

Models for indices



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I dedicate this thesis to my god-grandmother who did not quite live to
see this.

Abstract

We consider a market index model for a large portfolio of risky assets traded in the stock market where the correlation is due to a market factor. By taking the limit of a simple systems of stochastic differential equations (SDEs), we obtain a limit stochastic differential equation (SDE) for the index price. We also investigated the limit empirical measure for the infinite system. The density evolves according to a stochastic partial differential equation (SPDE) which we also solve in some special cases. Using our limit SDE, we also try to compare its accuracy numerically with the “real index” calculated from summing the individual stock prices. Lastly, we make use of our limit SDE and empirical density to price different derivatives such as the European call & put option on the index and the European call & put option on the maximum of the constituent stock in the index.

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Chapter 1

Introduction

A stock market index is a method of measuring the performance of a portfolio of stocks in a stock market. Stock indices have often been used as an indicator or benchmark to evaluate the performance of certain particular stock market. Most commonly known indices are the ones that have since been associated with the performance of the stock market of a particular country or sector. Two of the best known ones include the Dow Jones Industrial Average (DJIA) and Standard and Poor's (S&P) 500 Index which are often considered as a bellwether for the American economy. NASDAQ Composite is another index which is widely cited for the performance of stocks of technology companies and growth companies. The most regularly quoted market indices are national indices composed of the stocks of large companies listed on a nation's largest stock exchanges, such as the American S&P 500, the Japanese Nikkei 225, the Russian RTSI, the Indian SENSEX and the British FTSE 100.¹ These are often referred to as the 'broad-based' index since they are broad and cover a larger number of stocks. They are usually used to represent the performance of a "whole market".

There are another type of stock index called the "sector index". These are usually indices which are more specialised and only track the performance of a particular sector of the stock market. Some example include the Dow Jones U.S. Oil & Gas Index which measures the performance of the oil and gas sector of the United States equity market and the Wilshire US REIT which tracks more than 80 American real estate investment trusts.²

There are also 'world' or 'global' stock market index that includes (typically large) companies without regard for where they are domiciled or traded. Two examples are MSCI World and S&P Global 100.³ Market indices are deemed important due to

¹http://en.wikipedia.org/wiki/Stock_market_index

²<http://www.learn-stock-options-trading.com/stock-market-index.html>

³http://en.wikipedia.org/wiki/Stock_market_index

several reasons. They are often used⁴

- to gauge the performance of the stock market (as mentioned above)
- as a benchmark to compare the performance of investors' individual portfolio, mutual funds, and exchange traded funds
- as a forecasting tool for future trends
- as the underlying for options, futures and other derivative products

Therefore it is important for the investors to understand the relationship between the indices and the individual stocks in order to utilise the information provided by the stock indices in a useful way whether it is to evaluate the relative performances of their portfolios or to estimate the general trends in the securities market.

Mathematically speaking, a stock market is a number computed to measure and track the value of a portfolio of stocks used to represent the stock market.⁵ There are numerous ways of constructing an index, and each of which has its own advantages and disadvantages.⁶ Since an index is supposed to be created by selecting a group of stocks that are capable of representing the whole market or a specified sector or segment of the market by summarising hundreds of price movements, it is unavoidable that much information is lost.

There are mainly 3 principal weighting schemes used: ⁷⁸

1. Price-weighted index
2. Capitalisation-weighted index (otherwise known as Value-weighted index)
3. Equally-weighted index (otherwise known as unweighted index)

A price-weighted index is calculated by summing the prices of the constituent stocks and dividing by a divisor.⁹

$$\text{Price-weighted Index} = \frac{\sum_i S_t^i}{\text{index divisor}}$$

⁴page 74, All About Stocks, 3E By Esme Faerber

⁵Page 110, Commodity and Financial Derivatives by S. Kevin

⁶Page 4, Stock Index Futures

⁷Page 155-159, Investments analysis and portfolio management, 6th Edition by F. Reilly and K. Brown

⁸Page 100, Schaum's Outline of Investments By Jack Clark Francis, Richard W. Taylor

⁹http://faculty.msb.edu/bodurthj/teaching/optinpt/S&P_on_IndexCalculations.pdf

where S_t^i is the price of the constituent stock at time t . The divisor begins as the number of constituent stocks but changes over time to take into account of stock splits, changes in constituent stocks, etc.¹⁰ The Dow Jones Industrial Average is price weighted since in 1890 before computers were invented, the easiest thing to do was to sum up the prices and divide by a divisor.¹¹ Other examples include Amex Major Market Index, and the NYSE ARCA Tech 100 Index. Since the price of each individual stock is the only factor determining the value of the index, price movement of a single security will heavily influence the value of the index and it also ignores the relative size of the company as a whole. Thus, the higher the price, the more weight the stock has in the index. A common criticism towards this type of indices is usually surrounding the fact that there is no reason to believe that higher priced stocks are more important than lower priced stocks (especially when stock splits, reverse stock splits, etc occur)

A capitalisation-weighted (or value-weighted) index is calculated by totalling the market capitalisation of the constituent stocks,

$$\text{Capitalisation-weighted Index} = \frac{\sum_i (\eta_i(t) S_t^i)}{\sum_i (\eta_i(0) S_0^i)} \times \text{Beginning Index Value} = \frac{\sum_i (\eta_i(t) S_t^i)}{\text{index divisor}}$$

where $\eta_i(t)$ and S_t^i are the number of shares and respectively the price of constituent stocks at time t . So, we can see that the larger stocks would have a greater influence on the index as compared to a smaller market cap company. The best known capitalisation-weighted index are the S&P 500, Hang Seng Index, etc.

The above two calculations for the indices index divisor involve a division by an *index divisor*. An index divisor is not constant but can change as the capitalisation of the index changes. In the event of additions and deletions to the index, rights issues, stock splits, declaration of special dividends, share buybacks and issuance's, and spin-offs, the index value might change. In order to account for this and to ensure continuity in index values, the divisor will change accordingly to reflect the relative value of the index.

An equally-weighted index (or unweighted index) would consist of giving each stock an equal weight in the index irrespective of their market capitalisation (market values) or stock prices.^{12,13} In other words, one can understand unweighted index as investing the same proportion of money in all the constituent stocks of the index

¹⁰clem.mscd.edu/~mayest/FIN3600/Files/FIN3600.5.ppt

¹¹http://faculty.msb.edu/bodurthj/teaching/optinpt/S&P_on_IndexCalculations.pdf

¹²Page 75, All About Stocks, 3E By Esme Faerber

¹³<http://www.investorconcepts.com/premium-content/understanding-stock-market-indices>

irrespective of price or market value. Some unweighted indices use arithmetic average, while others use geometric average to compute returns. In general the returns computed using geometric average is usually lower than arithmetic average. An example is Value Line index which is composed of 1626 publicly traded stocks on the NYSE, NASDAQ, American Stock Exchange, and Toronto Stock Exchange.

For arithmetic average,

$$\text{Equally-weighted index}_t^{(A)} = \text{Equally-weighted index}_{t-1} \times \frac{1}{N} \sum_i \frac{S_t^i}{S_{t-1}^i}$$

For geometric average,

$$\text{Equally-weighted index}_t^{(G)} = \text{Equally-weighted index}_{t-1} \times \sqrt[N]{\prod_i \frac{S_t^i}{S_{t-1}^i}}$$

that is to say the percentage increase in the index is the same as the (arithmetic or geometric) average return of all its constituent stocks. This type of index is comparatively less common as compared to the capitalisation-weighted or price-weighted index. This index differs from the other two indexes, where some stocks are given more weight than others, since all the stocks are equally weighted and one stock's performance will not have a dramatic effect on the performance of the index as a whole.¹⁴

Currently in the market, there are numerous tradable index options and futures. One of the reason for the popularity of index options is the fact that they are options on the market as a whole. Many investors prefer analysing the market as a whole and use index options to act on their forecasts¹⁵ They are traded on Chicago Board Options Exchange (CBOE).

We have thus seen why stock indices are important and how they can be constructed. In this thesis, we will explore more deeply into a particular model of the index and try to derive certain results from there. In the model that we will be looking into, we will be dealing with the price-weighted and capitalisation-weighted index. In Chapter 2, we will lay out the some mathematical tools and introduce the general mathematical model that we will be looking into. In Chapter 3, we will be looking at the model with constant parameters (drift, volatility and correlation), developing further extensions from previous results obtained by [6]. Then, in Chapter 4, we consider extending this model to random parameters. Chapter 4 is divided into different sections each dealing with one randomised parameter. We summarise our

¹⁴<http://www.investopedia.com/terms/u/unweightedindex.asp>

¹⁵Page 42, Introduction to Derivatives and Risk Management By Don M. Chance, Robert Brooks

result for a more general model in Chapter 5. Next, we consider a specific model with time dependent weights in Chapter 6. In Chapter 7, we look at the empirical measure (empirical density), by describing the limit of a simple systems of stochastic partial differential equation (SDEs) as a stochastic partial differential equation (SPDE). In Chapter 8, we will investigate numerically some of the approximations using the limit SDEs presented in the previous chapters. Finally, we look at how we can price some index options using our limit SDE and empirical measure in Chapter 9.

Chapter 2

Mathematical Setup

2.1 Some Preliminary Mathematical Background

In this thesis, we will be requiring a general form of the strong law of large numbers. We will need to refer to the fundamental result of Jamison, Orey and Pruitt for the Convergence of Weighted Averages of Independent Random Variables. We refer to the papers [7, 9] for the following:

Theorem 2.1.1 (The Strong Law). *Define the total weight by $A_N = \sum_{i=1}^N \alpha_i$ and the weighted sum $S_N = \sum_{i=1}^N \alpha_i X_i$. Let $\{X_i : i \geq 1\}$ be a sequence of pairwise independent, identically distributed random variables such that $\mathbb{E}|X_1| < \infty$. Let $\{\alpha_i : i \geq 1\}$ be a set of positive weights such that $A_N = \sum_{i=1}^N \alpha_i \rightarrow \infty$ and $\frac{\alpha_N}{A_N} \rightarrow 0$. Let $N(x) = \#\{n : \frac{A_n}{\alpha_n} \leq x\}$ be the distribution function of $\{\frac{A_n}{\alpha_n} : n \geq 1\}$. If*

$$\int x^2 \int_{y \geq |x|} \frac{N(y)}{y^3} dy dF(x) = \int_0^1 x \mathbb{E}N\left(\frac{|X_1|}{x}\right) dx < \infty$$

Then

$$\Rightarrow \frac{1}{A_N} \sum_{i=1}^N \alpha_i X_i \xrightarrow{a.e.} \mathbb{E}X_1 \quad (2.1)$$

2.2 Model Setup

In what follows, our model is based on the model used in [5, 8, 10, 6]. Let $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ denote a probability space for a market consisting of N different companies whose asset values (stock price values) S_t at time t evolve under the historical measure $\mathbb{P}^N$ according to a diffusion process given by

$$dS_t^i = \mu^i S_t^i dt + \sigma^i \sqrt{1 - \rho^i} S_t^i dW_t^i + \sigma^i \sqrt{\rho^i} S_t^i dM_t, \quad i = 1, \dots, N \quad (2.2)$$

and a risk free asset, S_t^0 , with risk free rate r evolving according to

$$dS_t^0 = rS_t^0 dt$$

where μ^i , ρ^i and σ^i are all i.i.d. random variables, $\sigma^i > 0$ and $\rho^i \in [0, 1)$ is the correlation. Note the co-dependence between asset prices is provided solely by the Brownian motion M_t which can be thought of as a market wide factor influencing all of the assets. We also assume W_t^i and M_t are independent Brownian Motion satisfying

$$d[W_t^i, M_t] = 0 \quad \forall i = 1, \dots, N$$

and

$$d[W_t^i, W_t^j] = \delta_{ij} dt \quad \forall i, j = 1, \dots, N$$

where we have written $[.,.]$ for the quadratic covariation and will use $[.]$ for the quadratic variation.

We will assume that $\{S_0^1, \dots, S_0^N\}$ is a family of exchangeable, $[0, \infty)$ -valued random variables with $\mathbb{E}(S_0^i) < \infty$ since it does not matter how we label our assets. Unless otherwise stated, throughout this thesis, we will always consider a quantity of the form

$$I_t^N = \frac{1}{A_N(t)} \sum_{i=1}^N \alpha_i(t) S_t^i \quad (2.3)$$

where $\alpha_i(t) > 0$, $\forall i = 1, \dots, N$ is a function of time and represents the weight allocated to the underlying asset S^i at time t whereas $A_N(t) = \sum_{i=1}^N \alpha_i(t)$.

