Bicategorical Aspects of Topological Quantum Computation



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Abstract

Topological quantum computation offers a paradigm shift for fault-tolerant quantum computation. It proposes encoding qubits in the mutual states of quasiparticles (called anyons) of 2-dimensional topologically ordered matter. Unitary ribbon fusion categories describe the algebraic theory of anyons. Although the category of an anyonic theory specifies suitable computational spaces, qubits, and quantum gates, it does not precisely formalise the category of topological quantum computation (TQC). We describe the TQC category using the elements of categorical quantum mechanics (CQM). Given the qubit-based graphical calculus accompanying CQM, called the ZX-calculus, we present the details of the ZX-representation of Fibonacci and Ising anyons. Finally, we propose a bicategorical approach to convene anyonic theory and TQC in one (bi)category.

In the second part of the thesis, we discuss how the bicategory of module categories formalises boundary types and their excitations. We show theorems that reduce the data to physically meaningful data. A 2-category capable of describing topological quantum computation with point-like and string-like excitations is speculated to be a fusion 2-category. We discuss the available definition in the literature. The paper particularly lacks a rigorous definition for additive 2-categories, in this thesis we give a definition. We propose the idea of typing higher-ranked matrices in additive 2-categories. Lastly, we overview the complete definition of monoidal 2-categories and recover Kapranov and Voevodsky's definition from the algebraic definition of tricategory by Gurski. This definition and the detailed list of data are the complete ingredients physicists need for quantum computation with higher dimensional excitations.

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Choosing a poem was a puzzling task, I changed it multiple times; from Saadi to Whitman, Larkin, Lorca and Bukowski, but I found nothing better than Lady Lazarus by Plath, so one can read between its lines!

Ash, ash—
You poke and stir.
Flesh, bone, there is nothing there——
A cake of soap,
A wedding ring,
A gold filling.
Herr God, Herr Lucifer
Beware
Beware.
Out of the ash
I rise with my red hair [...]

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Chapter 1

Introduction

Quantum computation, using quantum mechanics for computation, was proposed by Yuri Mannin in [44], and quantum computing as a means of simulating complex quantum systems by Richard Feynman [49]. The two, almost revolutionary, ideas underpin the quest for building a quantum computer; a device capable of exploiting quantum features to simulate and compute. However, encoding and computational errors are still the main obstacle [48].

To address this issue, various approaches have been pursued, including one that was jointly proposed by Alexei Kitaev and Michael Freedman [30, 23]. Their proposal represents a radical shift in two key aspects: firstly, constructing the hardware using topologically ordered systems, and secondly, performing computations using point-like (quasi)particles known as Anyons within these systems. This approach offers the promise of robust encoding and computation, capable of withstanding local errors and perturbations. In a recent publication by Microsoft Figure 1.1, practical applications of quantum computers are discussed, highlighting the need for millions of qubits to solve problems like Shor's factorization [9], a task currently infeasible within the NISQ regime [48]. As we will explain later, topological qubits, unlike other schemes, are scalable, that makes a strong case for topological quantum computation.

Another rather interesting aspect of topological quantum computation is its location at the junction of three modern areas of active research fields, topological condensed matter physics, category theory, and quantum computation. To summarise topological quantum computation in one line, topological quantum computation uses topologically ordered matter to do quantum computation whose algebraic theory is described by category theory. In this thesis, we are interested in the last part, categorical description.

The categorical depiction of topological quantum computation is inspired from the categorical representation of topological quantum field theories [5]. Category theory

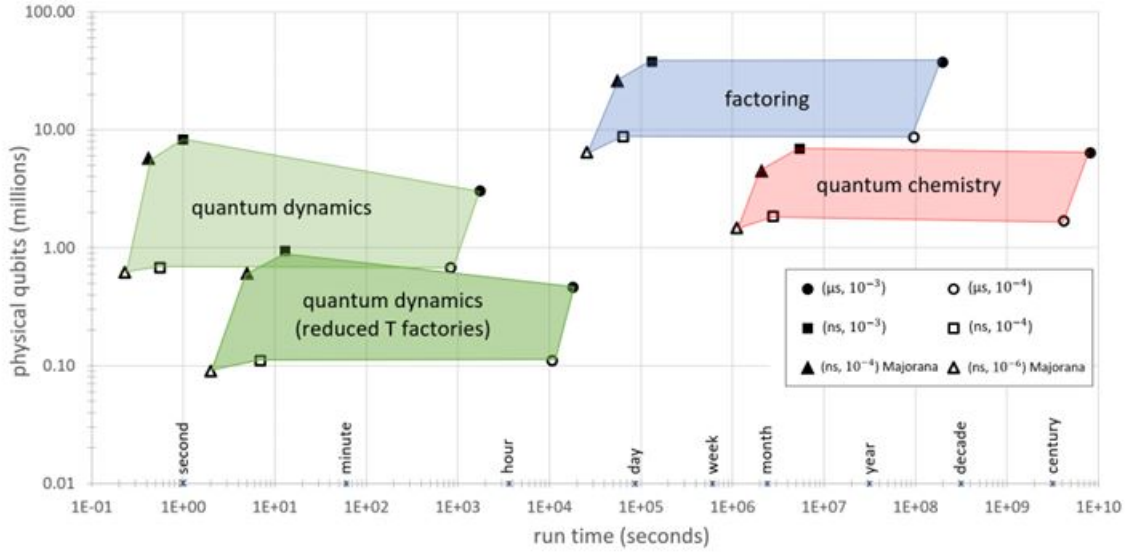


Figure 1.1: Estimates of the resources required to implement three applications [9].

is useful for comprehending and formalizing peculiar characteristics of anyons, such as fractional spins and charges, which may seem counter-intuitive when compared to the familiar bosonic and fermionic statistics. The shift in language occurs naturally because category theory, being concerned primarily with morphisms' properties, remains largely indifferent to the nature of objects involved [41]. In this framework, anyons are regarded as objects within a category, while processes like creation/annihilation operators are viewed as morphisms. Essentially, the categorical formalism allows us to bypass considerations of anyons' wave functions or their Hilbert spaces, leading to a comprehensive depiction of topological quantum computation based on processes transitioning between mutual states of anyons (highlighting the paradigm shift from encoding in anyonic states to morphisms between them). This categorical framework encompasses additional structures and properties that capture various aspects of anyons, such as a monoidal product to formalize their mutual statistics, braiding to capture their statistical behaviour, and unitarity of processes or morphisms to represent physical operations [63, 53].

Abramsky and Coecke introduced an alternative categorical viewpoint on quantum mechanics [3], where the emphasis shifts from quantum states of physical systems to solely examining processes and their compositions. This novel approach seeks to reconcile classical and quantum physics, offering insights into long-standing paradoxes such as causality and non-locality. One immediate question concerns with the relationship between categorical quantum mechanics and the category of topological

quantum computation.

While categorical quantum mechanics (CQM) aims to describe quantum mechanical systems, topological quantum computation (TQC) represents a computational model. Therefore, one expects to view the latter as a subcategory of the category of Hilbert spaces or **Hilb**. However, this does not seem to be straightforward with the current categorical description of topological quantum computation. In the current picture, anyon types are objects of the category and morphisms between them are creation/annihilation operators, and the computational spaces are sets of these morphisms. Therefore, any map from TQC to **Hilb** that sends hom-sets to objects is not functorial.

In the first part of the thesis, we delve into this inquiry and establish a clear distinction between a category that describes an anyonic theory and a TQC (topological quantum computation) category. Starting with an anyonic theory, we create the computation category by considering fusion spaces as objects and fusion and braiding matrices as morphisms. With this differentiation in mind, we observe that the TQC category, as defined, aligns with categorical quantum mechanics, and the functor between this category and **Hilb** (the category of Hilbert spaces) can be precisely characterized as an inclusion.

A notable outcome of categorical quantum mechanics is graphical calculi, including the ZX-calculus. Within this framework of TQC, we employ the ZX-calculus to depict elements of two renowned models of topological quantum computing: Fibonacci anyons and Ising anyons. By utilizing the ZX-representation of Fibonacci anyons, we showcase how braids can be simulated and simplified in a purely graphical fashion, eliminating the need to explicitly handle the numerical value associated with the Golden ratio. This graphical approach offers a convenient and intuitive means of working with these models, providing a visual representation that facilitates computational tasks.

An unanswered question is whether the categorical depiction of an anyonic theory and the TQC category, as defined by us, can be reconciled in a categorical manner. To address this question, the appropriate framework is that of a bicategory. We suggest that if one treats anyon types as objects, then Hom-categories can be used to describe various models of topological quantum computation. Within this bicategory, the 1-morphisms correspond to fusion/splitting spaces, while the 2-morphisms represent fusion/braiding matrices. By adopting this perspective, we can explore the relationship between the categorical description of an anyonic theory and the TQC category in a unified framework.

The overview provided thus far only scratches the surface of a much larger framework. Anyons, being point-like quasiparticles within two-dimensional topologically ordered systems, represent a subset of the possibilities. Higher-dimensional systems have the potential to exhibit higher-dimensional excitations. Furthermore, distinctions can arise between bulk and boundary excitations, each possessing unique characteristics. To adequately capture these intricacies, a shift towards higher categories becomes necessary [64]. This transition allows for a more comprehensive representation of the diverse phenomena that can arise in topological systems.

In recent years, there has been a growing effort to extend the categorical formalism of 2+1-dimensional topological orders to higher-dimensional n-categories, aiming to describe higher-dimensional topological field theories and topological orders. However, some foundational aspects are still missing or not rigorously defined. While practical questions, such as finding concrete examples of n-dimensional topologically ordered systems, need to be addressed, there are also fundamental mathematical concerns. These include developing a complete and physically "useful" definition of monoidal n-categories, establishing a rigorous description of n-limits, and the lack of well-developed categorical and algebraic examples.

In this work, we focus specifically on the bicategorical aspects of topological quantum computation, which manifest in various contexts such as bulk-boundary conditions in the Kitaev model or as a suitable general framework for computation with point-like and string-like anyons [32]. Our goal is to address and construct some of the missing foundational blocks in the existing literature.

We discuss the bicategory of module categories and present theorems that allow us to reduce the necessary data to more physically relevant data. We propose an algorithmic approach for calculating module categories over fusion categories, and we demonstrate how working with module categories with strict unitors reduces the number of pentagonal equations for modules and their associators.

In the final chapter, we delve into the current definition of fusion 2-categories as outlined in the literature [19]. Through a close analysis of the definition, we identify certain gaps, such as the neglect of additivity, discussion on limits, and finiteness conditions. Additionally, while the definition accommodates semistrict monoidal 2-categories, for a topological quantum computation model, the inclusion of the 3-D pentagonal equation becomes necessary.

To bridge these gaps, we propose a definition for additive bicategories and explore how this category can be useful for typing tensor calculus. While monoidal 2-categories have been discussed in previous works, a comprehensive definition without

strictness on some data has been lacking, and we provide it here. By combining the current definition of fusion 2-categories with our proposed definitions for additive and monoidal 2-categories, we establish a complete structure that captures the necessary conditions for computing with point- and string-like excitations in higher-dimensional topological quantum computation models.

Chapter 2

Topological Quantum Computation

Elements of a quantum computer, similar to a classical computer, are grouped under two lists; the device or hardware and the process of computation [2]. Moreover, similar to a classical computer, computation, to some extent, can be thought of as a purely mathematical process, meaning one can eliminate all elements of actualising each component and boil down everything to the similar questions in the classical case [46]. For example, what are the inputs, and how do we encode them? What is the process of computation, and how do we represent computation? What is the output, and how do we read the output? Is the output reliable enough? Furthermore, if not, how do we correct them to reliable enough data?

That is precisely the procedure quantum computation has gone through [2]. For each question, one needs an answer, and one needs to eliminate physical components and transfer the answers to the most fundamental theoretical model. This procedure guarantees the universality of computation, no matter the physical device's nature. The theoretical version of quantum computation has been well-developed, from its first baby steps to toy algorithms such as Deutsch-Joza's or Simon's algorithms to helpful and practical ones such as Shor's.

However, building a physical device capable of answering these questions "reliably" remains the main challenge of quantum computing [47]. To be more precise, the "reliable manner" is the main challenge. By reliability, we mean a quantum device capable of encoding, computing, and readout with minimum correctable errors and a scalable device. Scalability mainly refers to the fact that each of these processes and specific encoding can be multiplied without substantially increasing undetectable errors; otherwise, the system, in most cases, loses its quantumness (eg. quantum features such superposition and entanglement) and subsequently, the promised extra computational power.

This burden is an engineering problem as much as an inquiry about the nature of the physical hardware or device. In other words, of course, one might be able to improve the design and features of physical components. However, one might also be able to build an intrinsically robust device which fulfils this task, similar to transistors in the history of classical computers [45]. The second approach summarises the aim of topological quantum computing.

The initial idea of topological quantum computation was put forth by Alexei Kitaev in 1997 [30]. The paper includes, among other proposals, at least four main ideas:

- an error-correcting architecture with 4-body interactions,
- an exactly solvable toy model for topologically ordered systems; similar to the Ising model for quantum systems,
- ground states of topologically ordered systems as robust quantum memories,
- fault-tolerant quantum computation with excitations of these systems, i.e. anyons.

Excitations of topologically ordered systems are quasiparticles with fractional spins and electric charges. Because of various statistics, Frank Wilczek named these quasiparticles anyone without e, i.e. anyon [65]!

Anyonic statistics were well-known and well-studied in statistical mechanics, and fractional quantum Hall systems [65]. Even a Gedankenexperiment on 2- and 3-dimensional N-particle configuration spaces shows the variety of particle types in 2-dimensional versus three-dimensional space [37]. It also demonstrates statistical behaviour of particles can be analysed and understood by the first homotopy group of paths that particles create while moving in space. For example, if one considers two point-like particles in the 2-dimensional space, one of them acts as a hole, because that specific position in the space is occupied and cannot be crossed or occupied again. Thus, moving the other particle around the hole creates non-contractible loops or, in other words, $\pi_1(C_2) = \mathbb{Z}$. However, one can easily use the extra third dimension in the three-dimensional space and contract loops to zero. So the statistics of particles in 2-dimensional space are described by Braid groups, while in three dimensional space, it is the permutation group. The magic happens because 1-dimensional representations of the braid group are specified by a general complex factor, $e^{i\theta}$ where $\theta \in [0, \pi]$, while 1-dimensional representations of permutation are only ± 1 or $\theta = 0, \pi$, with plus sign classifying bosons and minus sign fermions. So in the 2-dimensional space, in principle, we can have some other statistics in addition to bosons and fermions [37, 55].

Before the Kitaev model, the only well-studied physical system to actualise anyons were fractional quantum Hall systems. An essentially 2-dimensional electronic gas formed at the interface of a semiconductor and an insulator subject to high magnetic fields ($10T$) and low temperatures ($10mK$). Under these conditions, the system creates coupled fluxes and charges which exhibit anyonic statistics. The Kitaev model and quantum Hall system represent prototypical examples of topological orders [64, 65, 11].

Studying a comprehensive definition for topological quantum orders is out of the scope of this thesis. However, the definition we are interested in, and to some extent is aligned with our perspective, states: quantum orders occur in systems whose "effective" background fields are topological (TQFT) [55]. A relatively intuitive definition describes TQFTs as quantum field theories without any dependency on metric or geometrical features of a system; they are only responsive to the topology. Other easy-to-check criteria to detect topological phases include anyonic excitations, dependency of the degeneracy of ground states' to topology, and non-zero topological entropy. These criteria are **iff** conditions [64, 66, 33].

Michael Freedman proposed that topological quantum computers can be a simulator for topological quantum field theories [23]. Moreover, since braids and knots are native to these systems, one can solve intricate knot problems by simulating knots on a topological quantum computer. Although it is shown that any other quantum computer can essentially solve those problems [4]; the nativity of braids and knots to these systems mainly inspired thinking about solving complex open problems in knot theory with quantum computers, among which the Jones polynomial [4]. Given a topological quantum field theory, a polynomial is a certain correlation function of the field.

Another older piece of history that we ought to mention is the categorical formalisation of topological quantum field theories by Michael Atiyah [5]. Atiyah, inspired by Graeme Segal's categorised version of conformal field theories, defined TQFTs as certain functors from the category of cobordisms to the category of vector spaces [59, 5]. So one can naively assume a field is nothing more than a specific functor which assigns to a time-space bordism a vector space [34].

$$\mathcal{F} : \mathbf{Bord_3} \longrightarrow \mathbb{V}_{\mathbb{C}} \tag{2.1}$$

Thinking about the combination of anyonic features and the categorical formalism of TQFT and considering one of the mottoes of category theory, i.e. agnostic to the properties of objects, one can appreciate the reasoning behind formalising topological

quantum computation with category theory. Categorical description washes out the bizarre anyonic features such as fractional spin and charge and boils down everything to morphisms between them which are essentially qubits and gates for quantum computation [63, 52]. This category, under certain conditions, will be able to capture braiding, twist, and unitarity in a well-defined unified manner.

In the next section, we revisit the categorical description of topological quantum computing. Although the theory is well-established, there are some gaps in the literature, such as the rigorous spelling of conditions. We intend to address those gaps in this thesis.

In addition, in the literature, a category of anyonic theory is often convoluted with the category of topological quantum computation. The latter has never been spelt out explicitly and only briefly mentioned in Appendix E of [31]. This construction appears here. The other assumed-to-be-obvious not-spelt-out property is the skeletally of fusion categories physicists work with. We explicitly spell out this property and show how taking this equivalent subcategory reduces the data for specifying an anyonic theory.

2.1 Categorical Description of TQC

Before discussing the details of categorical description, we summarise some key properties of anyons and topological quantum computation one needs to know to grasp better the underlying categorical description. For gaining a better intuition, the details of physical systems can be helpful, so we introduce the Kitaev $\mathbb{Z}_2$ model or Toric code and carry this example along the way to illuminate each part. Later, when we discuss boundary conditions and module categories, we will recall this example.

2.1.1 Kitaev Model

Consider a grid of spins $\frac{1}{2}$; each spin is located on the edge of each square of the lattice. Spins have 4-body interactions via two operators, stars, A_s, acting on each vertex and plaquette operators, B_p, acting on each plaquette [30].

$$A_s = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x, \qquad B_p = \sigma_5^z \sigma_6^z \sigma_7^z \sigma_8^z, \qquad H = -\sum_s A_s - \sum_p B_p \tag{2.2}$$

The choice of 2×2 Pauli operators is not coincident, given that spins take values from Z_2. In the extension of the model to Z_n, spins take values from 0 to $n-1$ and plaquette, and star operators are $n \times n$ Pauli operators [10].

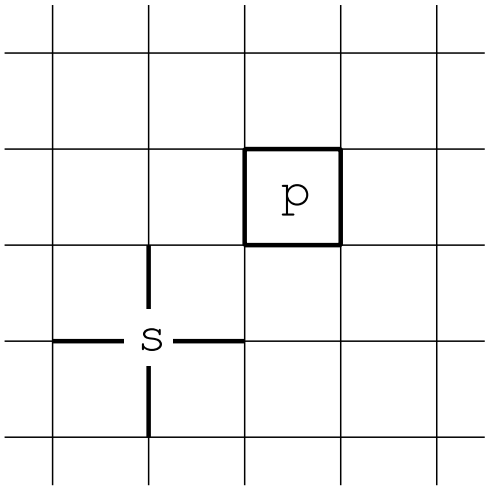


Figure 2.1: Square lattice on the torus [30].

One can easily check all star and plaquette operators commute with each other. The Hamiltonian is a sum of these operators, which also commutes with all star and plaquette operators.

$$[A_s, A_{s'}] = 0, \qquad [B_p, B_{p'}] = 0, \qquad [\hat{H}, A_s] = 0, \qquad [\hat{H}, B_p] = 0 \tag{2.3}$$

It is not difficult to check any closed loop, i.e. combination of star and plaquette operators, also commute with each other and the Hamiltonian. The non-commutativity happens when one has an open string of Paulis along vertices or plaquettes. For example, given an open string of σ_x operators, we can check since the two ends only cross one edge of each plaquette, the open string does not commute with plaquette operators at both ends of the string. The same happens for open strings of σ_z operators and star operators at both ends of the string.

We assign to the violation of commutation relation at both ends, quasi-particles, due to the energy increase of the system. Particles created in plaquettes are called magnetic charge m, and particles created at vertices, electric charge e, and we call the combination of an e and m, epsilon, ϵ. We recommend keeping the properties of this model in mind for future referrals.

2.1.2 Anyons and Braid Group

In this section, we aim to take a closer look at anyon properties and braid group. As we mentioned, anyons are the excitations of 2-dimensional systems whose statistics are described by representations of the braid group. To see a more intuitive picture

instead of the abstract picture by Leinaas and Myrheim, that we described earlier. Consider a combination of N particles located in a 2-dimensional space and assume one changes the location of these particles, so it changes the initial configuration at $t = t_0$ to another configuration at time $t = t_1$. If one sees the picture of this exchange in $2+1$-dimensional space, the world lines resemble braids [55].

Definition 1. *A **Braid group** on $n-$strands consists of $n-1$ generators, $< \sigma_1, \sigma_2, ..., \sigma_{n-1} >$, such that the generator σ_i, braid strands i and $i+1$ subject to the following relations.*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j| \geq 2, \tag{2.4}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \forall i \in \{1, .., n\}. \tag{2.5}$$

The interesting point about the braid group is its rich representation theory [29]. The braid group on n-anyons, B_n, can generally have non-trivial 1- to n-dimensional representations. 1-dimensional representations classify **abelian anyons**, and higher dimensional representations **non-abelian anyons**.

Similar to other particles, anyons obey selection and super-selection rules, meaning anyons are created and annihilated in pairs, i.e. particle and antiparticle. Furthermore, there is not any process which transfers a single anyon of, for example, Type A to Type B, and one cannot make a coherent state from different super-selection sectors [31]. To understand the latter part, try to build a coherent state of a boson and a fermion. Bosonic and fermionic states, by definition, are living in two different perpendicular sectors of Hilbert's space. Moreover, bosonic states are symmetric and fermionic states anti-symmetric, so making a coherent state is impossible [31].

Another important property of anyons, similar to any other theory of particles, is their mutual statistics which we describe as **fusion** of anyons. But note that the term fusion is rather general and specifies the behaviour of mutual statistics, not a collision between particles. In other words, given a box containing two anyons, fusion rules specify the outcome statistics of the box.

To better understand the fusion of anyons, let's go back again to the Kitaev model. One can conceive of fusion by enclosing two particles inside a loop created by a test particle moving around them. For instance, if we have two $e's$ and enclose them by the loops created by e or m, we do not see any change, or any sign will not be picked up by the Hamiltonian. But if we have e and m particles and enclose them by e- or m-loops, both paths pick up a minus sign because moving a particle around another particle means two exchanges, the exchange phase is $\frac{\pi}{2}$. Hence, this shows the combination or fusion of an m and e creates another particle, which we earlier

called ϵ, and this is a fermion. To summarise, the fusion rules of excitations of the Kitaev model are as follows,

$$e \otimes e = 1, \qquad m \otimes m = 1, \qquad e \otimes m = \epsilon \tag{2.6}$$

Where 1 represents the vacuum. One rule that should be immediately obvious is the fusion of 1 with any particle, which results in that particle and keeps the particle and state of the system intact. Another interesting property of the Kitaev model is particles and antiparticles are the same.

The associativity of fusion rules is another physical expectation. Intuitively, we do not expect the box distinguishes between $(a \times b) \times c$ and $a \times (b \times c)$.

We can also explicitly check the braiding of particles in the Kitaev model. Note that due to the nature of paths which create different particle types, e-particles living on vertices and m-particles living in plaquettes, we cannot change their locations. But we can move one particle around the other one to pick up a phase which is twice of the phase they would pick up, had we changed their locations directly.

Observe that $e's$ with respect to each others are bosons, and so do $m's$. But when we braid an e around m and vice versa, the state picks up a phase of $\frac{\pi}{2}$. That essentially proves the Kitaev model represents a topological order, and the nature of quasiparticles is anyonic. Because as we mentioned earlier, the observation of anyonic statistics in a system implies the system is topological.

Although the picture we have sketched so far gives us some intuition about the bizarre features of anyons, it still lacks the generality of the complexity of non-abelian anyons. For example, if one considers open strings responsible for producing pairs of anyons as creation operators, for each pair of anyon and anti-anyon, we only had one creation or annihilation of a single string. However, in general, for non-abelian anyons, either there is more than one version of a string type or more than one type.

Another way of looking at this could be through the reverse process of creation, i.e. annihilation. Imagine you have a box containing moving anyons and anti-anyons; the probability of these particles meeting each other head-to-head to annihilate depends on the different possible ways or the number of annihilation operators [47]. If a pair has more than one way of creation or annihilation, the probability increases obviously. If it still seems rather difficult to digest, there is no way other than running the above analogy with the Kitaev model with a non-abelian group symmetry, for example, S_3, or studying the fractional quantum hall effect. This, however, is out of the scope of this thesis, and we trust the power of mental generalisation! Otherwise, the reader can consult the last section of [30], Bombin's review [10] or Simon's lecture notes [60].

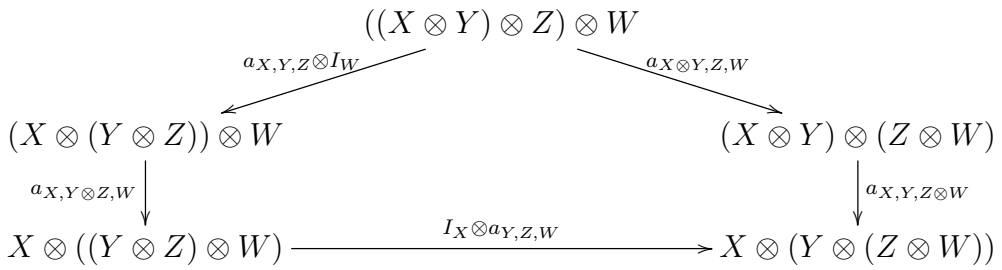


Figure 2.2: Pentagonal equation [35]

Keeping in mind the physical expectations of an anyonic model, i.e. fractional charges and spins, fusion and braiding properties, in the next section, we discuss the categorification of the model.

2.1.3 Categorical Formalism of TQC

We assume the reader has an elementary knowledge of category theory at the level of Leinster's Basic Category Theory [39].

As mentioned above, the algebraic description of TQC was developed using category theory, but some unspelled gaps have remained in the literature. We intend to sketch a comprehensive fully categorical picture as Kitaev wished to provide in Appendix E of [31]. Note that the categories we work with are all small, meaning the category has a set of objects and morphisms between each pair of objects form sets.

We first provide some necessary background on monoidal, semi-additive, semi-simple, braided, ribbon, and unitary categories. We, then, outline a description of TQC given these structures and provide some theorems to reduce the amount of data one needs to calculate to fully specify an anyonic theory.

Definition 2. *A* ***monoidal category*** $(\mathcal{C}, \otimes, a, l, r, I)$ *is a category with the unit object,* I*, a functor* $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *and a natural transformation* $a : (- \otimes -) \otimes - \longrightarrow - \otimes (- \otimes -)$ *as associator and left- and right-unitors* $l : I \otimes - \longrightarrow -$ *and* $r : - \otimes I \longrightarrow -$ *such that they satisfy pentagonal and triangle equations. If* a*,* r*, and* l *are identity morphisms, the category is* ***strict****.*

A well-developed example of a monoidal category is the category of vector spaces with either direct sum or tensor product. But as we see later, while direct sum is a property of this category, tensor product is a structure, i.e. the nature of direct sum is a property of category and stems from limits, but tensor is a functor on the category.

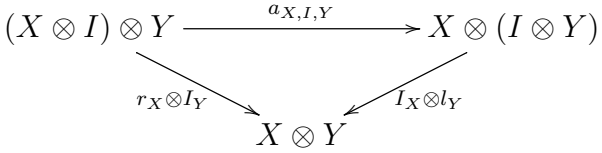


Figure 2.3: Triangle equation [35].

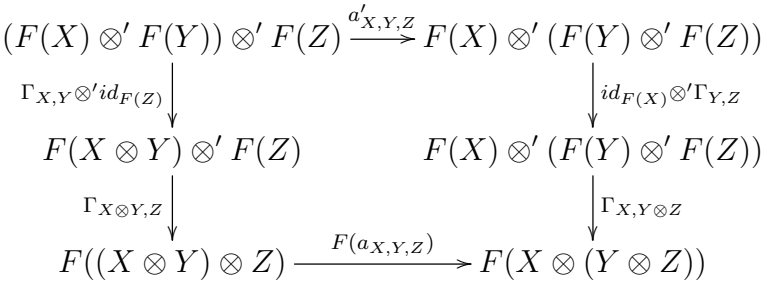


Figure 2.4: Monoidal functor condition.

Another example is a discrete group as a category, taking elements of a group as objects and multiplication as tensor product. Understanding the full description of these examples, one can build algebraic examples; such as the categories of modules or representations of a group or an algebra.

Definition 3. *A (strong) monoidal functor between two monoidal categories is* $(F : \mathcal{C} \longrightarrow \mathcal{C}', \Gamma : F(X \otimes Y) \longrightarrow F(X) \otimes' F(Y), \gamma : F(I) \longrightarrow I')$ *where* γ *and* Γ *are natural isomorphisms, and they satisfy Condition 2.4.*

Definition 4. *A category is called skeletal if isomorphic objects are necessarily identical.*

In other words, one can skeletalise a category by keeping only one object from each set of isomorphic objects and throwing away all morphisms to/from omitted objects. Note that a skeleton of a category is the smallest subcategory containing all information of the main category. Here, we state a proof for monoidal categories; which guarantees a monoidal category is monoidally equivalent to its skeleton. The proof we bring here is mainly based on [57]. We then use this proof sketch to prove similar theorems for module categories and bicategories.

Lemma 1. *Let* $(\mathcal{C}, \otimes, a, I, r, l)$ *be a monoidal category. Suppose we are given another monoidal product* $\otimes' : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *via natural isomorphisms* $\xi_{X,Y} : X \otimes Y \longrightarrow$

$X \otimes' Y$, *a unit object* I' *isomorphic to* I *via* $\xi_0 : I \xrightarrow{\cong} I'$. *Then, there is a unique monoidal structure on* $\mathcal{C}$ *such that* $(id, \xi, \xi_0) : (\mathcal{C}, \otimes, a, I, r, l) \longrightarrow (\mathcal{C}, \otimes', a', I', r', l')$ *is a monoidal functor.*

Proof. To prove this theorem, we define the associator a' and unitors r' and l' in 2.7. Then, we have to show they are natural and satisfy pentagonal and triangle equations.

$$a'_{X,Y,Z} = \xi_{X,Y\otimes Z} \circ (id_X \otimes \xi_{Y,Z}) \circ (a_{X,Y,Z}) \circ (\xi^{-1}_{X,Y} \otimes id_Z) \circ (\xi^{-1}_{X\otimes' Y,Z}) \tag{2.7}$$

$$r'_X = r_X \circ (id_X \otimes \xi_0^{-1}) \circ \xi^{-1}_{X,I}, \qquad l'_X = l_X \circ (id_X \otimes \xi_0^{-1}) \circ \xi^{-1}_{I,X} \tag{2.8}$$

Pentagonal equation: In the pentagonal equation for a', the outer diagram of the Figure 2.5, is commutative because the central diagram is the commutative pentagonal equation for a and due to the definition of a' five squares around are also commutative.

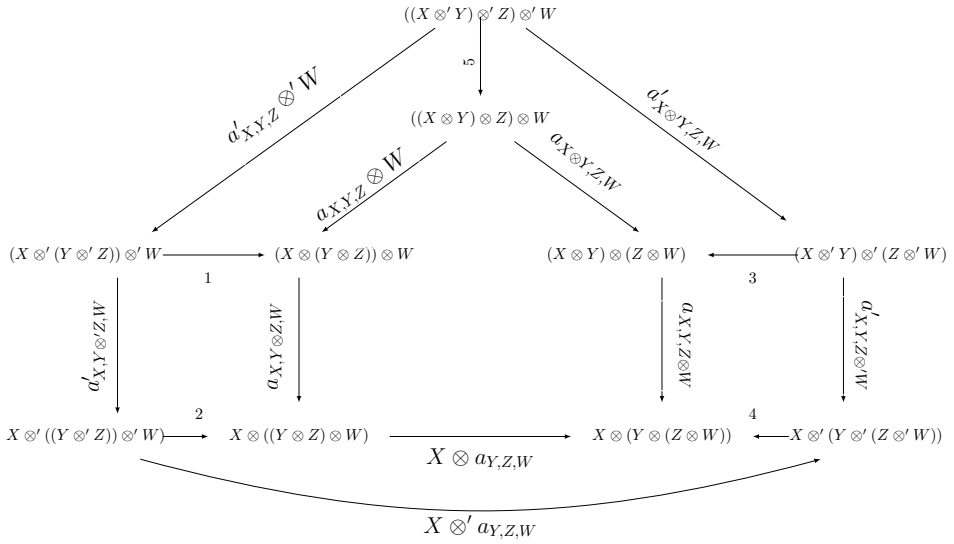


Figure 2.5: Proof of Lemma 1.

The missing arrows are as below,

1. $(id_X \otimes \xi^{-1}_{Y,Z} \otimes id_W)(\xi^{-1}_{X,Y\otimes' Z} \otimes id_W)(\xi^{-1}_{X\otimes'(Y\otimes' Z),W})$
2. $(id_X \otimes \xi^{-1}_{Y,Z} \otimes id_W)(id_X \otimes \xi^{-1}_{Y\otimes' Z,W})(\xi^{-1}_{X,(Y\otimes' Z)\otimes' W})$
3. $(\xi^{-1}_{X,Y} \otimes \xi^{-1}_{Z,W})(\xi^{-1}_{X\otimes' Y,Z\otimes' W})$
4. $(id_X \otimes id_Y \otimes \xi^{-1}_{Z,W})(id_X \otimes \xi^{-1}_{Y,Z\otimes' W})(\xi^{-1}_{X,Y\otimes'(Z\otimes' W)})$
5. $((\xi^{-1}_{X,Y} \otimes id_Z) \otimes id_W)(\xi^{-1}_{X\otimes' Y,Z} \otimes id_W)(\xi^{-1}_{(X\otimes' Y)\otimes' Z,W}$

To see the commutativity of squares, we explicitly show the upper left diagram in Figure 2.6.

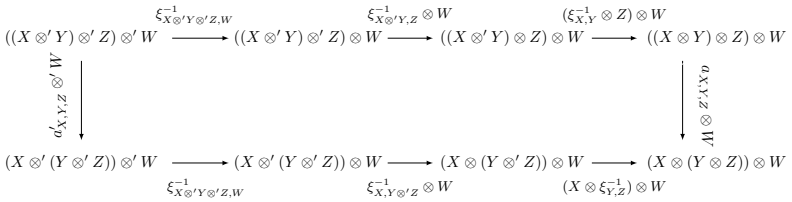


Figure 2.6: Upper left square in Figure 2.5, proof of Lemma 1

Triangle equation: For checking the compatibility of the new definitions of unitors and associators and to see if they satisfy the triangle equation, we need to check if all parts of 2.7 commute. We check explicitly the commutativity of the left square of Figure 2.7 in Figure 2.8.

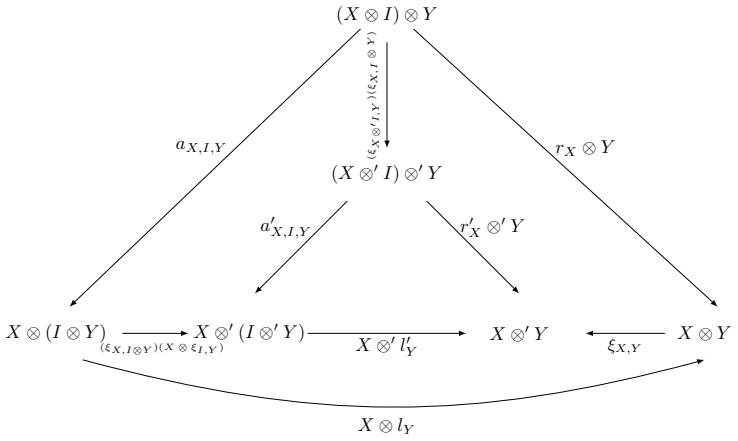


Figure 2.7: Checking triangle equation, proof of Lemma 1.

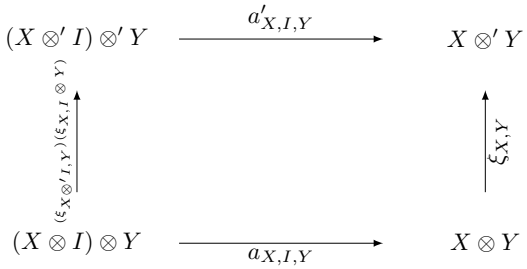


Figure 2.8: Right square of triangle equation 2.7.

