\sigma(x) - u_\sigma(y)| \\ &= \mathbb{E}'_{a,b} |w(a) - w(b)| \cdot \mathbb{E}'_{a,b} |u(a) - u(b)| \\ &= W_1 U_1, \end{aligned}$$

as required.

For $k \geq 2$, we prove Theorem 4 by induction. Consider

$$\mathbb{E}_{\pi, \sigma} |\langle (w_\pi)^{xy}, (u_\sigma)^{xy} \rangle|.$$

For fixed x, y we shall (as usual) write \mathbb{E}_{ρ^*} for the expectation over permutations ρ^* of $[n] \setminus x, y$; we will identify each such permutation ρ^* with the

corresponding permutation ρ of $[n]$ that fixes x and y and otherwise acts as ρ^* . Note that if $\pi \in \text{Sym}(V)$ and $\rho^* \in \text{Sym}(V \setminus \{x, y\})$ are both uniformly distributed, then so is $\rho\pi$. So

$$\mathbb{E}_{\pi, \sigma} |\langle (w_\pi)^{xy}, (u_\sigma)^{xy} \rangle| = \mathbb{E}_{\pi, \rho^*, \sigma} |\langle (w_{\rho\pi})^{xy}, (u_\sigma)^{xy} \rangle|$$

Consider first the case when σ is the identity. We know by induction that

$$\mathbb{E}_{\rho^*} |\langle (w^{xy})_{\rho^*}, u^{xy} \rangle| \geq c_{k-1} \sum_{i=0}^{k-1} W_i^{xy} U_i^{xy} n^{k-1-i/2},$$

where $(W_0^{xy}, \dots, W_{k-1}^{xy})$ and $(U_0^{xy}, \dots, U_{k-1}^{xy})$ are the W -vectors of w^{xy} and u^{xy} respectively. Now note that if ρ fixes x and y then $(w^{xy})_{\rho^*} = (w_\rho)^{xy}$. It follows that

$$\begin{aligned} \mathbb{E}_\pi |\langle w_\pi, u \rangle - \langle w_{\tau\pi}, u \rangle| &= \mathbb{E}_\pi \mathbb{E}_{\rho^*} |\langle w_{\rho\pi}, u \rangle - \langle w_{\tau\rho\pi}, u \rangle| \\ &= \mathbb{E}_\pi \mathbb{E}_{\rho^*} |\langle (w_{\rho\pi})^{xy}, u^{xy} \rangle| \\ &= \mathbb{E}_\pi \mathbb{E}_{\rho^*} |\langle (w_\pi)^{xy}_{\rho^*}, u^{xy} \rangle| \\ &\geq c_{k-1} \mathbb{E}_\pi \sum_{i=0}^{k-1} (W_\pi^{xy})_i U_i^{xy} n^{k-1-i/2}, \end{aligned} \quad (46)$$

where we have used (35), and the fact that $\pi, \sigma\pi, \sigma^*\pi$ are all uniformly distributed over $\text{Sym}(V)$. But w_π^{xy} is isomorphic to $w^{\pi^{-1}(x)\pi^{-1}(y)}$, and so $(W_\pi^{xy})_i = (W^{\pi^{-1}(x)\pi^{-1}(y)})_i$. It follows that

$$\mathbb{E}_\pi (W_\pi^{xy})_i = \mathbb{E}'_{v,w} (W^{vw})_i = W_{i+1} \quad (47)$$

and so, by (46),

$$\begin{aligned} \mathbb{E}_\pi |\langle w_\pi, u \rangle - \langle w_{\tau\pi}, u \rangle| &\geq c_{k-1} \sum_{i=0}^{k-1} W_{i+1} U_i^{xy} n^{k-1-i/2} \\ &= c_{k-1} \sum_{i=1}^k W_i U_{i-1}^{xy} n^{k-1/2-i/2}. \end{aligned}$$

But now, applying the same argument to u_σ over random σ gives

$$\begin{aligned} \mathbb{E}_{\pi, \sigma} |\langle w_\pi, u_\sigma \rangle - \langle w_{\tau\pi}, u_\sigma \rangle| &\geq \mathbb{E}_\sigma c_{k-1} \sum_{i=1}^k W_i (U_\sigma^{xy})_{i-1} n^{k-1/2-i/2} \\ &\geq c'_{k-1} \sum_{i=1}^k W_i U_i n^{k-1/2-i/2}, \end{aligned}$$

where we have used (47) with u instead of w in the final line. This proves inequality (45), and therefore (13). \square

Proof of Theorem 3. This now follows easily from Lemma 11 and (45), as we have

$$\begin{aligned} \text{disc}^+(w, u) \text{disc}^-(w, u) &\geq c^2 \gamma(w, u)^2 n^2 \\ &\geq c' n^{2k+1} \left(\sum_{i=1}^k n^{-i/2} W_i U_i \right)^2 \\ &\geq c' n^{2k+1} \sum_{i=1}^k n^{-i} W_i^2 U_i^2, \end{aligned}$$

since all terms in the sum are nonnegative. \square

Theorems 3 and 4 also allow us to prove the following result.

