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OxCarre Research Paper 227

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OPTIMAL CARBON PRICING IN GENERAL EQUILIBRIUM: Temperature caps and stranded assets in an extended annual DSGE model

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OPTIMAL CARBON PRICING IN GENERAL EQUILIBRIUM:

Temperature caps and stranded assets in an extended annual DSGE model ♣

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Abstract

The general equilibrium model developed by Golosov et al. (2014), GHKT for short, is modified to allow for additional negative impacts of global warming on utility and productivity growth, mean reversion in the ratio of climate damages to production, labour-augmenting technical progress, and population growth. We also replace the GHKT assumption of full depreciation of capital each decade by annual logarithmic depreciation. Furthermore, we allow the government to use a lower discount rate than the private sector. We derive a tractable rule for the optimal carbon price for each of these extensions. We then simplify the GHKT model by modelling temperature as cumulative emissions and calibrating it to Burke et al. (2015) damages. Finally, we consider how the rule for the optimal carbon price must be modified to allow for a temperature cap, and what this implies for stranded oil and gas reserves. We illustrate our analytical results with a range of optimal policy simulations.

Keywords: carbon price, tractable rule, general equilibrium, utility and growth damages, technical progress, population growth, logarithmic depreciation, differential discount rates, temperature cap, stranded oil and gas reserves

JEL codes: H21, Q51, Q54

This version: January 2021

♣ We have benefited from the incisive comments of Lint Barrage, David von Below, Elisa Belfiori, Cees Withagen, and two anonymous referees on an earlier version of this paper entitled "Stranded assets, the social cost of carbon, and directed technical change: Macroeconomic dynamic of optimal climate policy", which has been extended to have a yearly time scale and allow a more general production function, logarithmic depreciation and differential discounting for public and private sector.

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1. Introduction

Carbon emissions are at the root of the most important global externality (Stern, 2007). The first-best response is to price carbon, either via a carbon tax or an emissions market, at a uniform price throughout the globe and rebate the revenue as lump-sum rebates. The tractable and influential general equilibrium model of growth and climate change developed by Golosov et al. (2014), denoted GHKT from hereon, presents a simple and intuitive rule for the optimal pricing of carbon in general equilibrium: the carbon price should be set a fixed proportion of output with the proportionality factor depending on the discount rate, the severity of damages, and the dynamics of atmospheric carbon. The GHKT model has become very popular due to its analytical versatility. It has been extended to allow for climate model uncertainty (Li et al. 2016; Anderson et al., 2017; Gerlagh and Liski, 2017a), abrupt climate change and tipping points (Engström and Gars, 2016), hyperbolic discounting (Gerlagh and Liski, 2017b; Iverson and Karp, 2020), and multi-country analysis (Hassler and Krusell, 2011). Our aim is to offer various extensions of the GHKT model, some more important than others, and to offer a simplification of the model where temperature is driven by cumulative emissions and damages are calibrated to Burke et al. (2015) rather than to Nordhaus (2016).

We first extend the GHKT model to allow for logarithmic depreciation. Using theoretical and empirical arguments, Anderson and Brock (2019) demonstrate that logarithmic depreciation is better able to fit the aggregate data than geometric depreciation and they suggests its tractability lends itself for tractable expressions for optimal policy. Full depreciation each decade, as is in GHKT, is a special case of logarithmic depreciation. We adopt their depreciation model and find a tractable expression for the optimal carbon price. This allows us to use finer time resolutions, opening the model to the inclusion of business-cycle interactions.

Global warming damages in the model of GHKT are proportional to aggregate production. We generalize these damages to include disutility from global warming and climate damages to the growth rate (rather than the level) of total factor productivity. These extensions also lead to a tractable expression for the optimal carbon price. Using recent empirical studies, we show that the optimal carbon price in 2019 rises from $64 to $456 per ton of carbon if damages affect productivity growth permanently.

We further extend the GHKT model to allow for positive rates of population growth and labour-augmenting technical progress, with important implications for the rates of growth and interest and obtain a generalized tractable expression for the optimal carbon price. A one percent growth rate in population increases the optimal carbon price per ton of carbon from $64 to $164 per ton of carbon (and to $2,039 per ton of carbon if damages affect productivity growth). We also allow for mean reversion in the process of total factor productivity growth and show how this affects the rule for the optimal carbon price.

Following Belfiori (2016) and Barrage (2018) we allow the government to have a lower discount rate than the private sector. This implies that the carbon pricing policy must be complemented with a subsidy for natural and man-made capital to offset the relative myopia of the private sector. In this case, the social cost of carbon increases as the government places greater weight on future generations. Given that a handout to the owners of capital and, more importantly, oil reserves is politically unlikely, we show by how much the carbon price would be increased to offset the missing subsidy. This increase is a second-best way to compensate future generations for the lower material wealth left to them in the form of capital stocks and fossil fuel reserves. We derive the tractable rules for each of these policy instruments in general equilibrium and illustrate why such an equilibrium is time consistent. In our numerical simulations the second-best carbon price is up to 10% above its first-best level.

We also apply two simplifications of the GHKT model based on recent developments in atmospheric science and economics, which lead to a general-equilibrium extension of the framework set out in van der Ploeg (2020). First, recent atmospheric science insights suggest that temperature is driven by cumulative emissions (Allen et al., 2009; Matthews et al., 2009; van der Ploeg, 2018; Dietz and Venmans, 2018; Dietz et al., 2020). Second, the GHKT model models the ratio of damages to output as a negative exponential function of the atmospheric carbon stock and calibrates the function to the DICE-13 model of Nordhaus (2008). We have calibrated our damages to the GHKT model. Since these damages are very modest compared to those found econometrically in Burke et al. (2015), we also calibrated damages to these much higher estimates. Both simplifications increase the realism of the GHKT further and bring it up to date with the most recent atmospheric insights and empirical evidence on global warming damages. They yield a simpler expression for the optimal carbon price and a much higher optimal carbon price of $1,507 instead of $64 per ton of carbon.

Finally, we add realism to our welfare analysis by adding a ceiling on temperature or equivalent a cap on cumulative emissions. This captures the 2015 Paris Climate Accord where countries agreed to limit global warming to 2°C while aiming to keep warming at or below 1.5 degrees Celsius from pre-industrial levels. We show that, if the temperature cap bites, this requires adding a Hotelling term that rises at a rate equal to the market rate of interest to the welfare-maximizing carbon price. The size of this term increases for tighter temperature caps with the initial carbon prices $281 and $1,152 for the 2°C and 1.5°C targets. We also show what this implies for stranded oil and gas reserves.

Section 2 presents the original GHKT model and our extensions of it. Section 3 derives the social optimum and the decentralized equilibrium. Sections 4 discusses the case when the government discounts the future at a lower rate than private agents do. Section 5 simplifies the GHKT model by letting temperature be driven by cumulative emissions and damages be calibrated on Burke et al. (2015). Section 6 explores how our results are modified by a temperature cap. Section 7 calibrates our annual version of the GHKT model and discusses the baseline optimal policy and business as usual results. Section 8 demonstrates the numerical sensitivity of the optimal policy simulation with respect to each of our extensions of the GHKT model. Section 9 concludes.

2. Five Extensions of the GHKT Model

We build on the familiar Brock-Mirman (1972) and Golosov et al. (2014) assumptions: logarithmic utility, Cobb-Douglas production function, full depreciation of capital within a decade, exponential climate damages in production, fossil fuel extraction and production of renewable energy requiring no capital, and a two-box carbon cycle with a part of carbon staying up permanently in the atmosphere and another part that gradually decays and is returned to the surface of the earth and oceans. Energy is derived from coal, oil/gas, and renewable sources.

Replacing the assumption of manmade capital depreciating fully in every decade, we allow for logarithmic depreciation as in Anderson and Brock (2019). Here, the case of full depreciation corresponds to a special case but importantly we can vary temporal resolution to yearly or even quarterly. We extend the GHKT model in four other directions by allowing for a direct negative effect of climate change on household utility and on productivity growth including allowing for mean reversion in total factor productivity, population growth, and growth in labour-augmenting technical progress. We assume that the public and private discount rates are the same but in section 4 we will investigate differential discount rates.

The social planner maximizes utilitarian social welfare, which consists of utility derived from per capita consumption, $\ln(C_t / N_t)$. Climate change lowers output via production damage and, in a first extension of the GHKT model, via instantaneous per capita welfare loss due to climate change, ψE_t where $\psi \geq 0$,

$$(1) \qquad \sum_{t=0}^{\infty} \beta^t N_t \left[\ln(C_t / N_t) - \psi E_t \right].$$

Subscripts denote periods of time $t = 0, 1, 2, \ldots$ The time impatience factor is constant and denoted by $0 < \beta < 1$. Population at time t is N_t and, in a second minor but quantitatively important extension to the GHKT model, its constant gross growth factor equals $N_{t+1}/N_t = \gamma$.[1] The stock of atmospheric carbon is E_t.

Production in the GHKT framework occurs in energy and final goods sectors and is given by a nested production function. Final goods output, Y_t, is produced by combining labour, L_t, capital, K_t, and energy from a Cobb-Douglas production function with a unit elasticity of substitution, while energy output is described by a CES production function and produced from (i) a finite stock of fossil fuel S_t which is utilized without cost, but subject to a Hotelling rent, (ii) an infinite stock of fossil fuel (i.e. coal), and (iii) renewable energy sources, with the latter two requiring only labour input, L_{2t} and L_{3t}, but no capital. A_{1t} and A_{2t} are the corresponding exogenous labour productivities. Production is thus given by $Y_t = A_t K_t^{\alpha} \left[\kappa_{1t} F_t^{\rho} + \kappa_{2t} \left(A_{2t} L_{2t} \right)^{\rho} + \kappa_{3t} \left(A_{3t} L_{3t} \right)^{\rho} \right]^{\upsilon/\rho} L_t^{1-\alpha-\upsilon}$, where F_t denotes fossil fuel

[1] We assume that the discount rate corrected for population growth is positive, i.e., $\beta\gamma < 1$.

use in period *t*. The elasticity of substitution between different energy types is constant and given by $1/(1-\rho)$. The share parameters in the energy sub-production function are positive and satisfy $\kappa_{1t}+\kappa_{2t}+\kappa_{3t}=1$.

In a third minor extension of the GHKT model, we allow positive constant gross growth in labour productivity, denoted by $\omega \geq 1$, and define the number of (exogenous) efficiency units in the economy as $M_t \equiv \omega^t N_t$. Labour (in efficiency units) is allocated either to final goods production L_t or to fossil or renewable energy production, L_{2t} and L_{3t}, respectively, so that $L_t+L_{2t}+L_{3t}=M_t$.

Our fourth extension is logarithmic depreciation of capital. This can be formulated as $K_{t+1}=\iota I_t^{\kappa}K_t^{1-\kappa}$, where $\iota>0$ and $0\leq\kappa\leq 1$. While the case of geometric depreciation is commonly used in growth theory due to its linearity, nonlinear specifications have often been applied, e.g. Uzawa (1969), Lucas and Prescott (1971), Hayashi (1982), and Abel and Blanchard (1983). Anderson and Brock (2019) compare logarithmic depreciation to the more conventional assumption of geometric depreciation of capital. Using theoretical and empirical arguments, they show that logarithmic depreciation is superior in explaining aggregate data and recommend it, therefore, in the formulation of economic policy. Using material balance, $C_t+I_t=Y_t$, we can write the dynamics of capital as

$$(2)\quad K_{t+1}=\iota\left(A_tK_t^{\alpha}\left[\kappa_{1t}F_t^{\rho}+\kappa_{2t}\left(A_{2t}L_{2t}\right)^{\rho}+\kappa_{3t}\left(A_{3t}L_{3t}\right)^{\rho}\right]^{\upsilon/\rho}\left(M_t-L_{2t}-L_{3t}\right)^{1-\alpha-\upsilon}-C_t\right)^{\kappa}K_t^{1-\kappa}.$$

The formulation in (2) captures GHKT model as a special case, since with $\iota=\kappa=1$ it boils down to full depreciation, $K_{t+1}=Y_t-C_t$.