As mentioned in our introduction, stock indices are most commonly either (1) price-weighted or (2) capitalisation-weighted. With this notation, the stock market index I_t will then be $A_N I_t^N$. The weights $\alpha_i(t)$ will then be either (1) equal to 1 or (2) the number of shares at time t .¹

By taking the limit as $N \rightarrow \infty$, the index can then be approximated by $A_N I_t^\infty$ where I_t^∞ is the limit of I_t^N as N tends to infinity.

By applying Itô's Lemma, we could solve (2.2) in the usual way to get

$$S_t^i = S_0^i e^{(\mu^i - \frac{1}{2}(\sigma^i)^2)t + \sigma^i \sqrt{\rho^i} M_t + \sigma^i \sqrt{1-\rho^i} W_t^i} \quad (2.4)$$

giving

$$I_t^N = \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_0^i e^{(\mu^i - \frac{1}{2}(\sigma^i)^2)t + \sigma^i \sqrt{\rho^i} M_t + \sigma^i \sqrt{1-\rho^i} W_t^i} \quad (2.5)$$

¹Although as seen in our introduction, it is also divided by a scaling factor, called the Index Divisor, to rescale it to a smaller number which is easier to handle

Chapter 3

Model with all constant parameters

We will consider first a simple model in which all assets have the same constant volatility, drift and are correlated via a single market factor, i.e. $\mu^i = \mu$, $\rho^i = \rho$ and $\sigma^i = \sigma$ for all i . We also take the weights α_i as fixed constants.

$$I_t^N = \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i \quad (3.1)$$

where α_i is the weight allocated to the underlying assets S^i and $A_N = \sum_{i=1}^N \alpha_i$. Hence, the stock index $I_t = A_N I_t^N = \sum_{i=1}^N \alpha_i S_t^i$.

This model was considered in [6]. In the following, we will present a different method and derive further results from this.

3.1 Literature review

We will attempt to re-derive some of the results from [6] using a different approach in the next section. For now, we shall review some of the results proven in [6] and try to establish a relationship between the approach [6] and our new approach. From (2.3), we can differentiate and substitute (2.2) in to get the following:

$$dI_t^N = \mu I_t^N dt + \sigma \sqrt{\rho} I_t^N dM_t + \sigma \sqrt{1-\rho} \left(\frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i dW_t^i \right) \quad (3.2)$$

We have the following theorem, which is Theorem 1.1 in [6],

Theorem 3.1.1. *Under the following assumptions*

$$\sum_{N=1}^{\infty} \frac{1}{A_N^2} < \infty \quad (3.3)$$

$$\lim_{N \rightarrow \infty} \frac{1}{A_N} \sum_{i=1}^N \alpha_i^2 (S_0^i)^2 = C \quad (3.4)$$

where C is a constant, the idiosyncratic term tends to zero

$$d\Psi_N = \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i \phi'(S_t^i) dW_t^i \rightarrow 0$$

as $N \rightarrow \infty$

Proof. Please refer to [6] for proof. □

We also want to prove the following

Lemma 3.1.2. *Under the conditions in Theorem 2.1.1, with the condition that*

$$\frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} \rightarrow 0 \text{ as } N \rightarrow \infty$$

and assuming S_0^i and W^i are independent, we also have the idiosyncratic term tending to zero

$$d\Psi_N = \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i \phi'(S_t^i) dW_t^i \rightarrow 0$$

as $N \rightarrow \infty$

Proof.

$$\langle \Psi_N \rangle_t = \int_0^t \frac{1}{A_N^2} \sum_{i=1}^N \alpha_i^2 (S_u^i)^2 \phi'(S_u^i)^2 du$$

as $N \rightarrow \infty$, we have from Theorem 2.1.1 that

$$\frac{1}{\left(\sum_{i=1}^N \alpha_i^2\right)} \sum_{i=1}^N \alpha_i^2 (S_u^i)^2 \phi'(S_u^i)^2 \rightarrow \mathbb{E} \left((S_u^i)^2 \phi'(S_u^i)^2 \right)$$

moreover, we know that for $\alpha_i > 0$, by induction,

$$A_N^2 = \left(\sum_{i=1}^N \alpha_i \right)^2 > \left(\sum_{i=1}^N \alpha_i^2 \right) \quad (3.5)$$

$$0 < \frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} < 1 \quad (3.6)$$

therefore, we have

$$\begin{aligned}\langle \Psi_N \rangle_t &= \frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} \int_0^t \frac{1}{\sum_{i=1}^N \alpha_i^2} \sum_{i=1}^N \alpha_i^2 (S_u^i)^2 \phi'(S_u^i)^2 du \\ &\leq \frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} \int_0^t \mathbb{E}((S_u)^2 \phi'(S_u)^2) du\end{aligned}$$

As $\phi \in C_0^\infty(\mathbb{R})$, there exist a constant K_ϕ such that $|\phi'| \leq K_\phi$

$$\begin{aligned}&\leq \frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} K_\phi^2 \int_0^t \mathbb{E}_{S_0, W} \left(S_0^2 e^{2(\mu - \frac{1}{2}\sigma^2)u + 2\sigma\sqrt{\rho}M_u + 2\sigma\sqrt{1-\rho}W_u} \right) du \\ &= \frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} K_\phi^2 \int_0^t \mathbb{E}(S_0^2) \mathbb{E}_{W_u} \left(e^{2(\mu - \frac{1}{2}\sigma^2)t + 2\sigma\sqrt{\rho}M_u + 2\sigma\sqrt{1-\rho}W_u} \right) du\end{aligned}$$

for S_0^i, W_u^i independent

$$\begin{aligned}&= \frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} K_\phi^2 \int_0^t e^{2(\mu - \frac{1}{2}\sigma^2 - \sigma^2\rho)u + 2\sigma\sqrt{\rho}M_u} \mathbb{E}S_0^2 du \\ &\rightarrow 0 \quad \text{a.s. , by assumption that } \frac{\sum_{i=1}^N \alpha_i^2}{A_N^2} \rightarrow 0 \text{ as } N \rightarrow \infty\end{aligned}$$

We know that a martingale with quadratic variation which equals to 0 is almost surely constant and equal to its initial value (c.f. eg Chapter II in [14], [12]). Therefore, as $N \rightarrow \infty$, we have $\Psi_N(t) \rightarrow \Psi_N(0)$ which is equal to 0. $\square$

Therefore, by taking $\phi(x) = x$, we have $\frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i dW_t^i \xrightarrow{N \rightarrow \infty} 0$ and we obtained the ‘first approximation’ $I_t^{(1)}$ to I_t^∞ .

$$dI_t^{(1)} = \mu I_t^{(1)} dt + \sigma \sqrt{\rho} I_t^{(1)} dM_t \quad (3.7)$$

This is the form that is presented in [6]. Herein, $I^{(1)}$ follows a Black-Scholes model with a different volatility which depends on the correlation (market) factor ρ .

3.2 Further results

Next, we attempt to study further the idiosyncratic term $d\Psi_N = \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i dW_t^i$ in (3.2). We have

$$d[\Psi]_N = \frac{1}{A_N^2} \sum_{i=1}^N \alpha_i^2 (S_t^i)^2 dt \quad (3.8)$$

Let

$$dB_t^1 = \frac{\sum_{i=1}^N \alpha_i S_t^i dW_t^i}{\left(\sum_{i=1}^N \alpha_i^2 (S_t^i)^2\right)^{\frac{1}{2}}}$$

Then B_t^1 is a Brownian motion by Lévy's characterisation since $d[B^1]_t = dt$. Substituting this back, we have

$$d\Psi_N = \sqrt{\theta} dB_t^1 \quad (3.9)$$

where $\theta = \frac{1}{A_N^2} \sum_{i=1}^N \alpha_i^2 (S_t^i)^2$

Next, we define

$$\theta_k = \frac{1}{A_N^k} \sum_{i=1}^N \alpha_i^k (S_t^i)^k$$

and process B_t^k

$$dB_t^k = \frac{\sum_{i=1}^N \alpha_i^k (S_t^i)^k dW_t^i}{\left(\sum_{i=1}^N \alpha_i^{2k} (S_t^i)^{2k}\right)^{\frac{1}{2}}}$$

which are again Brownian motion by Lévy's characterisation.

Applying Itô's lemma,

$$\begin{aligned} d\theta_k &= \frac{1}{A_N^k} \sum_{i=1}^N \alpha_i^k \left(k(S_t^i)^{k-1} dS_t^i + \frac{1}{2} k(k-1) (S_t^i)^{k-2} d[S^i]_t \right) \\ &= \left(k\mu + \frac{k(k-1)}{2} \sigma^2 \right) \theta_k dt + k\sigma \sqrt{\rho} \theta_k dM_t + k\sigma \sqrt{1-\rho} \frac{1}{A_N^k} \sum_{i=1}^N \alpha_i^k (S_t^i)^k dW_t^i \\ &= \left(k\mu + \frac{k(k-1)}{2} \sigma^2 \right) \theta_k dt + k\sigma \sqrt{\rho} \theta_k dM_t + k\sigma \sqrt{1-\rho} \frac{1}{A_N^k} \left(\sum_{i=1}^N \alpha_i^{2k} (S_t^i)^{2k} \right)^{\frac{1}{2}} dB_t^k \\ \Rightarrow d\theta_k &= \left(k\mu + \frac{k(k-1)}{2} \sigma^2 \right) \theta_k dt + k\sigma \sqrt{\rho} \theta_k dM_t + k\sigma \sqrt{1-\rho} (\theta_{2k})^{\frac{1}{2}} dB_t^k \quad (3.10) \end{aligned}$$

Thus, we have obtained a recurrence equation. We summarise our results as follow:

$$\therefore d\Psi_N = \sqrt{\theta_2} dB_t^1 \quad (3.11)$$

where θ_k for $k \geq 2$ follows the above recurrence relation.

$$dI_t^N = \mu I_t^N dt + \sigma \sqrt{\rho} I_t^N dM_t + \sigma \sqrt{1-\rho} \sqrt{\theta_2} dB_t^1 \quad (3.12)$$

Applying (3.10) repeatedly, we can obtain approximations for I_t^N . For the first few approximations, we have

First approximation

$$dI_t^{(1)} = \mu I_t^{(1)} dt + \sigma \sqrt{\rho} I_t^{(1)} dM_t \quad (3.13)$$

Second approximation

$$\begin{aligned} dI_t^{(2)} &= \mu I_t^{(2)} dt + \sigma \sqrt{\rho} I_t^N dM_t + \sigma \sqrt{1-\rho} \sqrt{\theta_2} dB_t^1 \\ \text{where } d\theta_2 &= (2\mu + \sigma^2)\theta_2 dt + 2\sigma \sqrt{\rho} \theta_2 dM_t \end{aligned} \quad (3.14)$$

Third approximation

$$\begin{aligned} dI_t^{(3)} &= \mu I_t^{(3)} dt + \sigma \sqrt{\rho} I_t^N dM_t + \sigma \sqrt{1-\rho} \sqrt{\theta_2'} dB_t^1 \\ \text{where } d\theta_2' &= (2\mu + \sigma^2)\theta_2' dt + 2\sigma \sqrt{\rho} \theta_2' dM_t + 2\sigma \sqrt{1-\rho} (\theta_4)^{\frac{1}{2}} dB_t^2 \\ \text{and } d\theta_4 &= (4\mu + 6\sigma^4)\theta_4 dt + 4\sigma \sqrt{\rho} \theta_4 dM_t \\ &\vdots \end{aligned} \quad (3.15)$$

whereby for each approximation we have $(\theta_{2k})^{\frac{1}{2}} dB_t^k = \frac{1}{A_N^k} \sum_{i=1}^N \alpha_i^k (S_t^i)^k dW_t^i \rightarrow 0$ for all k , using Theorem 3.1.1 repeatedly with a different weight α_i and a suitable $\phi(x)$.

