

Pricing of European, Bermudan and American Options under the Exponential Variance Gamma Process



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This dissertation is dedicated to my family, and especially to my wife
Carol, who have given me so much encouragement and support while
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Abstract

This dissertation reports on a study of the pricing of European, Bermudan and American put options where the stock price is driven by an exponential Variance Gamma process. Closed-form solutions (for European options), Monte Carlo methods (for Bermudan options) and Finite Difference methods (for all three types) were implemented to value the options. The performance of the various algorithms was investigated and the option values produced by the different algorithms were compared. Finally, the delta and gamma greeks were calculated for these options using finite difference and Monte Carlo methods.

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Chapter 1

Introduction

It is well-known that the classic Black-Scholes equation for calculating the price of European options is unable to match observed prices over a range of strikes and maturities if a constant volatility is assumed. Indeed, it has been observed that if *implied volatilities* are derived from market prices using the classic Black-Scholes model, they display distinct smiles or skews.

Various extensions to the classic Black-Scholes model have been proposed to improve the fit to observed option prices. These include the use of *local volatility* and *stochastic volatility* models.

Another popular approach has been to replace the Brownian Motion noise source in the Black-Scholes model by a stochastic process that includes *jumps*, one such family of models being the *exponential Lévy processes*.

In this study, an example of such a process, driven by the *Variance Gamma* process, is used to price European, Bermudan and American put options. The Variance Gamma process was introduced by Madan, Carr and Chang ([2]) in 1998 and a substantial literature on the application of this process to financial modelling has developed.

The pricing methods used in this study range from closed-form solutions (for European options) through Monte Carlo methods (for European and Bermudan options) to Finite Difference methods (for all three option types).

The objectives of the study were to implement and compare the results from the various pricing methods and also compare the results for the American options with published reference data.

Chapter 2

Exponential Lévy models of the stock price

In this report, an exponential Lévy model is used for option pricing. This describes the stock price S_t at time t as follows

$$S_t = \exp(at + X_t)$$

where at is a drift term and X_t is a Lévy process.

A *Lévy process* is a stochastic process with the following properties (see [16] for further details):

- increments of the process are independent random variables
- the increments are stationary, i.e. $X_{t+s} - X_t$ and X_s have the same distribution, and
- the paths of the process have left limits and are right continuous (càdlàg paths)

The stationarity condition implies that the probability distribution of the increments is *infinitely divisible* [16] which restricts the set of probability distributions that can feature in Lévy processes.

In addition, any Lévy process has a *Lévy-Kintchine representation* [16] for the logarithm of its characteristic function, $\psi(u) = \ln(\phi(u))$, as follows

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (\exp(iux) - 1 - iux\mathbf{1}_{|x|<1})\nu(dx)$$

where γ is a drift coefficient, σ^2 is the variance of the Brownian component and the Lévy measure $\nu(dx)$ describes the jump process. A Lévy process is then defined by

its Lévy triplet $(\gamma, \sigma^2, \nu(dx))$ but the triplet may not be unique as the drift coefficient depends on the form of the indicator function used to truncate the last term in the integral.

Schoutens ([14], Chapters 5,6 and 7) gives details of a wide range of Lévy processes and describes how they can be used to develop models of stock price dynamics which can also be extended to stock price models including stochastic volatility.

In this report, the process X_t is taken to be a *Variance Gamma* process. The choice of this process for study was partly motivated by the availability (in [7]) of numerical results that could be used as reference data for the algorithms investigated and described in this report. In addition, the Variance Gamma process is perhaps slightly more tractable than some of the other Lévy processes described in [14].

The Variance Gamma process will now be described in some detail.

Chapter 3

The Variance Gamma process

The Variance Gamma (VG) process can be defined in a number of ways. Clearly, it can be defined by giving its Lévy triplet (see below). However, it can also be defined in terms of other stochastic processes. One way of doing this is to define it as the difference between two Gamma processes ([16], p.26).

The $Gamma(a, b)$ process is a pure jump Lévy process (it has no Brownian component) where the size of a (positive) jump in a small interval of time (Δt) follows the $Gamma(a\Delta t, b)$ distribution. The two Gamma processes used to generate a VG process have the same 'a' parameter but may have different 'b' parameters. When the VG process is defined in this way, it is written as $VG(C, G, M)$ and is equal to

$$G_{(1)} - G_{(2)}$$

where $G_{(1)}$ is a $Gamma(C, M)$ process and $G_{(2)}$ is a $Gamma(C, G)$ process. In this parametrisation of the VG process, the Lévy triplet is given by ([14], pp.153-154)

$$\gamma = C(MG)^{-1}(G(\exp(-M) - 1) - M(\exp(-G) - 1))$$

$$\sigma^2 = 0$$

$$\nu(dx) = C|x|^{-1}(\exp(Gx)\mathbf{1}_{x<0} + \exp(-Mx)\mathbf{1}_{x>0})dx$$

so it can be seen that the VG process is a pure jump process, having no Brownian component.

Another way of defining the VG process is as a *subordinated Brownian Motion* [6, p.57], where the parameters of a Brownian Motion (the drift and variance) are driven by a Gamma process. This version of the VG process is written as $VG(\theta, \nu, \sigma)$. In a small interval of time (Δt), the process increment is given by

$$\Delta X_t = \theta\chi + \sigma\sqrt{\chi}Z$$

where χ is sampled from the $\text{Gamma}(\frac{\Delta t}{\nu}, \frac{1}{\nu})$ distribution and Z is sampled from the normal $N(0,1)$ distribution.

As would be expected there are relationships between the two parametrisations and these can be established by evaluating and matching the characteristic function of X_1 for the two versions. The result is the following set of relationships ([14], p.57)

$$C = \frac{1}{\nu}$$

$$G = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu \right)^{-1}$$

$$M = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu \right)^{-1}$$

In the remainder of this report, only the representation of the Variance Gamma process as subordinated Brownian motion will be used.

The VG process is an infinite activity process, i.e. the process has infinitely many jumps in any finite interval as the Lévy measure has infinite mass. However, any VG process path is of finite variation.

The following formulae give results on the first four moments of X_t

$$\text{Mean} = \theta t$$

$$\text{Variance} = (\sigma^2 + \nu\theta^2)t$$

$$\text{Skew} = (\theta\nu(3\sigma^2 + 2\nu\theta^2))/((\sigma^2 + \nu\theta^2)^{\frac{3}{2}})/t^{\frac{1}{2}}$$

$$\text{Excess Kurtosis} = 3(2\nu - \nu\sigma^4)/(\sigma^2 + \nu\theta^2)^{-\frac{1}{2}}/t$$

(derived from [14], p.58).

As the normal distribution (zero skew and excess kurtosis) does not provide a good fit to log stock returns, the extra flexibility offered by the parameters of the VG process provides scope to improve the matching of historic returns and implied volatilities of option prices. However, it can be seen that for longer times to maturity, the statistics of X_t tend to those of a normal distribution and the implied volatility surface tends to flatten in a way that may not match observed implied volatilities.

Chapter 4

Risk-Neutral Pricing

The First Fundamental Theorem of Asset Pricing states that a market is arbitrage-free if and only if there exists an *equivalent martingale measure* under which the discounted stock price is a martingale.

The Second Fundamental Theorem of Asset Pricing states that a market is complete if and only if the equivalent martingale measure is *unique*. Market completeness (for a simple market consisting of one risky asset and one riskless asset) means that every contingent claim can be replicated by a self-financing trading strategy using a portfolio made up holdings in the two assets.

In an arbitrage-free but *incomplete* market, multiple equivalent martingale measures will exist and the no-arbitrage option price is no longer unique.

The existence of an equivalent martingale measure \mathbf{Q} leads to the risk-neutral pricing formula giving the value, $V(S_t, t)$, at time t of a European claim with payoff X_T , in the form

$$V(S_t, t) = E^{\mathbf{Q}}(e^{-r(T-t)}X_T|F_t)$$

where r is the risk-free rate of return and F_t is the filtration generated by the \mathbf{Q} -Brownian Motion $W_t^{\mathbf{Q}}$.

For an American option, this formula is replaced by

$$V(S_t, t) = \sup_{\tau} E^{\mathbf{Q}}(e^{-r(T-t)}X_T|F_t)$$

where the supremum is taken over all non-anticipating exercise strategies (stopping times) τ .

As the only exponential Lévy processes that result in a complete Lévy market (comprised of one risky and one riskless asset) are those built on a Brownian Motion or on a Poisson process ([16], p.49), the market modeled on the exponential VG process

is incomplete. Consequently, the equivalent martingale measure is non-unique and a choice of measure must be made.

In this report (and as in [7]) the approach taken is to assume that under the market measure the stock price follows the exponential VG process and that the discounted stock price is a martingale. Market data (the prices of European options) can then be used to fit the model parameters and the martingale condition requires that the drift parameter, a , in $S_t = \exp(at + X_t)$ must be given in terms of the VG parameters by

$$a = (r - q) + \omega$$

where

$$\omega = \frac{1}{\nu} \ln(1 - \nu\theta - \frac{1}{2}\sigma^2\nu)$$

(derived from [14], p.80) where r is the risk-free rate, and the continuous dividend yield, q , has now been incorporated. So the real-world stock price process is not considered directly.

Estimation of the VG parameters can be done by fitting the prices of European calls and puts to the exponential VG process making use of known closed-form solutions for these prices (see Chapter 5). This is the approach used in [14], (pp.82-83), where the VG parameters are estimated by minimising the mean-square error between a set of observed option prices and those generated by the model.

