Infinite Toeplitz and Laurent matrices with localized impurities

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Abstract

This paper is concerned with the change of the spectra of infinite Toeplitz and Laurent matrices under perturbations in a prescribed finite set of sites. The main result says that the spectrum of a Toeplitz matrix with a non-constant rational symbol is not affected by small localized impurities, while such impurities can nevertheless enlarge the spectrum of the corresponding Laurent matrix. We also study the spectra that may emerge when randomly perturbing Toeplitz or Laurent matrices in a randomly chosen single site.

1 Introduction and main results

Given a complex-valued continuous function a on the complex unit circle $\mathbf{T}$, $a \in \mathcal{C}$, we consider the matrices

$$T(a) = (a_{j-k})_{j,k=1}^{\infty} \quad \text{and} \quad L(a) = (a_{j-k})_{j,k=-\infty}^{\infty},$$

where

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{-\mathrm{i}k\theta}\ \mathrm{d}\theta, \quad k \in \mathbf{Z}.$$

The matrices $T(a)$ and $L(a)$ are referred to as the Toeplitz matrix and the Laurent matrix with the symbol a, respectively. These matrices induce bounded operators on $l^2(\mathbf{N})$ and $l^2(\mathbf{Z})$, respectively.

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The spectrum $\operatorname{sp} A$ of a bounded linear operator is defined as usual, that is, as the set of all $\lambda \in \mathbf{C}$ for which $A - \lambda I$ is not invertible. It is well known (see, e.g., [5] or [10]) that

$$\operatorname{sp} L(a) = a(\mathbf{T}), \tag{1}$$
$$\operatorname{sp} T(a) = a(\mathbf{T}) \cup \{\lambda \in \mathbf{C} \setminus a(\mathbf{T}) : \operatorname{wind}(a, \lambda) \neq 0\}, \tag{2}$$

where $\operatorname{wind}(a, \lambda)$ is the winding number of a on the positively oriented unit circle with respect to λ, and that

$$\dim \operatorname{Ker}(T(a) - \lambda I) = \max(0, -\operatorname{wind}(a, \lambda)) \quad \text{for} \quad \lambda \in \operatorname{sp} T(a) \setminus a(\mathbf{T}).$$

Moreover, if K is any compact operator, then $\operatorname{sp}(L(a) + K) \supset \operatorname{sp} L(a)$ and $\operatorname{sp}(T(a) + K) \supset \operatorname{sp} T(a)$; see [5,6,10].

We study the extent to which the spectra of Toeplitz and Laurent operators can increase as the result of (compact) perturbations that are either localized in a given site (j, k) of the matrix or in the square of all sites (j, k) with $j, k \in \{1, \ldots, m\}$. Our approach is based on the consideration of structured pseudospectra, and this paper was essentially stimulated by the works [9,19].

Structured perturbation problems arise in non-Hermitian quantum mechanics [8], linear systems theory [12,13], and small world networks [16]. The questions studied here, when asked of finite dimensional Toeplitz matrices, also have important applications to the theory of discretized initial–boundary value problems (see, e.g., [2]). We investigate such finite dimensional questions in our forthcoming paper [4].

For a study of instances where finite rank perturbations induce isolated eigenvalues in the spectra of Laurent matrices, see [1].

Let A be a matrix that generates a bounded operator on $l^2(\mathbf{N})$ or $l^2(\mathbf{Z})$. Given a subset S of $\mathbf{N} \times \mathbf{N}$ or $\mathbf{Z} \times \mathbf{Z}$ and a set $\Omega \subset \mathbf{C}$, we define $\operatorname{sp}_\Omega^S A$ as the union of the spectra $\operatorname{sp}(A + K)$ where K ranges over all matrices $K = (K_{jk})$ such that $K_{jk} = 0$ for $(j, k) \notin S$ and $K_{jk} \in \Omega$ for $(j, k) \in S$. We focus our attention on two (in a sense, extreme) cases:

$$S = \{1, \ldots, m\} \times \{1, \ldots, m\}, \quad \Omega = \varepsilon\overline{\mathbf{D}} := \{z \in \mathbf{C} : |z| \leq \varepsilon\}; \tag{3}$$
$$S = \{(j, k)\}, \quad \Omega = [-\varepsilon, \varepsilon] := \{x \in \mathbf{R} : |x| \leq \varepsilon\}. \tag{4}$$

In the settings (3) and (4), we abbreviate $\operatorname{sp}_\Omega^S A$ to $\operatorname{sp}_{\varepsilon\overline{\mathbf{D}}}^m A$ and $\operatorname{sp}_{[-\varepsilon,\varepsilon]}^{(j,k)} A$, respectively.

Suppose the symbol a is constant, $a(t) = a_0$ for all $t \in \mathbf{T}$. Then, obviously,

$$\operatorname{sp}^m_{\varepsilon\overline{\mathbf{D}}} T(a) = a_0 + \varepsilon\overline{\mathbf{D}}, \quad \operatorname{sp} T(a) = \{a_0\}.$$

Thus, $\operatorname{sp}^m_{\varepsilon\overline{\mathbf{D}}} T(a)$ is strictly larger than $\operatorname{sp} T(a)$ for every $\varepsilon > 0$. Let $\mathcal{R}$ be the set of all rational functions without poles on $\mathbf{T}$. We can think of $\mathcal{R}$ as a subset of $\mathcal{C}$, the functions continuous on $\mathbf{T}$. Note, in particular, that Toeplitz band matrices have symbols that are trigonometric polynomials, and thus in $\mathcal{R}$.

Theorem 1.1 *If $a \in \mathcal{R}$ is not constant, then there exists an $\varepsilon_0 > 0$, depending on a and m, such that*

$$\operatorname{sp}^m_{\varepsilon\overline{\mathbf{D}}} T(a) = \operatorname{sp} T(a) \text{ for all } \varepsilon \in (0, \varepsilon_0).$$

Thus, the spectrum of a non-scalar rationally-generated Toeplitz matrix is not affected by sufficiently small localized impurities. Of course, this robustness of the spectrum is caused by the rather rigid structure of Toeplitz matrices, which is in the case at hand not destroyed outside the upper-left $m \times m$ block. The phenomenon uncovered by Theorem 1.1 is nevertheless intriguing in view of the following two observations.

Proposition 1.2 *There is a dense subset $\mathcal{M}$ of $\mathcal{C}$ such that if $a \in \mathcal{M}$, then $\operatorname{sp}^m_{\varepsilon\overline{\mathbf{D}}} T(a)$ is strictly larger than $\operatorname{sp} T(a)$ for every $\varepsilon > 0$.*

Proposition 1.3 *Let $a \in \mathcal{C}$. Then $\operatorname{sp}^m_{\varepsilon\overline{\mathbf{D}}} L(a) = \operatorname{sp} L(a)$ for all sufficiently small $\varepsilon > 0$ if and only if*

$$\max_{|k| \le m-1} \sup_{\lambda \notin a(\mathbf{T})} \left| \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}k\theta}}{a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda} \, \mathrm{d}\theta \right| < \infty. \tag{5}$$

If $a \in \mathcal{R}$ is real-valued, then $\operatorname{sp}^m_{\varepsilon\overline{\mathbf{D}}} L(a)$ is strictly larger than $\operatorname{sp} L(a)$ for every $\varepsilon > 0$.

Proposition 1.2 tells us that Theorem 1.1 is not a consequence of some sort of general perturbation theory. The failure of Theorem 1.1 for Laurent matrices (Proposition 1.3) is undoubtedly connected with the circumstance that Laurent matrices have no "center", so that the impurity is in fact not "localized" but may be viewed as "drifting" like an ice floe.