3.3 Alternative method

In this section, we shall derive a similar result to that presented in the above section using an alternative approach. Using (2.5), we have

$$\begin{aligned} I_t^N &= \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_0^i e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \sqrt{\rho} M_t + \sigma \sqrt{1-\rho} W_t^i} \\ &= e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \sqrt{\rho} M_t} \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_0^i e^{\sigma \sqrt{1-\rho} W_t^i} \end{aligned}$$

If S_0^i and W^i are independent for each i and under the condition (2.1.1) as presented in Theorem 2.1.1, as $N \rightarrow +\infty$,

$$\begin{aligned} I_t^\infty &= e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \sqrt{\rho} M_t} \mathbb{E} \left(S_0 e^{\sigma \sqrt{1-\rho} W_t} \right) \\ &= \mathbb{E} (S_0) e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \sqrt{\rho} M_t} e^{\frac{1}{2}\sigma^2(1-\rho)t} \\ &= \mathbb{E} (S_0) e^{(\mu - \frac{1}{2}\sigma^2\rho)t + \sigma \sqrt{\rho} M_t} \end{aligned} \quad (3.16)$$

where we have used our knowledge of the moment generating function of $W_t^i \stackrel{i.i.d.}{\sim} N(0, t)$

By Itô's lemma, we obtain our 'first approximation' again.

$$\begin{aligned} dI_t^\infty &= \left(\mu - \frac{1}{2} \sigma^2 \rho \right) I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t + \frac{1}{2} \sigma^2 \rho I_t^\infty dt \\ &= \mu I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t \end{aligned} \tag{3.17}$$

Chapter 4

Models with random parameters

4.1 Model with random drift

Taking constant time-independent weights α_i , we will now consider a model in which all assets have the same constant volatility and are correlated via a single market factor, i.e. $\rho^i = \rho$ and $\sigma^i = \sigma$ for all i , but a different drift μ^i , each being i.i.d. random variables. There are two cases that we are particularly interested in, the drift being

1. normally distributed, i.e. $\mu^i \stackrel{iid}{\sim} N(\bar{\mu}, \xi^2)$, and
2. uniformly distributed, i.e. $\mu^i \stackrel{iid}{\sim} U[a, b]$

These two cases are particularly interesting not just because of the relative ease of getting a closed form solution for the first approximation of the SDE. The drift is the instantaneous rate of change of the mean of each constituent stock. (1) For the normality case, if we assume that we have a large sample, it is reasonable to assume that the drift will follow a normal distribution as well (2) For the uniform case, it will also be interesting to see in the event that it is equally likely to have different drifts for the constituent stocks.

4.1.1 Study on general drift conditions

Using Theorem 2.1.1 again, under the assumptions (2.1.1) and S_0^i , μ^i and W^i all being independent, then

$$\begin{aligned} I_t^N &= \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_0^i e^{(\mu^i - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t^i} \\ &= e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{\rho}M_t} \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_0^i e^{\mu^i t} e^{\sigma\sqrt{1-\rho}W_t^i} \end{aligned} \quad (4.1)$$

since $\frac{1}{A_N} \sum_{i=1}^N \alpha_i S_0^i e^{\mu^i t} e^{\sigma \sqrt{1-\rho} W_t^i} \xrightarrow{N \rightarrow \infty} \mathbb{E} (S_0 e^{\mu t} e^{\sigma \sqrt{1-\rho} W_t})$ using (2.1.1) and the independence condition gives

$$I_t^\infty = e^{-\frac{1}{2}\sigma^2 t + \sigma \sqrt{\rho} M_t} \mathbb{E} \left(S_0 e^{\mu t} e^{\sigma \sqrt{1-\rho} W_t} \right) \quad (4.2)$$

$$= \mathbb{E}(S_0) \mathbb{E}(e^{\mu t}) e^{-\frac{1}{2}\sigma^2 \rho t + \sigma \sqrt{\rho} M_t} \quad (4.3)$$

by applying Itô's lemma under certain conditions to take derivative under the integral (in the expectation $\mathbb{E}$) and simplifying, we obtain

$$\begin{aligned} dI_t^\infty &= \mathbb{E}(S_0) \mathbb{E}(\mu e^{\mu t}) e^{-\frac{1}{2}\sigma^2 \rho t + \sigma \sqrt{\rho} M_t} dt + \sigma \sqrt{\rho} I_t^\infty dM_t \\ \Rightarrow dI_t^\infty &= \frac{\mathbb{E}(\mu e^{\mu t})}{\mathbb{E}(e^{\mu t})} I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t \end{aligned} \quad (4.4)$$

To investigate the asymptotic behaviour of the drift condition in the short time limit, we do a Taylor series expansion,

$$\begin{aligned} \frac{\mathbb{E}(\mu e^{\mu t})}{\mathbb{E}(e^{\mu t})} &= \frac{\mathbb{E}(\mu) + \mathbb{E}(\mu^2)t + \frac{1}{2}\mathbb{E}(\mu^3)t^2 + \dots}{1 + \mathbb{E}(\mu)t + \frac{1}{2}\mathbb{E}(\mu^2)t^2 + \dots} \\ &\approx \left(\mathbb{E}(\mu) + \mathbb{E}(\mu^2)t + \frac{1}{2}\mathbb{E}(\mu^3)t^2 + \dots \right) \left(1 - \mathbb{E}(\mu)t - \frac{1}{2}\mathbb{E}(\mu^2)t^2 + \dots \right) \quad , \text{for small } t \\ &= \mathbb{E}(\mu) + (\mathbb{E}(\mu^2) - (\mathbb{E}(\mu))^2)t + \mathcal{O}(t^2) \\ &= \bar{\mu} + \xi^2 t + \dots \end{aligned} \quad (4.5)$$

Therefore in the short time limit as $t \rightarrow 0$, we have the asymptotic form

$$\Rightarrow dI_t^\infty \sim (\bar{\mu} + \xi^2 t) I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t \quad (4.6)$$

In the limit of $\xi \rightarrow 0$, that is the case when μ is constant, the expression $\frac{\mathbb{E}(\mu e^{\mu t})}{\mathbb{E}(e^{\mu t})} = \frac{\mu e^{\mu t}}{e^{\mu t}} = \mu$, whereby we recovered our 'first approximation' SDE in (3.7).

4.1.2 Normally distributed drift

We now proceed to consider the first case. We continue from (4.3)

$$I_t^N = \mathbb{E}(S_0) \mathbb{E}(e^{\mu t}) e^{-\frac{1}{2}\sigma^2 \rho t + \sigma \sqrt{\rho} M_t} \quad (4.7)$$

$$= \mathbb{E}(S_0) e^{\bar{\mu} t + \frac{1}{2}\xi^2 t^2} e^{-\frac{1}{2}\sigma^2 \rho t + \sigma \sqrt{\rho} M_t} \quad (4.8)$$

using our knowledge of the moment generating function of normal distribution, $\mathbb{E}(e^{\mu t}) = e^{\bar{\mu} t + \frac{1}{2}\xi^2 t^2}$ and $\mathbb{E}(e^{\sigma \sqrt{1-\rho} W_t}) = e^{\sigma^2(1-\rho)t}$, and then applying Itô's lemma, we obtain

$$\Rightarrow dI_t^\infty = (\bar{\mu} + \xi^2 t) I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t \quad (4.9)$$

Alternatively, from (2.3), and substituting in (2.2), we get

$$\begin{aligned}
dI_t^N &= \frac{1}{A_N} \sum_{i=1}^N \alpha_i \mu^i S_t^i dt + \sigma \sqrt{1-\rho} \left(\frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i dW_t^i \right) + \sigma \sqrt{\rho} I_t^N dM_t \\
&= \frac{1}{A_N} \sum_{i=1}^N \alpha_i \mu^i S_0^i e^{(\mu^i - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t^i} dt \\
&\quad + \sigma \sqrt{1-\rho} \left(\frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i dW_t^i \right) + \sigma \sqrt{\rho} I_t^N dM_t
\end{aligned} \tag{4.10}$$

Under conditions in Theorem 2.1.1 & 3.1.1 mentioned before, $\frac{1}{A_N} \sum_{i=1}^N \alpha_i S_t^i dW_t^i \xrightarrow{N \rightarrow \infty} 0$ and $\frac{1}{A_N} \sum_{i=1}^N \alpha_i \mu^i S_0^i e^{(\mu^i - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t^i} \xrightarrow{N \rightarrow \infty} \mathbb{E} \left(\mu S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t} \right)$,

$$\begin{aligned}
dI_t^\infty &= \mathbb{E}(S_0) \mathbb{E}(\mu e^{\mu t}) \mathbb{E}(\sigma \sqrt{1-\rho} W_t) e^{-\frac{1}{2}\sigma^2 t dt + \sigma\sqrt{\rho}M_t} dt + \sigma \sqrt{\rho} I_t^\infty dM_t \\
&= (\bar{\mu} + \xi^2 t) I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t
\end{aligned} \tag{4.11}$$

where we used the fact $\mathbb{E}(\mu e^{\mu t}) = \frac{\partial}{\partial t} \mathbb{E}(e^{\mu t}) = (\bar{\mu} + \xi^2 t) e^{\bar{\mu}t + \frac{1}{2}\xi^2 t^2}$ and the result (4.8) to obtain our previous result as in (4.9).

We see that I_t^∞ now follows an SDE with a time-dependent drift. The time dependence increases with the increase in the variance of μ . In the limit that variance of the drift of the constituent stocks becomes zero, $\xi \rightarrow 0$, the drift in the limit SDE becomes constant, $\mu \rightarrow \bar{\mu}$, and we recover (3.7).

We also note that the drift is exactly of the asymptotic form in the short time limit (4.6) where we don't have the higher order terms.

4.1.3 Uniformly distributed drift

In a similar fashion, we now consider the case when the drift of each individual stock is uniformly distributed between a & b , i.e. $\mu \sim U[a, b]$. Proceeding from (4.3)

$$\begin{aligned}
I_t^\infty &= \mathbb{E}(S_0) \mathbb{E}(e^{\mu t}) e^{-\frac{1}{2}\sigma^2 \rho t + \sigma\sqrt{\rho}M_t} \\
&= \mathbb{E}(S_0) e^{-\frac{1}{2}\sigma^2 \rho t + \sigma\sqrt{\rho}M_t} \left(\frac{e^{tb} - e^{ta}}{t(b-a)} \right)
\end{aligned}$$

using $\mathbb{E}(e^{\mu t}) = \frac{e^{tb} - e^{ta}}{t(b-a)}$ and $\mathbb{E}(e^{\sigma\sqrt{1-\rho}W_t}) = e^{\sigma^2(1-\rho)t}$, and then applying Itô's lemma, we obtain

$$\Rightarrow dI_t^\infty = \sigma \sqrt{\rho} I_t^\infty dM_t + \left(-\frac{1}{t} \right) I_t^\infty dt + \mathbb{E}(S_0) e^{-\frac{1}{2}\sigma^2 \rho t + \sigma\sqrt{\rho}M_t} \left(\frac{be^{tb} - ae^{ta}}{t(b-a)} \right) dt \tag{4.12}$$

$$= \left(-\frac{1}{t} + \frac{be^{tb} - ae^{ta}}{e^{tb} - e^{ta}} \right) I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t \tag{4.13}$$

Let $\frac{1}{2}(b - a) = d$, $\bar{\mu} = \mathbb{E}(\mu) = \frac{1}{2}(b + a)$ and $\xi^2 = \text{Var}(\mu) = \frac{1}{12}(b - a)^2 = \frac{1}{3}d^2$, then $a = \bar{\mu} - d$ and $b = \bar{\mu} + d$

$$\begin{aligned} \frac{be^{tb} - ae^{ta}}{e^{tb} - e^{ta}} &= \frac{(\bar{\mu} + d)e^{t(\bar{\mu}+d)} - (\bar{\mu} - d)e^{t(\bar{\mu}-d)}}{e^{t(\bar{\mu}+d)} - e^{t(\bar{\mu}-d)}} \\ &= \frac{(\bar{\mu} + d)e^{td} - (\bar{\mu} - d)e^{-td}}{e^{td} - e^{-td}} \\ &= \bar{\mu} + d \coth(td) \\ &= \bar{\mu} + \sqrt{3}\xi \coth(t\sqrt{3}\xi) \end{aligned}$$

$$\therefore dI_t^\infty = \left(\bar{\mu} + \sqrt{3}\xi \coth(t\sqrt{3}\xi) - \frac{1}{t} \right) I_t^\infty dt + \sigma\sqrt{\rho}I_t^\infty dM_t \quad (4.14)$$

We note that we again obtain an SDE with a time-dependent drift for I_t^∞ . As in the above case when μ is normally distributed, to see the behaviour of the drift in the limit SDE when $\xi \rightarrow 0$, we look at the Laurent series of $\coth(x) = x^{-1} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \dots$, $0 < |x| < \pi$. Therefore, our drift term can be expanded locally around the origin to get

$$\begin{aligned} \bar{\mu} + \sqrt{3}\xi \coth(t\sqrt{3}\xi) - \frac{1}{t} &= \bar{\mu} + \sqrt{3}\xi \left(\frac{1}{t\sqrt{3}\xi} + \frac{t\sqrt{3}\xi}{3} - \frac{(t\sqrt{3}\xi)^3}{45} + \dots \right) - \frac{1}{t} \\ &= \bar{\mu} + \sqrt{3}\xi \left(\frac{1}{t\sqrt{3}\xi} + \frac{t\sqrt{3}\xi}{3} - \frac{(t\sqrt{3}\xi)^3}{45} + \dots \right) - \frac{1}{t} \\ &= \bar{\mu} + \xi^2 t + \dots \end{aligned} \quad (4.15)$$

hence we see that again the limit SDE reduces to (3.7) when the drift, μ , is constant (i.e. $\xi = 0$). Also, in the limit of $t \rightarrow 0$,

$$dI_t^\infty \sim (\bar{\mu} + \xi^2 t) I_t^\infty dt + \sigma\sqrt{\rho}I_t^\infty dM_t \quad (4.16)$$

which is the asymptotic form in 4.6 which we obtained earlier in our consideration of the general drift condition.

