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Essays on Financial Frictions, Collateral and Default

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Abstract

The first model of my thesis introduces a monetary Real Business Cycle model with incomplete markets, a durable good, collateralized securities and endogenous default. We establish that equilibria are inefficient since agents underinvest due to the existence of endogenous default. Moreover, the Pareto optimal equilibrium allocation cannot be achieved through monetary policy. Our analysis suggests the need of an additional policy tool to generate optimal investment and improve equilibrium allocations. In the second model of my thesis, we propose an integrated tractable framework that incorporates endogenous default in a continuous-time setting. Productive experts that face leverage constraints and aggregate risk, borrow from households and choose to partially default subject to a penalty that decreases their objective function. Our results show that default increases borrowing costs, lowers expert's capital holdings, suppresses investment and subsequently leading to a recession. We show that the path of inflation under quantitative easing policies that target interest rates, is determinate in the presence of default. We achieve this through different payoff profiles that a collateralised defaultable bond achieves in different states of nature with distinct default outcomes. In the model, heterogeneous households trade this bond and other shorter maturity risk-free bonds to maximise their intertemporal utility of consumption and labour. The differentiated payoffs of the collateralised bond, in an equilibrium with active default, span the full state space giving determinacy of prices and inflation as an outcome.

Introduction

This introductory part is a joint paper with Charles Goodhart, Dimitrios Tsomocos and Martin Shubik. The reference can be found in [42].

Why Mainstream Macro-Modelling is Insufficient

When the Queen came to the London School of Economics, to open a new academic building, in November, 2008, shortly after the recent financial crisis had struck, she turned to one of her hosts, and asked "Why did no one warn us of this?" One reason for this failure to foresee the coming financial debacle was that mainstream economic models have mostly assumed away all such financial frictions. Standard DSGE models were in essence real (Real Business Cycle, RBC) models, with the addition of price/wage stickiness (frictions).

All models involve simplification, and such simplification has both benefits, in enabling focus and a capacity to manipulate and solve, and costs, in the guise of departure from reality. The assumption of no default by any agent, as represented by the transversality condition for example, allowed for much simplification. The whole financial sector could be eliminated from such models; all agents could borrow, or lend, at the same riskless interest rate(s) since there was no credit (counter-party) or liquidity risk; it greatly facilitated the adoption of the 'representative agent' as a modelling device (this also has the incidental property that it rules out any examination of changes in the wealth structure of an economy).

Under normal conditions the treatment of finance as a 'veil', and the assumption that default is constant and low may be a reasonable approximation to reality. But then, in such normal times almost any form of simple modelling, such as a vector auto-regression, will forecast as well as or better than the more complex macro-models. What, however, appears clear from the historical records is that really serious disturbances to the economy, both deflation and depression on the one hand, and inflation on the other, are closely connected to monetary and financial malfunctions. In particular, many of the most severe depressions have been set off by bank defaults and the concomitant collapse of asset prices. Amongst such occasions are 1907, failure of the Knickerbocker Trust; 1929-32, failure of Credit Anstalt and the Bank of New York; 2008, failure of Lehman Bros. The failure, and the expectation of potential failure, of major financial entities have been key drivers of macro-economic weakness throughout modern history.

While the standard DSGE model does not include financial intermediaries, let alone the rare instances of their default, it does normally include (outside) money. Fluctuations in monetary growth, which may be caused in part by financial failure, can, with sticky prices, lead to depression in such models when monetary growth is abruptly cut back, and of course to inflation when monetary growth surges.

But the inclusion of money in such a model is anomalous and internally inconsistent, as Hahn [51] originally argued in 1965, and as has been repeatedly underlined by the authors here (Quint and Shubik [79], Goodhart and Tsomocos [47]). If no one ever defaults, anyone's personal IOU will be perfectly acceptable, and perfectly liquid, in purchasing any good, service or other (financial) asset from anyone else. There is no need for any specific money, and devices such as cash in advance constraints, money

in the utility function, etc., are not only artificial but inconsistent with the other assumptions of the model (see for example Fischer Black [10] and the experimental work of Huber, Shubik and Sunder [55]).

Moreover, the 'no default' assumption of mainstream DSGE models is internally inconsistent in other respects. There is no explicit mechanism in such models to enforce repayment (plus interest). Homo economicus in such models is supposed to be rationally selfish. So if she can default in circumstances when the benefit of so doing (i.e. more consumption) outweighs the cost, she should do so. Yet she is assumed not to do so, an internal inconsistency. In essence if individuals, or institutions, are meant to have any strategic power whatsoever, while lending plays any role whatsoever, bankruptcy and default rules are required as a logical necessity, not just an institutional addition.

The minimum addition to standard macro-models that is necessary to introduce money, liquidity, credit risk, financial intermediaries, (such as banks), and financial frictions in general, (and bank and other agents' failures in particular), is the possibility of default. It has been a major part of the work of the fourth author here to model default, and to show how such default affects the working of the financial system and the economy more widely (Shubik [83], Shubik and Wilson [90]). It has also been the largest part of our analytical endeavour to extend this work towards the construction, calibration and estimation of macro-models in which default plays a central role, and in which, therefore, monetary and banking developments have real effects, (money and banks are non-neutral). The development of Strategic Market Games (Shubik [83], Shapley and Shubik [82] and [89]) came about in order to turn the essentially static general equilibrium model into a system with fully and properly delineated processes

for dynamic adjustment. Money played a critical role in providing the extra degree of freedom that enabled the system to be defined out of equilibrium as well as in equilibrium; but money alone, unless in sufficiency and appropriately distributed, did not guarantee efficient market outcomes. Credit had to be introduced. The introduction of credit leads to the need to specify default conditions[1]. Furthermore, if through error, intent or chance, credit cannot be repaid, the default laws keep the rules of motion well-defined.

Since the original introduction of default into the economic system, there has been a considerable growth in the literature on bankruptcy and default, both in an open or partial equilibrium setting and a closed or general equilibrium setting, as represented by the works of Hellwig [54], Dubey, Geanakoplos and Shubik [29], Zame [97], Geanakoplos and Zame [39] and in a dynamic programming setting Geanakoplos, Karatzas, Shubik and Sudderth [37].

Our approach may be compared and contrasted with the classical Walras-Arrow-Debreu (W-A-D) general economic equilibrium model and its extension by Lucas and his followers (e.g. Lucas and Stokey [65]). The W-A-D model is essentially non-institutional and static. No institutions are present in this model and the market structure of the economy is ad hoc in the sense that the "invisible hand" brings the system to a resting point. All the real world phenomena that pertain to credit, liquidity, default and financial intermediation are absent from the classical paradigm. In addition, the W-A-D is static and does not detail the route whereby equilibrium is attained, i.e., it exists only in equilibrium. Through the operation of the invisible hand, one can study the

[1]Unexpected inflation results into monetization of debt obligation but it is not equivalent to default since each economic agent chooses optimally his own repayment rates.

equilibrium properties of the model, provided that one such equilibrium is selected arbitrarily, because the classical model as developed by Arrow and Debreu (see, Debreu [21]) manifests a great degree of nominal indeterminacy. In other words, since there are more unknowns than equations, the equilibrium specification requires additional ad hoc assumptions to be well defined. Thus, the process that leads to an equilibrium outcome is undefined and patently absent from that model.

Our approach is not subject to any of the above criticisms. In particular, both the minimal necessary institutions emerge as logical necessities to close our model and the process of adjustment is an integral part of our framework. The degrees of freedom that actually exist, even in the classical model, are so many that institutions have to emerge to generate a determinate solution. Unless one is prepared to make the following implausible assumptions:

- complete financial markets,
- no transactions costs,
- personal complete commitment, (no strategic default),

then money (fiat or commodity) has to be introduced as a means of payment. Money and the associated transactions technology remove the nominal indeterminacy which the classical W-A-D model (even without uncertainty) manifests. In addition, the transactions technology that we introduce underscores the importance and the real effects of liquidity in the economy and, consequently, of monetary policy. The aforementioned analysis is incorporated in the institutional economics as it has been developed by Martin Shubik (see, Martin Shubik [84]). In this framework, the liquidity, credit, default and

financial intermediaries codetermine the real and the nominal variables of the economy. In our approach there exists a genuine interaction of the real and the financial sectors of the economy. The study of monetary policy must be done in a framework where credit, money and liquidity matter. Analysis of monetary policy and, more generally of fiscal and regulatory policy in a non-monetary world is akin to performing surgery on the operating table without the patient being present!

The real world is uncertain, as well as risky; there are unknown unknowns, as well as known unknowns (U.S. Secretary of Defense, Donald Rumsfeld, 2002). It is too costly to hedge all eventualities, even when their probabilities may be assessed. When people can get away with not paying their debts, they will do so. So the first step in our analysis is to consider the possibility of default on the contractual obligations that an agent undertakes and, therefore, indicates the necessity for some form of a cash-in-advance constraint. The interplay of liquidity and default justifies (fiat) money as the stipulated mean of exchange. Otherwise, the mere presence of a monetary sector without any possibility of endogenous default or any other friction in equilibrium would become a veil without affecting real trade and, eventually, the final equilibrium allocation. Indeed, cash-in-advance constraints are the minimal institutional arrangement to capture a fundamental aspect of liquidity and how it interacts with default to affect the real economy.

The introduction of institutions in our framework and the specific price formation mechanisms that we analyse explicitly require process oriented models that are well defined not only in equilibrium but also out of equilibrium. The class of models that we have produced are all "playable games" in the sense that they can be implemented, e.g. in a

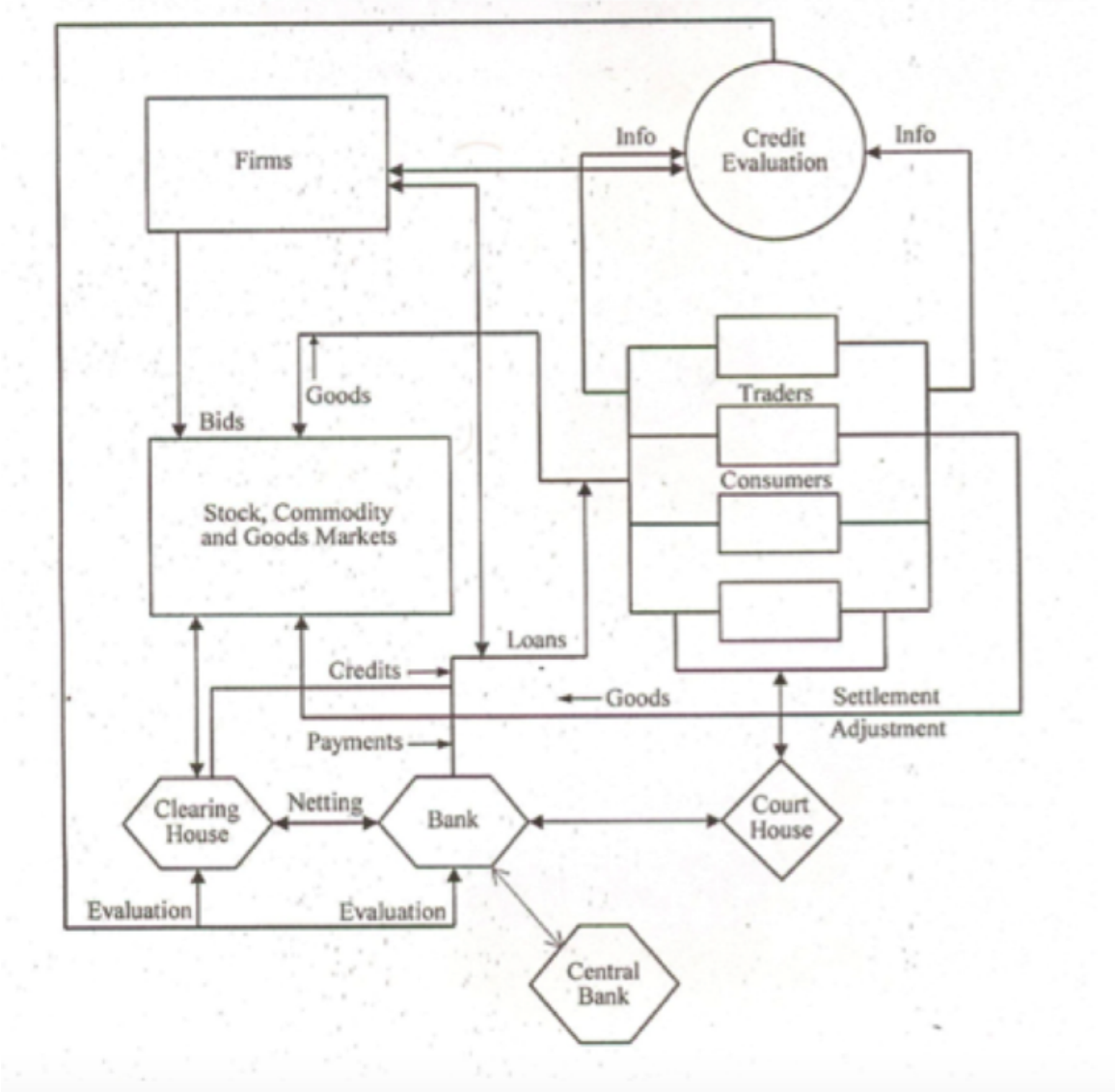
classroom, with clear and well articulated rules of the game. This is definitely useful as it provides a detailed setup that ensures valid conclusions. Is that useful? In conclusion, we argue that the introduction of institutions and of financial frictions in an economic model provides for the guidance of the economy as a loosely coupled economic system and acts as its "neural network". The inclusion of default, money, financial institutions, liquidity, etc., allow policy to be explicitly addressed and connect the different players, including differing generations, in a coherent fashion. It is only by constructing a mathematical institutional economics that one can study the economic system in a rigorous and analytical manner.

This should be done. But if it was easy, it would already have been done. It is not easy because in a world with default, money and credit risk, there are numerous (evolving) institutions and financial instruments (not just a single (credit riskless) interest rate at which everyone can borrow or lend, (see for example, Woodford [96])). Moreover agents differ in their risk aversion and probability of default; hence there is no room for the 'representative agent' simplification in a satisfactory model. To give a brief illustration of the complexity of the scale of the kind of minimum model, (n.b. it is without housing, fiscal policy, non-bank financial intermediaries, and is a single, closed economy), see Chart 1. Where the arrows represent behavioral interactions among the various sectors and agents of the economy. We have to add that the complete model is far from being fully developed. Nevertheless, the work that we describe in this chapter constitutes an initial step to this direction.

Finally, with missing financial markets, since we depart from the complete markets paradigm, there are important welfare consequences. As Geanakoplos and Polemar-

chakis [38] argued, the general equilibrium with incomplete markets model leads to constrained inefficient equilibria. Not only does the first welfare theorem not hold, as it does in the standard W-A-D model, but also the second best is not obtained. The consequence of this is that policy and government intervention may generate welfare improving effects for consumers and firms without necessarily removing the frictions in the short run. Indeed, in Goodhart, Kashyap, Tsomocos and Vardoulakis [45] we produce one such example whereby multiple regulations Pareto improve the equilibrium allocation.

Chart 1


Firms
Info
Credit Evaluation
Info
Goods
Bids
Traders
Consumers
Stock, Commodity and Goods Markets
Loans
Credits
Goods
Settlement
Adjustment
Payments
Clearing House
Netting
Bank
Court House
Evaluation
Evaluation
Central Bank

Modelling a Macro-economic System with Default

The possibility of default amongst economic agents, companies, and above all, financial institutions is not only pervasive, but also at times can have a dominating impact on macro-economic developments. Trying to remove default from macro-economic models will mean that they are at best partial, and at worst, as in recent years since 2007, almost completely useless. But, unfortunately, the attempt to incorporate default as a central, key, component of such models makes their construction far more difficult.

In particular, it makes the dimensionality of the model so much larger. It is no longer feasible to use the useful fiction of a representative agent. Not only do agents, in practice, vary in risk aversion, (and such risk aversion may vary over time in response to events), but also if all agents were required to move in lockstep as a representative agent then either none defaults (so the model reverts to the default free form) or they all default simultaneously, which may lead to the collapse of the model.

Next, the number of sectors in the model will increase. With default, there is a two-fold function for banks. On the liability side, bank deposits, especially when insured, are less subject to default (a lower probability of default (PD)) than the IOUs of (most) other members of the private sector, and hence become used as acceptable means of payment. On the asset side banks should have a comparative advantage in assessing the credit-worthiness of other agents' IOUs (borrowing proposals) and monitoring them to insure that PD is kept lower[2]. Likewise with incomplete financial markets, and asymmetric

[2]After those occasions when (some) banks fail in function 2, and hence amass non-performing loans (npls) and then fail (or approach failure), there is usually a call to split these two functions, into a narrow bank which provides payments services on retail deposits against safe (government) assets on

information, there is room for a large assortment of other financial intermediaries.

In the standard DSGE model there are two assets, (though as argued above, one of these, i.e. outside money, zero interest-bearing currency, is actually redundant and should play no role). The other asset in the model is riskless debt, which all agents can issue for borrowing, or buy for saving. In equilibrium, when the model settles down, all borrowers in the final period just have enough income to pay off their entire debt with certainty; and all net savers just obtain sufficient repayment in that final session to finance their planned expenditures. Hence there is no residual availability of funds, or equity finance, in such models.

By contrast, once default is allowed to occur, a much wider menu of financial assets naturally occurs, including currency, bank deposits and bank loans, debt with a wide range of credit ratings, as well as a whole range of derivatives, other intermediary liabilities and assets, and equity. In other words it is default that gives a meaning, and an explanation, for the wide assortment of financial assets/liabilities that we observe in practice. Put this all together, and this means that a model with default at its centre will have more agents per sector, more sectors and more assets. The problem immediately arises of a potentially exploding dimensionality, (besides the additional complexity of maximising subject to bankruptcy costs). This can be, and has been, partially resolved by refocusing the model in each case to deal with a specific issue, and omitting those agents/sectors/assets not strictly necessary for the question at hand. But this means that models with default have not yet been 'general' in the sense of

the one hand, and a lending institution financed by some other (non-payment) set of liabilities on the other. The general idea is to increase safety, even at the expense of efficiency. The arguments, pro and con, are perennial and outside the scope of this paper.

encompassing a complete macro-economy for all questions. Yet good application calls for context specific models of specific economies designed to answer specific real world questions with a general theoretical basis that can be applied to the case at hand.

A main challenge is still how to extend such general models with default, such as Shubik [85], Dubey and Geanakoplos [25], Tsomocos [95], Dubey, Geanakoplos and Shubik [29] and Goodhart, Sunirand and Tsomocos [46], in a dynamic setting, preserving the main properties of a parsimonious model, without the inclusion of redundant or unrealistic features. Default in the aforementioned models is introduced as an endogenous choice variable whereby agents weigh the marginal benefits vs the cost of defaulting either in the form of loss of collateral or non-pecuniary costs by subtracting a term proportional to the level of default. The initial endeavours, by de Walque et al [20], Leao [60] and Iacoviello [56], that introduced those concepts into the DSGE framework, have not taken into account simultaneously liquidity, agent heterogeneity, money and default risk. Nevertheless, those models are valuable efforts in the development of a plausible explanation of the phenomena observed after the credit crisis. The interplay of liquidity and default justifies the presence of fiat money as the stipulated medium of exchange. Otherwise, the mere presence of a monetary sector without the possibility of endogenous default or any other friction in equilibrium may become a veil without affecting real trade and, eventually, the final equilibrium allocation. Indeed, liquidity constraints, or their extreme version of cash-in-advance constraints, are the minimal institutional arrangement to capture a fundamental aspect of liquidity and how it interacts with default to affect the real economy. It is also worth noting that renegotiation is a first cousin of default. Renegotiation is often used euphemistically to tell the public that no

default has taken place.

There is, perhaps, an even more fundamental problem with models with default. Such default can happen at any time, from the interaction of stochastic events with agents' risk aversion. Thus there is no sense in which such models can ever settle down into some long-run steady state distributions of equilibrium variables. Indeed, the underlying problem is even worse. Laws, market structures, and economic institutions are themselves subject to endogenous change as a result of experience, witness the changes to financial regulation as a consequence of the recent crisis. Most of us find it difficult to countenance a system which does not revert to an equilibrium, supported by some set of fixed social, legal and institutional structures. Yet the reality of default and crises actually means that we, and the macro-economy, are in the middle of a process. A reading of Keynes suggests that he was talking about an economy tending towards equilibrium; not one that actually attained the misleading equilibrium system imposed by Hicks. If one were to add in this context the amendments of Schumpeter on innovation, the appropriate picture at best is one of a system that may follow some adjustment towards a perceived equilibrium, but it will be constantly knocked off course by every innovation[3] and political/institutional change, such as Brexit..

[3]Hayek and Marshall with the "trees and forest" analogy and Smith (see, Shubik and Smith [86]) offer the same argument.

Some Implications of Default as a Process

What are the costs of default?

One, rather obvious, reason why default has not been made a central feature of macroeconomic finance, is that it is quite hard to do. Default is a binary condition. Either an entity has declared bankruptcy, or not. It is, therefore, difficult to fit into the standard models of continuous variables, manipulated and solved via calculus. This problem can be treated by turning it around and focusing not on the occasion of default but on the expectation of repayment, (Shubik [84], Ch 12). This is akin to the inverse of the combination of the probability of default (PD) and expected loss given default (LGD) that play such a central role in finance.

If the costs of default were infinite (e.g. death for the debtor and her family), no one would ever borrow. If the costs of default were zero, no one would ever repay, and so no one would ever lend. So, given a positive marginal efficiency of investment and intertemporal preferences over consumption, an interior optimum must exist with positive, but not infinite, costs of default and positive borrowing and lending. Another problem of modelling default is that its costs depend on institutional conditions and social norms which can change over time, and can, indeed, vary in response to economic conditions, such as the frequency and severity of prior defaults.

There are two main categories of default costs. The first is that the borrower may be required to put the ownership claim to some asset, which may be, but need not always be, the same asset bought with the borrowed money, in forfeit to the lender,

should the borrower not meet her contractual repayment obligations. This is known as the collateral for the loan, and loans protected by such collateral are termed secured loans. Normally the market value of the collateral exceeds the value of the loan, with the difference being the required margin, or down payment, that the borrower has to make to get the initial loan. Geanakoplos [36], and Shubik and Yao [91], have focused primarily on this aspect of the cost of default.

Naturally the required margin will be greater, the greater is the perceived risk of the collateralised asset falling (volatility) and the greater the deviation between the value of the asset to the lender, should foreclosure occur, and to the borrower, e.g. houses during a housing price collapse. Thus this aspect of the borrower/lender relationship, the extent of down payment, margin or leverage available is highly cyclical, with the costs of borrowing (margins, down-payments) going up sharply during financial crises and busts, and going down during expansionary booms. Since there is no need for collateral in a world without default, this aspect of the cost of borrowing is omitted from standard DSGE models without financial frictions.

Monetary policy aims to stabilise the macro-economy by lowering (raising) riskless interest rates during depressions (booms), but, without having a better grasp of the impact on borrowing/lending decisions of the changing terms and conditions of borrowing, which generally fluctuate inversely to interest rates, it is impossible to forecast at all accurately what the net effect on private sector behaviour may be. Particularly after a financial crisis, policies which focus solely on official rates without concern with spreads and margins facing private sector borrowers are likely to be inefficient and insufficient. The difficulty of assessing the changing effects of terms of borrowing is one

reason why focusing on quantities of lending or monetary growth may be often a better indicator of the effects of monetary policy, than is looking at the time path of some chosen interest rate.

Besides having to give up (to the lender) such collateral as the borrower may have pledged, or can be seized by the lender, e.g. in the case of sovereign default, there are other costs to default. These depend, inter alia, on the form of the bankruptcy laws, (see La Porta, Shleifer et al. [59] for a useful overview). Some such costs can be non-pecuniary, e.g. debtor's prison, social shame, loss of use of asset (house, car, etc.); some may be pecuniary, e.g. the present value of future exclusion from financial markets, legal costs, the reduction in income as the bankrupt has to shift from her preferred role and occupation to a less-preferred activity.

Modelling both the time-varying collateral (margin/down-payment) costs and the more direct costs of bankruptcy is an under-developed aspect of macro-economics, and one that needs much more attention. In much of our own work to date we have assumed that the direct costs of bankruptcy were fixed in linear proportion to the rate of repayment failure[4] (Shubik [84]), but more analytical and empirical work would be useful to try to explore this key element in the model.

[4]The parameter may be regarded as an approximation for the marginal disutility of default. A more general functional form would add little to the qualitative understanding, but is mathematically considerably more complex.

DSGE models and default

In a number of recent models (Bernanke, Gertler and Gilchrist [7], Carlstrom and Fuerst [15] and Christiano, Motto and Rostagno [18]) default occurs, however in an oversimplified way. It is implicitly assumed that a fraction of entrepreneurs goes bankrupt and this probability of bankruptcy can be made partially endogenous to the macroeconomic context. There can be both ex ante and ex post default, operating at the firm level. Some of the firms do default in equilibrium, as their return to capital is subject to both idiosyncratic and aggregate risk. However, in aggregate, the bank is assumed to be able to diversify its loan portfolio, so that it repays depositors the riskless rate of interest. Such idiosyncratic risk can affect the firm's equity through the cost channel, since expected default probability directly affects the interest rate paid by firms. Default in their model is not, however, genuinely endogenous. Banks and firms do not choose explicitly what amount to default upon. Instead, the firm chooses its capital demand and cutoff productivity level where the firm will start to default in order to maximize its expected profits subject to a bank's participation constraint. Hence, by optimally raising the cutoff level, it subjects itself to a higher probability that it will default in the case that the realized productivity level turns out to be low. It may, or may not, default in the end, given the realization of the shock. As it turns out, there is no default in equilibrium in the financial sector. Banks charge a default premium to borrowers so that on average they break even!

Curdia and Woodford [19] and Meh and Moran [70] address default in a completely ad hoc fashion. In the former impatient households default on their loans every period by an exogenous amount that is known to the banks. In the latter, in each period, a

fraction of the (risk-neutral) entrepreneurs and a fraction of (risk-neutral) bankers exit the economy and they are replaced by new ones with zero assets. The aforementioned fractions are exogenously given.