Our proofs of the above results are based on comparing $\operatorname{sp}^m_{\varepsilon\overline{\mathbf{D}}} A$ with the set

$$\widetilde{\operatorname{sp}}^m_{\varepsilon\mathbf{D}} A = \bigcup_{\|K\| < \varepsilon} \operatorname{sp}(A + P_m K P_m), \tag{6}$$

where P_m is the projection on $l^2(\mathbf{N})$ or $l^2(\mathbf{Z})$ that sends a sequence $x = \{x_k\}$ to the sequence given by

$$(P_m x)_k = \begin{cases} x_k \text{ if } k \in \{1, \ldots, m\}; \\ 0 \text{ otherwise,} \end{cases}$$

and where $\|K\|$ is the norm of K as an operator on $l^2(\mathbf{N})$ or $l^2(\mathbf{Z})$. Sets like (6) are called *structured pseudospectra* or *spectral value sets*; see, e.g., [9,12,13]. Theory and examples of conventional (unstructured) pseudospectra are discussed in [17,18]. For applications of pseudospectral theory to Toeplitz operators, see [3,5,14,15]. A recent result by Gallestey, Hinrichsen, and Pritchard [9] implies that

$$\widetilde{\mathrm{sp}}^m_{\varepsilon\mathbf{D}} A = \mathrm{sp}\, A \cup \{\lambda \in \mathbf{C} \setminus \mathrm{sp}\, A : \|P_m(A - \lambda I)^{-1} P_m\| > 1/\varepsilon\}. \tag{7}$$

Equality (7) reduces the calculation of structured pseudospectra of infinite dimensional Toeplitz and Laurent matrices to the estimation of finitely many entries of the resolvent of such matrices.

For $m \geq 2$, there is no simple analogue of (7) if the values of the perturbations (impurities) are restricted to the real line (see [13]). However, for perturbations in a single site it is elementary to see that if $0 \in \Omega$, then

$$\mathrm{sp}^{(j,k)}_{\Omega} A = \mathrm{sp}\, A \cup \{\lambda \notin \mathrm{sp}\, A : 1 + ((A - \lambda I)^{-1})_{kj}\omega = 0 \text{ for some } \omega \in \Omega\}. \tag{8}$$

In particular,

$$\mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} A = \mathrm{sp}\, A \cup \{\lambda \notin \mathrm{sp}\, A : ((A - \lambda I)^{-1})_{kj} \in (-\infty, -1/\varepsilon] \cup [1/\varepsilon, \infty)\}.$$

This reveals that unless the (k, j) entry $((A - \lambda I)^{-1})_{kj}$ of the resolvent matrix $(A - \lambda I)^{-1}$ is a nonzero constant on some open component of $\mathbf{C} \setminus \mathrm{sp}\, A$, the intersection of $\mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} A$ with this open component is either empty or an at most countable union of analytic arcs. Such arcs arise in our examples and are visualized in Figures 1, 3, 6, 10, and 11.

Once the sets $\mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} A$ are available, we have the sets

$$\bigcup_{(j,k)\in S} \mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} A \tag{9}$$

at our disposal. Clearly, (9) is the set of all eigenvalues that may emerge by perturbing A in a single randomly chosen site in S by a random number supported on $[-\varepsilon, \varepsilon]$.

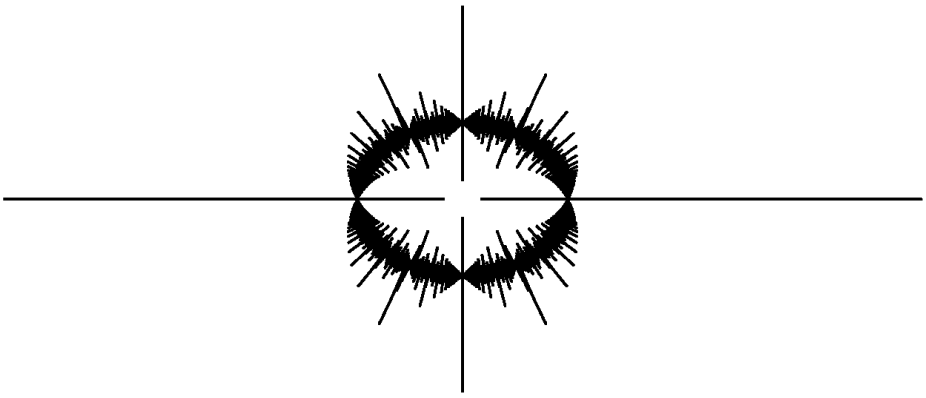


Fig. 1. The set $\cup_{(j,k)\in S} \operatorname{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a)$ for $S = \{(j,k) \in \mathbf{Z} \times \mathbf{Z} : j - k \neq 1\}$ and $a(t) = t + \alpha^2 t^{-1}$ with $\alpha = \frac{2}{5}$ and $\varepsilon = 5$.

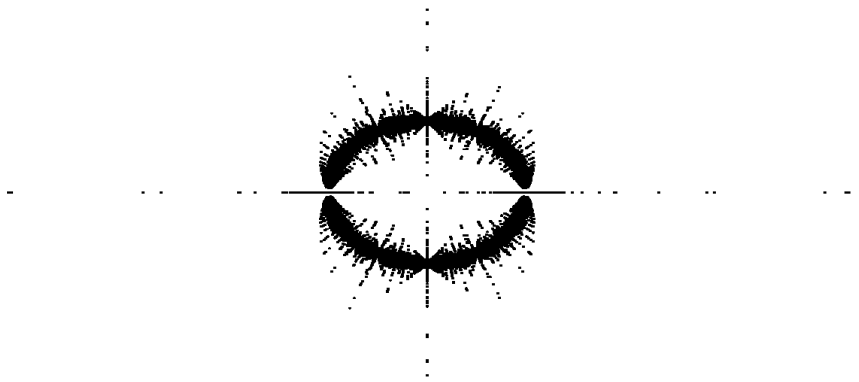


Fig. 2. Computed eigenvalues of 1000 circulant matrices of dimension 50, each perturbed in a single random entry by a random number uniformly distributed in $[-5, 5]$.

Let $a(t) = t + \alpha^2 t^{-1}$ $(t \in \mathbf{T})$ with $\alpha \in (0, 1]$. The range $a(\mathbf{T})$ is the ellipse

$$\left\{ x + \mathrm{i}y \in \mathbf{C} : \frac{x^2}{(1+\alpha^2)^2} + \frac{y^2}{(1-\alpha^2)^2} = 1 \right\},$$

which collapses to the line segment $[-2, 2]$ for $\alpha = 1$. Let the sets E_+ and E_- denote the points in the interior and exterior of this ellipse, respectively. From (1) and (2) we deduce that

$$\operatorname{sp} L(a) = a(\mathbf{T}) \quad \text{and} \quad \operatorname{sp} T(a) = a(\mathbf{T}) \cup E_+.$$

The sets

$$\bigcup_{(j,k)\in\mathbf{Z}\times\mathbf{Z}} \operatorname{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a) \quad \text{and} \quad \bigcup_{(j,k)\in\mathbf{N}\times\mathbf{N}} \operatorname{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} T(a)$$

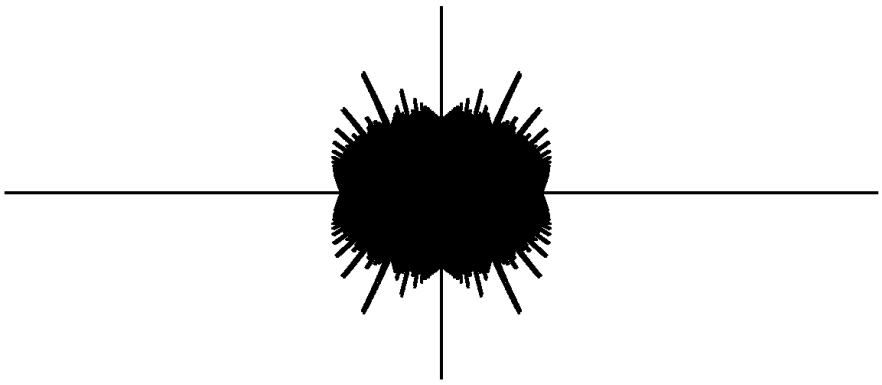


Fig. 3. The set $\cup_{(j,k)\in\mathbf{N}\times\mathbf{N}} \mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} T(a)$ for $a(t) = t + \alpha^2 t^{-1}$ with $\alpha = \frac{2}{5}$ and $\varepsilon = 5$.