Note also that in the limit of $t \rightarrow \infty$, the term $\coth(t\sqrt{3}\xi) \xrightarrow{t \rightarrow \infty} 1$, we get

$$dI_t^\infty \approx (\bar{\mu} + \sqrt{3}\xi) I_t^\infty dt + \sigma\sqrt{\rho}I_t^\infty dM_t$$

So, in the long run, the limit SDE behaves like the usual geometric brownian motion with constant drift, $\bar{\mu} + \sqrt{3}\xi$, and volatility, $\sigma\sqrt{\rho}$.

We summarise our results as follows:

$\begin{cases} dI_t^\infty = (\bar{\mu} + \xi^2 t) I_t^\infty dt + \sigma\sqrt{\rho}I_t^\infty dM_t \\ dI_t^\infty = (\bar{\mu} + \sqrt{3}\xi \coth(t\sqrt{3}\xi) - \frac{1}{t}) I_t^\infty dt + \sigma\sqrt{\rho}I_t^\infty dM_t \end{cases}$	$\begin{matrix} \text{when } \mu \stackrel{iid}{\sim} N(\mu, \xi^2) \\ \text{when } \mu \stackrel{iid}{\sim} U[a, b] \end{matrix}$
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4.2 Model with random volatility

We assume time-independent weights α_i as before and consider now all parameters to be constant except the volatility σ_i . Again, we consider the volatility of the constituent assets being:

1. normally distributed, $\sigma \stackrel{iid}{\sim} N(m, s^2)$.
2. uniform distributed, $\sigma \stackrel{iid}{\sim} U[a, b]$ where $a, b > 0$

There is obvious pitfalls and drawbacks for our consideration of the individual volatility following a Normal Distribution, i.e. $\sigma^i \stackrel{iid}{\sim} N(m, s^2)$, one of which being the fact that the volatility must be positive. However, this simplification shall ease our discussion of some close-formed results. The idea of considering a uniform distribution is again similar to the case of random drift.

4.3 Study on general volatility conditions

To investigate the general volatility condition, we look at the expression (2.5)

$$I_t^N = \frac{1}{A_N} \sum_{i=1}^N \alpha_i S_0^i e^{(\mu - \frac{1}{2}(\sigma^i)^2)t + \sigma^i \sqrt{\rho} M_t + \sigma^i \sqrt{1-\rho} W_t^i}$$

since $\frac{1}{A_N} \sum_{i=1}^N \alpha_i e^{(-\frac{1}{2}(\sigma^i)^2)t + \sigma^i \sqrt{\rho} M_t + \sigma^i \sqrt{1-\rho} W_t^i} \xrightarrow{N \rightarrow \infty} \mathbb{E}_{\sigma, W_t} \left(e^{-\frac{1}{2}\sigma^2 t + \sigma \sqrt{\rho} M_t + \sigma \sqrt{1-\rho} W_t} \right) = J_t$ using Theorem 2.1.1

$$\therefore I_t^\infty = e^{\mu t} \mathbb{E}(S_0) J_t \quad (4.17)$$

and the fact that S_0^i , σ^i & W_t^i are independent.

By applying Itô's lemma again, we obtain

$$dI_t^\infty = \mu I_t^\infty dt + e^{\mu t} \mathbb{E}(S_0) dJ_t$$

We have, by simplifying,

$$J_t = \mathbb{E}_\sigma \left(e^{-\frac{1}{2}\sigma^2 t + \sigma \sqrt{\rho} M_t + \frac{1}{2}\sigma^2 (1-\rho)t} \right)$$

Using Itô under certain conditions for us to differentiate under the integral, and writing $Y_t = e^{-\frac{1}{2}\sigma^2\rho t + \sigma\sqrt{\rho}M_t}$

$$\begin{aligned} dJ_t &= \mathbb{E}_\sigma \left(-\frac{1}{2}\sigma^2\rho Y_t dt + \sigma\sqrt{\rho}Y_t dM_t + \frac{1}{2}\sigma^2\rho Y_t dt \right) \\ &= \sqrt{\rho}\mathbb{E}_\sigma(\sigma Y_t) dM_t \end{aligned}$$

Therefore, we have in general,

$$\Rightarrow dI_t^\infty = \mu I_t^\infty dt + \sqrt{\rho} \frac{\mathbb{E}_\sigma(\sigma Y_t)}{\mathbb{E}_\sigma(Y_t)} dM_t \quad (4.18)$$

and once again, $\frac{\mathbb{E}_\sigma(\sigma Y_t)}{\mathbb{E}_\sigma(Y_t)} \rightarrow \sigma$ as σ becomes constant, and we recover (3.7).

4.3.1 Normally distributed volatility

For the case $\sigma^i \stackrel{iid}{\sim} N(m, s^2)$, we proceed as before from (4.17). We first evaluate

$$\begin{aligned} J_t &= \mathbb{E} \left(e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t} \right) \\ J_t &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi s\sqrt{t}} e^{-\frac{1}{2}x^2 t + x\sqrt{\rho}M_t + x\sqrt{1-\rho}y} e^{-\frac{(x-m)^2}{2s^2}} e^{-\frac{y^2}{2t}} dx dy \end{aligned}$$

using completing the square on the exponential terms

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi s^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 t + x\sqrt{\rho}M_t} e^{-\frac{(x-m)^2}{2s^2}} e^{\frac{1}{2t}(tx\sqrt{1-\rho})^2} dx \\ &= \frac{1}{\sqrt{1+s^2\rho t}} e^{-\frac{m^2}{2s^2} + \frac{(m+s^2\sqrt{\rho}M_t)^2}{2s^2(1+s^2\rho t)}} \end{aligned} \quad (4.19)$$

Therefore, we have the following

$$I_t^\infty = e^{\mu t} \mathbb{E}(S_0) \frac{1}{\sqrt{1+s^2\rho t}} e^{-\frac{m^2}{2s^2} + \frac{(m+s^2\sqrt{\rho}M_t)^2}{2s^2(1+s^2\rho t)}} \quad (4.20)$$

by applying Itô's lemma again, we obtain

$$\begin{aligned} dI_t^\infty &= \left(\mu - \frac{s^2\rho}{2(1+s^2\rho t)} - \frac{\rho(m+s^2\sqrt{\rho}M_t)^2}{2(1+s^2\rho t)^2} \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{s^2\rho}{1+s^2\rho t} + \frac{\rho(m+s^2\sqrt{\rho}M_t)^2}{(1+s^2\rho t)^2} \right) \right) I_t^\infty dt \\ &\quad + \sqrt{\rho} \left(\frac{m+s^2\sqrt{\rho}M_t}{1+s^2\rho t} \right) I_t^\infty dM_t \\ \Rightarrow dI_t^\infty &= \mu I_t^\infty dt + \sqrt{\rho} \left(\frac{m+s^2\sqrt{\rho}M_t}{1+s^2\rho t} \right) I_t^\infty dM_t \end{aligned} \quad (4.21)$$

Although an unrealistic model it may seem, we still notice that the drift in the limit SDE remains the same as before in (3.7) and it is only the volatility in the limit SDE that changes.

This in turn suggests that the consideration of drift being i.i.d. random variables changes the drift of the SDE of I_t^∞ in the case when all parameters are constant whereas the consideration of volatility being i.i.d. random variables changes the volatility of the SDE of I_t^∞ in the case when all parameters are constant. We shall investigate this further in Chapter 5.

We also note that similar to the model with random drift case, in the limit that $s \rightarrow 0$, when the variance tends to zero, we recover the case when we have all parameters constant, (3.7). I_t^∞ now follows an SDE with a volatility that depends on both the time and the Brownian motion, M_t driven by the Market factor.

4.3.2 Uniformly distributed volatility

We now consider the case when the volatility is uniformly distributed, with $\sigma^i \stackrel{iid}{\sim} U[a, b]$, where a and b are strictly positive. As before, from (4.17),

$$I_t^\infty = e^{\mu t} \mathbb{E}(S_0) J_t$$

where $J_t = \mathbb{E} \left(e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{\rho} M_t + \sigma\sqrt{1-\rho} W_t} \right)$

We try to evaluate the term

$$J_t = \frac{1}{b-a} \int_a^b \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}x^2 t + x\sqrt{\rho} M_t + x\sqrt{1-\rho} y} e^{-\frac{y^2}{2t}} dy dx$$

by completing the square, we get

$$\begin{aligned} &= \frac{1}{b-a} \int_a^b e^{-\frac{1}{2}x^2 t + x\sqrt{\rho} M_t} e^{\frac{1}{2}t(1-\rho)x^2} dx \\ &= \frac{1}{b-a} e^{\frac{1}{2t}M_t^2} \int_a^b e^{-\frac{1}{2}\rho t(x - \frac{M_t}{t\sqrt{\rho}})^2} dx \end{aligned}$$

So, by applying Itô's lemma and simplifying, we get

$$\begin{aligned}
I_t^\infty &= e^{\mu t} \mathbb{E}(S_0) \frac{1}{b-a} e^{\frac{1}{2t} M_t^2} \int_a^b e^{-\frac{1}{2} \rho t (x - \frac{M_t}{t\sqrt{\rho}})^2} dx \\
dI_t^\infty &= \mu I_t^\infty dt + \left(\frac{M_t}{t} I_t^\infty + e^{\mu t} \mathbb{E}(S_0) \frac{1}{b-a} e^{\frac{1}{2t} M_t^2} \frac{1}{t\sqrt{\rho}} \left(e^{-\frac{1}{2} \rho t (a - \frac{M_t}{t\sqrt{\rho}})^2} - e^{-\frac{1}{2} \rho t (b - \frac{M_t}{t\sqrt{\rho}})^2} \right) \right) dM_t \\
&= \mu I_t^\infty dt + \left(\frac{M_t}{t} + \frac{\frac{1}{b-a} e^{\frac{1}{2t} M_t^2} \frac{1}{t\sqrt{\rho}} \left(e^{-\frac{1}{2} \rho t (a - \frac{M_t}{t\sqrt{\rho}})^2} - e^{-\frac{1}{2} \rho t (b - \frac{M_t}{t\sqrt{\rho}})^2} \right)}{\frac{1}{b-a} e^{\frac{1}{2t} M_t^2} \int_a^b e^{-\frac{1}{2} \rho t (x - \frac{M_t}{t\sqrt{\rho}})^2} dx} \right) I_t^\infty dM_t
\end{aligned} \tag{4.22}$$