Checking naturality: The naturality of a' is shown in Figure 2.9. Proof of naturality of new unitors is similar. □

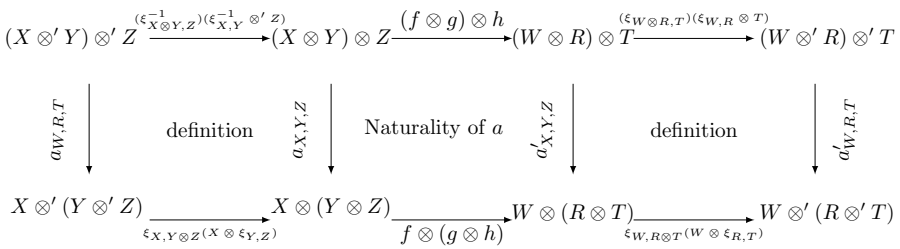


Figure 2.9: Check naturality of a'.

□

Theorem 1. *Every monoidal category is monoidally equivalent to a monoidal category with strict unitors.*

Proof. Using the last lemma, we now define a new monoidal product and corresponding isomorphisms which make unitors identity.

$$X \otimes' Y = \begin{cases} X & \text{if } Y = I, \\ Y & \text{if } X = I, \\ X \otimes Y & \text{otherwise.} \end{cases}$$

We define the tensor product on morphisms defined as below. This further guarantees the uniqueness of identity maps, for example, $id_{X\otimes I} = id_X$.

$$f \otimes' g = \begin{cases} f & \text{if } g = id_I, \\ g & \text{if } f = id_I, \\ f \otimes g & \text{otherwise.} \end{cases}$$

Then, we define $I' = I$ and $\xi_0 = id_I$, so:

$$\xi_{X,Y} = \begin{cases} r_X & \text{if } Y = I, \\ l_X & \text{if } X = I, \\ id_{X\otimes Y} & \text{otherwise.} \end{cases}$$

These choices result in the strict unitors $r' = r_X(id_X \otimes id_I)r_X^{-1} = id_X$. We know from the last lemma that this satisfies triangle and pentagonal equations. □

Theorem 2. *Every monoidal category is monoidally equivalent to a skeletal monoidal category.*

Proof. To prove this theorem, for each class of isomorphic objects $[X_i]$, we choose a representative X_i, and we fix a choice of isomorphisms, $\xi_{i,j} : X_i \otimes X_j \longrightarrow X_{i,j}$, for each pair of representative objects. Note that with this notation, $X_0 = I$.

Now if we take the full subcategory of $\mathcal{C}$ whose objects are representative objects X_i and define a new monoidal product $X \otimes' Y = X_{i,j}$, and if we assign to morphisms

between the tensor product of objects, $f \otimes g : X_i \otimes X_j \longrightarrow X_k \otimes X_l$, morphisms, $\xi_{kj}(f \otimes g)\xi_{ij}^{-1}$, that guarantees the naturality of ξ_{ij}.

Given Lemma 1, if we define the functor $(id, \xi, id) : (\mathcal{C}, \otimes, a, I, r, l) \longrightarrow (\mathcal{C}, \otimes', a', I, r, l)$, that proves the theorem. □

Corollary 1. *Every monoidal category is monoidally equivalent to a skeletal monoidal category with strict unitors.*

Proof. In the last theorems, 1 and 2, let $\xi_{i,0} = r_i$ and $\xi_{0,j} = l_j$. □

Definition 5. *Let $\mathcal{C}$ be a monoidal category and V be an object in $\mathcal{C}$. A right dual to V is an object V^* with two morphisms*

$$e_V : V^* \otimes V \longrightarrow I \tag{2.9}$$

$$i_V : I \longrightarrow V \otimes V^* \tag{2.10}$$

such that $(id_V \otimes e_V) \circ (i_V \otimes id_V) = id_V$ and $(e_V \otimes id_{V^*}) \circ (id_{V^*} \otimes i_V) = id_{V^*}$.

A left dual correspondingly can be defined.

Definition 6. *A monoidal category is **rigid** if every object has right and left duals, and they are the same object.*

One can prove dual objects are unique up to a unique isomorphism. We defined a more restrictive version of rigidity; in general, left and right duals can be different. In the next part, we revisit some lemmas the proof of which will be helpful for our cause. From now on, we restrict ourselves to categories with the same right and left duals.

Lemma 2. *Suppose that V has a dual V^*. Then, there exist canonical isomorphisms,*

$$hom(U \otimes V, W) \cong hom(U, W \otimes V^*) \tag{2.11}$$

$$hom(U, V \otimes W) \cong hom(V^* \otimes U, W) \tag{2.12}$$

This lemma is proved in Page 31 of [7]. Another structure the desired category ought to have stems from exchange statistics between anyons; which as mentioned earlier is captured by braid group rather than permutation group. Braids can be defined as natural transformations between two functors; tensor and opposite-tensor.

An opposite-tensor functor is a functor which first swap the ordered pairs of objects or morphisms, then tensor them.

$$\otimes^{opp} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \tag{2.13}$$

$$\otimes^{opp}(A, B) = \otimes(B, A) = B \otimes A, \qquad \otimes^{opp}(f, g) = \otimes(g, f) = g \otimes f \tag{2.14}$$

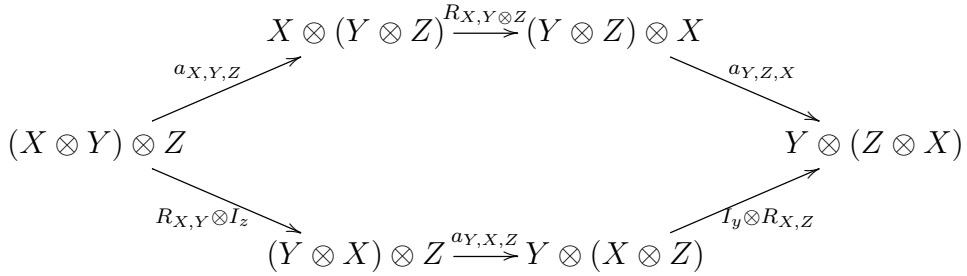


Figure 2.10: Hexagonal equation.

$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\Gamma_{X,Y}} & F(X) \otimes' F(Y) \\ {\scriptstyle F(R_{X,Y})}\downarrow & & \downarrow{\scriptstyle R'_{FX,FY}} \\ F(Y \otimes X) & \xrightarrow{\Gamma_{Y,X}} & F(Y) \otimes' F(X) \end{array}$$

Figure 2.11: Condition on braided functor.

Definition 7. *A monoidal category $(\mathcal{C}, \otimes, a, l, r, I)$ is **braided**, if there exists a natural transformation $R : \otimes \longrightarrow \otimes^{opp}$ which satisfies the hexagonal equation 2.10.*

Take the example of a group G as a category, the category is braided only if group G is abelian. Note that in a braided tensor category, between braid and left/right unitors, the following equalities hold. The last equation results from substituting the first equation in the second one.

$$l_X \circ R_{X,I} = r_X, \qquad r_X \circ R_{I,X} = l_X, \qquad R_{I,X} = R^{-1}_{X,I} \tag{2.15}$$

If the category is strict, $R_{I,X} = R^{-1}_{X,I} = id_X$. Note also the equality $R \circ R = id$ does not hold in general case. But if it holds, the category is symmetric and swapping of particles is captured by permutation group rather than braid group. An example of symmetric categories is the category of vector spaces, **Vec** over field $\Bbbk$.

Definition 8. *A **braided tensor functor** is a tensor functor (F, γ, Γ) between two braided categories $(C, \otimes, a, r, l, I, R)$ and $(C', \otimes', a', r', l', I', R')$ such that for each pair of objects (A, B) Diagram 2.11 commutes .*

So far we have discussed two seemingly separate notions, braided monoidal categories and braid group. One can ask how these two notions meet each other. The connection is straightforward, and the middle connecting step is the Yang-Baxter equation. To see the Yang-Baxter equation for braid group, one needs to only consider the Artin conditions $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. The next theorem shows explicitly how

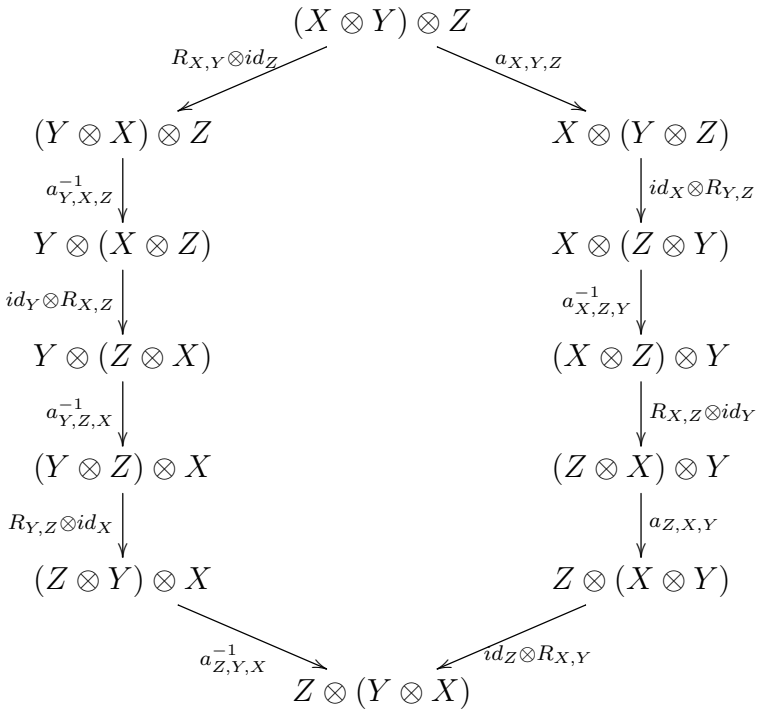


Figure 2.12: Yang-Baxter equation for braided monoidal categories [29].

we can obtain the Yang-Baxter equation from the definition of braid monoidal category [29], and therefore how we can connect Braid group to braided categories.

An interesting connection between the Yang-Baxter equation and exactly solvable models was proposed after the introduction of the Ising model by Onsager [8]. A model is called exactly solvable if the dynamic is 2-body reducible. The Yang-Baxter equation is a sufficient condition for solvability of a model. Given the close connection of these models with conformal field theories, one can also observe the connection between their solutions and CFT conditions [59]. The proof below demonstrates how braided monoidal categories can classify exactly solvable models.

Theorem 3. *For any three objects (X, Y, Z) in a braided tensor category the dodecagon identity commutes.*

Proof. If you closely check Diagram 2.12, the two side inner diagrams, $(X \otimes Y) \otimes Z$ to $Z \otimes (X \otimes Y)$ and $(Y \otimes X) \otimes Z$ to $Z \otimes (Y \otimes X)$ are commutative due to the hexagonal identity and the middle square is the naturality condition. □

The next structure we need to define is a kind of trace, but before that we need to define an isomorphism between an object X and its double star $(X^*)^*$.

Definition 9. *In a rigid monoidal category if there exist isomorphisms $\phi_X : X \longrightarrow X^{**}$ such that they satisfy the following conditions, then the monoidal category is* ***pivotal****.*

$$\phi_{X\otimes Y} = \phi_X \otimes \phi_Y \tag{2.16}$$

$$f^{**} = f \tag{2.17}$$

Having these isomorphisms, we are able to define left and right traces which are not equivalent in general. We are, however, interested in categories with equivalent left and right traces.

Definition 10. *In a pivotal category, given $f : X \longrightarrow X$, one can define left and right* ***traces*** *as appears below,*

$$tr^r(f) = e_{X^*} \circ (\phi_X \otimes id_{X^*}) \circ (f \otimes id_{X^*}) \circ i_X \tag{2.18}$$

$$tr^l(f) = e_X \circ (id_X \otimes f) \circ (id_{X^*} \otimes \phi_X^{-1}) \circ i_{X^*} \tag{2.19}$$

Definition 11. *In a pivotal category if for every morphism f, the left and right traces are equal, the category is called* ***spherical****.*

So far the pivotal and spherical structures are only defined in rigid monoidal categories, but one can further examine the interaction between ϕ and braids in a braided rigid monoidal category. This results in the definition of twist.

Definition 12. *A braiding is compatible with a pivotal structure if isomorphisms $\theta_X = \psi_X \phi_X$ where $\psi_X = (id_X \otimes e_{X^*}) \circ (R_{X^{**},X} \otimes id_{X*}) \circ (id_{X^{**}} \otimes i_X)$ satisfies $\theta_{X^*} = \theta_X^*$. A* ***ribbon*** *category is a spherical braided category with compatible braiding.*

Next we need to define a notion of addition between objects, which can, furthermore, demand an addition between morphisms. Since in the following chapter we intend to build the 2-categorical version of such, we study additive categories with complete details.

2.1.4 Additive Categories

This section is devoted to a review of definitions and propositions of semiadditive categories. The review part is based on [13, 14].

Constant morphisms in categories

A constant morphism is a particular morphism whose left and right composites for all composable morphisms are equal. Generally, a category can have constant morphisms for some pairs of objects, but not all of them. If all hom-sets, however, have constant morphisms, then they should be compatible and unique.

Definition 13. *A morphism $f : X \longrightarrow Y$ is left constant if for every object W and every pair of morphisms $h, g : W \longrightarrow X$, the right composition with f is the same, $fg = fh$.*

Definition 14. *A morphism $f : X \longrightarrow Y$ is right constant if for every object Y and for every pair of morphisms $h, g : Y \longrightarrow Z$, the left composition with f is the same, $gf = hf$.*

Definition 15. *A morphism is constant if it is both left and right constant.*

The following definition indicates the compatibility condition between constant morphisms in hom-sets.

Definition 16. *A locally pointed category is a category whose hom-sets for every pair of objects (A, B) have constant morphisms $\star_{B,A} : A \longrightarrow B$ such that for an arbitrary pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, Diagram 2.20 commutes.*

$$\begin{array}{ccc} A & \xrightarrow{\star_{B,A}} & B \\ {\scriptstyle f}\downarrow & \searrow^{\star_{C,A}} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[\star_{C,B}]{} & C \end{array} \tag{2.20}$$

Proposition 1. *In a locally pointed category, the family of constant morphisms is unique.*

Proof. Consider two different families of constant morphisms $\{\star\}$ and $\{\star'\}$. If $A \xrightarrow{\star_{B,A}} B \underset{\star'_{C,B}}{\overset{\star_{C,B}}{\longrightarrow}} C$, then from left constantness of $\star'$, we have $\star'_{C,B}\star_{B,A} = \star'_{C,A}$ and from right constantness of $\star$, we have $\star'_{C,B}\star_{B,A} = \star_{C,A}$ therefore, $\star_{C,A} = \star'_{C,A}$ for every (A, C).

Definition 17. *A zero object is an object which is simultaneously initial and terminal.*

Definition 18. *A pointed category is a category with a zero object.*

Definition 19. *In a category with a zero object, a zero morphism is a morphism which factorizes through the zero object.*

Remark 1. *In a pointed category, the family of zero morphisms is the only family of constant morphisms thanks to Proposition 1.*

Remark 2. *A category might have constant morphisms but not a zero object. A class of examples can be constructed by considering any category with a zero object, and removing that.*

Biproducts in categories

In the limit-form definition, a biproduct in a category is an addition between objects that induces an addition between morphisms. The algebraic definition takes two forms. One without any enrichment condition on the initial category [28], and for the other one, one needs to start from a **CMon**-enriched category. We restrict our attention to the second case, i.e. with enrichment. The limit-form and the algebraic form with enrichment shake hands when we examine enrichment in Proposition 2.

Definition 20. *A **CMon**-enriched category is a category whose* hom*-sets are commutative monoids and the composition of morphisms is bilinear.*

Lemma 3. *Every **CMon**-enriched category is a locally pointed category.*

Proof. Consider $A \xrightarrow[e_{B,A}]{f} B \xrightarrow{g} C$, such that e is the unit element. From the linearity of composition, we have $g(f + e_{B,A}) = gf + ge_{B,A}$. Since $e_{B,A}$ is the unit element of $hom(A, B)$, we have $g(f+e_{B,A}) = gf$. Note that the unit element of monoid is unique. Since this equality holds for any g and f, and from the uniqueness condition of the unit morphism, we conclude $ge_{B,A}$ is the unit element of monoid $hom(A, C)$. □

A pair of objects in a category can have non-isomorphic products and coproducts, but if there exists a natural isomorphism between them, we say this pair of objects has a biproduct [36]. Note that the cited paper shows the isomorphism between products and coproducts does not need to be the canonical isomorphism.

Another definition is in the algebraic form. The algebraic definitions, whenever they exist, are more desirable, since for instance: checking whether an algebraic definition is preserved by a particular functor, boils down to checking functoriality.

Definition 21. *In a category, a product of a pair of objects (A, B) is a tuple $(A \times B, p_A, p_B)$ such that for an arbitrary cone $(X, f : X \longrightarrow A, g : X \longrightarrow B)$ shown in*

Figure 2.13, there exists a unique morphism $b : X \longrightarrow A \times B$ which satisfies $p_A b = f$ and $p_B b = g$. Morphisms p_A and p_B are called projections.

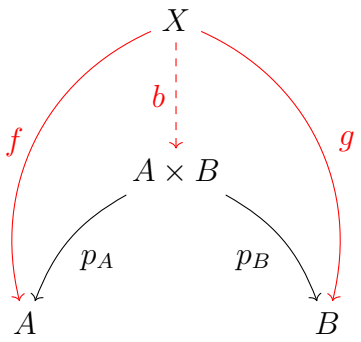


Figure 2.13: Product of (A, B).

Definition 22. *In a category, a coproduct of a pair of objects (A, B) is a tuple $(A \sqcup B, i_A, i_B)$ such that for an arbitrary (co)cone $(X, f : A \longrightarrow X, g : B \longrightarrow X)$ there exists a unique morphism $b : A \sqcup B \longrightarrow X$ which satisfies $bi_A = f$ and $bi_B = g$. orphisms i_A and i_B are called injections.*

Definition 23. *In a locally pointed category, the canonical morphism between a coproduct of a pair of objects A_1, A_2, i.e. $A_1 \sqcup A_2$, and a product $A_1 \times A_2$ satisfies $p_k r i_j = \delta_{k,j}$ (kronecker delta $\delta_{ij} = 0$ if $i \neq j$ and 1 otherwise).*

$$A_j \xrightarrow{i_j} A_1 \sqcup A_2 \xrightarrow{r} A_1 \times A_2 \xrightarrow{p_k} A_k, \qquad \text{if} \quad i, j \in \{1, 2\}$$

Definition 24. *In a locally pointed category, a pair of objects A and B has a biproduct if the canonical morphism r is an isomorphism.*

Definition 25. *A semiadditive category is a category with finite biproducts.*

The next proposition states if a category has biproducts with the limit-form definition, then the category is enriched over monoids. In the proof of the proposition, we use the notation of co/diagonal functions. Given the product of objects, $A \times A$, the diagonal function is the universal map for products; $A \xrightarrow{\Delta_A} A \times A$. The co-diagonal map is the universal map for the coproduct of objects $A \sqcup A \xrightarrow{\nabla_A} A$. In other words, the diagonal function is the copy map, and the co-diagonal is the erasing map.

Proposition 2. *A semi-additive category is a **CMon**-enriched category.*

Proof. Define a monoid addition in $\hom(A, B)$ as below:

$$f + g : A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\nabla_B} B$$

where $\triangle_A = \begin{pmatrix} id_A \\ id_A \end{pmatrix}$ and $\triangledown_B = \begin{pmatrix} id_B & id_B \end{pmatrix}$. Commutativity and associativity of $+$ follow from associativity and commutativity of biproduct $\oplus$:

$$f + g = \triangledown_B (f \oplus g) \triangle_A = \triangledown_B (g \oplus f) \triangle_A = g + f$$
$$(f + g) + h = \triangledown_B p_{12}((f \oplus g) \oplus h) i_{12} \triangle_A = \triangledown_B p_{12}(f \oplus (g \oplus h)) i_{12} \triangle_A = f + (g + h)$$

□

In the next definition, we present the algebraic form of biproducts.

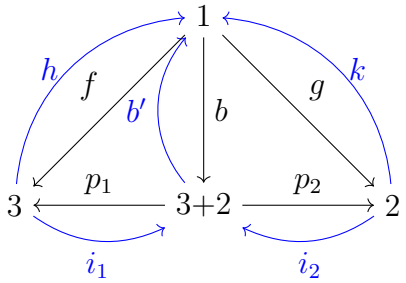
Definition 26. *In a **CMon**-enriched category, a biproduct of a pair of objects* (A, B) *is a tuple* $(A \oplus B, p_A, p_B, i_A, i_B)$ *such that:*

$$p_A i_A = id_A, \quad p_B i_B = id_B, \quad p_A i_B = 0_{A,B}, \quad p_B i_A = 0_{B,A}$$

$$i_A p_A + i_B p_B = id_{A \oplus B}$$

Matrix notation for biproducts

As stated by Mac Lane [42], and reappeared in Ref [43], a category with biproducts offers a suitable framework for matrix calculus. To observe a transparent example, consider the category of matrices over a field, $\mathbf{Mat}_{\Bbbk}$. Objects of this category are natural numbers, and morphisms are matrices whose entries belong to the field. For example, $hom(2, 3)$ consists of all 3×2 matrices over Field $\Bbbk$. The composition of morphisms is matrix multiplication $A.B$. Since hom-sets with matrix addition form abelian groups, this category is semiadditive. Ref [43] gives a full account of $\mathbf{Mat}_{\Bbbk}$, and exploits its structure to type linear algebra. To see the role of biproducts in matrix calculus, take the example below in which $b = \begin{pmatrix} f \\ g \end{pmatrix}$ and $b' = \begin{pmatrix} h & | & k \end{pmatrix}$.



Projections are $p_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and injections $i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $i_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We have:

$$p_1 \begin{pmatrix} f \\ \hline g \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ \hline g \end{pmatrix} = f, \qquad \begin{pmatrix} h & | & k \end{pmatrix} i_1 = \begin{pmatrix} h & | & k \end{pmatrix} \begin{pmatrix} 1 \\ \hline 0 \end{pmatrix} = h$$

$$p_2 \begin{pmatrix} f \\ \hline g \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} f \\ \hline g \end{pmatrix} = g \qquad \begin{pmatrix} h & | & k \end{pmatrix} i_2 = \begin{pmatrix} h & | & k \end{pmatrix} \begin{pmatrix} 0 \\ \hline 1 \end{pmatrix} = k$$

This partitioning of matrices via biproducts is utilized in Ref [43] to implement a divide-and-conquer algorithm for matrix multiplication.

Our intention to use matrix notation is the other way around. We write projections, injections, and morphisms to/from biproducts in the matrix form to facilitate calculations and gain an intuition while working with 1- and 2-morphisms in 2-categories. As we will discuss later, in a similar fashion to Ref [43], one could expect to type matrices of higher dimensions or do tensor calculus up to rank 4 in a 2-category with 2-biproducts, Section 4.1.2.

As should be clear from the explicit form of projections and injections above, projections and injections play the role of basis elements which determines the entries of morphisms to/from biproducts. For instance, assume the following list of morphisms

$$f : A_1 \longrightarrow A'_1, g : A_1 \longrightarrow A'_2, h : A_1 \longrightarrow A'_3$$

$$f' : A_2 \longrightarrow A'_1, g' : A_2 \longrightarrow A'_2, h' : A_2 \longrightarrow A'_3$$

It is not difficult to show that $A_1 \oplus A_2 \xrightarrow{t} A'_1 \oplus A'_2 \oplus A'_3$ is $t = \begin{pmatrix} f & f' \\ g & g' \\ h & h' \end{pmatrix}$, projections $p_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $p_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, $p_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ and injections $i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $i_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Our intuition to guess the correct order of matrices comes from the category of matrices $\mathbf{Mat_k}$. Henceforth, we represent morphisms in matrix notation when it eases our calculations.

To have a category with morphisms with the structure of vector space, we define additive categories.

Definition 27. *An **additive category** is a semi-additive category enriched over the category of vector spaces over* $\Bbbk$.

Definition 28. *An **abelian category** is an additive category if every morphism has a kernel and a cokernel, and every monomorphism is a kernel and every epimorphism is a cokernel.*

One can also define an abelian category as a category enriched over Ab, the category of abelian groups. But it is not necessary a priori [7]. In the next section, we define semi-simple categories. The semi-simplicity ensures objects in the desirable category are restricted to only anyon types.

Definition 29. *A **sub-object** of an object X is an isomorphism class of monomorphisms.*

$$i : Y \hookrightarrow X$$

Definition 30. *In an abelian category, a **simple object** is an object whose sub-objects are only zero object and itself.*

Let us denote simple objects with some indices, X_i.

Definition 31. *An abelian category is **semi-simple** if any object X is isomorphic to a direct sum of simple objects, where $N_i's$ are natural numbers.*

$$X \cong \bigoplus_{i \in I} N_i X_i$$

The semi-simplicity condition guarantees a notion for anyon types. Semi-simplicity as a condition for anyonic category was first proposed by Wang in [63]. From the definition of simple objects, we conclude the only morphisms between two simple objects are the identity and zero morphisms, so-called **Schur's Lemma** [63].

The non-zero integer coefficients behind simple objects, N_i, count the number of inequivalent projections and injections in hom-sets such that $N_i = dim(hom(X, X_i))$, and if Y is not a sub-object of X, the hom-set is the zero vector space,

$$hom(X, Y) \cong \mathbf{0}. \tag{2.21}$$

The definitions laid out in the previous parts provide ample structures for fusion categories. As the name suggests, they are categories with enough structure to capture fusion rules.

Definition 32. *A category $\mathcal{C}$ is a **fusion category** if,*

- *$\mathcal{C}$ is a semi-simple category over complex numbers, $\mathbb{C}$,*
- *the cardinality of the set of objects is finite, and hom-sets are finite dimensional vector spaces,*
- *$\mathcal{C}$ is monoidal,*
- *$\mathcal{C}$ is rigid,*
- *for every simple object X, $hom(X_i, X_j) \cong \delta_{ij}\mathbb{C}$,*
- *the unit object I is simple, we assign index 0 to the unit object, $X_0 = I$.*

If the semi-simplicity and monoidality conditions imply any non-simple object including the tensor product of two simple ones can be re-written as a direct sum of simple objects.

$$X_i \otimes X_j \cong \bigoplus_k N_{ij}^k X_k \tag{2.22}$$

The above equations specify an important property of any anyonic theory, **fusion rules**, and N_{ij}^k are called **fusion coefficients**. So one can write,

$$N_{ij}^k = dim(hom(X_i \otimes X_j, X_k)) \tag{2.23}$$

We also call these hom-sets as **fusion space** and denote them by V_{ij}^k. In the similar fashion, we call $V_k^{ij} = hom(X_k, X_i \otimes X_j)$ **decomposition space**. If one inspects hom-sets of fusion categories, one realises they are either fusion or decomposition spaces or

$$hom(X_i, X_j) \cong \delta_{ij}\mathbb{C}$$

Associators and unitors correspondingly can be indexed in a fusion category. So associators between the tensor product of three consecutive objects will be indexed by four labels, F_{ijk}^l. Why four? because as we mentioned earlier, the zero morphism is the only morphism between different simple objects, so non-zero morphisms are F_{ijk}^l between the same outcome X_l.

$$\underbrace{(X_i \otimes X_j) \otimes X_k}_{X_l} \overset{F_{ijk}^l}{\cong} \underbrace{X_i \otimes (X_j \otimes X_k)}_{X_l} \tag{2.24}$$

and unitors are indexed in the same manner,

$$X_i \otimes X_0 \overset{r_i}{\cong} X_i, \qquad\qquad X_0 \otimes X_i \overset{l_i}{\cong} X_i \tag{2.25}$$

We continue and add necessary structures the desirable category demands such as; braiding, ribbon, unitarity.

Definition 33. *A **braided fusion category** is a fusion category equipped with braiding,*

$$X_i \otimes X_j \overset{R_{ij}}{\cong} X_j \otimes X_i \tag{2.26}$$

Definition 34. *A spherical braided fusion category with compatible braiding is **ribbon fusion category** (RFC).*

We now summarise some of the properties of fusion coefficients, which are often necessary and rather insightful for solving pentagonal and hexagonal equations and further calculation with F and R. As mentioned earlier, fusion coefficients, are fusion space's dimension, i.e. $N_{ij}^k = dim(V_{ij}^k)$. Note that some of these identities proved in [7], and some left as exercise for readers. We address the proofs of all of them here.

- $N_{ij}^k = N_{ji}^k$, since $X_i \otimes X_j \cong X_j \otimes X_i$ so any morphism in $f \in hom(X_i \otimes X_j, X_k)$ can be written as $R_{ij}^k \circ f \in hom(X_j \otimes X_i, X_k)$, and vice versa.
- $N_{ij}^k = N_k^{ij}$, since $hom(X_i \otimes X_j, X_k) \cong hom(X_k, X_i \otimes X_j)$.
- $N_{ij}^k = N_{i^*k}^j$. from previous equality $hom(X_i \otimes X_j, X_k) \cong hom(X_k, X_i \otimes X_j)$ and given Lemma 2, we have $hom(X_i \otimes X_j, X_k) \cong hom(X_i^* \otimes X_k, X_j)$.
- $N_{ij}^k = N_{kj*}^i$, same as previous equality.
- $N_{ij}^k = N_{k*ji}^0$, same as previous equality.
- $N_{ij}^k = N_{k*i}^{j*}$, we move indices, k, j, according to previous equalities.
- $N_{ij}^k = N_{i*j*}^{k*} = N_{j*i*}^{k*}$.
- $N_{ij}^0 = \delta_{ij^*}$, this is helpful and can be easily obtained by using previous equalities.

One of the computationally insightful properties of objects in a RFC is their dimension. For every object X_i, we define $d_i = tr(id_i)$. This essentially shows how computational space grows. An important question to answer is the sign of d_i. We show under unitarity condition for a RFC, $d_i \geq 0$. Hermitian and unitarity definitions agree with the definitions appeared in [62].

Definition 35. *A **dagger category** or †-category $\mathcal{C}$, is a category with a function $()^\dagger : hom(A, B) \longrightarrow hom(B, A)$ for every pair of objects, $A, B \in Obj(\mathcal{C})$, such that*

- *For every object, $A \in Obj(\mathcal{C})$, $(1_A)^\dagger = 1_A$.*
- *For every $A, B \in Obj(\mathcal{C})$ and every $f \in hom(A, B)$ and $g \in hom(B, C)$, $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.*
- *For every $A, B \in Obj(\mathcal{C})$ and every $f \in hom(A, B)$, $((f)^\dagger)^\dagger = f$.*

Definition 36. *A dagger RFC is hermitian if for every morphism $f \in hom(X, Y)$ the dagger of every morphism $f^\dagger \in hom(Y, X)$ satisfies the conditions below,*

$$(f^\dagger)^\dagger = f, \qquad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger, (f \circ g)^\dagger = g^\dagger \circ f^\dagger, \qquad (id_i)^\dagger = id_i \tag{2.27}$$

and braid and twist satisfies,

$$(R_{ij})^\dagger = R_{ij}^{-1}, \qquad (\theta_i)^\dagger = \theta_i^{-1} \tag{2.28}$$

Additionally, dual morphisms are compatible with †,

$$(i_j)^\dagger = e_j \circ R_{jj*} \circ (\theta_i \otimes id_{j*}), \qquad (e_j)^\dagger = (id_{j*} \otimes \theta_j^{-1}) \circ R_{j*j}^{-1} \circ i_j \tag{2.29}$$

In a RFC, a dagger assigns to each morphism in the fusion space, a morphism in decomposition space, meaning, $\dagger : V_{ij}^{k} \longrightarrow V_{k}^{ij}$. Or you can also think of it as, the dagger of a projection operator is an injection operator. We are now in a proper position to define an inner product on a hermitian RFC.

Definition 37. *An inner product between two vectors is defined as* $\langle , \rangle : V^* \times V \longrightarrow \mathbb{C}$*, such that it satisfies the following conditions,*

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
- $\langle x, x \rangle \geq 0$ *the equality happens when* $x = 0$.

Definition 38. *In a hermitian RFC, an inner product for a pair of morphisms* (f, g) *is defined as,*

$$\langle , \rangle : hom(Y, X) \times hom(X, Y) \longrightarrow \mathbb{C}, \quad \langle f, g \rangle = \frac{1}{\sqrt{dim(X)dim(Y)}} tr(f^{\dagger} g) \tag{2.30}$$

we can check all conditions of inner product. In general, $tr(f^{\dagger} f)$ *takes positive and negative values but if it is always positive definite then the category is* ***unitary****.*

One can observe in a Unitary Ribbon Fusion category, URFC, the dimension of each object $d_i = tr(id_{i*} id_i)$ is positive. Without unitarity condition, we could not conclude the following,

$$tr(id_{i*} id_i) \geq 0 \tag{2.31}$$

A URFC describes essentially an anyonic theory for computational purposes. Simple objects of the category are anyon types and Schur's lemma guarantees super-selection rule. Fusion rules are defining rules of the theory and fusion matrices F solutions of pentagonal equations, braiding matrices R result from solving hexagonal equations.

An important and undiscussed point is the necessity of modularity condition as physical requirement for anyonic models. Modular categories ensure the theory has a corresponding topological background field. We did not discuss modularity condition because we are solely interested in TQC. We refer you to [7] and [62] for further discussion on this topic.

Skelelality of URFC has been often assumed in the literature. While physically it may seem obvious that one does not keep isomorphic anyons/objects in the set of objects, one needs to explicitly states this condition. Hence, the category we are interested in is the skeleton of a URFC. The baffling next question is whether associators F and braids R are identity morphisms. In other words, if a skeletal category is also strict. But it turns out two conditions will not occur at the same

time. Keeping an object from each class of isomorphic objects holds us from letting associators and unitors be identity. Nonetheless, we can work with skeleton of a URFC with strict unitors. This condition reduces the number of pentagonal and hexagonal equations one needs to solve. Because those F-solutions which include the unit object as an index also have to satisfy triangle equations, which in this case will be a composition of identity morphisms.

Corollary 2. *Every fusion category is monoidally equivalent to a skeletal fusion category with strict unitors.*

Proof. Refer to the proof of Theorem 1. □

Lemma 4. *For a fusion category with strict unitors, every associator, F, with at least one zero index is identity.*

Proof. Take a variation of triangle equations, $(X_i \otimes l_j) \circ (r_i \otimes X_j) = F^k_{i0j}$ let unitors be identity, so F^k_{i0j}. Others are obtainable in the similar fashion. □

So far we have discussed how elements of a URFC capture expected properties of an anyonic theory. However, as we mentioned, the category describing an anyonic theory is often convoluted with TQC category. In other words, it is often assumed the category of computation is similar to the category of anyons. Here, we argue that a better perspective to make sense of computation coherent with categorical quantum mechanics is thinking of TQC as a subcategory of the category of Hilbert spaces **Hilb**. In this framework, the matrix representation of F and R matrices are isomorphisms between fusion and decomposition spaces.

2.2 Categorical Quantum Mechanics (CQM)

The idea of categorical quantum mechanics was proposed by Abramsky and Coecke [3]. It was further developed in [16]. Categorical quantum mechanics essentially attempts to shift the perspective from quantum states to processes. In other words, from the lens of CQM everything in a unified manner represents a process and the composition between processes. To explore further the relevancy of this picture to the usual Schrödinger's picture see [15].

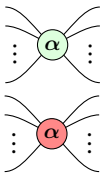
In addition to foundational purposes, CQM comes equipped with a graphical calculus, and a more convenient calculus for quantum circuits called the ZX-calculus sprang from this calculus later. We outline a short summary of CQM and the ZX-calculus here and in the next section, we explain how TQC fits this picture.

As we mentioned earlier, CQM aims to unify classical and quantum structures, so in essence, we have a dagger symmetric monoidal category which includes classical and quantum structures. Well-developed examples of such a category are the category of relations, **Rel**, and the category of finite dimensional Hilbert spaces, **FdHilb**. **FdHilb** practically captures all properties of quantum systems; so we only discuss this category and to shorten the acronym, we will show it by **Hilb**.

The general categorical calculus is applicable to any quantum system irrespective of their dimensions or specific properties. For example, be a system in a mixed state or pure state, 2-dimensional or higher-dimensional, categorical calculus can be meaningfully applied to gain an intuition and sometimes even to predict the outcome.

Quantum computation essentially works with 2-dimensional systems, i.e. qubits. A specific graphical representation analogous to classical computation circuits has been devised for this purpose. CQM also invented its own version called the ZX-calculus [20].