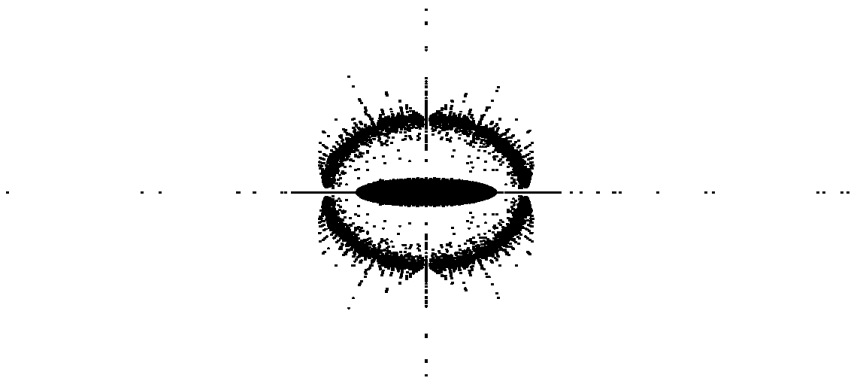


Fig. 4. Computed eigenvalues of 1000 Toeplitz matrices of dimension 50, each perturbed in a single random entry by a random number uniformly distributed in $[-5, 5]$.

are shown in Figures 1 through 4. (Actually, Figure 1 omits perturbations to entries on the first subdiagonal, i.e., the entries $(j+1, j)$ for $j \in \mathbf{Z}$. If these entries are also perturbed, the entire interior of the ellipse in Figure 1 would be filled.)

The paper is organized as follows. In Section 2 we prove Propositions 1.2 and 1.3 and discuss some instructive examples that explain Figures 1 and 3. The discussion of Figures 2 and 4, which concern large finite matrices, will be the subject of [4]. Section 3 contains the proof of Theorem 1.1.

2 Examples

The operators considered in Examples 2.1 and 2.2 have attracted considerable attention in the field of non-Hermitian quantum mechanics, where random perturbations from some set Ω are added in a set of sites S, generally taken to

be the diagonal [7,8,11,19]. These examples can be viewed as generalizations of the "single impurity" model of Feinberg and Zee [8], and are related to the spectral analysis of Davies [7]. The following example (taking $\alpha = 1$ and $\varepsilon = 1$ below) has also been studied recently by Gilbert Strang and his students in their investigation of small world networks [16].

In what follows we write $L^{-1}(\cdot)$ and $T^{-1}(\cdot)$ for $(L(\cdot))^{-1}$ and $(T(\cdot))^{-1}$.

We begin with Laurent matrices. If $\lambda \in \mathbf{C} \setminus a(\mathbf{T})$, then $L^{-1}(a-\lambda)$ equals $L((a-\lambda)^{-1})$, and hence $(L^{-1}(a-\lambda))_{kj}$ is nothing but the $(k-j)$th Fourier coefficient of $(a-\lambda)^{-1}$.

Example 2.1 Let $a(t) = t + \alpha^2 t^{-1}$ $(t \in \mathbf{T})$ with $\alpha \in (0,1]$. For $\varrho > 0$, put $a_\varrho(t) = \varrho t + \alpha^2 \varrho^{-1} t^{-1}$. It is readily verified that

$$[-2\alpha, 2\alpha] = a_\alpha(\mathbf{T}), \quad E_+ = \bigcup_{\varrho \in [\alpha,1)} a_\varrho(\mathbf{T}), \quad E_- = \bigcup_{\varrho \in (1,\infty)} a_\varrho(\mathbf{T}) \tag{10}$$

(where $E_+ = \emptyset$ for $\alpha = 1$). Fix $\lambda \in \mathbf{C} \setminus a(\mathbf{T})$. Then define z_1 and z_2 according to the factorization

$$a(t) - \lambda = t^{-1}(t^2 - \lambda t + \alpha^2) = t^{-1}(t - z_1)(t - z_2). \tag{11}$$

Suppose first that $\alpha \in (0,1)$ and $\lambda \in E_+$. Then $|z_1| < 1$ and $|z_2| < 1$ and thus

$$\frac{1}{a(t)-\lambda} = \frac{1}{t} + \frac{1}{t^2}(z_1 + z_2) + \frac{1}{t^3}(z_1^2 + z_1 z_2 + z_2^2) + \cdots. \tag{12}$$

It follows from (8) that $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(j,k)} L(a)$ and $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(j,k)} L(a)$ have no points in E_+ whenever $j \le k$. That is, perturbations of any magnitude on or above the main diagonal cannot add components to the spectrum in the interior of the ellipse. By virtue of (10), we can represent λ in the form

$$\lambda = \varrho\, \mathrm{e}^{\mathrm{i}\theta} + \alpha^2 \varrho^{-1}\, \mathrm{e}^{-\mathrm{i}\theta} \tag{13}$$

with $\varrho \in [\alpha, 1)$ and $\theta \in [0, 2\pi)$, and (11) is clearly satisfied by

$$z_1 = \alpha^2 \varrho^{-1}\, \mathrm{e}^{-\mathrm{i}\theta} \quad \text{and} \quad z_2 = \varrho\, \mathrm{e}^{\mathrm{i}\theta}. \tag{14}$$

One can use (8), (12), (13) and (14) to compute the sets $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(j,k)} L(a) \cap E_+$ and $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(j,k)} L(a) \cap E_+$ for $j > k$. For example,

$$\mathrm{sp}^{(2,1)}_{[-\varepsilon,\varepsilon]} L(a) \cap E_+ = \begin{cases} \emptyset \text{ if } \varepsilon < 1, \\ E_+ \text{ if } \varepsilon \geq 1; \end{cases}$$

$$\mathrm{sp}^{(3,1)}_{[-\varepsilon,\varepsilon]} L(a) \cap E_+ = \begin{cases} \emptyset \text{ if } \varepsilon \leq 1/(1+\alpha^2), \\ (-1\text{–}\alpha^2, -1/\varepsilon] \cup [1/\varepsilon, 1\text{+}\alpha^2) \text{ if } \varepsilon > 1/(1+\alpha^2). \end{cases}$$

The portion of Figure 1 located inside the ellipse shows the intersection of E_+ and $\cup_{(j,k)\in S} \mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a)$ with $S = \{(j,k) \in \mathbf{Z} \times \mathbf{Z} : j - k \neq 1\}$ for a particular choice of ε and α.