The dynamics of fossil fuel depletion are given by

$$(3)\qquad S_{t+1}=S_t-F_t,\qquad \sum_{t=0}^{\infty}F_t\leq S_0,\quad S_0 \text{ given.}$$

The carbon dynamics of the GHKT model are as follows. Burning fossil fuel leads to carbon emissions of which a fraction stays permanently in the atmosphere, $0<\varphi_L<1$. Of the transitory emissions, a fraction $0<\varphi_0<1$ is still there at the end of the period (i.e. a decade in the GHKT model), and a fraction $1-\varphi_0$ is absorbed by carbon sinks within each period. The decay factor of the stock of atmospheric carbon is $0 < \varepsilon < 1$. With E_t^p and E_t^t denoting the permanent and transient stocks of atmospheric carbon (the sum of which is E_t), the dynamics of the stock of carbon in the atmosphere are

$$(4) \qquad E_t^p = E_{t-1}^p + \varphi_L (F_t + A_{2t} L_{2t}), \quad E_t^t = \varepsilon E_{t-1}^t + \varphi_0 (1 - \varphi_L)(F_t + A_{2t} L_{2t}), \quad E_t = E_t^p + E_t^t.$$ [2]

In our numerical simulations, we annualized the decadal calibration of (4) in GHKT.

Total factor productivity in final goods production, A_t, falls as the stock of atmospheric carbon increases. The instantaneous damage of the stock of atmospheric carbon to total factor productivity is denoted by $\chi > 0$. In our fifth extension of the GHKT model we allow for mean reversion around an exogenous trend growth path for total factor productivity, $\overline{A}_t$. With mean reversion in the log of total factor productivity denoted by $0 \le 1 - \delta \le 1$, the development of total factor productivity is given by

$$(5) \qquad \ln(A_t) = \delta \ln(A_{t-1}) + (1 - \delta) \ln(\overline{A}_t) - \chi (E_t - \overline{E}).$$

The GHKT model effectively has $\delta = 0$ in which case climate change (i.e. atmospheric carbon concentrations in excess of pre-industrial level $\overline{E}$) lowers the level of total factor productivity. However, if $\delta = 1$, climate change affects the growth of total factor productivity permanently. Intermediate values of δ allow for transitory effects, i.e. mean reversion in the effects, of climate change on total factor productivity.

3. The Social Optimum

The social optimum maximizes utilitarian social welfare, eqn. (1), subject to equations (2)-(5) and the non-negativity constraints $F_t \ge 0$, $L_t, L_{2t}, L_{3t} \ge 0$, and $S_t \ge 0$ for all $t \ge 0$, and satisfies the properties stated in the following proposition.

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[2] The carbon cycle can be extended to include arbitrary many boxes. The model of Gerlagh and Liski (2017a) has 3 boxes. Joos et al. (2013) and Aengenheyster et al. (2018) have, respectively, a deterministic and a stochastic carbon dynamics model with 4 boxes. Temperature is modelled indirectly, assuming an equilibrium relationship between carbon stocks and global mean temperature $T_t = ECS \ln(E_t / 596.4) / \ln(2)$, with the equilibrium climate sensitivity ECS equal to 3. This gives temperature in 2010 as 1.3°C, with the GHKT calibration of $E_{2010} = 802$. Rezai and van der Ploeg (2016) introduce a lagged response of temperature to emissions and derive a tractable expression for the optimal SCC which is akin to the one given in (8) of Proposition 1. In the GHKT notation $\varepsilon = 1 - \varphi$.

Proposition 1: *The socially optimal saving and consumption functions are*

(6)
$$K_{t+1} = \iota\left(\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{\kappa} Y_t^{\kappa} K_t^{1-\kappa} \quad \text{and} \quad C_t = c(\beta)Y_t$$
$$\text{with} \quad c(\beta) \equiv \left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right).$$

Demand for the three energy types follow from the efficiency conditions

(7)
$$\left.\begin{matrix}\frac{\partial Y_t}{\partial F_t} \le h_t + \tau_t \\ F_t \ge 0\end{matrix}\right\}\text{c.s.,} \qquad \left.\begin{matrix}\frac{\partial Y_t}{\partial A_2 L_{2t}} \le \frac{w_t}{A_2} + \tau_t \\ L_{2t} \ge 0\end{matrix}\right\}\text{c.s.,} \qquad \left.\begin{matrix}\frac{\partial Y_t}{\partial A_3 L_{3t}} \le \frac{w_t}{A_3} \\ L_{3t} \ge 0\end{matrix}\right\}\text{c.s.,}$$

where $h_t, \tau_t, w_t \equiv (1-\alpha-\upsilon)Y_t / L_t,$ *and* b_t *are the scarcity rent of fossil fuel, the social cost of carbon (SCC) and the social wage (i.e. the marginal product of labour), respectively, all in units of final goods. The first-best optimal SCC is*

$$\text{(8)} \quad \tau_t = \varphi(\beta)Y_t \quad \text{with} \quad \varphi(\beta) \equiv \left(\frac{\varphi_L}{1-\beta\gamma} + \frac{\varphi_0(1-\varphi_L)}{1-\beta\gamma\varepsilon}\right)\left[\psi\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right) + \frac{\chi}{1-\beta\gamma\delta}\right] \ge 0.$$

The scarcity rents on fossil fuel follow from

$$\text{(9)} \qquad h_t = j_t h_{t-1},$$

where $j_t \equiv \iota\left((1-\kappa)\frac{Y_t - C_t}{K_t} + \kappa\alpha\frac{Y_t}{K_t}\right)\left(\frac{Y_{t-1} - C_{t-1}}{K_{t-1}}\right)^{\kappa-1}$ *is the user cost of capital, i.e. interest plus depreciation charges. In the case of full depreciation,* $\iota = \kappa = 1,$ *this user cost of capital boils down to the marginal product of capital,* $\alpha Y_t / K_t$.

Proof: see Appendix A.

The saving and consumption functions (6) follow from the Euler equation and the capital accumulation equation (2). They modify Brock and Mirman (1972) and Anderson and Brock (2019) to allow for population growth and logarithmic depreciation. They indicate that a higher capital share, more patience, higher population growth, and lower depreciation boost incentives for aggregate investment and, hence, curb the propensity to consume. If capital depreciates fully and population growth is absent (i.e. $\iota = \kappa = 1$ and $\gamma = 0$), the consumption share equals $(1-\alpha\beta)$ and the equations in (6) reduce to that in the GHKT model. As in the GHKT model, the propensity to consume, c, is constant and independent of assumptions on the climate and its interactions with the economy.

Equations (7) indicate that an energy good is not used in the production of final goods if its marginal product is less than its social marginal cost. For the scarce fossil fuel type (oil and gas), this cost consists of the scarcity rent plus the SCC. If fossil fuel use is used in production, its marginal product exactly equals its social marginal cost. As fossil fuel reserves are fully depleted (asymptotically), the marginal product of fossil fuel must rise indefinitely. Similarly, the abundant fossil fuel type (coal) is only used if its marginal product equals unit labour cost (i.e., the wage divided by sector-specific labour productivity) plus the SCC. The marginal social cost of renewable energy consists of the wage divided by (potentially) endogenous labour productivity. If the marginal product of renewable energy is less than its marginal social cost, it is not used in production.

Equation (8) implies that the optimal SCC is a constant proportion of aggregate output and this proportion is bigger if the damage parameters χ and ψ are large and carbon resides in the atmosphere for longer (i.e. if the fraction staying permanently in the atmosphere, φ_L, is large or the fraction absorbed by carbon sinks within each period, $1-\varphi_0$, and the decay factor, ε, are low). The component of the SCC due to production damages increases if society has more patience (i.e., has a large discount factor β), population growth γ is high, and the mean reversion in the process of total factor productivity is slow (δ is high). Deviations from full depreciation of capital, i.e. $\kappa < 1$, depress the effect of utility damages on the SCC. As in the original GHKT model, technical assumptions about substitutability of fossil energy sources influences the SCC only indirectly via their effect on the level of output.

It is easy to see that the effect of utility damages on the SCC is positive. This effect can increase or decrease since shifts in parameters discussed above induce a simultaneous revaluation of marginal utility, potentially leading to offsetting effects. If we therefore calibrate the utility damage parameter, ψ, to a given amount of utility lost in today's dollar terms, i.e. $\psi^{\$} \equiv \psi / C_0$, the proportionality factor in the rule for the optimal carbon price becomes $\varphi(\beta) = \left(\frac{\varphi_L}{1-\beta\gamma} + \frac{\varphi_0(1-\varphi_L)}{1-\beta\gamma\varepsilon} \right) \left[\frac{\psi^{\$}}{Y_0} + \frac{\chi}{1-\beta\gamma\delta} \right]$. The component of the SCC due to utility damages calibrated in this way increases if the utility discount factor is small (i.e. society has more patience and population grows rapidly) and if carbon resides in the atmosphere for a long period.

Equation (9) is the Hotelling rule. It states that the growth rate of the scarcity rent must equal the rate of interest as only then will society be indifferent between depleting an extra unit of oil or gas and getting a return equal to the rate of interest and leaving this unit in the ground and getting the capital gains.

The GHKT model supposes that population is constant and climate damages only affect the *level* of total factor productivity ($\gamma = 1,\ \delta = \psi = 0$) in which case the optimal SCC simplifies to $\tau_t = \chi\left(\frac{\varphi_L}{1-\beta} + \frac{\varphi_0(1-\varphi_L)}{1-\beta\varepsilon}\right)Y_t$.[3] If climate damages affect the *rate of growth* of total factor productivity ($\gamma = 1$, $\psi = 0$ and $\delta = 1$), the optimal SCC becomes $(1-\beta)^{-1}$ times bigger: $\tau_t = \frac{\chi}{1-\beta}\left(\frac{\varphi_L}{1-\beta} + \frac{\varphi_0(1-\varphi_L)}{1-\beta\varepsilon}\right)Y_t$. The general elasticity of the SCC with respect to the productivity damage parameter δ is $\frac{\partial \tau_t}{\partial \delta} \Big/ \frac{\tau_t}{\delta} = (1-\beta\gamma\delta)^{-1} - 1 > 0$ which increases in β, γ, and δ. If the growth in population is higher (larger γ), we can see from equation (8) that the SCC increases for production damages but falls for utility damages. The former is intuitive, the latter is the result of two opposing effects: higher population growth increases the attractiveness of capital accumulation due to higher consumption possibilities in the future but also increases damage created by carbon. The utility cost of damages to utility is constant in our formulation. This leaves only the effect on the attractiveness of capital accumulation and, hence, the SCC decreases as population growth increases. All other parameters affect the SCC as in the GHKT model. As discussed above, the SCC is higher under a lower discount rate and a longer residence time of carbon emission in the atmosphere, either because of a larger fraction staying permanently or a lower dissipation rate of the transient fraction (higher $\beta, \varphi_L, \varphi_0, \varepsilon$).

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[3] The carbon tax should be proportional to the unobservable, socially optimal level of GDP, not to its observable, business-as-usual level. In practice, this difference is small (Rezai and van der Ploeg, 2015).

Note that the coefficients of the production functions for final goods and for energy outputs and the coefficients driving the process of logarithmic depreciation (2) do not affect the expression for the optimal SCC (8) directly. They do affect energy use and world GDP, and thus only affect the expression for the optimal SCC indirectly.

Under the assumption of the model, all components of the efficiency conditions in equation (7) scale to output and, thus, the energy system decouples from the rest of the economy. To see this, note that Cobb-Douglas technology in the final goods sector implies proportionality of the marginal products of energy and labour to GDP. Further, the combination of logarithmic utility and logarithmic depreciation ensures that the inverse of marginal utility equals GDP times the constant consumption share. Given the equilibrium outcomes of the energy sector, the evolution of climate change is fully determined. Aggregate sector variables, such as total factor productivity, output, and capital stock, follow from those. This sequence of equilibria depends on the fact that the energy system is solely using labour as an input. We summarize this general finding of the model in the following corollary.

