Large Spin Perturbation Theory in CFT

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We consider conformal field theories around points of large twist-degeneracy. Examples of this are theories with weakly broken higher spin symmetry and perturbations around generalised free fields. At the degenerate point we introduce twist conformal blocks. These are eigenfunctions of certain quartic operators and encode the contribution, to a given four-point correlator, of the whole tower of intermediate operators with a given twist. As we perturb around the degenerate point, the twist-degeneracy is lifted. In many situations this breaking is controlled by inverse powers of the spin. In such cases the twist conformal blocks can be decomposed into a sequence of functions which we systematically construct. Decomposing the four-point correlator in this basis turns crossing symmetry into an algebraic problem. Our method can be applied to a wide spectrum of conformal field theories in any number of dimensions and at any order in the breaking parameter. As an example, we compute the spectrum of various theories around generalised free fields.

INTRODUCTION AND SUMMARY

A conformal field theory (CFT) is characterised by a set of local primary operators $\mathcal{O}_{\Delta,\ell}(x)$ labelled by their scaling dimension Δ and Lorentz spin ℓ. These operators satisfy an algebra, whose structure constants are denoted OPE coefficients. The spectrum of scaling dimensions and OPE coefficients constitute the *CFT data*.

Many analytic results for the CFT data can be obtained in regimes with small parameters. These computations are in essence perturbative.The conformal bootstrap [1, 2] consists in using instead associativity of the operator algebra to constraint the CFT data. In higher dimensions this was first implemented in [3], leading to extensive numerical results for vast families of CFTs. This also motivated the search for analytic methods using the same idea. A set of results along these lines involves the large spin sector, first studied in [4] and then systematically from the bootstrap perspective in [5, 6] for generic CFT and [7, 8] for weakly coupled CFT [9]. A remarkable conclusion is that the large spin sector of CFTs is universal and essentially free. Other analytic results from the bootstrap perspective involve expansions around small parameters, including large-N gauge theories [15] and the $\epsilon-$expansion [16].

Our aim is to connect these developments. We consider a CFT around a point of large, actually infinite, twist-degeneracy: at the degenerate point we assume the spin for each twist is unbounded. We then introduce twist conformal blocks (TCB) $H_\tau^{(0)}(u,v)$ in which four-point correlators decompose

$$\mathcal{G}^{(0)}(u,v)=\sum_\tau H_\tau^{(0)}(u,v) \quad (1)$$

As we move from the degenerate point, operators acquire anomalous dimensions and the twist-degeneracy is lifted. We then introduce a sequence of functions $H_\tau^{(m)}(u,v)$, where m measures the departure from the degenerate value such that [12]

$$\mathcal{G}(u,v)=\sum_{\tau,m} H_\tau^{(m)}(u,v) \quad (2)$$

A great advantage of this decomposition is that the functions $H_\tau^{(\rho)}(u,v)$ have well understood behaviour around $u,v\sim 0$. This makes the crossing equations algebraic! Our method can be applied to vast families of CFTs: theories with weakly broken HS symmetry, large-N theories, etc. As an example, we compute the spectrum of various theories around generalised free fields (GFF).

DEGENERATE POINT

Consider the four-point correlator of identical scalar operators in a CFT in $d-$dimensional Minkowski space

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle=\frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} \quad (3)$$

with $u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$. Crossing symmetry implies

$$v^{\Delta_\phi}\mathcal{G}(u,v)=u^{\Delta_\phi}\mathcal{G}(v,u). \quad (4)$$

The correlator can be decomposed in conformal blocks

$$\mathcal{G}(u,v)=1+\sum_{\Delta,\ell}a_{\Delta,\ell}u^{\frac{\tau}{2}}g_{\tau,\ell}(u,v) \quad (5)$$

with $\tau=\Delta-\ell$ the twist. This notation makes manifest the small u behaviour of conformal blocks. Assume the CFT has a small parameter g, such that at $g=0$ the spectrum of twists is highly degenerate. Namely for each twist τ there is an infinite tower of operators of unbounded spin ℓ. Consider the functions

$$\sum_\ell a_{\tau,\ell}^{(0)}u^{\tau/2}g_{\tau,\ell}(u,v)=H_\tau^{(0)}(u,v) \quad (6)$$

with $a^{(0)}_{\tau,\ell}$ the squared OPE coefficients at $g = 0$. $H^{(0)}_\tau(u,v)$ encodes the contribution from a given twist to the correlator at $g=0$. Hence

$$\mathcal{G}^{(0)}(u,v) = \sum_\tau H^{(0)}_\tau(u,v) \tag{7}$$

We call these functions twist conformal blocks (TCB).