Now assume that $\alpha \in (0,1]$ and $\lambda \in E_-$. Write λ in the form (13) with $\varrho \in (1,\infty)$ and $\theta \in [0, 2\pi)$, and define z_1 and z_2 by (14). Then (11) holds, and since $|z_1| < 1$ and $|z_2| > 1$, we get

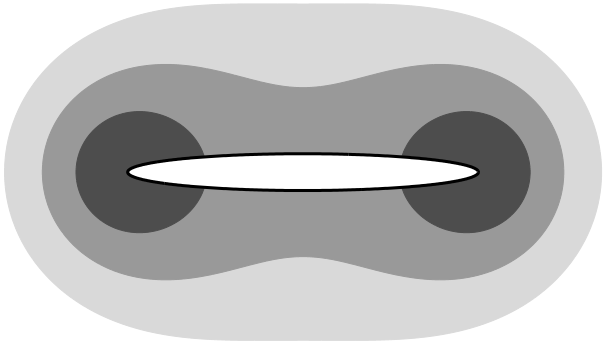
$$\begin{aligned} \frac{1}{a(t) - \lambda} &= -\frac{1}{z_2}\left(1 + \frac{z_1}{t} + \frac{z_1^2}{t^2} + \cdots\right)\left(1 + \frac{t}{z_2} + \frac{t^2}{z_2^2} + \cdots\right) \\ &= \frac{1}{z_1 - z_2} + \sum_{n=1}^{\infty} \frac{1}{t^n} \frac{z_1^n}{z_1 - z_2} + \sum_{n=1}^{\infty} t^n \frac{1}{z_1 - z_2} \frac{1}{z_2^n}. \end{aligned} \tag{15}$$

We can now compute $\mathrm{sp}^{(j,k)}_{\varepsilon\overline{\mathbf{D}}} L(a) \cap E_-$ and $\mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a) \cap E_-$ using (8), (13), (14) and (15). For instance,

$$\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}} L(a) \cap E_- = \begin{cases} \emptyset \text{ if } \varepsilon \leq 1 - \alpha^2, \\ \{\lambda \in E_- : |\lambda - 2\alpha|\,|\lambda + 2\alpha| \leq \varepsilon^2\} \text{ if } \varepsilon > 1 - \alpha^2; \end{cases}$$

$$\mathrm{sp}^{(1,1)}_{[-\varepsilon,\varepsilon]} L(a) \cap E_- = \begin{cases} \emptyset \text{ if } \varepsilon \leq 1 - \alpha^2, \\ [-\sqrt{\varepsilon^2 + 4\alpha^2}, -1 - \alpha^2) \cup (1 + \alpha^2, \sqrt{\varepsilon^2 + 4\alpha^2}\,] \\ \qquad\qquad \text{if } \varepsilon > 1 - \alpha^2. \end{cases}$$

Thus, for $\varepsilon > 1 - \alpha^2$, $\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}} L(a)$ is the union of the ellipse $a(\mathbf{T})$ and two eventually merging "buds", while $\mathrm{sp}^{(1,1)}_{[-\varepsilon,\varepsilon]} L(a)$ is the union of $a(\mathbf{T})$ and two "wings" (see Figure 5). The intersection of E_- and $\cup_{(j,k)\in\mathbf{Z}\times\mathbf{Z}} \mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a)$ is the part of Figure 1 lying outside the ellipse. Figure 6 contains some more information about the structure of Figure 1, and Figure 2 demonstrates that similar behavior is observed for finite dimensional problems. ■



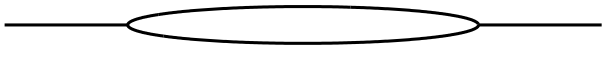


Fig. 5. The sets $\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}} L(a)$ and $\mathrm{sp}^{(1,1)}_{[-\varepsilon,\varepsilon]} L(a)$ for $a(t) = t + \alpha^2 t^{-1}$ with $\alpha = \frac{9}{10}$. (In the top plot, black corresponds to $\varepsilon = 0$, dark gray to $\varepsilon = \frac{3}{2}$, etc.)

Proof of Proposition 1.3 If $A = (A_{jk})_{j,k=1}^m$ is an $m \times m$ matrix, then

$$\max_{j,k} |A_{jk}| \le \|A\| \le m \max_{j,k} |A_{jk}|, \tag{16}$$

where $\|A\|$ is the norm of A as an operator on $\mathbf{C}^n$ with the l^2 norm. From (6) we therefore see that $\mathrm{sp}^m_{\varepsilon\overline{\mathbf{D}}} L(a) = \mathrm{sp}\, L(a)$ for all sufficiently small $\varepsilon > 0$ if and only if $\widetilde{\mathrm{sp}}^m_{\varepsilon\mathbf{D}} L(a) = \mathrm{sp}\, L(a)$ for all $\varepsilon > 0$ small enough, which, by (7), is equivalent to the condition

$$\{\lambda \notin a(\mathbf{T}) : \|P_m L((a-\lambda)^{-1}) P_m\| > 1/\varepsilon\} = \emptyset \tag{17}$$

for all sufficiently small $\varepsilon > 0$. Again taking into account (16) we arrive at the conclusion that (17) holds exactly if

$$\sup_{\lambda \notin a(\mathbf{T})} |((a-\lambda)^{-1})_{k-j}| < \infty$$

for all $k, j \in \{1, \dots, m\}$, which is obviously equivalent to (5).

Now suppose $a \in \mathcal{R}$ is real-valued and $a(\mathbf{T}) = [\mu, \nu]$. If a is constant, then $\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}} L(a)$ is clearly strictly larger than $\mathrm{sp}\, L(a)$ for all $\varepsilon > 0$. So assume that a is not constant. Then $a - \mu$ has a finite number of zeros $\mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_N}$ on $\mathbf{T}$

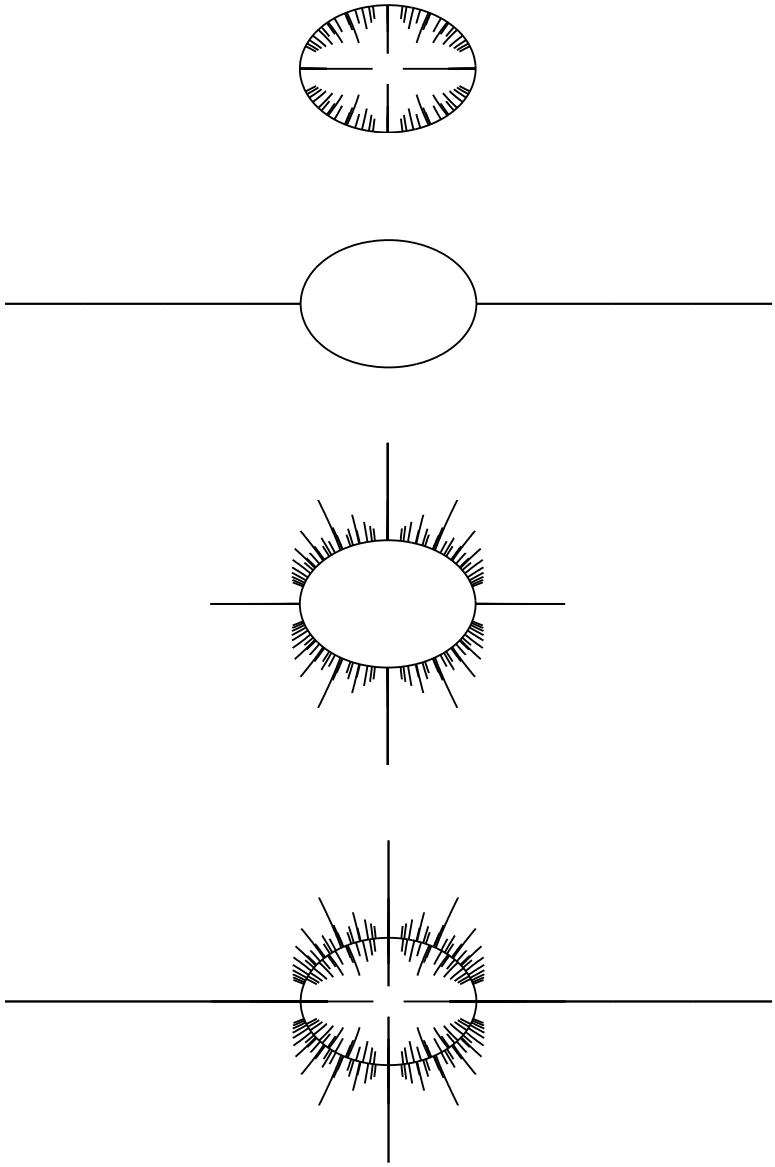


Fig. 6. $\mathrm{sp}^S_{[-\varepsilon,\varepsilon]} L(a)$ for $\varepsilon = 5$, $a(t) = t + \alpha^2 t^{-1}$, and $\alpha = 2/5$. Here S consists of: a) the second through tenth subdiagonals; b) the main diagonal; c) the first ten superdiagonals; d) the union of a), b), and c).