Corollary 1: The evolution of the energy sector only depends on the stock of fossil fuel reserves and determines the evolution of temperature and of the permanent and atmospheric stocks of carbon. The evolution of capital and total factor productivity follows from fossil fuel reserves and the stocks of atmospheric carbon.

Proof: see Appendix A.

Proposition 1 and Corollary 1 characterize the command optimum. Proposition B.1 in Appendix B shows that this command optimum can be sustained in the decentralized market economy if carbon is priced at a level equal to the SCC, via levying a carbon tax or setting up a competitive market for carbon permits, and the revenue is rebated as lump sums to the private sector. These correspond to the first-best climate policies. Decentralization of the command optimum is only feasible if all other externalities and market failures are appropriate dealt with. If not, it is necessary to consider second-best climate policies (e.g. Bovenberg and van der Ploeg, 1994; Bovenberg and Goulder, 2002; Kalkuhl et al., 2013; van der Ploeg, 2016; Rezai and van der Ploeg, 2017; Barrage, 2019).

4. Different Discount Rates for the Private and Public Sector

There has been a debate on what the appropriate choice of discount rate for designing climate change policies is. On the one hand, Nordhaus (2008) adopts a utility discount rate of 1.5% per annum ($\beta = 0.985$). With trend growth of 2% per annum and an elasticity of intertemporal substitution (EIS) of 1/1.45, this gives a consumption discount rate of 4.4% per annum. Like many others, this tries to make the consumption discount rate reflect market returns on assets. Stern (2007) takes the stance that it is unethical to discount the welfare of future generations and therefore chooses EIS = 1 and very small utility discount rate of 0.1% per annum (corresponding to the risk of a meteorite ending the earth as we know it). This reflects ethical preferences, which lead to a much higher SCC. Rather than trying to reconcile the "descriptive" and the "normative" approaches, it seems best to use a high discount rate for the private sector and a low one for the government. Belfiori (2018) shows that this approach can be viewed as an extension of the literature that analyses lower public discount rates as equivalents to greater Pareto weights on future generations (Farhi and Werning, 2007) to the problem of climate change.

Von Below (2012), Belfiori (2017), and Barrage (2018) show that the first-best solution for the optimal carbon prices requires a capital income subsidy alongside as an additional instrument.[4] The reason is that lower social discount rates lead to insufficient saving by private agents with relatively high private discount rates, even in the absence of climate change, and can warrant action to overcome the excessive consumption bias as an additional policy goal. We thus assume that the government is more patient than the private sector who has discount factor $\beta^P > 0,$ hence we assume $\beta > \beta^P$. In Proposition 2 we extend the findings of von Below (2012) and Belfiori (2018) and state the optimal capital income subsidy, denoted by σ, and carbon tax for our model.

Proposition 2: *The social optimum is replicated in the decentralized market economy with a government that is more patient than private agents,* $\beta > \beta^P$, *if the capital subsidy is* $\sigma^{FB} = (\beta - \beta^P) / \beta^P > 0$ *and the carbon tax* $\tau_t^{FB} = \varphi(\beta) Y_t$ *is given by equation (8), where*

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[4] Von Below (2012) and Belfiori (2017, 2018) allow for scarce fossil fuel reserves and Hotelling price dynamics, but Barrage (2018) does not.

the superscript FB denotes first-best policies. The first-best optimal capital income subsidy depends on the gap between the public and private discount factors. The first-best optimal social cost of carbon increases in the public patience.

Proof: see Appendix B.

The interpretation of Proposition 2 is as follows. A constant capital income subsidy curbs the interest rate to its socially optimal level. The optimal capital income subsidy is independent of the carbon tax, and of the capital income tax when measured in utility units. Since households own all assets in the economy, i.e. capital and fossil fuel reserves, this amounts to an identical subsidy on capital and fossil fuel reserves.[5] The constrained-optimal or second-best carbon tax is relevant when the capital subsidy is fixed at a too low level. It decreases as the capital subsidy is raised to the socially optimal level. Society wants to compensate future generations, who are relatively worse off due to the suboptimal capital subsidy, by improving the climate and lowering damages from global warming. If the capital subsidy is set too high, the planner would set the carbon tax below the Pigouvian rate to take some of the pressure off current generations. Hence, we can determine second-best policies if governments are constrained in the choice of their instruments. Moreover, our rule for the second-best optimal carbon tax is unaffected by re-optimization and is therefore time consistent.

Proposition 3: *If the capital subsidy cannot be positive,* $\sigma = \bar{\sigma} \le 0,$ *the second-best optimal carbon tax is* $\tau_t^{SB} = \varphi^{SB} Y_t > \tau_t^{FB}$ *with* $\varphi^{SB} \equiv \varphi(\beta) c^{SB}(\beta^P, \bar{\sigma}) / c(\beta) > \varphi(\beta),$ *where revenue of the taxes on carbon and capital income is rebated as lump-sum transfers. This policy is time consistent.*

Proof: see Appendix B.

If each period capital fully depreciates as in the GHKT model ($\iota = \kappa = 1$), the equilibrium consumption share simplifies to $c^{SB}(\beta^P, \bar{\sigma}) = 1 - \alpha \beta^P \gamma (1 + \bar{\sigma})$ and the second-best consumption share as fraction of the first-best consumption share is $c^{SB}(\beta^P, \bar{\sigma}) / c(\beta) = \left[1 - \alpha \beta^P \gamma (1 + \bar{\sigma})\right] / (1 - \alpha \beta \gamma) > 1.$ We thus see that, if the subsidy for capital and fossil fuel reserves cannot be given, the second-best carbon tax has to be set

[5] The first-best policy in von Below (2012) and Belfiori (2017, 2018) includes separate subsidies for capital and fossil fuel assets. In assuming that households own both assets, we can simplify the policy mix.

higher than the first-best carbon tax to compensate. This holds also for the case of partial depreciation. If the government were to re-optimize, it does not have an incentive to deviate from its second-best optimal carbon tax path. The second-best policy is thus time consistent. This is quite a special result, which is due to the assumptions of the GHKT model muting all the intertemporal strategic considerations.

There are various other ways of implementing policies with different discount rates for the government and private sector. One is to subsidize the fossil fuel owners and the final goods producers instead of subsidizing the households. Another one is if to have a carbon tax that rises faster than the rate of growth of GDP as this would also force the economy to deplete less quickly (von Below, 2012; Belfiori, 2018).

5. Two Simplifications of the GHKT Model: Global Warming and Damages

Here we give two science-based simplifications of the GHKT model. First, we use recent atmospheric science insights that suggest that temperature is driven by cumulative emissions (Allen et al., 2009; Matthews et al., 2009; van der Ploeg, 2018; Dietz and Venmans, 2018). Hence, we replace the two difference equations for the temporary and permanent component of temperature (4) by an equation linking temperature to cumulative emission and one difference equation for the stock of cumulative emissions:

$$(4') \qquad T_t = T_0 + \zeta_1 E_t, \quad E_{t+1} = E_t + F_t,$$

where E_t denotes the stock of cumulative emissions, not the stock of atmospheric carbon, T_0 denotes initial temperature, and ζ_1 the transient climate response to cumulative emissions (TCRE). We set the former to 1.3°C and the TCRE to 2°C /TtC.

Second, we calibrate our damages to Burke et al. (2015) instead of taking them from the GHKT model. The GHKT model models the ratio of damages to output as a negative exponential function of the atmospheric carbon stock. Note that the GHKT model like the DICE-2013 and the most recent DICE-2016 models have very small damages compared to those found econometrically in Burke et al. (2015), which are almost linear in temperature. We replace equation (5) by $\ln(A_t) = \delta \ln(A_{t-1}) + (1-\delta)\ln(\overline{A}_t) - \zeta_2 T_t.$ and substitute for temperature from (4′) to obtain

(5′) $\ln(A_t) = \delta \ln(A_{t-1}) + (1-\delta)\ln(\overline{A}_t) - \chi E_t - \zeta_2 T_0,$

where $\chi \equiv \zeta_1 \zeta_2 > 0.$ Both extensions simplify the GHKT model and make it more realistic and up to date with the most recent atmospheric insights and empirical evidence on global warming damages.

Proposition 4: *With temperature given by (4′) and total factor productivity by (5′), the expression for the optimal carbon price is*

$$(8') \qquad \tau_t = \varphi(\beta) Y_t \quad \text{with} \quad \varphi(\beta) \equiv \frac{1}{1-\beta\gamma}\left[\psi\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right) + \frac{\chi}{1-\beta\gamma\delta}\right] \geq 0.$$

Proof: analogous to proof of Proposition 1. See also Appendix C.

Without utility damages from global warming and no mean reversion in total factor productivity (i.e. $\psi = 0$ and $\delta = 0$), this simplifies to $\tau_t = \chi Y_t / (1-\beta\gamma).$

We use Burke et al. (2015, panel d, Figure 5) to deduce that for every increase in temperature by 1 $^{\circ}$C, global warming damages increase by 12.5% of global economic activity. This is much higher than the damages in the DICE models (e.g. 2.35% of global GDP for each trillion ton of excess carbon in the atmosphere; see section 6). We thus calibrate $\zeta_2 = -ln(1-0.125) = 0.1333$ and $\chi = \zeta_1 \zeta_2 = 0.002 \times 0.133 = 2.66 \times 10^{-4}.$

6. Implications of a cap on temperature or cumulative emissions

Using the finding that temperature is driven by cumulative emissions and given that climate policy is often articulated in terms of temperature targets, we now suppose that there is a cap on cumulative emissions, $\overline{E},$ so that $E(t) \leq \overline{E}, \forall t \geq 0,$ in addition to the marginal damages of climate change on production and utility. Such caps can be formulated on a global level or may result from the nationally determined contributions to emissions reductions as allocated in the Paris Accord. Alternatively, it corresponds to a temperature cap which from (4′) implies a cap on cumulative emissions. E.g. a cap of 2 degrees Celsius implies a global cap on cumulative emissions of $\overline{E} = (2 - T_0) / \zeta_1.$ If T_0 = 1.3°C and ζ_1 = 2°C/TtC, we have a carbon budget of $\overline{E}$ = 350 GtC. Proposition 5 states how the carbon price needs to be modified to ensure that the temperature cap or cap on cumulative emissions is satisfied. An extra term is needed emissions are too high if the carbon tax if set according to equation (8′). In that case, the cap of cumulative emissions bites and the carbon tax must be adjusted upwards.

Proposition 5: *With temperature given by (4′) and total factor productivity by (5′), the expression for the optimal carbon price under a cap on cumulative emissions is given by*

(9) $\tau_t = \varphi(\beta)Y_t + \Delta(\beta\gamma)^{-(t-1)}Y_t, \quad \forall t \geq 0,$

where $\varphi(\beta)$ *is defined in (8′) and* $\Delta = \varpi\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)$. *The constant Δ is zero if the optimal trajectory does not fully exhaust the carbon budget,* $\lim_{t\to\infty} E_t < \bar{E}$. *And Δ > 0 if the optimal trajectory fully exhausts the carbon budget,* $\lim_{t\to\infty} E_t = \bar{E}$. *The constant Δ then has to be chosen so that the carbon budget is exactly exhausted.*

Proof: see Appendix C.

Equation (9) shows that the optimal carbon price consists of two terms. The first term is the usual term corresponding to the optimum without a (biting) temperature as in equation (8′) of Proposition 4. This term rises at a rate equal to the rate of growth of aggregate output. The second term is only present if the cumulative emissions constraint or the temperature cap bites. It corresponds to a Hotelling path along which this term rises at a rate equal to the rate of interest (i.e. the rate of time impatience plus the per-capita growth rate of the economy according to the Keynes-Ramsey rule or j_t as in Proposition 1), since the stock of carbon that can be emitted is now finite. Note that if there are no damages of global warming to utility or total factor productivity, then equation (9) for the optimal carbon price reduces to the Hotelling rule $\tau_t = \Delta(\beta\gamma)^{-(t-1)}Y_t, \forall t \geq 0.$ If the temperature cap bites, then tightening the temperature or cumulative emissions cap requires the Hotelling path for the second term to be lifted (i.e. Δ increases as $\bar{E}$ is cut).