We see that once again the drift of the limit SDE remains the same as in the case when we have constant volatility, $\sigma^i = \sigma$ and the volatility in the limit SDE now depends on both time and M_t . Writing $s = \frac{1}{2}(b+a)$ and $d = \frac{1}{2}(b-a)$, we see that the numerator of the second term

$$\begin{aligned}
&\frac{1}{2d} e^{\frac{1}{2t} M_t^2} \frac{1}{t\sqrt{\rho}} \left(e^{-\frac{1}{2} \rho t (a - \frac{M_t}{t\sqrt{\rho}})^2} - e^{-\frac{1}{2} \rho t (b - \frac{M_t}{t\sqrt{\rho}})^2} \right) \\
&= \frac{1}{d} e^{\frac{1}{2t} M_t^2} \frac{1}{t\sqrt{\rho}} e^{-\frac{1}{2} \rho t \left((s - \frac{M_t}{t\sqrt{\rho}})^2 + d^2 \right)} \sinh(\rho t (s - \frac{M_t}{t\sqrt{\rho}}) d) \\
&\longrightarrow \sqrt{\rho} (s - \frac{M_t}{t\sqrt{\rho}}) e^{-\frac{1}{2} \rho s^2 t + s \sqrt{\rho} M_t} \quad \text{since } \frac{\sinh(x)}{x} \rightarrow 1
\end{aligned}$$

as $d \rightarrow 0$, $a, b \rightarrow s$ and the denominator

$$\frac{1}{b-a} e^{\frac{1}{2t} M_t^2} \int_a^b e^{-\frac{1}{2} \rho t (x - \frac{M_t}{t\sqrt{\rho}})^2} dx \longrightarrow e^{\frac{1}{2t} M_t^2} e^{-\frac{1}{2} \rho t (s - \frac{M_t}{t\sqrt{\rho}})^2} = e^{-\frac{1}{2} \rho s^2 t + s \sqrt{\rho} M_t}$$

Combining these two results, we see that in the limit that the volatility in the limit SDE becomes constant, we recover (3.7)

$$dI_t^\infty = \mu I_t^\infty dt + s \sqrt{\rho} I_t^\infty dM_t$$

4.4 Model with random correlation

4.4.1 Study on general correlation conditions

Again, we shall start by looking at the general volatility condition arising from considering a random correlation model. As before, by Theorem 2.1.1, we have from (2.5)

$$\begin{aligned}
I_t^\infty &= e^{\mu t} \mathbb{E}(S_0) e^{-\frac{1}{2} \sigma^2 t} \mathbb{E}_{\rho, W_t} \left(e^{\sigma \sqrt{\rho} M_t + \sigma \sqrt{1-\rho} W_t} \right) \\
&= e^{\mu t} \mathbb{E}(S_0) e^{-\frac{1}{2} \sigma^2 t} \mathbb{E}_\rho \left(e^{\sigma \sqrt{\rho} M_t + \frac{1}{2} \sigma^2 (1-\rho) t} \right)
\end{aligned} \tag{4.23}$$

Let $Y_t = e^{\sigma\sqrt{\rho}M_t - \frac{1}{2}\sigma^2\rho t}$ Using Itô's lemma again, with $J_t = \mathbb{E}_\rho(Y_t)$ we shall get

$$\begin{aligned} I_t^\infty &= e^{\mu t} \mathbb{E}(S_0) J_t \\ \Rightarrow dI_t^\infty &= \mu I_t^\infty dt + e^{\mu t} \mathbb{E}(S_0) dJ_t \end{aligned}$$

and

$$dJ_t = \mathbb{E}_r \left(-\frac{1}{2}\sigma^2\rho Y_t dt + \sigma\sqrt{\rho}Y_t dM_t + \frac{1}{2}\sigma^2\rho Y_t dt \right) \quad (4.24)$$

thus giving

$$dI_t^\infty = \mu I_t^\infty dt + \sigma \frac{\mathbb{E}_\rho(\sqrt{\rho}Y_t)}{\mathbb{E}_\rho(Y_t)} I_t^\infty dM_t \quad (4.25)$$

We can actually see that in this form, the drift in the limit SDE is again the same as in the case when we have constant correlation ρ and the volatility now depends on the term $\frac{\mathbb{E}_\rho(\sqrt{\rho}Y_t)}{\mathbb{E}_\rho(Y_t)}$ which again depends on t and M_t .

4.5 Uniformly distributed correlation

We have $\rho^i \in [0, 1)$, so we can consider the case when $\rho^i = (r^i)^2$, whereby r^i follows a uniform distribution, $r^i \stackrel{iid}{\sim} U[a, b]$ whereby $[a, b] \subseteq [-1, 1]$. Denoting $\hat{r} = \frac{1}{2}(b + a)$ and $d = \frac{1}{2}(b - a)$

$$\begin{aligned} J_t &= \mathbb{E}_r(Y_t) \\ &= \frac{1}{2d} \int_a^b e^{\sigma r M_t - \frac{1}{2}\sigma^2 r^2 t} dr \\ &= \frac{1}{2d} e^{\frac{M_t^2}{2t}} \int_a^b e^{-\frac{1}{2}\sigma^2 t (r - \frac{M_t}{\sigma t})^2} dr \end{aligned} \quad (4.26)$$

and

$$\begin{aligned} \mathbb{E}_r(rY_t) &= \frac{1}{\sigma} \frac{\partial}{\partial x} \mathbb{E}_r(Y_t) \quad \text{writing } M_t = x \text{ in } Y_t \\ &= \frac{1}{2d\sigma} \frac{\partial}{\partial x} e^{\frac{x^2}{2t}} \int_a^b e^{-\frac{1}{2}\sigma^2 t (r - \frac{x}{\sigma t})^2} dr \\ &= \frac{x}{\sigma t} J_t + \frac{1}{2d} e^{\frac{x^2}{2t}} \int_a^b (r - \frac{x}{\sigma t}) e^{-\frac{1}{2}\sigma^2 t (r - \frac{x}{\sigma t})^2} dr \end{aligned} \quad (4.27)$$

and after much simplification

$$\mathbb{E}_r(rY_t) = \frac{x}{\sigma t} J_t - \frac{1}{d\sigma^2 t} e^{\frac{x^2}{2t}} e^{-\frac{1}{2}\sigma^2 t ((\hat{r} - \frac{x}{\sigma t})^2 + d^2)} \sinh \left(\left(\hat{r} - \frac{x}{\sigma t} \right) d\sigma^2 t \right) \quad (4.28)$$

and so, we have

$$dI_t^\infty = \mu I_t^\infty dt + \sigma \left(\frac{M_t}{\sigma t} - \frac{\frac{1}{d\sigma^2 t} e^{\frac{x^2}{2t}} e^{-\frac{1}{2}\sigma^2 t \left(\left(\hat{r} - \frac{x}{\sigma t} \right)^2 + d^2 \right)} \sinh \left(\left(\hat{r} - \frac{x}{\sigma t} \right) d\sigma^2 t \right)}{\frac{1}{2d} e^{\frac{M_t^2}{2t}} \int_a^b e^{-\frac{1}{2}\sigma^2 t \left(r - \frac{M_t}{\sigma t} \right)^2} dr} \right) I_t^\infty dM_t \quad (4.29)$$

As $d \rightarrow 0$, a & $b \rightarrow \hat{r}$,

$$\begin{aligned} \frac{1}{d\sigma^2 t} e^{\frac{x^2}{2t}} e^{-\frac{1}{2}\sigma^2 t \left(\left(\hat{r} - \frac{x}{\sigma t} \right)^2 + d^2 \right)} \sinh \left(\left(\hat{r} - \frac{x}{\sigma t} \right) d\sigma^2 t \right) &\rightarrow - \left(\hat{r} - \frac{x}{\sigma t} \right) e^{\frac{x^2}{2t}} e^{-\frac{1}{2}\sigma^2 t \left(\hat{r} - \frac{x}{\sigma t} \right)^2} \\ \frac{1}{2d} e^{\frac{M_t^2}{2t}} \int_a^b e^{-\frac{1}{2}\sigma^2 t \left(r - \frac{M_t}{\sigma t} \right)^2} dr &\rightarrow e^{\frac{x^2}{2t}} e^{-\frac{1}{2}\sigma^2 t \left(\hat{r} - \frac{x}{\sigma t} \right)^2} \end{aligned} \quad (4.30)$$

thereby giving in the limit that $r \rightarrow \hat{r}$ becomes constant, our usual (3.7)

$$dI_t^\infty = \mu I_t^\infty dt + \sigma \hat{r} I_t^\infty dM_t \quad (4.31)$$

Chapter 5

A more general model

Now that we have considered different models, we can summarise our results by considering similar arguments as before when dealing with the general drift, volatility and correlation.

$$I_t^\infty = \mathbb{E} \left(S_0 e^{\mu t} e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{\rho} M_t + \sigma\sqrt{1-\rho} W_t} \right)$$

If S_0^i , μ^i , σ^i and ρ^i are all independent

$$\begin{aligned} dI_t^\infty &= \mathbb{E}(S_0) \mathbb{E}(e^{\mu t}) \mathbb{E} \left(e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{\rho} M_t + \sigma\sqrt{1-\rho} W_t} \right) \\ dI_t^\infty &= \mathbb{E}(S_0) \mathbb{E}_\mu(e^{\mu t}) \mathbb{E}_{\sigma, \rho, W_t} \left(e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{\rho} M_t + \sigma\sqrt{1-\rho} W_t} \right) \\ &= \mathbb{E}(S_0) \mathbb{E}_\mu(e^{\mu t}) \mathbb{E}_{\sigma, \rho} \left(e^{-\frac{1}{2}\sigma^2 \rho t + \sigma\sqrt{\rho} M_t} \right) \end{aligned}$$

by applying Itô's lemma again, we have

$$dI_t^\infty = \frac{\mathbb{E}_\mu(\mu e^{\mu t})}{\mathbb{E}_\mu(e^{\mu t})} dt + \frac{\mathbb{E}_{\sigma, \rho}(\sigma\sqrt{\rho} Y_t)}{\mathbb{E}_{\sigma, \rho}(Y_t)} dM_t \quad (5.1)$$

where $Y_t = e^{-\frac{1}{2}\sigma^2 \rho t + \sigma\sqrt{\rho} M_t}$.

This is the general form for the limit SDE and we can again verify that as all parameters become constant, we recover (3.7).

Chapter 6

Model with time-dependent weights

We now consider the more general model where the weights allocated to each individual stock in a stock market index is time dependent. In the general case, as mentioned earlier, for a capitalisation-weighted index, α_i can be perceived as the number of stocks $\eta_i(t)$. We could have modelled this as a counting process, such as the Poisson process. However, this will not be considered in the following discussion. Instead, we try to look into a model that calculate the weighted average of the stock prices with the weights being their respective market capitalisation. The reason we investigate this crude model is that we hope to look into how an index can be constructed in a way whereby the index is very much dependent upon the market capitalisation of the stock since market capitalisation, in fact, represents the public opinion of a company's net worth and is a determining factor in stock valuation whereas stock prices show the current tradable price of a stock of the company. However, we should point out that this method of calculating an index is not being used in the market and is simply considered here for theoretical reasons.

$$I_t = \sum_{i=1}^N \alpha_i(t) S_i(t) \quad (6.1)$$

where $\alpha_i(t) = \frac{\eta_i S_t^i}{\sum_{j=1}^N \eta_j S_t^j}$. η_i may be perceived as the number of shares of the i th stock and $\eta_i S_t^i$ being the market capitalisation of the i th stock.