De Walque et al [20] introduces endogenous default probabilities for both firms and banks[5]. Following Goodhart et al [46], the borrowers choose the rate of repayment of loans optimally, taking into account penalties incurred. The penalties can be both non-pecuniary in terms of reputation cost, and pecuniary in terms of search costs to obtain new loans. Distortions arise as lenders price in such default probability into the interest rates, and in line with previous literature, contributes to the financial accelerator. Their model is then used to assess the role of capital regulations and liquidity injections on economic and financial stability. In response to the current crisis, where disruption in financial intermediation generates negative consequences on the real economy, some recent articles (e.g. Gertler and Kiyotaki [40]) have shifted focus to impose credit constraints on intermediaries, rather than on non-financial borrowers.

Most other models, despite their focus on default, do not generate any bank default in equilibrium.

The functions of liquidity

In the absence of default, it is hard to give much meaning to the concept of liquidity, since each agent can borrow, at the given riskless rate of interest, up to a limit given

[5]Martinez and Tsomocos [68] produce a related model that studies the interplay between liquidity and default.

by her need to repay with certainty, and is presumed not even to try to borrow beyond that limit. So, up to that limit each agent is fully liquid, and can buy whatever she wants. Up to that limit no agent is liquidity constrained. The implicit assumption of mainstream DSGE is that, in equilibrium, no agent is liquidity constrained, i.e. that no agent is attempting to borrow so much that it has a non-zero probability of default. This also removes any significant distinction between long and short term loans, and treats informational services such as accounting and evaluation as free goods.

If agents are perceived as capable of defaulting, then they will only be able to borrow at rising interest rates, or not at all. If so, their ability to spend will be constrained by their access to income and owned liquid assets. In this latter case agents are liquidity constrained.

As is surely obvious, in practice, the scale of economic activity, the volume of goods and services traded in a period, is endogenous. By contrast, in the model of Lucas and his followers the scale of economic activity is exogenously specified by the requirement that each agent sells the whole of his endowment in each period. A corollary of the quantity theory of money in a model with default, ceteris paribus, is that increases in trading activity in a good state, due perhaps to more productivity or lower interest rates will result in lower price levels, given the same money supply in that state because of variation in credit risk. Similarly, the volume of trade in asset markets affects prices and has second order effects on the inflation rate.

Non-neutrality of money

Given the existence of money, then government monetary policy (open market operations, i.e. changing the stock of bank money) typically has real effects on consumption. Similarly, government transfers of money to agents, no matter how it is distributed across the population, also necessarily have real effects. These non-neutrality conclusions are contrary to those derived by the "representative agent" paradigm. The explanation is that in a model with default, the outcome is not necessarily Pareto efficient, because of the distortion caused by trading via money borrowed at positive interest rates with a wedge between effective borrowing rate and the riskless rate. In any transactions based model with heterogeneous agents, positive interest rates generate a margin between selling and buying prices, thus, inducing non-neutral effects when monetary policy changes. When the government eases credit (by putting more money into the banks, e.g. by open market operations) it facilitates borrowing, reduces interest rates, and increases real activity. In an RBC model with wage/price frictions, the mainstream DSGE models, money is non-neutral in the short run, but becomes neutral in the long run as the wage/price frictions dissipate. Financial frictions provide another, and probably rather longer-lived source of non- neutrality. Financial developments, and monetary policy, will help to determine time-varying perceptions of the probability of default, and hence risk spreads and the liquidity constraints of agents. The present euro-crisis can be analysed in the context of models in which *endogenous* default is possible. It cannot be adequately analysed by using models without default. It has been the ubiquity of the latter kind of model that has led Central Bank Governors, such as Trichet and King, to complain that macro-economics provides little guide

to the present weak financial conjuncture.

There is also another deeper sense in which finance is non-neutral. In RBC models growth is driven by an exogenous (trend) increase in productivity, deriving from innovations, which is disturbed in the short run by a variety of shocks. In models with default, fluctuations in growth will also be driven by the willingness, and ability, of agents to assume risk. Virtually all projects are risky. So a very risk-averse society is likely to be safe, but slow growing. Levine [61] has documented the positive association between financial intermediation and growth. But the rapid expansion of credit expansion, and leverage can put the economy at greater risk, as evidenced since 2000 (Taylor [93]).

Society, and governments, can change the costs of default, and the costs and availability of access to borrowing (e.g. via intermediaries) by a variety of regulatory measures. What we should be trying to assess is the optimal structure of (regulatory) control over risk, by such instruments as Recovery and Resolution Programmes (RRPs), capital adequacy requirements (CARs), margin requirements, etc., to give the best balance between growth (more risk taking) and financial disturbances (less risk taking). Models without default cannot even begin to address this issue; models with default at their heart can now begin to do so (Goodhart, et al [45], Tsomocos [95]). Sadly, one notable feature of the exercises to reform and to improve financial regulation in recent decades, e.g. Basel I, II and III and the various bankruptcy codes is how little they have been informed by macro-economic theory (Goodhart [41], Chapter 16), at least until quite recently. This lacuna was, at least in part, owing to the implicit belief amongst macro-economists that such matters had no, or little, macro-economic effects. In a world

without default such a view would have been correct; but that world does not exist in practice.

The unification of macro-economics, micro-economics and finance

In recent decades the models and the focus of attention of macro-economists, micro-economists and financial theorists have drifted apart. In so far as such macro-economic models abstract from default, their sole focus on financial matters was on the determination of the official, short-term, interest rate, usually via a Taylor reaction function, and thence on the determination of the riskless yield curve, via expectations of the future path of official rates. So, such macro-economists had little interest in financial intermediation, which played no role in their models.

In contrast, of course, the probability of default played a major role in finance theory and in the pricing of assets. Even here, however, the existence and incidence of bankruptcy costs was often assumed away, as in the Modigliani-Miller theorem, or their wider impact on the economy, via externalities, ignored. In a different setting mathematical micro-economists and game theorists, as noted above, have written extensively on bankruptcy in a general equilibrium or strategic market game setting but with little connection to macro-economic literature, although one should regard the work of Diamond and Dybvig [22], Morris and Shin [74] and Smith and Shubik [92] on bank runs as connected with macro-economic stability.

Be that as it may, a greater appreciation, especially amongst macro-economists, of the importance of financial frictions in general, and of default in particular, may re-unify

these specialisations. The, often physical, separation of micro-, and macro-economists in Economics Departments and of finance theorists in Business Schools with almost entirely differing agenda and models has not been beneficial. The greater focus now on financial frictions as a key issue in money-macro analysis will have the side-advantage of reunifying these three strands of the subject, which should never have diverged as far as they did.

Also note that the bankruptcy process, like innovation, does not generally involve the destruction of real assets. It involves their reorganization. The only items that are destroyed are credit arrangements. One of the deepest misunderstanding of the public, in general, is the fear of a, largely mythical, bankruptcy process. The public often appears to believe that it is the assets that are destroyed, whereas in fact a well-executed bankruptcy may be the saviour of on-going enterprises[6]. The saving of General Motors from bankruptcy in the United States in the Great Financial Crisis, for example is presented as an act that saved over a million jobs (including many of the upper management of General Motors), though it is debatable whether an orderly bankruptcy of General Motors with a subsequent take over by a group led by, say, Toyota or Honda might have been better for all concerned other than the continuing G.M. management.

[6]The renegotiation of sovereign debt can be an euphemism for a less transparent and efficient process than a clean reorganization by bankruptcy.

Modelling Banks: An example of how, and how not, to undertake financial modelling

The failure to focus on financial frictions was partly responsible for a continuing, often egregious, failure in most macro work to model the behaviour of financial intermediaries, especially banks. If such intermediaries were part of a veil, with no real effect, why bother?

Even amongst those who most emphasised behavioural incentives and optimisation, the supply of money was often, and in most textbooks until recently, supposedly controlled by a deterministic, and tautological, formula, the monetary base multiplier, whereby the stock of money depended on the stock of base money and two ratios, the banks' desired reserve/deposit ratio and the public's desired currency/deposit ratio. Causation supposedly ran from the monetary base, supposedly set by the Central Bank, to the broader money supply. This accounting identity provided the analytical framework for the (otherwise) great historical book by Friedman and Schwartz.

But even in the decades prior to the 1970s, which had conditions most appropriate to this formulation with required cash, and sometimes required liquidity, ratios, the monetary base multiplier was upside down. As everyone, including prominently Milton Friedman, knew in practice, the policy choice variable of (almost all) Central Banks, almost all the time, was the level of the official short-term interest rate. Given that official interest rate, and other macro conditions (and their risk aversion), banks would decide on their own interest rate terms and conditions for lending (and for offering interest-bearing deposits), with such decisions in turn often dependent on various con-

trols applied by the authorities. That determined credit and monetary expansion, while the banks' and public's preferences for reserves and currency then determining the monetary base that the authorities had to create in order to maintain their separately decided policy choice over short-term interest rates[7].

The concept that individual commercial banks were cash, or liquidity, constrained became even harder to sustain after the 1970s. The emergence of the euro-dollar market in the late 1960s and early 1970s, and its extension into an international wholesale market for funds, meant that all credit-worthy banks, which meant most large banks most of the time, could raise large amounts of additional funding whenever they wanted on standard terms on such markets, e.g. the elasticity of supply of such wholesale funds to any one bank was high; banks were almost price takers in wholesale markets.

Funding liquidity was replacing market (asset) liquidity. At least, prior to the 1970s, when a bank treasurer was contemplating credit expansion, he had to consider how such loans might be funded. Once the wholesale markets got going, all the treasurer needed to compare was the expected risk-adjusted rate of return on credit expansion with the marginal interest rate on wholesale markets. Prior to the 1970s the money stock (broad money) and bank lending marched in lock-step together, in most countries, whereas after the 1970s they increasingly diverged, with credit expansion growing far faster than broad money (Schularick and Taylor [81]). Bank deposits had, increasingly, been insured after the Great Depression of 1929-33, in order to prevent runs; the new wholesale funding of credit expansion, via MMFs, Repos, etc., was not, but no one paid

[7]Technically when there are no constraints, control of the quantity of money or control of the rate of interest by the Central Bank give the same results as they are dual variables. This is not so when there are constraints present.

that much attention, since the credit-worthiness of bank borrowers in such markets was meant to be anchored by their adherence to the Basel Capital Adequacy Ratios (CARs).

Nowadays, with QE and the payment of interest on bank reserves at the Central Bank, the liquidity requirements of banks are satiated, (Greenwood, et al [50]). In this context, the main constraint on bank expansion is not the availability of cash reserves, but the availability of capital, and the profitability of banking business.

So where and how is this, perfectly well known, history of banking and financial developments reflected in a modern theory of banking and money (and near-money) supply determination? The blunt answer is that, by and large, it is not. One can, indeed, say that there is at present no comprehensive theoretical analysis of banking worthy of the name. This has unfortunate consequences. At the start of the unconventional monetary expansions, (QE, CE, LTROs, etc.), after the zero-lower bound to interest rates was hit, it was widely hoped that expansion of the monetary base (H) would lead to some large scale, if not fully commensurate, expansion in broad money and bank lending. It did not. In most key developed countries the monetary base quadrupled; bank lending and broad money barely budged. Besides some general arm-waving about a lack of demand for credit from the private sector, there has been a dearth of post-mortems, of analyses about why the banks behaved as they did.

Then in pursuit of the closure of the doors of stables, from which the horses had already bolted, the authorities, especially in Europe, mandated the early achievement of sharply higher ratios of equity to (risk-weighted) assets. But what factors would determine bank willingness to raise additional equity? Admati and Hellwig (Admati,

et al [2]) have shown that, especially where there is debt overhang and a perceived risk of insolvency, the benefits of additional equity go largely to the creditors, not to existing shareholders. So existing shareholders, and bank managers responsive to their shareholders, will eschew all methods of reinforcing equity, by new issues or profit retention, as far as possible. That implies that raising required equity ratios will be achieved by deleveraging. Of course, each country's monetary authorities will try to ensure that such deleveraging will not take the form of reductions in bank lending to the private sector by their own banks in their own countries. So cross-border bank lending gets decimated, (actually decimation was only one out of ten; this is in many cases nearer nine out of ten), and so-called 'non-core' assets are forcibly sold. This has led to the balkanisation and fragmentation of the European banking system.

Meanwhile most analytic studies of the effects of regulation on banking demonstrated that, in equilibrium, a banking system with a higher equity ratio would not require any much higher interest charge on bank lending, and would be much safer. All this is perfectly true. The problem has been that the present banking system is not in an equilibrium state but is in a transitional process under conditions of severe market stress. It has long been a key theme of ours, that the focus on equilibrium conditions in mainstream macro is insufficient, and even at times potentially misleading. What needs to be done instead is to model financial developments as a continuing process. Rarely has this argument been so clearly manifested as in the case of banking in the aftermath of the 2008 debacle.

In a series of excellent symposia reported in the Journal of Economic Perspectives there has been an attempt to look back at the debacle in 2008. In particular Gorton [49]

and Gorton and Metrick [48] trace the explosion of leveraging by the sophisticated utilization of instruments such as the Repo. Note that the utilization of the Repo is recorded by Bagehot [5] in an appendix in his masterful work, Lombard Street.

Those who study finance deal with the micro-economic realities of the arbitrage among markets, laws, customs, accounting conventions and cultures. They also deal with the hi-tech micro-micro-economic problems of mechanism design (see, as a nice example, Caccioli, Bouchaud and Farmer [14]), and with sophisticated studies in stochastic processes, which seek out even minor fleeting correlations in stock or foreign currency trading in some instances in order to trigger computerized trading. This work rarely interacts with either main stream micro- or macro-economics yet it is relevant to both in the sense that in its devotion to the understanding and profiting from trading, it attempts to capture, at least in the short run, actual financial dynamics.

Role and rationale of financial sector in general equilibrium models

In this section, we consider how the role and rationale of the financial sector is addressed in some general equilibrium models.

In Kiyotaki and Moore [58], Fostel and Geanakoplos [34] and Iacoviello and Neri [57] the financial sector is not modelled. The first model is a pure real model with two types of agents (credit constrained and non-credit constrained) and the non-credit constrained agents directly lend to the credit constrained ones. Thus, in this model there is no financial sector. And the second subsumes the role of the financial sector in the activities

of agents, whereas in the last the banking sector is also not present. Likewise, in Bianchi and Mendoza [9] there is no fully-fledged financial sector.

The Neo Keynesian models (e.g., Bernanke, Gertler and Gilchrist [7], Curdia and Woodford [19]) introduce a rudimentary financial sector to intermediate between the borrowers and savers, introducing a spread between deposit and lending rates. Moreover, state-contingent contracts are introduced directly without intermediation (Curdia and Woodford [19]) or through intermediation (Bernanke, Gertler and Gilchrist [7]) between the households in order to insure one another against both aggregate risk and the idiosyncratic risk associated with household types. Banks break even in equilibrium by charging an insurance premium to entrepreneurs. As a result, aggregate risk is fully absorbed by the entrepreneurial sector. Banks' only role is to intermediate funds in a rather mechanical fashion. For example, in Carlstrom and Fuerst [15] the entrepreneurs are involved in producing the investment good, receiving their external financing from households via intermediaries that are referred to as capital mutual funds. The production of capital is subject to an idiosyncratic productivity shock which incorporates innovation and capital goods are purchased at the end of the period with the assistance of intermediaries.

Notable exceptions to this family of models are Christiano, Motto and Rostagno [18] and Meh and Moran [70]. In Christiano, Motto and Rostagno [18], there is a fully integrated financial sector and competitive banks. They receive deposits from the households and make loans to the entrepreneurs as in Bernanke, Gertler and Gilchrist [7] and working capital loans to the intermediate good firms and other banks as in Christiano, Eichenbaum and Evans [17]. Banks bundle transactions services with their deposit

liabilities and produce transactions services with capital, labor and bank reserves. They hold a minimum of cash reserves against households deposits, consisting of base money and Central Bank liquidity injections. The modeling of the banking sector becomes non-trivial, partially due to the production function for bank deposits or liquidity creation. In particular, capital, labor and excess reserves are utilized. This gives rise to a liquidity constraint for banks, and hence allows certain types of shocks, such as funding liquidity and shifts in demand for excess reserves. In addition, there is also a central bank that sets the nominal interest rate according to a generalized Taylor Rule.

Meh and Moran [70] extend the basic framework introduced by Bernanke, Gertler and Gilchrist [7]. Loans are modeled in a similar fashion. However, an agency problem is posited between the bank depositors and the banking sector. Monitoring is costly and depositors cannot monitor the quality of entrepreneurs' projects. Since losses are mostly suffered by depositors, banks lack the incentive to monitor entrepreneurs. Therefore, investors require banks to invest their own capital as well. As a result, banks that are better capitalized attract more loanable funds. Finally, in Goodhart, Kashyap, Tsomocos and Vardoulakis [44], there is also a fully integrated banking system with a non-trivial role and furthermore a shadow banking system that during bad times triggers fire sales which further worsen the financial crises by causing further reduction in asset prices. Commercial Banks are risk-averse, take deposits, make mortgage loans, short-term loans to households and repo loans to the shadow banks. They also receive money from the Central Bank through a discount window. Moreover, they securitize mortgage loans and sell them and they are endowed with capital. On the other hand, Shadow Banks are less risk-averse, endowed with capital, receive repo loans to buy

securitized assets and use these assets as a collateral for the repo loan. All of them are profit maximizers.

Can such a Complex System ever be Adequately Modelled within a General Equilibrium (GE) Framework?

Many (mathematical) economists, approaching macro-dynamics from an (industrial) microeconomics viewpoint, have worried about the large number of degrees of freedom that seem to appear as soon as one tries to introduce default and money into a general equilibrium structure. As is often the case, a paradox is no paradox when examined from the appropriate stance. Following any attempt to describe the general equilibrium structure as a playable game with a monetary economy, the immediate critique is that not only is it unduly special but also that the construction could have been done in an astronomically large number of ways. We provide a framework, however, whereby we use institutions as minimally necessary to define the economic process and allow policy experiments to be conducted. Therefore, our analysis is robust to alternative institutional arrangements, since our emphasis is not so much on the specific institutions that we introduce but rather on the pecuniary and non-pecuniary externalities that we try to rectify given the frictions that are present in our models. For example, cash in advance formulation is the simplest possible way to capture the effects of liquidity in a transactions based model. Alternatively, we could utilise more complicated liquidity requirements, as in Espinoza and Tsomocos [32] and Martinez and Tsomocos [68], where not only fiat money but also commodities can be partly used for transactions purposes.

So this criticism is completely correct, but yet completely misses the positive message, which is the need for a link between static economics on one side and economic dynamics and institutional evolution on the other side. The plethora of degrees of freedom are there so that it is feasible to construct hosts of ad hoc micro-macro-economic models that fit special niches and are able to adjust with varying flexibility to the macro and the macro dynamics to which they are applied. In other words, given each particular friction and its associated externality, we are able to address it in multiple ways without necessarily modifying our results. We consider the fact that we offer many alternatives to address a particular externality a strength rather than a weakness of our approach. One should be aware of all the possible alternatives and subsequently select the one that fits current objectives best. In attempting to describe the general equilibrium structure as a playable game with a monetary economy the immediate critique of anyone who constructs a playable strategic market game is that not merely is it so special but that the construction could have been done in an astronomically large number of ways. One may argue that general equilibrium analysis is not useful because in such large scale models anything can happen when initial conditions are changed. We consider this attitude as a vindication of our analysis. We believe that until one knows how all the effects depend on the behaviour of the agents, one can not aim to comprehend fully any policy change.

In a modern economy one of the most important markets, is the market for firms, and firms are institutions with both an aggregation of assets and an organization. A good industry specialist trying to forecast the dynamics of the firm and its viability will investigate real and financial assets and liabilities, as well as inter alia, management

and software. A check-list includes:

1. Labor,

2. Land,

3. Buildings

4. Machinery,

5. Raw materials

6. Goods in process

7. Office supplies

8. Cash and receivables

9. Short term debt

10. Long term debt and

11. financial assets

12. Software, and off-balance sheet assets such as derivatives and

13. Management.

Each of these items may have a time profile attached to it involving properties such as the ages and length of service of management and the age structure of machinery.

Cautious accounting calls for leaving the valuation of management and organization off the balance sheets; but one of the key tasks of a good analyst is to attach estimates to the worth of this off balance sheet asset[8].

In general the larger institutions are all run by committees of fiduciaries using "Other Peoples' Money" (OPM) and buried in a Byzantine mixture of tax laws, regulation and accounting conventions that place considerable constraints on the manifestation of economic behavior. This observation is not a counsel of despair. It merely suggests that the diversity of time lags and organizational constraints, (that are the daily concern of a good industrial organization economist and any competent financial analyst doing due diligence), do not disappear into the maw of a one aggregate single dimension dynamic programming model. Economic dynamics is highly dependent on models aggregated from the bottom up. This is not an impossible, but it is an arduous task. A government in general and a central bank in particular, require evidence on the macro-economy far beyond the analogies of RBC. The challenges and the dangers of bridging the gap between theory and practice were eloquently noted in Edgeworth's inaugural address in 1891 (Edgeworth [30]). He commented:

• "It is worthwhile to consider why the path of applied economics is so slippery; and how it is possible to combine an enthusiastic admiration of theory with the coldest hesitation

[8]Given this listing, a little contemplation of the description in these terms of a bank, a life insurance company, a family restaurant, an automobile rental firm, a large oil company, a hedge fund or a funeral parlor immediately calls for a taxonomy of dynamic models at least as diverse as an optimal kingdom stretching from whales to ants or mites. The sizes in terms of both personnel and assets, the flexibility of the institution and their complexity differ so considerably that it is unlikely that "one shoe fits all" when one is trying to develop economic dynamics. This is not counter to general economic theory. It merely places two sets of empirical constraints on it. It implicitly suggests that the understanding of the dynamics of a specific firm in the steel industry requires that one knows considerable physical and financial facts about both the firm and its industry. Yet that alone is not sufficient to describe the dynamics.

in practice. The explanation may be partially given in the words of a distinguished logician who has well and quaintly said, that if a malign spirit sought to annihilate to whole fabric of useful knowledge with the least effort and change, it would by no means be necessary that he should abrogate the laws of nature. The links of the chain of causation need not be corroded. Like effects shall still follow like causes; only like causes shall no longer occur in collocation. Every case is to be singular; every species, like the fabled Phoenix, to be unique. Now most of our practical problems have this character of singularity; every burning question is a Phoenix in the sense of being sui generis." — F.Y. Edgeworth, 1891

In spite of the many successful applications of mathematics to equilibrium economics, the development of economics as a science has a considerable way to go. In particular, as is well known, there is no generally acceptable theory of dynamics. Yet the whole basis of macroeconomics requires economic dynamics.

In the broad sweep of the development of both pure and applied economics, techniques come and go, and fashions change.

This warning by Edgeworth is not an admission of defeat but a call for us to appreciate that the use of dynamics requires an understanding of structure and a realisation that control in a loosely coupled dynamic system is not the same as prediction, but the intelligent placement of constraints to prevent an unpredictable dynamics from getting out of control.

In other words, the manifold degrees of freedom in a modern economy require an intelligent appreciation of the missing institutional and/or political constraints that anchor

the political economy of modern societal organisations. In addition, these institutions permit comparative institutional dynamics and remove the indeterminacy that follows from the classical W-A-D model and its non-institutional structure. To sum up, we can build a political economy that can be flexible enough to survive robustness checks and yet also be analytically tractable.

Recently one of the authors, with colleagues, has been considering the problems involved in incorporating innovation into a financial control structure. Two out of four projected papers have been completed (Shubik and Sudderth [87], [88]). Even in highly simplified low dimensional models a clean proof of the convergence of a low dimensional dynamic programming model is difficult to obtain and the length of time to equilibrium may be of any length, however some simulation results show extreme instability with the introduction of a banking system (Miller and Shubik [71]).

So, in comparison with the standard dynamic stochastic general equilibrium models, models which face up to the reality of default will surely be dynamic, will generally be stochastic, but can only approach generality and do not converge to an equilibrium. Going down this road has some severe disadvantages. For those who find abandoning both the G and the E features of their familiar DSGE models too much of a stretch, we would suggest that the best way forward is that suggested by Curdia and Woodford [19], which is to add an ad hoc, exogenous, but time-varying credit-risk spread to the official interest and then proceed as before. It is a simple addition, and incorporates some of the effects of financial frictions (in an ad hoc manner) into the model.

But it does not incorporate all such effects, being unable, for example, to account for the changing terms and conditions of borrowing, notably the cyclical path of collateral,

margins and leverage, emphasized by Geanakoplos and Fostel [34] Also it fails to account for the, sometimes self-amplifying, interactions between the financial system and the rest of the economy; thus weakness in, say, the housing sector can weaken the banks, and their fragility can impinge back onto the economy (and onto government finance) and so on. Finally, taking financial frictions as exogenously given provides no help or guidance whatsoever for the monetary authorities in their role as guardians of financial stability. By and large economic theory has played little, or no, role to date in informing the design of financial regulation (Goodhart [41]). One of the key functions that Tsomocos and Goodhart have seen for such models is to provide a top-down approach for assessing stresses in financial markets and in helping to design regulatory reform. Our hope is that Central Banks will come to use such models on a regular basis, besides or instead of models which assume away default, financial intermediation and money, just those variables which should lie at the centre of their concerns. One of the consequences of the recent financial crisis was the rehabilitation of the reputation of Hyman Minsky. He had great insight into the workings of the financial system, but he never formalised these into a model which could be developed as a basis for analysis and for regulatory reform. Our own aim is <u>not</u> to revert only to a narrative and historical approach towards finance, but to extend rigorous and mathematical modeling to encompass a wider set of real and behavioural phenomena.

Where do we go from here?

We start from the premise that financial systems and markets are not static (with reversion to some constant equilibrium) but evolving processes, subject to institutional control mechanisms which are themselves subject to socio/political development.

The concept of Mathematical Institutional Economics as an appropriate way to develop the dynamics of a financially controlled economy is consistent with Edgeworth's warning, with precise question formulation and an empirically based construction of ad hoc models dealing with the important specific goals of government regulation of an economy with a high level of functionally useful decentralized market activity. The flexible, but logically consistent and complete, process-oriented models of strategic market games translate, with care, into stochastic process models incorporating default and boundary constraints. In special simple examples, it is possibly to obtain interior solutions and even more rarely analytical solutions, but, in general, even the parsimonious models require approximation or simulation methods to trace the dynamics. But because there is no royal road, this does not mean that there is no road.