Twist conformal blocks

Let's understand the properties of TCB. The small u behaviour of conformal blocks implies

$$H^{(0)}_\tau(u,v) \sim u^{\tau/2} \quad \text{at small } u \tag{8}$$

The small v limit of TCB is more subtle, since the sum over the spin can enhance the divergence of a single conformal block. The behaviour can be determined following [4–7]. The scaling relevant for GFF is

$$H^{(0)}_\tau(u,v) \sim \frac{1}{v^{\Delta_\phi}} \quad \text{at small } v \tag{9}$$

As we will briefly comment later, a more general behaviour is also possible.

Conformal blocks are eigenfunctions of Casimir operators [17, 18]

$$\mathcal{D}_2 = D + \bar{D} + (d-2)\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\partial - (1-\bar{z})\bar{\partial}\right) \tag{10}$$

$$\mathcal{D}_4 = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{d-2}(D-\bar{D})\left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2-d}(D-\bar{D}) \tag{11}$$

where $u = z\bar{z}, v = (1-z)(1-\bar{z})$ and $D = (1-z)z^2\partial^2 - z^2\partial$, with eigenvalues

$$\lambda_2 = \frac{1}{2}\left(\ell(\ell+d-2) + (\tau+\ell)(\tau+\ell-d)\right) \tag{12}$$

$$\lambda_4 = \ell(\ell+d-2)(\tau+\ell-1)(\tau+\ell-d+1) \tag{13}$$

This allows to construct a quartic eigenoperator of TCB:

$$\mathcal{H}_\tau H^{(0)}_\tau(u,v) = \lambda H^{(0)}_\tau(u,v) \tag{14}$$

with eigenvalue

$$\lambda = \frac{1}{4}\tau(\tau-d)(\tau-d+2)(\tau-2d+2), \tag{15}$$

given by a combination of the Casimir operators such that the spin dependence disappears:

$$\mathcal{H}_\tau = \mathcal{D}_4 - \mathcal{D}_2^2 + (d^2 - d(2\tau+3) + \tau^2 + 2\tau + 2)\mathcal{D}_2 \tag{16}$$

In obtaining these properties little information was needed about the explicit form of conformal blocks, or OPE coefficients at $g=0$. The eigenvalue equation (14), together with the behaviour at small u, v and some information about the theory at $g=0$ suffices to fix the TCB. Let's see this in detail. Around $v=0$ we expect

$$H^{(0)}_\tau(u,v) = \frac{1}{v^{\Delta_\phi}}\left(h^{(0)}_\tau(u) + h^{(1)}_\tau(u)v + \cdots\right) \tag{17}$$

Plugging this into (14) we obtain a sequence of second order differential equations for the functions $h^{(i)}_\tau(u)$. The equation for $h^{(0)}_\tau(u)$ has two independent solutions. Imposing the correct behaviour at small u we obtain

$$h^{(0)}_\tau(u) = c_0(1-u)^{1-\frac{d}{2}+\Delta_\phi}u^{\frac{\tau}{2}}F_{\frac{2+\tau-d}{2}}(u) \tag{18}$$

where $F_\beta(u) = {}_2F_1(\beta,\beta,2\beta;u)$ is the standard hypergeometric function. Plugging this into the next equation we obtain a second-order equation for $h^{(1)}_\tau(u)$. The correct small u behaviour leave us with another arbitrary coefficient, c_1, and so on. The situation is particularly simple in $d=2$. The eigenvalue equation can be solved to all orders and the solution takes the factorised form

$$H^{(0)}_\tau(z,\bar{z}) = \bar{H}^{(0)}_\tau(\bar{z})z^{\frac{\tau}{2}}F_{\frac{\tau}{2}}(z) \tag{19}$$

where $\bar{H}^{(0)}_\tau(\bar{z}) \sim (1-\bar{z})^{-\Delta_\phi}$ for $\bar{z} \sim 1$. To understand how to fix $H^{(0)}_\tau(z,\bar{z})$ completely, let's look at a specific example. Consider GFF [15]:

$$\mathcal{G}^{(0)}(u,v) = 1 + u^{\Delta_\phi} + \left(\frac{u}{v}\right)^{\Delta_\phi} \tag{20}$$