We can simplify the above expression to

$$I_t = \sum_{i=1}^N \frac{\eta_i (S_i(t))^2}{A(S_i(t))} \quad (6.2)$$

where $A(S_i(t)) = \sum_{j=1}^N \eta_j S_j(t)$

we have

$$dA = \sum_{j=1}^N \eta_j dS_t^j$$

$$dA = \mu A dt + \sigma \sqrt{1-\rho} \sum_{j=1}^N \eta_j S_t^j dW_t^j + \sigma \sqrt{\rho} A dM_t \quad (6.3)$$

$$\Rightarrow d[A]_t = \sigma^2(1-\rho) \sum_{j=1}^N \eta_j^2 (S_t^j)^2 dt + \sigma^2 \rho A^2 dt, \text{ and} \quad (6.4)$$

$$d[S_i, A] = \sigma^2(1-\rho) \eta_i S_i^2 dt + \sigma^2 \rho A S_i dM \quad (6.5)$$

and then applying Itô's lemma to (6.2), writing S_t^i as S_i .

$$\begin{aligned} dI_t &= \sum_{i=1}^N \frac{1}{A} (2\eta_i S_i dS_i) + \sum_{i=1}^N -\frac{1}{A^2} (\eta_i S_i^2) dA \\ &+ \frac{1}{2} \sum_{i=1}^N \frac{1}{A} (2\eta_i d[S_i]) + \frac{1}{2} \sum_{i=1}^N \left(\frac{2}{A^3} \right) \eta_i S_i^2 d[A] \\ &- \sum_{i=1}^N \frac{2\eta_i S_i}{A^2} d[S_i, A] \end{aligned}$$

substituting in and after some simplification, we obtain

$$\begin{aligned} &= (\mu + \sigma^2(1-\rho)) I_t dt + \sigma^2(1-\rho) \frac{\sum_{i=1}^N \eta_i^2 S_i^2}{A_N^2} I_t dt - 2 \frac{\sigma^2(1-\rho)}{A_N^2} \sum_{i=1}^N \eta_i^3 S_i^3 dt \\ &+ \sigma \sqrt{\rho} dM_t + 2 \frac{\sigma \sqrt{1-\rho}}{A_N} \sum_{i=1}^N \eta_i S_i^2 dW_t^i \\ &- \frac{\sigma \sqrt{1-\rho} I_t}{A_N} \sum_{i=1}^N \eta_i S_i dW_t^i \quad (6.6) \end{aligned}$$

As we take the limit as $N \rightarrow \infty$, we notice that the last two terms tend to zero by Theorem 3.1.1 as before. However, we see that the above expression is not as simple as the ones we obtained before. The main difficulty here lies with the fact that we factored in the price of the stock into the weights, α_i . This resulted in us having a quadratic S_i factor in the expression for the index. This particular case exhibits the need for a careful choice of weights, not just from economic theory, but also looking at it from the mathematical point of view.

Chapter 7

Empirical density

In this chapter, we shall look at the empirical measure. (See [2, 13] for reference) We will begin by describing the system (2.2) by a measure valued process and showing that there is a limit empirical measure for the infinite system. We will also show that its evolution can be captured by an SPDE.

7.1 The limit empirical density

We proceed in a similar fashion as shown in [5, 6]. However, this time, we shall try to solve the Stochastic PDE for the empirical density and derive further results.

Using our previous definitions as set up in Section 2.2 with time-independent weights $\alpha_i(t) = \alpha_i$, we define the empirical measure

$$\nu_{N,t}^\alpha = \frac{1}{A_N} \sum_{i=1}^N \alpha_i \delta_{S_t^i} \quad (7.1)$$

where $\delta_{S_t^i}$ is the Dirac Delta distribution in S_t^i . In distributional terms, $\delta_{S_t^i}$ is a linear functional on $C_0^\infty(\mathbb{R})$, the space of the infinitely differentiable continuous functions with compact support, such that for $\phi \in C_0^\infty(\mathbb{R})$

$$\langle \delta_{S_t^i}, \phi \rangle = \phi(S_t^i)$$

For a measure ν_t and integrable function ϕ we write

$$\langle \phi, \nu_t \rangle = \int \phi(x) \nu_t(dx) \quad (7.2)$$

Using the empirical measure (7.1), the action of $\nu_{N,t}^\alpha$ on ϕ is given by

$$F_t^{N,\phi} = \langle \nu_{N,t}^\alpha, \phi \rangle = \frac{1}{A_N} \sum_{i=1}^N \alpha_i \phi(S_t^i) \quad (7.3)$$

Applying Itô's formula to $F_t^{N,\phi}$, we have:

$$\begin{aligned}
F_t^{N,\phi} - F_0^{N,\phi} &= \frac{1}{A_N} \sum_{i=1}^N \alpha_i \int_0^t \left(\phi'(S_u^i) dS_u^i + \frac{1}{2} \phi''(S_u^i) d[S_u^i] \right) \\
&= \frac{1}{A_N} \sum_{i=1}^N \alpha_i \int_0^t \left[\phi'(S_u^i) \mu^i S_u^i du + \phi'(S_u^i) \sigma^i \sqrt{1 - \rho^i} S_u^i dW_u^i + \phi'(S_u^i) \sigma^i \sqrt{\rho^i} S_u^i dM_u \right. \\
&\quad \left. + \frac{1}{2} \phi''(S_u^i) (\sigma^i)^2 (S_u^i)^2 du \right] \\
\Rightarrow F_t^{N,\phi} &= F_0^{N,\phi} + \int_0^t \left[\langle \nu_{N,u}^{\alpha\mu}, x\phi' \rangle du + \frac{1}{2} \langle \nu_{N,u}^{\alpha\sigma^2}, x^2\phi'' \rangle du + \langle \nu_{N,u}^{\alpha\sigma\sqrt{\rho}}, x\phi' \rangle dM_u \right] \\
&\quad + \int_0^t \frac{1}{A_N} \sum_{i=1}^N \alpha_i \phi'(S_u^i) \sigma^i \sqrt{1 - \rho^i} S_u^i dW_u^i
\end{aligned} \tag{7.4}$$

We now pass to the limit by letting $N \rightarrow \infty$.

In order to see this, we focus on the idiosyncratic term in (7.4)

$$\int_0^t \frac{1}{A_N} \sum_{i=1}^N \alpha_i \phi'(S_u^i) \sigma^i \sqrt{1 - \rho^i} S_u^i dW_u^i \tag{7.5}$$

This term goes to zero as N goes to infinity. The proof is very similar to the proof laid out for Theorem 3.1.1.

We note that as ϕ', ϕ'' are bounded and $\nu_{N,u}^\alpha$ is a probability measure, we can apply the dominated convergence theorem to take the limit under the integrals in the other terms in (7.4). We summarise our result in the following.

Theorem 7.1.1. *The sequence of empirical measures $\nu_{N,u}^\alpha$ on $(0, \infty)$ satisfies for all $\phi \in C_0^\infty$,*

$$F_t^{N,\phi} \longrightarrow F_t^\phi = \langle \nu_t^\alpha, \phi \rangle \text{ as } N \longrightarrow \infty, \text{ a.s.}$$

The evolution of the limit empirical measure in the weak sense is given by

$$\langle \nu_t^\alpha, \phi \rangle = \langle \nu_0^\alpha, \phi \rangle + \int_0^t \left[\left(\langle \nu_u^{\alpha\mu}, x\phi' \rangle + \frac{1}{2} \langle \nu_u^{\alpha\sigma^2}, x^2\phi'' \rangle \right) du + \langle \nu_u^{\alpha\sigma\sqrt{\rho}}, x\phi' \rangle dM_u \right] \tag{7.6}$$

In order to recharacterise the evolution obtained in (7.4) as a stochastic PDE, we need the measure ν_t to be absolutely continuous with respect to the Lebesgue measure to write $\nu_t(dx) = \nu(t, x)dx$ for some density ν . For the following discussion, we shall assume the uniqueness and existence of this density.

Substituting the Lebesgue representation for the empirical measure in to (7.4), and integrating by parts, we get

$$\begin{aligned}
\int \phi(x) \nu^\alpha(x, t) dx &= \int \phi(x) \nu^\alpha(x, 0) dx + \int_0^t \int \left(\nu^{\alpha\mu}(x, u) x \phi'(x) + \frac{1}{2} \nu^{\alpha\sigma^2}(x, u) x^2 \phi''(x) \right) dx du \\
&\quad + \int_0^t \int \nu^{\alpha\sigma\sqrt{\rho}}(x, u) x \phi'(x) dx dM_u \\
&= \int \phi(x) \nu^\alpha(x, 0) dx \\
&\quad + \int_0^t \int \left(-\phi(x) \frac{\partial}{\partial x} (x \nu^{\alpha\mu}(x, u)) + \frac{1}{2} \phi(x) \frac{\partial^2}{\partial x^2} (x^2 \nu^{\alpha\sigma^2}(x, u)) \right) dx du \\
&\quad - \int_0^t \int \phi(x) \frac{\partial}{\partial x} (x \nu^{\alpha\sigma\sqrt{\rho}}(x, u)) dx dM_u \\
&= \int \phi(x) \left(\nu^\alpha(x, 0) + \int_0^t \left(-\frac{\partial}{\partial x} (x \nu^{\alpha\mu}(x, u)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (x^2 \nu^{\alpha\sigma^2}(x, u)) \right) du \right. \\
&\quad \left. - \int_0^t \frac{\partial}{\partial x} (x \nu^{\alpha\sigma\sqrt{\rho}}(x, u)) dM_u \right) dx
\end{aligned}$$

As this holds for $\forall \phi \in C_0^\infty$ we have shown that we have a weak solution to the SPDE given by

$$\begin{aligned}
\nu^\alpha(x, t) &= \nu^\alpha(x, 0) + \int_0^t \left(-\frac{\partial}{\partial x} (x \nu^{\alpha\mu}(x, u)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (x^2 \nu^{\alpha\sigma^2}(x, u)) \right) du \\
&\quad - \int_0^t \frac{\partial}{\partial x} (x \nu^{\alpha\sigma\sqrt{\rho}}(x, u)) dM_u
\end{aligned} \tag{7.7}$$

with $\nu^\alpha(t, 0) = 0$ for all $t \in [0, T]$ and the initial condition

$$\nu^\alpha(x, 0) = \frac{1}{A_N} \sum_{i=1}^N \alpha_i \delta_{S_0^i}(x) \tag{7.8}$$

Alternatively, we can write this in its differential form

$$d\nu^\alpha(x, t) = \left(-\frac{\partial}{\partial x} (x \nu^{\alpha\mu}(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (x^2 \nu^{\alpha\sigma^2}(x, t)) \right) dt - \frac{\partial}{\partial x} (x \nu^{\alpha\sigma\sqrt{\rho}}(x, t)) dM_t \tag{7.9}$$

Hence, we also have the approximation

$$I_t^N \sim A_N \langle \nu_t^\alpha, x \rangle = A_N \int_{\mathbb{R}} x \nu(x, t) dx \tag{7.10}$$

7.2 Solving the SPDE

7.2.1 Model with constant parameters and fixed weights

As before, we consider the case when all assets have the same constant volatility, drift and are correlated via a single market factor. The weight factor is also taken to be time-independent. Then, (7.9) reduces to

$$d\nu(x, t) = \left(-\mu \frac{\partial}{\partial x} (x\nu(x, t)) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} (x^2 \nu(x, t)) \right) dt - \sigma \sqrt{\rho} \frac{\partial}{\partial x} (x\nu(x, t)) dM_t \quad (7.11)$$

We can solve this SPDE by considering in the following:

Let $\nu(x, t) = Y_t f(t, xY_t)$ where $Y_t = e^{-\sigma \sqrt{\rho} M_t}$,

$$\begin{aligned} dY_t &= -\sigma \sqrt{\rho} Y_t dM_t \\ d[Y]_t &= \sigma^2 \rho Y_t^2 dt \end{aligned}$$

Using Itô's lemma, with $z = xY_t$

$$\begin{aligned} d\nu(x, t) &= (f + xY_t \partial_z f) dY_t + (Y_t \partial_t f) dt + \frac{1}{2} (2x \partial_z f + x^2 Y_t \partial_{zz} f) d[Y]_t \\ &= Y_t \left(\partial_t f + \sigma^2 \rho x Y_t \partial_z f + \frac{1}{2} \sigma^2 \rho x^2 Y_t^2 \partial_{zz} f \right) dt - \sigma \sqrt{\rho} Y_t (f + xY_t \partial_z f) dM_t \end{aligned} \quad (7.12)$$

We can match (7.12) and (7.11)

$$\begin{aligned} \frac{\partial}{\partial x} (x\nu(x, t)) &= \frac{\partial}{\partial x} (zf(t, z)) = Y_t \frac{\partial}{\partial z} (zf) = Y_t f + zY_t \partial_z f \\ \frac{\partial^2}{\partial x^2} (x^2 \nu(x, t)) &= \frac{\partial^2}{\partial z^2} (x^2 Y_t f) (Y_t)^2 = Y_t \frac{\partial^2}{\partial z^2} (z^2 f) \end{aligned}$$

The volatilities are matched and now we look at the drift condition

$$\begin{aligned} -\mu \frac{\partial}{\partial x} (x\nu(x, t)) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} (x^2 \nu(x, t)) &= Y_t \left(-\mu \frac{\partial}{\partial z} (zf) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial z^2} (z^2 f) \right) \\ \therefore -\mu \frac{\partial}{\partial z} (zf) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial z^2} (z^2 f) &= \partial_t f + \frac{1}{2} \sigma^2 \rho z (2\partial_z f + z\partial_{zz} f) \\ &= \partial_t f + \frac{1}{2} \sigma^2 \rho z \frac{\partial^2}{\partial z^2} (zf) \end{aligned} \quad (7.13)$$

Rearranging,

$$\frac{\partial f}{\partial t} = -\mu \frac{\partial}{\partial z} (zf) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial z^2} (z^2 f) - \frac{1}{2} \sigma^2 \rho z \frac{\partial^2}{\partial z^2} (zf)$$

since $\frac{\partial^2}{\partial z^2}(z^2 f) = z \frac{\partial^2}{\partial z^2}(zf) + 2 \frac{\partial}{\partial z}(zf)$

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\mu \frac{\partial}{\partial z}(zf) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial z^2}(z^2 f) - \frac{1}{2} \sigma^2 \rho \left(\frac{\partial^2}{\partial z^2}(z^2 f) - 2 \frac{\partial}{\partial z}(zf) \right) \\ \therefore \frac{\partial f}{\partial t} &= (-\mu + \sigma^2 \rho) \frac{\partial}{\partial z}(zf) + \frac{1}{2} \sigma^2 (1 - \rho) \frac{\partial^2}{\partial z^2}(z^2 f) \end{aligned} \quad (7.14)$$

This is the *Kolmogorov Forward Equation* or *Fokker-Planck Equation* which we can solve (see page 78 of [3]).