Suboptimal Monetary Policy in the Presence of Default

Introduction

I intend to analyze the effects of default in the real economy and its connection to the conduct of monetary policy and macro-prudential policy. I aim to answer the question of whether or not default is a significant factor that can lead to bigger fluctuations in the business cycle when a crisis occurs and when default can be strategic choice by some investors. For this purpose, I aim to build a special case model that explicitly and endogenously incorporates default.

During the crisis of 2008, default turned out to be a major characteristic that played an important role in exacerbating the business cycle and in tightening the borrowing constraints that the agents were facing. In the recent literature of DSGE models that try to explain the business cycle and propose policies to better guard the economy against future crises, default is either absent or it exists but in a rather trivial way. The reason that, most models in a DSGE framework abstract from default is because modelling default in such a framework is not easy and complicates things when you have a lot of other features in the model such as heterogeneous agents, collateral and financial frictions. However, by not assuming the possibility of default, you can never be sufficiently certain that the various policies and regulations that you propose will work as well as you expected or even work at all.

The model builds upon the paper by Peiris and Vardoulakis [77] and consists of a representative household and a central bank. Nominal default is a possible outcome. The households borrow due to a cash-in-advance constraint using capital as collateral, and produce. monetary policy is conducted by the central bank through setting the nominal interest rates. Peiris and Vardoulakis [77] present a model where default is not allowed due to a collateral constraint. In this model, I remove this constraint and seek to see whether the presence of default can affect the efficiency of monetary policy in certain scenarios. I, moreover, compare the two models, when default marginally occurs in my model compared to the case that the collateral constraint just binds in the model by Peiris and Vardoulakis [77].

Related Literature

The importance of default in the economy is the subject of a large literature in macroeconomics. Pradeep Dubey, John Geanakoplos, and Martin Shubik [28] incorporate default in a model of incomplete markets and calculate optimal punishment rates and default rates in a similar vein I model default. Raphäel Espinoza, Charles Goodhart, and Dimitrios Tsomocos [31] show, in a monetary exchange economy, that asset prices in a complete markets general equilibrium are a function of the supply of liquidity by the Central Bank, through its effect on default and interest rates. Inflated asset prices are emerging in an equivalent way that expected inflation increases in my model under default. The following papers are exploring various policy techniques to combat a recession. I mostly verify that indeed default removes the efficiency of monetary policy, and thus, a broader policy mix is required to moderate a recession. Charles Goodhart,

Dimitrios Tsomocos, and Alexandros Vardoulakis [43] explore financial instability due to a housing crisis and defaults on mortgages and conclude that government support to banks in crises stabilises the economy and risk-taking behaviour of banks is procyclical. Charles Goodhart, Anil Kashyap, Dimitrios Tsomocos, and Alexandros Vardoulakis [44] incorporate households that sometimes default on their loans and assess how different types of financial regulation could combat many of the phenomena that were observed in the financial crisis of 2007 to 2009. Charles Goodhart, Pojanart Sunirand, and Dimitrios Tsomocos [46] set out a tractable model which illuminates problems relating to individual bank behaviour, to possible contagious inter-relationships between banks, and to the appropriate design of prudential requirements and incentives to limit 'excessive' risk-taking. They find that both default and liquidity premia co-determine the interest rates. Li Lin, Dimitrios Tsomocos, and Alexandros Vardoulakis [62] incorporate collateral default to examine the effects of monetary policy on total output within a framework of fully flexible prices. The underlying friction is that they allow agents to (endogenously) default on their long-term loan obligations. Thus, the need for collateral to back these loans arises. Finally, Dimitrios Tsomocos [95] extends the canonical General Equilibrium with Incomplete Markets (GEI) model with money and default to allow for competitive banking and financial instability and introduces capital requirements for the banking sector to assess the short and medium term macroeconomic consequences of the proposed New Basel Accord.

The remainder of the paper proceeds as follows. Section 2 introduces the broad features of the model, describes the agents' preferences and optimization problem, introduces the notation used to describe each agent's choices and defines equilibrium. Section

3 solves for the optimization and presents the main equilibrium results of the model. Section 4 concludes.

Model Economy

I consider a stochastic monetary economy as in Peiris and Vardoulakis [77], Nakajima and Polemarchakis [76] and Lucas and Stokey [65]. All markets are competitive and prices are flexible. Monetary policy is conducted through trading a risk-free nominal bond to achieve predetermined interest rate targets. As in Peiris and Vardoulakis [77], I include a durable capital good used for production and we require that bond sales by households are backed by collateral. However, in contrast with Peiris and Vardoulakis [77], there is no collateral constraint and default can occur in the next period if the nominal value of collateral is less that the repayment value on the bond. Moreover, markets are incomplete in this model economy.

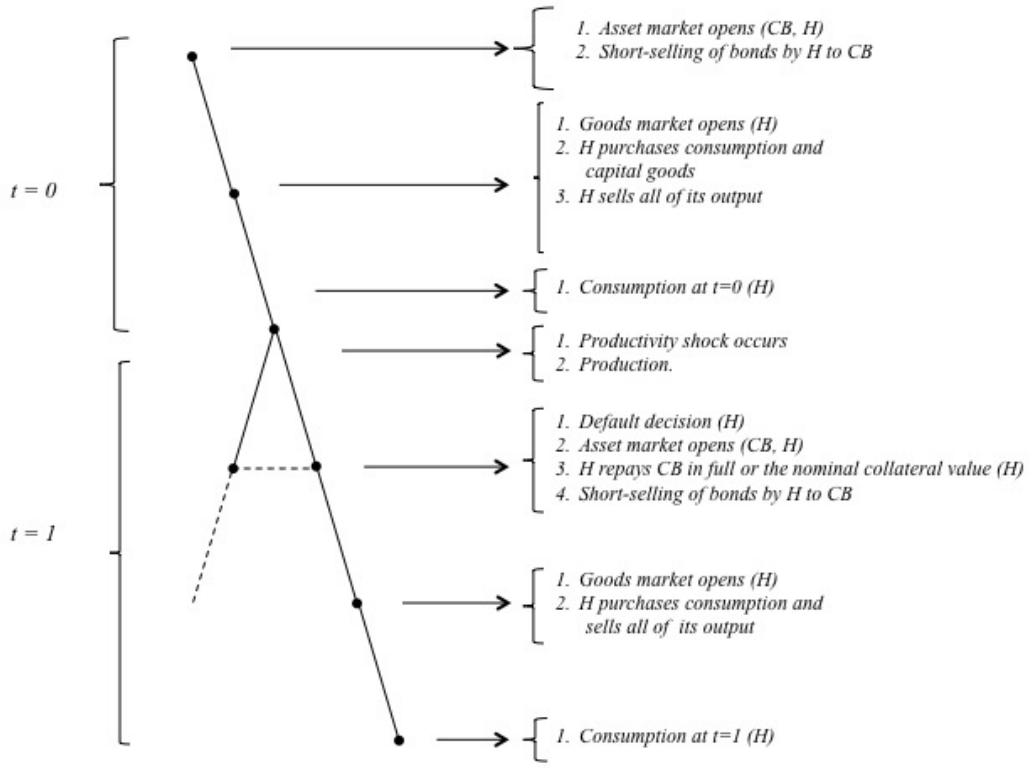
The model consists of three periods, namely, $t = \{0, 1, 2\}$. There are S possible states that can be realized in the second period, each of them, with probability $\pi(s) > 0$, where $s \in \{1, \ldots, S\}$. A stochastic productivity shock, $A_1(s)$, which occurs at the beginning of the second period, differentiates these states. Production takes place in the second period. The third period is added for households and Central Bank to redeem their debt and money (fiat) to move out of the economy.

There is a continuum of identical households, distributed uniformly over [0,1]. Households are endowed with a single homogeneous good in the first period, $y(0)$, and produce

this same good in the second period and at every state, $y_1(s)$, using a decreasing to scale production function. In the first period agents consume some portion of their endowment and invest the rest as capital to be used for production in the second period. Capital and consumption goods are the same good and have the same nominal price. In fact, capital is used as means of transferring wealth to the next period. As production is not present in the third period and the collateral represents a nominal value, this does not affect the variables of the model. Capital depreciates fully after production. Capital is the only factor of production.

Finally, there asset markets and goods markets. In the model, there is a cash-in-advance constraint in operation. Households are required to purchase consumption and capital goods using money and to sell all of their real output in the goods markets only after monetary expenditures on consumption and capital goods, as in Peiris and Vardoulakis [77] and Nakajima and Polemarchakis [76]. This creates a need for money in our model and as a result for asset markets, where money is exchanged for bonds by both the households and the central bank which conduct open market operations. Asset markets open before goods markets and after uncertainty is resolved as in Lucas and Stokey [65]. The only demand for money in the model is transactions demand. As a result, households will hold an end-of-period money balance at least as large as their real output (or endowment) sales because they are required to purchase consumption and capital goods before they receive any cash from their output sales. This is basically the effective cash-in-advance constraint.

The timeline of the model is as follows:



Households

The preferences of the representative household are described by a linear lifetime utility as shown below.

The maximization problem for the households is as follows:

$$max_{c(0),c_1(s),k(1),b_1,b_2(s),m(0),m_1(s)}\{c\left(0\right)+\beta\sum_{s}\pi\left(s\right)c_1\left(s\right)\} \tag{1}$$

subject to

$$\hat{m}(0) \leq w(0) + \frac{b_1}{1 + r(0)} \tag{2}$$

that is, amount of cash required for transactions at $t = 0 \leq$ initial private monetary wealth + date 0 value of the amount of the bond short-sold by the household,

$$p(0) c(0) + p(0) k(1) \leq \hat{m}(0) \tag{3}$$

that is, consumption purchases + capital purchases $\leq$ amount of cash required for transactions at $t = 0$ i.e. this is the cash-in-advance constraint,

$$m(0) = p(0) y(0) + \hat{m}(0) - p(0) c(0) - p(0) k(1) \tag{4}$$

that is, amount of cash transferred by the household to the next period = real output sold sales + amount of cash required for transactions - consumption purchases - capital purchases,

$$\hat{m}_1(s) \leq m(0) - min\{p_1(s) k(1), b_1\} + \frac{b_2(s)}{1 + r_1(s)} \tag{5}$$

that is, amount of cash required for transactions at $t = 1$ and state $s \leq$ amount of cash transferred from period 0 - minimum of monetary value of capital and repayment on the bond + debt purchased at period 1 and state s,

$$\hat{m}_1(s) \geq p_1(s) c_1(s) \tag{6}$$

that is, amount of cash required for transactions at $t = 1$ and state $s \geq$ consumption

purchases in period 1 and state s,

$$m_1(s) = p_1(s)\, y_1(s) + \hat{m}_1(s) - p_1(s)\, c_1(s) \tag{7}$$

that is, amount of cash transferred by the household to the next period (period 2) = real output sales + amount of cash required for transactions at $t = 1$ and state s - consumption purchases in period 1 and state s,

$$b_2(s) \leq m_1(s) \tag{8}$$

that is, debt repayment at period 2 $\leq$amount of cash transferred by the household to the next period (period 2),

$$y_1(s) = A_1(s)\, k(1)^{\alpha} \tag{9}$$

that is, real output at state s = decreasing-to-scale production function ($0 < \alpha < 1$), where

$\hat{m}(0) \equiv$ amount of cash required for transactions at $t = 0$,

$m(0) \equiv$ amount of cash transferred by the household to period 1,

$w(0) \equiv$ initial private monetary wealth at $t = 0$,

$p(0) \equiv$ nominal price of goods and capital in period 0,

$p_1(s) \equiv$ nominal price of goods and capital in state s,

$b_1 \equiv$ face-value of bond,

$\hat{m}(s) \equiv$ amount of cash required for transactions at state s,

$m_1(s) \equiv$ amount of cash transferred by the household to period 2,

$b_2(s) \equiv$ face-value of second period nominally riskless bond,

$r(0) \equiv$ nominal interest rate in period 0,

$r_1(s) \equiv$ nominal interest rate in state s,

$c(0) \equiv$ consumption in period 0,

$c_1(s) \equiv$ consumption in state s,

$y(0) \equiv$real endowment in period 0,

$y_1(s) \equiv$ real output in state s,

$k(1) \equiv$capital purchased in period 0,

$\beta \equiv$ discount factor of the households for next's period consumption.

Substituting for $\hat{m}(0)$ in (2) and (3), using (4), we get

$$m(0) + p(0)\, c(0) + p(0)\, k(1) \leq w(0) + \frac{b_1}{1 + r(0)} + p(0)\, y(0) \tag{10}$$

which is the household period 0 budget constraint and

$$m(0) \geq p(0)\, y(0) \tag{11}$$

which is the effective period 0 cash-in-advance constraint. By doing the same for $\hat{m}_1(s)$, we get

$$m_1(s) + p_1(s) c_1(s) + min\{p_1(s) k(1), b_1\} \leq m(0) + \frac{b_2(s)}{1 + r_1(s)} + p_1(s) y_1(s) \quad (12)$$

which is the household budget constraint in state s and

$$m_1(s) \geq p_1(s) y_1(s) \quad (13)$$

which is the effective period 1 cash-in-advance constraint. Note here, that (8), (11) and (13) will hold with equality given positive interest rates.

Central Bank

The budget constraints for the central bank are defined as follows.

$$M(0) = \frac{B_1}{1 + r(0)} + W(0) \quad (14)$$

that is, money supply in te first period = loans extended to the households in the first period + initial money supply.

$$M_1(s) + min\{p_1(s) k(1), B_1\} = M(0) + \frac{B_2(s)}{1 + r_1(s)} \quad (15)$$

that is, because in period 1, state s, the change in the money supply, $M_1(s) - M(0)$, is funded by the change in value of the portfolio of bonds the central bank holds,

$\frac{B_2(s)}{1+r_1(s)} - min\left\{p_1\left(s\right)k\left(1\right), B_1\right\}$.

$$M_1\left(s\right) = B_2\left(s\right) \tag{16}$$

i.e. central bank receives repayment on its asset holdings, $B_2(s)$, in period 2 after state s has been realized and cancels its existing liability $M_1\left(s\right)$. Namely, money moves out of the economy.

As in Peiris and Vardoulakis [77], we consider that the central bank targets the $s+1$ one period nominal interest rates. Open market operations here are when the central bank trades money for bonds at pre-specified interest rates.

Equilibrium Conditions

The market clearing conditions are as described below.

$$c\left(0\right) + k\left(1\right) = y\left(0\right)$$

that is, consumption in period 0 + investment = period 0 endowment (households are identical of measure 1).

$$c_1\left(s\right) = y_1\left(s\right)$$

that is, consumption in period 1 = production in period 1 (no investment in period 1).

$$m(0) = M(0) \qquad m_1(s) = M_1(s)$$

that is, money balance in the end of each period = money supply in the corresponding period.

$$b_1(s) = B_1(s) \qquad b_2(s) = B_2(s)$$

that is, bond sales by households = bond purchases by the central bank in every period.

Finally, consistency requires that $w(0) = W(0)$.

Definition. Given initial nominal wealth, $w(0) = W(0)$, interest rates policy, $\{r(0), r_1(s)\}$, a competitive equilibrium consists of an allocation, $\{c(0), c_1(s)\}$, a portfolio of households, $\{m(0), m_1(s)\}$, a portfolio of the central bank, $\{M(0), M_1(s), B_1, B_2(s)\}$ and spot-market prices, $\{p(0), p_1(s)\}$, such that

1. The central bank accommodates the money demand, $M(0) = m(0)$ and $M_1(s) = m_1(s)$,
2. given interest rates, $r(0)$, $r_1(s)$, spot-market prices, $p(0)$, $p_1(s)$, the household's problem is solved by $c(0)$, $c_1(s)$, $k(1)$, $m(0)$, $m_1(s)$, b_1 and $b_2(s)$,
3. all markets clear.

Equilibrium Results

In what follows, we will examine the case in which the future states are two and we will consider an equilibrium where default occurs in one of the states. We assume that $S = 2$ and default occurs at $s = 2$, while at $s = 1$ bonds are repaid in full. Namely,

$$p_1(2)\,k(1) \leq b_1 < p_1(1)\,k(1) \tag{17}$$

For this condition to hold, we need strictly positive interest rates. Otherwise, the Friedman's rule obtains.

Lemma 1. *The first order conditions with respect to the bonds and for the optimal investment of capital are given by*

$$\beta\frac{\pi(1)}{p_1(1)} = \frac{1}{p(0)\,[1+r(0)]} \qquad k(1)^{1-\alpha} = \frac{\alpha\beta}{1+\beta\pi(2)}\left\{\frac{\pi(1)\,A_1(1)}{1+r_1(1)} + \frac{\pi(2)\,A_1(2)}{1+r_1(2)}\right\} \tag{18}$$

Proof. Consider the Lagrangian for the problem

$$L = c(0)+\beta\pi(1)\,c_1(1)+\beta\pi(2)\,c_1(2)-\lambda_0\left[p(0)\,c(0)+p(0)\,k(1)-w(0)-\frac{b_1}{1+r(0)}\right]-$$

$$-\lambda_1\left[p_1(1)\,c_1(1)+b_1-p(0)\,y(0)-\frac{p_1(1)\,y_1(1)}{1+r_1(1)}\right]-$$

$$-\lambda_2\left[p_1(2)\,c_1(2)+p_1(2)\,k(1)-p(0)\,y(0)-\frac{p_1(2)\,y_1(2)}{1+r_1(2)}\right]$$

Substituting for $y_1(1)$, and $y_1(2)$ in the Lagrangian and differentiating with respect to consumptions, bond and capital we get

$$1 - \lambda_0 p(0) = 0$$

$$\beta\pi(1) - p_1(1)\lambda_1 = 0$$

$$\beta\pi(2) - p_1(2)\lambda_2 = 0$$

$$\frac{\lambda_0}{1 + r(0)} - \lambda_1 = 0$$

$$-\lambda_0 p(0) + \alpha\lambda_1 k(1)^{\alpha-1} \frac{A_1(1) p_1(1)}{1 + r_1(1)} - \lambda_2 p_1(2) + \alpha\lambda_2 k(1)^{\alpha-1} \frac{A_1(2) p_1(2)}{1 + r_1(2)} = 0$$

which after solving it completes the proof. □

We need to make some assumptions to ensure that the solutions are interior. The assumptions are:

- $A_1(L) = A_1(1) < A_1(2) = A_1(H)$
- $\{\frac{\alpha\beta}{1+\beta\pi(2)}\left[\frac{\pi(1)A_1(1)}{1+r_1(1)} + \frac{\pi(2)A_1(2)}{1+r_1(2)}\right]\}^{\frac{1}{1-\alpha}} < y(0)$

- $\frac{\alpha\beta}{1+\beta\pi(2)}\left[\frac{\pi(1)A_1(1)}{1+r_1(1)}+\frac{\pi(2)A_1(2)}{1+r_1(2)}\right]<A_1(L)$

This first assumption implies that default actually occurs in the high productivity state. This might seem counter-intuitive at the first glance but it's not, in our case, since the collateral is monetary. For the households to decide to default, prices need to fall significantly for the value of capital to fall and become less than the face-value of the bonds. Due to the quantity theory of money, this is more likely to occur in the high productivity state where real output is higher. If we assume that capital is the same, and money supply of similar magnitude, in the higher productivity state, output will be higher and thus, prices lower.

In this model there is no indeterminacy, due to the fact that we have a single bond. The model can be solved and give a unique path of prices and money supplies.

Now, let us consider the first order condition for capital

$$k(1)^{1-\alpha}=\frac{\alpha\beta}{1+\beta\pi(2)}\left\{\frac{\pi(1)A_1(1)}{1+r_1(1)}+\frac{\pi(2)A_1(2)}{1+r_1(2)}\right\}$$

The above equation implies clearly (by assumption interest rates are strictly positive) that

$$k(1)^{1-\alpha}<\alpha\beta\left[\pi(1)A_1(1)+\pi(2)A_1(2)\right]$$

where the right hand-side of the above equation is the optimal value for $k(1)^{1-\alpha}$ in an economy without cash. This means basically that nominal default causes real effects in our economy. Households underinvest due to possibility of default, which in turn makes borrowing more expensive. Moreover, monetary policy through controlling the interest

rate cannot fix for this, since even by setting the interest rates of the second period to 0 capital will still be less than its optimal value.

Proposition 1. *The central bank cannot achieve the first best allocation.*

What this result shows, is that monetary policy is insufficient when default is present. Namely, we need another tool to prevent the negative effects of default.

As we have mentioned earlier, interest rates should be strictly positive for default to happen. In this example that we solve, we consider an equilibrium that default occurs, and we can do this only because interest rates are positive. In a different case, the unique equilibrium that exists is the one that Friedman's rule obtains. So, monetary policy is insufficient, given the positive interest rates.

Proposition 2. *Default can be an equilibrium outcome, only when interest rates are positive. When the interest rates are zero, default cannot happen and Friedman's rule obtains.*

In what follows, I aim to compare the above model at the point where agents start to default marginally at the second state with the model by Peiris and Vardoulakis [77] where their collateral constraint just binds. We assume in this comparison that all of the parameters and the nominal interest rates are the same so as the comparison to make sense. Note, also, that parameters need to be suitably picked such that all of the equations are satisfied.

Proposition 3. *Denote by $I_d(s)$ inflation in period 1, state s $(s = 1, 2)$ when households default marginally at state 2 and by $I(s)$, inflation in period 1, state s $(s = 1, 2)$ when the collateral constraint just binds (no default) at state 2. Then,*

$$I_d(s) > I(s) \quad for \quad s = 1, 2$$

Proof. To prove the above, we aggregate budget constraints and first order conditions for both models and then we solve for prices by playing around with equations. Denote with a d superscript different variables for the default equilibrium.

In the case without default and a collateral constraint that just binds, we have

$$k(1)^{1-\alpha} = \alpha\beta \left[\frac{\pi(1)A_1(1)}{1 + r_1(1)} + \frac{\pi(2)A_1(2)}{1 + r_1(2)} \right]$$

which is the first order condition for capital in the just-binding case.

$$b_1 = p_1(2)k(1)$$

which holds since the constraint just binds. The budget constraints after substituting suitably, take the following form

$$p(0)y(0) = w(0) + \frac{b_1}{1 + r(0)}$$

$$\frac{r_1(1)p_1(1)A_1(1)}{1 + r_1(1)} k(1)^{\alpha} + b_1 = p(0)y(0)$$

$$\frac{r_1(2)p_1(2)A_1(2)}{1 + r_1(2)} k(1)^{\alpha} + b_1 = p(0)y(0)$$

Denote

$$K = \frac{\pi(1)A_1(1)}{1 + r_1(1)} + \frac{\pi(2)A_1(2)}{1 + r_1(2)}$$

and let

$$X_1 = \frac{(1 + r_1(1))\,(1 + r(0))\,w(0)}{r_1(1)A_1(1)\,(\alpha\beta * K)^{\frac{\alpha}{1-\alpha}}}, \qquad Y_1 = \frac{r(0)y(0)\,(1 + r_1(1))}{r_1(1)A_1(1)\,(\alpha\beta * K)^{\frac{\alpha}{1-\alpha}}}$$

and

$$X_2 = \frac{(1 + r_1(2))\,(1 + r(0))\,w(0)}{r_1(2)A_1(2)\,(\alpha\beta * K)^{\frac{\alpha}{1-\alpha}}}, \qquad Y_2 = \frac{r(0)y(0)\,(1 + r_1(2))}{r_1(2)A_1(2)\,(\alpha\beta * K)^{\frac{\alpha}{1-\alpha}}}$$

Then, by using the above equations, we have that

$$p(0) = \frac{X_2\,(\alpha\beta * K)^{\frac{1}{1-\alpha}} + w(0)\,(1 + r(0))}{y(0)\,(1 + r(0)) + Y_2\,(\alpha\beta * K)^{\frac{1}{1-\alpha}}}, \quad p_1(s) = X_s - p(0)Y_s,\ for\ s = 1, 2 \tag{19}$$

and hence

$$I(s) = \frac{X_s}{p(0)} - Y_s, \quad for\ s = 1, 2 \tag{20}$$

In the model with default where agents start to default marginally, thinngs are similar although slighly different. We have

$$k^d(1)^{1-\alpha} = \frac{\alpha\beta}{1 + \beta * \pi(2)} \left[\frac{\pi(1)A_1(1)}{1 + r_1(1)} + \frac{\pi(2)A_1(2)}{1 + r_1(2)} \right]$$

which is the first order condition for capital.

$$b_1^d = p_1^d(2)k^d(1)$$

which holds since the agent defaults marginally. The budget constraints after substituting suitably, take the following form

$$p^d(0)y(0) = w(0) + \frac{b_1^d}{1 + r(0)}$$

$$\frac{r_1(1)p_1^d(1)A_1(1)}{1 + r_1(1)}k^d(1)^\alpha + b_1^d = p^d(0)y(0)$$

$$\frac{r_1(2)p_1^d(2)A_1(2)}{1 + r_1(2)}k(1)^\alpha + b_1^d = p^d(0)y(0)$$

Denote

$$\lambda = 1 + \beta * \pi(2)$$

Again, now, by substituting for capital in the budget constraints and solving for prices, we have

$$p^d(0) = \frac{X_2\left(\alpha\beta * K\right)^{\frac{1}{1-\alpha}} + \lambda * w(0)\left(1 + r(0)\right)}{\lambda * y(0)\left(1 + r(0)\right) + Y_2\left(\alpha\beta * K\right)^{\frac{1}{1-\alpha}}}, \quad p_1^d(s) = \lambda^{\frac{\alpha}{1-\alpha}}\left(X_s - p^d(0)Y_s\right), \ for\ s = 1, 2 \tag{21}$$

and hence

$$I^d(s) = \lambda^{\frac{\alpha}{1-\alpha}}\left(\frac{X_s}{p^d(0)} - Y_s\right), \quad for \quad s = 1, 2 \tag{22}$$

Now, from (19) and (21), we can easily see that

$$p^d(0) < p(0)$$

since

$$\lambda > 1$$

which implies that

$$\frac{1}{p^d(0)} > \frac{1}{p(0)}$$

and thus, from (20) and (22), we have that

$$I^d(s) > I(s) \quad for \quad s = 1, 2$$

as desired. □

The above proposition namely tells us that due to the the positive probability of default, inflation rises. The welfare of the representative agent is negatively affected. Thus, the strategic default choice is reducing welfare. The central bank is not suffering any losses, because they set sufficiently high margins. Only the households are negatively affected due to default because of the suboptimal inter-temporal allocation of resources between the two periods. The agents, though, are still defaulting since they cannot commit not to default. It is the lack of the ability to commit that creates the distortion from the first best allocation in the model.