The elements of ZX-calculus consist of green and red nodes, called spiders, and wires. Spiders have attached wires from left and right; wires entering from left are inputs and those existing from right are outputs. The composition of spiders is done through gluing outputs of a diagram to inputs of another; and tensoring diagrams means stacking them vertically. Generally, there is no condition on the number of inputs and outputs. A spider with only one output represents a state and with only one input represent an effect or measurement or the dual of a state. We show green spiders with white and red spiders with black for readability.



$$:= \ |0...0\rangle\langle 0...0| + e^{i\alpha}|1...1\rangle\langle 1...1| \qquad (2.32)$$

$$:= \ |+...+\rangle\langle +...+| + e^{i\alpha}|-...-\rangle\langle -...-|$$

The ZX-calculus comes with a set of rules; they can be applied while transforming diagrams back and forth to each other. Before delving into relationship between topological quantum computation and categorical quantum mechanics, we attract your attention to two extra rules we will use throughout next sections, Hadamard ZX-transformation and P-rule [17].

Definition 39. *(P-rule) Given a combination of (red-green-red) spiders, by using relation between angles below, one can find an equivalent (up to $\pm\pi$) combination of (green-red-green) spiders and vice versa.*



(2.33)

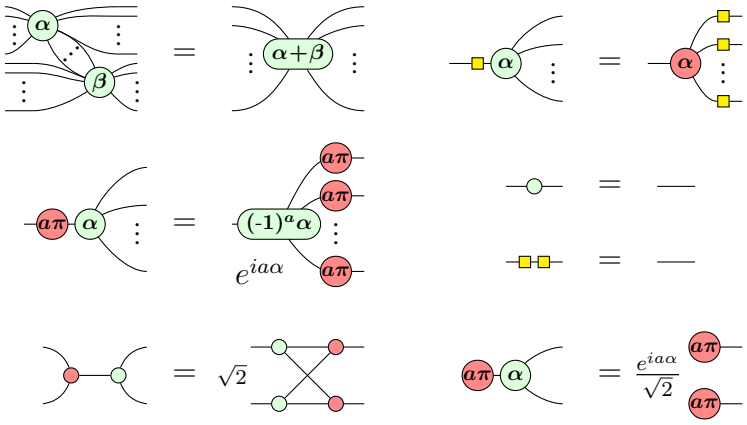


Figure 2.14: A convenient presentation for the ZX-calculus. These rules hold for all $\alpha, \beta \in [0, 2\pi)$ and $a \in \{0, 1\}$. They also hold with the colours (red and green) interchanged, and with the inputs and outputs permuted arbitrarily.

$$z = cos(\frac{\beta}{2})cos(\frac{\alpha+\gamma}{2}) + isin(\frac{\beta}{2})cos(\frac{\alpha-\gamma}{2}) \tag{2.34}$$

$$z' = cos(\frac{\beta}{2})sin(\frac{\alpha+\gamma}{2}) - isin(\frac{\beta}{2})sin(\frac{\alpha-\gamma}{2}) \tag{2.35}$$

The equivalent three angles are:

$$(\alpha', \beta', \gamma') = \begin{cases} \alpha' = arg(z) + arg(z') \\ \beta' = 2arg(\frac{|z|}{|z'|} + i) \\ \gamma' = arg(z) - arg(z') \end{cases}$$

2.3 The Connection Between TQC and CQM

In Section 2.1, we discussed the structures and properties of a suitable category which captures anyonic theories, are summarised in unitary ribbon fusion categories. Also, we briefly touched on the fact that the category for TQC, in literature, is convoluted with the category describing anyonic theories. Given the structure of categorical quantum computing, the question of interaction between TQC and CQM arises. We show TQC category is a subcategory of **Hilb**.

Knowing this fact, another exciting area to explore is the representation of TQC models with the ZX-calculus. If the TQC category is a subcategory of **Hilb**, then, in principle, graphical calculus for CQM also should work for TQC models. We show this explicitly for the Fibonacci and Ising models, introduce new relations, and conclude that while the Ising model is only the Clifford fraction of the ZX-calculus, Fibonacci represents a new fraction. This further results in a new P-rule.

Let us again review the elements of TQC. For quantum computation, we need a well-defined anyonic theory. This means we have a set of anyon types, fusion rules which essentially specify the outcome of the tensor product between each pair of objects, and solutions of pentagonal and hexagonal equations.

The computational space is not an anoynic space; instead, it is the fusion space, V_{ij}^k. Physically, we are encoding the possibility of different fusion outcomes between anyons. So if the outcome of $(X_i \otimes X_j) \otimes X_k$ has two possibilities, either with a single outcome with a multiplicity coefficients $N_{ij}^{=}2$ or with more than one outcome, then one can take V_{ij}^k as 2-dimensional computational space.

Having this picture in mind, then $F's$ solutions of pentagonal equations are represented as matrices and linear isomorphisms between two isomorphic vector spaces $V_{(ij)k}^l \cong V_{i(jk)}^l$. However, primary spaces are fusion spacesV_{ij}^k, so any other space has an equivalent sum written as a direct sum of these fusion spaces.

$$V_{(ij)k}^l \cong \bigoplus_{ek} V_{ij}^e \otimes V_{ek}^l \tag{2.36}$$

Similar to F matrices, it should not be surprising that hexagonal equations' solutions, R's, are linear isomorphisms between two isomorphic spaces $V_{ij}^k \cong V_{ji}^k$. So basically, the objects of the desired category are vector spaces and the TQC category is a subcategory of **Hilb**.

Definition 40. *A **TQC category** consists of the following:*

- ***Objects** fusion spaces V_{ij}^k, decompositions spaces V_k^{ij}, and their tensor products and direct sums.*
- ***Morphisms** R matrices are isomorphisms between V_{ij}^k and V_{ji}^k. F matrices are isomorphisms between $V_{(ij)k}^l$ and $V_{i(jk)}^l$. Plus their tensor products and direct sums.*

Pentagonal and hexagonal equations can be rewritten with vector spaces; one must solve these equations to find F and R solutions practically. The category is semi-simple; simple objects are fusion spaces, and any other space has an equivalent direct sum of fusion spaces.

$$V_{i_1..i_n}^j \cong \bigoplus_{k_1...k_{n-1}} V_{i_1 i_2}^{k_1} \otimes V_{k_1 i_3}^{k_2} \otimes ... \otimes V_{k_{n-1} i_n}^j \tag{2.37}$$

Note that fusion spaces are hom-sets of a URFC. Hence, they come equipped with a well-defined inner product, making them Hilbert space. So the relationship between CQM and TQC should be evident by now. The described category is a subcategory

of **Hilb**. Given the definition above, this category is also closed under the tensor product, and the direct sum and the F and R matrices are well-behaved. Therefore, the natural next step is to develop the ZX-representation of TQC. We show our attempt for two cases and leave the general case of $SU(2)_k$ for future works. $SU(2)_k$ represents a class of anyonic models, and fusion rules are given by truncated spins $\frac{1}{2}$ multiplication, Appendix of [11].

2.3.1 The ZX-Representation of Fibonacci and Ising anyons

Ising Model

Ising anyons, also known as Majorana fermions, are the most straightforward realization of non-Abelian anyons [56]. Discussion of lab-realization of these anyons is far from the scope of this thesis, for further information about different proposal, review [54].

The Ising model includes two non-trivial anyons $\{1, \sigma, \psi\}$ and non-trivial fusion rules are as below,

$$\sigma \otimes \sigma = 1 + \psi, \qquad \sigma \otimes \psi = \sigma, \qquad \psi \otimes \psi = 1 \tag{2.38}$$

As it should be clear from these rules, σ is the non-Abelian anyon called Ising. To encode qubits, we need three Ising anyons, so that based on their internal charge; we can map them to qubits, $\{|1\psi\rangle = |0\rangle, |\psi\sigma\rangle = |1\rangle\}$. Fixing this basis, the computational Hilbert space is $H = \oplus_x V^x_{\sigma\sigma} \otimes V^\sigma_{x\sigma}$. One can solve pentagonal and hexagonal equations to find F and R matrices,

$$F^{isg} = H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad R_1^{isg} = -e^{\frac{\pi i}{8}}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}, R_2^{isg} = -\frac{e^{\frac{\pi i}{8}}}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}, \tag{2.39}$$

As it should be clear, R_1^{isg} matrix is just a $\frac{-\pi}{2}$ green spider phase. We mentioned before that Hadamard also has a ZX-representation as a combination of three $\frac{\pi}{2}$ angles. So $R_2^{isg} = HR_1^{isg}H$ gives the other braid generator.

For braid groups on 6-strands or 2-qubit gates, we need to find matrix representations of the braid group's generators, B_6. Based on the general relation between Temperley-Lieb algebra and braid groups, [22] presents a 4-dimensional representation of the Ising braid group on six strands or anyons.

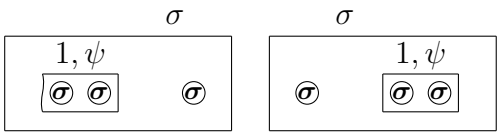


Figure 2.15: The braid group of six Ising qubits, B_6.

$$\rho^{isg}(\sigma_1) = exp(\frac{\pi i}{8}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad \rho^{isg}(\sigma_2) = -\frac{exp(\frac{-\pi i}{8})}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}$$

$$\rho^{isg}(\sigma_3) = exp(\frac{\pi i}{8}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho^{isg}(\sigma_4) = -\frac{exp(\frac{-\pi i}{8})}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix}$$

$$\rho^{isg}(\sigma_5) = exp(\frac{\pi i}{8}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

We extract two basic generators for constructing these matrices, called U-generators.

$$U_1^{isg} = exp(\frac{\pi i}{4}) \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = Z_{\frac{\pi}{2}}, \qquad U_2^{isg} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = X_{\frac{\pi}{2}}$$

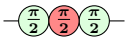
Given U-generators, we restate the 4-dimensional representations of the Ising braid group on 6-anyons as below,

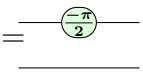
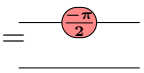
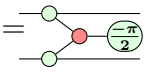
$$\begin{aligned}
&\rho^{isg}(\sigma_1) = -exp(\frac{i\pi}{8})(U_1^{isg} \otimes I) \approx (Z_{\frac{\pi}{2}} \otimes I), \\
&\rho^{isg}(\sigma_2) = -exp(\frac{-i\pi}{8})(U_2^{isg} \otimes I) \approx (X_{\frac{\pi}{2}} \otimes I), \\
&\rho^{isg}(\sigma_3) \approx ((X \otimes X)(CZ_{12})(X \otimes X))(PhaseGadget_1(\frac{\pi}{2}))(CZ_{12}), \\
&\rho^{isg}(\sigma_4) = -exp(\frac{i\pi}{8})(I \otimes U_2^{isg}) \approx (I \otimes X_{\frac{\pi}{2}}) \\
&\rho^{isg}(\sigma_5) = -exp(\frac{-i\pi}{8})(I \otimes U_1^{isg}) \approx (I \otimes Z_{\frac{\pi}{2}})
\end{aligned}$$

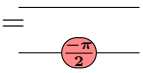
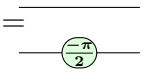
One can clearly deduce commutation relations between braids generators. They stem from the braid group definition but are also observable from the above explicit ma-

trices.

$$
\begin{aligned}
&[\rho^{isg}(\sigma_1), \rho^{isg}(\sigma_i)] = 0, \forall i \neq 2, \quad && \rho^{isg}(\sigma_1)\rho^{isg}(\sigma_2)\rho^{isg}(\sigma_1) = \rho^{isg}(\sigma_2)\rho^{isg}(\sigma_1)\rho^{isg}(\sigma_2) \\
&[\rho^{isg}(\sigma_2), \rho^{isg}(\sigma_i)] = 0, \forall i \neq 1, 3, && \rho^{isg}(\sigma_2)\rho^{isg}(\sigma_3)\rho^{isg}(\sigma_2) = \rho^{isg}(\sigma_3)\rho^{isg}(\sigma_2)\rho^{isg}(\sigma_3) \\
&[\rho^{isg}(\sigma_3), \rho^{isg}(\sigma_i)] = 0, \forall i \neq 2, 4, && \rho^{isg}(\sigma_4)\rho^{isg}(\sigma_3)\rho^{isg}(\sigma_4) = \rho^{isg}(\sigma_3)\rho^{isg}(\sigma_4)\rho^{isg}(\sigma_3) \\
&[\rho^{isg}(\sigma_4), \rho^{isg}(\sigma_i)] = 0, \forall i \neq 3, 5, && \rho^{isg}(\sigma_4)\rho^{isg}(\sigma_5)\rho^{isg}(\sigma_4) = \rho^{isg}(\sigma_5)\rho^{isg}(\sigma_4)\rho^{isg}(\sigma_5)
\end{aligned}
$$

$R_1^{isg} =$  , $R_2^{isg} =$  , $F^{isg} =$ 

$\rho^{isg}(\sigma_1) =$  , $\rho^{isg}(\sigma_2) =$  , $\rho^{isg}(\sigma_3) =$  (2.40)

$\rho^{isg}(\sigma_4) =$  , $\rho^{isg}(\sigma_5) =$ 

The Ising model is proved to be a non-universal model of quantum computation [63]. The above relations show they represent the Clifford fraction of quantum computation, hence, non-universality.

Fibonacci Model

Fibonacci anyons present the simplest, most elegant, and at the same time, a universal model of topological quantum computation. The label set has only one non-trivial anyon called the Fibonacci anyon, τ. There is only one non-trivial Fibonacci rule, namely when the Fibonacci anyon fuses with another Fibonacci anyon, $\tau \otimes \tau = 1 \oplus \tau$.

So one will be able to construct a universal gate set with braiding Fibonacci anyons. An approximately efficient result for building single qubit gates and $CNOT$ was proposed in [12]. Here, we take a somewhat different approach; our concern is the representation of 1- and 2-dimensional representations of the Fibonacci braid group. To encode qubits, we have to consider three Fibonacci anyons when their

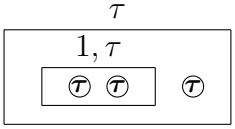


Figure 2.16: Fibonacci qubits or the braid group on three strands, B_3.

total charge is τ. The internal charge x determines two basis states. We map $\{|x = 1\rangle, |x = \tau\rangle\}$ to $\{|0\rangle, |1\rangle\}$ respectively. Solving pentagonal and hexagonal equations in

this Hilbert space, $H = Span(\{|0\rangle, |1\rangle\})$, we obtain the following solutions for F and R_1^{Fib} matrices, if $\phi = \frac{\sqrt{5}+1}{2}$.

$$\phi^2 = \phi + 1, 1 = \phi^{-1} + \phi^{-2}, \tag{2.41}$$

$$F = \begin{pmatrix} \phi^{-1} & \phi^{\frac{-1}{2}} \\ \phi^{\frac{-1}{2}} & -\phi^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \phi^{-1} & -\phi^{\frac{-1}{2}} i \\ -\phi^{\frac{-1}{2}} i & \phi^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{2.42}$$

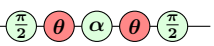
$$R_1^{Fib} = exp(\frac{-4\pi i}{5}) \begin{pmatrix} 1 & 0 \\ 0 & exp(\frac{7\pi i}{5}) \end{pmatrix}, R = exp(\frac{7\pi i}{5}) \tag{2.43}$$

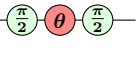
$$R_2^{Fib} = F^{Fib} R_1^{Fib} F^{Fib} = \begin{pmatrix} \phi^{-2} + R\phi^{-1} & \phi^{\frac{-3}{2}}(1-R) \\ \phi^{\frac{-3}{2}}(1-R) & \phi^{-1} + R\phi^{-2} \end{pmatrix} \tag{2.44}$$

One can easily observe $F = Z_{-\frac{\pi}{2}} X_\theta Z_{-\frac{\pi}{2}}$ and $F^{-1} = Z_{\frac{\pi}{2}} X_{-\theta} Z_{\frac{\pi}{2}}$, and R_1^{Fib} is a $\frac{7\pi i}{5}$ green spider. Considering the general form of X_θ-rotation,

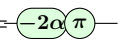
$$X_\theta = exp(\frac{\theta i}{2}) \begin{pmatrix} cos(\frac{\theta}{2}) & -isin(\frac{\theta}{2}) \\ -isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix}$$

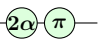
we can find θ for F^{Fib}, which is $\theta = 2Arccos(\phi^{-1})$ if $\frac{\pi}{2} \prec \theta \prec \pi$. Numerically, it is $\theta \approx \frac{129\pi}{224}$.

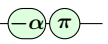
$R_1^{Fib} =$  , R_2^{Fib}  (2.45)

$F =$  (2.46)

Based on the explicit representation of R_1 and R_2, we can see the following equalities which somehow simplify our next equations.

$3R_1 =$  , $4R_1 =$  , $5R_1 =$  , (2.47)

$6R_1 =$  , $7R_1 =$  , $8R_1 =$  ,

$9R_1 =$ 

To find 4-dimensional representations of Fibonacci anyons, we need to create two pairs of three anyons; the total charge of these six anyons is 1. However, the charge of a group of three anyons can be either 1 or τ. If it is τ, we are in the computational basis, and if it is 1, we are in the non-computational basis. As it should be clear from the graph, the computational space is 4-dimensional, and the non-computational space is one-dimensional.

Unlike Ising anyons, Fibonacci anyons have the possibility of leakage into the non-computational space. That is, while swapping the middle anyons, Anyons number 3 and 4, the state may change to non-computational state. Note that this is admissible

since it preserves the total charge 1 (parent node). So representations of braid group on six anyons, should be actually mapped into this 5-dimensional space of $H = H_{NC} \oplus H_C$, or $H = Span(\{|NC\rangle, |11\rangle, |1\tau\rangle, |\tau 1\rangle, |\tau\tau\rangle, \})$.

The same method of Temperly-Lieb algebra [18] gives representation of Fibonacci B_6. If $R = exp(\frac{7\pi i}{5})$.

$$\rho^{Fib}(\sigma_1) = exp(\frac{-4\pi i}{5})[R \oplus (R_1^{Fib} \otimes I_2)] \tag{2.48}$$

$$\rho^{Fib}(\sigma_2) = exp(\frac{-4\pi i}{5})[R \oplus (FR_1^{Fib}F \otimes I_2)] \tag{2.49}$$

$$\rho^{Fib}(\sigma_3) = exp(\frac{-4\pi i}{5})[P_{14}(R \oplus R_1^{Fib} \oplus FR_1^{Fib}F)P_{14}] \tag{2.50}$$

$$\rho^{Fib}(\sigma_4) = exp(\frac{-4\pi i}{5})[R \oplus (I_2 \otimes FR_1^{Fib}F)] \tag{2.51}$$

$$\rho^{Fib}(\sigma_5) = exp(\frac{-4\pi i}{5})[R \oplus (I_2 \otimes R_1)] \tag{2.52}$$

Unlike the 2-dimensional generators, the 5-dimensional ones do not give us any clues to their ZX diagrams. However, as one can observe from their matrix form, they are block diagonal and easily representable by quantum circuits. In the following part, we intend to find these circuits and then transform them into the ZX diagrams.

Because Fibonacci generators are 5-dimensional, we first need to embed them into 8-dimensional space and then find their corresponding quantum circuits. Next, we add three garbage qubits and map the previous basis into a new basis, as appeared in Equation 2.53.

$$|000\rangle = |\overline{1}\rangle, |001\rangle = |\overline{2}\rangle, |010\rangle = |\overline{3}\rangle, |011\rangle = |\overline{4}\rangle, \tag{2.53}$$

$$|100\rangle = |\overline{5}\rangle, |101\rangle = |\overline{6}\rangle, |110\rangle = |\overline{7}\rangle, |111\rangle = |\overline{8}\rangle, \tag{2.54}$$

$$|NC\rangle \longrightarrow |\overline{4}\rangle, |11\rangle \longrightarrow |\overline{5}\rangle, |1\tau\rangle \longrightarrow |\overline{6}\rangle, |\tau 1\rangle \longrightarrow |\overline{7}\rangle, |\tau\tau\rangle \longrightarrow |\overline{8}\rangle \tag{2.55}$$

Note that this mapping sends the computational bases to states whose first qubit is 1 and garbage and non-computational bases to states whose first qubit is 0. We still need to perform another simplification step. By adding three garbage qubits, I_3 will be added to generators. However, to preserve the block-diagonal forms, we instead add RI_3. That does not cause any issues since we throw away garbage qubits at the end of computation. The embedded matrices allow us to identify the building blocks or U-generators.

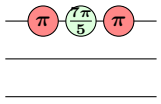
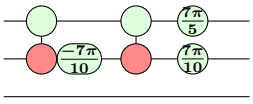
$U_1 =$  $U_2 =$ 

Figure 2.17: The first two U-generators of Fibonacci B_6

$$U_1^{Fib} = \begin{pmatrix} RI_4 & 0 \\ 0 & I_4 \end{pmatrix}, U_2^{Fib} = \begin{pmatrix} I_6 & 0 \\ 0 & RI_2 \end{pmatrix}, U_3^{Fib} = \begin{pmatrix} I_2 & 0 \\ 0 & R_2^{Fib} \end{pmatrix} \otimes I_2,$$
$$U_4^{Fib} = \begin{pmatrix} I_4 & 0 & 0 \\ 0 & R_1^{Fib} & 0 \\ 0 & 0 & I_2 \end{pmatrix}, U_5^{Fib} = \begin{pmatrix} I_6 & 0 \\ 0 & R_1^{Fib} \end{pmatrix}, U_6 = \begin{pmatrix} I_4 & 0 & 0 \\ 0 & R_2^{Fib} & 0 \\ 0 & 0 & I_2 \end{pmatrix},$$
$$U_7^{Fib} = \begin{pmatrix} I_6 & 0 \\ 0 & R_2^{Fib} \end{pmatrix}, P_{14} = (CCX_2)(CCX_0)(CCX_2), CCX_i : i - \text{target qubit}$$

Fibonacci anyons' braid generators satisfy the following commutation relations, which are also provable by graphical calculus. Consider, $\{1, 2, ..., 7\}$,

$$\begin{aligned}
&[U_1, U_i] = 0 && \forall i \in l, && \\
&[U_2, U_i] = 0 && \forall i \in l/\{3\}, && U_2U_3U_2 = U_3U_2U_3, \\
&[U_3, U_i] = 0 && \forall i, && \\
&[U_4, U_i] = 0 && \forall i \in l/\{3, 6\}, && U_4U_6U_4 = U_6U_4U_6, \\
&[U_5, U_i] = 0 && \forall i \in l/\{3, 7\}, && U_5U_7U_5 = U_7U_5U_7, \\
&[U_6, U_i] = 0 && \forall i \in l/\{3, 4\}, &&
\end{aligned}$$

We break down generators of $\mathcal{B}_6$ to the multiplication of U-generators as follows;

$$\rho^{Fib}(\sigma_1) = U_1U_2, \quad \rho^{Fib}(\sigma_2) = U_1U_3, \quad \rho^{Fib}(\sigma_3) = P_{14}(U_1U_4U_7)P_{14},$$
$$\rho^{Fib}(\sigma_4) = U_1U_6U_7, \quad \rho^{Fib}(\sigma_5) = U_1U_4U_5,$$

It is, therefore, enough to find quantum circuits of the U-generators and transform them to ZX-diagrams. One can clearly observes how the expression of U_6 and U_7 are related to each other by taking $U_7CCR_2^{Fib}$ and $U_6 = (I \otimes X \otimes I)U_7(I \otimes X \otimes I)$. We represent $U_7 = (CCF)U_5(CCF)$, that means the ZX-representation of U_7 reduces to representing CCF or $(CCZ_{\frac{-\pi}{2}})(CCX_\theta)(CCZ_{\frac{-\pi}{2}})$.

So far, we have explored the full description of braid generators up to six strands. Given these ZX diagrams, one can create any braid of up to six strands and exploit rewrite rules to simplify them. Using anyon library of $PyZX$, this process is

automatic [1]. However, due to the current limitation of $PyZX$, after simplification, $PyZX$ does not extract a circuit diagram from a simplified graph since the Fibonacci angle is an arbitrary non-Clifford angle.

Note that $PyZX$ does an acceptable job when dealing with single qubit gates and braids on three strands. However, we make this simplification more precise by introducing specific P-rules for Fibonacci and Ising. The specific P-rules not only help to simplify braids, but also it is helpful for topological quantum compiling of single-qubit gates; one can use an intermediate step, switch to the ZX-calculus, optimise them exactly, and come back to the circuit representation.

Theorem 4. *(**Fibonacci P-rule**) A triplet of angles $(\theta, \frac{2\pi}{5}, \theta)$, if $\theta = 2Arccos(\phi^{-1})$, is equivalent to $(\frac{3\pi}{5}, \theta, \frac{3\pi}{5})$.*



The proof uses the main P-rule, the fact that ϕ is the golden ratio and satisfies $\phi^2 - \phi - 1 = 0$, also $R = \frac{7\pi}{5}$, and that R and ϕ are related to each other through hexagonal equation. As Coecke and Wang mentioned in [17], if side angles of a triplet are equivalent, the resultant side angles also match. To prove the above rule, we need a lemma.

Lemma 5. *If $\alpha = \frac{2\pi}{5}$ and ϕ is the golden ration, then $cos(\alpha) = \frac{\phi^{-1}}{2}$.*

Proof. We define a general $R_m = \begin{pmatrix} R_1 & 0 \\ 0 & R_\tau \end{pmatrix}$, and if we substitute it in the hexagonal equation, we have:

$$\phi^{-1} + R_\tau = R_1^2, \qquad \phi^{-1} - \phi^{-1}R_\tau = R_1 R_\tau, \qquad R_\tau^2 + \phi^{-1}R_\tau + 1 = 0$$

Considering this set of equations and $cos(\xi) = \frac{-\phi^{-1}}{2}$, we obtain $\beta = \frac{\pi}{2} - \frac{\xi}{2}$ and $R = \frac{R_\tau}{R_1} = \xi - \beta = \frac{3\xi}{2} - \frac{\pi}{2}$. □

Proof of Theorem 4. Let us substitute initial angles in (z, z'),

$$z = cos(\frac{\alpha}{2})cos(\theta) + isin(\frac{\alpha}{2})$$
$$z' = cos(\frac{\alpha}{2})sin(\theta)$$

Using previous Lemma, we have:

$$cos(\frac{\alpha}{2}) = \frac{\sqrt{\phi^{-1}+2}}{2} = \frac{\phi}{2}, \qquad sin(\frac{\alpha}{2}) = \frac{\sqrt{2-\phi^{-1}}}{2} = \frac{\phi}{2},$$
$$cos(\frac{\theta}{2}) = \phi^{-1}, sin(\frac{\theta}{2}) = \phi^{\frac{-1}{2}}, \qquad sin(\theta) = 2\phi^{\frac{-3}{2}}, cos(\theta) = \sqrt{1-4\phi^{-3}}$$

We can check $\sin(\frac{\gamma}{2}) = \phi^{\frac{-1}{2}}$, which incidently proves $\gamma = \theta \pm \pi$. For side angles. we need to only compute argument of z as clearly $arg(z') = 0$. We compute $cos(arg(z))$ instead,

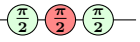
$$cos(arg(z)) = \frac{\frac{\phi}{2}\sqrt{1 - 4\phi^{-3}}}{\phi^{-1}} = -cos(\alpha)$$

□

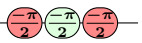
Single-qubit Braid Relations

In this part, we can check the non-trivial braid relations on three anyons, namely Yang-Baxter equation. We show that explicitly by using P-rule for Ising and Fibonacci anyons. We see that the Ising Yang-Baxter is exactly an instance of the P-rule. In fact, it can be obtained directly from the Hadamard rule by taking the adjoint of both sides.

- For **Ising anyons**, we have

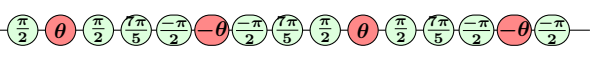
$$R_2R_1R_2 =$$  (2.56)

The other side, $R_1R_2R_1$ results in an equivalent chain of angles,

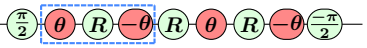
$$R_1R_2R_1 =$$  (2.57)

This shows Yang-Baxter equation is an example of the P-Rule for Ising anyons.

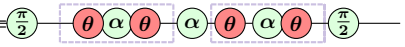
- For **Fibonacci anyons**, from combination of the ZX-equivalent of R_1 and R_2, we have

$$R_2R_1R_2 =$$  (2.58)

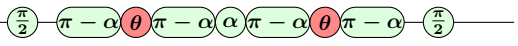
We fuse adjacent phases,

$$R_2R_1R_2 =$$  (2.59)

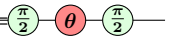
Applying the Fibonacci P-rule to the boxed angles, we have

$$R_2R_1R_2 =$$  (2.60)

We again apply the P-rule to obtain,

$$R_2R_1R_2 =$$  (2.61)

We fuse the middle angles,

$$R_2R_1R_2 =$$  (2.62)

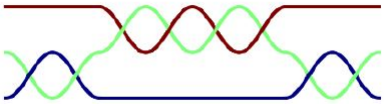
One can explicitly write the left hand side, $R_1R_2R_1$, and obtain the same equality.

Remark 3. *Observe that the above results are completely exact. We did not use ϕ explicitly. However, anyon library of PyZX works explicitly with angles and give an acceptable approximate for braids on three strands or a composition of Fibonacci single qubit gates.*

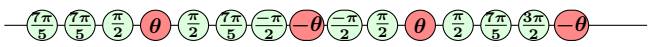
As mentioned before, having a P-rule for Fibonacci anyons, one can create a long chain of braids on three strands, and by consecutive application of P-rule for Fibonacci, plus other ZX-calculus rules, we are able to find a simple equivalent braid or circuit.The following braid is built on the Braid word

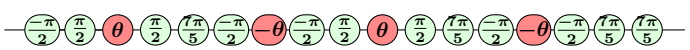
$$B = [R1, R1, R2, R2, R2, R2, R1, R1]$$

The braid is drawn in Figure below.



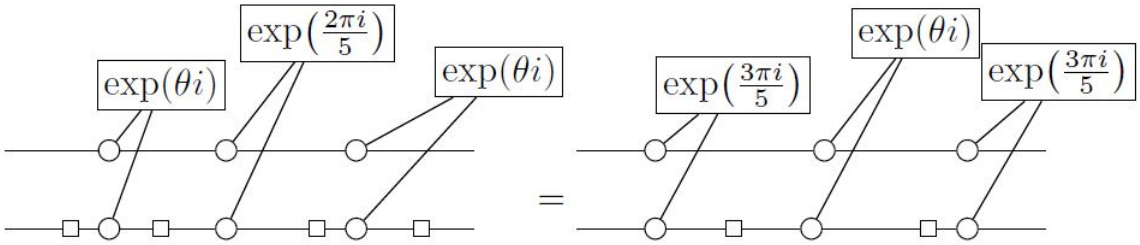
The same braid in the ZX-representation is as follows. In general, to find out the outcome matrix for this braid, one needs to multiply braid matrices. But we consider coloured lines as Fibonacci anyons, we are able to simplify the braid graphically exactly. (Note that the dotted lines are edges containing a Hadamard box.) The same relations hold for CZ- and CX-phases on 2-qubits.

$B =$  $\cdots$

$\cdots$ 

$B =$  (2.63)

Theorem 5. *(**Fibonacci P-Rule for Control Phases**) For a triple of Fibonacci CZ and CX gates, if $\theta = 2Arccos(\phi^{-1})$, the following equality holds.*



We stop expanding on this section, but one can potentially find the P-rules and general ZX-representation for anyonic model, $SU(2)_k$. We are postponing this part for future works.

We have demonstrated that the category describing topological quantum computation is actually a subcategory of **Hilb**. This perspective clears the confusion around the category of anyonic theory vs the category of topological quantum computation. But we can demand for a more unified picture, a structure which not only describes anyonic theory, but also topological quantum computation. We believe the correct picture is a bicategory. We try to illuminate further on this picture in the following chapter.

Definition 41 (**Bicategory of Topological Quantum Computing**). *A bicategory describing an anyonic theory and its corresponding model of topological quantum computation consists of:*

- *anyon types as 0-cells or objects.*
- *fusion spaces between anyons as 1-cells or 1-morphisms.*
- *braiding and fusion matrices between fusion matrices/1-morphisms as 2-cells or 2-morphisms.*

You can think of anyon types or objects as a labelling set for vector spaces or fusion spaces, and fusion rules as deciding rules set for decomposition of vector spaces or fusion of more than two anyons. Note that we assume the bicategory is skeletal meaning $X_i \otimes X_j = X_j \otimes X_i$ hence, if the outcome of two anyons is X_k, we have, $V_{ij}^k \overset{R_{ij}^k}{\cong} V_{ji}^k$.

The same happens for three simple anyons $(X_i \otimes X_i) \otimes X_k = X_i \otimes (X_j \otimes X_k)$ hence, $V_{(ij)k}^l \overset{F_{ijk}^l}{\cong} V_{i(jk)}^l$ is a 2-morphism between two 1-morphisms.

Note that anyons are still point-like excitations, but the next section and chapter are focused on 1-dimensional excitations including domain walls and boundary lines. The categorical description of a theory with boundary lines is 2-dimensional,

bicategories, and a category describing topological quantum computation with both objects and 1-morphisms, 1-dimensional excitations, is a tricategory.

Chapter 3

Bicategories and Topological Quantum Computing

3.1 Bulk-Boundary Relation

In this chapter, we are interested in the mathematical structures of a more general version of the Kitaev model, called the Levin-Wen model [40]. This model is again a spin network, with spins on a grid. We do not need the details of model. However, by bulk, we mean spin network and boundaries are the edges of the grid, a domain wall is a boundary between two grids or the intersection of two boundaries.

Kitaev and Kong proposed a categorical formalism for boundaries classification [32]. Given a fusion category as a bulk, module categories over the fusion category classify all different possible types of boundaries. Excitations on boundaries are not necessarily the excitations of the bulk, and they are endo-functors, $Hom(M, M)$, if M is a module category over the bulk. Domain walls are module categories between two fusion categories, C and D; excitations are bimodule functors. Defects of codimension 2 are functors between two boundaries or modules $Hom(M, N)$, and defects of codimension 3 are natural transformations. We unpack this picture further in this section.

One can trace back the idea to the classification of boundaries of conformal field theories. Likewise, there is a duality relation between boundaries and bulk. The boundary theory uniquely specifies the bulk theory by discussing the monoidal centre. Because one can think of the excitations of the bulk as excitations over a domain wall when both sides have the same background categories and the wall is transparent. However, the bulk theory can specify boundaries (up to a Morita equivalence).

Similar to the previous chapter where the Kitaev model was our guiding example; we carry the Kitaev model with domain walls and boundary conditions as a simple

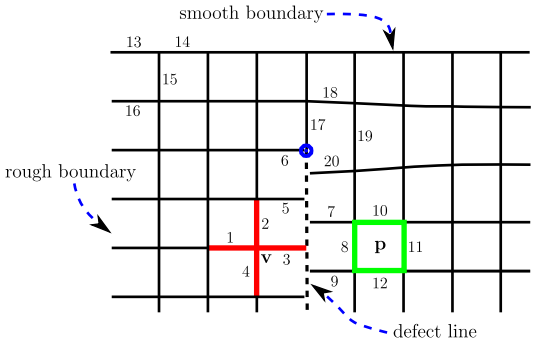


Figure 3.1: Toric codes boundaries and defect [32]

example for further analysis. We analyse the Z_2-Kitaev model without considering the module picture, and at the end of this section, we explicitly use the module formalism and demonstrate compatibility.

Consider the Toric code or the Kitaev model with Z_2 symmetry again. If we lift an array of vertical or horizontal spins along a line, the junction of a shifted site creates a kind of anomaly in the lattice which acts as the domain wall [35]. The domain wall also acts as a transformation map changing e excitations to m and vice versa. By folding the bulk along the wall, one can also consider domain walls as boundaries.

Now let us check boundaries; we can have two types: a complete edge with all spins on the lattice called a smooth boundary, or a boundary with deleted arrays of spins called a rough boundary. Observe that the domain wall is the composition of these two boundaries.

The excitations on boundaries are interesting; if we move a magnetic charge, m, to a smooth boundary, it vanishes completely, while an electric charge, e, gets stuck on the boundary. For a rough boundary, the story happens reversely, m gets stuck, but e disappears. Since ϵ is a combination of m and e, we can check its situation based on what happens to m and e.

$$\begin{aligned} &\text{smooth boundary :} && 1, m \longrightarrow 1, && e, \epsilon \longrightarrow e \\ &\text{rough boundary :} && 1, e \longrightarrow 1, && m, \epsilon \longrightarrow m \end{aligned}$$

The "vanishing" process is called **condensation**, and one can realise that with functors from bulk to boundaries. In the next section, we review module categories over monoidal categories and fusion categories. First, we skeletalise module categories which help us to index their associators. We then prove that the unitors can be deleted as unnecessary data.