and the orders $2\beta_1, \ldots, 2\beta_N$ of these zeros are even natural numbers,

$$a(\mathrm{e}^{\mathrm{i}\theta}) - \mu = \prod_{j=1}^{N} (\theta - \theta_j)^{2\beta_j} b(\theta), \quad \theta \in [0, 2\pi),$$

for some real-valued function $b \in \mathcal{C}[0, 2\pi]$ that has no zeros. For $y \in (0, 1)$, put $\Delta_y = \{\theta \in [0, 2\pi) : a(\mathrm{e}^{\mathrm{i}\theta}) < \mu + y\}$. It is not difficult to show that there

is a constant $c > 0$ independent of y such that

$$|\Delta_y| \geq c y^{1/\min(2\beta_1,\ldots,2\beta_N)} \geq c\sqrt{y},$$

where $|\Delta_y|$ denotes the length of Δ_y. For $\lambda_n = \mu - 1/n$ we therefore obtain

$$\left| \int_0^{2\pi} \frac{\mathrm{d}\theta}{a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda_n} \right| = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda_n} \geq \int\limits_{\Delta_{1/n}} \frac{\mathrm{d}\theta}{a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda_n} \geq \frac{nc}{2} \sqrt{\frac{1}{n}},$$

which tells us that (5) is not satisfied. Hence, $\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}} L(a)$ is strictly larger than $\mathrm{sp}\, L(a)$ for every $\varepsilon > 0$. This implies that $\mathrm{sp}^{n}_{\varepsilon\overline{\mathbf{D}}} L(a)$ is all the more strictly larger than $\mathrm{sp}\, L(a)$. ■

We now proceed to Toeplitz matrices. Let $a \in \mathcal{R}$ and $\lambda \in \mathbf{C} \setminus \mathrm{sp}\, T(a)$. The inverse of $T(a-\lambda)$ can be written down provided we have a so-called Wiener–Hopf factorization of $a - \lambda$. This is a representation

$$a(t) - \lambda = a_-(t) a_+(t) \quad (t \in \mathbf{T})$$

where $a_\pm \in \mathcal{R}$ and

$$\begin{aligned} a_-(t) &= a_0^- + a_1^- \frac{1}{t} + a_2^- \frac{1}{t^2} + \cdots, \quad a_-(z) \neq 0 \text{ for } z \in \overline{\mathbf{D}}_-, \\ a_+(t) &= a_0^+ + a_1^+ t + a_2^+ t^2 + \cdots, \quad a_+(z) \neq 0 \text{ for } z \in \overline{\mathbf{D}}, \end{aligned}$$

with $\overline{\mathbf{D}}_- = \{z \in \mathbf{C} : |z| \geq 1\} \cup \{\infty\}$ and $\overline{\mathbf{D}} = \{z \in \mathbf{C} : |z| \leq 1\}$. We have

$$a_-^{-1}(t) = b_0 + b_1 \frac{1}{t} + b_2 \frac{1}{t^2} + \cdots, \quad a_+^{-1}(t) = c_0 + c_1 t + c_2 t^2 + \cdots,$$

and straightforward computations with Toeplitz matrices show that

$$T^{-1}(a-\lambda) = T(a_+^{-1}) T(a_-^{-1}) = \begin{pmatrix} c_0 & & \\ c_1 & c_0 & \\ c_2 & c_1 & c_0 \\ \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots \\ & b_0 & b_1 & \cdots \\ & & b_0 & \cdots \\ & & & \cdots \end{pmatrix} \tag{18}$$

(see, e.g., [5, Section 1.5]).

Example 2.2 Let $a(t) = t + \alpha^2 t^{-1}$ with $\alpha \in (0,1]$ as in Example 2.1. Pick $\lambda \in E_-$, write λ in the form (13) with $\varrho \in (1,\infty)$ and $\theta \in [0,2\pi)$, and define z_1 and z_2 by (14). Since $|z_1| < 1$ and $|z_2| > 1$, we infer from (11) that the representation

$$a(t) - \lambda = (1 - z_1/t)(t - z_2) =: a_-(t)a_+(t)$$

is a Wiener–Hopf factorization, and from (18) we therefore deduce that

$$T^{-1}(a-\lambda) = -\frac{1}{z_2} \begin{pmatrix} 1 & & & \\ 1/z_2 & 1 & & \\ 1/z_2^2 & 1/z_2 & 1 & \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots \\ & 1 & z_1 & \cdots \\ & & 1 & \cdots \\ & & & \cdots \end{pmatrix}. \tag{19}$$

Combining (8), (13), (14) and (19), we can compute the parts of $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(j,k)} T(a)$ and $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(j,k)} T(a)$ in E_-. For example,

$$\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(1,1)} T(a) \cap E_- = \begin{cases} \emptyset \text{ if } \varepsilon \le 1, \\ \cup_{\varrho\in(1,\varepsilon]} a_\varrho(\mathbf{T}) \text{ if } \varepsilon > 1; \end{cases}$$

$$\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(1,1)} T(a) \cap E_- = \begin{cases} \emptyset \text{ if } \varepsilon \le 1, \\ [-\varepsilon - \alpha^2/\varepsilon, -1-\alpha^2) \cup (1+\alpha^2, \varepsilon + \alpha^2/\varepsilon] \\ \qquad\qquad\qquad\qquad \text{if } \varepsilon > 1. \end{cases}$$

Figure 7 shows examples of $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(1,1)} T(a)$ and $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(1,1)} T(a)$: an elliptic "halo" emerges in the former spectrum and two "wings" arise in the latter. In Figures 8 and 9, we illustrate $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(2,2)} T(a)$ and $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(2,2)} T(a)$, and $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(3,3)} T(a)$ and $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(2,2)} T(a)$ for the same values of ε. Notice that $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(2,2)} T(a)$ and $\mathrm{sp}_{\varepsilon\overline{\mathbf{D}}}^{(3,3)} T(a)$ contain holes that disappear for larger values of ε. Computing the sets $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(j,k)} T(a)$ for all (j,k) with $\varepsilon = 5$, we arrive at Figure 3, which is examined in more detail in the three close-ups of Figure 11. Figure 10 shows $\mathrm{sp}_{[-\varepsilon,\varepsilon]}^{(j,k)} T(a)$ for perturbations in smaller sets of entries. ■