The last proposition gives rise to the following corollaries.

Corollary 1. *Expected inflation is higher under the presence of default.*

Proof. We have

$$\mathbb{E}[I] = \pi(1)I(1) + \pi(2)I(2), \quad \mathbb{E}\left[I^d\right] = \pi(1)I^d(1) + \pi(2)I^d(2)$$

and thus, by proposition 3, we conclude that

$$\mathbb{E}\left[I\right] < \mathbb{E}\left[I^d\right]$$

which completes the proof. □

Again, the positive probability of default, gives rise to higher expected inflation compared to the model by Peiris and Vardoulakis [77]. The main contribution of the paper, thus, is the fact that default, even when it is nominal, and there is only a representative agent, can affect negatively agents' welfare by creating a suboptimal inter-temporal allocation, due to the lack of agents' ability to commit not to default.

Corollary 2. *Real interest rates in the second period are lower under the presence of default.*

Proof. Denote by capital R the real interest rates, then by Fisher's rule, we have that

$$r_1(s) = R(s) + I(s)$$

and

$$r_1(s) = R^d(s) + I^d(s)$$

Since $I^d(s) > I(s) \quad for \quad s = 1, 2$, we have that

$$R^d(s) < R(s) \quad for \quad s = 1, 2$$

as desired. □

This final result, is related to the fact that, due to default, agents underinvest. Note here, that the suboptimal allocation rises through underinvestment in the first period as we have shown earlier. This underinvestment, in turn, results in lower real interest rates in the second period.

Concluding Remarks

I have presented a model and I examined the effects that monetary policy can have in the economy under the presence of default and whether it can achieve the optimal allocation. Peiris and Vardoulakis [77] have found that in a collateral constrained economy, monetary policy actually can anticipate the effects of underinvestment caused by the collateral constraint and achieve the optimal allocation by setting the interest rate. Increasing the interest rate relaxes the collateral constraint and decreasing it causes it to bind and increases investment to the optimal level. However, in my model with collateral and default, we saw that monetary policy cannot achieve the optimal investment levels even if they set the interest rates equal to zero, meaning that default has indeed real effects in a monetary economy. This basically happens because there is the probability of a high productivity shock which in turn will push prices down via a quantity theory of money argument. At the same time, agents repay the monetary value of their capital because they choose to default (which at that time is optimal), the amount that they repay is increasing in capital and thus, they have less cash for consumption. Because they purchase consumption before they sell all of their output, they need to borrow more and this pushes interest rates up due to the funding need of

agents and money supplies being exogenous. Thus, this creates a trade off for capital investment for the agents. Monetary policy cannot fix for this trade off because agents cannot commit to repay.

Endogenous Default in Continuous Time Finance

Introduction

The collapse of Lehman Brothers in September 15, 2008, and the subprime loan crisis in the US, put into doubt the existing mainstream macroeconomic literature and have re-established interest in studying financial stability. In the modern literature, most mainstream macro models are built in a discrete time framework as well as they have assumed away financial frictions such as default [42]. The main reason that most dynamic stochastic general equilibrium models do not incorporate default is due to the fact that modelling default along with different elements such as financial frictions, heterogeneity etc. amplifies the computational complexity. However, if we decide to eliminate default, policy makers and regulators can never be sure that their economic proposals will have the expected effect on the economy, or even have an effect at all. Furthermore, the absence of default in the classical DSGE literature highlights further inconsistencies, such as the lack of any process to enforce repayment [42]. Rationality

must provide the right to all agents to default in cases where the utility advantage from doing so are greater than the default value (strategic default). In continuous time finance, endogenous default is absent. As researchers, there is no doubt that a continuous time setting provides a variety of advantages over discrete time. There are three main advantages that continuous time provides over discrete time in modelling default. First, it ensures neat first order optimality conditions. In particular, a continuous time setting ensures, at large, the benefit of tractability. Tractability here means that we might also be capable to derive more analytical steps and reach a characterization of the equilibrium, which conceptually is closer to a closed-form characterization, before moving on to perform simulations and numerical approximations. Most discrete-time models require log-linear approximation around the steady state. Thus, it can be argued that continuous time modelling would possibly also replicate reality in a better way. For instance, agents do not consume solely at the cease of the month or the cease of the quarter. Linear time aggregation within a quarter and a non-linear throughout quarters is implicitly assumed in continuous time modelling [13]. That means that inside a quarter, we obtain a limitless inter-temporal elasticity of consumption, however across quarters we obtain it by way of discovering the curvature of the objective function. In continuous time setting, each unit of time is handled in the same way. Thus, log-linearization can be avoided. Second, this framework allows for a more comprehensive study and comparative statics for the default amplification effect on the economy. Third, the default amplification effect leads to a market inefficiency, which then calls for policy and regulation which, can be well evaluated, under this framework in future research. In our model, we incorporate endogenous default, in the form of a repayment rate, at a cost of a quadratic penalty that is increasing in leverage and interest rate and

decreasing in the repayment rate. We conclude that default is non-neutral, suppresses investment, increases borrowing costs and leads to a recession. We model the cyclical effect that default can have in the economy. When default probability increases, borrowing rates go up, which in turn drives the capital excess return negative, which leads to less output and more default. We manage to quantify the default's impact on investment, as well as, capture its second order effects. Continuous time gives also the ability to price CDS (Credit Default Swaps) as well as evaluate the effects of monetary policy and macro-prudential regulation. In the subsequent sections, we will present the model, obtain its solution, and run simulations to obtain results and compare and contrast our model to the no default case. More specifically, section 12 presents the model, section 13 solves for the equilibrium and provides some important theorems, section 14 provides an algorithm to solve a numerical example and presents and discusses the results and finally, section 15 concludes. Before moving to section 12, we will present a short related literature review.

Related Literature

We will present a model which extends the work of Brunnermeier and Sannikov [11] via introducing default on a real bond that is exchanged between the two main agents present in our economy. In our model, we differentiate, by way of additionally incorporating a leverage constraint as in Jing Chen [16], and endogenising the interest rate. Default comes with a cost which is quadratic in leverage, repayment rate and interest rate. Our goal is to address the relationship between default and the business cycle. Below, we will present some literature that is related to our model, either through

default or through a similar continuous time setting.

Default Literature

Regarding the default literature, the following have been researched so far. Bernanke, Gertler and Gilchrist [7], default is modelled but in a naive way. In particular, a fraction $(1-a)$ of entrepreneurs defaults in the model, where a is exogenously given. In this model, the most important reason that companies default in equilibrium is that return to capital can be hit by means of both combination and idiosyncratic risk. In Curdia and Woodford [19], every period, every impatient household chooses no longer to repay the mortgage to the bank. The whole quantity of default is exogenous in the model as well as banks are aware of it. In Christiano, Motto and Rostagno [18], failure to repay a loan has comparable penalties as in the aforementioned BGG model. Goodhart, Kashyap, Tsomocos and Vardoulakis [44], incorporate a genuinely endogenous default mechanism. In this model, households that have benefited from personal loans in the past, fail to fulfil their price tasks if the unfavourable conditions occur. As a result, assets are offered at decreased prices and there is the possibility of non-payment on the repo loans awarded through the shadow banks. Bianchi and Mendoza [9] use a collateral default framework. Consequentially, failure to repay the loan is present only as an opportunity, which does not in fact occur. Our modelling idea for default (which is additionally called "lambda" default) was first introduced in Shubik [83]. The difference in our model is that default is quadratic. This assists us to model it better in continuous time. Shubik and Wilson [90] first introduced default into a general equilibrium model. They also introduced default penalties to represent the

idea that the banking sector might only imperfectly be able to enforce the repayment of its loans. In this formulation, an agent suffered a loss of utility proportional to the number of dollars of his unpaid bank loans. Dubey and Geanakoplos [24] considers a banking sector in a monetary economy and investigates the existence of equilibrium when there are finite bankruptcy penalties. Dubey, Geanakoplos and Shubik [27] and Geanakoplos [35] explain that GEI (General Equilibrium with Incomplete markets) can accommodate default and bankruptcy, while preserving perfect competition, rational expectations, and the general existence of equilibrium. Martinez and Tsomocos [68] developed a model that includes agent heterogeneity, liquidity and endogenous default in a pure exchange economy and they address financial stability and policy questions.

Continuous Time Literature

In regards to continuous time macro-finance, the following areas have been explored so far. He and Krishnamurthy [52, 53] are two very crucial papers in the literature as Basak and Cuoco [6] where they form the basis where the following papers on the literature build upon. Di Tella [94] investigates the shrinkage of the stability sheet precipitated by exogenous uncertainty shocks, even in cases when contracting is possible. A framework in which the pro-cyclicality of leverage is possible and in which banks issue equity, is viewed via Phelan [78]. Liquidity preference shocks, is the key thing to gain pro-cyclical leverage in Adrian and Shin [4]. The interplay of two types of intermediaries, namely banks and nonbanks, is considered via Adrian and Boyarchenko [3]. Money-like claims are allowed to be issued through shadow banks in Moreira and Savov [73]. In recessions, their activity has to be scaled down, which decreases the recovery rate and a scarcity

in collateral is created. Misallocation of capital is found in Moll [72] because collateral constraints restrict the productive agents to lever up. There is also a collection of research papers in continuous time macro-finance concerning monetary economics. For instance, in Brunnermeier and Sannikov [12], money is behaving like a bubble such as in Bewley [8] and Samuelson [80]. In this paper, the banking sector creates inside cash endogenously and there is an interplay between monetary and macroprudential policy. Furthermore, in Achdou [1], an algorithm that solves Bewley models with uninsurable endowment risk in continuous time is presented. Finally, Drechsler [23] suggests how risk premia is affected through economic coverage in a monetary economy had been banks are less risk averse. Jing Chen [16] investigates the addition of leverage constraint to a continuous time setting following Brunnermeier and Sannikov [11].

The Model

In this section we consider a simple baseline model, which extends brunnermeier and Sannikov [11] and Jing Chen [16], by introducing endogenous default, with two types of agents: households and firms managed by experts, in which productive agents, experts, can finance their projects only by issuing risky debt. Both types of agents can own capital, but firms are able to manage it more productively. There are two type of goods in this economy, one that can be used for consumption or investment and physical capital that is used for production. All prices are expressed in terms of the consumption good that serves as numeraire.

Technology

The agents of the economy use physical capital to produce output in the form of consumption goods. The production function utilised by an expert at any point, t, in time is given by

$$y_t = Ak_t \tag{23}$$

where A is a positive constant.

Physical capital held by firms incurs adjustment costs and evolves as follows

$$\frac{dk_t}{k_t} = (\phi(\iota_t) - \delta)\, dt + \sigma dZ_t \tag{24}$$

where $\{Z_t,\, t \geq 0\}$ is an exogenously given standard brownian motion which acts as a shock process that can be thought of changes in expectations about future productivity of capital and ϕ is a function satisfying

$$\phi(x) = \frac{1}{\theta}\left(\sqrt{1 + 2\theta x} - 1\right) \qquad \phi(0) = 0,\ \phi'(x) > 0,\ and\ \phi''(x) < 0 \tag{25}$$

ι_t is the investment rate i.e. the firm decides to reinvest an amount of $\iota_t k_t dt$ in the interval of time $(t,\, t + dt)$ and δ is the depreciation rate of capital. The above functional form satisfies the required convexity assumptions and makes the model tractable. Proposition 5 and later Proposition 8 use this functional form to calculate investment and the price of capital in a closed form.

The production function utilised by a household at any point, t, in time is given by

$$\underline{y}_t = \underline{A}\,\underline{k}_t \tag{26}$$

with $\underline{A} \leq A$.

Finally, physical capital held by households incurs adjustment costs and evolves as follows

$$\frac{d\underline{k}_t}{\underline{k}_t} = \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta}\right) dt + \sigma dZ_t \tag{27}$$

Firms, that are superior in technology, since they are managed by experts who are more productive, produce A units of consumption good per unit of time and per unit of capital. Households that are less productive than the firms, can produce $\underline{A}$ units of consumption good per unit of time and per unit of capital.

Firms

Firms are managed by experts i in the economy, and are of unit measure lying on the interval $[0, 1]$. For simplicity, we drop the indices i throughout the paper unless needed. Since they are managed by experts, they are able to utilize the full capacity of the production technology and produce at a rate of $Ak_t dt$ at the interval of time $(t,\, t + dt)$ where k_t is the capital held by a representative firm. To do so, firms need to raise funds and they do this either internally or externally. There is plenty of literature like Myers and Majluf [75] that illustrates that firms prefer internal financing to external

financing due to substantial costs that firms face when they raise external financing due to asymmetric information between the agents. In this model, the firms will retain part of their earnings in order to reinvest them, due to the credit constraint that they face. They will raise external funds from the households by selling risky bonds to them. Let assume that d_t is the amount of bonds that the firm sells to the households, and let n_t denote the firm's networth then the following identity holds

$$q_t k_t = d_t + n_t \tag{28}$$

where q_t is the price of capital.

In sum, firms retain some of their earnings to reinvest in capital, sell risky bonds to the households to obtain leverage and use their networth along with the leverage obtained by the households to purchase further physical capital and produce.

The evolution of the firm's networth depends on the earnings that the firm made, the investment, the evolution of capital (i.e. the new capital created and the debt interest payments. The firm might decide to default partially on its debt face value and interest payments. if it does default then the firm faces a penalty that is subtracted from its objective function. This will be discussed further below.

At every time interval $(t,\, t + dt)$, the firm has to repay some accrued interest. Based on equation (28), the accrued interest is

$$(q_t k_t - n_t)\, dR_t \tag{29}$$

where dR_t is the interest payment per unit of consumption good borrowed by the firm. However, the firm might decide to partially default and repay instead, on interest

$$v_t \left(q_t k_t - n_t\right) dR_t \tag{30}$$

where v_t is the repayment rate satisfying $0 \leq v_t \leq 1$. If the repayment rate equals zero, then the firm defaults fully on the interest payment and if it is equal to one then, the firm repays the full amount. The firm also defaults on the face value of the bonds by using the same repayment rate and in this case it repays $v_t d_t$ instead of d_t.

It's worth mentioning here, that the model is fully equivalent to the case of a perpetual bond instead of an instantaneous bond. The equivalence is immediate, as the extra term emerging by allowing default on the face value adds only a bounded multiple of d_t, thus rescaling the face value. The reason, we use this instead of a perpetual bond is that perpetual bonds are rarely issued in practice, and when they do, they are usually called earlier.

Given the above, we can conclude that the firm's networth evolves as follows

$$dn_t = Ak_t dt - \iota_t k_t dt + d\left(q_t k_t\right) - v_t \left(q_t k_t - n_t\right) dR_t - v_t d_t dt + d_t dt \tag{31}$$

We postulate that the price of capital evolves as follows

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t \tag{32}$$

Now, by applying Ito's product rule[9] to $d(q_t k_t)$ and using equation (24) and equation (32) we obtain that

$$d\left(q_t k_t\right) = q_t dk_t + k_t dq_t = \left(\phi\left(\iota_t\right) - \delta\right) q_t k_t dt + k_t dq_t + \sigma q_t k_t dZ_t =$$

$$= q_t k_t \left[\left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t\right] \tag{33}$$

Now, this is the capital gains rate of the firms. The total risk the face consists both of the fundamental risk of capital σ and of the endogenous risk due to the financial frictions in the economy σ_t^q. Capital also generates an extra dividend term due to the output remaining after reinvestment. This in addition to the capital gains rate gives the total return on capital. Hence, we obtain,

Corollary 3. *The total return on capital for firms is given by*

$$dr_t^k = \frac{A - \iota_t}{q_t} dt + \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t \tag{34}$$

Now, if we define $x_t = \frac{q_t k_t}{n_t}$ as the leverage ratio of the firm and $dR_t = r_t dt$, then, the firm's networth evolves as follows

$$\frac{dn_t}{n_t} = x_t dr_t^k - v_t\left(x_t - 1\right)\left(1 + r_t\right) dt + \left(x_t - 1\right) dt \tag{35}$$

Furthermore, the firm faces a credit constraint ala Kiyotaki and Moore [58] and is given

[9] Ito's product rule states that $d(XY) = XdY + YdX + dXdY$.

by

$$mq_t k_t \leq min\{n_t,\, mq_t K_t\} \tag{36}$$

where K_t is the total capital in the economy at time t i.e. $K_t = k_t + \underline{k}_t$ (its evolution is given in the clearing section)[10] and if we rearrange and use also the leverage definition we obtain

$$x_t \leq min\{\frac{1}{m},\, \frac{q_t K_t}{n_t}\} \tag{37}$$

where m is positive, exogenously given and less or equal than one.

Finally, we need to state the objective function that the firm maximizes. Following He and Krishnamurthy [52, 53] and Adrian and Boyachenko [3] we postulate that the firm is choosing its strategy such as to maximize the rate of return on its wealth at every period minus a return penalty which can be thought of as a partial confiscation of wealth. Basically, this means, that the experts re-invest all of their wealth in the firm and they do not consume anything (E.g. Amazon). Hence, firms maximize

$$max_{\iota_t,\, x_t,\, v_t} \left\{ E\left[\frac{dn_t}{n_t}\right] - \lambda max\{0,\, (1-v_t)\}^2 I\left(x_t \geq 1\right) x_t^2 \left(1+r_t\right)^2 dt \right\} \tag{38}$$

subject to the dynamic budget constraint i.e. networth's evolution given by equation (35) and the leverage constraint (37).

[10]The total capital of the economy is known to the expert before the clearing. This is because the shocks for the individual capital evolution for both agents are fixed to the same value. and the different parameters in this evolution are know and fixed as well. The investment is also known because of rational expectations and is not specified through clearing. Moreover, the price of capital is taken as given in the maximisation of both agents. Thus, our constraint is valid. This constraint, only, says that once the expert has enough wealth to purchase all the capital, she can not leverage any further.

Therefore, experts maximize their instantaneous return minus the default penalty. This is not the standard way of defining a default penalty. Normally, it would be a cost subtracted from a lifetime utility or from the absolute value of utility rather than a return. However, it is a more realistic way of modelling and at the same time, allows for tractability in the continuous time setting. A penalty deducted from a return is being charged in the same way as an income tax. We cannot measure robustness, as, most other methods are not tractable, however, this objective function provides a rather accurate depiction of the dynamics of the business cycle.

where λ is an exogenously given positive constant. The penalty here is convex in the interest rate, the total amount and the default (i.e. concave in the repayment rate). This means that while the amount upon which thr agent defaults is increasing, the marginal penalty is also increasing. The quadratic form of the penalty also assists in getting as an equilibrium solution a continuum of possible repayment rates, which measure the size of default. A linear form would lead to a repayment rate of either one or zero, which would remove the dimension of default's size from the model and therefore its impact. Thus, a quadratic penalty is more realistic. Of course, any other non-linear functional form would satisfy the above, but for the sake of simplicity as well, we pick the quadratic form. Furthermore, we assume there are some fixed costs associated with default. We guarantee this by making sure, that, once, the repayment rate becomes different than 1, the penalty has a positive value for a fixed interest rate even if the leverage is fixed at 1, namely, the firm has not issued any bonds to the household. We interpret these costs as fixed default fees i.e. court fees. We will later prove that in equilibrium, we must have $x_t \geq 1$, i.e. the firm never lends the household.

The goal of this model is to extend Brunnermeier and Sannikov [11] by firstly, endogenising the interest rate by adding a leverage constraint in a similar fashion as in Jing Chen [16], and secondly, by adding default and examining its effects on the economy relative to the no default equilibrium.

Households

In the model economy, there is a second group of agents, namely, the households j that act as the credit providers. Households are of unit mass and lie in the interval $(1, 2]$. Again, for simplicity, we drop the indices j throughout the paper except if needed. Households are distinguished from firms, due to the fact that they lack the good technology to produce. As a result, they are less productive and this motivates them to invest their funds in the firms in the form of debt. Households, therefore can do two things: either invest their in the firms by extending credit to them or produce directly themselves at the cost of lower productivity. Households, when they decide to produce, produce at a rate of $\underline{A}\underline{k}_t dt$. As for the firms, households decide to reinvest a fraction of their capital i.e. $\underline{\iota}_t \underline{k}_t dt$ over the time interval $(t, t + dt)$. Now, we assume that an amount $\phi(\underline{\iota}_t)\underline{k}_t$ of new capital is generated in the time period $(t, t + dt)$ and thus, the capital that a representative household owns evolves according to

$$\frac{d\underline{k}_t}{\underline{k}_t} = \left(\underline{\phi}\left(\underline{\iota}_t\right) - \underline{\delta}\right) dt + \sigma dZ_t \tag{39}$$

In equilibrium, households are always credit providers and thus, we have the following

identity

$$\underline{n}_t = \underline{d}_t + q_t \underline{k}_t \tag{40}$$

where $\underline{d}_t$ is the face value of the bonds that the household purchases.

The household's networth evolves as follows

$$d\underline{n}_t = \underline{A}\underline{k}_t dt - \underline{\iota}_t \underline{k}_t dt + d(q_t \underline{k}_t) + v_t \left(\underline{n}_t - q_t \underline{k}_t\right) dR_t + v_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \tag{41}$$

where $\underline{A}\underline{k}_t dt$ is the output that the household produced in the period $(t, t+dt)$, $\underline{\iota}_t \underline{k}_t dt$ is what the household decides to invest in the same period, $d\left(q_t \underline{k}_t\right)$ is the change in the value of capital held by the household, $v_t \left(\underline{n}_t - q_t \underline{k}_t\right) dR_t$ is the amount of money that the household receives as an interest repayment on the loan that it extends to the firms (which is subject to a partial default), $v_t \underline{d}_t dt$ is the amount that the household receives on the face value of the bonds as a repayment (which is also subject to the same partial default), $\underline{d}_t dt$ is the face value extended and $\underline{c}_t dt$ is consumption.

Now, in a similar fashion as for the firms, by applying Ito's product rule[11] to $d(q_t \underline{k}_t)$ and using equation (27) and (32), we obtain that

$$d(q_t \underline{k}_t) = q_t d\underline{k}_t + \underline{k}_t dq_t = \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta}\right) q_t \underline{k}_t dt + \underline{k}_t dq_t + \sigma q_t \underline{k}_t dZ_t =$$

$$= q_t \underline{k}_t \left[\left(\phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t\right] \tag{42}$$

which is the capital gains rate for households and if we add the dividend term as well

[11] Ito's product rule states that $d(XY) = XdY + YdX + dXdY$.

we obtain the total return on capital.

Corollary 4. *The total return on capital for the households is given by*

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t} dt + \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t \tag{43}$$

We assume that households are utility maximizers, they plan their consumption and investment decisions accordingly and they maximize a log utility that is exponentially discounted with a discount rate of γ. Namely, they maximize

$$max_{\underline{\iota}_t, \underline{x}_t, \underline{c}_t} E\left[\int_0^{+\infty} e^{-\gamma t} log\left(\underline{c}_t\right) dt\right] \tag{44}$$

subject to

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t d\underline{r}_t^k + v_t(1 - \underline{x}_t)(1 + r_t)dt - \left(1 - \underline{x}_t\right) dt - \frac{\underline{c}_t}{\underline{n}_t} dt \tag{45}$$

where $\underline{x}_t = \frac{q_t \underline{k}_t}{\underline{n}_t}$ is the leverage ratio of the household (which is less or equal to 1 in equilibrium). The households are rational agents and take the repayment rate from the experts as given.

Equilibrium

In this section, we will proceed by stating the market clearing conditions and defining the equilibrium. We will, then, proceed into analysing the equilibrium properties and the repercussions for the economy.

Market Clearing

There are three markets in the model that need to clear, capital, consumption and debt market.

Capital Market

The total capital held by the firms plus the total capital held by the households should be equal to the total supply of capital i.e.

$$\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj = K_t \tag{46}$$

where K_t is the total supply of capital. We should mention here that the total supply of capital in the model is not fixed since both agents create new capital through their production. The following equation describes the evolution of the total supply of capital.

$$dK_t = \left(\int_0^1 \left(\phi \left(\iota_t^i \right) - \delta \right) k_t^i di + \int_1^2 \left(\phi \left(\underline{\iota}_t^j \right) - \underline{\delta} \right) \underline{k}_t^j dj \right) dt + \sigma K_t dZ_t \tag{47}$$

Debt Market

Debt market clearing condition says that the face value of the debt that the firm receives should equal to the face value of the debt that the household extends, namely,

$$\int_0^1 \left(n_t^i - q_t k_t^i \right) di + \int_1^2 \left(\underline{n}_t^j - q_t \underline{k}_t^j \right) dj = 0 \tag{48}$$

Consumption Market

Finally, if the capital and debt market clear then the consumption market also clears, i.e.

$$\int_0^1 k_t^i \left(A - \iota_t^i\right) di + \int_1^2 \underline{k}_t^j \left(\underline{A} - \underline{\iota}_t^j\right) dj = \int_1^2 \underline{c}_t^j dj \tag{49}$$

Agents in this economy are atomistic perfectly competitive price-takers, and thus the distribution of wealth among themselves does not matter as previously implied by dropping the i's and j's. however, the the fraction of the wealth of the firms over the total wealth in the economy matters and is actually very crucial in the model. Finally, default via the repayment rate v_t and the default penalty λ is implicitly present in all of the above market clearing conditions.

Definition. An equilibrium in this economy consists of a set of stochastic processes: the process of physical capital price $\{q_t, t \geq 0\}$, the interest rate $\{R_t, t \geq 0\}$, the investment, consumption and default decisions for both agents i.e. $\left\{\left(k_t^i, \iota_t^i, v_t^i\right), t \geq 0\right\}$ for the experts (firms) and $\left\{\left(\underline{k}_t^j, \underline{\iota}_t^j, \underline{c}_t^j\right), t \geq 0\right\}$ for $i \in [0,1]$ and $j \in (1,2]$. These processes should satisfy the following two conditions:

1. Given the price processes q_t and R_t, each firm $i \in [0,1]$ and each household $j \in (1,2]$ maximize their objective functions (38) and (44) respectively with the set of choice variables $\left\{\left(k_t^i, \iota_t^i, v_t^i\right), t \geq 0\right\}$ and $\left\{\left(\underline{k}_t^j, \underline{\iota}_t^j, \underline{c}_t^j\right), t \geq 0\right\}$, respectively.