The first two conditions of (7) give efficient use for coal and gas, $\frac{\partial Y_t}{\partial F_t} / \frac{\partial Y_t}{\partial A_2 L_{2t}} = \frac{\kappa_{1t} F_t^{\rho-1}}{\kappa_{2t} A_{2t}^{\rho} L_{2t}^{\rho-1}} = \frac{h_t + \tau_t}{w_t / A_2 + \tau_t}$, so that carbon emissions from coal relative to those of fossil fuel are $\frac{A_{2t}L_{2t}}{F_t} = \left(\frac{\kappa_{2t}A_{2t}}{\kappa_{1t}} \frac{h_t + \tau_t}{w_t / A_2 + \tau_t} \right)^{\varepsilon}$. The price of the oil-gas aggregate is currently less than that of coal, so that a higher carbon tax curbs at least initially emissions from coal relative to that of oil and gas. As time increases, the scarcity rent on oil and gas increases exponentially and thus eventually a carbon tax boosts emissions from coal relative to that of oil and gas.

As time goes to infinity, the carbon tax and wage rise in line with aggregate output which is at a smaller rate than the growth rate of the scarcity rent on oil and gas which equals the interest rate. It follows that as time goes to infinity and with asymptotic depletion of oil and gas, the ratio of emissions of coal to that of fossil fuel goes to infinity. However, as time goes to infinity, coal use and oil/gas use tend to zero, for else the cap on cumulative emissions will be violated, while renewable energy grows to a positive asymptote and grows forever if the economy grows. It is not immediately clear yet how the cap on cumulative emissions, $\lim_{t\to\infty}\left(\sum_{t=1}^{t} (F_t + A_2 L_{2t}) \right) = \overline{E} > 0,$ is met by depleting oil/gas so that $S_t \to \overline{S} \geq 0$ as $t \to \infty$ with $S_0 - \overline{S} \leq \overline{E},$ or by curbing the amount of coal use, cumulative emissions approach $\overline{E}$ only asymptotically (but see section 8.5).

7. Matching the GHKT Model: Logarithmic Depreciation on an Annual Time Scale

Our model is an extension of the original version presented in GHKT. We mean by this that if capital depreciates fully, population and TFP growth are absent, damages only impact production rather than utility or productivity growth, private and public discount rates are the same, and time is on a decadal scale, our extension reproduces the findings of GHKT exactly. However, since our analytical solution allows for logarithmic depreciation, we can abandon the restrictive assumption of a decadal time scale. In this section, we show that our model reproduces the numerical findings of GHKT even on an annual scale with partial logarithmic depreciation. We then illustrate how the equilibrium trajectories under business-as-usual and optimal policy change if we allow for the various extensions of our model and estimates for depreciation in Anderson and Brock (2019). In our baseline numerical simulations, we adopt the following parameter values from the calibration of GHKT given in Table 1.

This implies a capital share of 30% and an energy share of 4% of value added and a rate of time impatience of 1.5% per annum. The carbon cycle is calibrated to the following points: 20% of carbon emissions stay up forever in the atmosphere, of the remainder 60% is absorbed by oceans and the surface of the earth within a year and the rest has a mean life of 300 years. Half of a carbon emissions impulse is removed from the atmosphere after thirty years. Production damages from global warming are 2.35% of global GDP for each trillion ton of excess carbon in the atmosphere. Initial output is calibrated to match $700 trillion per decade.[6] Using these parameters and a decadal time scale, setting population and TFP constant, depreciation of capital to 100% ($\iota = \kappa = 1$), and with only production damages ($\gamma = 1$, $\psi = \delta = 0$), the optimal carbon price directly follows from expression (8) and equals $56.5 per ton of carbon (tC) or $15.4 per ton of CO_2 in 2010.

Table 1: Benchmark - GHKT calibration of the model

Decadal GHKT model	
Final goods production function	Share of capital = α = 0.3, share of energy = υ = 0.04, Initial world GDP per decade = Y_0 = 700 T$
Energy production function	$\kappa_1 = 0.5008$, $\kappa_2 = 0.08916$, $\kappa_3 = 0.41004$, $\rho = -0.058$
Process of logarithmic depreciation	$\iota = \kappa = 1$
Population growth and technical progress	$N_t = 1$, $A_{2,0} = 7683$, $A_{3,0} = 1311$, $A_{2,t+1}/A_{2,t} = A_{3,t+1}/A_{3,t} = 1.02^{10}$
Gross growth labour productivity	$\omega = 1$
Dynamics of atmospheric carbon	$\varphi_L = 0.2$, $\varphi_0 = 0.393$, $\varepsilon = 0.0228$, $S_0 = 253.8$ GtC, $E_0^p = 684$ GtC, $E_0^t = 118$ GtC,
Global warming damages and TFP	$\chi = 2.379\ 10^{-5}$, $\delta = 0$
Time impatience and utility damages	$\beta = 0.985^{10}$, $\psi = 0$
Gross population growth	$\gamma = 1$
Annual model with log depreciation	Y_0 = 70 T$, $\iota = 1.26$, $\kappa = 0.1$
Recalibration to annual time scale and matching of above decadal model	$N_t = 1$, $A_{2,0} = 768.3$, $A_{3,0} = 131.1$, $A_{2,t+1}/A_{2,t} = A_{3,t+1}/A_{3,t} = 1.02$ $\omega = 1$, $\varphi_L = 0.2$, $\varphi_0 = 0.401$, $\varepsilon = 0.0023078$, $S_0 = 253.8$ GtC $\chi = 2.379\ 10^{-5}$, $\delta = 0$, $\beta = 0.985$, $\psi = 0$, $\gamma = 1$

[6] In gauging the effect of parameter changes on the initial carbon price, we follow GHKT in assuming that Y_0 is given. This is, however, only a first-order approximation as Y_0 will be affected by climate policy. Since energy is only a small share in value added, this is not a bad approximation.

Population is constant and normalized to unity, $N_t = 1$. GHKT set the elasticity of substitution between energy types to 0.945 < 1 (i.e. $\rho = -0.058$). This implies that all energy factors are essential to production and can never be phased out completely. Climate policy, therefore, solely aims at depressing fossil energy use rather than a transition to a carbon-free economy where emissions are zero. Relative prices and demand of different energy types (in GtC) and extraction costs of coal are used to calibrate the energy share parameter $\kappa_1 = 0.5008$, $\kappa_2 = 0.08916$ and $\kappa_3 = 0.41004$, and the initial labour-efficiency parameters for coal, $A_{2,0} = 7683$ and renewables, $A_{3,0} = 1311$. The efficiency of labour in coal and renewables is assumed to grow at 2% per annum (i.e. $A_{2,t+1}/A_{2,t} = A_{3,t+1}/A_{3,t} = 1.02^{10}$). The finite stock of oil is set to 300 Gt of oil which converts to 253 GtC. It is assumed that there is no productivity increase in the aggregate goods sector, $\omega = 1$. Together with the initial conditions for the atmospheric stocks of carbon, $E_0^p = 684$, $E_0^t = 118$, the equilibrium trajectories of energy use and climate change can be computed.[7] Using energy inputs, total factor productivity is calibrated to reproduce initial output Y_0 with K_0 = \$128,922 billion from Barrage (2014). This gives $A_0 = 18{,}298$.

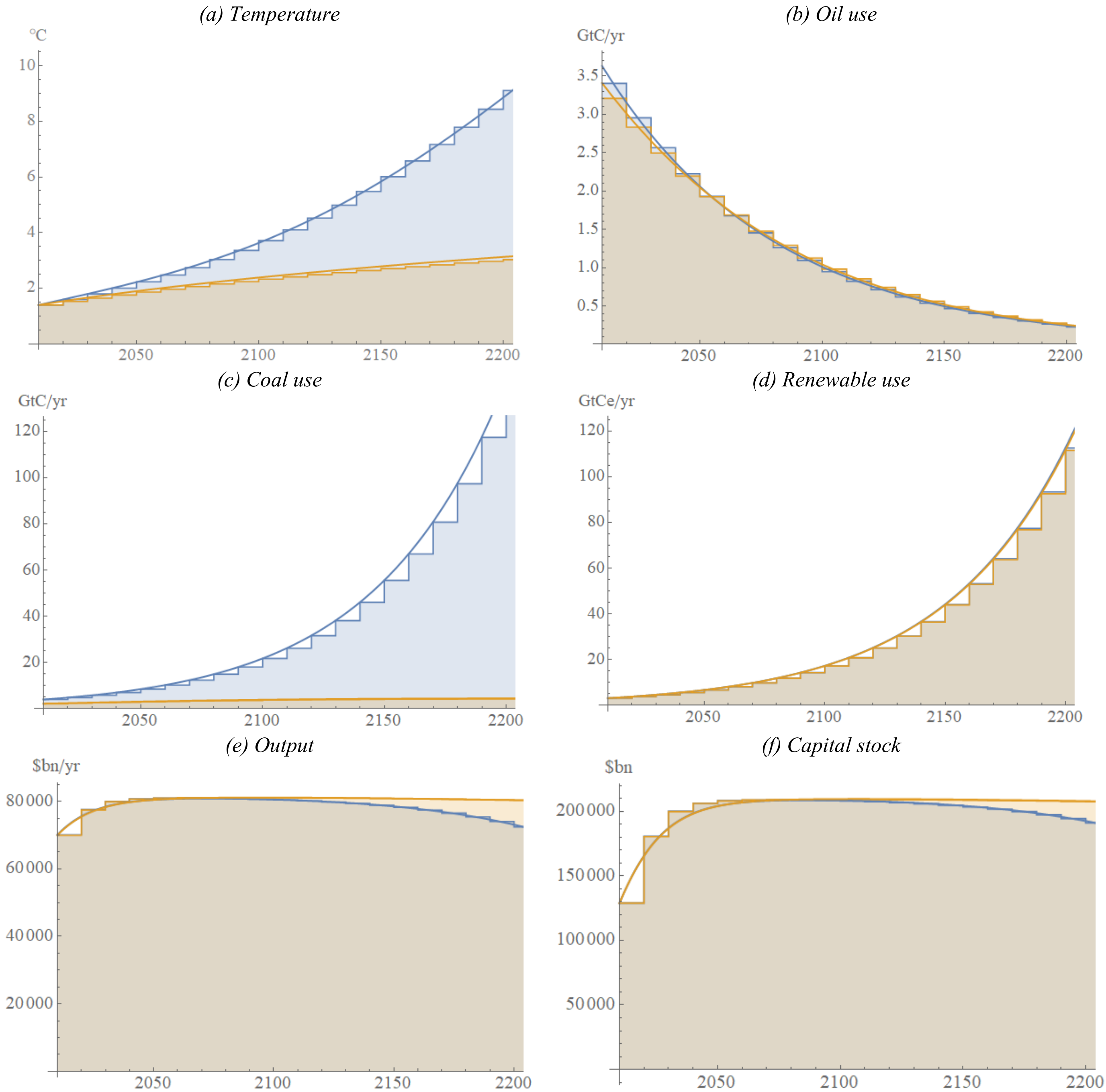
The equations of Proposition 1 can readily be solved numerically using standard routines.[8] Figure 1 reproduces the findings of GHKT for the optimal policy (orange) and business-as-usual (blue) cases. The decadal time scale is visible by the stepwise increments. We compare these with annual version of our model (smooth solid lines). Adjustment of all time-dependent parameters to the annual scale is reported in Table 1 (see also Appendix D). We change depreciation from 100% in each decade to annual partial (logarithmic) depreciation with $\iota = 1.26$ and $\kappa = 0.1$ to match the output and capital dynamics of the decadal GHKT model with 100% depreciation.

The top two panels of Figure 1 show the effects of pricing carbon on global equilibrium temperature and coal use. As described in detail in GHKT, pricing carbon effectively avoids the worst effects of climate change, virtually exclusively through a reduction in coal use as this is the most carbon-intensive fossil fuel. Temperature increases beyond 2°C at the end of this century and 3°C at the end of next century even under carbon pricing. Since oil can be used without cost, its stock of reserves will always be fully

[7] Note that in the supplementary material to GHKT Barrage (2014) reports initial values of 699GtC for permanent and 103 GtC for transitory atmospheric carbon and $\kappa_1 = 0.5429$ and $\kappa_2 = 0.1015$.
[8] The source code containing our solution routines is available upon request.

depleted (asymptotically) in this model. Policy intervention hardly affects the time profiles of oil use and renewable energy output, but coal use falls significantly albeit it takes a century or so for this to occur.