Proof of Proposition 1.2 Let $a \in \mathcal{C}$. By (2), the intersection of $a(\mathbf{T})$ and the boundary $\partial \mathrm{sp}\, T(a)$ of $\mathrm{sp}\, T(a)$ is not empty. Pick any $\alpha = a(t_0) \in a(\mathbf{T}) \cap \partial \mathrm{sp}\, T(a)$, where $t_0 = \mathrm{e}^{\mathrm{i}\theta_0}$. We can approximate a in $\mathcal{C}$ as closely as desired by continuous functions that assume the value α throughout some open

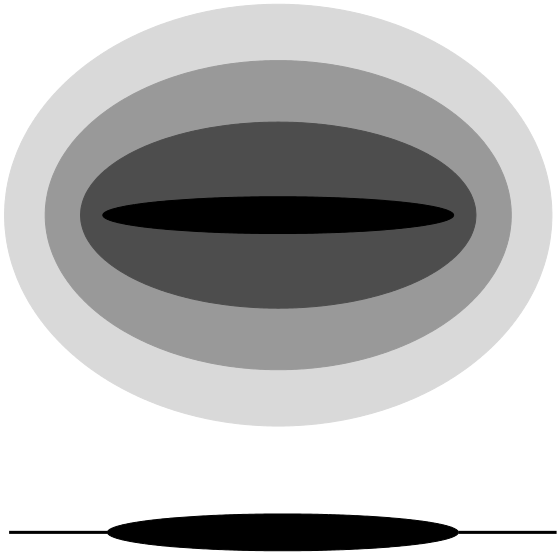


Fig. 7. The sets $\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}} T(a)$ and $\mathrm{sp}^{(1,1)}_{[-\varepsilon,\varepsilon]} T(a)$ for $a(t) = t + \alpha^2 t^{-1}$ with $\alpha = \frac{9}{10}$.

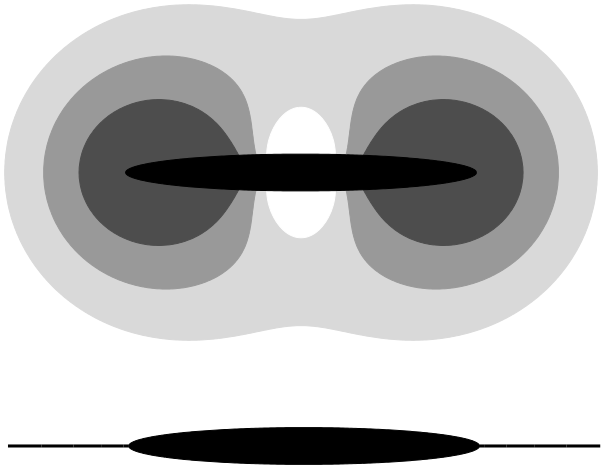


Fig. 8. The sets $\mathrm{sp}^{(2,2)}_{\varepsilon\overline{\mathbf{D}}} T(a)$ and $\mathrm{sp}^{(2,2)}_{[-\varepsilon,\varepsilon]} T(a)$ for $a(t) = t + \alpha^2 t^{-1}$ with $\alpha = \frac{9}{10}$.

neighborhood of t_0. To avoid new notation, suppose a itself is constant in an open neighborhood $\gamma \subset \mathbf{T}$ of t_0, $a(t) = \alpha$ for all $t \in \gamma$. We show that then $\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}} T(a)$ is strictly larger than $\mathrm{sp}\, T(a)$, which clearly implies Proposition 1.2.

The $(1,1)$ entry of $T^{-1}(a - \lambda)$ never vanishes and is traditionally denoted by

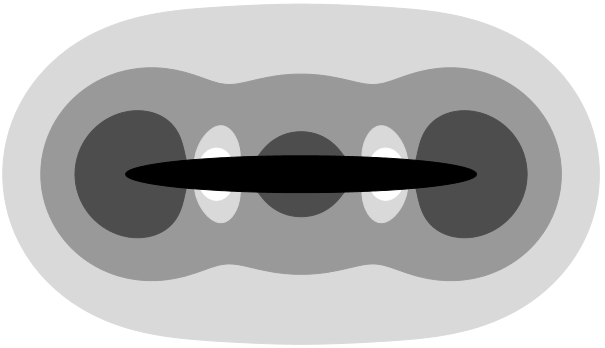




Fig. 9. The sets $\mathrm{sp}^{(3,3)}_{\varepsilon\overline{\mathbf{D}}}T(a)$ and $\mathrm{sp}^{(3,3)}_{[-\varepsilon,\varepsilon]}T(a)$ for $a(t) = t + \alpha^2 t^{-1}$ with $\alpha = \frac{9}{10}$.

$1/G(a-\lambda)$. It is well known (see, e.g., [5, Prop. 5.4]) that

$$G(a-\lambda) = \exp((\log(a-\lambda))_0) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda)\,\mathrm{d}\theta\right),$$

where $\mathrm{e}^{\mathrm{i}\theta} \mapsto \log(a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda)$ is any continuous branch of the logarithm, which exists by virtue of (2). Since $|G(a-\lambda)| = G(|a-\lambda|)$, we infer from (8) that

$$\mathrm{sp}^{(1,1)}_{\varepsilon\overline{\mathbf{D}}}T(a) = \mathrm{sp}\,T(a) \cup \{\lambda \notin \mathrm{sp}\,T(a) : G(|a-\lambda|) \le \varepsilon\}. \tag{20}$$

Fix any $\varepsilon > 0$. In view of (20), we are left to show that there is a $\lambda \in \mathbf{C}\backslash\mathrm{sp}\,T(a)$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda|\;\mathrm{d}\theta \le \log\varepsilon.$$

Pick a point $\lambda \in \mathbf{C} \setminus \mathrm{sp}\,T(a)$ sufficiently close to α and write γ in the form $\{\mathrm{e}^{\mathrm{i}\theta} : |\theta - \theta_0| < \delta\}$. We have

$$\begin{aligned}\int_0^{2\pi} \log|a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda|\,\mathrm{d}\theta &= \int_{|\theta-\theta_0|<\delta} \log|a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda|\;\mathrm{d}\theta + \int_{|\theta-\theta_0|\ge\delta} \log|a(\mathrm{e}^{\mathrm{i}\theta}) - \lambda|\;\mathrm{d}\theta \\ &\le \log|\alpha-\lambda|\,(2\delta) + \log\|a-\lambda\|_\infty\,(2\pi),\end{aligned} \tag{21}$$