Consider the SDE

$$dX_t = (\mu - \sigma^2 \rho) dt + \sigma \sqrt{1 - \rho} dW_t \quad (7.15)$$

X_t has log-normal distribution and we can easily write down the transition density as the following

$$p(s, y; t, z) = \frac{1}{z \sqrt{2\pi \sigma^2 (1 - \rho)(t - s)}} \exp \left(-\frac{\left(\log \left(\frac{z}{y} \right) - (\mu - \frac{1}{2} \sigma^2)(t - s) \right)^2}{2\sigma^2 (1 - \rho)(t - s)} \right) \quad (7.16)$$

Therefore we can write our solution to the SPDE (7.11) as

$$\begin{aligned} \nu(x, t) &= Y_t f(t, x Y_t) \text{ where } Y_t = e^{-\sigma \sqrt{\rho} M_t} \text{ and} \\ f(t, x Y_t) &= \frac{1}{x Y_t \sqrt{2\pi \sigma^2 (1 - \rho)(t - s)}} \exp \left(-\frac{\left(\log \left(\frac{x Y_t}{y} \right) - (\mu - \frac{1}{2} \sigma^2)(t - s) \right)^2}{2\sigma^2 (1 - \rho)(t - s)} \right) \end{aligned} \quad (7.17)$$

$$\Rightarrow \nu(x, t) = \frac{1}{x \sqrt{2\pi \sigma^2 (1 - \rho)(t - s)}} \exp \left(-\frac{\left(\log \left(\frac{x}{y} \right) - \sigma \sqrt{\rho} M_t - (\mu - \frac{1}{2} \sigma^2)(t - s) \right)^2}{2\sigma^2 (1 - \rho)(t - s)} \right) \quad (7.18)$$

Alternatively, we could have proceeded from (7.3). Under suitable conditions as mentioned in our Theorem 2.1.1, we have

$$\langle \nu_{N,t}, \phi \rangle = \frac{1}{A_N} \sum_{i=1}^N \alpha_i \phi(S_t^i) \xrightarrow{N \rightarrow \infty} \mathbb{E}(\phi(S_t)) = \int \nu(x, t) \phi(x) dx \quad (7.19)$$

We substituted (2.4) with constant parameters into the LHS of the equation

$$\mathbb{E}(\phi(S_t)) = \mathbb{E}_{S_0, W_t} \left[\phi \left(S_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma \sqrt{\rho} M_t + \sigma \sqrt{1 - \rho} W_t} \right) \right]$$

consider first the case if $S_0 = y$ is a constant

$$\begin{aligned} &= \int_{-\infty}^{\infty} \phi \left(y e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma \sqrt{\rho} M_t + \sigma \sqrt{1 - \rho} z} \right) f_{W_t}(z) dz \\ &= \int_{-\infty}^{\infty} \phi \left(y e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma \sqrt{\rho} M_t + \sigma \sqrt{1 - \rho} z} \right) f_{W_t}(z) dz \\ &= \int_{-\infty}^{\infty} \phi(x) f_{W_t} \left(\frac{\log(x/y) - (\mu - \frac{1}{2} \sigma^2)t - \sigma \sqrt{\rho} M_t}{\sigma \sqrt{1 - \rho}} \right) \frac{1}{x \sigma \sqrt{1 - \rho}} dx \end{aligned}$$

where f_{W_t} is the probability density function for W_t .

Therefore, by comparing with (7.19), we see that we have

$$\begin{aligned}\nu(x, t) &= \frac{1}{x\sigma\sqrt{1-\rho}} f_{W_t} \left(\frac{\log(x/y) - (\mu - \frac{1}{2}\sigma^2)t - \sigma\sqrt{\rho}M_t}{\sigma\sqrt{1-\rho}} \right) \\ &= \frac{1}{x\sigma\sqrt{1-\rho}} \frac{1}{\sqrt{2\pi t}} \exp \left\{ -\frac{1}{2t} \left(\frac{\log(x/y) - (\mu - \frac{1}{2}\sigma^2)t - \sigma\sqrt{\rho}M_t}{\sigma\sqrt{1-\rho}} \right)^2 \right\} \quad (7.20)\end{aligned}$$

which agrees with (7.18) with initial time $s = 0$.

7.2.2 General Model

We can consider the following transformation

$$U_t(x, \theta) = \sum_{i,j,k} \frac{\theta_1^i \theta_2^j \theta_3^k}{i!j!k!} \nu^{\alpha\mu^i \sigma^{2j}(\sigma\sqrt{\rho})^k}(x, t)$$

Using (7.9), we get

$$\begin{aligned}d\nu^{\alpha\mu^i \sigma^{2j}(\sigma\sqrt{\rho})^k} &= \left(-\frac{\partial}{\partial x} \left(x\nu^{\alpha\mu^{i+1} \sigma^{2j}(\sigma\sqrt{\rho})^k}(x, t) \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(x^2 \nu^{\alpha\mu^i \sigma^{2(j+1)}(\sigma\sqrt{\rho})^k}(x, t) \right) \right) dt \\ &\quad - \frac{\partial}{\partial x} \left(x\nu^{\alpha\mu^i \sigma^{2j}(\sigma\sqrt{\rho})^{k+1}}(x, t) \right) dM_t \\ dU_t &= \sum_{i,j,k} \frac{\theta_1^i \theta_2^j \theta_3^k}{i!j!k!} d\nu^{\alpha\mu^i \sigma^{2j}(\sigma\sqrt{\rho})^k}(x, t) \\ &= \left(-\frac{\partial}{\partial x} \left(x \frac{\partial U}{\partial \theta_1} \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(x^2 \left(\frac{\partial U}{\partial \theta_2} \right) \right) \right) dt - \frac{\partial}{\partial x} \left(x \frac{\partial U}{\partial \theta_3} \right) dM_t \quad (7.21)\end{aligned}$$

Next, we take Laplace transform with respect to $x \rightarrow \psi$ with $U(x, 0) = 0$, we get

$$d\hat{U} = \left(-\psi \partial_{\psi\theta_1} \hat{U} + \frac{1}{2} \psi^2 \partial_{\psi\psi\theta_2} \hat{U} \right) dt - \psi \partial_{\psi\theta_3} \hat{U} dM_t \quad (7.22)$$

There are a few approaches to analyse this SPDE. However, it is not clear at this moment if we would obtain a solution from this SPDE.

Chapter 8

Numerical result

In order to investigate further some of our theoretical results, we shall consider performing some numerical simulations. In what follows, we shall work with a *price weighted index*, taking $I_t^N = \frac{1}{N} \sum_{i=1}^N S_t^i$. We will calculate the European call option price in the physical measure (investor's perception of the call price value) using with our limit SDE and compare it with the case (real index) when we generate the individual stock price paths to calculate the index price directly.

We will be mainly concerned with pricing the model with random drifts (normal) owing to the relative ease of their limit SDE.

$$I_t^\infty = I_0^\infty \exp(\bar{\mu}t + \frac{1}{2}\bar{\xi}^2 t^2 - \frac{1}{2}\sigma^2 \rho t) \exp(\sigma \sqrt{\rho} M_t) \text{ when } \mu \stackrel{iid}{\sim} N(\mu, \xi^2)$$

A European call option on the index with maturity T and strike K has the payoff at T given by

$$\max(I_T^N - K, 0) = \max\left(\left(\frac{1}{N} \sum_{i=1}^N S_0 e^{(\mu^i - \frac{1}{2}\sigma^2)t + \sigma^i \sqrt{\rho} M_t + \sigma \sqrt{1-\rho} W_t^i}\right) - K, 0\right)$$

For the following simulation, we assume the risk-free rate, $r = 0.08$. We will be using Monte Carlo method with 10,000 simulations, unless otherwise stated. $S_0^i = 50$, $\forall i = 1, \dots, N$, $\sigma = 0.3$, $\rho = 0.5$, $T = 1$

8.0.3 Model with normally distributed drift

N	100	200	300	400	500	600	700	800	900	1000
Real Index	6.9591	6.8976	6.9351	7.1403	6.9129	7.1169	7.0929	7.0002	6.9402	7.1368
Approx	7.1659	7.2121	7.2322	6.8855	6.9883	7.0323	7.0288	7.0320	6.9832	7.1396
Error	0.2067	0.3145	0.2971	0.2548	0.0755	0.0847	0.0640	0.0318	0.0430	0.0028

Table 8.1: “At the money call price” in the physical measure and error between the Real Index and Limit SDE approximation, $K = 50$

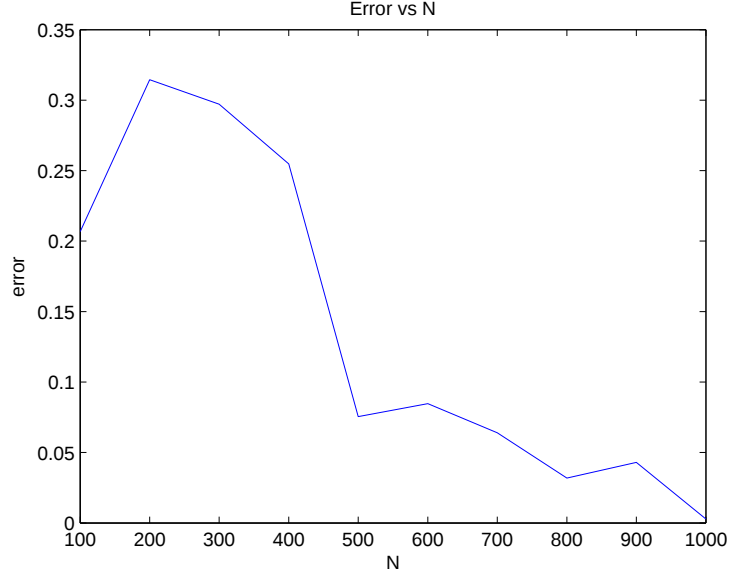


Figure 8.1: Plot of error against N for the random drift case with $\mu \sim (0.1, 0.05^2)$

We see that although the error started off quite large, it soon decreases as N gets larger and eventually, it is converging to 0, as we would expect.

8.0.4 Model with constant drift

In the following simulations, we continue taking $S_0^i = 50$, $\forall i = 1, \dots, N$, and $\sigma = 0.3$, $\rho = 0.5$, $T = 1$, so as to give us a comparison to the result above as well.