2. Capital, consumption and debt market clears.

Equilibrium Analysis

There are three main frictions that drive the entire dynamics of the economy. The first one is is the different productivity between experts and households and the fact that the experts will start with zero endowment of capital and the households with all the capital. Thus, this creates a need for lending. The second, is the leverage constraint that the experts face. When the experts are relatively poor compared to the entire economy, the cannot borrow as much as they would ideally like Brunnermeier and Sannikov [11] and thus won't hold the total amount of capital, which would be optimal as they are more productive. The third friction in the economy is default. Default is strategic in our model and is driven both by the relative wealth of experts and thus, by experts' leverage and by the level of the interest rate. From the above, it turns out that the wealth of experts relative to the entire economy plays a crucial role in the dynamics. We continue with some definitions.

Definition 1. The total wealth of the experts and the households is given, respectively, by

$$N_t = \int_0^1 n_t^i di \qquad \underline{N}_t = \int_1^2 \underline{n}_t^j dj$$

Let's further denote $\underline{\varepsilon}_t^j = \frac{\underline{c}_t^j}{\underline{n}_t^j}$. Now, it's easy to see that the the clearing conditions (46), (48) and (49) become

$$\int_0^1 x_t^i n_t^i di + \int_1^2 \underline{x}_t^j \underline{n}_t^j dj = q_t K_t \tag{50}$$

$$N_t + \underline{N}_t = q_t K_t \tag{51}$$

$$\int_0^1 x_t^i n_t^i \left(A - \iota_t^i\right) di + \int_1^2 \underline{x}_t^j \underline{n}_t^j \left(\underline{A} - \underline{\iota}_t^j\right) dj = q_t \int_1^2 \underline{n}_t^j \underline{\varepsilon}_t^j dj \tag{52}$$

Now, we proceed by defining the main state variable of the economy, η_t. In equilibrium, every other variable, is calculated as a function of η_t and therefore, its dynamics drive the entire economy.

Definition 2. We define the wealth of the experts relative to the entire economy as

$$\eta_t = \frac{N_t}{N_t + \underline{N}_t} \tag{53}$$

We further postulate that the dynamics of η_t evolve as follows

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t \tag{54}$$

Hence, in equilibrium, by using (51) and (53), we have

$$\eta_t = \frac{N_t}{q_t K_t} \tag{55}$$

The whole dynamics of the model are being driven by the value of η_t. Recall that equation (37) is the leverage constraint that the expert faces. In equlibrium, once everything has been settled, by using equation (55), (37) becomes

$$x_t \leq min\{\frac{1}{m}, \frac{1}{\eta_t}\} \tag{56}$$

So, when η_t is small, less than m, the leverage constraint will bind at the value $\frac{1}{m}$ and thus, the expert won't purchase all the capital, that is available in the economy. Thus, the household will be required to hold some capital too, which will cause the price of capital to fall, since the households are less efficient producers than the experts. When, η_t is large, larger than m, then $x_t = \frac{1}{\eta_t}$ in equilibrium, which through equation (55), implies that the expert is holding all the capital as it is optimal and is the case in Brunnermeier and Sannikov. When default, enters, this is further affected and even in the region, that the expert is not constrained, and can lever up to purchase all the capital, she won't do it and we will have a further recession. In the simulations, I compare both models, the one with the leverage constraint only (Jing Chen [16]) and the one with default as well, and illustrate the default impact.

Before, we proceed to solving the expert's and household's maximization problem, we need to prove the following proposition.

Proposition 4. *In equilibrium, the expert will never lend to the household i.e.* $0 \leq \underline{x}_t \leq 1$ *and* $1 \leq x_t \leq U_t$, *where* $U_t = min\{\frac{1}{m}, \frac{q_t K_t}{n_t}\}$.

Proof. The first statement is immediate once we prove the second. The second branch of the second inequality follows directly, from the leverage constraint. We need to prove the first. Assume, that in equilibrium, at an instant t, the expected return on bonds (accounting for default) is ρ_t if the household is the lender and $\underline{\rho}_t$ if the expert is the lender. Then, for the expert to prefer lending (the expert is risk neutral), it should be the case that

$$\underline{\rho}_t - E\left[dr_t^k\right] > E\left[dr_t^k\right] - \rho_t$$

where $E\left[dr_t^k\right]$ is the expected return on capital. For, the household to prefer borrowing, it should be the case that

$$E\left[d\underline{r}_t^k\right] - \underline{\rho}_t > \rho_t - E\left[d\underline{r}_t^k\right]$$

Actually, the fact that the household is risk averse means that its excess return on capital should be much higher than the excess return on lending, in order to prefer to borrow, since default is also strategic, and thus, anticipated due to rational expectations. For credit trade to take place, both of these equations need to be satisfied, which leads to a contradiction since $E\left[dr_t^k\right] > E\left[d\underline{r}_t^k\right]$ or to $x_t = 1 = \underline{x}_t$[12] (no lending takes place) and concludes the proof.[13] □

Now, we can start deriving the optimal choices for the expert.

Expert's problem

The expert is maximizing her objective function (38) subject to her networth evolution (35) and the leverage constraint she faces (37). Before, we proceed to solving the maximization problem, we need to prove the following proposition.

Proposition 5. *The optimal investment for the expert is given by*

$$\iota_t = \frac{1}{2\theta}\left(q_t^2 - 1\right) \tag{57}$$

[12] We will later see that this cannot constitute an equilibrium.

[13] We have proved this proposition under the more general assumption that if the expert lends the household, the household can also default. In the case where the household cannot default (which is the actual assumption of the model), the proof follows in the same way but now $\underline{\rho}_t = r_t$.

The on-the-verge condition for default is given by

$$v_t = 1 - \frac{x_t - 1}{2\lambda x_t^2 (1 + r_r)} \tag{58}$$

and v_t should satisfy $0 < v_t \leq 1$.

Proof. In the objective function of the expert, the investment ι_t, appears only in the term $E\left[\frac{dn_t}{n_t}\right]$. Recall that

$$\frac{dn_t}{n_t} = x_t dr_t^k - v_t \left(x_t - 1\right)\left(1 + r_t\right)dt + \left(x_t - 1\right)dt$$

and

$$dr_t^k = \frac{A - \iota_t}{q_t}dt + \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right)dt + \left(\sigma + \sigma_t^q\right)dZ_t$$

Thus,

$$E\left[\frac{dn_t}{n_t}\right] = x_t(\frac{A - \iota_t}{q_t} + \phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)dt - v_t\left(x_t - 1\right)\left(1 + r_t\right)dt + \left(x_t - 1\right)dt$$

To solve for the optimal value for the investment, we need to differentiate the above equation with respect to ι_t and equate the resulting equation to 0 (since ϕ is concave, this ensures we obtain the maximum). Hence, the optimal investment satisfies

$$\phi^{'}(\iota_t) = \frac{1}{q_t} \Longrightarrow \frac{1}{\sqrt{1 + 2\theta\iota_t}} = \frac{1}{q_t} \Longrightarrow \iota_t = \frac{1}{2\theta}\left(q_t^2 - 1\right)$$

To obtain the first order condition for default, we firstly need to observe that the

objective function is a quadratic function in v_t with a negative coefficient of v_t^2. Thus, it is concave and its maximum is obtained at the interior point where its derivative is equal to zero, i.e.

$$-\left(x_t-1\right)\left(1+r_t\right)dt+2\lambda\left(1-v_t\right)x_t^2\left(1+r_t\right)^2dt=0\Rightarrow v_t=1-\frac{x_t-1}{2\lambda x_t^2\left(1+r_r\right)}$$

Finally, this is clearly less or equal than 1, since from proposition 4, $x_t \geq 1$, and for suitable values of λ, and since x_t is bounded by $\frac{1}{m}$, it is clearly greater than 0.[14][15] □

We, next proceed to the main theorem of the expert's problem which establishes the expert's leverage choice, x. As, we have seen from proposition 4, the solution will lie in the interval $[1,\, U_t]$. The following theorem, establishes when such a solution is a corner solution or an interior one, and will be used later to establish the equilibrium domain of the values of η.

Theorem 1. *One of the following must hold.*

1. *There exists $x_t \in (1,\, U_t)$, such that, $x_t \geq 2$ and*

$$\frac{E\left[dr_t^k\right]}{dt}-r_t+\frac{x_t-1}{2\lambda x_t^3}=0 \tag{59}$$

then the expert's leverage choice is x_t.

[14] In fact, in the simulations, it turns out, it is significantly greater than 0, for a great range of values for λ.

[15] Note here, that if the optimal choice for the expert, turned out to be 0 repayment, this would imply no borrowing as due to rational expectations, the household would prefer to maintain its capital rather than lending to the expert.

2. *If 1 does not hold then if U_t satisfies the following two equations*

$$\frac{E\left[dr_t^k\right]}{dt} - r_t + \frac{U_t - 1}{2\lambda U_t^3} > 0 \qquad \frac{E\left[dr_t^k\right]}{dt} - r_t + \frac{U_t - 1}{4\lambda U_t^2} \geq 0 \tag{60}$$

then the expert's leverage choice is $x_t = U_t$.

3. *If none of the above is satisfied, then there is no $x_t \in [1, U_t]$ that consists an equilibrium.*

Proof. From proposition 5, we have that the optimal choices of the expert for the investment and repayment rate are given by

$$\iota_t = \frac{q_t^2 - 1}{2\theta} \qquad v_t = 1 - \frac{x_t - 1}{2\lambda x_t^2(1 + r_t)}$$

Substituting these into the objective function of the expert given by equation (38), and simplifying the expression, we get

$$x_t\left(E\left[dr_t^k\right] - r_r dt\right) + r_t dt + \frac{(x_t - 1)^2}{4\lambda x_t^2} dt \tag{61}$$

Now, the expert is maximizing the above with respect to x_t. The derivative of the above equation, with respect to x_t, (note that, after the substitution of ι_t, $E\left[dr_t^k\right] - r_t dt$, depends only on variables, that are taken as given by the expert), we get

$$E\left[dr_t^k\right] - r_r dt + \frac{x_t - 1}{2\lambda x_t^3} dt \tag{62}$$

and the second derivative of the equation (61), with respect to x_t is

$$\frac{3-2x_t}{2\lambda x_t^4} \tag{63}$$

Now, assume that there exists an $x_t \in (1, U_t)$ such that $x_t \geq 2$ and equation (62) is zero. It is easy to see that since $x_t \geq 2$, equation (63) is negative and thus since the derivative is zero at this point, x_t is a local maximum of the objective function (61). It is further evident that at $x_t = 1$, the derivative is negative since, the derivative there is

$$E\left[dr_t^k\right] - r_r dt = -\frac{x_t - 1}{2\lambda x_t^3} dt < 0$$

since $x_t \geq 2$. From equation (63) we know that the first derivative of the objective is increasing in the interval $[1, \frac{3}{2})$ and decreasing in the interval $\left[\frac{3}{2}, U_t\right]$. Since we already know that equation (62) has a root in the interval $(\frac{3}{2}, U_t]$, namely, x_t, and that equation (62) is negative at 1, by continuity, we conclude that the first derivative of the objective has exactly two roots, and changes sign twice, from negative to positive and then back to negative. Let's denote the two roots by s_1 and $s_2 = x_t$.

Thus, the objective is decreasing in $[1, s_1)$, increasing in $[s_1, x_t]$ and decreasing in $(x_t, U_t]$. Therefore, the global maximum of the objective in the interval $[1, U_t]$ is achieved, either at 1 or at x_t. The difference between the value of the objective at x_t and the value of the objective at 1 is given by

$$(x_t - 1)\left(E\left[dr_t^k\right] - r_r dt\right) + \frac{(x_t-1)^2}{4\lambda x_t^2} dt = -\frac{(x_t-1)^2}{2\lambda x_t^3} + \frac{(x_t-1)^2}{4\lambda x_t^2} \geq 0$$

where the equality follows from the fact that equation (62) is zero at x_t and the inequality from the fact that $x_t \geq 2$. Thus, the global maximum is achieved at x_t. This concludes the proof for part 1 of the theorem.

Now, let's assume that there is no x_t as in 1, and the conditions of part 2 of the theorem are satisfied. that implies that the derivative has at most one root in the interval $[1, U_t]$. Firstly, let's assume that the derivative of the objective is negative at 1. Then since, at U_t, the derivative is positive by assumption, it must have exactly one root, say s. Hence, the objective is decreasing in $[1, s)$ and increasing in $[s, U_t]$. Thus, the global maximum is either at 1 or at U_t. The difference between the value of the objective at U_t and the value of the objective at 1 is given by

$$(U_t - 1)\left(E\left[dr_t^k\right] - r_r dt\right) + \frac{(U_t - 1)^2}{4\lambda U_t^2} dt$$

Dividing through by $\frac{(U_t - 1)}{dt}$ gives

$$\frac{E\left[dr_t^k\right]}{dt} - r_t + \frac{U_t - 1}{4\lambda U_t^2}$$

which is greater or equal than 0 by assumption and thus, U_t is the global maximum. In the case, that the derivative is non-negative at 1, we clearly have that the derivative is non-negative in the whole interval $[1, U_t]$ and thus again, the global maximum is achieved at U_t.

Finally, assume that neither part 1 nor part 2 of the theorem are satisfied fully. Assume further that an equilibrium exists. We can easily see that if an equilibrium exists and

it's interior then the assumptions of 1 should both hold. The derivative should be zero by definition and x_t should be greater or equal than 2 else the expert has a reason to deviate to zero borrowing. Thus, if an equilibrium exists, the expert's leverage choice should be at a corner. If the optimal x_t is U_t then clearly the objective at U_t should be higher than the objective at 1 and thus,

$$\frac{E\left[dr_t^k\right]}{dt} - r_t + \frac{U_t - 1}{4\lambda U_t^2} \geq 0$$

by definition. Now, if the derivative is non-positive at U_t then it has to be negative at 1. Hence, the derivative, due to its shape, should have either two roots or none in the $[1, U_t]$. if it has none, then the derivative is always negative, implying that the objective is decreasing which is a contradiction. If it has two roots, then the objective decreases initially, then increases and then decreases again implying that U_t cannot be the maximum. Again a contradiction.

Therefore, we are left with two possibilities. Either there exists an equilibrium, where the expert's leverage choice is 1 or there does not exist at all.

We only need to prove that $x_t = 1$ cannot consist an equilibrium and the proof of the theorem is then completed. Assume that $x_t = 1$ is an equilibrium. Then, after solving for the full equilibrium, we will see that

$$\frac{E\left[dr_t^k\right]}{dt} - r_t > 0$$

Intuitively, if there is no trade, and the expected repayment is at 1, we expect that since

the household is risk-averse they prefer to lend, and thus, they offer an interest rate less than their own expected return on capital which is less than the expert's expected return on capital since the household is less productive.

Now if the above equation is true, then this implies that the derivative of the objective is always positive and thus, the objective is increasing, contradicting the fact that $x_t = 1$ is optimal. This concludes the proof. □

We can already, make some comments regarding the dynamics of the model. The above theorem, really, says that the expert is trading off benefits between investing in capital and defaulting in some circumstances that the interest rate is high enough and her expected profitability from the capital investment low. This can happen now, since the interest rate is including the default premium which is not present in the case without default. This suppresses the capital price as we will later see, causing the profitability of capital to increase. Note, here that in the no default case, where only the leverage constraint is present, the expert's leverage choice is always U_t, thus, the leverage constraint is always binding. This is because, the excess return on capital is always positive, due to the lack of default premium, and thus, the expert always prefer to invest in capital.

From now on, we denote by x_t the expert's leverage choice in equilibrium. After, we calculate all endogenous variables as functions of x_t and our state variable, η_t, we will use theorem 1 to find the equilibrium domain and to build the algorithm that numerically computes x_t and therefore, all the endogenous variables.

Household's problem

The household is maximizing its utility function (44) subject to its networth evolution (45). The following theorem solves for the household's optimal choices for consumption $\underline{c}_t$, leverage $\underline{x}_t$ (here is lending), and investment $\underline{\iota}_t$ using the Hamilton-Jacobi-Bellman (HJB) method.

Theorem 2. *The household's optimal choices for consumption, leverage and investment are given by the following equations*

$$\underline{x}_t = \frac{E\left[d\underline{r}_t^k\right]/dt - v_t r_t - v_t + 1}{\left(\sigma + \sigma_t^q\right)^2} \quad \underline{\iota}_t = \frac{1}{2\theta}\left(q_t^2 - 1\right) \quad \underline{c}_t = \gamma \underline{n}_t \tag{64}$$

Proof. Recall that

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t$$

and $\underline{\varepsilon}_t = \frac{\underline{c}_t}{\underline{n}_t}$. Let

$$V\left(t,\, \underline{n}_t,\, \eta_t\right) = sup_{\underline{x}_t \geq 0,\, \underline{\iota}_t \geq 0,\, \underline{\varepsilon}_t \geq 0} E_t\left[\int_t^{+\infty} e^{-\gamma s}\left[log\left(\underline{\varepsilon}_t \underline{n}_t\right)\right] ds\right] \tag{65}$$

The HJB equation that V must satisfy to be optimized is given by

$$\gamma V = sup_{\underline{x}_t \geq 0,\, \underline{\iota}_t \geq 0,\, \underline{\varepsilon}_t \geq 0}\{e^{-\gamma t} log\left(\underline{c}_t\right) + \frac{dV}{d\underline{n}_t}\left\{drift\, of\, \underline{n}_t\right\} + \frac{d^2V}{d\underline{n}_t^2}\frac{\left\{vol\, of\, \underline{n}_t\right\}}{2} +$$

$$+ \frac{dV}{d\eta_t}\left\{drift\, of\, \eta_t\right\} + \frac{d^2V}{d\eta_t^2}\frac{\left\{vol\, of\, \eta_t\right\}}{2}\} \tag{66}$$

We speculate that $V = e^{-\gamma t}\left(f\left(\eta_t\right) + \frac{log(\underline{n}_t)}{\gamma}\right)$ for some function f. After substituting for

V, the drift and the volatility of the household's wealth and the drift and the volatility of the state variable η_t and simplifying we obtain

$$\gamma f(\eta_t) + log(\underline{n}_t) = sup_{\underline{x}_t \geq 0, \underline{\iota}_t \geq 0, \underline{\varepsilon}_t \geq 0} \{log(\underline{c}_t) +$$

$$+ \frac{1}{\gamma \underline{n}_t} \left[\underline{x}_t \underline{n}_t \left({}^{E[d\underline{r}_t^k]}/_{dt} - v_t r_t - v_t + 1 \right) + (v_t r_t + v_t - 1) \underline{n}_t - \underline{c}_t \right]$$

$$- \frac{1}{\gamma} \left[\frac{((\sigma + \sigma_t^q) \underline{x}_t)^2}{2} \right] + \eta_t f'(\eta_t) \alpha(\eta_t) + \frac{1}{2} \eta_t^2 f''(\eta_t) \beta^2(\eta_t) \} \tag{67}$$

Clearly, when maximizing with respect to $\underline{\iota}_t$ we obtain $\phi'(\underline{\iota}_t) = \frac{1}{q_t}$ which similarly to the expert's problem gives

$$\underline{\iota}_t = \frac{1}{2\theta} \left(q_t^2 - 1 \right)$$

Set $\underline{b}_t = {}^{E[d\underline{r}_t^k]}/_{dt} - v_t r_t - v_t + 1$. Re-arranging equation (67) and taking into account that f should satisfy this for all values of η_t we have that f should satisfy

$$\gamma f(\eta_t) - \frac{(v_t r_t + v_t - 1)}{\gamma} - \eta_t f'(\eta_t) \alpha(\eta_t) - \frac{1}{2} \eta_t^2 f''(\eta_t) \beta^2(\eta_t) -$$

$$- sup_{\underline{x}_t \geq 0} \frac{1}{\gamma} \left[\underline{b}_t \underline{x}_t - \frac{1}{2} ((\sigma + \sigma_t^q) \underline{x}_t)^2 \right] - sup_{\underline{\varepsilon} \geq 0_t} \left[-\frac{\underline{\varepsilon}_t}{\gamma} + log(\underline{\varepsilon}_t) \right] = 0 \tag{68}$$

Now, solving the two maximization problems that are embedded in equation (68) gives clearly that

$$\underline{\varepsilon}_t = \gamma \Longrightarrow \underline{c}_t = \gamma \underline{n}_t$$

To maximize $\underline{b}_t \underline{x}_t - \frac{1}{2} ((\sigma + \sigma_t^q) \underline{x}_t)^2$, for $\underline{x}_t \geq 0$ we consider its derivative and equate it

to zero, so we have that

$$\underline{x}_t = \frac{\underline{b}_t}{\left(\sigma + \sigma_t^q\right)^2} = \frac{{}^{E\left[d\underline{r}_t^k\right]}/_{dt} - v_t r_t - v_t + 1}{\left(\sigma + \sigma_t^q\right)^2}$$

which concludes the proof. □

We will now proceed into calculating the full equilibrium. We will start by deriving the evolution of the state variable η_t. Define the fraction of capital held by firms by

$$\psi_t = \frac{\int_0^1 k_t^i di}{K_t}$$

Then, in equilibrium, $1 - \psi_t = \frac{\int_1^2 k_t^j dj}{K_t}$, is the fraction of capital held by households.

Proposition 6. *We have that, in equilibrium*

$$x_t = \frac{\psi_t}{\eta_t} \qquad \underline{x}_t = \frac{1 - \psi_t}{1 - \eta_t} \tag{69}$$

Proof. It is clear to see that for an expert we have

$$x_t n_t = q_t k_t$$

Aggregating over all experts and dividing through by $q_t K_t$ we get

$$x_t \eta_t = \psi_t \Longrightarrow x_t = \frac{\psi_t}{\eta_t}$$

Finally from equation (50), we have

$$x_t N_t + \underline{x}_t \underline{N}_t = q_t K_t$$

where $\underline{N}_t = q_t K_t - N_t$ market clearing. Dividing through by $q_t K_t$ we get

$$x_t \eta_t + \underline{x}_t (1 - \eta_t) = 1$$

which gives the desired result after substituting for $x_t = \frac{\psi_t}{\eta_t}$ and concludes the proof. □

Proposition 7. *The evolution of the expert's wealth relative to the entire economy is given by*

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t} \left(dr_t^k - r_t dt - (\sigma + \sigma_t^q)^2 \, dt \right) + \left(\frac{A - \iota_t}{q_t} + (1 - \psi_t) \left(\underline{\delta} - \delta \right) \right) dt +$$

$$+ \frac{\psi_t - \eta_t}{\eta_t} \frac{x_t - 1}{2\lambda x_t^2} dt \tag{70}$$

Proof. Aggregating the wealth evolution over all experts and using proposition 6 we get

$$dN_t = \psi_t q_t K_t dr_t^k - v_t \psi_t q_t K_t \left(1 + r_t\right) dt + v_t N_t \left(1 + r_t\right) dt + \psi_t q_t K_t dt - N_t dt \tag{71}$$

By Ito's quotient rule[16], we have

$$\frac{d\eta_t}{\eta_t} = \frac{dN_t}{N_t} - \frac{d\left(q_t K_t\right)}{q_t K_t} + \left(\frac{d\left(q_t K_t\right)}{q_t K_t}\right)^2 - \frac{dN_t}{N_t}\frac{d\left(q_t K_t\right)}{q_t K_t} \tag{72}$$

From equation (34), equation (47), the definition of ψ_t, Ito's product rule and the fact that the optimal investment choice is the same for both the experts and the households, we get

$$\frac{d\left(q_t K_t\right)}{q_t K_t} = dr_t^k - \frac{A - \iota_t}{q_t}dt - \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right)dt \tag{73}$$

where the first two terms of the right-hand side are the expert capital gains and the third term is the adjustment for the household held capital. Substituting (73) and (71) into (72) and using the stochastic calculus rules gives

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k - \left(\sigma + \sigma_t^q\right)^2 dt\right) + \left(\frac{A - \iota_t}{q_t} + \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right)\right)dt -$$

$$-v_t\frac{\psi_t}{\eta_t}\left(1 + r_t\right)dt + v_t\left(1 + r_t\right)dt + \frac{\psi_t}{\eta_t}dt - dt$$

Finally, substituting the optimal choice for the repayment rate, v_t given by equation (58) yields the desired result. □

Corollary 5. *The drift and volatility of η_t are given by*

$$\alpha\left(\eta_t\right) = \frac{\psi_t - \eta_t}{\eta_t}\left(\frac{E\left[dr_t^k\right]}{dt} - r_t - \left(\sigma + \sigma_t^q\right)^2\right) + \frac{A - \iota_t}{q_t} + \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right) +$$

[16]Ito's quotient rule states that $\frac{d(X/Y)}{X/Y} = \frac{dX}{X} - \frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2 - \frac{dX}{X}\frac{dY}{Y}$.

$$+\frac{\psi_t - \eta_t}{\eta_t}\frac{x_t - 1}{2\lambda x_t^2} \tag{74}$$

and

$$\beta(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q) \tag{75}$$

Proof. It follows immediately from proposition 7 and the fact that dr_t^k has volatility $\sigma + \sigma_t^q$. □

The following proposition calculates the equilibrium price of capital, q_t, as a function of the state variable η_t and the expert's leverage choice $x_t = x(\eta_t)$.