Figure 1: Simulating the GHKT model and our extension for no and optimal policy

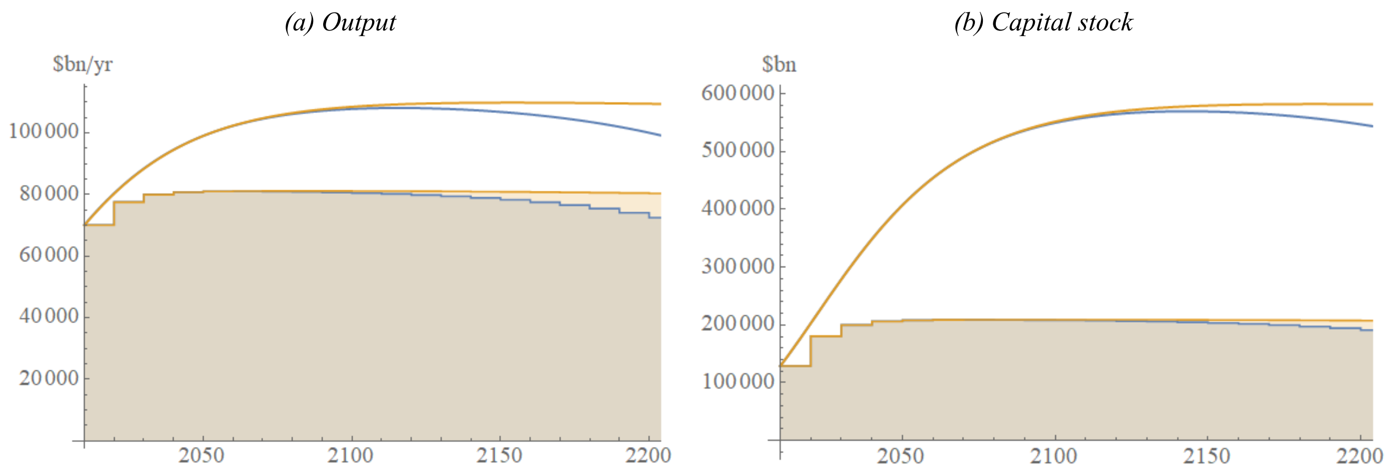


Key: Optimal policy (orange) and business-as-usual (blue) simulations for decadal/full depreciation (step-bars) as in the GHKT model and annual/partial logarithmic depreciation to match GHKT model (smooth lines).

Capital stock and output (net of damages) are reported in the bottom panels of Figure 1. Both increase rapidly within the first decades but due to the absence of any growth engines, both converge to their steady state level around 2050. This steady-state level is falling over time due to the continued emissions of carbon and increases in damages. Given that carbon-based energy inputs are essential in production, this is an unavoidable feature of the GHKT model. Our calibrated annual version of the model can match the dynamics of the decadal version very well. The initial carbon price in 2010 is slightly lower at \$52.5/tC but taking account of output growth over the whole decade, the carbon price averages at \$56.3/tC which matches the decadal number of \$56.5/tC closely.

Thus far, our annualized model was calibrated to reproduce the dynamics of the baseline of GHKT which features a decadal time scale and full depreciation. Anderson and Brock (2019) compare empirically the cases of geometric and logarithmic depreciation and present econometric estimates from a world panel with $\iota = 1.17$ and $\kappa = 0.05$. Adopting these two parameters affects the consumption and investment decisions, as lower depreciation (relative to the complete case) increases the return to capital. The trajectories of capital and output change, while all other variables, most notably those of the energy sector, are unaffected.

Figure 2: Logarithmic depreciation (using Anderson and Brock (2019) estimates)



Key: Calibrated annual model (lines) versus decadal GHKT (bars) with optimal policy (orange) and business-as-usual (blue) simulations. None of the other variables are affected by this process of logarithmic depreciation (see Corollary 1). Bars are same as in Figure 1. Lines correspond to the outcomes with the Anderson and Brock (2019) estimates of logarithmic depreciation.

Figure 2 shows that reducing κ from 0.1 to 0.05 increases the consumption share given in equation (6) in addition to lowering depreciation. With the latter effect dominating, this leads to a substantially higher accumulation of capital and a prolonged period of output growth. With no exogenous growth drivers, output growth starts at 1.6% per annum and falls below 1% by 2025. In the optimal policy scenario, positive economic growth is maintained until the mid of next century while growth turns negative by 2115 in the business-as-usual scenario. Capital accumulation is the only growth engine in the GHKT model (together with exogenous progress in the coal and renewable sectors which affects output growth by less than a twentieth of a percentage point at most).

8. Pricing Carbon in the Extended GHKT Model

We use our annualized version of GHKT model as a baseline to compare how extensions regarding long-run growth in population and productivity and damages to productivity and utility change the model's predictions. Table 1 presents key environmental variables for these scenarios. The reported social cost of carbon are updated to 2019's GDP level of $85 trillion in constant 2010 US$ (World Bank, 2020). This increases the baseline carbon tax from $56 to $64 and increases to grow at the rate of (real) GDP.

8.1. Population growth and technological progress

The introduction of population growth lowers the population-adjusted discount rate and increases the saving rate and the social cost of carbon as the current generations are more willing to provide for a larger population in the future and to make sacrifices to curb future global warming. With annual population growth of 1% ($\gamma = 1.01$), the optimal carbon price more than doubles to $164/tC in 2019 and temperature at the end of the century falls by half a degree to 1.9°C. Peak warming falls from 3.7 to 2.7°C. In contrast, TFP growth of 1% per annum does not affect the initial carbon price. However, over time the economy grows at a faster pace and thus the optimal price of carbon grows at a faster pace too. As a result, energy output increases slightly and temperature by 0.1°C. Bohn and Stuart (2015) consider endogenous population size, calculate the externality from an extra birth on climate change and find it too be large. This requires policy to be less pronatalist and may require a sizeable Pigouvian tax on having children. We abstract from these issues here.

Table 1: Sensitivity of optimal policy simulations

	SCC in 2019 ($2010/tC)	Temperature in 2100 (°C)	Temperature max (°C)
Baseline annualized GHKT model	$64	2.4	3.7
(1) Population grows 1% p.a. (γ = 1.01)	$164	1.9	2.7
(2) TFP growth of 1% p.a.	$64	2.5	3.8
(3) Damages to utility	$63	2.4	3.8
(4) Damage effects on TFP growth	$100	2.2	3.1
(1) & (4): population growth and TFP growth damages	$259	1.8	2.4
(2) & (4): TFP growth and TFP growth damages	$100	2.2	3.2
(3) & (1): Utility damages and population growth	$162	1.9	2.8
(3) & (2): Utility damages and TFP growth	$63	2.4	3.8
(3) & (4): utility damages and TFP growth damages	$90	2.2	3.2
(1), (2), (3), and (4)	$232	1.8	2.5

8.2. Utility damages and damages to productivity growth

To allow for utility damages we follow the careful calibration of Barrage (2018) and assign 74% of damages at 2.5°C to production and 26% of damages to utility ($\chi = 1.806 \times 10^{-5}$ and $\psi = 7.376 \times 10^{-6}$ but $\delta = 0$). The optimal carbon price falls only slightly to $63/tC, which causes temperature to increase by up to 0.1°C.

In the GHKT formulation damages only affect the current level of TFP but not its growth rate. Dell et al. (2012) find that a temperature increase of 1°C lowers per capita income growth of an economy on a balanced growth path by 1.171%-points in poor and 0.152%-points in rich countries (although the latter result is not statistically significant) which requires setting $\delta = 0.367$ in our model. The resulting optimal carbon price increases significantly from $64 to $100/tC. Pricing emissions at this level is still insufficient to be consistent with the ambitions of the Paris Agreement to keep the temperature increase by the end of the century well below 2°C, because temperature increases at the end of the century to 2.2°C and peak temperature is 3.1°C.

Table 1 also reports combinations of the effects discussed so far. Most notably the effects of population growth (1) and damages to productivity growth (4) compound to an initial carbon price of \$259/tC and temperature of 1.8°C in 2100 and a peak temperature of 2.4°C. Adding the effects of TFP growth (2) to the effects of damages to productivity growth (4) does not change the optimal carbon price: it stays at \$100/tC.

Allowing for damages to utility (3) does not alter scenarios (1) or (2) much, except when combined with damages affecting economic growth (4). Here it lessens the lasting effect of productivity damages as these only affect the damage component affecting production. When utility damages (3) are added to damages to the growth rate (4), the optimal SCC falls from \$100/tC to \$90/tC. Finally, adding effects (1), (2), (3) and (4) leads to an optimal SCC of \$232/tC and a temperature increase of 1.8°C at the end of the century.

8.3. Different discount rates for the private and public sector

If the public sector applies a lower discount rate, the policy makers can reproduce the social optimum if they price carbon and simultaneously subsidize saving. If the government discounts the future at 0.1% per year while the private sector maintains the baseline of 1.5% per year, the SCC increases to \$601/tC in 2019 and the required capital income subsidy is 1.4%. If the government cannot subsidize capital income, the second-best policy is to increase the carbon tax by 9% to \$650/tC to compensate the future for the inefficiently low savings rate. Note that the missing capital subsidy also makes saving in the form of fossil fuel less attractive. Oil use is brought forward. Temperature at the end of the century increases by 0.2°C as a result. If the government uses a discount rate of 1%, the optimal carbon tax is \$93/tC, the capital subsidy 0.5%, and the second-best carbon tax \$96/tC.

8.4. Temperature driven by cumulative emissions and damages of Burke et al. (2015)

We can simplify the climate model of GHKT by assuming that cumulative emissions drive temperature, as in section 5. This is equivalent to assuming that $\varphi_L = 1.$ If carbon remains in the atmosphere permanently, the social cost of emitting increases to \$135/tC (see also the discussion of equation (8)). Temperature based on this model increases significantly compared to 2.9°C at the end of the century and 4.6°C at the end of the simulation period or, if temperature is derived from the cumulative emissions approach, by 2.0°C and 3.1°C. Adopting the estimates of Burke et al. (2015) increases the damage parameter to $\chi = 2.66\times 10^{-4}$ and the SCC tenfold to \$1,507/tC. Temperature in increases by 2.0°C by 2100 and at most 2.8°C using the radiative-forcing-based temperature formula in footnote 2 or 1.6°C in 2100 and at most 2.0°C if formula (4') based on cumulative emissions is used. This huge difference with the optimal carbon price under GHKT or Nordhaus damages suggests that much more econometric work is needed on the effects of climate change on economic damages.

Table 2: Sensitivity of optimal policy simulations

	SCC in 2019 ($2010/tC)	Temperature in 2100 (°C)	Temperature max (°C)
Baseline annualized GHKT model	$64	2.4	3.7
(1) social discounting 0.1%: first-best	$601	1.6	2.0
(2) public 0.1% private 1.5%: no subsidy	$650	1.8	2.0
(3) social discounting 1%: first-best	$93	2.2	3.3
(4) public 1% private 1.5%: no subsidy	$95	2.2	3.2
(5) Cumulative emissions	$135	2.9 (2.0)	4.6 (3.1)
(5) & Damages from Burke et al. (2015)	$1,507	2.0 (1.6)	2.8 (2.0)
(6) Cap on temperature, 2°C	$281	1.8	1.9
(7) Cap on temperature, 1.5°C	$1,152	1.5	1.5

Note: Temperature follows from radiative forcing (see in fn. 2); values in (.) follow from equation (4′).