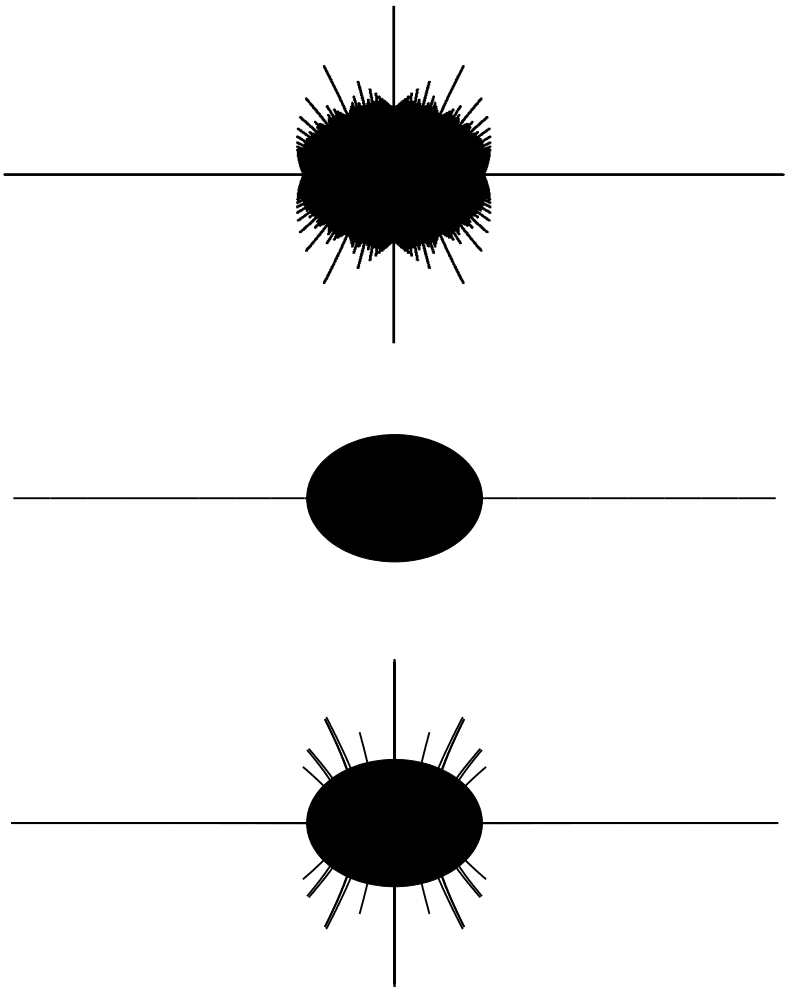


Fig. 10. $\mathrm{sp}^S_{[-\varepsilon,\varepsilon]}\, T(a)$ for $\varepsilon = 5$, $a(t) = t + \alpha^2 t^{-1}$, and $\alpha = 2/5$. Here S consists of: a) the first column; b) the first row; c) the upper-left 5×5 block.

and it is clear that (21) is smaller than $2\pi \log \varepsilon$ provided $|\alpha - \lambda|$ is sufficiently small. ■

3 General rational symbols

Proof of Theorem 1.1 By virtue of (16), it suffices to show that $\widetilde{\mathrm{sp}}^m_{\varepsilon \mathbf{D}}\, T(a) = \mathrm{sp}\, T(a)$ for all sufficiently small $\varepsilon > 0$. Due to (7), our goal is therefore to prove that

$$\sup_{\lambda \notin \mathrm{sp}\, T(a)} \|P_m T^{-1}(a - \lambda) P_m\| < \infty.$$

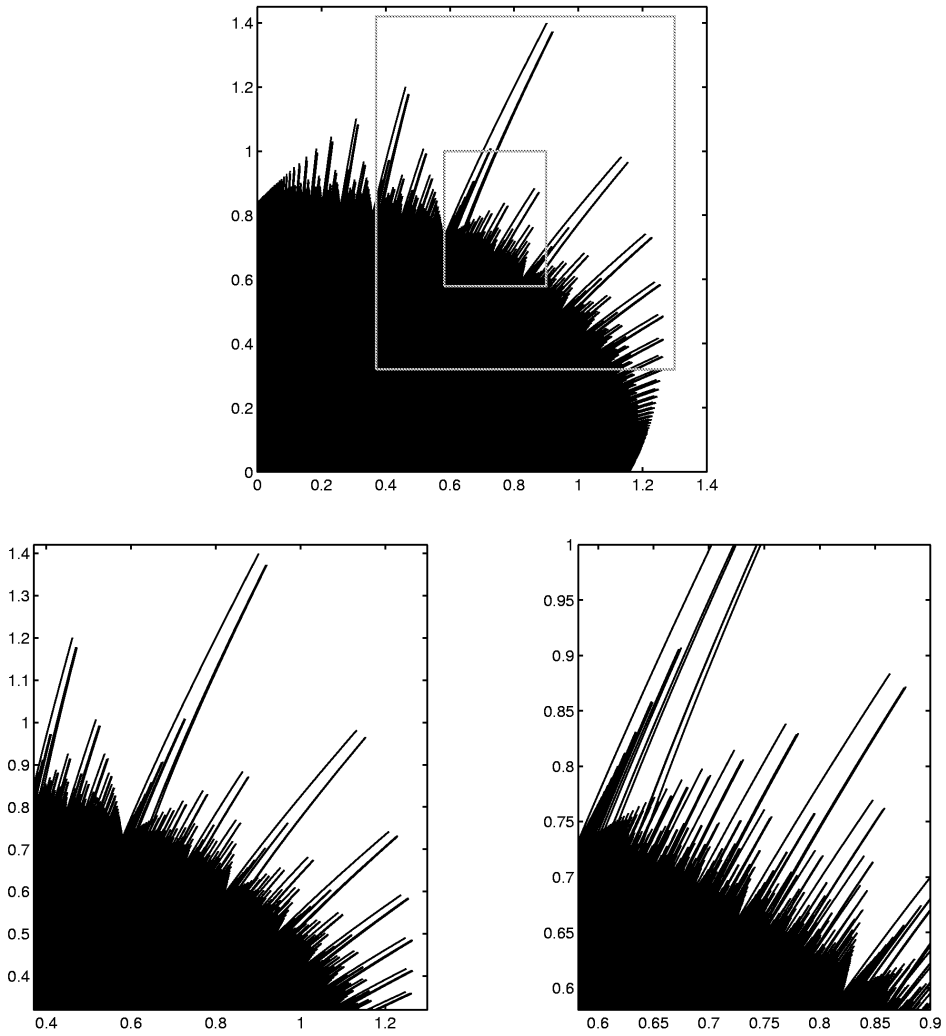


Fig. 11. Close-ups of the set shown in Figure 3; the gray boxes in the top image indicate the axes of the images below.

Let $a(t) = p(t)/q(t)$, where

$$p(t) = p_0 + \cdots + p_s t^s, \quad q(t) = q_0 + \cdots + q_r t^r;$$
$$s \geq 0,\ r \geq 0,\ s + r > 0,\ p_s \neq 0,\ q_r \neq 0;$$
$$p \text{ and } q \text{ have no common zeros.}$$

Assume first that $s < r$ and $0 \in \operatorname{sp} T(a)$. Pick $\lambda \in \mathbf{C} \setminus \operatorname{sp} T(a)$. Then $\lambda \neq 0$ and we can write

$$a(t) - \lambda = \frac{p(t) - \lambda q(t)}{q(t)} = -\lambda \frac{\prod_{j=1}^{L}(t - \nu_j)\ \prod_{j=1}^{K}(t - \mu_j)}{\prod_{j=1}^{l}(t - \delta_j)\ \prod_{j=1}^{N}(t - \gamma_j)} \tag{22}$$

where $|\nu_j| < 1, |\delta_j| < 1, |\mu_j| > 1, |\gamma_j| > 1$. Since $0 = \operatorname{wind}(a, \lambda) = L - l$, it follows that $l = L$. Notice that $L = l$, $K = r - L$, N, δ_j, and γ_j are independent

of λ, while ν_j and μ_j depend on λ. Taking into account that $L = l$, we get a Wiener–Hopf factorization (with an additional scalar factor)

$$a(t) - \lambda = -\lambda(-1)^{N+K} \frac{\mu_1 \cdots \mu_K}{\gamma_1 \cdots \gamma_N} a_-(t) a_+(t),$$

where

$$a_-(t) = \prod_{j=1}^{L} \left(1 - \frac{\nu_j}{t}\right) \prod_{j=1}^{L} \left(1 - \frac{\delta_j}{t}\right)^{-1},$$

$$a_+(t) = \prod_{j=1}^{K} \left(1 - \frac{t}{\mu_j}\right) \prod_{j=1}^{N} \left(1 - \frac{t}{\gamma_j}\right)^{-1}.$$

Formula (18) gives

$$P_m T^{-1}(a - \lambda) P_m = -\frac{1}{\lambda}(-1)^{N+K} \frac{\gamma_1 \cdots \gamma_N}{\mu_1 \cdots \mu_N} P_m T(a_+^{-1}) P_m \, P_m T(a_-^{-1}) P_m.$$