Proposition 8. *The equilibrium price of capital is given by*

$$q_t = -\theta\gamma(1-\eta_t) + \sqrt{\theta^2\gamma^2(1-\eta_t)^2 + 2\theta A x_t \eta_t - 2\theta\underline{A}x_t\eta_t + 2\theta\underline{A} + 1} \tag{76}$$

Proof. From the market clearing condition from consumption, we have

$$x_t N_t(A - \iota_t) + \underline{x}_t\underline{N}_t(\underline{A} - \underline{\iota}_t) = q_t\underline{c}_t$$

Using that $x_t N_t = \psi_t q_t K_t$, $\underline{x}_t\underline{N}_t = (1-\psi_t)q_t K_t$, $\underline{c}_t = \gamma\underline{N}_t$, $\underline{N}_t = q_t K_t - N_t$ and $\eta_t = \frac{N_t}{q_t K_t}$ we get

$$(A - \iota_t)\psi_t + (\underline{A} - \underline{\iota}_t)(1-\psi_t) = \gamma q_t(1-\eta_t)$$

We further substitute for $\iota_t = \underline{\iota}_t = \frac{1}{2\theta}(q_t^2 - 1)$ and we get

$$A\psi_t - \frac{1}{2\theta}\left(q_t^2 - 1\right)\psi_t + \underline{A} - \frac{1}{2\theta}\left(q_t^2 - 1\right) - \underline{A}\psi_t + \frac{1}{2\theta}\left(q_t^2 - 1\right)\psi_t = \gamma q_t(1-\eta_t) \Longrightarrow$$

$$\Longrightarrow q_t^2 + 2\theta\gamma q_t\left(1-\eta_t\right) - 2\theta A\psi_t + 2\theta\underline{A}\psi_t - 2\theta\underline{A} - 1 = 0$$

By solving the above quadratic equation, keeping the positive root and substituting for $\psi_t = x_t\eta_t$ we get the desired equation

$$q_t = -\theta\gamma\left(1-\eta_t\right) + \sqrt{\theta^2\gamma^2\left(1-\eta_t\right)^2 + 2\theta A x_t\eta_t - 2\theta\underline{A}x_t\eta_t + 2\theta\underline{A} + 1}$$

□

Later on, in this paper, we will need to calculate the drift and volatility of the price of capital, and for that purpose, we need to calculate the first and second derivative of the price of capital with respect to η_t.

Corollary 6. *The first derivative of the price of capital, q_t, with respect to η_t, is given by*

$$q^{'}\left(\eta_t\right) = \theta\gamma +$$
$$+\frac{-\theta^2\gamma^2\left(1-\eta_t\right) + \theta A x\left(\eta_t\right) - \theta\underline{A}x\left(\eta_t\right) + \theta A x^{'}\left(\eta_t\right)\eta_t - \theta\underline{A}x^{'}\left(\eta_t\right)\eta_t}{\sqrt{M\left(\eta_t\right)}} \tag{77}$$

The second derivative of the price of capital, q_t, with respect to η_t, is given by

$$q^{''}\left(\eta_t\right) = \frac{\theta^2\gamma^2 + 2\theta A x^{'}\left(\eta_t\right) - 2\theta\underline{A}x^{'}\left(\eta_t\right) + \theta A x^{''}\left(\eta_t\right)\eta_t - \theta\underline{A}x^{''}\left(\eta_t\right)\eta_t}{\sqrt{M\left(\eta_t\right)}} -$$

$$-\frac{\left(-\theta^2\gamma^2\left(1-\eta_t\right) + \theta A x\left(\eta_t\right) - \theta\underline{A}x\left(\eta_t\right) + \theta A x^{'}\left(\eta_t\right)\eta_t - \theta\underline{A}x^{'}\left(\eta_t\right)\eta_t\right)^2}{M\left(\eta_t\right)\sqrt{M\left(\eta_t\right)}} \tag{78}$$

where

$$M\left(\eta_t\right) = \theta^2\gamma^2\left(1-\eta_t\right)^2 + 2\theta A x\left(\eta_t\right)\eta_t - 2\theta\underline{A}x\left(\eta_t\right)\eta_t + 2\theta\underline{A} + 1$$

Proof. It follows directly from proposition 8 after performing the calculations. □

Next, we calculate the interest rate, r_t, as a function of the price of capital drift, μ_t^q, the price of capital volatility, σ_t^q, the expert's optimal leverage choice, x_t and the expert's wealth relative to the entire economy, η_t.

Proposition 9. *The interest rate, r_t, is given by*

$$r_t = \frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1-\eta_t} + \frac{x_t - 1}{2\lambda x_t^2} \tag{79}$$

Proof. From theorem 2, we have that

$$\underline{x}_t = \frac{{}^{E\left[d\underline{r}_t^k\right]}/_{dt} - v_t r_t - v_t + 1}{\left(\sigma + \sigma_t^q\right)^2} \tag{80}$$

From corollary 4, we have that

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t}dt + \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right)dt + \left(\sigma + \sigma_t^q\right)dZ_t$$

From proposition 5, we have that

$$v_t = 1 - \frac{x_t - 1}{2\lambda x_t^2(1 + r_r)}$$

By capital market clearing and debt market clearing, we got equations (50) and (51). After we divide through by $q_t K_t$, we get

$$x_t \eta_t + \underline{x}_t \underline{\eta}_t = 1 \quad \eta_t + \underline{\eta}_t = 1$$

which, after substituting for $\underline{\eta}_t$ in the first equation and rearrange, we get that

$$\underline{x}_t = \frac{1 - x_t \eta_t}{1 - \eta_t}$$

Substituting for $\underline{x}_t$, v_t and $E\left[d\underline{r}_t^k\right]$ into equation (80), we get

$$\frac{1 - x_t \eta_t}{1 - \eta_t} = \frac{\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - r_t + \frac{x_t - 1}{2\lambda x_t^2}}{\left(\sigma + \sigma_t^q\right)^2}$$

Rearranging the above equation gives the desired result. □

Finally, we need to calculate μ_t^q and σ_t^q as functions of leverage x_t and the relative wealth η_t. Then, we will have obtained all the endogenous variables as functions of x_t and η_t and by using theorem 1, we can find the equation that x_t satisfies where the terms involved are either exogenous variables or the state variable η_t.

Proposition 10. *The drift of the price of capital is given by*

$$\mu_t^q = \frac{\frac{q'(\eta_t)}{q_t}\left(x_t\eta_t - \eta_t\right)\left(\frac{A - \iota_t}{q_t} + \phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2\right) + \frac{q'(\eta_t)}{q_t}\frac{A - \iota_t}{q_t}\eta_t}{1 + x_t\eta_t - \eta_t - \left(x_t\eta_t - \eta_t\right)\frac{q'(\eta_t)}{q_t}} +$$

$$+\frac{\frac{q^{'}(\eta_t)}{q_t}\left(1-x_t\eta_t\right)\left(\underline{\delta}-\delta\right)\eta_t-\frac{q^{'}(\eta_t)}{q_t}\left(x_t\eta_t-\eta_t\right)\left(\frac{A-\iota_t}{q_t}+\phi\left(\underline{\iota}_t\right)-\underline{\delta}+\sigma\sigma_t^q\right)}{1+x_t\eta_t-\eta_t-\left(x_t\eta_t-\eta_t\right)\frac{q^{'}(\eta_t)}{q_t}}+$$

$$+\frac{-\frac{q^{'}(\eta_t)}{q_t}\left(x_t\eta_t-\eta_t\right)\frac{1-x_t\eta_t}{1-\eta_t}\left(\sigma+\sigma_t^q\right)^2+\frac{q^{''}(\eta_t)}{q_t}\beta\left(\eta_t\right)^2\eta_t^2}{1+x_t\eta_t-\eta_t-\left(x_t\eta_t-\eta_t\right)\frac{q^{'}(\eta_t)}{q_t}} \tag{81}$$

where $\beta(\eta_t)$ is given by equation (75) and the volatility of the price of capital, σ_t^q, is given by

$$\sigma_t^q=\frac{\left(x_t\eta_t-\eta_t\right)\frac{q^{'}(\eta_t)}{q_t}\sigma}{1-\left(x_t\eta_t-\eta_t\right)\frac{q^{'}(\eta_t)}{q_t}} \tag{82}$$

Proof. To prove this, we will use Ito's lemma.[17]

We apply Ito's lemma to the function $q_t=q(\eta_t)$ and we get

$$dq_t=q^{'}(\eta_t)\,d\eta_t+\frac{1}{2}q^{''}(\eta_t)\,(d\eta_t)^2$$

Recall that

$$\frac{d\eta_t}{\eta_t}=\alpha(\eta_t)dt+\beta(\eta_t)dZ_t$$

From this last equation, using the stochastic calculus rules, we obtain that

$$(d\eta_t)^2=\eta_t^2\beta(\eta_t)^2dt$$

[17]Ito's formula states that if you have a function f of a random process S_t then the evolution of f is given by

$$df(S_t)=\frac{\vartheta f(S_t)}{\vartheta S_t}dS_t+\frac{1}{2}\frac{\vartheta^2 f(S_t)}{(\vartheta S_t)^2}(dS_t)^2$$

and thus, substituting the last two equations into the first one and dividing by q_t, we obtain

$$\frac{dq_t}{q_t} = \left(\eta_t \alpha(\eta_t) \frac{q^{'}(\eta_t)}{q_t} + \frac{\eta_t^2 \beta(\eta_t)^2}{2} \frac{q^{''}(\eta_t)}{q_t} \right) dt + \eta_t \beta(\eta_t) \frac{q^{'}(\eta_t)}{q_t} dZ_t \tag{83}$$

From proposition 6 and corollary 5, we have that

$$\alpha(\eta_t) = \frac{x_t \eta_t - \eta_t}{\eta_t} \left(\frac{E\left[dr_t^k\right]}{dt} - r_t - (\sigma + \sigma_t^q)^2 \right) + \frac{A - \iota_t}{q_t} + (1 - x_t \eta_t)(\underline{\delta} - \delta) +$$

$$+ \frac{x_t \eta_t - \eta_t}{\eta_t} \frac{x_t - 1}{2\lambda x_t^2}$$

$$\beta(\eta_t) = \frac{x_t \eta_t - \eta_t}{\eta_t} (\sigma + \sigma_t^q)$$

Thus, from equation (32), equation (83) and the above for $\beta(\eta_t)$, we have that

$$\sigma_t^q = \frac{(x_t \eta_t - \eta_t) \frac{q^{'}(\eta_t)}{q_t} \sigma}{1 - (x_t \eta_t - \eta_t) \frac{q^{'}(\eta_t)}{q_t}}$$

Finally, combining, the above equation for $\alpha(\eta_t)$, corollary 3, proposition 9, equation (32) and equation (83) and performing a series of tedious calculations, we obtain equation (81) which gives the drift of the price of capital as a function of exogenous variables, the state variable η_t and the leverage x_t. This concludes the proof. □

We are finally ready, to derive the equation that the leverage satisfies, using all of the above theorems and propositions.

Corollary 7. *One of the following must hold.*

1. *The leverage, x_t, satisfies: $x_t \in (1,\, U_t)$, $x_t \geq 2$ and*

$$g\left(x_t\right) = \frac{A - \underline{A}}{q_t\left(x_t,\, \eta_t\right)} + \underline{\delta} - \delta + \left(\sigma + \sigma_t^q\left(x_t,\, \eta_t\right)\right)^2 \left(\frac{1 - x_t\eta_t}{1 - \eta_t}\right) - \frac{x_t - 1}{2\lambda x_t^2} + \frac{x_t - 1}{2\lambda x_t^3} = 0 \tag{84}$$

2. *The leverage, x_t, satisfies: $x_t = U_t$ and*

$$g\left(x_t\right) = \frac{A - \underline{A}}{q_t\left(x_t,\, \eta_t\right)} + \underline{\delta} - \delta + \left(\sigma + \sigma_t^q\left(x_t,\, \eta_t\right)\right)^2 \left(\frac{1 - x_t\eta_t}{1 - \eta_t}\right) - \frac{x_t - 1}{2\lambda x_t^2} + \frac{x_t - 1}{2\lambda x_t^3} > 0 \tag{85}$$

$$g\left(x_t\right) = \frac{A - \underline{A}}{q_t\left(x_t,\, \eta_t\right)} + \underline{\delta} - \delta + \left(\sigma + \sigma_t^q\left(x_t,\, \eta_t\right)\right)^2 \left(\frac{1 - x_t\eta_t}{1 - \eta_t}\right) \geq 0 \tag{86}$$

3. *If none of the above is satisfied, then there is no $x_t \in [1,\, U_t]$ that consists an equilibrium.*

Moreover the parameters of the function $g(x_t)$ are functions only of exogenous variables and the state variable η_t.

Proof. The proof of the first part follows immediately from theorem 1, corollary 3 and proposition 9. The fact that the parameters of the function $g\left(x_t\right)$ are functions only of exogenous variables and the state variable η_t follows from proposition 8, corollary 6 and proposition 10. □

Equation (84) is a first order non-linear differential equation in x_t. We cannot obtain a closed form solution and therefore, we will use a numerical method to solve it. In the next section, we will provide an algorithm to solve it, which is based on a variation of the Newton method. Then, we will run the algorithm for specific values of the exogenous

variables, obtain the results and discuss the economic intuition. we will further compare the results to the model that does not allow for default and includes only a leverage constraint a variant of Brunnermeier and Sannikov [11]. As we will see, the default impact is significant in specific regions of the state variable η_t.

As a final note, before we proceed to the next section, we postulate the proposition below. The following proposition shows that once , we remove default, we obtain exactly the model of Brunnermeier and Sannikov [11], but with an a leverage constraint and an endogenous interest rate which is the case of Jing Chen [16].

Proposition 11. *(Equivalence theorem) Keeping all else equal, but removing default from our model, the equilibrium value of x_t is always equal to its upper limit i.e. U_t.*

Proof. The proof follows easily from observing that once default is excluded, then the objective function of the expert becomes linear in x_t and should therefore be either U_t or 1. It can easily be checked given what we have shown up to now, that irrespectively of the value of x_t, the expert's excess return over the interest rate r_t, in the no default case is always positive i.e.

$$\frac{E\left[dr_t^k\right]}{dt} - r_t > 0$$

and hence, the only stable equilibrium can be the one that satisfies $x_t = U_t$ for all t.

The above proposition is a variant of Jing Chen [16] which shows the same as true in equilibrium, however, it models slightly differently the leverage constraint. □

Simulations and Results

In this section, there are two subsections. In the first, we provide an algorithm to solve equation (84), which is based on a variation of the Newton method. In the second, we run a numerical example, provide the results and discuss their economic significance. We further compare the results to the model that does not allow for default and includes only a leverage constraint a variant of Brunnermeier and Sannikov [11]. The default impact is significant in a specific region of the relative wealth η_t. We further discuss the economic significance of our results.

Algorithm

The algorithm uses a shooting method to solve equation (84). We use MATLAB to run it. The main steps of the algorithm are the following:

1. Initially, the algorithm assumes that the derivative vector of x_t (The entries of the vector correspond to the number of η values we use to run it in the interval [0, 1]. We split this interval into 100 equal parts and calculate the equilibrium there. Therefore, the vector of x_t has 100 entries), is the zero vector.

2. It then proceeds into solving equation (84), as if the derivative vector of x_t is the last obtained derivative vector. To do this we use the following sub-algorithm:

   (a) we calculate the values of $g(x_t)$ at 2 and U_t. if the have opposite signs, we conclude there is a root (We have already verified by splitting the function

$g$into two parts and sketching it, that it will have at most one root in the interval [2, U_t] given that $U_t \leq 4$).

(b) If there is a root, we split the interval [2, U_t] into a 1000 smaller intervals, calculate the function there and once it becomes less than a very small value (we use 0.001), we assign this value to the corresponding entry of the vector of x_t.

(c) If there is no root, we test the second condition of corollary 7 and, if it is true we assign to the corresponding entry of the vector of x_t the value of U_t.[18]. After this step, we have constructed, our first vector of x_t.

3. Once we have a vector for x_t, we re-calculate its derivative vector as follows:

   (a) We subtract the n-th entry of that vector from the $(n+1)$-th entry and divide by the step we used to dissect the interval [0, 1] and obtain the initial values of η.[19]

   (b) Then to correct for errors and avoid spikes in our plots, we use the MATLAB smooth function to smooth this vector and thus, obtain our new derivative vector of x_t.

4. Then, we check the distance of the new derivative vector from the old one using the metric $|A-B| = max_{i=1}^{k}|a_i - b_i|$ where $A = (a_1, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_k)$. If this distance is less than a very small value (we use the value 0.05), we stop, and keep the last obtained vector of x_t as the solution. Then, we approximate the

[18]It turns out that there is always an equilibrium and therefore, the third condition of corollary 7 never applies.

[19]For this step we always assume that $x^{'}(0) = 0$ i.e. the derivative of x_t at $\eta = 0$ is always zero.

second derivative using the method discussed in (a) and (b) above applied on the derivative vector. Once these are done, we calculate the rest of the endogenous variables and obtain the equilibrium.

5. Finally, if this distance is not less than the very small value we have picked, the algorithm enters a loop and repeats all of the above. Eventually, it reached a point where the loop stops and gives the final solution.

In the next subsection, we run the above algorithm in MATLAB and obtain our results. We further run a much simpler algorithm, to obtain the equilibrium when the model includes a leverage constraint but does not allow for default. This algorithm is simpler, since based on proposition 11, we know in advance the value of x_t, and therefore, we do not need to solve any complex differential equation. We just need to calculate the equilibrium using the closed form functions of x_t and η_t that correspond to the rest of the endogenous variables.

Results

Numerical Example

Since we have solved fully for the model dynamics, we can now use the above algorithm and run a numerical example. The various exogenous variables and other important parameters are specified in the table below.

Parameters	Parameter value assigned
Exogenous penalty - λ	0.6
Productivity of Experts - A	1.1
Productivity of households - $\underline{A}$	0.5
Discount factor of households - γ	0.06
Exogenous shock - σ	0.025
Depreciation for experts - δ	0.03
Depreciation for households - $\underline{\delta}$	0.08
Leverage constraint - m	0.4

Table 1: Parameter values

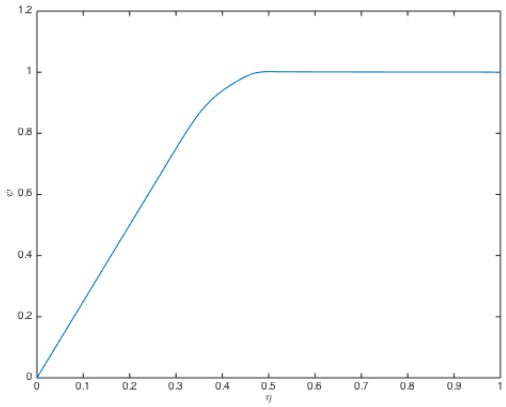
The parameter value of m being 0.4 implies an absolute leverage bound of 2.5 for the expert. Results discussed below are robust for all values of λ. Note here that for very large values of λ, default never occurs and the model becomes the same as in the no default case, with the exception of the interest rate that still includes the default premium and thus it's higher in our model. The reason that large values of λ, exclude default is due to the fixed positive term, $\delta - \underline{\delta}$, of equation (84), which captures the difference between the depreciation rate of capital for experts and households respectively. Hence, once λ becomes very large, equation (84) does not have a solution and condition 2 of corollary 5 obtains.

Discussion of Results

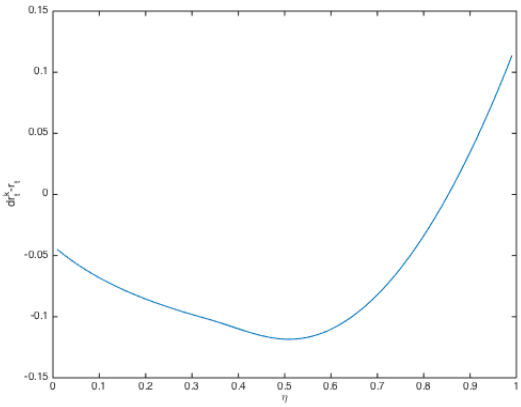
The following figures provide our results. Figure 1, graphs (a) - (j) give the results for the endogenous variables of our model. Figure 2, graphs (k) - (q), compare the default

model to te model that includes a leverage constraint but does not allow for default. In this way, we can isolate the effects of default in our economy.

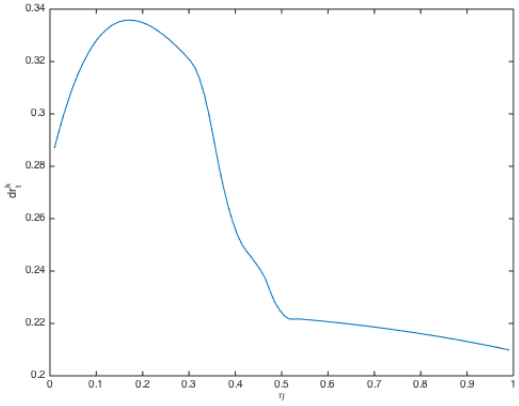
The inefficiencies in our model, as in Brunnermeier and Sannikov [11], are capital misallocation, since less productive households end up managing capital for lower values of $\eta$and underinvestment since the investment is increasing in the price of capital, and a less than the highest possible price of capital (which starts obtaining when experts manage all the available capital of the economy.) gives rise to a lower investment. Consumption distortion is not part of the inefficiencies, as experts re-invest all of their wealth. Contrary to Brunnermeier and Sannikov [11], that the above inefficiencies arise only due to a specific financial friction i.e. a productivity differentiation among agents, in our model, they also arise and are further exacerbated by the presence of default and leverage constraints. Jing Chen [16] has studied the effect of leverage constraints. We will focus on the effects of default on top of leverage constraints.



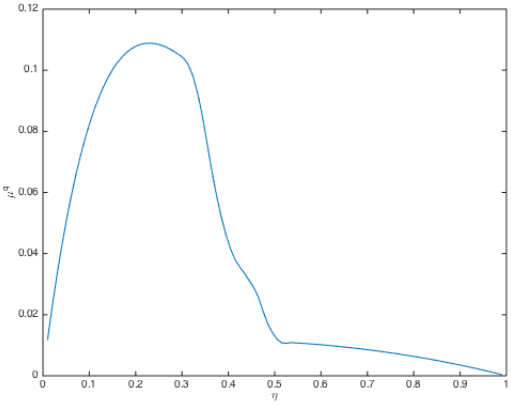
(a) Proportion of capital held by expert



(b) Expert's excess return over the interest rate

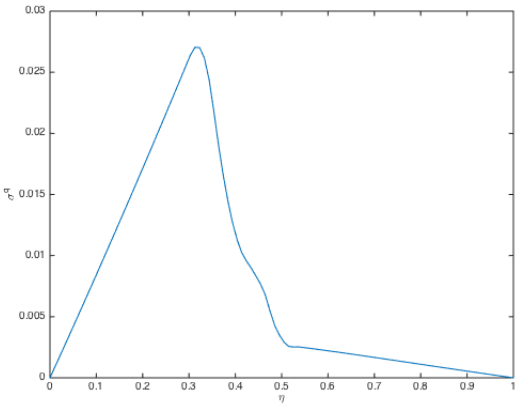


(c) Expert's return on capital

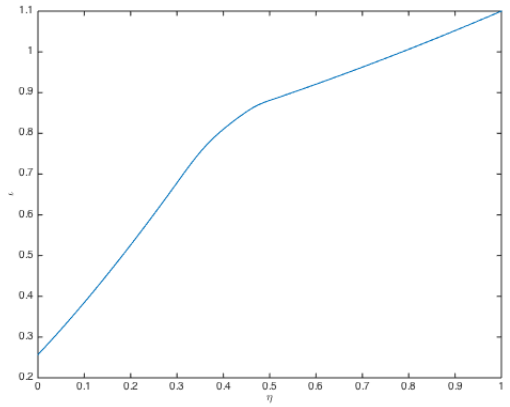


(d) Price of capital drift

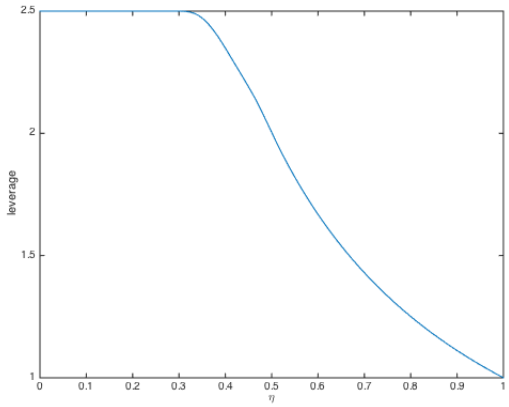
Figure 1: Results



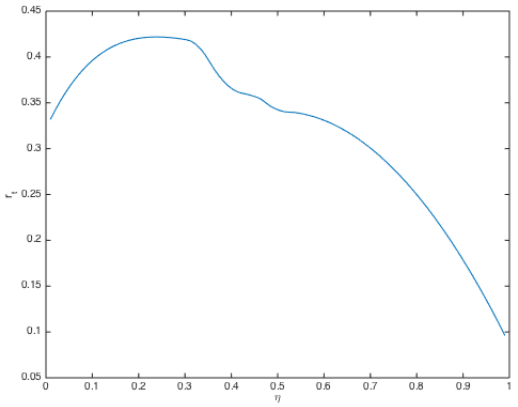
(e) Price of capital volatility- Endogenous risk



(f) Investment



(g) Expert's leverage

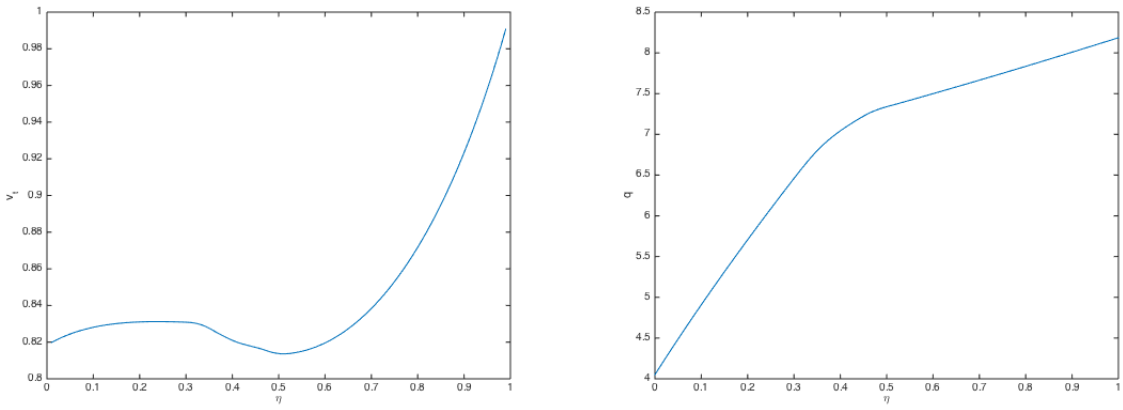


(h) Interest rate

Figure 1: Results

Firstly, we will discuss the results of our model independently, and then in reference to the model where default is not present. The interest rate, as we see in in figure 1, graph (h), is initially increasing, makes a small hump and then decreases. This is because, the leverage constraint binds, when the expert is relatively poor, the price thus, remains low, causing higher returns on capital, lower demand for bonds due to

the binding constraint, and thus, a higher interest rate demanded from the household to compensate both for the lower demand of bonds and the increasing returns on capital. Once the expert's leverage does not hit the absolute upper limit $\frac{1}{m}$, then the expert is able to acquire all the capital available in the economy, returns on capital fall, and thus, the compensation required by the household. The hump shape around the middle is attributed to default which causes a recession during that time as we will discuss later. Endogenous risk, follows a similar pattern as in Brunnermeier and Sannikov [11], namely it amplifies before the steady state $\eta = 1$, and slowly adjusts as it approaches the steady state. This is contrary to papers like BGG [7]and KM [58] where the distinction of endogenous risk between normal times and crisis times is not captured.