In this report, the parameters for the VG process have been taken from [7] in which they were derived from a set of observed European option prices. In [7], these parameters were then used to calculate American option prices and the results have been used as reference values in this report. Clearly, the parameters used are not of general application; they are only used in this study to illustrate the pricing methods. The parameters used are shown in Table 4.1

Table 4.1: Parameters used in the numerical calculations

Parameter	Value
S_0	1369.41
T	0.56164
r	0.0541
q	0.012
ν	0.50215
θ	-0.22898
σ	0.20722

Chapter 5

Pricing Methods - Theory

In this dissertation, three methods have been used to price put options. For European options, there are closed-form solutions for the option prices, two of which have been implemented to provide data for comparison with the other pricing methods. For European and Bermudan options, Monte Carlo methods have been used to price the options. Finally, a formulation in terms of a partial integral differential equation (PIDE) has been used to price European and Bermudan option values and solve the related linear complementarity problem for American options.

5.1 Closed-form solutions for European options

Two closed-form solutions have been used to calculate the values of European options under the exponential Variance Gamma process.

Firstly, a method based on the Fourier Transform described in [9] was implemented. This technique is based on an application of the Fourier transform and its inverse and takes advantage of the fact that the Fourier transform of the payoff function for a call or put has a particularly simple form. This leads to the fairly simple representation of the option price as an integral (which is actually a Fourier transform). For a European call option the price, $\text{Call}^E(S_t, K, t)$, can be written as

$$\text{Call}^E(S_t, K, t) = S \exp^{-qT} - \frac{\sqrt{SK}}{\pi} e^{-(r+q)T/2} \int_0^{\infty} \text{Re} \left[e^{iuk} \phi_T \left(u - \frac{i}{2} \right) \right] \frac{du}{u^2 + \frac{1}{4}}$$

where $\phi_T(u)$ is the characteristic function given by

$$\phi_{T(z)} = \exp(i\omega z T) (1 - iz\nu\theta + \frac{1}{2}\sigma^2\nu\theta^2)^{-\frac{T}{\nu}}$$

where

$$k = \ln \left(\frac{S}{K} \right) + (r - q)T$$

and

$$\omega = \frac{1}{\nu} \ln(1 - \nu\theta - \frac{1}{2}\sigma^2\nu)$$

as previously defined.

The second closed-form solution is given in [2] (and will subsequently be referred to as the CMC method). This solution is essentially derived from the classic Black-Scholes formula, taking account of the conditional normality of the process (i.e. viewing the VG process as drifting Brownian motion with time sampled from a Gamma distribution) which finally gives the value of a European call option as

$$\text{Call}^E(S_t, K, t) = S_t e^{-r(T-t)} \Psi(a_1, b_1, c) - K e^{-r(T-t)} \Psi(a_2, b_2, c)$$

where

$$c = \frac{(T-t)}{\nu}$$

and where, after defining

$$\zeta = \frac{(\ln(S_t/K) + \omega(T-t))}{\sigma}$$

and

$$v = 1 - \nu(\theta + \frac{1}{2}\sigma^2)$$

we have

$$a_1 = \zeta \sqrt{\frac{v}{\nu}}$$

$$b_1 = \frac{1}{\sigma}(\theta + \sigma^2) \sqrt{\frac{\nu}{v}}$$

$$a_2 = \zeta \sqrt{\frac{1}{\nu}}$$

$$b_2 = \frac{1}{\sigma} \theta \sqrt{\nu}$$

The function Ψ is defined in terms of the modified Bessel function of the second kind, K_λ , and the degenerate hypergeometric function of two variables, Φ , as

$$\Psi(a, b, c) = \frac{d^{c+\frac{1}{2}} \exp(\text{sign}(a)d)(1+u)^c}{c\sqrt{2\pi}\Gamma(c)} K_{c+\frac{1}{2}}(d) \Phi(c, 1-c, 1+c; \frac{1+u}{2}, -\text{sign}(a)d(1+u))$$

$$-\text{sign}(a) \frac{d^{c+\frac{1}{2}} \exp(\text{sign}(a)d)(1+u)^{1+c}}{(1+c)\sqrt{2\pi}\Gamma(c)} K_{c-\frac{1}{2}}(d) \Phi(1+c, 1-c, 2+c; \frac{1+u}{2}, -\text{sign}(a)d(1+u))$$

$$\text{sign}(a) \frac{d^{c+\frac{1}{2}} \exp(\text{sign}(a)d)(1+u)^c}{c\sqrt{2\pi}\Gamma(c)} K_{c-\frac{1}{2}}(d) \Phi(c, 1-c, 1+c; \frac{1+u}{2}, -\text{sign}(a)d(1+u))$$

where

$$d = |a|\sqrt{2+b^2}$$

and

$$u = \frac{b}{\sqrt{2+b^2}}$$

Finally, the the degenerate hypergeometric function of two variables, $\Phi(\alpha, \beta, \gamma; x, y)$, is itself defined by ([8])

$$\Phi(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 v^{\alpha-1} (1-v)^{\gamma-\alpha-1} (1-vx)^{-\beta} \exp(vy) dv$$

where $\Gamma(x)$ is the gamma function.

Once the value of the European call has been calculated, the value of the European put follows from put-call parity, i.e.

$$\text{Put}^E(S_t, K, t) = \text{Call}^E(S_t, K, t) - S_t e^{-q(T-t)} + K e^{-r(T-t)}$$

In the section on numerical results, the results from these two calculation methods are compared.

5.2 Monte Carlo - Longstaff-Schwartz and Low-/High-bias algorithms.

In order to calculate option prices using Monte Carlo simulation, it is necessary to generate sample paths for the asset price process. This is a straightforward task in the case of the VG process represented as subordinated Brownian motion as MATLAB provides functions to sample from both the Gamma distribution and the Normal distribution. For European options, calculation of option prices by Monte Carlo is straightforward given a method of simulating the asset price paths ([15]). For Bermudan options, the Longstaff-Schwartz ([11]) method has been used. The implementation has been based on the approach and the MATLAB code described in ([10]). In

theory, the algorithm only values Bermudan options but it is to be expected that as the number of (equally-spaced) Bermudan exercise opportunities increases, the result should tend to an estimate of the value of an American option.

The algorithm uses a dynamic programming approach working backwards from the final expiry date T in m steps of size Δt to reach the time at which the option is to be valued, $T - m\Delta t$, defined as time t_0 .

There are two stages to the algorithm and for the first stage, the steps are as follows:

1. A set of sample paths is generated for the price process from time t_0 to time T .
2. At maturity (time= T), an exercise flag is set to 1 for each path and the *discounted continuation value* of each path at time $T - \Delta t$ is set to its payoff at time T , discounted back to time t_0 .
3. k is set to 1.
4. At time $T - k\Delta t$, a regression is performed of the discounted continuation value at that time against selected basis functions of the asset price at that time. The regression equations are used to calculate an *expected discounted continuation value* for each path and this is compared with the discounted value of the immediate exercise of the option, i.e. its intrinsic value, also discounted back to t_0 .
5. If the expected discounted continuation value for a path is less than the discounted intrinsic value, the exercise flag for the path is set to 1 at time $T - k\Delta t$ and any later exercise flags along the path are set to 0.
6. If the exercise flag was set at time $T - k\Delta t$, the discounted continuation value for the path at time $T - (k + 1)\Delta t$ is set to the discounted intrinsic value at time $T - k\Delta t$. Otherwise the discounted continuation value at time $T - (k + 1)\Delta t$ is set to the discounted continuation value at time $T - k\Delta t$.
7. k is increased by 1, the algorithm returns to step 4 and continues until $T - k\Delta t$ becomes t_0 at which point the option cannot be exercised so the iterations stop and the calculations are complete.
8. The output of the algorithm is a set of regression equations at each time $T - k\Delta t$ for $k = 1, 2, \dots, (m - 1)$ which define an exercise strategy that can be used to calculate an option value.

Note that, in the implementation of the algorithm, all intrinsic and continuation values are discounted back to time t_0 to simplify the treatment of discounting.

When the first stage of the algorithm is complete, each path has an exercise flag set to 1 at the time when the option would have been exercised. These flags can be used to calculate an estimate of the option price by identifying where on each path the flag is set and then discounting the payoff at that time back to time 1. However, (see [4], pp.449-450) this estimate may have bias of uncertain sign.

To ensure that a low-biased result is obtained, the algorithm has a second stage in which a separate Monte Carlo run is performed with an independent set of paths, making use of the exercise policy defined by the regression equations at each time step. The exercise policy will be to exercise the option at time t (where the asset price is S_t) if the discounted continuation value calculated from the asset price using the regression equations for time t is less than the discounted intrinsic value of the option. The asset price where the two values agree is called the 'critical asset value' and, as a function of time, is called the 'exercise boundary'.

So in the low-bias (LB) algorithm which is the second stage of the Longstaff-Schwartz algorithm, new sample paths for the asset price are generated and the value of the option for a particular path is found by determining the point along the path where the path crosses the exercise boundary.

In their paper, Longstaff-Schwartz recommend using only 'in the money' data in the regression calculations, i.e. for a put option, at time t , only the subset of paths where $S_t < K$ are included in the regression. It is possible to use all the paths in the regression but then a different set of basis functions may be required to get reasonable results. This is described further in the results section.

To provide bounds on the option value, a high-bias (HB) estimate for the option value can be generated using a dual formulation of the pricing problem due to Rogers ([13]) and Haugh and Kogan ([6]). The implementation, described in full in [10], of this martingale-based method makes use of the Longstaff-Schwartz regression equations to generate the high-bias estimate.