We have

$$\prod_{j=1}^{K} \left(1 - \frac{t}{\mu_j}\right)^{-1} = \prod_{j=1}^{K} \left(1 + \frac{t}{\mu_j} + \frac{t^2}{\mu_j^2} + \cdots\right) = \sum_{n=0}^{\infty} d_n t^n$$

with

$$|d_n| = \left| \sum_{j_1 + \cdots + j_K = n} \frac{1}{\mu_{j_1} \cdots \mu_{j_K}} \right| \le \left| \left(\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_K}\right)^n \right| \le K^n \le r^n,$$

whence

$$a_+^{-1}(t) = \left(\sum_{n=0}^{\infty} d_n t^n\right) \prod_{j=1}^{N} \left(1 - \frac{t}{\gamma_j}\right) = \sum_{n=0}^{\infty} c_n t^n$$

with

$$c_n = d_n - d_{n-1}\left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_N}\right) + d_{n-2} \sum_{1 \le i \le j \le N} \frac{1}{\gamma_i \gamma_j} - \cdots.$$

It follows that

$$|c_n| \le r^n + N r^{n-1} + \binom{N}{2} r^{n-2} + \cdots = (1 + r)^n.$$

Since $T(a_+^{-1})$ is the lower triangular matrix on the right hand side of (18), we get

$$\begin{aligned}\|P_m T(a_+^{-1}) P_m\|^2 &\le m|c_0|^2 + (m-1)|c_1|^2 + \cdots + |c_{m-1}|^2 \\ &\le m + (m-1)(1+r)^2 + \cdots + (1+r)^{2(m-1)} =: D^2.\end{aligned}$$

Analogously, $\|P_m T(a_-^{-1}) P_m\|^2 \le D^2$. Thus, in summary, we obtain

$$\|P_m T^{-1}(a-\lambda) P_m\| \le \frac{1}{|\lambda|} \frac{|\gamma_1 \cdots \gamma_N|}{|\mu_1 \cdots \mu_K|} D^2. \tag{23}$$

Recall that $\mu_j = \mu_j(\lambda)$ actually depends on λ. We claim that there is an $\eta_0 > 0$ such that

$$M := \sup_{\substack{\lambda \in \mathbf{C} \setminus \operatorname{sp} T(a) \\ |\lambda| < \eta_0}} \frac{1}{|\lambda|\,|\mu_1(\lambda) \cdots \mu_K(\lambda)|} < \infty. \tag{24}$$

Notice that (24) is trivially satisfied if the origin is an interior point of $\operatorname{sp} T(a)$. Since $|\mu_j(\lambda)| > 1$ for all j and λ, we have

$$\sup_{|\lambda| \ge \eta_0} \frac{1}{|\lambda|\,|\mu_1(\lambda) \cdots \mu_K(\lambda)|} \le \frac{1}{\eta_0}. \tag{25}$$

Combining (23), (24), (25) we obtain that

$$\|P_m T^{-1}(a-\lambda) P_m\| \le \max(M, 1/\eta_0)\, |\gamma_1 \cdots \gamma_N| D^2, \tag{26}$$

which, by virtue of the norm bound (16), gives Theorem 1.1 with $1/\varepsilon_0$ equal to $1/m$ times the right hand side of (26).

We are left to prove (24). Contrary to our objective, assume that there exists a sequence $\{\lambda_n\}$ such that

$$|\lambda_n| \to 0 \ \text{ and } \ |\lambda_n|\,|\mu_1(\lambda_n) \cdots \mu_K(\lambda_n)| \to 0. \tag{27}$$

If $|\mu|$ is sufficiently large, then $p(\mu) \ne 0$, $q(\mu) \ne 0$, and the winding number with respect to the origin of the function

$$\frac{p(\mu)}{q(\mu)} = \mu^{s-r} \left(\frac{p_s + p_{s-1}/\mu + \cdots + p_0/\mu^s}{q_r + q_{r-1}/\mu + \cdots + q_0/\mu^r} \right)$$

on the counter-clockwise oriented circle $|\mu| = \textit{constant}$ is $s - r < 0$. Consequently, for every natural number n there exists an $\alpha_n \in (0,1)$ with the

following property: if $|\lambda| < \alpha_n$, then the equation $p(\mu)/q(\mu) = \lambda$ has exactly $r-s$ solutions $\mu_1(\lambda), \dots, \mu_{r-s}(\lambda)$ such that $|\mu_j(\lambda)| \geq n$ for all $j = 1, \dots, r-s$. For every n, we can find a natural number $\varphi(n)$ such that $|\lambda_{\varphi(n)}| < \alpha_n$. Thus, there exist $\mu_1(\lambda_{\varphi(n)}), \dots, \mu_{r-s}(\lambda_{\varphi(n)})$ satisfying

$$n \leq |\mu_1(\lambda_{\varphi(n)})| \leq \cdots \leq |\mu_{r-s}(\lambda_{\varphi(n)})|, \quad \frac{p(\mu_j(\lambda_{\varphi(n)}))}{q(\mu_j(\lambda_{\varphi(n)}))} = \lambda_{\varphi(n)}.$$

It follows that

$$\begin{aligned}
|\lambda_{\varphi(n)}|\,|\mu_1(\lambda_{\varphi(n)}) \cdots \mu_K(\lambda_{\varphi(n)})| &\geq |\lambda_{\varphi(n)}|\,|\mu_1(\lambda_{\varphi(n)}) \cdots \mu_{r-s}(\lambda_{\varphi(n)})| \\
&\geq |\lambda_{\varphi(n)}|\,|\mu_1(\lambda_{\varphi(n)})|^{r-s} = \left|\frac{p(\mu_1(\lambda_{\varphi(n)}))}{q(\mu_1(\lambda_{\varphi(n)}))}\right| |\mu_1(\lambda_{\varphi(n)})|^{r-s} \\
&= \frac{p_s + O(1/\mu_1(\lambda_{\varphi(n)}))}{q_r + O(1/\mu_1(\lambda_{\varphi(n)}))} = \frac{p_s + O(1/n)}{q_r + O(1/n)},
\end{aligned}$$

which contradicts (27) and hence gives (24).

At this point we have proved the assertion provided $s < r$ and $0 \in \operatorname{sp} T(a)$. The case where $0 \notin \operatorname{sp} T(a)$ can be reduced to this case because

$$\operatorname{sp} T(a-\lambda) = \operatorname{sp} T(a) - \lambda, \quad \text{and} \quad \operatorname{sp}_\varepsilon^m T(a-\lambda) = \operatorname{sp}_\varepsilon^m T(a) - \lambda.$$

Now suppose $s > r$. In that case (22) is valid with the $-\lambda$ on the right replaced by p_s/q_r, which gives estimate (23) with $1/|\lambda|$ replaced by $|q_r/p_s|$ and thus the assertion with $1/\varepsilon_0 = |q_r/p_s|\,|\gamma_1 \cdots \gamma_N| D^2$. Finally, if $r = s > 0$, we can write

$$a(t) = \frac{p_r}{q_r} + \frac{u_{r-1}t^{r-1} + \cdots + u_0}{q_r t^r + \cdots + q_0} =: \frac{p_r}{q_r} + b(t),$$

and since

$$\operatorname{sp} T(a) = p_r/q_r + \operatorname{sp} T(b), \quad \operatorname{sp}_\varepsilon^m T(a) = p_r/q_r + \operatorname{sp}_\varepsilon^m T(b),$$

we have reduced the problem to the case $s < r$. This completes the proof of Theorem 1.1. ∎

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