(i) Repayment rate

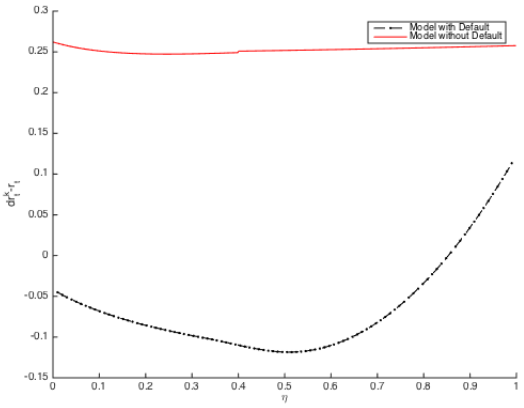
(j) Price of capital

Figure 1: Results

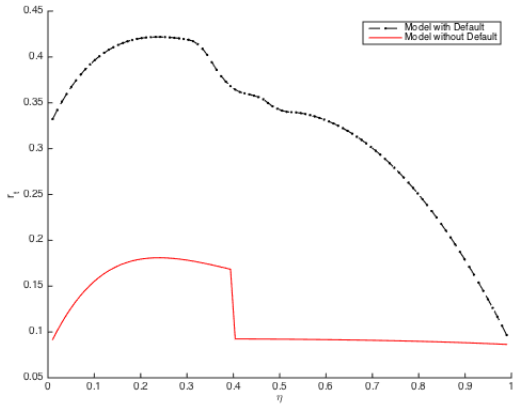
To discuss the economic interpretation of the repayment rate, we firstly need to understand what drives it. There are two main forces that drive the behaviour of the repayment rate, through the trade off condition between defaulting or not defaulting,

captured by equation (58). The first is the level of leverage and the second the level of the interest rate. The higher the interest rate, the less prone, the expert is to default as the costs of defaulting are quadratic on the proportion of the total value of the loan plus interest that the expert chooses to default upon. Leverage impact is not so straightforward as, the expert faces fixed costs upon starting defaulting, meaning that when leverage is high, default is more attractive to the expert, as fixed costs are relatively small, but when leverage lower, the expert might prefer not to default so much as fixed costs become relatively large. On the other hand, if we ignore the fixed costs, a higher leverage implies higher costs. Based on these observations, we see that when the leverage constraint binds, default is high as the interest rate dependence drives it. The fact that the repayment reaches a low around the middle of the graph is being driven by the fact at this region, the interest rate becomes higher than the returns on capital, making investment less attractive, and in turn this suppresses leverage and increases default for the reasons mentioned earlier.

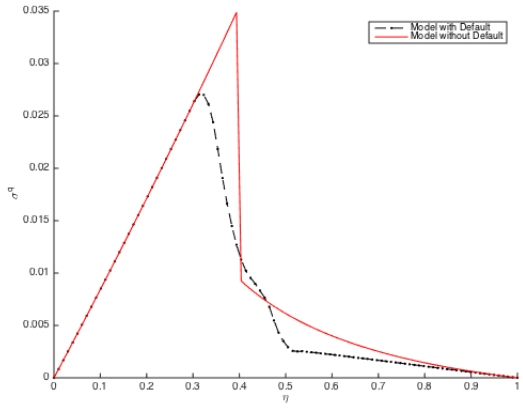
Note here, that the fact the excess return over the interest rate becomes negative in some region of the relative wealth η, is caused by the default premium, which is captured in figure 2, graph (l). The default premium arises since the household anticipates default by the expert due to rational expectations. This, in turn, makes the expert in some instances, as discussed above, better off to obtain less debt as the returns on capital are relatively small to the interest rate. The benefits of leveraging up to the limit are no longer good enough, and thus, the inefficiency in allocation increases, causing the less productive household to hold less capital in that region. This drives the recession.



(k) Expert's excess return over interest rate



(l) Interest rate

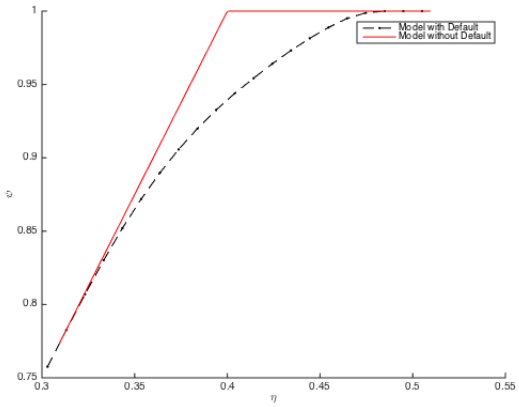


(m) Endogenous risk

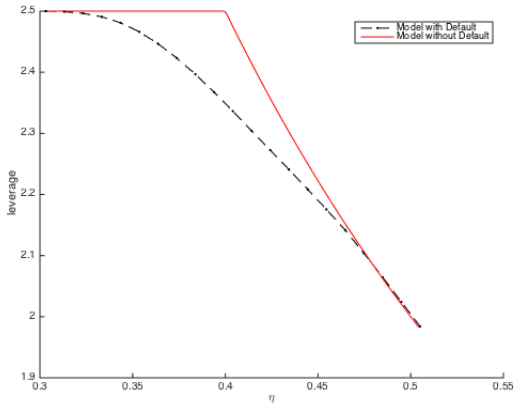
Figure 2: Default vs no default model

Figure 2, graph (k) shows the difference of the excess return between the model with default and the model with no default. The black dashed line represents the default case and the red line the no default case. As, it is clear, the default causes the excess return to be much lower due to the default premium. Figure 2, graph (l), illustrates the interest rate in the two cases, and depicts the difference which is explained by the

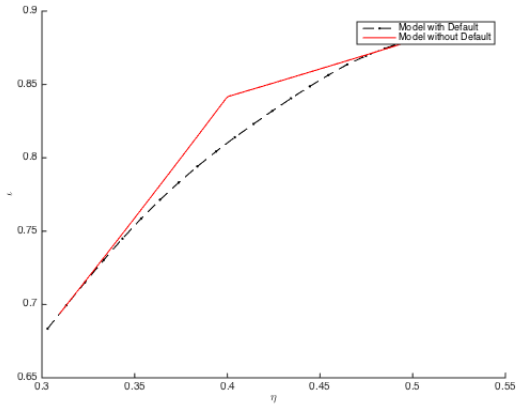
default premium. In figure 2, graph (m), the difference in the endogenous risk is being captured. It's important to note here, that default, as it can be seen in the graph, helps the economy to breathe by releasing some risk initially. We can see that in the fact that the endogenous risk picks lower than the no default case. However, the economy adjusts at a slower pace when default is present, while approaching the steady state. This is explained by the recession that was described above. Although, the expert could purchase all the capital and decrease faster the endogenous risk, she doesn't and thus risk adjusts slower.



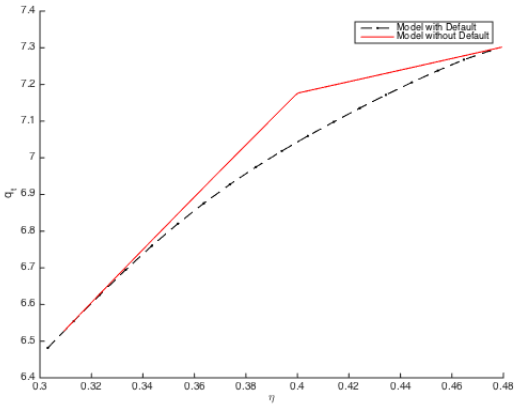
(n) Proportion of capital held by expert



(o) Expert's leverage



(p) Investment



(q) Price of capital

Figure 2: Default vs no default model

Finally, graphs, (n), (o), (p) and (q) illustrate the impact of default in the recession region where allocations are being altered, on the proportion of capital held by expert, ψ, the expert's leverage x, the investment, ι, and the price of capital, q. In these , graphs we see, that the expert holds less capital and leverage, investment and the price of capital are lower than they would have been if no default occurred. The breaking

point in the no default case arises at the point where $\eta = m$, namely, the point, where the expert's leverage does no longer hit the absolute upper limit $\frac{1}{m}$ since the expert starts being able to purchase all the available capital.

Note here, that the interest rate has a hump same in the same region, and this is because economic activity is negatively impacted and as a result interest rate is lower. However, default premium sustains it at a higher level. We close this section by stating the following theorem.

Theorem 3. *Default has real effects on the allocation of capital. It impacts investment, and overall economic activity, causing a recession.*

Proof. It follows immediately by corollary 4 and the above results. □

The proposition below quantifies the effect of default on investment during a recession.

Proposition 12. *The presence of default decreases investment by up to 4.2% in a recession.*

Proof. It follows immediately by the above simulation results. □

Concluding Remarks

We have built a continuous time model that extends Brunnermeier and Sannikov [11] and Jing Chen [16] by incorporating default. We modelled default through incorpoating

a repayment rate and adding a penalty to the objective function of the agent as it has been modelled by Dubey, Geanakoplos and Shubik [29]. We solved the dynamics of the model and all the endogenous variables in terms of the leverage of the productive agent and the state variable of our model, the relative wealth of the productive agent to the entire economy. We ran a numerical example using MATLAB and produced the results. It turned out that default has real effects on the allocations and impacts overall economic activity, causing a recession. The contribution to the literature is that it models default in continuous time for the first time and provides a setting where monetary policy and regulation can be better examined due to the nature of the modelling which allows for closed form solutions relative to the leverage that is being estimated numerically. It, moreover, provides a framework where, CDS (Credit Default Swaps) can be priced due to the continuous time nature of our modelling, giving a new approach. Our future research plans, include to use the above framework, to examine the impact of monetary policy and macro-prudential regulation, and thus, deriving the optimal policy mix for mitigating recessions. To implement this, we would introduce demand for money through the agents' need to insure against adverse shocks. Therefore, money would have a store value role rather than a means of exchange one. In this setting, monetary policy would operate by redistributing wealth, i.e., by affecting the value of η_t. A similar procedure can be found in [12].

Default and Determinacy under Quantitative Easing

This part is a joint paper with my supervisor Professor Dimitrios Tsomocos.

Introduction

The interconnection between the real and monetary parts of the economy creates the question of how price levels and inflation are being determined in equilibrium. In real business cycle models (RBC), where the means of exchange is a real good, the price level is indeterminate as prices can only be pinned down in relative terms due to Walras law. In models, where money is introduced, price levels not only remain indeterminate but the introduction of money can also lead to indeterminacy of real allocations among agents. Lin, Tsomocos and Vardoulakis ([64]) show that the introduction of default remove the indeterminacy of price levels and inflation through the implementation of conventional monetary policy.

Conventional monetary policy puts limits in the central bank's balance sheet and allows short dated government bonds only. Unconventional monetary policy, also known as, quantitative easing (QE) removes this restriction on the assets the central bank can hold in its balance sheet and allows for assets of different credit risk and duration profiles. McMahon, Peiris and Polemarchakis ([69]) show that in a cash-in-advance monetary

economy with flexible prices, unconventional monetary policy, cannot determine the path of inflation.

The main goal of this paper is to show, using an analogy of Lin, Tsomocos and Vardoulakis ([64]) model, that the introduction of default in the setting of McMahon, Peiris and Polemarchakis ([69]), resolves the indeterminacy of the price level and thus, the inflation path under quantitative easing (QE). This, thus, implies that the QE that was implemented by the European Central Bank (ECB) in the past years can be efficient in achieving the economic outcomes it seeks, part of which consists of determining the inflation path. Magill and Quinzii ([67]) show, by using a variant of the Lucas and Stokey ([66]) model, that the central bank can determine the inflation path, if it purchases the whole endowment of agents in exchange of a fixed supply of money, which the agents then, use to purchase back goods. In this setup, by construction, the money supply is determined, the quantity theory of money holds and the central bank can choose the price level through this fixed money supply. McMahon, Peiris and Polemarchakis ([69]) show that, under quantitative easing, the inflation path is indeterminate when the supply of money is not fixed. They point out that indeterminacy can be lifted under different forms of quantitative easing. Dubey and Geanakoplos ([26]) show that with the presence of outside money, monetary policy can achieve determinacy. We will focus on showing that without the presence of outside money but with positive interest rates, which stop agents from transferring money from one period to the next, and there is no outside money present, the introduction of default raises the indeterminacy of the inflation path under unconventional monetary policy.

The way, this paper is replicating quantitative easing is through a set of collateralised

long term loans. When default is present in every state in period 1, for some of these collateralised long term loans, the payoff of profile of these loans changes creating a set of payoffs that can span the full state space and combined with the distinct collateral profiles, these loans consist of a set of loans with different risk profiles which can replicate the different durations of bonds being purchased by the ECB in the economy. At the same time, the lost wealth from defaulting on these long term loans is being used to repay positive interest rates back to the central bank and thus, support the presence of of positive interest rates and lifting the need for outside money endowments since, thee is a cost of carrying money to the next period. Thus, the quantity theory of money obtains and the central bank can determine the price level and inflation path by targeting the interest rates of oth the short term and long term loans and in addition, the collateral requirements on the long term loans, ensuring that there is default in every state in period 1. Under interest rate targeting, the central bank targets positive interest rates, the households use some of the money obtained by long term loans to repay short term loans at the end of the first period. Therefore, endogenously you transfer less money than the long term loan obtained to the second period. This supports default.

Goodhart, Tsomocos and Vardoulakis ([43]) modelled default in a similar manner and studied the exacerbating effects of default in the business cycle during a housing crisis. Lin, Tsomocos and Vardoulakis ([63]) modeled default in a similar vein and examined its impact on debt deflation.

The rest of the paper proceeds as follows: In section 16.1, we present a summary of the various forms of QE that was implemented by the major central banks around the world. Section 17, presents the model, the agents' optimization problem and provides

the market clearing conditions. Section 18 defines the equilibrium and provides the main propositions that are used to prove determinacy of price levels and inflation path under positive interest rate targets. Section 19 proves the main result of the paper, and shows determinacy. Finally, section 20, concludes. Finally, Peiris and Vardoulakis ([77]) showed that monetary policy is efficient by introducing a collateral constraint instead of allowing collateral default. In their model, they ensured determinacy through the presence of outside money.

Implementation of Quantitative Easing (QE) by Central Banks

The way that quantitative easing has been put in practise varies among major central banks.

First, The Fed approached quantitative easing in a similar way that this paper models it, by targeting both the total level of purchases but also setting restrictions in the characteristics of assets that it purchases. They did this through their Large Scale Asset Purchases program (LSAP).

Second, The Bank of Japan (BOJ), during the QE undertaken between 2001 and 2006, set new operational targets, and to achieve these targets, it purchased long term Japanese government bonds, stocks and asset backed securities (ABSs) but the specific levels of its portfolio did not matter. It cared about the total level of balances. More recent QE policies by BOJ have targeted specific levels of lending towards banks rather than total level of balances. Fawley and Christopher ([33]) argue that BOJ still

did not care much about the individual characteristics of assets and provided limited restrictions.

Third, The Bank of England set similar targets as the BOJ, targeting an overall level of the amount of assets purchased through the Asset Purchase Facility (APF), rather than the individual characteristics of assets. There were though some restrictions making it more like the Fed's QE.

Finally, European Central Bank (E.C.B.), through, Public Sector Purchasing Program (PSPP) and Corporate Sector Purchasing Program (CSPP), targeted the overall levels of balances but also the banks that was lending and the characteristics of assets it purchased in terms of risk profile and collateral accepted. E.C.B.'s QE is similar to the way this paper models QE.

The Model

We extend the model by McMahon, Peiris and Polemarchakis ([69]) by including heterogeneous agents and default. Our goal is to show that default raises the indeterminacy of prices and inflation under quantitative easing and gives rise to a unique inflation and price path. The idea is motivated by Tsomocos and Vardoulakis ([64]). We consider a three-period model where the third period serves for accounting purposes and debts are settled in this date. There is uncertainty in the second period and each one of S states can obtain. Nature selects one of the states $s_1 \in \{1, \ldots, S\}$ which occur in period 1. State s_0 occurs in period 0. There is one perishable good in the economy. There is also

one durable real asset in the economy that pays out X_{s_1} units or perishable good at state s_1 in period 1. The asset payoffs can be viewed as a vector, $\mathbb{X}_{+}^{\mathrm{S}}$, which is a $S \times 1$ vector.

Let the price for the perishable good be denoted by $p(s_0)$ in period 0 and by $p(s_1|s_0) = p(s_1)$ in period 1. Let also the price of the real asset be denoted by $q(s_0)$ in period 0. There is a set of households, $\mathbb{H} = \{1, \ldots, H\}$, where each agent is endowed with an initial endowment of the real asset at each period. Denote by $e^h(s_0)$ the endowment of real asset at period 0 by household h.

There is also production in the economy as in McMahon, Peiris and Polemarchakis (2018). At period t, a household, h, provides $l^h(s_t)$ units of labour to produce perishable goods,

$$y^h(s_t) = l^h(s_t) \tag{87}$$

Utility of consumption and leisure is defined by

$$U^h = u^h\left(c^h(s_0), 1 - l^h(s_0)\right) + \beta \sum_{s_1} u^h\left(c^h(s_1), 1 - l^h(s_1)\right) \tag{88}$$

where $c^h(s_t)$ denotes consumption at period t and u^h is continuously differentiable, strictly increasing, and strictly concave.

In the model, households, trade in both the perishable good and the real asset. We denote by $\theta_1^h(s_t)$ the quantity of perishable goods purchased by household h at period t and by $\theta_2^h(s_t)$, the quantity of perishable goods sold by the household at period t.

We denote by $\phi_1^h(s_0)$ the quantity of real assets purchased by household h at period 0, and by $\phi_2^h(s_0)$, the quantity of real assets sold by household h, at period 0.

In the model, demand for money is introduced through a cash-in-advance constraint. All trades of goods and real assets are made in fiat money. Money can also be carried over to the next period. Households need to have cash at hand to make purchases at the beginning of each period. Households do not have initial nominal wealth (monetary endowments) but they obtain cash through short-term and long-term borrowing from the central bank. Short-term borrowing in our case is replacing the monetary endowments in the McMahon, Peiris and Polemarchakis ([69]) model. Long-term loans are collateralized and subject to default.

In particular, short term loans are issued by the central bank at the beginning of each period t, and promise a repayment of $\mu^h(s_t)$ at the end of period t. Long-term loans are issued only at period 0 by the central bank, and promise a repayment of $\mu^h(d)$ at the beginning of period 1 and before any commodity trade. These long term loans are collateralised, and are indexed by $d \in \mathbb{D} = \{1, \ldots, D\}$, where each of these loans correspond to a unique collateral requirement δ_d, which represents the nominal value of real assets required as collateral, per unit of loan d. The interest rate, on each of these loans, is denoted by $r(d)$ and on the short term loans by $r(s_t)$ at period t. That means, that in order for a household h, to obtain a long-term loan of size $\mu^h(d)$, the household needs to pledge $\frac{\delta_d}{q(s_0)}$ units of the real asset as collateral. Therefore, each household needs to satisfy a collateral constraint based on the amount of real asset it pledges i.e.

$$\sum_d \mu^h(d) \frac{\delta_d}{q(s_0)} \leq e^h(s_0) + \phi_1^h(s_0) - \phi_2^h(s_0) \tag{89}$$

At the beginning of period 1, each household decides whether to default on the loan or not. The household defaults if the collateral value is less than the size of the loan and does not default otherwise. Thus, household, h, repays at state s, period 1,

$$\mu^h(d) \times min\left[1, \delta_d X_{s_1} \frac{p(s_1)}{q(s_0)}\right] \tag{90}$$

Define $R^d(s_1) = min\left[1, \delta_d X_{s_1} \frac{p(s_1)}{q(s_0)}\right]$. The collateral requirements, δ_d, are exogenous and set by the central bank.

The timeline of the model is as follows:

1. In the beginning of period 0, households supply labour and produce, borrow short term and long term and trade in real assets and perishable goods.
2. At the end of period 0, households repay their short term loan and consume.
3. In the beginning of period 1, households supply labour and produce, borrow short term, decide whether to default or not on their long term loan obligations and trade in goods.
4. At the end of period 1, households repay their short term loans and consume.
5. In period 2, all outstanding liabilities have been repaid and money has exited the economy.

Households

Define

$$B(h) = \left\{ l^h(s_t), \theta_1^h(s_t), \theta_2^h(s_t), \phi_1^h(s_0), \phi_2^h(s_0), \mu^h(s_t), \mu^h(d), c^h(s_t), m^h(s_t) \right\} \quad (91)$$

where $m^h(s_t)$ is money left unused at the end of period t and transferred to next period. Household $h \in \mathbb{H}$, maximises its utility,

$$max_{B(h)} U^h = u^h\left(c^h(s_0), 1 - l^h(s_0)\right) + \beta \sum_{s_1} u^h\left(c^h(s_1), 1 - l^h(s_1)\right) \quad (92)$$

subject to the following constraints:

$$\hat{m}^h(s_0) = \frac{\mu^h(s_0)}{1 + r(s_0)} + \sum_d \frac{\mu^h(d)}{1 + r(d)} \quad (93)$$

where $\hat{m}^h(s_0)$ is the cash used to purchase real assets and goods at the beginning of period 0. This constraint says that this cash should be equal to the cash available which is the loans obtained.

$$\hat{m}^h(s_0) \geq p(s_0)\theta_1^h(s_0) + q(s_0)\phi_1^h(s_0) \quad (94)$$

This is the cash-in-advance constraint i.e. cash at hand is greater or equal than the

total purchases in real assets and goods by household h.

$$m^h(s_0) = \hat{m}^h(s_0) - p(s_0)\,\theta^h(s_0) - q(s_0)\,\phi^h(s_0) - \mu^h(s_0) \geq 0 \tag{95}$$

where $\theta^h(s_0) = \theta_1^h(s_0) - \theta_2^h(s_0)$ and $\phi^h(s_0) = \phi_1^h(s_0) - \phi_2^h(s_0)$. This is the money left at the end of period 0, i.e. total cash minus net purchases of goods, minus net purchases of real assets, minus short term loan repayment. This must be greater or equal than 0.

$$\sum_d \mu^h(d)\,\frac{\delta_d}{q(s_0)} \leq e^h(s_0) + \phi_1^h(s_0) - \phi_2^h(s_0) \tag{96}$$

This is the collateral constraint that each household faces i.e. real assets pledged as collateral must be less that real assets endowed plus net purchases of real assets.

$$c^h(s_0) = y^h(s_0) + \theta_1^h(s_0) - \theta_2^h(s_0) \tag{97}$$

This is the consumption of household h which should be equal to its output in period 0 plus its net purchases of perishable goods.

$$p(s_1)\,\theta_1^h(s_1) + \sum_d \mu^h(d)\,\mathbb{I}^d(s_1) \leq m^h(s_0) + \frac{\mu^h(s_1)}{1 + r(s_1)} \tag{98}$$

where $\mathbb{I}^d(s_1)$ equals to 1 if household h decides to repay and equals to 0 if the household decides to default. This constraint says that in period 1 ans state s_1, money carried forward plus money borrowed short term are greater or equal than purchases of goods

in state s_1 plus repayment on long term loan.

$$m^h(s_1) = m^h(s_0) + \frac{\mu^h(s_1)}{1 + r(s_1)} - p(s_1)\theta^h(s_1) - \sum_d \mu^h(d)\mathbb{I}^d(s_1) - \mu^h(s_1) \geq 0 \quad (99)$$

where $\theta^h(s_1) = \theta_1^h(s_1) - \theta_2^h(s_1)$. This is the money left at the end of period 1, i.e. money carried over plus money borrowed short term, minus net purchases of goods, minus long term debt repayment, minus short term loan repayment. This must be greater or equal than 0.

$$e^h(s_1) = e^h(s_0) + \phi^h(s_0) - \sum_d \left(1 - \mathbb{I}^d(s_1)\right)\mu^h(d)\frac{\delta_d}{q(s_0)} \quad (100)$$

This is the real assets held by household h in period 1, state s_1. This is equal to real assets endowed in the beginning of period 0, plus net purchases of real assets by household h, minus real assets seized by the central bank upon default.

$$c^h(s_1) = y(s_1) + \theta^h(s_1) + e^h(s_1)X_{s_1} \quad (101)$$

This is consumption of household h in period 1, state s_1. It is equal to output produced in period 1, state s_1 by household h, plus net purchases of perishable goods by household h, plus payoff of real assets held by household h in period 1, state s_1.

$$y^h(s_t) = l^h(s_t) \quad (102)$$

This is production by the household in both periods.

Central Bank

The central bank extends short-term and long-term loans. It sets the interest rates and chooses the collateral requirements.