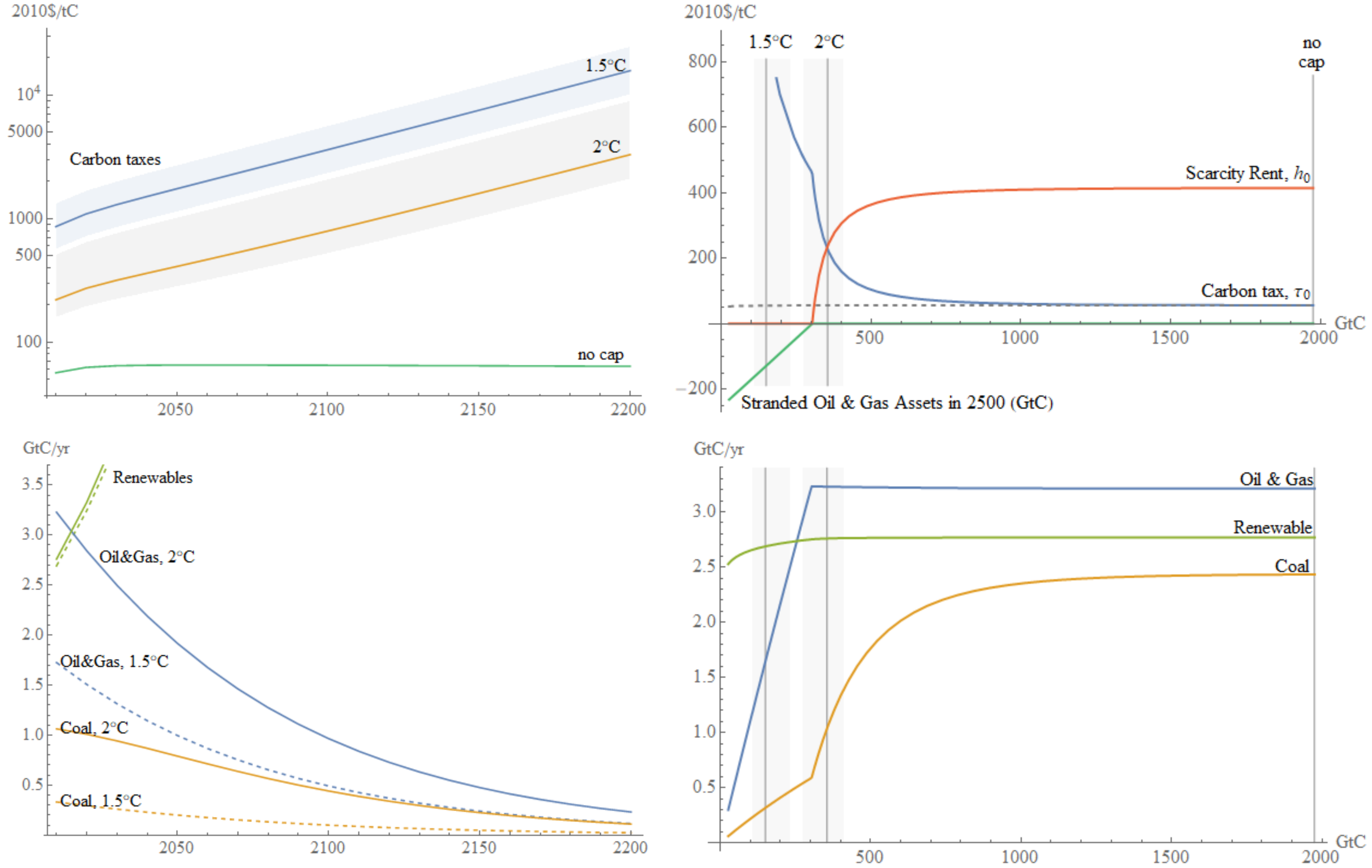
8.5. Cap on temperature or on cumulative emissions and stranded oil and gas reserves

The finding that temperature is driven by cumulative emissions has given rise to temperature targets being expressed in carbon budgets. Using the figures presented in Table 2.2 of the IPCC's Fifth Assessment Report, we can add temperature targets as discussed in section 6 and quantify the 1.5°C and 2°C as a cap on cumulative emissions of 150 GtC and 355 GtC, respectively. This is consistent with our calibration on cumulative emissions for the 2°C target. These constraints on cumulative emissions are based on the baseline GHKT rather than the model of cumulative emissions to make the figures reported below consistent with those displayed in Figures 1 and 2.

The introduction of emission caps increases the optimal carbon tax to $281 and $1,152 for the more stringent target but also augments the carbon tax in that a fraction grows at the interest rate, and therefore each year 1.5% (using $\left(\beta\gamma\right)^{-1}=1.015$) faster than GDP. This permits a low temperature peak around 2100 after which temperature falls. Panel (a) in Figure 3 plots the time profiles of the carbon taxes for temperature caps and compares them to the baseline case of no cap. Since in the baseline calibration, all exogenous growth engines are turned off, the optimal carbon tax is fairly flat in the case of no cap. Following equation (9), however, the carbon taxes increase exponentially if the cap is binding (note the log-scale in the plot). Under the budgets used, 50% of the models used by the IPCC remain within the respective temperature target. The IPCC also reports carbon budgets for the 33% and 66% mark, which are captured by the shaded areas in Figure 3.

Figure 3: Caps on cumulative emissions and temperature

(a) Time paths for 1.5° and 2°C targets *(b) Initial values in 2010 across cumulative emissions*



Coal use is most affected by carbon taxation. As shown in Figure 1, the use of renewable sources and oil and gas only decrease by 0.3% and 6%, while coal use nearly halves from 4 GtC to 2 GtC per annum initially. Given that oil and gas are available without cost (save the intertemporal scarcity rent), all reserves are generally used up eventually, slashing coal use is the primary function of the carbon tax. Similar results hold for carbon taxation under a 2°C cap. The lower panel in Figure 3(a) shows that renewable use and oil and gas extraction barely change, while initial coal use drops to 1 GtC. The more forceful pricing of carbon under the 1.5°C target has a similar effect, again reducing coal to 10% of its business-as-usual level and less than half its use under the 2°C target. While renewable use falls only marginally, pricing emissions at $1,152 per tC also hits oil and gas, cutting emissions from its use in half.

The qualitative difference in oil and gas use between the 1.5°C and 2°C caps is illustrated in Figure 2.b, which plots initial carbon taxes and scarcity rents and long-run stranded oil and gas assets as a function of the carbon budget. In the baseline case of no cap, where carbon is only taxed to avoid marginal damages, cumulative emissions amount to roughly 2,000 GtC. As the budget becomes more smaller, climate policy must become more stringent and the carbon tax gradually deviates from the baseline tax of $56/tC (gray dashed). As the budget approaches the level of 300 GtC the initial carbon tax rises rapidly to above $400/tC, after which it continues to increase at a lower rate. The behaviour of the carbon tax is mirrored by the scarcity rent on oil and gas reserves. Since a lower carbon budget implies less carbon-intensive use, the Hotelling rent declines as the carbon budget increases. Below 500 GtC of cumulative emissions, the decline accelerates before reaching zero at 300 GtC. This is the budget of cumulative emissions, which directly equates supply and use of oil and gas reserves. Lower levels of cumulative emissions keep total use below supply and some oil and gas assets will be abandoned underground (see green slope in Figure 2.(b)). The (natural resource) wealth effects of temperature caps can be measured in terms of write-off due to depressed prices or in the amount of unburnable fossil fuel. At the introduction of a 2°C cap the value of oil and gas reserves drops from $105bn to $62bn. This is $43bn reduction is 128 times the $334mn drop when carbon prices based on baseline damages are introduced. The cap for 1.5°C makes reserves worthless as half (or 128 GtC) will never be used and the scarcity rent is, therefore, zero.

The lower panel of Figure 2.(b) plots energy use across the carbon budget. Oil and gas use is flat as long as the scarcity rent is positive. From equation (7) this implies that the carbon tax is set to offset the decrease in the scarcity rent and only coal use is affected at carbon budgets above 300 GtC, which is consistent with the Herfindahl rule. At lower levels, coal and oil and gas use also drop linearly, although reduction in cumulative use are proportionally larger for oil and gas (due the greater slope) which contradicts the Herfindahl rule and stems from the fact that coal and oil and gas use are regulated by a uniform tax. Carbon taxation also lowers renewable energy use, since the GHKT calibration assumes that all energy types are cooperative factors of production. These reductions are small, reaching at most 10% at very high levels of carbon taxation.

9. Conclusion

The GHKT model (Golosov et al., 2014) provides a simple rule for the social cost of carbon and the price of carbon emissions: it should be proportional to aggregate output and thus grow in line with trend growth. This rule is very intuitive and easy to calculate but has been criticized for the set of limiting assumptions necessary to derive it. Our contribution is to relax some of them and still obtain tractable expressions for the optimal carbon price in the following ways. First, we adopt the model of logarithmic depreciation put forward by Anderson and Brock (2019). This allows us to have a finer time resolution with an annual time scale. Second, we also allow global warming to negatively affect utility and the rate of growth of total factor productivity (as estimated in Dell et al. (2012) and studied in Dietz and Stern (2015)), and also allow more generally for mean reversion in total factor productivity. Third, we follow Von Below (2012), Belfiori (2017, 2018) and Barrage (2018, 2019) and allow policy makers to be more patient than the private sector. This requires the carbon tax to be complemented with a capital subsidy. We also consider the second-best optimal carbon tax in case a capital subsidy is infeasible and discuss its time consistency. We also show that the rule can easily be adapted to allow for positive long-run growth by introducing population growth and labour-augmenting technical progress.

We also offer two simplifications of the GHKT model. Firstly, we replace their carbon cycle and implicit temperature function by a simple linear relationship between temperature and cumulative emissions. Secondly, we replace the calibration to the GHKT model based on the DICE-2013 model by a calibration based on the detailed econometric estimates of Burke et al. (2015). We then use our model to derive a tractable expression for the optimal carbon price under a temperature cap. If the cap bites, this requires adding a term to the unconstrained optimal carbon price that rises at a rate equal to the rate of interest, and will lead to stranded oil and gas reserves. We illustrate all our results with numerical simulations which can be done with a simple programme. We thus think that the extended GHKT model is more realistic and can be used for teaching purposes.

The GHKT model has thus turned out to be very versatile. It can also be extended to allow the cost of fossil fuel extraction to increase as reserves are depleted. This allows for partial exhaustion of reserves. Hence, policy can affect how fast to deplete fossil fuel reserves and how much of the reserves to abandon, introducing considerations of economic obsolescence to the discussion of stranded assets. Moreover, even under stringent climate policy, scarcity rents could remain positive as depleting chap reserves is intertemporal costly. As Golosov et al. (2014) point out, their model can also be extended to have learning by doing in the production of renewable energy. This requires that the carbon price is complemented with a renewable energy subsidy, which is set equal to the present discounted value of all present and future benefits of using one unit of renewable energy for short or the social benefit of learning for short (van der Zwaan et al., 2002; Popp, 2004; Rezai and van der Ploeg, 2017). This leads to a spike in renewable energy subsidies and a gradual ramp in carbon prices (cf. Acemoglu et al. (2012) who show this in a context with directed technical change).[9]

As the discussion of carbon budgets has shown, all energy sources are assumed essential to production in the baseline calibration of the GHKT model and energy can fairly easily be substituted for labour and capital at the macroeconomic level. Empirical evidence points in the opposite direction with energy sources, at least coal and renewable, close substitutes and energy at the macroeconomic level essential to production. The substitutability of energy sources increases in the long run. With our annual calibration, this distinction becomes relevant and could be represented by separate innovations, similar to the ones for renewable energy discussed above.

Although the GHKT model has been extended in various directions, we think we have come to the limits of this popular and tractable general equilibrium model for at least three reasons. First, logarithmic utility implies a unit coefficient of relative risk aversion, a unit coefficient of intertemporal substitution and a unit coefficient of intergenerational inequality aversion which is too restrictive from an ethical and preferences point of

[9] These extensions are discussed for a model where energy types are perfect substitutes in van der Ploeg and Rezai (2016).

view.[10] Furthermore, it implies that stochastic shocks do not lead to a precautionary effect on the optimal price of carbon. Effectively, a higher volatility of the growth rate then does not affect the optimal social cost of carbon at all. Second, the GHKT model is effectively static so that issues of time inconsistency play a limited role. Third, similarly, strategic interactions between countries are severed as the pre-commitment and subgame-perfect outcomes coincide (e.g. Hambel and Kraft, 2020).