Theorem 15. *For every $k \geq 1$ there are constants $c_1, c_2 > 0$ such that the following holds. For every $n \geq 2k$, and every family \mathcal{F} of k -uniform hypergraphs with vertex set $[n]$ such that $w([n]) = 0$ for all $w \in \mathcal{F}$, there is a partition $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ such that, for every $1 \leq i \leq k$ and all distinct pairs $w, u \in \mathcal{F}_i$, we have*

$$\text{disc}(w, u) \geq c_1 \|w\|_1 \|u\|_1 / n^{i/2-1/2+k},$$

and,

$$\mathbb{E}_\pi |\langle w_\pi, u \rangle| \geq c_2 \|w\|_1 \|u\|_1 / n^{i/2+k}.$$

Proof. Suppose $w \in \mathcal{F}$ has W -vector (W_0, \dots, W_k) . Since $w(V) = 0$ we have $W_0 = 0$, and so by Lemma 9 there is $i \geq 1$ with $W_i \geq \|w\|_1 / n^k$: we choose such an i and place w in \mathcal{F}_i . Now for $w, u \in \mathcal{F}_i$, with W -vectors (W_0, \dots, W_k) and (U_0, \dots, U_k) , we have by Theorem 3

$$\text{disc}^+(w, u) \text{disc}^-(w, u) \geq c n^{2k+1-i} W_i^2 U_i^2 \geq c' n^{1-i} \|w\|_1^2 \|u\|_1^2 / n^{2k}.$$

The first bound now follows, and the second follows similarly by applying Theorem 4. \square

6 Orthogonal sets of weightings

Consider integers n, k with $n \geq 2k \geq 1$ and a set $V = \{v_1, \dots, v_n\}$. Let us choose a sequence $\mathbf{s} = (x_1, y_1, \dots, x_k, y_k)$ of distinct elements of V and define the weightings ϕ_i on $V^{(k)}$ as in (19).

We also define the subspace V_i of $\mathbb{R}^{V^{(k)}}$ to be the linear span

$$V_i = \langle (\phi_i)_\pi : \pi \in \text{Sym}(V) \rangle.$$

Note that V_i is independent of our choice of \mathbf{s} .

Theorem 16. (a) For $i \geq 1$, $\text{disc}(w, \phi_i) = 0$ if and only if $w \in V_i^\perp$.

(b) If $u \in V_i$ and $w \in V_j$, where $i \neq j$, then $\text{disc}(u, w) = 0$.

(c) $\mathbb{R}^{V^{(k)}}$ is the direct sum $V_0 \oplus \dots \oplus V_k$.

(d) Suppose that $u = u_0 + \dots + u_k$, with $u_i \in V_i$ for each i , and let (U_0, \dots, U_k) be the W -vector of u . For $i = 0, \dots, k$, we have $U_i = 0$ if and only if $u_i = \mathbf{0}$.

(e) If w_1, \dots, w_t are nonzero and satisfy $\text{disc}(w_i, w_j) = 0$ for all $i \neq j$ then there is a partition $[k] = I_1 \cup \dots \cup I_t$ such that we have

$$w_i \in V_0 \oplus \bigoplus_{h \in I_i} V_h$$

for each i .

Proof. (a) Note first that $\phi_i(V) = 0$, so ϕ has density $d(\phi_i) = 0$. But then

$$\begin{aligned} \text{disc}(u, \phi_i) = 0 &\iff \forall \pi, \quad \langle u_\pi, \phi_i \rangle = 0 \\ &\iff \forall \pi, \quad \langle u, (\phi_i)_\pi \rangle = 0 \\ &\iff \forall \text{ sequences } (\lambda_\pi), \quad \langle u, \sum_{\pi} \lambda_\pi (\phi_i)_\pi \rangle = 0 \\ &\iff u \in V_i^\perp. \end{aligned}$$

(b) We may assume that $i, j \neq 0$ or else the result is trivial. We may therefore assume $d(w) = d(u) = 0$. It is then sufficient to show that, for any choice of permutations π and ρ , we have $\langle (\phi_i)_\pi, (\phi_j)_\rho \rangle = 0$. So, let us choose π and ρ , and set $\psi_i = (\phi_i)_\pi$ and $\psi_j = (\phi_j)_\rho$.