It can be shown ([10] following [13]) that there exists an optimal martingale M_t such that the value of the option at time t is given by

$$V(S_0) = \max_{k=1, \dots, m} (\tilde{h}_k(S_k) - M_k)$$

where $\tilde{h}_k(S_k)$ is the discounted intrinsic value of the option at time t_k . The HB algorithm works by calculating an estimate of the optimal martingale. Based on the

code described in [10], the algorithm generates a set of single-step forward (mini) paths at each point of a main simulation path and uses the estimated continuation values along the paths to calculate the required martingale increments.

In the section on numerical results, the behaviour of the low- and high-bias algorithms is described as the number and form of basis functions and number of timesteps is varied.

5.3 The Finite Difference method - the PIDE.

In the classic Black-Scholes world, the partial differential equation describing the price of a European option is well-known, i.e.

$$\frac{\partial V}{\partial t} + (r - q)V \frac{\partial V}{\partial S} + \frac{1}{2}S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

This formula can be derived from the requirement that the discounted option price must be a martingale under the risk-neutral measure making use of the underlying asset price dynamics and Ito's lemma. This approach extends readily to the situation where the stock price dynamics are driven by an exponential Lévy process and derivation of the corresponding results makes use of Ito's formula for processes including jumps. In [12], this method is used to derive a *partial integral differential equation (PIDE)* for the value of an option under an exponential Lévy process which, when there is no Brownian component (as in the case of the VG process), can be written as

$$\int_{-\infty}^{\infty} \left[V(S_{t-}e^y, t) - V(S_{t-}, t) - \frac{\partial V}{\partial S}(S_{t-}, t)S_{t-}(e^y - 1) \right] k(y)dy \\ + \frac{\partial V}{\partial t}(S_t, t) + (r - q)S_t \frac{\partial V}{\partial S}(S_t, t) - rV(S_t, t) = 0$$

where $V(S_{t-}, t)$ is the option value at time t where the asset price just prior to t is S_{t-} and $k(y)$ is the Lévy density

$$k(y) = \frac{e^{-\lambda_p y}}{\nu y} \mathbf{1}_{y>0} + \frac{e^{-\lambda_n y}}{\nu |y|} \mathbf{1}_{y<0}$$

where

$$\lambda_p = \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu} \right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2}$$

and

$$\lambda_n = \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu} \right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2}$$

Making the change of variables $x = \ln(S)$ and $\tau = T - t$, this leads to the PIDE with x and τ as independent variables

$$-\frac{\partial w}{\partial \tau}(x, \tau) + (r - q)\frac{\partial w}{\partial x}(x, \tau) - rw(x, \tau) + \int_{-\infty}^{\infty} \left[w(x + y, \tau) - w(x, \tau) - \frac{\partial w}{\partial \tau}(e^y - 1) \right] k(y) dy$$

or, equivalently,

$$-\frac{\partial w}{\partial \tau}(x, \tau) + (r - q + \omega)\frac{\partial w}{\partial x}(x, \tau) - rw(x, \tau) + \int_{-\infty}^{\infty} [w(x + y, \tau) - w(x, \tau)] k(y) dy$$

as it can be shown by direct calculation that

$$\int_{-\infty}^{\infty} (e^y - 1)k(y) dy = -\omega$$

Now this PIDE holds only in the continuation region for an American option (for a European or Bermudan option it holds in the entire (x, τ) domain). However, as shown in [7], the PIDE can be extended to the entire (x, τ) domain for an American option by considering the result of applying the operator L given by

$$Lf = \left[\frac{\partial}{\partial t} + (r - q + \omega + \int_{-\infty}^{\infty} xk(x) dx) \frac{\partial}{\partial Y} \right] f + \int_{-\infty}^{\infty} \left[f(Y + x, t) - f(Y, t) - x \frac{\partial}{\partial Y} f \right] k(x) dx$$

to the known value of the option in the exercise region. For the American put, this value is $(K - S)$ or $(K - e^x)$. Applying the operator L to this function leads to a value

$$\delta(x) = -rK + qe^x + \int_{x(\tau)-x}^{\infty} [w(x + y, \tau) - (K - e^{x+y})] k(y) dy$$

where $x(\tau)$ represents the exercise boundary of the option. This expression can be incorporated into the original PIDE to give the extended PIDE

$$\frac{\partial w}{\partial \tau}(x, \tau) - (r - q + \omega)\frac{\partial w}{\partial x}(x, \tau) + rw(x, \tau)$$

$$\begin{aligned}
& - \int_{-\infty}^{\infty} [w(x+y, \tau) - w(x, \tau)] k(y) dy \\
& - \mathbf{1}_{x < x(\tau)} \left[rK - qe^x - \int_{x(\tau)-x}^{\infty} [w(x+y, \tau) - (K - e^{x+y})] k(y) dy \right] = 0
\end{aligned}$$

This is the equation to be discretised and solved by the finite difference method when valuing the American option. For the European and Bermudan options, the final term (the Heaviside term) which includes the indicator function is not required. Note that although the PIDE has been extended to the whole (x, τ) domain, there is still a free boundary to be located.

In the section on numerical results, the behaviour of the PIDE solver is demonstrated as the discretisation is varied and as the solver is used to value European options and Bermudan options with varying intervals between exercise opportunities.

Chapter 6

Pricing Methods - Implementation

6.1 Closed-form solutions

The formula based on the method in [9] has been evaluated in MATLAB to price call options (and then put options by put-call parity) by numerically evaluating the integral. This is done by writing the integral as

$$\int_0^X \operatorname{Re} \left[e^{iuk} \phi_T \left(u - \frac{i}{2} \right) \right] \frac{du}{u^2 + \frac{1}{4}} + \int_X^\infty \operatorname{Re} \left[e^{iuk} \phi_T \left(u - \frac{i}{2} \right) \right] \frac{du}{u^2 + \frac{1}{4}}$$

and, by estimating an upper bound on the integrand, identifying the value of X that makes the second integral sufficiently small. The first integral is then calculated by Simpson's rule.

For the CMC method, the formulae in [2] have been evaluated in MATLAB to price call options (and then put options by put-call parity). Although there is a series representation for the degenerate Hypergeometric function of two variables, in this study, the function was calculated by direct evaluation of the integral in its definition,

$$\Phi(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 v^{\alpha-1} (1-v)^{\gamma-\alpha-1} (1-vx)^{-\beta} \exp(vy) dv$$

In this application, $x = (1+u)/2$ where $u < 1$, so $x < 1$ and $(1-vx) > 0$. Also, the values of α and γ are such that $\gamma - \alpha - 1 = 0$. Finally, if the time to maturity satisfies $T > \nu$ then $\alpha > 1$ so the integrand is well-behaved and the integral can be calculated by Simpson's rule. The numerical method can be easily modified to handle the situation where $T \leq \nu$ as the parameters used ensure that the singularity of the integrand is not severe at $v = 0$ because $\alpha > 0$.

The results from the two different closed form solutions have been compared and found to be in close agreement as shown by the differences (for a strike of 1380) plotted shown in Figure 6.1. A comparison with the results from the Monte Carlo

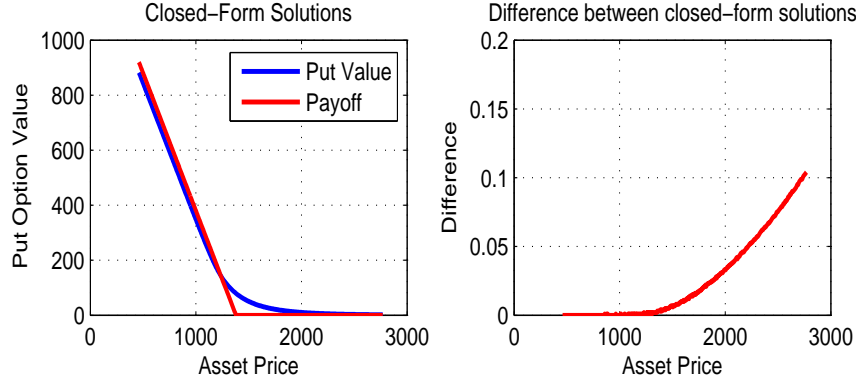


Figure 6.1: Comparison of closed-form solutions for European Put

and PIDE methods is given below.

6.2 Monte Carlo

The complete Longstaff-Schwartz algorithm including the low- and high-bias estimators has been implemented in MATLAB. The code accepts as input the VG parameters, the parameters of the put option (strike, maturity), the current asset price and the required control parameters (e.g. number of simulation paths for the regression, number of time steps, whether all the data or just 'in the money data' is used in the regression and a description of the basis functions to use, etc.). Simulation runs have been performed to investigate the effect of

1. varying the number of basis functions (and including or excluding 'out of the money' paths from the regression)
2. varying the number of timesteps (as this number increases, the (Bermudan) option value would be expected to converge to the American option value).