The intermediate operators are double-trace operators $\phi\Box^n\partial_{\mu_1}\cdots\partial_{\mu_\ell}\phi$ with twist

$$\tau_n = 2\Delta_\phi + 2n \tag{21}$$

The OPE coefficients can be found in [15]. Their explicit form will not be used here. Let us now consider the decomposition in TCB

$$\mathcal{G}^{(0)}(u,v) = \sum_{n=0}^{\infty} H^{(0)}_{\tau_n}(u,v) \tag{22}$$

The functions $H^{(0)}_{\tau_n}(u,v)$ can be fixed as follows. Consider them in a small u, v expansion

$$H^{(0)}_{\tau_n}(u,v) = \frac{u^{\Delta_\phi+n}\left(c^{(0)}_n + \cdots\right)}{v^{\Delta_\phi}} + \frac{u^{\Delta_\phi+n}\left(c^{(1)}_n + \cdots\right)}{v^{\Delta_\phi-1}} + \cdots$$

As discussed above, the eigenvalue equation fixes all the coefficients in terms of the leading ones $c^{(0)}_n, c^{(1)}_n, \cdots$. Focus in the leading term $H_{\tau_0}(u,v)$. The explicit divergence of $\mathcal{G}^{(0)}(u,v)$ as $\bar{z} \to 1$ leads to

$$c^{(0)}_0 = 1, \quad c^{(1)}_0 = c^{(2)}_0 = \cdots = 0 \tag{23}$$

Fixing completely $H_{\tau_0}(u,v)$. This function contains subleading terms in u. Canceling them fixes all $c^{(i)}_1$, and so on. In carrying out this procedure it is convenient to think of Δ_ϕ as arbitrarily large, and then analytically continue in it. In $d=2$ we find the following closed expression for the TCB:

$$H_\tau^{(0)}(u,v)=c_\tau\left(\frac{\bar z}{1-\bar z}\right)^{\Delta_\phi}z^{\tau/2}F_{\tau/2}(z) \tag{24}$$

for $\tau=2\Delta_\phi+2n$ and

$$c_\tau=\frac{\sqrt{\pi}2^{2-\tau}\Gamma\left(\frac{\tau}{2}\right)\Gamma\left(\Delta_\phi+\frac{\tau}{2}-1\right)}{\Gamma(\Delta_\phi)^2\Gamma\left(\frac{\tau-1}{2}\right)\Gamma\left(\frac{\tau}{2}+1-\Delta_\phi\right)} \tag{25}$$

24 contains all terms around $z=0,\bar z=1$ for large enough Δ_ϕ. The full expression for the TCB is actually the sum of two terms (one of which is not divergent as $v\to 0$) and can be recovered by imposing the symmetry $u\to u/v, v\to 1/v$ which corresponds to the exchange of operators at x_1,x_2. At any rate, for our purposes it will suffice to focus in the divergent part of TCBs.

BREAKING THE TWIST-DEGENERACY

As we turn on g the spectrum and OPE coefficients acquire a small correction. To deal with this problem we introduce a shifted Casimir $\mathcal{C}$:

$$\mathcal{C}_\tau=\mathcal{D}_2+\frac{1}{4}\tau(2d-\tau-2) \tag{26}$$

Conformal blocks are eigenfunctions of this operator, with the conformal spin

$$J^2_{\tau,\ell}=\frac{1}{4}(2\ell+\tau)(2\ell+\tau-2), \tag{27}$$

as eigenvalue. We will assume corrections to the spectrum admit the following expansion around large spin ℓ

$$\tau_\ell=\tau+2g\sum_\rho\frac{B_{\tau,m}}{\left(J^2_{\tau,\ell}\right)^\rho} \tag{28}$$

In this letter we will be interested in corrections to the spectrum. OPE coefficients can be treated in similar way. The precise range of ρ will be dictated by crossing and will be fixed later.