Market clearing and equilibrium

Assets market

For assets markets to clear, total purchases should be equal to total sales and thus,

$$\sum_h \phi_1^h(s_0) = \sum_h \phi_2^h(s_0) \tag{103}$$

Goods market

For goods markets to clear, total purchases should be equal to total sales, and hence,

$$\sum_h \theta_1^h(s_0) = \sum_h \theta_2^h(s_0) \tag{104}$$

and at state s_1,

$$\sum_h \theta_1^h(s_1) = \sum_h \theta_2^h(s_1) + \sum_h \sum_d \left(1 - \mathbb{I}^d(s_1)\right) \mu^h(d)\, \delta_d \frac{X_{s_1}}{q(s_0)} \tag{105}$$

Short-term money market

Demand for short-term loans must be equal to money supply,

$$1 + r(s_t) = \frac{\sum_h \mu^h(s_t)}{M(s_t)} \tag{106}$$

Long-term money market

Demand for long-term loans must be equal to money supply,

$$1 + r(d) = \frac{\sum_h \mu^h(d)}{M(d)} \quad \forall d \in \mathbb{D} \tag{107}$$

Definition. $\left\{(r(s_t), r(d), p(s_t), q(s_0)), \left(M(s_t), M(d)_{d\in\mathbb{D}}\right), B(h)_{h\in\mathbb{H}}\right\}$ is a Monetary Collateral Equilibrium (MCE) for the economy $\mathbb{E} = \left\{\left(U^h, e^h(s_0)\right)_{h\in\mathbb{H}}, \mathbb{X}_+^S, (\delta_d)_{d\in\mathbb{D}}\right\}$ if and only if all markets clear and expectations are rational, i.e., future prices and interest rates are correctly anticipated, and agents optimize given their budget sets.

Equilibrium Analysis

In this section, we will derive some preliminary results in order to prove the main goal of this paper which is determinacy of prices. First of all, we need to define Quantitative Easing (QE) in our setting and prove that this definition is equivalent to the definition by McMahon, Peiris and Polemarchakis ([69]). Once this is done, we can proceed into showing that the presence of default can raise price indeterminacy.

Definition 3. A balance sheet policy is unrestricted and termed as "Quantitative Easing", if the shares of collateralised long-term loans issued are endogenously determined and the realised payoffs of the collateralised long-term loans span the whole state space.

According to McMahon, Peiris and Polemarchakis ([69]), Quantitative Easing is defined as a balance sheet policy where the shares of Arrow securities held by agents, are endogenously determined. We will later show that this definition is equivalent to definition 3.

We, now, proceed into showing a set of propositions that we will need to show determinacy in the next section.

Proposition 13. *Short-term and long-term interest rates are nonnegative in any monetary equilibrium i.e.,* $r(s_t) \geq 0$ *and* $r(d) \geq 0$ *for all* $d \in \mathbb{D}$ *and* $s_t \in S \cup \{s_0\}$.

Proof. $r(s_t)$ is clearly greater or equal than zero, since, if it was negative, then the households could indefinitely borrow from the central bank and thus, the no arbitrage condition would fail. Now, lets assume that

$$r(d) < r(s_0)$$

for some $d \in \mathbb{D}$. Assume also that for this d, there is no default. If there was, then the interest rate would be certainly higher. Since demand is bounded, due to the need of posting collateral, $r(d)$ must be higher than $r(s_0)$. Else, the household would be willing to post more collateral to acquire further long-term loans. Thus, a contradiction. And therefore, $r(s_t) \geq 0$ and $r(d) \geq 0$ for all $d \in \mathbb{D}$ and $s_t \in S \cup \{s_0\}$, which concludes the proof. □

Proposition 14. *There is no cash left at the end of period 1 i.e.,* $m^h(s_1) = 0$.

Proof. We will prove this by contradiction. Assume that household h has some unspent cash at the end of period 1 i.e., $m^h(s_1) > 0$. Then by equation (99), the household could have borrowed a bit more at the beginning of period 1 in the short-term money market, purchase more goods and hence, increase its consumption, and still have some money left. Since the agent maximizes its utility, this is a contradiction and thus,

$$m^h(s_1) = 0$$

□

Proposition 15. *The term structure of interest rates is given by*

$$r(s_0) M(s_0) + r(s_1) M(s_1) + \sum_d \left[R^d(s_1) M(d)(1 + r(d)) - M(d)\right] = 0 \quad \forall\, s_1 \in S \tag{108}$$

Proof. Summing equation (93) and equation (95) over all households, and using the market clearing conditions (103), (104), (106) and (107) we get that

$$\sum_d M(d) = r(s_0) M(s_0) + m(s_0) \tag{109}$$

where $m(s_0) = \sum_h m^h(s_0)$.

In a similar fashion, summing equation (99) over all households, using market clearing

conditions (104), (105) and (106) and also the result of proposition 15, we obtain

$$r(s_1) M(s_1) + \sum_d R^d(s_1) M(d) (1 + r(d)) = m(s_0) \tag{110}$$

Finally, subtracting equation (109) from equation (110) and re-arranging gives the desired result. □

The following proposition investigates whether a household transfers money within each period.

Proposition 16. *Assume that* $r(s_t) > 0 \quad \forall s_t \in S \cup \{s_0\}$. *Then, constraint (94) binds if the household borrows short-term in period 0 i.e.,* $\mu^h(s_0) > 0$ *and constraint (98) binds always. However, constraint (94) might be slack if the household does not borrow short-term in period 0.*

Proof. Assume that $\mu^h(s_0) > 0$ and that constraint (94) is slack. Then, the household can reduce its short-term borrowing by $\varepsilon(1 + r(s_0))$, without violating (94) and reduce its sales of good by $\frac{\varepsilon r(s_0)}{p(s_0)}$, without violating constraint (95). This again increases its utility since it increases its consumption. Thus, our initial assumption cannot be valid since there is a utility improving strategy. In case, that $\mu^h(s_0) = 0$, then the above proof does not work and thus, the result might fail.

Same logic applies for period 1 and constraint (98). In this case short-term borrowing is the only borrowing available and thus, the household will borrow. This implies that our proof above works always. This concludes the proof. □

Proposition 17. *If the household is selling goods in period 1, then the household must have borrowed short-term in that period. However, if a household is selling goods in period 0, then the household has not borrowed short-term necessarily.*

Proof. The household would never sell in the last period if it had not borrowed short-term since cash at the end is worthless. However, in the first period, the household might sell in order to save money for the next period. Furthermore, the household could have obtained the cash required for transactions through long-term loans only. Therefore, sales at the end of period 0 do not imply short-term borrowing necessarily. □

Proposition 18. *Households either sell or buy goods and either sell or buy real assets i.e.*

$$\theta_1^h(s_t)\,\theta_2^h(s_t) = 0 \text{ and } \phi_1^h(s_t)\,\phi_2^h(s_t) = 0 \quad \forall\, s_t \in S \cup \{s_0\}$$

Proof. Suppose, on the contrary, that $\theta_1^h(s_t)\,\theta_2^h(s_t) > 0$ for some $s_t \in S \cup \{s_0\}$. Assume further that $\mu^h(s_t) > 0$. Then, the household can reduce the amount of short term loan in that state by ε, spend ε less in the purchase of the good and sell $\frac{\varepsilon(1+r(s_t))}{p(s_t)}$ less of the good. Overall, this strategy increases the household's utility since it increases its consumption without violating any constraint. A contradiction.

Now assume that $\mu^h(s_t) = 0$. That implies that $s_t = s_0$, since if $s_t \in S$, then the amount the household would borrow in the short-term money market would be strictly positive in order to make sense for the household to sell goods. That implies that $\mu^h(d) > 0$ for some long-term loan d. Now, the household can decrease its long-term borrowing by ε, decrease its purchase of goods by ε and sell $\frac{\varepsilon(1+r(d))}{p(s_t)}$ less of the good.

That again would increase the household's utility since it increases its consumption and does not violate any of the constraints.

In a similar vein, we show that households either sell or buy real assets. That concludes the proof. □

Proposition 19. *If* $\delta_d X_{s_1} \frac{p(s_1)}{q(s_0)} > 1 \ \forall d \in \mathbb{D}$ *and* $\forall s_1 \in S$*, then*

$$r(s_t) = r(d) = 0 \quad \forall s_t \in S \cup \{s_0\}, \ d \in \mathbb{D}$$

Proof. By the assumption and equation (90), we conclude that there is no default on any contract d and state s. Thus, equation (108) becomes

$$r(s_0) M(s_0) + r(s_1) M(s_1) + \sum_d M(d) r(d) = 0 \quad \forall \, s_1 \in S$$

which means, due to the positivity of money supplies, that

$$r(s_t) = r(d) = 0 \quad \forall s_t \in S \cup \{s_0\}, \ d \in \mathbb{D}$$

□

The above proposition says that if no default occurs in equilibrium on any state, then all interest rates are zero. That would lead to indeterminacy of prices and shows the importance of default, in our model, in determining the prices.

Proposition 20. *The central bank can choose the collateral requirements (i.e., $\exists \delta_d \, d \in \mathbb{D}$), in such a way that default occurs in every state and the payoffs of these collateralised long-term loans span the entire state space.*

Proof. The central bank has to pick collateral requirements $\delta_d d \in \mathbb{D}$, such that $\delta_d X_{s_1} \frac{p(s_1)}{q(s_0)} < 1$ for any d and the payoffs of these contracts span the entire state space. First, $\delta_d \, d \in \mathbb{D}$ need to be positive so that the contract to be bounded.[20]We can choose the real asset's payoffs X_{s_1} for $s_1 \in S$ such that $X_{s_1} p\left(s_1\right)$ for $s_1 \in S$ are distinct and $\frac{q(s_0)}{p(s_1) X_{s_1}}$ for $s_1 \in S$ are in an increasing order. Now, since the total number of contracts is at least S, we can select δ_1 to be smaller than all of the ordered numbers, δ_2 to be larger than the smallest of the ordered numbers and smaller than the rest, δ_3 to be larger than the two smallest ordered numbers and smaller than the rest and so forth.

Hence, we ensure that the span of these payoffs is the same as the span of the following vectors

$$\{(1,1,\ldots,1),(0,1,\ldots,1),\ldots,(0,0,\ldots,1)\}$$

and therefore, span the whole space, and default is present in every state. □

The above proposition is crucial for our results. Default allows a set of collateralised bonds to be able to replicate the entire space of different payoffs which is spanned by the full set of Arrow securities. It, thus, enables quantitative easing, (that in reality is the purchase of bonds with different credit profiles and duration), to span the full set of

[20]Prices are bounded since positive collateral requirements ensure that the demand for these loans is bounded.

payoffs and thus, determine prices, as we will later prove. This is achieved, through the selection of exogenous collateral requirements by the central bank, that will ensure that at least one contract defaults in each state. In essence, this is equivalent to the central bank exogenously deciding the weights of the Arrow securities, as in McMahon, Peiris and Polemarchakis ([69]). According to McMahon, Peiris and Polemarchakis ([69]), the case, where the weights are exogenous, constitutes Credit Easing. Credit Easing is a subset of the more general definition for Quantitative Easing and it's the way that QE is implemented in practice. In our model, we achieve further, through the introduction of default, the above selection to be applied indirectly through the selection of collateral requirements rather than by directly setting the central bank's portfolio weights.

Corollary 8. *Assume that households trade in Arrow securities instead of collateralised long term loans. Let, Quantitative Easing to be a balance sheet policy where, the shares of Arrow securities held by households are endogenously determined. Then this definition is equivalent to definition 1, where households trade in collateralised long-term loans.*

Proof. First of all, note that in both definitions the shares of the specified contracts are endogenously determined.[21] Thus, in order to show the above, it is enough to show that there is a set of contracts i.e. collateralised loans, that span the the whole state space. But, this follows directly from the definitions and proposition 10. □

The above corollary confirms that definition 3 applies and ensures that the interest

[21]Note here, that since there is not a representative agent, endogeneity does not follow directly. However, under an interest rate target, the total issuance of each of the collateralised long-term loans is endogenously determined, and hence since each contract clears in a different market, the shares are also endogenous.

rates are positive (we will later show that default is a necessary and sufficient condition for the interest rates to be positive).

We will now show that households can obtain collateralised long-term loans and at the same time transfer cash from one period to the next. This is required in order to show that default is a necessary and sufficient condition for the interest rates to be positive.

Proposition 21. *Household h may transfer cash from one period to the next i.e. $m^h(s_0) > 0$ and also borrow in the collateralised long term money market i.e. $\mu^h(d) > 0$ for some $d \in \mathbb{D}$, simultaneously.*

Proof. First of all, household h will not carry cash into the next period if the household does not default on its long-term loan obligations. To illustrate this, assume that $m^h(s_0) > 0$. Since, $r(d) >> r(s_0)$, $\forall d \in \mathbb{D}$, household h would be better off to decrease the level of cash carrying forward by ε, borrow therefore $\frac{\varepsilon}{1+r(s_0)}$ more in the short term money market and borrow $\frac{\varepsilon}{1+r(s_0)}$ less in the long term money market. Then, in the beginning of period 1, household h will hold εless cash and also will repay $\frac{\varepsilon(1+r(d))}{1+r(s_0)} >> \varepsilon$, less on the long term loan and therefore, household h will be better off.

However, in the case that the household does default on its long term loan obligations in period 1, then it might carry cash forward from period 0. To illustrate this consider the first order condition for $m^h(s_0)$, using equations (93), (94), (95), (97) and (101),

$$\sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} = \frac{1}{1+r(s_0)} \frac{\vartheta U^h / \vartheta c^h(s_0)}{p(s_0)}. \tag{111}$$

for $m^h(s_0) > 0$, if the household borrows short term in period 0, and

$$\sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} = \frac{\vartheta U^h / \vartheta c^h(s_0)}{p(s_0)}. \tag{112}$$

$m^h(s_0) > 0$, if the household does not borrow short term in period 0.

The first order condition for long term loans, using (87)-(101), is,

$$(1 + r(d)) \sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} R^d(s_1) + \lambda^h \frac{\delta_d}{q(s_0)} = \frac{\vartheta U^h / \vartheta c^h(s_0)}{p(s_0)} \quad \forall d \in \mathbb{D}. \tag{113}$$

where, λ^h is the lagrange multiplier linked to the binding collateral constraint for the real asset.

Thus, combining (111), (112) and (113) gives that

$$(1 + r(d)) \sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} R^d(s_1) + \lambda^h \frac{\delta_d}{q(s_0)} = (1 + r(s_0)) \sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)}.$$

or

$$(1 + r(d)) \sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} R^d(s_1) + \lambda^h \frac{\delta_d}{q(s_0)} = \sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)}.$$

For the household to transfer cash to the next period and to borrow long term one of the above equations needs to be satisfied. Since $R^d(s_1) < 1$, for defaulted loans, we have that the second from the above equations can always be satisfied, which concludes the proof. □

Finally, in this chapter we proceed into showing that that positive interest rates can be supported under the presence of default.

Proposition 22. *There exist interest rates* $r(d) > 0$ *and collateral requirements* δ_d, $\forall d \in \mathbb{D}$, *such that all short term interest rates* $r(s_t)$ *are positive when* $0 < m(s_0) = \sum_h m^h(s_0) < \sum_d M(d)$.

Proof. Since $0 < m(s_0) = \sum_h m^h(s_0) < \sum_d M(d)$, we have by equation (23) that $r(s_0) M(s_0) > 0$. Thus, $r(s_0) > 0$. Moreover, by equation (24), we have that

$$r(s_1) M(s_1) + \sum_d R^d(s_1) M(d) (1 + r(d)) = m(s_0)$$

Therefore, by suitably choosing collateral requirements, δ_d, $\forall d \in \mathbb{D}$, the central bank can decide which loans will be repaid in full i.e. $R^d(s_1) = 1$ and which not such that the above equation holds for positive $r(s_1)$, which can happen in equilibrium, since $0 < m(s_0) = \sum_h m^h(s_0)$. □

Determinacy

Having proved the propositions in the previous section, we are ready to prove our main result which is the determinacy of prices and inflation under quantitative easing (as defined in definition 3 in the presence of default. Under interest rate targets, the central bank sets the interest rates and the collateral requirements. As we have already seen, in proposition 20, the central bank can set the collateral requirements such that there

is active default in every state. Moreover, due to proposition 22, active default suffices to support positive interest rates in every state and period in equilibrium. Finally, due to proposition 16, households do not transfer money within periods, i.e. constraint (94) binds.

The above suffice to show determinacy of prices and inflation. To proceed with the proof, we need to show that there is an equal number of equations and unknown variables to be determined as this suffices for uniqueness. Finally, the existence of the monetary equilibrium through proving the transvesality theorem, can be found in Tsomocos 2003 paper, Equilibrium Analysis, Banking and Financial Instability [95].

The central bank, as mentioned above, targets the collateral requirements, $(\delta_d)_{d\in\mathbb{D}}$, and the interest rates, $\left((r(d))_{d\in\mathbb{D}}, (r(s_t))_{s_t\in\{s_0\}\cup S}\right)$. The endogenous variable of the economy are $((M(d))_{d\in\mathbb{D}}, (M(s_t))_{s_t\in\{s_0\}\cup S}, (p(s_t))_{s_t\in\{s_0\}\cup S}, q(s_0), \left(c^h(s_t)\right)_{s_t\in\{s_0\}\cup S}$,

$\left(\theta_1^h(s_t)\right)_{s_t\in\{s_0\}\cup S}, \left(\theta_2^h(s_t)\right)_{s_t\in\{s_0\}\cup S}, \phi_1^h(s_0), \phi_2^h(s_0), \left(\mu^h(s_t)\right)_{s_t\in\{s_0\}\cup S}, \left(\mu^h(d)\right)_{d\in\mathbb{D}}$,

$, \left(l^h(s_t)\right)_{s_t\in\{s_0\}\cup S}, \lambda^h)_{h\in\mathbb{H}}$.

These variables will be determined by the budget constraints (93)-(102) and the clearing conditions (103)-(107), along with the first order conditions for $m^h(s_0)$, $\mu^h(d)$, $\phi_1^h(s_0)$, $\phi_2^h(s_0)$ and the complementarity slackness condition for the collateral constraint.

We have that, the first order condition for $m^h(s_0)$ is

$$\sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} = \frac{1}{1+r(s_0)} \frac{\vartheta U^h / \vartheta c^h(s_0)}{p(s_0)}. \tag{114}$$

if the household borrows short term in period 0, and

$$\sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} = \frac{\vartheta U^h / \vartheta c^h(s_0)}{p(s_0)}. \tag{115}$$

if the household does not borrow short term in period 0.

The first order condition for long term loans is,

$$(1 + r(d)) \sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} R^d(s_1) + \lambda^h \frac{\delta_d}{q(s_0)} = \frac{\vartheta U^h / \vartheta c^h(s_0)}{p(s_0)} \quad \forall d \in \mathbb{D}. \tag{116}$$

The first order condition for purchases of the real asset is given by

$$\sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} X_{s_1} p(s_1) + \lambda^h - \frac{q(s_0)}{p(s_0)} \vartheta U^h / \vartheta c(s_0) = 0. \tag{117}$$

and for sales is given by

$$(1 + r(s_0)) \sum_{s_1} \frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)} X_{s_1} p(s_1) + \lambda^h - \frac{q(s_0)}{p(s_0)} \vartheta U^h / \vartheta c(s_0) = 0. \tag{118}$$

The complementary slackness condition for the collateral constraint is

$$\lambda^h \left(\sum_d \mu^h(d) \frac{\delta_d}{q(s_0)} - e^h(s_0) - \phi_1^h(s_0) + \phi_2^h(s_0) \right) = 0. \tag{119}$$

By proposition 15, we know that the endogenous variables

$$\left(\left(M\left(d\right)\right)_{d\in\mathbb{D}},\left(M\left(s_t\right)\right)_{s_t\in\{s_0\}\cup S},\left(p\left(s_t\right)\right)_{s_t\in\{s_0\}\cup S},q\left(s_0\right),\left(c^h\left(s_t\right)\right)_{s_t\in\{s_0\}\cup S}\right)$$

will be determined by equations (97), (101) and the clrearing conditions. The variable $\left(\left(l^h\left(s_t\right)\right)_{s_t\in\{s_0\}\cup S}\right)_{h\in\mathbb{H}}$ by equation (102). Denote the set of the remaining free endogenous variables by Ω and their range by $I\left(\Omega\right)$.

Free variable	Equation
$\phi_1^h\left(s_t\right),\phi_2^h\left(s_t\right)\ s_t\in S\ h\in\mathbb{H}$	Equations (93), (94), (98), (99), (114), (115)
$\phi_1^h\left(s_0\right),\phi_2^h\left(s_0\right)\ h\in\mathbb{H}$	Equations (118), (119)
$\mu^h\left(s_t\right)\ s_t\in\{s_0\}\cup S\ h\in\mathbb{H}$	Equations (95), (99)
$\mu^h\left(d\right)\ d\in\mathbb{D}\ h\in\mathbb{H}$	Equation (116)
$\lambda^h\ h\in\mathbb{H}$	Equation (119)
$m^h\left(s_0\right)$	Equations (114), (115)
$\hat{m}^h\left(s_0\right),\ m^h\left(s_1\right)\ s_1\in S$	Proposition 16 or equation (95)

Define the set of utility functions for household $h\in\mathbb{H}$, as

$$\mathbb{U}^h=\left\{u^h,\ u^h\left(c,\,1-l\right)=\hat{u}^h\left(c,\,1-l\right)+\alpha c|\ \alpha\in\mathbb{R}^{S^*}\right\} \tag{120}$$

and

$$\mathbb{U}=\times_{h\in\mathbb{H}}\mathbb{U}^h \tag{121}$$

Every free endogenous variable in Ω is derived by an equation from the remaining equations. In each of these equations move the RHS to the LHS so as LHS-RHS=0 in equilibrium. Define globally the following map

$$\Pi : \mathbb{U} \times I(\Omega) \to \mathbb{R}^k \qquad \Pi(u, \Omega) = RHS \tag{122}$$

where k is the dimension of $I(\Omega)$ and RHS refers to the RHS of the re-arranged equations. Thus, if $\Omega \in I(\Omega)$ represents an equilibrium, then $\Pi(u, \Omega) = 0$. Thus, for an equilibrium to exist and be unique we must have that $\Pi_u^{-1}(0)$ must be a zero-dimensional manifold. The uniqueness is implied by the equal number of unknowns and equations and the existence by the fact that $\Pi_u : I(\Omega) \to \mathbb{R}^k$ is transverse to 0. The following theorem proves that this holds when the central bank targets strictly positive interest rates and sets the collateral requirements in such a way such that there is default in every state on some long term loan in period 1.

Theorem 4. *The full set of equilibrium outcomes is determinate for a generic u in $\mathbb{U}$ when the central bank targets strictly positive interest rates $\left((r(d))_{d\in\mathbb{D}}, (r(s_t))_{s_t\in\{s_0\}\cup S}\right)$ and sets collateral requirements δ_d, $d \in \mathbb{D}$ such that there is default in exactly one long term loan d in every state s, assuming that for every agent h and for all collateralised long term loans d, the real payoff of a riskless bond $\frac{1}{q(s_0)}$, the real payoff of the long term collateralised loan $\frac{R^d}{q(s_0)}\, d \in \mathbb{D}$ and the real payoff of the real asset $\frac{X_{s_1} p(s_1)}{q(s_0)}$ are linearly independent.*

Proof. We start by considering the first order conditions for $m^h(s_0)$, the long term loans and the asset. By following proposition 21, equations (115) and (116) can either

be considered together or not. Based on the above assumption that the vectors of the real payoff of a riskless bond $\frac{1}{q(s_0)}$, the real payoff of the long term collateralised loan $\frac{R^d}{q(s_0)}\, d \in \mathbb{D}$ and the real payoff of the real asset $\frac{X_{s_1} p(s_1)}{q(s_0)}$ are linearly independent, we can adjust $\frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)}$ in a direction which is orthogonal all excpet one of the above vectors. Thus, we can unilaterally perturb any from the equations in this set.

To perturb equation (96), choose δ_d without changing $\mathbb{I}^d(s_1)$. To perturb equations (93) and (94) increase $\mu^h(s_0)$ by $\varepsilon > 0$. Therefore, h spends $\frac{\varepsilon}{1+r(s_0)}$ more in her purchases. This disturbs equation (95) since some other agents receive ε^h more. Assuming that one other agent receives them all, we must have that $\varepsilon^h = \frac{\varepsilon}{1+r(s_0)}$. We proceed by considering two different cases.

Suppose that this subsequent agent that receives all, increases her borrowing, and thus, can increase her purchases by $\frac{\varepsilon}{(1+r(s_0))^2}$. Continuing with this logic, and the fact that each subsequent agent can borrow at the same interest rate $r(s_0)$, the total increase in borrowing is $\sum_0^{\infty} \frac{\varepsilon}{(1+r(s_0))^n} = \frac{(1+r(s_0))\varepsilon}{r(s_0)}$, and hence, the total increase in spending goes up by $\frac{\varepsilon}{r(s_0)}$ which is finitely small.

Suppose that the subsequent agent h does not increase her borrowing. Then, this means, that she would carry the amount ε^h into the next period and will increase her purchases in the next period by this amount which will be transferred to another agent h^1 and hence, $\varepsilon^{h^1} = \frac{\varepsilon}{1+r(s_1)}$. From this point we return to case 1, and thus the total borrowing and purchases will increase by a finitely small amount.

It follows that $p(s_0)$, $p(s_1)$, $c^h(s_0)$ and $c^h(s_1)$ change also by a finitely small amount for all $h \in \mathbb{H}$. Perturbing utility functions can restore the ratio $\frac{\vartheta U^h / \vartheta c^h(s_1)}{p(s_1)}$. The proofs

for the remaining equations follow the same logic. To perturb equation (119), adjust $\mu^h(d)$ for some $d \in \mathbb{D}$. We re-adjust as above.

Therefore, the mapping $\Pi_u : I(\Omega) \to \mathbb{R}^k$ is transverse to 0, proving the determinacy of monetary outcomes, i.e., prices and inflation path. □

Concluding Remarks

This paper presented a model that features trade in fiat money, one physical good, one real asset, riskless short term loans and collateralised risky long term loans. It was showed that under Quantitative easing policies under which the central bank purchases a variety of credit risk and duration bonds, the price level and inflation path is determined when the central bank targets positive interest rates and decides the collateral requirements of the bonds in such way that default is present in every state of the world in the final period of the model. Default is an outcome when the value of the collateral is less that the repayment on the long term collateralised loan. The support of positive interest rates comes from the presence of default since the extra loss of monetary wealth by the E.C.B. is replaced by interest rate payments. This, in turn, makes agents not to transfer money from the first period to the second since there is a positive cost associated with obtaining loans. Thus, the bank makes losses in its long term lending but makes profits in its short term lending trough the positive interest rates.

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