The basis functions used in this study consist of powers of the asset price (scaled by the strike) along with the payoff function itself. The results of these runs are shown in Figures 6.2, 6.3 and 6.4. Figure 6.2 shows that if only 'in the money' data is used in the regression, the results are not highly dependent on the order of the polynomial used (the payoff function is not used in these regressions). In addition it can be seen that the low-bias and high-bias estimates are quite close with the ± 3

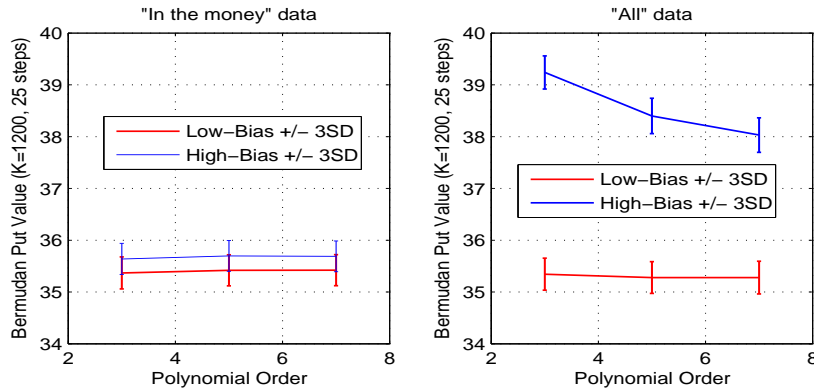


Figure 6.2: Monte Carlo values versus polynomial order - in the money data

s.d. intervals overlapping (clearly the size of these intervals depends on the number of Monte Carlo simulations used). Figure 6.2 also shows that if all the path data is used in the regression the high-bias results are poor for a polynomial of order 3 but do improve somewhat as the order of the polynomial increases. However, as shown in Figure 6.3 the results are improved to some extent by including the payoff function in the basis set as shown by the solid lines in the plot which are for a polynomials combined with the payoff function. However, it does appear that much better results are obtained by only using 'in the money' data for the regressions.

Finally, Figure 6.4 shows how the low- and high-bias option values (using cubic polynomials regressed on 'in the money' data) behave as the number of Bermuda steps increases. The calculation of the regression coefficients used 3×10^5 to 1×10^6 paths and the LB algorithm used 8×10^5 paths while the HB algorithm used 3×10^3 main paths each with 500 mini-paths.

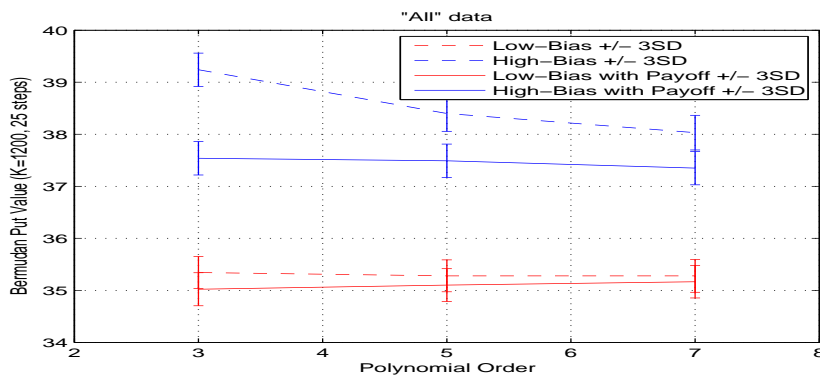


Figure 6.3: Monte Carlo values (all data) versus polynomial order

A comparison of the Monte Carlo simulation results with those the closed-form and PIDE methods is given below.

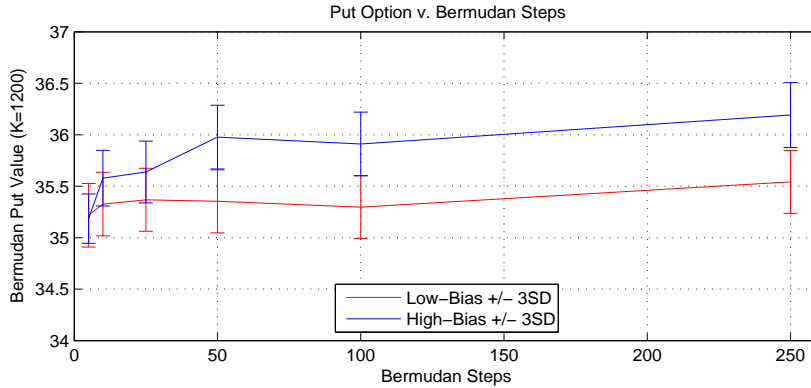


Figure 6.4: Monte Carlo values versus number of Bermudan time steps

6.3 The PIDE

To solve the (extended) PIDE by the finite difference (FD) method it must first be discretised. The main difficulty here is with the integral term which is 'non-local', i.e. the option values for all asset prices contribute to the PIDE at a particular asset price (this can be contrasted with the classic Black-Scholes equation which is purely local).

The discretisation method used (as in [7]) is to break the integral into (essentially) three parts, giving integrals over three subsets of the real line. One integral is over a small interval containing the origin, where the probability density has a singularity. The second integral is over the complement of a finite interval around the origin, where this interval is chosen so that the option value is known outside the interval. Finally, the third integral is over what remains of the real line and after discretisation of the x -domain, this integral can be written as the sum of integrals. Note that, for the VG process, there are actually six integrals to evaluate as the probability density function is different for positive and negative arguments. The evaluation of some of the integrals is performed as follows (following [7]).

Firstly, note that the x domain is discretised using a space step of Δx and the τ domain is discretised using a time step of $\Delta \tau$, so that the grid points are at (x_i, τ_j) where $x_i = x_0 + i\Delta x$ ($i=0, \dots, N$) and $\tau_j = \tau_0 + j\Delta \tau$ ($j=0, \dots, M$). τ_0 is set to zero. In the MATLAB implementation, x_0 has been chosen to correspond to an asset price of Strike/3 and x_N to 2 x Strike.

For one of the integrals around the singularity of $k(y)$

$$\int_0^{\Delta x} (w(x_i + y, \tau_j) - w(x_i, \tau_j))k(y)dy$$

$$= \int_0^{\Delta x} (w(x_i + y, \tau_j) - w(x_i, \tau_j)) \frac{e^{-\lambda_p y}}{\nu y} dy$$

or, by using the approximation

$$\frac{(w(x_i + y, \tau_j) - w(x_i, \tau_j))}{y} \approx \frac{(w(x_{i+1}, \tau_j) - w(x_i, \tau_j))}{\Delta x}$$

the integral becomes

$$\frac{(w(x_{i+1}, \tau_j) - w(x_i, \tau_j))}{\lambda_p \nu \Delta x} (1 - e^{-\lambda_p x})$$

For the integrals over the upper part of the discretised x -domain

$$\begin{aligned} & \int_{\Delta x}^{x_N - x_i} (w(x_i + y, \tau_j) - w(x_i, \tau_j)) k(y) dy \\ &= \sum_{k=1}^{N-i-1} \int_{k\Delta x}^{(k+1)\Delta x} (w(x_i + y, \tau_j) - w(x_i, \tau_j)) \frac{e^{-\lambda_p y}}{\nu y} dy \end{aligned}$$

or, by using the approximation

$$(w(x_i + y, \tau_j) - w(x_i, \tau_j)) \approx (w(x_{i+k}, \tau_j) + \frac{(w(x_{i+k+1}, \tau_j) - w(x_{i+k}, \tau_j))}{\Delta x} (y - k\Delta x) - w(x_i, \tau_j))$$

the integral becomes

$$\begin{aligned} &= \sum_{k=1}^{N-i-1} \int_{k\Delta x}^{(k+1)\Delta x} (w(x_{i+k}, \tau_j) + \frac{(w(x_{i+k+1}, \tau_j) - w(x_{i+k}, \tau_j))}{\Delta x} (y - k\Delta x) - w(x_i, \tau_j)) \frac{e^{-\lambda_p y}}{\nu y} dy \\ &= \sum_{k=1}^{N-i-1} \frac{1}{\nu} (w(x_{i+k}, \tau_j) - w(x_i, \tau_j) - k(w(x_{i+k+1}, \tau_j) - w(x_{i+k}, \tau_j))) \\ &\quad \times (\text{expint}(k\Delta x \lambda_p) - \text{expint}((k+1)\Delta x \lambda_p)) \\ &\quad + \sum_{k=1}^{N-i-1} \frac{(w(x_{i+k+1}, \tau_j) - w(x_{i+k}, \tau_j))}{\lambda_p \nu \Delta x} (e^{-k\Delta x \lambda_p} - e^{-(k+1)\Delta x \lambda_p}) \end{aligned}$$

where x_N is largest of the discretised x-values. Here $\text{expint}(z)$ is the exponential integral defined by

$$\text{expint}(z) = \int_z^{\infty} \frac{e^{-u}}{u} du$$

The corresponding two integrals for negative y are treated a similar fashion.

The integral over the remainder of the positive real line is

$$\int_{x_N - x_i}^{\infty} (w(x_i + y, \tau_j) - w(x_i, \tau_j))k(y)dy$$

and for the put option, we assume that $w(x, \tau_j) = 0$ for $x > x_N$ so $w(x_i + y, \tau_j) = 0$ for $y > x_N - x_i$, i.e. the upper boundary of the x-grid is chosen so that this condition is satisfied.

Then the integral becomes

$$\begin{aligned} & \int_{(N-i)\Delta x}^{\infty} (w(x_i + y, \tau_j) - w(x_i, \tau_j))k(y)dy \\ &= -\frac{1}{\nu} w(x_i, \tau_j) \text{expint}((N-1)\Delta x \lambda_p) \end{aligned}$$

Finally, the integral over the remainder of the negative real line is

$$\int_{-\infty}^{x_0 - x_i} (w(x_i + y, \tau_j) - w(x_i, \tau_j))k(y)dy$$

which, by a change of variable ($y \rightarrow -y$), becomes

$$\int_{i\Delta x}^{\infty} (w(x_i - y, \tau_j) - w(x_i, \tau_j)) \frac{e^{-\lambda_n y}}{\nu y} dy$$

Then for the American put option, we assume that $w(x, \tau_j) = K - e^x$ for $x < x_0$ so that $w(x_i - y, \tau_j) = K - e^{x_i - y}$ for $y > i\Delta x$, i.e. the lower boundary of the x-grid is chosen so that this condition is satisfied.