The crossing condition becomes algebraic after defining:

$$\sum_\ell a^{(0)}_{\tau,\ell}\frac{u^{\tau/2}}{\left(J^2_{\tau,\ell}\right)^m}g_{\tau,\ell}(u,v)=H_\tau^{(m)}(u,v) \tag{29}$$

$H_\tau^{(0)}(u,v)$ coincides with the TCB introduced above, while m "measures" the departure from the degenerate point. The functions $H_\tau^{(m)}(u,v)$ satisfy the recursion relations

$$\mathcal{C}H_\tau^{(m+1)}(u,v)=H_\tau^{(m)}(u,v),\quad m=0,1,\cdots \tag{30}$$

Namely, the operator $\mathcal{C}$ move us along the sequence of functions $H_\tau^{(m)}(u,v)$. Furthermore, we have the following behaviour for small u,v

$$H_\tau^{(m)}(u,v)\sim u^{\frac{\tau}{2}},\qquad H_\tau^{(m)}(u,v)\sim\frac{1}{v^{\Delta_\phi-m}} \tag{31}$$

For $\Delta_\phi-m$ an integer, a $\log^2 v$ behaviour can also arise. As we turn on the coupling, the four-point function becomes

$$\mathcal{G}(u,v)=\mathcal{G}^{(0)}(u,v)+g\,\mathcal{G}^{(1)}(u,v)+\cdots \tag{32}$$

with

$$\mathcal{G}^{(1)}(u,v)=\sum_{\tau,\rho}B_{\tau,\rho}H_\tau^{(\rho)}(u,v)\log u+\cdots \tag{33}$$

τ runs over the twist-spectrum at $g=0$ while ρ turns out to run over the twist-spectrum plus integers. To compute corrections to the spectrum only the piece proportional to $\log u$ in a small u expansion will be relevant. Now we make the following powerful observation. The functions $H_\tau^{(\rho)}(u,v)$ have a well understood/computable expansion around $u,v=0$. The form of this expansion is such that crossing symmetry can be solved order by order, becoming an algebraic problem! Let's analyse some examples in detail.

Example

Consider GFF in $d=2$. For large enough Δ_ϕ and to all orders in $(1-\bar z)$ the 2d-TCB are

$$H_\tau^{(0)}(u,v)=c_\tau\left(\frac{\bar z}{1-\bar z}\right)^{\Delta_\phi}z^{\tau/2}F_{\frac{\tau}{2}}(z) \tag{34}$$

Plugging this into (30) results in a factorised form also for $H_\tau^{(m)}(u,v)$:

$$H_\tau^{(m)}(u,v)=c_\tau\bar H_\tau^{(m)}(\bar z)z^{\tau/2}F_{\frac{\tau}{2}}(z) \tag{35}$$

with

$$\bar D\bar H_\tau^{(m+1)}(\bar z)=\bar H_\tau^{(m)}(\bar z),\quad H_\tau^{(0)}(\bar z)=\left(\frac{\bar z}{1-\bar z}\right)^{\Delta_\phi} \tag{36}$$

together with $\bar H_\tau^{(m)}(\bar z)\sim(1-\bar z)^{-(\Delta_\phi-m)}$ this allows to find $\bar H_\tau^{(m)}(\bar z)$ as an expansion in $(1-\bar z)$. For the first few cases this expansion can be re-summed. Note that in (36) the dependence on the twist τ has completely dropped out. As a result, the functions $H_\tau^{(m)}(z,\bar z)$ in 2d have the following factorised form

$$H_\tau^{(m)}(z,\bar z)=c_\tau\bar H^{(m)}(\bar z)z^{\tau/2}F_{\frac{\tau}{2}}(z) \tag{37}$$