3.2 Module Categories

Similar to fusion categories which are a categorification of rings, module categories categorise modules over rings. Although studying structures and properties of module categories is interesting, we only focus on module categories as a suitable structure for formalising boundary conditions and domain walls.

Definition 42. *A **left module category** over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with an action $\Box : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$, and a natural isomorphism,*

$$m_{X,Y,M} : (X \otimes Y)\Box M : X\Box(Y\Box M), \qquad X, Y \in \mathcal{C}, M \in \mathcal{M} \tag{3.1}$$

called module associator which satisfies the pentagonal identity.

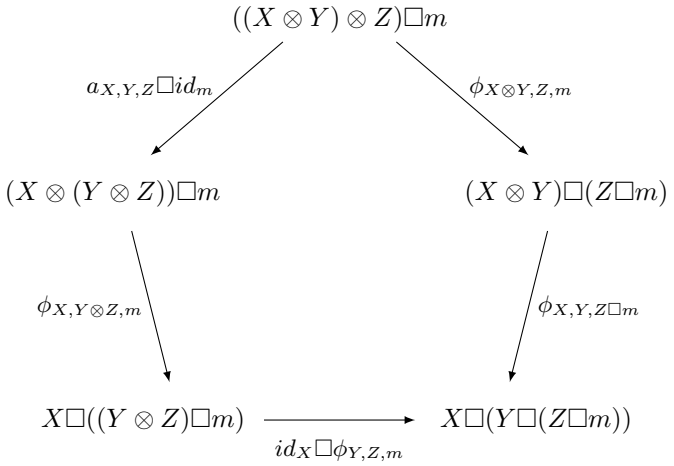


Figure 3.2: Pentagonal equation for module categories

$$l_M : 1\Box M \longrightarrow M$$

For every object $M \in \mathcal{M}$, there is also a natural isomorphism subject to triangle identity.

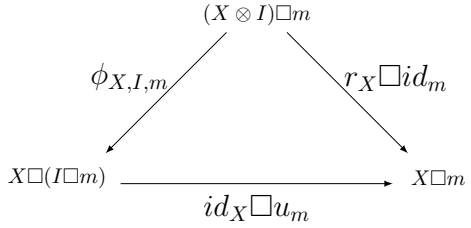


Figure 3.3: Triangle equation for module categories

Right module category can be defined similarly with action acting from right plus corresponding associators and unitors [21].

Definition 43. *Let $\mathcal{M}$ and $\mathcal{N}$ be two module categories over $\mathcal{C}$ with associativity constraints m and n, a* ***$\mathcal{C}$-module functor*** *from $\mathcal{M}$ to $\mathcal{N}$ consists of a functor $F : \mathcal{M} \longrightarrow \mathcal{N}$, and a natural transformation,*

$$s_{X,M} : F(X \square M) \longrightarrow X \square' F(M)$$

such that the following diagram commutes.

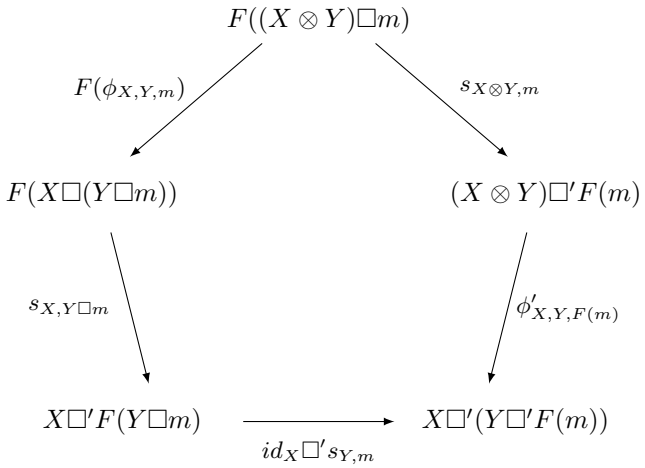


Figure 3.4: Pentagonal equation for module functor

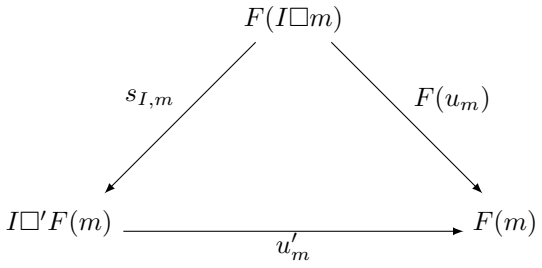


Figure 3.5: Triangle equation for module functor

Definition 44. *A morphism of $\mathcal{C}$-module functors (F, s) to (G, t) is a natural transformation ν between F and G such that the following diagram commutes for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$:*

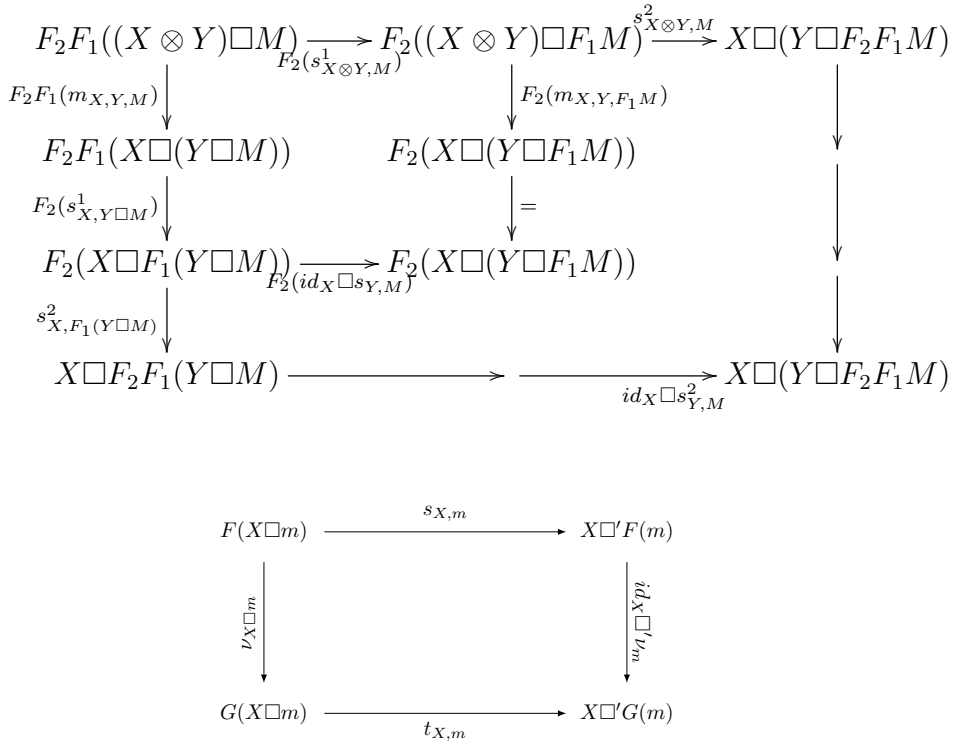


Figure 3.6: Natural transformation condition

Lemma 6. *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ be $\mathcal{C}$-module categories and let $F_1 : \mathcal{M} \longrightarrow \mathcal{N}$ and $F_2 : \mathcal{N} \longrightarrow \mathcal{L}$ be $\mathcal{C}$-module functors. The composition of functors $F_2 \circ F_1$ has a canonical structure of a C-module functor.*

Proof. Let module categories and associators be (M, m, l_M), (N, n, l_N), (L, p, l_L). We need first to build a natural transformation for the composition, then check if it satisfies the pentagonal identity. Define $s_2 s_1$ to be the natural transformation. So we have,

$$(s_2)_{X,F_1(M)} \circ (s_1)_{X,M} : F_2F_1(X\Box_M M) \longrightarrow X\Box_L F_2F_1(M)$$

We, furthermore, need to prove the pentagonal identity commutes. As it appears in the above diagram, the upper left pentagon is commutative because it is essentially $F_2(pentagon)$, and the right lower pentagon commutes as well, so the outer one commutes too. □

As mentioned earlier, we are particularly interested in module categories over the bulk category, which is a fusion category. So in the next section, we introduce module categories over fusion categories. Since objects of module categories represent

labels on boundaries, they should be simple objects. Hence, we start with semisimple module categories.

In this section, exploiting the properties of fusion categories and considering the above assumption, we revisit module categories over a fusion category. Although it is not conceptually different from Definition 42, it is insightful and allows us to calculate the module associators and unitors.

Definition 45. *A **module category over a fusion category** is a semisimple category with the following data:*

- *A set of simple objects=*$\{m_i, m_j, m_k, ...\}$,
- *Associators show in Figure 3.7, such that* $1, 2, 3$ *are simple objects of the fusion category* $\mathcal{C}$*, and* i, j, k *are simple objects of a module category* $\mathcal{M}$*. The Greek indices represent multiplicities,*

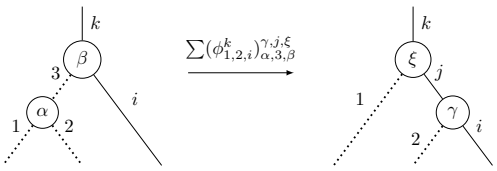


Figure 3.7: Associators of module.

- *The associator and unitor satisfy Pentagonal 3.8 and Triangle equations 3.9,*

Remark: From now on, we assume all multiplicity coefficients are 1 or $hom(m_i, m_j) \cong \mathbb{C}\delta_{ij}$.

Strict unitor for module categories

Recall that in the previous chapter, we showed any fusion category is monoidally equivalent to a fusion category with strict unitor2. Let us check the consequence of this theorem for Fibonacci anyon, and examine if it reduces the number of pentagonal equations. Given Theorem 2, one knows which fusion matrices are identity and which pentagonal equations should be solved to find non-trivial ones. For example, for the Fibonacci category, the number of pentagonal equations is $2^4 = 16$, and the number of the fusion matrices is $2^5 = 32$, but at least 31 of fusion matrices are identities, and also 16 of pentagonal equations are trivial. So to find the non-trivial matrices, we need to solve only one equation.

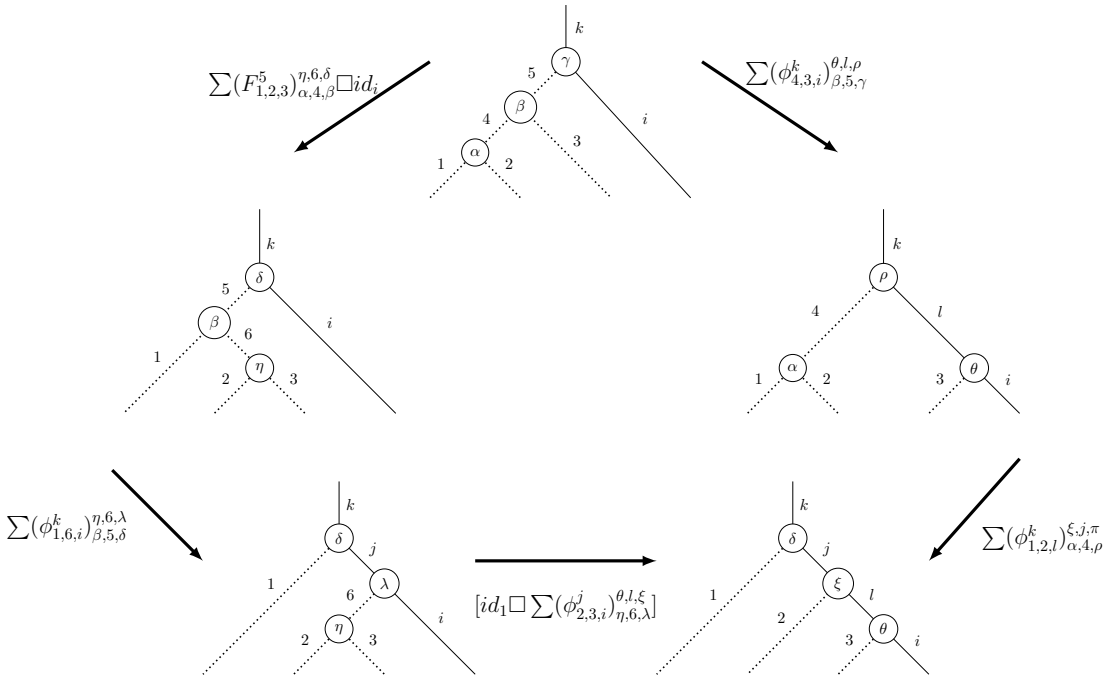


Figure 3.8: Pentagonal equation for module categories over a fusion category.

	Total	Trivial	Non-trivial
F matrices	2^5	2^5-1	2^5-2^5+1
Pentagonal equations	2^4	2^4-1	2^4-2^4+1

Table 3.1: Number of F matrices and pentagonal equations for Fibonacci category.

The same reason motivates us to state similar theorems for module categories. The proof technique is the same, but one should note here our concern is the module action rather than the tensor product.

Lemma 7. *Let $(\mathcal{C},\otimes,I,a,r,l)$ be a monoidal category and $(\mathcal{M},\Box,\phi,u)$ be a left module category over $\mathcal{C}$. Suppose we are given another left action $\Box':\mathcal{C}\times\mathcal{M}\longrightarrow\mathcal{M}$ and a set of isomorphisms in $\mathcal{M}$, $\xi_{X,m}:X\Box m\longrightarrow X\Box' m$. Then, there is a unique module structure $(\mathcal{M},\Box',\phi',u')$ such that $(id,\xi):(\mathcal{M},\Box,\phi,u)\longrightarrow(\mathcal{M},\Box',\phi',u')$ is a module functor.*

Proof. Using ξ's, we define the new structure's associators and unitor as follows:

$$\phi_{X,Y,m}=(id_X\Box'\xi_{Y,m})\xi_{X,Y\Box m}\phi_{X,Y,m}\xi^{-1}_{X\otimes Y,m}, \tag{3.2}$$

$$u'_m=u_M\circ\xi^{-1}_{I,m} \tag{3.3}$$

Now we need to show they satisfy module-pentagonal (3.11), and -triangle equations, (3.10), and that they are natural (3.12, 3.13).

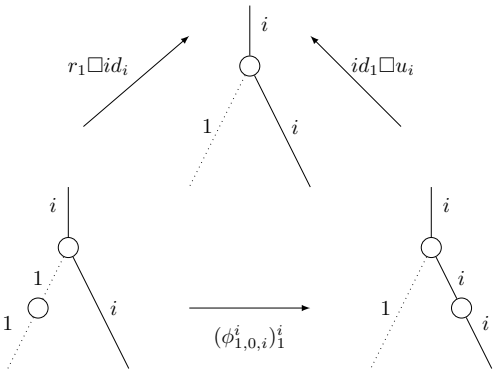


Figure 3.9: Triangle equation for module categories over a fusion category.

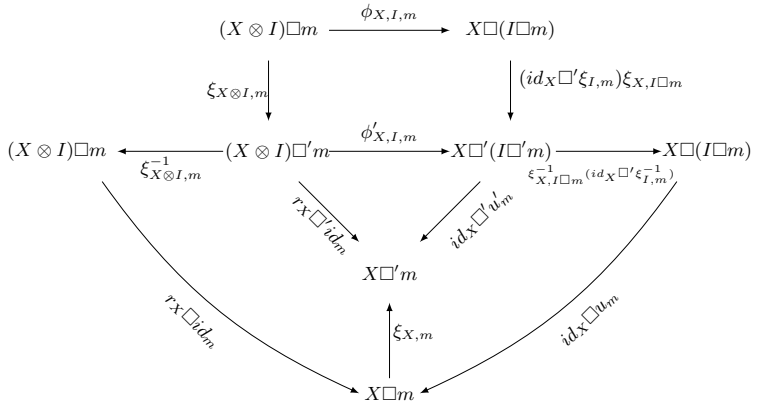


Figure 3.10: Checking triangle equation for u'. Proof of Lemma 7.

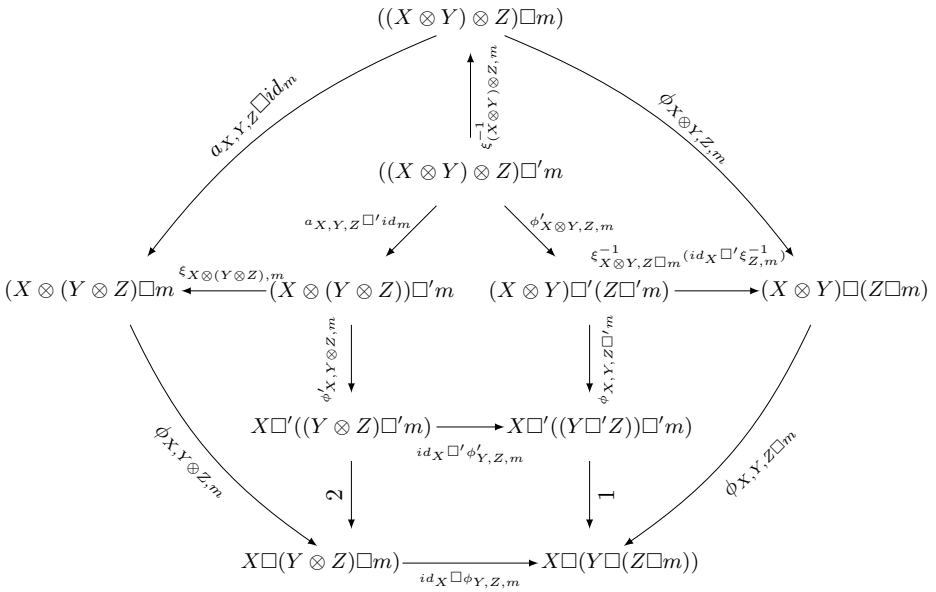


Figure 3.11: Checking pentagonal equation for ϕ', proof of Lemma 7.

In Figure 3.11, 1 and 2 are compositions of two ξ isomorphisms.

$$1 = [id_X \square (id_Y \square \xi_{Z,m})] \circ (id_X \square \xi_{Y,Z\square' m}) \circ (id_X \square \phi'_{Y,Z,m}) \circ (\xi_{X,(Y\square' Z)\square' m})$$
$$2 = \xi_{X,(Y\otimes Z)\square' m} \circ [id_X \square \xi_{Y\otimes Z,m}]$$

□

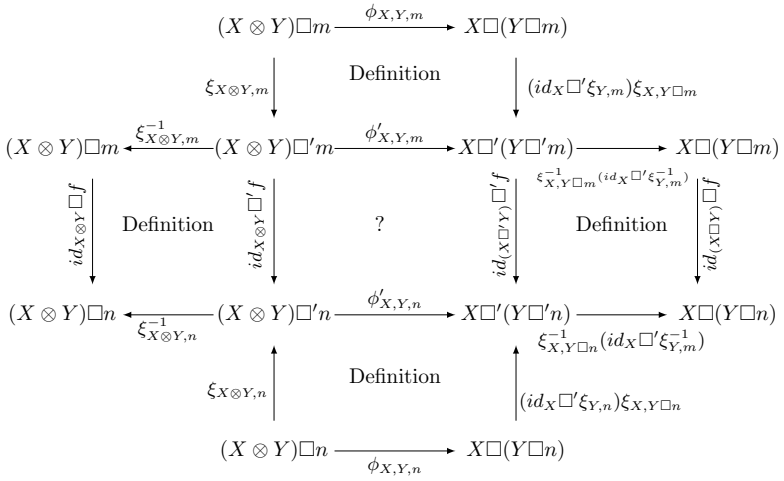


Figure 3.12: Checking naturality of ϕ'. Proof of Lemma 7.

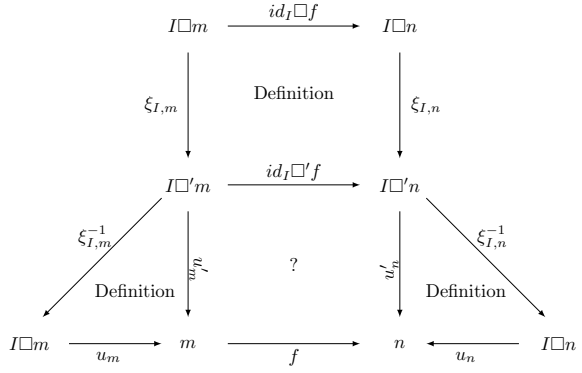


Figure 3.13: Checking naturality of u'. Proof of Lemma 7.

Theorem 6. *Let $(\mathcal{C}, \otimes, a, I, r, l)$ be a monoidal category and $(\mathcal{M}, \square, \phi, u)$ be a (left) module category over it. Then, there is an isomorphic (left) module over $\mathcal{C}$, such that $u_m = id_m$ for every object $m \in Obj(\mathcal{M})$.*

Proof. Inspired by the previous lemma, we define a new module action in the following way: $X\square' m = \begin{cases} m & if X = I \\ X\square m & otherwise \end{cases}$. $\xi_{I,m} = u_m$, otherwise, it is identity.

□

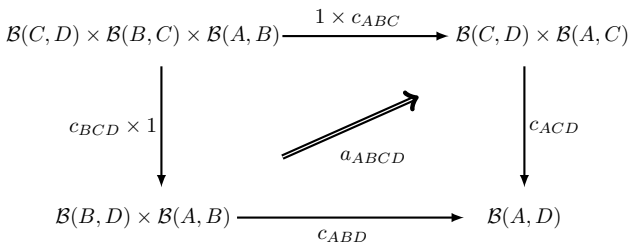


Figure 3.14: Bicategory associator

3.3 Bicategory of Module Categories

In this section, we define bicategories and prove a theorem, which shows every bicategory is equivalent to a bicategory with strict unitors. We see the structure of a bicategory of module categories over fusion categories.

Definition 46. *A **bicategory** $\mathcal{B}$ has the following information:*

- ***0-cells**: $Obj(\mathcal{B}) = \{A, B, C, ...\}$*

  For each pair of 0-cells there is a category $\mathcal{B}(A, B)$ whose objects are 1-cells and morphisms are 2-cells.

- ***1-cells**: objects of $\mathcal{B}(A, B)$ which are $f, g, ... : A \longrightarrow B$*

- ***2-cells**: morphisms of $\mathcal{B}(A, B)$ which look like* $A \overset{f}{\underset{g}{\Downarrow \alpha}} B$

- ***Functors**: c_{ABC} and I_A such that*

$$c_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \longrightarrow \mathcal{B}(A, C)$$
$$(g, f) \longmapsto g \circ f = gf$$
$$(\beta, \alpha) \longmapsto \beta \circ \alpha = \beta\alpha$$

  and $I_A : 1 \longrightarrow \mathcal{B}(A, A)(I_A : A \longrightarrow A$ which is the identity morphism as a $1-$cell in bicategory).

- ***Natural isomorphisms:** Associators a 3.14, left unitor l and right unitor r defined in the figure 3.15 such that they satisfy the axioms of figure 3.17 and 3.16, [38].*

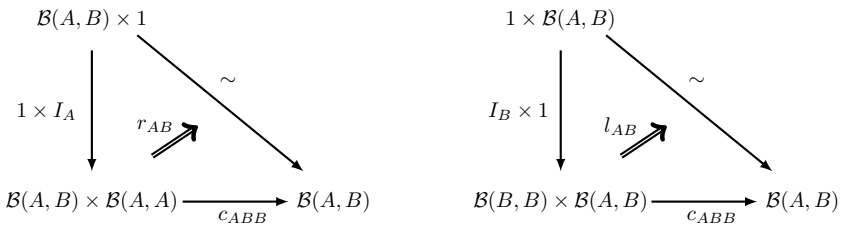


Figure 3.15: Bicategory right and left unitors

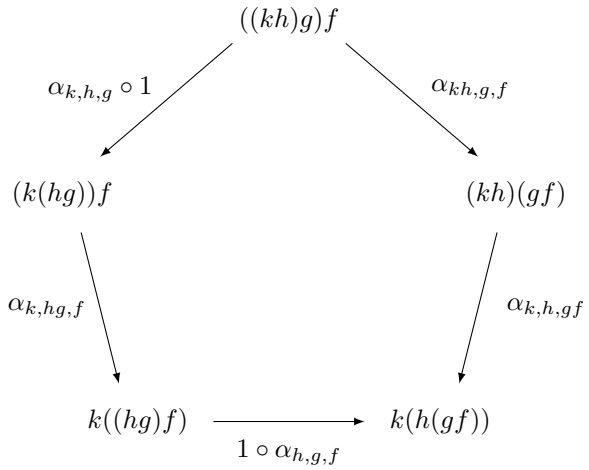


Figure 3.17: Pentagonal equation for associators a of bicategory

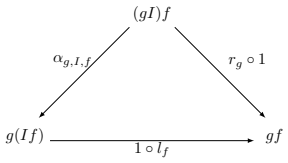


Figure 3.16: Triangle equation for unitors of bicategory

Now we define bifunctors between two bicategories. Generally, there are three types of bifunctors: lax, weak and strict [39]. Here, we define only the weak one.

Definition 47. *A **bifunctor** $F:\mathcal{B}\longrightarrow\mathcal{B}'$ between two bicategories consists of:*

- *a function between 0-cells such that for every A in $\mathcal{B}$, it assigns a 0-cell $F(A)$ in $\mathcal{B}'$;*
- *for each hom category in $\mathcal{B}$, a functor $F_{A,B}:\mathcal{B}(A,B)\longrightarrow\mathcal{B}'(A,B)$;*
- *for each 0-cell A in $\mathcal{B}$, an invertible 2-cell: $F_{I_A}:I_{F(A)}\Rightarrow F_{A,A}(I_A)$;*
- *for each triple of 0-cells (A,B,C) in $\mathcal{B}$, an isomorphism $\xi_{A,B,C}(f,g):F_{A,B}(f)\circ' F_{B,C}(g)\Rightarrow F_{A,C}(f\circ g)$ such that for every hom category $\mathcal{B}(A,B)$ satisfies the diagrams in Figure 3.18, 3.19, [38].*

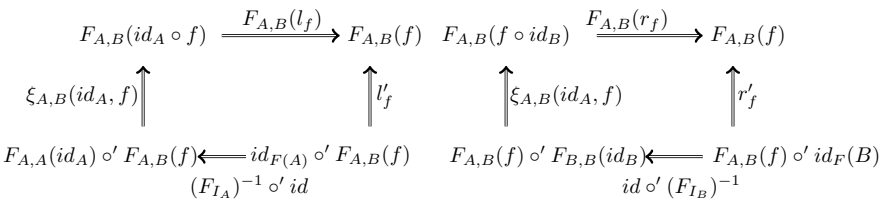


Figure 3.18: Conditions of bifunctor I, II

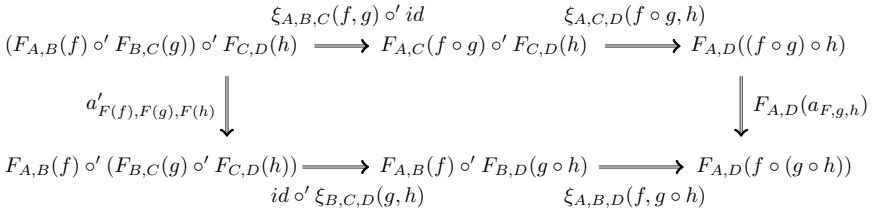


Figure 3.19: Conditions of bifunctor III

Definition 48. *A weak* ***natural bitransformation*** $\sigma : F \longrightarrow G$ *between two bifunctors* $F, G : \mathcal{B} \longrightarrow \mathcal{B}'$ *consists of:*

- *for each 0-cell* A *in* $\mathcal{B}$*, a 1-cell* $\sigma_A : F(A) \longrightarrow G(A)$ *;*
- *for each 1-cell* $f \in \mathcal{B}(A, B)$ *a 2-cell isomorphism* σ_f *which is 3.20,*

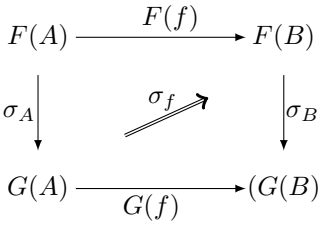


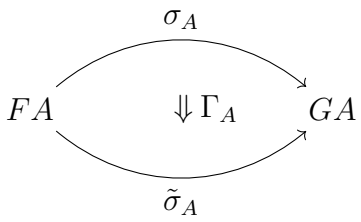
Figure 3.20: Natural bitransformation

such that σ_f *satisfies two axioms mentioned in part 1.2 of [38].*

Definition 49. *A bifucntor* F *is a* ***biequivalence*** *if its function part on objects is essentially surjective and its functor part on morphisms* $F_{A,B} : \mathcal{B}(A, B) \longrightarrow \mathcal{B}'(A, B)$ *is an equivalence [38].*

Definition 50. *A morphism between two natural morphisms* $(\sigma, \tilde{\sigma})$ *between two 2-functors is called a* ***modification*** Γ*. For every object* A*, it consists of a 2-morphism*

Γ_A, *subjects to the modification square (3.4) for every morphism* f.



$$\begin{array}{ccc} Gf\sigma_A & \xRightarrow{1_{Gf} * \Gamma_A} & Gf\tilde{\sigma_A} \\ {\scriptstyle \sigma_f}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \tilde{\sigma_f}} \\ \sigma_B Ff & \xRightarrow[\Gamma_B * 1_{Ff}]{} & \tilde{\sigma_B} Ff \end{array} \tag{3.4}$$

Although bicategories of module categories are strict, we bring our proof here because we aim to skeletise this category finally, so this theorem will act similar to the previous theorems and reduce equations to the non-trivial ones. In addition, it might be a useful result for the general case of fusion bicategories.

Lemma 8. *Let* $\mathcal{B}$ *be a bicategory. Suppose we are given two other functors* c', I'_A *and a set of natural isomorphisms* ξ *such that:*

$$c'_{A,B,C} : \mathcal{B}(B,C) \times \mathcal{B}(A,B) \longrightarrow \mathcal{B}(A,C) \qquad I'_A : 1 \longrightarrow \mathcal{B}(A,A)$$

$$\xi : c'_{A,B,C} \longrightarrow c_{A,B,C} \qquad \xi_{f,g} : f \circ' g \longmapsto f \circ g$$

$$\xi_A : I'_A \longrightarrow I_A$$

Then, there is a unique bicategorical structure on $\mathcal{B}$ *such that* $(id, \xi_{f,g}, \xi_A)$ *is the equivalence functor between bicategories.*

Proof. Inspired by the axiom (3.19), we define new category's associators and unitors as follows:

$$\boxed{a'_{f,g,h} = \xi^{-1}_{f,g\circ' h} * (id_f \circ \xi_{g,h}) * a_{f,g,h} * (\xi_{f,g} \circ id_h)\xi_{f\circ' g,h}}$$

$$\boxed{r'_f = r_f * (\xi_{f,I'_A}) * (id_f \circ \xi_A)}$$

$$\boxed{l'_f = l_f * (\xi_{I'_A,f}) * (\xi_A \circ id_f)}$$

□

The new definition should satisfy Figures (3.12) and (3.13). Because the diagrams are similar to our previous theorems and proofs, we skip them and refer to 2.5.

Theorem 7. *Let* $\mathcal{B}$ *be a bicategory. There is a bicategory equivalent to* $\mathcal{B}$ *whose unitors are stricts.*

Proof. Take $I'_A = I_A$ and $\xi_A = id_{I_A}$. Using the above lemma, we define functor d' as follows:

$$g \circ' f = \begin{cases} f & if \quad g = I_B \\ g & if \quad f = I_A \\ g \circ f & otherwise \end{cases}$$

and we define

$$\xi_{g,f} = \begin{cases} r_g^{-1} & if \quad f = I_A \\ l_f^{-1} & if \quad g = I_B \\ id_{g \circ f} & otherwise \end{cases}$$

Substituting the new definition in $r'_f = r_f(\xi_{f,I'_A})(id_f \circ \xi_A)$, we have $r'_f = r_f \circ (r_f^{-1})(id_f \circ id_{I_A}) = id_f$. □

Because proofs of the following theorems are exactly the same as previous theorems, we only mention theorems' statements .

Theorem 8. *For a bicategory described in Theorem 7, associators a with at least one index I are identity.*

Theorem 9. *For a bicategory described in Theorem 7, all pentagonal equations with at least one I are trivial.*

As we mentioned in the summary part, the main reason we look at the bicategories of module categories is to get an idea of what a fusion bicategory looks like. Here, we define this bicategory.

Definition 51. *The* ***bicategory of module categories over a monoidal category****, $\mathcal{C}$ has module categories as 0-cells. 1-cells are functors between module categories that respect the module structure, and natural isomorphisms between module functors are 2-cells. The $\circ$-functor is the composition between functors and because it is strict, associators a and unitors are identity*

Definition 52. *The* ***bicategory of module categories over a fusion category*** *$\mathcal{C}$, has indecomposable semisimple module categories as 0-cells. 1-cells are simple functors between module categories that respect module structure, and natural isomorphisms between module functors are 2-cells which are complex coefficients of $id_{\mathbb{C}}$. the $\circ$-functor is the composition between functors and because it is strict, associators a and unitors are identity*

Because we are looking for a semisimple bicategory, we took the bicategory of module categories over a fusion category. We restricted modules to indecomposable semisimple ones. By doing so, we semi-simplified this bicategory in the level of 0-cells. We then went further and did the same in the level of 1-cells, that is, we took simple functors between module categories and assumed all functors can be written

as a biproduct of these functors. As you will see explicitly in the next chapter, in this levelling up procedure, you have a bicategory which is closed under composition in the level of 1-cells, so it seems we can define a kind of fusion rules in this bicategory. However, the major dissimilarities are:

- We cannot compose each pair of functors.
- This category has two unit 1-cells, so it is a multifusion category rather than fusion.

To make it clear, in the next section we describe in detail the bicategory of module categories over the category of representations of Z_2.

3.3.1 Modules Over $Rep(Z_2)$

In this section, we work out the bicategory of semisimple module categories over the category of representations of Z_2. Simple objects are the two one-dimensional representations T_0 and T_1 of $Rep(Z_2)$. T_0 is the trivial representation, that is, $T_0(0) = T_0(1) = id_{\mathbb{C}}$, and T_1 is the non-trivial one, which is $T_1(0) = id_{\mathbb{C}}$ and $T_1(1) = -id_{\mathbb{C}}$. The monoidal product is defined through the tensor product between representations:

$$(T_i \otimes T_j)(\alpha) = T_i(\alpha) \otimes T_j(\alpha) \qquad \alpha \in Z_2 \tag{3.5}$$

Each object is self-dual so the category is rigid, and biproduct is the usual direct sum:

$$(T_i \oplus T_j)(\alpha) = T_i(\alpha) \oplus T_j(\alpha) \qquad \alpha \in Z_2 \tag{3.6}$$

Considering skeletality and strict unitor property, all fusion associators/matrices are identity. So we have, in sum, table 3.2.

Simple objects	$\{T_0, T_1\}$
Fusion rules	$T_0 \otimes T_i = T_i \otimes T_0 = T_i, T_1 \otimes T_1 = T_0$
Fusion matrices	$F^l_{i,j,k} = 1$ for all $i, j, k, l \in \{0, 1\}$

Table 3.2: Category of $Rep(Z_2)$

Now we want to find the number of modules over this fusion category. Based on Ostrik's paper [21], there is a relation between indecomposable algebras in a monoidal category and the number of module categories over it. However, here we use a more explicit way to find modules for fusion categories. This approach is rather more algorithmic, and can be employed to find modules over any other well-defined fusion category.

Number of semisimple modules

Since we only consider semisimple module categories, any morphism between objects is either zero or an isomorphism. The former happens when objects belong to two different classes and the latter when they belong to the same class. To find the number of modules, we only need to check the non-trivial action of 1 on objects. So if we generally have: $1\Box m_i \cong \oplus_k N_{1,i}^k m_k$. Acting 1 on the both sides of this equality results in:

$$1\Box(1\Box m_i) \cong \oplus_k N_{1,i}^k (1\Box m_k)$$

Let the right hand side be $1\Box m_k \cong \oplus_j N_{1,k}^j m_j$. The left hand side is also isomorphic to $(1 \otimes 1)\Box m_i$. So,

$$0\Box m_i \cong \oplus_{k,j} N_{1,i}^k N_{1,k}^i m_j$$

But they are isomorphic only when $j = i$:

$$1 = \oplus_{k,i} N_{1,i}^k N_{1,k}^i$$

Solving this equation, we have two possibilities. When we only have one module object, m_i and when we have two distinct module objects, $\{m_i, m_k\}$. So the number of possible semisimple modules are two with following actions,

$$\forall i, 1\Box m_i = m_i, \qquad \text{or} \qquad (1\Box m_i = m_k, \quad 1\Box m_k = m_i) \tag{3.7}$$

The second one is the trivial module which is the fusion category itself, so the non-trivial module is the first one. Now we investigate properties of each module category.