Then the integral becomes

$$\begin{aligned} & \int_{i\Delta x}^{\infty} (K - e^{x_i - y} - w(x_i, \tau_j)) \frac{e^{-\lambda_n y}}{\nu y} dy \\ &= \frac{1}{\nu} (K - w(x_i, \tau_j)) \text{expint}(i\Delta x \lambda_n) - \frac{1}{\nu} e^{x_i} \text{expint}(i\Delta x (\lambda_n + 1)) \end{aligned}$$

A similar method is used to discretise the Heaviside term, taking account of the index L that identifies the current position of the critical asset price (on the exercise boundary) as $x(\tau) = x_L$. The result of discretising the main integral term can be expressed as follows. First define the following series

$$\begin{aligned} d_k &= \text{expint}(k\Delta x\lambda_n) - \text{expint}((k+1)\Delta x\lambda_n) \\ e_k &= e^{-k\Delta x\lambda_n} - e^{(k+1)\Delta x\lambda_n} \\ f_k &= \text{expint}(k\Delta x\lambda_p) - \text{expint}((k+1)\Delta x\lambda_p) \\ g_k &= e^{-k\Delta x\lambda_p} - e^{(k+1)\Delta x\lambda_p} \end{aligned}$$

Then the main integral contribution for the discretisation at x_i can then be written as

$$\begin{aligned} & (- (i-1)d_{i-1} + e_{i-1})w_0 \\ & + \sum_{k=2}^{i-1} \{(1+k)d_k - e_k - (k-1)d_{k-1} + e_{k-1}\} w_{i-k} \\ & \quad + (2d_1 - e_1)w_{i-1} \\ & \quad + \frac{1}{\nu\Delta x\lambda_n} (1 - e^{-\lambda_n\Delta x})w_{i-1} \\ & \quad - \left(\sum_{k=1}^{i-1} d_k + \sum_{k=1}^{N-i-1} f_k \right) w_i \\ & \quad - \left(\frac{1}{\nu} \text{expint}(i\Delta x\lambda_n) + \frac{1}{\nu\Delta x\lambda_n} (1 - e^{-\lambda_n\Delta x}) \right) w_i \\ & \quad - \left(\frac{1}{\nu} \text{expint}((N-i)\Delta x\lambda_p) + \frac{1}{\nu\Delta x\lambda_p} (1 - e^{-\lambda_p\Delta x}) \right) w_i \\ & \quad + (2f_1 - g_1)w_{i+1} \\ & \quad + \frac{1}{\nu\Delta x\lambda_p} (1 - e^{-\lambda_p\Delta x})w_{i+1} \\ & + \sum_{k=2}^{N-i-1} \{(1+k)f_k - g_k - (k-1)f_{k-1} + g_{k-1}\} w_{i+k} \\ & \quad + (-(N-1-i)f_{N-i-1} + g_{N-i-1})w_N \\ & + \frac{1}{\nu} (K \text{expint}(i\Delta x\lambda_n) - e^{x_i} \text{expint}(i\Delta x(\lambda_n + 1))) \end{aligned}$$

and there is a similar, but shorter formula for the Heaviside integral approximation. These formulae show how the discretisation of the PIDE at x_i depends on all the grid points.

Once the integrals have been approximated in this way and the $\frac{\partial w}{\partial t}$ and the $\frac{\partial w}{\partial x}$ terms have been replaced by their finite difference approximations (central difference for x , forward difference for τ), the equations needed for the time-stepping can be assembled, in the form

$$Aw_{i-1,j+1} + Bw_{i,j+1} - Cw_{i+1,j+1} = RHS_i$$

and at this point, there are many choices for where to place the integral contribution. In [7], the only integral terms to feature on the left-hand side of the equation come from the integral approximations around $y = 0$ and the left-hand side matrix is then tridiagonal. However, many other choices are possible with additional integral terms being treated implicitly by being placed on the left-hand side, leading to a left-hand side matrix that is more dense. A fully implicit treatment of the integral terms will lead to a fully-populated left-hand side matrix.

The discretisation in [7] which has been used in the main implementation of the method is thus

$$Aw_{i-1,j+1} + B_i w_{i,j+1} - Cw_{i+1,j+1} = w_{i,j} + \frac{\Delta\tau}{\nu} R_{i,j} + \Delta\tau 1_{x_i < x(\tau_L)} H_{i,j}$$

where

$$\begin{aligned} A &= a - B_n \\ B_i &= 1 + r\Delta\tau + B_n + B_p + \frac{\Delta\tau}{\nu} (\text{expint}(i\Delta x\lambda_n) + \text{expint}((N-i)\Delta x\lambda_p)) \\ C &= a + B_p \end{aligned}$$

$H_{i,j}$ comes from the Heaviside integral terms, $R_{i,j}$ contains all the remaining integral terms and

$$\begin{aligned} a &= (r - q - \omega) \frac{\Delta t}{2\Delta x} \\ B_n &= \frac{\Delta t}{\nu\Delta x\lambda_n} (1 - e^{-\lambda_n\Delta x}) \\ B_p &= \frac{\Delta t}{\nu\Delta x\lambda_p} (1 - e^{-\lambda_p\Delta x}) \end{aligned}$$

As mentioned above, the integral approximation the low x -values needs to match the known option values in that region. For the extended PIDE including the Heaviside term the appropriate value is $K - e^x$, as used above. However, when the PIDE is used to model Bermudan and European options (with no Heaviside term), the appropriate value is $w(x_i - y, \tau_j) = Ke^{-t\Delta\tau_j} - e^{x_i - y} e^{-q\Delta\tau_j}$ where $\Delta\tau_j$ is the time that has elapsed

since the previous exercise date (noting that backward time-stepping is being used). If this requirement is not honoured for the Bermudan and European option model, the option values will be over-estimated for small values of the spot price.

Once the matrix has been created, the equations can be solved after the boundary conditions have been specified. In the implementation in MATLAB, the boundary conditions for both large and small x are taken to be the 'zero gamma' boundary conditions.

When solving the system for European or Bermudan options, the PIDE is solved between exercise times by time-stepping. Whenever an exercise time for the Bermudan option is reached, the option value is compared to the intrinsic option value (immediate payoff) and any option values below the intrinsic values are reset to the intrinsic values.

When solving the system for American options, the extended PIDE is solved at each timestep and the new critical asset value is determined by identifying the asset value at which the calculated option value meets the payoff. This identifies the value of L to be used in the Heaviside term at the next timestep. The 'exercise boundary' located in this way is also displayed by the MATLAB code.

The PIDE solver has been run using VG parameters taken from [7] and the performance of the solver has been investigated as the space/time discretisation is refined. The solver has also been run to calculate European put option values and Bermudan put option values for a range of equally-spaced Bermudan time steps. The variation of the PIDE with the number of Bermudan steps is shown in Figure 6.5 and the variation with discretisation and steps is shown in Table 6.1.

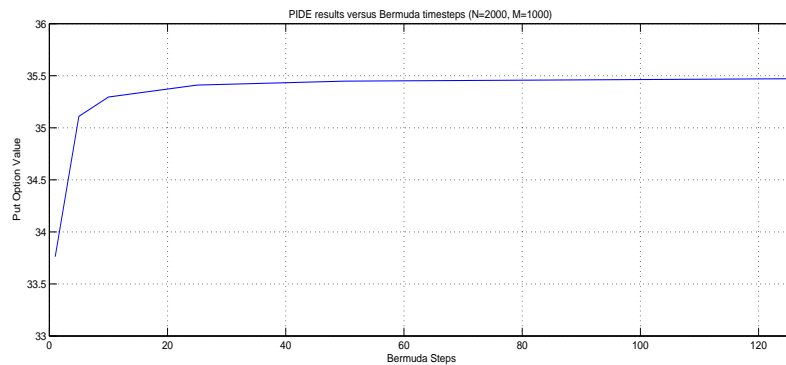


Figure 6.5: PIDE option values versus number of Bermudan steps

Table 6.1: Variation of PIDE option values versus discretisation and number of Bermudan steps

European, Bermudan and American Put Values							
Grid↓ Steps→	European	5	10	25	50	125	American
N/M = 500/250	33.752	35.095	35.281	35.394	35.432	35.454	35.460
N/M = 1000/500	33.759	35.106	35.292	35.405	35.443	35.466	35.477
N/M = 2000/1000	33.762	35.110	35.296	35.410	35.448	35.471	35.484

Chapter 7

Comparison of results from the three estimation methods

Figure 7.1 shows the results from the three calculation methods in a single plot (for $K=1200$) and it can be seen that the three methods are in reasonable agreement. The Monte Carlo simulation results were based on a cubic regression on 10^6 'in the money' paths - the low-bias estimate used $\approx 10^6$ paths and the high-bias estimate used $\approx 3 \times 10^4$ main paths each with 500 mini-paths. The Monte Carlo estimates are only expected to bracket the PIDE solutions for the Bermudan and American option as a non-optimal exercise policy is being used in the calculations. The most inaccurate estimate is the high-bias Monte Carlo estimate which overestimates the option value by about 2%.