N	100	200	300	400	500	600	700	800	900	1000
Real Index	6.8695	7.1021	6.8994	6.9566	7.0467	6.8221	7.0121	7.0760	6.8686	6.9814
Approx	7.0305	7.1268	6.9727	6.9403	7.0237	6.7593	6.9620	7.0501	6.8421	6.9876
Error	0.1610	0.0247	0.0733	0.0163	0.0230	0.0628	0.0500	0.0259	0.0265	0.0061

Table 8.2: “At the money call price” in the physical measure and error between the Real Index and Limit SDE approximation, $K = 50$

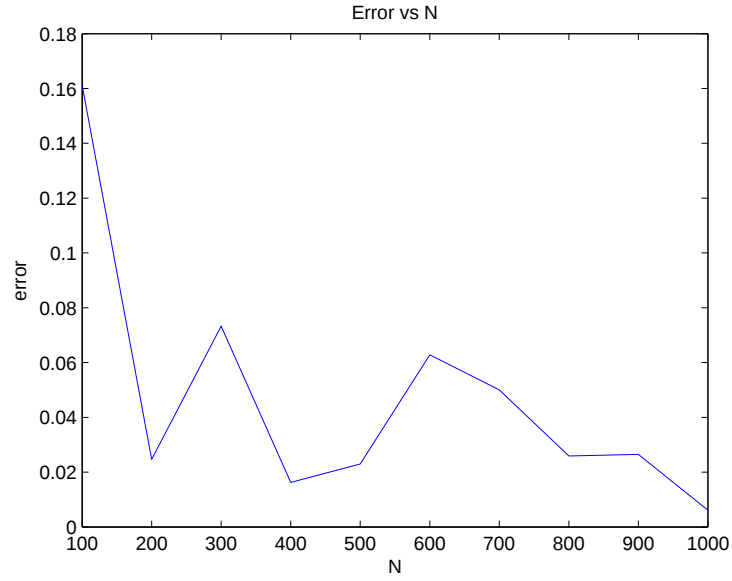


Figure 8.2: Plot of error against N for constant μ

In the case when we have constant drift, we should expect the convergence to be quicker as there is less randomness in the SDE. As we can see from comparing Table 8.1 and 8.2. The errors in the constant drift case is generally smaller than the random drift case. Although there is more fluctuations in error when N is small, this is most likely due to numerical error or the use of Monte Carlo method. Therefore, we do see a nice convergence to the call price on the ‘real’ index value for both the random drift and constant drift case.

Chapter 9

Index options

We shall focus on pricing some common index options using our limit SDE or empirical distribution $\nu(x, t)$. We shall continue working with a *price weighted index*, taking $I_t^N = \frac{1}{N} \sum_{i=1}^N S_t^i$

9.1 European call & put option on the index

We shall focus on pricing the model with random drifts.

$$\begin{aligned} dI_t^\infty &= (\bar{\mu} + \xi^2 t) I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t \text{ when } \mu \stackrel{iid}{\sim} N(\mu, \xi^2) \\ dI_t^\infty &= \left(\bar{\mu} + \sqrt{3} \xi \coth(t\sqrt{3}\xi) - \frac{1}{t} \right) I_t^\infty dt + \sigma \sqrt{\rho} I_t^\infty dM_t \text{ when } \mu \stackrel{iid}{\sim} U[a, b] \end{aligned}$$

They both follow an SDE with time dependent drift but time independent volatility. Since we can always use the change of measure (by Girsanov's Theorem), we can price the options using Black-Scholes formula. Therefore the call option price at initial time $t = 0$ is

$$V = I_0^\infty \Phi(d_+) - K e^{-rT} \Phi(d_-) \quad (9.1)$$

where

$$d_\pm = \frac{\log(\frac{I_0^\infty}{K}) + (r \pm \frac{\sigma^2 \rho}{2})T}{\sigma \sqrt{\rho T}} \quad \text{and} \quad I_0^\infty = \frac{1}{N} \sum_{i=1}^N S_0^i$$

The put option price can be calculated similarly

$$V = K e^{-rT} \Phi(-d_-) - I_0^\infty \Phi(-d_+) \quad (9.2)$$

9.2 European call option on the maximum or minimum of the constituent stock of index

We will now consider pricing a European call option on the maximum or minimum of the index with maturity T and strike K , that is to say that the payoff at T is given by

$$(\max(S_1^N, \dots, S_T^N) - K)^+ \quad \text{or} \quad (\min(S_1^N, \dots, S_T^N) - K)^+$$

For the case of independent assets, there are closed form solutions for such options [11]. However, we shall try to derive some results based on our model, making use of our empirical density. The idea is that once we have an our empirical density, the proportions of observations between an interval (a, b) is approximated by $\int_a^b f(x) dx$, where $f(x)$ is the empirical distribution (see Chapter 4, [13]). Thus we can divide our empirical distribution into portions with an equal area of $\frac{1}{N}$ under the graph, each corresponding to the proportion of stocks (in this case 1 stock). Mathematically, denoting F^{-1} as the inverse cumulative distribution function, we can write the k th order order statistic as $S_{(k)}$ (c.f. order statistics),

$$S_{(k+1)} \in \left(F^{-1} \left(\frac{k-1}{N} \right), F^{-1} \left(\frac{k}{N} \right) \right)$$

then we can approximate it by picking the midpoint of the interval

$$S_{(k+1)} \approx F^{-1} \left(\frac{2k-1}{2N} \right)$$

therefore, for the stock with the maximum or minimum of stock price, that is when $k = N - 1$ or $k = 1$

$$\begin{aligned} S_{(N)} &\approx F^{-1} \left(\frac{2(N-1)-1}{2N} \right) = F^{-1} \left(1 - \frac{1}{2N} \right) \\ S_{(1)} &\approx F^{-1} \left(\frac{1}{2N} \right) \end{aligned}$$

Let us define $\bar{S}_N = \max(S_1^N, \dots, S_T^N)$ and $\underline{S}_N = \min(S_1^N, \dots, S_T^N)$. Then, we can approximate the maximum $\bar{S}_N$ or minimum $\underline{S}_N$ by

$$\int_{\bar{S}_N}^{\infty} \nu(x, t) dx = \frac{1}{2N} \quad \text{and} \quad \int_{-\infty}^{\underline{S}_N} \nu(x, t) dx = \frac{1}{2N} \quad (9.3)$$

Log-normal initial stock price with constant parameters

To ease our discussion further, we will continue from (7.19) but this time making another assumption that S_0^i each follows a log-normal distribution, that is

$$S_0^i \stackrel{iid}{\sim} e^X, \text{ where } X \sim N(\mu_0, \sigma_0^2)$$

So, we have again from (7.19)

$$\begin{aligned} \mathbb{E}(\phi(S_t)) &= \mathbb{E}_{S_0, W_t} \left[\phi \left(S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t} \right) \right] \\ &= \mathbb{E}_{X, W_t} \left[\phi \left(e^X e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t} \right) \right] \\ &= \mathbb{E}_{X, W_t} [\phi(e^Z)] \end{aligned} \quad (9.4)$$

where $Z = X + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sigma\sqrt{1-\rho}W_t$, that is to say Z follows a normal distribution, $Z \sim N(\mu_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t, \sigma_0^2 + \sigma^2(1-\rho)t)$

$$\begin{aligned} &= \int \phi(e^\xi) f_Z(\xi) d\xi \\ &= \int \phi(x) f_Z(\log x) \frac{1}{x} dx \end{aligned} \quad (9.5)$$

where f_Z is the probability density function of Z .

Comparing with the RHS of (7.19), then we get

$$\nu(x, t) = \frac{1}{x} f_Z(\log x) \quad (9.6)$$

Then, we see that for $\bar{S}_N$,

$$\begin{aligned} &\int_{\bar{S}_N}^\infty \frac{1}{x} f_Z(\log x) dx = \frac{1}{2N} \\ &\Rightarrow \int_{\log \bar{S}_N}^\infty f_Z(y) dy = \frac{1}{2N} \\ &\Rightarrow 1 - \Phi \left(\frac{\log \bar{S}_N - (\mu_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t)}{\sqrt{\sigma_0^2 + \sigma^2(1-\rho)t}} \right) = \frac{1}{2N} \end{aligned}$$

$$\Rightarrow \log \bar{S}_N = \mu_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sqrt{\sigma_0^2 + \sigma^2(1-\rho)t} \left(\Phi^{-1} \left(1 - \frac{1}{2N} \right) \right)$$

so we have explicitly the expression for $\bar{S}_N$

$$\bar{S}_N = \exp \left(\mu_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{\rho}M_t + \sqrt{\sigma_0^2 + \sigma^2(1-\rho)t} \left(\Phi^{-1} \left(1 - \frac{1}{2N} \right) \right) \right) \quad (9.7)$$

Similarly, for $\underline{S}_N$,

$$\underline{S}_N = \exp \left(\mu_0 + \left(\mu - \frac{1}{2}\sigma^2 \right)t + \sigma\sqrt{\rho}M_t + \sqrt{\sigma_0^2 + \sigma^2(1-\rho)t} \left(\Phi^{-1} \left(\frac{1}{2N} \right) \right) \right) \quad (9.8)$$

This also imply that

$$\frac{d\bar{S}_N}{\bar{S}_N} = \left(\mu - \frac{1}{2}\sigma^2(1-\rho) + \frac{\sigma^2(1-\rho)}{2\sqrt{\sigma_0^2 + \sigma^2(1-\rho)t}} \Phi^{-1} \left(1 - \frac{1}{2N} \right) \right) dt + \sigma\sqrt{\rho} dM_t \quad (9.9)$$

and

$$\frac{d\underline{S}_N}{\underline{S}_N} = \left(\mu - \frac{1}{2}\sigma^2(1-\rho) + \frac{\sigma^2(1-\rho)}{2\sqrt{\sigma_0^2 + \sigma^2(1-\rho)t}} \Phi^{-1} \left(\frac{1}{2N} \right) \right) dt + \sigma\sqrt{\rho} dM_t \quad (9.10)$$

We see that $\bar{S}_N$, and similarly $\underline{S}_N$, follow SDEs with time dependent drift but time independent volatility. By pricing under the risk-neutral measure, we have the usual Black-Scholes formula for the European call price for both maximum and minimum of the constituent stocks of the index at initial time, $t = 0$ to be

$$\bar{V} = \bar{S}_N(0)\Phi(d_+) - Ke^{-rT}\Phi(d_-) \quad (9.11)$$

where

$$d_{\pm} = \frac{\log(\frac{\bar{S}_N(0)}{K}) + (r \pm \frac{\sigma^2\rho}{2})T}{\sigma\sqrt{\rho T}} \quad \text{and} \quad \bar{S}_N(0) = e^{\mu_0 + \sigma_0^2\Phi^{-1}(1-\frac{1}{2N})}$$

and

$$\underline{V} = \underline{S}_N(0)\Phi(d_+) - Ke^{-rT}\Phi(d_-) \quad (9.12)$$

where

$$d_{\pm} = \frac{\log(\frac{\underline{S}_N(0)}{K}) + (r \pm \frac{\sigma^2\rho}{2})T}{\sigma\sqrt{\rho T}} \quad \text{and} \quad \underline{S}_N(0) = e^{\mu_0 + \sigma_0^2\Phi^{-1}(\frac{1}{2N})}$$

The put option price can be obtained as a Black-Scholes type formula via similar arguments.

Before we end this section, we just note that if the maximum/minimum is not tradable, the model is incomplete, i.e. you cannot hedge all the risk, as there are $N+1$ Brownian motion (noises) driving the N assets - thus there should be a market price of risk appearing in the calculations.

Chapter 10

Conclusion

Through the above analysis, we first presented and developed more on the limit SDE for the model with constant parameters. Later on, we then moved on to consider models with random parameters and obtain similar limit SDEs in those cases. We also summarised our result with the consideration of a more general model. Next, we considered a model with time-dependent weights. In Chapter 8, we tried to compare numerically the accuracy of our limit SDE. In addition to these, we also tried to look at the empirical density by considering this infinite system of SDEs. With constant parameters and some additional assumptions, we were able to obtain an explicit solution to this empirical density. Lastly, we try to price some index options using our knowledge of the limit SDE and the empirical density.

Since we have obtained an explicit formula for our empirical density for the constant parameter case, a natural extension of this is to find the solution to the SPDE for the general case (7.22). Also, we could use our empirical density to price more exotic options, such as the mountain range options, which are based on a basket of stocks. One of which is the Himalayan option where the payout is based on the average performance of the best-performing constituent stock in the basket on specified dates. The number of components in the Basket reduces over the life of the option as the best performer on each calculation date is removed for future calculations. The idea to price this option using the empirical density would be to remove a portion of the density corresponding to the removal of a stock and use (7.10) to price it.

In this simplified model of the index that we have considered here, we conclude that we could obtain several mathematically elegant and explicit results for the different cases that we consider. These formulas gave us more insights into the model that we are considering and provided a good approximation to the real index. Nevertheless,

the practicality of these approximations are being neglected for the moment during our consideration. There are still much work to be done in this aspect, especially in the numerical work to see the convergence rate of our limit SDE to the real index.

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