Module category $\mathcal{M}_1$

This category has one simple object C and all morphisms are isomorphisms in $Hom(C, C)$. Solutions of triangle and pentagonal equations are:
Module unitors:

$$U_C = u_C id_C \quad \text{s.t.} \quad u_C \in \mathbb{C}$$

Module associators:

$$(\phi_{0,0,C}^C = \phi_{1,0,C}^C = \phi_{0,1,C}^C = \frac{id_C}{u_C}, \phi_{1,1,C}^C = did_C) \quad \text{s.t.} \quad d \in \mathbb{C}$$

However, according to the theorem 6 which states every module category is equivalent to a module category with strict unitor, we let $u_C = 1$. It is also easy to show that this category with a non-trivial associator $\phi_{1,1,C}^C = did_C$ is isomorphic to a module category, all of whose all associators are trivial i.e. $d = 1$. So the summarised data is in table (3.3).

Simple objects	C
Morphisms	$f = \alpha id_C$ s.t. $\alpha \in \mathbb{C}$ and $id_C = id_{\mathbb{C}}$
Module actions	$1 \square C = 0 \square C = C$
Module unitor	$id_C = id_{\mathbb{C}}$
Module associators	$(\phi^C_{0,0,C} = \phi^C_{1,0,C} = \phi^C_{0,1,C} = \phi^C_{1,1,C} = id_C)$

Table 3.3: Module category $\mathcal{M}_1$.

Module category $\mathcal{M}_2$

This category is the trivial module with two simple objects A and B. Morphisms are either in $Hom(A, A) \cong \mathbb{C}$ or $Hom(B, B) \cong \mathbb{C}$. Again here before applying theorems of strict unitor, unitors and associators are generally non trivial:
Module unitors:

$$U_A = u_A id_A, \quad \text{and} \quad U_B = u_B id_B, \quad \text{s.t.} \quad id_A = id_B = id_{\mathbb{C}}$$

Module associators:

$$[(\phi^A_{0,0,A})^A_0 = \frac{id_A}{u_A}, (\phi^B_{1,0,A})^A_1 = \frac{id_B}{u_A}, (\phi^B_{0,1,A})^B_1 = \frac{id_B}{u_B}, (\phi^A_{1,1,A})^B_1 = did_A]$$

$$[(\phi^B_{0,0,B})^B_0 = \frac{id_B}{u_B}, (\phi^A_{1,0,B})^B_1 = \frac{id_A}{u_B}, (\phi^A_{0,1,B})^A_1 = \frac{id_A}{u_A}, (\phi^B_{1,1,B})^A_1 = \frac{du_B id_B}{u_A}]$$

Here again regarding skeletality and strict unitor properties and being isomorphic to a category with trivial associators, the data of this category is shown in Table (3.4).

Simple objects	$\{A, B\}$
Morphisms	$f = \alpha id_A$ or $f = \alpha id_B$ s.t. $\alpha \in \mathbb{C}$ and $id_A = id_B = id_{\mathbb{C}}$
Non-trivial module actions	$1 \square A = B, 1 \square B = A$
Module unitor	$U_A = id_A = id_{\mathbb{C}}$ and $U_B = id_B = id_{\mathbb{C}}$
Module associators	$[(\phi^A_{0,0,A})^A_0 = (\phi^A_{1,1,A})^B_1 = id_A, (\phi^B_{1,0,A})^A_1 = (\phi^B_{0,1,A})^B_1 = id_B]$ $[(\phi^B_{0,0,B})^B_0 = (\phi^B_{1,1,B})^A_1 = id_B, (\phi^A_{1,0,B})^B_1 = (\phi^A_{0,1,B})^A_1 = id_A]$

Table 3.4: Module category $\mathcal{M}_2$

Module morphisms

To gain an insight into the bicategorical structure of the category of semisimple module categories over a fusion category, in this section, we first find functors, then we look at the composition rules similar to fusion rules.

Endofunctors $\in Hom(\mathcal{M}_1, \mathcal{M}_1)$

A general functor F can take simple object m and gives a finite direct sum of them:

$$F(m) = \underbrace{m \oplus m \oplus ... \oplus m}_{n}$$

So the isomorphism part of this functor is actually a non-trivial n by n matrix with non-zero off diagonal elements $F(1\square m) \xrightarrow{f_{1,m}} 1\square F(m)$ so:

$$\underbrace{m \oplus m \oplus ... \oplus m}_{n} \xrightarrow{f_{1,m}} \underbrace{m \oplus m \oplus ... \oplus m}_{n}$$

But we assume that they are all diagonal, so then it is not hard to see that they are block diagonal matrices whose diagonal elements are ± 1. More explicitly, if $F(m) = m$ then solutions of pentagonal equations are:$f_{0,m} = id_m$ and $f_{1,m} = \pm id_m$. Therefore, we have two endofunctors shown in table 3.5,

Endofunctor	$(F, f_{0,m}, f_{1,m})$
$I(m) = m$	$(id_{\mathcal{M}_1}, i^m_{0,m} = id_m, i^m_{1,m} = id_m)$
$H(m) = m$	$(id_{\mathcal{M}_1}, h^m_{0,m} = id_m, h^m_{1,m} = -id_m)$

Table 3.5: Endofunctors $\in Hom(\mathcal{M}_1, \mathcal{M}_1)$

Endofunctors $\in Hom(\mathcal{M}_2, \mathcal{M}_2)$

The general functor F is:

$$F(A) = nA \oplus pB \qquad F(B) = qA \oplus tB$$

However, two sides of $F(1\square A) \xrightarrow{\cong} 1\square F(A)$ need to be isomorphic. Using the definition of F and the result of action of 1, we have:

$$\underbrace{F(B)}_{qA \oplus tB} \xrightarrow{\cong} \underbrace{1\square(nA \oplus pB)}_{nB \oplus pA}$$

So $q = p$ and $t = n$ and,

$$F(A) = nA \oplus pB \qquad F(B) = pA \oplus nB$$

Thus, F is actually the sum of two functors E and S, $F = nE \oplus pS$ such that

$$(E(A) = A, E(B) = B) \qquad (S(A) = B, S(B) = A)$$

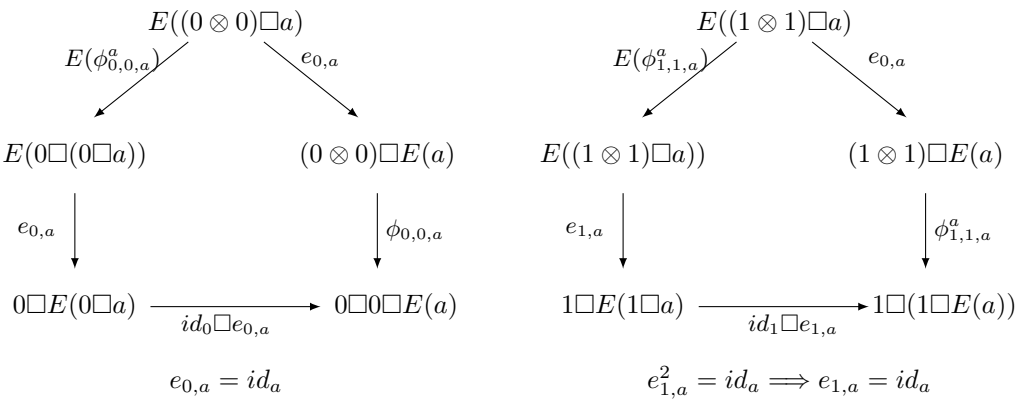


Figure 3.21: Pentagonal equation for endofunctors $\in Hom(\mathcal{M}_2, \mathcal{M}_2)$

To complete functors' description, we still need to find the isomorphism part of each functor. In Figure **??**, we calculate them. But because these equations are symmetric with respect to A and B, let's solve them for $a \in \{A, B\}$ such that if $a = A$, $\overline{a} = B$ and vice versa , check Figure 3.21. The data of these functors summarised is in Table 3.6.

Endofunctors	$(F, f_{0,a}, f_{1,a})$
$E(A) = A, E(B) = B$	$(E, e^A_{0,A} = e^A_{1,B} = id_A, e^B_{0,B} = e^B_{1,A} = id_B$
$S(A) = B, S(B) = A$	$(S, s^A_{0,B} = s^A_{1,A} = id_A, s^B_{0,A} = s^B_{1,B} = id_B$

Table 3.6: Functors $\in Hom(\mathcal{M}_2, \mathcal{M}_2)$

;

Functors $\in Hom(\mathcal{M}_1, \mathcal{M}_2)$

A general functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is as follows $F(m) = nA \oplus pB$. But again from applying the isomorphic condition:

$$\underbrace{F(1\Box m)}_{nA\oplus pB} \xrightarrow{\cong} \underbrace{1\Box F(m)}_{nB\oplus pA}$$

we have $n = p$ and $F(m) = n(A \oplus B)$ so it is the direct sum of a simpler functor $K(m) = A \oplus B$. Solving pentagonal equation for this, yields Table 3.7

Functors $\in Hom(\mathcal{M}_2, \mathcal{M}_1)$

A general functor $F : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ is as follows: $F(A) = nm, F(B) = pm$ The isomorphic condition:

$$\underbrace{F(1\Box A)}_{pm} \xrightarrow{\cong} \underbrace{1\Box F(A)}_{nm}$$

Thus, $n = p$ and $F(A) = nm$ and $F(B) = nm$. So it is the direct sum of the simpler functor $L(A) = L(B) = m$. The isomorphism part is in Table 3.7.

$K : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ $K(m) = A \oplus B$	$(K, k^A_{0,m} = id_A, k^B_{0,m} = id_B; k^A_{1,m} = \xi id_A, k^B_{1,m} = \xi^{-1} id_B)$
$L : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ $L(A) = m$ and $L(B) = m$	$(L, l^m_{0,A} = l^m_{0,B} = id_m; l^m_{1,A} = \eta id_m, l^m_{1,B} = \eta^{-1} id_m$

Table 3.7: Functors between $\mathcal{M}_1$ and $\mathcal{M}_2$

Composition of functors

In this section, since the composition of functors is actually the tensor product in the level of 1−cells, we compose them to get the fusion rules between simple functors.

The composition of module functors has two parts. The easy part is the composition of the functorial part, but it is the isomorphism part which determines the final result. Let's look at the general case. If we have three module categories $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ over monoidal category $\mathcal{C}$, and functors $(F, f) : \mathcal{X} \longrightarrow \mathcal{Y}$ and $(G, g) : \mathcal{Y} \longrightarrow \mathcal{Z}$, then their composition is $(G \circ F, g \circ G(f))$. One can show this resultant isomorphism satisfies the pentagonal equation.

Considering the fact that not every two functors are composable, the fusion rules of functors when $|\eta| = |\xi|^{-1} = 1$ are:

$$I_+ \circ \begin{cases} H_- &= H_- \\ I_+ &= I_+ \\ L_\eta &= L_\eta \end{cases} \qquad H_- \circ \begin{cases} H_- &= I_+ \\ I_+ &= H_- \\ L_\eta &= L_{-\eta} \end{cases} \qquad L_\eta \circ \begin{cases} E_\alpha &= L_{\eta\alpha} \\ S_\beta &= L_{\eta^{-1}\beta} \\ K_\xi &= I_+ \oplus H_- \end{cases}$$

$$E_\alpha \circ \begin{cases} E_{\alpha'} &= E_{\alpha\alpha'} \\ S_\beta &= S_{\beta\alpha^{-1}} \\ K_\xi &= K_{\xi\alpha^{-1}} \end{cases} \qquad S_\eta \circ \begin{cases} E_\alpha &= S_{\alpha\eta} \\ S_{\eta'} &= E_{\eta^{-1}\eta'} \\ K_\xi &= K_{\xi^{-1}\eta} \end{cases} \qquad K_\xi \circ \begin{cases} L_\eta &= E_{\eta\xi^{-1}} \oplus S_{\xi\eta} \\ I_+ &= K_\xi \\ H_- &= K_{-\xi} \end{cases}$$

Chapter 4

Fusion Bicategories

The bicategory of module categories is a an example for higher dimensional excitations, for instance, string-like excitations. However, it is far from a comprehensive structure to describe all theories of this sort. Let us remember what we have seen so far.

Unitary ribbon fusion categories describe point-like anyonic theories in the 2-dimensional gapped systems. Having the modularity on top of it, it specifies a 2+1-d topological field theory. We further saw how this category determines fusion, essentially computational spaces. We then went on and categorified the boundary conditions by module categories.

However, one can ask what if we consider a 3+1-d lattice, which category specifies excitations of such a system, or given a 2+1-d system, which category best classifies all boundary types and excitations on the boundaries. The answer to these questions has long been speculated to be a fusion 2-category [53].

Intuitively, one expect to generalise some notions, properties, and structures, such as simple objects, additions between objects and morphisms, tensor product, and fusion and braiding matrices. The other expectation is the recovery of fusion categories from fusion 2-categories. In other words, a fusion 2-category with one object should be a fusion category.

Such a definition was proposed by authors of [19]; this definition, while useful for the purpose of introducing state sum, is far from a complete and rigorous definition for topological quantum computation. In the following, we summarise some comments and points about the definition.

First and foremost, lack of general enough examples. Indeed, the category of semisimple module categories or **2Vec** or 2-groups can be suitable examples. However, they are too well-structured and are not general enough. Meaning, they do not have general enough properties and structures.

The 2+1-d conformal field theories heavily inspired the categorification of fusion categories and topological quantum field theory. One familiar with CFT can observe the utility of a unified picture, while $6j$ symbols or equations may seem ad-hoc. Unfortunately, the paper lacks any comparison or recovery of familiar features of 3+1-d conformal field theories.

The other serious problem is the absence of a well-defined direct sum between objects. While one probably can assume such a property could exist in a 2-category, as we show in the next section, this is not trivial. Questions such as the type of enrichment, the interaction between 2-limits and limits, and coherence conditions do not seem straightforward and obvious. The paper uses some sort of limit while working with terminal objects, but it does not specify precisely what type of limits or 2-limits.

Moreover, the last issue we address here is the definition of monoidal 2-categories. The definition addressed in the paper is semistrict. We provide a more explicit definition that has appeared in the literature but in a scattered manner, and we show how these definitions need to be modified. Our approach takes Gurski's algebraic definition of tricategory and recovers Kapranov and Voevodsky's definition. KV's definition has been addressed in the literature [6, 61], so our recovery procedure systematically convenes all of them.

An issue neglected in the paper is a concrete discussion on which coherence conditions, natural coherence isomorphisms, axioms, and identities are meaningful physical data. This specifically comes to one's attention when one attempts to exploit this category for higher dimensional anyonic theories and topological quantum computation. Essentially, to characterise different physical theories, one need to solve the coherency equations and axioms.

Since I was simultaneously working on a definition for fusion 2-categories, this chapter is devoted to works towards that goal. To summarise, to define a fusion 2-category one needs to specify and define the following parts:

- **Semisimplicity** Define simple objects and simple morphisms in fusion 2-categories. The authors of Paper [19] *addressed* this.
- **Additivity** Define a direct sum between objects in 2-categories. Address limits- and 2-limits in fusion 2-categories, subsequently address the finiteness condition. The authors of Paper [19] *did not* address this. We discuss it here.
- **Monoidality** The authors work with semi-strict definition of monoidal 2-categories, while similar to fusion categories to specify a TQC theory one needs to solve

the pentagonal equation. The complete definition of monoidal 2-categories one needs to consider appears here.

Therefore, in some sense, to have a more rigorous definition, one can consider the definition of Paper [19], plus the next two sections of this chapter. What still remains to be addressed is the **finiteness condition**.

4.1 Additivity Property in Bicategories

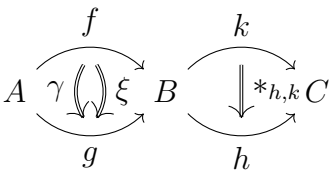
Semiadditive 2-categories

To define semiadditive 2-categories, a 2-category requires enough structures in both object and Hom-categories levels. This section is devoted to developing these structures. Similar to Chapter one, we start with constant morphisms.

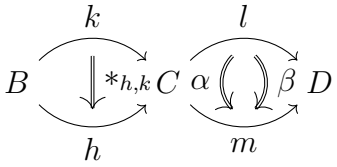
Constant morphisms in 2-categories

A 2-category can have constant 1- or 2-morphisms; we start from the top level. Constant 2-morphisms exist independently for horizontal or vertical compositions. For example, it is possible for Hom-categories to be locally pointed with respect to the vertical composition of 2-morphisms. It is also conceivable to have a 2-category with horizontally constant 2-morphisms. Hence, it seems constant 2-morphisms in both directions exist completely independently of each other, but if a 2-category happens to have two different families of horizontally and vertically constant 2-morphisms, then they collapse, due to the interchange law.

Definition 53. *A 2-morphism $*_{h,k} : k \Rightarrow h$ is left constant if for every object A, every pair of 1-morphisms f, g, and every pair of 2-morphisms, $\xi, \gamma : f \Rightarrow g$, horizontal composition on the right $*_{h,k}\xi = *_{h,k}\gamma$ is the same.*

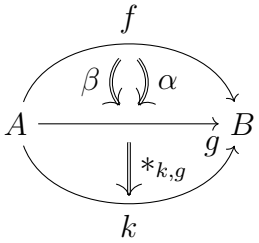


Definition 54. *A 2-morphism $*_{h,k} : k \Rightarrow h$ is right constant if for every object D, every pair of 1-morphisms l, m, and every pair of 2-morphisms, $\alpha, \beta : l \Rightarrow m$, horizontal composition on the left $\beta *_{h,k} = \alpha *_{h,k}$ is the same.*

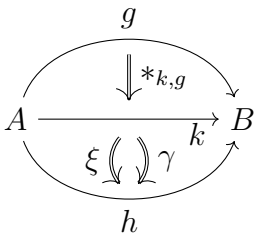


Definition 55. *In a 2-category, a horizontally constant 2-morphism is a 2-morphism which is left and right constant.*

Definition 56. *A 2-morphism $*_{k,g} : k \Rightarrow g$ in a 2-category is down constant if for every 1-morphism f, and every pair of 2-morphisms $\beta, \alpha : f \Rightarrow g$, the vertical composition from above $*_{k,g} \odot \ \beta = *_{k,g} \odot \ \alpha$ is the same.*



Definition 57. *A 2-morphism $*_{k,g} : k \Rightarrow g$ in a 2-category is up constant if for every 1-morphism f, and every pair of 2-morphisms, $\xi, \gamma : k \Rightarrow h$, vertical composition from below $\xi \odot *_{k,g} = \gamma \odot *_{k,g}$ is the same.*

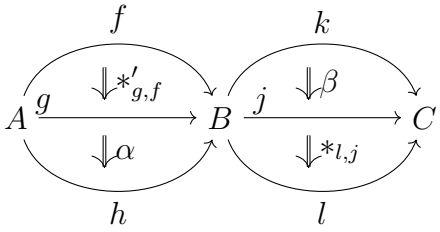


Definition 58. *A vertically constant 2-morphism is a 2-morphism which is up and down constant.*

We now present the proposition which guarantees that zero 2-morphisms in Hom-categories are not only vertically but also horizontally constant.

Proposition 3. *If a 2-category has a family of horizontally constant 2-morphisms $\{*\}$ and a family of vertically constant 2-morphisms $\{*'\}$, then they are equal.*

Proof. Suppose we are given 1- and 2-morphisms shown in the following figure,



Due to the interchange law in 2-categories, we expect to have:

$$(*_{l,j} \odot \beta)\ (\alpha \odot *'_{g,f}) = (*_{l,j}\ \alpha) \odot (\beta\ *'_{g,f}) \tag{4.1}$$

Because $*'$ is vertically constant. The left hand side outcome is $(*_{l,j} \odot \beta)*'_{h,f}$. Because $*$ is horizontally constant. The right hand side is, $*_{lh,jg} \odot (\beta *'_{g,f})$. Now let $B = C$, $\beta = 1_{id_B}$, and $k = j = l = id_B$. Then, we have:

$$LHS : (*_{id_B,id_B} \odot 1_{id_B})*'_{h,f} = *_{h,f} \qquad RHS : *_{h,g} \odot (1_{id_B} *'_{g,f}) = *'_{h,f}$$

Thus, for every pair of 1-morphisms (h, f), horizontally and vertically constant 2-morphisms are equal. □

Remark 4. *Generally, the 2-identity 1_f for vertical composition is not also a horizontal 2-identity, unless f is a 1-identity. That is, $1^{horizontal}_{id_B} = 1^{vertical}_{id_B}$. Because consider composable $\alpha : g \Rightarrow h$ and 1_f, then α and $\alpha f : gf \Rightarrow hf$ do not have even the same source and target [51, 50].*

The presence of 2-morphisms requires redefining constant 1-morphisms because a category can have both weak and strict versions. We only define the weak version here; the strict case is determined by letting 2-isomorphisms be 2-identities.

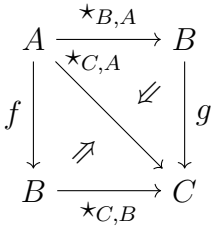
Definition 59. *A 1-morphism $f : X \longrightarrow Y$ is weakly left constant if for every object W and for every pair of 1-morphisms, $h, g : W \longrightarrow X$, there exists a 2-isomorphism such that $\eta : fg \Rightarrow fh$.*

Definition 60. *A 1-morphism $f : X \longrightarrow Y$ is weakly right constant if for every object Y and for every pair of 1-morphisms, $h, g : Y \longrightarrow Z$, there exists a 2-isomorphism such that $\eta' : gf \Rightarrow hf$.*

Definition 61. *A 1-morphism is weakly constant if it is both weakly left and right constant.*

Remark 5. *From now on, by constant morphisms we mean weakly constant.*

Definition 62. *A 2-category with constant 1-morphisms is a 2-category for which Diagram 4.2 commutes up to 2-isomorphisms.*



(4.2)

Proposition 4. *In a 2-category with constant 1-morphisms, the family of constant 1-morphisms is unique up to a 2-isomorphism.*

Proof. : Consider two different families of constant 1-morphisms with their characteristic weakening 2-isomorphisms, $\{\star, \eta\}$ and $\{\star', \beta\}$. Given $\eta^{-1} : \star \Rightarrow \star\star'$ and $\beta : \star\star' \Rightarrow \star'$, we have $\beta \odot \eta^{-1} : \star \Rightarrow \star'$. □

Interaction between limits and 2-limits

In 2-categories, 2-limits in 2-categories and limits in Hom-categories are coexisting. Hence, in principle, two different families of zero 1-morphisms could exist but how do they interact with each other? Explicitly, when do these two families of 1-morphisms coincide?

- zero 1-morphisms with limit-based definition in Hom-categories.
- 1-morphisms that factorise through zero objects based on 2-limit definition.

The short answer is: when a 2-category is compositionally distributive.

In a 2-category, a Hom-category might have a zero 1-morphism, which means isomorphic initial and terminal 1-morphisms in that Hom-category. Note also this zero 1-morphism generally can exist independently of zero objects or constant 1-morphisms. In other words, having some sorts of limits in a Hom-category does not imply anything about limits on the level of 0-morphisms.

Assume that each Hom-category has a zero 1-morphism; are they constant 1-morphisms as well? Constantness is the matter of composition, so it might be the case that zero 1-morphisms do not talk to each other. However, we prove later, if a 2-category is compositionally distributive; then, zero 1-morphisms are, indeed, constant. Moreover, given that the family of constant 1-morphisms is unique up to 2-isomorphisms, canonical constant 1-morphisms are those that factorise through the zero object. Therefore, in such a 2-category, up to 2-isomorphisms, 1-morphisms which factorise through the zero object, are zero 1-morphisms.

Definition 63. *An object T is weakly terminal, if for every object A, there exists a 1-morphism from A to T, $t_A : A \longrightarrow T$ such that for any other 1-morphism $t'_A : A \longrightarrow T$ there exists a unique 2-isomorphism between them $\gamma_A : t_A \Rightarrow t'_A$.*

Definition 64. *A zero object is a simultaneous weakly initial and terminal object.*

Remark 6. *From now on, by initial/terminal objects, and left/right constant 1-morphisms, we mean the weak versions.*

Proposition 5. *In a 2-category, a zero object is unique up to an equivalence.*

Proof. Consider two distinct zero objects, 0 and $0'$, for which there exist 1-morphisms f and f' such that $0 \xrightarrow{f} 0'$ and $0' \xrightarrow{f'} 0$. Due to the uniqueness of the terminal and initial objects up to a 2-isomorphism, there exist 2-isomorphisms $\eta : f'f \Rightarrow id_0$ and $\epsilon : id_{0'} \Rightarrow ff'$. □

Definition 65. *A pointed 2-category is a 2-category with a zero object.*

Lemma 9. *Every 1-morphism from the initial object, $I \xrightarrow{u_A} A$ is a weakly left constant 1-morphism.*

Proof. Consider $I \xrightarrow{u_A} A \underset{f}{\overset{g}{\rightrightarrows}} B$, the composition of these two 1-morphisms fu_A and gu_A should be unique up to a 2-isomorphism, because 1-morphisms from the initial object are unique up to 2-morphisms. Thus, there exists a 2-isomorphism $\gamma : fu_A \Rightarrow gu_A$; this proves u_A is weakly left constant. □

Proposition 6. *Every pointed 2-category is a locally pointed 2-category.*

Proof. In each Hom-category, there exists a 1-morphism $\star_{B,A} : A \longrightarrow B$, which factorizes uniquely up to a 2-isomorphism through the zero object, $\star_{B,A} \cong u_B t_A$. It is constant because u_B is left constant by Lemma 9, and t_A is right constant by the colemma of 9. □

4.1.1 Biproducts in 2-Categories

In order to define 2-biproducts, we replace categories with 2-categories and limits with 2-limits. One can consider four possible limits in 2-categories: strict, weak, lax, and oplax [13]. We restrict our attention to weak 2-limits; the strict version is obtained by letting all weakening 2-isomorphisms be identities. The following definition is by Borceux in Categorical Algebra volume I [13].

Definition 66. *In a 2-category, a weak 2-product of a pair of objects A, B is an object $A \times B$ equipped with 1-morphism projections $(p_A : A \times B \longrightarrow A, p_B : A \times B \longrightarrow B)$ such that:*

- *for every cone $(X, f : X \longrightarrow A, g : X \longrightarrow B)$, there exist a 1-morphism $b : X \longrightarrow A \times B$ and 2-isomorphisms $\{\xi\}$ such that $(\xi_A : p_A b \Longrightarrow f, \xi_B : p_B b \Longrightarrow g)$ (the red cone in Figure 4.1).*
- *Moreover, for any other cone $(X, f' : X \longrightarrow A, g' : X \longrightarrow B)$ with a corresponding 1-morphism $b' : X \longrightarrow A \times B$ and 2-isomorphisms $(\xi'_A : p_A b' \Longrightarrow f', \xi'_B : p_B b' \Longrightarrow g')$ (the blue cone in Figure 4.1) and given 2-morphisms $(\Sigma_A : f \Longrightarrow f', \Sigma_B : g \Longrightarrow g')$,*

there exists a unique 2-morphism $\gamma : b \Longrightarrow b'$ which satisfies the following condition:

$$(p_A \gamma) = (\xi'_A)^{-1} \odot \Sigma_A \odot (\xi_A) \tag{4.3}$$

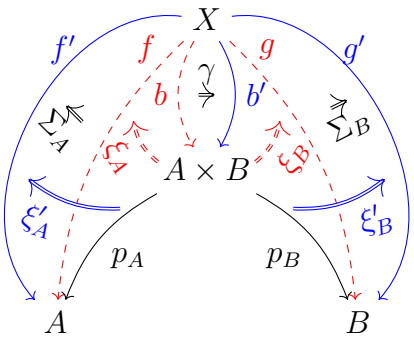


Figure 4.1: Weak 2-product in 2-categories.

Definition 67. *In a 2-category, a strict 2-product of a pair of objects is a weak 2-product whose weakening 2-isomorphisms ξ_A and ξ_B are identities.*

Remark 7. *Observe that unlike 2-limits, because a 1-morphism from X to $A \times B$ is not unique, corresponding to each of these 1-morphisms, there exists a cone with the same apex X. Also, for weak 2-limits, each of the side triangles commutes up to a 2-isomorphism.*

Definition 68. *A locally semiadditive 2-category is a 2-category whose Hom-categories are semiadditive (have finite biproducts).*

Definition 69. *A locally semiadditive and compositionally distributive 2-category is a locally semiadditive 2-category whose 2-morphisms distribute over addition of 2-morphisms. That is Equations 4.4 and 4.5 hold in a compositionally distributive 2-category:*

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta \tag{4.4}$$

$$\alpha \odot (\beta + \gamma) = \alpha \odot \beta + \alpha \odot \gamma \tag{4.5}$$

Remark 8. *In a locally semiadditive 2-category, biproducts in Hom-categories are biproducts of 1-morphisms26. In the proceeding sections, projections π and injections ν are 2-morphisms indexed by 1-morphisms.*

Proposition 7. *In a locally semiadditive and compositionally distributive 2-category, the composition of 1-morphisms is distributive over the biproducts of 1-morphisms.*

$$f(g \oplus h) \cong fg \oplus fh$$

Proof. From universality of products of 1-morphisms shown in Figure 4.2, we have:

$$(f\pi_h) \odot \alpha = \pi_{fh}, \qquad (f\pi_g) \odot \alpha = \pi_{fg} \tag{4.6}$$

and universality of coproducts of 1-morphisms results in:

$$\alpha' \odot (f\nu_h) = \nu_{fh}, \qquad \alpha' \odot (f\nu_g) = \nu_{fg} \tag{4.7}$$

Composing Equations 4.6 and 4.7, we have:

$$\alpha' \odot (f\nu_h) \odot (f\pi_h) \odot \alpha = \nu_{fh} \odot \pi_{fh} \tag{4.8}$$

$$\alpha' \odot (f\nu_g) \odot (f\pi_g) \odot \alpha = \nu_{fg} \odot \pi_{fg} \tag{4.9}$$

Considering distributivity conditions 4.4 and 4.5 and adding two sides of Equations 4.8 and 4.9, we obtain:

$$\alpha' \odot (f[(\nu_g \odot \pi_g) + (\nu_h \odot \pi_h)] \odot \alpha = \nu_{fg} \odot \pi_{fg} + \nu_{fh} \odot \pi_{fh}$$

which is: $\alpha' \odot (1_f \circ 1_{g\oplus h}) \odot \alpha = 1_{(fg\oplus fh)}$, hence, $\alpha' \odot \alpha = 1_{fg\oplus fh}$. One can also show that $\alpha \odot \alpha' = 1_{f(g\oplus h)}$ by using the equations above.

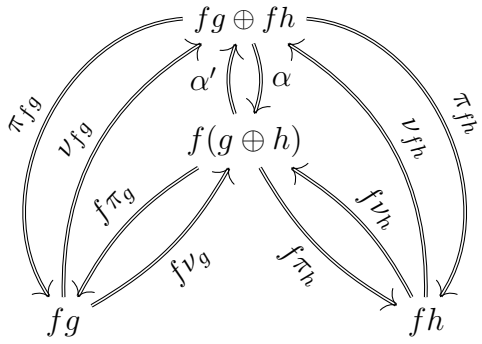


Figure 4.2: Proof of Proposition 7.

□

Proposition 8. *In a locally semiadditive and compositionally distributive 2-category, zero 1-morphisms are constant.*

Proof. Consider the proposition above, for every g we have $g(f \oplus 0) \cong gf \oplus g0$. On the other hand, $f \oplus 0 \cong f$ which shows $g0 \cong 0$ because zero 1-morphisms are unique up to 2-isomorphisms. □

We have developed enough structures to be able to introduce the definitionof 2-biproducts:

Definition 70. *In a locally semiadditive and compositionally distributive 2-category, a weak 2-biproduct of a pair of objects (A, B) is a tuple*

$$(A \boxplus B, p_A, p_B, i_A, i_B, \theta_A, \theta_B, \theta_{AB}, \theta_{BA}, \theta_P)$$

such that:

- *1-Morphism projections and injections:*

$$p_A : A \boxplus B \longrightarrow A, \qquad p_B : A \boxplus B \longrightarrow B$$
$$i_A : A \longrightarrow A \boxplus B, \qquad i_B : B \longrightarrow A \boxplus B$$

- *Weakening 2-isomorphisms:*

$$\theta_A : p_A i_A \Rightarrow id_A, \qquad \theta_B : p_B i_B \Rightarrow id_B,$$
$$\theta_{BA} : p_B i_A \Rightarrow 0_{B,A}, \qquad \theta_{AB} : p_A i_B \Rightarrow 0_{A,B},$$

$$\theta_P : i_A p_A \oplus i_B p_B \Rightarrow id_{A \boxplus B}$$

- *Conditions for 2-biproducts:*

$$p_A\theta_P i_A = \begin{pmatrix} (p_A i_A)\theta_A & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_B\theta_P i_B = \begin{pmatrix} 0 & 0 \\ 0 & (p_B i_B)\theta_B \end{pmatrix} \tag{4.10}$$

Definition 71. *In a locally semiadditive and compositionally distributive 2-category, a strict 2-biproduct is a weak 2-biproduct whose weakening 2-isomorphisms $\{\theta\}$ are 2-identities.*

Definition 72. *A semiadditive 2-category is a locally semiadditive and compositionally distributive 2-category with weak binary 2-biproducts and a zero object.*

To prove the consistency of this definition with weak 2-products, we need the following lemmas.

Lemma 10. *In a 2-category with 2-biproducts 70, if $p_A i_A : A \longrightarrow A$ and $\theta_A : p_A i_A \Rightarrow id_A$, then $(p_A i_A)\theta_A = \theta_A(p_A i_A)$.*

Proof. Given $f, g : B \longrightarrow A$ and $\lambda : f \Rightarrow g$, one can write $(p_A i_A)\lambda = [\theta_A^{-1} g] \odot \lambda \odot [\theta_A f]$. If we let $f = p_A i_A$ and $g = id_A$ and $\lambda = \theta_A$, we obtain the desired equality. □

Lemma 11. *In a locally semiadditive 2-category, $\theta_{BA} : p_B i_A \Rightarrow 0_{B,A}$ and $\theta_{AB} : p_A i_B \Rightarrow 0_{A,B}$ are zero 2-morphisms.*

Proof. Because $0_{A,B}$ is a zero object in Hom-category $Hom(A, B)$ and because each set of 2-morphisms should have a zero 2-morphism and θ_{AB} is unique, it is indeed a zero 2-morphism. □

Lemma 12. *In a locally semiadditive 2-category, $1_{p_A i_B} = 0$ and $1_{p_B i_A} = 0$.*

Proof. From Diagram 4.11, we have $\theta_{AB} \odot 1_0 \odot \theta_{AB} = 1_{p_A i_B}$. Because each hom-set has a zero 1-morphism, $1_{0_{B,A}} = 0_{0_{B,A}}$, also from the previous lemma $\theta_{AB} = 0$. Therefore, $1_{p_A i_B} = 0$.

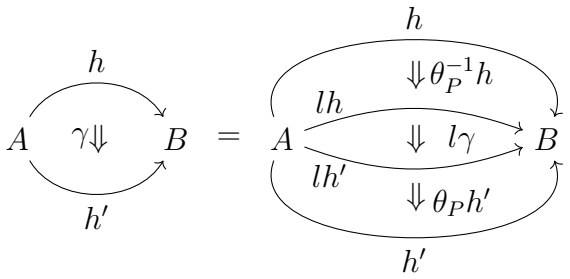
$$\begin{array}{ccc} p_B i_A & \xRightarrow{1_{p_B i_A}} & p_B i_A \\ {\scriptstyle \theta_{BA}}\Downarrow & & \Uparrow{\scriptstyle \theta_{BA}^{-1}} \\ 0_{B,A} & \xRightarrow[1_0]{} & 0_{B,A} \end{array} \tag{4.11}$$

□

Lemma 13. *In a 2-category, for a unique 2-morphism $\gamma : h \Rightarrow h'$, if there exists a 2-isomorphism $\theta_P : l \Rightarrow id_B$, then γ has an equivalent expression as follows:*

$$(\theta_P h') \odot (l\gamma) \odot (\theta_P^{-1} h) = \gamma \tag{4.12}$$

Proof. Follows from the figure below.