For $1 \leq s \leq i$, we let $(a_s, b_s) = (\pi^{-1}(x_s), \pi^{-1}(y_s))$, and, for $1 \leq t \leq j$, we let $(c_t, d_t) = (\rho^{-1}(x_t), \rho^{-1}(y_t))$. Then we have

$$\psi_i(e) = \begin{cases} 0 & \text{if } |e \cap \{a_s, b_s\}| \neq 1 \text{ for some } 1 \leq s \leq i \\ (-1)^{|A \cap \{b_1, \dots, b_i\}|} & \text{otherwise,} \end{cases} \quad (48)$$

and

$$\psi_j(e) = \begin{cases} 0 & \text{if } |e \cap \{c_t, d_t\}| \neq 1 \text{ for some } 1 \leq t \leq j \\ (-1)^{|A \cap \{d_1, \dots, d_j\}|} & \text{otherwise.} \end{cases} \quad (49)$$

Now consider the multigraph G with vertex set V and edge set given by $a_1b_1, \dots, a_ib_i, c_1d_1, \dots, c_jd_j$. As $E(G)$ is the union of two matchings, it contains no odd cycles and so is the vertex-disjoint union of paths and even cycles (possibly including double edges). Even cycles and paths with an even number of edges meet $\{a_1b_1, \dots, a_ib_i\}$ and $\{c_1d_1, \dots, c_jd_j\}$ in the same number of edges, so (as $i \neq j$) there must be a path $P = x_1 \cdots x_{2t}$ with an odd number of edges. Let $X = V(P)$. If $e \in V^{(k)}$ is such that $\psi_i(e)$ and $\psi_j(e)$ are both nonzero, it follows from (48) and (49) that either $e \cap X = \{x_1, x_3, \dots, x_{2t-1}\}$ or $e \cap X = \{x_2, x_4, \dots, x_{2t}\}$, as each edge of P must contain exactly one vertex of e . Furthermore, as P has $2t - 1$ edges, if we write $e' = e \triangle X$ then

$$\psi_i(e')\psi_j(e') = (-1)^{2t-1}\psi_i(e)\psi_j(e) = -\psi_i(e)\psi_j(e).$$

It follows that $\psi_i(e)\psi_j(e) + \psi_i(e')\psi_j(e') = 0$. But now, pairing off such edges, we see that $\sum_e \psi_i(e)\psi_j(e) = 0$ and so

$$\langle \psi_i, \psi_j \rangle = 0 \quad (50)$$

as required. Note also that (50) holds if $i = 0$ or $j = 0$.

(c) It follows from (50) and linearity that there is no linear dependence among sets of vectors chosen from distinct V_i and thus that $V_0 + \cdots + V_k$ is a direct sum. Now suppose that $u \in (V_0 \oplus \cdots \oplus V_k)^\perp$. For $i = 0, \dots, k$, and any π , we have $\langle u, (\phi_i)_\pi \rangle = 0$. It follows that $\text{disc}(u, \phi_i) = 0$ for every i , and so, by Lemma 8, u has W-vector $(0, \dots, 0)$. But by Lemma 9, this implies that $u = \mathbf{0}$. It follows that $V_0 \oplus \cdots \oplus V_k = \mathbb{R}^{V^{(k)}}$.

(d) For $j \neq i$, we have $V_j \subseteq V_i^\perp$ and so $\langle (u_i)_\pi, \phi_j \rangle = 0$ for every π . Thus, for any j ,

$$U_j = \mathbb{E}_\pi |\langle (\sum_i u_i)_\pi, \phi_j \rangle| = \mathbb{E}_\pi |\langle (u_j)_\pi, \phi_j \rangle|. \quad (51)$$

Now let (U_0^j, \dots, U_k^j) be the W -vector of the weighted hypergraph u_j , so (51) implies $U_j = U_j^j$. Clearly $U_i^j = 0$ if $i \neq j$. By Lemma 9 we have $U_j^j \neq 0$ if and only if $u_j \neq \mathbf{0}$.

(e) It is enough to prove this for two k -uniform weighted hypergraphs u, w , say with W -vectors (U_0, \dots, U_k) and (W_0, \dots, W_k) . By (c), we can write $u = u_0 + \dots + u_k$ and $w = w_0 + \dots + w_k$, where $u_i, w_i \in V_i$ for each i . By (d) we have $U_i > 0$ whenever u_i is nonzero, and $W_i > 0$ whenever w_i is nonzero. Since $\text{disc}(u, w) = 0$, it follows from Theorem 3 that $U_i W_i = 0$ for every $i \geq 1$ and so we deduce that u_i and w_i cannot both be nonzero. The result follows. \square

Note in particular, that part (e) proves Theorem 1. Indeed it has the following stronger corollary.