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[10] Barrage (2014) and Rezai and van der Ploeg allow for more general preferences numerically, but the latter also derive an approximate, tractable expression for the optimal social carbon price.

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Appendix A: Proof of Proposition 1:

The Lagrangian for this problem is

(A1)

$$\sum_{t=0}^{\infty}\beta^t N_t\{\ln(C_t/N_t)$$

$$-\lambda_t\left[K_{t+1}-\iota\left(\frac{A_tK_t^{\alpha}\left[\kappa_{1t}F_t^{\rho}+\kappa_{2t}\left(A_{2t}L_{2t}\right)^{\rho}+\kappa_{3t}\left(A_{3t}L_{3t}\right)^{\rho}\right]^{\upsilon/\rho}\left(M_t-L_{2t}-L_{3t}\right)^{1-\alpha-\upsilon}-C_t}{K_t}\right)^{\kappa}K_t\right]$$

$$-\psi(E_t^p+E_t^t)-\mu_t(S_{t+1}-S_t+F_t)-\phi_t\left[\ln(A_t)-\delta\ln(A_{t-1})-(1-\delta)\ln(\bar{A})+\chi E_t\right]$$

$$-\left[\eta_t^p(E_{t+1}^p-E_t^p-\varphi_L F_t)+\eta_t^t\left(E_{t+1}^t-\varepsilon E_t^t-\varphi_0(1-\varphi_L)F_t\right)\right]\},$$

where λ_t, μ_t, and ϕ_t denote the shadow values of capital and fossil fuel reserves and the shadow value of the log of total factor productivity at time t, respectively, and the η_t^t and η_t^p denote the shadow disvalues of the transitory and permanent components of atmospheric carbon, respectively.

The first-order conditions for C_t and K_t are $\frac{1}{C_t}=\lambda_t\iota\kappa\left(\frac{Y_t-C_t}{K_t}\right)^{\kappa-1}\equiv x_t$ and $\iota\left(\left(1-\kappa\right)\frac{Y_t-C_t}{K_t}+\alpha\kappa\frac{Y_t}{K_t}\right)\left(\frac{Y_t-C_t}{K_t}\right)^{\kappa-1}\lambda_t=\frac{\lambda_{t-1}}{\beta\gamma}$. In choosing consumption, the marginal benefit of using output for consumption today has to equal the benefit of transferring output into the future, the shadow price of capital. Under geometric depreciation this term is constant, but here depreciation depends on the level of investment and, therefore, consumption. Let $r_t\equiv\iota\left(\left(1-\kappa\right)\frac{Y_t-C_t}{K_t}+\kappa\alpha\frac{Y_t}{K_t}\right)\left(\frac{Y_{t-1}-C_{t-1}}{K_{t-1}}\right)^{\kappa-1}$ be the interest rate in the case of logarithmic depreciation, we have $\frac{C_t}{C_{t-1}}=\beta\gamma r_t$ and thus the growth rate of per-capita consumption at time t must equal the product of the discount factor β and the user cost of capital which consists of the net interest and deprecation

$$\text{(A2)}\quad \frac{C_t/N_t}{C_{t-1}/N_{t-1}}=\beta r_t,\quad\text{where}\quad r_t\equiv\iota\left(\left(1-\kappa\right)\frac{Y_t-C_t}{K_t}+\kappa\alpha\frac{Y_t}{K_t}\right)\left(\frac{Y_{t-1}-C_{t-1}}{K_{t-1}}\right)^{\kappa-1}.$$

In case $\kappa = 1,$ we have $r_t = \alpha \dfrac{Y_t}{K_t}$ and thus per-capita consumption grows at the gross interest as the (geometric) depreciation rate 100% per period. The Euler equation (A2) and (2) form a difference equation in the consumption share which is saddle-point stable, since $\beta\gamma\left(\alpha + (1-\kappa)\right) < 1.$ The stable manifold is given by $C_t = \left(1 - \dfrac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right) Y_t$ in equation (6).

The first-order optimality condition for the log of total factor productivity gives $x_t N_t \dfrac{Y_t}{A_t} - \dfrac{N_t\phi_t}{A_t} + \beta\delta \dfrac{N_{t+1}\phi_{t+1}}{A_t} = 0$ or, using (6), $\left(1 - \dfrac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1} - \phi_t + \beta\gamma\delta\phi_{t+1} = 0.$ The only non-explosive solution of this difference equation gives a constant:

(A3) $$\phi_t = \frac{1}{(1-\beta\gamma\delta)}\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1}.$$

The first-order optimality conditions for the transient and permanent components of the atmospheric carbon stock give $N_t\eta_t^t - \beta\varepsilon N_{t+1}\eta_{t+1}^t - \psi N_t - \chi\phi_t N_t = 0$ and $N_t\eta_t^p - \beta N_{t+1}\eta_{t+1}^p - \psi N_t - \chi\phi_t N_t = 0.$ Using (A3) this boils down to

$$\eta_{t+1}^t = \frac{1}{\beta\gamma\varepsilon}\eta_t^t - \frac{1}{\beta\gamma\varepsilon}(\psi + \chi\phi_t) = \frac{1}{\beta\gamma\varepsilon}\eta_t^t - \frac{1}{\beta\gamma\varepsilon}\left[\psi + \frac{\chi}{(1-\beta\gamma\delta)}\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1}\right]$$ and

$$\eta_{t+1}^p = \frac{1}{\beta\gamma}\eta_t^p - \frac{1}{\beta\gamma}(\psi + \chi\phi_t) = \frac{1}{\beta\gamma}\eta_t^p - \frac{1}{\beta\gamma}\left[\psi + \frac{\chi}{(1-\beta\gamma\delta)}\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1}\right].$$

Since $\beta\gamma < 1,$ this difference equation satisfies the saddle-point condition, so the only non-explosive solution equation are the following positive constants:

(A4) $$\eta_t^t = \frac{1}{1-\beta\gamma\varepsilon}\left[\psi + \frac{\chi}{(1-\beta\gamma\delta)}\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1}\right] > 0,$$
$$\eta_t^p = \frac{1}{1-\beta\gamma}\left[\psi + \frac{\chi}{(1-\beta\gamma\delta)}\left(1 - \frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1}\right] > 0.$$

Hence, using (6) and $\tau_t \equiv \dfrac{\varphi_L \eta_t^p + \varphi_0(1-\varphi_L)\eta_t^t}{x_t}$, we get equation (8) of Proposition 1. The first-order optimality conditions for fossil fuel and renewables give rise to the Kuhn-Tucker conditions stated in (7), where $w_t \equiv (1-\alpha-\upsilon)Y_t / L_t$ and $h_t \equiv \mu_t / x_t$. The first-order optimality condition for reserves is $N_{t-1}\mu_{t-1} = \beta N_t \mu_t$ and recalling that $h_t \equiv \mu_t / x_t$, it follows (using (A2)) that scarcity rents must satisfy equation (9). □

Appendix B: Implementing the social optimum in decentralized market economy

The discussion of the social optimum conceals the underlying market dynamics of our economy. In a competitive market economy households receive wage, capital, and resource income and government transfers, T_t, and choose consumption, investment, and natural resource use to maximize their utility, $\sum_{t=0}^{\infty} (\beta^P)^t N_t \left[\ln(C_t / N_t) - \psi E_t\right]$, subject to their household budget constraint $Z_{t+1} = (1+i_{t+1})(1+\sigma_t)Z_t + w_t L_t + (p_t - \tau_t)F_t - C_t + T_t$, and the depletion constraint $S_{t+1} = S_t - F_t$ or $\sum_{t=0}^{\infty} F_t \le S_0$, where Z_t denotes household assets, w_t the wage rate, τ_t the carbon tax, i_{t+1} the interest rate on assets, and σ_t the capital income subsidy rate. Households take prices, tax and subsidy rates, and transfers as given. The Euler equation for the representative household (A2) becomes $\dfrac{1}{\tilde{\beta}} \dfrac{C_t / N_t}{C_{t-1} / N_{t-1}} = (1+i_t)(1+\sigma_t)$. Households provide natural resources according to $p_t - \tau_t = h_t$, i.e. households need to be indifferent between selling the resource and keeping it under the ground with the Hoteling rent evolving according to the standard Hoteling rule corrected for the subsidy on capital income, $h_t = (1+i_t)(1+\sigma_t)h_{t-1}$. The capital gain on the resource wealth has to match the return a household can make on the capital market.

Production occurs under perfect competition and firms choose production to maximize profits. In the final goods sector, firm hire labour and capital and use energy to maximize profits, $Y_t - w_t L_t - (i_{t+1} + \delta)K_t - p_{1t}F_t - p_{2t}A_{2t}L_{2t} - p_{3t}A_{3t}L_{3t}$, taking the wage rate w_t, the market interest rate i_{t+1}, the market prices for fossil and renewable energy p_{it} for

$i = 1, 2, 3,$ and level of productivity as given. Asset market and final good market equilibrium require $Z_t = K_t$. Capital accumulation follows from (2). The finite resource is provided by households. The firm using the abundant fossil fuel type maximizes its profits $\sum_{t=0}^{\infty} \Delta_t \left[(p_{2t} - \tau_t) A_{2t} L_{2t} - w_t L_{2t} \right]$ with $\Delta_t \equiv \prod_{s=0}^{t} (1 + i_{s+1})^{-1}$, and renewable energy producers $\sum_{t=0}^{\infty} \Delta_t \left[p_{3t} A_{3t} L_{3t} - w_t L_{3t} \right]$, choosing energy output (i.e. labour employed in either sector) and taking the market prices of energy p_{it} (with $i = 2, 3$), the carbon tax τ_t, and the wage rate w_t as given. Since Ricardian debt neutrality holds, there is no loss of generality in assuming that the government balances its books in each period, $T_t = \tau_t (F_t + A_{2t} L_{2t}) - \sigma_t r_t K_t$.

Given that all agents have perfect information and all markets are complete, the first fundamental theorem of welfare economics applies. The first-best optimum for the command economy can thus be sustained in a market economy.

Proposition B.1: *The social optimum is replicated in the decentralized market economy if* $\beta = \beta^P$ *when* $\sigma_t = 0$, *and* $\tau_t = \varphi Y_t$ *following from (8).*

Proof: The optimality conditions of the agents and the equilibrium conditions in the decentralized economy reduce to the social optimality conditions if the capital income subsidy is set to zero, the carbon tax is set to the SCC and the government's net revenue is distributed in lump-sum fashion. □

The government might want to choose to discount the future at a lower rate than private agents, so $\beta > \beta^P$. The first best solution can only be implemented if a capital income subsidy, $\sigma_t > 0$, is introduced. As in the social optimum, the saving/consumption rate is constant, if the capital income subsidy is constant (which is the case for the socially optimal outcome as we show below), and equals

(B1) $$c^{SB}(\beta^P, \sigma) = \left(1 - \frac{\alpha \beta^P \gamma \kappa (1 + \sigma)}{1 - \beta^P \gamma (1 - \kappa)(1 + \sigma)} \right).$$

The government maximizes aggregate utility $\sum_{t=0}^{\infty}\beta^t N_t\left[\ln(C_t/N_t)-\psi E_t\right]$ subject to equations (2)-(5) and the decentralized equilibrium conditions above. With a capital income subsidy, the first-best optimum for the command economy can be sustained in a market economy.

Proof of Proposition 2: *The social optimum is replicated in the decentralized market economy even under differing discount rates if* $\sigma^{FB}=(\beta-\beta^P)/\beta^P>0$ *and* $\tau_t^{FB}=\varphi(\beta)Y_t$.