□

Lemma 14. *In a locally semiadditive 2-category, considering the definition of 2-biproducts and $r = i_A p_A \oplus i_B p_B$, the 2-morphisms $\Sigma_A : p_A r \Rightarrow p_A$ and $\Sigma_B : p_B r \Rightarrow p_B$ are given by the following row matrices:*

$$\Sigma_A = \begin{pmatrix} \theta_A p_A & 0 \end{pmatrix}, \qquad \Sigma_B = \begin{pmatrix} 0 & \theta_B p_B \end{pmatrix} \tag{4.13}$$

Proof. Given Diagram 4.3, we have: $\lambda = \begin{pmatrix} \theta_A p_A & 0 \\ 0 & \theta_{AB} p_B \end{pmatrix}$. To obtain, Σ_A, we must calculate $\pi_1 \odot \lambda$, which is

$$\pi_1 \odot \lambda = \begin{pmatrix} 1 & 0 \end{pmatrix} \odot \begin{pmatrix} \theta_A p_A & 0 \\ 0 & \theta_{AB} p_B \end{pmatrix} = \begin{pmatrix} \theta_A p_A & 0 \end{pmatrix}$$

One can similarly show that for $\pi_2 \odot \lambda' = \Sigma_B$, if $\lambda' = \begin{pmatrix} \theta_{B,A} p_A & 0 \\ 0 & \theta_B p_B \end{pmatrix}$.

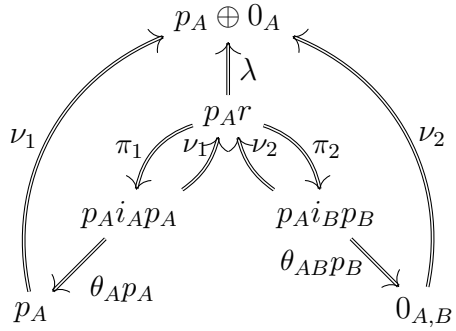


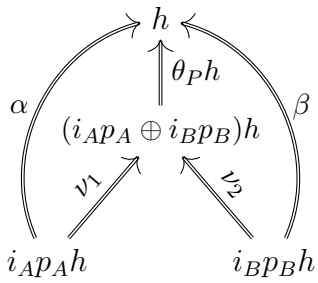
Figure 4.3: The 2-morphism λ in the figure is $\lambda = \nu_1 \odot \theta_A p_A \odot \pi_1 + \nu_2 \odot \theta_{AB} p_B \odot \pi_2$.

□

Lemma 15. *In a semiadditive 2-category, for every 1-morphism written as* $h = i_A f \oplus i_B g$ *and given a 2-isomorphism* $\theta_P : i_A p_A \oplus i_B p_B \Rightarrow id_{A \boxplus B}$ *we have:*

$$\theta_P h = \begin{pmatrix} i_A \theta_A f & 0 \\ 0 & i_B \theta_B g \end{pmatrix} \tag{4.14}$$

Proof. The essence of the proof is in the figure below, in which $\alpha = \theta_P h \odot \nu_1$ and $\beta = \theta_p h \odot \nu_2$. Because α and β determine $\theta_P h$, we only need to find them.



We can determine α by writing the explicit forms of $i_A p_A h$ and $i_B p_B h$ to find the 2-morphism from $i_A p_A h$ to h.

$$i_A p_A h = i_A p_A i_A f \oplus i_A p_A i_B g \xrightarrow{\pi_1} i_A p_A i_A f \xrightarrow{i_A \theta_A f} i_A f \xrightarrow{\nu_1} h$$

Starting from $i_B p_B h$ and following the same procedure, we can find β.

$$\alpha = \nu_1 \odot i_A \theta_A f \odot \pi_1, \qquad \beta = \nu_2 \odot i_B \theta_B g \odot \pi_2$$

□

We now present the main theorem of this chapter. Similar to the 1-dimensional case, which was discussed in References [13], we show the consistency of the algebraic definition of weak 2-biproducts with weak 2-products/2-coproducts.

Theorem 10. *In a locally semiadditive and compositionally distributive 2-category, the following conditions for a pair of objects* A *and* B *are equivalent:*

1. *the weak 2-product* (P, p_A, p_B) *of* A, B *exists.*
2. *the weak 2-coproduct* (P, i_A, i_B) *of* A, B *exists.*
3. *the weak 2-biproduct* $(P, p_A, p_B, i_A, i_B, \theta_A, \theta_B, \theta_{AB}, \theta_{BA}, \theta_P)$ *of* A, B *exists.*

Proof. $1 \Longrightarrow 3$: Assuming a pair of objects A, B has a weak 2-product P, we want to show P is also the weak 2-biproduct. To this end, check the universal property of weak 2-products for $(A, id_A : A \longrightarrow A, 0_{B,A} : A \longrightarrow B, i_A : A \longrightarrow P)$ and $(B, id_B : B \longrightarrow B, 0_{A,B} : B \longrightarrow A, i_B : B \longrightarrow P)$. From the definition of 2-products, we know that there exist two 2-morphisms $(\gamma_A : p_A i_A \Longrightarrow id_A, \gamma_B : p_B i_B \Longrightarrow id_B)$. We let $\theta_A := \gamma_A$ and $\theta_B := \gamma_B$. To find θ_P, we check the universality condition for P and two 2-cones:

1. $(P, p_A l : P \longrightarrow A, p_B l : P \longrightarrow B, l : P \longrightarrow P), \quad l := i_A p_A \oplus i_B p_B$
2. $(P, p_A : P \longrightarrow A, p_B : P \longrightarrow B, id_P : P \longrightarrow P)$

By Lemma 14, 2-morphisms between these two cones are as below:

$$\Sigma_A = \begin{pmatrix} \gamma_A p_A & 0 \end{pmatrix} : p_A i_A p_A \oplus p_A i_B p_B \Rightarrow p_A,$$
$$\Sigma_B = \begin{pmatrix} 0 & \gamma_B p_B \end{pmatrix} : p_B i_A p_A \oplus p_B i_B p_B \Rightarrow p_B$$

Therefore, $\theta_P = \begin{pmatrix} i_A \Sigma_A \\ i_B \Sigma_B \end{pmatrix} = \begin{pmatrix} i_A \gamma_A p_A & 0 \\ 0 & i_B \gamma_B p_B \end{pmatrix}$.
Considering Lemmas 12 and 10, also the point that $(1_{p_A i_A})^2 = 1_{p_A i_A}$, we can show θ_P satisfies necessary conditions.

$$p_A \theta_P i_A = \begin{pmatrix} (p_A i_A)\theta_A (p_A i_A) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (p_A i_A)\theta_A & 0 \\ 0 & 0 \end{pmatrix}.$$

$3 \Longrightarrow 1$: Now if we have the algebraic definition of 2-biproduct for a pair of objects, we want to demonstrate P is also a 2-product. Because we already have projections, it is enough to show it satisfies the universality condition. To this end, check the universality condition for an arbitrary cone $(X, f : X \longrightarrow A, g : X \longrightarrow B)$ and let a 1-morphism from X to P be $h = i_A f \oplus i_B g$:

$$\xi_A : p_A h = p_A (i_A f \oplus i_B g) \Rightarrow f, \qquad \xi_A = \theta_A f \odot \pi_1$$

Similarly, we obtain $\xi_B = \theta_B g \odot \pi_2$. Given another cone $(X, f' : X \longrightarrow A, g' : X \longrightarrow B)$, Letting $h' = i_A f' \oplus i_B g'$, we have $\xi'_A = \theta_A f' \odot \pi_1$ and $\xi'_B = \theta_B g' \odot \pi_2$. If there exist 2-morphisms $(\Sigma_A : f \Longrightarrow f', \Sigma_B : g \Longrightarrow g')$, we should show that there is a unique 2-morphism $\gamma : i_A f \oplus i_B g \Longrightarrow i_A f' \oplus i_B g'$ such that $(p_A \gamma) = (\xi'_A)^{-1} \odot \Sigma_A \odot \xi_A$. Suppose γ in the explicit form is $\gamma = \nu_1 \odot i_A \Sigma_A \odot \pi_1 + \nu_2 \odot i_B \Sigma_B \odot \pi_2$ and in the matrix form is as follows:

$$\gamma = \begin{pmatrix} i_A \Sigma_A & 0 \\ 0 & i_B \Sigma_B \end{pmatrix}$$

Multiplying γ by p_A, we have:

$$p_A\gamma = \nu_1 \odot p_A i_A \Sigma_A \odot \pi_1 + \nu_2 \odot p_A i_B \Sigma_B \odot \pi_2$$

Note that $p_A i_A \Sigma_A = (\theta_A^{-1} f') \odot \Sigma_A \odot (\theta_A f)$ and $p_A i_B \cong 0_{A,B}$, so:

$$p_A\gamma = \nu_1 \odot (\theta_A^{-1} f') \odot \Sigma_A \odot (\theta_A f) \odot \pi_1$$

Additionally, the inverse of $\xi_A = (\theta_A f) \odot \pi_1$ is $(\xi_A)^{-1} = \nu_1 \odot (\theta_A^{-1} f)$. Substitution of these terms results in the desired condition:

$$p_A\gamma = \xi_A^{-1} \odot \Sigma_A \odot \xi_A$$

Uniqueness of γ: Assume that there exists another 2-morphism $\gamma' : i_A f \oplus i_B g \Rightarrow i_A f' \oplus i_B g'$ such that it satisfies all necessary conditions. Using Lemma 13 and Lemma 15, we prove γ' is equivalent to γ:

$$\gamma' = (id_p)\gamma' \overset{\text{lemma 13}}{=} (\theta_P h') \odot [(i_A p_A \oplus i_B p_B)\gamma'] \odot ({\theta_P}^{-1} h) \xrightarrow{\text{Equation 4.3, lemma 15}}$$

$$\gamma' = \begin{pmatrix} i_A(\theta_A f') & 0 \\ 0 & i_B(\theta_B g') \end{pmatrix} \odot \begin{pmatrix} i_A[(\theta_A^{-1} f') \odot \Sigma_A \odot (\theta_A f)] & 0 \\ 0 & i_B[(\theta_B^{-1} g') \odot \Sigma_B \odot (\theta_B g)] \end{pmatrix}$$

$$\odot \begin{pmatrix} i_A(\theta_A^{-1} f) & 0 \\ 0 & i_B(\theta_B^{-1} g) \end{pmatrix} = \begin{pmatrix} i_A\Sigma_A & 0 \\ 0 & i_B\Sigma_B \end{pmatrix} = \gamma.$$

In the second line, the first and third matrices are written by using Lemma 15, and the middle matrix is obtained by condition of weak 2-products Equation 4.3.

Note that $\theta_P h = \nu_1 \odot (i_A \theta_A f) \odot \pi_1 + \nu_2 \odot (i_B \theta_B g) \odot \pi_2$. If one lets $X = P$, $f = id_A$, and $g = 0_{B,P}$; this yields $h = i_A$ and $\xi_A = \theta_A id_A$, hence, $\theta_P h = \theta_P i_A$. Multiplying $\theta_P h$ with p_A, we obtain one of the conditions of 2-biproducts.

$$p_A \theta_P i_A = \begin{pmatrix} (p_A i_A)\theta_A & 0 \\ 0 & 0 \end{pmatrix}$$

□

As mentioned earlier, the categorification of the canonical definition of biproducts is achievable in a rather simple way by only using the notion of weak 2-limits in 2-categories. In the following, we first state the definition obtained from the limit notion and examine the compatibility of this definition with our algebraic version.

Definition 73. *In a locally semiadditive and compositionally distributive 2-category, a canonical 1-morphism between a 2-coproduct of a pair of objects A, B, i.e. $A \sqcup B$ and a 2-product $A \times B$ is a 1-morphism which satisfies $\theta_{k,j} : p_k r i_j \Rightarrow \delta_{k,j} id_j$, if $\theta_{k,j}$ are 2-isomorphisms.*

$$A_j \xrightarrow{i_j} A_1 \sqcup A_2 \xrightarrow{r} A_1 \times A_2 \xrightarrow{p_k} A_k, \quad \text{if} \quad j,k \in \{1,2\}$$

Remark 9. *A canonical 1-morphism r is unique up to a unique 2-isomorphism. That is, for every pair of such 1-morphisms, there exists a unique 2-isomorphism $\gamma : r \Rightarrow r'$ such that $p_k \gamma i_j = (\theta'_{k,j})^{-1} \odot \theta_{k,j}$.*

Remark 10. *The 2-isomorphisms $\theta_{i,j}$ for $i \neq j$ are zero 2-morphisms, according to Lemma 11.*

Definition 74. *In a locally semiadditive and compositionally distributive 2-category, a pair of objects A and B has a weak 2-biproduct if the canonical 1-morphism r is an equivalence and satisfies the following conditions:*

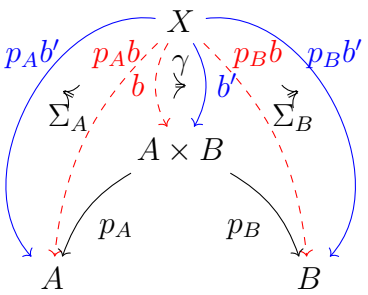
$$A_1 \sqcup A_2 \xrightarrow{r} A_1 \times A_2, \qquad A_1 \sqcup A_2 \xleftarrow{r'} A_1 \times A_2$$
$$\xi_{A\times B} : rr' \Rightarrow id_{A\times B}, \qquad \xi_{A\sqcup B} : id_{A\sqcup B} \Rightarrow r'r$$
$$(r'\xi_{A\times B}) \odot (\xi_{A\sqcup B} r') = 1_{r'}, \qquad (\xi_{A\times B} r) \odot (r\xi_{A\sqcup B}) = 1_r$$

To show the consistency of this definition with the algebraic version, we first need to prove 1-morphism projections are weakly monic.

Definition 75. *In a 2-category, a 1-morphism $f : B \longrightarrow C$ is weakly monic if for every pair of 1-morphisms $g, h : A \longrightarrow B$, if fg is isomorphic to fh, i.e. $fg \cong fh$. Then g and h are isomorphic $g \cong h$.*

Lemma 16. *In a 2-category with binary 2-products, 1-morphism projections are weakly monomorphic.*

Proof. To prove p_A is mono, we need to show that if $p_A b \overset{\Sigma_A}{\cong} p_A b'$ there exists a 2-isomorphism $\gamma : b \Rightarrow b'$. With 1-morphisms b and b', we can make two 2-cones $(X, p_A b, p_B b, b)$ and $(X, p_A b', p_B b', b')$. Due to our assumption, there exists 2-isomorphisms $\Sigma_A : p_A b \Rightarrow p_A b'$ and $\Sigma_B : p_B b \Rightarrow p_B b'$, hence, a unique 2-isomorphism $\gamma : b \Rightarrow b'$ such that $p_A \gamma = \Sigma_A$ and $p_B \gamma = \Sigma_B$. □



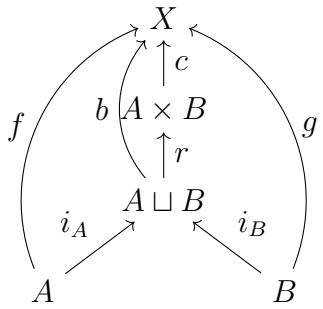
Proposition 9. *In a locally semiadditive and compositionally distributive 2-category, a canonical 1-morphism r, between a 2-coproduct $A \sqcup B$ and a 2-product $A \times B$ of a pair of objects is an equivalence if and only if projections and injections satisfy the conditions of Definition 70.*

Proof. Suppose we have the algebraic definition of 2-biproducts, that is $\theta_{\alpha,\beta} : p_\alpha i_\beta \Rightarrow \delta_{\alpha,\beta} id_\alpha$ and $\theta_p : \oplus_\alpha i_\alpha p_\alpha \Rightarrow id_{A \boxplus B}$. If $\theta_{A,A} = \theta_A$ and $\theta_{B,B} = \theta_B$. To prove a canonical 1-morphism r, which satisfies $\xi_{\alpha,\beta} : p_\alpha r i_\beta \Rightarrow \delta_{\alpha,\beta}$, has an inverse, we first need to show that $A \boxplus B$ is indeed a 2-product and a 2-coproduct, i.e. it satisfies the universal property, which is exactly the first part of Theorem 10. Therefore, it remains to show that r is an equivalence. Let the inverse be $r' = i_A p_A \oplus i_B p_B$. Using $\theta_{\alpha,\beta} : p_\alpha i_\beta \Rightarrow \delta_{\alpha,\beta} id_\alpha$, we have:

$$p_A r r' = p_A r (i_A p_A \oplus i_B p_B) \overset{\lambda}{\Rightarrow} p_A \oplus 0 \overset{\pi_1}{\Longrightarrow} p_A$$

which shows $\lambda = \begin{pmatrix} \xi_{A,A} & 0 \\ 0 & \xi_{A,B} \end{pmatrix}$. Since p_A is weakly monic, there exists a 2-isomorphism $\xi_{A\times B} : rr' \Rightarrow id_{A\times B}$ such that $p_A \xi_{A\times B} = \pi_1 \odot \lambda$. One can also show that $r'r \Rightarrow id_{A\sqcup B}$ by checking $r'ri_A$ and using epicitiy of i_A.

$\Leftarrow$ Now given r and r' which satisfy $\xi_{\alpha,\beta} : p_\alpha r i_\beta \Rightarrow \delta_{\alpha,\beta}$, $\xi_{A\sqcup B} : r'r \Rightarrow id_{A\sqcup B}$ and $\xi_{A\times B} : rr' \Rightarrow id_{A\times B}$, to obtain the necessary equations and the condition for 2-biproducts, we define new injections $i'_A = ri_A$ and claim the 2-product of two objects with these injections $(A \times B, i'_A, i'_B)$ is a 2-coproduct .i.e. it satisfies the universal property.



Because $(A \sqcup B, i_A, i_B)$ is a 2-coproduct, there exists a 1-morphism b such that $\eta_A : bi_A \Rightarrow f$ and $\eta_B : bi_B \Rightarrow g$. Let $c = br'$, therefore, for f and similarly for g:

$$ci'_A = br'ri_A \overset{b\xi_{A\sqcup B} i_A}{\Longrightarrow} b\ id_{A\sqcup B}\ i_A = bi_A \overset{\eta_A}{\Longrightarrow} f$$

Moreover, we will not show that $(A\times B, i'_\alpha, p_\alpha, \alpha = A, B)$ is a 2-biproduct of A and B. We need to find 2-isomorphisms $\theta_{\alpha,\beta} : p_\alpha i'_\beta \Rightarrow \delta_{\alpha,\beta} id_\alpha$ and $\theta_p : i'_A p_A \oplus i'_B p_B \Rightarrow id_{A\times B}$ which satisfy appropriate conditions. By the condition on r, we define 2-isomorphisms

$\theta_{\alpha,\beta} := \xi_{\alpha,\beta}$, Finally, $\theta_p : i'_A p_A \oplus i'_B p_B \Rightarrow id_{A\times B}$ be the result of the universal property of 2-products and monicity of projections.

$$p_A[i'_A p_A \oplus i'_B p_B] = p_A(ri_A)p_A \oplus p_A(ri_B)p_B \overset{\rho}{\Rightarrow} p_A \oplus 0$$

$$p_B[i'_A p_A \oplus i'_B p_B] = p_B(ri_A)p_A \oplus p_B(ri_B)p_B \overset{\rho'}{\Rightarrow} 0 \oplus p_B$$

such that $\rho = \begin{pmatrix} \xi_{A,A}p_A & 0 \\ 0 & \xi_{A,B}p_B \end{pmatrix}$ and $\rho' = \begin{pmatrix} \xi_{A,B}p_A & 0 \\ 0 & \xi_{B,B}p_B \end{pmatrix}$. Note that $p_A\theta_P = \rho$ and $p_B\theta_P = \rho'$. So we can check the condition on θ_p, by
$p_A\theta_p i_A = \begin{pmatrix} \xi_{A,A}p_A i_A & 0 \\ 0 & \xi_{A,B}p_B i_A \end{pmatrix}$. □

4.1.2 Typing Tensor Calculus

We have used matrix notation through this chapter to make the difficult calculations easier and to gain an intuition about the difference between biproducts in categories and 2-categories. For instance, we have represented the condition on 2-biproducts (4.10) as 2×2 matrices. But the conditions on 2-morphism θ_P become matrix only after applying projections and injections, this observation clearly displays the type of 2-morphisms in semiadditive 2-categories is different from the type of 1-morphisms.

Tensor is an umbrella term for mathematical objects including scalars, vectors, matrices, and so on. Each tensor has a rank specified by their indices, for example, scalars are rank zero tensors, vectors rank one, matrices are rank two tensors, and this list can go on. In linear algebra, they are represented as below, when e_{α_j} is the jth basis vector:

$$T = \sum_{\alpha_1,\alpha_2,\alpha_3,\ldots} t_{\alpha_1,\alpha_2,\ldots}(e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots) \tag{4.15}$$

Hence, in this notation,

Scalar: $n = t$ Vector: $v = \sum_{\alpha_1} t_{\alpha_1} e_{\alpha_1}$

Matrix: $M = \sum_{\alpha_1,\alpha_2} t_{\alpha_1,\alpha_2}(e_{\alpha_1} \otimes e_{\alpha_2})$

Pay attention to entries of tensors above. Entries of vectors are scalar. If we fix only one index, then entries of a matrix become vectors. If we fix both indices of a matrix, then entries are scalar.

Remark 11. *We conjecture in a semiadditive n-category, n-morphisms are tensors of rank $2n$.*

4.1.3 Category 2Vec

Similar to Chapter one that carried the guiding example of the category of matrices $\mathbf{Mat}_{\mathbb{k}}$, 2-category **2Vec** in the sense of Kapranov and Voevodsky [27] inspires us throughout this section. Kapranov and Voevodsky in their seminal work, introduced **2Vec** as an example of symmetric monoidal 2-categories. Objects of **2Vec** are natural numbers, 1-morphisms are matrices whose entries are finite dimensional vector spaces, and 2-morphisms are matrices of linear maps between vector spaces. **2Vec** is certainly monoidal but due to lacking a rigorous definition for 2-biproducts, the folklore claim that it is also semiadditive has not been built on sound ground.

From the **2Vec** perspective, one can observe some properties of semiadditive 2-categories. For instance, in **2Vec** , there are two collections of bases: global bases which are 1-morphism projections and injections $\{p, i\}$ indexed by objects (natural numbers) and local bases, which are the internal bases of each vector space $\{\pi, \nu\}$. Correspondingly, a semiadditive 2-category has a collection of global bases, and for each Hom-category a collection of local bases, which are 2-morphisms projections and injections indexed by 1-morphisms. Global bases are projections p and injections i of 2-biproducts $\boxplus$, and local bases are projections π and injections ν of biproducts $\oplus$ in Hom-categories. Generally, a 2-morphism is a tensor of rank up to four, or in other words, it is a 4-dimensional matrix. Thus, each entry of a matrix has four indices: Latin indices are global and Greek indices are local.

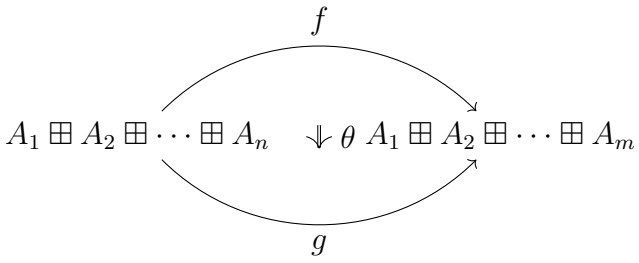
$$\theta = \sum_{\alpha,\beta} \sum_{k,l} (\theta_j^k)_\alpha^\beta [(\nu_\alpha \otimes \pi_\beta) \otimes (i_j \otimes p_k)] \tag{4.16}$$

Applying the global projections and injections, one selects an entry which itself is a matrix with entries indexed by local bases. In a 4-dimensional picture, by applying a 1-morphism projection, one picks a 3-dimensional matrix, a cube. Applying a 1-morphism injection to that cube, we select a square matrix. To pick an entry from this square matrix, one has to apply 2-morphism projections and injections. To unpack Equation (4.16), consider a 2-morphism $\theta : f \Rightarrow g$ and

$$f, g : \boxplus_{j=0}^{n} A_j \longrightarrow \boxplus_{k=0}^{m} A_k$$

Explicitly, f, g have the following expressions:

$$f = \bigoplus_{k=0}^{m} \bigoplus_{j=0}^{n} f_j^k (i_j \otimes p_k), \qquad g = \bigoplus_{k=0}^{m} \bigoplus_{j=0}^{n} g_j^k (i_j \otimes p_k) \tag{4.17}$$



(4.18)

Note that entries of f and g are in the same Hom-categories as f and g, that is, $f_i^k, g_j^k \in Hom(\boxplus A_j, \boxplus A_k)$. We intentionally use $\oplus$ instead of $\sum$ to emphasise what we are summing over are 1-morphisms, so the correct sum is actually a biproduct. Now if entries of f and g are a sum of 1-morphisms,

$$f_j^k = \bigoplus_{\alpha=0}^{p} h_\alpha \qquad\qquad g_j^k = \bigoplus_{\beta=0}^{z} h_\beta \tag{4.19}$$

The unpacked form of a 2-morphism $\theta_j^k : f_j^k \Rightarrow g_j^k$ is as follows:

$$\theta_j^k = \sum_{\alpha=0}^{p} \sum_{\beta=0}^{z} (\theta_j^k)_\alpha^\beta (\nu_\alpha \otimes \pi_\beta) \tag{4.20}$$

For the horizontal composition of 1-morphisms in matrix notation, we first calculate the usual multiplication of matrices, then we compose the corresponding entries of matrices. In the example above, consider a 1-morphism $r : \boxplus_{k=0}^{m} A_k \longrightarrow \boxplus_{l=0}^{w} A_l$, the composition of r with f is:

$$f = \bigoplus_{k=0}^{m} \bigoplus_{j=0}^{n} f_j^k (i_j \otimes p_k), \qquad\qquad r = \bigoplus_{l=0}^{w} \bigoplus_{k'=0}^{m} r_{k'}^l (i_{k'} \otimes p_l)$$

$$r \circ f = \bigoplus_{k=0}^{m} \bigoplus_{j=0}^{n} \bigoplus_{l=0}^{w} \bigoplus_{k'=0}^{m} [(f_j^k \circ r_{k'}^l)][(i_j \otimes p_k) \otimes (i_{k'} \otimes p_l)] \xrightarrow{p_k \otimes i_{k'} = \delta_{k,k'}}$$

$$r \circ f = \bigoplus_{j=0}^{n} \bigoplus_{l=0}^{w} \bigoplus_{k,k'=0}^{m} [(f_j^k \circ r_{k'}^l)][\delta_{kk'}(i_j \otimes p_l)] = \bigoplus_{j=0}^{n} \bigoplus_{l=0}^{w} \bigoplus_{k=0}^{m} (f_j^k \circ r_k^l)(i_j \otimes p_l)$$

The reader should not be surprised by the tensor sign between projection and injection $p_k \otimes i_{k'}$. This convention agrees with our knowledge of tensors calculus in algebra. The horizontal composition of 2-morphisms is similar to 1-morphisms composition.

$$\theta \circ \xi = \bigoplus_{j,k,m} (\theta_j^k \otimes \xi_k^m)(i_j \otimes p_m)$$

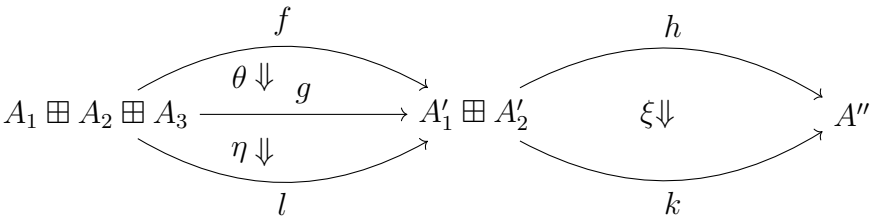
The vertical composition is obtained first by Hadamard product or entry-wise matrix multiplication, and then by usual matrix multiplication between corresponding terms.

Consider 2-morphisms $f \overset{\theta}{\Rightarrow} g \overset{\xi}{\Rightarrow} q$, note that f, g, q all have the same source and target; so the same orders, i.e. order $= m \times n$. To do Hadamard product, we need to apply 1-morphism projections and injections $p_k \theta i_j = \theta^k_j$ and $p_k \xi i_j = \xi^k_j$. Then the vertical composition of θ and ξ is obtained by matrix multiplication of corresponding entries. The $\sum$ sign here is the monoid addition.

$$\theta^k_j \odot \xi^k_j = \sum_{\alpha'=0}^{x} \sum_{\beta'=0}^{z} \sum_{\alpha=0}^{p} \sum_{\beta=0}^{z} (\theta^k_j)^\beta_\alpha (\xi^k_j)^{\alpha'}_{\beta'} [(\nu_\alpha \otimes \pi_\beta) \otimes (\nu_{\beta'} \otimes \pi_{\alpha'})] \xRightarrow{\pi_\beta \otimes \nu_{\beta'} = \delta_{\beta,\beta'}}$$

$$\theta^k_j \odot \xi^k_j = \sum_{\alpha'=0}^{x} \sum_{\alpha=0}^{p} \sum_{\beta=0}^{z} (\theta^k_j)^\beta_\alpha (\xi^k_j)^{\alpha'}_{\beta} (\nu_\alpha \otimes \pi_{\alpha'})$$

We summarize the content of this section in an example in **2Vec** . Consider 1- and 2-morphisms in the following figure:



We write 1-morphisms as matrices of orders 2×3 and 1×2. Orders of 1-morphisms are specified by the number of objects in the target times the number of objects in the source.

$$f = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{pmatrix}, g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{pmatrix}, l = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{pmatrix},$$

$$h = \begin{pmatrix} h_{11} & h_{12} \end{pmatrix}, k = \begin{pmatrix} k_{11} & k_{12} \end{pmatrix}$$

Entries of 2-morphisms are 2-morphisms between 1-morphisms $\theta_{ij} : f_{ij} \Rightarrow g_{ij}$ written in the following. So the order of θ is the same as f and g, but orders of entries are determined by the number of components in entries of f and g.

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{pmatrix}, \eta = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_{11} & \xi_{12} \end{pmatrix},$$

Assume that some of the entries of 1-morphisms are biproducts of 1-morphisms in the same Hom-category, and other entries do not have any further components:

$$f_{11} = f^1_{11} \oplus f^2_{11}, \qquad g_{12} = g^1_{12} \oplus g^2_{12} \oplus g^3_{12}, \qquad l_{22} = l^1_{22} \oplus l^2_{22},$$
$$h_{11} = h^1_{11} \oplus h^2_{11} \qquad k_{11} = k^1_{11} \oplus k^2_{11}$$

Thus, the entries of 2-morphisms have the explicit matrix form as follows:

$$\theta_{11} = \begin{pmatrix} \theta_{11}^1 & \theta_{11}^2 \end{pmatrix}, \theta_{12} = \begin{pmatrix} \theta_{12}^1 \\ \theta_{12}^2 \\ \theta_{12}^3 \end{pmatrix}, \xi_{11} = \begin{pmatrix} \xi_{11}^1 & \xi_{11}^2 \\ \xi_{11}^3 & \xi_{11}^4 \end{pmatrix}, \eta_{22} = \begin{pmatrix} \eta_{22}^1 \\ \eta_{22}^2 \end{pmatrix},$$
$$\eta_{12} = \begin{pmatrix} \eta_{12}^1 & \eta_{12}^2 & \eta_{12}^3 \end{pmatrix}$$

- *Composition of 1-morphisms*

$$h \circ f = \begin{pmatrix} h_{11}f_{11} \oplus h_{12}f_{21} & h_{11}f_{12} \oplus h_{12}f_{22} & h_{11}f_{13} \oplus h_{12}f_{23} \end{pmatrix}$$

- *Vertical composition of 2-morphisms*

$$\eta \odot \theta = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \end{pmatrix} \odot \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{pmatrix} = \begin{pmatrix} \eta_{11}.\theta_{11} & \eta_{12}.\theta_{12} & \eta_{13}.\theta_{13} \\ \eta_{21}.\theta_{21} & \eta_{22}.\theta_{22} & \eta_{23}.\theta_{23} \end{pmatrix}$$
$$= \begin{pmatrix} \begin{pmatrix} \eta_{11}\theta_{11}^1 & \eta_{11}\theta_{11}^2 \end{pmatrix} & \eta_{12}^1\theta_{12}^1 + \eta_{12}^2\theta_{12}^2 + \eta_{12}^3\theta_{12}^3 & \eta_{13}\theta_{13} \\ \eta_{21}\theta_{21} & \begin{pmatrix} \eta_{22}^1\theta_{22} \\ \eta_{22}^2\theta_{22} \end{pmatrix} & \eta_{23}\theta_{23} \end{pmatrix}$$

- *Horizontal composition of 2-morphisms*

$$\xi \circ \theta = \begin{pmatrix} \xi_{11} & \xi_{12} \end{pmatrix} \circ \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{pmatrix}$$
$$= \begin{pmatrix} \xi_{11} \otimes \theta_{11} \oplus \xi_{12} \otimes \theta_{21} & \xi_{11} \otimes \theta_{12} \oplus \xi_{12} \otimes \theta_{22} & \xi_{11} \otimes \theta_{13} \oplus \xi_{12} \otimes \theta_{23} \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$$

$$\alpha = \begin{pmatrix} \begin{pmatrix} \xi_{11}^1 \begin{pmatrix} \theta_{11}^1 & \theta_{11}^2 \end{pmatrix} & \xi_{11}^2 \begin{pmatrix} \theta_{11}^1 & \theta_{11}^2 \end{pmatrix} \\ \xi_{11}^3 \begin{pmatrix} \theta_{11}^1 & \theta_{11}^2 \end{pmatrix} & \xi_{11}^4 \begin{pmatrix} \theta_{11}^1 & \theta_{11}^2 \end{pmatrix} \end{pmatrix} & \\ & \xi_{12} \otimes \theta_{21} \end{pmatrix},$$

$$\beta = \begin{pmatrix} \begin{pmatrix} \xi_{11}^1 \begin{pmatrix} \theta_{12}^1 \\ \theta_{12}^2 \\ \theta_{12}^3 \end{pmatrix} & \xi_{11}^2 \begin{pmatrix} \theta_{12}^1 \\ \theta_{12}^2 \\ \theta_{12}^3 \end{pmatrix} \\ \xi_{11}^3 \begin{pmatrix} \theta_{12}^1 \\ \theta_{12}^2 \\ \theta_{12}^3 \end{pmatrix} & \xi_{11}^4 \begin{pmatrix} \theta_{12}^1 \\ \theta_{12}^2 \\ \theta_{12}^3 \end{pmatrix} \end{pmatrix} & \\ & \xi_{12} \otimes \theta_{22} \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \begin{pmatrix} \xi_{11}^1\theta_{13} & \xi_{11}^2\theta_{13} \\ \xi_{11}^3\theta_{13} & \xi_{11}^4\theta_{13} \end{pmatrix} & \\ & \xi_{12} \otimes \theta_{23} \end{pmatrix}$$

Enrichment over semiadditive categories

To algebraically define 2-biproducts in a 2-category, the additions of all morphisms are required. Hence, one should top-down equip the 2-category with additions for 1-

and 2-morphisms. Conversely, one expects that similar to categories (Proposition 2) if a 2-category happens to have additions of all objects, then one should be able to show this category is enriched over the 2-category of semiadditive categories. However, this expectation for our definition fails. Mainly because the existence of limits of lower morphisms does not guarantee having corresponding limits for higher morphisms.

For an example, take the 2-category **CAT** which is complete, and consider a category without products A. If T is the terminal category, then $\mathbf{Cat}(T, A) \cong A$ which does not have products. For the specific case of biproducts, take the category of relations **Rel** as a 2-category with Hom-categories as posets. **Rel** has as 2-biproducts disjoint unions of sets; however, posets do not have biproducts since joins(coproducts) and meets(products) are not the same.

The next structure we intend to delve into is monoidal 2-categories. Of course, the definition is not new, and it has appeared in multiple works before [6], but previous definitions always lack some parts of the data, either without tensorators, or associators or both. The definition appeared in fusion 2-category [19], is semi-strict; meaning all monoidal associators and unitors are identity. However, it should be rather obvious that for specification of an anyonic theory and further computation with point-like and string-like excitations, one needs to have solutions of three-dimensional version of pentagonal equation, hence, one needs to work in a more general case.

Since Kapranov and Voevodsky's definition was the inspiring starting point of other definition, in the next section, we systematically produce the definition from the algebraic definition of tricategory by Gurski [25]. By doing so, not only we find a complete list of data, but we are able to point out mistakes in previous definitions.

4.2 Monoidal Structure in Bicategories

Kapranov and Voevodsky, in their seminal paper [27], introduced monoidal 2-categories. Their definition comprises a list of axioms, the categorical origin of which does not seem transparent.