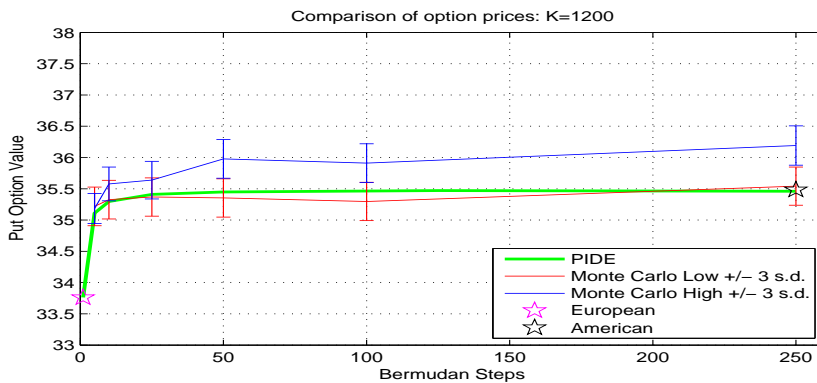


Figure 7.1: Comparison of option values from the three pricing methods

Finally, Table 7.1 shows the PIDE American option estimates versus the values reported in [7]. Although the PIDE implementation has followed the method set out in [7], there are some remaining uncertainties about the implementation that could

have caused the results to differ from those in [7]. In particular, the minimum and maximum values of the stock price grid used are not specified in the paper.

Table 7.1: Comparison of PIDE American option values with [7]

American Put Values			
Discretisation	Strike	PIDE	From [7]
N/M = 2000/1000	1200	35.484	35.531
N/M = 2000/1000	1260	48.734	48.798
N/M = 2000/1000	1320	65.906	65.993
N/M = 2000/1000	1380	87.880	87.992

Chapter 8

The Exercise Boundary

As mentioned above, both the Monte Carlo and PIDE code generate an estimate of the exercise boundary. Figure 8.1 shows the estimated boundaries for $K=1300$ along with the boundary shown in [7]. The Monte Carlo boundary used cubic regression on 'in the money' data, 250 timesteps and 4×10^5 sample paths. It can be seen that the three boundaries are quite different. In [1] it is demonstrated that for certain choices of parameters, American options under the VG process can exhibit an exercise boundary that does not tend to the strike at maturity which is the behaviour seen in this study.

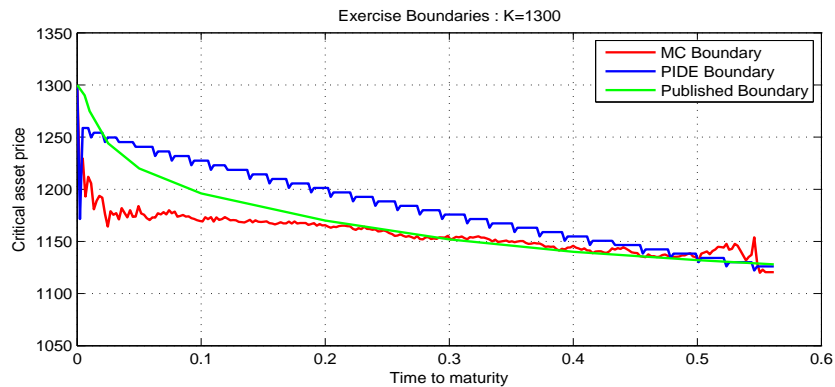


Figure 8.1: Exercise boundaries - 250 timesteps

To investigate the sensitivity of the calculated option prices to these boundaries, a Monte Carlo simulation was carried out where the exercise policies were determined by these boundaries. The simulation used 1.5×10^7 paths to calculate the low-bias estimate for each exercise boundary and the results are displayed in Figure 8.2 where the error bars correspond to ± 3 s.d.

It can be seen that the PIDE American put value falls in the centre of its simulation range and is not significantly different from the value derived from the boundary shown in [7]. The value of the option derived from the Longstaff-Schwartz boundary

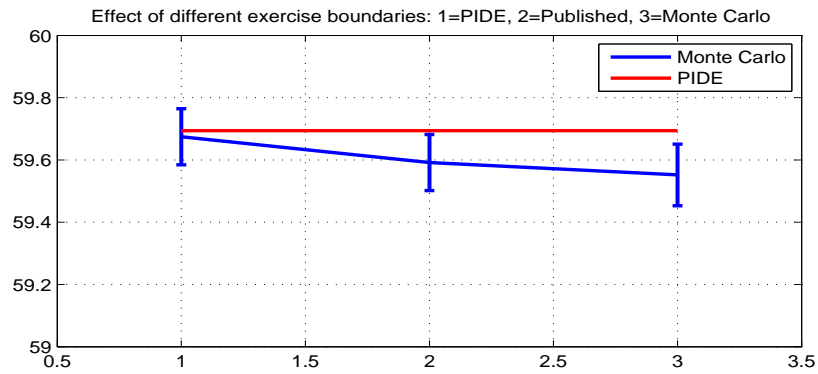


Figure 8.2: Sensitivity of put option price to the exercise boundary

is expected to be lower than the other two values as it represents a sub-optimal exercise policy but actually its error bars overlap those of the other two estimates, so more simulation paths would be required to discriminate between the estimates. Overall, the option value is not very sensitive to the precise shape of the boundary. It would be interesting to know whether the authors of [7] had to modify their numerical scheme to achieve the smooth boundary shown in their paper.

Chapter 9

Greeks

As the market is incomplete for the model being studied, hedging using the asset alone is not possible and so the significance of the standard hedging parameters is not clear. Nevertheless, to compare the PIDE and Monte Carlo option pricing methods, some parameter sensitivities (Greeks) have been evaluated, i.e. the Delta ($\frac{\partial V}{\partial S}$) and Gamma ($\frac{\partial^2 V}{\partial S^2}$). The Delta has been calculated in three ways; using the results of the PIDE, using the pathwise sensitivity method in the Monte Carlo method and finally by 'bumping' the Monte Carlo estimates. The Gamma has been estimated from the PIDE output and by 'bumping' the Monte Carlo estimates. No greeks have been calculated from the closed form solutions. For the PIDE method, the Delta and Gamma are estimated from the output grid of option values by using finite differences. Noting that the grid is uniformly-spaced in terms of x ($=\ln(S)$) and t , so the Delta and Gamma were estimated by

$$\begin{aligned}\text{Delta} &= \frac{\partial V}{\partial S} = \frac{1}{S} \frac{\partial V}{\partial x} \\ &\approx \frac{(V(x + \Delta x) - V(x - \Delta x))}{2S\Delta x}\end{aligned}$$

and

$$\begin{aligned}\text{Gamma} &= \frac{\partial^2 V}{\partial S^2} = \frac{1}{S^2} \left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \right) \\ &\approx \frac{1}{S^2} \left(\frac{(V(x + \Delta x) - 2V(x) + V(x - \Delta x)))}{\Delta x^2} - \frac{(V(x + \Delta x) - V(x - \Delta x))}{2S\Delta x} \right)\end{aligned}$$

Surface plots of Delta and Gamma for European, Bermudan and American options are shown as Figures 9.1, 9.2 and 9.3. In each case the strike is at 1200.

Note the numerical instability at maturity where the payoff has discontinuous derivative. Also note that for the Bermudan and American options, there appears to be some numerical instability at the exercise boundary away from maturity, for

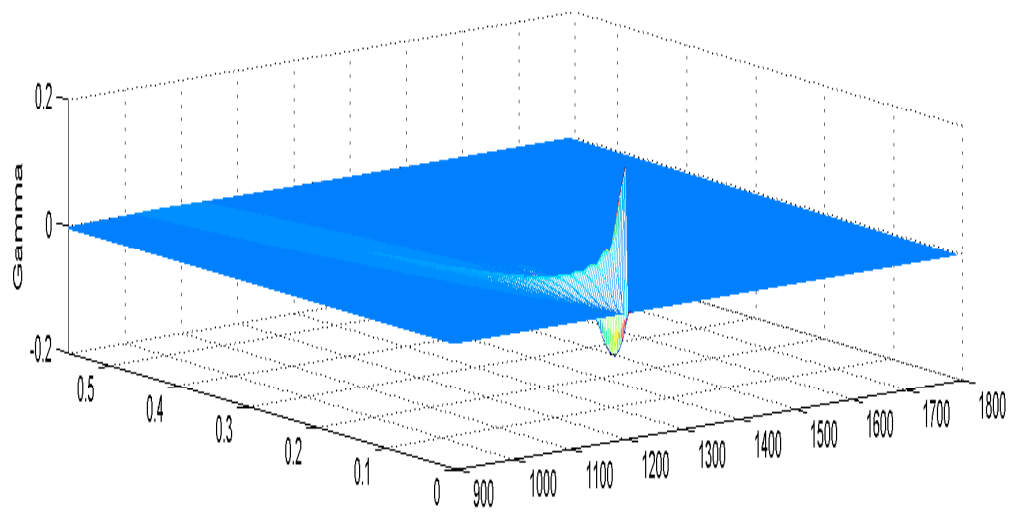
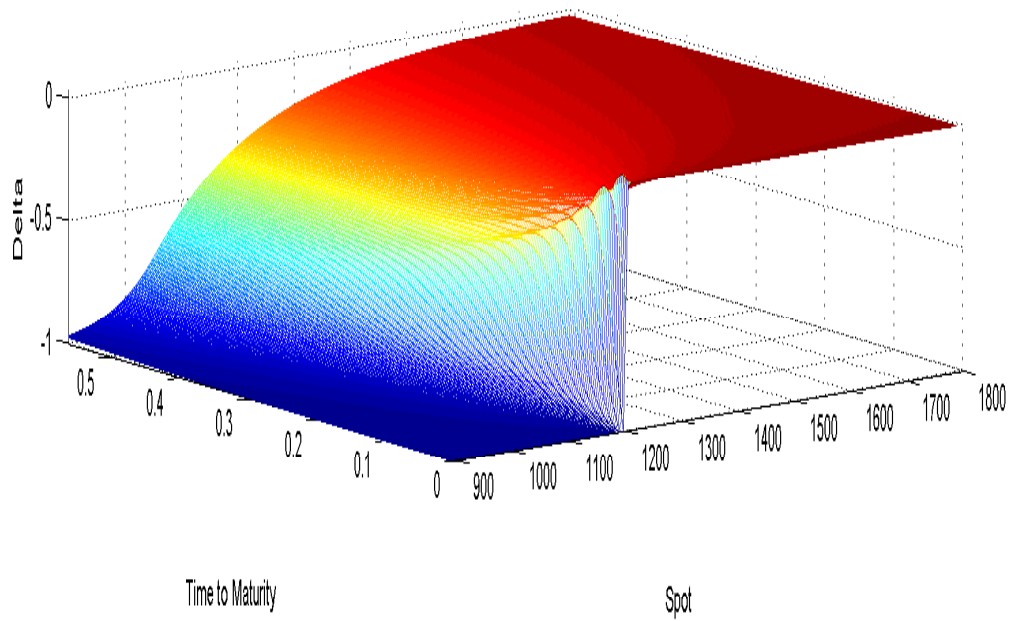