Integer Δ_ϕ

A nice structure arises for integer but not necessarily large Δ_ϕ . As before, the divergent part of the TCB is captured by

$$H_\tau^{(0)}(z,\bar{z}) = c_\tau \left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\phi} z^{\tau/2} F_{\frac{\tau}{2}}(z) \tag{38}$$

where $\tau = 2\Delta_\phi + 2n$. Let us construct explicitly the functions $H_\tau^{(m)}(z,\bar{z})$. In doing so we will keep only the pieces with enhanced divergence, as $\bar{z} \to 1$, with respect to a single conformal block, or which become divergent upon applying the Casimir $\mathcal{C}$ a finite number of times. Examples are negative powers of $(1-\bar{z})$ or $(1-\bar{z})^p \log^2(1-\bar{z})$ for any p. The factorised form of $H_\tau^{(m)}(z,\bar{z})$ in 2d allows to only deal with $\bar{H}^{(m)}(\bar{z})$. From (36), we can compute the sequence of functions $\bar{H}^{(m)}(\bar{z})$ for different values of Δ_ϕ. For instance

$\Delta_\phi = 2$	$\Delta_\phi = 3$
$\bar{H}^{(0)}(\bar{z}) = \left(\frac{\bar{z}}{1-\bar{z}}\right)^2$	$\bar{H}^{(0)}(\bar{z}) = \left(\frac{\bar{z}}{1-\bar{z}}\right)^3$
$\bar{H}^{(1)}(\bar{z}) = \frac{1}{1-\bar{z}}$	$\bar{H}^{(1)}(\bar{z}) = \frac{4\bar{z}-3}{4(1-\bar{z})^2}$
$\bar{H}^{(2)}(\bar{z}) = \frac{1}{2}\log^2(1-\bar{z})$	$\bar{H}^{(2)}(\bar{z}) = \frac{1}{4(1-\bar{z})} - \frac{1}{4}\log^2(1-\bar{z})$

The general structure is as follows. $\bar{H}^{(m)}(\bar{z})$ contains power-law divergent terms for $m = 0, \cdots, \Delta_\phi - 1$. $\bar{H}^{(m)}(\bar{z})$ contains $\log^2(1-\bar{z})$ terms for $m > 1$, and for $m \geq \Delta_\phi$ it is of the form $\bar{H}^{(m)}(\bar{z}) = g_m(\bar{z})\log^2(1-\bar{z})$ with $g_m(\bar{z}) \sim (1-\bar{z})^{m-\Delta_\phi}$.

What are the consequences of this for the spectrum of the theory at order g? First assume we have only double-trace operators in the OPE $\phi \times \phi$. At order g

$$\mathcal{G}^{(1)}(z,\bar{z}) = \sum_{\tau,\rho} B_{\tau,\rho} H_\tau^{(\rho)}(z,\bar{z}) \log z\bar{z} + \cdots \tag{39}$$

Write the crossing equation as

$$\left(\frac{1-z}{z}\right)^{\Delta_\phi} \mathcal{G}^{(1)}(z,\bar{z}) = \left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\phi} \mathcal{G}^{(1)}(1-\bar{z},1-z), \tag{40}$$

where crossing takes $z \leftrightarrow 1-\bar{z}$. We now make the following observation. Since all intermediate operators have twist $\tau = 2\Delta_\phi + 2n$, all terms on the r.h.s. behave as $(1-\bar{z})^{-\Delta_\phi}(1-\bar{z})^{\Delta_\phi+n}$ as $\bar{z} \to 1$. Hence the r.h.s. does not have power law divergences at $\bar{z} = 1$. Given the behaviour of $\bar{H}^{(m)}(\bar{z})$ around $\bar{z} = 1$ we see that all functions $\bar{H}^{(m)}(\bar{z})$ with $m = 0, 1, \cdots, \Delta_\phi - 1$ are forbidden. Otherwise they would produce a divergence not present on the r.h.s. Functions with higher m are also forbidden, since they would lead to terms containing $\log^2(1-\bar{z})$, also not present on the r.h.s. at one loop. We arrive to the following remarkable conclusion: at first order in g only solutions with finite support in the spin (or which decay faster than any power!) are allowed. A similar argument can be carried out also in $d = 4$, with the same conclusions. This justifies, for instance, some of the claims made in [15].