A monoidal n-category, by the categorical procedure, can be obtained by taking a one-object weak $n+1$-category. For instance, a monoidal category is defined as one object weak 2-category(or bicategory). Due to the lack of a complete definition for weak n-categories, n cannot be any arbitrary natural number. However, thanks to the algebraic definition of weak 3-category (or tricategory) introduced by Gordon, Power and Street [24] and Gurski [25], one can define a monoidal 2-category as a tricategory with one object.

After a full list of necessary data, Gurski's thesis gives two main axioms, namely, Stasheff and unit polytopes: the 2-dimensional correspondence of pentagonal and triangle equations for monoidal categories. However, it does not explicitly spell out all diagrams obtainable by naturality, modification, and (2-)functoriality conditions. Unpacking these conditions, one can recover KV's axioms, which underlines the goal of this section.

The recovery procedure needs to consider two main issues. The first one is the difference between tensorators in KV's version and our version. Kapranov and Voevodsky defined three tensorators which can be obtained by a single tensorator of the tricategory approach. The second difference is an extra piece of data given by KV which can be written based on other data: a 2-morphism ϵ between left and right 1-unitors indexed by the unit object, i.e., $r_I, l_I : I \otimes I \longrightarrow I$. To show it is redundant, we present a proof sketch in the comparison section 4.2.3.

Discussion 1. *Baez and Neuchl [6] reviewed the semi-strict definition of monoidal 2-category. Stay, one the other hand, spelled out the definition but without tensorators [61]. Schommer Pries cited Stay [58]. The combination of both constructs the full description of a monoidal 2-category.*

Moreover, Stay listed four unit polytopes by alternating the location of the unit object, but only two of them are necessary, namely, when the unit object is the second or third object in the polytope. As Gurski proved the other two are corollaries of these axioms. Note also Stay's diagrams, in our case, need to be revised, as in the presence of tensorators, filling 2-morphisms will be modified based on the modified tensor product.

Remark 12. *For a concise review of bicategories consult Leinster's paper [38]. By a 2-category, we mean a strict 2-category. In other words, a bicategory whose associators and unitors are identities.*

Remark 13. *The section is organized in the following way: having in mind both KV's and Gurski's definitions, Section 4.2 presents the definition of monoidal 2-category. So you will see the same structures and diagrams as KV whose origins are explained in Gurski's tricategory language. We first list the data (4.2.1); then in Subsection (4.2.2), we give all conditions under three different groups: Naturality Conditions (4.2.2), Modification Conditions (4.2.2) and Axioms (4.2.2). The diagrams of these subsections are depicted in Appendix (4.2.4). The section concludes with Section (4.2.3) in which we compare the produced definition with the original definition given by Kapranov and Voevodsky.*

Notation 1. *Capital letters, $A, B, ..$ are reserved for objects, small letters $f, g, ...$ for 1-morphisms and Greek letters $\alpha, \beta, ...$ for 2-morphisms. Horizontal composition of 1- and 2-morphisms is denoted by juxtaposition and vertical composition of 2-morphisms by $\odot$. For every object A, the identity 1-morphism or 1-identity is shown by id_A, but in the tensor product of id_A with 1-morphisms $id_A \otimes f$, we leave out id and denote it by $A \otimes f$. For every 1-morphism f, the identity 2-morphism or 2-identity is represented by 1_f . Whenever it is clear from the context, in the horizontal composition of 1-identities with 2-morphisms, we leave out 1 and use the whiskering convention $1_f \circ \alpha = f\alpha$.*

Convention 1. *Following the style of the stunning figures of Stay's paper, we colour diagrams. Shapes filled by pink are penetrators π, those by blue are tensorators ϕ, and those filled by brown are 2-unitors. Unfilled ones are natural 2-isomorphisms.*

Definition 76. *A monoidal 2-category is a 2-category $\mathcal{C}$ with a 2-functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and a list of data given in Section 4.2.1 subject to some conditions presented in Section 4.2.2.*

4.2.1 Data of Monoidal 2-Categories

1. A *unit object* $I \in Obj(\mathcal{C})$.

2. For every pair of objects A, B in $Obj(\mathcal{C})$, the *tensor product of objects* is an object $A \otimes B \in Obj(\mathcal{C})$ denoted by juxtaposition AB.

3. For 1-morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, the *tensor product of 1-morphisms* defined as $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ exists.

4. For every pair of 2-morphisms, $\alpha : f \Rightarrow h$, $\beta : g \Rightarrow k$, and 1-morphisms $f, h : A \longrightarrow B$ and $g, k : C \longrightarrow D$, there exists a *tensor product of 2-morphism* $\alpha \otimes \beta : f \otimes g \Rightarrow h \otimes k$ such that $f \otimes g, h \otimes k \in Hom(A \otimes C, B \otimes D)$.

5. Functoriality of 2-functor $\otimes$ holds up to natural 2-isomorphisms, meaning, it preserves composition of 1-morphisms and identity 1-morphisms up to natural 2-isomorphisms.

    - For every quadruple of 1-morphisms (f, f', g, g'), such that $f : A \longrightarrow A', f' : A' \longrightarrow A'', g : B \longrightarrow B', g' : B' \longrightarrow B''$, a natural 2-isomorphism called *tensorator*:

$$\phi_{f',g',f,g} : (f' \otimes g') \circ (f \otimes g) \Rightarrow (f' \circ f) \otimes (g' \circ g) \tag{4.21}$$

In addition to naturality condition, it has to satisfy tensorator's axiom. Note that KV's definition introduces three different tensorators which the tricategory approach packs in one single tensorator denoted by ϕ, for further discussion check Section 4.2.3.

- For every pair of objects A, B, we also need a natural 2-isomorphism:

$$\phi_{A,B} : id_A \otimes id_B \Rightarrow id_{A\otimes B} \tag{4.22}$$

For our purpose of recovering KV's definition, we let $\phi_{A,B}$ be identity 2-morphisms. This assumption has a consequence which we elaborate on in Section 4.2.3.

6. A natural isomorphism called *1-associator*:

$$a : \otimes(\otimes \times \mathbf{Id}) \longrightarrow \otimes(\mathbf{Id} \times \otimes)$$

Since a is between two 2-functors, it consists of two pieces of data: a 1-isomorphism indexed by For every triplet of objects (A, B, C) and a natural 2-isomorphism indexed by at least one 1-morphism. For instance, for $f : A \longrightarrow A'$ there exists a natural 2-isomorphism $\alpha_{f,B,C}$ subject to the naturality conditions in the 1-morphism f, objects B and C, and also the axiom of natural transformation.

$$\begin{array}{ccc} (A'B)C & \xrightarrow{a_{A',B,C}} & A'(BC) \\ \uparrow{\scriptstyle (f\otimes B)\otimes C} & \overset{\alpha_{f,B,C}}{\nwarrow} & \uparrow{\scriptstyle f\otimes(B\otimes C)} \\ (AB)C & \xrightarrow[a_{A,B,C}]{} & A(BC) \end{array} \tag{4.23}$$

7. A natural isomorphism called *left 1-unitor*,

$$l : I \otimes - \longrightarrow \mathbf{Id}$$

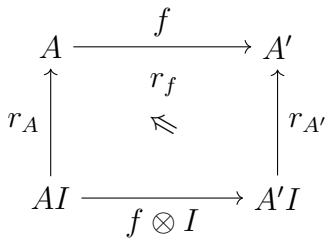
Similar to 1-associators, because l is a natural 2-transformation between two 2-functors $I \otimes -$ and $\mathbf{Id}$, there should be a natural 2-isomorphism indexed by a 1-morphism, l_f, which further satisfies the naturality condition and the axiom of natural transformation. Note that $I \otimes A = IA$ in the figure below.

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \uparrow{\scriptstyle l_A} & \overset{l_f}{\nwarrow} & \uparrow{\scriptstyle l_{A'}} \\ IA & \xrightarrow[I\otimes f]{} & IA' \end{array} \tag{4.24}$$

8. A natural isomorphism called *right 1-unitor*:

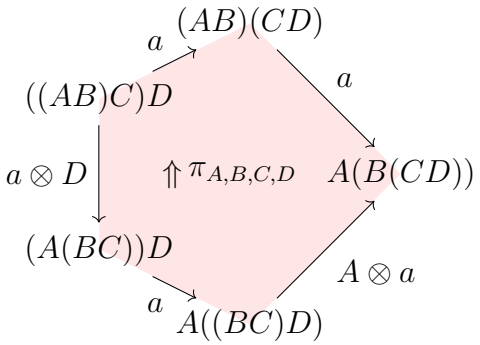
$$r : - \otimes I \longrightarrow \mathbf{Id}$$

This includes a natural 1-isomorphism $r_A : A \otimes I \longrightarrow A$, and a natural 2-isomorphism r_f subject to the naturality conditions and the axiom of natural transformation.



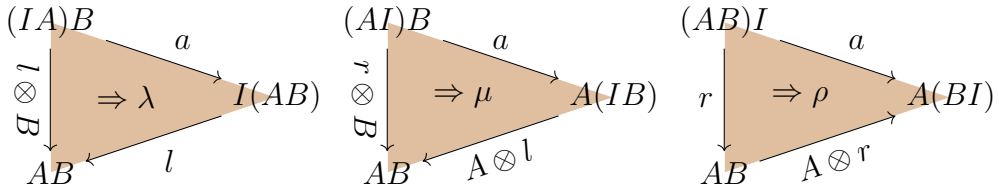
(4.25)

9. For every four objects A, B, C, D, there is a modification between composition of 1-associators called pentagonator $\pi_{A,B,C,D}$ shown in (4.26). Note that π is not natural but it should satisfy the modification condition (3.4).



(4.26)

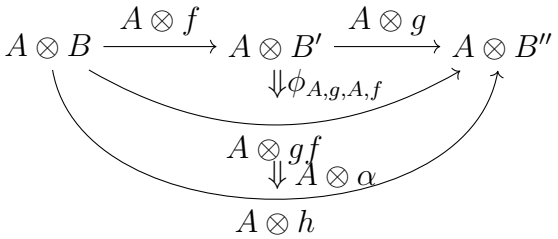
10. There exist three 2-isomorphisms called 2-unitors Equation 4.27. They are modifications, hence, subject to the modification condition (3.4).



(4.27)

Remark 14. *Before listing the conditions on data, we should notice that the existence of tensorators changes the tensor product of an object with a 2-morphism. We denote the new tensor product with* $\hat{\otimes}$*. We show a more comprehensible example first, then restate KV's example in Figure 4.4 for* $\pi_{A,B,C,D}\hat{\otimes}E$*.*

- *Assume that there exists a 2-morphism $\alpha : gf \Rightarrow h$, now if we tensor each object from the left with an object A, the filling 2-morphism will not be modified as $A \otimes \alpha$, but it will be $(A \otimes \alpha) \circ \phi_{A,g,A,f}$.*



- *Now take the example presented by Kapranov and Voevodsky Figure 4.4: consider the pentagonator if each object is tensored by an object E from right. One should observe that the filling 2-morphism is $\pi_{A,B,C,D} \hat{\otimes} E$ which is different from $\gamma = \phi \otimes E$.*

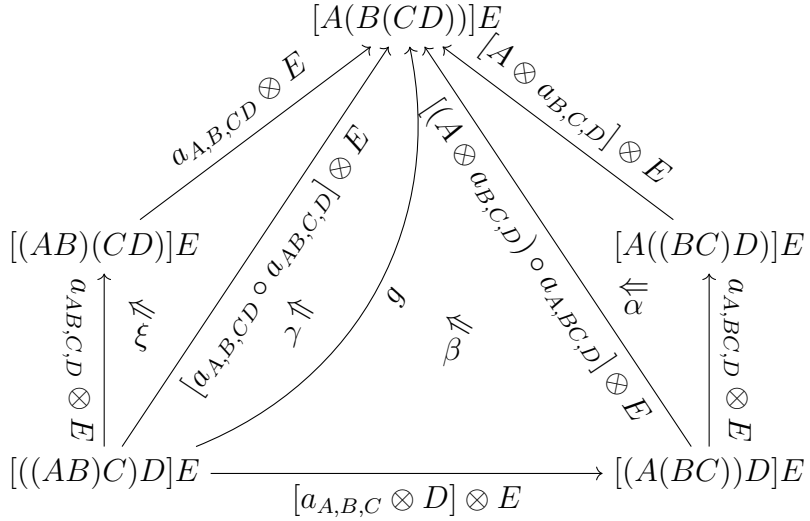


Figure 4.4: Revised tensor product.

The missing 1-morphisms in the figure is $g = [(A \otimes a_{B,C,D}) \circ a_{A,BC,D} \circ (a_{A,B,C} \otimes D)] \otimes E$, and the missing 2-morphisms are $\alpha = \phi_{A \otimes a_{B,C,D}, a_{A,BC,D}, E}$, $\gamma = \pi \otimes E$, $\beta = \phi_{[(A \otimes a_{B,C,D}) \circ a_{A,BC,D}], E, [a_{A,B,C} \otimes D], E}$, $\xi = \phi^{-1}_{a_{A,B,CD}, E, a_{AB,C,D}, E}$

Remark 15. *As mentioned briefly in the discussion, Stay did not use the modified tensor product. Since Stay assumes tensorators are identity 2-morphisms. However, in the following, we shall work with the modified tensor product when it is necessary.*

4.2.2 Conditions on the Data of Monoidal 2-Category

Each of the data above should further satisfy some conditions either due to naturality or axioms of 2-transformation or modification. However, similar to monoidal categories, in addition to the conditions arising from the nature of the data, monoidal 2-categories are further subject to two axioms called Stasheff and unit polytopes. Due to the coherence theorems, these two axioms results in other coherence diagrams built on six objects and more.

Naturality Conditions

1. **Naturality of 1-associators**: weakening 2-isomorphisms $\alpha_{f,B,C}$ for 1-associators should be natural in its indices f, B, C.

   - Naturality in 1-Morphism f: if $\gamma : f \Rightarrow f'$, the cylinder shown in Equation (4.29) commutes. Similar cylinders when the 1-morphism is the first or second index should commute.
   - Naturality in Object B: to check the naturality in B, assume another object B', 1-morphisms $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$, then, the naturality squares of α result in the white squares depicted in the cube (4.31). Gluing squares results in the cube.
   - Naturality axiom: for any composable pair of 1-morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A''$, Diagram (4.32) commutes.

2. **Naturality of 1-unitors**: weakening 1-unitors l_f and r_f need to satisfy naturality conditions and axiom. We only present the conditions for the left unitors l_f . Corresponding conditions should hold for the right unitors r_f .

   - Naturality in 1-Morphism f: the left and right 1-unitors are natural, and their natural 2-isomorphisms l_f and r_f are indexed by a 1-morphism. Hence, to check naturality of l_f and r_f we alternate f; that is, for $\gamma : f \Rightarrow f'$, the cylinder shown in Figure (4.33) commutes.
   - Naturality axiom For any composable pair of 1-morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A''$, the triangle prism shown in Figure (4.34) commutes.

3. **Naturality of tensorators:** Observing and finding appropriate diagrams are easier if one tries to alternate indices of KV's tensorators Section (4.2.3). We list three classes of them and explicitly mention how you can find these diagrams. One might prefer to check Section (4.2.3) before proceeding further.

- The first condition is obtained by checking the naturality of tensorator $\otimes_{f,g}$ in f and g (this is the symbol that KV used). If one alternates f or g in $\otimes_{f,g}$ or equally in $\phi^{-1}_{A',g,f,B} \odot \phi_{f,B',A,g}$, for $\gamma : f \longrightarrow f'$, the cylinder in Figure (4.35) should commute.
- If one alternates one of the 1-morphisms of $\otimes_{g,f,B}$ or in $\phi_{g,B,f,B}$, then for every α and g, 0

$$0\ A \overset{f}{\underset{f'}{\Downarrow \alpha}} A' \xrightarrow{g} A'' \tag{4.28}$$

  Figure (4.36) commutes.
- Now if we check the naturality of $\otimes_{f',f,B}$ in Object B, for $A \xrightarrow{f} A' \xrightarrow{f'} A''$ and $g : B \longrightarrow B'$, the triangle prism commutes Figure (4.37).
- Axiom of natural transformations for tensorators: For any composable triplet of 1-morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A'' \xrightarrow{f''} A'''$ and an object B, Tetrahedron (4.38) commutes.

Modification Data

Pentagonator and 2-unitors are modifications. Hence, the axiom of modification (3.4) should hold for them. We present the details of obtaining the diagrams below later.

- **2-Unitors:** The modification square for every $f : A \longrightarrow A'$ results in a two dimensional diagram. Gluing the boundaries of the 2-dimensional picture gives us the prism (4.39).
- **Pentagonator:** for every 1-morphism $f : A \longrightarrow A'$ and objects B, C, D the pentagonal prism (4.40) commutes.

Axioms

- For every five objects, (A, B, C, D, E), the Stasheff polytope (4.41) should commute.
- For every triplet of objects A, B, C, the unit polytopes (4.42) and (4.43) commute.

Definition 77. *A monoidal 2-category is **strict** if all weakening 1- and 2-morphisms are identities. It is **semi-strict** if all weakening 1- and 2-morphisms except tensorators ϕ are identities.*

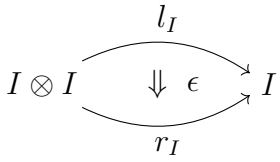
4.2.3 Comparison Between Definitions of Monoidal 2-Categories

Isomorphism or (adjoint) equivalence

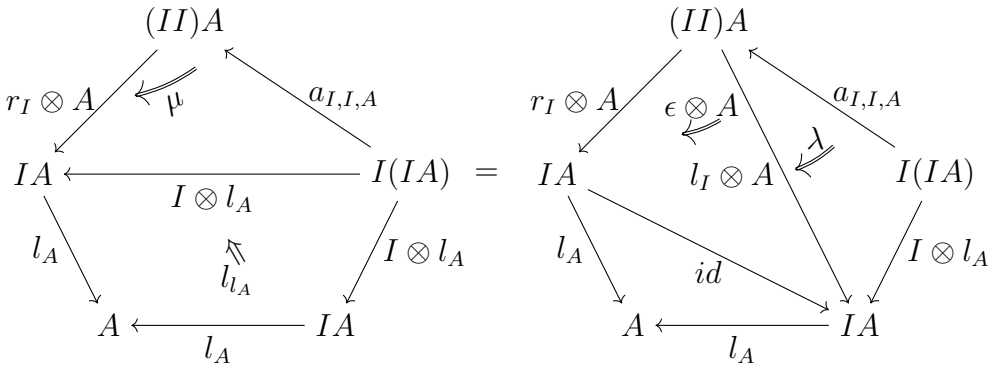
Gordon, Power and Street [24] defined tricategories whose structure 1-morphisms are all equivalences. However, Gurski replaced them with adjoint equivalence. To recover KV's definition, we make them even stronger, and substitute them with isomorphisms.

An extra datum

KV's definition included an extra piece of data which is unnecessary and can be obtained form the 2-morphisms listed above. They defined a special 2-morphism between the left and right unitors for the unit object,



such that it satisfies the diagram below,



Although tiresome, one can write ϵ based on other 2-morphisms by taking the figure given on Page 58 of [26]. The cited figure is used to prove the equality of r_I and l_I for monoidal categories. All inner diagrams can be filled by weakening 2-morphisms to obtain the result. The figure above, regardless of considering ϵ, commutes, which is shown by Gurski in Proposition 4.24 [25].

Tensorators

Kapranov and Voevodsky enumerated three types of tensorators as data because they did not mention that the tensor product $\otimes$ is a 2-functor at heart. They are denoted as $\otimes_{f,g}$, $\otimes_{f',f,B}$ and $\otimes_{A,g',g}$ and defined in Figure 4.5.

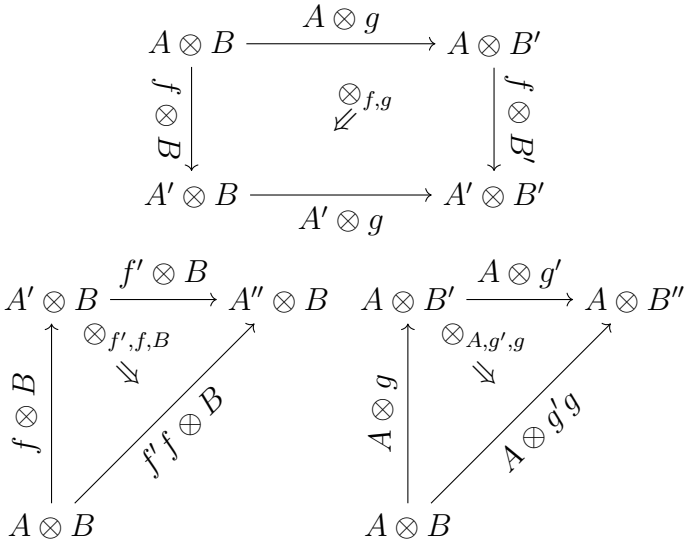


Figure 4.5: KV's tensorators

We can translate them based on our tensorator ϕ. For the first case, $\otimes_{f,g} = \phi^{-1}_{A',f,g,B} \odot \phi_{f,B',A,g}$.

$$(f \otimes B') \circ (A \otimes g) \xRightarrow{\phi_{f,B',A,g}} (f \circ A) \otimes (B' \circ g) = (A' \circ f) \otimes (g \circ B) \xRightarrow{\phi^{-1}_{A',f,g,B}} (A' \otimes g) \circ (f \otimes B)$$

The second and third KV's tensorators are labelled by two 1-morphisms and an object, based on ϕ, they are $\otimes_{f',f,B} = \phi_{f',B,f,B}$ and $\otimes_{A,g',g} = \phi_{A,g',A,g}$.

Consequence of $\phi_{A,B} = 1$

1-Associators and 1-unitors, apart from the naturality axioms, should further satisfy naturality axiom, which seemingly we have ignored throughout the paper. This axiom is built on the assumption that $id_{A \otimes B}$ and $id_A \otimes id_B$ are isomorphic not equivalent, $id_{A \otimes B} \overset{\phi_{A,B}}{\cong} id_A \otimes id_B$. However, to obtain Kapranov and Voevodsky's definition, here we assume $\phi_{A,B}$ are identities, which results in the proceeding corollaries.

Corollary 1. *In a monoidal 2-category with the definition given above, 2-unitors are identity 2-morphisms if they are indexed by identity 1-morphisms.*

Proof. In Equation (4.25), let $F = - \otimes I$ and $G = Id$, $\sigma_A = r_A$ and $\sigma_f = r_f^{-1}$. We have $r_{id_A} = (1 * \phi_{A,I})$, which means $r_{id_A} = 1$. □

Corollary 2. *In a monoidal 2-category with the definition given above, 2-associators are identities if at least one of their indices is an identity 1-morphism.*

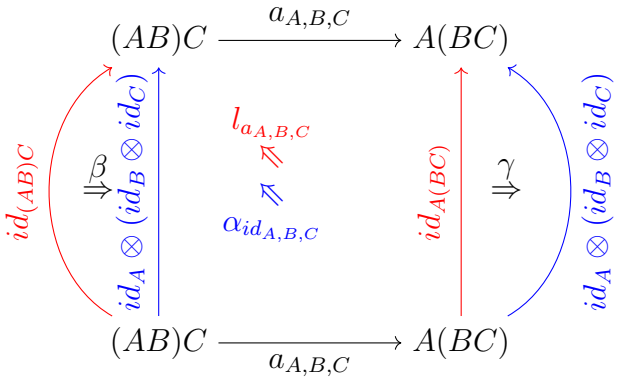


Figure 4.6: Figure of naturality section

Proof. Check Figure (4.2.3), if $\beta = \phi_{AB,C} \odot (\phi_{A,B} \otimes C)$ and $\gamma = \phi_{A,BC} \odot (A \otimes \phi_{B,C})$. The blue square is naturality and the filling 2-associator is,

$$(a_{A,B,C} \circ \beta^{-1}) \odot (\gamma \circ a_{A,B,C}) = \alpha_{id_A,B,C}$$

If one lets $\phi_{A,B}$ identity, then 2-associators become identities if at least one of their indices is identity. □

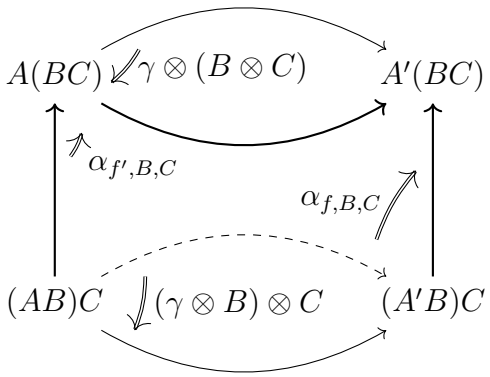
Remark 16. *Note that since we are working with 2-categories not bicategories,* $a_{A,B,C} \circ id_{(AB)C} = id_{A(BC)} \circ a_{A,B,C} = a_{A,B,C}$.

4.2.4 Recovery of Kapranov and Voevodsky's Axioms

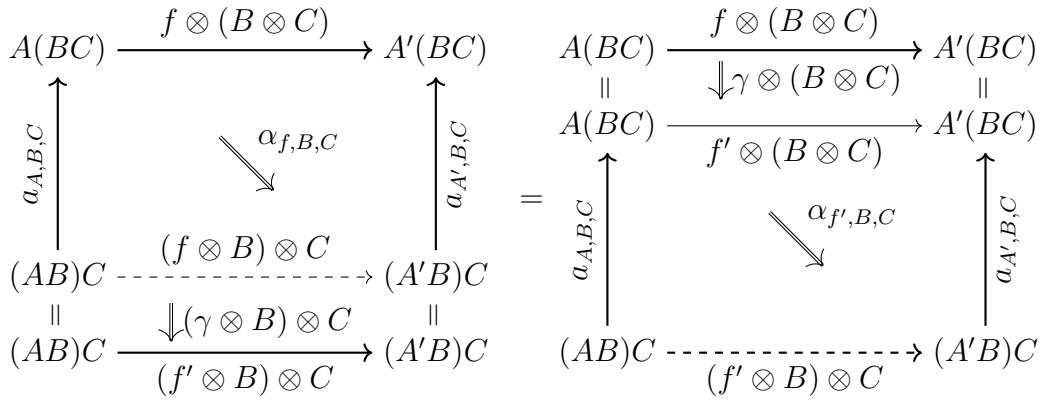
Diagrams of Naturality

Naturality of 1-Associators

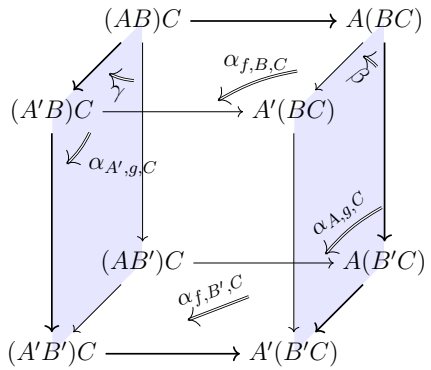
- Naturality of 1-associators in 1-morphism f



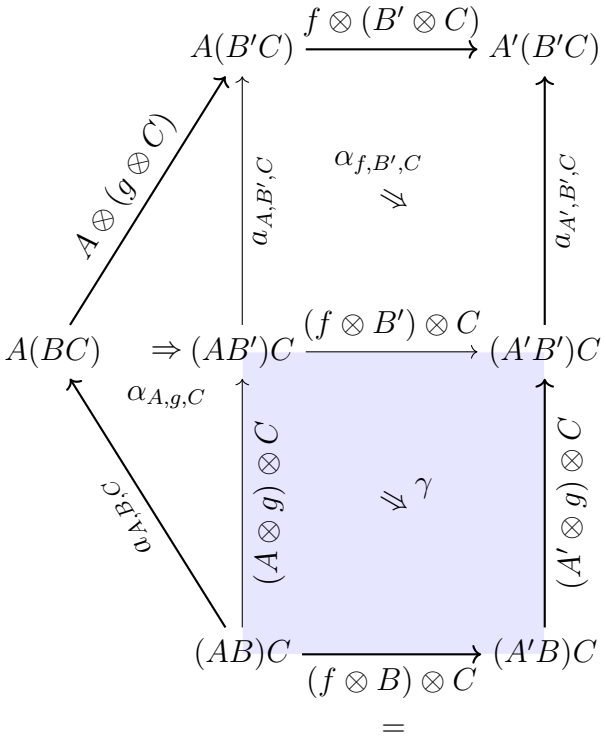
(4.29)

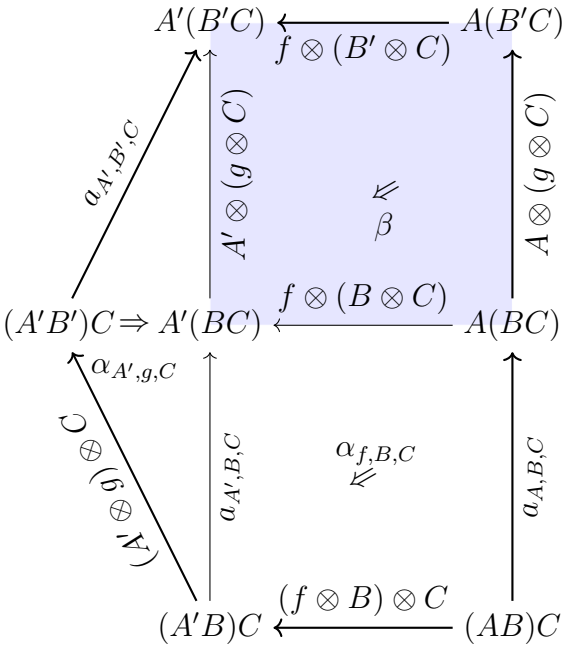


- Naturality of 1-associators in Object B: the missing 2-morphisms in the blue squares are $\beta = \phi^{-1}_{A',g\otimes C,f,B\otimes C} \odot \phi^{-1}_{g,c,B,C} \odot (f \otimes \phi_{B',c,g,C}) \odot \phi_{f,B'\otimes C,A,g\otimes C}$ and $\gamma = (\phi^{-1}_{A',g,f,B} \odot \phi_{f,B',A,g})\hat{\otimes}C$.



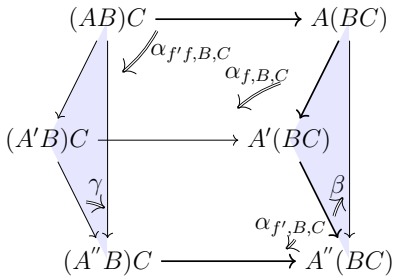
(4.30)

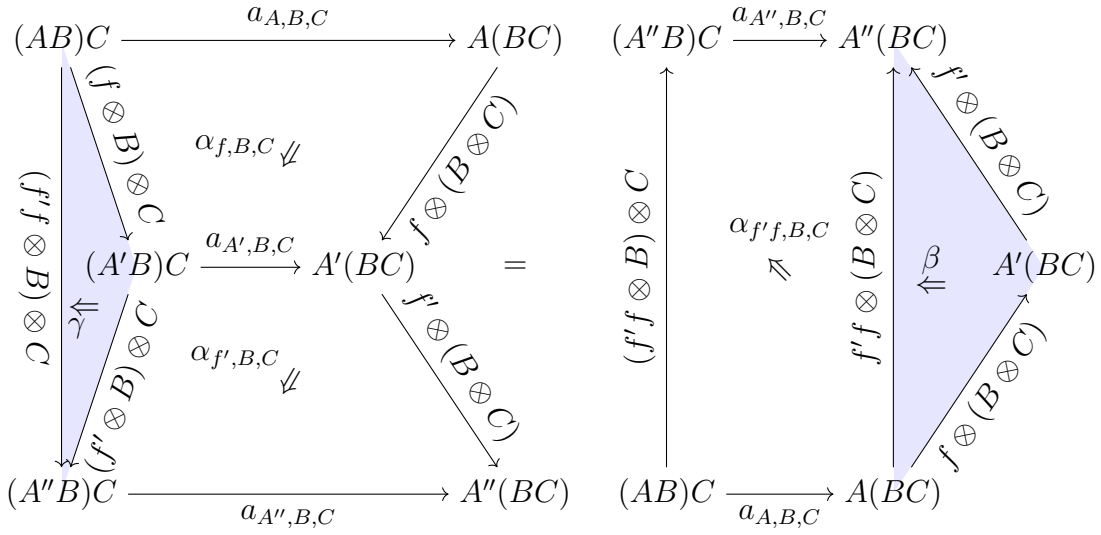




(4.31)

- Naturality axiom for associators: let $F(A,B,C) = (AB)C$, $G(A,B,C) = A(BC)$, $\sigma_{A,B,C} = a_{A,B,C}$, $\sigma_f = \alpha_{f,B,C}$ and $\kappa = \psi = \phi$, it is straightforward to first obtain the planar diagrams, then from shared boundaries glue them to obtain the 3-dimensional diagram of Kapranov and Voevodsky in Page 219. The missing 2-morphisms in the blue triangles are, $\beta = \phi_{f',BC,f,BC}$ and $\gamma = \phi_{f',B,f,B}\hat{\otimes}C$.

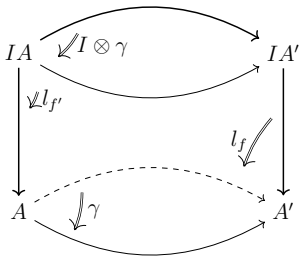


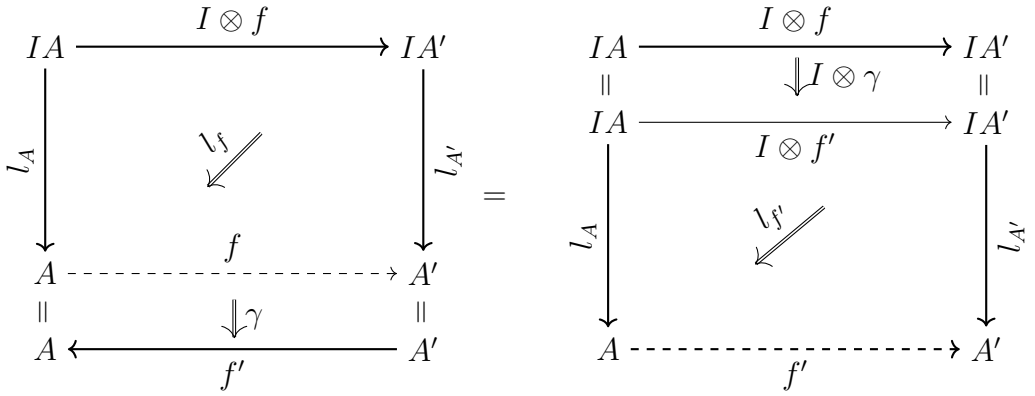


(4.32)

Naturality of 1-Unitors

- Naturality of 1-unitors in 1-morphism f:

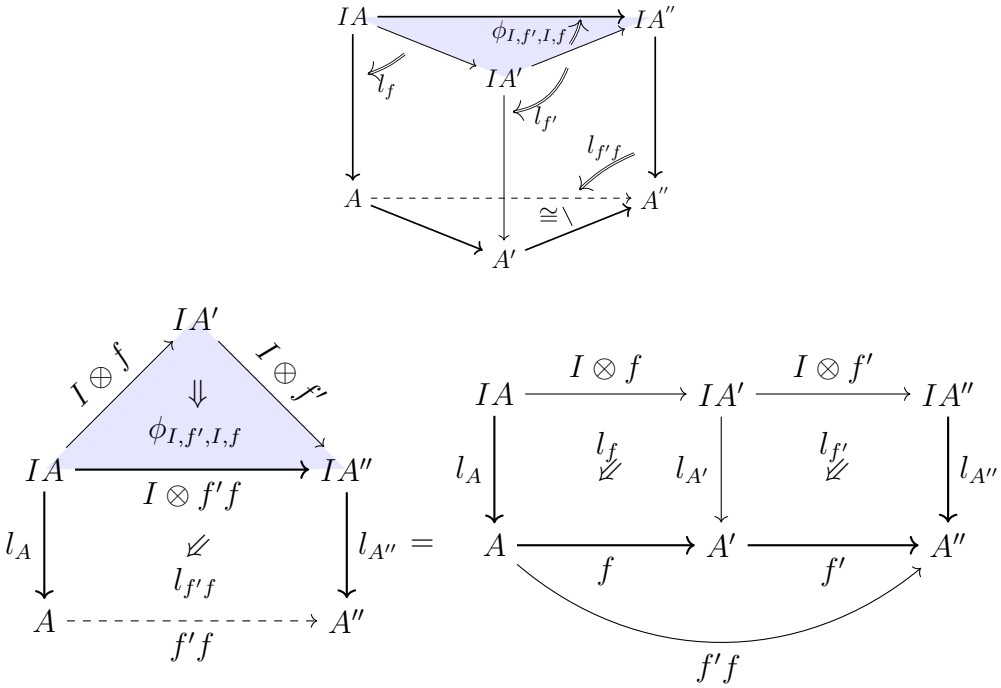




(4.33)

- Naturality axiom for unitors let $GA = A$, $FA = IA$, $\sigma_A = l_A$, $\sigma_f = l_f$, $\kappa = 1$,

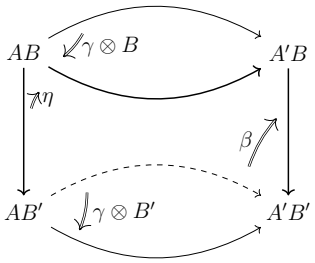
$\psi = \phi$.