Figure 9.1: European Greeks

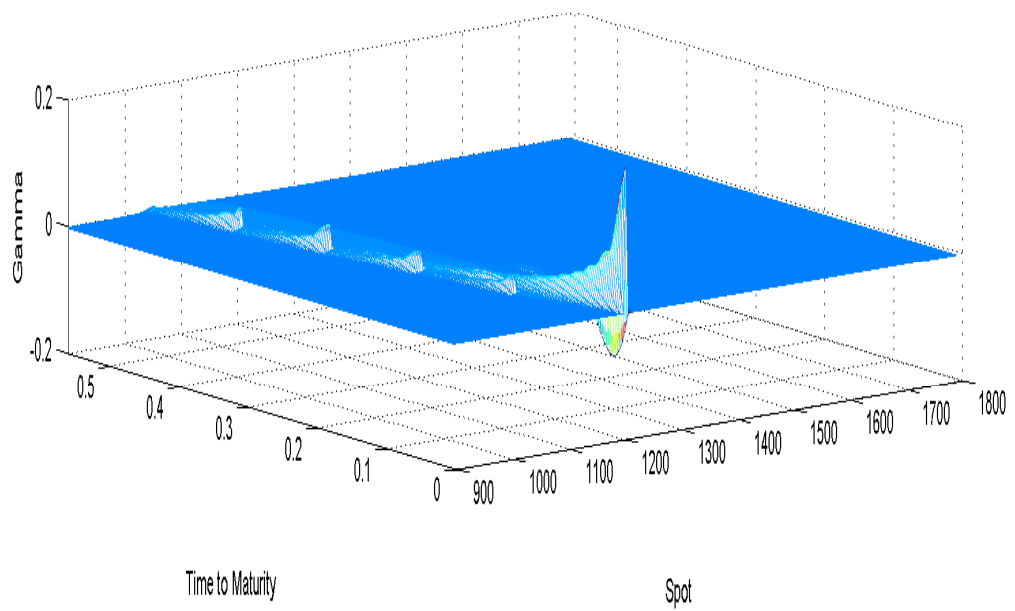
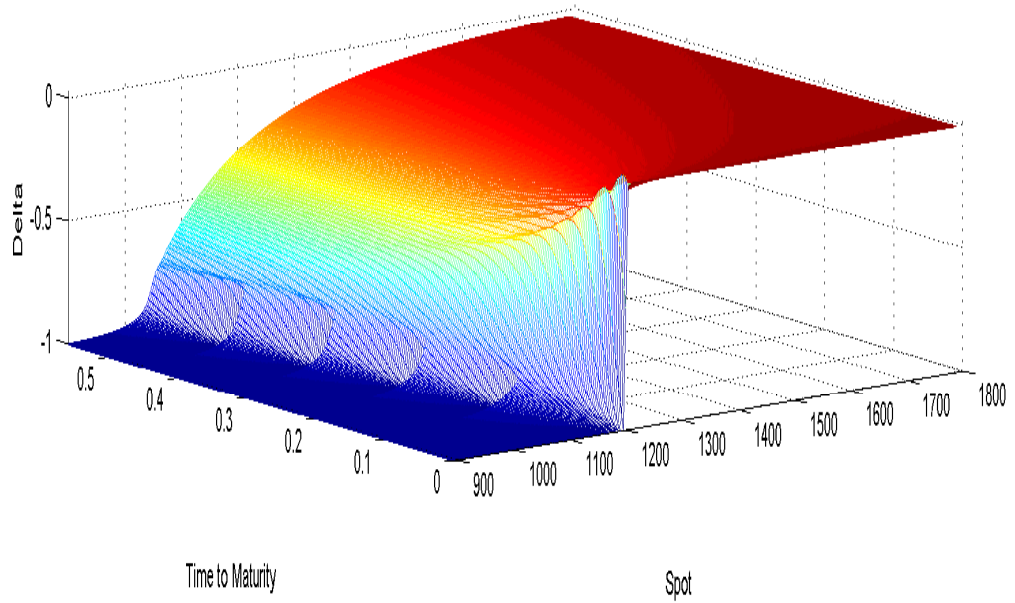


Figure 9.2: Bermudan Greeks - 5 Exercise Dates

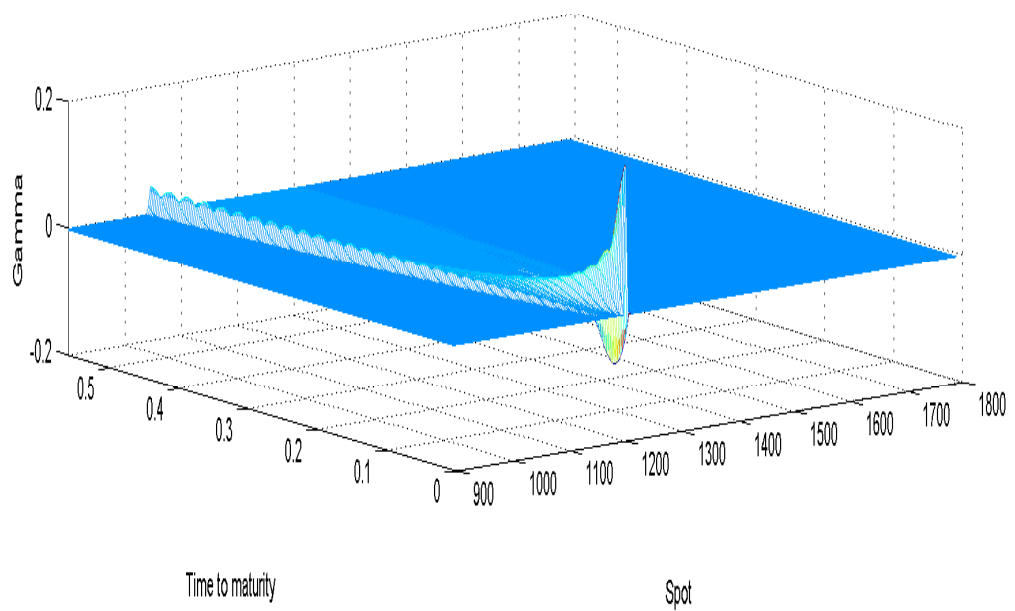
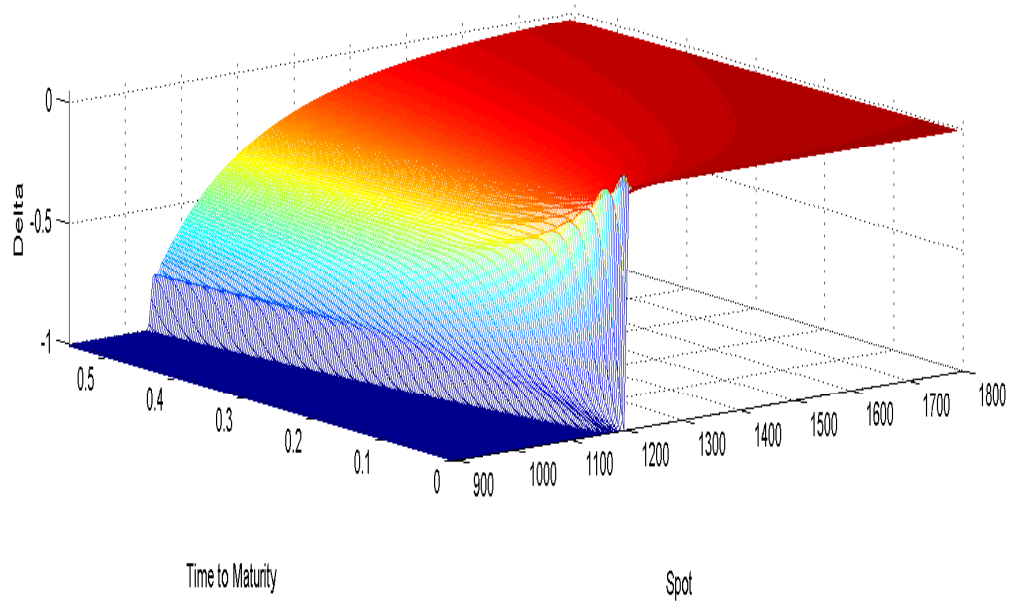


Figure 9.3: American Greeks

example as seen in Figure 9.4 (the European option Delta and Gamma do not show this behaviour).