Proof: The optimality conditions of the agents and the equilibrium conditions in the decentralized economy reduce to the social optimality conditions if the capital income subsidy is set to $(\beta-\beta^P)/\beta^P$ from equation (B1), the carbon tax is set to the SCC and the government's net revenue is distributed in lump-sum fashion. □

Proof of Proposition 3: *If the capital subsidy cannot be positive,* $\sigma=\bar{\sigma}\leq 0$, *the second-best optimal carbon tax is* $\tau_t^{SB}=\varphi^{SB}Y_t>\tau_t^{FB}$ *with* $\varphi^{SB}\equiv\varphi(\beta)c^{SB}(\beta^P,\bar{\sigma})/c(\beta)>\varphi(\beta)$, *where revenue of the taxes on carbon and capital income is rebated as lump-sum transfers. This policy is time consistent.*

Proof: If capital income cannot be subsidies, e.g. due to political resistance, the second-best policy is derived under fixed subsidy rate $\bar{\sigma}\leq 0$ by applying consumption share $c^{SB}(\beta^P,\bar{\sigma})$ when converting equilibrium variables from utils into final good units. We have $c^{SB}(\beta^P,\bar{\sigma})>c(\beta)$, since $\bar{\sigma}\leq 0$ and $\beta>\beta^P$. As shown in equation (A4), the SCC in utils is independent of consumption and income. The change in the consumption rule due to differing discount rates matters only when converting the SCC into units of the consumption good, i.e. $\tau_t^{SB}=\varphi^{SB}Y_t$ with

$$\varphi^{SB}\equiv\frac{c^{SB}(\beta^P,\bar{\sigma})}{c(\beta)}\varphi(\beta)$$

$$=\left(\frac{\varphi_L}{1-\beta\gamma}+\frac{\varphi_0(1-\varphi_L)}{1-\beta\gamma\varepsilon}\right)\left[\psi+\frac{\chi}{(1-\beta\gamma\delta)}\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)\right]\left(1-\frac{\alpha\beta^P\gamma\kappa(1+\sigma)}{1-\beta^P\gamma(1-\kappa)(1+\sigma)}\right).$$

□

Appendix C: Proofs of Propositions 4 and 5

The aim is to solve for the optimal SCC under a cap on cumulative emissions. Damages are based on Burke et al. (2015), total factor productivity follows (5′), and temperature depends on cumulative emissions E_t as in (4′). The Lagrangian for this problem is

(C1)

$$\sum_{t=0}^{\infty}\beta^t N_t\{\ln(C_t/N_t)-\omega_t(E_t-\bar{E})$$

$$-\lambda_t\left[K_{t+1}-\iota\left(\frac{A_tK_t^{\alpha}\left[\kappa_{1t}F_t^{\rho}+\kappa_{2t}\left(A_{2t}L_{2t}\right)^{\rho}+\kappa_{3t}\left(A_{3t}L_{3t}\right)^{\rho}\right]^{\upsilon/\rho}\left(M_t-L_{2t}-L_{3t}\right)^{1-\alpha-\upsilon}-C_t}{K_t}\right)^{\kappa}K_t\right]$$

$$-\psi E_t-\mu_t(S_{t+1}-S_t+F_t)-\phi_t\left[\ln(A_t)-\delta\ln(A_{t-1})-(1-\delta)\ln(\bar{A})+\chi E_t\right]-\eta_t(E_{t+1}-E_t-F_t-A_{2t}L_{2t})\},$$

where η_t denotes the shadow disvalue of cumulative emissions and ω_t the Kuhn-Tucker multiplier for the cap on cumulative emissions at time t. The first-order optimality conditions for C_t, K_t and $\ln(A_t)$ are the same as in Appendix A. They give the Euler equation (A2), consumption $C_t=\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)Y_t$ as in (6), and equation (A3) for ϕ_t.

The optimality conditions for fossil fuel and renewables give rise to the Kuhn-Tucker conditions stated in (7). The optimality condition for reserves is again $N_{t-1}\mu_{t-1}=\beta N_t\mu_t$, so scarcity rents must satisfy equation (9). The first-order condition for E_t is $N_t\eta_t-\beta N_{t+1}\eta_{t+1}-\psi N_t-\chi\phi_t N_t-\omega_t N_t=0$. Using (A3) this condition boils down to

(C2)

$$\begin{aligned}\eta_{t+1}&=\frac{1}{\beta\gamma}\eta_t-\frac{1}{\beta\gamma}(\psi+\chi\phi_t+\omega_t)\\&=\frac{1}{\beta\gamma}\eta_t-\frac{1}{\beta\gamma}\left[\psi+\frac{\chi}{(1-\beta\gamma\delta)}\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1}+\omega_t\right].\end{aligned}$$

The complementary slackness conditions are $E_t\le\bar{E}$ and $\omega_t\ge 0$. It is optimal to asymptotically deplete fossil fuel reserves from $S_0>0$ down to $S_t\to\bar{S}\ge 0$ as $t\to\infty$ with $S_0-\bar{S}\le\bar{E}$ and the remainder of the carbon budget used up by coal. The cumulative emissions constraint never bites, $E_t < \bar{E}, \forall t \geq 0,$ but with $\lim_{t\to\infty}\left(\sum_{t=1}^{t}(F_t + A_2 L_{2t})\right) = \bar{E} > 0.$

Hence, $\omega_t = 0, \forall t \geq 0.$ Define $\bar{\eta} \equiv \frac{1}{1-\beta\gamma}\left[\psi + \frac{\chi}{1-\beta\gamma\delta}\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)^{-1}\right] > 0$ and write

(C2) as $\beta\gamma\eta_{t+1} = \eta_t - (1-\beta\gamma)\bar{\eta}.$ This gives $\eta_t = (\beta\gamma)^{-(t-1)}\eta_1 - (1-\beta\gamma)\bar{\eta}\sum_{s=1}^{t-1}(\beta\gamma)^{-s}.$ It can easily be verified that the general solution to (C2) equals

(C3) $\quad \eta_t = \bar{\eta} + \varpi(\beta\gamma)^{-(t-1)},$

where $\varpi \geq 0$ is a constant that is to be determined and $\eta_t = \bar{\eta}$ is the saddle-point solution to (C2). Using (6) and $\tau_t = \eta_t C_t = \eta_t\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)Y_t,$ we get (9) in Proposition 5 or

(C4) $\quad \tau_t = \varphi(\beta)Y_t + \Delta(\beta\gamma)^{-(t-1)}Y_t, \quad \forall t \geq 0,$

where $\varphi(\beta) \equiv \frac{1}{1-\beta\gamma}\left[\psi\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right) + \frac{\chi}{1-\beta\gamma\delta}\right]$ and $\Delta = \varpi\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right).$

The constant Δ is zero if the optimal trajectory does not fully exhaust the carbon budget, $\lim_{t\to\infty} E_t < \bar{E}.$ This case also pertains if there is no temperature cap and thus corresponds to Proposition 4. However, $\Delta > 0$ if the optimal trajectory fully exhausts the carbon budget, $\lim_{t\to\infty} E_t = \bar{E}.$ In that case, the constant Δ must be chosen so that the carbon budget is exactly exhausted. This completes the proof of Proposition 5.

Appendix D: Details of Calibration

Our model allows for damages to utility and productivity growth drawing on previous studies by Barrage (2017) and Dell et al. (2012). The adaptation to our model formulation is described below.

Annual time scale: Adjustment of the time steps from a decadal to an annual scale requires changes in the time-sensitive parameters β, φ_0, $\varphi = 1-\varepsilon,$ and the levels and growth rates of A, A_2 and A_3. Converting β and the growth rates of A_2, and A_3 is easily achieved by taking decadal value to the power 1/10. Productivity levels are adjusted by dividing by 10. (Adjusting A is equivalent to adjusting Y_0). To recalibrate the parameters describing the carbon cycle, we use the two calibrations points used by GHKT, i.e. that the temporary component of atmospheric carbon has a mean lifetime of 300 years and that half of emissions is removed after 30 years, or $\frac{1}{2}=(1-\varphi)^{30t}$ and $\frac{1}{2}=\varphi_L+(1-\varphi_L)\varphi_0(1-\varphi)^{3t-1}$ where $t=1$ on a decadal and $t=10$ on an annual grid. With $t=10$, we have $\varphi_0=0.401$ and $\varphi=1-\varepsilon=0.0023078$.

Utility damages: We follow Barrage (2017) who carefully attributed the aggregate damage function of DICE (which was used for the calibration of damages in GHKT) to utility and production damages, finding that 26% of damages should be attributed to utility damages at 2.5°C. To (re-)calibrate our production and utility damage parameters χ and ψ we assume that damages affect the level of productivity only (i.e. δ = 0). With ϖ = 0.76, the share of GHKT damages attributable to production, we calibrate the new damage parameter for production damage $\tilde{\chi}$ according to

$$\left(1-e^{-\tilde{\chi}(E_{2.5^\circ C}-\bar{E})}\right)=\varpi\left(1-e^{-\chi^{GHKT}(E_{2.5^\circ C}-\bar{E})}\right).$$

Remaining damages $(1-\varpi)\left(1-e^{-\chi^{GHKT}E_{2.5^\circ C}}\right)$ are damages to utility. So we have for utility damages, converted into dollars (by dividing by the marginal utility of consumption and expressed as a fraction of Y_t:

$$\frac{\psi(E_{2.5°C}-E_{0°C})}{u'(C)Y_t}=\psi(E_{2.5°C}-E_{0°C})\left(1-\frac{\alpha\beta\gamma\kappa}{1-\beta\gamma(1-\kappa)}\right)=(1-\varpi)\left(1-e^{-\chi^{GHKT}E_{2.5°C}}\right).$$

Using the standard parameter values and the stock level for the pre-industrial and 2.5°C carbon stock (581 GtC and 1035 GtC, respectively), we solve both equations for $\chi = 1.806\ 10^{-5}$ and $\psi = 7.376\ 10^{-6}$.

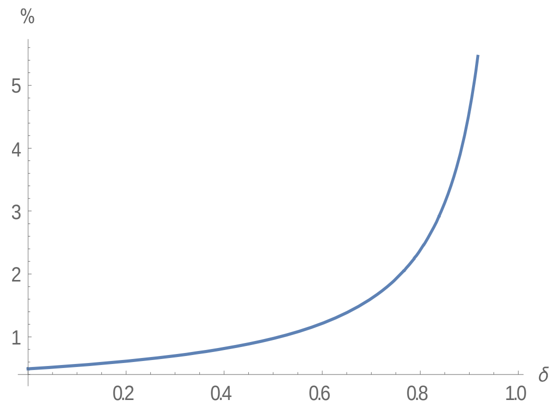
Productivity damages: We calibrate the growth damage, the parameter δ, to the empirical findings of Dell, Jones, and Olken (DJO, 2012) using their linear long-run relationship between income per capita growth and temperature changes (see equation A1.6 in their online appendix). If the economy is on a balanced growth path and temperature increases by 1°C, they find that per capita income growth falls by 1.171

percentage points in poor and 0.152 pp. in rich countries (although the latter result is not statistically significant). To calibrate our parameter δ we derive an analogue of this long-run relationship in our model. If we assume that damages are time-invariant and that initial state is without damages, we have

$$A_t = e^{-\chi(E-\bar{E})} A_{t-1}^{\delta} \bar{A}^{1-\delta} = \bar{A}\left(e^{-\chi(E-\bar{E})}\right)^{(1-\delta^t)/(1-\delta)} \text{ and } A_\infty = \bar{A}\left(e^{-\chi(E-\bar{E})}\right)^{1/(1-\delta)}$$

DJO use a linear relationship between income p.c. growth rate and temperature. To relate to this formulation, we need to convert a 1°C increase at current concentration (with implies an equilibrium temperature increase of 1.3°C) into an increase in concentration using the equilibrium relationship $T_{eq} = 3\ln(E/581)/\ln(2)$. A 1°C increase corresponds to a concentration of atmospheric carbon of 1010 GtC (or an equilibrium increase of 2.3°C). Figure D.1 plots the difference in percentage losses in aggregate productivity in the long run, $100(1-A_\infty)$, as a function of δ.

Figure D.1: Losses in aggregate productivity versus δ



DJO find that in poor countries, allowing for up to 10 annual lags, an increase of 1°C leads to a long-run reduction in the annual income per capita growth rate of 1.171 percentage points. To convert this into a growth rate of TFP, we multiply by the labour share which equals $1 - \alpha - v = 0.66$. This gives a calibrated $\delta = 0.367$.