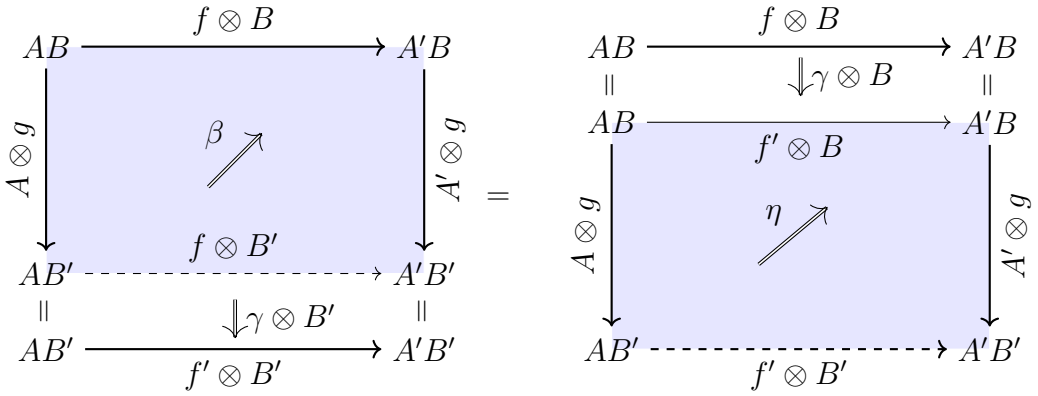


(4.34)

Naturality of tensorators

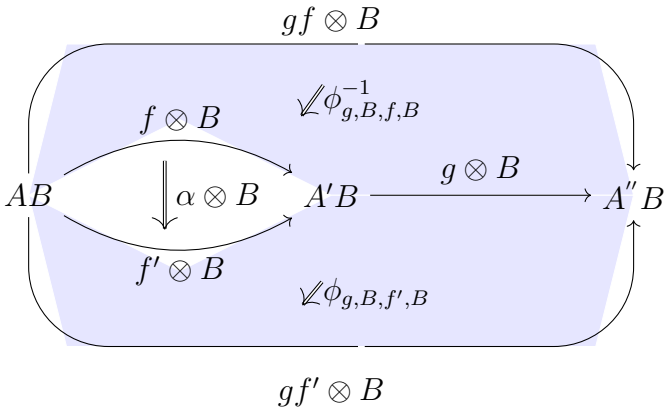
- Naturality of tensorator 1 The missing 2-morphism is $\beta = \phi^{-1}_{A',g,f,B} \odot \phi_{f,B',A,g}$, $\eta = \phi^{-1}_{A',g,f',B} \odot \phi_{f',B',A,g}$.





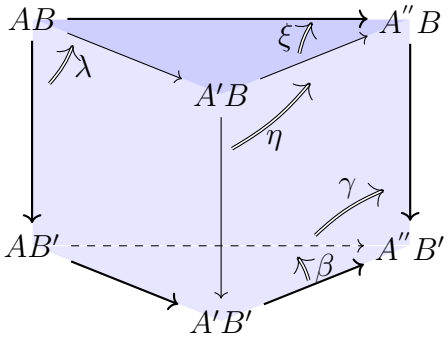
(4.35)

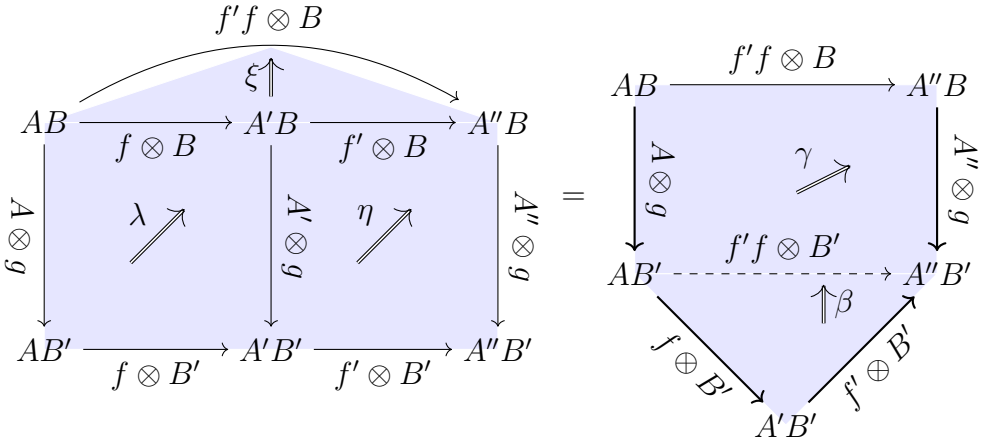
- Naturality of tensorator 2:



(4.36)

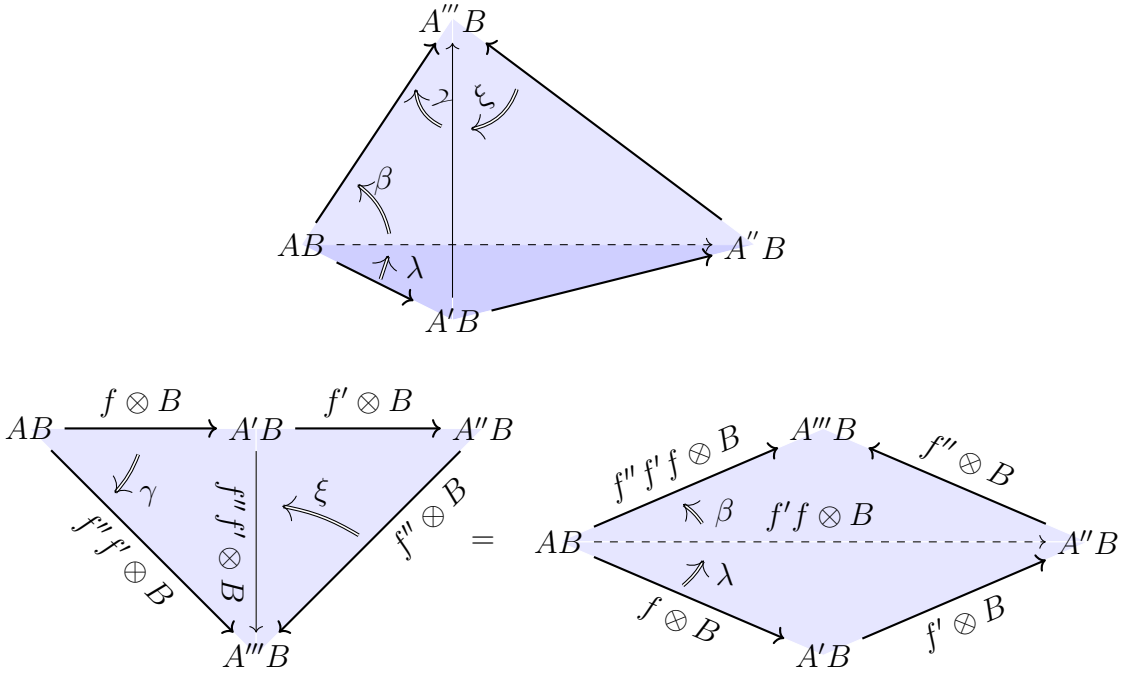
- Naturality of tensorator 3: the missing 2-morphisms in the figure 4.37 are $\xi = \phi_{f',B,f,B}, \beta = \phi_{f',B',f,B'}, \gamma = \phi^{-1}_{A'',g,f'f,B} \odot \phi_{f'f,B',A,g}, \eta = \phi^{-1}_{A'',g,f',B} \odot \phi_{f',B',A',g}, \lambda = \phi^{-1}_{A',g,f,B} \odot \phi_{f,B',A,g}$.





(4.37)

- Naturality axiom for tensorators Let $F(A) = AB$. The filling 2-morphisms are $\lambda = \phi_{f',B,f,B}$, $\beta = \phi_{f'',B,f'f,B}$, $\gamma = \phi_{f''f',B,f,B}$, $\xi = \phi_{f'',B,f',B}$.

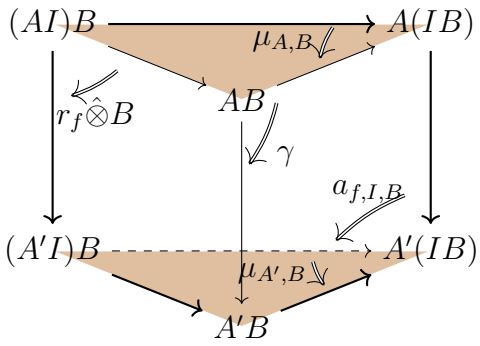


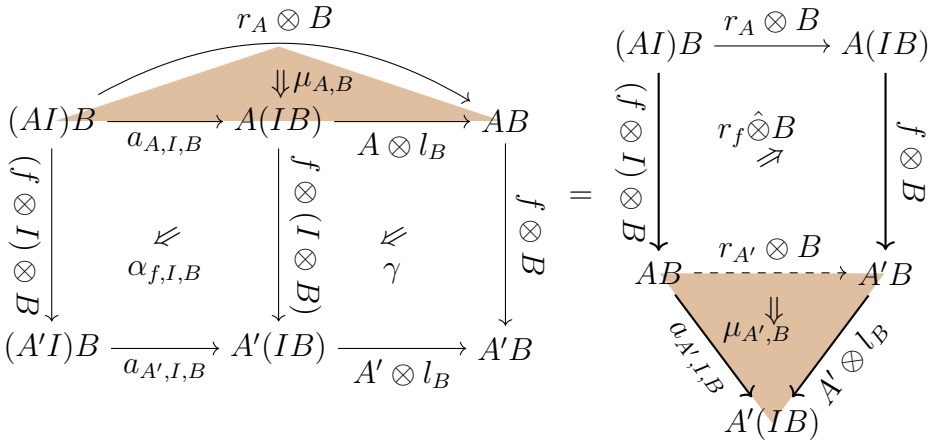
(4.38)

Diagrams of modification

2-Unitors

Modification condition for 2-unitors: We only describe the condition on η, other ones are similar. The missing 2-morphism is $\gamma = \phi^{-1}_{A',l_B,f,IB} \odot \phi_{f,B,A,l_B}$.

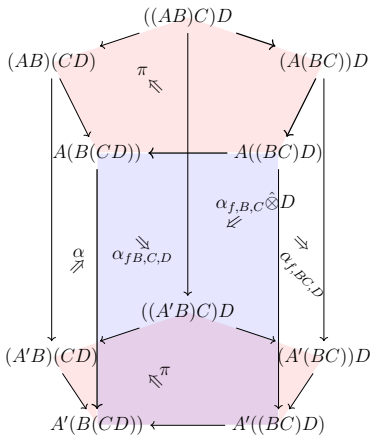


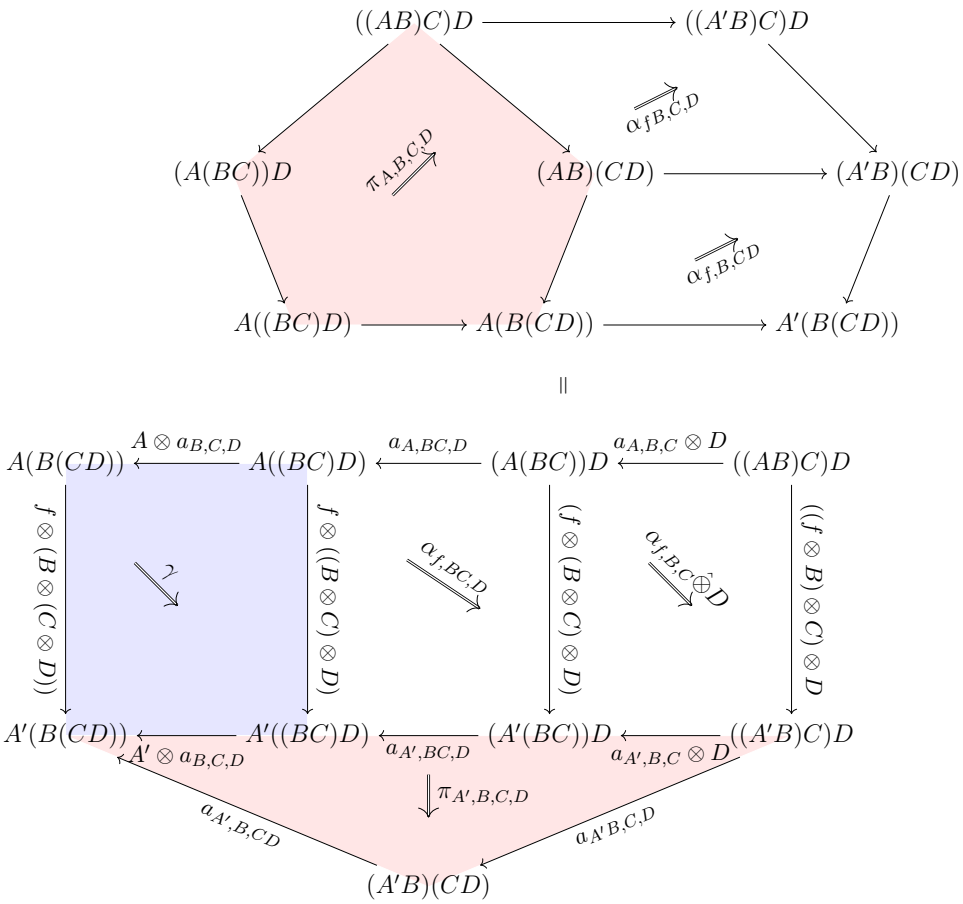


(4.39)

Pentagonator

Modification condition for pentagonator: the 2-isomorphism filling the front square of the 3D picture is $\gamma = \phi^{-1}_{A',a_{B,C,D},f,((BC)D)} \odot \phi_{f,B(CD),A,a_{B,C,D}}$



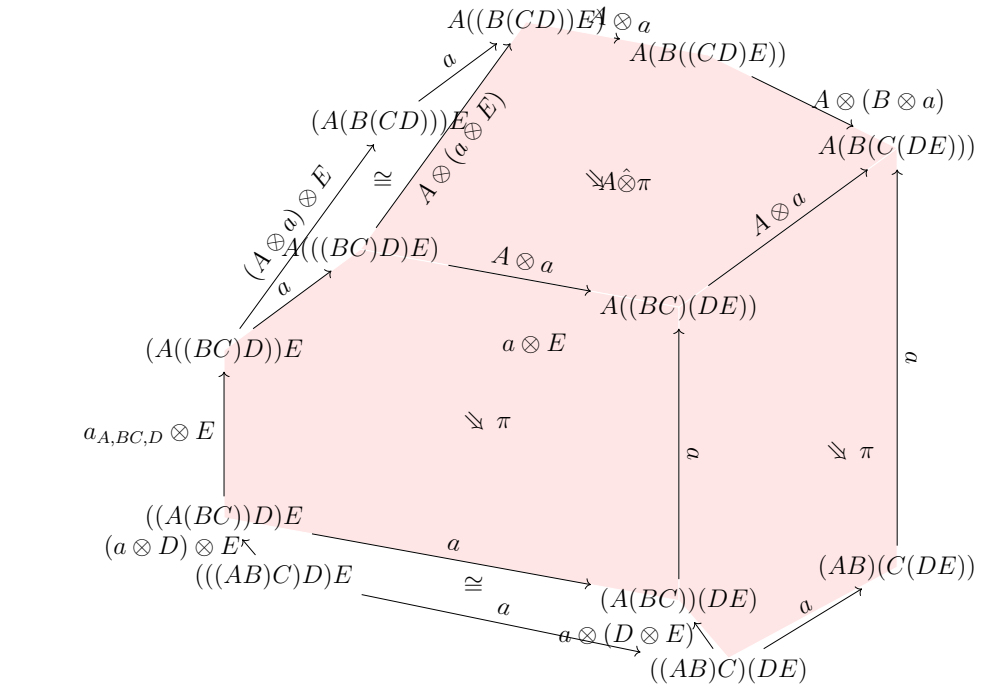

((AB)C)D
((A′B)C)D
(A(BC))D
(AB)(CD)
(A′B)(CD)
A((BC)D)
A(B(CD))
A′(B(CD))
=
A(B(CD))
A((BC)D)
(A(BC))D
((AB)C)D
A′(B(CD))
A′((BC)D)
(A′(BC))D
((A′B)C)D
(A′B)(CD)

(4.40)

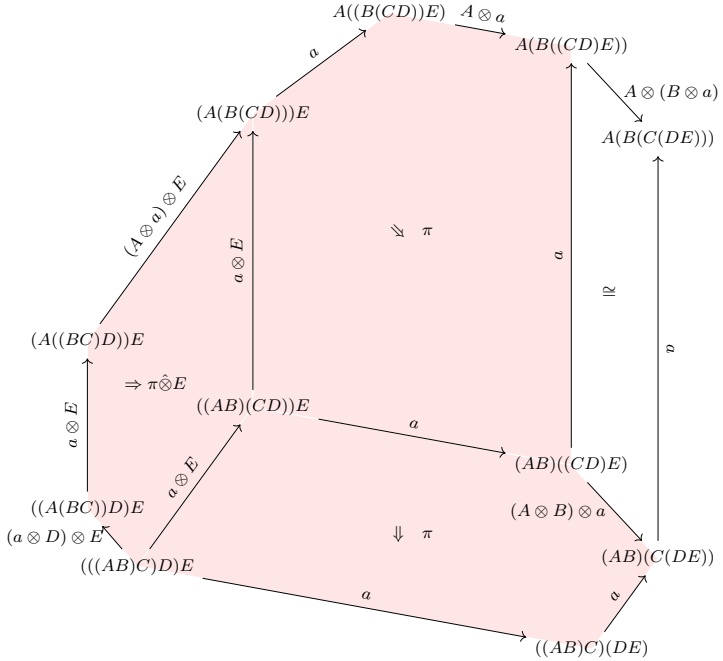
4.2.5 Axioms of Monoidal Categories

Stasheff Polytope

Stasheff Polytope for pentagonator: To draw this diagram, we used the coordinates of Stasheff Polytope in Stay's paper [61].

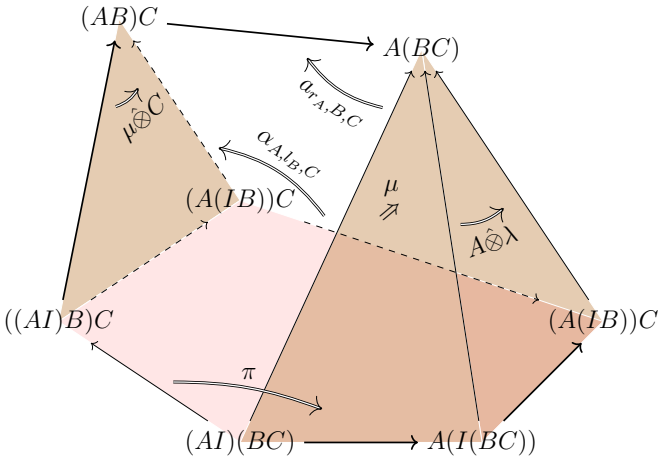


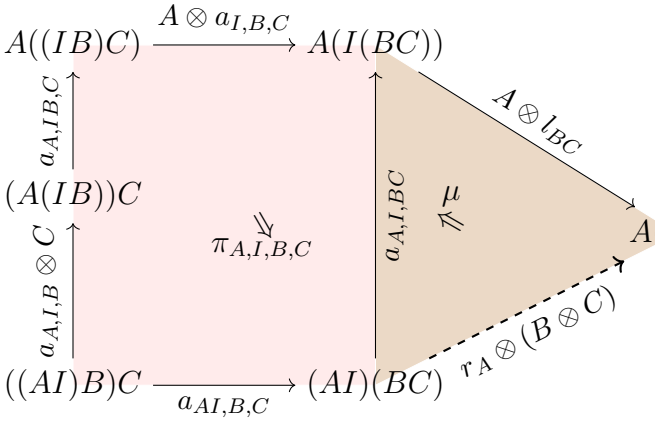
(4.41)



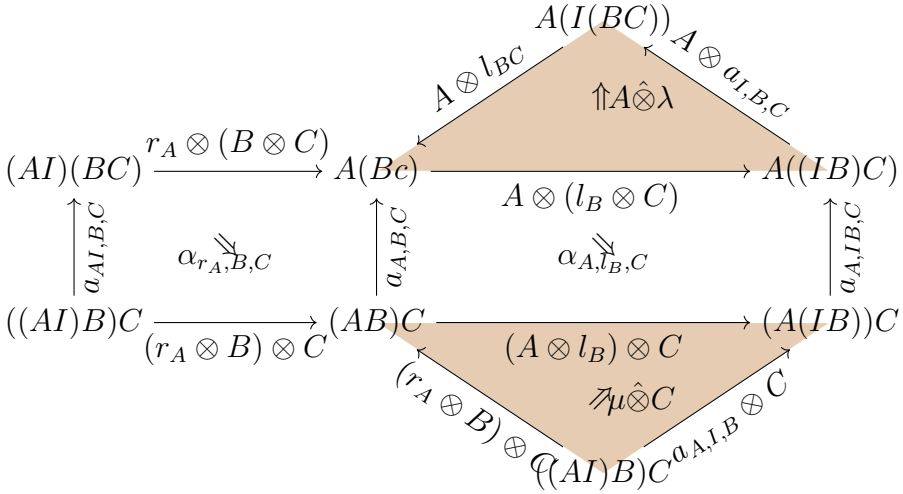
Unit Polytopes

- Unit polytope I

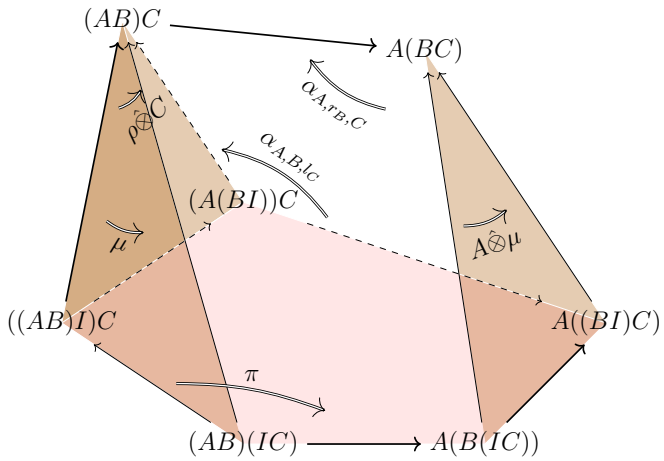




(4.42)



- Unit polytope II



(4.43)

Details of modification diagram for 2-unitors

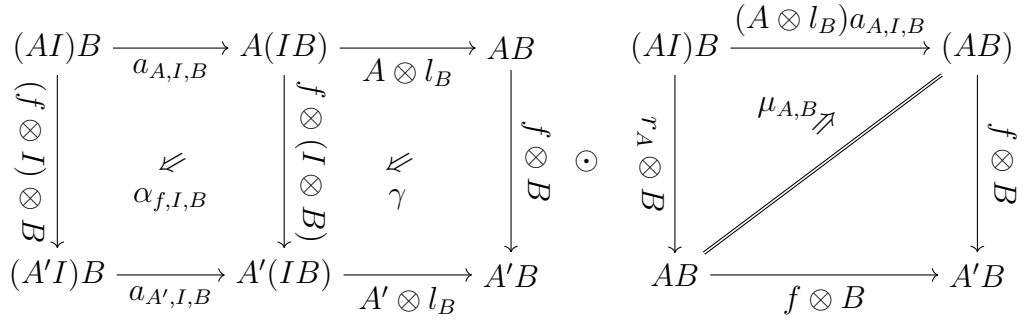
We only describe the details of our procedure for obtaining the prism for one of the 2-unitors, namely, μ. The recipe works for other 2-unitors as well. In square (3.4), let $G(A,B) = (AI)B$ and $F(A,B) = AB$, $\sigma_{A,B} = (A \otimes l_B)a_{A,I,B}$ and $\tilde{\sigma}_{A,B} = r_A \otimes B$. Now, we find the edges of Diagram 4.44 and compose the 2-morphisms.

$$\begin{array}{ccc} Gf\sigma_{A,B} & \xRightarrow{1_{GF} * \mu_{A,B}} & Gf\tilde{\sigma}_{A,B} \\ {\scriptstyle \sigma_f}\Downarrow & & \Downarrow{\scriptstyle \tilde{\sigma}_f} \\ \sigma_{A',B}Ff & \xRightarrow[\mu_{A,B} * 1_{Ff}]{} & \tilde{\sigma}_{A',B}Ff \end{array} \tag{4.44}$$

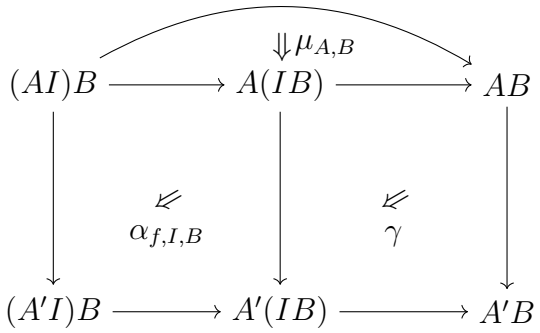
- $(\tilde{\sigma}_f \odot (1_{Gf} * \mu_{A,B}))$

$$\begin{array}{ccc} (AI)B & \xrightarrow{(f\otimes I)\otimes B} & (A'I)B \\ {\scriptstyle (f\otimes I)\otimes B}\downarrow & \swarrow \mu_{A',B} & \downarrow{\scriptstyle r_{A'}\otimes B} \\ (A'I)B & \xrightarrow[l_B a_{A',B,C}]{} & A'B \end{array} \odot \begin{array}{ccc} (AI)B & \xrightarrow{r_A\otimes B} & (A'I)B \\ {\scriptstyle (f\otimes I)\otimes B}\downarrow & r_f\otimes B \swarrow & \downarrow{\scriptstyle f\otimes B} \\ (A'I)B & \xrightarrow[r_{A'}\otimes B]{} & A'B \end{array} = \begin{array}{ccc} (AI)B & \longrightarrow & (A'I)B \\ \downarrow & \swarrow & \downarrow \\ (A'I)B & \longrightarrow & A'B \\ & \searrow \Downarrow \nearrow & \\ & A'(IB) & \end{array}$$

- $(\mu_{A,B} * 1_{Ff}) \odot \sigma_f$



$=$



Details of modification diagram for pentagonator

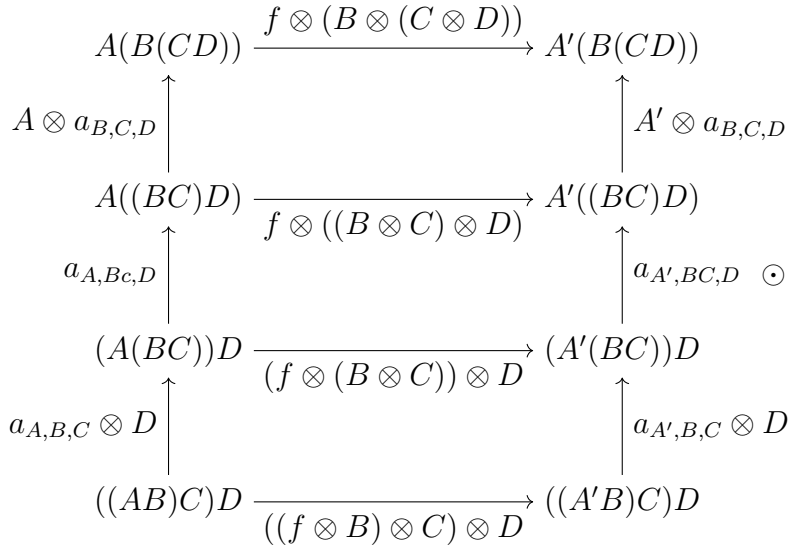
To unpack the modification square, let 2-functors be $G(A, B, C, D) = ((AB)C)D$, and $F(A, B, C, D) = A(B(CD))$, natural transformations are,

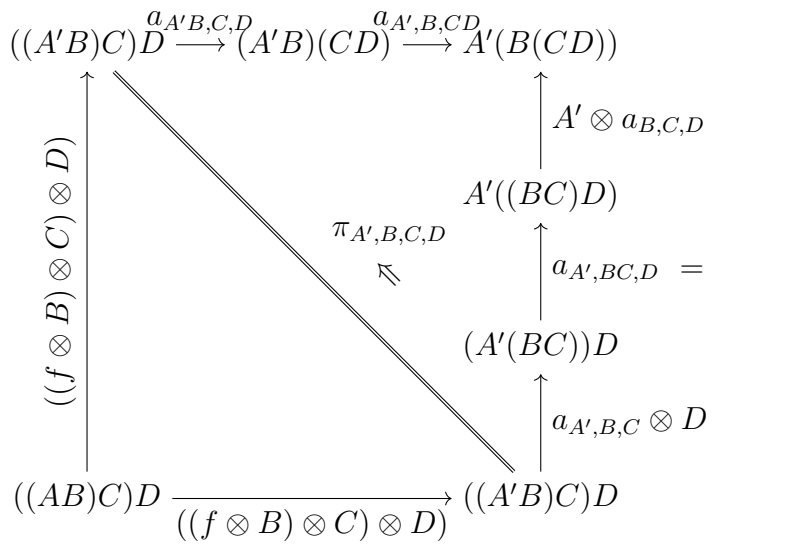
$$\sigma_{A,B,C,D} = (A \otimes a_{B,C,D})a_{A,BC,D}(a_{A,B,C} \odot D)$$

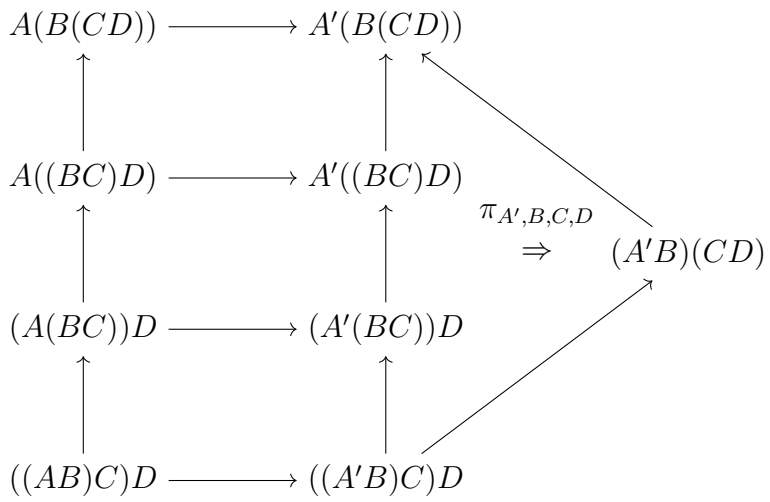
and $\tilde{\sigma}_{A,B,C,D} = a_{A,B,CD}a_{AB,C,D}$.

$$\begin{array}{ccc} Gf\sigma_{A,B,C,D} & \xRightarrow{1_{GF} * \pi_{A,B,C,D}} & Gf\tilde{\sigma}_{A,B,C,D} \\ \Big\Downarrow{\scriptstyle \sigma_f} & & \Big\Downarrow{\scriptstyle \tilde{\sigma}_f} \\ \sigma_{A',b,C,D}Ff & \xRightarrow[\pi_{A,B,C,D} * 1_{Ff}]{} & \tilde{\sigma}_{A',B,C,D}Ff \end{array} \tag{4.45}$$

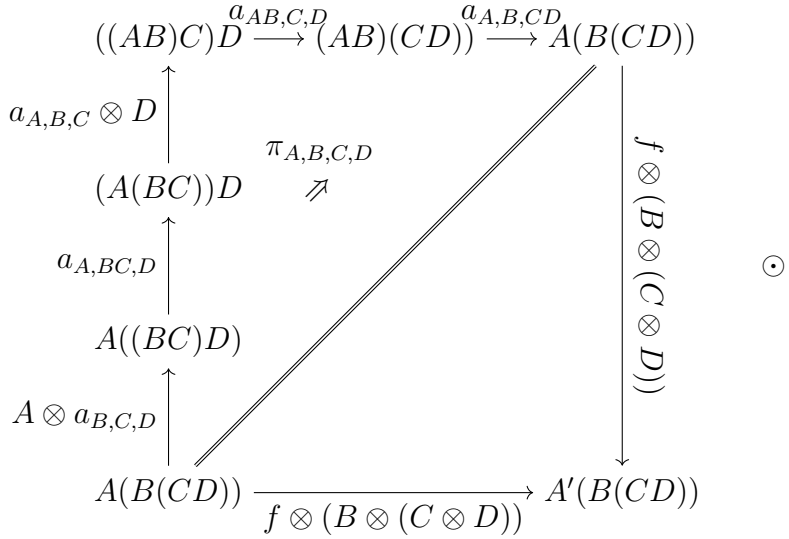
- $(\tilde{\sigma}_f \odot (1_{Gf} * \mu_{A,B}))$



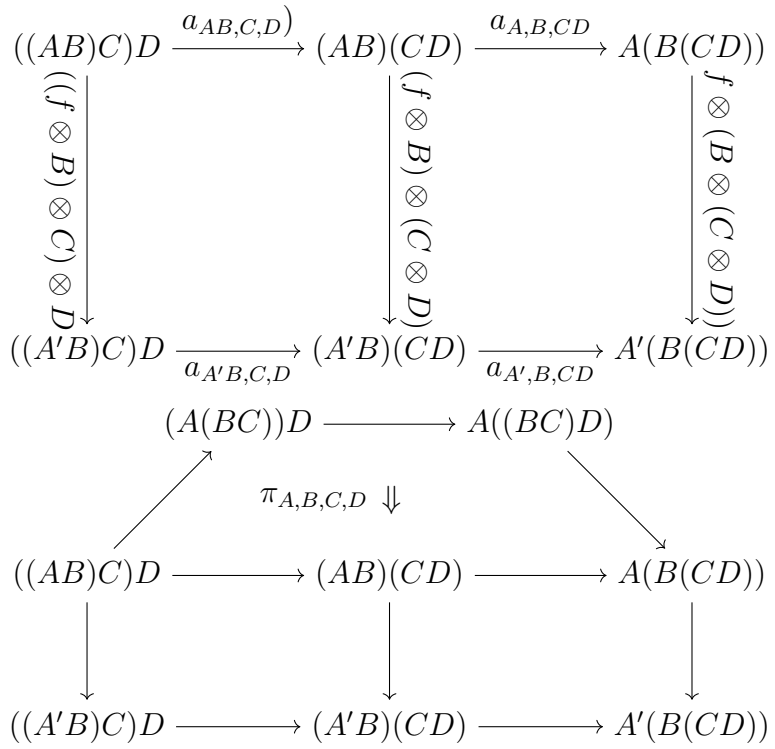




- $(\mu_{A,B} * 1_{Ff}) \odot \sigma_f$



$\odot$



=

4.3 Conclusion and Future Works

In this thesis, we highlighted some bicategorical aspects of topological quantum computation. We presented some theorems to fill the gap in the literature, and we have spelled out the proofs of some known results such as skeletality. We clarified the confusion around a category describing an anyonic theory and TQC category. We showed how the latter is a subcategory of **Hilb**, so the relation between categorical quantum mechanics and TQC is rather obvious. We represented elements of Fibonacci and Ising models of topological quantum computation with the ZX-calculus.

We discussed module categories as categories formalising boundary conditions and domain walls in Kitaev and Levin-Wen models. We explained the elements of the bicategory of module categories and presented similar theorems to skeletalise and reduce the data to necessary ones.

We gave an analysis of the current definition of fusion 2-categories. In this vein, we proposed a definition for 2-biproducts and additive 2-categories. We, furthermore, proposed typing tensors in these 2-categories. We also spelled out a complete definition of monoidal 2-categories, which include necessary date while working with fusion 2-categories for topological quantum computation.

An area for further exploration and currently under development is the ZX-representation of $SU(2)_k$ models. Additionally, one can consider to define 2-biproducts in 2-categories enriched over symmetric monoidal categories instead of additive categories. Another further work under development is skelelality of monoidal 2-categories and reduction of equations to necessary data.

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