Consider now a more interesting situation. Imagine ϕ itself is present in the OPE $\phi \times \phi$ at order g. In this case $\mathcal{G}^{(1)}(z,\bar{z})$ contains the following piece:

$$\mathcal{G}^{(1)}(z,\bar{z}) \supset a_\phi (z\bar{z})^{\Delta_\phi/2} g_{\Delta_\phi,0}(z,\bar{z}) \tag{41}$$

where a_ϕ is the (squared)OPE coefficient with which ϕ appears. This term acts as a source in the crossing equations. Now

$$\left(\tfrac{1-z}{z}\right)^{\Delta_\phi} \sum_{\tau,\rho} B_{\tau,\rho} H_\tau^{(\rho)}(z,\bar{z})\Big|_{div} = a_\phi \bar{z}^{\Delta_\phi} \left(\tfrac{1-z}{1-\bar{z}}\right)^{\Delta_\phi/2} F_{\frac{\Delta_\phi}{2}}(1-z) F_{\frac{\Delta_\phi}{2}}(1-\bar{z})\Big|_{\log z} \tag{42}$$

The sum on the l.h.s. of (42) has to reproduce the divergence on the r.h.s. This implies the sum over ρ starts at $\rho = \Delta_\phi/2$ and is such that the precise power law divergence is reproduced for all values of z. Moreover also terms containing $\log^2(1-\bar{z})$ should be absent. To extract the $\log z$ piece on the r.h.s use

$$F_{\frac{\Delta_\phi}{2}}(1-z) = -\frac{\Gamma(\Delta_\phi)}{\Gamma^2(\Delta_\phi/2)}\, {}_2F_1\!\left(\frac{\Delta_\phi}{2}, \frac{\Delta_\phi}{2}, 1; z\right) \log z \tag{43}$$

up to an holomorphic function at $z = 0$. This leads to

$$\sum_{\tau,\rho} B_{\tau,\rho} c_\tau H^{(\rho)}(\bar{z}) z^{\tau/2} F_{\tau/2}(z) = -a_\phi \frac{(z\bar{z})^{\Delta_\phi}}{((1-z)(1-\bar{z}))^{\Delta_\phi/2}} \frac{\Gamma(\Delta_\phi)}{\Gamma^2(\Delta_\phi/2)} F_{\Delta_\phi/2}(1-\bar{z})\, {}_2F_1\!\left(\frac{\Delta_\phi}{2}, \frac{\Delta_\phi}{2}, 1; z\right) \tag{44}$$

The crossing equation has become completely algebraic as both sides can be expanded around $z = 0, \bar{z} = 1$! Let us solve (44) in some examples.

Case $\Delta_\phi = 2$

In this case (44) becomes

$$\sum_{\tau,\rho} B_{\tau,\rho} c_\tau \bar{H}^{(\rho)}(\bar{z}) z^{\tau/2} F_{\tau/2}(z) = -\frac{a_\phi z^2}{(1-z)^2(1-\bar{z})} \quad (45)$$

The sum over twists runs over $\tau = 4 + 2n$. To reproduce the divergence on the r.h.s. the sum over ρ should start at $\rho = 1$. Not to produce $\log^2(1-\bar{z})$ the sum over ρ should stop also at $\rho = 1$. Hence the expansion of the anomalous dimensions in inverse powers of the conformal spin has exactly one term! This result is valid to all orders in inverse powers of the spin, for all values of the twist. Setting $\rho = 1$ we obtain

$$\sum_{\tau} B_{\tau,1} c_\tau z^{\tau/2} F_{\tau/2}(z) = -\frac{a_\phi z^2}{(1-z)^2} \quad (46)$$

Which implies

$$B_{2\Delta_\phi+2n,1} = -a_\phi \quad (47)$$

leading to the following anomalous dimensions for double-trace operators at first order in g and to all orders in $1/\ell$

$$\gamma_{n,\ell} = -\frac{2a_\phi}{(\ell+n+2)(\ell+n+1)} \quad (48)$$

This result is obtained in [22] by more standard methods.