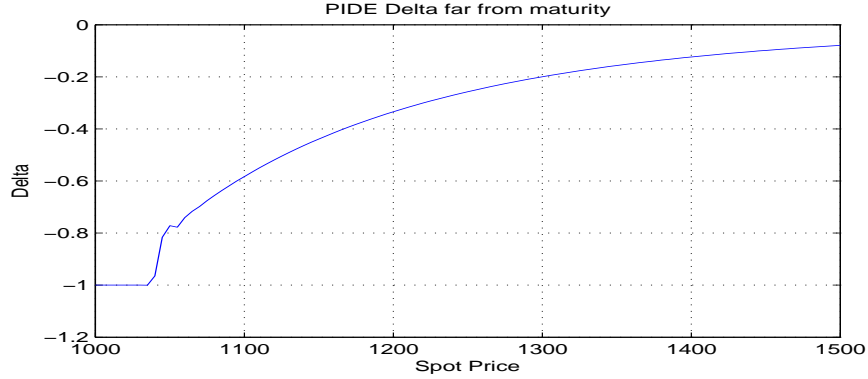


Figure 9.4: Instability of the American PIDE Delta at the exercise boundary

This instability has also been observed in [1] where it is shown that a more sophisticated finite difference scheme can eliminate it. The plots also suggest that the option Delta for the American option is not continuous across the exercise boundary which is consistent with the further observation in [1] that 'smooth-pasting' may not apply to American options under the VG process (depending on the process parameters). The European option Gamma is smooth far from maturity but this is not the case for the American option Gamma.

For the Monte Carlo method, the Delta has been estimated using the pathwise sensitivity method during the Low-Bias valuation. This method calculates the average, by Monte Carlo simulation, of

$$\frac{\partial X_t}{\partial \phi}$$

where X_t is the payoff and ϕ is the parameter being sensitised. This can then be factored into

$$\frac{\partial X_t}{\partial S_t} \frac{\partial S_t}{\partial \phi}$$

Note that when dealing with the American option, t refers to the time of exercise for a particular Monte Carlo path. If the option is exercised before expiry, it must be in the money and so, for a put option, the term $\frac{\partial X_t}{\partial S_t}$ is equal to -1. If the option is only exercised at maturity then the term $\frac{\partial X_t}{\partial S_t}$ is $-\mathbf{1}_{(X_T < K)}$. For the Delta, the term $\frac{\partial S_t}{\partial \phi}$ is simply $\frac{S_t}{S_0}$ as the asset price at time t scales directly with S_0 as $S_t = S_0 \exp(at + X_t)$.

Selected values of the Pathwise Sensitivity Deltas (again for $K=1200$) are plotted against the PIDE Deltas in Figure 9.5 and the agreement can be seen to be very good.

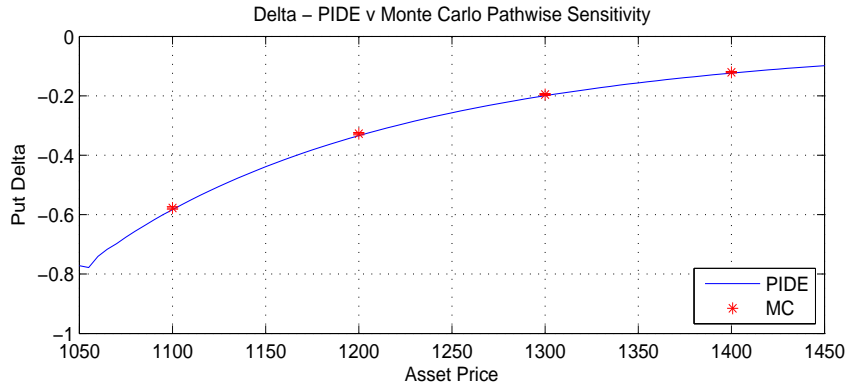


Figure 9.5: Pathwise Sensitivity Monte Carlo Delta versus PIDE Delta

There are errors bars around the Pathwise Sensitivity Deltas but they are very close together.

The option Gamma cannot be calculated using the pathwise sensitivity method unless the payoff is smoothed in some fashion.

An alternative method of calculating greeks which does allow Gamma to be calculated is the likelihood ratio method which requires the calculation of

$$\frac{\partial(\ln(p))}{\partial\phi}$$

where p is the transition density for the process. In the case of geometric Brownian motion, this leads to a simple method for calculating Gamma, as differentiating the transition density with respect to S_0 is fairly straightforward and the result can be expressed as a simple function of the underlying normal random variable used in the simulation which makes the computation of Gamma by this method tractable.

In the case of the VG process, with $S_t = S_0 \exp(at + X_t)$, the transition density (from S_0 at time 0) is given by ([3], p.117)

$$p_t(x) = C|x|^{\frac{t}{\nu}-\frac{1}{2}}e^{Ax}K_{\frac{t}{\nu}-\frac{1}{2}}(B|x|)$$

where

$$A = \frac{\theta}{\sigma^2}$$

$$B = \frac{\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}}}{\sigma^2}$$

and

$$C = \sqrt{\frac{\sigma^2\nu}{2\pi}} \frac{(\theta^2\nu + 2\sigma^2)^{\frac{1}{4}} - \frac{\theta}{2\nu}}{\Gamma(\frac{t}{\nu})}$$

$$x = \ln\left(\frac{S_t}{S_0}\right) - at$$

and differentiating this with respect to S_0 is difficult and the result is unlikely to be a simple function of the underlying gamma and normal random variables used in the simulation.

Finally, for the Monte Carlo method, the Delta and Gamma have also been estimated by 'bumping'. In this method, the parameter of interest, here $\phi = S_0$, is perturbed up and down by an amount ΔS_0 and two additional Monte Carlo estimates V_{up} and V_{down} bracketing the base value, V , of the option price are calculated. To reduce the variance of the 'bumped' sensitivity, the same sample paths are used for each of the three valuations. Then, the 'bumped' Delta and Gamma are estimated as

$$\text{Delta} = \frac{(V_{up} - V_{down})}{2\Delta S_0}$$

and

$$\text{Gamma} = \frac{(V_{up} - 2V + V_{down})}{(\Delta S_0)^2}$$

Selected values of the 'bumped' Monte Carlo Deltas and Gamma are plotted against the PIDE values in Figure 9.6 and the agreement is again very good for the Deltas, but somewhat less so for the Gammas. There are errors bars around the 'bumped' Deltas and Gammas but for the Deltas they are very close together.

Note that the validity of the pathwise sensitivity and bumping methods for Bermudan and American options depends on the fact that the exercise boundary (optimal stopping rule) does not depend on the value of the spot, S_0 , so that when differentiating inside the expectation, for the pathwise sensitivity method, it is not necessary to account for changes in the position of the boundary as S_0 changes and, in the bumping method, it is not necessary to repeat the first stage of the Longstaff-Schwartz algorithm, the regression step, as S_0 is flexed. The same regression equations can be used for the base, up and down option price estimates. A proof that the exercise boundary is not a function of S_0 is given in an unpublished note [5].

In practice, although the exercise boundary should not depend on S_0 , when running Monte Carlo simulations with a limited number of paths from different initial spot prices, it would be likely that different parts of the (spot,time) space would be explored more thoroughly, and the regression equations may vary with the spot price due to these practical limitations. So in the calculation of the pathwise sensitivities above (starting from spot prices of 1100, 1200, 1300 and 1400) the regression equations were calculated afresh for each run. For the bumping method with a fairly small

perturbation of the spot (± 25) the same regression equations were used for the base, up and down valuations.

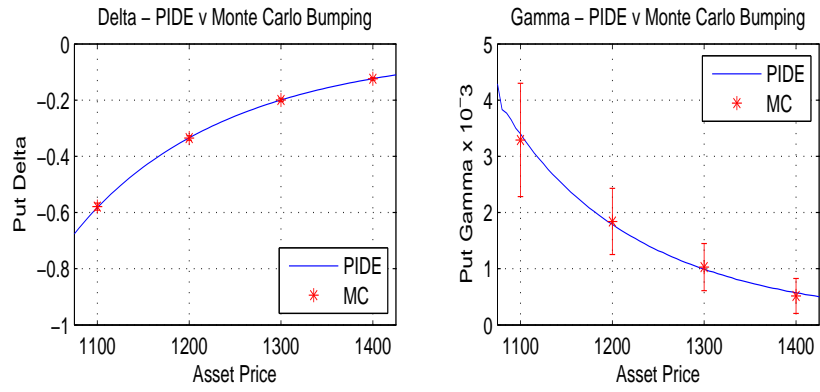


Figure 9.6: Bumped Monte Carlo Greeks versus PIDE Greeks

Chapter 10

Conclusions

In this study of option pricing under the exponential Variance Gamma process, a range of numerical techniques have been used to price European, Bermudan and American options and, subject to the unavoidable uncertainty in Monte Carlo estimates, the results have been shown to be consistent.

The Longstaff-Schwartz method has been shown to give much better results in these calculations if only 'in the money' data is used in its regression calculations.

The exercise boundaries for an American put generated by the PIDE and the Monte Carlo methods are dissimilar in shape and, for the parameters used in the study, do not trend to the strike at maturity, which is behaviour observed in other studies [1]. As this observation is in conflict with the shape of the boundary published in [7] an analysis was carried to test the sensitivity of the option price to the shape of the boundary.

A possibility for further work would be to determine if the model parameters used in this report would be expected to lead to this behaviour, based on the analysis in [1].

The Delta and Gamma greeks have been calculated by a variety of methods which have been shown to produce consistent results.

For the parameters used in this work, it has been observed that the 'smooth-pasting' condition does not apply for the American put, i.e. the gradient of the value function is discontinuous across the exercise boundary.

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