Corollary 17. *Suppose that u_1, \dots, u_k are weighted k -uniform hypergraphs on vertex set V such that $u_i(V) = 0$ for all i , and $\text{disc}(u_i, u_j) = 0$ for all $1 \leq i < j \leq k$. Then there is a relabelling such that $w_i \in V_i$ for $i = 1, \dots, k$.*

7 Further questions

In this paper, we have proved some results on the discrepancy of pairs of weighted k -uniform hypergraphs. However, many interesting questions remain.

- What can we say about discrepancy of directed graphs, or more generally of directed k -uniform hypergraphs (in which edges are ordered k -tuples of distinct vertices)? More simply, what about oriented graphs, or tournaments? What can be said about functions from $X \times Y$ to \mathbb{R} , where we are allowed to permute both X and Y ?
- It is interesting to note what Theorem 3 says about the expected discrepancy of pairs of random hypergraphs. Let us fix $p \in (0, 1)$ and let w be a random k -uniform hypergraph with vertex set $[n]$, where each edge is present independently with probability p . For $i \leq k$, consider a sequence $x_1, y_1, \dots, x_i, y_i$ of $2i$ distinct vertices. It follows from (19) that ϕ_i is nonzero on $\Theta(n^{k-i})$ k -sets, while $\phi_i([n]) = 0$. Thus $\langle \phi_i, w \rangle$ is the difference of two binomial random variables with (the

same) distribution with parameters $\Theta(n^{k-i})$ and p . It follows that $\mathbb{E}|\langle \phi_i, w \rangle| = \Theta(\sqrt{p(1-p)}n^{(k-i)/2})$ and so

$$\mathbb{E}W_i = \Theta(\sqrt{p(1-p)}n^{(k-i)/2}/n^{k-i}) = \Theta(\sqrt{p(1-p)}n^{-(k-i)/2}).$$

If w and u are random graphs with densities p, q respectively, it then follows from Theorem 3 that $\mathbb{E}\text{disc}(w, u) \geq \sqrt{p(1-p)q(1-q)}n^{(k+1)/2}$, where the main contribution comes from the final components W_k, U_k of the W -vectors of w, u .

The problem of determining the expected value of $\text{disc}(w, u)$ for pairs of random hypergraphs was raised (for graphs) in [11]. It has since been shown in Bollobás and Scott [12] and Ma, Naves and Sudakov [24] that in fact it is possible to improve on the bound above by a $\sqrt{\log n}$ factor. For further related work, see [9].

- What about the sharpness of everything? For instance, when are the bounds in Theorem 2 and Theorem 3 sharp to within a constant factor? Can better bounds be proved?
- The results in this paper are concerned with weighted k -uniform hypergraphs, but what happens if we restrict ourselves to the unweighted case? We can always generate a pair of k -uniform hypergraphs with discrepancy 0 by letting G be the hypergraph with all edges containing a fixed vertex, and letting H be any regular k -uniform hypergraph, but what if we want G and H to have density bounded away from 0 and 1? For $k = 2$, the lower bound (11) (from [11]) shows that the discrepancy must be large; but for $k = 3$, as noted in the introduction, there is a pair of dense unweighted hypergraphs with discrepancy 0. What happens for $k \geq 4$? Could a version of Conjecture 10 from [11] hold in this case? In the opposite direction, it would be very interesting to characterize zero discrepancy pairs of unweighted hypergraphs. In light of Theorem 3, one line of attack would be to consider which components of the W -vector can be 0 for an unweighted hypergraph. More generally, which subsets of components can support the W -vector of an unweighted k -uniform hypergraph? And is there a set of three unweighted hypergraphs that pairwise have discrepancy 0?
- Can we say anything about the *distribution* of $\langle w_\pi, u \rangle$ for $k \geq 3$? It seems natural to hope for some form of Central Limit Theorem, as in

the cases $k = 1, 2$. Perhaps less ambitiously: we have a lower bound on $\mathbb{E}|\langle w_\pi, u \rangle|$, but what about an upper bound? Maybe it is possible to determine this expectation up to a $\Theta(1)$ factor.

- To what extent do the results above extend to the continuous setting, when we have measurable functions from $[0, 1]^k$ to \mathbb{R} ?
- We have worked with real weights in this paper. What happens if we work with complex functions (with the usual inner product $\langle u, v \rangle = \sum_e u(e)\overline{w(e)}$)?
- It would be interesting to consider different group actions. As a starting point, what happens if we take the action of the cyclic group on itself, or of \mathbb{Z}_2^n on itself? In the case of the cyclic group, it would be natural to work with complex weights, as the Fourier basis gives a set of pairwise orthogonal weightings.

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