Case $\Delta_\phi = 4$

This case is more interesting and, to our knowledge, the results unknown. (44) becomes

$$\sum_{\tau,\rho} B_{\tau,\rho} c_\tau \bar{H}^{(\rho)}(\bar{z}) z^{\tau/2} F_{\tau/2}(z) \quad (49)$$
$$= -a_\phi \frac{6z^4(1+z)}{(1-z)^5}\left(\frac{1}{(1-\bar{z})^2} - \frac{3}{1-\bar{z}}\right)$$

The factorisation into holomorphic and anti-holomorphic functions allows to solve the problem in two steps. More precisely $B_{\tau,\rho}$ factorises into a function of τ times a function of ρ. First focus in the ρ dependence. To reproduce the correct divergence around $\bar{z} = 1$ the sum over ρ should include $\rho = 2$ and $\rho = 3$. This in turn will produce a term proportional to $\log^2(1-\bar{z})$. In order to cancel this term we must include $\rho \geq 4$ with

$$\frac{1}{(1-\bar{z})^2} - \frac{3}{1-\bar{z}} = \sum_{\rho=2} \alpha_\rho \bar{H}^{(\rho)}(\bar{z}) \quad (50)$$

the coefficients α_ρ can be found recursively by applying $\bar{D}$ repeatedly to both sides of the equation.[19] Note that to carry out this procedure we don't need to know the explicit form of the functions $\bar{H}^{(\rho)}(\bar{z})$. A similar procedure works in higher dimensions. One finds

$$\alpha_\rho = 2^{\rho-2}(5 \times 3^\rho - 9) \quad (51)$$

To fix the dependence on n we need to solve the following problem

$$\sum_{\tau=2\Delta_\phi+2n} B_{\tau,\rho} c_\tau z^{\tau/2} F_{\tau/2}(z) = -a_\phi \frac{6z^4(1+z)}{(1-z)^5}, \quad (52)$$

solved by

$$B_{2\Delta_\phi+2n,\rho} = -\frac{3a_\phi}{4}(n^2+7n+8), \quad (53)$$

which leads to

$$\gamma_{n,\ell} = -54a_\phi(n^2+7n+8)\frac{J^2-1}{J^2(J^2-2)(J^2-6)}, \quad (54)$$

where $J^2 = (\ell+4+n)(\ell+3+n)$. This prediction is valid to all orders in $1/\ell$.

OUTLOOK

We have proposed a new method to study CFT around points of large twist-degeneracy. This method transforms the crossing equations into an algebraic problem and allows to solve the theory perturbatively around large spin. The method does not rely on a Lagrangian description and has a wide range of applicability. As an example we computed the anomalous dimensions for scalar models around GFF in 2d. For $\Delta_\phi = 2$ our method offers a simple explanation of why the expansions in inverse powers of the conformal spin truncate after a single term at order g. Although we have shown how this works in $d = 2$, this result generalises to higher dimensions and indeed this truncation also holds for the $O(N)$ model in $d = 4 - \epsilon$, see *e.g.* [20, 21]. Our method explains the reason!

We have got some milage by assuming (9). This assumption was motivated by GFF but is not always true. For other cases the correct behaviour can be inferred once we select a specific CFT. Then it is straightforward to apply the machinery developed here.

Although features of the method have been shown in simple examples, the range of applicability is much wider. In general g can be any small parameter: a coupling or $1/N$ or ϵ. Some possible applications are:

CFT in various dimensions. We have shown how to systematically construct (as series expansions) the functions $H^m_\tau(u,v)$ in any number of dimensions. Furthermore, the behaviour around $u, v \sim 0$ is universal and defined by the theory at $g = 0$, so that the method can be readily applied to CFT in general dimensions.

Higher orders in g. At higher orders the correlator will contain terms proportional to $\log^2 v, \cdots$, which can be computed from the CFT-data at previous orders. Reproducing these divergences will again fix the CFT-data as an expansion in $1/\ell$. This is used in [22] to compute $1/N^4$ corrections to anomalous dimensions in large N CFTs. Even in the non-perturbative regime, the method proposed here generalises [14] to arbitrary twist ($\gamma_{n,\ell}$ as opposed to $\gamma_{0,\ell}$).

Weakly coupled conformal gauge theories. These contain single-trace operators whose anomalous dimension grows logarithmically with the spin. Logarithmic insertions in our set-up can be studied by inserting $1/J^{2m}$, analytically continuing in m and then taking derivatives with respect to this parameter. Again we will obtain algebraic equations. The approach of this letter offers a gauge invariant on-shell method to study weakly coupled gauge theories.

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